NSS Neutrosophic Sets and Systems, Vol. 95, 2026

University of New Mexico

Interval-Valued Neutrosophic Fuzzy n-Fold Positive Implicative Ideals of BCK-Algebras: Properties and Characterizations

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Abstract. This paper investigation elucidates the notion of interval-valued neutrosophic fuzzy n-fold ideals in BCK-algebras, with a specific focus on interval-valued neutrosophic fuzzy n-fold BCK-ideal and interval-valued neutrosophic fuzzy n-fold positive implicative ideals. A comprehensive examination of the characterization theorems and extension properties of these ideals is undertaken. Notably, it is demonstrated that every interval-valued neutrosophic fuzzy n-fold BCK-ideal is an interval-valued neutrosophic fuzzy ideal, and every interval-valued neutrosophic fuzzy n-fold positive implicative ideal is an interval-valued neutrosophic fuzzy ideal. Furthermore, counterexamples are constructed to delineate the limitations of these concepts.

Keywords: Interval-valued neutrosophic fuzzy n-fold Positive implicative ideals; interval-valued neutrosophic fuzzy n-fold BCK-ideal; interval-valued neutrosophic fuzzy set; BCK-algebra.

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1. **Introduction**

BCK-algebras and BCI-algebras are two classes of abstract algebras introduced by Imai and Ise´ki [1] in 1966. The BCK-algebra class is recognized as a legitimate subclass of the BCI-algebra class. Researchers have extensively studied various properties of BCK-algebras. In 1999, Huang and Chen [2] introduced the notion of "n-fold implicative ideals and n-fold (weak) commutative ideals." Fuzzy sets were first introduced by Zadeh [13] in 1965, have been applied to various fields of pure and applied mathematics, computer science, and engineering. “Interval-valued fuzzy sets, an extension of fuzzy sets,” were introduced by Zadeh. “Interval-valued fuzzy sets have been used to study fuzzy ideals and fuzzy subalgebras in various algebraic structures,” including semigroups, groups, rings, BCK/BCI-algebras, and more. In 2001 Jun and Kim [3] further pursued this in constructing “n-fold fuzzy positive implicative ideals in BCK-algebras” and researched its corresponding properties. “Intuitionistic fuzzy sets are an extension of fuzzy sets,” initially introduced by Atanassov [4]. Thereafter, Atanassov and Gargov [5] extended the term “intuitionistic fuzzy sets and interval-valued fuzzy sets” further by introducing the concept of “interval-valued intuitionistic fuzzy sets.” Smarandache [12] introduced neutrosophic set which consists of “the degree of truth membership function, the degree of indeterminacy function and the degree of falsity membership function” these three are independent functions. Satyanarayana et al. (see [6], [7], [11]) studied new concepts of “foldness of interval-valued intuitionistic fuzzy implicative, commutative, positive implicative ideals of BCK-algebras”.

Several variations have been introduced into the world of neutrosophic structures to help us better understand indeterminacy and uncertainty. In 2005, Wang [14], extended the idea with the introduction of “interval-valued neutrosophic sets.” Following this, Jun et al. [15] applied “interval-valued neutrosophic sets to the ideals in BCI/BCK algebras.” Takalo et al. [16] introduced “the mBJ-neutrosophic structures as a generalisation of the neutrosophic sets,” in which the indeterminacy function is represented by “interval-valued fuzzy sets.” Smarandache et al. [17] introduced neutrosophic N-structures in which the truth, uncertainty, and falsity membership functions are negatively valued functions. Later, neutrosophic N-structures were applied to positive implicative ideals [18] and commutative ideals [19] in BCK-algebras. Additionally, Song et al., [20] introduced the concept of generalised NSS. Subsequently, [21] Borzooei et al., proposed the idea of commutative generalised neutrosophic ideals in BCI/BCK-algebras.

Interval-valued neutrosophic fuzzy n-fold positive implicative ideals, interval-valued neutrosophic fuzzy n-fold BCK-ideals are provide a framework for handling uncertainty and imprecision in BCK-algebras. This concept generalizes existing concepts of “fuzzy ideals and neutrosophic fuzzy ideals in BCK-algebras.” Interval-valued neutrosophic fuzzy n-fold positive implicative ideals, interval-valued neutrosophic fuzzy n-fold BCK-ideals can be applied to real-world problems that involve uncertainty and imprecision, such as decision-making, optimization, and control systems. Interval-valued neutrosophic fuzzy n-fold positive implicative ideals can be complex and difficult to compute. There is limited existing research on interval-valued neutrosophic fuzzy n-fold BCK-ideals, interval-valued neutrosophic fuzzy n-fold positive implicative ideals, which can make it difficult to apply and generalize. The results obtained from interval-valued neutrosophic fuzzy n-fold BCK-ideals, interval-valued neutrosophic fuzzy n-fold positive implicative ideals can be difficult to interpret and understand. Interval-valued neutrosophic fuzzy n-fold BCK-ideals, interval-valued neutrosophic fuzzy n-fold positive implicative ideals can contribute to advancements in fuzzy mathematics and its applications. This concept can be applied to decision-making problems that involve uncertainty and imprecision, leading to improved decision-making. Interval-valued neutrosophic fuzzy n-fold BCK-ideals, interval-valued neutrosophic fuzzy n-fold positive implicative ideals can be applied to new areas in engineering and computer science, such as image processing, pattern recognition, and machine learning.

Recently, Satyanarayana et al.(see [8], [9], [10], [22]) introduced the idea of BS-neutrosophic structure in BCI/BCK-algebra, where false membership is represented by an interval-valued fuzzy set, and studied "interval-valued neutrosophic fuzzy implicative ideal, commutative ideal, positive implicative ideals in BCK-algebras, and also introduced interval-valued neutrosophic fuzzy n-fold implicative, commutative ideals in the context of BCK-algebras". In [25] Hazim et al., studied fuzzy metric spaces of the Two-fold fuzzy algebra and Tahsin et al., [24] introduced Neutrosophic N-structures on Sheffer stroke Hilbert algebras.

motivation and novelties: The study of BCK-algebras and their ideals has been an active area of research in recent years. However, the existing literature on BCK-algebras has several limitations, including: (a) The inability to handle uncertainty and imprecision in a comprehensive manner. (b) The lack of a framework for studying the algebraic properties of BCK-algebras in a fuzzy environment. Interval-valued neutrosophic fuzzy sets provide a powerful tool for handling uncertainty and imprecision in algebraic structures. However, the application of interval-valued neutrosophic fuzzy sets to BCK-algebras is still in its infancy. In this way we motivated and introduce in this work. The existing literature on BCK-algebras and their ideals has several research gaps, including: (c) The lack of a comprehensive framework for studying the algebraic properties of BCK-algebras in a fuzzy environment. (d) The limited availability of methods for studying the ideals of BCK-algebras in a fuzzy environment. (e) The need for new mathematical tools for handling uncertainty and imprecision in BCK-algebras. The novelties of this research is (f) The introduction of a new concept of interval-valued neutrosophic fuzzy n-fold positive implicative ideals of BCK-algebras. (g) The development of new methods for studying the algebraic properties of BCK-algebras in a fuzzy environment. (h) The application of interval-valued neutrosophic fuzzy sets to BCK-algebras, which provides a new perspective on the study of algebraic structures. Applying interval-valued neutrosophic fuzzy sets to BCK-algebras, which provides a new perspective on the study of algebraic structures.

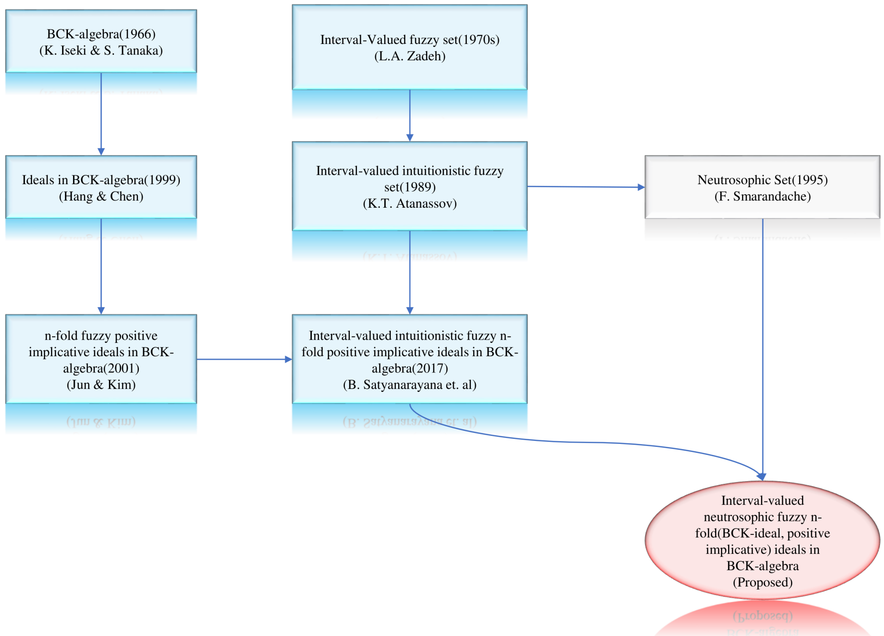

Figure 1. Flowchart of IVNFnPII in BCK-algebras

In this paper, the relations between "interval-valued neutrosophic fuzzy sets (IVNFS)" and "n-fold BCK-ideals, n-fold positive implicative ideals in BCK-algebras" are explained. By describing relations among these notions and investigating corresponding features, we define new notions of "interval-valued neutrosophic fuzzy n-fold BCK-ideals and interval-valued neutrosophic fuzzy n-fold positive implicative ideals (IVNFnPII)." We also study extension properties of IVNFnPII and provide characterizations of them.

This paper, we used the following abbreviations:

- BCK-A (or) $\mathfrak{A}$: BCK-algebras
- n-BCK-I : n-fold BCK-ideal
- IVNF-n-BCK-I : interval valued neutrosophic fuzzy n-fold BCK-ideal
- IVNFS : interval valued neutrosophic fuzzy sets
- IVNFI : interval valued neutrosophic fuzzy ideal
- IVN-n-PII : interval valued neutrosophic n-fold positive implicative ideal
- IVNF-n-PII : interval valued neutrosophic fuzzy n-fold positive implicative ideal
- NFSA : neutrosophic fuzzy sub-algebra
- NFI : neutrosophic fuzzy ideal

2. Preliminaries

In this part, we give some preliminary definitions which are basic in this research article. For the purpose of this paper, $\mathfrak{A}$ will be assumed to be a $\mathcal{BCK}$ - algebras, unless we say otherwise.

Definition 2.1. [1] Let $\mathfrak{A}$ be a ($\neq \phi$) set with a binary operation "$*$" and a constant "0". Then $(\mathfrak{A}, *, 0)$ is called BCK-A, if it fulfils the below conditions
(BCK1) $((\kappa_1 * \kappa_2) * (\kappa_1 * \kappa_3)) * (\kappa_3 * \kappa_2) = 0$,
(BCK2) $(\kappa_1 * (\kappa_1 * \kappa_2)) * \kappa_2 = 0$,
(BCK3) $\kappa_1 * \kappa_1 = 0$,
(BCK4) $0 * \kappa_1 = 0$,
(BCK5) $\kappa_1 * \kappa_2 = 0$, and $\kappa_2 * \kappa_1 = 0 \Rightarrow \kappa_1 = \kappa_2$, for any κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

A binary relation "$\leq$" on $\mathfrak{A}$ is defined as $\kappa_1 \leq \kappa_2$ iff $\kappa_1 * \kappa_2 = 0$. This relation (i.e.,$(\mathfrak{A}, \leq)$) induces a partial order on $\mathfrak{A}$, with "0" as the least element.
Furthermore, $(\mathfrak{A}, *, 0)$ is a BCK-A iff the following properties hold:
$\forall$ κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.
(i) $((\kappa_1 * \kappa_2) * (\kappa_1 * \kappa_3)) \leq (\kappa_3 * \kappa_2)$,
(ii) $(\kappa_1 * (\kappa_1 * \kappa_2)) \leq \kappa_2$,
(iii) $\kappa_1 \leq \kappa_1$,
(iv) $0 \leq \kappa_1$,
(v) $\kappa_1 \leq \kappa_2$ and $\kappa_2 \leq \kappa_1 \Rightarrow \kappa_1 = \kappa_2$,
Within the framework of a BCK-A $(\mathfrak{A}, *, 0)$, the below properties hold:
(P1) $\kappa_1 * 0 = \kappa_1$,
(P2) $\kappa_1 * \kappa_2 \leq \kappa_1$,
(P3) $(\kappa_1 * \kappa_2) * \kappa_3 = (\kappa_1 * \kappa_3) * \kappa_2$,
(P4) $(\kappa_1 * \kappa_3) * (\kappa_2 * \kappa_3) \leq \kappa_1 * \kappa_2$,
(P5) $\kappa_1 * (\kappa_1 * (\kappa_1 * \kappa_2)) = \kappa_1 * \kappa_2$,
(P6) $\kappa_1 \leq \kappa_2 \Rightarrow \kappa_1 * \kappa_3 \leq \kappa_2 * \kappa_3$ and $\kappa_3 * \kappa_2 \leq \kappa_3 * \kappa_1$,
(P7) $\kappa_1 * \kappa_2 \leq \mathrm{h} \Rightarrow \kappa_1 * \kappa_3 \leq \kappa_2$, $\forall$ κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

Definition 2.2. A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be a "sub-algebra" of $\mathfrak{A}$, if for κ_1,$\kappa_2 \in \Im$ $\Rightarrow \kappa_1 * \kappa_2 \in \Im$.

Definition 2.3. A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be an "ideal," if ($\Im$-1) $0 \in \Im$, ($\Im$-2)$\kappa_1 * \kappa_2$, and $\kappa_2 \in \Im \Rightarrow \kappa_1 \in \Im$, for every κ_1,$\kappa_2 \in \mathfrak{A}$.

Definition 2.4. [3] A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be an "n-fold implicative ideal," if ($\Im$-1), and ($\Im$-3) $\exists$ a fixed $n \in \mathfrak{A} \ni (\kappa_1 * (\kappa_2 * {\kappa_1}^n) * \kappa_3 \in \Im$, and $\kappa_3 \in \Im \Rightarrow \kappa_1 \in \Im$,
for every κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

Definition 2.5. A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be an "n-fold positive implicative ideal," if ($\Im$-1), and ($\Im$-4) $\exists$ a fixed $n \in \mathfrak{A} \ni (\kappa_1 * \kappa_2) * {\kappa_3}^n \in \Im$ and $\kappa_2 * {\kappa_3}^n \in \Im \Rightarrow \kappa_1 * {\kappa_3}^n \in \Im$, for every κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

Definition 2.6. [3] A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be an "n-fold commutative ideal," if ($\Im$-1), and ($\Im$-5)$\exists$ a fixed $n \in \mathfrak{A} \ni (\kappa_1 * \kappa_2) * \kappa_3 \in \Im$ and $\kappa_3 \in \Im \Rightarrow \kappa_1 *(\kappa_2 *(\kappa_2 * {\kappa_1}^n)) \in \Im$, for every κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

Definition 2.7. [3] A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be an "n-fold *weak* commutative ideal," if ($\Im$-1), and ($\Im$-6) $\exists$ a fixed $n \in \mathfrak{A} \ni (\kappa_1 *(\kappa_1 * {\kappa_2}^n) * \kappa_3) \in \Im$ and $\kappa_3 \in \Im$
$\Rightarrow (\kappa_2 *(\kappa_2 * \kappa_1)) \in \Im$, for every κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

Definition 2.8. A ($\neq \phi$) sub-set $\Im$ of $\mathfrak{A}$ is said to be an "n-fold BCK-ideal," if ($\Im$-1), and ($\Im$-7) $\exists$ a fixed $n \in \mathfrak{A} \ni (\kappa_1 * {\kappa_2}^(n+1)\ ) * \kappa_3 \in \Im$ and $\kappa_3 \in \Im \Rightarrow \kappa_1 * {\kappa_2}^n \in \Im$, for every κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$.

For any elements κ_1 and κ_2 of $\mathfrak{A}$, $\kappa_1 * {\kappa_2}^n$ denotes $(...((\kappa_1 * \kappa_2) * \kappa_2) *...) * \kappa_2$ in which 'κ_2' occurs n-times.

Definition 2.9. [23] An "interval-valued fuzzy set (IVFS)" $\widetilde{\mathcal{P}}$ over $\mathfrak{A}$ is characterized by the expression $\widetilde{\mathcal{P}}$: $\mathfrak{A} \to [\Im]$, is called "interval-valued fuzzy set" in $\mathfrak{A}$. Let $[\Im]^{\mathfrak{A}}$ stand for "the set of all interval-valued fuzzy sets" in $\mathfrak{A}$. For every $\widetilde{\mathcal{P}} \in [\Im]^{\mathfrak{A}}$, and $\kappa_1 \in \mathfrak{A}$, $\widetilde{\mathcal{P}}\ (\kappa_1) = [\mathcal{P}^-(\kappa_1), \mathcal{P}^+(\kappa_1)]$ is called "the degree of truth membership" of an element $\kappa_1 \in \mathfrak{A}$, where $\mathcal{P}^- : \mathfrak{A} \to [\Im]$, and $\mathcal{P}^+ : \mathfrak{A} \to [\Im]$ are "fuzzy sets" in $\mathfrak{A}$ which are called "a lower fuzzy set and an upper fuzzy set" in $\mathfrak{A}$. For simplicity, we denote the symbol $\widetilde{\mathcal{P}} = [\mathcal{P}^-,\ \mathcal{P}^+]$.

Definition 2.10. [12] A "neutrosophic fuzzy set (NFS)" $\mathcal{M}$ over $\mathfrak{A}$ is characterized by the expression $\mathcal{M} = \{\ (\kappa_1, \xi_{\mathcal{M}}(\kappa_1), \zeta_{\mathcal{M}}(\kappa_1), \varpi_{\mathcal{M}}(\kappa_1)) : \kappa_1 \in \mathfrak{A}\ \}$, where $\xi_{\mathcal{M}}(\kappa_1) : \mathcal{A} \to [0,1]$, $\zeta_{\mathcal{M}}(\kappa_1) : \mathcal{A} \to [0,1]$, and $\varpi_{\mathcal{M}}(\kappa_1) : \mathcal{A} \to [0,1]$, We denote $\xi_{\mathcal{M}}(\kappa_1)$, $\zeta_{\mathcal{M}}(\kappa_1)$, and $\varpi_{\mathcal{M}}(\kappa_1)$ as "the membership, indeterminacy, and non-membership intervals," respectively of element κ_1 with respect to set $\mathcal{M}$, For the purpose of clarity, we introduce the notation $\mathcal{M} = (\xi_{\mathcal{M}}, \zeta_{\mathcal{M}}, \varpi_{\mathcal{M}})$. For the neutrosophic set $\mathcal{M} = \{(\kappa_1, \xi_{\mathcal{M}}(\kappa_1), \zeta_{\mathcal{M}}(\kappa_1), \varpi_{\mathcal{M}}(\kappa_1)) : \kappa_1 \in \mathfrak{A}\}$.

Definition 2.11. An "interval-valued neutrosophic fuzzy set (IVNFS)" $\widetilde{\mathcal{M}}$ over $\mathfrak{A}$ is characterized by the expression $\widetilde{\mathcal{M}}$=$\{(\kappa_1, \widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}}) : \kappa_1 \in \mathfrak{A}\}$, where $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) : \mathfrak{A} \to \mathcal{A}[0,1]$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) : \mathfrak{A} \to \mathcal{A}[0,1]$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) : \mathfrak{A} \to \mathcal{A}[0,1]$, We denote $\widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$ as "the membership, indeterminacy, and non-membership intervals," respectively of element κ_1 with respect to set $\widetilde{\mathcal{M}}$, where $\widetilde{\xi_{\mathcal{M}}}$(f)=$[\xi^-_{\mathcal{M}}(\kappa_1), \xi^+_{\mathcal{M}}(\kappa_1)]$,$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) = [\zeta^-_{\mathcal{M}}(\kappa_1), \zeta^+_{\mathcal{M}}(\kappa_1)]$ and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) = [\varpi^-_{\mathcal{M}}(\kappa_1), \varpi^+_{\mathcal{M}}](\kappa_1)]$, $\forall\ \kappa_1 \in \mathfrak{A}$ with the condition $[0,0] \leq \widetilde{\xi_{\mathcal{M}}}(\kappa_1) + \widetilde{\zeta_{\mathcal{M}}}(\kappa_1) + \widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq [3,3]$, $\forall\ \kappa_1 \in \mathfrak{A}$. For the purpose of clarity, we introduce the notation $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$.

Definition 2.12. [9] An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an "interval-valued neutrosophic fuzzy ideal (IVNFI)" of $\mathfrak{A}$, if it satisfies

(IVNFI1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$,

(IVNFI2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq \min\{\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2),\widetilde{\xi_{\mathcal{M}}}(\kappa_1)\}$,

(IVNFI3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2),\widetilde{\zeta_{\mathcal{M}}}(\kappa_1)\}$, and

(IVNFI4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2),\widetilde{\varpi_{\mathcal{M}}}(\kappa_1)\}$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.

Definition 2.13. [10] An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an "interval-valued neutrosophic fuzzy n-fold implicative ideal (IVNFnII)" of $\mathfrak{A}$, if it fulfils

(IVNFnII1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$, and $\exists$ a fixed $n \in \mathfrak{A}$ such that

(IVNFnII2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * (\kappa_2 * {\kappa_1}^n\ )) * \kappa_3),\widetilde{\xi_{\mathcal{M}}}(\kappa_3)\}$,

(IVNFnII3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * (\kappa_2 * {\kappa_1}^n\ )) * \kappa_3),\widetilde{\zeta_{\mathcal{M}}}(\kappa_3)\}$, and

(IVNFnII4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * (\kappa_2 * {\kappa_1}^n\ )) * \kappa_3),\widetilde{\varpi_{\mathcal{M}}}(\kappa_3)\}$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Definition 2.14. [10] An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an "interval-valued neutrosophic fuzzy n-fold commutative ideal (IVNFnCI)" of $\mathfrak{A}$, if it fulfils

(IVNFnCI1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$,$\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$, and $\exists$ a fixed $n \in \mathfrak{A}$ such that

(IVNFnCI2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * (\kappa_2 * {\kappa_1}^n\ ))) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * \kappa_3),\widetilde{\xi_{\mathcal{M}}}(\kappa_3)\}$,

(IVNFnCI3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * (\kappa_2 * {\kappa_1}^n\ ))) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * \kappa_3),\widetilde{\zeta_{\mathcal{M}}}(\kappa_3)\}$, and

(IVNFnCI4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * (\kappa_2 * {\kappa_1}^n\ ))) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * \kappa_3),\widetilde{\varpi_{\mathcal{M}}}(\kappa_3)\}$,
$\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Definition 2.15. [10] An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an "interval-valued neutrosophic fuzzy n-fold *weak* commutative ideal (IVNF-n-W-CI)" of $\mathfrak{A}$, if it fulfils

(IVNFnWCI1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$,$\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$, and $\exists$ a fixed $n \in \mathfrak{A}$ such that

(IVNFnWCI2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_2 * (\kappa_2 * \kappa_1)) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * {\kappa_2}^n\ )) * \kappa_3),\widetilde{\xi_{\mathcal{M}}}(\kappa_3)\}$,

(IVNFnWCI3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * (\kappa_2 * \kappa_1)) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * {\kappa_2}^n\ )) * \kappa_3),\widetilde{\zeta_{\mathcal{M}}}(\kappa_3)\}$, and

(IVNFnWCI4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * (\kappa_2 * \kappa_1)) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * {\kappa_2}^n\ )) * \kappa_3),\widetilde{\varpi_{\mathcal{M}}}(\kappa_3)\}$,
$\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Definition 2.16. [8] An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an "interval-valued neutrosophic fuzzy positive implicative ideal (IVNFPII)" of $\mathfrak{A}$, if it fulfils

(IVNFPII1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$,

(IVNFPII2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_3) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * \kappa_3),\widetilde{\xi_{\mathcal{M}}}(\kappa_2 * \kappa_3)\}$,

(IVNFPII3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_3) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * \kappa_3),\widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * \kappa_3)\}$, and

(IVNFPII4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_3) \leq \max\{\ \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * \kappa_3),\widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * \kappa_3)\}$,$\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Theorem 2.17. *[9] Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFI of $\mathfrak{A}$, if $\kappa_1 \leq \kappa_2$ in $\mathfrak{A}$, then $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_2)$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_2)$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_2)$, that is, $\widetilde{\xi_{\mathcal{M}}}$, $\widetilde{\zeta_{\mathcal{M}}}$ are order reversing and $\widetilde{\varpi_{\mathcal{M}}}$ is order preserving.*

Theorem 2.18. *[9] Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFI of $\mathfrak{A}$, $\Longleftrightarrow$ the inequality $\kappa_1 * \kappa_2 \leq \kappa_3$ holds in $\mathfrak{A}$, for $\kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$, then $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq min\{\widetilde{\xi_{\mathcal{M}}}(\kappa_2),\widetilde{\xi_{\mathcal{M}}}(\kappa_3)\}$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq min\{\widetilde{\zeta_{\mathcal{M}}}(\kappa_2),\widetilde{\zeta_{\mathcal{M}}}(\kappa_3)\}$ and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq max\{\widetilde{\varpi_{\mathcal{M}}}(\kappa_2),\widetilde{\varpi_{\mathcal{M}}}(\kappa_3)\}$.*

Theorem 2.19. *[10] Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFI of $\mathfrak{A}$, then*
(i) $\widetilde{\mathcal{M}}$ is an IVNFn-CI of $\mathfrak{A} \Longleftrightarrow \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * (\kappa_2 * {\kappa_1}^n))) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2)$,
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * (\kappa_2 * {\kappa_1}^n))) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2)$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * (\kappa_2 * {\kappa_1}^n))) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2)$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.
(ii) $\widetilde{\mathcal{M}}$ is an IVNFn-weak-CI of $\mathfrak{A} \Longleftrightarrow \widetilde{\xi_{\mathcal{M}}}(g * (\kappa_2 * \kappa_1)) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * {\kappa_2}^n))$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * (\kappa_2 * \kappa_1)) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * {\kappa_2}^n))$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * (\kappa_2 * \kappa_1)) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * {\kappa_2}^n))$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Theorem 2.20. *[10] Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFI of $\mathfrak{A}$. Then $\widetilde{\mathcal{M}}$ is an IVNFnII of $\mathfrak{A} \Longleftrightarrow$ it satisfies the inequalities $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.*

3. Interval-Valued Neutrosophic Fuzzy n-Fold BCK-ideals of BCK-algebra

In this section we apply the concept of 'interval-valued neutrosophic fuzzy sets to n-fold BCK-ideal of BCK-algebras and introduce the notions of interval-valued neutrosophic fuzzy n-fold BCK-ideal of BCK-algebras (IVNF-n-BCK-I)' and investigate some of its related properties.

Definition 3.1. An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an IVNF-n-BCK-I of $\mathfrak{A}$, if it satisfies
(IVNFnBCKI1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$, and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$, $\exists$ a fixed $n \in \mathfrak{A}$ $\ni$
(IVNFnBCKI2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_2}^(n+1)) * \kappa_3),\widetilde{\xi_{\mathcal{M}}}(\kappa_3)\}$,
(IVNFnBCKI3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_2}^(n+1)) * \kappa_3),\widetilde{\zeta_{\mathcal{M}}}(\kappa_3)\}$, and
(IVNFnBCKI4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_2}^(n+1)) * \kappa_3),\widetilde{\varpi_{\mathcal{M}}}(\kappa_3)\}$,
$\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Theorem 3.2. *Every IVNF-n-BCK-I of $\mathfrak{A}$ is an IVNFI of $\mathfrak{A}$.*

Proof. Let $\widetilde{\mathcal{M}}$ be an IVNF-n-BCK-I of $\mathfrak{A}$. Put $\kappa_2 = 0$ in (IVNFnBCKI2), (IVNFnBCKI3) and (IVNFnBCKI4). Consequently, we obtain the proof of the required result. □

Note. We provide a counterexample to the converse of Theorem 3.2, which is presented below.

Example 3.3. Consider the set $\mathfrak{A} = \mathbf{N} \cup \{0\}$, where $\mathbf{N}$ denotes the set of natural numbers, we define the operation " $*$ " on $\mathfrak{A}$ by $\kappa_1 * \kappa_2 = \max\{0, \kappa_1 - \kappa_2\}$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$. Then $\mathfrak{A}$ is a BCK-A.
Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS in $\mathfrak{A}$ given by $\widetilde{\xi_{\mathcal{M}}}(0) = [0.6, 0.7] > [0.14, 0.15] = \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(0) = [0.2, 0.3] > [0.14, 0.15] = \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$ and $\widetilde{\varpi_{\mathcal{M}}}(0) = [0.12, 0.13] < [0.14, 0.15] = \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$, $\forall\ \kappa_1(\neq 0) \in \mathfrak{A}$. Then $\widetilde{\mathcal{M}}$ is an IVNFI of $\mathfrak{A}$, but it is not an IVNF-2-BCK-I of $\mathfrak{A}$, because
$\widetilde{\xi_{\mathcal{M}}}(10 * 3^2) = \widetilde{\xi_{\mathcal{M}}}(1) = [0.14, 0.15] < [0.6, 0.7] = \widetilde{\xi_{\mathcal{M}}}(0) = \min\{\widetilde{\xi_{\mathcal{M}}}((10 * 3^3) * 0),\widetilde{\xi_{\mathcal{M}}}(0)\}$,
$\widetilde{\zeta_{\mathcal{M}}}(10 * 3^2) = \widetilde{\zeta_{\mathcal{M}}}(1) = [0.14, 0.15] < [0.2, 0.3] = \widetilde{\zeta_{\mathcal{M}}}(0) = \min\{\widetilde{\zeta_{\mathcal{M}}}((10 * 3^3) * 0),\widetilde{\zeta_{\mathcal{M}}}(0)\}$ and
$\widetilde{\varpi_{\mathcal{M}}}(10 * 3^2)=\widetilde{\varpi_{\mathcal{M}}}(1)=[0.14, 0.15] > [0.12, 0.13]=\widetilde{\varpi_{\mathcal{M}}}(0)= \max\{\widetilde{\varpi_{\mathcal{M}}}((10 * 3^3) * 0),\widetilde{\varpi_{\mathcal{M}}}(0)\}$.

We investigate the conditions under which an IVNFI is an IVNF-n-BCK-I.

Proposition 3.4. *Let* $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ *be an IVNFI of* $\mathfrak{A}$. *Then* $\widetilde{\mathcal{M}}$ *is an IVNF-n-BCK-I of* $\mathfrak{A} \iff$ *it satisfies the following inequalities*
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$, *and*
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.

Proof. Assume that $\widetilde{\mathcal{M}}$ is an IVNF-n-BCK-I of $\mathfrak{A}$. Put $\kappa_3 = 0$ in (IVNFnBCKI2), (IVNFnBCKI3) and (IVNFnBCKI4), we obtain
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^(n + 1)\ ) * 0),\widetilde{\xi_{\mathcal{M}}}(0)\} = \min\{\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ ),\widetilde{\xi_{\mathcal{M}}}(0)\}$
$= \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$,
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^(n + 1)\ ) * 0),\widetilde{\zeta_{\mathcal{M}}}(0)\} = \min\{\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ ),\widetilde{\zeta_{\mathcal{M}}}(0)\}$
$= \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^(n{+}1)\ ) * 0),\widetilde{\varpi_{\mathcal{M}}}(0)\} = \max\{\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n{+}1)\ ),\widetilde{\varpi_{\mathcal{M}}}(0)\}$
$= \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$.
Therefore, $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1))$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ )$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.
Conversely, since (IVNFnBCKI2), (IVNFnBCKI3) and (IVNFnBCKI4) we obtain
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ ) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^(n + 1)\ ) * \kappa_3),\widetilde{\xi_{\mathcal{M}}}(\kappa_3)\}$,
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ ) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^(n + 1)\ ) * \kappa_3),\widetilde{\zeta_{\mathcal{M}}}(\kappa_3)\}$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + 1)\ ) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^(n + 1)\ ) * \kappa_3),\widetilde{\varpi_{\mathcal{M}}}(\kappa_3)\}$,
$\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$. Thus, $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-BCK-I of $\mathfrak{A}$. □

Corollary 3.5. *Every IVNF-n-BCK-I* $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ *of* $\mathfrak{A}$ *satisfies the inequalities*
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + k)\ )$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + k)\ )$, *and*
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^(n + k)\ )$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$, *and* $k \in \mathbf{N}$.

Proof. By invoking Proposition 3.4 and employing induction, the proof becomes straightforward. □

4. Interval-Valued Neutrosophic Fuzzy n-Fold Positive Implicative Ideals of BCK-algebra

In this section, we apply the concept of 'interval-valued neutrosophic fuzzy sets to n-fold positive implicative ideals of BCK-algebras and introduce the notions of interval-valued neutrosophic fuzzy n-fold positive implicative ideals of BCK-algebras (IVNF-n-PII)' and investigate some of its related properties.

Definition 4.1. An IVNFS $\widetilde{\Updownarrow} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ is an IVNF-n-PII of $\mathfrak{A}$, if it satisfies
(IVNFnPII1) $\widetilde{\xi_{\mathcal{M}}}(0) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(0) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1)$ and $\widetilde{\varpi_{\mathcal{M}}}(0) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1)$, $\exists$ a fixed $n \in \mathfrak{A} \ni$
(IVNFnPII2) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n), \widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$,
(IVNFnPII3) $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n), \widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$, and
(IVNFnPII4) $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n), \widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$,
$\forall\ \kappa_1, \kappa_2, \kappa_3 \in \mathfrak{A}$.

Example 4.2. Let's consider the BCK-algebra $\mathfrak{A} = \{0, a, m\}$, and its Cayley table is as follows:

TABLE 1. BCK-algebra

$*$	0	a	m
0	0	0	0
a	a	0	0
m	m	m	0

Define an IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ by $\widetilde{\xi_{\mathcal{M}}}(0) = [0.5, 0.6]$, $\widetilde{\xi_{\mathcal{M}}}(a) = [0.3, 0.4]$, $\widetilde{\xi_{\mathcal{M}}}(m) = [0.1, 0.2]$, $\widetilde{\zeta_{\mathcal{M}}}(0) = [0.15, 0.25]$, $\widetilde{\zeta_{\mathcal{M}}}(a) = [0.13, 0.24]$, $\widetilde{\zeta_{\mathcal{M}}}(m) = [0.12, 0.23]$ and $\widetilde{\varpi_{\mathcal{M}}}(0) = [0.02, 0.05]$, $\widetilde{\varpi_{\mathcal{M}}}(a) = [0.03, 0.06]$, $\widetilde{\varpi_{\mathcal{M}}}(m) = [0.4, 0.5]$.
Then $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-PII of $\mathfrak{A}$, $\forall\ n \in \mathfrak{A}$.

The following Theorem we give the relation between IVNF-n-PII and IVNFI.

Theorem 4.3. *Every IVNF-n-PII of* $\mathfrak{A}$ *must be IVNFI of* $\mathfrak{A}$.

Proof. Let $\widetilde{\mathcal{M}}$ be IVNF-n-PII of $\mathfrak{A}$. Put $\kappa_3 = 0$ in (IVNFnPII2), (IVNFnPII3) and (IVNFnPII4), Consequently, we obtain the proof of the stated Theorem. □

Now we present an example that shows the converse of Theorem 4.3 does not hold in general.

Example 4.4. Consider $\mathfrak{A} = \{0,x_p,y_q,z_f,s_w\}$ with the Cayley table below

Table 2. BCK-algebra

$*$	0	x_p	y_q	z_f	s_w
0	0	0	0	0	0
x_p	x_p	0	x_p	0	0
y_q	y_q	y_q	0	0	0
z_f	z_f	z_f	z_f	0	0
s_w	s_w	s_w	s_w	z_f	0

Define an IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ in $\mathfrak{A}$ by $\widetilde{\xi_{\mathcal{M}}}(0) = \tilde{l_0}$, $\widetilde{\xi_{\mathcal{M}}}\ (x_p) = \tilde{l_1}$, $\widetilde{\xi_{\mathcal{M}}}\ (y_q) = \widetilde{\xi_{\mathcal{M}}}\ (z_f) = \widetilde{\xi_{\mathcal{M}}}\ (s_w) = \tilde{l_2}$, $\widetilde{\zeta_{\mathcal{M}}}(0) = \tilde{r_0}$, $\widetilde{\zeta_{\mathcal{M}}}\ (x_p) = \tilde{r_1}$, $\widetilde{\zeta_{\mathcal{M}}}\ (y_q) = \widetilde{\zeta_{\mathcal{M}}}\ (z_f) = \widetilde{\zeta_{\mathcal{M}}}\ (s_w) = \tilde{r_2}$ and $\widetilde{\varpi_{\mathcal{M}}}(0) = \tilde{n_0}$, $\widetilde{\varpi_{\mathcal{M}}}\ (x_p) = \tilde{n_1}$, $\widetilde{\varpi_{\mathcal{M}}}\ (y_q) = \widetilde{\varpi_{\mathcal{M}}}\ (z_f) = \widetilde{\varpi_{\mathcal{M}}}\ (s_w) = \tilde{n_2}$, where $\tilde{l_0} > \tilde{l_1} > \tilde{l_2}$, $\tilde{r_0} > \tilde{r_1} > \tilde{r_2}$ and $\tilde{n_0} < \tilde{n_1} < \tilde{n_2}$ and $[0,0] \leq \tilde{l_i} + \tilde{r_i} + \tilde{n_i} \leq [1,1]$ for i = 0,1,2. By simple calculations one can give that $\widetilde{\mathcal{M}}$ is an IVNFI of $\mathfrak{A}$, but it is not an IVNF-n-PII of $\mathfrak{A}$, because
$\widetilde{\xi_{\mathcal{M}}}(s_w * {z_f}^n\ ) = \tilde{l_2} < \tilde{l_0} = \min\{\widetilde{\xi_{\mathcal{M}}}((s_w * z_f\ ) * {z_f}^n\ ), \widetilde{\xi_{\mathcal{M}}}(z_f * {z_f}^n\ )\}$,
$\widetilde{\zeta_{\mathcal{M}}}(s_w * {z_f}^n\ ) = \tilde{r_2} < \tilde{r_0} = \min\{\widetilde{\zeta_{\mathcal{M}}}((s_w * z_f\ ) * {z_f}^n\ ), \widetilde{\zeta_{\mathcal{M}}}(z_f * {z_f}^n\ )\}$, and
$\widetilde{\varpi_{\mathcal{M}}}(s_w * {z_f}^n\ ) = \tilde{n_2} > \tilde{n_0} = \max\{\widetilde{\varpi_{\mathcal{M}}}((s_w * z_f\ ) * {z_f}^n\ ), \widetilde{\varpi_{\mathcal{M}}}(z_f * {z_f}^n\ )\}$.

Theorem 4.5. *Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS of $\mathfrak{A}$. Then $\widetilde{\Updownarrow}$ is an IVNF-n-PII of $\mathfrak{A}$ $\Leftrightarrow$ it satisfies the inequalities $\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$,*
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, and
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.

Proof. Let $\kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$, $a = \kappa_1 * (\kappa_2 * {\kappa_3}^n\ )$ and $b = \kappa_1 * \kappa_2$. Since, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$,
$((\kappa_1 * (\kappa_2 * {\kappa_3}^n\ )) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n \leq (\kappa_2 * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n$,
By BCK1 and Theorem 2.17, we have
$\widetilde{\xi_{\mathcal{M}}}(((\kappa_1 * (\kappa_2 * {\kappa_3}^n\ )) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ )$,
$\widetilde{\zeta_{\mathcal{M}}}(((\kappa_1 * (\kappa_2 * {\kappa_3}^n\ )) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ )$, and
$\widetilde{\varpi_{\mathcal{M}}}(((\kappa_1 * (\kappa_2 * {\kappa_3}^n\ )) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ )$.
Then $\widetilde{\xi_{\mathcal{M}}}((a * b) * {\kappa_3}^n\ ) = \widetilde{\xi_{\mathcal{M}}}(((\kappa_1 * (\kappa_2 * {\kappa_3}^n\ )) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n\ )$
$\geq \widetilde{\xi_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ ) = \widetilde{\xi_{\mathcal{M}}}((\kappa_2 * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ ))$ [By P3]
$= \widetilde{\xi_{\mathcal{M}}}\ (0)$. [By BCK3]
and so $\widetilde{\xi_{\mathcal{M}}}((a * b) * {\kappa_3}^n\ ) = \widetilde{\xi_{\mathcal{M}}}(0)$.
$\widetilde{\zeta_{\mathcal{M}}}((a * b) * {\kappa_3}^n\ ) = \widetilde{\zeta_{\mathcal{M}}}(((\kappa_1 * (\kappa_2 * {\kappa_3}^n\ )) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n\ )$

$\geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_3}^n)) * {\kappa_3}^n) = \widetilde{\zeta_{\mathcal{M}}}((\kappa_2 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n))$ [By P3]
$= \widetilde{\zeta_{\mathcal{M}}}(0)$. [By BCK3]
and so $\widetilde{\zeta_{\mathcal{M}}}((a * b) * {\kappa_3}^n) = \widetilde{\zeta_{\mathcal{M}}}(0)$. And
$\widetilde{\varpi_{\mathcal{M}}}((a * b) * {\kappa_3}^n) = \widetilde{\varpi_{\mathcal{M}}}(((\kappa_1 * (\kappa_2 * {\kappa_3}^n)) * (\kappa_1 * \kappa_2)) * {\kappa_3}^n)$
$\leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_3}^n)) * {\kappa_3}^n) = \widetilde{\varpi_{\mathcal{M}}}((\kappa_2 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n))$ [By P3]
$= \widetilde{\varpi_{\mathcal{M}}}(0)$. [By BCK3]
and so $\widetilde{\varpi_{\mathcal{M}}}((a * b) * {\kappa_3}^n) = \widetilde{\varpi_{\mathcal{M}}}(0)$.
Using (P3), (IVNFnPII2), (IVNFnPII3) and (IVNFnPII4) we obtain
$\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) = \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * (\kappa_2 * {\kappa_3}^n)) * {\kappa_3}^n) = \widetilde{\xi_{\mathcal{M}}}(a * {\kappa_3}^n)$
$\geq \min\{\widetilde{\xi_{\mathcal{M}}}((a * b) * {\kappa_3}^n),\widetilde{\xi_{\mathcal{M}}}(b * {\kappa_3}^n)\} = \min\{\widetilde{\xi_{\mathcal{M}}}(0),\widetilde{\xi_{\mathcal{M}}}(b * {\kappa_3}^n)\} = \widetilde{\xi_{\mathcal{M}}}(b * {\kappa_3}^n)$
$= \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$,
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) = \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * (\kappa_2 * {\kappa_3}^n)) * {\kappa_3}^n) = \widetilde{\zeta_{\mathcal{M}}}(a * {\kappa_3}^n)$
$\geq \min\{\widetilde{\zeta_{\mathcal{M}}}((a * b) * {\kappa_3}^n),\widetilde{\zeta_{\mathcal{M}}}(b * {\kappa_3}^n)\} = \min\{\widetilde{\zeta_{\mathcal{M}}}(0),\widetilde{\zeta_{\mathcal{M}}}(b * {\kappa_3}^n)\} = \widetilde{\zeta_{\mathcal{M}}}(b * {\kappa_3}^n)$
$= \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, and
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) = \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * (\kappa_2 * {\kappa_3}^n)) * {\kappa_3}^n) = \widetilde{\varpi_{\mathcal{M}}}(a * {\kappa_3}^n)$
$\leq \max\{\widetilde{\varpi_{\mathcal{M}}}((a * b) * {\kappa_3}^n),\widetilde{\varpi_{\mathcal{M}}}(b * {\kappa_3}^n)\} = \max\{\widetilde{\varpi_{\mathcal{M}}}(0),\widetilde{\varpi_{\mathcal{M}}}(b * {\kappa_3}^n)\} = \widetilde{\varpi_{\mathcal{M}}}(b * {\kappa_3}^n)$
$= \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$.
Thus $\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$,
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, and
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.
Conversely, for any $\kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$. By (IVNFI2), (IVNFI3) and (IVNFI4), we obtain:
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)),\widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$
$\geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n),\widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$,
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)),\widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$
$\geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n),\widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)),\widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$
$\leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n),\widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n)\}$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.
Thus, $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-PII of $\mathfrak{A}$. □

Proposition 4.6. *Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS of $\mathfrak{A}$. Then $\widetilde{\mathcal{M}}$ is an IVNF-n-PII of $\mathfrak{A}$ then it satisfies the inequalities $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n)$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n)$ and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n)$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.*

Proof. Put $\kappa_3 = \kappa_2$ in (IVNFnPII2), (IVNFnPII3) and (IVNFnPII4), Consequently, we obtain the proof of the stated proposition. □

The following Theorem we give the relation between IVNF-n-BCK-I and IVNF-n-PII.

Proposition 4.7. *Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS of $\mathfrak{A}$. Then $\widetilde{\mathcal{M}}$ is an IVNF-n-PII of $\mathfrak{A}$ $\Leftrightarrow$ it is an IVNF-n-BCK-I of $\mathfrak{A}$.*

Proof. Assuming $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-PII of $\mathfrak{A}$.
Putting $\kappa_3 = \kappa_2$ in (IVNFnPII2), (IVNFnPII3) and (IVNFnPII4), we get:
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n\ ) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n\ ), \widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_2}^n\ )\} = \min\{\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ ), \widetilde{\xi_{\mathcal{M}}}(0)\}$
$= \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ )$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n\ ) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n\ ), \widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_2}^n\ )\}$
$= \min\{\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ ), \widetilde{\zeta_{\mathcal{M}}}(0)\} = \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ )$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n), \widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_2}^n\ )\}$
$= \max\{\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)), \widetilde{\varpi_{\mathcal{M}}}(0)\} = \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ )$.
Therefore, $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ )$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ )$ and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * \kappa_2{}^{(}n+1)\ )$, $\forall\ \kappa_1, \kappa_2 \in \mathfrak{A}$.
Using 3.4 Proposition, $\widetilde{\Updownarrow}$ is an IVNF-n-BCK-I of $\mathfrak{A}$.
Conversely, (P3) and (P4) give us
$\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^{2n}\ ) * (\kappa_2 * {\kappa_3}^n\ )) = \widetilde{\xi_{\mathcal{M}}}(((\kappa_1 * {\kappa_3}^n\ ) * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ ))$
$= \widetilde{\xi_{\mathcal{M}}}(((\kappa_1 * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n\ )$
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^{2n}\ ) * (\kappa_2 * {\kappa_3}^n\ )) = \widetilde{\zeta_{\mathcal{M}}}(((\kappa_1 * {\kappa_3}^n\ ) * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ ))$
$= \widetilde{\zeta_{\mathcal{M}}}(((\kappa_1 * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n\ )$
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^{2n}\ ) * (\kappa_2 * {\kappa_3}^n\ )) = \widetilde{\varpi_{\mathcal{M}}}(((\kappa_1 * {\kappa_3}^n\ ) * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ ))$
$= \widetilde{\varpi_{\mathcal{M}}}(((\kappa_1 * {\kappa_3}^n\ ) * (\kappa_2 * {\kappa_3}^n\ )) * {\kappa_3}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n\ )$.
Using Corollary 3.5,(IVNFI2), (IVNFI3) and (IVNFI4) we get:
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n\ ) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^{2n}\ ) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^{2n}\ ) * (\kappa_2 * {\kappa_3}^n\ )), \widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n\ )\}$
$\geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n\ ), \widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n\ )\}$,
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n\ ) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^{2n}\ ) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^{2n}\ ) * (\kappa_2 * {\kappa_3}^n\ )), \widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n\ )\}$
$\geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n\ ), \widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n\ )\}$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^n\ ) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_3}^{2n}\ ) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^{2n}\ ) * (\kappa_2 * {\kappa_3}^n\ )), \widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n\ )\}$
$\leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n\ ), \widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_3}^n\ )\}$.
Thus, $\widetilde{\mathcal{M}}$ is an IVNF-n-PII of $\mathfrak{A}$. □

Theorem 4.8. *Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS of $\mathfrak{A}$. Then the subsequent conditions are interchangeable:*
(i) $\widetilde{\mathcal{M}}$ is an IVNF-n-PII of $\mathfrak{A}$.
(ii) The ($\neq \phi$) sets $\mathcal{U}(\widetilde{\xi_{\mathcal{M}}};[s_1, s_2])$, $\mathcal{U}(\widetilde{\zeta_{\mathcal{M}}};[t_1, t_2])$ and $\mathcal{L}(\widetilde{\varpi_{\mathcal{M}}};[v_1, v_2])$ are IV-n-PIIs of $\mathfrak{A}$, $\forall$ $[s_1, s_2], [t_1, t_2], [v_1, v_2] \in \mathfrak{A}[0, 1]$.

Proof. Verification is straightforward and can be easily established. □

Theorem 4.9. *Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS of $\mathfrak{A}$. Then the subsequent conditions are interchangeable:*
(i) $\widetilde{\mathcal{M}}$ is an IVNF-n-PII of $\mathfrak{A}$.
(ii) The $(\neq \phi)$ sets $\mathcal{U}(\widetilde{\xi_{\mathcal{M}}};[s_1, s_2])$, $\mathcal{U}(\widetilde{\zeta_{\mathcal{M}}};[t_1, t_2])$, and $\mathcal{L}(\widetilde{\varpi_{\mathcal{M}}};[v_1, v_2])$ are IV-n-PIIs of $\mathfrak{A}$, $\forall\ [s_1, s_2], [t_1, t_2], [v_1, v_2] \in \mathfrak{A}[0, 1]$.

The following Theorem we give the relation between IVNF-n-BCK-I and IVNF-n-PII.

Theorem 4.10. *Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ be an IVNFS of $\mathfrak{A}$. Then the subsequent conditions are interchangeable:*
(i) $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-PII.
(ii) $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-BCK-I of $\mathfrak{A}$.
(iii) $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^(n+1))$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^(n+1))$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^(n+1))$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.
(iv) $\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$,
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, and
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n)$, $\forall\ \kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$.
(v) $\mathcal{U}(\widetilde{\xi_{\mathcal{M}}};[s_1, s_2])$, $\mathcal{U}(\widetilde{\zeta_{\mathcal{M}}};[t_1, t_2])$, and $\mathcal{L}(\widetilde{\varpi_{\mathcal{M}}};[v_1, v_2])$ are "n-fold positive implicative ideals" of $\mathfrak{A}$, $\forall\ [s_1, s_2], [t_1, t_2], [v_1, v_2] \in \mathfrak{A}[0, 1]$.

Theorem 4.11. *If $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-PII of $\mathfrak{A}$ then*
(i) $\forall\ \kappa_1,\kappa_2,a,b \in \mathfrak{A}$, $((\kappa_1 * \kappa_2) * {\kappa_2}^n) * a \leq b \Rightarrow \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq min\{\widetilde{\xi_{\mathcal{M}}}(a),\widetilde{\xi_{\mathcal{M}}}(b)\}$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq min\{\widetilde{\zeta_{\mathcal{M}}}(a),\widetilde{\zeta_{\mathcal{M}}}(b)\}$ and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq max\{\widetilde{\varpi_{\mathcal{M}}}(a),\widetilde{\varpi_{\mathcal{M}}}(b)\}$.
(ii) $\forall\ \kappa_1,\kappa_2,\kappa_3,a,b \in \mathfrak{A}, ((\kappa_1 * \kappa_2)*{\kappa_3}^n)*a \leq b \Rightarrow \widetilde{\xi_{\mathcal{M}}}((\kappa_1*{\kappa_3}^n)*(\kappa_2*{\kappa_3}^n)) \geq min\{\widetilde{\xi_{\mathcal{M}}}(a),\widetilde{\xi_{\mathcal{M}}}(b)\}$,
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq min\{\widetilde{\zeta_{\mathcal{M}}}(a),\widetilde{\zeta_{\mathcal{M}}}(b)\}$, and
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \leq max\{\widetilde{\varpi_{\mathcal{M}}}(a),\widetilde{\varpi_{\mathcal{M}}}(b)\}$.

Proof. (i) Consider, $\kappa_1,\kappa_2,\kappa_3 \in \mathfrak{A}$ be such-that $((\kappa_1 * \kappa_2) * {\kappa_2}^n) * a \leq b$. Using Theorem 2.18, we have $\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}(a),\widetilde{\xi_{\mathcal{M}}}(b)\}$, $\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}(a),\widetilde{\zeta_{\mathcal{M}}}(b)\}$, and $\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}(a),\widetilde{\varpi_{\mathcal{M}}}(b)\}$. Put $\kappa_3 = \kappa_2$ in (IVNFnPII2), (IVNFnPII3), and (IVNFnPII4), we get:
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n),\widetilde{\xi_{\mathcal{M}}}(\kappa_2 * {\kappa_2}^n)\} = \min\{\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n),\widetilde{\xi_{\mathcal{M}}}(0)\}$
$= \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}(a),\widetilde{\xi_{\mathcal{M}}}(b)\}$
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n),\widetilde{\zeta_{\mathcal{M}}}(\kappa_2 * {\kappa_2}^n)\} = \min\{\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n),\widetilde{\zeta_{\mathcal{M}}}(0)\}$
$= \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}(a),\widetilde{\zeta_{\mathcal{M}}}(b)\}$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n),\widetilde{\varpi_{\mathcal{M}}}(\kappa_2 * {\kappa_2}^n)\}$
$= \max\{\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n),\widetilde{\varpi_{\mathcal{M}}}(0)\} = \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_2}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}(a),\widetilde{\varpi_{\mathcal{M}}}(b)\}$.
Therefore, $\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}(a),\widetilde{\xi_{\mathcal{M}}}(b)\}$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}(a),\widetilde{\zeta_{\mathcal{M}}}(b)\}$, and $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * {\kappa_2}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}(a),\widetilde{\varpi_{\mathcal{M}}}(b)\}$.

(ii) Consider κ_1,κ_2,$\kappa_3 \in \mathfrak{A}$ be such-that $((\kappa_1 * \kappa_2) * {\kappa_3}^n) * a \leq b$. From Theorem 4.5, Theorem 4.6, and the preceding result (i), it follows that
$\widetilde{\xi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n) \geq \min\{\widetilde{\xi_{\mathcal{M}}}(a),\widetilde{\xi_{\mathcal{M}}}(b)\}$,
$\widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}(a),\widetilde{\zeta_{\mathcal{M}}}(b)\}$, and
$\widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * {\kappa_3}^n) * (\kappa_2 * {\kappa_3}^n)) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2) * {\kappa_3}^n) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}(a),\widetilde{\varpi_{\mathcal{M}}}(b)\}$.
This concludes our proof. □

The following Theorem we give the relation between IVNF-n-PII, IVNF-n-W-CI and IVNFnII.

Theorem 4.12. *An IVNFS $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is both an IVNF-n-PII and IVNF-n-W-CI of $\mathfrak{A}$, then it is an IVNFnII of $\mathfrak{A}$.*

Proof. Let κ_1,$\kappa_2 \in \mathfrak{A}$. Using Theorem 2.19(ii), Theorem 2.17, (P_3) and (BCK3), we have
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * (\kappa_2 * {\kappa_1}^n))) \geq \widetilde{\xi_{\mathcal{M}}}((\kappa_2 * {\kappa_1}^n) * ((\kappa_2 * {\kappa_1}^n) * {\kappa_1}^n))$
$\geq \widetilde{\xi_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_1}^n)) * {\kappa_1}^n) = \widetilde{\xi_{\mathcal{M}}}((\kappa_2 * {\kappa_1}^n) * (\kappa_2 * {\kappa_1}^n)) = \widetilde{\xi_{\mathcal{M}}}(0)$,
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * (\kappa_2 * {\kappa_1}^n))) \geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_2 * {\kappa_1}^n) * ((\kappa_2 * {\kappa_1}^n) * {\kappa_1}^n))$
$\geq \widetilde{\zeta_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_1}^n)) * {\kappa_1}^n) = \widetilde{\zeta_{\mathcal{M}}}((\kappa_2 * {\kappa_1}^n) * (\kappa_2 * {\kappa_1}^n)) = \widetilde{\zeta_{\mathcal{M}}}(0)$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * (\kappa_2 * {\kappa_1}^n))) \leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_2 * {\kappa_1}^n) * ((\kappa_2 * {\kappa_1}^n) * {\kappa_1}^n))$
$\leq \widetilde{\varpi_{\mathcal{M}}}((\kappa_2 * (\kappa_2 * {\kappa_1}^n)) * {\kappa_1}^n) = \widetilde{\varpi_{\mathcal{M}}}((\kappa_2 * {\kappa_1}^n) * (\kappa_2 * {\kappa_1}^n)) = \widetilde{\varpi_{\mathcal{M}}}(0)$
Since, $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNFI of $\mathfrak{A}$ from this we have
$\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq \min\{\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * (\kappa_2 * {\kappa_1}^n))),\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))\}$
$\geq \min\{\widetilde{\xi_{\mathcal{M}}}(0),\widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))\} = \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$
$\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq \min\{\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * (\kappa_2 * {\kappa_1}^n))),\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))\}$
$\geq \min\{\widetilde{\zeta_{\mathcal{M}}}(0),\widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))\} = \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq \max\{\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_1 * (\kappa_2 * {\kappa_1}^n))),\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 *(\kappa_2 * {\kappa_1}^n))\}$
$\leq \max\{\widetilde{\varpi_{\mathcal{M}}}(0),\widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))\} = \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$.
Hence, $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\xi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\zeta_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, and
$\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \leq \widetilde{\varpi_{\mathcal{M}}}(\kappa_1 * (\kappa_2 * {\kappa_1}^n))$, $\forall\ \kappa_1$,$\kappa_2 \in \mathfrak{A}$. So, from Theorem 2.20, $\widetilde{\mathcal{M}}$ is an IVNFnII of $\mathfrak{A}$. □

Theorem 4.13. *("Extension Theorem of Interval-valued neutrosophic fuzzy n-fold positive impliocative ideals"). Let $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ and $\widetilde{\mathcal{N}} = (\widetilde{\xi_{\mathcal{N}}}, \widetilde{\zeta_{\mathcal{N}}}, \widetilde{\varpi_{\mathcal{N}}})$ be an IVNFI of $\mathfrak{A} \ni \widetilde{\mathcal{M}}(0) = \widetilde{\mathcal{N}}(0)$ and $\widetilde{\mathcal{M}} \subseteq \widetilde{\mathcal{N}}$, that is, $\widetilde{\xi_{\mathcal{M}}}(0) = \widetilde{\xi_{\mathcal{N}}}(0)$, $\widetilde{\zeta_{\mathcal{M}}}(0) = \widetilde{\zeta_{\mathcal{N}}}(0)$, $\widetilde{\varpi_{\mathcal{M}}}(0) = \widetilde{\varpi_{\mathcal{N}}}(0)$ and $\widetilde{\xi_{\mathcal{M}}}(\kappa_1) \leq \widetilde{\xi_{\mathcal{N}}}(\kappa_1)$, $\widetilde{\zeta_{\mathcal{M}}}(\kappa_1) \leq \widetilde{\zeta_{\mathcal{N}}}(\kappa_1)$, $\widetilde{\varpi_{\mathcal{M}}}(\kappa_1) \geq \widetilde{\varpi_{\mathcal{N}}}(\kappa_1)$, $\forall\ \kappa_1 \in \mathfrak{A}$. If $\widetilde{\mathcal{M}} = (\widetilde{\xi_{\mathcal{M}}}, \widetilde{\zeta_{\mathcal{M}}}, \widetilde{\varpi_{\mathcal{M}}})$ is an IVNF-n-PII of $\mathfrak{A}$ then so is $\widetilde{\mathcal{N}} = (\widetilde{\xi_{\mathcal{N}}}, \widetilde{\zeta_{\mathcal{N}}}, \widetilde{\varpi_{\mathcal{N}}})$.*

Proof. It is enough to verify that $\widetilde{\mathcal{N}} = (\widetilde{\xi_{\mathcal{N}}}, \widetilde{\zeta_{\mathcal{N}}}, \widetilde{\varpi_{\mathcal{N}}})$ obeys the given inequalities
$\widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \geq \widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, $\widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \geq \widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, and
$\widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \leq \widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.
Let $\kappa_1,\kappa_2 \in \mathfrak{A}$. Using (BCK3), (P3) and Proposition 3.4, we get
$\widetilde{\xi_{\mathcal{N}}}(0) = \widetilde{\xi_{\mathcal{M}}}(0) = \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * \kappa_2{}^{(n+1)})) * \kappa_2{}^{(n+1)})$
$\leq \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * \kappa_2{}^{(n+1)})) * \kappa_2{}^{n}) = \widetilde{\xi_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$
$\leq \widetilde{\xi_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$,
$\widetilde{\zeta_{\mathcal{N}}}(0) = \widetilde{\zeta_{\mathcal{M}}}(0) = \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * \kappa_2{}^{(n+1)})) * \kappa_2{}^{(n+1)})$
$\leq \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * \kappa_2{}^{(n+1)})) * \kappa_2{}^{n}) = \widetilde{\zeta_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$
$\leq \widetilde{\zeta_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$, and
$\widetilde{\varpi_{\mathcal{N}}}(0) = \widetilde{\varpi_{\mathcal{M}}}(0) = \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * \kappa_2{}^{(n+1)})) * \kappa_2{}^{(n+1)})$
$\geq \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * (\kappa_1 * \kappa_2{}^{(n+1)})) * \kappa_2{}^{n}) = \widetilde{\varpi_{\mathcal{M}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$
$\geq \widetilde{\varpi_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$
Therefore, $\widetilde{\xi_{\mathcal{N}}}(0) \leq \widetilde{\xi_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$,
$\widetilde{\zeta_{\mathcal{N}}}(0) \leq \widetilde{\zeta_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$ and $\widetilde{\varpi_{\mathcal{N}}}(0) \geq \widetilde{\varpi_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)}))$,
$\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$. It follows from (IVNFI2), (IVNFI3) and (IVNFI4) that
$\widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \geq \min\{\widetilde{\xi_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)})), \widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})\}$
$\geq \min\{\widetilde{\xi_{\mathcal{N}}}(0), \widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})\} = \widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$
$\widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \geq \min\{\widetilde{\zeta_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)})), \widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})\}$
$\geq \min\{\widetilde{\zeta_{\mathcal{N}}}(0), \widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})\} = \widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, and
$\widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \leq \max\{\widetilde{\varpi_{\mathcal{N}}}((\kappa_1 * \kappa_2{}^{n}) * (\kappa_1 * \kappa_2{}^{(n+1)})), \widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})\}$
$\leq \max\{\widetilde{\varpi_{\mathcal{N}}}(0), \widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})\} = \widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$.
Therefore, $\widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \geq \widetilde{\xi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, $\widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \geq \widetilde{\zeta_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, and
$\widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{n}) \leq \widetilde{\varpi_{\mathcal{N}}}(\kappa_1 * \kappa_2{}^{(n+1)})$, $\forall\ \kappa_1,\kappa_2 \in \mathfrak{A}$.
Thus, $\widetilde{\mathcal{N}} = (\widetilde{\xi_{\mathcal{N}}}, \widetilde{\zeta_{\mathcal{N}}}, \widetilde{\varpi_{\mathcal{N}}})$ is an IVNF-n-PII of $\mathfrak{A}$. □

5. **Discussion**

Lastly, this work pioneers the application of interval-valued neutrosophic fuzzy sets to BCK-algebras, yielding new results on n-fold positive implicative ideals and n-fold BCK-ideals. Furthermore, we establish the extension theorem for interval-valued neutrosophic n-fold fuzzy positive implicative ideals. In future research work, the concept will be used on algebraic structures like BCI, B, BCH, d, UP, G, etc.

6. **Conclusions**

This paper presents a comprehensive investigation into the notion of interval-valued neutrosophic fuzzy n-fold ideals in BCK-algebras, with a particular emphasis on interval-valued neutrosophic fuzzy n-fold BCK-ideals and interval-valued neutrosophic fuzzy n-fold positive implicative ideals. Through a rigorous examination of characterization theorems and extension properties, we have established the fundamental properties of these ideals. This research contributes to the advancement of fuzzy mathematics and its applications, particularly in the realm of BCK-algebras. The outcomes of this study have far-reaching implications for future research in this field, and we anticipate that our findings will inspire further exploration into the properties and applications of interval-valued neutrosophic fuzzy n-fold ideals in BCK-algebras.

Acknowledgments: The National Fellowship and Scholarship for Higher Education of ST Students, under Agreement No. 11019/07/2018-Sch (Award No-202223-NFST-AND-00689), Government of India ministry of Tribal Affairs, provided funding for Anjaneyulu Naik Kalavath's research.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: June 30, 2025. Accepted: Sep 9, 2025