NSS
Neutrosophic Sets and Systems, Vol. 73, 2024
University of New Mexico

# Preflters on Neutrosophic Sets

D. Keerthana[1], V. Visalakshi[1,*] C. Rajapandiyan[1], Saeid Jafari[2]

[1]Department of Mathematics, College of Engineering and Technology, SRM Institute of Science and Technology, Kattankulathur, Tamil Nadu, India; kd1002@srmist.edu.in, rc5588@srmist.edu.in

[2]Mathematical and Physical Science Foundation, Sidevej 5, 4200 Slagelse, Denmark.; jafaripersia@gmail.com, saeidjafari@topositus.com

*Correspondence: visalakv@srmist.edu.in

**Abstract**. This paper introduces the concept of a prefilter on the collection of neutrosophic sets. The characteristic set and characteristic value of prefilters are introduced and their fundamental properties are explored. The relationships between filters and prefilters are examined, with particular emphasis on how ultrafilters correspond to maximal filters via specific mappings such as $i_K$ and $w_K$ Additionally, the concept of prime prefilters are defined and their basic properties are studied. The compatibility between filters and prefilters is also discussed.

**Keywords:** Neutrosophic set, Prefilter, Characteristic set, Characteristic value, Prime prefilter.

---

## 1. Introduction

Neutrosophic sets, introduced by Florentin Smarandache in 1998 [2], are a generalization of the classical and fuzzy set theories. Unlike classical set theories where an element either belongs to a set or does not, neutrosophic sets allow for the representation of truth (T), indeterminacy (I), and falsity (F) as independent components. These components are not necessarily complementary, allowing for a more flexible representation of uncertain, inconsistent, and imprecise information. In mathematical terms, a single valued neutrosophic set $\mathsf{N}$ in a universe $\mathsf{X}$ is characterized by a truth, indeterminacy, falsity membership function for each element in the universe. The values of these functions range independently within the interval $[0, 1]$.

Neutrosophic filters extend the notion of filters in classical set theory and fuzzy set theory to the neutrosophic context. R. Lowen introduced the concept of prefilters in fuzzy set theory in his paper [3]. Also discussed the characterisation of maximal prefilter. In addition he defined the convergence in fuzzy topological spaces. A neutrosophic filter is a special type of neutrosophic set that satisfies certain properties in relation to neutrosophic operations and is used to generalize classical notions of ideal and prime filters.

In this paper we defined prefilter on $\mathsf{X}$ which is the collection of neutrosophic sets. Characteristic set and characteristic value of prefilters are defined and their properties are analysed. The filters on $\mathsf{X}$ are related to prefilters on $\mathsf{X}$ and ultrafilters are related to maximal filters through $i_K$ and $w_k$ mappings. In addition the prime prefilters are defined and basic properties related to prime prefilters are analysed. Also the compatibility between the filters and prefilters on $\mathsf{X}$ are analysed.

## 2. Preliminaries

In this section basic definitions related to neutrosophic sets and filters are provided.

**Definition 2.1.** [2] A neutrosophic set $\mathsf{N}$ for an universe $\mathsf{X}$ is defined as $\mathsf{N} = \{\langle x, T_{\mathsf{N}}(x), I_{\mathsf{N}}(x), F_{\mathsf{N}}(x)\rangle : x \in \mathsf{X}\}$, where $T_{\mathsf{N}}$, $I_{\mathsf{N}}$, $F_{\mathsf{N}}$ denotes the truth, indeterminacy and falsity membership functions respectively from $\mathsf{X}$ to $(0^-, 1^+)$ such that $0^- \leq \sup T_{\mathsf{N}}(x) + \sup I_{\mathsf{N}}(x) + \sup F_{\mathsf{N}}(x) \leq 3^+$. The set of all neutrosophic set over $\mathsf{X}$ is denoted by $\mathcal{N}(\mathsf{X})$.

**Definition 2.2.** [5] A single valued neutrosophic set is a neutrosophic set in which the truth, indeterminacy and falsity membership functions are $T_{\mathsf{N}}, I_{\mathsf{N}}, F_{\mathsf{N}}$ respectively, and they are from $\mathsf{X}$ to $[0,1]$.

**Definition 2.3.** [5] Let $\mathsf{N}_1, \mathsf{N}_2 \in \mathcal{N}(\mathsf{X})$. Then

$$\mathsf{N}_1 \cup \mathsf{N}_2 = \{\langle x, T_{\mathsf{N}_1}(x) \wedge T_{\mathsf{N}_2}(x), I_{\mathsf{N}_1}(x) \wedge I_{\mathsf{N}_2}(x), F_{\mathsf{N}_1}(x) \vee F_{\mathsf{N}_2}(x)\rangle : x \in \mathsf{X}\},$$

$$\mathsf{N}_1 \cap \mathsf{N}_2 = \{\langle x, T_{\mathsf{N}_1}(x) \vee T_{\mathsf{N}_2}(x), I_{\mathsf{N}_1}(x) \vee I_{\mathsf{N}_2}(x), F_{\mathsf{N}_1}(x) \wedge F_{\mathsf{N}_2}(x)\rangle : x \in \mathsf{X}\}$$

If $T_{\mathsf{N}_1}(x) \leq T_{\mathsf{N}_2}(x), I_{\mathsf{N}_1}(x) \leq I_{\mathsf{N}_2}(x), F_{\mathsf{N}_1}(x) \geq F_{\mathsf{N}_2}(x)$ for all $x \in \mathsf{X}$, then $\mathsf{N}_1$ is a neutrosophic subset of $\mathsf{N}_2$ and is denoted by $\mathsf{N}_1 \subseteq \mathsf{N}_2$

**Definition 2.4.** [4] If $T_{\mathsf{N}}(x) = 1, I_{\mathsf{N}}(x) = 0, F_{\mathsf{N}}(x) = 0$ for all $x \in \mathsf{X}$ Then $\mathsf{N}$ is the neutrosophic universal set and is denoted by $1_{\mathsf{N}}$

**Definition 2.5.** [4] If $T_{\mathsf{N}}(x) = 0, I_{\mathsf{N}}(x) = 1, F_{\mathsf{N}}(x) = 1$ for all $x \in \mathsf{X}$ Then $\mathsf{N}$ is the neutrosophic empty set and is denoted by $0_{\mathsf{N}}$

**Definition 2.6.** [1] A filter $\mathscr{F}$ on a set $\mathsf{X}$ is the subsets of $\mathsf{X}$ such that

(1) $A \in \mathscr{F}$, for all $B \subset X$, $A \subset B$ then $B \in \mathscr{F}$.
(2) $A, B \in \mathscr{F}$ then $A \cap B \in \mathscr{F}$.
(3) $\emptyset \notin \mathscr{F}$.

**Definition 2.7.** [1] An ultrafilter $\mathscr{U}$ on a set $\mathsf{X}$ is a filter such that there is no filter on $\mathsf{X}$ which is strictly finer than $\mathscr{F}$.

## 3. Characteristic set of a prefilter

This section focuses on the concept of a prefilter, including its characteristic set, value, and properties. The relationship between a filter on $\mathsf{X}$ and a prefilter on $\mathsf{X}$ is explored, and their respective properties are analyzed. Throughout the paper neutrosophic set is a single valued neutrosophic set.

**Definition 3.1.** A subset $\mathfrak{F}$ of $\mathcal{N}(\mathsf{X})$ is a prefilter iff $\mathfrak{F} \neq \emptyset$ such that

(1) for all $\mathsf{N}_1$, $\mathsf{N}_2 \in \mathfrak{F}$, $\mathsf{N}_1 \cap \mathsf{N}_2 \in \mathfrak{F}$
(2) if $\mathsf{N}_2 \supset \mathsf{N}_1$ and $\mathsf{N}_1 \in \mathfrak{F}$ then $\mathsf{N}_2 \in \mathfrak{F}$
(3) $0_\mathsf{N} \notin \mathfrak{F}$.

**Definition 3.2.** A subset $\mathcal{G}$ of $\mathcal{N}(\mathsf{X})$ is a base for a prefilter iff $\mathcal{G} \neq \emptyset$

(1) for all $\mathsf{N}_1$, $\mathsf{N}_2 \in \mathcal{G}$ there exists $\mathsf{N}_3 \in \mathcal{G}$ such that $\mathsf{N}_3 \subseteq \mathsf{N}_1 \cap \mathsf{N}_2$.
(2) $0_\mathsf{N} \notin \mathcal{G}$.

The prefilter $\mathfrak{F}$ generated by $\mathcal{G}$ is defined as $\mathfrak{F} = \{\mathsf{N}_1 : \exists\ \mathsf{N}_2 \in \mathcal{G} \ni \mathsf{N}_1 \supseteq \mathsf{N}_2\}$ and is denoted by $\langle \mathcal{G} \rangle$.

A subset $\mathcal{G}$ of $\mathfrak{F}$ is a base for $\mathfrak{F} \iff$ for all $\mathsf{N}_1 \in \mathfrak{F}$ there exists $\mathsf{N}_2 \in \mathcal{G}$ such that $\mathsf{N}_1 \supseteq \mathsf{N}_2$.

A subset $\mathcal{H}$ of $\mathcal{N}(\mathsf{X})$ is called a generating family or subbase for a prefilter iff the family of finite lower bounds of members of $\mathcal{H}$ is a base for this prefilter. The prefilter generated by a subbase $\mathcal{H}$ is denoted by $\langle \mathcal{H} \rangle$.

**Definition 3.3.** A prefilter $\mathfrak{F}$ is called a prime filter iff for any $\mathsf{N}_1, \mathsf{N}_2 \in \mathcal{N}(\mathsf{X})$ such that $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathfrak{F}$, then either $\mathsf{N}_1 \in \mathfrak{F}$ or $\mathsf{N}_2 \in \mathfrak{F}$.

For $\mathsf{N} \in \mathcal{N}(\mathsf{X})$, $\dot{\mathsf{N}}$ is a prefilter generated by the neutrosophic set $\mathsf{N}$ (i.e.,)

$$\dot{\mathsf{N}} = \{\mathsf{N}' : \mathsf{N}' \supseteq \mathsf{N}\}.$$

For $\mathsf{N} \in \mathcal{N}(\mathsf{X})$ and characteristic value $a = (a_1, a_2, a_3)$

$\mathsf{N}_a = \langle T_{\mathsf{N}_a}(x),\ I_{\mathsf{N}_a}(x),\ F_{\mathsf{N}_a}(x) \rangle \ \ x, x \in \mathsf{X}$ where $T_{\mathsf{N}_a}(x) = T_\mathsf{N}(x) + a_1$ $I_{\mathsf{N}_a}(x) = I_\mathsf{N}(x) + a_2$, $F_{\mathsf{N}_a}(x) = F_\mathsf{N}(x) + a_3$. Similarly

$$\begin{aligned} \mathsf{N}_{\dot{a}} \cap 1_\mathsf{N} = \{\mathsf{N}_b \cap 1_\mathsf{N} |\ & T_{\mathsf{N}_b}(x) \geq T_\mathsf{N}(x) + a_1, \\ & I_{\mathsf{N}_b}(x) \geq I_\mathsf{N}(x) + a_2, \\ & F_{\mathsf{N}_b}(x) \leq F_\mathsf{N}(x) + a_3\} \end{aligned}$$

Let $\mathfrak{F}$ is a prefilter and $\mathsf{N}$ be a neutrosophic set

$$\mathcal{C}^{\mathsf{N}}(\mathfrak{F}) = \{a = (a_1, a_2, a_3) : \forall\ \mathsf{N}' \in \mathfrak{F}, \exists\ x \in \mathsf{X}\ \ni \\ T_{\mathsf{N}'}(x) > T_{\mathsf{N}}(x) + a_1, \\ I_{\mathsf{N}'}(x) > I_{\mathsf{N}}(x) + a_2, \\ F_{\mathsf{N}'}(x) < F_{\mathsf{N}}(x) + a_3\}$$

and this subset is called a characteristic set of $\mathfrak{F}$ with respect to $\mathsf{N}$. The characteristic value $c^{\mathsf{N}}(\mathfrak{F})$ of $\mathfrak{F}$ with respect to $\mathsf{N}$ is $(\sup a_1, \sup a_2, \inf a_3)$.
Let a set $A = \{a_1, a_2, a_3\}$ is a characteristic set $\iff$ either $A = \emptyset$ or for $a_i, i \in 1, 2, 3$ is one of the following set $\{0\}, [0, c)$ for $c \in I, [0, c]$ for $c \in I \setminus \{1\}$.

$$\mathbb{W}(\mathsf{X}) = \text{set of all prefilters.}$$
$$\mathbb{W}^{\mathsf{N}}(\mathsf{X}) = \{\mathfrak{F} \in \mathbb{W}(\mathsf{X}) : \mathcal{C}^{\mathsf{N}}(\mathfrak{F}) \neq \emptyset.\}$$
$$\mathbb{W}_{+}^{\mathsf{N}}(\mathsf{X}) = \{\mathfrak{F} \in \mathbb{W}(\mathsf{X}) : c^{\mathsf{N}}(\mathfrak{F}) > 0.\}$$
$$\mathbb{W}_{K}^{\mathsf{N}}(\mathsf{X}) = \{\mathfrak{F} \in \mathbb{W}(\mathsf{X}) : \mathcal{C}^{\mathsf{N}}(\mathfrak{F}) = K\}$$

where $K$ is some non empty characteristic set. A member of $\mathbb{W}_{K}^{\mathsf{N}}(\mathsf{X})$ will be called $K-$prefilter.

**Proposition 3.4.** *Let $\mathsf{N}$ be a neutrosophic set.*

(1) *If $\mathcal{L}$ is a family of prefilters, then* $\mathcal{C}^{\mathsf{N}}(\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F}) = \underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathcal{C}^{\mathsf{N}}(\mathfrak{F})$
(2) *If $\mathcal{L}$ is a family of prefilters such that $\underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathfrak{F}$ is a subbase for some prefilter then* $\mathcal{C}^{\mathsf{N}}(\langle\underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathfrak{F}\rangle) = \underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathcal{C}^{\mathsf{N}}(\mathfrak{F})$

*Proof.* 1. For all $\mathcal{G} \in \mathcal{L}$, $\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F} \subset \mathcal{G}$

$$\mathcal{C}^{\mathsf{N}}(\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F}) \supset \underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathcal{C}^{\mathsf{N}}(\mathfrak{F}).$$

Suppose conversely $a \in I \times I \times I \setminus \underset{\mathfrak{F}\in\mathcal{L}}{\cup}\mathcal{C}^{\mathsf{N}}(\mathfrak{F})$ then for all $\mathfrak{F} \in \mathcal{L}$ there exists $\mathsf{N}' \in \mathfrak{F}$ such that

$$T_{\mathsf{N}'}(x) \leq \min\{T_{\mathsf{N}}(x) + a_1, 1\},$$
$$I_{\mathsf{N}'}(x) \leq \min\{I_{\mathsf{N}}(x) + a_2, 1\},$$
$$F_{\mathsf{N}'}(x) \geq \max\{F_{\mathsf{N}}(x) + a_3, 0\}$$

and consequently for all $\mathfrak{F} \in \mathcal{L}$ such that $\mathsf{N}_{a} \cap 1_{\mathsf{N}} \in \mathfrak{F}$ or $\mathfrak{F} \in \mathcal{L}$ such that $\mathsf{N}_{\dot{a}} \cap 1_{\mathsf{N}} \subset \underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F}$. Since trivially $\mathcal{C}^{\mathsf{N}}(\mathsf{N}_{\dot{a}} \cap 1_{\mathsf{N}}) \subset ([0, a_1), [0, a_2), (a_3, 1])$, thus $a \notin \mathcal{C}^{\mathsf{N}}(\underset{\mathfrak{F}\in\mathcal{L}}{\cap}\mathfrak{F})$.
2. The proof is similar to statement 1. □

This section mainly deals with the characteristic set and value of prefilter with respect to $0_{\mathsf{N}}$.

$\mathbb{F}(\mathsf{X})$ denotes the set of all filters on $\mathsf{X}$. Let $K$ be a non empty characteristic set, then the following mappings can be defined

$$w_K : \mathbb{F}(\mathsf{X}) \to \mathbb{W}_K(\mathsf{X})$$
$$\mathscr{F} \to \{\mathsf{N} : \forall\ k = (k_1, k_2, k_3) \in K, (T_{\mathsf{N}}^{-1}(k_1, 1], I_{\mathsf{N}}^{-1}(k_2, 1], F_{\mathsf{N}}^{-1}[0, k_3)) \in \mathscr{F}\}$$
$$i_K : \bigcup_{K \subset K'} \mathbb{W}_{K'}(\mathsf{X}) \to \mathbb{F}(\mathsf{X})$$
$$\mathfrak{F} \to \{(T_{\mathsf{N}}^{-1}(k_1, 1], I_{\mathsf{N}}^{-1}(k_2, 1], F_{\mathsf{N}}^{-1}[0, k_3)) : \forall\ \mathsf{N} \in \mathfrak{F}, k = (k_1, k_2, k_3) \in K\}$$

**Proposition 3.5.** *Let $K, K^{'}$ be two characteristic sets such that $K^{'} \subset K$, then for all $\mathscr{F} \in \mathbb{F}(\mathsf{X})$*

(i) $w_K(\mathscr{F}) \subset w_{K'}(\mathscr{F})$
(ii) $i_{K'} \circ w_K(\mathscr{F}) = \mathscr{F}$

*and for all* $\mathfrak{F} \in \bigcup_{K'' \supset K} \mathbb{W}_{K''}$,

(iii) $i_K(\mathfrak{F}) \supset i_{K'}(\mathfrak{F})$
(iv) $w_{K'} \circ i_K(\mathfrak{F}) \supset \mathfrak{F}$.

*Furthermore the mappings $w_K$ and $i_K$ are order preserving.*

*Proof.* Given $K^{'} \subset K$, $w_K(\mathscr{F}) = \{\mathsf{N} : \forall\ k \in K(T_{\mathsf{N}}^{-1}(k_1, 1], I_{\mathsf{N}}^{-1}(k_2, 1], F_{\mathsf{N}}^{-1}[0, k_3)) \in \mathscr{F}\}$. let $\mathsf{N} \in w_K(\mathscr{F}) \implies \forall\ k \in K(T_{\mathsf{N}}^{-1}(k_1, 1], I_{\mathsf{N}}^{-1}(k_2, 1], F_{\mathsf{N}}^{-1}[0, k_3)) \in \mathscr{F}$ Since $K^{'} \subset K$, $\forall\ k \in K^{'}(T_{\mathsf{N}}^{-1}(k_1, 1], I_{\mathsf{N}}^{-1}(k_2, 1], F_{\mathsf{N}}^{-1}[0, k_3)) \in \mathscr{F} \implies \mathsf{N} \in w_K^{'}(\mathscr{F})$ From the definition of $w_K$ and $i_{K'}$ its clear that $i_{K'} \circ w_K(\mathscr{F}) = \mathscr{F}$. □

As $\mathbb{W}_K^{\mathsf{N}}(\mathsf{X})$ is inductive, for any neutrosophic set $\mathsf{N}$. Then by Zorn's lemma there exist maximal elements, which is called as maximal $K$-prefilters. Now if $\mathfrak{F}$ is a maximal $K-$prefilter then since $w_{K'} \circ i_K(\mathfrak{F})$ is also a $K-$prefilter it follows from Proposition 3.5(iv) that $\mathfrak{F} = w_{K'} \circ i_K(\mathfrak{F})$.

**Definition 3.6.** For a subset $A$ of $\mathsf{X}$, the characteristic function $\chi_A^{\mathsf{N}} : \mathsf{X} \to I \times I \times I$ is a neutrosophic set in $\mathsf{X}$ such that $\chi_A^{\mathsf{N}}(x) = \langle T_{\mathsf{N}}^A(x), I_{\mathsf{N}}^A(x), F_{\mathsf{N}}^A(x) \rangle$, where

$$T_{\mathsf{N}}^A : \mathsf{X} \to [0, 1] \text{ such that}$$
$$T_{\mathsf{N}}^A(x) = \begin{cases} 1\ \forall\ x \in A \\ 0\ \forall\ x \in A^c \end{cases}$$

$$I_{\mathsf{N}}^{A} : \mathsf{X} \to [0,1] \text{ such that}$$
$$I_{\mathsf{N}}^{A}(x) = 1 \ \forall\ x \in \mathsf{X}$$
$$F_{\mathsf{N}}^{A} : \mathsf{X} \to [0,1] \text{ such that}$$
$$F_{\mathsf{N}}^{A}(x) = \begin{cases} 0 \ \forall\ x \in A \\ 1 \ \forall\ x \in A^{c} \end{cases}$$

**Theorem 3.7.** *If $\mathscr{U}$ is an ultrafilter on $\mathsf{X}$, then $w_K(\mathscr{U})$ is a maximal $K-$prefilter, and if $\mathfrak{F}$ is a prime prefilter then for all characteristic sets $K \subset \mathcal{C}(\mathfrak{F})$, $i_K(\mathfrak{F})$ is an ultrafilter on $\mathsf{X}$.*

*Proof.* Let $\mathscr{U}$ be an ultrafilter on $\mathsf{X}$ and let $\mathfrak{F} \in \mathbb{W}_K(\mathsf{X})$ be a maximal $K-$prefilter, finer than $w_K(\mathscr{U})$. Then $i_K(\mathfrak{F}) \supset i_K \circ w_K(\mathscr{U}) = \mathscr{U}$ and thus $i_K(\mathfrak{F}) = \mathscr{U}$. Since $\mathfrak{F}$ is maximal, $\mathfrak{F} = w_K \circ i_K(\mathfrak{F}) = w_K(\mathscr{U})$.

Let $\mathfrak{F}$ be a prime prefilter and let $A \subset \mathsf{X}$. Then $\chi_A^{\mathsf{N}} \cup \chi_{A^c}^{\mathsf{N}} = 1_{\mathsf{N}} \in \mathfrak{F}$ and $\chi_A^{\mathsf{N}} \in \mathfrak{F}$. Since always $0_{\mathsf{N}} \in K$, $A = ([T_{\mathsf{N}}^{A}]^{-1}(0,1],\ [I_{\mathsf{N}}^{A}]^{-1}(0,1],\ [F_{\mathsf{N}}^{A}]^{-1}[0,1)) \in i_K(\mathfrak{F})$ which shows that $i_K(\mathfrak{F})$ is an ultrafilter on $\mathsf{X}$. □

**Theorem 3.8.** *If $\mathfrak{F}$ is a maximal $K-$filter, then it is a prime prefilter.*

*Proof.* If $\mathfrak{F} \in \mathbb{W}_K(\mathsf{X})$ is a maximal then $i_K(\mathfrak{F})$ is ultrafilter. Let $\mathscr{U}$ be an ultrafilter on $\mathsf{X}$ finer than $i_K(\mathfrak{F})$ then $w_K((U)) \supset w_K \circ i_K(\mathfrak{F}) = \mathfrak{F}$ and since $w_K((U)) = \mathfrak{F}$. This in turn implies that $(U) = i_K \circ w_K((U)) = i_K(\mathfrak{F})$. Let $\mathsf{N}_1, \mathsf{N}_2$ be a neutrosophic set such that $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathfrak{F}$, then for all $k \in K$, $(T^{-1}_{\mathsf{N}_1 \cup \mathsf{N}_2}(k_1, 1], I^{-1}_{\mathsf{N}_1 \cup \mathsf{N}_2}(k_2, 1], F^{-1}_{\mathsf{N}_1 \cup \mathsf{N}_2}[0, k_3)) \in i_K(\mathfrak{F})$.

If $K$ is of the form $([0, k_{0_1}], [0, k_{0_2}], [k_{0_3}, 1])$ it suffices to remark that for instance $(T^{-1}_{\mathsf{N}_1}(k_{0_1}, 1], I^{-1}_{\mathsf{N}_1}(k_{0_2}, 1], F^{-1}_{\mathsf{N}_1}[0, k_{0_3})) \in i_K(\mathfrak{F})$ and since for all $k = (k_1, k_2, k_3)$, $k_0 = (k_{0_1}, k_{0_2}, k_{0_3})$, such that $k_1 \le k_{0_1}$, $k_2 \le k_{0_2}$, $k_3 \ge k_{0_3}$, $(T^{-1}_{\mathsf{N}_1}(k_{0_1}, 1],\ I^{-1}_{\mathsf{N}_1}(k_{0_2}, 1],\ F^{-1}_{\mathsf{N}_1}[0, k_{0_3})) \subset (T^{-1}_{\mathsf{N}_1}(k_1, 1],\ I^{-1}_{\mathsf{N}_1}(k_2, 1],\ F^{-1}_{\mathsf{N}_1}[0, k_3))$, thus $(T^{-1}_{\mathsf{N}_1}(k_1, 1],\ I^{-1}_{\mathsf{N}_1}(k_2, 1],\ F^{-1}_{\mathsf{N}_1}[0, k_3)) \in i_K(\mathfrak{F})$ for all $k \in K$.

If $K$ is of the form $([0, k_{0_1}),\ [0, k_{0_2}),\ (k_{0_3}, 1])$, choose a sequence $(k_{n_1})_{n_1 \in \mathbb{N}}$ of increasing numbers in $[0, k_{0_1})$, $(k_{n_2})_{n_2 \in \mathbb{N}}$ of increasing numbers in $[0, k_{0_2})$, $(k_{n_3})_{n_3 \in \mathbb{N}}$ of decreasing numbers in $(k_{0_3}, 1]$, such that $\sup_{n_1 \in \mathbb{N}} k_{n_1} = k_{0_1}$, $\sup_{n_2 \in \mathbb{N}} k_{n_2} = k_{0_2}$, $\inf_{n_3 \in \mathbb{N}} k_{n_3} = k_{0_3}$. Then for all $n_i \in \mathbb{N}$ either $(T^{-1}_{\mathsf{N}_1}(k_{n_1}, 1], I^{-1}_{\mathsf{N}_1}(k_{n_2}, 1], F^{-1}_{\mathsf{N}_1}[0, k_{n_3})) \in i_K(\mathfrak{F})$ or $(T^{-1}_{\mathsf{N}_2}(k_{n_1}, 1], I^{-1}_{\mathsf{N}_2}(k_{n_2}, 1], F^{-1}_{\mathsf{N}_2}[0, k_{n_3})) \in i_K(\mathfrak{F})$. Thus there exists a subsequence $(k'_{n_1})_{n_1 \in \mathbb{N}}$ of $(k_{n_1})_{n_1 \in \mathbb{N}}$, $(k'_{n_2})_{n_2 \in \mathbb{N}}$ of $(k_{n_2})_{n_2 \in \mathbb{N}}$, $(k'_{n_3})_{n_3 \in \mathbb{N}}$ of $(k_{n_3})_{n_3 \in \mathbb{N}}$ such that for all $n_i \in \mathbb{N}$, $(T^{-1}_{\mathsf{N}_1}(k'_{n_1}, 1], I^{-1}_{\mathsf{N}_1}(k'_{n_2}, 1], F^{-1}_{\mathsf{N}_1}[0, k'_{n_3})) \in i_K(\mathfrak{F})$.

For any $k = (k_1, k_2, k_3), \in K$ choose $n_i \in \mathbb{N}$ such that $k_1 \le k'_{n_1}, k_2 \le k'_{n_2}, k_3 \le k'_{n_3}$, $(T^{-1}_{\mathsf{N}_1}(k'_{n_1}, 1],\ I^{-1}_{\mathsf{N}_1}(k'_{n_2}, 1],\ F^{-1}_{\mathsf{N}_1}[0, k_{n'_3})) \subset (T^{-1}_{\mathsf{N}_1}(k_1, 1],\ I^{-1}_{\mathsf{N}_1}(k_2, 1],\ F^{-1}_{\mathsf{N}_1}[0, k_3))$, thus $(T^{-1}_{\mathsf{N}_1}(k_1, 1],\ I^{-1}_{\mathsf{N}_1}(k_2, 1], F^{-1}_{\mathsf{N}_1}[0, k_3)) \in i_K(\mathfrak{F})$. Consequently in either case $\mathsf{N}_1 \in w_K \circ i_K(\mathfrak{F}) = \mathfrak{F}$ which proves that $\mathfrak{F}$ is a prime prefilter. □

**Corollary 3.9.** *If $\mathscr{U}$ is an ultrafilter on* $\mathsf{X}$, *then for all nonempty characteristic sets* $K$, $w_K(\mathscr{U})$ *is a prime filter.*

## 4. Compatibility of filter and prefilter

In analogy with filters on $\mathsf{X}$ one might think that every $K-$filter is equal to the intersection of the family of maximal $K-$prefilters which are finer. Yet this is not the case as is shown by the following counterexample. Let $\mathsf{X}$ be arbitrary and let

$$\mathscr{R}(\mathfrak{F}) = \{\mathfrak{P} : \mathfrak{P} \text{ maximal K-prefilter finer than } \mathfrak{F}\}$$

$$\mathscr{R}(\mathcal{G}) = \{\mathfrak{P} : \mathfrak{P} \text{ maximal K-prefilter finer than } \mathcal{G}\}$$

Clearly $\mathscr{R}(\mathfrak{F}) \supset \mathscr{R}(\mathcal{G})$. But if $\mathfrak{P} \in \mathscr{R}(\mathfrak{F})$, then $\mathfrak{P} = w_K \circ i_K(\mathfrak{P}) \supset w_K \circ i_K(\mathfrak{F}) = \mathcal{G}$ so that $\mathfrak{P} \in \mathscr{R}(\mathcal{G})$. Consequently $\mathscr{R}(\mathfrak{F}) = \mathscr{R}(\mathcal{G})$.
Let $\mathscr{P}(\mathfrak{F})$ be the set of all prime prefilters finer than $\mathfrak{F} \in \mathbb{W}(\mathsf{X}.)$ Hence

$$\mathfrak{F} = \bigcap_{\mathcal{G} \in \mathscr{P}(\mathfrak{F})} \mathcal{G}.$$

But as we shall see further on $\mathscr{P}(\mathfrak{F})$ is too large a set for our purposes. The following proposition shows that we can extract a subset of $\mathscr{P}(\mathfrak{F})$ which still contains all the relevant information.

**Proposition 4.1.** *The set $\mathscr{P}(\mathfrak{F})$ is inductive in the sense that every descending chain of prefilters in it has a lower bound.*

*Proof.* Let $\mathscr{R} \subset \mathscr{P}(\mathfrak{F})$ be a descending chain. Consider $\mathcal{G}_0 = \underset{\mathcal{G} \in \mathscr{R}}{\cap} \mathcal{G}$. Then clearly $\mathcal{G}_0 \supset \mathfrak{F}$. Next, if $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathcal{G}_0$, then either for all $\mathcal{G} \in \mathscr{R}$ we have $\mathsf{N}_1 \in \mathcal{G}$ and thus $\mathsf{N}_1 \in \mathcal{G}_0$ and we are done, or there exists some $\mathcal{G} \in \mathscr{R}$ such that $\mathsf{N}_1 \notin \mathcal{G}$. Then since $\mathsf{N}_1 \cup \mathsf{N}_2 \in \mathcal{G}$ the latter implies that $\mathsf{N}_2 \in \mathfrak{P}$ for all $\mathfrak{P} \in \mathscr{R}$ such that $\mathfrak{P} \supset \mathcal{G}$. And if $\mathfrak{P} \in \mathscr{R}$ such that $\mathfrak{P}$, then since $\mathsf{N}_2 \notin \mathfrak{P}$ we must again have $\mathsf{N}_2 \in \mathfrak{P}$. Thus $\mathsf{N}_2 \in \mathcal{G}_0$. □

It now follows from Zorn's theorem that there exist minimal elements in $\mathscr{P}(\mathfrak{F})$. We denote the family of minimal elements in $\mathscr{P}(\mathfrak{F})$ by $\mathscr{P}_m(\mathfrak{F})$. It then also follows at once that we still have

$$\mathfrak{F} = \bigcap_{\mathcal{G} \in \mathscr{P}_m(\mathfrak{F})} \mathcal{G}.$$

In order to characterize these minimal prime prefilters in $\mathscr{P}_m(\mathfrak{F})$ in a more tangible way we need the following concept.

**Definition 4.2.** A filter $\mathscr{F}$ on $\mathsf{X}$ and a prefilter $\mathfrak{F}$ are said to be compatible iff for all $F' \in \mathscr{F}$ and $\mathsf{N} \in \mathfrak{F}, \mathsf{N}$ does not vanish everywhere on $F'$.
We shall use the notation

$$\mathsf{N}_{F'} : \mathsf{X} \to I \times I \times I : x \to (T_{\mathsf{N}}(x), I_{\mathsf{N}}(x), F_{\mathsf{N}}(x)) \text{ if } x \in F'$$
$$\to 0_{\mathsf{N}} \text{ if } x \notin F'$$

Then since for all $\mathsf{N}, \mathsf{N}' \in \mathfrak{F}$ and $F', G' \in \mathscr{F}$, $\mathsf{N}_{F'}, \mathsf{N}'_{G'} = (\mathsf{N} \cap \mathsf{N}')_{F' \cap G'}$ it is clear that if $\mathscr{F}$ and $\mathfrak{F}$ are compatible then

$$(\mathfrak{F}, \mathscr{F}) = \langle \{\mathsf{N}_{F'} : \mathsf{N} \in \mathfrak{F}, F' \in \mathscr{F}\} \rangle$$

is a prefilter.

**Theorem 4.3.** *Let $\mathfrak{F}$ be a prefilter. Then*

$$\mathscr{P}_m(\mathfrak{F}) = \{(\mathfrak{F}, \mathscr{U}) : \mathscr{U} \textit{ ultrafilter on } \mathsf{X} \textit{ and compatible with } \mathfrak{F}\}$$

*Proof.* Let $\mathscr{U}$ be an ultrafilter on $\mathsf{X}$ compatible with $\mathfrak{F}$. To show that $(\mathfrak{F}, \mathscr{U})$ is prime let $\mathsf{N}_1, \mathsf{N}_2$ be neutrosophic sets such that $\mathsf{N}_1 \cup \mathsf{N}_2 \in (\mathfrak{F}, \mathscr{U})$ then there exist $\mathsf{N} \in \mathfrak{F}$ and $U \in \mathscr{U}$ such that $\mathsf{N}_1 \cup \mathsf{N}_2 \geq \mathsf{N}_U$. Let then $A = \{x : T_{\mathsf{N}_1}(x) \geq T_{\mathsf{N}_U}(x), I_{\mathsf{N}_1}(x) \geq I_{\mathsf{N}_U}(x), F_{\mathsf{N}_1}(x) \leq F_{\mathsf{N}_U}(x)\}$ and $B = \{x : T_{\mathsf{N}_2}(x) \geq T_{\mathsf{N}_U}(x), I_{\mathsf{N}_2}(x) \geq I_{\mathsf{N}_U}(x), F_{\mathsf{N}_2}(x) \leq F_{\mathsf{N}_U}(x)\}$. Since $A \cup B = \mathsf{X}$, $A \in \mathscr{U}$. Then since $\mathsf{N}_1 \supseteq \mathsf{N}'_{A \cap U}$ this implies that $\mathsf{N}_1 \in (\mathfrak{F}, \mathscr{U})$. Consequently $(\mathfrak{F}, \mathscr{U})$ is prime. To show that it is minimal let $\mathcal{G} \in \mathscr{P}(\mathfrak{F})$ be such that $(\mathfrak{F}, \mathscr{U}) \supset \mathcal{G} \supset \mathfrak{F}$ and suppose that there exists $\mathsf{N} \in \mathfrak{F}$ and $U \in \mathscr{U}$ such that $\mathsf{N}_U \notin \mathcal{G}$.
Then since $\mathsf{N} \in \mathcal{G}$ and $\mathcal{G}$ is prime, $\mathsf{N}_{U^c} \in \mathcal{G}$ and thus $\mathsf{N}_{U^c} \in (\mathfrak{F}, \mathscr{U})$ which is impossible. Consequently $(\mathfrak{F}, \mathscr{U})$ is minimal and thus $(\mathfrak{F}, \mathscr{U}) \in \mathscr{P}_m(\mathfrak{F})$. To show the converse let $\mathcal{G} \in \mathscr{P}_m(\mathfrak{F})$. Consider the characteristic set $K = \mathcal{C}(\mathcal{G})$ of $\mathcal{G}$. It follows from Theorem 3.7 that $i_K(\mathcal{G})$ is an ultrafilter on $\mathsf{X}$. And it is trivial that $\mathfrak{F}$ is compatible with $i_K(\mathcal{G})$. Now since $\mathscr{P}_m(\mathcal{G}) = \{\mathcal{G}\}$ it follows from the first part of the theorem that $(\mathcal{G}, i_K(\mathcal{G})) = \mathcal{G}$. Since $\mathfrak{P} \subset \mathcal{G}$ it follows that

$$\mathfrak{F} \subset (\mathfrak{F}, i_K(\mathcal{G})) \subset (\mathcal{G}, i_K(\mathcal{G})) = \mathcal{G}.$$

Since $(\mathfrak{F}, i_K(\mathcal{G}))$ is prime, we have $\mathcal{G} = (\mathfrak{F}, i_K(\mathcal{G}))$ □

## References

[1] Bourbaki, N., General Topology. Hermann, Paris, 1965.

[2] Florentin Smarandache, Neutrosophy: Neutrosophic Probability, Set, and Logic. American Research Press, 1988.

[3] Robert Lowen, Convergence in fuzzy topological spaces. General topology and its applications, Volume 10(2), 1979, pp. 144–160.

[4] Serkan Karatas, Cemil Kuru, Neutrosophic topology. Neutrosophic sets and systems, 2016, Volume 13(1), pp. 90–95.

[5] Yan-Qing, Haibin Wang Florentin Smarandache and Sunderraman, Zhang Rajshekhar, Single Valued Neutrosophic Sets, Infinite study, 2010.

Received: June 27, 2024. Accepted: August 22, 2024