Determinant Theory of Quadri-Partitioned Neutrosophic Fuzzy Matrices and its Application to Multi-Criteria Decision-Making Problems

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Abstract: We explore the determinant theory for Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs), investigating their properties. In this study, we establish that $\det(Padj(P)) = \det(P) = \det(adj(P)P)$. Additionally, we propose a refined method to compute the determinant of matrices with a higher number of rows and columns. Furthermore, an algorithm is developed to address decision-making problems based on QPNFMs. An illustrative example is provided to demonstrate the effectiveness of the proposed method.

Keywords: Quadri Partitioned Neutrosophic fuzzy sets , Adjoint, Determinant, Decision-Making.

1. Introduction

The introduction of fuzzy sets by Zadeh [1] provided a foundational framework to handle uncertainty by representing degrees of membership. This framework has since evolved into a crucial tool for managing imprecise information. Zadeh later advanced this concept by introducing linguistic variables to facilitate approximate reasoning [3]. Pawlak [2] further contributed to the field with rough set theory, offering an alternative approach for representing uncertainty through boundary regions. Atanassov [4] extended fuzzy set theory by proposing IFSs, incorporating an additional parameter to capture the degree of hesitation, which is not present in traditional fuzzy sets. He explored various theoretical aspects of IFS, including applications [5] and implications, such as fuzzy modus ponens and types of negations [6, 10]. Anandhkumar,et al [7,8,9] have studied Pseudo Similarity of NFM, On various Inverse of NFM, Reverse Sharp and Left-T Right-T Partial Ordering on NFM. These advancements have paved the way for a deeper understanding of uncertainty in decision-making contexts. Research has also applied fuzzy and intuitionistic fuzzy theories to matrices. Kim and Roush [14] explored generalized fuzzy matrices, and Thomason [15] investigated convergence properties in fuzzy matrices. Later, Xu and Yager [13] developed geometric operators based on intuitionistic fuzzy sets, which have been useful in multi-criteria decision-making scenarios, enhancing the applicability of IFS in complex systems.

In addition to foundational works on fuzzy sets and IFSs , significant advancements have been made in fuzzy matrix theory. Kim [16] examined idempotents and inverses in fuzzy matrices, which led to further exploration of matrix properties such as transitivity and sub-inverses in fuzzy matrices, as studied by Mishref and Eman [17]. The determinant and adjoint concepts for square fuzzy matrices were later developed by Ragab and Eman [18], adding depth to the algebraic study of fuzzy matrices.Further research by Kim [19, 20] addressed T-type fuzzy idempotent matrices and established a determinant theory specific to square fuzzy matrices. Punithavalli [21] has discussed Kernel and K-Kernel Symmetric IFM. Anandhkumar et al [22, 23] have present Partial orderings, Characterizations and Generalization of k-idempotent NFM, Reverse Tilde (T) and Minus Partial Ordering on IFM. Lun [24] expanded this by exploring determinant theory for D01 lattice matrices, adding a novel perspective within the fuzzy matrix domain.

For IFM, Atanassov [25, 26] proposed the concept of generalized index matrices and applied it to represent intuitionistic fuzzy graphs. This was followed by work by Pal [27], who introduced an intuitionistic fuzzy determinant, and by Im, Lee, and Park [28], who analyzed determinants in square intuitionistic fuzzy matrices. Pal, Khan, and Shyamal [29] further contributed by exploring general properties of intuitionistic fuzzy matrices, expanding the theoretical framework for applications in multi-criteria decision-making.The body of work surrounding intuitionistic fuzzy matrices has continued to grow, contributing to both theoretical foundations and practical applications. Pal, Khan, and Shyamal [30] elaborated on the characteristics of intuitionistic fuzzy matrices, enhancing the understanding of their structures in various mathematical contexts. Meenakshi and Gandhimathi [31] investigated intuitionistic fuzzy relational equations, highlighting their applications in fuzzy logic and decision-making. Sriram and Murugadas focused on the concept of sub-inverses of IFM [32], expanding the algebraic framework, while also examining semi-rings associated with these matrices [33]. Meenakshi's work on Fuzzy Matrix Theory and Applications [34] provided a comprehensive resource for understanding both the theoretical and applied aspects of fuzzy matrices.

Research on distances between intuitionistic fuzzy matrices was conducted by Shyamal and Pal [35], offering new metrics for comparison. Bhowmik and Pal [36, 37] explored various properties of intuitionistic fuzzy matrices, including generalized forms and circulant structures, which are important for complex systems in fuzzy logic. Im [38] analyzed determinants of square intuitionistic fuzzy matrices, which is crucial for many mathematical applications. Atanassov [39] provided insights into index matrices and their relevance to augmented matrix calculus, further contributing to the theoretical landscape. Anandhkumar et al [40] have present Secondary K-Range Symmetric Neutrosophic Fuzzy Matrices. Punithavalli [41] has studied Reverse Sharp and Left-T Right-T Partial Ordering On Intuitionistic Fuzzy Matrices. Adak and colleagues [42, 43] also focused on generalized IFM, investigating their applications in multi-criteria decision-making and exploring properties of generalized fuzzy nilpotent matrices, reinforcing the practical significance of these concepts in real-world scenarios.

The research on intuitionistic fuzzy matrices continues to expand with various contributions, focusing on their structures, properties, and applications.Lee and Jeong [44] examined the canonical form of transitive intuitionistic fuzzy matrices, contributing to a deeper understanding of the properties that define these matrices. Mondal and Pal's studies [45, 46] explored similarity relations, invertibility, and eigenvalues of intuitionistic fuzzy matrices, along with their determinants, providing essential insights for both theoretical exploration and practical implementation in fuzzy systems. Radhika [47,48] et al have presented On Schur Complement in k-Kernel Symmetric Block Quadri Partitioned Neutrosophic Fuzzy Matrices and Interval Valued Secondary k-Range Symmetric Quadri Partitioned Neutrosophic Fuzzy Matrices with Decision Making. Prathab [49] et al have studied Interval Valued Secondary k-Range Symmetric Fuzzy Matrices with Generalized Inverses. Pradhan and Pal [50] contributed significantly to the understanding of intuitionistic fuzzy matrices by exploring convergence properties of different arithmetic means of intuitionistic fuzzy matrices [51], linear transformations [52]. The study by Im, Lee, and Park [53] on the determinant of square IFMs is significant in understanding the mathematical properties and applications of intuitionistic fuzzy sets. Smarandache [56] has studied Neutrosophic set, a generalization of the IFs. Murugadas and Padder [54] and Uma [55] their research focuses on defining and calculating the determinants of these matrices, which are essential for various applications in decision-making, optimization, and fuzzy logic systems. The authors explore the characteristics of determinants in the context of intuitionistic fuzzy matrices, examining how the unique features of intuitionistic fuzzy sets—such as membership and non-membership degrees—affect the determinant's computation and interpretation. Their findings contribute to the broader field of fuzzy mathematics by providing foundational knowledge that can be applied in both theoretical research and practical applications where uncertainty and vagueness are present.

In recent years, advancements in neutrosophic fuzzy matrices have facilitated deeper analysis in fields involving uncertainty and indeterminate information. Symmetric NSMs, as explored by Anandhkumar et al. [57], have proven valuable in complex decision-making and data classification. Further work by Anandhkumar et al. [58] introduced secondary k-column symmetric NFM, while Anandhkumar et al. [59] expanded this framework with interval-valued secondary k-range symmetric neutrosophic fuzzy matrices, enhancing models with interval-based evaluations. Additionally, Anandhkumar et al. [60] extended these concepts to generalized symmetric Fermatean neutrosophic fuzzy matrices, emphasizing their applicability in systems requiring nuanced symmetrical handling. The development of quadripartitioned neutrosophic soft sets has enriched the analysis of complex data sets under uncertainty. Mary [61] introduced foundational concepts in this area, providing a framework that allows for a more nuanced approach to handling uncertain information. Chatterjee, Majumdar, and Samanta [62] furthered this research by exploring similarity measures and entropy within quadripartitioned single-valued neutrosophic sets, which facilitate enhanced classification and decision-making. Subsequent analysis by Smith and Doe [63] examined the properties and applications of quadri-partitioned neutrosophic soft sets, contributing to their growing utility in handling intricate data structures.

1.1 Abbrivations

IFM: Intuitionistic Fuzzy Matrices

IFSs: Intuitionistic Fuzzy Sets

NFM: Neutrosophic fuzzy matrices.
QPNFM: Quadri Partitioned Neutrosophic fuzzy matrices.
MCDMP: Multi-Criteria Decision- Making Problem.

2. The structure of this article is arranged as follows: In Section 3 presents the objectives of the present work, laying the foundation for the study. Section 4 delves into the motivation behind this research, where a comparative analysis of the Quadri Partitioned Neutrosophic Fuzzy Matrix (QPNFM) model with existing soft models is provided to showcase its advantages. Section 5 identifies the research gap, emphasizing the limitations of current approaches and the need for the proposed model. Section 6 highlights the novelty of the QPNFM model, underscoring its unique contributions to the field. Section 7 introduces the preliminaries necessary for understanding the model, followed by Section 8, which formally defines Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs) and their mathematical structure. In Section 9 present Properties of the Quadri-Partitioned Neutrosophic Fuzzy Matrices In Section 10, relevant theorems and results are presented to establish the theoretical foundations of the model. Section 11 describes an algorithm based on QPNSS specifically designed for decision-making problems, while Section 12 demonstrates a practical application of this algorithm through a detailed example. This structure facilitates a systematic exploration of the model, providing insights into its theoretical background, methodological rigor, and real-world applicability in complex decision-making scenarios.

3. The objectives of the present work are given:

- **To develop determinant theory for Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs):** Establish fundamental principles and properties of determinants specific to QPNFMs, focusing on unique characteristics within the neutrosophic fuzzy domain.

- **To investigate determinant relationships in QPNFMs:** Prove key determinant relationships, such as $\det\left(Padj\left(P\right)\right)=\det\left(P\right)=\det\left(adj\left(P\right)P\right)$. thereby expanding mathematical understanding within QPNFMs.
- **To propose an efficient method for computing determinants in larger QPNFMs:** Introduce a novel technique for calculating determinants of QPNFMs with high dimensionality, aimed at simplifying computation for matrices with more rows and columns.
- **To construct an algorithm for solving decision-making problems:** Develop a systematic approach leveraging QPNFM properties to address complex decision-making scenarios, enhancing practical applications of QPNFMs.
- **To validate the proposed methods with an illustrative example:** Demonstrate the effectiveness and applicability of the determinant theory, methods, and algorithm through a practical example, solidifying the proposed study's relevance.

4. Motivation (Comparative of QPNFM model with the existing soft models).

In earlier research works, the concept of Fuzzy soft sets, Intuitioniasic Fuzzy soft sets, Interval valued Fuzzy soft sets, Interval valued Intuitioniasic Fuzzy soft sets, Neutrosophic Fuzzy soft sets Interval valued Neutrosophic Fuzzy soft sets , Quadri partitions Neutrosophic Fuzzy soft sets, etc. are used successfully to solve decision-making problems that contain parametric uncertain, incomplete, inconsistent, hesitant or indeterminate data. There is no such work that has been done so far where the indeterminacy can be handled parametrically under the neutrosophic environment by keeping T, F, C, and U as dependent quadripartitioned neutrosophic components. So, the present work is devoted to developing a new methodology to handle indeterminacy parametrically by introducing the quadripartitioned neutrosophic Fuzzy Matrices . This study surely provides a more flexible framework for the decision-makers to explore new decision-making approaches to address the issues under the quadripartitioned neutrosophic soft environment with the inherent restrictions. To make the proposed model more visible in the real-life scenario, we give a comparative analysis in the following Table 1 and 2:

Table 1. Comparative of IVQPNFM model with the existing soft models

Types of soft set	Uncertainty	Falsity	Hesitation	Indeterminacy	Indeterminacy is bifurcated	Indeterminacy is bifurcated and restricted
FSS [46]	✓	×	×	×	×	×
IVFSS [47]	✓	×	×	×	×	×
IFSS [48]	✓	✓	✓	×	×	×

IVIFSS [49]	✓	✓	✓	×	×	×
NSS [50]	✓	✓	×	✓	×	×
INSS [51]	✓	✓	×	✓	×	×
QNSS [52]	✓	✓	×	✓	✓	×
QPNFM (Proposed)	✓	✓	✓	✓	✓	✓

Table 2. Validation and Comparison of QPNFM with Existing Models

Aspect	Fuzzy Matrices	Intuitionistic Fuzzy Matrices	Neutrosophic Fuzzy Matrices	QPNFM (Proposed Model)
Advantages	Easy to implement and interpret; suitable for basic uncertainty problems.	Lacks indeterminacy handling.	Basic indeterminate handling; limited for complex data structures.	Handles high indeterminacy with quadri-partitioned structure; suitable for complex decision-making.
Results	Limited to systems where indeterminacy and dynamics are minimal.	Suitable for moderate uncertainty but limited in layered decision analysis.	Adequate in simpler datasets but inconsistent in highly uncertain scenarios.	Improved accuracy in uncertain environments.
Applications	Suitable for straightforward uncertainty handling and decision problems.	Useful in moderate uncertainty; limited for high indeterminacy tasks.	Best for low-complexity decisions.	Ideal for complex, uncertain scenarios.

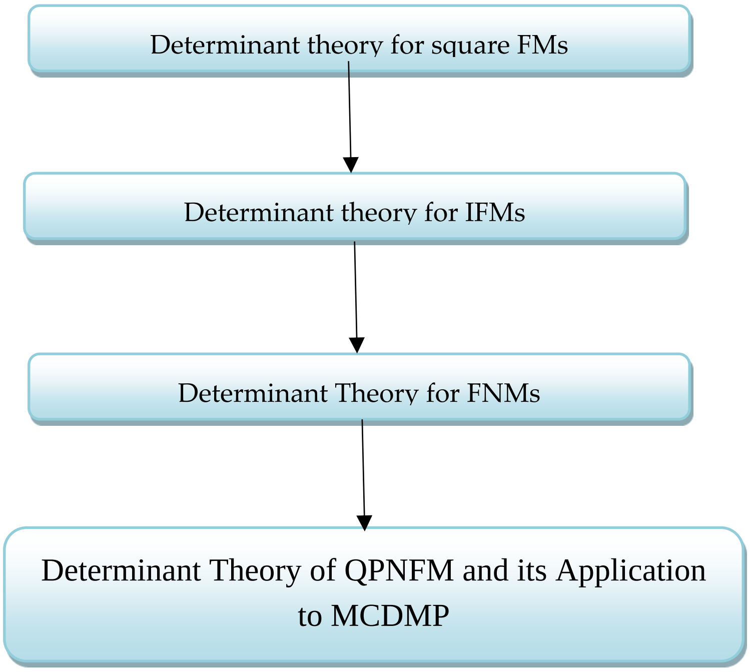
5. Research Gap

The research gap identified in these references centers on the need for a comprehensive framework that extends determinant theory to more complex structures such as Quadri Partitioned Neutrosophic Fuzzy Matrices (QPNFMs) for applications in uncertain and multi-dimensional decision-making environments. While foundational studies by Zadeh [1, 3] and Atanassov [4–6] laid the groundwork for fuzzy and intuitionistic fuzzy set theories, these frameworks do not fully address the complexity introduced in quadri-partitioned systems where neutrosophic uncertainty plays a significant role. Furthermore, Kim and Roush [14] and Lun [24] explored determinant theory in basic fuzzy matrices, and Pal [27] and Im, Lee, and Park [28] extended this to intuitionistic fuzzy matrices, but none of these studies account for the additional partitioning in neutrosophic matrices.

The work of Padder and Murugadas [54] and Uma, Murugadas, and Sriram [55] introduced determinant theories in intuitionistic fuzzy and fuzzy neutrosophic matrices, yet there remains a gap in addressing QPNFMs, where multiple layers of truth, indeterminacy, and falsity in quadri-partitioned setups impact both the matrix structure and the determinant calculation methods. Additionally, existing algorithms for multi-criteria decision-making (e.g., Adak, Bhowmik, and Pal [43]) utilize simpler fuzzy matrix approaches, lacking algorithms tailored for QPNFMs that could leverage their unique partitioning to provide more nuanced decision-support solutions. Addressing this gap could yield a robust determinant theory for QPNFMs, alongside efficient algorithms for high-dimensional decision-making problems involving complex, uncertain data.

Table:1 Review of the Extension of QPNFM.

Ref	**Journal Name**	**Authors Name**	**Extension of NFM.**	**Year**
[20]	Fuzzy Sets Systems	Kim	Determinant theory for square FMs	1989
[55]	Progress in Nonlinear Dynamics and Chaos	Uma et al.	Determinant Theory for NFMs	2016
[54]	Afrika Matematika	Riyaz Ahmad Padder et al	Determinant theory for IFMs	2019
Proposed	Neutrosophic Sets and Systems	Anandhkumar et al.	Determinant Theory of QPNFM and its Application to MCDMP	2024



6. Novelty

The referenced works collectively contribute to the advancement of fuzzy and intuitionistic fuzzy matrix theory, supporting a range of applications in uncertainty modeling and multi-criteria decision-making. Zadeh [1, 3] laid the foundation of fuzzy set theory by introducing partial membership, which became the basis for further developments in fuzzy systems, and later extended this with linguistic variables for approximate reasoning. Atanassov [4–6] extended fuzzy theory with IFSs, which incorporate both membership and non-membership degrees, creating a richer framework for handling uncertainty. Researchers like Kim and Roush [14], Thomason [15], Kim [16, 19, 20], Ragab and Eman [17], and Lun [24] contributed to the mathematical structure of fuzzy matrices by investigating properties such as idempotence, inverses, determinants, and convergence, which are crucial for stability and transformations in fuzzy systems.

Further developments came from Pal, Bhowmik, and their collaborators [27, 29, 30, 35, 36, 37], who explored operations and properties specific to intuitionistic fuzzy matrices, such as determinants, eigenvalues, and similarity relations. Their work added depth to the theoretical understanding and practical application of these matrices in complex decision-making. Adak, Bhowmik, and Pal [42, 37] applied IFMs to multi-criteria decision-making, highlighting their relevance in real-world applications. Studies by Lee and Jeong [36], Murugadas and Padder [48–49], and Pradhan and Pal [50-52] introduced forms, reductions, and convergence criteria for intuitionistic fuzzy matrices, aiding in simplification and predictability. Collectively, these works by Zadeh, Atanassov, Pal, and others have enriched the mathematical framework and utility of fuzzy and intuitionistic fuzzy matrices, making them valuable tools in areas requiring nuanced handling of uncertainty and complex decision parameters.

7. Preliminaries

Definition 7.1 [53] Let X is an initial universe set and E is a set of parameters. Consider a non-empty set A where A ⊆ E. Let P(X) denote the set of all QPNSS of X. The collection (F, A) is termed the (QPNSS) over X, where F is a mapping given by F : A⟶ P(X). Here,
$A = \{< x, T_A(x), C_A(x), U_A(x), FA(x) >: x \in U\}$ with T_A , F_A, C_A ,$U_A : X \longrightarrow [0,1]$ and $0 \le T_A(x) + C_A(x) + U_A(x) + F_A(x) \le 4$. In this context

- $T_A(x)$ is the truth membership (TM),
- $C_A(x)$ is contradiction membership (CM),
- $U_A(x)$ is ignorance membership (IM),
- $F_A(x)$ is the false membership (FM).

8. Quadri-Partitioned Neutrosophic Fuzzy Matrices

Definition 8.1 Let $P = < p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} >, Q = < q^T{}_{ij}, q^C{}_{ij}, q^U{}_{ij}, q^F{}_{ij} > \in (QPNFM)_n$

Component-wise addition and multiplication are defined as follows

(i) $$P \oplus Q = \left(\sup\left\{p^T{}_{ij}, q^T{}_{ij}\right\}, \sup\left\{p^C{}_{ij}, q^C{}_{ij}\right\}, \inf\left\{p^U{}_{ij}, q^U{}_{ij}\right\}, \inf\left\{p^F{}_{ij}, q^F{}_{ij}\right\}\right)$$

(ii) $$P\square\, Q = \left(\inf\left\{p^T{}_{ij}, q^T{}_{ij}\right\}, \inf\left\{p^C{}_{ij}, q^C{}_{ij}\right\}, \sup\left\{p^U{}_{ij}, q^U{}_{ij}\right\}, \sup\left\{p^F{}_{ij}, q^F{}_{ij}\right\}\right)$$

Definition 8.2. Let $P = < p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} >, Q = < q^T{}_{ij}, q^C{}_{ij}, q^U{}_{ij}, q^F{}_{ij} > \in (QPNFM)_n$

the composition of P and Q is well-defined as

$$P \circ Q = \left(\sum_{k=1}^{n}\left(p^T{}_{ij} \wedge q^T{}_{ij}\right), \sum_{k=1}^{n}\left(p^C{}_{ij} \wedge q^C{}_{ij}\right), \prod_{k=1}^{n}\left(p^U{}_{ij} \vee q^U{}_{ij}\right), \prod_{k=1}^{n}\left(p^F{}_{ij} \vee q^F{}_{ij}\right)\right)$$

consistently we can write the same as

$$P \circ Q = \left(\bigcup_{k=1}^{n}\left(p^T{}_{ij} \wedge q^T{}_{ij}\right), \bigcup_{k=1}^{n}\left(p^C{}_{ij} \wedge q^C{}_{ij}\right), \bigcap_{k=1}^{n}\left(p^U{}_{ij} \vee q^U{}_{ij}\right), \bigcap_{k=1}^{n}\left(p^U{}_{ij} \vee q^U{}_{ij}\right)\right)$$

The product $P \circ Q$ is defined only when the number of columns in P equals the number of rows in Q. When this condition is met, matrices P and Q are considered conformable for multiplication. For simplicity, we denote the product as $P \circ Q$.

Definition 8.3 The determinant $|P|$ of nxn QPNFM $P = < p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} > \in (QPNFM)_n$ is defined as follows

$$|P| = < \bigcup_{\sigma \in S_n} p^T{}_{1\sigma(1)} \bigcap \cdots \bigcap p^T{}_{n\sigma(n)}, \bigcup_{\sigma \in S_n} p^C{}_{1\sigma(1)} \bigcap \cdots \bigcap p^C{}_{n\sigma(n)}, \bigcap_{\sigma \in S_n} p^U{}_{1\sigma(1)} \bigcup \cdots \bigcup p^U{}_{n\sigma(n)},$$
$$\bigcap_{\sigma \in S_n} p^F{}_{1\sigma(1)} \bigcup \cdots \bigcup p^F{}_{n\sigma(n)} >$$

Here, S_n represents the symmetric group consisting of all possible permutations of the indices (1,2,... n).

Definition 8.4 The adjoint of an n xn QPFNSM P denoted by adj P, is defined as follows

$q_{ij} = |P_{ji}|$ is the determinant of the (n-1)x(n-1) QPFNSM formed by removing row j and column i from P and Q = adjP

Definition:8.5 Let $P = < p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} > \in (QPNFM)_n$ and let Q be a matrix from P by striking out e_1, row e_2,… row e_k and column r_1, column r_2,…, column r_k. we define $P\begin{pmatrix} e_1 & e_2 & \cdots & e_k \\ r_1 & r_2 & \cdots & r_k \end{pmatrix} = \det(H)$.

Remark:8.1 We can write the element q_{ij} of adjP = Q = (q_{ij}) as follows:
$q_{ij} = \sum_{\pi \in S_{n_j n_i}} \prod_{t \in n_j} < p^T{}_{t\pi(t)}, p^C{}_{t\pi(t)}, p^U{}_{t\pi(t)}, p^F{}_{t\pi(t)} >$ where n_j = {1,2,3,…n}\{j} and $S_{n_j n_i}$ is the set of all permutation of set n_j over the set n_i.

9.Properties of the Quadri-Partitioned Neutrosophic Fuzzy Matrices

(i) The determinant's value remains unchanged if any two rows or any two columns are swapped.

(ii) For a Quadri-Partitioned Neutrosophic Fuzzy Matrix (QPNFM), the determinant value is preserved when rows and columns are interchanged.

(iii) For two QPNFMs, P and Q, the property det(PQ)≠det(P)·det(Q) holds.

(iv) If the elements of one row (or column) are added to the corresponding elements of another row (or column), the determinant’s value remains the same as the original.

10. Theorems and Results

Theorem:10.1 $P \in (QPNFM)_n$,then

(i) $$\det(P) = |P| = \sum_{t=1}^{n} < p^T{}_{it}, p^C{}_{it}, p^U{}_{it}, p^F{}_{it} > P_{it}, i \in \{1,2,...,n\}.$$

(ii) $$\det(P) = \sum_{e<f} \begin{vmatrix} < p^T{}_{1e}, p^C{}_{1e}, p^U{}_{1e}, p^F{}_{1e} > & < p^T{}_{1f}, p^C{}_{1f}, p^U{}_{1f}, p^F{}_{1f} > \\ < p^T{}_{2e}, p^C{}_{2e}, p^U{}_{2e}, p^F{}_{2e} > & < p^T{}_{2f}, p^C{}_{2f}, p^U{}_{2f}, p^F{}_{2f} > \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix}$$

where the summation is taken over all e and f in {1,2,...,n} such that e < f.

Theorem: 10.2

$$\det(P) = \begin{pmatrix} < p^T{}_{1r_1}, p^C{}_{1r_1}, p^U{}_{1r_1}, p^F{}_{1r_1} > & \cdots & \cdots & < p^T{}_{1r_k}, p^C{}_{1r_k}, p^U{}_{1r_k}, p^F{}_{1r_k} > \\ < p^T{}_{2r_1}, p^C{}_{2r_1}, p^U{}_{2r_1}, p^F{}_{2r_1} > & \cdots & \cdots & < p^T{}_{2r_k}, p^C{}_{2r_k}, p^U{}_{2r_k}, p^F{}_{2r_k} > \\ \cdots & \cdots & \cdots & \cdots \\ < p^T{}_{kr_1}, p^C{}_{kr_1}, p^U{}_{kr_1}, p^F{}_{kr_1} > & \cdots & \cdots & < p^T{}_{kr_k}, p^C{}_{kr_k}, p^U{}_{kr_k}, p^F{}_{kr_k} > \end{pmatrix} P\begin{pmatrix} 1 & \cdots & k \\ r_1 & \cdots & r_k \end{pmatrix}$$

where the summation is taken over all $r_1, r_2, ..., r_k \in \{1,2,...,n\}$, such that $r_1 < r_2 < ... < r_k$.

Proof: Let $S(r_1, r_2, ..., r_k) = \{\sigma : \{1,2,...,k\} \to \{r_1, r_2, ..., r_k\} / \sigma$ is a bijection}. Then

$$\det(P) = \sum_{\sigma \in S_n} < p^T{}_{1\sigma(1)}, p^C{}_{1\sigma(1)}, p^U{}_{1\sigma(1)}, p^F{}_{1\sigma(1)} > ... < p^T{}_{n\sigma(n)}, p^C{}_{n\sigma(n)}, p^U{}_{n\sigma(n)}, p^F{}_{n\sigma(n)} >$$

$$= \sum_{r_1<r_2<...<r_k} \left(\sum_{\sigma\{1,2,...,k\} \in S(r_1,r_2,...,r_k)} < p^T{}_{1\sigma(1)}, p^C{}_{1\sigma(1)}, p^U{}_{1\sigma(1)}, p^F{}_{1\sigma(1)} > ... < p^T{}_{n\sigma(n)}, p^C{}_{n\sigma(n)}, p^U{}_{n\sigma(n)}, p^F{}_{n\sigma(n)} > \right)$$

$$= \sum_{r_1<r_2<...<r_k} \left(\sum_{\sigma'\{1,2,...,k\} \in S(r_1,r_2,...,r_k)} < p^T{}_{1\sigma'(1)}, p^C{}_{1\sigma'(1)}, p^U{}_{1\sigma'(1)}, p^F{}_{1\sigma'(1)} > ... < p^T{}_{n\sigma'(n)}, p^C{}_{n\sigma'(n)}, p^U{}_{n\sigma'(n)}, p^F{}_{n\sigma'(n)} > \right)$$
$$P\begin{pmatrix} 1 & \cdots & k \\ r_1 & \cdots & r_k \end{pmatrix}$$

$$= \det \sum_{r_1<r_2<...<r_k} \begin{pmatrix} < p^T{}_{1r_1}, p^C{}_{1r_1}, p^U{}_{1r_1}, p^F{}_{1r_1} > & \cdots & \cdots & < p^T{}_{1r_k}, p^C{}_{1r_k}, p^U{}_{1r_k}, p^F{}_{1r_k} > \\ < p^T{}_{2r_1}, p^C{}_{2r_1}, p^U{}_{2r_1}, p^F{}_{2r_1} > & \cdots & \cdots & < p^T{}_{2r_k}, p^C{}_{2r_k}, p^U{}_{2r_k}, p^F{}_{2r_k} > \\ \cdots & \cdots & \cdots & \cdots \\ < p^T{}_{kr_1}, p^C{}_{kr_1}, p^U{}_{kr_1}, p^F{}_{kr_1} > & \cdots & \cdots & < p^T{}_{kr_k}, p^C{}_{kr_k}, p^U{}_{kr_k}, p^F{}_{kr_k} > \end{pmatrix} P\begin{pmatrix} 1 & \cdots & k \\ r_1 & \cdots & r_k \end{pmatrix}$$

Hence the theorem.

Lemma 10.1 Let $P=\begin{pmatrix} <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \\ <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \end{pmatrix}$ be a QPNFM.

Then $\det\begin{pmatrix} <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \\ <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \end{pmatrix}\det\begin{pmatrix} <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \\ <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \end{pmatrix}$

$$=\begin{vmatrix} <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \\ <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \end{vmatrix}\begin{vmatrix} <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \\ <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \end{vmatrix}\le \det(P)$$

Proof: We see that

$$\det\begin{pmatrix} <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \\ <p^T,p^C,p^U,p^F> & <q^T,q^C,q^U,q^F> \end{pmatrix}\det\begin{pmatrix} <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \\ <r^T,r^C,r^U,r^F> & <s^T,s^C,s^U,s^F> \end{pmatrix}$$

$$=<p^T,p^C,p^U,p^F><q^T,q^C,q^U,q^F><r^T,r^C,r^U,r^F><s^T,s^C,s^U,s^F>$$

$$\le\left(<p^T,p^C,p^U,p^F><s^T,s^C,s^U,s^F>+<q^T,q^C,q^U,q^F><r^T,r^C,r^U,r^F>\right)$$

$$\le \det(P)$$

Hence the theorem.

Theorem:10.3 Let $P\in\left(QPNFM\right)_n$,then

(i) $\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(1\Rightarrow 2\right)\right)\le\det\left(P\right)$.

(ii) $\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(3\Rightarrow 2\right)\right)\le\det\left(P\right)$.

(iii) $\det\left(P\left(p\Rightarrow q\right)\right)\det\left(P\left(q\Rightarrow k\right)\right)\le\det\left(P\right)$.

Proof: To prove (i) $\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(1\Rightarrow 2\right)\right)\le\det\left(P\right)$.

$$=\left(\begin{matrix} \begin{vmatrix} <p^T{}_{11},p^C{}_{11},p^U{}_{11},p^F{}_{11}> & <p^T{}_{12},p^C{}_{12},p^U{}_{12},p^F{}_{12}> \\ <p^T{}_{11},p^C{}_{11},p^U{}_{11},p^F{}_{11}> & <p^T{}_{12},p^C{}_{12},p^U{}_{12},p^F{}_{12}> \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \\ +\ldots+\begin{vmatrix} <p^T{}_{1e},p^C{}_{1e},p^U{}_{1e},p^F{}_{1e}> & <p^T{}_{1f},p^C{}_{1f},p^U{}_{1f},p^F{}_{1f}> \\ <p^T{}_{1e},p^C{}_{1e},p^U{}_{1e},p^F{}_{1e}> & <p^T{}_{1f},p^C{}_{1f},p^U{}_{1f},p^F{}_{1f}> \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ e & f \end{pmatrix} \\ +\ldots+\begin{vmatrix} <p^T{}_{1n-1},p^C{}_{1n-1},p^U{}_{1n-1},p^F{}_{1n-1}> & <p^T{}_{1n},p^C{}_{1n},p^U{}_{1n},p^F{}_{1n}> \\ <p^T{}_{1n-1},p^C{}_{1n-1},p^U{}_{1n-1},p^F{}_{1n-1}> & <p^T{}_{1n},p^C{}_{1n},p^U{}_{1n},p^F{}_{1n}> \end{vmatrix} P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix} \end{matrix}\right)$$

$$\left(\begin{vmatrix} <p^T{}_{21},p^C{}_{21},p^U{}_{21},p^F{}_{21}> & <p^T{}_{22},p^C{}_{22},p^U{}_{22},p^F{}_{22}> \\ <p^T{}_{21},p^C{}_{21},p^U{}_{21},p^F{}_{21}> & <p^T{}_{22},p^C{}_{22},p^U{}_{22},p^F{}_{22}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} \\ +\ldots+\begin{vmatrix} <p^T{}_{2e},p^C{}_{2e},p^U{}_{2e},p^F{}_{2e}> & <p^T{}_{2f},p^C{}_{2f},p^U{}_{2f},p^F{}_{2f}> \\ <p^T{}_{2e},p^C{}_{2e},p^U{}_{2e},p^F{}_{2e}> & <p^T{}_{2f},p^C{}_{2f},p^U{}_{2f},p^F{}_{2f}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix} \\ +\ldots+\begin{vmatrix} <p^T{}_{2n-1},p^C{}_{2n-1},p^U{}_{2n-1},p^F{}_{2n-1}> & <p^T{}_{2n},p^C{}_{2n},p^U{}_{2n},p^F{}_{2n}> \\ <p^T{}_{2n-1},p^C{}_{2n-1},p^U{}_{2n-1},p^F{}_{2n-1}> & <p^T{}_{2n},p^C{}_{2n},p^U{}_{2n},p^F{}_{2n}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}\right)$$

$$=\left(\sum_{e<f}\begin{vmatrix} <p^T{}_{1e},p^C{}_{1e},p^U{}_{1e},p^F{}_{1e}> & <p^T{}_{1f},p^C{}_{1f},p^U{}_{1f},p^F{}_{1f}> \\ <p^T{}_{1e},p^C{}_{1e},p^U{}_{1e},p^F{}_{1e}> & <p^T{}_{1f},p^C{}_{1f},p^U{}_{1f},p^F{}_{1f}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}\right)$$

$$\left(\sum_{g<h}\begin{vmatrix} <p^T{}_{2g},p^C{}_{2g},p^U{}_{2g},p^F{}_{2g}> & <p^T{}_{2h},p^C{}_{2h},p^U{}_{2h},p^F{}_{2h}> \\ <p^T{}_{2g},p^C{}_{2g},p^U{}_{2g},p^F{}_{2g}> & <p^T{}_{2h},p^C{}_{2h},p^U{}_{2h},p^F{}_{2h}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}\right)$$

$$\leq\left(\sum_{\substack{e<f\\ g<h}}\begin{vmatrix} <p^T{}_{1e},p^C{}_{1e},p^U{}_{1e},p^F{}_{1e}> & <p^T{}_{1f},p^C{}_{1f},p^U{}_{1f},p^F{}_{1f}> \\ <p^T{}_{2g},p^C{}_{2g},p^U{}_{2g},p^F{}_{2g}> & <p^T{}_{2h},p^C{}_{2h},p^U{}_{2h},p^F{}_{2h}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix} P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}\right)$$

We now introduce symbols $\Omega_1,\Omega_2,\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and Ω. Define

$$\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\begin{vmatrix} <p^T{}_{1e},p^C{}_{1e},p^U{}_{1e},p^F{}_{1e}> & <p^T{}_{1f},p^C{}_{1f},p^U{}_{1f},p^F{}_{1f}> \\ <p^T{}_{2g},p^C{}_{2g},p^U{}_{2g},p^F{}_{2g}> & <p^T{}_{2h},p^C{}_{2h},p^U{}_{2h},p^F{}_{2h}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix} P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}$$

$$\Omega_1=\sum_{(e,f)=(g,h)}\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\sum_{e<f}\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix},$$

$$\Omega_2=\sum_{(e,f)\neq(g,h)}\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix} \quad \text{and} \quad \Omega=\Omega_1+\Omega_2$$

Then we see that

$$\Omega\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}=\begin{vmatrix} <p^T{}_{11},p^C{}_{11},p^U{}_{11},p^F{}_{11}> & <p^T{}_{12},p^C{}_{12},p^U{}_{12},p^F{}_{12}> \\ <p^T{}_{21},p^C{}_{21},p^U{}_{21},p^F{}_{21}> & <p^T{}_{22},p^C{}_{22},p^U{}_{22},p^F{}_{22}> \end{vmatrix} P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$$

$$\Omega_1=\det(P)$$

$$\det\left(P(2\Rightarrow 1)\right)\det\left(P(1\Rightarrow 2)\right)\leq\Omega=\det(P)+\Omega_2.$$

We show that $\Omega_2\leq\det(P)$

We consider two separate cases.

Case 1. We consider $p = \Omega\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$, a term of Ω_2.

Let $p_1 =< p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} >< p^T{}_{23}, p^C{}_{23}, p^U{}_{23}, p^F{}_{23} > P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

$p_2 =< p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} >< p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$

Then $p = p_1 + p_2$,

$$p_1 \le \begin{vmatrix} < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} > & < p^T{}_{13}, p^C{}_{13}, p^U{}_{13}, p^F{}_{13} > \\ < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{23}, p^C{}_{23}, p^U{}_{23}, p^F{}_{23} > \end{vmatrix} P\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} \le \det(P)$$

$$p_2 \le \begin{vmatrix} < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} > & < p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} > \\ < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22} > \end{vmatrix} P\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \le \det(P)$$

and $\Omega\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \le \det(P)$.

Case 2. We take $\Omega\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$

Let $q_1 =< p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} >< p^T{}_{2n}, p^C{}_{2n}, p^U{}_{2n}, p^F{}_{2n} > P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$ and

$q_2 =< p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} >< p^T{}_{2n-1}, p^C{}_{2n-1}, p^U{}_{2n-1}, p^F{}_{2n-1} > P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$.

Then $\Omega\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix} = q_1 + q_2$.

To show that $q_1 \le \det(P)$ and $q_2 = \det(P)$ we observe all coordinates of the elements p_{ij}

involved in $P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$

The coordinates of the elements p_{ij} involved in these determinants are all coordinates of the elements of the k th – row P_k of P, for $k \geq 3$. Therefore, if we let $q = p_{3n-1} p_{4n-2} \cdots p_{k+2n-k} \cdots p_{nn-2}$, then we see that

$$q_1 \leq \left(< p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} >< p^T{}_{2n}, p^C{}_{2n}, p^U{}_{2n}, p^F{}_{2n} > \right) c \leq \det\left(P\right).$$

For q_2, let $c = p_{3n} p_{4n-2} p_{5n-3} \cdots p_{n-13} p_{2n-1}$, then we see that

$$q_2 \leq \left(< p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} >< p^T{}_{2n-1}, p^C{}_{2n-1}, p^U{}_{2n-1}, p^F{}_{2n-1} > \right) c \leq \det(P).$$

For any $\Omega \begin{pmatrix} e & f \\ g & h \end{pmatrix}_{(e,f) \neq (g,h)}$, we apply either the case 1 or the case 2 and we can deduce that $\Omega \begin{pmatrix} e & f \\ g & h \end{pmatrix} \leq \det(P).$ Thus (i) holds. (ii).

First we consider

$$\begin{vmatrix} < q^T{}_{11}, q^C{}_{11}, q^U{}_{11}, q^F{}_{11} > & < q^T{}_{12}, q^C{}_{12}, q^U{}_{12}, q^F{}_{12} > \\ < p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} > & < p^T{}_{32}, p^C{}_{32}, p^U{}_{23}, p^F{}_{32} > \end{vmatrix} \begin{vmatrix} < q^T{}_{21}, q^C{}_{21}, q^U{}_{21}, q^F{}_{21} > & < q^T{}_{22}, q^C{}_{22}, q^U{}_{22}, q^F{}_{22} > \\ < q^T{}_{21}, q^C{}_{21}, q^U{}_{21}, q^F{}_{21} > & < q^T{}_{22}, q^C{}_{22}, q^U{}_{22}, q^F{}_{22} > \end{vmatrix}$$

$$\leq \begin{vmatrix} < q^T{}_{21}, q^C{}_{21}, q^U{}_{21}, q^F{}_{21} > & < q^T{}_{32}, q^C{}_{32}, q^U{}_{32}, q^F{}_{32} > \\ < p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} > & < p^T{}_{32}, p^C{}_{32}, p^U{}_{23}, p^F{}_{32} > \end{vmatrix}$$

$$K \begin{pmatrix} g & h \\ e & f \end{pmatrix} = \begin{vmatrix} < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{32}, p^C{}_{32}, p^U{}_{32}, p^F{}_{32} > \\ < p^T{}_{31}, p^C{}_{31}, p^U{}_{31}, p^F{}_{31} > & < p^T{}_{32}, p^C{}_{32}, p^U{}_{23}, p^F{}_{32} > \end{vmatrix} P \begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} P \begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$\det\left(P\left(2 \Rightarrow 1\right)\right) \det\left(P\left(3 \Rightarrow 2\right)\right)$$

$$= \sum_{\substack{e<f \\ g<h}} \begin{vmatrix} < p^T{}_{1e}, p^C{}_{1e}, p^U{}_{1e}, p^F{}_{1e} > & < p^T{}_{1f}, p^C{}_{1f}, p^U{}_{1f}, p^F{}_{1f} > \\ < p^T{}_{3e}, p^C{}_{3e}, p^U{}_{3e}, p^F{}_{3e} > & < p^T{}_{3f}, p^C{}_{3f}, p^U{}_{3f}, p^F{}_{3f} > \end{vmatrix} P \begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$\begin{vmatrix} < p^T{}_{2g}, p^C{}_{2g}, p^U{}_{2g}, p^F{}_{2g} > & < p^T{}_{2h}, p^C{}_{2h}, p^U{}_{2h}, p^F{}_{2h} > \\ < p^T{}_{2g}, p^C{}_{2g}, p^U{}_{2g}, p^F{}_{2g} > & < p^T{}_{2h}, p^C{}_{2h}, p^U{}_{2h}, p^F{}_{2h} > \end{vmatrix} P \begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix}$$

$$\leq \sum_{\substack{g<h \\ e<f}} \begin{vmatrix} < p^T{}_{2g}, p^C{}_{2g}, p^U{}_{2g}, p^F{}_{2g} > & < p^T{}_{2h}, p^C{}_{2h}, p^U{}_{2h}, p^F{}_{2h} > \\ < p^T{}_{3e}, p^C{}_{3e}, p^U{}_{3e}, p^F{}_{3e} > & < p^T{}_{3f}, p^C{}_{3f}, p^U{}_{3f}, p^F{}_{3f} > \end{vmatrix} P \begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} P \begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$=\sum_{\substack{g<h\\e<f}} P\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

$$=\sum_{(g,h)=(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix}+\sum_{(g,h)\neq(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

Next we prove that

$$K\begin{pmatrix} g & h \\ e & f \end{pmatrix}_{(g,h)\neq(e,f)} \leq \det(P)$$

.We consider two separate cases.

Case 1. We take $K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. We see that

$$K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}=\left(<p^{T}{}_{21},p^{C}{}_{21},p^{U}{}_{21},p^{F}{}_{21}><p^{T}{}_{33},p^{C}{}_{33},p^{U}{}_{33},p^{F}{}_{33}>\right)P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$+\left(<p^{T}{}_{22},p^{C}{}_{22},p^{U}{}_{22},p^{F}{}_{22}><p^{T}{}_{31},p^{C}{}_{31},p^{U}{}_{31},p^{F}{}_{31}>\right)P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$\leq\begin{vmatrix} <p^{T}{}_{21},p^{C}{}_{21},p^{U}{}_{21},p^{F}{}_{21}> & <p^{T}{}_{23},p^{C}{}_{23},p^{U}{}_{23},p^{F}{}_{23}> \\ <p^{T}{}_{31},p^{C}{}_{31},p^{U}{}_{31},p^{F}{}_{31}> & <p^{T}{}_{33},p^{C}{}_{33},p^{U}{}_{33},p^{F}{}_{33}> \end{vmatrix}P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$+\begin{vmatrix} <p^{T}{}_{21},p^{C}{}_{21},p^{U}{}_{21},p^{F}{}_{21}> & <p^{T}{}_{22},p^{C}{}_{22},p^{U}{}_{22},p^{F}{}_{22}> \\ <p^{T}{}_{31},p^{C}{}_{31},p^{U}{}_{31},p^{F}{}_{31}> & <p^{T}{}_{32},p^{C}{}_{32},p^{U}{}_{32},p^{F}{}_{32}> \end{vmatrix}P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$\leq \det(P)+\det(P)=\det(P)$$

Case 2. We take $K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix}$ We see that

$$K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix}=\left(<p^{T}{}_{2n-1},p^{C}{}_{2n-1},p^{U}{}_{2n-1},p^{F}{}_{2n-1}><p^{T}{}_{32},p^{C}{}_{32},p^{U}{}_{32},p^{F}{}_{32}>\right)P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$$

$$+\left(<q^{T}{}_{2n},q^{C}{}_{2n},q^{U}{}_{2n},q^{F}{}_{2n}><p^{T}{}_{31},p^{C}{}_{31},p^{U}{}_{31},p^{F}{}_{31}>\right)P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}.$$

Considering the coordinates of the elements p_{ij} involved in $P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$, we claim that

$$\left(< p^{T}{}_{2n-1}, p^{C}{}_{2n-1}, p^{U}{}_{2n-1}, p^{F}{}_{2n-1} >< p^{T}{}_{32}, p^{C}{}_{32}, p^{U}{}_{32}, p^{F}{}_{32} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \leq \det\left(P\right)$$

and $\left(< q^{T}{}_{2n}, q^{C}{}_{2n}, q^{U}{}_{2n}, q^{F}{}_{2n} >< p^{T}{}_{31}, p^{C}{}_{31}, p^{U}{}_{31}, p^{F}{}_{31} >\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \leq \det\left(P\right)$

Similarly we can prove (iii).

Hence the theorem.

Theorem 10.4. Let $P = \left(p^{T}{}_{ij}, p^{C}{}_{ij}, p^{U}{}_{ij}, p^{F}{}_{ij}\right), Q = \left(q^{T}{}_{ij}, q^{C}{}_{ij}, q^{U}{}_{ij}, q^{F}{}_{ij}\right), R = \left(r^{T}{}_{ij}, r^{C}{}_{ij}, r^{U}{}_{ij}, r^{F}{}_{ij}\right)$ $\in (QPNFM)_n$. Then

(i) If $\left(p^{T}{}_{ii}, p^{C}{}_{ii}, p^{U}{}_{ii}, p^{F}{}_{ii}\right) \geq \left(p^{T}{}_{ik}, p^{C}{}_{ik}, p^{U}{}_{ik}, p^{F}{}_{ik}\right)$ (k= 1,2,3,…,n) for all $1 \leq i \leq n$, then

$$\det(P) = < p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} >< p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \ldots < p^{T}{}_{nn}, p^{C}{}_{nn}, p^{U}{}_{nn}, p^{F}{}_{nn} >.$$

(ii) $\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} \det(P)\det(Q)$ where $(< 0,0,1,1 >)_n \in (QPNFM)_n$.

(iii) $\det\left(PP^{T}\right) \geq \det\left(P\right)$

Proof: (i). We have

$$\left(< p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} >< p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \ldots < p^{T}{}_{nn}, p^{C}{}_{nn}, p^{U}{}_{nn}, p^{F}{}_{nn} >\right) \geq$$

$$\left(< p^{T}{}_{1\sigma(1)}, p^{C}{}_{1\sigma(1)}, p^{U}{}_{1\sigma(1)}, p^{F}{}_{1\sigma(1)} >< p^{T}{}_{2\sigma(2)}, p^{C}{}_{2\sigma(2)}, p^{U}{}_{2\sigma(2)}, p^{F}{}_{2\sigma(2)} > \ldots < p^{T}{}_{n\sigma(n)}, p^{C}{}_{n\sigma(n)}, p^{U}{}_{n\sigma(n)}, p^{F}{}_{n\sigma(n)} >\right)$$

for every $\sigma \in S_n$.

Since $\left(p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11}\right) \geq \left(p^{T}{}_{ik}, p^{C}{}_{ik}, p^{U}{}_{ik}, p^{F}{}_{ik}\right)$ $\left(k = 1, 2, 3, \ldots, n\right)$ for all $1 \leq i \leq n$.

Hence

$$\det\left(P\right) = \sum_{\sigma \in S_n} < p^{T}{}_{1\sigma(1)}, p^{C}{}_{1\sigma(1)}, p^{U}{}_{1\sigma(1)}, p^{F}{}_{1\sigma(1)} > \ldots < p^{T}{}_{n\sigma(n)}, p^{C}{}_{n\sigma(n)}, p^{U}{}_{n\sigma(n)}, p^{F}{}_{n\sigma(n)} >$$

$$= < p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} >< p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \ldots < p^{T}{}_{nn}, p^{C}{}_{nn}, p^{U}{}_{nn}, p^{F}{}_{nn} >.$$

This proves (i)

(ii) $\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \left(< s^T{}_{ij}, s^C{}_{ij}, s^U{}_{ij}, s^F{}_{ij} > \right)_{2n}$.

$$\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \sum_{\sigma\in S_{2n}} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} >$$

$$= \sum_{\sigma\in S_{2n}, \sigma(i)\le n\,(if\; i\le n)} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} >$$

$$= \sum_{\sigma\in S_{2n}, \sigma(i)\le n\,(if\; i\le n)} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} > + < 0,0,1,1 >$$

$$+ \sum_{\sigma\in S_{2n}, \exists k>n,\, if\; \sigma(k)\le n} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{2n\sigma(2n)}, s^C{}_{2n\sigma(2n)}, s^U{}_{2n\sigma(2n)}, s^F{}_{2n\sigma(2n)} >$$

$$= \sum_{\sigma'\in S_n} < s^T{}_{1\sigma'(1)}, s^C{}_{1\sigma'(1)}, s^U{}_{1\sigma'(1)}, s^F{}_{1\sigma'(1)} > \ldots < s^T{}_{n\sigma'(n)}, s^C{}_{n\sigma'(n)}, s^U{}_{n\sigma'(n)}, s^F{}_{n\sigma'(n)} > \det(Q)$$

$$= \left(\sum_{\sigma\in S_n} < s^T{}_{1\sigma(1)}, s^C{}_{1\sigma(1)}, s^U{}_{1\sigma(1)}, s^F{}_{1\sigma(1)} > \ldots < s^T{}_{n\sigma(n)}, s^C{}_{n\sigma(n)}, s^U{}_{n\sigma(n)}, s^F{}_{n\sigma(n)} > \right) \det(Q)$$

$$= \det(P)\det(Q).$$

This proves (ii)

(iii) $PP^T = \left(< h^T{}_{ij}, h^C{}_{ij}, h^U{}_{ij}, h^F{}_{ij} > \right)_n$,

We have , for every $\sigma \in S_n$.

$$< h^T{}_{ij}, h^C{}_{ij}, h^U{}_{ij}, h^F{}_{ij} > = \sum_{k=1}^{n} < p^T{}_{ik}, p^C{}_{ik}, p^U{}_{ik}, p^F{}_{ik} >< p^T{}_{kj}, p^C{}_{kj}, p^U{}_{kj}, p^F{}_{kj} > .$$

$$< h^T{}_{11}, h^C{}_{11}, h^U{}_{11}, h^F{}_{11} >< h^T{}_{22}, h^C{}_{22}, h^U{}_{22}, h^F{}_{22} > \ldots < h^T{}_{nn}, h^C{}_{nn}, h^U{}_{nn}, h^F{}_{nn} >$$

$$= \left(\sum_{k=1}^{n} < p^T{}_{ik}, p^C{}_{ik}, p^U{}_{ik}, p^F{}_{ik} > \right) \ldots \left(\sum_{k=1}^{n} < p^T{}_{nk}, p^C{}_{nk}, p^U{}_{nk}, p^F{}_{nk} > \right)$$

$$\ge \left(< p^T{}_{1\sigma(1)}, p^C{}_{1\sigma(1)}, p^U{}_{1\sigma(1)}, p^F{}_{1\sigma(1)} > \ldots < p^T{}_{n\sigma(n)}, p^C{}_{n\sigma(n)}, p^U{}_{n\sigma(n)}, p^F{}_{n\sigma(n)} > \right)$$

Hence

$$\det\left(PP^T\right) \ge \left(< h^T{}_{11}, h^C{}_{11}, h^U{}_{11}, h^F{}_{11} >< h^T{}_{22}, h^C{}_{22}, h^U{}_{22}, h^F{}_{22} > \ldots < h^T{}_{nn}, h^C{}_{nn}, h^U{}_{nn}, h^F{}_{nn} > \right)$$

$$\geq \sum_{\sigma\in S_n} < p^T{}_{1\sigma(1)}, p^C{}_{1\sigma(1)}, p^U{}_{1\sigma(1)}, p^F{}_{1\sigma(1)} > \ldots < p^T{}_{n\sigma(n)}, p^C{}_{n\sigma(n)}, p^U{}_{n\sigma(n)}, p^F{}_{n\sigma(n)} >$$

$$= \det(P).$$

This proves (iii)

Hence the Theorem

Theorem 10.5 Let P = (p_{ij}) be a QPNFM. Then we have the following

$$\det(Padj(P)) = \det(P) = \det(adj(P)P).$$

Proof: We prove that $\det(Padj(P)) = \det(P)$.

We first consider n = 2.

Let $P = \begin{pmatrix} < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} > & < p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} > \\ < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22} > \end{pmatrix}$

$$adj(P) = \begin{pmatrix} < p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22} > & < p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} > \\ < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} > & < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} > \end{pmatrix}$$

$$\det(Padj(P)) =$$

$$\begin{vmatrix} \det(P) & < p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} >< p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} > \\ < p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} >< p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22} > & \det(P) \end{vmatrix}$$

$$= \det(P) + \left(< p^T{}_{11}, p^C{}_{11}, p^U{}_{11}, p^F{}_{11} >< p^T{}_{12}, p^C{}_{12}, p^U{}_{12}, p^F{}_{12} >\right)$$
$$\left(< p^T{}_{21}, p^C{}_{21}, p^U{}_{21}, p^F{}_{21} >< p^T{}_{22}, p^C{}_{22}, p^U{}_{22}, p^F{}_{22} >\right)$$
$$\leq \det(P)$$

Next consider $n > 2$. We can see that

$$Padj(P) =$$

$$\begin{pmatrix} \sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{1t} & \sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{2t} & \ldots & \sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{nt} \\ \sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{1t} & \sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{2t} & \ldots & \sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{nt} \\ \ldots & \ldots & \ldots & \ldots \\ \sum < p^T{}_{nt}, p^C{}_{nt}, p^U{}_{nt}, p^F{}_{nt} > P_{1t} & \sum < p^T{}_{nt}, p^C{}_{nt}, p^U{}_{nt}, p^F{}_{nt} > P_{2t} & \ldots & \sum < p^T{}_{nt}, p^C{}_{nt}, p^U{}_{nt}, p^F{}_{nt} > P_{nt} \end{pmatrix}$$

$$= \left(\sum < p^T{}_{it}, p^C{}_{it}, p^U{}_{it}, p^F{}_{it} > P_{jt}\right).$$

$$\det(Padj(P)) = \sum_{\pi\in S_n} \left(\sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{\pi(2)t}\right)$$

$$...\left(\sum < p^T{}_{nt}, p^C{}_{nt}, p^U{}_{nt}, p^F{}_{nt} > P_{\pi(n)t}\right).$$

It is evident that each diagonal entry of the matrix Padj(P) equals det(P). We demonstrate this result as follows.

(i) Let us define

$$T_\pi = \left(\sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{\pi(2)t}\right)$$
$$...\left(\sum < p^T{}_{nt}, p^C{}_{nt}, p^U{}_{nt}, p^F{}_{nt} > P_{\pi(n)t}\right).$$

for $\pi \in S_n$. Let e be the identity of the group S_n. If $\pi = e$, then $T_\pi = \det(P)$.

Suppose that there exists $k \in \{1, 2, ..., n\}$ such that $\pi(k) = k$. Then we see that

$$= \sum < p^T{}_{kt}, p^C{}_{kt}, p^U{}_{kt}, p^F{}_{kt} > P_{\pi(k)t} = \sum < p^T{}_{kt}, p^C{}_{kt}, p^U{}_{kt}, p^F{}_{kt} > P_{kt}$$

$$= \det(P)$$

$$\Omega_n = \left(\sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{\pi(2)t}\right)...\det(P)...$$
$$\left(\sum < p^T{}_{nt}, p^C{}_{nt}, p^U{}_{nt}, p^F{}_{nt} > P_{\pi(n)t}\right) \le \det(P).$$

(ii) Let π be a permutation in S_n. Assume that $\pi(k) \neq k$. for all $k \in \{1, 2, ..., n\}$. We know that every permutation π can be written as a product of disjoint cycles π_i and let $\pi = \pi_1\pi_2...\pi_k$

We further assume that $\pi_1 = (1\ \ 2)$ transposition. Then Ω_π has two factors $\sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{\pi(1)t}$ and $\sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{\pi(2)t}$, and from these we see that

$$\left(\sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{\pi(1)t}\right)\left(\sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{\pi(2)t}\right)$$

$$= \left(\sum < p^T{}_{1t}, p^C{}_{1t}, p^U{}_{1t}, p^F{}_{1t} > P_{2t}\right)\left(\sum < p^T{}_{2t}, p^C{}_{2t}, p^U{}_{2t}, p^F{}_{2t} > P_{1t}\right)$$

$$\det\left(P(2 \Rightarrow 1)\right)\det\left(P(1 \Rightarrow 2)\right) \le \det(P)$$

(iii) If $\pi = \pi_1\pi_2...\pi_k$ and $\pi_1(s,t)$, then we can prove that $\Omega_n \le \det(P)$ by an argument used in (ii). Consider Ω for $\pi = \pi_1\pi_2...\pi_k$. If $\pi = (k, e, f, ...)$, then we see that

$$\Omega_n = \left(\sum < p^T{}_{kt}, p^C{}_{kt}, p^U{}_{kt}, p^F{}_{kt} > P_{\pi(k)t}\right)\left(\sum < p^T{}_{et}, p^C{}_{et}, p^U{}_{et}, p^F{}_{et} > P_{\pi(e)t}\right)...$$

$$= \left(\sum < p^T{}_{kt}, p^C{}_{kt}, p^U{}_{kt}, p^F{}_{kt} > P_{et}\right)\left(\sum < p^T{}_{et}, p^C{}_{et}, p^U{}_{et}, p^F{}_{et} > P_{ft}\right)...$$

$$= \det\left(P\left(e \Rightarrow k\right)\right)\det\left(P\left(f \Rightarrow e\right)\right)...$$

we obtain that $\det\left(P\left(e \Rightarrow k\right)\right)\det\left(P\left(f \Rightarrow e\right)\right) \leq \det\left(P\right)$ and so that $\Omega_n \leq \det\left(P\right)$. This proves that $\det\left(Padj\left(P\right)\right) = \det\left(P\right)$. Equally, we can prove that $\det\left(adj\left(P\right)P\right) = \det\left(P\right)$.

Hence the theorem.

Theorem 10.6 Let $P, Q \in (QPNFM)_n$. Then

(i) $\det(PQ) \geq \det(P)\det(Q)$

(ii) $\det(PQ) \leq \det(P+Q)$,

Where $\det(PQ) = \det\left(\sum_{k=1}^{n} p^T{}_{ik} \wedge q^T{}_{kj}, \sum_{k=1}^{n} p^C{}_{ik} \wedge q^C{}_{kj}, \prod_{k=1}^{n} p^U{}_{ik} \vee q^U{}_{kj}, \prod_{k=1}^{n} p^F{}_{ik} \vee q^F{}_{kj}\right)$

Proof: $P + Q = \left(\sup\left\{p_{ij}{}^T, q_{ij}{}^T\right\}, \sup\left\{p_{ij}{}^C, q_{ij}{}^C\right\}, \inf\left\{p_{ij}{}^U, q_{ij}{}^U\right\}, \inf\left\{p_{ij}{}^F, q_{ij}{}^F\right\}\right)$

$$= \sum_{\sigma \in S_n}\left(\sum_{k=1}^{n} p^T{}_{1k} \wedge q^T{}_{k\sigma(1)}, \sum_{k=1}^{n} p^C{}_{1k} \wedge q^C{}_{k\sigma(1)}, \prod_{k=1}^{n} p^U{}_{1k} \vee q^U{}_{k\sigma(1)}, \prod_{k=1}^{n} p^F{}_{1k} \vee q^F{}_{k\sigma(1)}\right), ...,$$

$$\left(\sum_{k=1}^{n} p^T{}_{nk} \wedge q^T{}_{k\sigma(n)}, \sum_{k=1}^{n} p^C{}_{nk} \wedge q^C{}_{k\sigma(n)}, \prod_{k=1}^{n} p^U{}_{nk} \vee q^U{}_{k\sigma(n)}, \prod_{k=1}^{n} p^F{}_{nk} \vee q^F{}_{k\sigma(n)}\right)$$

$$= \sum_{\sigma \in S_n}\left(\sum_{k_1, k_2, ..., k_n} p^T{}_{1k_1} \wedge p^T{}_{2k_2} ... \wedge p^T{}_{nk_n} \wedge q^T{}_{k_1\sigma(1)} \wedge q^T{}_{k_2\sigma(2)} ... \wedge q^T{}_{k_n\sigma(n)}\right)$$

$$\left(\sum_{k_1, k_2, ..., k_n} p^C{}_{1k_1} \wedge p^C{}_{2k_2} ... \wedge p^C{}_{nk_n} \wedge q^C{}_{k_1\sigma(1)} \wedge q^C{}_{k_2\sigma(2)} ... \wedge q^C{}_{k_n\sigma(n)}\right)$$

$$\left(\prod_{k_1, k_2, ..., k_n} p^U{}_{1k_1} \vee p^U{}_{2k_2} ... \vee p^U{}_{nk_n} \vee q^U{}_{k_1\sigma(1)} \vee q^U{}_{k_2\sigma(2)} ... \vee q^U{}_{k_n\sigma(n)}\right)$$

$$\left(\prod_{k_1, k_2, ..., k_n} p^F{}_{1k_1} \vee p^F{}_{2k_2} ... \vee p^F{}_{nk_n} \vee q^F{}_{k_1\sigma(1)} \vee q^F{}_{k_2\sigma(2)} ... \vee q^F{}_{k_n\sigma(n)}\right)$$

$$\geq \sum_{(k_1, k_2, ..., k_n) \in S_n} < p^T{}_{1k_1}, p^C{}_{1k_1}, p^U{}_{1k_1}, p^F{}_{1k_1} > ... < p^T{}_{nk_n}, p^C{}_{nk_n}, p^U{}_{nk_n}, p^F{}_{nk_n} >$$

$$\sum_{\sigma\in S_n} < q^T{}_{1k_1}, q^C{}_{1k_1}, q^U{}_{1k_1}, q^F{}_{1k_1} > \ldots < q^T{}_{nk_n}, q^C{}_{nk_n}, q^U{}_{nk_n}, q^F{}_{nk_n} >$$

$$= \left(\sum_{(k_1,k_2,\ldots,k_n)\in S_n} < p^T{}_{1k_1}, p^C{}_{1k_1}, p^U{}_{1k_1}, p^F{}_{1k_1} > \ldots < p^T{}_{nk_n}, p^C{}_{nk_n}, p^U{}_{nk_n}, p^F{}_{nk_n} > \right) \det(Q)$$

$$= \det(P)\det(Q)$$

(iii) We know that

$$\det(PQ) = \det\left(\sum_{k=1}^{n} p^T{}_{ik} \wedge q^T{}_{kj}, \sum_{k=1}^{n} p^C{}_{ik} \wedge q^C{}_{kj}, \prod_{k=1}^{n} p^U{}_{ik} \vee q^U{}_{kj}, \prod_{k=1}^{n} p^F{}_{ik} \vee q^F{}_{kj} \right)$$

$$\sum_{\sigma\in S_n} (\sum_{k=1}^{n} p^T{}_{ik} \wedge q^T{}_{k\sigma(1)}, \sum_{k=1}^{n} p^C{}_{ik} \wedge q^C{}_{k\sigma(1)}, \prod_{k=1}^{n} p^U{}_{ik} \vee q^U{}_{k\sigma(1)}, \prod_{k=1}^{n} p^F{}_{ik} \vee q^F{}_{k\sigma(1)} \cdots$$

$$\sum_{k=1}^{n} p^T{}_{nk} \wedge q^T{}_{k\sigma(n)}, \sum_{k=1}^{n} p^C{}_{nk} \wedge q^C{}_{k\sigma(n)}, \prod_{k=1}^{n} p^U{}_{nk} \vee q^U{}_{k\sigma(n)}, \prod_{k=1}^{n} p^F{}_{nk} \vee q^F{}_{k\sigma(n)})$$

$$= \sum_{\sigma\in S_n} (\bigcap_{t\le s,t\le n} \left(p^T{}_{1s} \vee q^T{}_{t\sigma(1)}\right), \bigcap_{t\le s,t\le n} \left(p^C{}_{1s} \vee q^C{}_{t\sigma(1)}\right), \bigcup_{t\le s,t\le n} \left(p^U{}_{1s} \wedge q^U{}_{t\sigma(1)}\right), \bigcup_{t\le s,t\le n} \left(p^F{}_{1s} \wedge q^F{}_{t\sigma(1)}\right) \ldots$$

$$\bigcap_{t\le s,t\le n} \left(p^T{}_{ns} \vee q^T{}_{t\sigma(n)}\right), \bigcap_{t\le s,t\le n} \left(p^C{}_{ns} \vee q^C{}_{t\sigma(n)}\right), \bigcup_{t\le s,t\le n} \left(p^U{}_{ns} \wedge q^U{}_{t\sigma(n)}\right), \bigcup_{t\le s,t\le n} \left(p^F{}_{ns} \wedge q^F{}_{t\sigma(n)}\right))$$

$$\le \sum_{\sigma\in S_n}^{n} \left(p^T{}_{1\sigma(1)} \vee q^T{}_{1\sigma(1)}\right) \wedge \left(p^C{}_{1\sigma(1)} \vee q^C{}_{1\sigma(1)}\right) \wedge \left(p^U{}_{1\sigma(1)} \vee q^U{}_{1\sigma(1)}\right) \wedge \left(p^F{}_{1\sigma(1)} \vee q^F{}_{1\sigma(1)}\right) \ldots.$$

$$\bigcup_{t\le s,t\le n} \left(p^T{}_{n\sigma(n)} \wedge q^T{}_{n\sigma(n)}\right) \wedge \left(p^C{}_{n\sigma(n)} \wedge q^C{}_{n\sigma(n)}\right) \wedge \left(p^U{}_{n\sigma(n)} \wedge q^U{}_{n\sigma(n)}\right) \wedge \left(p^F{}_{n\sigma(n)} \wedge q^F{}_{n\sigma(n)}\right)$$

$$= \det\left(\left(\left(p^T{}_{ij}, p^C{}_{ij}, p^U{}_{ij}, p^F{}_{ij} \right) + \left(q^T{}_{ij}, q^C{}_{ij}, q^U{}_{ij}, q^F{}_{ij} \right) \right)_n \right)$$

$$= \det\left(P+Q\right)$$

Hence the theorem.

Corollary 10.1. Let A be a QPNFM, $P_r = \left(p_{ij}\right) \in (QPNFM)_n$ (r = 1,2,3,…,m).Then

(i) $\det(P_1)\det(P_1)\ldots\det(P_m) \le \det\left(\sum_{r=1}^{m} P_r\right)$ whrere $\sum_{r=1}^{m} P_r = \left(\sum_{r=1}^{m} a^r{}_{ij}\right)_n \in (QPNFM)_n$.

(ii) $\det(P^{(r)}) = \det(P)$, where $P = \left(p_{ij}\right) \in (QPNFM)_n$ and $r \in N$.

Example 10.1. Consider the 4x4 QPNFM

$$P=\begin{pmatrix} < p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} > & < p^{T}{}_{12}, p^{C}{}_{12}, p^{U}{}_{12}, p^{F}{}_{12} > & \cdots & < p^{T}{}_{14}, p^{C}{}_{14}, p^{U}{}_{14}, p^{F}{}_{14} > \\ < p^{T}{}_{21}, p^{C}{}_{21}, p^{U}{}_{21}, p^{F}{}_{21} > & < p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > & \cdots & < p^{T}{}_{24}, p^{C}{}_{24}, p^{U}{}_{24}, p^{F}{}_{24} > \\ < p^{T}{}_{31}, p^{C}{}_{31}, p^{U}{}_{31}, p^{F}{}_{31} > & < p^{T}{}_{31}, p^{C}{}_{31}, p^{U}{}_{31}, p^{F}{}_{31} > & \cdots & < p^{T}{}_{34}, p^{C}{}_{34}, p^{U}{}_{34}, p^{F}{}_{34} > \\ < p^{T}{}_{41}, p^{C}{}_{41}, p^{U}{}_{41}, p^{F}{}_{41} > & < p^{T}{}_{42}, p^{C}{}_{42}, p^{U}{}_{42}, p^{F}{}_{42} > & \cdots & < p^{T}{}_{44}, p^{C}{}_{44}, p^{U}{}_{44}, p^{F}{}_{44} > \end{pmatrix}$$

We compute the determinant of the matrix above using the following QPNFM.

$$=\begin{vmatrix} < p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} > & < p^{T}{}_{12}, p^{C}{}_{12}, p^{U}{}_{12}, p^{F}{}_{12} > \\ < p^{T}{}_{21}, p^{C}{}_{21}, p^{U}{}_{21}, p^{F}{}_{21} > & < p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > \end{vmatrix}_{1<2}$$

$$\begin{vmatrix} < p^{T}{}_{33}, p^{C}{}_{33}, p^{U}{}_{33}, p^{F}{}_{33} > & < p^{T}{}_{34}, p^{C}{}_{34}, p^{U}{}_{34}, p^{F}{}_{34} > \\ < p^{T}{}_{43}, p^{C}{}_{43}, p^{U}{}_{43}, p^{F}{}_{43} > & < p^{T}{}_{44}, p^{C}{}_{44}, p^{U}{}_{44}, p^{F}{}_{44} > \end{vmatrix}$$

$$+\begin{vmatrix} < p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} > & < p^{T}{}_{13}, p^{C}{}_{13}, p^{U}{}_{13}, p^{F}{}_{13} > \\ < p^{T}{}_{21}, p^{C}{}_{21}, p^{U}{}_{21}, p^{F}{}_{21} > & < p^{T}{}_{23}, p^{C}{}_{23}, p^{U}{}_{23}, p^{F}{}_{23} > \end{vmatrix}_{1<3}$$

$$\begin{matrix} < p^{T}{}_{32}, p^{C}{}_{32}, p^{U}{}_{32}, p^{F}{}_{32} > & < p^{T}{}_{34}, p^{C}{}_{34}, p^{U}{}_{34}, p^{F}{}_{34} > \\ < p^{T}{}_{42}, p^{C}{}_{42}, p^{U}{}_{42}, p^{F}{}_{42} > & < p^{T}{}_{44}, p^{C}{}_{44}, p^{U}{}_{44}, p^{F}{}_{44} > \end{matrix}$$

$$+\begin{vmatrix} < p^{T}{}_{11}, p^{C}{}_{11}, p^{U}{}_{11}, p^{F}{}_{11} > & < p^{T}{}_{14}, p^{C}{}_{14}, p^{U}{}_{14}, p^{F}{}_{14} > \\ < p^{T}{}_{21}, p^{C}{}_{21}, p^{U}{}_{21}, p^{F}{}_{21} > & < p^{T}{}_{24}, p^{C}{}_{24}, p^{U}{}_{24}, p^{F}{}_{24} > \end{vmatrix}_{1<4}$$

$$\begin{vmatrix} < p^{T}{}_{32}, p^{C}{}_{32}, p^{U}{}_{32}, p^{F}{}_{32} > & < p^{T}{}_{33}, p^{C}{}_{33}, p^{U}{}_{33}, p^{F}{}_{33} > \\ < p^{T}{}_{42}, p^{C}{}_{42}, p^{U}{}_{42}, p^{F}{}_{42} > & < p^{T}{}_{43}, p^{C}{}_{43}, p^{U}{}_{43}, p^{F}{}_{43} > \end{vmatrix}$$

$$+\begin{vmatrix} < p^{T}{}_{12}, p^{C}{}_{12}, p^{U}{}_{12}, p^{F}{}_{12} > & < p^{T}{}_{13}, p^{C}{}_{13}, p^{U}{}_{13}, p^{F}{}_{13} > \\ < p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > & < p^{T}{}_{23}, p^{C}{}_{23}, p^{U}{}_{23}, p^{F}{}_{23} > \end{vmatrix}_{2<3}$$

$$\begin{vmatrix} < p^{T}{}_{31}, p^{C}{}_{31}, p^{U}{}_{31}, p^{F}{}_{31} > & < p^{T}{}_{34}, p^{C}{}_{34}, p^{U}{}_{34}, p^{F}{}_{34} > \\ < p^{T}{}_{41}, p^{C}{}_{41}, p^{U}{}_{41}, p^{F}{}_{41} > & < p^{T}{}_{44}, p^{C}{}_{44}, p^{U}{}_{44}, p^{F}{}_{44} > \end{vmatrix}$$

$$+\begin{vmatrix} < p^{T}{}_{12}, p^{C}{}_{12}, p^{U}{}_{12}, p^{F}{}_{12} > & < p^{T}{}_{14}, p^{C}{}_{14}, p^{U}{}_{14}, p^{F}{}_{14} > \\ < p^{T}{}_{22}, p^{C}{}_{22}, p^{U}{}_{22}, p^{F}{}_{22} > & < p^{T}{}_{24}, p^{C}{}_{24}, p^{U}{}_{24}, p^{F}{}_{24} > \end{vmatrix}_{2<4}$$

$$\begin{vmatrix} < p^{T}{}_{31}, p^{C}{}_{31}, p^{U}{}_{31}, p^{F}{}_{31} > & < p^{T}{}_{33}, p^{C}{}_{33}, p^{U}{}_{33}, p^{F}{}_{33} > \\ < p^{T}{}_{41}, p^{C}{}_{41}, p^{U}{}_{41}, p^{F}{}_{41} > & < p^{T}{}_{43}, p^{C}{}_{43}, p^{U}{}_{43}, p^{F}{}_{43} > \end{vmatrix}$$

$$+\begin{vmatrix} < p^{T}{}_{13}, p^{C}{}_{13}, p^{U}{}_{13}, p^{F}{}_{13} > & < p^{T}{}_{14}, p^{C}{}_{14}, p^{U}{}_{14}, p^{F}{}_{14} > \\ < p^{T}{}_{23}, p^{C}{}_{23}, p^{U}{}_{23}, p^{F}{}_{23} > & < p^{T}{}_{24}, p^{C}{}_{24}, p^{U}{}_{24}, p^{F}{}_{24} > \end{vmatrix}_{2<4}$$

$$\begin{vmatrix} <p^T{}_{31},p^C{}_{31},p^U{}_{31},p^F{}_{31}> & <p^T{}_{32},p^C{}_{32},p^U{}_{32},p^F{}_{32}> \\ <p^T{}_{41},p^C{}_{41},p^U{}_{41},p^F{}_{41}> & <p^T{}_{42},p^C{}_{42},p^U{}_{42},p^F{}_{42}> \end{vmatrix}$$

By applying this method, we can determine the determinant of the given QPNFM.

$$P=\begin{pmatrix} <0.8,0.7,0.3,0.5> & <0.7,0.5,0.4,0.2> & <0.7,0.8,0.3,0.2> & <0.3,0.6,0.3,0.1> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> & <0.4,0.7,0.8,0.2> & <0.1,0.3,0.5,0.9> \\ <0.4,0.6,0.8,0.7> & <0.2,0.3,0.4,0.5> & <0.6,0.3,0.5,0.7> & <0.9,0.8,0.5,0.4> \\ <0.5,0.3,0.4,0.8> & <0.4,0.8,0.5,0.2> & <0.1,0.3,0.6,0.8> & <0.7,0.3,0.2,0.1> \end{pmatrix}$$

$$=\begin{vmatrix} <0.8,0.7,0.3,0.5> & <0.7,0.5,0.4,0.2> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> \end{vmatrix}\begin{vmatrix} <0.6,0.3,0.5,0.7> & <0.9,0.8,0.5,0.4> \\ <0.1,0.3,0.6,0.8> & <0.7,0.3,0.2,0.1> \end{vmatrix}$$

$$+\begin{vmatrix} <0.8,0.7,0.3,0.5> & <0.7,0.8,0.3,0.2> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> \end{vmatrix}\begin{vmatrix} <0.2,0.3,0.4,0.5> & <0.9,0.8,0.5,0.4> \\ <0.4,0.8,0.5,0.2> & <0.7,0.3,0.2,0.1> \end{vmatrix}$$

$$+\begin{vmatrix} <0.8,0.7,0.3,0.5> & <0.3,0.6,0.3,0.1> \\ <0.5,0.9,0.4,0.3> & <0.1,0.3,0.5,0.9> \end{vmatrix}\begin{vmatrix} <0.2,0.3,0.4,0.5> & <0.6,0.3,0.5,0.7> \\ <0.4,0.8,0.5,0.2> & <0.1,0.3,0.6,0.8> \end{vmatrix}$$

$$+\begin{vmatrix} <0.7,0.5,0.4,0.2> & <0.7,0.8,0.3,0.2> \\ <0.1,0.3,0.5,0.9> & <0.4,0.7,0.8,0.2> \end{vmatrix}\begin{vmatrix} <0.4,0.6,0.8,0.7> & <0.9,0.8,0.5,0.4> \\ <0.5,0.3,0.4,0.8> & <0.7,0.3,0.2,0.1> \end{vmatrix}$$

$$+\begin{vmatrix} <0.7,0.5,0.4,0.2> & <0.3,0.6,0.3,0.1> \\ <0.1,0.3,0.5,0.9> & <0.1,0.3,0.5,0.9> \end{vmatrix}\begin{vmatrix} <0.4,0.6,0.8,0.7> & <0.6,0.3,0.5,0.7> \\ <0.5,0.3,0.4,0.8> & <0.1,0.3,0.6,0.8> \end{vmatrix}$$

$$+\begin{vmatrix} <0.7,0.8,0.3,0.2> & <0.3,0.6,0.3,0.1> \\ <0.4,0.7,0.8,0.2> & <0.1,0.3,0.5,0.9> \end{vmatrix}\begin{vmatrix} <0.4,0.6,0.8,0.7> & <0.2,0.3,0.4,0.5> \\ <0.5,0.3,0.4,0.8> & <0.2,0.3,0.4,0.5> \end{vmatrix}$$

$$=\left[<0.1,0.3,0.5,0.9>+<0.5,0.5,0.4,0.3>\right]\left[<0.6,0.3,0.5,0.7>+<0.1,0.3,0.6,0.8>\right]+$$

$$\left[<0.1,0.3,0.5,0.9>+<0.5,0.8,0.4,0.3>\right]\left[<0.2,0.3,0.4,0.8>+<0.1,0.3,0.6,0.8>\right]+$$

$$\left[<0.1,0.3,0.5,0.9>+<0.3,0.6,0.4,0.3>\right]\left[<0.1,0.3,0.6,0.8>+<0.4,0.3,0.5,0.7>\right]+$$

$$\left[<0.4,0.5,0.8,0.2>+<0.1,0.3,0.5,0.9>\right]\left[<0.4,0.3,0.8,0.7>+<0.5,0.3,0.8,0.7>\right]+$$

$$\left[<0.1,0.3,0.5,0.9>+<0.1,0.3,0.5,0.9>\right]\left[<0.1,0.3,0.8,0.8>+<0.5,0.3,0.5,0.8>\right]+$$

$$\left[<0.1,0.3,0.5,0.9>+<0.3,0.6,0.8,0.2>\right]\left[<0.2,0.3,0.8,0.7>+<0.2,0.3,0.4,0.8>\right]$$

$$=<0.5,0.5,0.4,0.3><0.6,0.3,0.5,0.7>+<0.5,0.8,0.4,0.3><0.2,0.3,0.4,0.8>+$$

$< 0.3, 0.6, 0.4, 0.3 >< 0.4, 0.3, 0.5, 0.7 > + < 0.4, 0.5, 0.5, 0.2 >< 0.5, 0.3, 0.8, 0.7 > +$
$< 0.1, 0.3, 0.5, 0.9 >< 0.5, 0.3, 0.5, 0.8 > + < 0.3, 0.6, 0.5, 0.2 >< 0.2, 0.3, 0.4, 0.7 >$
$=< 0.5, 0.3, 0.5, 0.7 > + < 0.2, 0.3, 0.4, 0.8 > + < 0.3, 0.3, 0.5, 0.7 > + < 0.4, 0.3, 0.8, 0.7 > +$
$< 0.1, 0.3, 0.5, 0.9 > + < 0.2, 0.3, 0.5, 0.7 >$
$=< 0.5, 0.3, 0.4, 0.7 >$

11. An Algorithm Based on QPNFM in a Decision-Making Problem

Definition 11.1. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. Then, for an QPNSS (η,E) over Y the degree of TM and the degree of − CM of an element y_i to $\eta(e_j)$ denoted by $T_{\eta(e_j)}(y_i)$ and $C_{\eta(e_j)}(y_i)$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{T_{\eta(e_j)}}(y_i) = \sum_{k=1}^{N}\left[T_{\eta(e_j)}(y_i) - T_{\eta(e_j)}(y_k)\right]$$

$$S_{C_{\eta(e_j)}}(y_i) = \sum_{k=1}^{N}\left[C_{\eta(e_j)}(y_i) - C_{\eta(e_j)}(y_k)\right]$$

Definition 11.2. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. Then, for an QPNSS (η,E) over Y the degree of UM and the degree of − FM of an element xi to $\eta(e_j)$ denoted by $U_{\eta(e_j)}(y_i)$ and $F_{\eta(e_j)}(y_i)$ respectively. Then, their corresponding score functions are denoted and defined by the following:

$$S_{U_{\eta(e_j)}}(y_i) = -\sum_{k=1}^{N}\left[U_{\eta(e_j)}(y_i) - U_{\eta(e_j)}(y_k)\right]$$

$$S_{F_{\eta(e_j)}}(y_i) = \sum_{K=1}^{N}\left[F_{\eta(e_j)}(y_i) - F_{\eta(e_j)}(y_k)\right]$$

Definition 11.3. Let Y = {$y_1, y_2, \ldots, y_n$} be an initial universe and $E = \{e_1, e_2, \ldots, e_m\}$ be a set of parameters. For an QPNSS (η,E) over Y, the scores of the TM, CM, UM, and FM of y_i for each e_j be denoted by $S_{T_{\eta(e_j)}}(y_i)$, $S_{C_{\eta(e_j)}}(y_i)$, $S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ respectively. Then, the total score of x_i for each e_j is denoted by $\square_{T_{\eta(e_j)}}(y_i) = S_{T_{\eta(e_j)}}(y_i) + S_{C_{\eta(e_j)}}(y_i) + S_{U_{\eta(e_j)}}(y_i) + S_{F_{\eta(e_j)}}(y_i)$

Based on the above definitions, we give the steps of the proposed algorithm as follows:

Algorithm:

Step 1. For the universal set Y = {$y_1, y_2, \ldots, y_n$} and the parameter set $E = \{e_1, e_2, \ldots, e_m\}$ input the matrix representation of an QPNSS (η,E) in tabular form, according to a decision-maker.

Step 2. Reference to the input matrix obtained in step 1 and using the Definitions 11.1 and 11.2, we compute $S_{T_{\eta(e_j)}}(y_i), S_{C_{\eta(e_j)}}(y_i), S_{U_{\eta(e_j)}}(y_i)$ and $S_{F_{\eta(e_j)}}(y_i)$ for xi for each ej where i = 1 to n; j = 1 to m.

Step 3. Taking the results obtained in step 2 and using the Definition 11.3, compute the score $\square_{T_{\eta(e_j)}}(x_i)$ of xi for each $ej$where i = 1 to n; j = 1 to m.

Step 4. Compute the overall score ui for xiin such a way that

$$v_i = \square_{T_{\eta(e_1)}}(y_i) + \square_{C_{\eta(e_2)}}(y_i) + \square_{U_{\eta(e_3)}}(y_i) + \ldots + \square_{F_{\eta(e_m)}}(y_i)$$

Step 5. Find k, for which $v_k = \max_{xi \in X}\{v_i\}$. Then, $x_k \in X$ is the optimal choice.

Step 6. In case of a tie, either we take both as an optimal choice or we reassess all the values with the expert's advice and repeat all the previous steps.

12. To demonstrate the practical application of the algorithm, we present the following example.

To implement the proposed algorithm successfully in a real-life context, we consider the following problem:

An agricultural planner wants to select the most suitable crop for cultivation in a specific region. However, limited expertise in agronomy complicates the decision-making process due to various conflicting factors involved in crop selection. To address this challenge, the planner consults a group of experts and decision makers (DMs) who specialize in agricultural sciences. The DMs evaluate five alternative crops according to a set of defined parameters.

Define Parameters

First, the five critical parameters for evaluating the crop options are identified as follows:

1. **Soil Fertility**: Measures the nutrient content and suitability of the soil for crop growth.
2. **Water Requirements**: Assesses the water needs of each crop relative to available irrigation and rainfall.
3. **Climate Suitability**: Evaluates how well the local climate matches the optimal growing conditions for each crop.
4. **Pest Resistance**: Indicates the crop's resilience against local pests and diseases, impacting yield stability.
5. **Economic Viability**: Considers market price, demand, and yield potential to determine the profitability of each crop.

Evaluation Procedure

To select the best crop alternative, the assessment procedure is executed as follows:

Step 1: Consider a set of five crops represented as C={ c_1,c_2,c_3,c_4,c_5}, where:

c_1 = Rice, c_2 = Wheat, c_3 = Maize, c_4 = Millet and c_5 = Barley

And define the set of parameters as P={p_1, p_2, p_3, p_4, p_5}, where:

p_1= Soil Fertility, p_2= Water Requirements, p_3 = Climate Suitability, p_4 = Pest Resistance

p_5 = Economic Viability

Based on the evaluations from the DMs, the decision matrix reflecting the five crops and five evaluation criteria under the Multi-Criteria Decision-Making framework is presented in Table 2.

Table 2. Tabular illustration of QPNFM to describe the set of five crops

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	<0.8,0.8,0.9,0.6>	<0.9,0.5,0.4,0.1>	<0.6,0.4,0.5,0.2>	<0.9,0.7,0.2,0.3>	<0.6,0.2,0.1,0.3>
y_2	<0.8,0.4,0.3,0.2>	<0.6,0.1,0.2,0.3>	<0.8,0.3,0.7,0.6>	<0.4,0.1,0.3,0.2>	<0.5,0.4,0.3,0.1>
y_3	<0.9,0.8,0.9,0.6>	<0.4,0.8,0.9,0.1>	<0.3,0.8,0.1,0.6>	<0.5,0.2,0.3,0.6>	<0.5,0.8,0.1,0.3>
y_4	<0.7,0.1,0.2,0.3>	<0.5,0.7,0.1,0.6>	<0.9,0.8,0.2,0.3>	<0.4,0.1,0.3,0.5>	<0.7,0.3,0.2,0.1>
y_5	<0.6,0.3,0.5,0.1>	<0.5,0.4,0.3,0.2>	<0.9,0.2,0.3,0.4>	<0.8,0.5,0.2,0.6>	<0.2,0.4,0.1,0.2>

Step 2. The score of the truth-membership degrees $S_{T_{\eta(e_j)}}(y_i)$ for (η ,E) is exposed in Table 3.

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	0.2	1.6	-0.5	1.5	0.5
y_2	0.2	0.1	0.5	-1	0
y_3	0.7	-0.9	-2	-0.5	0
y_4	-0.3	-0.4	1	-1	1
y_5	-0.8	-0.4	1	1	-1.5

The score of the contradiction-membership degrees $S_{C_{\eta(e_j)}}(y_i)$ for (η ,E) is exposed in Table 4.

Table 4. Tabular illustration of the score of contradiction-membership degree

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	1.6	0	-0.5	1.9	-1.1
y_2	-0.4	-2	-1	-1.1	-0.1
y_3	1.6	1.5	1.5	-0.6	1.9
y_4	-1.9	1	1.5	-1.1	-0.6
y_5	-0.9	-0.5	-1.5	0.9	-0.1

The score of the unknown-membership degrees $S_{U_{\eta(e_j)}}(y_i)$ for (η ,E) is exposed in Table 5.

Table 5. Tabular illustration of the score of unknown-membership degree

Y/E	e1	e2	e3	e4	e5
y_1	1.7	0.1	0.7	-0.3	-0.3
y_2	-1.3	-0.9	1.7	0.2	0.7
y_3	1.7	2.6	-1.3	0.2	-0.3
y_4	-1.8	-1.4	-0.8	0.2	0.2
y_5	-0.3	-0.4	-0.3	-0.3	-0.3

The score of the false-membership degrees $S_{F_{\eta(e_j)}}(y_i)$ for (η ,E) is exposed in Table 6.

Table 6. Tabular illustration of the score of false-membership degree

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	1.2	-0.8	-1.1	-0.8	0.5
y_2	-0.8	0.2	0.9	-1.3	-0.5
y_3	1.2	-0.8	0.9	0.7	0.5
y_4	-0.3	1.7	-0.6	0.2	-0.5
y_5	-1.3	-0.3	-0.1	0.7	0

Step 3. By using Table 3 to Table 6, the score $\eta_{(e_j)}(x_i)$ for (η ,E) is exhibited in Table 7.

Table 7. Tabular illustration of the score $\eta_{(e_j)}(x_i)$.

Y/E	e_1	e_2	e_3	e_4	e_5
y_1	4.7	0.9	-0.5	2.3	-0.4
y_2	-2.3	-2.6	2.1	-3.2	0.1
y_3	5.2	3.4	-0.9	-0.2	2.1
y_4	-4.3	0.9	1.1	-1.7	0.1
y_5	-3.3	-1.6	-0.9	2.3	-1.9

Step 4. Now, we calculate the overall score given as:

$v_1 = 7$, $v_2 = -5.9$, $v_3 = 9.6$, $v_4 = -3.9$, $v_5 = -5.4$.

Step 5. Thus, $v_k = \max_{y_i \in Y}\{v_1, v_2, v_3, v_4, v_5\} = v_3$. Therefore, v_3 is the optimal choice object for the decision maker.

13. Conclusion and Future Work

In this study, we investigated determinant theory for Quadri-Partitioned Neutrosophic Fuzzy Matrices (QPNFMs), establishing foundational properties and proofs for key determinant relationships, such as det(Padj(P)) = det(P) = det(adj(P)P). Additionally, we proposed an efficient method to calculate determinants for large QPNFMs, addressing computational challenges posed by matrices with higher dimensions. The introduction of a structured algorithm for decision-making problems further enhances the applicability of QPNFMs in complex decision scenarios. Our illustrative example demonstrates the practical relevance of our approach, verifying the effectiveness and robustness of our methodology in real-world applications.

Future research could expand on this work by exploring determinant properties in other classes of neutrosophic fuzzy matrices, including different partitioning schemes beyond quadri-partitioned structures. Investigating QPNFMs under various types of operations, such as matrix inversions and generalized matrix functions, could also deepen understanding of their theoretical and practical utility. Additionally, developing more sophisticated algorithms for multi-criteria decision-making using QPNFMs could broaden the scope of applications across different domains, such as engineering, economics, and artificial intelligence. Finally, computational optimizations leveraging parallel processing could make determinant calculation methods for QPNFMs more efficient, especially for high-dimensional matrices, further enhancing their viability in large-scale applications.

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