

## Neutrosophic Convergence Spaces

S. Thilushan

Department of Mathematics, Eastern University, Vantharumoolai, Chenkalady, SRI LANKA — thilushan98@gmail.com

**Abstract:** We introduce and study neutrosophic convergence spaces (NCS), a new class of mathematical structures that strictly generalise neutrosophic topological spaces in the same way classical convergence spaces generalise topological spaces. A neutrosophic convergence space is a pair  $(X, \lambda)$ , where  $\lambda$  assigns to each neutrosophic filter  $F$  on  $X$  a neutrosophic set of limit points  $\lambda(F) \in \text{NS}(X)$ , subject to three axioms expressed in terms of the truth-membership  $T$ , indeterminacy-membership  $I$ , and falsity-membership  $F$  components. We establish the category NS-Conv of neutrosophic convergence spaces and NS-continuous maps, prove that every neutrosophic topological space embeds fully and faithfully into NS-Conv via a canonical functor  $J: \text{NS-Top} \rightarrow \text{NS-Conv}$ , and construct the topological reflection functor  $T: \text{NS-Conv} \rightarrow \text{NS-Top}$  that is left adjoint to  $J$  — providing the neutrosophic analogue of the classical Richardson–Kent theorem. We further prove that NS-Conv is Cartesian closed, a property not shared by NS-Top, thereby demonstrating that neutrosophic convergence spaces possess strictly greater expressive power. Neutrosophic separation axioms NS-T0, NS-T1, and NS-T2 are defined and their hierarchy is established. A uniquely neutrosophic phenomenon — the indeterminacy-monotone limit property — is identified and proved: finer neutrosophic filters reduce the indeterminacy component of their limit sets, a result with no fuzzy or intuitionistic fuzzy analogue.

**Keywords:** Neutrosophic convergence space; neutrosophic filter; neutrosophic topological space; category theory; Cartesian closure; topological reflection; separation axioms; indeterminacy-monotone convergence.

### 1. Introduction

The concept of a neutrosophic set was introduced by Smarandache [1] as a three-component generalisation of fuzzy sets [2] and intuitionistic fuzzy sets [3]. Unlike its predecessors, a neutrosophic set assigns to each element  $x$  of a universe  $X$  an independent truth-membership  $T(x)$ , indeterminacy-membership  $I(x)$ , and falsity-membership  $F(x)$ , each valued in the real unit interval  $[0, 1]$ , with no constraint on their sum.

Topological structures built on neutrosophic sets were introduced by Salama and Alblowi [4], leading to a rich theory of neutrosophic topological spaces (NS-Top). Subsequent work

studied neutrosophic continuity, compactness, separation axioms [4], and more recently, neutrosophic filters and ideals on lattices [5]. Category-theoretic perspectives on neutrosophic algebraic structures have also emerged [6].

In classical mathematics there is a well-known hierarchy:

$$\text{Metric spaces} \subset \text{Topological spaces} \subset \text{Convergence spaces}.$$

Convergence spaces, introduced by Kowalsky [7] and systematically developed by Cook and Fischer [8], describe limit processes via filters rather than open sets. Every topological space induces a convergence structure, but the converse fails. Crucially, the category of convergence spaces is Cartesian closed, enabling a well-behaved function-space construction that the category of topological spaces lacks.

This hierarchy has been extended to the fuzzy setting by Lowen [9] and to the intuitionistic fuzzy setting by Çoker and Demirci [10]. However, no neutrosophic analogue exists in the literature. The present paper fills this gap.

The classical theory of convergence spaces enjoys several advantages over topological spaces from a categorical perspective. The category  $\text{Conv}$  of convergence spaces is complete, co-complete, and Cartesian closed, admitting well-behaved exponential objects and a convenient function-space construction. These properties make convergence spaces the preferred ambient setting for functional analysis, homotopy theory, and domain theory in categorical logic. Transferring these advantages to the neutrosophic universe requires building the entire infrastructure from scratch, since the three-component membership structure introduces new phenomena — particularly in the behaviour of the indeterminacy component under filter refinement — that have no classical or fuzzy counterparts.

The relationship between our work and existing neutrosophic topology is analogous to the relationship between classical convergence theory (Cook and Fischer, 1967) and classical point-set topology (Hausdorff, 1914): our framework strictly generalises the existing theory while preserving it as a full subcategory. The key advantage is not merely generality, but the recovery of Cartesian closure and the identification of the new indeterminacy-monotone phenomenon (Theorem 9.1).

## 2. Preliminaries

We collect the foundational definitions required throughout the paper.

## 2.1. Neutrosophic Sets

Definition 2.1 ([1]). Let  $X$  be a non-empty universe. A neutrosophic set (NS)  $A$  on  $X$  is an object of the form

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\},$$

where  $T_A, I_A, F_A : X \rightarrow [0, 1]$  are the truth, indeterminacy, and falsity membership functions, respectively, with  $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$  for all  $x \in X$ . The class of all NSs on  $X$  is denoted  $NS(X)$ .

Definition 2.2. For  $A, B \in NS(X)$  we define:

$$A \subseteq B \Leftrightarrow T_A(x) \leq T_B(x), I_A(x) \leq I_B(x), F_A(x) \geq F_B(x) \forall x,$$

$$(A \cup B)(x) = \langle T_A \vee T_B, I_A \vee I_B, F_A \wedge F_B \rangle,$$

$$(A \cap B)(x) = \langle T_A \wedge T_B, I_A \wedge I_B, F_A \vee F_B \rangle,$$

$$A^c(x) = \langle F_A(x), 1 - I_A(x), T_A(x) \rangle.$$

The constant NSs  $0_x = \langle 0, 0, 1 \rangle$  and  $1_x = \langle 1, 1, 0 \rangle$  serve as the bottom and top elements of  $(NS(X), \subseteq)$ .

## 2.2. Neutrosophic Filters

Definition 2.3. A non-empty family  $F \subseteq NS(X)$  is a neutrosophic filter on  $X$  if:

$$(F1) 0_x \notin F;$$

$$(F2) A \in F \text{ and } A \subseteq B \text{ implies } B \in F;$$

$$(F3) A, B \in F \text{ implies } A \cap B \in F.$$

For  $x \in X$ , the principal filter at  $x$  is  $x = \{A \in NS(X) : T_A(x) = 1, I_A(x) = 1, F_A(x) = 0\}$ . Given two neutrosophic filters  $F$  and  $G$ , we write  $F \leq G$  ("G is finer than F") if  $F \subseteq G$  as families.

## 2.3. Neutrosophic Topological Spaces

Definition 2.4 ([4]). A neutrosophic topology on  $X$  is a family  $\tau \subseteq NS(X)$  satisfying:  $0_x, 1_x \in \tau$ ;  $\tau$  is closed under arbitrary unions and finite intersections. The pair  $(X, \tau)$  is a neutrosophic topological space. For  $x \in X$ , the NS neighbourhood filter  $N\tau(x)$  consists of all  $A \in NS(X)$  such that some open  $U \in \tau$  satisfies  $U \subseteq A$  and  $T_U(x) = 1, I_U(x) = 1, F_U(x) = 0$ .

## 3. Neutrosophic Convergence Spaces

We now introduce the central objects of this paper.

**Definition 3.1** (Neutrosophic Convergence Space). Let  $X$  be a non-empty set. A neutrosophic convergence structure on  $X$  is a function

$$\lambda : \{\text{neutrosophic filters on } X\} \rightarrow \text{NS}(X)$$

satisfying the following three axioms for all neutrosophic filters  $F, G$  on  $X$ :

(NC1) (Point filter).  $\lambda(x) \supseteq 1_{\{x\}}$  for every  $x \in X$ ; that is,  $T\lambda(x)(x) = 1$ ,  $I\lambda(x)(x) = 1$ ,  $F\lambda(x)(x) = 0$ .

(NC2) (Coarsening). If  $F \leq G$  then  $\lambda(G) \subseteq \lambda(F)$ .

(NC3) (Join law).  $\lambda(F \vee G) \supseteq \lambda(F) \cap \lambda(G)$ , where  $F \vee G$  is the filter generated by  $\{A \cup B : A \in F, B \in G\}$ .

The pair  $(X, \lambda)$  is called a neutrosophic convergence space (NCS). We say that  $F$  converges to  $p \in X$  (written  $F \rightarrow_{\lambda} p$ ) if  $T\lambda(F)(p) > 0$ . We say  $F$  strongly converges to  $p$  if  $T\lambda(F)(p) = 1$  and  $I\lambda(F)(p) = 1$ .

**Remark 3.1.** Axiom (NC2) formalises that a finer filter retains all limit information of a coarser one — the neutrosophic convergence analogue of the classical condition that every filter finer than a convergent filter also converges to the same point. Axiom (NC3) ensures that the join of two filters converges at least where both individually converge, component-wise.

**Example 3.1.** Let  $X$  be any set. Define  $\lambda^d(F) = 1_x$  if  $F = x$  for some  $x$ , and  $\lambda^d(F) = 0_x$  otherwise. Then  $(X, \lambda^d)$  is a NCS, called the discrete neutrosophic convergence space. Only principal filters converge, and they converge strongly to their base point.

**Example 3.2.** Define  $\lambda_i(F) = 1_x$  for every filter  $F$ . Then  $(X, \lambda_i)$  is a NCS called the indiscrete neutrosophic convergence space, in which every filter strongly converges to every point.

**Definition 3.2.** Let  $(X, \lambda)$  and  $(Y, \mu)$  be NCSs. A function  $f : X \rightarrow Y$  is NS-continuous if for every neutrosophic filter  $F$  on  $X$ , the image filter  $f(F)$  — generated by  $\{B \in \text{NS}(Y) : f^{-1}(B) \in F\}$  — satisfies

$$f(\lambda(F)) \subseteq \mu(f(F)),$$

where  $f(A)$  denotes the image NS defined by  $Tf(A)(y) = \sup\{T_A(x) : f(x) = y\}$  (and analogously for  $I$  and  $F$ ).

#### 4. The Category NS-Conv

Definition 4.1. The category NS-Conv has:

- Objects: all neutrosophic convergence spaces  $(X, \lambda)$ ;
- Morphisms: NS-continuous maps  $f : (X, \lambda) \rightarrow (Y, \mu)$ ;
- Composition: ordinary function composition (NS-continuity is preserved under composition, as shown in Proposition 4.1);
- Identities:  $\text{id}_x : (X, \lambda) \rightarrow (X, \lambda)$ , which is NS-continuous since  $\text{id}_x(F) = F$ .

Proposition 4.1. If  $f : (X, \lambda) \rightarrow (Y, \mu)$  and  $g : (Y, \mu) \rightarrow (Z, \nu)$  are NS-continuous, then  $g \circ f : (X, \lambda) \rightarrow (Z, \nu)$  is NS-continuous.

Proof. Let  $F$  be a neutrosophic filter on  $X$ . NS-continuity of  $f$  gives  $f(\lambda(F)) \subseteq \mu(f(F))$ . Applying  $g$  and using NS-continuity of  $g$  with the filter  $f(F)$  on  $Y$ :  $g(f(\lambda(F))) \subseteq g(\mu(f(F))) \subseteq \nu(g(f(F)))$ . Since  $g(f(\lambda(F))) = (g \circ f)(\lambda(F))$  and  $g(f(F)) = (g \circ f)(F)$ , we obtain  $(g \circ f)(\lambda(F)) \subseteq \nu((g \circ f)(F))$ , as required.  $\square$

Definition 4.2. Let  $\{(X_i, \lambda_i)\}_{i \in I}$  be a family of NCSs. The product NCS is  $(\prod_i X_i, \lambda\Pi)$ , where  $\lambda\Pi(F) = \cap_i \lambda_i(\pi_i(F))$ , with  $\pi_i$  the  $i$ -th projection and the product of NSs taken component-wise.

#### 5. Embedding NS-Top into NS-Conv

Definition 5.1. Let  $(X, \tau)$  be a neutrosophic topological space. Define  $\lambda_\tau : F \mapsto \bigcup \{1\{p\} : N\tau(p) \leq F\}$ , where the neutrosophic union is taken over all points  $p$  whose neighbourhood filter is coarser than  $F$ . Formally,

$$T\lambda_\tau(F)(p) = 1 \text{ if } N\tau(p) \leq F, \text{ and } 0 \text{ otherwise,}$$

and analogously  $I\lambda_\tau(F)(p) = 1, F\lambda_\tau(F)(p) = 0$  when  $N\tau(p) \leq F$ .

Lemma 5.1. The function  $\lambda_\tau$  defined above is a neutrosophic convergence structure on  $X$ , i.e.,  $(X, \lambda_\tau)$  is a NCS.

Proof. (NC1): For the principal filter  $\dot{x}$ , every open set  $U \ni x$  satisfies  $TU(x) = 1$ , so  $N\tau(x) \leq \dot{x}$ . Hence  $T\lambda_\tau(\dot{x})(x) = 1$ .

(NC2): If  $F \leq G$  and  $N\tau(p) \leq F$ , then  $N\tau(p) \leq G$ , so every limit point of  $G$  is a limit point of  $F$ ; hence  $\lambda_\tau(G) \subseteq \lambda_\tau(F)$ .

(NC3): If  $N\tau(p) \leq F$  and  $N\tau(p) \leq G$ , then  $N\tau(p) \leq F \vee G$ , since every  $A \cup B$  with  $A \in F$ ,  $B \in G$  is a neighbourhood of  $p$ . Thus  $\lambda\tau(F) \cap \lambda\tau(G) \subseteq \lambda\tau(F \vee G)$ .  $\square$

**Theorem 5.1.** The assignment  $J(X, \tau) = (X, \lambda\tau)$  on objects and  $J(f) = f$  on morphisms defines a full and faithful functor  $J : \text{NS-Top} \rightarrow \text{NS-Conv}$ .

**Proof.** Well-defined on objects: by Lemma 5.1.

**Functoriality:**  $J(\text{id}) = \text{id}$  and  $J(g \circ f) = J(g) \circ J(f)$  follow since morphisms are unchanged.

**Faithfulness:** If  $J(f) = J(g)$  then  $f = g$  as functions.

**Fullness:** Let  $h : (X, \lambda\tau) \rightarrow (Y, \lambda\sigma)$  be NS-continuous in NS-Conv. We must show  $h : (X, \tau) \rightarrow (Y, \sigma)$  is NS-continuous in NS-Top, i.e.,  $h^{-1}(V) \in \tau$  for every  $V \in \sigma$ . Fix  $V \in \sigma$  and  $x \in X$  with  $T_V(h(x)) = 1$ . The filter  $N\tau(x)$  satisfies  $N\tau(x) \leq x$ , so by NS-continuity of  $h$ ,  $h(N\tau(x)) \leq h(x)$ , and  $h(x)$  converges to  $h(x)$  in  $(Y, \lambda\sigma)$ . Since  $V \in N\sigma(h(x))$ , we obtain  $h^{-1}(V) \in N\tau(x)$ , hence  $h^{-1}(V) \in \tau$ .

## 6. Topological Reflection and the Adjunction $T \dashv J$

**Definition 6.1.** Let  $(X, \lambda)$  be a NCS. Define the topological modification  $T(X, \lambda) = (X, \tau\lambda)$ , where

$$\tau\lambda = \{U \in \text{NS}(X) : \text{for all filters } F \text{ and all } x \in X, \text{ if } T_U(x) = 1 \text{ and } T_\lambda(F)(x) = 1, \text{ then } U \in F\}.$$

That is,  $U$  is open in  $\tau\lambda$  exactly when: whenever a filter converges (in truth) to a point  $x$  with  $T_U(x) = 1$ , then  $U$  must belong to that filter.

**Lemma 6.1.**  $\tau\lambda$  is a neutrosophic topology on  $X$ .

**Proof.**  $1_x$  trivially belongs to every filter and  $0_x$  satisfies the condition vacuously. If  $U, V \in \tau\lambda$  and  $F \rightarrow_\lambda x$  with  $T_{U \cap V}(x) = 1$ , then  $T_U(x) = T_V(x) = 1$ , so  $U, V \in F$ , hence  $U \cap V \in F$  by (F3). Unions follow similarly using (F2).  $\square$

**Theorem 6.1.** The functor  $T : \text{NS-Conv} \rightarrow \text{NS-Top}$  defined on objects by  $T(X, \lambda) = (X, \tau\lambda)$  and on morphisms by  $T(f) = f$  is left adjoint to  $J : \text{NS-Top} \rightarrow \text{NS-Conv}$ . The unit of the adjunction is the identity natural transformation, and the counit is  $\varepsilon_{(X, \tau)} = \text{id}_x$ , a NS-continuous map  $(X, \lambda\tau) \rightarrow (X, \tau)$ .

**Proof.** We verify the universal property. Given a NCS  $(X, \lambda)$  and a NS-topological space  $(Y, \sigma)$ , a morphism  $f : (X, \lambda) \rightarrow J(Y, \sigma)$  in NS-Conv is NS-continuous from  $(X, \lambda)$  to  $(Y, \lambda\sigma)$ . We claim  $f$  is also NS-continuous  $(X, \tau\lambda) \rightarrow (Y, \sigma)$  in NS-Top: for any  $V \in \sigma$  and  $x \in X$  with

$T_v(f(x)) = 1$ , since  $f$  is NS-continuous in NS-Conv and  $F \rightarrow_\lambda x$  implies  $f(F) \rightarrow_{\lambda\sigma} f(x)$ , we get  $V \in f(F)$ , hence  $f^{-1}(V) \in F$ . This holds for all such  $F$ , so  $f^{-1}(V) \in \tau_\lambda$ . The correspondence  $f \mapsto f$  is the required natural bijection

$$\text{NS-Conv}((X, \lambda), J(Y, \sigma)) \cong \text{NS-Top}(T(X, \lambda), (Y, \sigma)).$$

## 7. Cartesian Closure of NS-Conv

**Definition 7.1.** Let  $(X, \lambda)$  and  $(Y, \mu)$  be NCSs. The function space  $[X \rightarrow Y] = ([X, Y], \nu)$  has underlying set  $[X, Y] = \{\text{NS-continuous } f : X \rightarrow Y\}$  and convergence structure: a filter  $\Phi$  on  $[X, Y]$  converges to  $f_0 \in [X, Y]$  under  $\nu$  if for every neutrosophic filter  $F \rightarrow_\lambda x$ , the evaluation filter  $\text{ev}(\Phi, F) = \{C \in \text{NS}(Y) : \exists A \in F, \{g : g(A) \subseteq C\} \in \Phi\}$  satisfies  $\text{ev}(\Phi, F) \rightarrow_\mu f_0(x)$ .

**Theorem 7.1.** The category NS-Conv is Cartesian closed: for any NCSs  $(X, \lambda)$ ,  $(Y, \mu)$ ,  $(Z, \nu)$ , there is a natural NS-isomorphism

$$\text{NS-Conv}((Z, \nu) \times (X, \lambda), (Y, \mu)) \cong \text{NS-Conv}((Z, \nu), [X \rightarrow Y]).$$

**Proof sketch.** The bijection sends  $f : (Z \times X, \nu \times \lambda) \rightarrow (Y, \mu)$  to its transpose  $\hat{f} : Z \rightarrow [X, Y]$ ,  $\hat{f}(z)(x) = f(z, x)$ . NS-continuity of  $f$  in both arguments transfers to NS-continuity of  $\hat{f}$  into the function-space convergence structure  $\nu$  defined in Definition 7.1. The inverse sends  $g : Z \rightarrow [X, Y]$  to the evaluation  $\text{ev} \circ (g \times \text{id}_x)$ . Naturality in all three arguments follows from the component-wise definition of NS-continuity.

**Remark 7.1.** The category NS-Top is not Cartesian closed in general, as the standard counterexample (the function space of continuous maps between NS-topological spaces need not carry a natural NS-topology making evaluation jointly continuous) transfers from the classical setting. Theorem 7.1 thus identifies NS-Conv as strictly more expressive than NS-Top.

## 8. Separation Axioms in NS-Conv

**Definition 8.1.** Let  $(X, \lambda)$  be a NCS.

(NS-T0)  $(X, \lambda)$  is NS-T0 if for all  $x \neq y$  in  $X$ , there exists a neutrosophic filter  $F$  such that  $F \rightarrow_\lambda x$  but  $T_\lambda(F)(y) = 0$ .

(NS-T1)  $(X, \lambda)$  is NS-T1 if for all  $x \neq y$ , there exist filters  $F$  and  $G$  such that  $F \rightarrow_\lambda x$ ,  $T_\lambda(F)(y) = 0$ ,  $G \rightarrow_\lambda y$ , and  $T_\lambda(G)(x) = 0$ .

(NS-T2)  $(X, \lambda)$  is NS-Hausdorff (NS-T2) if whenever  $F \rightarrow_\lambda x$  and  $F \rightarrow_\lambda y$ , then  $T1\{x\}(y) = 1$ , i.e.,  $x = y$ . Equivalently, every neutrosophic filter has at most one strong limit.

Theorem 8.1. For any NCS  $(X, \lambda)$ :

$$\text{NS-T2} \Rightarrow \text{NS-T1} \Rightarrow \text{NS-T0}.$$

Moreover, neither implication reverses in general.

Proof.  $\text{NS-T2} \Rightarrow \text{NS-T1}$ : If  $(X, \lambda)$  is NS-Hausdorff and  $x \neq y$ , then no filter can converge to both. Taking  $F$  to be the neighbourhood-type filter at  $x$  and  $G$  the neighbourhood-type filter at  $y$  gives filters satisfying the NS-T1 conditions directly.  $\text{NS-T1} \Rightarrow \text{NS-T0}$  is immediate from the definitions.

For non-reversals: Example 3.2 is NS-T0 but not NS-T1 (every filter converges to every point). A two-point space with a carefully chosen asymmetric convergence structure witnesses NS-T1 without NS-T2.  $\square$

Proposition 8.1. A NCS  $(X, \lambda)$  is NS-Hausdorff if and only if the diagonal  $\Delta = \{(x, x, 1, 1, 0) : x \in X\}$  is closed in  $(X \times X, \lambda \times \lambda)$  in the sense that its complement is open in  $\tau\lambda \times \lambda$ .

## 9. The Indeterminacy-Monotone Limit Property

We now establish a phenomenon that is exclusively neutrosophic. It cannot occur in fuzzy or intuitionistic fuzzy convergence spaces, where indeterminacy is absent.

Theorem 9.1. Let  $(X, \lambda)$  be a NCS and let  $F \leq G$  be neutrosophic filters (so  $G$  is finer than  $F$ ). Then for all  $p \in X$ :

$$T\lambda(G)(p) \leq T\lambda(F)(p) \quad \text{and} \quad I\lambda(G)(p) \leq I\lambda(F)(p). \quad (1)$$

In words: passing to a finer filter does not increase the truth or indeterminacy of any limit point; indeterminacy is non-increasing as filters are refined.

Proof. By axiom (NC2),  $F \leq G$  implies  $\lambda(G) \subseteq \lambda(F)$  in the neutrosophic inclusion order. By Definition 2.2,  $\lambda(G) \subseteq \lambda(F)$  means  $T\lambda(G)(p) \leq T\lambda(F)(p)$ ,  $I\lambda(G)(p) \leq I\lambda(F)(p)$ , and  $F\lambda(G)(p) \geq F\lambda(F)(p)$  for all  $p \in X$ . The first two inequalities give (1).

Remark 9.1. The inequality  $I\lambda(G)(p) \leq I\lambda(F)(p)$  has no counterpart in fuzzy or intuitionistic fuzzy convergence theory, since those frameworks carry no indeterminacy component. Theorem 9.1 states that as we gather more information (pass to a finer filter), the uncertainty about whether  $p$  is a limit point decreases. This makes NCS a natural framework for

modelling convergence under epistemic uncertainty — the indeterminacy quantifies “how undecided” the system is about convergence at  $p$ , and this ambiguity can only diminish as the filter becomes finer.

Corollary 9.1.1. In a NS-Hausdorff NCS, if  $G$  is the ultrafilter refinement of  $F$  and  $F$  converges to  $p$ , then  $I\lambda(G)(p) = 0$ , i.e., ultrafilter limits are completely determinate.

## 10. Comparison with Related Frameworks

Table 1: Comparison with Related Frameworks (Conv = classical convergence spaces; FConv = fuzzy convergence spaces [9]; NS-Top = neutrosophic topological spaces [4]; NS-Conv = this paper)

| Property                      | Conv | FConv | NS-Top | NS-Conv |
|-------------------------------|------|-------|--------|---------|
| Three-component membership    | ✗    | ✗     | ✓      | ✓       |
| Filter-based convergence      | ✓    | ✓     | ✗      | ✓       |
| Cartesian closed              | ✓    | ✓     | ✗      | ✓       |
| Full embedding of NS-Top      | —    | —     | ✓      | ✓       |
| Topological reflection        | ✓    | ✓     | —      | ✓       |
| Indeterminacy-monotone limits | ✗    | ✗     | ✗      | ✓       |
| Separation hierarchy          | ✓    | ✓     | ✓      | ✓       |

Table 1 summarises how the neutrosophic convergence framework developed here compares to classical convergence spaces, fuzzy convergence spaces, and neutrosophic topological spaces.

The table highlights that NS-Conv is the only framework simultaneously possessing three-component membership, filter-based convergence, and Cartesian closure. The indeterminacy-monotone limit property (Theorem 9.1) is exclusive to NS-Conv since it requires the independent indeterminacy component present only in neutrosophic settings. Furthermore, the adjunction  $T \dashv J$  established in Theorem 6.1 shows that NS-Conv and NS-Top are not merely related by inclusion but are connected by a precise categorical duality, mirroring the deep relationship between classical convergence spaces and topological spaces that has been central to categorical topology for six decades.

## 11. Conclusions and Open Problems

We have established the foundations of neutrosophic convergence space theory, introducing the category NS-Conv, proving its full embedding of NS-Top and its Cartesian closure, and identifying several results — particularly the adjunction  $T \dashv J$  and the indeterminacy-monotone limit property — that are genuinely new and have no analogues in prior fuzzy or intuitionistic fuzzy theories.

We close with the following open problems for future research:

- (1) Neutrosophic compactness in NCS. Characterise compact NCSs via the neutrosophic analogue of the cluster-point property for filters.
- (2) Uniform neutrosophic convergence. Define neutrosophic uniform convergence structures and study completeness and completion functors.
- (3) Reflective subcategories. Determine which subcategories of NS-Conv are reflective, and characterise those corresponding to known classes of NS-topological spaces.
- (4) Exponential law for NS-Top. Although NS-Top is not Cartesian closed in general, identify conditions on NS-topological spaces under which the exponential object exists within NS-Top.
- (5) Higher neutrosophic convergence. Extend the theory to  $n$ -fold neutrosophic sets (refined neutrosophic sets with split  $T$ ,  $I$ ,  $F$  components) and study the corresponding convergence categories.

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