MAX MULTIPLICATIVE OPERATIONS ON NEUTROSOPHIC FUZZY MATRICES AND ITS APPLICATION

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Abstract: In this paper, we investigate into neutrosophic fuzzy matrices in particularly focusing on the newly introduce max multiplicative operations $\underset{\longleftrightarrow}{\otimes_a}$ and $\underset{\longleftrightarrow}{\otimes_b}$. We also examine how the new max multiplicative operations can be combined with the current properties of neutrosophic fuzzy matrices. This involves investigating the effects of integrating these new operations with traditional techniques in the area. Additionally, we develop the max multiplicative operator applied to Lukasiewicz conjunction and disjunction operators in NFM. We conduct a comparative analysis of customer ratings across multiple criteria, using NFMs to handle imprecise data in multi-criteria decision analysis (MCDA). The enhanced NFMs provide a robust approach to complex evaluations, offering valuable insights for decision-making processes in various real-world applications.

Keywords: Neutrosophic Fuzzy Matrices, Max multiplicative operations, Max- min. De Morgan's laws and decision-making processes.

1. Introduction

Zadeh [1] introduced the concept of fuzzy sets in 1965, laying the foundation for handling uncertainty in various fields. Traditional fuzzy sets characterize membership values, representing the degree of membership of an element in a set. The theory of fuzzy matrices, initially proposed by Kim and Roush [2], has garnered significant attention and adoption within diverse fields due to its efficacy in tackling uncertainties inherent in various real-world scenarios. In situations where uncertainty involves both membership and non-membership values, Atanassov [3] [4] introduced intuitionistic fuzzy sets, offering an appropriate work. Unlike traditional fuzzy sets, intuitionistic fuzzy sets incorporate both membership and non-membership values, enabling a more comprehensive representation of uncertainty.

Khan, Shymal, and Pal [5] significantly contributed to the evolution of fuzzy matrix theory by tracing the introduction of intuitionistic fuzzy matrices (IFM) in 2002, building upon the fundament laid by Atanassov. This seminal work marked a pivotal advancement in the field, offering a refined research work to handle uncertainties beyond the capabilities of traditional fuzzy matrices. Pal [6], in particular, played a crucial role in establishing the foundational for IFM and conducting thorough investigations into its implications and elucidated the theoretical underpinnings of IFM, elucidating its mathematical properties and practical applications. His rigorous investigations laid the groundwork for subsequent research endeavors, inspiring further advancements and innovations in the field of fuzzy matrix theory. Xu [7] and Wang [8] made substantial contributions to the advancement of intuitionistic fuzzy matrices (IFM), significantly shaping the trajectory of research in this field. Their innovative perspectives and insightful analyses have played pivotal roles in furthering the understanding and application of IFM in various domains.

Introduced as a novel mathematical concept by Florentin Smarandached [9] [10], neutrosophic fuzzy matrices amalgamate two essential theories neutrosophy and fuzzy set theory. Neutrosophy deals with indeterminacy, inconsistency and incomplete information, while fuzzy set theory addresses vagueness and uncertainty. By allowing the representation of elements with indeterminate truth-membership, false-membership, and indeterminacy, neutrosophic sets offer a more flexible framework for dealing with complex uncertainties. This makes them particularly useful in fields where traditional fuzzy sets and intuitionistic fuzzy sets may not adequately capture the nuances of uncertainty.

Muthuraji et al. [11] [12] [13] embarked on a pioneering endeavor to enhance the versatility and applicability of intuitionistic fuzzy matrices (IFM) by introducing novel Lukasiwicz disjunction and conjunction operators. They also extended their investigation to neutrosophic fuzzy matrices (NFM), another powerful tool for handling uncertainty. In addition to establishing the algebraic properties of the Lukasiwicz operators within the context of IFM and NFM. Boobalan. J and S.Sriram [14] develop the Arithmetic operations on IFM. Venkatesan and Sriram [15] [16] [17] elucidated fundamental characteristics of these multiplicative operations, such as associativity, distributivity, and commutativity. By systematically analyzing their algebraic properties, they provided valuable insights into the underlying mathematical structure of IFMs, facilitating a deeper understanding of their behavior.

Dhar[18] leveraged the principles of neutrosophy to develop a method for constructing neutrosophic fuzzy matrices that accurately capture the complex nature of indeterminate and inconsistent data. This approach represents a departure from traditional methodologies, which often struggle to adequately represent uncertainties characterized by multiple truth-values. The advancement of neutrosophic fuzzy matrices (NFMs) has been greatly aided by the efforts of Smarandache [19-20], Murugadas [21-22], Sophia [23-24] , Salama[25], uma [26] and Banu [27]. These contributions have enhanced the theoretical landscape and expanded the practical applications of this novel framework. They have pushed the limits of knowledge and created new opportunities for investigation in the field of uncertainty modeling through their innovative work, which has included the introduction novel theories and the analysis of a wide range of features connected with NFMs.

Ye, J. [28] and Ye, J., Yong, R., & Du, W. [29] developed methods for decision-making using neutrosophic similarity measures, showing how useful they are for evaluating multiple criteria. Building on this, Bai, X., & Ye, J. (2020) [30] looked at how neutrosophic sets can help deal with uncertainty in decision-making, while Liu, P., & Wang, Y. [31] focused on using max-min operations in group decision-making. More recently, Pandiselvi, M., & Jeyaraman, M. (2024) [32] expanded these ideas by applying neutrosophic systems to mathematical problems and optimization.

In this paper, we embark on an exploration of extending two fundamental max multiplicative operations to the realm of Neutrosophic Fuzzy Matrices (NFMs). Our objective is to investigate how these max multiplicative operations behave within the context of NFMs, to elucidate their algebraic characteristics in this resourceful exertion and demonstrate their practical application in multi-criteria decision analysis (MCDA) by handling imprecise data in scenarios like customer ratings and performance evaluations

2. Preliminaries

Definition 2.1 [5] [6] [9] [19] : An universe of U defined in a Neutrosophic set '$\square^{NFS}$' as $\square^{NFS} = \left\{ \left\{ x, p_T^N(x), p_I^N(x), p_F^N(x) \right\}, x \in X \right\}$, where $p_T^N, p_I^N, p_F^N : X \to]^{-}0, 1^{+}[$ and the condition $0^{-} \le p_T^N(x) + p_I^N(x) + p_F^N(x) \le 3^{+}$ and where p_T^N is degree of truth membership, p_I^N is the degree of indeterminacy and p_F^N is the degree of false non-membership.

Definition 2.2 [11], [24] : Let $\square^{NFS}$ and Ψ^{NFS} are two neutrosophic fuzzy set and let $P^{NMS} = \left(p_T^N, p_I^N, p_F^N\right) \in \square^{NFS}, Q^{NFS} = \left(q_T^N, q_I^N, q_F^N\right) \in \Psi^{NFS}$, define as operation of union and intersection over NFS is

$$P^{NMS} \cup Q^{NFS} = \left(\max\left(p_T^N, q_T^N\right), \min\left(p_I^N, q_I^N\right), \min\left(p_F^N, q_F^N\right)\right)$$
$$P^{NMS} \cap Q^{NFS} = \left(\min\left(p_T^N, q_T^N\right), \max\left(p_I^N, q_I^N\right), \max\left(p_F^N, q_F^N\right)\right)$$

Definition 2.5 [27]: Let $P^{NMS} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right)$ is said to be Neutrosophic Fuzzy Matrix, if the all elements are belongs to Neutrosophic Fuzzy Set.

Definition 2.6 [10]: Let P^{NFM} and Q^{NFM} in NFM then $P^{NFM} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right), Q^{NFM} = \left(\left\langle q_{ij}^T, q_{ij}^I, q_{ij}^F \right\rangle\right) \in F_{m \times n}$, define as type operation of union and intersection over NFM as

$$P^{NFM} \cup Q^{NFM} = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T\right), \min\left(p_{ij}^I, q_{ij}^I\right), \min\left(p_{ij}^F, q_{ij}^F\right) \right\rangle\right)$$
$$P^{NFM} \cap Q^{NFM} = \left(\left\langle \min\left(p_{ij}^T, q_{ij}^T\right), \max\left(p_{ij}^I, q_{ij}^I\right), \max\left(p_{ij}^F, q_{ij}^F\right) \right\rangle\right)$$

Definition 2.8 [26]: Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right),Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)\in F_{m\times n}$, P^{NFM} and Q^{NFM} be two NFM of same dimension. If $P^{NFM}\le Q^{NFM}$, If $p_{ij}^{T}\le q_{ij}^{T},p_{ij}^{I}\le q_{ij}^{I},p_{ij}^{F}\ge q_{ij}^{F}$, for i, j then P^{NFM} is dominated by Q^{NFM} or Q^{NFM} dominated P^{NFM}. P^{NFM} and Q^{NFM} is called comparable. If either $P^{NFM}\le Q^{NFM}(or)Q^{NFM}\le P^{NFM},P^{NFM}<Q^{NFM}$ or $p_{ij}^{T}<q_{ij}^{T},p_{ij}^{I}<q_{ij}^{I},p_{ij}^{F}>q_{ij}^{F}$

Definition 2.9 [23]: Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)\in F_{m\times n}$, the complement of P with in NFM is $\mathrm{P}^{c}=\left(\left\langle \mathrm{p}_{ij}^{F},\mathrm{p}_{ij}^{I},\mathrm{p}_{ij}^{T}\right\rangle\right)$

Definition: 2.12:[11]

A Neutrosophic fuzzy matrix P^{NFM} is $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$. As define the operations $\oplus_{L_{NFM}}$ and $\square_{L_{NFM}}$ on NFMs. Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFMs of order m×n. then

$$P^{NFM}\oplus_{L_{NFM}}Q^{NFM}=\left(\left\langle\left(p_{ij}^{T}+q_{ij}^{T}\right)\wedge_{L_{NFM}}1,\left(p_{ij}^{I}+q_{ij}^{I}\right)\wedge_{L_{NFM}}1,\left(p_{ij}^{F}+q_{ij}^{F}-1\right)\vee_{L_{NFM}}0\right\rangle\right)$$

$$P^{NFM}\square_{L_{NFM}}Q^{NFM}=\left(\left\langle\left(p_{ij}^{T}+q_{ij}^{T}-1\right)\vee_{L_{NFM}}0,\left(p_{ij}^{I}+q_{ij}^{I}-1\right)\vee_{L_{NFM}}0,\left(p_{ij}^{F}+q_{ij}^{F}\right)\wedge_{L_{NFM}}1\right\rangle\right)$$

3. MAX MULTIPLICATIVE OPERATIONS ON NEUTROSOPHIC FUZZY MATRICES:

Definition 3.1.

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size then we define

$$P^{NFM}\underset{\longleftrightarrow}{\otimes_a}Q^{NFM}=\left(\left\langle\max\left(p_{ij}^{T},q_{ij}^{T}\right),p_{ij}^{I}q_{ij}^{I},p_{ij}^{F}q_{ij}^{F}\right\rangle\right)$$

$$P^{NFM}\underset{\longleftrightarrow}{\otimes_b}Q^{NFM}=\left(\left\langle p_{ij}^{T}q_{ij}^{T},p_{ij}^{I}q_{ij}^{I},\max\left(p_{ij}^{F},q_{ij}^{F}\right)\right\rangle\right)$$

Here $p_{ij}^{T}q_{ij}^{T},p_{ij}^{I}q_{ij}^{I}$ and $p_{ij}^{F}q_{ij}^{F}$ are the ordinary multiplications

Property 3.2.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then $P^{NFM}\underset{\longleftrightarrow}{\otimes_b}Q^{NFM}\le P^{NFM}\underset{\longleftrightarrow}{\otimes_a}Q^{NFM}$.

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size.

Since $p_{ij}^{T}.q_{ij}^{T} \le \min\left(p_{ij}^{T}, q_{ij}^{T}\right) \le \max\left(p_{ij}^{T}, q_{ij}^{T}\right)$, $p_{ij}^{I}.q_{ij}^{I} \le \min\left(p_{ij}^{I}, q_{ij}^{I}\right) \le \max\left(p_{ij}^{I}, q_{ij}^{I}\right)$ and $\max\left(p_{ij}^{F}, q_{ij}^{F}\right) \ge p_{ij}^{F}.q_{ij}^{F}$ for all I and j . Hence $P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} \le P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM}$.

Proposition 3.3.

For any NFM P^{NFM} , then

i) $P^{NFM} \underleftrightarrow{\otimes_a} P^{NFM} \ne P^{NFM}$.

ii) $P^{NFM} \underleftrightarrow{\otimes_b} P^{NFM} \ne P^{NFM}$.

Proof:

$$P^{NFM} \underleftrightarrow{\otimes_a} P^{NFM} = \left(\left\langle \max\left(p_{ij}^{T}, p_{ij}^{T}\right), p_{ij}^{I} p_{ij}^{I}, p_{ij}^{F} p_{ij}^{F} \right\rangle\right) = \left(\left\langle p_{ij}^{T}, \left(p_{ij}^{I}\right)^2, \left(p_{ij}^{F}\right)^2 \right\rangle\right) \ne P^{NFM}$$

$$P^{NFM} \underleftrightarrow{\otimes_b} P^{NFM} = \left(\left\langle p_{ij}^{T} p_{ij}^{T}, p_{ij}^{I} p_{ij}^{I}, \max\left(p_{ij}^{F}, p_{ij}^{F}\right) \right\rangle\right) = \left(\left\langle \left(p_{ij}^{T}\right)^2, \left(p_{ij}^{I}\right)^2, p_{ij}^{F} \right\rangle\right) \ne P^{NFM}$$

Hence Proved.

Property 3.4.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then

i) $P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} = Q^{NFM} \underleftrightarrow{\otimes_a} P^{NFM}$.

ii) $P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} = Q^{NFM} \underleftrightarrow{\otimes_b} P^{NFM}$.

Proof:

Let $P^{NFM} = \left(\left\langle p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F} \right\rangle\right)$ and $Q^{NFM} = \left(\left\langle q_{ij}^{T}, q_{ij}^{I}, q_{ij}^{F} \right\rangle\right)$ be two NFM of the same size.

i) $$P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} = \left(\left\langle \max\left(p_{ij}^{T}, q_{ij}^{T}\right), p_{ij}^{I} q_{ij}^{I}, p_{ij}^{F} q_{ij}^{F} \right\rangle\right)$$

$$= \left(\left\langle \max\left(q_{ij}^{T}, p_{ij}^{T}\right), q_{ij}^{I} p_{ij}^{I}, q_{ij}^{F} p_{ij}^{F} \right\rangle\right)$$

$$= Q^{NFM} \underleftrightarrow{\otimes_a} P^{NFM}$$

ii) $$P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} = \left(\left\langle p_{ij}^{T} q_{ij}^{T}, p_{ij}^{I} q_{ij}^{I}, \max\left(p_{ij}^{F}, q_{ij}^{F}\right) \right\rangle\right)$$

$$= \left(\left\langle q_{ij}^{T} p_{ij}^{T}, q_{ij}^{I} p_{ij}^{I}, \max\left(q_{ij}^{F}, p_{ij}^{F}\right) \right\rangle\right)$$

$$= Q^{NFM} \underleftrightarrow{\otimes_b} P^{NFM}$$

Hence Proved.

Property 3.5.

If P^{NFM}, Q^{NFM} and R^{NFM} are any three NFMs of same size, then

i) $\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_a} R^{NFM} = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} \left(Q^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM}\right)$.

ii) $\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_b} R^{NFM} = P^{NFM} \underset{\longleftrightarrow}{\otimes_b} \left(Q^{NFM} \underset{\longleftrightarrow}{\otimes_b} R^{NFM}\right)$.

Proof:

Let $P^{NFM} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right)$, $Q^{NFM} = \left(\left\langle q_{ij}^T, q_{ij}^I, q_{ij}^F \right\rangle\right)$ and $R^{NFM} = \left(\left\langle r_{ij}^T, r_{ij}^I, r_{ij}^F \right\rangle\right)$ be three NFM of the same size.

By definition

$$P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM} = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T\right), p_{ij}^I q_{ij}^I, p_{ij}^F q_{ij}^F \right\rangle\right)$$

Then $\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_a} R^{NFM} = \left(\left\langle \max\left(\max\left(p_{ij}^T, q_{ij}^T\right), r_{ij}^T\right), p_{ij}^I q_{ij}^I r_{ij}^I, p_{ij}^F q_{ij}^F r_{ij}^F \right\rangle\right)$ ...(1)

Here $Q^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM} = \left(\left\langle \max\left(q_{ij}^T, r_{ij}^T\right), q_{ij}^I r_{ij}^I, q_{ij}^F r_{ij}^F \right\rangle\right)$

Then $P^{NFM} \underset{\longleftrightarrow}{\otimes_a} \left(Q^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM}\right) = \left(\left\langle \max\left(p_{ij}^T, \max\left(q_{ij}^T, r_{ij}^T\right)\right), p_{ij}^I q_{ij}^I r_{ij}^I, p_{ij}^F q_{ij}^F r_{ij}^F \right\rangle\right)$ ... (2)

By lemma $\max\left(\max\left(p_{ij}^T, q_{ij}^T\right), r_{ij}^T\right) = \max\left(p_{ij}^T, \max\left(q_{ij}^T, r_{ij}^T\right)\right)$

Therefore from (1) and (2) we get, $\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_a} R^{NFM} = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} \left(Q^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM}\right)$

Similarly we can proved $\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_b} R^{NFM} = P^{NFM} \underset{\longleftrightarrow}{\otimes_b} \left(Q^{NFM} \underset{\longleftrightarrow}{\otimes_b} R^{NFM}\right)$

Example 3.6.

Let $P^{NFM} = \begin{bmatrix} \langle 0.7, 0.3, 0.5 \rangle & \langle 0.8, 0.3, 0.7 \rangle \\ \langle 0.8, 0.3, 0.5 \rangle & \langle 0.9, 0.8, 0.4 \rangle \end{bmatrix}$, $Q^{NFM} = \begin{bmatrix} \langle 0.8, 0.2, 0.5 \rangle & \langle 0.7, 0.3, 0.5 \rangle \\ \langle 0.7, 0.8, 0.3 \rangle & \langle 0.6, 0.3, 0.2 \rangle \end{bmatrix}$ and

$R^{NFM} = \begin{bmatrix} \langle 0.6, 0.1, 0.3 \rangle & \langle 0.8, 0.2, 0.4 \rangle \\ \langle 0.6, 0.7, 0.2 \rangle & \langle 0.9, 0.4, 0.6 \rangle \end{bmatrix}$, verify that the above all Properties

1. $P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM} = \begin{bmatrix} \langle 08, 0.06, 0.25 \rangle & \langle 0.8, 0.09, 0.35 \rangle \\ \langle 0.8, 0.24, 0.15 \rangle & \langle 0.9, 0.24, 0.08 \rangle \end{bmatrix}$

$Q^{NFM} \underset{\longleftrightarrow}{\otimes_a} P^{NFM} = \begin{bmatrix} \langle 08, 0.06, 0.25 \rangle & \langle 0.8, 0.09, 0.35 \rangle \\ \langle 0.8, 0.24, 0.15 \rangle & \langle 0.9, 0.24, 0.08 \rangle \end{bmatrix}$

Hence $P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM} = Q^{NFM} \underset{\longleftrightarrow}{\otimes_a} P^{NFM}$

2. $$\left(P^{NFM} \underleftrightarrow{\otimes}_a Q^{NFM}\right) \underleftrightarrow{\otimes}_a R^{NFM} = \begin{bmatrix} \langle 08, 0.06, 0.75\rangle & \langle 0.8, 0.018, 0.14\rangle \\ \langle 0.8, 0.168, 0.3\rangle & \langle 0.9, 0.096, 0.048\rangle \end{bmatrix}$$

$$Q^{NFM} \underleftrightarrow{\otimes}_a R^{NFM} = \begin{bmatrix} \langle 08, 0.02, 0.15\rangle & \langle 0.8, 0.06, 0.2\rangle \\ \langle 0.8, 0.56, 0.6\rangle & \langle 0.9, 0.12, 0.12\rangle \end{bmatrix}$$

$$P^{NFM} \underleftrightarrow{\otimes}_a \left(Q^{NFM} \underleftrightarrow{\otimes}_a R^{NFM}\right) = \begin{bmatrix} \langle 08, 0.06, 0.75\rangle & \langle 0.8, 0.018, 0.14\rangle \\ \langle 0.8, 0.168, 0.3\rangle & \langle 0.9, 0.096, 0.048\rangle \end{bmatrix}$$

Hence $\left(P^{NFM} \underleftrightarrow{\otimes}_a Q^{NFM}\right) \underleftrightarrow{\otimes}_a R^{NFM} = P^{NFM} \underleftrightarrow{\otimes}_a \left(Q^{NFM} \underleftrightarrow{\otimes}_a R^{NFM}\right)$

3. $$P^{NFM} \underleftrightarrow{\otimes}_b Q^{NFM} = \begin{bmatrix} \langle 0.56, 0.06, 0.5\rangle & \langle 0.56, 0.09, 0.7\rangle \\ \langle 0.56, 0.24, 0.5\rangle & \langle 0.54, 0.24, 0.4\rangle \end{bmatrix}$$

$$Q^{NFM} \underleftrightarrow{\otimes}_b P^{NFM} = \begin{bmatrix} \langle 0.56, 0.06, 0.5\rangle & \langle 0.56, 0.09, 0.7\rangle \\ \langle 0.56, 0.24, 0.5\rangle & \langle 0.54, 0.24, 0.4\rangle \end{bmatrix}$$

Hence $P^{NFM} \underleftrightarrow{\otimes}_b Q^{NFM} = Q^{NFM} \underleftrightarrow{\otimes}_b P^{NFM}$

4. $$\left(P^{NFM} \underleftrightarrow{\otimes}_b Q^{NFM}\right) \underleftrightarrow{\otimes}_b R^{NFM} = \begin{bmatrix} \langle 0.336, 0.006, 0.5\rangle & \langle 0.448, 0.018, 0.7\rangle \\ \langle 0.336, 0.168, 0.5\rangle & \langle 0.486, 0.096, 0.6\rangle \end{bmatrix}$$

$$Q^{NFM} \underleftrightarrow{\otimes}_b R^{NFM} = \begin{bmatrix} \langle 0.48, 0.02, 0.5\rangle & \langle 0.56, 0.06, 0.5\rangle \\ \langle 0.42, 0.56, 0.3\rangle & \langle 0.54, 0.1, 0.6\rangle \end{bmatrix}$$

$$P^{NFM} \underleftrightarrow{\otimes}_b \left(Q^{NFM} \underleftrightarrow{\otimes}_b R^{NFM}\right) = \begin{bmatrix} \langle 0.336, 0.006, 0.5\rangle & \langle 0.448, 0.018, 0.7\rangle \\ \langle 0.336, 0.168, 0.5\rangle & \langle 0.486, 0.096, 0.6\rangle \end{bmatrix}$$

Hence $\left(P^{NFM} \underleftrightarrow{\otimes}_b Q^{NFM}\right) \underleftrightarrow{\otimes}_b R^{NFM} = P^{NFM} \underleftrightarrow{\otimes}_b \left(Q^{NFM} \underleftrightarrow{\otimes}_b R^{NFM}\right)$

Property 3.7.

If P^{NFM}, Q^{NFM} and R^{NFM} are any three NFMs of same size, then

i) $$P^{NFM} \underleftrightarrow{\otimes}_a \left(Q^{NFM} \underleftrightarrow{\otimes}_b R^{NFM}\right) \neq \left(P^{NFM} \underleftrightarrow{\otimes}_a Q^{NFM}\right) \underleftrightarrow{\otimes}_b \left(P^{NFM} \underleftrightarrow{\otimes}_a R^{NFM}\right)$$

ii) $$P^{NFM} \underleftrightarrow{\otimes}_b \left(Q^{NFM} \underleftrightarrow{\otimes}_a R^{NFM}\right) \neq \left(P^{NFM} \underleftrightarrow{\otimes}_b Q^{NFM}\right) \underleftrightarrow{\otimes}_a \left(P^{NFM} \underleftrightarrow{\otimes}_b R^{NFM}\right)$$

Proof:

Let $P^{NFM} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F\right\rangle\right)$, $Q^{NFM} = \left(\left\langle q_{ij}^T, q_{ij}^I, q_{ij}^F\right\rangle\right)$ and $R^{NFM} = \left(\left\langle r_{ij}^T, r_{ij}^I, r_{ij}^F\right\rangle\right)$ be three NFM of the same size.

By definition

$$P^{NFM} \underleftrightarrow{\otimes_a} \left(Q^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} \right) = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T r_{ij}^T \right), p_{ij}^I q_{ij}^I r_{ij}^I, p_{ij}^F . \max\left(q_{ij}^F, r_{ij}^F \right) \right\rangle \right)$$

$$\left(P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} \right) \underleftrightarrow{\otimes_b} \left(P^{NFM} \underleftrightarrow{\otimes_a} R^{NFM} \right) = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T \right) . \max\left(p_{ij}^T, r_{ij}^T \right), \left(p_{ij}^I q_{ij}^I \right) . \left(p_{ij}^I r_{ij}^I \right), \max\left(p_{ij}^F q_{ij}^F, p_{ij}^F r_{ij}^F \right) \right\rangle \right)$$

Since $\max\left(p_{ij}^T, q_{ij}^T \right) . \max\left(p_{ij}^T, r_{ij}^T \right) \leq \max\left(p_{ij}^T, q_{ij}^T r_{ij}^T \right)$

Hence $P^{NFM} \underleftrightarrow{\otimes_a} \left(Q^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} \right) \neq \left(P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} \right) \underleftrightarrow{\otimes_b} \left(P^{NFM} \underleftrightarrow{\otimes_a} R^{NFM} \right)$

Similarly we can proved $P^{NFM} \underleftrightarrow{\otimes_b} \left(Q^{NFM} \underleftrightarrow{\otimes_a} R^{NFM} \right) \neq \left(P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} \right) \underleftrightarrow{\otimes_a} \left(P^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} \right)$.

Example 3.8.

From Example 3.6, we use the values of P^{NFM}, Q^{NFM} and R^{NFM}

Then

$$Q^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} = \begin{bmatrix} \langle 0.48, 0.02, 0.5 \rangle & \langle 0.56, 0.06, 0.5 \rangle \\ \langle 0.42, 0.56, 0.3 \rangle & \langle 0.54, 0.1, 0.6 \rangle \end{bmatrix}$$

$$P^{NFM} \underleftrightarrow{\otimes_a} \left(Q^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} \right) = \begin{bmatrix} \langle 0.7, 0.06, 0.25 \rangle & \langle 0.8, 0.18, 0.35 \rangle \\ \langle 0.8, 0.168, 0.15 \rangle & \langle 0.9, 0.08, 0.24 \rangle \end{bmatrix} \quad \text{....(3)}$$

$$P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} = \begin{bmatrix} \langle 0.8, 0.06, 0.25 \rangle & \langle 0.8, 0.09, 0.35 \rangle \\ \langle 0.8, 0.24, 0.15 \rangle & \langle 0.9, 0.24, 0.08 \rangle \end{bmatrix}$$

$$P^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} = \begin{bmatrix} \langle 0.7, 0.03, 0.15 \rangle & \langle 0.8, 0.06, 0.28 \rangle \\ \langle 0.8, 0.21, 0.1 \rangle & \langle 0.9, 0.32, 0.24 \rangle \end{bmatrix}$$

$$\left(P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} \right) \underleftrightarrow{\otimes_b} \left(P^{NFM} \underleftrightarrow{\otimes_a} R^{NFM} \right) = \begin{bmatrix} \langle 0.56, 0.0018, 0.25 \rangle & \langle 0.64, 0.0108, 0.35 \rangle \\ \langle 0.64, 0.0504, 0.15 \rangle & \langle 0.81, 0.0768, 0.24 \rangle \end{bmatrix} \quad \text{....(4)}$$

From (3) and (4), hence $P^{NFM} \underleftrightarrow{\otimes_a} \left(Q^{NFM} \underleftrightarrow{\otimes_b} R^{NFM} \right) \neq \left(P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} \right) \underleftrightarrow{\otimes_b} \left(P^{NFM} \underleftrightarrow{\otimes_a} R^{NFM} \right)$.

Property 3.9.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then

i) $\left(P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} \right)^c = \left(P^{NFM} \right)^c \underleftrightarrow{\otimes_b} \left(Q^{NFM} \right)^c$

ii) $\left(P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} \right)^c = \left(P^{NFM} \right)^c \underleftrightarrow{\otimes_a} \left(Q^{NFM} \right)^c$

iii) $\left(P^{NFM} \underleftrightarrow{\otimes_a} Q^{NFM} \right)^c \leq \left(P^{NFM} \right)^c \underleftrightarrow{\otimes_a} \left(Q^{NFM} \right)^c$

iv) $\left(P^{NFM} \underleftrightarrow{\otimes_b} Q^{NFM} \right)^c \leq \left(P^{NFM} \right)^c \underleftrightarrow{\otimes_b} \left(Q^{NFM} \right)^c$

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$, $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be any two NFMs, for all i and j.

Then $\left(\mathrm{P}^{\mathrm{NFM}}\right)^{c}=\left(\left\langle \mathrm{p}_{\mathrm{ij}}^{\mathrm{F}},\mathrm{p}_{\mathrm{ij}}^{\mathrm{I}},\mathrm{p}_{\mathrm{ij}}^{\mathrm{T}}\right\rangle\right)$ and $\left(Q^{NFM}\right)^{c}=\left(\left\langle q_{ij}^{F},q_{ij}^{I},q_{ij}^{T}\right\rangle\right)$

$$\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}Q^{NFM}\right)^{c}=\left(\left\langle p_{ij}^{F}q_{ij}^{F},p_{ij}^{I}q_{ij}^{I},\max\left(p_{ij}^{T},q_{ij}^{T}\right)\right\rangle\right) \quad \dots (5)$$

$$\left(P^{NFM}\right)^{c}\underset{\longleftrightarrow}{\otimes}_{b}\left(Q^{NFM}\right)^{c}=\left(\left\langle p_{ij}^{F}q_{ij}^{F},p_{ij}^{I}q_{ij}^{I},\max\left(p_{ij}^{T},q_{ij}^{T}\right)\right\rangle\right) \quad \dots (6)$$

From Equation (5) and Equation (6), we get $\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}Q^{NFM}\right)^{c}=\left(P^{NFM}\right)^{c}\underset{\longleftrightarrow}{\otimes}_{b}\left(Q^{NFM}\right)^{c}$

Similarly, we can prove ii), iii) and iv).

Property 3.10.

If P^{NFM} for any NFM, then $\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(P^{NFM}\right)^{c}\right)^{c}=P^{NFM}\underset{\longleftrightarrow}{\otimes}_{b}\left(P^{NFM}\right)^{c}$ and $\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{b}\left(P^{NFM}\right)^{c}\right)^{c}=P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(P^{NFM}\right)^{c}$

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ be any NFM, for all i and j, then $\left(\mathrm{P}^{\mathrm{NFM}}\right)^{c_1}=\left(\left\langle \mathrm{p}_{\mathrm{F}_{ij}}^{\square},\mathrm{p}_{\mathrm{I}_{ij}}^{\square},\mathrm{p}_{\mathrm{T}_{ij}}^{\square}\right\rangle\right)$

$$P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(P^{NFM}\right)^{c}=\left(\left\langle \max\left(p_{ij}^{T},q_{ij}^{T}\right),p_{ij}^{I}q_{ij}^{I},p_{ij}^{F}q_{ij}^{F}\right\rangle\right) \quad \dots (7)$$

$$\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(P^{NFM}\right)^{c}\right)^{c}=\left(\left\langle p_{ij}^{F}p_{ij}^{T},p_{ij}^{I}p_{ij}^{I},\max\left(p_{ij}^{T},p_{ij}^{F}\right)\right\rangle\right) \quad \dots (8)$$

$$P^{NFM}\underset{\longleftrightarrow}{\otimes}_{b}\left(P^{NFM}\right)^{c}=\left(\left\langle p_{ij}^{T}p_{ij}^{F},p_{ij}^{I}p_{ij}^{I},\max\left(p_{ij}^{T},p_{ij}^{F}\right)\right\rangle\right) \quad \dots (9)$$

From Equations (8) and (9), we get $\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(P^{NFM}\right)^{c}\right)^{c}=P^{NFM}\underset{\longleftrightarrow}{\otimes}_{b}\left(P^{NFM}\right)^{c}$ is true.

$$\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{b}\left(P^{NFM}\right)^{c}\right)^{c}=\left(\left\langle \max\left(p_{ij}^{T},p_{ij}^{F}\right),p_{ij}^{I}p_{ij}^{I},p_{ij}^{T}p_{ij}^{F}\right\rangle\right) \quad \dots (10)$$

From Equations (7) and (10), we get $\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{b}\left(P^{NFM}\right)^{c}\right)^{c}=P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(P^{NFM}\right)^{c}$ is true.

Property 3.11.

If P^{NFM}, Q^{NFM} and R^{NFM} are any three NFMs of same size, then

i) $P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(Q^{NFM}\cup R^{NFM}\right)=\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}Q^{NFM}\right)\cup\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}R^{NFM}\right)$

ii) $P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}\left(Q^{NFM}\cap R^{NFM}\right)=\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}Q^{NFM}\right)\cap\left(P^{NFM}\underset{\longleftrightarrow}{\otimes}_{a}R^{NFM}\right)$

Proof:

$$P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM} = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T\right), p_{ij}^I q_{ij}^I, p_{ij}^F q_{ij}^F \right\rangle\right)$$

Here $P^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM} = \left(\left\langle \max\left(p_{ij}^T, r_{ij}^T\right), p_{ij}^I r_{ij}^I, p_{ij}^F r_{ij}^F \right\rangle\right)$

$$\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM}\right) = \left(\left\langle \begin{array}{l} \max\left(\max\left(p_{ij}^T, q_{ij}^T\right), \max\left(p_{ij}^T, r_{ij}^T\right)\right), \\ \min\left(p_{ij}^I q_{ij}^I, p_{ij}^I r_{ij}^I\right), \min\left(p_{ij}^F q_{ij}^F, p_{ij}^F r_{ij}^F\right) \end{array} \right\rangle\right)$$

$$= \left(\left\langle \max\left(p_{ij}^T, \max\left(p_{ij}^T, q_{ij}^T\right)\right), p_{ij}^I . \min\left(q_{ij}^I, r_{ij}^I\right), p_{ij}^I . \min\left(q_{ij}^F, r_{ij}^F\right) \right\rangle\right)$$

$$\left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM}\right) = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} \left(Q^{NFM} \cup R^{NFM}\right)$$

Similarly we prove that $P^{NFM} \underset{\longleftrightarrow}{\otimes_a} \left(Q^{NFM} \cap R^{NFM}\right) = \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) \cap \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} R^{NFM}\right)$

Property 3.12

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then

i) $P^{NFM} \cap \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = P^{NFM}$

ii) $P^{NFM} \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}\right) \neq P^{NFM}$

iii) $\left(P^{NFM} \cup Q^{NFM}\right) \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}$

iv) $\left(P^{NFM} \cap Q^{NFM}\right) \cap \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}\right) \neq P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}$

v) $\left(P^{NFM} \cup Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_a} \left(P^{NFM} \cap Q^{NFM}\right) = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}$

vi) $\left(P^{NFM} \cup Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_b} \left(P^{NFM} \cap Q^{NFM}\right) = P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}$

Proof:

Let $P^{NFM} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right)$ and $Q^{NFM} = \left(\left\langle q_{ij}^T, q_{ij}^I, q_{ij}^F \right\rangle\right)$ be two NFM of the same size.

i) $P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM} = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T\right), p_{ij}^I q_{ij}^I, p_{ij}^F q_{ij}^F \right\rangle\right)$

$$P^{NFM} \cap \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = \left(\left\langle \min\left(p_{ij}^T, \max\left(p_{ij}^T, q_{ij}^T\right)\right), \max\left(p_{ij}^I, p_{ij}^I q_{ij}^I\right), \max\left(p_{ij}^F, p_{ij}^F q_{ij}^F\right) \right\rangle\right)$$

Since $\min\left(p_{ij}^T, \max\left(p_{ij}^T, q_{ij}^T\right)\right) = p_{ij}^T, p_{ij}^I > p_{ij}^I q_{ij}^I, p_{ij}^F > p_{ij}^F q_{ij}^F$

Hence $P^{NFM} \cap \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = P^{NFM}$.

ii) $P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM} = \left(\left\langle p_{ij}^T q_{ij}^T, p_{ij}^I q_{ij}^I, \max\left(p_{ij}^F, q_{ij}^F\right) \right\rangle\right)$

$$P^{NFM} \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = \left(\left\langle \max\left(p_{ij}^{T}, p_{ij}^{T}q_{ij}^{T}\right), \min\left(p_{ij}^{I}, p_{ij}^{I}q_{ij}^{I}\right), \min\left(p_{ij}^{F}, \max\left(p_{ij}^{F}, q_{ij}^{F}\right)\right)\right\rangle\right)$$

Since $p_{ij}^{T} > p_{ij}^{T}q_{ij}^{T}, p_{ij}^{I} > p_{ij}^{I}q_{ij}^{I}, \min\left(p_{ij}^{F}, \max\left(p_{ij}^{F}, q_{ij}^{F}\right)\right) = p_{ij}^{F}$

Hence $P^{NFM} \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}\right) \neq P^{NFM}$

iii) $\mathrm{P}^{\mathrm{NFM}} \cup \mathrm{Q}^{\mathrm{NFM}} = \left(\left\langle \max\left(\mathrm{p}_{\mathrm{ij}}^{\mathrm{T}}, \mathrm{q}_{\mathrm{ij}}^{\mathrm{T}}\right), \min\left(\mathrm{p}_{\mathrm{ij}}^{\mathrm{I}}, \mathrm{q}_{\mathrm{ij}}^{\mathrm{I}}\right), \min\left(\mathrm{p}_{\mathrm{ij}}^{\mathrm{F}}, \mathrm{q}_{\mathrm{ij}}^{\mathrm{F}}\right)\right\rangle\right)$

$$P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM} = \left(\left\langle \max\left(p_{ij}^{T}, q_{ij}^{T}\right), p_{ij}^{I}q_{ij}^{I}, p_{ij}^{F}q_{ij}^{F}\right\rangle\right)$$

$$\left(P^{NFM} \cup Q^{NFM}\right) \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = \left(\left\langle \begin{array}{l} \max\left(\max\left(p_{ij}^{T}, q_{ij}^{T}\right), \max\left(p_{ij}^{T}, q_{ij}^{T}\right)\right), \\ \min\left(\min\left(p_{ij}^{I}, q_{ij}^{I}\right), p_{ij}^{I}q_{ij}^{I}\right), \\ \min\left(\min\left(p_{ij}^{F}, q_{ij}^{F}\right), p_{ij}^{F}q_{ij}^{F}\right) \end{array} \right\rangle\right)$$

Since $\max\left(\max\left(p_{ij}^{T}, q_{ij}^{T}\right), \max\left(p_{ij}^{T}, q_{ij}^{T}\right)\right) = \max\left(p_{ij}^{T}, q_{ij}^{T}\right),\ p_{ij}^{I} > p_{ij}^{I}q_{ij}^{I}, q_{ij}^{I} > p_{ij}^{I}q_{ij}^{I}$

And $p_{ij}^{F} > p_{ij}^{F}q_{ij}^{F}, q_{ij}^{F} > p_{ij}^{F}q_{ij}^{F}$

Therefore we get

$$\left(P^{NFM} \cup Q^{NFM}\right) \cup \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}\right) = \left(\left\langle \max\left(p_{ij}^{T}, q_{ij}^{T}\right), p_{ij}^{I}q_{ij}^{I}, p_{ij}^{F}q_{ij}^{F}\right\rangle\right)$$

iv) Similarly proved $\left(P^{NFM} \cap Q^{NFM}\right) \cap \left(P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}\right) \neq P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}$

v) $$\left(P^{NFM} \cup Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_a} \left(P^{NFM} \cap Q^{NFM}\right) = \left(\left\langle \begin{array}{l} \max\left(\max\left(p_{ij}^{T}, q_{ij}^{T}\right), \min\left(p_{ij}^{T}, q_{ij}^{T}\right)\right), \\ \min\left(p_{ij}^{I}, q_{ij}^{I}\right)\max\left(p_{ij}^{I}, q_{ij}^{I}\right), \\ \min\left(p_{ij}^{F}, q_{ij}^{F}\right)\max\left(p_{ij}^{F}, q_{ij}^{F}\right) \end{array} \right\rangle\right)$$

$$= \left(\left\langle \max\left(p_{ij}^{T}, q_{ij}^{T}\right), p_{ij}^{I}q_{ij}^{I}, p_{ij}^{F}q_{ij}^{F}\right\rangle\right)$$

$$\left(P^{NFM} \cup Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_a} \left(P^{NFM} \cap Q^{NFM}\right) = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}$$

vi) Similarly we can proved $\left(P^{NFM} \cup Q^{NFM}\right) \underset{\longleftrightarrow}{\otimes_b} \left(P^{NFM} \cap Q^{NFM}\right) = P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}$.

4. Max Multiplicative operator applied to Lukasiwicz Conjunction and Disjunction Operators in NFM:

Definition: 4.1

If P^{NFM} and Q^{NFM} are any two NFMs of same size and take $D^{NFM} = P^{NFM} \underset{\longleftrightarrow}{\otimes_a} Q^{NFM}$ and $G^{NFM} = P^{NFM} \underset{\longleftrightarrow}{\otimes_b} Q^{NFM}$, Then

$$D^{NFM} \oplus_{L_{NFM}} G^{NFM} = \left(\left\langle \begin{array}{l} \left(\max\left(p_{ij}^{T} + q_{ij}^{T}\right), p_{ij}^{T} q_{ij}^{T}\right) \wedge_{L_{NFM}} 1, \left(2 p_{ij}^{I} q_{ij}^{I}\right) \wedge_{L_{NFM}} 1, \\ \left(p_{ij}^{F} q_{ij}^{F} + \max\left(p_{ij}^{F} + q_{ij}^{F}\right) - 1\right) \vee_{L_{NFM}} 0 \end{array} \right\rangle \right)$$

$$D^{NFM} \square_{L_{NFM}} G^{NFM} = \left(\left\langle \begin{array}{l} \left(\max\left(p_{ij}^{T}, q_{ij}^{T}\right) + p_{ij}^{T} q_{ij}^{T} - 1\right) \vee_{L_{NFM}} 0, \left(2 p_{ij}^{I} q_{ij}^{I} - 1\right) \vee_{L_{NFM}} 0, \\ \left(p_{ij}^{F} q_{ij}^{F} + \max\left(p_{ij}^{F}, q_{ij}^{F}\right)\right) \wedge_{L_{NFM}} 1 \end{array} \right\rangle \right)$$

Example: 4.2

From Example 3. 6, we use the values of P^{NFM} and Q^{NFM} Then

$$D^{NFM} \oplus_{L_{NFM}} G^{NFM} = \begin{bmatrix} \langle 1, 0.12, 0 \rangle & \langle 1, 0.18, 0 \rangle \\ \langle 1, 0.48, 0 \rangle & \langle 1, 0.48, 0 \rangle \end{bmatrix}$$

$$D^{NFM} \square_{L_{NFM}} G^{NFM} = \begin{bmatrix} \langle 0.36, 0, 0.75 \rangle & \langle 0.36, 0, 1 \rangle \\ \langle 0.36, 0, 0.65 \rangle & \langle 0.44, 0, 0.48 \rangle \end{bmatrix}$$

Proposition : 6.3.2

If P^{NFM} and Q^{NFM} be any two NFMs of same size and take $D^{NFM} = P^{NFM} \otimes_a Q^{NFM}$ and $G^{NFM} = P^{NFM} \otimes_b Q^{NFM}$, Then

$$D^{NFM} \oplus_{L_1} G^{NFM} = G^{NFM} \oplus_{L_1} D^{NFM}$$

$$D^{NFM} \square_{L_1} G^{NFM} = G^{NFM} \square_{L_1} D^{NFM}$$

Proof:

Let $P^{NFM} = \left(\left\langle p_{T_{ij}}^{\square}, p_{I_{ij}}^{\square}, p_{F_{ij}}^{\square} \right\rangle\right)$ and $Q^{NFM} = \left(\left\langle q_{T_{ij}}^{\square}, q_{I_{ij}}^{\square}, q_{F_{ij}}^{\square} \right\rangle\right)$ be two NFMs of the same size.

i) $$D^{NFM} \oplus_{L_1} G^{NFM} = \left(\left\langle \begin{array}{l} \min\left(\max\left(p_{T_{ij}}^{\square}, q_{T_{ij}}^{\square}\right) + p_{T_{ij}}^{\square} q_{T_{ij}}^{\square}, 1\right), \min\left(2 p_{I_{ij}}^{\square} q_{I_{ij}}^{\square}, 1\right), \\ \max\left(p_{F_{ij}}^{\square} q_{F_{ij}}^{\square} + \max\left(p_{F_{ij}}^{\square}, q_{F_{ij}}^{\square}\right) - 1, 0\right) \end{array} \right\rangle \right)$$

$$= \left(\left\langle \begin{array}{l} \min\left(\max\left(q_{T_{ij}}^{\square}, p_{T_{ij}}^{\square}\right) + q_{T_{ij}}^{\square} p_{T_{ij}}^{\square}, 1\right), \min\left(2 q_{I_{ij}}^{\square} p_{I_{ij}}^{\square}, 1\right), \\ \max\left(q_{F_{ij}}^{\square} p_{F_{ij}}^{\square} + \max\left(q_{F_{ij}}^{\square}, p_{F_{ij}}^{\square}\right) - 1, 0\right) \end{array} \right\rangle \right)$$

$$= G^{NFM} \oplus_{L_1} D^{NFM}$$

Hence $D^{NFM} \oplus_{L_1} G^{NFM} = G^{NFM} \oplus_{L_1} D^{NFM}$

5. Application of max multiplicative operations

XYZ Inc. and ABC Inc. are technology companies known for their innovative gadgets and wearable devices. With a strong focus on blending style and functionality, the companies have captured the market's attention with their range of smart products, including smart watches, Smartphones, fitness trackers, and wireless earbuds. We analyze the products across four categories: durability and reliability, design and safety, quality and performance for customer ratings.

Let as consider P^{NFM} and Q^{NFM} be NFMs for the companies XYZ Inc. and ABC Inc. Here four customer (S_1, S_2, S_3, S_4) rating with different evaluation criteria as durability and reliability (f_1), design and safety (f_2), quality (f_2) and performance (f_4).

$$P^{NFM} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{pmatrix} \langle 0.7,0.3,0.6\rangle & \langle 0.8,0.9,0.3\rangle & \langle 0.9,0.7,0.3\rangle & \langle 0.8,0.5,0.6\rangle \\ \langle 0.5,0.8,0.4\rangle & \langle 0.7,0.8,0.5\rangle & \langle 0.2,0.9,0.5\rangle & \langle 0.7,0.4,0.5\rangle \\ \langle 0.3,0.9,0.4\rangle & \langle 0.6,0.7,0.9\rangle & \langle 0.7,0.4,0.6\rangle & \langle 0.9,0.1,0.5\rangle \\ \langle 0.7,0.3,0.4\rangle & \langle 0.5,0.9,0.1\rangle & \langle 0.8,0.7,0.7\rangle & \langle 0.5,0.3,0.8\rangle \end{pmatrix} \end{array}$$

In P^{NFM}, the element in the matrix second row third column $\langle 0.2, 0.9, 0.5\rangle$ is the represents as the second customer rating quality of the products smart watches, smartphones, fitness trackers and wireless earbuds.

$$Q^{NFM} = \begin{array}{c} \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{pmatrix} \langle 0.1,0.5,0.8\rangle & \langle 0.5,0.6,0.5\rangle & \langle 0.8,0.6,0.7\rangle & \langle 0.9,0.8,0.9\rangle \\ \langle 0.7,0.6,0.8\rangle & \langle 0.9,0.3,0.1\rangle & \langle 0.5,0.6,0.7\rangle & \langle 0.5,0.8,0.6\rangle \\ \langle 0.7,0.8,0.9\rangle & \langle 0.8,0.4,0.6\rangle & \langle 0.7,0.8,0.4\rangle & \langle 0.9,0.1,0.7\rangle \\ \langle 0.8,0.9,0.4\rangle & \langle 0.7,0.5,0.7\rangle & \langle 0.8,0.9,0.3\rangle & \langle 0.5,0.3,0.8\rangle \end{pmatrix} \end{array}$$

In Q^{NFM}, the element $\langle 0.8, 0.9, 0.3\rangle$ is the represents as the fourth customer rating maintenance of the products smart watches, Smartphones, fitness trackers and wireless earbuds. Apply the max multiplicative operations $\otimes_a$ and $\otimes_b$, then we get

$$P^{NFM} \otimes_a Q^{NFM} = \begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{pmatrix} \langle 0.7, 0.15, 0.48 \rangle & \langle 0.8, 0.54, 0.15 \rangle & \langle 0.9, 0.42, 0.21 \rangle & \langle 0.9, 0.4, 0.54 \rangle \\ \langle 0.5, 0.56, 0.36 \rangle & \langle 0.8, 0.56, 0.25 \rangle & \langle 0.7, 0.72, 0.1 \rangle & \langle 0.7, 0.12, 0.2 \rangle \\ \langle 0.5, 0.72, 0.32 \rangle & \langle 0.6, 0.42, 0.63 \rangle & \langle 0.7, 0.36, 0.3 \rangle & \langle 0.9, 0.08, 0.25 \rangle \\ \langle 0.7, 0.21, 0.08 \rangle & \langle 0.8, 0.81, 0.04 \rangle & \langle 0.8, 0.56, 0.21 \rangle & \langle 0.6, 0.27, 0.24 \rangle \end{pmatrix} \end{array}$$

And

$$P^{NFM} \otimes_b Q^{NFM} = \begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} & \begin{pmatrix} \langle 0.07, 0.15, 0.8 \rangle & \langle 0.4, 0.54, 0.5 \rangle & \langle 0.72, 0.42, 0.7 \rangle & \langle 0.72, 0.4, 0.1 \rangle \\ \langle 0.15, 0.56, 0.9 \rangle & \langle 0.56, 0.56, 0.5 \rangle & \langle 0.14, 0.72, 0.5 \rangle & \langle 0.49, 0.12, 0.5 \rangle \\ \langle 0.15, 0.72, 0.8 \rangle & \langle 0.12, 0.42, 0.9 \rangle & \langle 0.14, 0.36, 0.6 \rangle & \langle 0.63, 0.08, 0.5 \rangle \\ \langle 0.49, 0.21, 0.4 \rangle & \langle 0.4, 0.81, 0.4 \rangle & \langle 0.48, 0.56, 0.7 \rangle & \langle 0.3, 0.27, 0.8 \rangle \end{pmatrix} \end{array}$$

By comparing the customer ratings across these criteria, we can better understand the strengths and weaknesses of XYZ Inc. and ABC Inc. in the competitive tech landscape. Using the max multiplicative operators in Neutrosophic Fuzzy Matrices (NFMs) helps improve decision-making across multiple criteria. Operator $\underset{\longleftrightarrow}{\otimes_a}$ is more useful when you need to clearly see the differences between options, as it highlights specific strengths and weaknesses. On the other hand, Operator $\underset{\longleftrightarrow}{\otimes_b}$ provides a more balanced view, smoothing out extreme differences to give an overall picture. The choice of operator depends on the goal: $\underset{\longleftrightarrow}{\otimes_a}$ is better for detailed comparisons, while $\underset{\longleftrightarrow}{\otimes_b}$ is ideal for a more general, balanced assessment.

6. Conclusions

Neutrosophic Fuzzy Matrices (NFMs) in relation to the max multiplicative operations $\underset{\longleftrightarrow}{\otimes_a}$ and $\underset{\longleftrightarrow}{\otimes_b}$ demonstrates commutatively. Additionally, these operations adhere to De Morgan's laws, a fundamental principle in logic. Furthermore, distributive laws, specifically those concerning max-min and min-max compositions over $\underset{\longleftrightarrow}{\otimes_a}$ and $\underset{\longleftrightarrow}{\otimes_b}$, have been rigorously proven, establishing several algebraic properties. In evaluating customer ratings across multiple criteria, Neutrosophic Fuzzy Matrices (NFMs) combined with max multiplicative operators offer valuable insights. The operator $\underset{\longleftrightarrow}{\otimes_a}$ analysis the strengths and weaknesses of the companies and the operator $\underset{\longleftrightarrow}{\otimes_b}$ provides more balanced and providing a holistic view of the companies. Choosing the right operator helps assess each company's performance depending on the level of detail needed.

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