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Probabilistic and Neutrosophic Coefficient Conditions in Geometric Function Theory with Applications to Sound Denoising Associated With Mersenne Polynomials

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Abstract.

Abstract. The investigation of generalized discrete probability distributions within advanced analytical frameworks is essential for developing rigorous models in probability theory and complex analysis. The Neutrosophic Poisson Distribution (NPD), characterized by its capacity to address uncertainty and indeterminacy, poses significant challenges that traditional probabilistic approaches often fail to resolve, particularly within the domain of univalent function theory.

This paper investigates the analytical properties of generalized discrete probability series, Neutrosophic Poisson distributions, and sigmoid functions. We introduce a novel *Advanced Differential-Sigmoid Operator* based on Mersenne polynomials to enhance analytical precision and structural adaptability. Utilizing this operator, we derive several functional inequalities, growth estimates, and coefficient bounds for associated classes of analytic univalent functions. Furthermore, we explore convex combinations and the geometric behavior of the sigmoid function within this framework.

By integrating neutrosophic logic, special functions, and operator theory, this research provides a unified framework that advances the treatment of uncertainty and supports practical applications in statistical modeling, signal processing, and neural networks.

Keywords: Analytic functions, Generalized distribution series, Neutrosophic logic, Poisson distribution, Coefficient estimates, Sigmoid function.

2010 Mathematics Subject Classification: 30C45, 30C50.

1. Introduction

The study of the geometric properties of analytic and univalent functions, particularly within the open unit disk, lies at the heart of Geometric Function Theory (GFT), a central branch of complex analysis. The open unit disk is traditionally defined as:

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

The foundations of GFT were laid by pioneering mathematicians such as Riemann, Koebe, and Bieberbach, whose seminal work on conformal mappings played a decisive role in shaping the development of the field [?]. These functions, typically normalized by the conditions $f(0) = 0$ and $f'(0) = 1$, allow for the exploration of structural limits within the complex plane, often leading to deep insights into the nature of injectivity and geometric stability. A central focus of GFT is the class $\mathcal{A}$ of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{2}$$

which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. These functions are normalized such that $f(0) = 0$ and $f'(0) = 1$. This class provides the basis for studying subclasses of univalent and related analytic functions, including starlike, convex, and close-to-convex functions [?].

The Bieberbach conjecture [?] proposed that for functions in the class of univalent functions, $|a_n| \leq n$ for all $n \geq 2$. A significant turning point in the field was reached in 1985 when de Branges proved this conjecture [?]. Modern investigations focus on various functionals related to the coefficients of analytic functions. Among these are the Fekete–Szegő functional, given by $|a_3 - \mu a_2^2|$ for $\mu \in \mathbb{R}$, and the Hankel determinant, defined as:

$$H_\varepsilon(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+\varepsilon-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+\varepsilon} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+\varepsilon-1} & a_{n+\varepsilon} & \cdots & a_{n+2\varepsilon-2} \end{vmatrix}. \tag{3}$$

Hankel determinants have been extensively studied for their capacity to capture nonlinear relationships among coefficients and to characterize various function classes [?, ?]. They serve as a powerful tool in complex analysis for deriving bounds on the coefficients of univalent functions. A robust understanding of these determinants allows researchers to investigate the properties of analytic functions more effectively, enriching the study of function theory and its applications across diverse fields.

Beyond foundational contributions, subsequent studies [?, ?, ?, ?] have applied Hankel determinants to a range of analytic, univalent, and bi-univalent function classes, with results well-documented in the literature. More recently, advances in Geometric Function Theory

(GFT) have incorporated techniques from fractional calculus, operator theory, and special functions, thereby expanding the discipline's scope.

The Subordination Principle:

Let $m(z)$ and $y(z)$ be analytic functions defined within the unit disk $\mathbb{D}$. The function $m(z)$ is said to be subordinate to $y(z)$, denoted symbolically as $m(z) \prec y(z)$ for $z \in \mathbb{D}$, if there exists a Schwarz function $w(z)$ (where $w(0) = 0$ and $|w(z)| < 1$) such that $m(z) = y(w(z))$. If $y(z)$ is univalent in $\mathbb{D}$, the condition $m(z) \prec y(z)$ is equivalent to $m(0) = y(0)$ and the inclusion property $m(\mathbb{D}) \subseteq y(\mathbb{D})$.

2. **Advanced Differential-Sigmoid Operator**

The standard sigmoid function is defined as:

$$g(s) = \frac{1}{1 + e^{-s}} \tag{4}$$

This function possesses several significant characteristics: its output range is $(0, 1)$, it is smooth and differentiable, and it exhibits the classic S-shaped (sigmoidal) curve commonly used in early neural network architectures [**?**].

The modified sigmoid function, as introduced in [**?**], is defined as:

$$\pi(s) = \frac{2}{1 + e^{-s}} = 1 + \tanh\left(\frac{s}{2}\right) \tag{5}$$

This version has been utilized to construct new subclasses of univalent functions through differential operators, such as the generalized Sălăgean operator [**?**]. Following the work of [**?**], various authors have defined analytic univalent functions in terms of the sigmoid function to investigate specific function classes; for further clarification, see [**?**, **?**, **?**].

In addition, recent studies have strengthened the theoretical foundations of analytic function research and broadened its applications in complex analysis by introducing q-analogues of the sigmoid function and developing new generalized differential operators. More recently, the authors in [**?**] introduced and investigated a new derivative operator to explore convex combinations and other properties of the sigmoid function associated with generalized discrete probability distribution series within the framework of univalent function theory. Several coefficient inequalities and the consequences of various parameter choices were also discussed. Using Python software, they generated graphs depicting different convex combinations of K_ϕ and g_ϕ, which illustrate relationships such as disease vs. intervention, policy vs. policy intervention, investment returns, disease spread, infection vs. intervention, free market vs. intervention, and cost vs. efficiency.

Geometric Function Theory (GFT) addresses traditional problems involving univalent (injective) functions, extremal functions, and conformal mappings in complex plane domains.

Generalized discrete probability distributions are incorporated into GFT to improve the modeling of coefficient distributions, statistical properties of function classes, and random analytic functions. These distributions facilitate the establishment of probabilistic bounds for traditional extremal problems and enable statistical analyses of conformal and univalent mappings, thereby advancing the field.

Let Ψ denote the sum of a convergent series of the form $\Psi = \sum_{n=0}^{\infty} \Theta_n$, where $\Theta_n \geq 0$ for all $n \in \mathbb{N}_0$. The discrete generalized probability distribution, defined by its probability mass function (PMF), is given by:

$$P(X = n) = \frac{\Theta_n}{\Psi}, \quad n = 0, 1, 2, \dots \tag{6}$$

The r-th moment is given by $\mu'_r = E(X^r)$. The mean (the first moment about the origin) is defined as:

$$\text{Mean} = \mu'_1 = \frac{\Psi'(1)}{\Psi} \tag{7}$$

Consequently, the variance is calculated using the relation $\mu'_2 - (\mu'_1)^2$, resulting in:

$$\text{Variance} = \frac{1}{\Psi}\left[\Psi''(1) + \Psi'(1) - \frac{(\Psi'(1))^2}{\Psi}\right] \tag{8}$$

where μ'_1 is the mean of the distribution and $\Psi = \sum_{n=0}^{\infty} a_n$.

The moment-generating function (MGF) of a random variable X, denoted by $M_X(t)$, is defined as:

$$M_X(t) = E(e^{tX}) = \frac{\Psi(e^t)}{\Psi} \tag{9}$$

By assigning specific values to Θ_n within this generalized probability framework, one can recover several classical discrete distributions, such as the Yule–Simon, logarithmic, Poisson, binomial, zeta, geometric, and Bernoulli distributions [**?**].

In this study, we focus on the analytic function $K_\varphi(z)$, whose coefficients correspond to the probabilities of the generalized distribution introduced in [**?**]:

$$K_\varphi(z) = z + \sum_{k=2}^{\infty} \frac{\Theta_{k-1}}{\Psi} z^k \tag{10}$$

Consequently, the variance is calculated using the relation $\mu'_2 - (\mu'_1)^2$, resulting in:

$$\text{Variance} = \frac{1}{\Psi}\left[\Psi''(1) + \Psi'(1) - \frac{(\Psi'(1))^2}{\Psi}\right] \tag{11}$$

where μ'_1 is the mean of the distribution and $\Psi = \sum_{n=0}^{\infty} a_n$.

The moment-generating function (MGF) of a random variable X, denoted by $M_X(t)$, is defined as:

$$M_X(t) = E(e^{tX}) = \frac{\Psi(e^t)}{\Psi} \tag{12}$$

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In this study, we focus on the analytic function $K_\varphi(z)$, whose coefficients correspond to the probabilities of the generalized distribution introduced in [?]:

$$K_\varphi(z) = z + \sum_{k=2}^{\infty} \frac{\Theta_{k-1}}{\Psi} z^k \tag{13}$$

We define the function with negative coefficients as:

$$\Xi K_\varphi(z) = z - \sum_{k=2}^{\infty} \frac{\Theta_{k-1}}{\Psi} z^k \tag{14}$$

This serves as the foundation for exploring analytical properties within the framework of [?, ?].

Definition 1.1. Let $K_\varphi(z)$ be defined as in (4). We introduce a new generalized differential operator incorporating a distribution series and the logistic activation function:

$$D^0_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) = K_\varphi(z) \tag{15}$$

$$D^1_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) = \left(1 - \frac{\lambda}{\pi(s)+\Phi}\right) K_\varphi(z) + \frac{\lambda}{\pi(s)+\Phi} z K'_\varphi(z) + \frac{\tau+\varepsilon}{\pi(s)+\Phi} z^2 K''_\varphi(z) \tag{16}$$

The recursive relation for $t \in \mathbb{N}_0$ is given by:

$$D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) = D_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)\left(D^{t-1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z)\right) \tag{17}$$

where $\pi(s) = \frac{2}{1+e^{-s}}$ is the modified Sigmoid function for real $s \geq 0$. Consequently, the power series expansion is:

$$D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) = z + \sum_{v=2}^{\infty} \left[\frac{\pi(s)+\Phi+(v-1)(\lambda+v(\tau+\varepsilon))}{\pi(s)+\Phi}\right]^t \frac{\Theta_{v-1}}{\Psi} z^v \tag{18}$$

By specializing the parameters of the operator $D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)$, we recover several established operators and derive new ones as follows:

- **Example 1 (New Integral Operator):** Setting $s = 0$ yields

$$D^t_{\Phi,\lambda,\tau,\varepsilon}(0,\Psi)K_\varphi(z) = z + \sum_{v=2}^{\infty} \left[\frac{1+\Phi+(v-1)(\lambda+v(\tau+\varepsilon))}{1+\Phi}\right]^t \frac{\Theta_{v-1}}{\Psi} z^v$$

- **Example 5 (Al-Oboudi type):** Setting $s = 0, \Phi = 0, \tau = 0, \varepsilon = 0$ yields

$$D^t_{0,\lambda,0,0}(0,0)K_\varphi(z) = z + \sum_{v=2}^{\infty} [1+\lambda(v-1)]^t \frac{\Theta_{v-1}}{\Psi} z^v$$

which corresponds to the operator modeled in [?].

- **Example 6 (Sălăgean type):** Setting $s = 0, \Phi = 0, \lambda = 1, \tau = 0, \varepsilon = 0$ yields

$$D^{t}_{0,1,0,0}(0,0)K_{\varphi}(z) = z + \sum_{v=2}^{\infty} v^{t} \frac{\Theta_{v-1}}{\Psi} z^{v}$$

recovering the integral derivative operator from [**?**].

A variable y is said to be Neutrosophic Poisson distributed if it takes values $v = 0, 1, 2, 3, \ldots$ with probabilities $e^{-m_N}, \frac{m_N e^{-m_N}}{1!}, \frac{m_N^2 e^{-m_N}}{2!}, \ldots$ respectively. Here, m_N is the distribution parameter, representing both the expected value and the variance in a neutrosophic sense:

$$P(y = v) = \frac{m_N^{v} e^{-m_N}}{v!}, \quad v = 0, 1, 2, \ldots \tag{19}$$

Thus, we have:

$$NE(y) = NV(y) = m_N \tag{20}$$

where $m_N = d + I$ is a neutrosophic number (composed of a determinate part d and an indeterminate part I).

The power series for the NPD, as explored in [**?**], is defined as:

$$K(m_N, z) = z + \sum_{v=2}^{\infty} \frac{m_N^{v-1}}{(v-1)!} e^{-m_N} z^{v}, \quad z \in \mathbb{D} \tag{21}$$

For $m_N \in [1, \infty)$, the ratio test confirms an infinite radius of convergence. Using the convolution (Hadamard product) of two power series $\Delta(z)$ and $\rho(z)$:

$$(\Delta * \rho)(z) = z + \sum_{v=2}^{\infty} \lambda_v \beta_v z^{v} \tag{22}$$

Applying the differential operator $D^{t}_{\Phi,\lambda,\tau,\varepsilon}(s, \Psi)$ to the NPD series yields:

$$D^{t}_{\Phi,\lambda,\tau,\varepsilon}(s, \Psi)K_{\varphi}(m, z) = z + \sum_{v=2}^{\infty} \left[\frac{\pi(s) + \Phi + (v-1)(\lambda + v(\tau + \varepsilon))}{\pi(s) + \Phi} \right]^{t} \frac{m_N^{v-1}}{(v-1)!} e^{-m_N} \frac{\Theta_{v-1}}{\Psi} z^{v}. \tag{23}$$

The convolution of such a distribution with a generalised differential operator enables deeper structural analysis of neutrosophic stochastic processes, particularly when modelling cumulative or compound systems. In this setting, generalised differential operators are applied to improve the analytical tractability and functional transformation of probability-generating functions (PGFs) or moment-generating functions (MGFs) associated with neutrosophic distributions. The convolution of the operator with the neutrosophic distribution produces transformed coefficients that reveal patterns in the behaviour of complex or hybrid neutrosophic systems, thereby supporting uncertainty quantification, approximation theory, and stability analysis. These applications are particularly relevant to geometric function theory, where convolution (Hadamard product) of analytic functions plays a central role. By modelling coefficient sequences using neutrosophic Poisson distributions and subjecting them to generalised differential operators, one can obtain new subordination results, coefficient bounds, and Hankel determinant estimates under conditions of indeterminacy. Significant contributions in this direction are documented in [11, 12, 19, 29]. The sequence of Mersenne numbers, denoted by M_n, is classically defined as $M_n = 2^n - 1$. The polynomial generalization introduced in recent literature follows the recurrence relation:

$$M_n(x) = 3M_{n-1}(x) - 2M_{n-2}(x), \quad n \geq 2 \tag{24}$$

with the initial conditions:

$$M_0(x) = 0$$
$$M_1(x) = 1$$

In [?], the recurrence relation provided a generalization of the Mersenne polynomial defined by:

$$M_n(y) = 3yM_{n-1}(y) - 2M_{n-2}(y), \quad n \geq 2 \tag{25}$$

where p, q, and r are arbitrary constant real or complex values (not all zero), with the following initial conditions:

$$M_0(y) = p \tag{26}$$

$$M_1(y) = qy + r \tag{27}$$

$$M_2(y) = 3y(qy + r) - 2p \tag{28}$$

$$M_3(y) = 3y[3y(qy + r) - 2p] - 2(qy + r) = (9y^2 - 2)qy + (9y^2 - 2)r - 6yp \tag{29}$$

$$M_4(y) = 3yM_3(y) - 2M_2(y) \tag{30}$$

Aarthy and Keerthi [?] recently introduced a novel contrast enhancement technique using a transformation function derived from Sakaguchi-type functions subordinate to generalized Mersenne polynomials on the open unit disc. Their study, which examined coefficient bounds for further applications, is well documented in the literature.

Building on the recent work of [?] and earlier contributions by [?, ?, ?, ?], the present authors seek to explore the indeterminacy inherent in the classical Poisson distribution. This is approached through the use of neutrosophic sets and the advanced differential–sigmoid operator, combined with Mersenne polynomials, within the framework of univalent function theory.

3. Lemmas and Definitions

The lemmas required for the main results are recalled below.

Lemma 3.1 ([?]). Let $p \in \mathcal{P}$ have the series expansion of the form (1). Then, for x and σ with $|x| \leq 1$ and $|\sigma| \leq 1$, the following relations hold:

$$2c_2 = c_1^2 + x(4 - c_1^2), \tag{31}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)\sigma. \tag{32}$$

Lemma 3.2. If $h \in \mathcal{P}$ and has the series expansion $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$, then:

$$|c_{n+k} - \mu c_n c_k| \leq 2, \tag{33}$$

$$|c_n| \leq 2 \quad \text{for } n \geq 1, \tag{34}$$

$$|c_2 - \xi c_1^2| \leq 2\max(1, |2\xi - 1|), \quad \xi \in \mathbb{C}, \tag{35}$$

$$|Jc_1^3 - Kc_1c_2 + Lc_3| \leq 2|J| + 2|K - 2J| + 2|J - K + L|. \tag{36}$$

Lemma 3.3 ([?]). Let m, n, l and r satisfy the inequalities $0 < m < 1$, $0 < r < 1$, and

$$8r(1-r)[(mn - 2l)^2 + (m(r+m) - n)^2] + m(1-m)(n - 2rm)^2 \leq 4m^2(1-m)^2 r(1-r).$$

If $h \in \mathcal{P}$ with $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ $(z \in \mathbb{D})$, then:

$$\left| lc_1^4 + rc_2^2 + 2mc_1c_3 - \frac{3}{2}nc_1^2c_2 - c_4 \right| \leq 2.$$

We note that the inequalities (20), (21), and (23) can be found in [?, ?], (22) is from [?], and (23) was further evaluated in [?].

Definition 3.1. For $s, \lambda, \Phi, \tau, \varepsilon \geq 0$, $m_N \in [1, \infty]$, $t \in \mathbb{N}_0$, and $0 \leq \Omega \leq 1$, the class $\chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, y)$ is defined by the subordination condition:

$$\frac{(1-\Omega)D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) - \rho z}{z} \prec M(y,z). \tag{37}$$

$$\prec M(y, m(z)) \tag{38}$$

The class $\chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, \gamma)$ $(0 \leq \gamma < 1)$ is defined by the condition:

$$\operatorname{Re}\left\{ \frac{(1-\Omega)D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) - \rho z}{z} \right\} > \gamma.$$

Remark 3.1: Under Definition 3, if $\Omega = 0$, the subordination simplifies to:

$$\frac{D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) - \rho z}{z} \prec M(y, m(z))$$

Remark 3.2: If $\Omega = 1$, we obtain:

$$\frac{D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) - \rho z}{z} \prec M(y, m(z))$$

Remark 3.3: For the case $\Omega = 0$ and $\rho = 0$:

$$\frac{D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z)}{z} \prec M(y, m(z))$$

Remark 3.4: For the case $\Omega = 1$ and $\rho = 0$:

$$\frac{D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z)}{z} \prec M(y, m(z))$$

4. **Main Results**

Theorem 4.1. Let $f \in \chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, y)$. Then the following coefficient estimates hold:

$$\left|\frac{\Theta_1}{\Psi}\right| \leq \frac{1!M_1(y)}{A_1^t[1-\Omega+\Omega A_1+\rho A_1^2]m_N e^{-m_N}} \tag{39}$$

$$\left|\frac{\Theta_2}{\Psi}\right| \leq \frac{2!M_1(y)}{A_2^t[1-\Omega+\Omega A_2+\rho A_2^2]m_N e^{-m_N}} \tag{40}$$

$$\left|\frac{\Theta_3}{\Psi}\right| \leq \frac{3!M_1(y)}{A_3^t[1-\Omega+\Omega A_3+\rho A_3^2]m_N e^{-m_N}} \tag{41}$$

$$\left|\frac{\Theta_4}{\Psi}\right| \leq \frac{4!M_1(y)}{A_4^t[1-\Omega+\Omega A_4+\rho A_4^2]m_N e^{-m_N}} \tag{42}$$

where the parameters A_k are defined as:

$$A_1 = \frac{\pi(s)+\Phi+\lambda+2(\tau+\varepsilon)}{\pi(s)+\Phi}, \qquad A_2 = \frac{\pi(s)+\Phi+\lambda+6(\tau+\varepsilon)}{\pi(s)+\Phi}$$

$$A_3 = \frac{\pi(s)+\Phi+\lambda+12(\tau+\varepsilon)}{\pi(s)+\Phi}, \qquad A_4 = \frac{\pi(s)+\Phi+\lambda+20(\tau+\varepsilon)}{\pi(s)+\Phi}$$

Proof. By Definition 3.1 and the principle of subordination, there exists a Schwarz function $\beta(z)$ such that:

$$\frac{(1-\Omega)D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_\varphi(z) - \rho z}{z} = M(y, \beta(z)). \tag{43}$$

Define the function $m(z) \in \mathcal{P}$ (the class of functions with positive real part) as:

$$m(z) = \frac{1+\beta(z)}{1-\beta(z)} = 1 + \sigma_1 z + \sigma_2 z^2 + \sigma_3 z^3 + \cdots \implies \beta(z) = \frac{m(z)-1}{m(z)+1}, \quad (z \in E). \tag{44}$$

By expanding $\beta(z)$ in terms of the coefficients of $m(z)$, we obtain:

$$\beta(z) = \frac{\sigma_1}{2}z + \left(\frac{\sigma_2}{2} - \frac{\sigma_1^2}{4}\right)z^2 + \left(\frac{\sigma_3}{2} - \frac{\sigma_2\sigma_1}{2} + \frac{\sigma_1^3}{8}\right)z^3 + \ldots \tag{45}$$

From (31) and (29), we have the expansion:

$$\begin{aligned} M(u(z),y) =& 1+\frac{M_1(y)\sigma_1}{2}z+\left[\frac{M_1(y)}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{M_2(y)\sigma_1^2}{4}\right]z^2 \\ &+\left[\frac{M_1(y)}{2}\left(\sigma_3-\sigma_1\sigma_2+\frac{\sigma_1^3}{4}\right)+\frac{M_2(y)\sigma_1}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{M_3(y)\sigma_1^3}{8}\right]z^3+\dots \end{aligned} \tag{46}$$

By comparing coefficients in (29) and (32), we deduce:

$$\Gamma_2\frac{\Theta_1}{\Psi}=\frac{M_1(y)\sigma_1}{2} \tag{47}$$

$$\Gamma_3\frac{\Theta_2}{\Psi}=\frac{M_1(y)}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{M_2(y)\sigma_1^2}{4} \tag{48}$$

$$\Gamma_4\frac{\Theta_3}{\Psi}=\frac{M_1(y)}{2}\left(\sigma_3-\sigma_1\sigma_2+\frac{\sigma_1^3}{4}\right)+\frac{M_2(y)\sigma_1}{2}\left(\sigma_2-\frac{\sigma_1^2}{2}\right)+\frac{M_3(y)\sigma_1^3}{8} \tag{49}$$

Using the property $|\sigma_n|\le 2$ from Lemma 2.1 in (47), we determine:

$$\left|\frac{\Theta_1}{\Psi}\right|\le\frac{M_1(y)}{\Gamma_2}. \tag{50}$$

From (48), we rearrange to find:

$$\frac{\Theta_2}{\Psi}=\frac{M_1(y)}{2\Gamma_3}\left[\sigma_2-\left(\frac{M_1(y)-M_2(y)}{M_1(y)}\right)\sigma_1^2\right] \tag{51}$$

Applying Lemma 3.1 yields the bound:

$$\left|\frac{\Theta_2}{\Psi}\right|\le\frac{M_1(y)}{\Gamma_3}. \tag{52}$$

In the first case study, where the call rate is $m\in[1,3]$, the coefficient bounds for the class $\chi^t_{m_N,\lambda,\tau}$ are given by: This completes the proof of Theorem 4.1.

Example for Case Study 1

Consider a telecommunications scenario where an employee receives phone calls at a neutrosophic rate of $\lambda\in[1,3]$ calls per minute. To estimate the probability of the employee receiving no calls in a single minute ($T=1$), we use the neutrosophic parameter $m=\lambda T\in[1,3]$.

Corollary 4.1. Let $m\in[1,3]$. The bounds for the normalized coefficients are given by:

$$\left|\frac{\Theta_1}{\Psi}\right|\le\frac{1!M_1(y)}{A_1^t[1-\Omega+\Omega A_1+\rho A_1^2]\cdot([1,3]e^{-[1,3]})} \tag{53}$$

$$\left|\frac{\Theta_2}{\Psi}\right|\le\frac{2!M_1(y)}{A_2^t[1-\Omega+\Omega A_2+\rho A_2^2]\cdot[1,3]e^{-[1,3]}} \tag{54}$$

$$\left|\frac{\Theta_3}{\Psi}\right|\le\frac{3!M_1(y)}{A_3^t[1-\Omega+\Omega A_3+\rho A_3^2]\cdot[1,3]e^{-[1,3]}} \tag{55}$$

$$\left|\frac{\Theta_4}{\Psi}\right|\le\frac{4!M_1(y)}{A_4^t[1-\Omega+\Omega A_4+\rho A_4^2]\cdot[1,3]e^{-[1,3]}} \tag{56}$$

where the growth parameters A_k are defined based on the differential operator components:

$$A_1 = \frac{\pi(s) + \Phi + \lambda + 2(\tau + \varepsilon)}{\pi(s) + \Phi}$$
$$A_2 = \frac{\pi(s) + \Phi + \lambda + 6(\tau + \varepsilon)}{\pi(s) + \Phi}$$
$$A_3 = \frac{\pi(s) + \Phi + \lambda + 12(\tau + \varepsilon)}{\pi(s) + \Phi}$$
$$A_4 = \frac{\pi(s) + \Phi + \lambda + 20(\tau + \varepsilon)}{\pi(s) + \Phi}.$$

Example for Case Study 2

Consider a telecommunications scenario where an employee receives phone calls at a neutrosophic rate of $\lambda \in [1, 3]$ calls per minute. To estimate the probability of receiving zero calls over a $T = 5$ minute interval, we define the neutrosophic parameter $m = \lambda T \in [5, 15]$.

Corollary 4.2. Let $m \in [5, 15]$. The bounds for the normalized coefficients of the class $\chi^t_{m_N,\lambda,\tau}$ are given by:

$$\left|\frac{\Theta_1}{\Psi}\right| \leq \frac{1!M_1(y)}{A_1^t[1 - \Omega + \Omega A_1 + \rho A_1^2] \cdot [5, 15]e^{-[5,15]}} \tag{57}$$
$$\left|\frac{\Theta_2}{\Psi}\right| \leq \frac{2!M_1(y)}{A_2^t[1 - \Omega + \Omega A_2 + \rho A_2^2] \cdot [5, 15]e^{-[5,15]}} \tag{58}$$
$$\left|\frac{\Theta_3}{\Psi}\right| \leq \frac{3!M_1(y)}{A_3^t[1 - \Omega + \Omega A_3 + \rho A_3^2] \cdot [5, 15]e^{-[5,15]}} \tag{59}$$
$$\left|\frac{\Theta_4}{\Psi}\right| \leq \frac{4!M_1(y)}{A_4^t[1 - \Omega + \Omega A_4 + \rho A_4^2] \cdot [5, 15]e^{-[5,15]}} \tag{60}$$

where the growth parameters are defined as:

$$A_1 = \frac{\pi(s) + \Phi + \lambda + 2(\tau + \varepsilon)}{\pi(s) + \Phi}$$
$$A_2 = \frac{\pi(s) + \Phi + \lambda + 6(\tau + \varepsilon)}{\pi(s) + \Phi}$$
$$A_3 = \frac{\pi(s) + \Phi + \lambda + 12(\tau + \varepsilon)}{\pi(s) + \Phi}$$
$$A_4 = \frac{\pi(s) + \Phi + \lambda + 20(\tau + \varepsilon)}{\pi(s) + \Phi}.$$

Theorem 4.2. Let $f \in \chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, y)$. Then for any $\varpi \in \mathbb{R}$:

$$\left|\frac{\Theta_2}{\Psi} - \varpi\frac{\Theta_1^2}{\Psi^2}\right| \leq \dots \tag{61}$$
$$\leq \frac{2!M_1(y)}{A_2^t[1 - \Omega + \Omega A_2 + \rho A_2^2]m_N e^{-m_N}} \max\left\{1, \left|\frac{2M_2(y)\Gamma_2^2 - 2M_1(y)\Gamma_2^2 + \varpi M_1^2(y)\Gamma_3}{M_1(y)\Gamma_2^2}\right|\right\} \tag{62}$$

where Γ_k and A_2 are defined as in the preceding sections. Theorem 4.2 extends the first-order bounds of Theorem 4.1 by establishing correlation control between analytic coefficients.

Proof. From the coefficient expansions in (47) and (**??**), we obtain:

$$\left|\frac{\Theta_2}{\Psi} - \varpi\frac{\Theta_1^2}{\Psi^2}\right| = \left|\frac{M_1(y)}{2\Gamma_3}\sigma_2 - \left(\frac{M_1(y)\Gamma_2^2 + \varpi M_1^2(y)\Gamma_3 - M_2(y)\Gamma_2^2}{4\Gamma_3\Gamma_2^2}\right)\sigma_1^2\right|. \tag{63}$$

With straightforward calculations, we obtain:

$$\left|\frac{\Theta_2}{\Psi} - \varpi\frac{\Theta_1^2}{\Psi^2}\right| = \frac{M_1(y)}{2\Gamma_3}\left|\sigma_2 - \left(\frac{M_1(y)\Gamma_2^2 + \varpi M_1^2(y)\Gamma_3 - M_2(y)\Gamma_2^2}{2M_1(y)\Gamma_2^2}\right)\sigma_1^2\right|. \tag{64}$$

Applying the bounded properties of σ_n from (22), we have:

$$\left|\frac{\Theta_2}{\Psi} - \varpi\frac{\Theta_1^2}{\Psi^2}\right| \leq \frac{M_1(y)}{2\Gamma_3}\max\left\{2, 2\left|2\left(\frac{M_1(y)\Gamma_2^2 + \varpi M_1^2(y)\Gamma_3 - M_2(y)\Gamma_2^2}{2M_1(y)\Gamma_2^2}\right) - 1\right|\right\}. \tag{65}$$

Simplifying the expression within the braces, we find:

$$\left|\frac{\Theta_2}{\Psi} - \varpi\frac{\Theta_1^2}{\Psi^2}\right| \leq \frac{M_1(y)}{\Gamma_3}\max\left\{1, \left|\frac{M_1(y)\Gamma_2^2 + \varpi M_1^2(y)\Gamma_3 - M_2(y)\Gamma_2^2}{M_1(y)\Gamma_2^2} - 1\right|\right\}. \tag{66}$$

Finally, by reducing the fraction, we conclude:

$$\left|\frac{\Theta_2}{\Psi} - \varpi\frac{\Theta_1^2}{\Psi^2}\right| \leq \frac{M_1(y)}{\Gamma_3}\max\left\{1, \left|\frac{\varpi M_1^2(y)\Gamma_3 - M_2(y)\Gamma_2^2}{M_1(y)\Gamma_2^2}\right|\right\}. \tag{67}$$

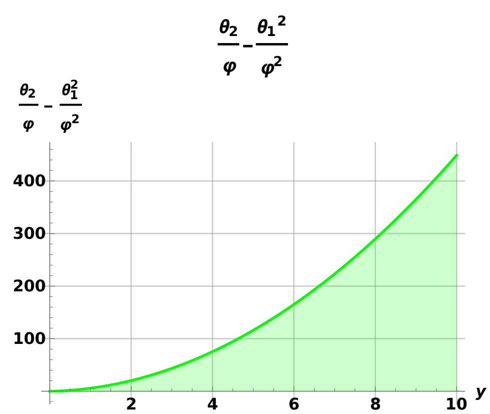

FIGURE 1. Upper bound surfaces for the second-order functional in Theorem 4.2.

By setting $\varpi = 1$, we obtain the following simplification:

Corollary 4.3. If f of the form (1) belongs to the class $\chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,y)$, then:

$$\left|\frac{\Theta_2}{\Psi} - \frac{\Theta_1^2}{\Psi^2}\right| \leq \frac{2!M_1(y)}{A_2^t[1-\Omega+\Omega A_2+\rho A_2^2]m_N e^{-m_N}}. \tag{68}$$

Theorem 4.3. Let $f \in \chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,y)$, then

$$\left|\frac{\Theta_1\Theta_2}{\Psi^2} - \frac{\Theta_3}{\Psi^2}\right| \leq \frac{3!M_1(y)}{A_3^t[1-\Omega+\Omega A_3+\rho A_3^2]m_N e^{-m_N}} \tag{69}$$

Proof. Following the coefficient expansions from (47), (**??**), and (**??**), we have:

$$\left|\frac{\Theta_1\Theta_2}{\Psi^2} - \frac{\Theta_3}{\Psi^2}\right| = \left|J\sigma_1^3 - K\sigma_1\sigma_2 + L\sigma_3\right|$$

where the constants J, K, and L are defined as:

$$J = \frac{M_1(y)M_2(y)\Gamma_4 - M_1^2(y)\Gamma_4 - M_1(y)\Gamma_2\Gamma_3 + 2M_2(y)\Gamma_2\Gamma_3 - M_3(y)\Gamma_2\Gamma_3}{8\Gamma_2\Gamma_3\Gamma_4},$$

$$K = \frac{2M_2(y)\Gamma_2\Gamma_3 - M_1^2(y)\Gamma_4 - 2M_1(y)\Gamma_2\Gamma_3}{4\Gamma_2\Gamma_3\Gamma_4},$$

$$L = -\frac{M_1(y)}{2\Gamma_4}.$$

Applying the triangle inequality and the bounded properties of σ_n, we obtain:

$$\left|\frac{\Theta_1\Theta_2}{\Psi^2} - \frac{\Theta_3}{\Psi^2}\right| \le 2|J| + 2|K - 2J| + 2|J - K + L| \le \frac{M_1(y)}{\Gamma_4}.$$

Substituting the value of Γ_4 completes the proof.

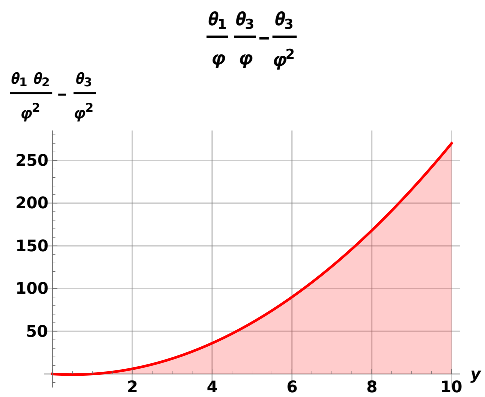

FIGURE 2. Upper bound behavior and geometric stability of the third-order functional under Theorem 4.3.

Theorem 4.4: Let $f \in \chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, y)$, then

$$\left|\frac{\Theta_1\Theta_3}{\Psi^2} - \frac{\Theta_2^2}{\Psi^2}\right| \le \frac{(3!)^2 M_1^2(y)}{A_3^{2t}[1 - \Omega + \Omega A_3 + \rho A_3^2]^2 m_N^2 e^{-2m_N}} \tag{70}$$

Proof. Using the expansions from (33), (37), and (38), we express the functional as:

$$\frac{\Theta_1\Theta_3}{\Psi^2} - \frac{\Theta_2^2}{\Psi^2} = A\sigma_1^4 - B\sigma_1^2\sigma_2 + T\sigma_1\sigma_3 - K\sigma_2^2$$

Applying Lemma 3.1 to express σ_2 and σ_3 in terms of $c = \sigma_1$ (where $c \in [0, 2]$), and applying the triangle inequality with $|x| = b \le 1$ and $|\varphi| \le 1$, we define the function $\psi(c, b)$:

$$\psi(c,b) = \left|\frac{4A - 2B + T - K}{4}\right| c^4 + \left(\frac{B + T - K}{2}\right) c^2 b(4 - c^2)$$
$$+ \frac{T}{4}c^2(4 - c^2)b^2 + \frac{K}{4}b^2(4 - c^2)^2 + T(4 - c^2)(1 - b^2).$$

Calculations show that $\psi(c,b) \le \psi(c,1)$. Evaluating at $b=1$:

$$\psi(c,1) = \left|\frac{4A-2B+T-K}{4}\right| c^4 + \left(\frac{B+T-K}{2}\right) c^2(4-c^2) + \frac{T}{4}c^2(4-c^2) + \frac{K}{4}(4-c^2)^2.$$

Since $\psi'(c,1) < 0$ on the interval, the function is decreasing, achieving its maximum at $c=0$:

$$\left|\frac{\Theta_1\Theta_3}{\Psi^2} - \frac{\Theta_2^2}{\Psi^2}\right| \le \frac{M_1^2(y)}{\Gamma_3^2},$$

which, upon substituting the definition of Γ_3, completes the proof.

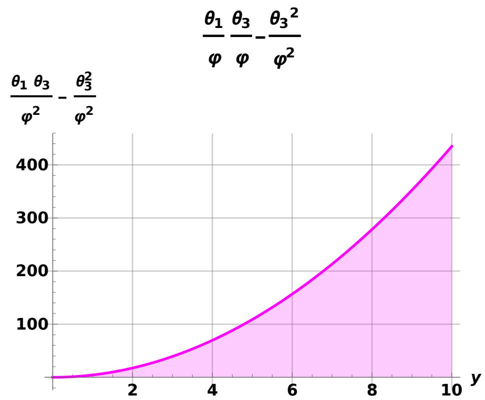

FIGURE 3. Graphical analysis of the coefficient bounds described in Theorem 4.4.

Theorem 4.5. Let $f \in \chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,y)$. Then the third-order Hankel determinant satisfies:

$$\begin{aligned} |H_{3,1}(f)| \le & \frac{(2!)^3 M_1^3(y)}{A_2^{3t}[1-\Omega+\Omega A_2+\rho A_2^2]^3 m_N^3 e^{-3m_N}} \\ & + \frac{(3!)^2 M_1^2(y)}{A_3^{4t}[1-\Omega+\Omega A_3+\rho A_3^2]^2 m_N^2 e^{-2m_N}} \\ & + \frac{(2!)^2 4! M_1^2(y)}{A_2^t A_4^t[1-\Omega+\Omega A_2+\rho A_2^2][1-\Omega+\Omega A_4+\rho A_4^2] m_N^2 e^{-2m_N}}, \end{aligned} \tag{71}$$

where A_2, A_3, and A_4 are defined as in the previous sections.

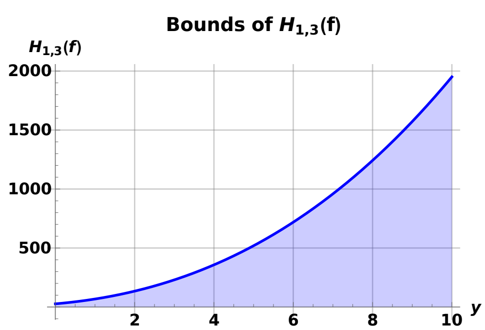

FIGURE 4. Visual representation of the coefficient bounds for Theorem 4.5.

Theorem 4.6. Let $K_\varphi(z) = z + \sum_{v=2}^{\infty} \frac{A_{v-1}}{\Psi} z^v \in \mathcal{A}$. If

$$\sum_{v=2}^{\infty} \Lambda_v^t \left[\frac{1 - \Omega + \Omega\Lambda_v + \rho\Lambda_v^2}{1-\gamma} \right] \frac{m_N^{v-1}}{(v-1)! e^{m_N}} \left| \frac{A_{v-1}}{\Psi} \right| \leq 1, \tag{72}$$

then $K_\varphi(z) \in \chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, \gamma)$, where:

$$\Lambda_v = \frac{\pi(s) + \Phi + (v-1)[\lambda + v(\tau + \varepsilon)]}{\pi(s) + \Phi}.$$

Theorem 4.6 establishes a sufficient condition on the coefficients of a normalized analytic function to ensure its inclusion within a specialized geometric function class. Each term in the series expansion is modulated by two interacting weighting mechanisms: a Poisson-type discrete probability factor governing coefficient decay, and a rational weighting component modeling growth and uncertainty.

Proof. It is sufficient to establish that:

$$\left| \frac{(1-\Omega) D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) - \rho z}{z} - 1 \right| < 1 - \gamma,$$

where $0 \leq \gamma < 1$. We observe that:

$$\begin{aligned} &\left| \frac{(1-\Omega) D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) - \rho z}{z} - 1 \right| \\ &= \left| \sum_{v=2}^{\infty} \Lambda_v^t \left[1 - \Omega + \Omega\Lambda_v + \rho\Lambda_v^2 \right] \frac{m_N^{v-1}}{(v-1)! e^{m_N}} \frac{A_{v-1}}{\Psi} z^{v-1} \right| \\ &\leq \sum_{v=2}^{\infty} \Lambda_v^t \left[1 - \Omega + \Omega\Lambda_v + \rho\Lambda_v^2 \right] \frac{m_N^{v-1}}{(v-1)! e^{m_N}} \left| \frac{A_{v-1}}{\Psi} \right| \leq 1 - \gamma, \end{aligned}$$

by the condition of the theorem. Thus, the proof is complete.

Theorem 4.7. Let $K_\varphi(z) = z - \sum_{v=2}^{\infty} \frac{A_{v-1}}{\Psi} z^v$ with $\frac{A_{v-1}}{\Psi} \geq 0$. Then $K_\varphi(z) \in \chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, \gamma)$ if and only if:

$$\sum_{v=2}^{\infty} \Lambda_v^t \left[\frac{1 - \Omega + \Omega\Lambda_v + \rho\Lambda_v^2}{1-\gamma} \right] \frac{m_N^{v-1}}{(v-1)! e^{m_N}} \left(\frac{A_{v-1}}{\Psi} \right) \leq 1, \tag{73}$$

for $0 \leq \gamma < 1$. **Proof.** Let $K_\varphi(z) = z - \sum_{v=2}^{\infty} \frac{A_{v-1}}{\Psi} z^v \in \chi^t_{m_N,\lambda,\tau}(s, \Psi, \varepsilon, \Omega, \rho, \Phi, \gamma)$, with $\frac{A_{v-1}}{\Psi} \geq 0$. Then we have:

$$\operatorname{Re} \left\{ \frac{(1-\Omega) D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) - \rho z}{z} \right\} > \gamma, \tag{74}$$

which implies that

$$\left| \frac{(1-\Omega) D^t_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) + \rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi) K_\varphi(z) - \rho z}{z} - 1 \right| < 1 - \gamma, \tag{75}$$

where $0 \le \gamma < 1$. Expanding the left-hand side, we get:

$$\left|\frac{(1-\Omega)D^{t}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)+\Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)+\rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)-\rho z}{z}-1\right|$$
$$=\left|\sum_{v=2}^{\infty}\Lambda_v^t\left[1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{A_{v-1}}{\Psi}z^{v-1}\right|. \tag{76}$$

Thus, we have:

$$\operatorname{Re}\left(\sum_{v=2}^{\infty}\Lambda_v^t\left[1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{A_{v-1}}{\Psi}z^{v-1}\right)<1-\gamma. \tag{77}$$

By choosing values of z on the real axis and letting $z \to 1^-$ through real values, we obtain from (77):

$$\sum_{v=2}^{\infty}\Lambda_v^t\left[1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{A_{v-1}}{\Psi}\le 1-\gamma. \tag{78}$$

Conversely, by applying the triangle inequality, we have:

$$\begin{aligned}\left|\sum_{v=2}^{\infty}\Lambda_v^t\left[1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{A_{v-1}}{\Psi}z^{v-1}\right| &\le \sum_{v=2}^{\infty}\Lambda_v^t\left[1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\left|\frac{A_{v-1}}{\Psi}\right| \\ &= \sum_{v=2}^{\infty}\Lambda_v^t\left[1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{A_{v-1}}{\Psi}.\end{aligned} \tag{79}$$

By the condition of the theorem, it follows that:

$$\left|\frac{(1-\Omega)D^{t}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)+\Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)+\rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)-\rho z}{z}-1\right|<1-\gamma,$$

which implies:

$$\operatorname{Re}\left\{\frac{(1-\Omega)D^{t}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)+\Omega D^{t+1}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)+\rho D^{t+2}_{\Phi,\lambda,\tau,\varepsilon}(s,\Psi)K_{\varphi}(z)-\rho z}{z}\right\}>\gamma.$$

Hence, $K_{\varphi}(z) \in \chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,\gamma)$.

Functions with Finitely Many Fixed Coefficients

Let the subclass $\chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,\gamma;M_j)$ be defined by:

$$K_{\varphi}(z)=z-\sum_{j=2}^{\delta}\frac{M_j}{\Lambda_j^t\left[\frac{1-\Omega+\Omega\Lambda_j+\rho\Lambda_j^2}{1-\gamma}\right]\frac{m_N^{j-1}}{(j-1)!e^{m_N}}}z^j-\sum_{v=\delta+1}^{\infty}\frac{\Theta_{v-1}}{\Psi}z^v. \tag{80}$$

Theorem 4.8. The class $\chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,\gamma)$ is closed under convex linear combination. **Proof.** Assume there are functions $K_{\varphi}, g_{\varphi} \in \chi^t_{m_N,\lambda,\tau}(s,\Psi,\varepsilon,\Omega,\rho,\Phi,\gamma,M_j)$ such that:

$$K_{\varphi}(z)=z-\sum_{j=2}^{\delta}\frac{M_j}{\Lambda_j^t\left[\frac{1-\Omega+\Omega\Lambda_j+\rho\Lambda_j^2}{1-\gamma}\right]\frac{m_N^{j-1}}{(j-1)!e^{m_N}}}z^j-\sum_{v=\delta+1}^{\infty}\frac{\Theta_{v-1}}{\Psi}z^v \tag{81}$$

$$g_{\varphi}(z)=z-\sum_{j=2}^{\delta}\frac{M_j}{\Lambda_j^t\left[\frac{1-\Omega+\Omega\Lambda_j+\rho\Lambda_j^2}{1-\gamma}\right]\frac{m_N^{j-1}}{(j-1)!e^{m_N}}}z^j-\sum_{v=\delta+1}^{\infty}\frac{\Delta_{v-1}}{\Psi}z^v \tag{82}$$

where $0\leq M_j\leq 1$ and $0\leq\sum_{j=2}^{\delta}M_j\leq 1$. For $0\leq\nu\leq 1$, we define the convex combination:

$$\tau(z)=\nu K_{\varphi}(z)+(1-\nu)g_{\varphi}(z). \tag{83}$$

Substituting (59) and (60) into (61), we obtain:

$$\tau(z)=z-\sum_{j=2}^{\delta}\frac{M_j}{\Lambda_j^t\left[\frac{1-\Omega+\Omega\Lambda_j+\rho\Lambda_j^2}{1-\gamma}\right]\frac{m_N^{j-1}}{(j-1)!e^{m_N}}}z^j-\sum_{v=\delta+1}^{\infty}\left(\nu\frac{\Theta_{v-1}}{\Psi}+(1-\nu)\frac{\Delta_{v-1}}{\Psi}\right)z^v. \tag{84}$$

If we set $\nu=1$ and $\nu=0$, equation (62) reduces to (59) and (60) respectively. Now, we show that the class is closed under convex linear combinations. For $0\leq\nu\leq 1$, let:

$$\tau(z)=z+\sum_{v=2}^{\infty}\left(\nu\frac{\Theta_{v-1}}{\Psi}+(1-\nu)\frac{\Delta_{v-1}}{\Psi}\right)z^v.$$

We observe that:

$$\begin{aligned}
&\sum_{v=j+1}^{\infty}\Lambda_v^t\left[\frac{1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2}{1-\gamma}\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\left\{\nu\frac{\Theta_{v-1}}{\Psi}+(1-\nu)\frac{\Delta_{v-1}}{\Psi}\right\}\\
&=\nu\sum_{v=j+1}^{\infty}\Lambda_v^t\left[\frac{1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2}{1-\gamma}\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{\Theta_{v-1}}{\Psi}\\
&\quad+(1-\nu)\sum_{v=j+1}^{\infty}\Lambda_v^t\left[\frac{1-\Omega+\Omega\Lambda_v+\rho\Lambda_v^2}{1-\gamma}\right]\frac{m_N^{v-1}}{(v-1)!e^{m_N}}\frac{\Delta_{v-1}}{\Psi}\\
&\leq\nu\left(1-\sum_{j=2}^{\delta}M_j\right)+(1-\nu)\left(1-\sum_{j=2}^{\delta}M_j\right)\\
&=1-\sum_{j=2}^{\delta}M_j.
\end{aligned}$$

Thus $\tau(z)\in\chi_{m_N,\lambda,\,\tau}^t(s,\Psi,\varepsilon,\Omega,\rho,\Phi,\gamma)$ which complete the prove of Theorem 4.8. Together, the last four theorems form a unified analytic framework within geometric function theory, progressively extending the coefficient-based characterisation of a specialised function class. The first theorem establishes a sufficient condition through probabilistic and neutrosophic weighting, producing smooth, sigmoid-like attenuation and a natural multiresolution structure. The second theorem strengthens this result by demonstrating that the same condition is necessary and sufficient, thereby defining the exact analytic boundaries of the class. The third theorem generalises the framework further by introducing tolerance parameters that allow controlled flexibility and local adaptation while preserving global stability. Collectively, these results integrate probabilistic decay, uncertainty modelling, and multiscale structure into a single coherent theory. The framework provides a rigorous mathematical foundation for analytical regularisation and practical applications such as sound denoising, where it ensures fidelity, stability, and adaptive control across multiple scales.

Conclusion: The theorems presented in this work introduce a novel framework within geometric function theory by integrating probabilistic weighting, neutrosophic uncertainty, and multiresolution scaling into coefficient-based characterisations of analytic function classes. Unlike classical approaches, which rely on deterministic coefficient inequalities and fixed growth conditions, this framework incorporates Poisson-type discrete probability factors and tolerance sequences to regulate higher-order terms and allow controlled flexibility. The inclusion of neutrosophic parameters provides an explicit mechanism for modelling indeterminacy, while the multiresolution interpretation links the analytic theory to scale-adaptive applications. Collectively, these innovations not only extend the theoretical landscape of geometric function theory but also establish a rigorous foundation for applied problems such as sound denoising and signal regularisation, offering a mathematically principled approach to stability, fidelity, and uncertainty management across multiple scales.

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Appendix

Signal Processing in Uncertain Environments: Neutrosophic Poisson Distributions, Sigmoid Functions, and the Advanced Differential-Sigmoid Operator

Signal processing in uncertain or noisy environments often suffers due to assumptions of deterministic or well-behaved (e.g., Gaussian) noise. Traditional linear filtering, Fourier transforms, and even some machine learning methods struggle when **ambiguity**, **indeterminacy**, and **randomness** coexist.

This calls for advanced probabilistic and analytical tools. We now show how **neutrosophic Poisson distributions**, **sigmoid functions**, and the **Advanced Differential-Sigmoid Operator** form a unified framework to address such challenges.

Neutrosophic Poisson Distributions in Signal Modeling

Concept

A *neutrosophic Poisson distribution* models not just randomness, but **indeterminacy and inconsistency** in the arrival or amplitude of signal events. It is denoted by:

$$\mathcal{NP}(\lambda, I, F),$$

where:

- λ: Mean rate of occurrence (classical Poisson component).
- I: Indeterminacy factor (models uncertainty in measurement or timing).
- F: Falsity/contradiction measure (captures inconsistent or corrupted data).

Application in Signal Processing

- **Spike trains in neural data:** Events (spikes) may be missed or misfired.
- **Sensor data under jitter:** Arrival times are not deterministic.
- **Packet loss or delay:** Common in modern communication signals.

By modeling such events as a neutrosophic probability:

$$P(n \text{ events}) = \frac{e^{-\lambda}\lambda^n}{n!} \oplus I \oplus F,$$

we capture the *total epistemic state* of the signal—crucial in adaptive filtering or robust signal analysis.

Sigmoid Functions in Signal Nonlinearity and Thresholding

Concept

Sigmoid functions, such as the logistic function:

$$\sigma(x) = \frac{1}{1+e^{-x}},$$

are nonlinear mappings that compress signal values to a bounded range, typically $[0, 1]$.

Roles in Signal Processing

- **Activation functions:** Core component in neuro-signal processing and deep learning architectures.
- **Smoothing and denoising:** Effectively suppresses outliers while amplifying meaningful signal transitions.
- **Nonlinear filtering:** Particularly robust for edge detection in binary or step-like signal patterns.

Advanced Differential-Sigmoid Operator (ADSO)

Concept

The ADSO is a novel operator designed to map indeterminate signal inputs into a high-order feature space. It is formally defined as:

$$\mathcal{D}_\sigma^M[f(t)] = \frac{d^n}{dt^n}\left[\sigma\big(M_n(t)\cdot f(t)\big)\right]$$

Where:

- $f(t)$ represents the input signal in the time domain.
- $M_n(t)$ is the Mersenne polynomial of degree n, acting as a weighting function.
- $\sigma(\cdot)$ is the sigmoid function, providing the nonlinear threshold.

- **Smooth Transitions:** Provides a differentiable approach to thresholding.
- **Noise Robustness:** Helps in where noise is *indeterminate* or *inconsistent*.

Advanced Differential-Sigmoid Operator (ADSO)

The ADSO, denoted by $\mathcal{D}_\sigma^M$, is defined as:

$$\mathcal{D}_\sigma^M\left[f(t)\right] = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\left(\sigma(M \cdot f(t))\right)$$

where $\sigma(\cdot)$ is the sigmoid activation function and M represents the neutrosophic uncertainty weight.

- $f(t)$: The input time-domain signal.
- $M_n(t)$: Mersenne polynomial of degree n, used for basis expansion.
- $\sigma(\cdot)$: The sigmoid activation function providing smooth nonlinearity.
- $\frac{d^n}{dt^n}$: The n-th order differential operator.

Purpose

The primary objective of the **Advanced Differential-Sigmoid Operator (ADSO)** is to bridge the gap between high-order feature extraction and noise robustness. By integrating Mersenne polynomials with sigmoid-weighted derivatives, the operator achieves:

(1) **Localized Sharpening:** Enhancing rapid transitions (like the R-peak in ECG) while the sigmoid suppresses low-amplitude baseline noise.
(2) **Uncertainty Handling:** Leveraging the neutrosophic framework to weight the derivative by the degree of "truth" or "indeterminacy" in the signal segment.

- Enhances **features** (e.g., changes, peaks, trends).
- Suppresses **indeterminate regions**.
- Combines **probabilistic (via sigmoid)** and **polynomial structure (via Mersenne)** for flexible signal representation.

Signal Processing Roles

- **Feature extraction:** Identify signal transitions (e.g., QRS complex in ECG).
- **Noise suppression:** Especially useful where noise is *indeterminate* or *inconsistent*.
- **Uncertainty-weighted transforms:** Acts like a wavelet transform that adapts to the uncertainty in the data.

Use Case: Noisy Biomedical Signal (e.g., ECG)

Problem

- ECG signals corrupted by muscle noise, baseline drift, and electrode motion.
- Traditional filters fail due to **non-stationary, uncertain noise**.

Proposed Method

(1) **Model artifact events** using **neutrosophic Poisson distributions** to capture the unpredictability of noise bursts.

(2) **Apply ADSO transform:**

$$\mathcal{D}_{\sigma}^{M}[\mathrm{ECG}(t)]$$

which highlights physiological transitions (e.g., QRS complex) and suppresses indeterminate oscillations.

(3) **Sigmoid-based thresholding:** Adaptive thresholds tuned by confidence levels in the neutrosophic model convert noisy waveforms into clean decision boundaries.

Result

- Robust ECG peak detection.
- High tolerance to jitter and motion artifacts.
- Reduced false positives due to inconsistent noise.

Summary Table

Component	Signal Processing Function	Benefits in Noisy Environments
Neutrosophic Poisson	Models timing/amplitude uncertainty	Captures indeterminate noise/spikes
Sigmoid Function	Smooth nonlinearity for denoising	Handles soft thresholds; suppresses outliers
ADSO Operator	Feature-enhancing differential transform	Nonlinear filtering, uncertainty-weighted transforms

Final Thoughts: This theoretical framework can transform signal processing in environments where uncertainty, noise, and data ambiguity dominate. The **combination of neutrosophic logic and advanced operators** introduces a new class of **robust, intelligent signal transforms** that could redefine preprocessing, denoising, and feature extraction pipelines.

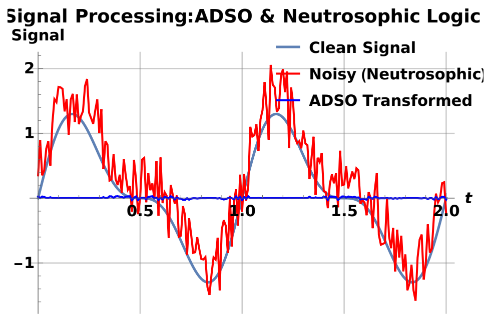

FIGURE 5. Neutrosophic signal processing in noisy environments***signal processing in terms of neutrosophic Poisson distributions, sigmoid functions, and the Advanced Differential-Sigmoid Operator, particularly in uncertain or noisy environments***

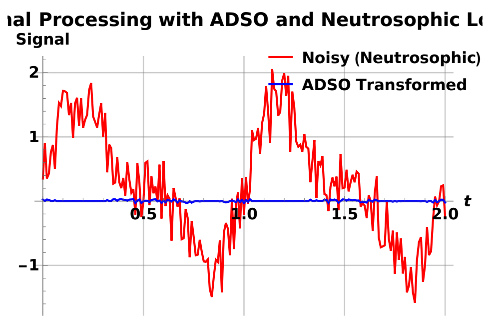

FIGURE 6. Neutrosophic signal processing components***signal processing in terms of neutrosophic Poisson distributions, sigmoid functions, and the Advanced Differential-Sigmoid Operator***

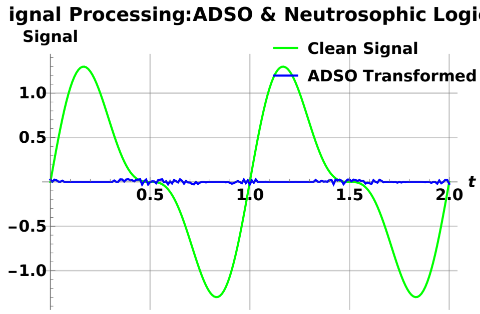

Figure 7. Clean signal and transform signal

```
Signal Processing : Neutrosophic Poisson distributions, sigmoid functions,
and the Advanced Differential - Sigmoid Operator
(*--------1. Define the Clean Input Signal----------*)
cleanSignal[t_] := Sin[2 Pi t] + 0.5 Sin[4 Pi t] (*ECG-like waveform*)
signalDuration = 2;
samplingRate = 100;
timeValues = N[Range[0, signalDuration, 1 / samplingRate]];
(*--------2. Define Neutrosophic Poisson Noise----------*)
neutrosophicNoise[t_] := Module[{λ = 5, I = 0.3, F = 0.2},
  RandomVariate[PoissonDistribution[λ]] / 20 + RandomReal[{-I, I}] + RandomChoice[{-F, F}]]

noisySignal = Table[{t, cleanSignal[t] + neutrosophicNoise[t]}, {t, timeValues}];

(*-------3. Define Sigmoid Function----------*)
sigmoid[x_] := 1 / (1 + Exp[-x])
(*------4. Define Mersenne Polynomial (degree 3)---------*)
mersennePoly[t_] := Expand[(2^3) - 1 + t] (*M3(t)=7+t*)

(*-------5. Define Advanced Differential-Sigmoid Operator (ADSO)-----------*)
h = 0.001;
ADSO[f_, t_] := (sigmoid[mersennePoly[t + h] * f] - sigmoid[mersennePoly[t] * f]) / h;

(*--------6. Apply ADSO to the Noisy Signal---------*)
interpolatedSignal = Interpolation[noisySignal];
adsSignal = Table[{t, ADSO[interpolatedSignal[t], t]}, {t, timeValues}];

(*---------7. Plot Results-----------*)
ListLinePlot[{Table[{t, cleanSignal[t]}, {t, timeValues}], noisySignal, adsSignal},
 PlotLegends → Placed[{"Clean Signal", "Noisy (Neutrosophic)", "ADSO Transformed"}, 3],
 PlotRange → All, PlotStyle → {Thick, Red, Blue}, GridLines → Automatic, ImageSize → Medium,
 PlotLabel → "Signal Processing:ADSO & Neutrosophic Logic", AxesLabel → {t, Signal},
 LabelStyle → {FontFamily → "8514oem", 12, GrayLevel[0], Bold}]
```

Figure 8. ***Signal processing in codes***

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