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Color Blending in Image Processing Using New Operation on Neutrosophic Fuzzy Matrices

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Abstract:

In this paper, we introduce a news operation on Neutrosophic Fuzzy Matrices (NFMs), which satisfies properties are absorption, commutative, and associative. The aim of this study is to research these operations in the environment of image processing, particularly for color blending into images. By using the mathematical properties of NFMs, the proposed operation helps strong and accurate processing of complex color transitions, these developments emphasize the possible of NFMs in solving recent image processing problems with effective manner. Color blending is one of the most commonly used tools today, developed using uncertainty operators in NFMs.

Keywords: Neutrosophic Fuzzy Matrices, operations, absorption, commutative, color blending and image processing.

1. Introduction

Zadeh [1] presented fuzzy sets in 1965 to handle uncertainty, later extended by Kim and Roush [2] through fuzzy matrices for real-world applications. Atanassov [3 - 6] further improved this with intuitionistic fuzzy sets, combine both membership and non-membership values for a more extensive representation. Khan, Shymal, and Pal [7] advanced fuzzy matrix theory by introducing intuitionistic fuzzy matrices (IFM) in 2002, expanding on Atanassov's foundation. Their work clarified methods for handling uncertainties beyond nomal fuzzy matrices, with Pal [8] playing a key role in establishing IFM's theoretical and practical significance. Xu [9] and Wang [10] further contributed by IFM research through innovative and applications in various domains. Florentin Smarandache [11-17] introduced neutrosophic fuzzy matrices, integrating neutrosophy and fuzzy set theory to address indeterminacy, inconsistency, and vagueness. Muthuraji et al. [18-27] explored operators and their properties in intuitionistic fuzzy matrices, extending their applications to neutrosophic fuzzy matrices and bipolar intuitionistic fuzzy matrices. The continuously contribute to the field of fuzzy and intuitionistic fuzzy matrices. Their research on decomposition, new operations, and commutative monoids provides valuable tools and theoretical perception that enhance the relevant of these matrices in various domains, including decision-making and data analysis.

Kaur and Garg [28] utilized the linguistic neutrosophic cubic set for grayscale image processing, employing three membership degrees and aggregation procedures. Their method enhances edge detection and minute object detailing, particularly in images like the Lena image, while also ensuring faster processing and lower memory usage compared to existing techniques. Salama et al.[29] and Ghanbari Talouki [30] introduce a neutrosophic approach for processing grayscale images, effectively handling uncertainty and noise. Their method converts images into a neutrosophic domain using three membership values, improving clarity and segmentation. The approach enhances image analysis, making it useful for various applications in image processing.

Smarandache et al. [32] introduced two offset-based methods for image processing, one using neutrosophic offsets for image segmentation and another leveraging neutrosophic transformation functions for edge detection. Chaira T [33] introduced a neutrosophic set-based clustering method to detect lesions in mammograms, reducing image uncertainty. The method quantifies uncertainty using Shannon entropy and standard deviation. A neutrosophic similarity function then improves the image quality. Finally, the approach groups regions to accurately identify lesions or tumors. Samia Mandour [34] highlights the growing need for image processing as manual handling becomes impractical in fields like security, healthcare, and remote sensing. Neutrosophic logic plays a key role in managing image ambiguity, enhancing retrieval, segmentation, noise reduction, and classification. This study provides a comprehensive review of its applications in image processing from 2019 to 2023. Liu et al. [35] introduce a method to reduce noise and improve segmentation in sonar images using NSCT and neutrosophic set clustering. It removes noise while keeping edges clear, enhances brightness and details, and uses spectral clustering for accurate object detection. The method is effective and reliable, avoiding mistakes in shadowed or noisy areas. AboElHamd et al. [36] explores neutrosophic logic, emphasizing its ability to handle uncertainty, indeterminacy and inconsistency in decision-making.

Koyama and Goto [37] introduced a method for image decomposition into semi-transparent layers with advanced color blending, improving editing flexibility. Abadpour et al. [38] introduce a fuzzy PCA-based color transfer method that preserves image texture while adjusting colors based on user-selected regions and blending ratios. Their approach shows greater robustness and faster processing than existing methods. It efficiently handles 512×512 images within seconds, making it ideal for human-perception-based image processing.

2. Literature Review

i. Foundations of Fuzzy Sets and Extensions

The concept of fuzzy sets was introduced by Zadeh [1] to handle uncertainty and vagueness in mathematical models. Kim & Roush [2] was foundational work laid the groundwork for subsequent extensions, such as generalized fuzzy matrices, which provided a framework for handling fuzzy data in matrix form. Atanassov [2,3 &5] further extended fuzzy sets by introducing intuitionistic fuzzy sets (IFS), which incorporate both membership and non-membership degrees, offering a more nuanced representation of uncertainty. Atanassov [6] also explored intuitionistic fuzzy implications, which are crucial for reasoning under uncertainty. Smarandache [11] introduced neutrosophic sets, which generalize intuitionistic fuzzy sets by incorporating an indeterminacy degree, providing a more comprehensive framework for dealing with uncertainty. Smarandache [17] further expanded this by showing that neutrosophic sets generalize various other fuzzy set extensions, including Pythagorean fuzzy sets and spherical fuzzy sets.

ii. Fuzzy and Intuitionistic Fuzzy Matrices

The application of fuzzy and intuitionistic fuzzy sets to matrices has been a significant area of research. Pal et al. [7] introduced intuitionistic fuzzy matrices, providing a new tool for handling complex data structures. Pal [8] extended this work by considering fuzzy matrices with fuzzy rows and columns, further enhancing their applicability. Xu and Yager [9] developed geometric aggregation operators based on intuitionistic fuzzy sets, which are essential for multi-criteria decision-making processes. Wang et al. [10] applied intuitionistic fuzzy sets to clustering analysis, demonstrating their effectiveness in handling uncertain data in clustering tasks.

iii. Neutrosophic Sets and Matrices

Neutrosophic sets, introduced by Smarandache [11], have been widely applied in various fields. Muthuraji and colleagues have made significant contributions to the study of neutrosophic fuzzy matrices. Their work includes the exploration of commutative monoids and monoid homomorphism on Lukasiewicz conjunction and disjunction operators over neutrosophic fuzzy matrices (Muthuraji et al., [20]), as well as their application to intuitionistic fuzzy matrices (Muthuraji & Sriram, [21 &22]). Muthuraji and PunithaElizabeth [23] further extended this work by exploring properties of operations over bipolar intuitionistic fuzzy matrices. Additionally, Muthuraji and Anbukkarasi [24] investigated properties of Lukasiewicz type operators over neutrosophic fuzzy matrices.

iv. Applications in Image Processing

Neutrosophic logic has found significant applications in image processing. Kaur and Garg [29] proposed a new method using generalized linguistic neutrosophic cubic aggregation operators, demonstrating their effectiveness in image processing tasks. Salama et al. [30] applied neutrosophic logic to grayscale image processing, highlighting its potential in handling uncertainty in image data. Ghanbari Talouki et al. [31] provided a comprehensive survey of neutrosophic logic applications in image processing, emphasizing its versatility. Smarandache et al. [32] explored the use of neutrosophic offsets in digital image processing, further expanding the applications of neutrosophic logic. Chaira [33] applied neutrosophic set-based clustering for segmenting abnormal regions in mammogram images, showcasing its utility in medical imaging. Mandour [34] provided an exhaustive review of neutrosophic logic in addressing image processing issues, summarizing the state-of-the-art in this field. Liu et al. [35] developed sonar image denoising and segmentation techniques based on neutrosophic sets, demonstrating their effectiveness in underwater imaging.

v. Color Image Processing and Fuzzy Logic

In the context of color image processing, Koyama and Goto [37] proposed a method for decomposing images into layers with advanced color blending, which is useful for image editing and enhancement. Abadpour and Kasaei [38] introduced a fast and efficient fuzzy color transfer method, which is valuable for color correction and image stylization. Moore et al. [43] developed a fuzzy logic classification scheme for selecting and blending satellite ocean color algorithms, which is crucial for remote sensing applications. Han and Ma [44] introduced fuzzy color histograms for color image retrieval, providing a new method for image search and classification. Huang et al. [46] improved color mood blending between images via fuzzy relationships, enhancing the aesthetic quality of image compositions.

vi. Extensions of Fuzzy Sets and New Operations

Peng [39] introduced new operations for interval-valued Pythagorean fuzzy sets, which have applications in decision-making and pattern recognition. Silambarasan and Sriram [40] extended this work by introducing new operations for Pythagorean fuzzy matrices, providing a new tool for handling complex data structures. Silambarasan [41] further explored new operators for Fermatean fuzzy matrices, demonstrating their utility in various applications. Adak et al. [42] introduced new operations on Pythagorean fuzzy sets, further enhancing their applicability.

vii. Theoretical Contributions and Surveys

AboElHamd et al. [36] discussed the theory and applications of neutrosophic logic, providing a broad overview of its potential. Mandour [49] provided an exhaustive review of neutrosophic logic in addressing image processing issues, summarizing the state-of-the-art in this field.

The literature demonstrates the evolution of fuzzy sets from their introduction by Zadeh to the development of more sophisticated frameworks like intuitionistic and neutrosophic sets. These advancements have significantly impacted various fields, including decision-making, image processing, and remote sensing, providing powerful tools for handling uncertainty and complexity in data. The application of these concepts to matrices and image processing has opened new avenues for research and practical applications, highlighting the versatility and robustness of fuzzy logic and its extensions.

3. Research gap

Despite significant advancements in fuzzy matrix theory, gaps continue in extending these concepts to neutrosophic fuzzy matrices. X. Peng [39] introduced operations for interval-valued Pythagorean fuzzy sets, while Silambarasan et al. [40, 41] developed operations for Pythagorean and Fermatean fuzzy matrices. More recently, Amal Kumar Adak et al. [42] proposed operations on Pythagorean fuzzy sets. However, research applying these advanced operations to neutrosophic fuzzy matrices remains limited.

Moore, Timothy S., Janet W. Campbell, and Hui Feng. [43] Introduces a fuzzy classification method for ocean color satellite images, blending retrievals from class-specific algorithms to improve accuracy and avoid the patchwork effect. Han, Ju, and Kai-Kuang Ma [44] introduces the fuzzy color histogram, which improves color histogram representation by addressing color similarity and dissimilarity through a fuzzy-set membership function, resulting in better image retrieval performance than conventional color histograms.

Abadpour et al. [45] proposes a novel fuzzy based color transfer method that preserves image features while efficiently transferring color content, showing improved robustness and speed compared to existing methods. Ming-Long Huang et al, [46] presents an improved color mood blending algorithm using fuzzy relationships and Gaussian Membership Functions, yielding more natural, vivid images efficiently without swatches. Ji et al. [47] propose a color image segmentation method combining superpixel-neutrosophic C-means clustering with gradient-structural similarity, enhancing segmentation accuracy. The approach effectively handles complex image data, improving results in terms of both visual quality and computational efficiency. Pratamasunu et al, [48] introduces an edge detection method using Ant Colony Optimization based on neutrosophic sets to enhance accuracy and reduce noise.

Mandour Samia [47] provides a comprehensive survey on the application of neutrosophic logic to image processing, covering key areas like segmentation, classification, and retrieval, and highlighting improvements. This study addresses this gap by extending and adapting these operations to NFMs, providing new insights and results to enhance their theoretical and practical applications in image processing.

Structure table for highlighting the research gap:

Study	Focus Area	Contributions
X. Peng (2019)	Interval-Valued Pythagorean Fuzzy Sets	Introduced operations for interval-valued Pythagorean fuzzy sets
I. Silambarasan & S. Sriram (2019)	Pythagorean Fuzzy Matrices	Developed operations for Pythagorean fuzzy matrices
I. Silambarasan (2021)	Fermatean Fuzzy Matrices	Proposed operators for Fermatean fuzzy matrices
Amal Kumar Adak et al. (2024)	Pythagorean Fuzzy Sets	Introduced additional operations for Pythagorean fuzzy sets
Moore, Timothy S., Janet W. Campbell, and Hui Feng. (2001)	Fuzzy logic	Selecting and Blending Satellite Ocean Color Algorithms
Mandour, Samia(2023).	Neutrosophic system	Novel of Neutrosophic Logic in Image Processing
Current Study	Neutrosophic Fuzzy Matrices	Extends and adapts existing fuzzy operations to NFMs and Using NFM in color blending

4. Article Structure

Section 1: Introduction – This section introduces the fundamental concepts of Neutrosophic Fuzzy Matrices (NFMs) and their significance application in image processing in the domain of color blending. We outline the motivation behind using NFMs for handling uncertainty in images processing and discuss how our approach improves upon existing methodologies.

Section 2: Literature Review – This section provides a comprehensive view of Neutrosophic fuzzy sets, Neutrosophic fuzzy matrices and their applications in image processing. We examine existing color blending techniques, identify their strengths and limitations, and discuss the challenges.

Section 3: Research Gap Identification – After analyzing existing methods, this section highlights the key limitations and gaps in current approaches. We emphasize the need for more effective, mathematically strong, and adaptable solutions

Section 4: Article Structure - This section provides an overview of the organization of the paper.

Section 5: Contribution of This Work – This section details the novel contributions of our study, which include: The introduction of new operations on NFMs that enhance color blending accuracy. A theoretical framework that improves uncertainty management in image processing. An algorithmic approach designed for faster processing and improved visual quality in images.

Section 6: Preliminaries – This section elaborates on the mathematical principles behind our operator. We discuss the newly introduced operations on NFMs, their algebraic properties, and their impact on color blending performance.

Section 7: New operations on Neutrosophic fuzzy matrices – This section presents a detailed explanation of the new operations, including their mathematical properties such as absorption, commutativity and associativity.

Section 8 Color Blending in Image Processing Using Neutrosophic Fuzzy Matrices Algorithm – We describe their practical application in color blending, including a step-by-step algorithm and its implementation in MATLAB, along with experimental results demonstrating its effectiveness.

Section 9: Conclusion and Future Work – This final section summarizes our findings, emphasizing the advantages of the proposed method in enhancing image color blending, efficiently handling uncertainty, and achieving faster processing speeds. We also outline potential future research directions, including its applications.

5. Contribution of Our Work

New Operation on Neutrosophic Fuzzy Matrices:

We introduce a novel mathematical operation on Neutrosophic Fuzzy Matrices (NFMs) that adheres to fundamental algebraic properties such as absorption, commutativity, and associativity. This operation extends the theoretical foundation of NFMs, enabling more robust and flexible handling of uncertain and complex data structures.

RGB to Neutrosophic Fuzzy Matrix (NFM) Transformation:

Introduced a novel method to convert an image's RGB values into Neutrosophic Fuzzy Matrices, incorporating truth, indeterminacy, and falsity components for each pixel.

Component-Based Color Blending:

Developed a unique approach to blend the T (truth), I (indeterminacy), and F (falsity) components of the Neutrosophic Fuzzy Matrix, enhancing the blending process and enabling more precise image manipulation.

Normalization and Mapping:

Successfully normalized the RGB channels to the range [0,1] and mapped them into the neutrosophic domain, allowing for a more effective and flexible blending process.

Improved Fused Image Generation and Defuzzification Process:

The use of a blending control factor enabled better fusion of images, producing a visually improved output compared to traditional methods. Implemented a defuzzification method to convert the Neutrosophic Fuzzy Matrix back into standard color values, which facilitates the display of the final image in MATLAB.

Application in Image Processing

Our method leverages NFMs in the domain of image processing, with a specific focus on smooth color blending within images. This application demonstrates the practical utility of NFMs in enhancing visual data. The new operation facilitates natural and realistic color transitions, significantly enhancing the overall aesthetic and quality of images. By utilizing the neutrosophic approach, our method effectively manages uncertain or mixed color regions, resulting in clearer and more refined images.

Comparative Analysis of the Proposed NFM-Based Color Blending Method

The proposed Neutrosophic Fuzzy Matrix (NFM)-based color blending method enhances image processing by addressing uncertainty and improving the smoothness of color transitions. Unlike traditional methods such as RGB interpolation, HSV-based blending, or alpha blending, which directly manipulate pixel values and often result in abrupt transitions or noticeable artifacts, the NFM-based approach models color variations using Truth , Indeterminacy and Falsity components. This unique representation allows for a more nuanced blending process that effectively manages uncertainty in color data, leading to smoother transitions and improved color consistency. One of the key advantages of this method is its ability to preserve edges and fine details in images, which is often a challenge in conventional blending techniques. Traditional linear interpolation methods may cause loss of detail or oversaturation, while the NFM approach maintains a balance between blended colors while ensuring that sharp edges remain well-defined. Additionally, this method is computationally more efficient than deep-learning-based blending techniques, as it relies on matrix-based operations rather than iterative training processes. However, the method does have certain limitations. Its effectiveness depends on the choice of membership functions for the Truth, Indeterminacy, and Falsity values, making it sensitive to parameter selection. While it is more efficient than deep-learning models, it is still more complex than basic linear interpolation techniques. Furthermore, the method needs to be tested across a broader range of images, including medical, satellite, and artistic images, to assess its robustness in different scenarios. To improve the method further, adaptive membership functions could be introduced, allowing the system to dynamically adjust based on image characteristics. Additionally, hybrid approaches combining NFMs with deep learning techniques could lead to more advanced color reconstruction

6. Preliminaries

Definition 6.1 [7] [8] [11] [17] : An universe of U defined in a Neutrosophic set '$\square^{NFS}$' as $\square^{NFS} = \left\{ \left\{ x, p_T^N(x), p_I^N(x), p_F^N(x) \right\}, x \in X \right\}$, where $p_T^N, p_I^N, p_F^N : X \to]^-0,1^+[$ and the condition $0^- \le p_T^N(x) + p_I^N(x) + p_F^N(x) \le 3^+$ and where p_T^N is degree of truth membership, p_I^N is the degree of indeterminacy and p_F^N is the degree of false non-membership.

Definition 6.2 [20] : Let $\square^{NFS}$ and Ψ^{NFS} are two neutrosophic fuzzy set and let $P^{NMS} = \left(p_T^N, p_I^N, p_F^N\right) \in \square^{NFS}, Q^{NFS} = \left(q_T^N, q_I^N, q_F^N\right) \in \Psi^{NFS}$, define as operation of union and intersection over NFS is

$$P^{NMS} \cup Q^{NFS} = \left(\max\left(p_T^N, q_T^N\right), \min\left(p_I^N, q_I^N\right), \min\left(p_F^N, q_F^N\right)\right)$$
$$P^{NMS} \cap Q^{NFS} = \left(\min\left(p_T^N, q_T^N\right), \max\left(p_I^N, q_I^N\right), \max\left(p_F^N, q_F^N\right)\right)$$

Definition 6.3 [51]: Let $P^{NMS} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right)$ is said to be Neutrosophic Fuzzy Matrix, if the all elements are belongs to Neutrosophic Fuzzy Set.

Definition 6.4 [17]: Let P^{NFM} and Q^{NFM} in NFM then

$P^{NFM} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right), Q^{NFM} = \left(\left\langle q_{ij}^T, q_{ij}^I, q_{ij}^F \right\rangle\right) \in F_{m\times n}$, define as type operation of union and intersection over NFM as

$$P^{NFM} \cup Q^{NFM} = \left(\left\langle \max\left(p_{ij}^T, q_{ij}^T\right), \min\left(p_{ij}^I, q_{ij}^I\right), \min\left(p_{ij}^F, q_{ij}^F\right) \right\rangle\right)$$
$$P^{NFM} \cap Q^{NFM} = \left(\left\langle \min\left(p_{ij}^T, q_{ij}^T\right), \max\left(p_{ij}^I, q_{ij}^I\right), \max\left(p_{ij}^F, q_{ij}^F\right) \right\rangle\right)$$

Definition 6.5 [52]: Let $P^{NFM} = \left(\left\langle p_{ij}^T, p_{ij}^I, p_{ij}^F \right\rangle\right), Q^{NFM} = \left(\left\langle q_{ij}^T, q_{ij}^I, q_{ij}^F \right\rangle\right) \in F_{m\times n}$, P^{NFM} and Q^{NFM} be two NFM of same dimension. If $P^{NFM} \le Q^{NFM}$, If $p_{ij}^T \le q_{ij}^T, p_{ij}^I \le q_{ij}^I, p_{ij}^F \ge q_{ij}^F$, for i, j then P^{NFM} is dominated by Q^{NFM} or Q^{NFM} dominated P^{NFM} . P^{NFM} and Q^{NFM} is called comparable. If either $P^{NFM} \le Q^{NFM} (or) Q^{NFM} \le P^{NFM}, P^{NFM} < Q^{NFM}$ or $p_{ij}^T < q_{ij}^T, p_{ij}^I < q_{ij}^I, p_{ij}^F > q_{ij}^F$

Definition 6.6 [52]: Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)\in F_{m\times n}$, the complement of P with in NFM is

$P^{c}=\left(\left\langle p_{ij}^{F},p_{ij}^{I},p_{ij}^{T}\right\rangle\right)$,

7. New operations on Neutrosophic fuzzy matrices

Definition 7.1.

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size then we define

$$P^{NFM}\sqrt{\$}\,Q^{NFM}=\left(\left\langle \sqrt{p_{ij}^{T}q_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}}\right\rangle\right)$$

$$P^{NFM}\sqrt{\#}\,Q^{NFM}=\left(\left\langle \frac{\sqrt{2}p_{ij}^{T}q_{ij}^{T}}{\sqrt{\left(p_{ij}^{T}\right)^{2}+\left(q_{ij}^{T}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{I}q_{ij}^{I}}{\sqrt{\left(p_{ij}^{I}\right)^{2}+\left(q_{ij}^{I}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{F}q_{ij}^{F}}{\sqrt{\left(p_{ij}^{F}\right)^{2}+\left(q_{ij}^{F}\right)^{2}}}\right\rangle\right)$$

If $p_{ij}^{T}=q_{ij}^{T}=0$ then $\frac{p_{ij}^{T}q_{ij}^{T}}{p_{ij}^{T}+q_{ij}^{T}}=0$, $p_{ij}^{I}=q_{ij}^{I}=0$ $\frac{p_{ij}^{I}q_{ij}^{I}}{p_{ij}^{I}+q_{ij}^{I}}=0$ and If $p_{ij}^{F}=q_{ij}^{F}=0$ and

$\frac{p_{ij}^{F}q_{ij}^{F}}{p_{ij}^{F}+q_{ij}^{F}}=0$.

Example: 7.2

Let P^{NFM} and Q^{NFM} be two NFM using temperature data from four different cities of two different state recorded at four different times of the day are

$$P^{NFM}=\begin{bmatrix}\langle 0.60,0.60,0.88\rangle & \langle 0.85,0.58,0.93\rangle & \langle 0.70,0.78,0.75\rangle & \langle 0.70,0.78,0.85\rangle\\ \langle 0.70,0.85,0.75\rangle & \langle 0.70,0.65,0.85\rangle & \langle 0.73,0.85,0.80\rangle & \langle 0.73,0.85,0.78\rangle\\ \langle 0.68,0.85,0.73\rangle & \langle 0.78,0.70,0.73\rangle & \langle 0.85,0.83,0.78\rangle & \langle 0.70,0.73,0.78\rangle\\ \langle 0.80,0.85,0.73\rangle & \langle 0.78,0.75,0.70\rangle & \langle 0.68,0.73,0.78\rangle & \langle 0.73,0.78,0.73\rangle\end{bmatrix}$$

$$Q^{NFM}=\begin{bmatrix}\langle 0.65,0.70,0.60\rangle & \langle 0.83,0.68,0.90\rangle & \langle 0.63,0.75,0.83\rangle & \langle 0.65,0.75,0.73\rangle\\ \langle 0.73,0.80,0.78\rangle & \langle 0.73,0.68,0.78\rangle & \langle 0.73,0.65,0.58\rangle & \langle 0.73,0.78,0.75\rangle\\ \langle 0.65,0.83,0.65\rangle & \langle 0.75,0.65,0.73\rangle & \langle 0.83,0.78,0.80\rangle & \langle 0.68,0.65,0.63\rangle\\ \langle 0.63,0.78,0.70\rangle & \langle 0.75,0.78,0.80\rangle & \langle 0.70,0.73,0.75\rangle & \langle 0.70,0.75,0.73\rangle\end{bmatrix}$$

In the NFM, Rows represent four different cities in a state. Columns represent four different times of the day (Morning, Noon, Evening, Night). Each entry T,I,F represents: T (Truth): How much the temperature is within a normal range. I (Indeterminacy): The uncertainty due to fluctuations or measurement errors. F (Falsity): How much the temperature deviates from the expected range.

Apply $\sqrt{\$}$ in the above matrix we get,

$$P^{NFM}\sqrt{\$}\,Q^{NFM}=\begin{bmatrix}\langle 0.62,0.65,0.72\rangle & \langle 0.84,0.62,0.91\rangle & \langle 0.66,0.76,0.79\rangle & \langle 0.67,0.76,0.79\rangle\\ \langle 0.71,0.82,0.76\rangle & \langle 0.71,0.66,0.81\rangle & \langle 0.73,0.74,0.68\rangle & \langle 0.73,0.81,0.76\rangle\\ \langle 0.66,0.84,0.69\rangle & \langle 0.76,0.67,0.73\rangle & \langle 0.84,0.80,0.79\rangle & \langle 0.69,0.69,0.70\rangle\\ \langle 0.71,0.81,0.71\rangle & \langle 0.76,0.76,0.75\rangle & \langle 0.69,0.73,0.76\rangle & \langle 0.71,0.76,0.73\rangle\end{bmatrix}$$

Property 7.3.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then $P^{NFM}\sqrt{\#}\;Q^{NFM}\le P^{NFM}\sqrt{\$}\;Q^{NFM}$.

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size.

Since $\dfrac{\sqrt{2}p_{ij}^{T}q_{ij}^{T}}{\sqrt{\left(p_{ij}^{T}\right)^{2}+\left(q_{ij}^{T}\right)^{2}}}\le\sqrt{p_{ij}^{T}q_{ij}^{T}}$, $\dfrac{\sqrt{2}p_{ij}^{I}q_{ij}^{I}}{\sqrt{\left(p_{ij}^{I}\right)^{2}+\left(q_{ij}^{I}\right)^{2}}}\le\sqrt{p_{ij}^{I}q_{ij}^{I}}$ and

$\dfrac{\sqrt{2}p_{ij}^{F}q_{ij}^{F}}{\sqrt{\left(p_{ij}^{F}\right)^{2}+\left(q_{ij}^{F}\right)^{2}}}\le\sqrt{p_{ij}^{F}q_{ij}^{F}}$ for all I and j . Hence $P^{NFM}\sqrt{\#}\;Q^{NFM}\le P^{NFM}\sqrt{\$}\;Q^{NFM}$.

Proposition 7.4.

For any NFM P^{NFM}, then

i) $P^{NFM}\sqrt{\$}\,P^{NFM}=P^{NFM}$.

ii) $P^{NFM}\sqrt{\#}\;P^{NFM}=P^{NFM}$.

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ be a NFM.

$$P^{NFM}\sqrt{\$}P^{NFM}=\left(\left\langle\sqrt{p_{ij}^{T}p_{ij}^{T}},\sqrt{p_{ij}^{I}p_{ij}^{I}},\sqrt{p_{ij}^{F}p_{ij}^{F}}\right\rangle\right)=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)=P^{NFM}$$

$$P^{NFM}\sqrt{\#}\;P^{NFM}=\left(\left\langle\frac{\sqrt{2}p_{ij}^{T}p_{ij}^{T}}{\sqrt{\left(p_{ij}^{T}\right)^{2}+\left(p_{ij}^{T}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{I}p_{ij}^{I}}{\sqrt{\left(p_{ij}^{I}\right)^{2}+\left(p_{ij}^{I}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{F}p_{ij}^{F}}{\sqrt{\left(p_{ij}^{F}\right)^{2}+\left(p_{ij}^{F}\right)^{2}}}\right\rangle\right)$$

$$=\left(\left\langle\frac{\sqrt{2}\left(p_{ij}^{T}\right)^{2}}{\sqrt{2}\left(p_{ij}^{T}\right)^{2}},\frac{\sqrt{2}\left(p_{ij}^{I}\right)^{2}}{\sqrt{2}\left(p_{ij}^{I}\right)^{2}},\frac{\sqrt{2}\left(p_{ij}^{F}\right)^{2}}{\sqrt{2}\left(p_{ij}^{F}\right)^{2}}\right\rangle\right)$$

$$=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)=P^{NFM}$$

Hence Proved.

Property 7.5.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then

i) $P^{NFM}\sqrt{\$}\,Q^{NFM}=Q^{NFM}\sqrt{\$}\,P^{NFM}$.

ii) $P^{NFM}\sqrt{\#}\,Q^{NFM}=Q^{NFM}\sqrt{\#}P^{NFM}$.

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size.

i)

$$P^{NFM}\sqrt{\$}\,Q^{NFM}=\left(\left\langle \sqrt{p_{ij}^{T}q_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}}\right\rangle\right)$$

$$=\left(\left\langle \sqrt{q_{ij}^{T}p_{ij}^{T}},\sqrt{q_{ij}^{I}p_{ij}^{I}},\sqrt{q_{ij}^{F}p_{ij}^{F}}\right\rangle\right)$$

$$=Q^{NFM}\sqrt{\$}\,P^{NFM}$$

ii)

$$P^{NFM}\sqrt{\#}\;Q^{NFM}=\left(\left\langle \frac{\sqrt{2}p_{ij}^{T}q_{ij}^{T}}{\sqrt{\left(p_{ij}^{T}\right)^{2}+\left(q_{ij}^{T}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{I}q_{ij}^{I}}{\sqrt{\left(p_{ij}^{I}\right)^{2}+\left(q_{ij}^{I}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{F}q_{ij}^{F}}{\sqrt{\left(p_{ij}^{F}\right)^{2}+\left(q_{ij}^{F}\right)^{2}}}\right\rangle\right)$$

$$=\left(\left\langle \frac{\sqrt{2}q_{ij}^{T}p_{ij}^{T}}{\sqrt{\left(q_{ij}^{T}\right)^{2}+\left(p_{ij}^{T}\right)^{2}}},\frac{\sqrt{2}q_{ij}^{I}p_{ij}^{I}}{\sqrt{\left(q_{ij}^{I}\right)^{2}+\left(p_{ij}^{I}\right)^{2}}},\frac{\sqrt{2}q_{ij}^{F}p_{ij}^{F}}{\sqrt{\left(q_{ij}^{F}\right)^{2}+\left(p_{ij}^{F}\right)^{2}}}\right\rangle\right)$$

$$P^{NFM}\sqrt{\#}\;Q^{NFM}=Q^{NFM}\sqrt{\#}\;P^{NFM}$$

Hence Proved.

Property 7.6.

If P^{NFM}, Q^{NFM} and R^{NFM} are any three NFMs of same size, then

i) $\left(P^{NFM}\sqrt{\$}\;Q^{NFM}\right)\sqrt{\$}\,R^{NFM}=P^{NFM}\sqrt{\$}\left(Q^{NFM}\sqrt{\$}\,R^{NFM}\right)$.

ii) $\left(P^{NFM}\sqrt{\#}\;Q^{NFM}\right)\sqrt{\#}\,R^{NFM}=P^{NFM}\sqrt{\#}\left(Q^{NFM}\sqrt{\#}\,R^{NFM}\right)$.

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$, $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ and $R^{NFM}=\left(\left\langle r_{ij}^{T},r_{ij}^{I},r_{ij}^{F}\right\rangle\right)$ be three NFM of the same size.

By definition,

$$P^{NFM}\sqrt{\$}\,Q^{NFM}=\left(\left\langle \sqrt{p_{ij}^{T}q_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}}\right\rangle\right)$$

Then $\left(P^{NFM}\sqrt{\$}\;Q^{NFM}\right)\sqrt{\$}\,R^{NFM}=\left(\left\langle \sqrt{p_{ij}^{T}q_{ij}^{T}r_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}r_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}r_{ij}^{F}}\right\rangle\right)$ ...(1)

Here $Q^{NFM}\sqrt{\$}\ R^{NFM}=\left(\left\langle\sqrt{q_{ij}^{F}r_{ij}^{F}},\sqrt{q_{ij}^{I}r_{ij}^{I}},\sqrt{q_{ij}^{F}r_{ij}^{F}}\right\rangle\right)$

Then $P^{NFM}\sqrt{\$}\left(Q^{NFM}\sqrt{\$}\,R^{NFM}\right)=\left(\left\langle\sqrt{p_{ij}^{T}q_{ij}^{T}r_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}r_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}r_{ij}^{F}}\right\rangle\right)$ ... (2)

Therefore from (1) and (2) we get, $\left(P^{NFM}\sqrt{\$}\ Q^{NFM}\right)\sqrt{\$}\ R^{NFM}=P^{NFM}\sqrt{\$}\left(Q^{NFM}\sqrt{\$}\ R^{NFM}\right)$

Similarly we can proved $\left(P^{NFM}\sqrt{\#}\ \ Q^{NFM}\right)\sqrt{\#}\ \ R^{NFM}=P^{NFM}\sqrt{\#}\left(Q^{NFM}\sqrt{\#}\ \ R^{NFM}\right)$.

Property 7.7.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then

i) $\left(P^{NFM}\sqrt{\$}\,Q^{NFM}\right)^{c}=\left(\left(P^{NFM}\right)^{c}\sqrt{\$}\left(Q^{NFM}\right)^{c}\right)$

ii) $\left(P^{NFM}\sqrt{\#}\,Q^{NFM}\right)^{c}=\left(\left(P^{NFM}\right)^{c}\sqrt{\#}\left(Q^{NFM}\right)^{c}\right)$

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size.

i) By definition,

$$P^{NFM}\sqrt{\$}\,Q^{NFM}=\left(\left\langle\sqrt{p_{ij}^{T}q_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}}\right\rangle\right)$$

$$\left(P^{NFM}\sqrt{\$}\,Q^{NFM}\right)^{c}=\left(\left\langle\sqrt{p_{ij}^{F}q_{ij}^{F}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{T}q_{ij}^{T}}\right\rangle\right) \quad \ldots(3)$$

$$\left(P^{NFM}\right)^{c}=\left(\left\langle p_{ij}^{F},p_{ij}^{I},p_{ij}^{T}\right\rangle\right),\ \left(Q^{NFM}\right)^{c}=\left(\left\langle q_{ij}^{F},q_{ij}^{I},q_{ij}^{T}\right\rangle\right)$$

$$\left(\left(P^{NFM}\right)^{c}\sqrt{\$}\left(Q^{NFM}\right)^{c}\right)=\left(\left\langle\sqrt{p_{ij}^{F}q_{ij}^{F}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{T}q_{ij}^{T}}\right\rangle\right) \quad \ldots(4)$$

From (3) and (4), we get

$$\left(P^{NFM}\sqrt{\$}\,Q^{NFM}\right)^{c}=\left(\left(P^{NFM}\right)^{c}\sqrt{\$}\left(Q^{NFM}\right)^{c}\right)$$

ii)

$$\left(P^{NFM}\sqrt{\#}\ \ Q^{NFM}\right)^{c}=\left(\left\langle\frac{\sqrt{2}\,p_{ij}^{F}q_{ij}^{F}}{\sqrt{\left(p_{ij}^{F}\right)^{2}+\left(q_{ij}^{F}\right)^{2}}},\frac{\sqrt{2}\,p_{ij}^{I}q_{ij}^{I}}{\sqrt{\left(p_{ij}^{I}\right)^{2}+\left(q_{ij}^{I}\right)^{2}}},\frac{\sqrt{2}\,p_{ij}^{T}q_{ij}^{T}}{\sqrt{\left(p_{ij}^{T}\right)^{2}+\left(q_{ij}^{T}\right)^{2}}}\right\rangle\right) \quad \ldots(5)$$

$$\left(P^{NFM}\right)^{c}=\left(\left\langle p_{ij}^{F},p_{ij}^{I},p_{ij}^{T}\right\rangle\right),\ \left(Q^{NFM}\right)^{c}=\left(\left\langle q_{ij}^{F},q_{ij}^{I},q_{ij}^{T}\right\rangle\right)$$

Apply

$$\left(\left(P^{NFM}\right)^{c}\sqrt{\#}\left(Q^{NFM}\right)^{c}\right)=\left(\left\langle \frac{\sqrt{2}p_{ij}^{F}q_{ij}^{F}}{\sqrt{\left(p_{ij}^{F}\right)^{2}+\left(q_{ij}^{F}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{I}q_{ij}^{I}}{\sqrt{\left(p_{ij}^{I}\right)^{2}+\left(q_{ij}^{I}\right)^{2}}},\frac{\sqrt{2}p_{ij}^{T}q_{ij}^{T}}{\sqrt{\left(p_{ij}^{T}\right)^{2}+\left(q_{ij}^{T}\right)^{2}}}\right\rangle\right) \quad \ldots(6)$$

From (5) and (6)

$$\left(P^{NFM}\sqrt{\#}Q^{NFM}\right)^{c}=\left(\left(P^{NFM}\right)^{c}\sqrt{\#}\left(Q^{NFM}\right)^{c}\right)$$

Property 7.8.

If P^{NFM} and Q^{NFM} are any two NFMs of same size, then

$$\left(\mathrm{P}^{\mathrm{NFM}}\cup\mathrm{Q}^{\mathrm{NFM}}\right)\sqrt{\$}\left(\mathrm{P}^{\mathrm{NFM}}\cap\mathrm{Q}^{\mathrm{NFM}}\right)=\mathrm{P}^{\mathrm{NFM}}\sqrt{\$}\,\mathrm{Q}^{\mathrm{NFM}}$$

Proof:

Let $P^{NFM}=\left(\left\langle p_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$ and $Q^{NFM}=\left(\left\langle q_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$ be two NFM of the same size.

$$\mathrm{P}^{\mathrm{NFM}}\cup\mathrm{Q}^{\mathrm{NFM}}=\left(\left\langle \max\left(p_{ij}^{T},q_{ij}^{T}\right),\min\left(p_{ij}^{I},q_{ij}^{I}\right),\min\left(p_{ij}^{F},q_{ij}^{F}\right)\right\rangle\right)$$

$$\mathrm{P}^{\mathrm{NFM}}\cap\mathrm{Q}^{\mathrm{NFM}}=\left(\left\langle \min\left(p_{ij}^{T},q_{ij}^{T}\right),\max\left(p_{ij}^{I},q_{ij}^{I}\right),\max\left(p_{ij}^{F},q_{ij}^{F}\right)\right\rangle\right)$$

Case: 1 If $p_{ij}^{T}<q_{ij}^{T}$, $p_{ij}^{I}<q_{ij}^{I}$, $p_{ij}^{F}<q_{ij}^{F}$ then

$$\mathrm{P}^{\mathrm{NFM}}\cup\mathrm{Q}^{\mathrm{NFM}}=\left(\left\langle q_{ij}^{T},p_{ij}^{I},p_{ij}^{F}\right\rangle\right)$$

$$\mathrm{P}^{\mathrm{NFM}}\cap\mathrm{Q}^{\mathrm{NFM}}=\left(\left\langle p_{ij}^{T},q_{ij}^{I},q_{ij}^{F}\right\rangle\right)$$

$$\left(\mathrm{P}^{\mathrm{NFM}}\cup\mathrm{Q}^{\mathrm{NFM}}\right)\sqrt{\$}\left(\mathrm{P}^{\mathrm{NFM}}\cap\mathrm{Q}^{\mathrm{NFM}}\right)=\left(\left\langle\sqrt{p_{ij}^{T}q_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}}\right\rangle\right)=\mathrm{P}^{\mathrm{NFM}}\sqrt{\$}\,\mathrm{Q}^{\mathrm{NFM}}$$

Case: 2 If $p_{ij}^{T}<q_{ij}^{T}$, $p_{ij}^{I}<q_{ij}^{I}$, $p_{ij}^{F}>q_{ij}^{F}$ then

$$\mathrm{P}^{\mathrm{NFM}}\cap\mathrm{Q}^{\mathrm{NFM}}=\left(\left\langle p_{ij}^{T},q_{ij}^{I},p_{ij}^{F}\right\rangle\right)$$

$$\mathrm{P}^{\mathrm{NFM}}\cup\mathrm{Q}^{\mathrm{NFM}}=\left(\left\langle q_{ij}^{T},p_{ij}^{I},q_{ij}^{F}\right\rangle\right)$$

$$\left(\mathrm{P}^{\mathrm{NFM}}\cup\mathrm{Q}^{\mathrm{NFM}}\right)\sqrt{\$}\left(\mathrm{P}^{\mathrm{NFM}}\cap\mathrm{Q}^{\mathrm{NFM}}\right)=\left(\left\langle\sqrt{p_{ij}^{T}q_{ij}^{T}},\sqrt{p_{ij}^{I}q_{ij}^{I}},\sqrt{p_{ij}^{F}q_{ij}^{F}}\right\rangle\right)=\mathrm{P}^{\mathrm{NFM}}\sqrt{\$}\,\mathrm{Q}^{\mathrm{NFM}}$$

Case: 3 If $p_{ij}^{T}<q_{ij}^{T}$, $p_{ij}^{I}>q_{ij}^{I}$, $p_{ij}^{F}>q_{ij}^{F}$ then

$$P^{NFM} \cup Q^{NFM} = \left(\left\langle q_{ij}^{T}, q_{ij}^{I}, q_{ij}^{F} \right\rangle\right)$$
$$P^{NFM} \cap Q^{NFM} = \left(\left\langle p_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F} \right\rangle\right)$$
$$\left(P^{NFM} \cup Q^{NFM}\right)\sqrt{\$}\left(P^{NFM} \cap Q^{NFM}\right) = \left(\left\langle \sqrt{p_{ij}^{T} q_{ij}^{T}}, \sqrt{p_{ij}^{I} q_{ij}^{I}}, \sqrt{p_{ij}^{F} q_{ij}^{F}} \right\rangle\right) = P^{NFM}\sqrt{\$}\,Q^{NFM}$$

Case: 4 If $p_{ij}^{T} < q_{ij}^{T}$, $p_{ij}^{I} > q_{ij}^{I}$, $p_{ij}^{F} < q_{ij}^{F}$ then

$$P^{NFM} \cup Q^{NFM} = \left(\left\langle q_{ij}^{T}, q_{ij}^{I}, p_{ij}^{F} \right\rangle\right)$$
$$P^{NFM} \cap Q^{NFM} = \left(\left\langle p_{ij}^{T}, p_{ij}^{I}, q_{ij}^{F} \right\rangle\right)$$
$$\left(P^{NFM} \cup Q^{NFM}\right)\sqrt{\$}\left(P^{NFM} \cap Q^{NFM}\right) = \left(\left\langle \sqrt{p_{ij}^{T} q_{ij}^{T}}, \sqrt{p_{ij}^{I} q_{ij}^{I}}, \sqrt{p_{ij}^{F} q_{ij}^{F}} \right\rangle\right) = P^{NFM}\sqrt{\$}\,Q^{NFM}$$

Case: 5 If $p_{ij}^{T} > q_{ij}^{T}$, $p_{ij}^{I} < q_{ij}^{I}$, $p_{ij}^{F} > q_{ij}^{F}$ then

$$P^{NFM} \cup Q^{NFM} = \left(\left\langle p_{ij}^{T}, p_{ij}^{I}, q_{ij}^{F} \right\rangle\right)$$
$$P^{NFM} \cap Q^{NFM} = \left(\left\langle q_{ij}^{T}, q_{ij}^{I}, p_{ij}^{F} \right\rangle\right)$$
$$\left(P^{NFM} \cup Q^{NFM}\right)\sqrt{\$}\left(P^{NFM} \cap Q^{NFM}\right) = \left(\left\langle \sqrt{p_{ij}^{T} q_{ij}^{T}}, \sqrt{p_{ij}^{I} q_{ij}^{I}}, \sqrt{p_{ij}^{F} q_{ij}^{F}} \right\rangle\right) = P^{NFM}\sqrt{\$}\,Q^{NFM}$$

Case: 6 If $p_{ij}^{T} > q_{ij}^{T}$, $p_{ij}^{I} > q_{ij}^{I}$, $p_{ij}^{F} > q_{ij}^{F}$ then

$$P^{NFM} \cup Q^{NFM} = \left(\left\langle p_{ij}^{T}, q_{ij}^{I}, q_{ij}^{F} \right\rangle\right)$$
$$P^{NFM} \cap Q^{NFM} = \left(\left\langle q_{ij}^{T}, p_{ij}^{I}, p_{ij}^{F} \right\rangle\right)$$
$$\left(P^{NFM} \cup Q^{NFM}\right)\sqrt{\$}\left(P^{NFM} \cap Q^{NFM}\right) = \left(\left\langle \sqrt{p_{ij}^{T} q_{ij}^{T}}, \sqrt{p_{ij}^{I} q_{ij}^{I}}, \sqrt{p_{ij}^{F} q_{ij}^{F}} \right\rangle\right) = P^{NFM}\sqrt{\$}\,Q^{NFM}$$

From the six case, we get

$$\left(P^{NFM} \cup Q^{NFM}\right)\sqrt{\$}\left(P^{NFM} \cap Q^{NFM}\right) = P^{NFM}\sqrt{\$}\,Q^{NFM}$$

8. Color Blending in Image Processing Using Neutrosophic Fuzzy Matrices Algorithm

The proposed method was implemented by using MATLAB software.

1. Converting the Image into RGB Matrices: The first step in the NFM-based color blending process involves converting the input image into a matrix representation of its pixel colors. Each pixel in the image is made up of three color channels: Red, Green, and Blue (RGB). These values are typically represented as numerical values, with each color channel having a range between 0 and 255. The normal color matric represented as "P"
2. Converting the RGB Matrix into Neutrosophic Fuzzy Matrix (NFM) Components
   - Each pixel's RGB value is transformed into a Neutrosophic Fuzzy Matrix
   - F or an RGB pixel (R,G,B), the corresponding Neutrosophic Fuzzy Matrix is represented as: NFM (T I F) = NFM (R G B)

     T (Truth): Red color channel.

I (Indeterminacy): Green color channel.
F (Falsity): Blue color channel.

- Each of these color channels is normalized to the range [0, 1]. Each P_{ij}^{NFM} is taken to the neutrosophic domain containing a triplet of elements corresponding to the truthfulness, indeterminacy and falsehood that is a pixel.

This is denoted by $P_{ij}^{NFM} = \left[\left\langle p^{T}(i,j), p^{I}(i,j), p^{F}(i,j)\right\rangle\right]$

3. Blending the Components

We blend the T, I, and F components using an $\sqrt{\$}$, which controls how much influence each Image has in the final fused image.

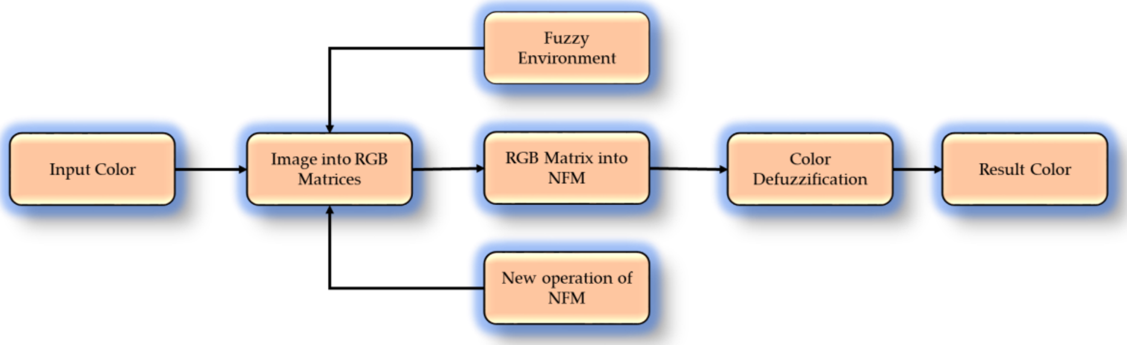
The formula used for blending with each component of NFM is

$$r^{T}(i,j) = \{p^{T}(k,l) / p^{T}(i,j)q^{T}(i,j), 0 \le i,j \le 1\}$$

$$r^{I}(i,j) = \{p^{I}(k,l) / p^{I}(i,j)q^{I}(i,j), 0 \le i,j \le 1\}$$ and

$$r^{F}(i,j) = \{p^{F}(k,l) / p^{F}(i,j)q^{F}(i,j), 0 \le i,j \le 1\}$$

4. Defuzzification and Converting the Neutrosophic Fuzzy Matrix Back into Standard Color Values in MATLAB will display the final image



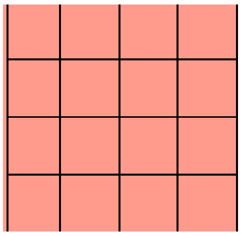
Flowchart for working Algorithm

The process of using Neutrosophic Matrix Operations for color images begins with an RGB image, where each pixel is represented by its Red, Green and Blue components. Instead of directly processing these values, the image is converted into a Neutrosophic Fuzzy Matrix, where each pixel is characterized by three properties: Truth, Falsity and Indeterminacy. This transformation allows for a more flexible and intelligent way of analyzing color variations, especially in complex images with uncertain or mixed colors. To improve clarity, a process called "Color Dehrazification" is applied, which reduces fuzziness and enhances the distinction between different color regions, making colors appear more defined and natural. After this refinement, the modified NFM is converted back into a standard RGB image, resulting in improved color blending with better sharpness and reduced uncertainty. Furthermore, additional advanced operations on the NFM can be performed to refine the image further, enhance details, or analyze specific features using Neutrosophic logic

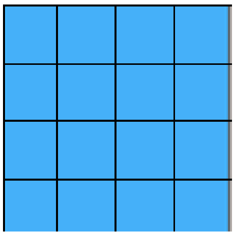
Apply this algorithm color image

Color image

Matrix with RGB values



$$P=\begin{bmatrix}\langle 255,155,140\rangle & \langle 255,156,156\rangle & \langle 255,145,143\rangle & \langle 255,143,156\rangle \\ \langle 255,156,156\rangle & \langle 250,143,156\rangle & \langle 248,138,143\rangle & \langle 249,145,157\rangle \\ \langle 252,145,143\rangle & \langle 255,156,157\rangle & \langle 249,145,156\rangle & \langle 253,148,156\rangle \\ \langle 255,156,157\rangle & \langle 255,145,143\rangle & \langle 249,145,156\rangle & \langle 253,148,156\rangle\end{bmatrix}$$



$$Q=\begin{bmatrix}\langle 69,176,249\rangle & \langle 75,170,240\rangle & \langle 70,176,255\rangle & \langle 55,180,236\rangle \\ \langle 73,156,240\rangle & \langle 68,186,236\rangle & \langle 57,167,240\rangle & \langle 58,177,240\rangle \\ \langle 73,156,240\rangle & \langle 64,165,248\rangle & \langle 69,170,244\rangle & \langle 65,187,249\rangle \\ \langle 71,179,245\rangle & \langle 67,166,245\rangle & \langle 67,176,250\rangle & \langle 77,176,236\rangle\end{bmatrix}$$

Converting the RGB Matrix into Neutrosophic Fuzzy Matrix (NFM) Components

$$P_{ij}^{NFM}=\begin{bmatrix}\langle 1,0.607843,0.54902\rangle & \langle 1,0.611765,0.611765\rangle & \langle 1,0.568627,0.560784\rangle & \langle 1,0.560784,0.611765\rangle \\ \langle 1,0.611765,0.611765\rangle & \langle 0.980392,0.560784,0.611765\rangle & \langle 0.972549,0.541176,0.560784\rangle & \langle 0.976471,0.568627,0.615686\rangle \\ \langle 0.988235,0.568627,0.560784\rangle & \langle 1,0.611765,0.615686\rangle & \langle 0.976471,0.568627,0.611765\rangle & \langle 0.992157,0.580392,0.611765\rangle \\ \langle 1,0.611765,0.615686\rangle & \langle 1,0.568627,0.560784\rangle & \langle 0.976471,0.568627,0.611765\rangle & \langle 0.992157,0.580392,0.611765\rangle\end{bmatrix}$$

$$Q_{ij}^{NFM}=\begin{bmatrix}\langle 0.270588, 0.690196, 0.976471\rangle & \langle 0.294118, 0.666667, 0.941176\rangle & \langle 0.27451,0.690196,1\rangle & \langle 0.215686,0.705882,0.92549\rangle \\ \langle 0.286275, 0.611765, 0.941176\rangle & \langle 0.266667,0.729412,0.92549\rangle & \langle 0.223529,0.654902,0.941176\rangle & \langle 0.227451, 0.694118, 0.941176\rangle \\ \langle 0.254902, 0.666667,0.956863\rangle & \langle 0.25098, 0.647059,0.972549\rangle & \langle 0.270588, 0.666667, 0.956863\rangle & \langle 0.254902,0.733333, 0.976471\rangle \\ \langle 0.278431, 0.701961, 0.960784\rangle & \langle 0.262745, 0.65098,0.960784\rangle & \langle 0.262745,0.690196,0.980392\rangle & \langle 0.301961,0.690196,0.92549\rangle\end{bmatrix}$$

Blending the Components

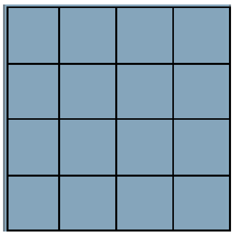
Apply operation for Neutrosophic Fuzzy Matrix P_{ij}^{NFM} and Q_{ij}^{NFM} , we get new Neutrosophic Fuzzy

Matrix as represented as R_{ij}^{NFM}

$$R_{ij}^{NFM}=\begin{bmatrix}\langle 0.520181,0.647712,0.73219\rangle & \langle 0.542326, 0.638626, 0.758801\rangle & \langle 0.523937, 0.62647, 0.7488551\rangle & \langle 0.46442, 0.629164, 0.752451\rangle \\ \langle 0.535046, 0.611765, 0.758801\rangle & \langle 0.51131, 0.639564, 0.752451\rangle & \langle 0.466255, 0.59533, 0.726496\rangle & \langle 0.471274, 0.628247, 0.761229\rangle \\ \langle 0.5019, 0.615699, 0.732526\rangle & \langle 0.500979, 0.629164, 0.773812\rangle & \langle 0.514025, 0.615699, 0.765098\rangle & \langle 0.502894, 0.652396, 0.772897\rangle \\ \langle 0.527666, 0.655313, 0.769117\rangle & \langle 0.512587, 0.608412, 0.734025\rangle & \langle 0.50652, 0.62647, 0.774448\rangle & \langle 0.54735, 0.632917, 0.752451\rangle\end{bmatrix}$$

Resultant Color image

Resultant Matrix with RGB values



$$R = \begin{bmatrix} \langle 133,165,187 \rangle & \langle 138,163,193 \rangle & \langle 134,160,191 \rangle & \langle 118,160,192 \rangle \\ \langle 136,156,193 \rangle & \langle 130,163,192 \rangle & \langle 119,152,185 \rangle & \langle 120,160,194 \rangle \\ \langle 128,157,187 \rangle & \langle 128,160,197 \rangle & \langle 131,157,195 \rangle & \langle 128,166,197 \rangle \\ \langle 135,167,196 \rangle & \langle 131,155,187 \rangle & \langle 129,160,197 \rangle & \langle 140,161,192 \rangle \end{bmatrix}$$

The result of the color image produced through the Neutrosophic Fuzzy Matrices (NFM) color blending process reveals notable improvements in both visual quality and computational efficiency. The final image exhibits more precise color reproduction, especially in regions with intricate gradients and transitions. The algorithm, implemented in MATLAB, generates clear and enhanced images, showcasing its effectiveness in improving overall image quality.

Result and future work

The results of this study demonstrate the effectiveness of the newly introduced operation on Neutrosophic Fuzzy Matrices, which satisfies the properties of absorption, commutativity, and associativity. The application of this operation in image processing, particularly in color blending, has shown notable improvements in handling complex color transitions. By leveraging the advanced mathematical properties of Neutrosophic Fuzzy Matrices, the proposed approach enhances visual coherence and computational efficiency, resulting in clearer and more refined images.

For future work, further exploration can be conducted to extend these operations to other image processing tasks, such as image enhancement, segmentation, and pattern recognition. Additionally, integrating deep learning techniques with Neutrosophic Fuzzy Matrices could open new possibilities for automated and adaptive image processing applications. The potential of this approach in medical imaging, remote sensing, and real-time image analysis will also be an interesting area for future research.

9. Conclusions

In this paper presents a new operation on Neutrosophic Fuzzy Matrices that satisfies the properties of absorption, commutativity, and associativity. The study investigated the application of these operations in image processing, particularly in color blending. By utilizing the advanced mathematical properties of Neutrosophic Fuzzy Matrices, the proposed operation enhances the processing of complex color transitions, ensuring both improved visual coherence and computational efficiency. These developments demonstrate the significant potential of Neutrosophic Fuzzy Matrices in overcoming challenges in modern image processing.

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Conflicts of Interest: The authors declare no conflict of interest.

References

1. Zadeh. Fuzzy Sets. *Journal of information and control*, **1965**, *Volume*. 8, pp. 338–353.

2. Ki Hang Kim and Fred W. Roush. Generalized Fuzzy Matrices. *Fuzzy Sets and Systems*, **1980**, *Volume*. 4, pp. 293–315.
3. Atanassov. K. T, (**1983**), Intuitionistic Fuzzy Sets, VII ITKR's Section, Sofia (Deposed inCentral Sci.-Techn. Library of Bulg.Acad. of Sci.,1697/84).
4. Atanassov. Intuitionistic Fuzzy Sets. *Fuzzy Sets and Systems*, **1986**, *Volume*. 20, pp. 87 – 96.
5. Atanassov. K. T, New operations defined over the Intuitionistic Fuzzy Sets, *Fuzzy sets and Systems*, **1994**, *Volume* 61, issues 2, pp. 137–142.
6. Atanassov. K. On some Intuitionistic Fuzzy Implications. *Mathematiques Sous – ensembles flouse*, **2006**, *Volume*. 1, no. 1, pp. 19–24.
7. Pal. M., S. K. Khan., A. K. Shymal. Intuitionistic fuzzy matrices. *Notes on Intuitionistic Fuzzy Sets*, **2002**, *Volume*. 8, pp. 51-62.
8. Pal M. Fuzzy matrices with fuzzy rows and columns. *Journal of Intelligent & Fuzzy Systems*, **2016**, *Volume*. 30(1), pp. 561-573.
9. Xu, Z., & Yager, R. R. Some geometric aggregation operators based on intuitionistic fuzzy sets. *International journal of general systems*, **2016**, *Volume*. 35, no. 4, pp. 417-433.
10. Zhong Wang., Zeshui Xu., Shoushing Liu., Jian Tang. A netting clustering analysis method under Intuitionistic fuzzy environment. *Applied Soft Computing*, **2011**, *Volume*. 11, pp. 5558–5564.
11. Smarandache F. A unifying Field in logics. Neutrosophy/ Neutrosophic Probability, Set and Logic. Rehoboth: *American Research Press*, **1998**.
12. Smarandach. F, Neutrosophic Set a Generalization of the Intuitionistic Fuzzy Set, *International Journal of Pure and Applied Mathematics*, **2005,** *Volume* 24, pp. 287–297.
13. Smarandache. F, Neutrosophic set–a generalization of the intuitionistic fuzzy set, *Journal of Defense Resources Management (JoDRM)*, **2020**, *Volume* 1(1), 107-116.
14. Smarandache. F, Introduction to neutrosophic measure, neutrosophic integral and neutrosophic probability, *Sitech*, **2013**.
15. Smarandache. F, Introduction to neutrosophic statistics, *Sitech,* **2014**.
16. Smarandache. F, Degree of Dependence and Independence of the subcomponents of Fuzzy Set and Neutrosophic Set, *Neutrosophic Sets and Systems*, **2016**, *Volume* 11, pp. 95- 97.
17. Smarandache. F, Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision (revisited), *Journal of New Theory*, **2019**, *Volume* 29, pp. 01-35.
18. Muthuraji. T and Sriram. S, Representation and Decomposition of an Intuitionistic Fuzzy Matrix Using Some (α, α') Cuts, *Applications and Applied Mathematics: An International Journal*, **2017**, *Volume* 12(1), 16, pp. 241–258.
19. Muthuraji. T and Lalitha. K, Some new operations and its properties on intuitionistic fuzzy matrices, *International Journal of Research and Analytical Reviews*, **2017**, *Volume* 4 (4), pp. 735-741.
20. Muthuraji. T., A. Anbukkarasi., T. Soupramanien. Commutative Monoids and Monoid homomorphism on Lukasiwicz conjunction and disjunction operators over neutrosophic fuzzy matrices. *AIP Conference Proceedings 2852*, 030003 (**2023**) https://doi.org/10.1063/5.0164778 .
21. Muthuraji. T., S. Sriram. Some remarks on Lukasiwicz disjunction and conjunction operators on intuitionistic fuzzy matrices. *Journal of Advances in Mathematics*, **2015**, *Volume*. 11(3), pp. 5000–5006.
22. Muthuraji. T., S. Sriram. Commutative monoids and monoid homomorphism on Lukasiwicz conjunction and disjunction operators over intuitionistic fuzzy matrices. *International Journal of Pure and Engineering Mathematics*, **2015**, *Volume*. 3(2), pp. 63–75.
23. Muthuraji, PunithaElizabeth, SOME PROPERTIES OF OPERATIONS CONJUNCTION, DISJUNCTION, AND IMPLICATION FROM LUKASIEWICZ'STYPE OVER BIPOLAR INTUITIONISTIC FUZZY MATRICES, *Journal of Nonlinear Analysis and Optimization*, **2024**, *Volume*. 15(1), pp. 57-64.
24. Muthuraji, Anbukkarasi, Some Properties on Lukasiwicz Type (1&2) Operators Over Neutrosophic Fuzzy Matrices, *Indian Journal of Natural Science*, **2025**, *Volume*. 15(85), pp. 5000–78503.
25. Muthuraji. T and B. Anitha, Lukasiewicz implication on Intuitionistic Fuzzy Tautological Matrices, *Journal of Emerging Technology and innavative Research*, **2019**, *Volume* 6(4), pp. 1-6.

26. Muthuraji. T and Punitha. R, Commutative monoids on algebraic sum and algebraic product over bipolar fuzzy set and bipolar fuzzy matrices, *Eur. Chem. Bull*, **2023**, *Volume* 12(4), pp. 18843-18852.
27. Muthuraji. T and Elizabeth. P. P, Commutative Monoid on Bounded Sum and Bounded Product over Bipolar Intuitionistic Fuzzy Matrices, *Chinese Journal of Computational Mechanics*, **2023**, *Volume* (5), pp. 524-533.
28. Murugadas. P, K. Balasubramaniyan and N. Vanmathi, Decomposition Of Neutrosophic Fuzzy Matrices, *Journal of Emerging Technologies and Innovative Research*, **2019**, *Volume* 6(3), pp. 201–211.
29. Kaur, G. and H. Garg, A new method for image processing using generalized linguistic neutrosophic cubic
30. aggregation operator. *Complex & Intelligent Systems,* **2022**, *Volume* 8(6): p. 4911-4937. https://doi.org/10.1007/s40747-022-00718-5.
31. Salama, A. A., Florentin Smarandache, and Hewayda ElGhawalby. "Neutrosophic approach to grayscale images domain, *Neutrosophic Sets and Systems*, **2018**, *Volume*. 21, pp. 13 -18.
32. Ghanbari Talouki, A., A. Koochari, and S. A. Edalatpanah. Applications of neutrosophic logic in image processing: a survey. *Journal of Electrical and Computer Engineering Innovations (JECEI),* **2022**, *Volume* 10, no. 1, pp. 243-258.
33. Smarandache, Florentin, Miguel A. Quiroz-Martínez, Jesús Estupiñan Ricardo, Noel Batista Hernández, and Maikel Y. Leyva Vázquez. Application of neutrosophic offsets for digital image processing. *Infinite Study*, **2020**.
34. Chaira, T., Neutrosophic set based clustering approach for segmenting abnormal regions in mammogram images. *Soft Computing*, **2022**. *Volume* 26(19): pp. 10423-10433. https:doi.org/ 10.1007/s00500-022-06882-7
35. Samia Mandour, An Exhaustive Review of Neutrosophic Logic in Addressing Image Processing Issues." *Neutrosophic Systems with Applications*, **2023**, *Volume* 12, pp. 36-55.
36. Liu, Biao, Guangyu Liu, Wei Feng, Enming Zhao, Wenxuan Liu, and Chunli Yang, Sonar image denoising and segmentation techniques based on neutrosophic set, *Signal, Image and Video Processing*, **2025**, *Volume* 19, no. 1, pp.143.
37. AboElHamd, Eman, Hamed M. Shamma, Mohamed Saleh, and Ihab El-Khodary. Neutrosophic logic theory and applications, *Neutrosophic sets and systems*, **2021**, *Volume* 41, no. 1, pp. 4.
38. Koyama, Yuki, and Masataka Goto, Decomposing images into layers with advanced color blending, *In Computer Graphics Forum*, **2018**, *Volume*. 37, no. 7, pp. 397-407.
39. Abadpour, Arash, and Shohreh Kasaei. A fast and efficient fuzzy color transfer method, *In Proceedings of the Fourth IEEE International Symposium on Signal Processing and Information Technology*, **2004**., pp. 491-494
40. Peng, Xindong, New operations for interval-valued Pythagorean fuzzy set, *Scientia iranica*, **2019**, *Volume* 26, no. 2, pp. 1049-1076.
41. Silambarasan, I., and S. Sriram, New operations for pythagorean fuzzy matrices, *Indian Journal of Science and Technology,* **2019**, *Volume* 12, no. 20, pp. 1-7.
42. Silambarasan, I, Some new operators for Fermatean fuzzy matrices, *Opern Journal of Discrete Applied Mathematics,* **2021**, *Volume* 4, no. 3, pp. 10-23.
43. Adak, Amal Kumar, Dheeraj Kumar, and Seyed Ahmad Edalatpanah, Some new operations on Pythagorean fuzzy sets, *Uncertainty discourse and applications* 1, **2024**, *Volume* no. 1, pp.11-19.
44. Moore, Timothy S., Janet W. Campbell, and Hui Feng, A fuzzy logic classification scheme for selecting and blending satellite ocean color algorithms, *IEEE Transactions on Geoscience and Remote sensing 39*, no. 8 ,**2001**, pp. 1764-1776.
45. Han, Ju, and Kai-Kuang Ma, Fuzzy color histogram and its use in color image retrieval, *IEEE Transactions on image Processing 11*, no. 8, **2002**, pp. 944-952.
46. Abadpour, Arash, and Shohreh Kasaei, A fast and efficient fuzzy color transfer method, *In Proceedings of the Fourth IEEE International Symposium on Signal Processing and Information Technology*, **2004**, pp. 491-494.
47. Ming-Long Huang, Yi-Cai Zhou, and Chung-Ming Wang, An improved color mood blending between images via fuzzy relationship, *In Computer Vision/Computer Graphics Collaboration Techniques: Third International Conference, MIRAGE 2007, Rocquencourt, France, March 28-30*, **2007**. Proceedings 3, pp. 1-11.
48. Ji, B, et al., An effective color image segmentation approach using superpixel-neutrosophic C-means clustering and gradient-structural similarity, *Optik*, **2022.**, *Volume 260*: pp. 169039. https://doi.org/10.1016/j.ijleo.2022.169039.

49. Pratamasunu, Gulpi Qorik Oktagalu, and Olief Ilmandira Ratu Farisi, A Neutrosophic Approach to Edge Detection Using Ant Colony Optimization, *International Journal on Electrical Engineering & Informatics*, **2021**, *Volume* 13, no. 3.
50. Mandour Samia, An Exhaustive Review of Neutrosophic Logic in Addressing Image Processing Issues, *Neutrosophic Systems with Applications*, **2023**, *Volume* 12, pp. 36-55.
51. Banu Pazar Varol., Vildan Cetkin and Halis Aygun. A New View on Neutrosophic Matrix. *Journal of Hyperstructures*, **2019**, *Volume* (1), pp. 48–57.
52. R. Uma., S. Sriram., P. Murugadas. Fuzzy Neutrosophic Soft Matrices of Type I and Type II. *Fuzzy Information and Engineering,* https://doi.org/10.1080/16168658.2021.1923621, (16 Jul **2021**).

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