NSS
Neutrosophic Sets and Systems, Vol. 95, 2026
University of New Mexico

Metrizability of Topology, Precompactness and Semi-Compatibal Mappings in Neutrosophic Metric Spaces

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Abstract: In this manuscript, we discuss the metrizability of the topology produced by arbitrary neutrosophic metric space. Further, we demonstrate that the resulting topology is completely metrizable if the neutrosophic metric space is complete and a neutrosophic metric space is precompact if and only if each sequence has a Cauchy subsequence. Furthermore, we present the idea of a non-Archimedean generalised neutrosophic metric space, analyse its P and Q properties, and get certain results for two semi-compatible mappings in this newly constructed space. In the end, we provide several nontrivial examples to support our main results.

Keywords: neutrosophic metric space; metrizable; non-Archimedean space; semi-compatible mapping.

1. Introduction

Fuzzy Sets, introduced by Zadeh [1], created a foundational work in the field of fixed point (FP) theory by addressing uncertainty. For the researchers to investigate, this idea opened up a new perspective. The most significant disadvantage of the fuzzy set was its lack of information on the non-membership function. Numerous researchers examined the suitable and consistent concept of a fuzzy metric space (FMS) presented by Kramosil and Michalek [2]. Additionally, they examined into this issue and developed and investigated a number of distinct concepts related to a FMS. FMSs are an important notion that George and Veeramani [3, 4] explored and demonstrated that all FMSs produce a Hausdorff first countable topology. Additionally, it has been demonstrated by Gregori and Romaguera [5] that any FMS can yield a metrizable topology and that if the FMS is complete, the generated topology can also be completely metrizable (CM). An intuitionistic fuzzy set, a generalisation of fuzzy sets that include membership and non-membership functions, was presented by Atanassov [6]. The intuitionistic fuzzy metric space (IFMS) was recently established by Park et al. [7], and Park et al. [8–12] investigated the numerous characteristics on IFMS. Saadati and park [13] derived the precompact set in IFMS and prove that every sub set of IFMS is precompact set as well as complete iff it is compact. Jeyaraman et al. [14] established the notion of non-Archimedean generalised IFMS. Park et al. [15] studied that every sequence has Cauchy subsequence if and only if IFMS is precompect.

Smarandache [16] presented the neutrosophic metric space (NMS) and expanded the concept of IFMS. Riaz et al. [17] developed the notion of neutrosophic cone metric space and ξ- chainable NMS and select three self-mappings for generalized the FP results in these spaces. Ishtiaq et al. [18] introduced the notion of orthogonal NMS and generalized the FP results in the sense of orthogonal NMS. Das and Tripathy [19] studied the neutrosophic simple b- open set, b- open cover, and b- compactness in neutrosophic topological space. Two fundamental theorems of classical analysis, Baire's and Cantor's intersection theorems, were established in the context of Neutrosophic 2-metric spaces (N2MS) by Ishtiaq et al. [20]. Das et al. [21] introduced the notion of neutrosophic ifni-open set (OS), ifni- semi-open set, ifni- pre-open set, ifni-b-OSin the sense of neutrosophic infi topological space. See [22-28] for more related literature.

In this work, we establish the metrizability of the topology produced by any NMS. Furthermore, we demonstrate that a NMS is precompact iff each sequence has a Cauchy subsequence, and that if the NMS is complete, the generated topology is completely metrizable. We propose the notion of non-Archimedean generalised neutrosophic metric space (NAGNMS) and generalized the FP results. Our results improve the results of jeyaraman et. al [14] and park et al. [15]. In section 2, of this manuscript we will discuss some basic definitions. In section 3, we will discuss second countablity, precompact, and Baire Space on NMS. In section 4, we generalized the NMS and developed the notion of NAGNMS and prove some interesting FP in this space also give some non-trivial examples which support our result. In section 5, we will discuss the P, Q properties.

2. Preliminaries

In this, we will study some definitions that are support our main result.

Definition 2.1: [12] A mapping $*: \Xi^2 \to \Xi$ is called continuous t-norm (CTN) if it is fulfilling the conditions which are given below:

(i) $*$ is continuous, commutative and associative,

(ii) $\mathfrak{a} * 1 = \mathfrak{a}$ for all $\mathfrak{a} \in \Xi$,

(iii) $\mathfrak{a} * \mathfrak{b} \le c * d$ whenever $\mathfrak{a} \le c$ and $\mathfrak{b} \le d$, for all $\mathfrak{a}, \mathfrak{b}, c, d \in \Xi$.

Definition 2.2: [12] A mapping $\Delta: \Xi^2 \to \Xi$ is said to be a continuous t-conorm (CTCN) if fulfilling the conditions which are given below:

(i) Δ is continuous, commutative and associative,

(ii) $\mathfrak{a}\Delta 0 = \mathfrak{a}$ for all $\mathfrak{a} \in \Xi$,

(iii) $\mathfrak{a}\Delta\mathfrak{b} \le c\Delta d$ whenever $\mathfrak{a} \le c$ and $\mathfrak{b} \le d$, for all $\mathfrak{a}, \mathfrak{b}, c, d \in \Xi$.

Definition 2.3: [7] A 5-tuple $(\Xi, P_i, N_i, *, \Delta)$ is called an IFMS if $*, \Delta$ are CTN and CTCN and P_i, N_i are fuzzy sets defined on $\Xi^2 \times (0, +\infty)$ fulfill below circumstances:

(i) $P_i(\mathfrak{e}, \mathfrak{z}, \tau) > 0, N_i(\mathfrak{e}, \mathfrak{z}, \tau) < 1$ and $0 \le P_i(\mathfrak{e}, \mathfrak{z}, \tau) + N_i(\mathfrak{e}, \mathfrak{z}, \tau) \le 1$,

(ii) $P_i(\mathfrak{e}, \mathfrak{z}, \tau) = 1$ iff $\mathfrak{e} = \mathfrak{z}$,

(iii) $N_i(\mathfrak{e}, \mathfrak{z}, \tau) = 0$ iff $\mathfrak{e} = \mathfrak{z}$,

(iv) $P_i(\mathfrak{e}, \mathfrak{z}, \tau) = P_i(\mathfrak{z}, \mathfrak{e}, \tau)$ and $N_i(\mathfrak{e}, \mathfrak{z}, \tau) = N_i(\mathfrak{z}, \mathfrak{e}, \tau)$,

(v) $P_i(\mathfrak{e}, \mathfrak{z}, \tau) * P_i(\mathfrak{z}, \omega, \lambda) \le P_i(\mathfrak{e}, \omega, \tau + \lambda)$,

(vi) $N_i(\mathfrak{e}, \mathfrak{z}, \tau)\Delta N_i(\mathfrak{z}, \omega, \lambda) \ge N_i(\mathfrak{e}, \omega, \tau + \lambda)$,

(vii) $P_i(\mathfrak{e}, \mathfrak{z}, \cdot), N_i(\mathfrak{e}, \mathfrak{z}, \cdot): (0, +\infty) \to (0, 1]$ are continuous.

for every $\mathfrak{e}, \mathfrak{z}, \omega \in \Xi$ and $\lambda, \tau > 0$.

Definition 2.4: [12] Let Ξ be an IFMS and let $0 < \mathfrak{r} < 1, \tau > 0$ and $\mathfrak{e} \in \Xi$. The set

$$\mathcal{B}(\mathfrak{e}, \mathfrak{r}, \tau) = \{\mathfrak{z} \in \Xi; P_i(\mathfrak{e}, \mathfrak{z}, \tau) > 1 - \mathfrak{r}, N_i(\mathfrak{e}, \mathfrak{z}, \tau) < \mathfrak{r}\},$$

is said to be an open ball, and the center of this open ball is $\mathfrak{e} \in \Xi$ with radius $\mathfrak{r}$. Let Ξ be an IFMS. Define

$$\tau = \{\mathcal{A} \subset \Xi; \mathfrak{e} \in \mathcal{A} \text{ iff } \exists\, \mathfrak{r}, \tau > 0, 0 < \mathfrak{r} < 1 \text{ such that } \mathcal{B}(\mathfrak{e}, \mathfrak{r}, \tau) \subset \mathcal{A}\}.$$

Then τ is a topology on Ξ.

Definition 2.5: [14] A 5-tuple $(\Xi, P, N, *, \Delta)$ is called a non-Archimedean generalized IFMS if Ξ is an arbitrary set $*, \Delta$ are CTN and CTCN and P, N are fuzzy sets defined on $\Xi^3 \times (0, +\infty)$ fulfill the below circumstances for all $\mathfrak{e}, \mathfrak{z}, \omega, \mathfrak{a} \in \Xi$ and $\lambda, \tau > 0$,

(i) $P(\mathfrak{e}, \mathfrak{z}, \omega, \tau) + N(\mathfrak{e}, \mathfrak{z}, \omega, \tau) \leq 1$,

(ii) $P(\mathfrak{e}, \mathfrak{z}, \omega, \tau) > 0$,

(iii) $P(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = 1$ iff $\mathfrak{e} = \mathfrak{z} = \omega$,

(iv) $P(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = P(\mathfrak{p}\{\mathfrak{e}, \mathfrak{z}, \omega\}, \tau)$, where $\mathfrak{p}$ is a permutation function (PF),

(v) $P(\mathfrak{e}, \mathfrak{z}, \mathfrak{a}, \tau) * P(\mathfrak{a}, \omega, \omega, \lambda) \leq P(\mathfrak{e}, \mathfrak{z}, \omega, \max\{\tau, \lambda\})$,

(vi) $P(\mathfrak{e}, \mathfrak{z}, \omega, \cdot): (0, +\infty) \to [0, 1]$ is continuous,

(vii) $N(\mathfrak{e}, \mathfrak{z}, \omega, \tau) > 0$,

(viii) $N(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = 1$ iff $\mathfrak{e} = \mathfrak{z} = \omega$,

(ix) $N(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = N(\mathfrak{p}\{\mathfrak{e}, \mathfrak{z}, \omega\}, \tau)$, where $\mathfrak{p}$ is a PF,

(x) $N(\mathfrak{e}, \mathfrak{z}, \mathfrak{a}, \tau)\, \Delta\, N(\mathfrak{a}, \omega, \omega, \lambda) \geq N(\mathfrak{e}, \mathfrak{z}, \omega, \min\{\tau, \lambda\})$,

(xi) $N(\mathfrak{e}, \mathfrak{z}, \omega, \cdot): (0, +\infty) \to [0, 1]$ is continuous.

The pair (P, N) is called a generalized intuitionistic fuzzy metric on Ξ.

Definition 2.6: [14] The class of continuous functions $\phi, \psi: [0, 1] \to [0, 1]$ are denoted by Φ, such that $\phi(\tau) > \tau$, where $\tau \in [0,1)$ and $\phi(1) = 1$. similarly, $\psi(\tau) < \tau$, where $\tau \in [0,1)$ and $\psi(0) = 0$.

Definition 2.7: [14] Let $\Sigma, \Upsilon: \Xi \to \Xi$, are self-mappings if $w = \Sigma\mathfrak{e} = \Upsilon\mathfrak{e}$, for some $\mathfrak{e} \in \Xi$, then w is said a coincidence point (CP) of these mappings Σ and Υ.

Preposition 2.1: [14] Assume that $\Sigma, \Upsilon: \Xi \to \Xi$, are self-mappings on Ξ if $w = \Sigma\mathfrak{e} = \Upsilon\mathfrak{e}$, for some $\mathfrak{e} \in \Xi$, then w is a unique FP of Σ and Υ.

Definition 2.8: [16] A 6-tuple $(\Xi, P, N, \mathcal{O}, *, \Delta)$ is called a NMS if Ξ is an arbitrary set, $*, \Delta$ are CTN and CTCN and P, N and $\mathcal{O}$ are neutrosophic sets defined on $\Xi^3 \times (0, +\infty)$ fulfill the circumstances which are given below for all $\mathfrak{e}, \mathfrak{z}, \omega, \mathfrak{a} \in \Xi$ and $\lambda, \tau > 0$,

(i) $P(\mathfrak{e}, \mathfrak{z}, \tau) + N(\mathfrak{e}, \mathfrak{z}, \tau) + \mathcal{O}(\mathfrak{e}, \mathfrak{z}, \tau) \leq 3$,

(ii) $P(\mathfrak{e}, \mathfrak{z}, \tau) > 0$,

(iii) $P(\mathfrak{e}, \mathfrak{z}, \tau) = 1$ iff $\mathfrak{e} = \mathfrak{z}$,

(iv) $P(\mathfrak{e}, \mathfrak{z}, \tau) = P(\mathfrak{z}, \mathfrak{e}, \tau)$,

(v) $P(\mathfrak{e}, \mathfrak{z}, \tau) * P(\mathfrak{z}, \omega, \lambda) \leq P(\mathfrak{e}, \omega, \tau + \lambda)$,

(vi) $P(\mathfrak{e}, \mathfrak{z}, \cdot): (0, +\infty) \to (0, 1]$ is continuous,

(vii) $N(\mathfrak{e}, \mathfrak{z}, \tau) < 1$,

(viii) $N(\mathfrak{e}, \mathfrak{z}, \tau) = 0$ iff $\mathfrak{e} = \mathfrak{z}$,

(ix) $N(\mathfrak{e}, \mathfrak{z}, \tau) = N(\mathfrak{z}, \mathfrak{e}, \tau)$,

(x) $N(\mathfrak{e}, \mathfrak{z}, \tau) \Delta N(\mathfrak{z}, \omega, \lambda) \geq N(\mathfrak{e}, \omega, \tau + \lambda)$,

(xi) $N(\mathfrak{e}, \mathfrak{z}, \cdot): (0, +\infty) \to (0, 1]$ is continuous,

(xii) $\mathcal{O}(\mathfrak{e}, \mathfrak{z}, \tau) < 1$,

(xiii) $\mathcal{O}(\mathfrak{e}, \mathfrak{z}, \tau) = 0$ iff $\mathfrak{e} = \mathfrak{z}$,

(xiv) $\mathcal{O}(\mathfrak{e}, \mathfrak{z}, \tau) = \mathcal{O}(\mathfrak{z}, \mathfrak{e}, \tau)$,

(xv) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\tau)\Delta\mathcal{O}(\mathfrak{z},\omega,\lambda) \geq \mathcal{O}(\mathfrak{e},\omega,\tau+\lambda)$,

(xvi) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\cdot):(0,+\infty)\to(0,1]$ is continuous.

Then, $(P,N,\mathcal{O})$ is called a neutrosophic metric on Ξ.

3. Some Properties and Baire Space on NMS

In this, we will study some interesting results in the sense of NMS.

Definition 3.1: Let $(\Xi,P,N,\mathcal{O},*,\Delta)$ be a NMS.

i. A convergent sequence $\{\mathfrak{e}_\mathfrak{n}\}\subset\Xi$, which convergent to $\mathfrak{e}$ in Ξ if $\lim_{\mathfrak{n}\to\infty} P(\mathfrak{e}_\mathfrak{n},\mathfrak{e},\tau)=1$, $\lim_{\mathfrak{n}\to\infty} N(\mathfrak{e}_\mathfrak{n},\mathfrak{e},\tau)=0$,

and $\lim_{\mathfrak{n}\to\infty}\mathcal{O}(\mathfrak{e}_\mathfrak{n},\mathfrak{e},\tau)=0$, for each $\tau>0$.

ii. A sequence $\{\mathfrak{e}_\mathfrak{n}\}\subset\Xi$, is said to be Cauchy sequence (CS) if for each $\mathfrak{r}, 0<\mathfrak{r}<1$, and $\tau>0$, there exists $\mathfrak{n}_0\in\mathbb{N}$ such that $P(\mathfrak{e}_\mathfrak{m},\mathfrak{e}_\mathfrak{n},\tau)>1-\mathfrak{r}$, $N(\mathfrak{e}_\mathfrak{m},\mathfrak{e}_\mathfrak{n},\tau)<\mathfrak{r}$, and $\mathcal{O}(\mathfrak{e}_\mathfrak{m},\mathfrak{e}_\mathfrak{n},\tau)<\mathfrak{r}$, for all $\mathfrak{m},\mathfrak{n}\geq\mathfrak{n}_0$.

iii. If every CS is convergent in Ξ, then Ξ is complete.

iv. If Ξ is a complete NMS, then $(P,N,\mathcal{O})$ complete neutrosophic metric on Ξ.

v. A dense OS which has intersection of a countable number is dense in Ξ, then Ξ is a Baire space.

vi. If for each $\tau>0$ and $\mathfrak{r}$ with $0<\mathfrak{r}<1$, there is a finite subset $\mathcal{A}$ of Ξ such that $\Xi=\cup_{\mathfrak{a}\in\mathcal{A}}\mathcal{B}(\mathfrak{a},\mathfrak{r},\tau)$, then Ξ is said to be precompact.

In this case, P,N and $\mathcal{O}$ are precompact neutrosophic metric on Ξ.

Lemma 3.1: A T_1 topological space (Ξ,τ) is metrizable iff it has an uniformly with a countable base.

Lemma 3.2: Suppose $(\Xi,P,N,\mathcal{O},*,\Delta)$ be an NMS. Then τ is a Hausdorff topology and for each $\mathfrak{e}\in\Xi$, $\left\{\mathcal{B}\left(\mathfrak{e},\frac{1}{\mathfrak{n}},\frac{1}{\mathfrak{n}}\right);\mathfrak{n}\in\mathbb{N}\right\}$.

Theorem 3.1: Let $(\Xi,P,N,\mathcal{O},*,\Delta)$ be an NMS. Then Ξ is a metrizable topological space.

Proof: For each $\mathfrak{n}\in\mathbb{N}$, we get

$$U_\mathfrak{n}=\left\{(\mathfrak{e},\mathfrak{z})\in\Xi^2; P\left(\mathfrak{e},\mathfrak{u},\frac{1}{\mathfrak{n}}\right)>1-\frac{1}{\mathfrak{n}}, N\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)<\frac{1}{\mathfrak{n}}, \mathcal{O}\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)<\frac{1}{\mathfrak{n}}\right\}.$$

Then, prove that $\{U_\mathfrak{n};\mathfrak{n}\in\mathbb{N}\}$, is a base for uniformity U on Ξ. Every $\mathfrak{n}\in\mathbb{N}$, $\{(\mathfrak{e},\mathfrak{e});\mathfrak{e}\in\Xi\}\subseteq U_\mathfrak{n}$, also, since

$$P\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)>P\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}+1}\right)>1-\frac{1}{\mathfrak{n}},$$

$$N\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)<N\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}+1}\right)<\frac{1}{\mathfrak{n}},$$

$$\mathcal{O}\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)<\mathcal{O}\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}+1}\right)<\frac{1}{\mathfrak{n}},$$

and

$$P\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)=P\left(\mathfrak{z},\mathfrak{e},\frac{1}{\mathfrak{n}}\right),$$

$$N\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)=N\left(\mathfrak{z},\mathfrak{e},\frac{1}{\mathfrak{n}}\right),$$

$$\mathcal{O}\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)=\mathcal{O}\left(\mathfrak{z},\mathfrak{e},\frac{1}{\mathfrak{n}}\right).$$

Hence $U_{\mathfrak{n}+1}\subseteq U_\mathfrak{n}$, and $U_\mathfrak{n}=U_\mathfrak{n}^{-1}$. Similarly, every $\mathfrak{n}\in\mathbb{N}$, we have, by the continuity of CTN and CTCN, an $\mathfrak{m}\in\mathbb{N}$, such that $\mathfrak{m}\geq 2\mathfrak{n}$,

$$\left(1-\frac{1}{\mathfrak{m}}\right)*\left(1-\frac{1}{\mathfrak{m}}\right)>1-\frac{1}{\mathfrak{n}},\frac{1}{\mathfrak{m}}\Delta\frac{1}{\mathfrak{m}}<\frac{1}{\mathfrak{n}}.$$

Let $(\mathfrak{e},\mathfrak{z}),(\mathfrak{z},\omega)\in U_{\mathfrak{m}}$. Then,

$$P\left(\mathfrak{e},\omega,\frac{1}{\mathfrak{n}}\right)\geq P\left(\mathfrak{e},\omega,\frac{2}{\mathfrak{m}}\right)\geq P\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{m}}\right)*P\left(\mathfrak{z},\omega,\frac{1}{\mathfrak{m}}\right)$$

$$\geq\left(1-\frac{1}{\mathfrak{m}}\right)*\left(1-\frac{1}{\mathfrak{m}}\right)>1-\frac{1}{\mathfrak{n}},$$

$$N\left(\mathfrak{e},\omega,\frac{1}{\mathfrak{n}}\right)\leq N\left(\mathfrak{e},\omega,\frac{2}{\mathfrak{m}}\right)\leq N\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{m}}\right)\Delta N\left(\mathfrak{z},\omega,\frac{1}{\mathfrak{m}}\right)$$

$$\leq\left(\frac{1}{\mathfrak{m}}\right)\Delta\left(\frac{1}{\mathfrak{m}}\right)<\frac{1}{\mathfrak{n}},$$

and

$$\mathcal{O}\left(\mathfrak{e},\omega,\frac{1}{\mathfrak{n}}\right)\leq\mathcal{O}\left(\mathfrak{e},\omega,\frac{2}{\mathfrak{m}}\right)\leq\mathcal{O}\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{m}}\right)\Delta\,\mathcal{O}\left(\mathfrak{z},\omega,\frac{1}{\mathfrak{m}}\right)$$

$$\leq\left(\frac{1}{\mathfrak{m}}\right)\Delta\left(\frac{1}{\mathfrak{m}}\right)<\frac{1}{\mathfrak{n}}.$$

Therefore $(\mathfrak{e},\omega)\in U_{\mathfrak{n}}$ and $U_{\mathfrak{m}}\circ U_{\mathfrak{n}}\subseteq U_{\mathfrak{n}}$. Thus $\{U_{\mathfrak{n}};\mathfrak{n}\in\mathbb{N}\}$, is a uniform base of U on Ξ. Therefore, every $\mathfrak{e}\in\Xi$, and $\mathfrak{n}\in\mathbb{N}$,

$$U_{\mathfrak{n}}(\mathfrak{e})=\left\{\mathfrak{z}\in\Xi;P\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)>1-\frac{1}{\mathfrak{n}},N\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)<\frac{1}{\mathfrak{n}},\mathcal{O}\left(\mathfrak{e},\mathfrak{z},\frac{1}{\mathfrak{n}}\right)<\frac{1}{\mathfrak{n}}\right\}=\mathcal{B}\left(\mathfrak{e},\frac{1}{\mathfrak{n}},\frac{1}{\mathfrak{n}}\right).$$

Which show that the topology induced by U coincides with τ and Ξ is metrizable topological space by Lemma 3.2.

Theorem 3.2: Every separable NMS is second countable.

Proof: Assume that $(\Xi,P,N,\mathcal{O},*,\Delta)$ be the given separable NMS and let $\mathcal{A}=\{\mathfrak{a}_{\mathfrak{n}};\mathfrak{n}\in\mathbb{N}\}$, be a countable dense subset of Ξ. Suppose that $\mathcal{B}=\left\{\mathcal{B}\left(\mathfrak{a}_j,\frac{1}{k},\frac{1}{k}\right);j,k\in\mathbb{N}\right\}$. Then $\mathcal{B}$ is countable. Therefore, we show that $\mathcal{B}\in\Xi$ is a basis for the family of all open sets. Assume that $G\in\Xi$ is an arbitrary OS and $\mathfrak{e}\in G$, then there exists $\mathfrak{r},\tau>0,0<\mathfrak{r}<1$, such that $\mathcal{B}(\mathfrak{e},\mathfrak{r},\tau)\subset G$. Since $\mathfrak{r}\in(0,1)$, there exist $\lambda\in(0,1)$, from the above result [7], such that $(1-\lambda)*(1-\lambda)>1-\mathfrak{r},\lambda\,\Delta\,\lambda<\mathfrak{r}$. Select any $\mathfrak{m}\in\mathbb{N}$ then $\frac{1}{\mathfrak{m}}<\min\{\lambda,\tau_2\}$, for $\mathcal{A}\in\Xi$ is dense there exists $\mathfrak{a}_j\in\mathcal{A}$, such that $\mathfrak{a}_j\in\mathcal{B}\left(\mathfrak{e},\frac{1}{\mathfrak{m}},\frac{1}{\mathfrak{m}}\right)$. Now, if $\mathfrak{z}\in\mathcal{B}\left(\mathfrak{e},\frac{1}{\mathfrak{m}},\frac{1}{\mathfrak{m}}\right)$, then,

$$P(\mathfrak{e},\mathfrak{z},\tau)\geq P\left(\mathfrak{e},\mathfrak{a}_j,\frac{\tau}{2}\right)*P\left(\mathfrak{a}_j,\mathfrak{z},\frac{\tau}{2}\right)\geq P\left(\mathfrak{e},\mathfrak{a}_j,\frac{1}{\mathfrak{m}}\right)*P\left(\mathfrak{a}_j,\mathfrak{z},\frac{1}{\mathfrak{m}}\right)$$

$$\geq\left(1-\frac{1}{\mathfrak{m}}\right)*\left(1-\frac{1}{\mathfrak{m}}\right)\geq(1-\lambda)*(1-\lambda)>1-\mathfrak{r},$$

$$N(\mathfrak{e},\mathfrak{z},\tau)\leq N\left(\mathfrak{e},\mathfrak{a}_j,\frac{\tau}{2}\right)\Delta N\left(\mathfrak{a}_j,\mathfrak{z},\frac{\tau}{2}\right)\leq N\left(\mathfrak{e},\mathfrak{a}_j,\frac{1}{\mathfrak{m}}\right)\Delta N\left(\mathfrak{a}_j,\mathfrak{z},\frac{1}{\mathfrak{m}}\right)$$

$$\leq\left(\frac{1}{\mathfrak{m}}\right)\Delta\left(\frac{1}{\mathfrak{m}}\right)\leq(\lambda)\Delta(\lambda)<\mathfrak{r},$$

and

$$\mathcal{O}(\mathfrak{e},\mathfrak{z},\tau)\leq\mathcal{O}\left(\mathfrak{e},\mathfrak{a}_j,\frac{\tau}{2}\right)\Delta\,\mathcal{O}\left(\mathfrak{a}_j,\mathfrak{z},\frac{\tau}{2}\right)\leq\mathcal{O}\left(\mathfrak{e},\mathfrak{a}_j,\frac{1}{\mathfrak{m}}\right)\Delta\mathcal{O}\left(\mathfrak{a}_j,\mathfrak{z},\frac{1}{\mathfrak{m}}\right)$$

$$\leq \left(\frac{1}{\mathfrak{m}}\right)\Delta\left(\frac{1}{\mathfrak{m}}\right) \leq (\lambda)\Delta(\lambda) < \mathfrak{r}.$$

Consequently, the topology τ has a base $\mathcal{B}$, and $\mathfrak{z} \in \mathcal{B}(\mathfrak{e}, \mathfrak{r}, \tau)$. Then, Ξ is a separable metrizable space by using Theorem 3.1. Hence Ξ is second countable.

Theorem 3.3: Suppose Ξ be a complete NMS. Then Ξ is completely metrizable (CM).

Proof: Assume that Ξ be an NMS. From Theorem 3.1, then uniformity U on Ξ has a base $\{U_{\mathfrak{n}}; \mathfrak{n} \in N\}$ where,

$$U_{\mathfrak{n}} = \left\{(\mathfrak{e}, \mathfrak{z}) \in \Xi^2; P\left(\mathfrak{e}, \mathfrak{z}, \frac{1}{\mathfrak{n}}\right) > 1 - \frac{1}{\mathfrak{n}}, N\left(\mathfrak{e}, \mathfrak{z}, \frac{1}{\mathfrak{n}}\right) < \frac{1}{\mathfrak{n}}, \mathcal{O}\left(\mathfrak{e}, \mathfrak{z}, \frac{1}{\mathfrak{n}}\right) < \frac{1}{\mathfrak{n}}\right\},$$

for every $\mathfrak{n} \in \mathbb{N}$. Then the induced uniformity of P, N and $\mathcal{O}$ on Ξ, coincides with U. A sequence $\{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathbb{N}}$ is a CS in Ξ fixed $\mathfrak{r}, \tau$ with $0 < \mathfrak{r} < 1$, and $\tau > 0$. Choosing $k \in \mathbb{N}$, such that $\frac{1}{k} \leq \min\{\tau, \mathfrak{r}\}$. Then there exist $\mathfrak{n}_0 \in \mathbb{N}$, then $(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}) \in U_k$, for every $\mathfrak{m}, \mathfrak{n} \geq \mathfrak{n}_0$. That is, for each $\mathfrak{m}, \mathfrak{n} \geq \mathfrak{n}_0$,

$$P(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \tau) \geq P\left(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \frac{1}{k}\right) > 1 - \frac{1}{k} \geq 1 - \mathfrak{r},$$

$$N(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \tau) \leq N\left(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \frac{1}{k}\right) < \frac{1}{k} \leq \mathfrak{r},$$

$$\mathcal{O}(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \tau) \leq \mathcal{O}\left(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \frac{1}{k}\right) < \frac{1}{k} \leq \mathfrak{r}.$$

Hence, $\{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n} \in \mathbb{N}}$ is a CS in complete NMS Ξ. so, $\{\mathfrak{e}_{\mathfrak{n}}\}$ is a convergent sequence and converges to $\mathfrak{e} \in \Xi$. Therefore, P, N and $\mathcal{O}$ are complete neutrosophic metric on Ξ. Hence, Ξ is CM.

Theorem 3.4: Let Ξ be a complete NMS. Then every complete space Ξ is a Baire space.

Proof: Let Ξ be given complete NMS and $\mathcal{B}_0 \neq \varphi$ be OS. Also, let $D_1, D_2, \cdots,$ be dense OSs in Ξ. So, $\mathcal{B}_0 \cap D_1 \neq \varphi$, from $D_1 = \Xi$. Let $\mathfrak{e}_1 \in \mathcal{B}_0 \cap D_1$, there exists $\tau_1 > 0, 0 < \mathfrak{r}_1 < 1$, such that $\mathcal{B}(\mathfrak{e}_1, \mathfrak{r}_1, \tau_1) \subset \mathcal{B}_0 \cap D_1$. Choosing $\mathfrak{r}'_1 < r_1$ and $\tau'_1 = \min\{\tau_1, 1\}$, such that

$$\mathcal{B}(\mathfrak{e}_1, \mathfrak{r}'_1, \tau'_1) \subset \mathcal{B}_0 \cap D_1.$$

Let $\mathcal{B}_1 = \mathcal{B}(\mathfrak{e}_1, \mathfrak{r}'_1, \tau'_1)$. Then $\mathcal{B}_1 \cap D_2 \neq \varphi$, from $\overline{D_2} = \Xi$. Let $\mathfrak{e}_2 \in \mathcal{B}_1 \cap D_2$ there exists $\tau_2 > 0, 0 < \mathfrak{r}_2 < 1$, such that $\mathcal{B}(\mathfrak{e}_2, \mathfrak{r}_2, \tau_2) \subset \mathcal{B}_1 \cap D_2$. Choosing $\mathfrak{r}'_2 < \mathfrak{r}_2$ and $\tau'_2 = \min\left\{\tau_2, \frac{1}{2}\right\}$, such that

$$\mathcal{B}(\mathfrak{e}_2, \mathfrak{r}'_2, \tau'_2) \subset \mathcal{B}_1 \cap D_2.$$

Let $\mathcal{B}_{\mathfrak{n}} = \mathcal{B}(\mathfrak{e}_2, \mathfrak{r}'_2, \tau'_2)$. Then by inductively methods, we can find $\mathfrak{e}_{\mathfrak{n}} \in \mathcal{B}_{\mathfrak{n}-1} \cap D_{\mathfrak{n}}$. Hence there exists $\tau_{\mathfrak{n}} > 0, 0 < \mathfrak{r}_{\mathfrak{n}} < \frac{1}{\mathfrak{n}}$, such that,

$$\mathcal{B}(\mathfrak{e}_{\mathfrak{n}}, \mathfrak{r}'_{\mathfrak{n}}, \tau'_{\mathfrak{n}}) \subset \mathcal{B}_{\mathfrak{n}-1} \cap D_{\mathfrak{n}}.$$

Let $\mathcal{B}_{\mathfrak{n}} = \mathcal{B}(\mathfrak{e}_{\mathfrak{n}}, \mathfrak{r}'_{\mathfrak{n}}, \tau'_{\mathfrak{n}})$. we show that $\{\mathfrak{e}_{\mathfrak{n}}\}$ is a CS. For given $\tau > 0, \epsilon > 0$, select any $\mathfrak{n}_0 \in \mathbb{N}$, such that $\frac{1}{\mathfrak{n}_0} < \tau, \frac{1}{\mathfrak{n}_0} < \epsilon$. Then for $\mathfrak{m} \geq \mathfrak{n} \geq \mathfrak{n}_0$,

$$P(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \tau) \geq P\left(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \frac{1}{\mathfrak{n}}\right) \geq 1 - \frac{1}{\mathfrak{n}} > 1 - \epsilon,$$

$$N(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \tau) \leq N\left(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \frac{1}{\mathfrak{n}}\right) \leq \frac{1}{\mathfrak{n}} < \epsilon,$$

$$\mathcal{O}(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \tau) \leq \mathcal{O}\left(\mathfrak{e}_{\mathfrak{m}}, \mathfrak{e}_{\mathfrak{n}}, \frac{1}{\mathfrak{n}}\right) \leq \frac{1}{\mathfrak{n}} < \epsilon.$$

Therefore, by using Definition 3.1 which shows that $\{\mathfrak{e}_{\mathfrak{n}}\}$ is a CS. Hence, Ξ is complete, $\mathfrak{e}_{\mathfrak{n}} \to \mathfrak{e}$, in Ξ. But $\mathfrak{e}_k \in \mathcal{B}(\mathfrak{e}_{\mathfrak{n}}, \mathfrak{r}'_{\mathfrak{n}}, \tau'_{\mathfrak{n}})$, for all $k \geq \mathfrak{n}$, and $\mathcal{B}(\mathfrak{e}_{\mathfrak{n}}, \mathfrak{r}'_{\mathfrak{n}}, \tau'_{\mathfrak{n}})$, is a closed set. Hence $\mathfrak{e} \in \mathcal{B}(\mathfrak{e}_{\mathfrak{n}}, \mathfrak{r}'_{\mathfrak{n}}, \tau'_{\mathfrak{n}}) \subset \mathcal{B}_{\mathfrak{n}-1} \cap D_{\mathfrak{n}}$, for all $\mathfrak{n} \in \mathbb{N}$. Therefore, $\mathcal{B}_0 \cap (\cap_{\mathfrak{n}=1}^{\infty} D_{\mathfrak{n}}) \neq \varphi$. Hence $\cap_{\mathfrak{n}=1}^{\infty} D_{\mathfrak{n}}$ is dense in Ξ.

Theorem 3.5: An NMS $(\Xi, P, N, \mathcal{O}, *, \Delta)$ is precompact iff every sequence in Ξ has a Cauchy subsequence.

Proof: Let Ξ is a precompact NMS. Let $\{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathbb{N}}$ be a sequence in Ξ. For each $\mathfrak{m} \in \mathbb{N}$ there is a finite subset $\mathcal{A}_{\mathfrak{m}}$ of Ξ then $\Xi = \cup_{\mathfrak{a}\in\mathcal{A}_{\mathfrak{m}}} \mathcal{B}\left(\mathfrak{a}, \frac{1}{\mathfrak{m}}, \frac{1}{\mathfrak{m}}\right)$. Hence, for $\mathfrak{m} = 1$, there exist an $\mathfrak{a}_1 \in \mathcal{A}_1$ and $\{\mathfrak{e}_{1(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}} \subset \{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathbb{N}}$, therefore $\mathfrak{e}_{1(\mathfrak{n})} \in \mathcal{B}(\mathfrak{a}_1, 1, 1)$, for every $\mathfrak{n} \in \mathbb{N}$. Also, for $\mathfrak{m} = 2$, there exist an $\mathfrak{a}_2 \in \mathcal{A}_2$ and $\{\mathfrak{e}_{2(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}} \subset \{\mathfrak{e}_{1(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}}$ then, $\mathfrak{e}_{2(\mathfrak{n})} \in \mathcal{B}\left(\mathfrak{a}_2, \frac{1}{2}, \frac{1}{2}\right)$. By inductively methods, for $\mathfrak{m} \in \mathbb{N}, \mathfrak{m} > 1$, there exist an $\mathfrak{a}_{\mathfrak{m}} \in \mathcal{A}_{\mathfrak{m}}$, and $\{\mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}} \subset \{\mathfrak{e}_{\mathfrak{m}-1(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}}$, such that $\mathfrak{e}_{\mathfrak{m}(\mathfrak{n})} \in \mathcal{B}\left(\mathfrak{a}_{\mathfrak{m}}, \frac{1}{\mathfrak{m}}, \frac{1}{\mathfrak{m}}\right)$, for every $\mathfrak{n} \in \mathbb{N}$. Now, consider $\{\mathfrak{e}_{\mathfrak{n}(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}} \subset \{\mathfrak{e}_{\mathfrak{n}-1(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}}$. Given $\tau > 0$, and $\mathfrak{r}$ with $0 < \mathfrak{r} < 1$, there exist $\mathfrak{n}_0 \in \mathbb{N}$, we get

$$\left(1 - \frac{1}{\mathfrak{n}_0}\right) * \left(1 - \frac{1}{\mathfrak{n}_0}\right) > 1 - \mathfrak{r}, \left(\frac{1}{\mathfrak{n}_0}\right) \Delta \left(\frac{1}{\mathfrak{n}_0}\right) < \mathfrak{r},$$

and $\frac{2}{\mathfrak{n}_0} < \tau$. Therefore,

$$P\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \tau\right) > P\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{2}{\mathfrak{n}_0}\right) \geq P\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{1}{\mathfrak{n}_0}\right) * P\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{1}{\mathfrak{n}_0}\right)$$

$$\geq \left(1 - \frac{1}{\mathfrak{n}_0}\right) * \left(1 - \frac{1}{\mathfrak{n}_0}\right) > 1 - \mathfrak{r},$$

$$N\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \tau\right) < N\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{2}{\mathfrak{n}_0}\right) \leq N\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{1}{\mathfrak{n}_0}\right) \Delta\, N\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{1}{\mathfrak{n}_0}\right)$$

$$\leq \left(\frac{1}{\mathfrak{n}_0}\right) \Delta \left(\frac{1}{\mathfrak{n}_0}\right) < \mathfrak{r},$$

and

$$\mathcal{O}\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \tau\right) < \mathcal{O}\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{2}{\mathfrak{n}_0}\right) \leq \mathcal{O}\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{1}{\mathfrak{n}_0}\right) \Delta\, \mathcal{O}\left(\mathfrak{e}_{k(k)}, \mathfrak{e}_{\mathfrak{m}(\mathfrak{n})}, \frac{1}{\mathfrak{n}_0}\right)$$

$$\leq \left(\frac{1}{\mathfrak{n}_0}\right) \Delta \left(\frac{1}{\mathfrak{n}_0}\right) < \mathfrak{r}.$$

Hence $\{\mathfrak{e}_{\mathfrak{n}(\mathfrak{n})}\}_{\mathfrak{n}\in\mathbb{N}}$,is a CS in Ξ. Conversely, let Ξ is a nonprecompact NMS. Hence, there exist $\tau > 0$, and $\mathfrak{r}$ with $0 < \mathfrak{r} < 1$, then for each finite subset $\mathcal{A}$ of Ξ, $\Xi = \cup_{\mathfrak{a}\in\mathcal{A}} \mathcal{B}(\mathfrak{a}, \mathfrak{r}, \tau)$. Fix $\mathfrak{e}_1 \in \Xi$, there exist $\mathfrak{e}_2 \in \Xi - \mathcal{B}(\mathfrak{e}_1, \mathfrak{r}, \tau)$. Moreover, there exist $\mathfrak{e}_3 \in \Xi - \cup_{k=1}^{2} \mathcal{B}(\mathfrak{e}_k, \mathfrak{r}, \tau)$. By inductively methods, we construct a sequence $\{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathbb{N}}$ of distinct points in Ξ such that $\mathfrak{e}_{\mathfrak{n}+1} \notin \cup_{k=1}^{\mathfrak{n}} \mathcal{B}(\mathfrak{e}_k, \mathfrak{r}, \tau)$, for every $\mathfrak{n} \in \mathbb{N}$. Hence $\{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathbb{N}}$, has no CS. Consequently, if $\{\mathfrak{e}_{\mathfrak{n}}\}_{\mathfrak{n}\in\mathbb{N}}$, has Cauchy subsequence in Ξ, hence Ξ is precompact NMS.

4. Non-Archimedean Generalized NMS

In this section, we discuss some FP results in of NAGNMS.

Definition 4.1: A 6-tuple $(\Xi, P, N, \mathcal{O}, *, \Delta)$ is called a NAGNMS if Ξ is an arbitrary set, $*, \Delta$ are CTN and CTCN and P, N and $\mathcal{O}$ are fuzzy sets defined on $\Xi^3 \times (0, +\infty)$, fulfill the below circumstances for all $\mathfrak{e}, \mathfrak{z}, \omega, \mathfrak{a} \in \Xi$, and $\lambda, \tau > 0$,

(i) $P(\mathfrak{e},\mathfrak{z},\omega,\tau)+N(\mathfrak{e},\mathfrak{z},\omega,\tau)+\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)\leq 3$,
(ii) $P(\mathfrak{e},\mathfrak{z},\omega,\tau)>0$,
(iii) $P(\mathfrak{e},\mathfrak{z},\omega,\tau)=1$ iff $\mathfrak{e}=\mathfrak{z}=\omega$,
(iv) $P(\mathfrak{e},\mathfrak{z},\omega,\tau)=P(\mathfrak{p}\{\mathfrak{e},\mathfrak{z},\omega\},\tau)$, where $\mathfrak{p}$ is a PF,
(v) $P(\mathfrak{e},\mathfrak{z},\mathfrak{a},\tau)*P(\mathfrak{a},\omega,\omega,\lambda)\leq P(\mathfrak{e},\mathfrak{z},\omega,\max\{\tau,\lambda\})$,
(vi) $P(\mathfrak{e},\mathfrak{z},\omega,\cdot):(0,+\infty)\to[0,1]$ is continuous,
(vii) $N(\mathfrak{e},\mathfrak{z},\omega,\tau)>0$,
(viii) $N(\mathfrak{e},\mathfrak{z},\omega,\tau)=1$ iff $\mathfrak{e}=\mathfrak{z}=\omega$,
(ix) $N(\mathfrak{e},\mathfrak{z},\omega,\tau)=N(\mathfrak{p}\{\mathfrak{e},\mathfrak{z},\omega\},\tau)$, where $\mathfrak{p}$ is a PF,
(x) $N(\mathfrak{e},\mathfrak{z},\mathfrak{a},\tau)\Delta N(\mathfrak{a},\omega,\omega,\lambda)\geq N(\mathfrak{e},\mathfrak{z},\omega,\min\{\tau,\lambda\})$,
(xi) $N(\mathfrak{e},\mathfrak{z},\omega,\cdot):(0,+\infty)\to[0,1]$ is continuous,
(xii) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)>0$,
(xiii) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)=1$ iff $\mathfrak{e}=\mathfrak{z}=\omega$,
(xiv) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)=\mathcal{O}(\mathfrak{p}\{\mathfrak{e},\mathfrak{z},\omega\},\tau)$, where $\mathfrak{p}$ is a PF,
(xv) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\mathfrak{a},\tau)\,\Delta\,\mathcal{O}(\mathfrak{a},\omega,\omega,\lambda)\geq\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\min\{\tau,\lambda\})$,
(xvi) $\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\cdot):(0,+\infty)\to[0,1]$ is continuous.

The pair $(P,N,\mathcal{O})$ is said to be a generalized neutrosophic metric on $\varXi$.

Definition 4.2: Let $\Sigma,\Upsilon:\varXi\to\varXi$, on a NAGNMS $(\varXi,P,N,*,\Delta)$. If these mappings are commute at their CP, that is, $\Sigma\mathfrak{e}=\Upsilon\mathfrak{e}$, implies that $\Sigma\Upsilon h=\Upsilon\Sigma h$, then mappings are called weakly compatible

Definition 4.3: A pair (Σ,Υ) of self-mappings of a NAGNMS is said to be semi-compatible if $\lim_{\mathfrak{n}\to\infty}\Sigma\Upsilon\mathfrak{e}_{\mathfrak{n}}=\Upsilon\mathfrak{e}_{\mathfrak{n}}$, whenever $\{\mathfrak{e}_{\mathfrak{n}}\}$ is a sequence in $\varXi$ such that $\lim_{\mathfrak{n}\to\infty}\Sigma\mathfrak{e}_{\mathfrak{n}}=\lim_{\mathfrak{n}\to\infty}\Upsilon\mathfrak{e}_{\mathfrak{n}}=\mathfrak{e}$, for some $\mathfrak{e}\in\varXi$.

Example 4.1: Let $\varXi=[0,1]$, with G-metric on $\varXi$ defined by $d(\mathfrak{e},\mathfrak{z},\omega)=|\mathfrak{e}-\mathfrak{z}|+|\mathfrak{z}-\omega|+|\omega-\mathfrak{e}|$. Denote $\mathfrak{a}*\mathfrak{b}=\mathfrak{a}\mathfrak{b}$, and $\mathfrak{a}\Delta\mathfrak{b}=\min\{1,\mathfrak{a}+\mathfrak{b}\}$, for all $\mathfrak{a},\mathfrak{b}\in[0,1]$. For all $\mathfrak{e},\mathfrak{z},\omega\in\varXi$, and $\tau>0$, define G on $\varXi^3\times(0,\infty)$, as follows:

$$P(\mathfrak{e},\mathfrak{z},\omega,\tau)=\left(\frac{\tau}{\tau+1}\right)^{d(\mathfrak{e},\mathfrak{z},\omega)},$$

$$N(\mathfrak{e},\mathfrak{z},\omega,\tau)=\left(\frac{1}{\tau+1}\right)^{d(\mathfrak{e},\mathfrak{z},\omega)},$$

$$\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)=1-\left(\frac{\tau}{\tau+1}\right)^{d(\mathfrak{e},\mathfrak{z},\omega)}.$$

Then, $(\varXi,P,N,\mathcal{O},*,\Delta)$ is a NAGNMS.

Lemma 4.1: Suppose that $(\varXi,P,N,\mathcal{O},*,\Delta)$ be a NAGNMS. Then $P(\mathfrak{e},\mathfrak{z},\omega,.)$ is non-decreasing, $N(\mathfrak{e},\mathfrak{z},\omega,.)$ is non-increasing and $\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,.)$ is neutral with respect to τ for all $\mathfrak{e},\mathfrak{z},\omega$ in $\varXi$.

In this manuscript, we suppose that $\lim_{\mathfrak{n}\to\infty}P(\mathfrak{e},\mathfrak{z},\omega,\tau)=1$, $\lim_{\mathfrak{n}\to\infty}N(\mathfrak{e},\mathfrak{z},\omega,\tau)=0$, and $\lim_{\mathfrak{n}\to\infty}\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)=0$, for all $\mathfrak{n}\in\mathbb{N}$.

Lemma 4.2: Suppose that $(\varXi,P,N,\mathcal{O},*,\Delta)$ be a NAGNMS. Let $\{\mathfrak{z}_{\mathfrak{n}}\}$ be a sequence in $\varXi$, where $*$ is a continuous t-norm, Δ is a continuous t-conorm satisfying $\tau*\tau\geq\tau$, and $(1-\tau)\Delta(1-\tau)\leq 1-\tau$, for all $\tau\in[0,1]$. If there exists $\tau>0$, and $\phi\in\Phi$, and $\psi\in\Psi$, such that

$$P(\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+2},\mathfrak{z}_{\mathfrak{n}+2},\tau)\geq\phi\big(P(\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\big),$$
$$N(\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+2},\mathfrak{z}_{\mathfrak{n}+2},\tau)\leq\psi\big(N(\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\big),$$

$$\mathcal{O}(\mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+2}, \mathfrak{z}_{\mathfrak{n}+2}, \tau) \leq \psi\big(\mathcal{O}(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+1}, \tau)\big),$$

$\mathfrak{n} \in \mathbb{N}$, then$\{\mathfrak{z}_{\mathfrak{n}}\}$, is a CS in Ξ.

Proof: If we define

$$\mathfrak{r}_{\mathfrak{n}} = P(\mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+2}, \mathfrak{z}_{\mathfrak{n}+2}, \tau),$$
$$\mathfrak{p}_{\mathfrak{n}} = N(\mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+2}, \mathfrak{z}_{\mathfrak{n}+2}, \tau),$$

and

$$\mathfrak{q}_{\mathfrak{n}} = \mathcal{O}(\mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+2}, \mathfrak{z}_{\mathfrak{n}+2}, \tau),$$

then

$$\mathfrak{r}_{\mathfrak{n}} \geq \phi(\mathfrak{r}_{\mathfrak{n}-1}) > \mathfrak{r}_{\mathfrak{n}-1}, \quad (1)$$
$$\mathfrak{p}_{\mathfrak{n}} \leq (\mathfrak{p}_{\mathfrak{n}-1}) < \mathfrak{p}_{\mathfrak{n}-1,} \quad (2)$$
$$\mathfrak{q}_{\mathfrak{n}} \leq (\mathfrak{q}_{\mathfrak{n}-1}) < \mathfrak{q}_{\mathfrak{n}-1,} \quad (3)$$

Accordingly, the sequences r_n tends to a limit $r \leq 1$ and represents an increasing sequence of positive real numbers in $[0, 1]$. In our contention that $r = 1$. We obtain $\mathfrak{r} \geq \phi(\mathfrak{r}) > \mathfrak{r}$, if $r < 1$, on taking $\mathfrak{n} \to \infty$ in (1), (2) and (3), which is a contradiction. Therefore $\mathfrak{r} = 1$. Hence, the sequence $\{\mathfrak{p}_{\mathfrak{n}}\}, \{\mathfrak{q}_{\mathfrak{n}}\} \in [0, 1]$ are non-increasing sequences of positive real numbers and tends to a limit $\mathfrak{p}, \mathfrak{q} \geq 0$. We claim that $\mathfrak{p}, \mathfrak{q} = 0$. If $\mathfrak{p}, \mathfrak{q} > 1$, on taking $\mathfrak{n} \to \infty$ in (1), (2) and (3) we get $\mathfrak{p} \leq \psi(\mathfrak{p}) < \mathfrak{p}$ and $\mathfrak{q} \leq \psi(\mathfrak{q}) < \mathfrak{q}$, which is contradiction. So, $\mathfrak{p}, \mathfrak{q} = 0$. Therefore, for some positive integers $\mathfrak{p}, \mathfrak{q}$, we obtain,

$$P\big(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big) \geq P(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+1}, \tau) * \cdots * P\big(\mathfrak{z}_{\mathfrak{n}+\mathfrak{p}-1}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big),$$
$$N\big(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big) \leq N(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+1}, \tau)\, \Delta \cdots \Delta\, N\big(\mathfrak{z}_{\mathfrak{n}+\mathfrak{p}-1}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big),$$
$$\mathcal{O}\big(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big) \leq \mathcal{O}(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+1}, \mathfrak{z}_{\mathfrak{n}+1}, \tau)\, \Delta \cdots \Delta\, \mathcal{O}\big(\mathfrak{z}_{\mathfrak{n}+\mathfrak{p}-1}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big).$$

as $\mathfrak{n} \to \infty$, we have

$$\lim_{\mathfrak{n}\to\infty} P\big(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big) = 1,$$

$$\lim_{\mathfrak{n}\to\infty} N\big(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big) = 0,$$

$$\lim_{\mathfrak{n}\to\infty} \mathcal{O}\big(\mathfrak{z}_{\mathfrak{n}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \mathfrak{z}_{\mathfrak{n}+\mathfrak{p}}, \tau\big) = 0.$$

Hence, $\{\mathfrak{z}_{\mathfrak{n}}\}$ is a CS.

Definition 4.4: Suppose that $(\Xi, P, N, \mathcal{O}, *, \Delta)$ is said to be a NAGNMS and a self-mapping $T : \Xi \to \Xi$ has FP set $\zeta(T) \neq \emptyset$. If $\zeta(T^{\mathfrak{n}}) = \zeta(T)$, so, T has property P for each $\mathfrak{n} \in \mathbb{N}$.

Definition 4.5: Suppose that $(\Xi, P, N, \mathcal{O}, *, \Delta)$ be a NAGNMS and $T, S: \Xi \to \Xi$ be two self-mappings with $\zeta(S) \cap \zeta(T) \neq \emptyset$. Hence, T and S have property Q if $\zeta(S^{\mathfrak{n}}) \cap \zeta(T^{\mathfrak{n}}) = \zeta(S) \cap \zeta(T)$, for each $\mathfrak{n} \in \mathbb{N}$.

Theorem 4.1: Assume $(\Xi, P, N, *, \Delta)$ is said to be a NAGNMS with $\tau \geq \tau$, and $(1 - \tau)\, \Delta\, (1 - \tau) \leq (1 - \tau)$. Suppose $\Sigma, \Upsilon: \Xi \to \Xi$ are two mappings fulfill below conditions for all $\mathfrak{e}, \mathfrak{z}, \omega \in \Xi$,

$$P(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \geq \phi\big(P(\Upsilon\mathfrak{e}, \Upsilon\mathfrak{z}, \Upsilon\omega, \tau)\big), \quad (3)$$
$$N(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \leq \psi\big(N(\Upsilon\mathfrak{e}, \Upsilon\mathfrak{z}, \Upsilon\omega, \tau)\big), \quad (4)$$
$$\mathcal{O}(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \leq \psi\big(\mathcal{O}(\Upsilon\mathfrak{e}, \Upsilon\mathfrak{z}, \Upsilon\omega, \tau)\big), \quad (5)$$

Where, $\phi \in \Phi, \psi \in \Psi, \tau > 0$. If $\Sigma(\Xi) \subset \Upsilon(\Xi)$, is a subspace of Ξ which is complete. Therefore, the coincident in Ξ of Σ and Υ is unique. Furthermore, if Σ and Υ are semi-complete, then Σ and Υ have a unique FP.

Proof: Suppose that $\mathfrak{e}_0 \in \Xi$ be any arbitrary point. Since $\Sigma(\Xi) \subset \Upsilon(\Xi)$, so we choose a point $\mathfrak{e}_1$ in Ξ such that $\Sigma(\mathfrak{e}_0) = \Upsilon(\mathfrak{e}_1)$. By continuing this process, we select $\mathfrak{e}_{\mathfrak{n}} \in \Xi$, we can find $\mathfrak{e}_{\mathfrak{n}+1} \in \Xi$, then

$$\mathfrak{z}_{\mathfrak{n}} = \Sigma\mathfrak{e}_{\mathfrak{n}} = \Upsilon\mathfrak{e}_{\mathfrak{n}+1}. \quad \mathfrak{n} = 0, 1, 2, \cdots \quad (6)$$

we prove that the sequence $\{\mathfrak{z}_{\mathfrak{n}}\}$ is a CS. So, by (3), (4) and (5) we have

$$P(\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau) = P(\Sigma\mathfrak{e}_{\mathfrak{n}},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau)$$
$$\geq \phi\big(P(\Upsilon\mathfrak{e}_{\mathfrak{n}},\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\tau)\big) = \phi\big(P(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}},\tau)\big),$$
$$N(\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau) = N(\Sigma\mathfrak{e}_{\mathfrak{n}},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau)$$
$$\leq \psi\big(N(\Upsilon\mathfrak{e}_{\mathfrak{n}},\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\tau)\big) = \psi\big(N(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}},\tau)\big),$$
$$\mathcal{O}(\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau) = \mathcal{O}(\Sigma\mathfrak{e}_{\mathfrak{n}},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau)$$
$$\leq \psi\big(\mathcal{O}(\Upsilon\mathfrak{e}_{\mathfrak{n}},\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\tau)\big) = \psi\big(\mathcal{O}(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_{\mathfrak{n}},\mathfrak{z}_{\mathfrak{n}},\tau)\big).$$

Then, Lemma (4.2), $\{\mathfrak{z}_{\mathfrak{n}}\}$ is a CS. Which implies that the sequence $\{\Upsilon\mathfrak{e}_{\mathfrak{n}}\}$ is a CS. Now $\Upsilon(\Xi)$ is complete, so there exists $\mathfrak{u}\in\Upsilon(\Xi)$, we obtain

$$\lim_{\mathfrak{n}\to\infty}\mathfrak{z}_{\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty}\Sigma\mathfrak{e}_{\mathfrak{n}} = \lim_{\mathfrak{n}\to\infty}\Upsilon\mathfrak{e}_{\mathfrak{n}} = \mathfrak{u}.$$

For $\mathfrak{u}\in\Upsilon(\Xi)$, so there exists $\mathfrak{p}\in\Xi$ then $\Upsilon\mathfrak{p} = \mathfrak{u}$. Suppose that $\Sigma\mathfrak{p} \neq \mathfrak{u}$. From (3), (4), and (5), we get

$$P(\Sigma\mathfrak{e}_{\mathfrak{n}},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \geq \phi\big(P(\Upsilon\mathfrak{e}_{\mathfrak{n}},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big),$$

as $\mathfrak{n}\to\infty$, we get

$$P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \geq \phi\big(P(\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big) = \phi(1) = 1.$$

Which implies that $P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) = 1$, and

$$N(\Sigma\mathfrak{e}_{\mathfrak{n}},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \leq \psi\big(N(\Upsilon\mathfrak{e}_{\mathfrak{n}},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big),$$

as $\mathfrak{n}\to\infty$, we get

$$N(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \leq \psi\big(N(\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big) = \psi(0) = 0.$$

This implies that $N(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) = 0$, and

$$\mathcal{O}(\Sigma\mathfrak{e}_{\mathfrak{n}},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \leq \psi\big(\mathcal{O}(\Upsilon\mathfrak{e}_{\mathfrak{n}},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big),$$

as $\mathfrak{n}\to\infty$, we get

$$\mathcal{O}(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \leq \psi\big(\mathcal{O}(\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big) = \psi(0) = 0.$$

This implies that $\mathcal{O}(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) = 0$. Which is contradiction, since $\Sigma\mathfrak{p} \neq \mathfrak{u}$. Thus, $\Sigma\mathfrak{p} = \Upsilon\mathfrak{p} = \mathfrak{u}$. Then, Σ and Υ has CP which is $\mathfrak{p}$. We will prove that Σ and Υ have unique CP. Suppose $\mathfrak{q}\in\Xi$, is another CP of Σ and Υ, then $\Sigma\mathfrak{q} = \Upsilon\mathfrak{q}$. If $\Sigma\mathfrak{p} \neq \Sigma\mathfrak{q}$, we have

$$P(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \geq \phi\big(P(\Upsilon\mathfrak{q},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big) = \phi\big(P(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\big) > P(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau),$$
$$N(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \leq \psi\big(N(\Upsilon\mathfrak{q},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big) = \psi\big(N(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\big) < N(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau),$$
$$\mathcal{O}(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau) \leq \psi\big(\mathcal{O}(\Upsilon\mathfrak{q},\Upsilon\mathfrak{p},\Upsilon\mathfrak{p},\tau)\big) = \psi\big(\mathcal{O}(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\big) < \mathcal{O}(\Sigma\mathfrak{q},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau).$$

which is a contradiction. Hence $\Sigma\mathfrak{p} = \Sigma\mathfrak{q}$. Furthermore, by proposition 2.1 if Σ and Υ are semi-compatible then have unique FP.

Corollary 4.1: Suppose that $(\Xi,P,N,\mathcal{O},*,\Delta)$ is said to be a NAGNMS with $\tau*\tau\geq\tau$, and $(1-\tau)\Delta(1-\tau)\leq(1-\tau)$. Suppose $\Sigma:\Xi\to\Xi$ a mapping which fulfill the below conditions for all $\mathfrak{e},\mathfrak{z},\omega\in\Xi$,

$$P(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \geq \phi\big(P(\mathfrak{e},\mathfrak{z},\omega,\tau)\big),$$
$$N(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \leq \psi\big(N(\mathfrak{e},\mathfrak{z},\omega,\tau)\big),$$
$$\mathcal{O}(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \leq \psi\big(\mathcal{O}(\mathfrak{e},\mathfrak{z},\omega,\tau)\big).$$

where $\tau>0$, and $\phi\in\Phi$, and $\psi\in\Psi$. So, Σ has a unique FP.

Theorem 4.2: Assume that $(\Xi,P,N,\mathcal{O},*,\Delta)$ is called a NAGNMS with $\tau*\tau\geq\tau$, and $(1-\tau)\Delta(1-\tau)\leq(1-\tau)$. If the mappings $\Sigma,\Upsilon:\Xi\to\Xi$ satisfy either

$$P(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \geq \phi(\min\{P(\Upsilon\mathfrak{e},\Sigma\mathfrak{e},\Sigma\mathfrak{e},\tau),P(\Upsilon\mathfrak{z},\Upsilon\mathfrak{z},\Sigma\mathfrak{z},\tau),P(\Upsilon\omega,\Sigma\omega,\Sigma\omega,\tau)\}), \quad (7)$$
$$N(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \leq \psi(\max\{N(\Upsilon\mathfrak{e},\Sigma\mathfrak{e},\Sigma\mathfrak{e},\tau),N(\Upsilon\mathfrak{z},\Upsilon\mathfrak{z},\Sigma\mathfrak{z},\tau),N(\Upsilon\omega,\Sigma\omega,\Sigma\omega,\tau)\}), \quad (8)$$
$$\mathcal{O}(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \leq \psi(\max\{\mathcal{O}(\Upsilon\mathfrak{e},\Sigma\mathfrak{e},\Sigma\mathfrak{e},\tau),\mathcal{O}(\Upsilon\mathfrak{z},\Upsilon\mathfrak{z},\Sigma\mathfrak{z},\tau),\mathcal{O}(\Upsilon\omega,\Sigma\omega,\Sigma\omega,\tau)\}), \quad (9)$$

or

$$P(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \geq \phi(\min\{P(\Upsilon\mathfrak{e},\Upsilon\mathfrak{e},\Sigma\mathfrak{e},\tau),P(\Upsilon\mathfrak{z},\Upsilon\mathfrak{z},\Sigma\mathfrak{z},\tau),P(\Upsilon\omega,\Upsilon\omega,\Sigma\omega,\tau)\}), \quad (10)$$
$$N(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau) \leq \psi(\max\{N(\Upsilon\mathfrak{e},\Upsilon\mathfrak{e},\Sigma\mathfrak{e},\tau),N(\Upsilon\mathfrak{z},\Upsilon\mathfrak{z},\Sigma\mathfrak{z},\tau),N(\Upsilon\omega,\Upsilon\omega,\Sigma\omega,\tau)\}), \quad (11)$$

$$\mathcal{O}(\Sigma\mathfrak{e},\Sigma\mathfrak{z},\Sigma\omega,\tau)\leq\psi(\max\{\mathcal{O}(\Upsilon\mathfrak{e},\Upsilon\mathfrak{e},\Sigma\mathfrak{e},\tau),\mathcal{O}(\Upsilon\mathfrak{z},\Upsilon\mathfrak{z},\Sigma\mathfrak{z},\tau),\mathcal{O}(\Upsilon\omega,\Upsilon\omega,\Sigma\omega,\tau)\}),\quad (12)$$

for all $\mathfrak{e},\mathfrak{z},\omega\in\Xi$, where $\phi\in\Phi$, and $\psi\in\Psi$, $\tau>0$. If $\Sigma(\Xi)\subset\Upsilon(\Xi)$, and $\Upsilon(\mathfrak{e})$ is a complete subspace of Ξ than Σ and Υ have a unique CP in Ξ. Furthermore, if Σ and Υ are semi-compatible, then Σ and Υ have a unique common FP.

Proof: Let Σ and Υ fulfill the conditions (10), (11) and (12). Let $\mathfrak{e}_0$ be an arbitrary point in Ξ. Since $\Sigma(\Xi)\subset\Upsilon(\Xi)$, so we choose a point $\mathfrak{e}_1$ in Ξ such that $\Sigma(\mathfrak{e}_0)=\Upsilon(\mathfrak{e}_1)$. By continuing this process, we select any $\mathfrak{e}_\mathfrak{n}\in\Xi$, then we can find $\mathfrak{e}_{\mathfrak{n}+1}\in\Xi$, therefore $\Sigma(\mathfrak{e}_\mathfrak{n})=\Upsilon(\mathfrak{e}_{\mathfrak{n}+1})$. Inductively, construct a sequence $\{\mathfrak{z}_\mathfrak{n}\}$ in Ξ we have

$$\mathfrak{z}_\mathfrak{n}=\Sigma\mathfrak{e}_\mathfrak{n}=\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\qquad \mathfrak{n}=0,1,\cdots\quad (13)$$

we prove that the sequence $\{\mathfrak{z}_\mathfrak{n}\}$ is a CS. So, by (10), (11) and (12) we have

$$P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)=P(\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma_{\mathfrak{n}+1},1)$$
$$\geq\phi(\min\{P(\Upsilon\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\tau),P(\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau),P(\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau)\})$$
$$\geq\phi(\min\{P(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau),P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau),P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\}),$$
$$N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)=N(\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma_{\mathfrak{n}+1},1)$$
$$\leq\psi(\max\{N(\Upsilon\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\tau),N(\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau),N(\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau)\})$$
$$\leq\psi(\max\{N(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau),N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau),N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\}),$$
$$\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)=\mathcal{O}(\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma_{\mathfrak{n}+1},1)$$
$$\leq\psi(\max\{\mathcal{O}(\Upsilon\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\tau),\mathcal{O}(\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau),\mathcal{O}(\Upsilon\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\Sigma\mathfrak{e}_{\mathfrak{n}+1},\tau)\})$$
$$\leq\psi(\max\{\mathcal{O}(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau),\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau),\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\}).$$

Thus, we obtain

$$P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\geq\phi(\min\{P(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau),P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1})\}),$$
$$N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\leq\psi(\max\{N(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau),N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1})\}),$$
$$\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\leq\psi(\max\{\mathcal{O}(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau),\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1})\}).$$

Without loss of generality, suppose that $\mathfrak{z}_\mathfrak{n}\neq\mathfrak{z}_{\mathfrak{n}+1}$, for each $\mathfrak{n}$. (If there exists an $\mathfrak{n}$ such that $\mathfrak{z}_\mathfrak{n}=\mathfrak{z}_{\mathfrak{n}+1}$, So $\mathfrak{z}_\mathfrak{n}=\Sigma\mathfrak{e}_\mathfrak{n}=\Upsilon\mathfrak{e}_{\mathfrak{n}+1}=\Sigma\mathfrak{e}_{\mathfrak{n}+1}=\Upsilon\mathfrak{e}_{\mathfrak{n}+2}$, which implies that, $\Upsilon\mathfrak{e}_{\mathfrak{n}+1}=\Sigma\mathfrak{e}_{\mathfrak{n}+1}$. Then, Σ and Υ have a CP.)

$$P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\geq\phi\big(P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\big)>P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau),$$
$$N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\leq\psi\big(N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\big)<N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau),$$
$$\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\leq\psi\big(\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\big)<\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau).$$

By Lemma (4.1), which is a contradiction. Hence,

$$P(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\geq\phi\big(P(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau)\big),$$
$$N(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\leq\psi\big(N(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau)\big),$$
$$\mathcal{O}(\mathfrak{z}_\mathfrak{n},\mathfrak{z}_{\mathfrak{n}+1},\mathfrak{z}_{\mathfrak{n}+1},\tau)\leq\psi\big(\mathcal{O}(\mathfrak{z}_{\mathfrak{n}-1},\mathfrak{z}_\mathfrak{n},\mathfrak{z}_\mathfrak{n},\tau)\big).$$

Thus, by Lemma (4.2), $\{\mathfrak{z}_\mathfrak{n}\}$ is a CS, which implies that this sequence $\{\Upsilon\mathfrak{e}_\mathfrak{n}\}$ is a CS. For, $\Upsilon(\Xi)$ is complete, so there exists $\mathfrak{u}\in\Upsilon(\Xi)$ we have

$$\lim_{\mathfrak{n}\to\infty}\mathfrak{z}_\mathfrak{n}=\lim_{\mathfrak{n}\to\infty}\Sigma\,\mathfrak{e}_\mathfrak{n}=\lim_{\mathfrak{n}\to\infty}\Upsilon\mathfrak{e}_\mathfrak{n}=\mathfrak{u}.$$

Since $\mathfrak{u}\in\Upsilon(\Xi)$, so there exists $\mathfrak{p}\in\Xi$, such that $\Upsilon\mathfrak{p}=\mathfrak{u}$. Assume that $\Upsilon\mathfrak{p}\neq\mathfrak{u}$. By (7), (8), and (9)

$$P(\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\geq\phi(\min\{P(\Upsilon\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\tau),P(\Upsilon\mathfrak{p},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau),P(\Upsilon\mathfrak{p},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\}),$$

as $\mathfrak{n}\to\infty$, we get

$$P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\geq\phi(\min\{P(\mathfrak{u},\mathfrak{u},\mathfrak{u},\tau),P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\})$$
$$\geq\phi(\min\{1,P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\}).$$

Now, if $P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\geq\phi(1)=1$, Which implies that $P(\mathfrak{u},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)=1$.

$$N(\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\leq\psi(\max\{N(\Upsilon\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\Sigma\mathfrak{e}_\mathfrak{n},\tau),N(\Upsilon\mathfrak{p},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau),N(\Upsilon\mathfrak{p},\Sigma\mathfrak{p},\Sigma\mathfrak{p},\tau)\}),$$

as $\mathfrak{n}\to\infty$, we get

$$N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(\max\{N(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau), N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\})$$
$$\leq \psi(\max\{1, N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\}).$$

Now, if $N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(1) = 1$, this implies that $N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) = 0$.

$$\mathcal{O}(\Sigma\mathfrak{e}_\mathfrak{n}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(\max\{\mathcal{O}(\Upsilon\mathfrak{e}_\mathfrak{n}, \Sigma\mathfrak{e}_\mathfrak{n}, \Sigma\mathfrak{e}_\mathfrak{n}, \tau), \mathcal{O}(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), \mathcal{O}(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\}),$$

as $\mathfrak{n} \to \infty$, we get

$$\mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(\max\{\mathcal{O}(\mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau), \mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\})$$
$$\leq \psi(\max\{1, \mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\}).$$

Now, if$\mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(1) = 1$,this implies that$\mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) = 0$, this is a contradiction, for $\mathfrak{p} \neq \mathfrak{u}$. Then

$$P(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \geq \phi\big(P(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\big) > P(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau),$$
$$N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi\big(N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\big) < N(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau),$$
$$\mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi\big(\mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\big) < \mathcal{O}(\mathfrak{u}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau).$$

which is absurd. Hence, $\Sigma\mathfrak{p} = \mathfrak{u}$. Thus, $\Sigma\mathfrak{p} = \Upsilon\mathfrak{p} = \mathfrak{u}$. Therefore Σ and Υ has CP which is $\mathfrak{p}$. We satisfy that Σ and Υ have unique CP in Ξ. Let $\mathfrak{q} \in \Xi$ is another CP of Σ and Υ then $\Sigma\mathfrak{q} = \Upsilon\mathfrak{q}$. If $\Sigma\mathfrak{p} \neq \Sigma\mathfrak{q}$, we have

$$P(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \geq \phi(\min\{P(\Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}, \tau), P(\Upsilon\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), P(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\})$$
$$\geq \phi(\min\{P(\Sigma\mathfrak{q}, \Sigma\mathfrak{q}, \Sigma\mathfrak{q}, \tau), P(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), P(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\}) \geq \phi(1) = 1.$$

This implies that $P(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) = 1$.

$$N(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(\max\{N(\Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}, \tau), N(\Upsilon\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), N(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\})$$
$$\leq \psi(\max\{N(\Sigma\mathfrak{q}, \Sigma\mathfrak{q}, \Sigma\mathfrak{q}, \tau), N(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), N(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\}) \leq \psi(0) = 0.$$

This implies that $N(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) = 0$.

$$\mathcal{O}(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) \leq \psi(\max\{\mathcal{O}(\Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}, \Upsilon\mathfrak{q}, \tau), \mathcal{O}(\Upsilon\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), \mathcal{O}(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\})$$
$$\leq \psi(\max\{\mathcal{O}(\Sigma\mathfrak{q}, \Sigma\mathfrak{q}, \Sigma\mathfrak{q}, \tau), \mathcal{O}(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau), \mathcal{O}(\Upsilon\mathfrak{p}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau)\}) \leq \psi(0) = 0.$$

This implies that $\mathcal{O}(\Sigma\mathfrak{q}, \Sigma\mathfrak{p}, \Sigma\mathfrak{p}, \tau) = 0$. By Lemma (4.1), which is a contradiction as $\Sigma\mathfrak{p} \neq \Sigma\mathfrak{q}$. Furthermore, if Σ and Υ are semi-compatible, so Σ and Υ have common FP which is unique by proposition 2.1.

Corollary 4.2: Suppose that $(\Xi, P, N, \mathcal{O}, *, \Delta)$ is called a complete NAGNMS space with $\tau * \tau \geq \tau$ and $(1 - \tau) \Delta (1 - \tau) \leq (1 - \tau)$. If the mappings $\Sigma: \Xi \to \Xi$ satisfy for all $\mathfrak{e}, \mathfrak{z}, \omega \in \Xi$, either

$$P(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \geq \phi(\min\{P(\mathfrak{e}, \Sigma\mathfrak{e}, \Sigma\mathfrak{e}, \tau), P(\mathfrak{z}, \Sigma\mathfrak{z}, \Sigma\mathfrak{z}, \tau), P(\omega, \Sigma\omega, \Sigma\omega, \tau)\}),$$
$$P(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \geq \phi(\min\{P(\mathfrak{e}, \mathfrak{e}, \Sigma\mathfrak{e}, \tau), P(\mathfrak{z}, \mathfrak{z}, \Sigma\mathfrak{z}, \tau), P(\omega, \omega, \Sigma\omega, \tau)\}),$$

and

$$N(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \leq \psi(\max\{N(\mathfrak{e}, \Sigma\mathfrak{e}, \Sigma\mathfrak{e}, \tau), N(\mathfrak{z}, \Sigma\mathfrak{z}, \Sigma\mathfrak{z}, \tau), N(\omega, \Sigma\omega, \Sigma\omega, \tau)\}),$$
$$N(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \leq \psi(\max\{N(\mathfrak{e}, \mathfrak{e}, \Sigma\mathfrak{e}, \tau), N(\mathfrak{z}, \mathfrak{z}, \Sigma\mathfrak{z}, \tau), N(\omega, \omega, \Sigma\omega, \tau)\}),$$

Similarly,

$$\mathcal{O}(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \leq \psi(\max\{\mathcal{O}(\mathfrak{e}, \Sigma\mathfrak{e}, \Sigma\mathfrak{e}, \tau), \mathcal{O}(\mathfrak{z}, \Sigma\mathfrak{z}, \Sigma\mathfrak{z}, \tau), \mathcal{O}(\omega, \Sigma\omega, \Sigma\omega, \tau)\}),$$
$$\mathcal{O}(\Sigma\mathfrak{e}, \Sigma\mathfrak{z}, \Sigma\omega, \tau) \leq \psi(\max\{\mathcal{O}(\mathfrak{e}, \mathfrak{e}, \Sigma\mathfrak{e}, \tau), \mathcal{O}(\mathfrak{z}, \mathfrak{z}, \Sigma\mathfrak{z}, \tau), \mathcal{O}(\omega, \omega, \Sigma\omega, \tau)\}).$$

Where $\tau > 0$, and $\phi \in \Phi, \psi \in \Psi$. Then Σ has a unique FP.

Example 4.2: Let $(\Xi, P, N, \mathcal{O}, *, \Delta)$ be a NAGNMS defined in the Example 4.1. Define $\Sigma, \Upsilon: \Xi \to \Xi$ as follows: $\Sigma\mathfrak{e} = \frac{\mathfrak{e}}{6}$, and $\Upsilon\mathfrak{e} = \frac{\mathfrak{e}}{3}$, and define $\phi: [0, 1] \to [0, 1]$, as $\phi(\tau) = \psi(\tau) = \sqrt{\tau}$.

Then all the conditions of Theorem 4.1 are satisfying. Furthermore, self-mappings Σ and Υ are also fulfill (3), (4) and (5) for all $\mathfrak{e}, \mathfrak{z}, \omega \in \mathbb{R}$, and Σ and Υ have a common FP 0, which is unique.

5. Properties P and Q

In this, we will discuss that properties P and Q are fulfill the circumstances of the Theorems 4.1, and 4.3 also satisfy Corollary 4.1.

Theorem 5.1: Two self-mappings Σ and Υ have the property Q, by the conditions of Theorem 4.1.

Proof: From Theorem 4.1, $\zeta(\Sigma) \cap \zeta(\Upsilon) \neq \phi$. Therefore, $\zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}}) \neq \phi$, for each positive integer $\mathfrak{n}$. Let $\mathfrak{n}$ be a fixed positive integer greater than 1 and suppose that $U \in \zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}})$. We claim that $\mathfrak{u} \in \zeta(\Sigma) \cap \zeta(\Upsilon)$. Let $\mathfrak{u} \in \zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}})$. Then, for any positive integers $i, j, k, \mathfrak{r}, l, \lambda$, satisfying $0 \leq i, j, \mathfrak{r}, k, l, \lambda \leq \mathfrak{n}$, we have

$$P\left(\Sigma^i \Upsilon^i \mathfrak{u}, \Sigma^{\mathfrak{r}} \Upsilon^l \mathfrak{u}, \Sigma^\lambda \Upsilon^k \mathfrak{u}, \tau\right) \geq \phi\left(P\left(\Upsilon\left(\Sigma^{i-1} \Upsilon^j \mathfrak{u}\right), \Upsilon\left(\Sigma^{i-1} \Upsilon^l \mathfrak{u}\right), \Upsilon\left(\Sigma^{\lambda-1} \Upsilon^k \mathfrak{u}\right), \tau\right)\right)$$

$$\geq \phi\left(P\left(\Sigma^{i-1} \Upsilon^{j+1} \mathfrak{u}, \Sigma^{\mathfrak{r}-1} \Upsilon^{j+1} \mathfrak{u}, \Sigma^{\lambda-1} \Upsilon^{k+1} \mathfrak{u}, \tau\right)\right),$$

$$N\left(\Sigma^i \Upsilon^i \mathfrak{u}, \Sigma^{\mathfrak{r}} \Upsilon^l \mathfrak{u}, \Sigma^\lambda \Upsilon^k \mathfrak{u}, \tau\right) \leq \psi\left(N\left(\Upsilon\left(\Sigma^{i-1} \Upsilon^j \mathfrak{u}\right), \Upsilon\left(\Sigma^{i-1} \Upsilon^l \mathfrak{u}\right), \Upsilon\left(\Sigma^{\lambda-1} \Upsilon^k \mathfrak{u}\right), \tau\right)\right)$$

$$\leq \psi\left(N\left(\Sigma^{i-1} \Upsilon^{j+1} \mathfrak{u}, \Sigma^{\mathfrak{r}-1} \Upsilon^{j+1} \mathfrak{u}, \Sigma^{\lambda-1} \Upsilon^{k+1} \mathfrak{u}, \tau\right)\right),$$

$$\mathcal{O}\left(\Sigma^i \Upsilon^i \mathfrak{u}, \Sigma^{\mathfrak{r}} \Upsilon^l \mathfrak{u}, \Sigma^\lambda \Upsilon^k \mathfrak{u}, \tau\right) \leq \psi\left(\mathcal{O}\left(\Upsilon\left(\Sigma^{i-1} \Upsilon^j \mathfrak{u}\right), \Upsilon\left(\Sigma^{i-1} \Upsilon^l \mathfrak{u}\right), \Upsilon\left(\Sigma^{\lambda-1} \Upsilon^k \mathfrak{u}\right), \tau\right)\right)$$

$$\leq \psi\left(\mathcal{O}\left(\Sigma^{i-1} \Upsilon^{j+1} \mathfrak{u}, \Sigma^{\mathfrak{r}-1} \Upsilon^{j+1} \mathfrak{u}, \Sigma^{\lambda-1} \Upsilon^{k+1} \mathfrak{u}, \tau\right)\right).$$

Define

$$\delta = \min_{0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}} P\left(\Sigma^i \Upsilon^j \mathfrak{u}, \Sigma^{\mathfrak{r}} \Upsilon^j \mathfrak{u}, \Sigma^\lambda \Upsilon^k \mathfrak{u}, \tau\right),$$

$$\delta = \max_{0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}} N\left(\Sigma^i \Upsilon^j \mathfrak{u}, \Sigma^{\mathfrak{r}} \Upsilon^j \mathfrak{u}, \Sigma^\lambda \Upsilon^k \mathfrak{u}, \tau\right),$$

$$\delta = \max_{0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}} \mathcal{O}\left(\Sigma^i \Upsilon^j \mathfrak{u}, \Sigma^{\mathfrak{r}} \Upsilon^j \mathfrak{u}, \Sigma^\lambda \Upsilon^k \mathfrak{u}, \tau\right).$$

where $\tau > 0$. Let $0 \leq \delta < 1$, then by using the conditions (3), (4), and (5) $\delta \geq \phi(\delta) > \delta, \delta \leq (\delta) < \delta$, this is contradiction then $\delta = 0$. In particular, $P(\Sigma\, \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 1$, an $P(\Upsilon \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 1, N(\Sigma\, \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$, and $N(\Upsilon \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$, same as, $\mathcal{O}(\Sigma\, \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$, and $\mathcal{O}(\Upsilon \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$, for each $\tau > 0$ and hence $\Sigma\, \mathfrak{u} = \Upsilon \mathfrak{u} = \mathfrak{u}$, which is implies that, $\mathfrak{u} \in \zeta(\Sigma) \cap \zeta(\Upsilon)$. Then, Σ and Υ have property Q.

Example 5.1: Let $\Xi = [0, 1]$, with G-metric on Ξ defined by $d(\mathfrak{e}, \mathfrak{z}, \omega) = |\mathfrak{e} - \mathfrak{z}| + |\mathfrak{z} - \omega| + |\omega - \mathfrak{e}|$. Denote $\mathfrak{a} * \mathfrak{b} = \mathfrak{a}\mathfrak{b}$, and $\mathfrak{a} \Delta \mathfrak{b} = \min\{1, \mathfrak{a} + \mathfrak{b}\}$, for all $\mathfrak{a}, \mathfrak{b} \in [0, 1]$. For all $\mathfrak{e}, \mathfrak{z}, \omega \in \Xi$, and $\tau > 0$, define G on $\Xi^3 \times (0, \infty)$, as follows:

$$P(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = \left(\frac{\tau}{\tau + 1}\right)^{d(\mathfrak{e}, \mathfrak{z}, \omega)},$$

$$N(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = \left(\frac{1}{\tau + 1}\right)^{d(\mathfrak{e}, \mathfrak{z}, \omega)},$$

$$\mathcal{O}(\mathfrak{e}, \mathfrak{z}, \omega, \tau) = 1 - \left(\frac{\tau}{\tau + 1}\right)^{d(\mathfrak{e}, \mathfrak{z}, \omega)}.$$

Then, $(\Xi, P, N, \mathcal{O}, *, \Delta)$ is a NAGNMS. Define $\Sigma, \Upsilon: \Xi \to \Xi$ as follows: $\Sigma \mathfrak{e} = \frac{\mathfrak{e}}{8}$, and $\Upsilon \mathfrak{e} = \frac{\mathfrak{e}}{2}$, and define $\phi: [0, 1] \to [0, 1]$, as $\phi(\tau) = \psi(\tau) = \tau$.

Then all the conditions of Theorem 4.1 are satisfying. Furthermore, self-mappings Σ and Υ are also fulfill (3), (4), and (5) for all $\mathfrak{e}, \mathfrak{z}, \omega \in \mathbb{R}$, and Σ and Υ have common FP 0, which is unique

$$\zeta(\Sigma(0)) \cap \zeta(\Upsilon(0)) = 0 \neq \phi, \quad (14)$$

and

$$\zeta\left(\Sigma\left(\Sigma\left(\Sigma \underbrace{\cdots}_{\mathfrak{n}-\tau ime\lambda} \Sigma(0)\right)\right)\right) \cap \zeta\left(\Upsilon\left(\Upsilon\left(\Upsilon \underbrace{\cdots}_{\mathfrak{n}-\tau ime\lambda} \Upsilon(0)\right)\right)\right) = 0 \neq \phi. \quad (15)$$

Then, from (14) and (15) $\zeta(\Sigma) \cap \zeta(\Upsilon) = \zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}})$. Hence Σ and Υ have property Q.

Corollary 5.1: A mapping Σ has the property P, by the conditions of Corollary 4.1.

Theorem 5.2: Two self-mappings Σ and Υ have the property Q, by the conditions of Theorem 4.2.

Proof: By Theorem 4.2, $\zeta(\Sigma) \cap \zeta(\Upsilon) \neq \emptyset$. Therefore, $\zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}}) \neq \emptyset$, for each positive integer $\mathfrak{n}$. Let $\mathfrak{n}$ be a fixed positive integer greater than 1 and suppose that $U \in \zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}})$. We claim that $\mathfrak{u} \in \zeta(\Sigma) \cap \zeta(\Upsilon)$. Let $\mathfrak{u} \in \zeta(\Sigma^{\mathfrak{n}}) \cap \zeta(\Upsilon^{\mathfrak{n}})$. Then, for positive integers $i, j, \mathfrak{r}, l, \lambda, k$, satisfying $0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}$, we have

$$P(\Sigma^i\Upsilon^i\mathfrak{u}, \Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}, \Sigma^\lambda\Upsilon^k\mathfrak{u}, \tau) \geq \phi\left(\min\left\{\begin{array}{c} P(\Upsilon(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \Sigma(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \Sigma(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \tau), \\ P(\Upsilon(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \Sigma(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \Sigma(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \tau), \\ P(\Upsilon(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \Sigma(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \Sigma(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \tau), \end{array}\right\}\right),$$

$$\geq \phi\left(\min\left\{\begin{array}{c} P\left((\Sigma^{i-1}\Upsilon^{j+1}\mathfrak{u}), (\Sigma^i\Upsilon^j\mathfrak{u}), (\Sigma^i\Upsilon^j\mathfrak{u}), \tau\right), \\ P\left((\Sigma^{\mathfrak{r}-1}\Upsilon^{l+1}\mathfrak{u}), (\Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}), (\Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}), \tau\right), \\ P\left((\Sigma^{\lambda-1}\Upsilon^{k+1}\mathfrak{u}), (\Sigma^\lambda\Upsilon^k\mathfrak{u}), (\Sigma^\lambda\Upsilon^k\mathfrak{u}), \tau\right), \end{array}\right\}\right),$$

$$N(\Sigma^i\Upsilon^i\mathfrak{u}, \Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}, \Sigma^\lambda\Upsilon^k\mathfrak{u}, \tau) \leq \psi\left(\max\left\{\begin{array}{c} N(\Upsilon(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \Sigma(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \Sigma(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \tau), \\ N(\Upsilon(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \Sigma(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \Sigma(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \tau), \\ N(\Upsilon(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \Sigma(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \Sigma(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \tau), \end{array}\right\}\right),$$

$$\leq \left(\max\left\{\begin{array}{c} N\left((\Sigma^{i-1}\Upsilon^{j+1}\mathfrak{u}), (\Sigma^i\Upsilon^j\mathfrak{u}), (\Sigma^i\Upsilon^j\mathfrak{u}), \tau\right), \\ N\left((\Sigma^{\mathfrak{r}-1}\Upsilon^{l+1}\mathfrak{u}), (\Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}), (\Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}), \tau\right), \\ N\left((\Sigma^{\lambda-1}\Upsilon^{k+1}\mathfrak{u}), (\Sigma^\lambda\Upsilon^k\mathfrak{u}), (\Sigma^\lambda\Upsilon^k\mathfrak{u}), \tau\right), \end{array}\right\}\right),$$

$$\mathcal{O}(\Sigma^i\Upsilon^i\mathfrak{u}, \Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}, \Sigma^\lambda\Upsilon^k\mathfrak{u}, \tau) \leq \psi\left(\max\left\{\begin{array}{c} \mathcal{O}(\Upsilon(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \Sigma(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \Sigma(\Sigma^{i-1}\Upsilon^j\mathfrak{u}), \tau), \\ \mathcal{O}(\Upsilon(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \Sigma(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \Sigma(\Sigma^{\mathfrak{r}-1}\Upsilon^l\mathfrak{u}), \tau), \\ \mathcal{O}(\Upsilon(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \Sigma(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \Sigma(\Sigma^{\lambda-1}\Upsilon^k\mathfrak{u}), \tau), \end{array}\right\}\right),$$

$$\leq \left(\max\left\{\begin{array}{c} \mathcal{O}\left((\Sigma^{i-1}\Upsilon^{j+1}\mathfrak{u}), (\Sigma^i\Upsilon^j\mathfrak{u}), (\Sigma^i\Upsilon^j\mathfrak{u}), \tau\right), \\ \mathcal{O}\left((\Sigma^{\mathfrak{r}-1}\Upsilon^{l+1}\mathfrak{u}), (\Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}), (\Sigma^{\mathfrak{r}}\Upsilon^l\mathfrak{u}), \tau\right), \\ \mathcal{O}\left((\Sigma^{\lambda-1}\Upsilon^{k+1}\mathfrak{u}), (\Sigma^\lambda\Upsilon^k\mathfrak{u}), (\Sigma^\lambda\Upsilon^k\mathfrak{u}), \tau\right), \end{array}\right\}\right).$$

Define,

$$\delta = \min_{0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}} P(\Sigma^i\Upsilon^j\mathfrak{u}, \Sigma^{\mathfrak{r}}\Upsilon^j\mathfrak{u}, \Sigma^\lambda\Upsilon^k\mathfrak{u}, \tau),$$

$$\delta = \max_{0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}} N(\Sigma^i\Upsilon^j\mathfrak{u}, \Sigma^{\mathfrak{r}}\Upsilon^j\mathfrak{u}, \Sigma^\lambda\Upsilon^k\mathfrak{u}, \tau),$$

$$\delta = \max_{0 \leq i, \mathfrak{r}, j, l, \lambda, k \leq \mathfrak{n}} \mathcal{O}(\Sigma^i\Upsilon^j\mathfrak{u}, \Sigma^{\mathfrak{r}}\Upsilon^j\mathfrak{u}, \Sigma^\lambda\Upsilon^k\mathfrak{u}, \tau),$$

where $\tau > 0$. Let $0 \leq \delta < 1$, by (7), (8), and (9), we have

$$\delta \geq \phi(\min\{\delta, \delta, \delta\}) = \phi(\delta) > \delta,$$
$$\delta \leq \psi(\max\{\delta, \delta, \delta\}) = \psi(\delta) < \delta.$$

This is a contradiction and then $\delta = 1$. In particular, $P(\Sigma\, \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 1$, and $P(\Upsilon \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 1$, $N(\Sigma \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$,and $N(\Upsilon \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$, $\mathcal{O}(\Sigma \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$,and $\mathcal{O}(\Upsilon \mathfrak{u}, \mathfrak{u}, \mathfrak{u}, \tau) = 0$, for each $\tau > 0$, so, $\Sigma \mathfrak{u} = \Upsilon \mathfrak{u} = \mathfrak{u}$, implies that, $\mathfrak{u} \in \zeta(\Sigma) \cap \zeta(\Upsilon)$. Hence Σ and Υ have property Q.

Corollary 5.2: A mapping Σ has the property P, by the conditions of Corollary 4.2.

6. Conclusions

In this work, we established the metrizability of the topology produced by any neutrosophic metric space. Further, we examined that the resulting topology is CM if the neutrosophic metric space is complete and that a neutrosophic metric space is precompact if and only if each sequence has a Cauchy subsequence. Furthermore, we introduced the concept of NAGNMS and proved several the FP results in this new introduced setting. This work is extendable in the setting of orthogonal NAGNMS, non-Archimedean generalized neutrosophic metric like spaces, non-Archimedean generalized neutrosophic partial metric spaces and many others.

Author Contributions: All authors contributed equally in this manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data will be available on demand from corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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Received: June 2, 2025. Accepted: Sep 6, 2025