Determinant Theory for Neutrosophic Hypersoft Rough Fuzzy Matrices

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Abstract: In this study, we examine the determinant theory associated with Neutrosophic Hypersoft Rough Fuzzy Matrices (NHSRFMs) and investigate several of their structural properties. An improved approach for computing the determinant is introduced, specifically designed to handle matrices of larger dimensions with greater efficiency. Beyond the theoretical development, the proposed method demonstrates significant potential for integration into software systems dealing with uncertain, vague, or multidimensional data. To highlight its practicality, we provide a detailed illustrative example along with an explanation of how the approach can enhance computational accuracy, reduce processing time, and improve decision-support capabilities in software applications.

Keywords: Neutrosophic Hyper soft Rough Fuzzy Matrices, Adjoint, Determinant.

1. Introduction

The representation and manipulation of uncertainty have been core to advancements in soft computing, especially through foundational theories like fuzzy sets by Zadeh [1] and rough sets by Pawlak [2]. These frameworks were designed to handle vagueness and indiscernibility, respectively, and have laid the groundwork for numerous hybrid models. Over the years, the evolving complexity of data has necessitated more expressive structures, leading to the development of intuitionistic fuzzy sets [6] and generalized fuzzy matrices [7, 8, 9]. Building on these paradigms, neutrosophic sets, introduced by Smarandache [25], generalize fuzzy and intuitionistic fuzzy sets by incorporating the degrees of truth, indeterminacy, and falsity independently. This has significantly enriched the capacity to model uncertainty in systems that are imprecise, incomplete, or even contradictory. The application of neutrosophic logic in matrix theory led to the formulation of neutrosophic fuzzy matrices, which are essential in decision-making, pattern recognition, and artificial intelligence.

Recent contributions have explored various algebraic properties and generalizations of these matrices. For example, Anandhkumar et al. [3, 4, 5] introduced novel operations such as pseudo similarity, inverses, and partial orderings for neutrosophic fuzzy matrices. These concepts have been crucial in defining matrix-based frameworks for handling complex data relations. Furthermore, works on k-idempotent and kernel-symmetric matrices [10, 11, 19] have extended the applicability of neutrosophic matrices in algebraic systems. Parallelly, determinant theories for fuzzy and intuitionistic fuzzy matrices have been studied extensively [8, 9, 13–15, 17, 18, 21–23], paving the way for determinant theories in neutrosophic soft matrices [24]. These contributions are instrumental in enabling more rigorous mathematical operations on uncertain data structures.

On another front, the integration of neutrosophic, soft, and rough set theories has created new hybrid frameworks. Al-Quran et al. [30] proposed a neutrosophic soft rough set model that enhances approximation under uncertainty. Similarly, Das et al. [31] and Kamacı [32] presented frameworks combining neutrosophic soft sets with rough and hypersoft structures, offering enriched semantics for real-world applications. Smarandache's [33] extension from soft sets to hypersoft and plithogenic hypersoft sets further broadens the spectrum for handling multi-attribute data with complex interdependencies. In particular, the work of Anandhkumar and collaborators [26–29] has systematically advanced the structure of secondary, interval-valued, and Fermatean neutrosophic fuzzy matrices. These contributions aim to capture multidimensional uncertainties and symmetries, laying the foundation for deeper algebraic investigations and practical implementations.

1.1 Abbrivations

IFM: Intuitionistic Fuzzy Matrices

IFSs: Intuitionistic Fuzzy Sets

NFM: Neutrosophic fuzzy matrices.

NHSRFM: Neutrosophic Hypersoft Rough Fuzzy Matrices

2. **The structure of this article is arranged as follows**: In Section 3 presents the objectives of the present work, laying the foundation for the study.. Section **4** highlights the **Comparative of NHSRFM model with the existing soft models**. Section **5** introduces the **Novelty of the work**, which formally defines **Neutrosophic Hyper Soft Rough Fuzzy Matrices** (NHSRFMs) and their mathematical structure. In Section **6** present **Preliminary** of the **Neutrosophic Hyper Soft Rough Fuzzy Matrices** In Section 7, relevant theorems and results are presented to establish the theoretical foundations of the model.

3. The objectives of the present work are given:

- **To develop determinant theory for Neutrosophic Hyper Soft Rough Fuzzy Matrices (NHSRFMs):** Establish fundamental principles and properties of determinants specific to NHSRFMs, focusing on unique characteristics within the neutrosophic fuzzy domain.
- **To investigate determinant relationships in NHSRFMs:** Prove key determinant relationships, such as $\det\left(Padj\left(P\right)\right)=\det\left(P\right)=\det\left(adj\left(P\right)P\right)$. thereby expanding mathematical understanding within NHSRFMs.

- **To propose an efficient method for computing determinants in larger NHSRFMs:** Introduce a novel technique for calculating determinants of NHSRFMs with high dimensionality, aimed at simplifying computation for matrices with more rows and columns.
- **To construct an algorithm for solving decision-making problems:** Develop a systematic approach leveraging NHSRFM properties to address complex decision-making scenarios, enhancing practical applications of NHSRFMs.
- **To validate the proposed methods with an illustrative example:** Demonstrate the effectiveness and applicability of the determinant theory, methods, and algorithm through a practical example, solidifying the proposed study's relevance.

4. Comparative of NHSRFM model with the existing soft models

Types of soft set	Uncertainty	Falsity	Hesitation	Indeterminacy	Indeterminacy is bifurcated	Indeterminacy is bifurcated and restricted
FSS [46]	✓	×	×	×	×	×
IVFSS [47]	✓	×	×	×	×	×
IFSS [48]	✓	✓	✓	×	×	×
IVIFSS [49]	✓	✓	✓	×	×	×
NSS [50]	✓	✓	×	✓	×	×
INSS [51]	✓	✓	×	✓	×	×
QNSS [52]	✓	✓	×	✓	✓	×
NHSRFM (Proposed)	✓	✓	×	✓	×	×

5. Novelty of the Work

Despite the significant progress in the study of fuzzy, intuitionistic fuzzy, and neutrosophic matrices, existing literature largely focuses on basic operations, inverse types, partial orderings, and determinant theories [3–5, 7–9, 14, 18, 21–24]. While hybrid models involving soft sets, rough sets, and neutrosophic frameworks have recently emerged [30–33], a formal integration of **quadri partitioning**, **interval-valued representation**, and **neutrosophic fuzzy logic** remains relatively unexplored. The novelty of this research lies in the **development and algebraic analysis of a new class of NHSRFM**, which introduces the following original contributions:

(i) **New Structural Paradigm**:
The proposed NHSRFM introduce a quadri-partitioning of the matrix structure, which allows separate modeling of membership, non-membership, indeterminacy, and hesitancy intervals. This is a significant extension over traditional and existing symmetric or k-symmetric neutrosophic fuzzy matrices [10, 11, 26].

(ii) **Integration with Hybrid Soft Set Models**:
While existing works such as [30–32] focus on neutrosophic soft sets and their hybridization with rough or hypersoft sets, our framework combines **neutrosophic fuzzy matrices** with **soft set-based parameterization**, facilitating applicability in complex decision-making environments.

(iii) **Generalization of Determinant and Ordering Concepts**:
The paper generalizes determinant concepts for the proposed matrices by extending works like [8, 9, 23, 24], and introduces new ordering relations suitable for NHSRFM which capture both structural and semantic dominance.

5.1 problem statement and motivation

In recent years, the study of uncertainty-based mathematical models has gained significant importance in computational mathematics, decision sciences, and artificial intelligence. Classical matrices and their determinant theories, while effective in crisp environments, often fail to address the challenges posed by uncertainty, vagueness, indeterminacy, and multi-parameterization in real-world problems. Existing frameworks based on fuzzy, rough, neutrosophic, and hypersoft sets provide partial solutions but remain insufficient when these characteristics coexist within large-dimensional data structures. Traditional determinant computation methods further encounter scalability and efficiency issues when applied to complex neutrosophic hypersoft rough fuzzy matrices (NHSRFMs).

This research is motivated by the urgent need for a unified and computationally feasible determinant theory tailored to NHSRFMs, capable of representing and processing higher-order uncertainty. By introducing an enhanced approach for determinant calculation, particularly designed for larger matrices, the study not only contributes to theoretical advancements in neutrosophic mathematics but also supports practical applications in software development. The proposed framework has the potential to improve computational accuracy, optimize processing efficiency, and strengthen decision-support capabilities in software systems dealing with uncertain and multidimensional datasets, thereby bridging the gap between abstract mathematical models and real-world computational applications.

5.2 Improved Flow of objectives, novelty, and theorems.

The main objective of this study is to establish a robust determinant theory for Neutrosophic Hypersoft Rough Fuzzy Matrices (NHSRFMs), particularly addressing the limitations of existing models in handling large-scale uncertain data. The novelty of the proposed approach lies in its ability to extend classical determinant computation to higher-dimensional neutrosophic environments while ensuring computational efficiency. This novelty is mathematically formalized through a series of theorems, which guarantee the stability and scalability of the method. Beyond their theoretical significance, these results have practical implications for software development, where improved determinant computation can enhance data processing accuracy, reduce computational costs, and strengthen decision-support capabilities under uncertainty.

*5.3 C*onsistency in mathematical notations

To maintain clarity and avoid confusion, consistency in mathematical notation is essential throughout this research. The truth, indeterminacy, and falsity components are uniformly denoted as T, I, and , respectively, and every neutrosophic element of the matrix is expressed in the form aij=(Tij,Iij,Fij). Subscripts are consistently used for row and column indices, while superscripts are reserved only for denoting levels or versions when necessary. The determinant of a Neutrosophic Hypersoft Rough Fuzzy Matrix is denoted det(A) or |A| and this notation is applied uniformly in all definitions, theorems, and examples. By standardizing the representation of sets, elements, and matrix operations, the proposed framework ensures readability, minimizes ambiguity, and facilitates seamless connections between the theoretical results and their practical applications.

6. Preliminary

Definition 6.1 (Addition) Let $P=\left[P_{ij}\right]$ *and* $Q=\left[Q_{ij}\right]$ be two NHSRFMs of same order, where

$\left[P_{ij}\right]=\left[\left(\underline{T}_{ij}^{P},\underline{I}_{ij}^{P},\underline{F}_{ij}^{P}\right);\left(\overline{T}_{ij}^{P},\overline{I}_{ij}^{P},\overline{F}_{ij}^{P}\right)\right]$ and $\left[Q_{ij}\right]=\left[\left(\underline{T}_{ij}^{Q},\underline{I}_{ij}^{Q},\underline{F}_{ij}^{Q}\right);\left(\overline{T}_{ij}^{Q},\overline{I}_{ij}^{Q},\overline{F}_{ij}^{Q}\right)\right]$

we define the addition in NHSRFMs as follows:

$$P+Q=\begin{bmatrix}\max\left(\underline{T}_{ij}^{P},\underline{T}_{ij}^{Q}\right),\max\left(\underline{I}_{ij}^{P},\underline{I}_{ij}^{Q}\right),\min\left(\underline{F}_{ij}^{P},\underline{F}_{ij}^{Q}\right);\max\left(\overline{T}_{ij}^{P},\overline{T}_{ij}^{Q}\right),\\ \max\left(\overline{I}_{ij}^{P},\overline{I}_{ij}^{Q}\right),\min\left(\overline{F}_{ij}^{P},\overline{F}_{ij}^{Q}\right)\end{bmatrix}$$

Definition 6.2 (Multiplication) Let $P=\left[P_{ij}\right]$ *and* $Q=\left[Q_{ij}\right]$ be two NHSRFMs of same order,

where and $\left[Q_{ij}\right]=\left[\left(\underline{T}_{ij}^{Q},\underline{I}_{ij}^{Q},\underline{F}_{ij}^{Q}\right);\left(\overline{T}_{ij}^{Q},\overline{I}_{ij}^{Q},\overline{F}_{ij}^{Q}\right)\right]$ we define the multiplication in NHSRFM as follows:

$$P.Q=\left[\min\left(\underline{T}_{ij}^{P},\underline{T}_{ij}^{Q}\right),\min\left(\underline{I}_{ij}^{P},\underline{I}_{ij}^{Q}\right),\max\left(\underline{F}_{ij}^{P},\underline{F}_{ij}^{Q}\right);\min\left(\overline{T}_{ij}^{P},\overline{T}_{ij}^{Q}\right),\min\left(\overline{I}_{ij}^{P},\overline{I}_{ij}^{Q}\right),\max\left(\overline{F}_{ij}^{P},\overline{F}_{ij}^{Q}\right)\right]$$

Definition: 6.3 Let $P=\left[\left(\underline{p}_{ij}{}^{T},\underline{p}_{ij}{}^{I},\underline{p}_{ij}{}^{F}\right);\left(\overline{p}_{ij}{}^{T},\overline{p}_{ij}{}^{I},\overline{p}_{ij}{}^{F}\right)\right]\in(NHSRFM)_{n}$ and let Q be a matrix from P by striking out row e1, row e2,… row ek and column r1, column r2,…, column rk. we define $P\begin{pmatrix} e_1 & e_2 & \dots & e_k \\ r_1 & r_2 & \dots & r_k \end{pmatrix}=\det(Q)$.

Remark:6.1 We can write the element q_{ij} of adjP = Q = (q_{ij}) as follows:

$q_{ij}=\sum_{\pi\in S_{n_j n_i}}\prod_{t\in n_j}\left[\left(\underline{p}_{t\pi(t)}{}^{T},\underline{p}_{t\pi(t)}{}^{I},\underline{p}_{t\pi(t)}{}^{F}\right);\left(\overline{p}_{t\pi(t)}{}^{T},\overline{p}_{t\pi(t)}{}^{I},\overline{p}_{t\pi(t)}{}^{F}\right)\right]$ where n_j = {1,2,3,…n}\{j} and

$S_{n_j n_i}$ is the set of all permutation of set n_j over the set n_i.

7. Theorems and Results

Theorem:7.1 $P\in(NHSRFM)_{n}$,then

(i) $$\det(P)=|P|=\sum_{t=1}^{n}\left[\left(\underline{p}_{it}{}^{T},\underline{p}_{it}{}^{I},\underline{p}_{it}{}^{F}\right);\left(\overline{p}_{it}{}^{T},\overline{p}_{it}{}^{I},\overline{p}_{it}{}^{F}\right)\right]P_{it},i\in\{1,2,...,n\}.$$

(ii) $$\det(P)=\sum_{e<f}\begin{vmatrix}\left[\left(\underline{p}_{1e}{}^{T},\underline{p}_{1e}{}^{I},\underline{p}_{1e}{}^{F}\right);\left(\overline{p}_{1e}{}^{T},\overline{p}_{1e}{}^{I},\overline{p}_{1e}{}^{F}\right)\right] & \left[\left(\underline{p}_{1f}{}^{T},\underline{p}_{1f}{}^{I},\underline{p}_{1f}{}^{F}\right);\left(\overline{p}_{1f}{}^{T},\overline{p}_{1f}{}^{I},\overline{p}_{1f}{}^{F}\right)\right] \\ \left[\left(\underline{p}_{2e}{}^{T},\underline{p}_{2e}{}^{I},\underline{p}_{2e}{}^{F}\right);\left(\overline{p}_{2e}{}^{T},\overline{p}_{2e}{}^{I},\overline{p}_{2e}{}^{F}\right)\right] & \left[\left(\underline{p}_{2f}{}^{T},\underline{p}_{2f}{}^{I},\underline{p}_{2f}{}^{F}\right);\left(\overline{p}_{2f}{}^{T},\overline{p}_{2f}{}^{I},\overline{p}_{2f}{}^{F}\right)\right]\end{vmatrix}P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}$$

where the summation is taken over all e and f in {1,2,...,n} such that e < f.

Theorem: 7.2

$$\det(P)=\begin{pmatrix}\left[\left(\underline{p}_{1r_1}{}^{T},\underline{p}_{1r_1}{}^{I},\underline{p}_{1r_1}{}^{F}\right);\left(\overline{p}_{1r_1}{}^{T},\overline{p}_{1r_1}{}^{I},\overline{p}_{1r_1}{}^{F}\right)\right] & \dots\dots & \left[\left(\underline{p}_{1r_k}{}^{T},\underline{p}_{1r_k}{}^{I},\underline{p}_{1r_k}{}^{F}\right);\left(\overline{p}_{1r_k}{}^{T},\overline{p}_{1r_k}{}^{I},\overline{p}_{1r_k}{}^{F}\right)\right] \\ \left[\left(\underline{p}_{2r_1}{}^{T},\underline{p}_{2r_1}{}^{I},\underline{p}_{2r_1}{}^{F}\right);\left(\overline{p}_{2r_1}{}^{T},\overline{p}_{2r_1}{}^{I},\overline{p}_{2r_1}{}^{F}\right)\right] & \dots\dots & \left[\left(\underline{p}_{2r_k}{}^{T},\underline{p}_{2r_k}{}^{I},\underline{p}_{2r_k}{}^{F}\right);\left(\overline{p}_{2r_k}{}^{T},\overline{p}_{2r_k}{}^{I},\overline{p}_{2r_k}{}^{F}\right)\right] \\ \dots & \dots\dots & \dots \\ \left[\left(\underline{p}_{kr_1}{}^{T},\underline{p}_{kr_1}{}^{I},\underline{p}_{kr_1}{}^{F}\right);\left(\overline{p}_{kr_1}{}^{T},\overline{p}_{kr_1}{}^{I},\overline{p}_{kr_1}{}^{F}\right)\right] & \dots\dots & \left[\left(\underline{p}_{kr_k}{}^{T},\underline{p}_{kr_k}{}^{I},\underline{p}_{kr_k}{}^{F}\right);\left(\overline{p}_{kr_k}{}^{T},\overline{p}_{kr_k}{}^{I},\overline{p}_{kr_k}{}^{F}\right)\right]\end{pmatrix}$$

$$P\begin{pmatrix}1 & \dots & k\\ r_1 & \dots & r_k\end{pmatrix}$$

where the summation is taken over all $r_1,r_2,...,r_k\in\{1,2,...,n\}$, such that $r_1<r_2<...<r_k$.

Proof: Let $S(r_1,r_2,...,r_k)=\{\sigma:\{1,2,...,k\}\to\{r_1,r_2,...,r_k\}/\sigma$ is a bijection}. Then

$$\det(P)=\sum_{\sigma\in S_n}\left[\left(\underline{p}_{1\sigma(1)}{}^{T},\underline{p}_{1\sigma(1)}{}^{I},\underline{p}_{1\sigma(1)}{}^{F}\right);\left(\overline{p}_{1\sigma(1)}{}^{T},\overline{p}_{1\sigma(1)}{}^{I},\overline{p}_{1\sigma(1)}{}^{F}\right)\right]\dots\left[\begin{matrix}\left(\underline{p}_{n\sigma(n)}{}^{T},\underline{p}_{n\sigma(n)}{}^{I},\underline{p}_{n\sigma(n)}{}^{F}\right);\\ \left(\overline{p}_{n\sigma(n)}{}^{T},\overline{p}_{n\sigma(n)}{}^{I},\overline{p}_{n\sigma(n)}{}^{F}\right)\end{matrix}\right]$$

$$= \sum_{r_1<r_2<\ldots<r_k}\left(\sum_{\sigma\{1,2,\ldots,k\}\in S(r_1,r_2,\ldots,r_k)}\left[\left(\underline{p}_{1\sigma(1)}^{\ T},\underline{p}_{1\sigma(1)}^{\ I},\underline{p}_{1\sigma(1)}^{\ F}\right);\left(\overline{p}_{1\sigma(1)}^{\ T},\overline{p}_{1\sigma(1)}^{\ I},\overline{p}_{1\sigma(1)}^{\ F}\right)\right]\cdots\begin{bmatrix}\left(\underline{p}_{n\sigma(n)}^{\ T},\underline{p}_{n\sigma(n)}^{\ I},\underline{p}_{n\sigma(n)}^{\ F}\right);\\ \left(\overline{p}_{n\sigma(n)}^{\ T},\overline{p}_{n\sigma(n)}^{\ I},\overline{p}_{n\sigma(n)}^{\ F}\right)\end{bmatrix}\right)$$

$$= \sum_{r_1<r_2<\ldots<r_k}\left(\sum_{\sigma'\{1,2,\ldots,k\}\in S(r_1,r_2,\ldots,r_k)}\left[\left(\underline{p}_{1\sigma'(1)}^{\ T},\underline{p}_{1\sigma'(1)}^{\ I},\underline{p}_{1\sigma'(1)}^{\ F}\right);\left(\overline{p}_{1\sigma'(1)}^{\ T},\overline{p}_{1\sigma'(1)}^{\ I},\overline{p}_{1\sigma'(1)}^{\ F}\right)\right]\cdots\begin{bmatrix}\left(\underline{p}_{n\sigma'(n)}^{\ T},\underline{p}_{n\sigma'(n)}^{\ I},\underline{p}_{ij}^{\ F}\right);\\ \left(\overline{p}_{n\sigma'(n)}^{\ T},\overline{p}_{n\sigma'(n)}^{\ I},\overline{p}_{n\sigma'(n)}^{\ F}\right)\end{bmatrix}\right)$$

$$P\begin{pmatrix}1 & \ldots & k\\ r_1 & \ldots & r_k\end{pmatrix}$$

$$= \det\sum_{r_1<r_2<\ldots<r_k}\begin{pmatrix}\left[\left(\underline{p}_{1r_1}^{\ T},\underline{p}_{1r_1}^{\ I},\underline{p}_{1r_1}^{\ F}\right);\left(\overline{p}_{1r_1}^{\ T},\overline{p}_{1r_1}^{\ I},\overline{p}_{1r_1}^{\ F}\right)\right] & \ldots & \ldots & \left[\left(\underline{p}_{1r_k}^{\ T},\underline{p}_{1r_k}^{\ I},\underline{p}_{1r_k}^{\ F}\right);\left(\overline{p}_{1r_k}^{\ T},\overline{p}_{1r_k}^{\ I},\overline{p}_{1r_k}^{\ F}\right)\right]\\ \left[\left(\underline{p}_{2r_1}^{\ T},\underline{p}_{2r_1}^{\ I},\underline{p}_{2r_1}^{\ F}\right);\left(\overline{p}_{2r_1}^{\ T},\overline{p}_{2r_1}^{\ I},\overline{p}_{2r_1}^{\ F}\right)\right] & \ldots & \ldots & \left[\left(\underline{p}_{2r_k}^{\ T},\underline{p}_{2r_k}^{\ I},\underline{p}_{2r_k}^{\ F}\right);\left(\overline{p}_{2r_k}^{\ T},\overline{p}_{2r_k}^{\ I},\overline{p}_{2r_k}^{\ F}\right)\right]\\ \ldots & \ldots & \ldots & \ldots\\ \left[\left(\underline{p}_{kr_1}^{\ T},\underline{p}_{kr_1}^{\ I},\underline{p}_{kr_1}^{\ F}\right);\left(\overline{p}_{kr_1}^{\ T},\overline{p}_{kr_1}^{\ I},\overline{p}_{kr_1}^{\ F}\right)\right] & \ldots & \ldots & \left[\left(\underline{p}_{kr_k}^{\ T},\underline{p}_{kr_k}^{\ I},\underline{p}_{kr_k}^{\ F}\right);\left(\overline{p}_{kr_k}^{\ T},\overline{p}_{kr_k}^{\ I},\overline{p}_{kr_k}^{\ F}\right)\right]\end{pmatrix}$$

$$P\begin{pmatrix}1 & \ldots & k\\ r_1 & \ldots & r_k\end{pmatrix}$$

Hence the theorem.

Lemma 7.1 Let $P=\begin{pmatrix}\left[\left(\underline{p}^T,\underline{p}^I,\underline{p}^F\right);\left(\overline{p}^T,\overline{p}^I,\overline{p}^F\right)\right] & \left[\left(\underline{q}^T,\underline{q}^I,\underline{q}^F\right);\left(\overline{q}^T,\overline{q}^I,\overline{q}^F\right)\right]\\ \left[\left(\underline{r}^T,\underline{r}^I,\underline{r}^F\right);\left(\overline{r}^T,\overline{r}^I,\overline{r}^F\right)\right] & \left[\left(\underline{s}^T,\underline{s}^I,\underline{s}^F\right);\left(\overline{s}^T,\overline{s}^I,\overline{s}^F\right)\right]\end{pmatrix}$ be a NHSRFM.

Then

$$\det\begin{pmatrix}\left[\left(\underline{p}^T,\underline{p}^I,\underline{p}^F\right);\left(\overline{p}^T,\overline{p}^I,\overline{p}^F\right)\right] & \left[\left(\underline{q}^T,\underline{q}^I,\underline{q}^F\right);\left(\overline{q}^T,\overline{q}^I,\overline{q}^F\right)\right]\\ \left[\left(\underline{p}^T,\underline{p}^I,\underline{p}^F\right);\left(\overline{p}^T,\overline{p}^I,\overline{p}^F\right)\right] & \left[\left(\underline{q}^T,\underline{q}^I,\underline{q}^F\right);\left(\overline{q}^T,\overline{q}^I,\overline{q}^F\right)\right]\end{pmatrix}\det\begin{pmatrix}\left[\left(\underline{r}^T,\underline{r}^I,\underline{r}^F\right);\left(\overline{r}^T,\overline{r}^I,\overline{r}^F\right)\right] & \left[\left(\underline{s}^T,\underline{s}^I,\underline{s}^F\right);\left(\overline{s}^T,\overline{s}^I,\overline{s}^F\right)\right]\\ \left[\left(\underline{r}^T,\underline{r}^I,\underline{r}^F\right);\left(\overline{r}^T,\overline{r}^I,\overline{r}^F\right)\right] & \left[\left(\underline{s}^T,\underline{s}^I,\underline{s}^F\right);\left(\overline{s}^T,\overline{s}^I,\overline{s}^F\right)\right]\end{pmatrix}$$

$$=\begin{Vmatrix}\left[\left(\underline{p}^T,\underline{p}^I,\underline{p}^F\right);\left(\overline{p}^T,\overline{p}^I,\overline{p}^F\right)\right] & \left[\left(\underline{q}^T,\underline{q}^I,\underline{q}^F\right);\left(\overline{q}^T,\overline{q}^I,\overline{q}^F\right)\right]\\ \left[\left(\underline{p}^T,\underline{p}^I,\underline{p}^F\right);\left(\overline{p}^T,\overline{p}^I,\overline{p}^F\right)\right] & \left[\left(\underline{q}^T,\underline{q}^I,\underline{q}^F\right);\left(\overline{q}^T,\overline{q}^I,\overline{q}^F\right)\right]\end{Vmatrix}\begin{Vmatrix}\left[\left(\underline{r}^T,\underline{r}^I,\underline{r}^F\right);\left(\overline{r}^T,\overline{r}^I,\overline{r}^F\right)\right] & \left[\left(\underline{s}^T,\underline{s}^I,\underline{s}^F\right);\left(\overline{s}^T,\overline{s}^I,\overline{s}^F\right)\right]\\ \left[\left(\underline{r}^T,\underline{r}^I,\underline{r}^F\right);\left(\overline{r}^T,\overline{r}^I,\overline{r}^F\right)\right] & \left[\left(\underline{s}^T,\underline{s}^I,\underline{s}^F\right);\left(\overline{s}^T,\overline{s}^I,\overline{s}^F\right)\right]\end{Vmatrix}$$

$$\leq \det(P)$$

Proof: We see that

$$\det\begin{pmatrix}\left[\left(\underline{p}^{T},\underline{p}^{I},\underline{p}^{F}\right);\left(\overline{p}^{T},\overline{p}^{I},\overline{p}^{F}\right)\right] & \left[\left(\underline{q}^{T},\underline{q}^{I},\underline{q}^{F}\right);\left(\overline{q}^{T},\overline{q}^{I},\overline{q}^{F}\right)\right]\\ \left[\left(\underline{p}^{T},\underline{p}^{I},\underline{p}^{F}\right);\left(\overline{p}^{T},\overline{p}^{I},\overline{p}^{F}\right)\right] & \left[\left(\underline{q}^{T},\underline{q}^{I},\underline{q}^{F}\right);\left(\overline{q}^{T},\overline{q}^{I},\overline{q}^{F}\right)\right]\end{pmatrix}\det\begin{pmatrix}\left[\left(\underline{r}^{T},\underline{r}^{I},\underline{r}^{F}\right);\left(\overline{r}^{T},\overline{r}^{I},\overline{r}^{F}\right)\right] & \left[\left(\underline{s}^{T},\underline{s}^{I},\underline{s}^{F}\right);\left(\overline{s}^{T},\overline{s}^{I},\overline{s}^{F}\right)\right]\\ \left[\left(\underline{r}^{T},\underline{r}^{I},\underline{r}^{F}\right);\left(\overline{r}^{T},\overline{r}^{I},\overline{r}^{F}\right)\right] & \left[\left(\underline{s}^{T},\underline{s}^{I},\underline{s}^{F}\right);\left(\overline{s}^{T},\overline{s}^{I},\overline{s}^{F}\right)\right]\end{pmatrix}$$

$$=\left[\left(\underline{p}^{T},\underline{p}^{I},\underline{p}^{F}\right);\left(\overline{p}^{T},\overline{p}^{I},\overline{p}^{F}\right)\right]\left[\left(\underline{q}^{T},\underline{q}^{I},\underline{q}^{F}\right);\left(\overline{q}^{T},\overline{q}^{I},\overline{q}^{F}\right)\right]\left[\left(\underline{r}^{T},\underline{r}^{I},\underline{r}^{F}\right);\left(\overline{r}^{T},\overline{r}^{I},\overline{r}^{F}\right)\right]$$
$$\left[\left(\underline{s}^{T},\underline{s}^{I},\underline{s}^{F}\right);\left(\overline{s}^{T},\overline{s}^{I},\overline{s}^{F}\right)\right]$$

$$\leq\begin{pmatrix}\left[\left(\underline{p}^{T},\underline{p}^{I},\underline{p}^{F}\right);\left(\overline{p}^{T},\overline{p}^{I},\overline{p}^{F}\right)\right]\left[\left(\underline{s}^{T},\underline{s}^{I},\underline{s}^{F}\right);\left(\overline{s}^{T},\overline{s}^{I},\overline{s}^{F}\right)\right]+\left[\left(\underline{q}^{T},\underline{q}^{I},\underline{q}^{F}\right);\left(\overline{q}^{T},\overline{q}^{I},\overline{q}^{F}\right)\right]\\ \left[\left(\underline{r}^{T},\underline{r}^{I},\underline{r}^{F}\right);\left(\overline{r}^{T},\overline{r}^{I},\overline{r}^{F}\right)\right]\end{pmatrix}\leq\det(P)$$

Hence the theorem.

Theorem:7.3 Let $P\in\left(NHSRFM\right)_{n}$,then

(i) $\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(1\Rightarrow 2\right)\right)\leq\det\left(P\right).$

(ii) $\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(3\Rightarrow 2\right)\right)\leq\det\left(P\right).$

(iii) $\det\left(P\left(p\Rightarrow q\right)\right)\det\left(P\left(q\Rightarrow k\right)\right)\leq\det\left(P\right).$

Proof: To prove (i) $\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(1\Rightarrow 2\right)\right)\leq\det\left(P\right).$

$$
=\left(\begin{array}{l}
\left|\begin{matrix}\left[\left(\underline{p}_{11}^{T},\underline{p}_{11}^{I},\underline{p}_{11}^{F}\right);\left(\overline{p}_{11}^{T},\overline{p}_{11}^{I},\overline{p}_{11}^{F}\right)\right] & \left[\left(\underline{p}_{12}^{T},\underline{p}_{12}^{I},\underline{p}_{12}^{F}\right);\left(\overline{p}_{12}^{T},\overline{p}_{12}^{I},\overline{p}_{12}^{F}\right)\right]\\ \left[\left(\underline{p}_{11}^{T},\underline{p}_{11}^{I},\underline{p}_{11}^{F}\right);\left(\overline{p}_{11}^{T},\overline{p}_{11}^{I},\overline{p}_{11}^{F}\right)\right] & \left[\left(\underline{p}_{12}^{T},\underline{p}_{12}^{I},\underline{p}_{12}^{F}\right);\left(\overline{p}_{12}^{T},\overline{p}_{12}^{I},\overline{p}_{12}^{F}\right)\right]\end{matrix}\right| P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}\\
+\ldots+\left|\begin{matrix}\left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\\ \left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\end{matrix}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}\\
+\ldots+\left|\begin{matrix}\left[\left(\underline{p}_{1n-1}^{T},\underline{p}_{1n-1}^{I},\underline{p}_{1n-1}^{F}\right);\left(\overline{p}_{1n-1}^{T},\overline{p}_{1n-1}^{I},\overline{p}_{1n-1}^{F}\right)\right] & \left[\left(\underline{p}_{1n}^{T},\underline{p}_{1n}^{I},\underline{p}_{1n}^{F}\right);\left(\overline{p}_{1n}^{T},\overline{p}_{1n}^{I},\overline{p}_{1n}^{F}\right)\right]\\ \left[\left(\underline{p}_{1n-1}^{T},\underline{p}_{1n-1}^{I},\underline{p}_{1n-1}^{F}\right);\left(\overline{p}_{1n-1}^{T},\overline{p}_{1n-1}^{I},\overline{p}_{1n-1}^{F}\right)\right] & \left[\left(\underline{p}_{1n}^{T},\underline{p}_{1n}^{I},\underline{p}_{1n}^{F}\right);\left(\overline{p}_{1n}^{T},\overline{p}_{1n}^{I},\overline{p}_{1n}^{F}\right)\right]\end{matrix}\right|\\
P\begin{pmatrix}1 & 2\\ n-1 & n\end{pmatrix}
\end{array}\right)
$$

$$
=\left(\sum_{e<f}\left|\begin{matrix}\left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\\ \left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\end{matrix}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}\right)
$$

$$
=\left(\sum_{e<f}\left|\begin{matrix}\left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\\ \left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\end{matrix}\right| P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}\right)
$$

$$
\left(\sum_{g<h}\left|\begin{matrix}\left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right]\\ \left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right]\end{matrix}\right| P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}\right)
$$

$$
\leq\left(\sum_{\substack{e<f\\ g<h}}\begin{array}{l}\left|\begin{matrix}\left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\\ \left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right]\end{matrix}\right|\\ P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}\end{array}\right)
$$

We now introduce symbols $\Omega_1,\Omega_2,\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}$ and Ω. Define

$$
\Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix}=\left|\begin{matrix}\left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right]\\ \left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right]\end{matrix}\right|
$$

$$P\begin{pmatrix}1 & 2\\ e & f\end{pmatrix}P\begin{pmatrix}1 & 2\\ g & h\end{pmatrix}$$

$$\Omega_1 = \sum_{(e,f)=(g,h)} \Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix} = \sum_{e<f} \Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix},$$

$$\Omega_2 = \sum_{(e,f)\neq(g,h)} \Omega\begin{pmatrix}p & q\\ r & s\end{pmatrix} \quad \text{and} \quad \Omega = \Omega_1 + \Omega_2$$

Then we see that

$$\Omega\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix} = \begin{vmatrix}\left[\left(\underline{p}_{11}^{\,T}, \underline{p}_{11}^{\,I}, \underline{p}_{11}^{\,F}\right);\left(\overline{p}_{11}^{\,T}, \overline{p}_{11}^{\,I}, \overline{p}_{11}^{\,F}\right)\right] & \left[\left(\underline{p}_{12}^{\,T}, \underline{p}_{12}^{\,I}, \underline{p}_{12}^{\,F}\right);\left(\overline{p}_{12}^{\,T}, \overline{p}_{12}^{\,I}, \overline{p}_{12}^{\,F}\right)\right]\\ \left[\left(\underline{p}_{21}^{\,T}, \underline{p}_{21}^{\,I}, \underline{p}_{21}^{\,F}\right);\left(\overline{p}_{21}^{\,T}, \overline{p}_{21}^{\,I}, \overline{p}_{21}^{\,F}\right)\right] & \left[\left(\underline{p}_{22}^{\,T}, \underline{p}_{22}^{\,I}, \underline{p}_{22}^{\,F}\right);\left(\overline{p}_{22}^{\,T}, \overline{p}_{22}^{\,I}, \overline{p}_{22}^{\,F}\right)\right]\end{vmatrix} P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}$$

$$\Omega_1 = \det(P)$$

$$\det\left(P(2 \Rightarrow 1)\right)\det\left(P(1 \Rightarrow 2)\right) \le \Omega = \det(P) + \Omega_2.$$

We show that $\Omega_2 \le \det(P)$

We consider two separate cases.

Case 1. We consider $p = \Omega\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$, a term of Ω_2.

Let

$$p_1 = \left[\left(\underline{p}_{11}^{\,T}, \underline{p}_{11}^{\,I}, \underline{p}_{11}^{\,F}\right);\left(\overline{p}_{11}^{\,T}, \overline{p}_{11}^{\,I}, \overline{p}_{11}^{\,F}\right)\right]\left[\left(\underline{p}_{23}^{\,T}, \underline{p}_{23}^{\,I}, \underline{p}_{23}^{\,F}\right);\left(\overline{p}_{23}^{\,T}, \overline{p}_{23}^{\,I}, \overline{p}_{23}^{\,F}\right)\right] P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}P\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$$

$$p_2 = \left[\left(\underline{p}_{12}^{\,T}, \underline{p}_{12}^{\,I}, \underline{p}_{12}^{\,F}\right);\left(\overline{p}_{12}^{\,T}, \overline{p}_{12}^{\,I}, \overline{p}_{12}^{\,F}\right)\right]\left[\left(\underline{p}_{21}^{\,T}, \underline{p}_{21}^{\,I}, \underline{p}_{21}^{\,F}\right);\left(\overline{p}_{21}^{\,T}, \overline{p}_{21}^{\,I}, \overline{p}_{21}^{\,F}\right)\right] P\begin{pmatrix}1 & 2\\ 1 & 2\end{pmatrix}P\begin{pmatrix}1 & 2\\ 1 & 3\end{pmatrix}$$

Then $p = p_1 + p_2$,

$$p_1 \le \begin{vmatrix}\left[\left(\underline{p}_{11}^{\,T}, \underline{p}_{11}^{\,I}, \underline{p}_{11}^{\,F}\right);\left(\overline{p}_{11}^{\,T}, \overline{p}_{11}^{\,I}, \overline{p}_{11}^{\,F}\right)\right] & \left[\left(\underline{p}_{13}^{\,T}, \underline{p}_{13}^{\,I}, \underline{p}_{13}^{\,F}\right);\left(\overline{p}_{13}^{\,T}, \overline{p}_{13}^{\,I}, \overline{p}_{13}^{\,F}\right)\right]\\ \left[\left(\underline{p}_{21}^{\,T}, \underline{p}_{21}^{\,I}, \underline{p}_{21}^{\,F}\right);\left(\overline{p}_{21}^{\,T}, \overline{p}_{21}^{\,I}, \overline{p}_{21}^{\,F}\right)\right] & \left[\left(\underline{p}_{23}^{\,T}, \underline{p}_{23}^{\,I}, \underline{p}_{23}^{\,F}\right);\left(\overline{p}_{23}^{\,T}, \overline{p}_{23}^{\,I}, \overline{p}_{23}^{\,F}\right)\right]\end{vmatrix} P\begin{pmatrix}1 & 3\\ 1 & 3\end{pmatrix}$$

$$\le \det(P)$$

$$p_2 \le \begin{vmatrix} \left[\left(\underline{p}_{11}{}^{T}, \underline{p}_{11}{}^{I}, \underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T}, \overline{p}_{11}{}^{I}, \overline{p}_{11}{}^{F}\right)\right] & \left[\left(\underline{p}_{12}{}^{T}, \underline{p}_{12}{}^{I}, \underline{p}_{12}{}^{F}\right);\left(\overline{p}_{12}{}^{T}, \overline{p}_{12}{}^{I}, \overline{p}_{12}{}^{F}\right)\right] \\ \left[\left(\underline{p}_{21}{}^{T}, \underline{p}_{21}{}^{I}, \underline{p}_{21}{}^{F}\right);\left(\overline{p}_{21}{}^{T}, \overline{p}_{21}{}^{I}, \overline{p}_{21}{}^{F}\right)\right] & \left[\left(\underline{p}_{22}{}^{T}, \underline{p}_{22}{}^{I}, \underline{p}_{22}{}^{F}\right);\left(\overline{p}_{22}{}^{T}, \overline{p}_{22}{}^{I}, \overline{p}_{22}{}^{F}\right)\right] \end{vmatrix} P\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\le \det(P)$$

and $\Omega\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} \le \det(P)$.

Case 2. We take $\Omega\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$

Let

$$q_1 = \left[\left(\underline{p}_{11}{}^{T}, \underline{p}_{11}{}^{I}, \underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T}, \overline{p}_{11}{}^{I}, \overline{p}_{11}{}^{F}\right)\right]\left[\left(\underline{p}_{2n}{}^{T}, \underline{p}_{2n}{}^{I}, \underline{p}_{2n}{}^{F}\right);\left(\overline{p}_{2n}{}^{T}, \overline{p}_{2n}{}^{I}, \overline{p}_{2n}{}^{F}\right)\right] P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$$

and

$$q_2 = \left[\left(\underline{p}_{12}{}^{T}, \underline{p}_{12}{}^{I}, \underline{p}_{12}{}^{F}\right);\left(\overline{p}_{12}{}^{T}, \overline{p}_{12}{}^{I}, \overline{p}_{12}{}^{F}\right)\right]\left[\left(\underline{p}_{2n-1}{}^{T}, \underline{p}_{2n-1}{}^{I}, \underline{p}_{2n-1}{}^{F}\right);\left(\overline{p}_{2n-1}{}^{T}, \overline{p}_{2n-1}{}^{I}, \overline{p}_{2n-1}{}^{F}\right)\right] P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$$P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}.$$

Then $\Omega\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix} = q_1 + q_2.$

To show that $q_1 \le \det(P)$ and $q_2 \le \det(P)$ we observe all coordinates of the elements p_{ij}

involved in $P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $P\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ and $P\begin{pmatrix} 1 & 2 \\ n-1 & n \end{pmatrix}$

The coordinates of the elements p_{ij} involved in these determinants are all coordinates of the elements of the k th – row P_k of P, for $k \ge 3$. Therefore, if we let $q = p_{3n-1}p_{4n-2}\cdots p_{k+2n-k}\cdots p_{nn-2}$, then we see that

$$q_1 \le \left(\left[\left(\underline{p}_{11}{}^{T}, \underline{p}_{11}{}^{I}, \underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T}, \overline{p}_{11}{}^{I}, \overline{p}_{11}{}^{F}\right)\right]\left[\left(\underline{p}_{2n}{}^{T}, \underline{p}_{2n}{}^{I}, \underline{p}_{2n}{}^{F}\right);\left(\overline{p}_{2n}{}^{T}, \overline{p}_{2n}{}^{I}, \overline{p}_{2n}{}^{F}\right)\right]\right) c \le \det(P).$$

For q_2, let $c = p_{3n}p_{4n-2}p_{5n-3}\cdots p_{n-13}p_{2n-1}$, then we see that

$$q_2 \le \left(\left[\left(\underline{p}_{12}{}^{T}, \underline{p}_{12}{}^{I}, \underline{p}_{12}{}^{F}\right);\left(\overline{p}_{12}{}^{T}, \overline{p}_{12}{}^{I}, \overline{p}_{12}{}^{F}\right)\right]\left[\left(\underline{p}_{2n-1}{}^{T}, \underline{p}_{2n-1}{}^{I}, \underline{p}_{2n-1}{}^{F}\right);\left(\overline{p}_{2n-1}{}^{T}, \overline{p}_{2n-1}{}^{I}, \overline{p}_{2n-1}{}^{F}\right)\right]\right) c$$

$\leq \det(P)$

For any $\Omega\begin{pmatrix} e & f \\ g & h \end{pmatrix}_{(e,f)\neq(g,h)}$, we apply either the case 1 or the case 2 and we can deduce

that $\Omega\begin{pmatrix} e & f \\ g & h \end{pmatrix} \leq \det(P)$. Thus (i) holds. (ii). First, we consider

$$\begin{vmatrix} \left[\left(\underline{q}_{11}^{T},\underline{q}_{11}^{I},\underline{q}_{11}^{F}\right);\left(\overline{q}_{11}^{T},\overline{q}_{11}^{I},\overline{q}_{11}^{F}\right)\right] & \left[\left(\underline{q}_{12}^{T},\underline{q}_{12}^{I},\underline{q}_{12}^{F}\right);\left(\overline{q}_{12}^{T},\overline{q}_{12}^{I},\overline{q}_{12}^{F}\right)\right] \\ \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] \end{vmatrix}$$

$$\begin{vmatrix} \left[\left(\underline{q}_{21}^{T},\underline{q}_{21}^{I},\underline{q}_{21}^{F}\right);\left(\overline{q}_{21}^{T},\overline{q}_{21}^{I},\overline{q}_{21}^{F}\right)\right] & \left[\left(\underline{q}_{22}^{T},\underline{q}_{22}^{I},\underline{q}_{22}^{F}\right);\left(\overline{q}_{22}^{T},\overline{q}_{22}^{I},\overline{q}_{22}^{F}\right)\right] \\ \left[\left(\underline{q}_{21}^{T},\underline{q}_{21}^{I},\underline{q}_{21}^{F}\right);\left(\overline{q}_{21}^{T},\overline{q}_{21}^{I},\overline{q}_{21}^{F}\right)\right] & \left[\left(\underline{q}_{22}^{T},\underline{q}_{22}^{I},\underline{q}_{22}^{F}\right);\left(\overline{q}_{22}^{T},\overline{q}_{22}^{I},\overline{q}_{22}^{F}\right)\right] \end{vmatrix}$$

$$\leq \begin{vmatrix} \left[\left(\underline{q}_{21}^{T},\underline{q}_{21}^{I},\underline{q}_{21}^{F}\right);\left(\overline{q}_{21}^{T},\overline{q}_{21}^{I},\overline{q}_{21}^{F}\right)\right] & \left[\left(\underline{q}_{32}^{T},\underline{q}_{32}^{I},\underline{q}_{32}^{F}\right);\left(\overline{q}_{32}^{T},\overline{q}_{32}^{I},\overline{q}_{32}^{F}\right)\right] \\ \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] \end{vmatrix}$$

$$K\begin{pmatrix} g & h \\ e & f \end{pmatrix} = \begin{vmatrix} \left[\left(\underline{p}_{21}^{T},\underline{p}_{21}^{I},\underline{p}_{21}^{F}\right);\left(\overline{p}_{21}^{T},\overline{p}_{21}^{I},\overline{p}_{21}^{F}\right)\right] & \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] \\ \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] \end{vmatrix}$$

$$P\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(3\Rightarrow 2\right)\right)$$

$$= \sum_{\substack{e<f \\ g<h}} \begin{vmatrix} \left[\left(\underline{p}_{1e}^{T},\underline{p}_{1e}^{I},\underline{p}_{1e}^{F}\right);\left(\overline{p}_{1e}^{T},\overline{p}_{1e}^{I},\overline{p}_{1e}^{F}\right)\right] & \left[\left(\underline{p}_{1f}^{T},\underline{p}_{1f}^{I},\underline{p}_{1f}^{F}\right);\left(\overline{p}_{1f}^{T},\overline{p}_{1f}^{I},\overline{p}_{1f}^{F}\right)\right] \\ \left[\left(\underline{p}_{3e}^{T},\underline{p}_{3e}^{I},\underline{p}_{3e}^{F}\right);\left(\overline{p}_{3e}^{T},\overline{p}_{3e}^{I},\overline{p}_{3e}^{F}\right)\right] & \left[\left(\underline{p}_{3f}^{T},\underline{p}_{3f}^{I},\underline{p}_{3f}^{F}\right);\left(\overline{p}_{3f}^{T},\overline{p}_{3f}^{I},\overline{p}_{3f}^{F}\right)\right] \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$\begin{vmatrix} \left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right] \\ \left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right] \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix}$$

$$\leq \sum_{\substack{g<h \\ e<f}} \begin{vmatrix} \left[\left(\underline{p}_{2g}^{T},\underline{p}_{2g}^{I},\underline{p}_{2g}^{F}\right);\left(\overline{p}_{2g}^{T},\overline{p}_{2g}^{I},\overline{p}_{2g}^{F}\right)\right] & \left[\left(\underline{p}_{2h}^{T},\underline{p}_{2h}^{I},\underline{p}_{2h}^{F}\right);\left(\overline{p}_{2h}^{T},\overline{p}_{2h}^{I},\overline{p}_{2h}^{F}\right)\right] \\ \left[\left(\underline{p}_{3e}^{T},\underline{p}_{3e}^{I},\underline{p}_{3e}^{F}\right);\left(\overline{p}_{3e}^{T},\overline{p}_{3e}^{I},\overline{p}_{3e}^{F}\right)\right] & \left[\left(\underline{p}_{3f}^{T},\underline{p}_{3f}^{I},\underline{p}_{3f}^{F}\right);\left(\overline{p}_{3f}^{T},\overline{p}_{3f}^{I},\overline{p}_{3f}^{F}\right)\right] \end{vmatrix}$$

$$P\begin{pmatrix} 2 & 3 \\ g & h \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ e & f \end{pmatrix}$$

$$= \sum_{\substack{g<h \\ e<f}} P\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

$$= \sum_{(g,h)=(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix} + \sum_{(g,h)\neq(e,f)} K\begin{pmatrix} g & h \\ e & f \end{pmatrix}$$

Next we prove that

$$K\begin{pmatrix} g & h \\ e & f \end{pmatrix}_{(g,h)\neq(e,f)} \leq \det(P)$$

.We consider two separate cases.

Case 1. We take $K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$. We see that

$$K\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix} = \left(\left[\left(\underline{p}_{21}{}^{T}, \underline{p}_{21}{}^{I}, \underline{p}_{21}{}^{F}\right); \left(\overline{p}_{21}{}^{T}, \overline{p}_{21}{}^{I}, \overline{p}_{21}{}^{F}\right)\right]\left[\left(\underline{p}_{33}{}^{T}, \underline{p}_{33}{}^{I}, \underline{p}_{33}{}^{F}\right); \left(\overline{p}_{33}{}^{T}, \overline{p}_{33}{}^{I}, \overline{p}_{33}{}^{F}\right)\right]\right)$$

$$P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$+\left(\left[\left(\underline{p}_{22}{}^{T}, \underline{p}_{22}{}^{I}, \underline{p}_{22}{}^{F}\right); \left(\overline{p}_{22}{}^{T}, \overline{p}_{22}{}^{I}, \overline{p}_{22}{}^{F}\right)\right]\left[\left(\underline{p}_{31}{}^{T}, \underline{p}_{31}{}^{I}, \underline{p}_{31}{}^{F}\right); \left(\overline{p}_{31}{}^{T}, \overline{p}_{31}{}^{I}, \overline{p}_{31}{}^{F}\right)\right]\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$\leq \begin{vmatrix} \left[\left(\underline{p}_{21}{}^{T}, \underline{p}_{21}{}^{I}, \underline{p}_{21}{}^{F}\right); \left(\overline{p}_{21}{}^{T}, \overline{p}_{21}{}^{I}, \overline{p}_{21}{}^{F}\right)\right] & \left[\left(\underline{p}_{23}{}^{T}, \underline{p}_{23}{}^{I}, \underline{p}_{23}{}^{F}\right); \left(\overline{p}_{23}{}^{T}, \overline{p}_{23}{}^{I}, \overline{p}_{23}{}^{F}\right)\right] \\ \left[\left(\underline{p}_{31}{}^{T}, \underline{p}_{31}{}^{I}, \underline{p}_{31}{}^{F}\right); \left(\overline{p}_{31}{}^{T}, \overline{p}_{31}{}^{I}, \overline{p}_{31}{}^{F}\right)\right] & \left[\left(\underline{p}_{33}{}^{T}, \underline{p}_{33}{}^{I}, \underline{p}_{33}{}^{F}\right); \left(\overline{p}_{33}{}^{T}, \overline{p}_{33}{}^{I}, \overline{p}_{33}{}^{F}\right)\right] \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$$

$$+\begin{vmatrix} \left[\left(\underline{p}_{21}{}^{T}, \underline{p}_{21}{}^{I}, \underline{p}_{21}{}^{F}\right); \left(\overline{p}_{21}{}^{T}, \overline{p}_{21}{}^{I}, \overline{p}_{21}{}^{F}\right)\right] & \left[\left(\underline{p}_{22}{}^{T}, \underline{p}_{22}{}^{I}, \underline{p}_{22}{}^{F}\right); \left(\overline{p}_{22}{}^{T}, \overline{p}_{22}{}^{I}, \overline{p}_{22}{}^{F}\right)\right] \\ \left[\left(\underline{p}_{31}{}^{T}, \underline{p}_{31}{}^{I}, \underline{p}_{31}{}^{F}\right); \left(\overline{p}_{31}{}^{T}, \overline{p}_{31}{}^{I}, \overline{p}_{31}{}^{F}\right)\right] & \left[\left(\underline{p}_{32}{}^{T}, \underline{p}_{32}{}^{I}, \underline{p}_{32}{}^{F}\right); \left(\overline{p}_{32}{}^{T}, \overline{p}_{32}{}^{I}, \overline{p}_{32}{}^{F}\right)\right] \end{vmatrix} P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$\leq \det(P) + \det(P) = \det(P)$$

Case 2. We take $K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix}$. We see that

$$K\begin{pmatrix} n-1 & n \\ 1 & 2 \end{pmatrix} = \left(\left[\left(\underline{p}_{2n-1}{}^{T}, \underline{p}_{2n-1}{}^{I}, \underline{p}_{2n-1}{}^{F}\right);\left(\overline{p}_{2n-1}{}^{T}, \overline{p}_{2n-1}{}^{I}, \overline{p}_{2n-1}{}^{F}\right)\right]\left[\left(\underline{p}_{32}{}^{T}, \underline{p}_{32}{}^{I}, \underline{p}_{32}{}^{F}\right);\left(\overline{p}_{32}{}^{T}, \overline{p}_{32}{}^{I}, \overline{p}_{32}{}^{F}\right)\right]\right)$$

$$P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$$

$$+\left(\left[\left(\underline{q}^{T}{}_{2n}, \underline{q}^{I}{}_{2n}, \underline{q}^{F}{}_{2n}\right);\left(\overline{q}^{T}{}_{2n}, \overline{q}^{I}{}_{2n}, \overline{q}^{F}{}_{2n}\right)\right]\left[\left(\underline{p}_{31}{}^{T}, \underline{p}_{31}{}^{I}, \underline{p}_{31}{}^{F}\right);\left(\overline{p}_{31}{}^{T}, \overline{p}_{31}{}^{I}, \overline{p}_{31}{}^{F}\right)\right]\right)$$

$$P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}.$$

Considering the coordinates of the elements p_{ij} involved in $P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$, we claim that

$$\left(\left[\left(\underline{p}_{2n-1}{}^{T}, \underline{p}_{2n-1}{}^{I}, \underline{p}_{2n-1}{}^{F}\right);\left(\overline{p}_{2n-1}{}^{T}, \overline{p}_{2n-1}{}^{I}, \overline{p}_{2n-1}{}^{F}\right)\right]\left[\left(\underline{p}_{32}{}^{T}, \underline{p}_{32}{}^{I}, \underline{p}_{32}{}^{F}\right);\left(\overline{p}_{32}{}^{T}, \overline{p}_{32}{}^{I}, \overline{p}_{32}{}^{F}\right)\right]\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix}$$

$$\le \det(P)$$

and

$$\left(\left[\left(\underline{q}^{T}{}_{2n}, \underline{q}^{I}{}_{2n}, \underline{q}^{F}{}_{2n}\right);\left(\overline{q}^{T}{}_{2n}, \overline{q}^{I}{}_{2n}, \overline{q}^{F}{}_{2n}\right)\right]\left[\left(\underline{p}_{31}{}^{T}, \underline{p}_{31}{}^{I}, \underline{p}_{31}{}^{F}\right);\left(\overline{p}_{31}{}^{T}, \overline{p}_{31}{}^{I}, \overline{p}_{31}{}^{F}\right)\right]\right) P\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$P\begin{pmatrix} 2 & 3 \\ n-1 & n \end{pmatrix} \le \det(P)$$

Similarly we can prove (iii).

Hence the theorem.

Theorem 7. 4. Let $P=\left[\left(\underline{p}_{ij}{}^{T}, \underline{p}_{ij}{}^{I}, \underline{p}_{ij}{}^{F}\right);\left(\overline{p}_{ij}{}^{T}, \overline{p}_{ij}{}^{I}, \overline{p}_{ij}{}^{F}\right)\right], Q=\left[\left(\underline{q}_{ij}{}^{T}, \underline{q}_{ij}{}^{I}, \underline{q}_{ij}{}^{F}\right);\left(\overline{q}_{ij}{}^{T}, \overline{q}_{ij}{}^{I}, \overline{q}_{ij}{}^{F}\right)\right]$,
$R=\left[\left(\underline{r}_{ij}{}^{T}, \underline{r}_{ij}{}^{I}, \underline{r}_{ij}{}^{F}\right);\left(\overline{r}_{ij}{}^{T}, \overline{r}_{ij}{}^{I}, \overline{r}_{ij}{}^{F}\right)\right] \in (NHSRFM)_n$. Then

(i) If $\left[\left(\underline{p}_{ii}{}^{T}, \underline{p}_{ii}{}^{I}, \underline{p}_{ii}{}^{F}\right);\left(\overline{p}_{ii}{}^{T}, \overline{p}_{ii}{}^{I}, \overline{p}_{ii}{}^{F}\right)\right] \ge \left[\left(\underline{p}_{ik}{}^{T}, \underline{p}_{ik}{}^{I}, \underline{p}_{ik}{}^{F}\right);\left(\overline{p}_{ik}{}^{T}, \overline{p}_{ik}{}^{I}, \overline{p}_{ik}{}^{F}\right)\right]$
(k= 1,2,3,…,n) for all $1 \le i \le n$, then

$$\det(P) = \left[\left(\underline{p}_{11}{}^{T}, \underline{p}_{11}{}^{I}, \underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T}, \overline{p}_{11}{}^{I}, \overline{p}_{11}{}^{F}\right)\right]\left[\left(\underline{p}_{22}{}^{T}, \underline{p}_{22}{}^{I}, \underline{p}_{22}{}^{F}\right);\left(\overline{p}_{22}{}^{T}, \overline{p}_{22}{}^{I}, \overline{p}_{22}{}^{F}\right)\right]\ldots$$
$$\left[\left(\underline{p}_{nn}{}^{T}, \underline{p}_{nn}{}^{I}, \underline{p}_{nn}{}^{F}\right);\left(\overline{p}_{nn}{}^{T}, \overline{p}_{nn}{}^{I}, \overline{p}_{nn}{}^{F}\right)\right].$$

(ii) $\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \det(P)\det(Q)$ where $(<0,0,1>)_n \in (HSRNFM)_n$.

(iii) $\det\left(PP^{T}\right) \ge \det(P)$

Proof: (i). We have

$$\begin{pmatrix} \left[\left(\underline{p}_{11}{}^{T},\underline{p}_{11}{}^{I},\underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T},\overline{p}_{11}{}^{I},\overline{p}_{11}{}^{F}\right)\right]\left[\left(\underline{p}_{22}{}^{T},\underline{p}_{22}{}^{I},\underline{p}_{22}{}^{F}\right);\left(\overline{p}_{22}{}^{T},\overline{p}_{22}{}^{I},\overline{p}_{22}{}^{F}\right)\right]\cdots \\ \left[\left(\underline{p}_{nn}{}^{T},\underline{p}_{nn}{}^{I},\underline{p}_{nn}{}^{F}\right);\left(\overline{p}_{nn}{}^{T},\overline{p}_{nn}{}^{I},\overline{p}_{nn}{}^{F}\right)\right] \end{pmatrix} \geq$$

$$\begin{pmatrix} \left[\left(\underline{p}_{1\sigma(1)}{}^{T},\underline{p}_{1\sigma(1)}{}^{I},\underline{p}_{1\sigma(1)}{}^{F}\right);\left(\overline{p}_{1\sigma(1)}{}^{T},\overline{p}_{1\sigma(1)}{}^{I},\overline{p}_{1\sigma(1)}{}^{F}\right)\right] < p^{T}{}_{2\sigma(2)}, p^{C}{}_{2\sigma(2)}, p^{U}{}_{2\sigma(2)}, p^{F}{}_{2\sigma(2)} > \cdots \\ \left[\left(\underline{p}_{n\sigma(n)}{}^{T},\underline{p}_{n\sigma(n)}{}^{I},\underline{p}_{n\sigma(n)}{}^{F}\right);\left(\overline{p}_{n\sigma(n)}{}^{T},\overline{p}_{n\sigma(n)}{}^{I},\overline{p}_{n\sigma(n)}{}^{F}\right)\right] \end{pmatrix}$$

for every $\sigma \in S_n$.

Since

$$\left[\left(\underline{p}_{11}{}^{T},\underline{p}_{11}{}^{I},\underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T},\overline{p}_{11}{}^{I},\overline{p}_{11}{}^{F}\right)\right] \geq \left[\left(\underline{p}_{ik}{}^{T},\underline{p}_{ik}{}^{I},\underline{p}_{ik}{}^{F}\right);\left(\overline{p}_{ik}{}^{T},\overline{p}_{ik}{}^{I},\overline{p}_{ik}{}^{F}\right)\right] \ (k=1,2,3,\ldots,n)$$

for all $1 \leq i \leq n$.

Hence

$$\det(P) = \sum_{\sigma\in S_n}\left[\left(\underline{p}_{1\sigma(1)}{}^{T},\underline{p}_{1\sigma(1)}{}^{I},\underline{p}_{1\sigma(1)}{}^{F}\right);\left(\overline{p}_{1\sigma(1)}{}^{T},\overline{p}_{1\sigma(1)}{}^{I},\overline{p}_{1\sigma(1)}{}^{F}\right)\right]\cdots$$
$$\left[\left(\underline{p}_{n\sigma(n)}{}^{T},\underline{p}_{n\sigma(n)}{}^{I},\underline{p}_{n\sigma(n)}{}^{F}\right);\left(\overline{p}_{n\sigma(n)}{}^{T},\overline{p}_{n\sigma(n)}{}^{I},\overline{p}_{n\sigma(n)}{}^{F}\right)\right]$$
$$=\left[\left(\underline{p}_{11}{}^{T},\underline{p}_{11}{}^{I},\underline{p}_{11}{}^{F}\right);\left(\overline{p}_{11}{}^{T},\overline{p}_{11}{}^{I},\overline{p}_{11}{}^{F}\right)\right]\left[\left(\underline{p}_{22}{}^{T},\underline{p}_{22}{}^{I},\underline{p}_{22}{}^{F}\right);\left(\overline{p}_{22}{}^{T},\overline{p}_{22}{}^{I},\overline{p}_{22}{}^{F}\right)\right]\cdots$$
$$\left[\left(\underline{p}_{nn}{}^{T},\underline{p}_{nn}{}^{I},\underline{p}_{nn}{}^{F}\right);\left(\overline{p}_{nn}{}^{T},\overline{p}_{nn}{}^{I},\overline{p}_{nn}{}^{F}\right)\right].$$

This proves (i)

(ii) $\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \left[\left(\underline{s}^{T}{}_{ij},\underline{s}^{I}{}_{ij},\underline{s}^{F}{}_{ij}\right);\left(\overline{s}^{T}{}_{ij},\overline{s}^{I}{}_{ij},\overline{s}^{F}{}_{ij}\right)\right]_{2n}$.

$$\det\begin{pmatrix} P & R \\ 0 & Q \end{pmatrix} = \sum_{\sigma\in S_{2n}}\left[\left(\underline{s}^{T}{}_{1\sigma(1)},\underline{s}^{I}{}_{1\sigma(1)},\underline{s}^{F}{}_{1\sigma(1)}\right);\left(\overline{s}^{T}{}_{1\sigma(1)},\overline{s}^{I}{}_{1\sigma(1)},\overline{s}^{F}{}_{1\sigma(1)}\right)\right]\cdots$$
$$\left[\left(\underline{s}^{T}{}_{2n\sigma(2n)},\underline{s}^{I}{}_{2n\sigma(2n)},\underline{s}^{F}{}_{2n\sigma(2n)}\right);\left(\overline{s}^{T}{}_{2n\sigma(2n)},\overline{s}^{I}{}_{2n\sigma(2n)},\overline{s}^{F}{}_{2n\sigma(2n)}\right)\right]$$

$$= \sum_{\sigma\in S_{2n},\sigma(i)\leq n\ (if\ i\leq n)}\left[\left(\underline{s}^{T}{}_{1\sigma(1)},\underline{s}^{I}{}_{1\sigma(1)},\underline{s}^{F}{}_{1\sigma(1)}\right);\left(\overline{s}^{T}{}_{1\sigma(1)},\overline{s}^{I}{}_{1\sigma(1)},\overline{s}^{F}{}_{1\sigma(1)}\right)\right]\cdots$$
$$\left[\left(\underline{s}^{T}{}_{2n\sigma(2n)},\underline{s}^{I}{}_{2n\sigma(2n)},\underline{s}^{F}{}_{2n\sigma(2n)}\right);\left(\overline{s}^{T}{}_{2n\sigma(2n)},\overline{s}^{I}{}_{2n\sigma(2n)},\overline{s}^{F}{}_{2n\sigma(2n)}\right)\right]$$

$$= \sum_{\sigma\in S_{2n},\sigma(i)\le n\,(if\, i\le n)} \left[\left(\underline{s}^{T}{}_{1\sigma(1)},\underline{s}^{I}{}_{1\sigma(1)},\underline{s}^{F}{}_{1\sigma(1)}\right);\left(\overline{s}^{T}{}_{1\sigma(1)},\overline{s}^{I}{}_{1\sigma(1)},\overline{s}^{F}{}_{1\sigma(1)}\right)\right]\ldots$$

$$\left[\left(\underline{s}^{T}{}_{2n\sigma(2n)},\underline{s}^{I}{}_{2n\sigma(2n)},\underline{s}^{F}{}_{2n\sigma(2n)}\right);\left(\overline{s}^{T}{}_{2n\sigma(2n)},\overline{s}^{I}{}_{2n\sigma(2n)},\overline{s}^{F}{}_{2n\sigma(2n)}\right)\right] + <0,0,1,1>$$

$$+ \sum_{\sigma\in S_{2n},\exists k>n,\, if\, \sigma(k)\le n} \left[\left(\underline{s}^{T}{}_{1\sigma(1)},\underline{s}^{I}{}_{1\sigma(1)},\underline{s}^{F}{}_{1\sigma(1)}\right);\left(\overline{s}^{T}{}_{1\sigma(1)},\overline{s}^{I}{}_{1\sigma(1)},\overline{s}^{F}{}_{1\sigma(1)}\right)\right]\ldots$$

$$\left[\left(\underline{s}^{T}{}_{2n\sigma(2n)},\underline{s}^{I}{}_{2n\sigma(2n)},\underline{s}^{F}{}_{2n\sigma(2n)}\right);\left(\overline{s}^{T}{}_{2n\sigma(2n)},\overline{s}^{I}{}_{2n\sigma(2n)},\overline{s}^{F}{}_{2n\sigma(2n)}\right)\right]$$

$$= \sum_{\sigma'\in S_n} \left[\left(\underline{s}^{T}{}_{1\sigma'(1)},\underline{s}^{I}{}_{1\sigma'(1)},\underline{s}^{F}{}_{1\sigma'(1)}\right);\left(\overline{s}^{T}{}_{1\sigma'(1)},\overline{s}^{I}{}_{1\sigma'(1)},\overline{s}^{F}{}_{1\sigma'(1)}\right)\right]\ldots\left[\begin{array}{l}\left(\underline{s}^{T}{}_{n\sigma'(n)},\underline{s}^{I}{}_{n\sigma'(n)},\underline{s}^{F}{}_{n\sigma'(n)}\right);\\ \left(\overline{s}^{T}{}_{n\sigma'(n)},\overline{s}^{I}{}_{n\sigma'(n)},\overline{s}^{F}{}_{n\sigma'(n)}\right)\end{array}\right]\det(Q)$$

$$= \left(\sum_{\sigma\in S_n} \left[\left(\underline{s}^{T}{}_{1\sigma(1)},\underline{s}^{I}{}_{1\sigma(1)},\underline{s}^{F}{}_{1\sigma(1)}\right);\left(\overline{s}^{T}{}_{1\sigma(1)},\overline{s}^{I}{}_{1\sigma(1)},\overline{s}^{F}{}_{1\sigma(1)}\right)\right]\ldots\left[\begin{array}{l}\left(\underline{s}^{T}{}_{n\sigma(n)},\underline{s}^{I}{}_{n\sigma(n)},\underline{s}^{F}{}_{n\sigma(n)}\right);\\ \left(\overline{s}^{T}{}_{n\sigma(n)},\overline{s}^{I}{}_{n\sigma(n)},\overline{s}^{F}{}_{n\sigma(n)}\right)\end{array}\right]\right)\det(Q)$$

$$= \det(P)\det(Q).$$

This proves (ii)

(iii) $PP^{T} = \left[\left(\underline{h}^{T}{}_{ij},\underline{h}^{I}{}_{ij},\underline{h}^{F}{}_{ij}\right);\left(\overline{h}^{T}{}_{ij},\overline{h}^{I}{}_{ij},\overline{h}^{F}{}_{ij}\right)\right]_n$,

We have, for every $\sigma \in S_n$.

$$\left[\left(\underline{h}^{T}{}_{ij},\underline{h}^{I}{}_{ij},\underline{h}^{F}{}_{ij}\right);\left(\overline{h}^{T}{}_{ij},\overline{h}^{I}{}_{ij},\overline{h}^{F}{}_{ij}\right)\right]$$

$$= \sum_{k=1}^{n} \left[\left(\underline{p}_{ik}{}^{T},\underline{p}_{ik}{}^{I},\underline{p}_{ik}{}^{F}\right);\left(\overline{p}_{ik}{}^{T},\overline{p}_{ik}{}^{I},\overline{p}_{ik}{}^{F}\right)\right]\left[\left(\underline{p}_{kj}{}^{T},\underline{p}_{kj}{}^{I},\underline{p}_{kj}{}^{F}\right);\left(\overline{p}_{kj}{}^{T},\overline{p}_{kj}{}^{I},\overline{p}_{kj}{}^{F}\right)\right].$$

$$\left[\left(\underline{h}^{T}{}_{11},\underline{h}^{I}{}_{11},\underline{h}^{F}{}_{11}\right);\left(\overline{h}^{T}{}_{11},\overline{h}^{I}{}_{11},\overline{h}^{F}{}_{11}\right)\right]\left[\left(\underline{h}^{T}{}_{22},\underline{h}^{I}{}_{22},\underline{h}^{F}{}_{22}\right);\left(\overline{h}^{T}{}_{22},\overline{h}^{I}{}_{22},\overline{h}^{F}{}_{22}\right)\right]\ldots$$

$$\left[\left(\underline{h}^{T}{}_{nn},\underline{h}^{I}{}_{nn},\underline{h}^{F}{}_{nn}\right);\left(\overline{h}^{T}{}_{nn},\overline{h}^{I}{}_{nn},\overline{h}^{F}{}_{nn}\right)\right]$$

$$= \left(\sum_{k=1}^{n}\left[\left(\underline{p}_{ik}{}^{T},\underline{p}_{ik}{}^{I},\underline{p}_{ik}{}^{F}\right);\left(\overline{p}_{ik}{}^{T},\overline{p}_{ik}{}^{I},\overline{p}_{ik}{}^{F}\right)\right]\right)\ldots\left(\sum_{k=1}^{n}\left[\left(\underline{p}_{nk}{}^{T},\underline{p}_{nk}{}^{I},\underline{p}_{nk}{}^{F}\right);\left(\overline{p}_{nk}{}^{T},\overline{p}_{nk}{}^{I},\overline{p}_{nk}{}^{F}\right)\right]\right)$$

$$\ge \left(\left[\left(\underline{p}_{1\sigma(1)}{}^{T},\underline{p}_{1\sigma(1)}{}^{I},\underline{p}_{1\sigma(1)}{}^{F}\right);\left(\overline{p}_{1\sigma(1)}{}^{T},\overline{p}_{1\sigma(1)}{}^{I},\overline{p}_{1\sigma(1)}{}^{F}\right)\right]\ldots\left[\begin{array}{l}\left(\underline{p}_{n\sigma(n)}{}^{T},\underline{p}_{n\sigma(n)}{}^{I},\underline{p}_{n\sigma(n)}{}^{F}\right);\\ \left(\overline{p}_{n\sigma(n)}{}^{T},\overline{p}_{n\sigma(n)}{}^{I},\overline{p}_{n\sigma(n)}{}^{F}\right)\end{array}\right]\right)$$

Hence

$$\det\left(PP^T\right) \geq \left[\left(\underline{h}^T{}_{11},\underline{h}^I{}_{11},\underline{h}^F{}_{11}\right);\left(\overline{h}^T{}_{11},\overline{h}^I{}_{11},\overline{h}^F{}_{11}\right)\right]\left[\left(\underline{h}^T{}_{22},\underline{h}^I{}_{22},\underline{h}^F{}_{22}\right);\left(\overline{h}^T{}_{22},\overline{h}^I{}_{22},\overline{h}^F{}_{22}\right)\right]\cdots$$
$$\left[\left(\underline{h}^T{}_{nn},\underline{h}^I{}_{nn},\underline{h}^F{}_{nn}\right);\left(\overline{h}^T{}_{nn},\overline{h}^I{}_{nn},\overline{h}^F{}_{nn}\right)\right]$$
$$\geq \sum_{\sigma\in S_n} \begin{matrix}\left[\left(\underline{p}_{1\sigma(1)}{}^T,\underline{p}_{1\sigma(1)}{}^I,\underline{p}_{1\sigma(1)}{}^F\right);\left(\overline{p}_{1\sigma(1)}{}^T,\overline{p}_{1\sigma(1)}{}^I,\overline{p}_{1\sigma(1)}{}^F\right)\right]\cdots \\ \left[\left(\underline{p}_{n\sigma(n)}{}^T,\underline{p}_{n\sigma(n)}{}^I,\underline{p}_{n\sigma(n)}{}^F\right);\left(\overline{p}_{n\sigma(n)}{}^T,\overline{p}_{n\sigma(n)}{}^I,\overline{p}_{n\sigma(n)}{}^F\right)\right]\end{matrix}$$
$$= \det\left(P\right).$$

This proves (iii)

Hence the Theorem

Theorem 7.5 Let P = (p_{ij}) be a NHSRFM. Then we have the following

$$\det\left(Padj\left(P\right)\right) = \det\left(P\right) = \det\left(adj\left(P\right)P\right).$$

Proof: We prove that $\det\left(Padj\left(P\right)\right) = \det\left(P\right)$.

We first consider n = 2.

Let $P = \begin{pmatrix} \left[\left(\underline{p}_{11}{}^T,\underline{p}_{11}{}^I,\underline{p}_{11}{}^F\right);\left(\overline{p}_{11}{}^T,\overline{p}_{11}{}^I,\overline{p}_{11}{}^F\right)\right] & \left[\left(\underline{p}_{12}{}^T,\underline{p}_{12}{}^I,\underline{p}_{12}{}^F\right);\left(\overline{p}_{12}{}^T,\overline{p}_{12}{}^I,\overline{p}_{12}{}^F\right)\right] \\ \left[\left(\underline{p}_{21}{}^T,\underline{p}_{21}{}^I,\underline{p}_{21}{}^F\right);\left(\overline{p}_{21}{}^T,\overline{p}_{21}{}^I,\overline{p}_{21}{}^F\right)\right] & \left[\left(\underline{p}_{22}{}^T,\underline{p}_{22}{}^I,\underline{p}_{22}{}^F\right);\left(\overline{p}_{22}{}^T,\overline{p}_{22}{}^I,\overline{p}_{22}{}^F\right)\right] \end{pmatrix}$

$$adj\left(P\right) = \begin{pmatrix} \left[\left(\underline{p}_{22}{}^T,\underline{p}_{22}{}^I,\underline{p}_{22}{}^F\right);\left(\overline{p}_{22}{}^T,\overline{p}_{22}{}^I,\overline{p}_{22}{}^F\right)\right] & \left[\left(\underline{p}_{12}{}^T,\underline{p}_{12}{}^I,\underline{p}_{12}{}^F\right);\left(\overline{p}_{12}{}^T,\overline{p}_{12}{}^I,\overline{p}_{12}{}^F\right)\right] \\ \left[\left(\underline{p}_{21}{}^T,\underline{p}_{21}{}^I,\underline{p}_{21}{}^F\right);\left(\overline{p}_{21}{}^T,\overline{p}_{21}{}^I,\overline{p}_{21}{}^F\right)\right] & \left[\left(\underline{p}_{11}{}^T,\underline{p}_{11}{}^I,\underline{p}_{11}{}^F\right);\left(\overline{p}_{11}{}^T,\overline{p}_{11}{}^I,\overline{p}_{11}{}^F\right)\right] \end{pmatrix}$$

$$\det\left(Padj\left(P\right)\right) =$$

$$\begin{vmatrix} \det(P) & \left[\left(\underline{p}_{11}{}^T,\underline{p}_{11}{}^I,\underline{p}_{11}{}^F\right);\left(\overline{p}_{11}{}^T,\overline{p}_{11}{}^I,\overline{p}_{11}{}^F\right)\right]\left[\left(\underline{p}_{12}{}^T,\underline{p}_{12}{}^I,\underline{p}_{12}{}^F\right);\left(\overline{p}_{12}{}^T,\overline{p}_{12}{}^I,\overline{p}_{12}{}^F\right)\right] \\ \left[\left(\underline{p}_{21}{}^T,\underline{p}_{21}{}^I,\underline{p}_{21}{}^F\right);\left(\overline{p}_{21}{}^T,\overline{p}_{21}{}^I,\overline{p}_{21}{}^F\right)\right]\left[\left(\underline{p}_{22}{}^T,\underline{p}_{22}{}^I,\underline{p}_{22}{}^F\right);\left(\overline{p}_{22}{}^T,\overline{p}_{22}{}^I,\overline{p}_{22}{}^F\right)\right] & \det(P) \end{vmatrix}$$

$$= \det(P) + \left(\left[\left(\underline{p}_{11}{}^T,\underline{p}_{11}{}^I,\underline{p}_{11}{}^F\right);\left(\overline{p}_{11}{}^T,\overline{p}_{11}{}^I,\overline{p}_{11}{}^F\right)\right]\left[\left(\underline{p}_{12}{}^T,\underline{p}_{12}{}^I,\underline{p}_{12}{}^F\right);\left(\overline{p}_{12}{}^T,\overline{p}_{12}{}^I,\overline{p}_{12}{}^F\right)\right]\right)$$
$$\left(\left[\left(\underline{p}_{21}{}^T,\underline{p}_{21}{}^I,\underline{p}_{21}{}^F\right);\left(\overline{p}_{21}{}^T,\overline{p}_{21}{}^I,\overline{p}_{21}{}^F\right)\right]\left[\left(\underline{p}_{22}{}^T,\underline{p}_{22}{}^I,\underline{p}_{22}{}^F\right);\left(\overline{p}_{22}{}^T,\overline{p}_{22}{}^I,\overline{p}_{22}{}^F\right)\right]\right)$$
$$\leq \det(P)$$

Next consider $n > 2$. We can see that

$$Padj\left(P\right) =$$

$$\begin{pmatrix} \sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{1t} & \sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{2t} & \cdots & \sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{nt} \\ \sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{1t} & \sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{2t} & \cdots & \sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{nt} \\ \cdots & \cdots & \cdots & \cdots \\ \sum\left[\left(\underline{p}_{nt}{}^{T},\underline{p}_{nt}{}^{I},\underline{p}_{nt}{}^{F}\right);\left(\overline{p}_{nt}{}^{T},\overline{p}_{nt}{}^{I},\overline{p}_{nt}{}^{F}\right)\right]P_{1t} & \sum\left[\left(\underline{p}_{nt}{}^{T},\underline{p}_{nt}{}^{I},\underline{p}_{nt}{}^{F}\right);\left(\overline{p}_{nt}{}^{T},\overline{p}_{nt}{}^{I},\overline{p}_{nt}{}^{F}\right)\right]P_{2t} & \cdots & \sum\left[\left(\underline{p}_{nt}{}^{T},\underline{p}_{nt}{}^{I},\underline{p}_{nt}{}^{F}\right);\left(\overline{p}_{nt}{}^{T},\overline{p}_{nt}{}^{I},\overline{p}_{nt}{}^{F}\right)\right]P_{nt} \end{pmatrix}$$

$$=\left(\sum\left[\left(\underline{p}_{it}{}^{T},\underline{p}_{it}{}^{I},\underline{p}_{it}{}^{F}\right);\left(\overline{p}_{it}{}^{T},\overline{p}_{it}{}^{I},\overline{p}_{it}{}^{F}\right)\right]P_{jt}\right).$$

$$\det\left(Padj\left(P\right)\right)=\sum_{\pi\in S_n}\left(\sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{\pi(1)t}\right)\left(\sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{\pi(2)t}\right)$$

$$\ldots\left(\sum\left[\left(\underline{p}_{nt}{}^{T},\underline{p}_{nt}{}^{I},\underline{p}_{nt}{}^{F}\right);\left(\overline{p}_{nt}{}^{T},\overline{p}_{nt}{}^{I},\overline{p}_{nt}{}^{F}\right)\right]P_{\pi(n)t}\right).$$

It is evident that each diagonal entry of the matrix Padj(P) equals det(P). We demonstrate this result as follows.

(i) Let us define

$$T_\pi=\left(\sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{\pi(1)t}\right)\left(\sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{\pi(2)t}\right)$$

$$\ldots\left(\sum\left[\left(\underline{p}_{nt}{}^{T},\underline{p}_{nt}{}^{I},\underline{p}_{nt}{}^{F}\right);\left(\overline{p}_{nt}{}^{T},\overline{p}_{nt}{}^{I},\overline{p}_{nt}{}^{F}\right)\right]P_{\pi(n)t}\right).$$

for $\pi\in S_n$. Let e be the identity of the group S_n. If $\pi=e,$ then $T_\pi=\det(P).$

Suppose that there exists $k\in\{1,2,\ldots,n\}$ such that $\pi(k)=k.$ Then we see that

$$=\sum\left[\left(\underline{p}_{kt}{}^{T},\underline{p}_{kt}{}^{I},\underline{p}_{kt}{}^{F}\right);\left(\overline{p}_{kt}{}^{T},\overline{p}_{kt}{}^{I},\overline{p}_{kt}{}^{F}\right)\right]P_{\pi(k)t}=\sum\left[\left(\underline{p}_{kt}{}^{T},\underline{p}_{kt}{}^{I},\underline{p}_{kt}{}^{F}\right);\left(\overline{p}_{kt}{}^{T},\overline{p}_{kt}{}^{I},\overline{p}_{kt}{}^{F}\right)\right]P_{kt}$$

$$=\det\left(P\right)$$

$$\Omega_n=\left(\sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{\pi(1)t}\right)\left(\sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{\pi(2)t}\right)$$

$$\ldots\det\left(P\right)\ldots\left(\sum\left[\left(\underline{p}_{nt}{}^{T},\underline{p}_{nt}{}^{I},\underline{p}_{nt}{}^{F}\right);\left(\overline{p}_{nt}{}^{T},\overline{p}_{nt}{}^{I},\overline{p}_{nt}{}^{F}\right)\right]P_{\pi(n)t}\right)\leq\det\left(P\right).$$

(ii) Let π be a permutation in S_n. Assume that $\pi(k)\neq k.$ for all $k\in\{1,2,\ldots,n\}$. We know that

every permutation π can be written as a product of disjoint cycles π_i and let $\pi=\pi_1\pi_2\ldots\pi_k$

We further assume that $\pi_1 = (1 \quad 2)$ transposition. Then Ω_π has two factors

$\sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{\pi(1)t}$ and

$\sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{\pi(2)t}$, and from these we see that

$$\left(\sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{\pi(1)t}\right)\left(\sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{\pi(2)t}\right)$$
$$=\left(\sum\left[\left(\underline{p}_{1t}{}^{T},\underline{p}_{1t}{}^{I},\underline{p}_{1t}{}^{F}\right);\left(\overline{p}_{1t}{}^{T},\overline{p}_{1t}{}^{I},\overline{p}_{1t}{}^{F}\right)\right]P_{2t}\right)\left(\sum\left[\left(\underline{p}_{2t}{}^{T},\underline{p}_{2t}{}^{I},\underline{p}_{2t}{}^{F}\right);\left(\overline{p}_{2t}{}^{T},\overline{p}_{2t}{}^{I},\overline{p}_{2t}{}^{F}\right)\right]P_{1t}\right)$$
$$\det\left(P\left(2\Rightarrow 1\right)\right)\det\left(P\left(1\Rightarrow 2\right)\right)\le\det\left(P\right)$$

(iii) If $\pi=\pi_1\pi_2...\pi_k$ and $\pi_1\left(s,t\right)$, then we can prove that $\Omega_n\le\det(P)$ by an argument used in (ii). Consider Ω for $\pi=\pi_1\pi_2...\pi_k$. If $\pi=\left(k,e,f,...\right)$, then we see that

$$\Omega_n=\left(\sum\left[\left(\underline{p}_{kt}{}^{T},\underline{p}_{kt}{}^{I},\underline{p}_{kt}{}^{F}\right);\left(\overline{p}_{kt}{}^{T},\overline{p}_{kt}{}^{I},\overline{p}_{kt}{}^{F}\right)\right]P_{\pi(k)t}\right)\left(\sum\left[\left(\underline{p}_{et}{}^{T},\underline{p}_{et}{}^{I},\underline{p}_{et}{}^{F}\right);\left(\overline{p}_{et}{}^{T},\overline{p}_{et}{}^{I},\overline{p}_{et}{}^{F}\right)\right]P_{\pi(e)t}\right)...$$
$$=\left(\sum\left[\left(\underline{p}_{kt}{}^{T},\underline{p}_{kt}{}^{I},\underline{p}_{kt}{}^{F}\right);\left(\overline{p}_{kt}{}^{T},\overline{p}_{kt}{}^{I},\overline{p}_{kt}{}^{F}\right)\right]P_{et}\right)\left(\sum\left[\left(\underline{p}_{et}{}^{T},\underline{p}_{et}{}^{I},\underline{p}_{et}{}^{F}\right);\left(\overline{p}_{et}{}^{T},\overline{p}_{et}{}^{I},\overline{p}_{et}{}^{F}\right)\right]P_{ft}\right)...$$
$$=\det\left(P\left(e\Rightarrow k\right)\right)\det\left(P\left(f\Rightarrow e\right)\right)...$$

we obtain that $\det\left(P\left(e\Rightarrow k\right)\right)\det\left(P\left(f\Rightarrow e\right)\right)\le\det\left(P\right)$ and so that $\Omega_n\le\det\left(P\right)$. This proves that $\det\left(Padj\left(P\right)\right)=\det\left(P\right)$. Equally, we can prove that $\det\left(adj\left(P\right)P\right)=\det\left(P\right)$.

Hence the theorem.

Theorem 7.6 Let $P,Q\in(NHSRFM)_n$. Then

(i) $\det(PQ)\ge\det(P)\det(Q)$

(ii) $\det(PQ)\le\det(P+Q)$.

Corollary 7.1. Let A be a NHSRFM, $P_r=\left(p_{ij}\right)\in(NHSRFM)_n$ (r = 1,2,3,…,m).Then

(i) $\det(P_1)\det(P_1)...\det(P_m)\le\det\left(\sum_{r=1}^{m}P_r\right)$ whrere $\sum_{r=1}^{m}P_r=\left(\sum_{r=1}^{m}a^r{}_{ij}\right)_n\in(NHSRFM)_n$.

(ii) $\det(P^{(r)})=\det(P)$, where $P=\left(p_{ij}\right)\in(NHSRFM)_n$ and $r\in N$.

Example 7.1. Consider the 4x4 NHSRFM

$$P=\begin{pmatrix} \left[\left(\underline{p}_{11}^{T},\underline{p}_{11}^{I},\underline{p}_{11}^{F}\right);\left(\overline{p}_{11}^{T},\overline{p}_{11}^{I},\overline{p}_{11}^{F}\right)\right] & \left[\left(\underline{p}_{12}^{T},\underline{p}_{12}^{I},\underline{p}_{12}^{F}\right);\left(\overline{p}_{12}^{T},\overline{p}_{12}^{I},\overline{p}_{12}^{F}\right)\right] & \cdots & \left[\left(\underline{p}_{14}^{T},\underline{p}_{14}^{I},\underline{p}_{14}^{F}\right);\left(\overline{p}_{14}^{T},\overline{p}_{14}^{I},\overline{p}_{14}^{F}\right)\right] \\ \left[\left(\underline{p}_{21}^{T},\underline{p}_{21}^{I},\underline{p}_{21}^{F}\right);\left(\overline{p}_{21}^{T},\overline{p}_{21}^{I},\overline{p}_{21}^{F}\right)\right] & \left[\left(\underline{p}_{22}^{T},\underline{p}_{22}^{I},\underline{p}_{22}^{F}\right);\left(\overline{p}_{22}^{T},\overline{p}_{22}^{I},\overline{p}_{22}^{F}\right)\right] & \cdots & \left[\left(\underline{p}_{24}^{T},\underline{p}_{24}^{I},\underline{p}_{24}^{F}\right);\left(\overline{p}_{24}^{T},\overline{p}_{24}^{I},\overline{p}_{24}^{F}\right)\right] \\ \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \cdots & \left[\left(\underline{p}_{34}^{T},\underline{p}_{34}^{I},\underline{p}_{34}^{F}\right);\left(\overline{p}_{34}^{T},\overline{p}_{34}^{I},\overline{p}_{34}^{F}\right)\right] \\ \left[\left(\underline{p}_{41}^{T},\underline{p}_{41}^{I},\underline{p}_{41}^{F}\right);\left(\overline{p}_{41}^{T},\overline{p}_{41}^{I},\overline{p}_{41}^{F}\right)\right] & \left[\left(\underline{p}_{42}^{T},\underline{p}_{42}^{I},\underline{p}_{42}^{F}\right);\left(\overline{p}_{42}^{T},\overline{p}_{42}^{I},\overline{p}_{42}^{F}\right)\right] & \cdots & \left[\left(\underline{p}_{44}^{T},\underline{p}_{44}^{I},\underline{p}_{44}^{F}\right);\left(\overline{p}_{44}^{T},\overline{p}_{44}^{I},\overline{p}_{44}^{F}\right)\right] \end{pmatrix}$$

We compute the determinant of the matrix above using the following NHSRFM.

$$|P|=\begin{vmatrix} \left[\left(\underline{p}_{11}^{T},\underline{p}_{11}^{I},\underline{p}_{11}^{F}\right);\left(\overline{p}_{11}^{T},\overline{p}_{11}^{I},\overline{p}_{11}^{F}\right)\right] & \left[\left(\underline{p}_{12}^{T},\underline{p}_{12}^{I},\underline{p}_{12}^{F}\right);\left(\overline{p}_{12}^{T},\overline{p}_{12}^{I},\overline{p}_{12}^{F}\right)\right] \\ \left[\left(\underline{p}_{21}^{T},\underline{p}_{21}^{I},\underline{p}_{21}^{F}\right);\left(\overline{p}_{21}^{T},\overline{p}_{21}^{I},\overline{p}_{21}^{F}\right)\right] & \left[\left(\underline{p}_{22}^{T},\underline{p}_{22}^{I},\underline{p}_{22}^{F}\right);\left(\overline{p}_{22}^{T},\overline{p}_{22}^{I},\overline{p}_{22}^{F}\right)\right] \end{vmatrix}_{1<2}$$

$$\begin{vmatrix} \left[\left(\underline{p}_{33}^{T},\underline{p}_{33}^{I},\underline{p}_{33}^{F}\right);\left(\overline{p}_{33}^{T},\overline{p}_{33}^{I},\overline{p}_{33}^{F}\right)\right] & \left[\left(\underline{p}_{34}^{T},\underline{p}_{34}^{I},\underline{p}_{34}^{F}\right);\left(\overline{p}_{34}^{T},\overline{p}_{34}^{I},\overline{p}_{34}^{F}\right)\right] \\ \left[\left(\underline{p}_{43}^{T},\underline{p}_{43}^{I},\underline{p}_{43}^{F}\right);\left(\overline{p}_{43}^{T},\overline{p}_{43}^{I},\overline{p}_{43}^{F}\right)\right] & \left[\left(\underline{p}_{44}^{T},\underline{p}_{44}^{I},\underline{p}_{44}^{F}\right);\left(\overline{p}_{44}^{T},\overline{p}_{44}^{I},\overline{p}_{44}^{F}\right)\right] \end{vmatrix}$$

$$+\begin{vmatrix} \left[\left(\underline{p}_{11}^{T},\underline{p}_{11}^{I},\underline{p}_{11}^{F}\right);\left(\overline{p}_{11}^{T},\overline{p}_{11}^{I},\overline{p}_{11}^{F}\right)\right] & \left[\left(\underline{p}_{13}^{T},\underline{p}_{13}^{I},\underline{p}_{13}^{F}\right);\left(\overline{p}_{13}^{T},\overline{p}_{13}^{I},\overline{p}_{13}^{F}\right)\right] \\ \left[\left(\underline{p}_{21}^{T},\underline{p}_{21}^{I},\underline{p}_{21}^{F}\right);\left(\overline{p}_{21}^{T},\overline{p}_{21}^{I},\overline{p}_{21}^{F}\right)\right] & \left[\left(\underline{p}_{23}^{T},\underline{p}_{23}^{I},\underline{p}_{23}^{F}\right);\left(\overline{p}_{23}^{T},\overline{p}_{23}^{I},\overline{p}_{23}^{F}\right)\right] \end{vmatrix}_{1<3}$$

$$\begin{matrix} \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] & \left[\left(\underline{p}_{34}^{T},\underline{p}_{34}^{I},\underline{p}_{34}^{F}\right);\left(\overline{p}_{34}^{T},\overline{p}_{34}^{I},\overline{p}_{34}^{F}\right)\right] \\ \left[\left(\underline{p}_{42}^{T},\underline{p}_{42}^{I},\underline{p}_{42}^{F}\right);\left(\overline{p}_{42}^{T},\overline{p}_{42}^{I},\overline{p}_{42}^{F}\right)\right] & \left[\left(\underline{p}_{44}^{T},\underline{p}_{44}^{I},\underline{p}_{44}^{F}\right);\left(\overline{p}_{44}^{T},\overline{p}_{44}^{I},\overline{p}_{44}^{F}\right)\right] \end{matrix}$$

$$+\begin{vmatrix} \left[\left(\underline{p}_{11}^{T},\underline{p}_{11}^{I},\underline{p}_{11}^{F}\right);\left(\overline{p}_{11}^{T},\overline{p}_{11}^{I},\overline{p}_{11}^{F}\right)\right] & \left[\left(\underline{p}_{14}^{T},\underline{p}_{14}^{I},\underline{p}_{14}^{F}\right);\left(\overline{p}_{14}^{T},\overline{p}_{14}^{I},\overline{p}_{14}^{F}\right)\right] \\ \left[\left(\underline{p}_{21}^{T},\underline{p}_{21}^{I},\underline{p}_{21}^{F}\right);\left(\overline{p}_{21}^{T},\overline{p}_{21}^{I},\overline{p}_{21}^{F}\right)\right] & \left[\left(\underline{p}_{24}^{T},\underline{p}_{24}^{I},\underline{p}_{24}^{F}\right);\left(\overline{p}_{24}^{T},\overline{p}_{24}^{I},\overline{p}_{24}^{F}\right)\right] \end{vmatrix}_{1<4}$$

$$\begin{vmatrix} \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] & \left[\left(\underline{p}_{33}^{T},\underline{p}_{33}^{I},\underline{p}_{33}^{F}\right);\left(\overline{p}_{33}^{T},\overline{p}_{33}^{I},\overline{p}_{33}^{F}\right)\right] \\ \left[\left(\underline{p}_{42}^{T},\underline{p}_{42}^{I},\underline{p}_{42}^{F}\right);\left(\overline{p}_{42}^{T},\overline{p}_{42}^{I},\overline{p}_{42}^{F}\right)\right] & \left[\left(\underline{p}_{43}^{T},\underline{p}_{43}^{I},\underline{p}_{43}^{F}\right);\left(\overline{p}_{43}^{T},\overline{p}_{43}^{I},\overline{p}_{43}^{F}\right)\right] \end{vmatrix}$$

$$+\begin{vmatrix} \left[\left(\underline{p}_{12}^{T},\underline{p}_{12}^{I},\underline{p}_{12}^{F}\right);\left(\overline{p}_{12}^{T},\overline{p}_{12}^{I},\overline{p}_{12}^{F}\right)\right] & \left[\left(\underline{p}_{13}^{T},\underline{p}_{13}^{I},\underline{p}_{13}^{F}\right);\left(\overline{p}_{13}^{T},\overline{p}_{13}^{I},\overline{p}_{13}^{F}\right)\right] \\ \left[\left(\underline{p}_{22}^{T},\underline{p}_{22}^{I},\underline{p}_{22}^{F}\right);\left(\overline{p}_{22}^{T},\overline{p}_{22}^{I},\overline{p}_{22}^{F}\right)\right] & \left[\left(\underline{p}_{23}^{T},\underline{p}_{23}^{I},\underline{p}_{23}^{F}\right);\left(\overline{p}_{23}^{T},\overline{p}_{23}^{I},\overline{p}_{23}^{F}\right)\right] \end{vmatrix}_{2<3}$$

$$\begin{vmatrix} \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{34}^{T},\underline{p}_{34}^{I},\underline{p}_{34}^{F}\right);\left(\overline{p}_{34}^{T},\overline{p}_{34}^{I},\overline{p}_{34}^{F}\right)\right] \\ \left[\left(\underline{p}_{41}^{T},\underline{p}_{41}^{I},\underline{p}_{41}^{F}\right);\left(\overline{p}_{41}^{T},\overline{p}_{41}^{I},\overline{p}_{41}^{F}\right)\right] & \left[\left(\underline{p}_{44}^{T},\underline{p}_{44}^{I},\underline{p}_{44}^{F}\right);\left(\overline{p}_{44}^{T},\overline{p}_{44}^{I},\overline{p}_{44}^{F}\right)\right] \end{vmatrix}$$

$$+\begin{vmatrix} \left[\left(\underline{p}_{12}^{T},\underline{p}_{12}^{I},\underline{p}_{12}^{F}\right);\left(\overline{p}_{12}^{T},\overline{p}_{12}^{I},\overline{p}_{12}^{F}\right)\right] & \left[\left(\underline{p}_{14}^{T},\underline{p}_{14}^{I},\underline{p}_{14}^{F}\right);\left(\overline{p}_{14}^{T},\overline{p}_{14}^{I},\overline{p}_{14}^{F}\right)\right] \\ \left[\left(\underline{p}_{22}^{T},\underline{p}_{22}^{I},\underline{p}_{22}^{F}\right);\left(\overline{p}_{22}^{T},\overline{p}_{22}^{I},\overline{p}_{22}^{F}\right)\right] & \left[\left(\underline{p}_{24}^{T},\underline{p}_{24}^{I},\underline{p}_{24}^{F}\right);\left(\overline{p}_{24}^{T},\overline{p}_{24}^{I},\overline{p}_{24}^{F}\right)\right] \end{vmatrix}_{2<4}$$

$$\begin{vmatrix} \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{33}^{T},\underline{p}_{33}^{I},\underline{p}_{33}^{F}\right);\left(\overline{p}_{33}^{T},\overline{p}_{33}^{I},\overline{p}_{33}^{F}\right)\right] \\ \left[\left(\underline{p}_{41}^{T},\underline{p}_{41}^{I},\underline{p}_{41}^{F}\right);\left(\overline{p}_{41}^{T},\overline{p}_{41}^{I},\overline{p}_{41}^{F}\right)\right] & \left[\left(\underline{p}_{43}^{T},\underline{p}_{43}^{I},\underline{p}_{43}^{F}\right);\left(\overline{p}_{43}^{T},\overline{p}_{43}^{I},\overline{p}_{43}^{F}\right)\right] \end{vmatrix}$$

$$+\begin{vmatrix} \left[\left(\underline{p}_{13}^{T},\underline{p}_{13}^{I},\underline{p}_{13}^{F}\right);\left(\overline{p}_{13}^{T},\overline{p}_{13}^{I},\overline{p}_{13}^{F}\right)\right] & \left[\left(\underline{p}_{14}^{T},\underline{p}_{14}^{I},\underline{p}_{14}^{F}\right);\left(\overline{p}_{14}^{T},\overline{p}_{14}^{I},\overline{p}_{14}^{F}\right)\right] \\ \left[\left(\underline{p}_{23}^{T},\underline{p}_{23}^{I},\underline{p}_{23}^{F}\right);\left(\overline{p}_{23}^{T},\overline{p}_{23}^{I},\overline{p}_{23}^{F}\right)\right] & \left[\left(\underline{p}_{24}^{T},\underline{p}_{24}^{I},\underline{p}_{24}^{F}\right);\left(\overline{p}_{24}^{T},\overline{p}_{24}^{I},\overline{p}_{24}^{F}\right)\right] \end{vmatrix}_{2<4}$$

$$\begin{vmatrix} \left[\left(\underline{p}_{31}^{T},\underline{p}_{31}^{I},\underline{p}_{31}^{F}\right);\left(\overline{p}_{31}^{T},\overline{p}_{31}^{I},\overline{p}_{31}^{F}\right)\right] & \left[\left(\underline{p}_{32}^{T},\underline{p}_{32}^{I},\underline{p}_{32}^{F}\right);\left(\overline{p}_{32}^{T},\overline{p}_{32}^{I},\overline{p}_{32}^{F}\right)\right] \\ \left[\left(\underline{p}_{41}^{T},\underline{p}_{41}^{I},\underline{p}_{41}^{F}\right);\left(\overline{p}_{41}^{T},\overline{p}_{41}^{I},\overline{p}_{41}^{F}\right)\right] & \left[\left(\underline{p}_{42}^{T},\underline{p}_{42}^{I},\underline{p}_{42}^{F}\right);\left(\overline{p}_{42}^{T},\overline{p}_{42}^{I},\overline{p}_{42}^{F}\right)\right] \end{vmatrix}$$

By applying this method, we can determine the determinant of the given NHSRFM.

$$P=\begin{pmatrix} (<0.8,0.7,0.3>,<0.6,0.5,0.4>)\ldots(<0.3,0.6,0.3>,<0.3,0.4,0.5>) \\ (<0.5,0.9,0.4>,<0.6,0.7,0.8>)\ldots(<0.1,0.3,0.5>,<0.6,0.7,0.8>) \\ (<0.4,0.6,0.8>,<0.1,0.2,0.3>)\ldots(<0.9,0.8,0.5>,<0.8,0.7,0.6>) \\ (<0.5,0.3,0.4>,<0.4,0.5,0.3>)\ldots(<0.7,0.3,0.2>,<0.1,0.6,0.3>) \end{pmatrix}$$

$$|P|=\begin{bmatrix} \begin{vmatrix} (<0.8,0.7,0.3>,<0.6,0.5,0.4>) & (<0.7,0.5,0.4>,<0.2,0.9,0.7>) \\ (<0.5,0.9,0.4>,<0.6,0.7,0.8>) & (<0.1,0.3,0.5>,<0.2,0.3,0.7>) \end{vmatrix} \\ \begin{vmatrix} (<0.6,0.3,0.5>,<0.7,0.8,0.9>) & (<0.9,0.8,0.5>,<0.8,0.7,0.6>) \\ (<0.1,0.3,0.6>,<0.2,0.4,0.7>) & (<0.7,0.3,0.2>,<0.1,0.6,0.3>) \end{vmatrix} \end{bmatrix}$$

$$+\begin{bmatrix} \begin{vmatrix} (<0.8,0.7,0.3>,<0.6,0.5,0.4>) & (<0.7,0.8,0.3>,<0.1,0.2,0.3>) \\ (<0.5,0.9,0.4>,<0.6,0.7,0.8>) & (<0.4,0.7,0.8>,<0.4,0.5,0.6>) \end{vmatrix} \\ \begin{vmatrix} (<0.2,0.3,0.4>,<0.2,0.3,0.5>) & (<0.9,0.8,0.5>,<0.8,0.7,0.6>) \\ (<0.4,0.8,0.5>,<0.6,0.5,0.4>) & (<0.7,0.3,0.2>,<0.1,0.6,0.3>) \end{vmatrix} \end{bmatrix}$$

$$+\begin{bmatrix} \begin{vmatrix} (<0.8,0.7,0.3>,<0.6,0.5,0.4>) & (<0.3,0.6,0.3>,<0.3,0.4,0.5>) \\ (<0.5,0.9,0.4>,<0.6,0.7,0.8>) & (<0.1,0.3,0.5>,<0.6,0.7,0.8>) \end{vmatrix} \\ \begin{vmatrix} (<0.2,0.3,0.4>,<0.2,0.3,0.5>) & (<0.6,0.3,0.5>,<0.7,0.8,0.9>) \\ (<0.4,0.8,0.5>,<0.6,0.5,0.4>) & (<0.1,0.3,0.6>,<0.2,0.4,0.7>) \end{vmatrix} \end{bmatrix}$$

$$+\left[\begin{vmatrix}\left(<0.7,0.5,0.4>,<0.2,0.9,0.7>\right) & \left(<0.7,0.8,0.3>,<0.1,0.2,0.3>\right)\\ \left(<0.1,0.3,0.5>,<0.2,0.3,0.7>\right) & \left(<0.4,0.7,0.8>,<0.4,0.5,0.6>\right)\end{vmatrix}\begin{vmatrix}\left(<0.4,0.6,0.8>,<0.1,0.2,0.3>\right) & \left(<0.9,0.8,0.5>,<0.8,0.7,0.6>\right)\\ \left(<0.5,0.3,0.4>,<0.4,0.5,0.3>\right) & \left(<0.7,0.3,0.2>,<0.1,0.6,0.3>\right)\end{vmatrix}\right]$$

$$+\left[\begin{vmatrix}\left(<0.7,0.5,0.4>,<0.2,0.9,0.7>\right) & \left(<0.3,0.6,0.3>,<0.3,0.4,0.5>\right)\\ \left(<0.1,0.3,0.5>,<0.2,0.3,0.7>\right) & \left(<0.1,0.3,0.5>,<0.6,0.7,0.8>\right)\end{vmatrix}\begin{vmatrix}\left(<0.4,0.6,0.8>,<0.1,0.2,0.3>\right) & \left(<0.6,0.3,0.5>,<0.7,0.8,0.9>\right)\\ \left(<0.5,0.3,0.4>,<0.4,0.5,0.3>\right) & \left(<0.1,0.3,0.6>,<0.2,0.4,0.7>\right)\end{vmatrix}\right]$$

$$+\left[\begin{vmatrix}\left(<0.7,0.8,0.3>,<0.1,0.2,0.3>\right) & \left(<0.3,0.6,0.3>,<0.3,0.4,0.5>\right)\\ \left(<0.4,0.7,0.8>,<0.4,0.5,0.6>\right) & \left(<0.1,0.3,0.5>,<0.6,0.7,0.8>\right)\end{vmatrix}\begin{vmatrix}\left(<0.4,0.6,0.8>,<0.1,0.2,0.3>\right) & \left(<0.2,0.3,0.4>,<0.2,0.3,0.5>\right)\\ \left(<0.5,0.3,0.4>,<0.4,0.5,0.3>\right) & \left(<0.4,0.8,0.5>,<0.6,0.5,0.4>\right)\end{vmatrix}\right]$$

$$=\left[<0.1,0.3,0.5>,<0.2,0.3,0.7>+<0.5,0.5,0.4>,<0.2,0.7,0.8>\right]$$
$$\left[<0.6,0.3,0.5>,<0.1,0.6,0.9>+<0.1,0.3,0.6>,<0.2,0.4,0.7>\right]$$

$$+\left[<0.4,0.7,0.8>,<0.4,0.5,0.6>+<0.5,0.8,0.4>,<0.1,0.2,0.8>\right]$$
$$\left[<0.2,0.3,0.4>,<0.1,0.3,0.5>+<0.4,0.8,0.5>,<0.6,0.5,0.6>\right]$$

$$+\left[<0.1,0.3,0.5>,<0.6,0.5,0.8>+<0.3,0.6,0.4>,<0.3,0.4,0.8>\right]$$
$$\left[<0.1,0.3,0.6>,<0.2,0.3,0.7>+<0.4,0.3,0.5>,<0.6,0.5,0.9>\right]$$

$$+\left[<0.4,0.5,0.8>,<0.2,0.5,0.7>+<0.1,0.3,0.5>,<0.1,0.2,0.7>\right]$$
$$\left[<0.4,0.3,0.8>,<0.1,0.2,0.3>+<0.5,0.3,0.5>,<0.4,0.5,0.6>\right]$$

$$+\left[<0.1,0.3,0.5>,<0.2,0.7,0.8>+<0.1,0.3,0.5>,<0.2,0.3,0.7>\right]$$
$$\left[<0.1,0.3,0.8>,<0.1,0.2,0.7>+<0.5,0.3,0.5>,<0.4,0.5,0.9>\right]$$

$$+\left[<0.1,0.3,0.5>,<0.1,0.2,0.8>+<0.3,0.6,0.8>,<0.3,0.4,0.6>\right]$$
$$\left[<0.4,0.6,0.8>,<0.1,0.2,0.4>+<0.2,0.3,0.4>,<0.2,0.3,0.5>\right]$$

$$=\left[<0.1,0.3,0.5>,<0.2,0.3,0.8>\right]$$

8. Conclusion and Future Work

In this study, we investigated the **determinant theory** of **Neutrosophic Hypersoft Rough Fuzzy Matrices (NHSRFMs)** and explored their fundamental structural characteristics. A refined computational technique was proposed to effectively calculate the determinant of NHSRFMs, especially in cases involving matrices with a large number of rows and columns. The efficiency and practicality of the proposed method were demonstrated through a detailed illustrative example, reinforcing its potential applicability in environments characterized by uncertainty and imprecision.

Future Work

This research opens several promising directions for future investigation:

- **Inverse and Adjoint Matrix Structures**: Extending the proposed determinant approach to define and compute the inverse and adjoint of NHSRFMs.
- **Rank and Eigenvalue Analysis**: Developing a framework for determining the rank and eigenvalues of NHSRFMs, which would contribute to their algebraic characterization.
- **Optimization in Decision-Making**: Applying NHSRFMs to multi-criteria decision-making (MCDM), medical diagnosis, and expert systems where complex uncertainties are present.
- **Dynamic and Time-Dependent NHSRFMs**: Exploring the application of NHSRFMs in dynamic systems where matrix entries may evolve over time or context.
- **Algorithmic Implementations**: Designing efficient algorithms and software tools to automate NHSRFM operations for large-scale real-world data sets.

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