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Advancing Neutrosophic Topology through Gamma Generalized Alpha Closed Sets

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Abstract: This paper introduces a novel class of Neutrosophic closed sets called *Neutrosophic γ-generalized $\boldsymbol{\alpha}$-closed sets* (Ne.(γG$\boldsymbol{\alpha}$)CS), along with their corresponding open sets (Ne.(γG$\boldsymbol{\alpha}$)OS), within the structure of Neutrosophic Topological Spaces (NTS). The motivation for this study arises from the limitations observed in existing Neutrosophic closed sets such as $\boldsymbol{\alpha}$-closed, semi-closed, and γ -closed sets, which often lack the flexibility to model hybrid structures involving partial membership and indeterminacy. To address this gap, we define the Ne.(γG$\boldsymbol{\alpha}$)CS using $\boldsymbol{\beta}$-closure operators and α-open supersets, offering a broader framework that unifies and extends several earlier concepts. The proposed sets are systematically analyzed through formal claims, and their behavior is demonstrated using counterexamples to confirm that reverse implications do not generally hold. Additionally, we explore their algebraic properties including union, intersection, and inclusion relationships. A comparative analysis illustrates how these sets generalize previously defined structures while preserving essential topological characteristics. The findings not only contribute to the advancement of Neutrosophic set theory but also offer a solid foundation for further research in uncertainty modeling, generalized topology, and decision-making systems. This work enhances the expressiveness of Neutrosophic topology and opens potential pathways for practical applications in fields requiring nuanced treatment of imprecision.

1. Introduction and Preliminaries

In recent decades, the limitations of classical set theory in handling real-world uncertainty have driven the development of more generalized mathematical frameworks. Among these, Smarandache's Neutrosophic Set theory stands out as a significant advancement . This enables the representation of uncertain, incomplete, inconsistent, and vague information with greater flexibility.

Building on this foundation, Neutrosophic Topology emerged as a natural extension of classical topology into the domain of indeterminacy. This new branch was initiated by A.A. Salama [10], who developed the concept of Neutrosophic Topological Spaces (NTS). In these spaces, the classical notions of open and closed sets are redefined to accommodate the presence of indeterminate and inconsistent information, which is especially relevant in areas such as artificial intelligence, decision support systems, and data analysis.

Since the introduction of NTS, many researchers have contributed to its advancement by proposing various types of Neutrosophic open and closed sets. These generalized forms have helped to build a more complete understanding of topological structures under uncertain conditions. For example, Arokiarani I. and colleagues [2] proposed the notion of Neutrosophic α-CS, which broadened the traditional concept of closedness in topological spaces by integrating indeterminacy and partial membership. Their work added depth to the exploration of closure operations in generalized topologies.

Similarly, Ishwarya P. et al. [7] studied Neutrosophic Semi-Open Sets, which represent a hybrid category between open and closed sets. This intermediate classification has provided new insights into how Neutrosophic sets behave with respect to topological boundaries, particularly when information is incomplete or partially defined.

Despite these advancements, existing classifications may not fully capture the intricate relationships between different types of Neutrosophic sets. To address this gap, the present study introduces a novel class of closed sets, termed Neutrosophic γ-Generalized α -Closed Sets (abbreviated as Ne.(γG$\boldsymbol{\alpha}$)CS, along with their corresponding open sets, known as Neutrosophic γ\gamma-Generalized α\alpha-Open Sets (Ne.(γG$\boldsymbol{\alpha}$)CS).

These newly defined set classes are proposed to further refine and generalize the concepts of closure and openness in Neutrosophic Topological Spaces. The key idea is that a set Λ_1 in a Neutrosophic topological space $(\chi_{n_{e.}}, \mathrm{N_{e.}}\tau)$ is said to be a Ne.(γG$\boldsymbol{\alpha}$)CS if $N_{e.}\mathrm{bcl}(\Lambda_1) \subseteq \Omega$ whenever $\Lambda_1 \subseteq \Omega$ and Ω is a Neutrosophic $\boldsymbol{\alpha}$- alpha-Open Set in the same space.

This framework incorporates both the γ-closure and α\alpha-openness concepts, leading to a more layered and flexible understanding of set boundaries. By doing so, it bridges the gap between multiple earlier notions and offers a unified structure to study more complex topological behaviors under uncertainty.

The objective of this research is threefold:

- To formally define and introduce the new classes of Ne.(γG$\boldsymbol{\alpha}$)CS closed and open sets;
- To analyze and prove their fundamental properties, including behavior under standard set operations like union, intersection, and complement;
- To explore their relationships with existing types of Neutrosophic sets, such as N(α\alpha)CS, N(G)CS, and N(GS)CS.

By addressing these goals, this paper aims to contribute both theoretical and structural value to the growing domain of Neutrosophic topology. These developments have the potential to enhance future investigations in topology, logic, and their interdisciplinary applications.

Moreover, this study lays the groundwork for further exploration into continuity, compactness, and separation axioms using the newly defined set types. It also opens the possibility for practical applications where vague, incomplete, or inconsistent information must be systematically analyzed.

In conclusion, the introduction of Neutrosophic γ −Generalized $\boldsymbol{\alpha}$-Closed and Open Sets represents a significant step forward in the evolution of Neutrosophic topology. It offers refined tools for topological analysis in the presence of indeterminacy and strengthens the theoretical foundation for further research in uncertainty modeling and applied mathematics.

1.1 Motivation for the Study

Many real-life problems involve situations where things are not fully true or false, and we face uncertainty or incomplete information. Traditional set theories like classical sets or fuzzy sets cannot properly deal with this kind of uncertainty. To solve this, Neutrosophic Set Theory was introduced, which allows us to separately consider truth, falsity, and indeterminacy. Building on this idea, Neutrosophic topology was developed to study open and closed sets in uncertain environments. Several types of Neutrosophic closed sets already exist, like $\boldsymbol{\alpha}$, γ and semi- CS. But these sets often work separately and don't give a complete picture when openness and closeness overlap. They are not flexible enough to handle all types of uncertain or mixed cases. This creates a need for a new, more general type of set that can combine and extend the features of the existing ones. That's why this paper introduces a new kind of set called the Neutrosophic γ-generalized α-closed set, which is designed to be broader and more useful in dealing with complex uncertain situations.

1.2 Research Gap

Although several classes of Neutrosophic closed sets have been introduced in recent years—such as Neutrosophic α-closed sets, semi-closed sets, pre-closed sets, and γ-closed sets—these concepts are limited in scope. Most of them address specific types of closure behavior and do not offer a unified structure that combines multiple closure and openness properties. As a result, they fall short in representing more complex topological structures that may arise in uncertain systems. Another issue is that the relationships between these different types of Neutrosophic closed sets are not fully explored in the literature. There is a lack of generalized set definitions that can include these existing types as special cases while offering new insights into how they interact or differ. Additionally, many existing models do not account for how sets behave under different closure operations, such as semi-closure or β-closure, in a combined or comparative manner. Therefore, there is a clear gap in developing a broader class of Neutrosophic closed sets that can generalize and unify various existing structures under a single theoretical framework. This limitation inspires the introduction and investigation of Neutrosophic *γ-generalized* -generalized α-CS in the present study.

1.3 Objective of this study

The main aim of this research is to introduce and explore a novel category of closed sets in Neutrosophic topology, referred to as Neutrosophic γ-generalized α-CS (Ne.(γGα)CS). This class is introduced to generalize and unify several existing Neutrosophic closed set types, such as α-closed, semi-closed, pre-closed, and γ-closed sets, under a broader and more inclusive framework. The study aims to establish the foundational properties of Ne.(γGα)CS, examine their algebraic behavior, and explore their interactions with other well-known closed sets. In addition, the paper provides formal proofs and counterexamples to demonstrate that while Ne.(γGα)CS include many existing classes as special cases, the reverse inclusions do not hold. Another key objective is to introduce the corresponding open sets, namely *Neutrosophic γ-generalized α-open sets*, and investigate their characteristics. Through this work, the paper seeks to enrich the structure of Neutrosophic topological spaces and support further theoretical development and practical application in fields that require refined treatment of uncertainty and imprecision.

1.4 Discussion of Existing Problems and Core Contributions

The study addresses a key limitation in Neutrosophic topology—namely, the lack of a unified structure that can generalize and relate various existing Neutrosophic closed sets such as α-closed, semi-closed, pre-closed, and γ-CS. These earlier set types are defined in isolated contexts and are often insufficient for representing the complex interplay between openness and closedness in uncertain systems. They do not capture all types of boundary behaviors, nor do they offer a general framework that allows comparison or inclusion among multiple closure concepts.

In response to this limitation, the paper introduces a new and more inclusive class called *Neutrosophic γ-generalized α-closed sets* (Ne.(γGα)CS), which incorporates β-closure operations with α-open supersets. This framework not only generalizes several known classes of Neutrosophic closed sets but also establishes their interrelationships through a series of logical claims. The paper rigorously proves that (Ne.(γGα)CS) includes all of these earlier classes as special cases and presents counterexamples to show that the converse is not generally true. This distinction is crucial for deepening the theoretical structure of Neutrosophic topology. The core contributions of the paper are as follows:

- Formal definition and development of the new class (Ne.(γG$\boldsymbol{\alpha}$)CS)and its corresponding open set (Ne.(γG$\boldsymbol{\alpha}$)OS).
- Establishment of inclusion relationships between (Ne.(γG$\boldsymbol{\alpha}$)CS)and existing Neutrosophic closed sets ($\boldsymbol{\alpha}$-closed, semi-closed, γ-closed, etc.).
- Presentation of multiple claims supported by proofs and counterexamples to clarify boundary conditions.
- Analysis of set operations (such as union and intersection) on (Ne.(γG$\boldsymbol{\alpha}$)CS)and their closure properties.
- Introduction of generalization theorems showing how (Ne.(γG$\boldsymbol{\alpha}$)CS) can serve as a broader framework for future topological investigations.

By resolving the fragmented nature of existing closed set definitions and offering a unified approach, this work significantly enhances the expressive power of Neutrosophic topological structures and provides a solid foundation for further applications and theoretical extensions.

1.5 Proposed Methodology

This study adopts a theoretical methodology to define and explore a new class of closed sets in Neutrosophic Topological Spaces (NTS), called *Neutrosophic γ-generalized $\boldsymbol{\alpha}$-closed sets* (Ne.(γG$\boldsymbol{\alpha}$)CS). The method begins with a review of existing closed set types—such as $\boldsymbol{\alpha}$-closed, semi-closed, pre-closed, and γ-closed sets—to highlight the need for a unifying structure. The new class is defined using β-closure and $\boldsymbol{\alpha}$-open sets: a set Λ_1 is (Ne.(γG$\boldsymbol{\alpha}$)CS) if its $\boldsymbol{\beta}$ -closure is contained in every $\boldsymbol{\alpha}$ -open superset that includes it. Several claims are then established to show that well-known Neutrosophic closed sets are special cases of (Ne.(γG$\boldsymbol{\alpha}$)CS), with counterexamples demonstrating that the converse is not generally true. Illustrative examples clarify the behavior of these sets, including their response to set operations like union and intersection. The study also introduces the corresponding open set class, *Neutrosophic γ -generalized $\boldsymbol{\alpha}$-open sets* (Ne.(γG$\boldsymbol{\alpha}$)OS), and explores their properties. Overall, this methodology provides a step-by-step generalization framework that strengthens and extends the theory of Neutrosophic topology.

The rationale for selecting a theoretical and axiomatic approach in this study stems from the need to generalize and unify multiple existing classes of Neutrosophic closed sets within a single framework. Traditional Neutrosophic closed sets—such as $\boldsymbol{\alpha}$ -closed, semi-closed, pre-closed, and γ-closed sets—are defined independently and lack a shared structure that allows for direct comparison or integration. By employing $\boldsymbol{\beta}$ -losure and α-open set operations, the proposed *Neutrosophic γ-generalized $\boldsymbol{\alpha}$ - closed sets* (Ne.(γG$\boldsymbol{\alpha}$)CS) offer a flexible yet rigorous extension that includes these existing sets as particular cases. This formal method ensures mathematical clarity, enables the derivation of inclusion relations, and allows the formulation of claims with both proofs and counterexamples. The goal was to address the structural gaps in current Neutrosophic topology and to enrich the theoretical landscape for future developments. The selected methodology thus provides a solid foundation for extending closure-based reasoning under uncertainty and lays the groundwork for potential applications in decision theory, data analysis, and soft computing.

2. Basic Definitions and Preliminaries

Definition 2.1 [5,6]
Consider a fixed non-empty set N^X. A Neutrosophic set V_1^* defined on N^X can be expressed as $V_1^* = \{\langle x, \mu_{V_1^*}(x), \sigma_{V_1^*}(x), \nu_{V_1^*}(x)\rangle | x \in N^x\}$,where $\mu_{V_1^*}(x)$:The membership degree is denoted by $N^X \rightarrow$, and the function $\nu_{V_1^*}(x): N^X \rightarrow [0,1]$ specifies the non-membership degree for the Neutrosophic set V_1^*, whereas $\sigma_{V_1^*}(x)$, represents the indeterminacy degree.

Definition 2.2 [10] A Neutrosophic topology (abbreviated as NT) on the set N^x defined as a collection $N^\tau$of Neutrosophic sets within N^x that satisfies the following conditions:
1. The null Neutrosophic set 0_N and the universal Neutrosophic set 1_N are elements of N^τ
2. The intersection $J_1 \cap J_2$ belongs to $N^\tau$for any two sets $J_1, J_2 \in N^\tau$
3. For any collection $\{J_i | i \in j\} \subseteq N^\tau$. In this situation, the couple $(N^X, N^\tau)(N^X, N^\tau)$ is denoted to as a NTS. A NOS is any subclass of N^X that fits to N^τ. The counterpart V_1^{*c} of a NOS V_1^* in the NTS (N^X, N^τ) is recognized as a NCS in N^X..

Claim 2. 3 [10]. For any NS V_1^* in (N^X, N^τ), we have
1. $N^{int}(0_N) = 0_N$ and $N^{cl}(0_N) = 0_N$
2. $(N^{int}(V_1^*))^c = N^{cl}(V_1^{*c})$
3. $(N^{cl}(V_1^*))^c = N^{int}(V_1^{*c})$
4. $N^{int}(1_N) = 1_N$ and $N^{cl}(1_N) = 1_N$

Definition 2.4 A NS $V_1^*$of a NTS (N^X, N^τ) is a
1. A Neutrosophic semi preclosed set (denoted as (N(γ)CS) is well-defined as a set V_1^* for which ∃an N(P) closed set V_2^* in N(P) Closed set and there exists a Neutrosophic preclosed set V_2^* in which contains the neutrosophic interior of V_2^* contains V_1^*.
2. In [15] (N(γ)OS ∃ N(P)OS V_2^* such that $V_1^* \subseteq (V_1^*) \subseteq N^{cl}(V_2^*)V_1^*$

Definition 2.5 Let V_1^*be an NS of a NTS (N^X, N^τ). Then
1. $N^{\alpha cl}(V_1^*) = \cap \{I | I$ is a $N(\alpha)CS$ in N^X and $V_1^* \subseteq I\}$
2. $N^{\alpha int}(V_1^*) = \cup \{I | I$ is a $N(\alpha)OS$ in N^X and $I \subseteq V_1^*\}$

Definition 2.6 Let V_1^* be a Neutrosophic set (NS) in the Neutrosophic Topological Space (NTS) (N^X, N^τ). Then: V_1^* is called a Neutrosophic Generalized Closed Set (abbreviated as N(G)CS if $N^{cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ is a Neutrosophic Open Set (NOS) in N^X..

1. V_1^* is called a Neutrosophic Generalized Semi Closed Set (abbreviated as N(GS)CS if $N^{cl}(V_1^*) \subseteq \Psi$ where Ψ is a NOS in N^X.

2. $V_1^*$is called an Alpha-Neutrosophic Generalized Closed Set (abbreviated as (N(α)GCS if $N^{cl}(V_1^*) \subseteq \Psi$, and Ψ is a NOS in N^X..
3. V_1^* is called a Neutrosophic Generalized Alpha Closed Set (abbreviated as (N(α)GCS if $N^{cl}(V_1^*) \subseteq \Psi$, and Ψ is a Neutrosophic Alpha Open Set (NαOS) in N^X

Remark 2.7 Let V_1^* be a NS in (N^X, N^τ). Then
1. $N^{S-cl}(V_1^*) = V_1^* \cap N^{int}(N^{cl}(V_1^*))$
2. $N^{S-int}(V_1^*) = V_1^* \cup N^{cl}(N^{int}(V_1^*))$
If V_1^* is a NS of N^X then $N^{Scl}(V_1^{*c}) = (N^{Scl}(V_1^*))^c$

3. $(N_{e.}(\gamma G\alpha)CS)$- *Neutrosophic γ generalized α - CS*

Definition 3.1

A Neutrosophic set Λ_1 in the Neutrosophic Topological Space $(\chi_{п_{e.}}, N_{e.}\tau)$ is called a Neutrosophic $(N_{e.}(\gamma G\alpha)CS)$ if $N_{e.}bcl(\Lambda_1) \subseteq \Omega$ whenever$\Lambda_1 \subseteq \Omega$and Ω is a $N_{e.}(\alpha)OS$ in $(\chi_{п_{e.}}, N_{e.}\tau)$ in the space $N_{e.}TS$ $(\chi_{п_{e.}}, N_{e.}\tau)$.

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Example 3.2:

Let $\chi_{п_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$, and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{п_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{п_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ stands an $N_{e.}s$ in $(\chi_{п_{e.}}, N_{e.}\tau)$.

Claim 3.3:

In the space $(\chi_{п_{e.}}, N_{e.}\tau)$ every $N_{e.}CS$ is also a$N_{e.}(\gamma G\alpha)CS$ but the converse does not generally hold.

Proof:

Assume Λ_1 is a $N_{e.}$ Closed set in $\chi_{п_{e.}}$ suppose $\Lambda_1 \subseteq \Omega$ where Ω is a $N_{e.}(\alpha)$ openset in $\chi_{п_{e.}}$. As given that $N_{e.}bcl(\Lambda_1) \subseteq N_{e.}cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ is follows that $N_{e.}bcl(\Lambda_1) \subseteq \Omega$. Then Λ_1 is in the space $(\chi_{п_{e.}})$ with the neutrosophic topology $N_{e.}\tau$ and is a a $N_{e.}(bG\alpha)$

Illustration 3.4:

Let $\chi_{п_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{п_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{п_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a neutrosophic topology is in $(\chi_{п_{e.}}, N_{e.}\tau)$, is a neutrosophic topology $(\gamma G\alpha)$ closed set nonetheless non $N_{e.}$closed in $(\chi_{п_{e.}}, N_{e.}\tau)$ as $N_{e.}cl(\Lambda_1) = {K_1^*}^C \neq \Lambda_1$.

Claim 3.5:

In the space $(\chi_{п_{e.}}, N_{e.}\tau)$ every $N_{e.}(S)CS$ is also $N_{e.}(bG\alpha)CS$ but the converse does not generally hold.

Proof:

Let Λ_1 be a $N_{e.}$SCS in $\chi_{\text{n}_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\text{n}_{e.}}$. As $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(S)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Then Λ_1 is a $N_{e.}(\gamma G\alpha)CS$ in $(\chi_{\text{n}_{e.}}, N\tau)$.

Illustration 3.6:

Let $\chi_{\text{n}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\text{n}_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\text{n}_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a $N_e.s$ in $(\chi_{\text{n}_{e.}}, N_{e.}\tau)$, stands a neutrosophi $(\gamma G\alpha)$ closed set nevertheless non a $N_{e.}(S)$ closed set in $(\chi_{\text{n}_{e.}}, N\tau)$ as $N_{e.}int(N_{e.}cl(\Lambda_1)) = N_{e.}int({K_1^*}^C) = K_1^* \nsubseteq \Lambda_1$.

Claim 3.7

Every Neutrosophic $N_{e.}(P)$ Closed Set in the space$(X, N_{e.}\tau)$ is also a Neutrosophic $N(\gamma G\alpha)$ Closed Set, In general, however, the converse is not necessarily true.

Proof:

Let Λ_1 is a $N_{e.}(P)CS$ in $\chi_{\text{n}_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\text{n}_{e.}}$. As $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(P)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Then Λ_1 is a $N_{e.}(\gamma G\alpha)CS$ in $(\chi_{\text{n}_{e.}}, N\tau)$.

Illustration 3.8:

Let $\chi_{\text{n}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\text{n}_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\text{n}_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a $N_e.s$ in $(\chi_{\text{n}_{e.}}, N_{e.}\tau)$, stands a $N_{e.}(bG\alpha)CS$ nevertheless not an$N_{e.}(P)CS$ in $(\chi_{\text{n}_{e.}}, N\tau)$ as $N_{e.}cl(N_{e.}int(\Lambda_1)) = N_{e.}cl(K_2^*) = {K_1^*}^C \nsubseteq \Lambda_1$.

Claim 3.9:

Every Neutrosophic α Closed Set in the space $(\chi_{\text{n}_{e.}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set, but the converse is not true in general.

Proof:

Let Λ_1 is a $N_{e.}(\alpha)CS$ in $\chi_{\text{n}_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\text{n}_{e.}}$. As $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(\alpha)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_e.bcl(\Lambda_1) \subseteq \Omega$. Therefore, in $(\chi_{\text{n}_{e.}}, N_e.\tau)$., Λ_1 is a $N_{e.}(\gamma G\alpha)CS$.

Illustration 3.10:

Let $\chi_{\text{n}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\text{n}_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\text{n}_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a $N_e.s$ in $(\chi_{\text{n}_{e.}}, N_e.\tau)$, stands a Neutrosophic$N_{e.}(bG\alpha$ closed set nevertheless non an $N_{e.}(\alpha)CS$ in $(\chi_{\text{n}_{e.}}, N_{e.}\tau)$ as $N_{e.}cl(N_{e.}int(N_{e.}cl(\Lambda_1))) = N_{e.}cl(N_{e.}int({K_1^*}^C)) = N_{e.}cl(K_1^*) = {K_1^*}^C \nsubseteq \Lambda_1$.

Claim 3.11:

Every Neutrosophic $\gamma-$ Closed Set in the space $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(\gamma G\alpha)$ Closed Set, but the converse does not hold in general

Proof:

Let Λ_1 Neutrosophic $\gamma-$ Closed Set in the space $(\chi_{\text{n}_{e.}})$. Let $\Lambda_1 \subseteq \Omega$ and Ω is aNeutrosophic$-\alpha$ open set in $\chi_{\text{n}_{e.}}$. while $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(\gamma)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, here consume $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Therefore, in $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$., Λ_1 is a Neutrosophic $(\gamma G\alpha)$ closed set.

Illustration 3.12:

Let $\chi_{\text{n}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\text{n}_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\text{n}_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right) \rangle$ is a $N_{e.}s$ in $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$, stands a Neutrosophic $N_{e.}(\gamma G\alpha)$ closed set but non an $N_{e.}(b)$ closed set in $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$ as $N_{e.}int(N_{e.}cl(\Lambda_1)) \cap N_{e.}cl(N_{e.}int(\Lambda_1)) = K_1^* \cap K_1^{*C} = K_1^* \not\subseteq \Lambda_1$.

Claim 3.13:

Every Neutrosophic $N_{e.}(R)$ Closed Set in the space in $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set, but the converse is not generally true.

Proof:

Let Λ_1 is a $N_{e.}(R)CS$ in $\chi_{\text{n}_{e.}}$. Since every $N_{e.}(R)CS$ is a $N_{e.}CS$., Λ_1 is a $N_{e.}CS$. Therefore by claim 2.3, Λ_1 is a $N_{e.}(\gamma G\alpha)CS$ in $(\chi_{\text{n}_{e.}}, N\tau)$.

Illustration 3.14:

Let $\chi_{\text{n}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\text{n}_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\text{n}_{e.}}$.Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ is a $N_{e.}(\gamma G\alpha)CS$ but not an $N_{e.}(R)CS$ in $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$ as $N_{e.}cl(N_{e.}int(\Lambda_1)) = N_{e.}cl(K_2^*) = K_1^{*C} \neq \Lambda_1$.

Claim 3.15:

Every Neutrosophic $N_{e.}(\gamma)$Closed Set in the space $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set; however, the converse does not necessarily hold.

Proof:

Let Λ_1 be a $N_{e.}(\gamma)CS$ in $\chi_{\text{n}_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\text{n}_{e.}}$. As $N_{e.}bcl(\Lambda_1) \subseteq N_{e.}(\gamma)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Therefore, in $(\chi_{\text{n}_{e.}},\ N_{e.}\tau)$., Λ_1 is a $N_{e.}(\gamma G\alpha)CS$.

Illustration 3.16:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)CS$ but not a $N_{e.}(\gamma)CS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$, as we could not find any $N_{e.}(P)CS$ Λ_2 such that $N_{e.}int(\Lambda_2) \subseteq \Lambda_1 \subseteq \Lambda_2$ in $\chi_{\pi_{e.}}$.

Claim 3.17:

Every Neutrosophic $N_{e.}(b)$ Closed Set in the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set, but the converse is not true in general.

Proof:

Let Λ_1 is a $N_{e.}(b)CS$ in $\chi_{\pi_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\pi_{e.}}$. Now $N_{e.}(\gamma)cl(\Lambda_1)=\Lambda_1\subseteq\Omega$, by hypothesis. Therefore we have $N_{e.}\ (b)cl(\Lambda_1)\subseteq\Omega$. Hence Λ_1 is a $N_{e.}(bG\alpha)CS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Illustration 3.18:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$, and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ is a $N_{e.}(bG\alpha)CS$ but not an $N_{e.}(b)CS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ as $N_{e.}int(N_{e.}cl(N_{e.}int(\Lambda_1))) = N_{e.}int(N_{e.}cl(K_2^*)) = N_{e.}int({K_1^*}^C) = K_1^* \nsubseteq \Lambda_1$

Remark 3.19:

In general, the union of two Neutrosophic $N_{e.}(bG\alpha)$ Closed Sets in the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is not necessarily a Neutrosophic $N_{e.}(bG\alpha)CS$ Closed Set, as demonstrated in the following example.

Illustration 3.20:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$, $K_2^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_3^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, K_3^*, 1_N\}$ is a $N_{e.}$ T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{1}{10}, \frac{5}{10}, \frac{9}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$, $\Lambda_2 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$, are $N_{e.}(bG\alpha)CSs$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$. But $\Lambda_1 \cup \Lambda_2$ is not an $N_{e.}(bG\alpha)CS$ as $\Lambda_1 \cup \Lambda_2 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle \subseteq K_1^*$ but $N_{e.}(bcl(\Lambda_1 \cup \Lambda_2) = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle \nsubseteq K_1^*$.

Remark 3.21:

The intersection of any two $N_{e.}(bG\alpha)CSs$ is not an $N_{e.}(bG\alpha)CS$ in general as seen in the following example.

Illustration 3.22:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$, $K_2^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_3^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, K_3^*, 1_N\}$ is a $N_{e.}$ T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 =$

⟨x, $\left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{8}{10},\frac{5}{10},\frac{2}{10}\right)$⟩, Λ_2 = ⟨x, $\left(\frac{8}{10},\frac{5}{10},\frac{2}{10}\right),\left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right)$⟩, are $N_{e.}(bG\alpha)CSs$ in ($\chi_{\text{n}_{\text{e.}}}$, $N_{e.}\tau$). But $\Lambda_1 \cap \Lambda_2$ is not an $N_{e.}(bG\alpha)CS$ as $\Lambda_1 \cap \Lambda_2$ = ⟨x, $\left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right),\left(\frac{5}{10},\frac{5}{10},\frac{5}{10}\right)$⟩⊆ K_1^* but $N_{e.}(bcl(\Lambda_1 \cap \Lambda_2)$ =⟨x, $\left(\frac{6}{10},\frac{5}{10},\frac{4}{10}\right),\left(\frac{7}{10},\frac{5}{10},\frac{3}{10}\right)$⟩ $\not\subseteq K_1^*$.

Claim 3.23:

Let ($\chi_{\text{n}_{\text{e.}}}$, $N_{e.}\tau$) is a $N_{e.}$TS. Then $\Lambda_1 \in N_{e.}$(γGα)C($\chi_{\text{n}_{\text{e.}}}$) and $\Lambda_2 \in N_{e.}$S($\chi_{\text{n}_{\text{e.}}}$), $\Lambda_1 \subseteq \Lambda_2 \subseteq N_{e.}$bcl($\Lambda_1$) ⇒$\Lambda_2 \in N_{e.}$(γGα)C($\chi_{\text{n}_{\text{e.}}}$).

Proof:

Let $\Lambda_2 \subseteq \Omega$ and Ω is a $N_{e.}$(α)OS in $\chi_{\text{n}_{\text{e.}}}$. Then since $\Lambda_1 \subseteq \Lambda_2$, $\Lambda_1 \subseteq \Omega$. By hypothesis $\Lambda_2 \subseteq N_{e.}$bcl($\Lambda_1$). Therefore $N_{e.}$bcl(Λ_2)⊆$N_{e.}$bcl($N_{e.}$bcl(Λ_1))=$N_{e.}$bcl(Λ_1)⊆Ω, since Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{\text{e.}}}$. Hence $\Lambda_2 \in N_{e.}$(γGα)C($\chi_{\text{n}_{\text{e.}}}$).

Claim 3.24:

Let Γ⊆ Λ_1⊆ $\chi_{\text{n}_{\text{e.}}}$ where Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{\text{e.}}}$. Then Γ is a $N_{e.}$(γGα)CS in Λ_1 if and only if Γ is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{\text{e.}}}$

Proof:

Necessity: Let Ω is a $N_{e.}$(α)OS in $\chi_{\text{n}_{\text{e.}}}$ and Γ⊆Ω. Λ_1 also let Γ is a $N_{e.}$(γGα)CS in Λ_1. Then clearly Γ ⊆ Λ_1∩Ω and Λ_1∩Ω is a $N_{e.}$(α)OS in Λ_1. Hence beta closure of Γ in Λ_1, $N_{e.}$bclΛ_1 (Γ)⊆Λ_1∩Ω and by claim 3.26: Λ_1 is a $N_{e.}$(γ) CS. Therefore $N_{e.}$b cl(Λ_1) = Λ_1 . Now beta closure of Γ in $\chi_{\text{n}_{\text{e.}}}$, $N_{e.}$bcl(Γ)⊆$N_{e.}$bcl(Γ)∩$N_{e.}$bcl(Λ_1)=$N_{e.}$bcl(Γ)∩Λ_1=$N_{e.}$bclΛ_1(Γ)⊆Λ_1∩Ω⊆Ω, that is $N_{e.}$bcl(Γ)⊆Ω, whenever Γ⊆Ω. Hence Γ is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{\text{e.}}}$.

Sufficiency: Let V is a Neutrosophic -α open set in Λ_1, such that Γ⊆V. Since Λ_1 is a Neutrosophic -α open set in $\chi_{\text{n}_{\text{e.}}}$, V is a Neutrosophic -α open set in $\chi_{\text{n}_{\text{e.}}}$. Therefore bcl(Γ) ⊆ V, as Γ is a Neutrosophic (γGα)CS in $\chi_{\text{n}_{\text{e.}}}$. Thus, $N_{e.}$bclΛ_1(Γ)= $N_{e.}$bcl(Γ)∩ Λ_1 ⊆V∩Λ_1⊆V. Hence Γ is a $N_{e.}(bG\alpha)CS$ in Λ_1.

Claim 3.25:

A $N_{e.}$S Λ_1 is both an $N_{e.}$OS and a $N_{e.}$(γGα)CS if and only if Λ_1 is a $N_{e.}$(R)OS in $\chi_{\text{n}_{\text{e.}}}$.

Proof:

Necessity: Let Λ_1 be both an $N_{e.}$OS and a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{\text{e.}}}$. Then Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}(bG\alpha)CS$. By claim 3.25, Λ_1 is a $N_{e.}$(γ) CS and $N_{e.}$ int($N_{e.}$ cl($N_{e.}$ int(Λ_1))) ⊆ Λ_1 . Since Λ_1 is a $N_{e.}$ OS, $N_{e.}$ int(Λ_1)= Λ_1 .Therefore $N_{e.}$ int($N_{e.}$ cl(Λ_1))⊆ Λ_1 .Since Λ_1 is a $N_{e.}$ OS, it is a $N_{e.}$ POS. Hence Λ_1⊆$N_{e.}$int($N_{e.}$cl(Λ_1)). Therefore Λ_1=$N_{e.}$int($N_{e.}$cl(Λ_1)) and Λ_1 is a $N_{e.}$(R)OS in $\chi_{\text{n}_{\text{e.}}}$.

Sufficiency: Let Λ_1 is a $N_{e.}$(R)OS in $\chi_{\text{n}_{\text{e.}}}$ then Λ_1= $N_{e.}$int($N_{e.}$cl(Λ_1)). Since every $N_{e.}$(R)OS is a $N_{e.}$OS, Λ_1 is a $N_{e.}$OS. We have $N_{e.}$int($N_{e.}$cl($N_{e.}$int(Λ_1))) = $N_{e.}$int($N_{e.}$cl(Λ_1))=Λ_1⊆Λ_1. Therefore Λ_1 is a $N_{e.}(b)CS$ in $\chi_{\text{n}_{\text{e.}}}$, and by claim 3.17, Λ_1 is a $N_{e.}(bG\alpha)CS$ in $\chi_{\text{n}_{\text{e.}}}$.

Claim 3.26:

For an $N_{e.}$OS Λ_1 in ($\chi_{\text{n}_{\text{e.}}}$, $N_{e.}\tau$), the following conditions are equivalent.

(i) Λ_1 is a $N_{e.}$CS

(ii) Λ_1 is a $N_{e.}$(γGα)CS and a $N_{e.}$Q set

Proof: (i) ⇒ (ii) Since Λ_1 is a $N_{e.}$CS, it is a $N_{e.}$(γGα)CS by claim 3. Now $N_{e.}$int($N_{e.}$cl(Λ_1)) =$N_{e.}$ int(Λ_1)=Λ_1=$N_{e.}$ cl(Λ_1) = $N_{e.}$cl($N_{e.}$int(Λ_1)), by hypothesis. Hence Λ_1 is a $N_{e.}$Q-set.

(ii) ⇒ (i) Since Λ_1 is a $N_{e.}$OS and a $N_{e.}$(γGα)CS, by claim 2.27, Λ_1 is a $N_{e.}$(R)OS. Therefore Λ_1 = $N_{e.}$int($N_{e.}$cl(Λ_1)) = $N_{e.}$cl($N_{e.}$int(Λ_1)) =$N_{e.}$ cl(Λ_1), by hypothesis. Hence Λ_1 is a $N_{e.}$CS in $\chi_{\pi_{e.}}$.

Claim 3.27:

Let ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$TS. Then $N_{e.}$bC($\chi_{\pi_{e.}}$) = $N_{e.}$(γGα)C($\chi_{\pi_{e.}}$) if every $N_{e.}$S in ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$(α)OS in $\chi_{\pi_{e.}}$.

Proof:

Suppose that every $N_{e.}$S in ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$(α)OS in $\chi_{\pi_{e.}}$. Let Λ_1∈$N_{e.}$(γGα)C($\chi_{\pi_{e.}}$). Then Λ_1 is also an $N_{e.}$(α)OS by hypothesis. Therefore by claim 2.25 Λ_1 is a $N_{e.}$(γ)CS.Therefore Λ_1 ∈ $N_{e.}$bC($\chi_{\pi_{e.}}$). Hence $N_{e.}$(γGα)C($\chi_{\pi_{e.}}$)⊑$N_{e.}$bC($\chi_{\pi_{e.}}$) (i) Let Λ_1∈$N_{e.}$bC($\chi_{\pi_{e.}}$). Then by claim 2.17, Λ_1 is a $N_{e.}$(γGα)CS and Λ_1 ∈ $N_{e.}$(γGα)C($\chi_{\pi_{e.}}$). Hence $N_{e.}$bC($\chi_{\pi_{e.}}$)⊑$N_{e.}$(γGα)C($\chi_{\pi_{e.}}$) (ii).From (i) and (ii) $N_{e.}$bC($\chi_{\pi_{e.}}$)=$N_{e.}$(γGα)C($\chi_{\pi_{e.}}$).

Claim 3.28:

Let Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}$(γGα)CS of ($\chi_{\pi_{e.}}$, $N_{e.}$τ). Then Λ_1⊓Γ is a $N_{e.}$(γGα)CS of ($\chi_{\pi_{e.}}$, $N_{e.}$τ) where Γ is a $N_{e.}$CS of $\chi_{\pi_{e.}}$.

Proof:

Consider that Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}$(γGα)CS of ($\chi_{\pi_{e.}}$, $N_{e.}$τ), then by claim 2.25, Λ_1 is a $N_{e.}$(γ)CS. But Γ is a $N_{e.}$CS in $\chi_{\pi_{e.}}$. Hence Λ_1⊓Γ is a $N_{e.}$(γ)CS as every $N_{e.}$CS is a $N_{e.}$(γ)CS. Therefore Λ_1⊓Γ is a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$, by claim 3.17.

Claim 3.29:

Let ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$TS, then for every Λ_1∈$N_{e.}$bC($\chi_{\pi_{e.}}$) and for every Λ_2 in $\chi_{\pi_{e.}}$, $N_{e.}$int(Λ_1)⊑Λ_2⊑Λ_1 implies Λ_2 ∈ $N_{e.}$(γGα)C($\chi_{\pi_{e.}}$).

Proof:

Let Λ_1 be Λ_1 $N_{e.}$(γ) CS in $\chi_{\pi_{e.}}$ .Then there exists an $N_{e.}$ (P)CS,(say) Λ_3 such that $N_{e.}$ int(Λ_3)⊑ Λ_1 ⊑ Λ_3 . By hypothesis, Λ_2 ⊑ Λ_1 . Therefore Λ_2 ⊑ Λ_3 . Since $N_{e.}int$(Λ_3)⊑ Λ_1 , $N_{e.}$int(Λ_3)⊑ $N_{e.}int$(Λ_1) and $N_{e.}int$(Λ_3)⊑ Λ_2, by hypothesis. Thus $N_{e.}int$(Λ_3)⊑ Λ_2⊑Λ_3 and Λ_2 ∈ $N_{e.}$bC($\chi_{\pi_{e.}}$). Hence by claim 3.15, Λ_2 ∈ $N_{e.}$(γGα)C($\chi_{\pi_{e.}}$)

Claim 3.30:

If a $N_{e.}$S Λ_1 of a $N_{e.}$TS ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is Neutrosophic nowhere dense, then it is a $N_{e.}$(γGα)CS in ($\chi_{\pi_{e.}}$, $N_{e.}$τ).

Proof:

If Λ_1 is Neutrosophic nowhere dense in $\chi_{\pi_{e.}}$, then $N_{e.}$int($N_{e.}$cl(Λ_1)) = 0_N. Let Λ_1 ⊑ Ω where Ω is a $N_{e.}$(α)OS in $\chi_{\pi_{e.}}$.Now $N_{e.}$bcl(Λ_1)⊑$N_{e.}$Scl(Λ_1)=Λ_1∪$N_{e.}$int($N_{e.}$cl(Λ_1))=Λ_1∪0_N = Λ_1⊑Ω and hence Λ_1 is a $N_{e.}$(γGα)CS in ($\chi_{\pi_{e.}}$, $N_{e.}$τ).

4. *γ -Generalized α -Type Open Sets within a Neutrosophic Framework*

In this section, various properties of Neutrosophic **γ** -generalized **α** -open sets have been examined and analyzed, leading to the development of several insightful characterization theorems.

Illustration 4.1:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$ is a $N_{e.}$(γGα)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Claim 4.2:

Every $N_{e.}$OS, $N_{e.}$(S)OS, $N_{e.}$(P)OS, $N_{e.}$(α)OS, $N_{e.}$(γ)OS, $N_{e.}$(R)OS, $N_{e.}$bOS, $N_{e.}$(γ)OS are $N_{e.}$(γGα)OS but not conversely in general in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Proof:

Straightforward.

Illustration 4.3:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.4:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$.

is a $N_{e.}(bG\alpha)OS$ but not $N_{e.}(S)OS$ in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.5:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right) \rangle$ is a $N_{e.}$(γGα)OS but not an $N_{e.}$(P)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.6:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$(α)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.7:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right) \rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$(γ)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.8:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}$T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$(R)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.9:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}$T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$

is a $N_{e.}$(γGα)OS but not $N_{e.}$bOS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Illustration 4.10:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}$T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$(γ)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Claim 4.11:

Let ($\chi_{\pi_{e.}}$, $N_{e.}\tau$) is a $N_{e.}$TS. Then for every $\Lambda_1 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$) and for every $\Lambda_2 \in N_{e.}$S($\chi_{\pi_{e.}}$), $N_{e.}$bint(Λ_1)⊆Λ_2⊆Λ_1⇒Λ_2∈$N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Proof:

Let Λ_1 is ay $N_{e.}$(γGα)OS of $\chi_{\pi_{e.}}$ and Λ_2 is any $N_{e.}$S of $\chi_{\pi_{e.}}$. Let $N_{e.}$bint(Λ_1)⊆Λ_2⊆Λ_1. Then Λ_1^C is a $N_{e.}$(γGα)CS and $\Lambda_1^C \subseteq \Lambda_2^C \subseteq N_{e.}$bcl($\Lambda_1^C$). Therefore $\Lambda_1^C$is a $N_{e.}$(γGα)CS by claim 2.23, which implies Λ_2 is a $N_{e.}$(γGα)OS in $\chi_{\pi_{e.}}$. Hence $\Lambda_2 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Sufficiency: Let Γ is a $N_{e.}$(α)CS such that Γ⊆Λ_1 and Γ⊆$N_{e.}$bint(Λ_1). Then $(N_{e.}bint(\Lambda_1))^C \subseteq \Gamma^C$ and $\Lambda_1^C \subseteq \Gamma^C$.This implies that bcl($\Lambda_1^C$)⊆$\Gamma^C$, where Γ^C is a $N_{e.}$(α)OS. Therefore $\Lambda_1^C$is a $N_{e.}$(γGα)CS. Hence Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\pi_{e.}}$.

Claim 4.12:

Let ($\chi_{\pi_{e.}}$, $N_{e.}\tau$) is a $N_{e.}$TS then for every $\Lambda_1 \in N_{e.}$bO($\chi_{\pi_{e.}}$) and for every $N_{e.}$S Λ_2 in $\chi_{\pi_{e.}}$, $\Lambda_1 \subseteq \Lambda_2 \subseteq$ cl(Λ_1) ⇒ $\Lambda_2 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Proof:

Let Λ_1 be a $N_{e.}$bOS in $\chi_{\pi_{e.}}$. Then there exists an $N_{e.}$POS, (say) Λ_3 such that $\Lambda_3 \subseteq \Lambda_1 \subseteq N_{e.}$cl($\Lambda_3$). By hypothesis, Λ_1⊆Λ_2. Therefore Λ_3⊆Λ_2. Since Λ_1⊆$N_{e.}$cl(Λ_3), $N_{e.}$cl(Λ_1)⊆ $N_{e.}$cl(Λ_3) and Λ_2⊆$N_{e.}$cl(Λ_3), by hypothesis. Therefore Λ_2 is Λ_1 $N_{e.}$bOS. As every $N_{e.}$bOS is a $N_{e.}$(γGα)OS by claim 3.2, $\Lambda_2 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Claim 4.13:

If Λ_1 is a $N_{e.}$(α)CS and a $N_{e.}$(γGα)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$), then Λ_1 is a $N_{e.}$(γ)OS in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$).

Proof: Since $\Lambda_1 \subseteq \Lambda_1$ and Λ_1 is a $N_{e.}$(α)CS, by hypothesis $\Lambda_1 \subseteq$ bint(Λ_1). But bint(Λ_1) $\subseteq \Lambda_1$. Therefore bint(Λ_1) = Λ_1. Hence Λ_1 is a $N_{e.}$(γ)OS in ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ).

Claim 4.14:

Let ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}$TS. Then $N_{e.}$bO($\chi_{\text{п}_{e.}}$) = $N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$) if every $N_{e.}$S in ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}$(α)CS in $\chi_{\text{п}_{e.}}$.

Proof:

Assume that every $N_{e.}$S in ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}$(α)CS in $\chi_{\text{п}_{e.}}$. Let $\Lambda_1 \in N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$). Then Λ_1 is also an $N_{e.}$(α)CS, by hypothesis. Therefore by claim 3.15 Λ_1 is a $N_{e.}$(γ)OS. Therefore $\Lambda_1 \in N_{e.}$bO($\chi_{\text{п}_{e.}}$). Hence $N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$) $\subseteq N_{e.}$bO($\chi_{\text{п}_{e.}}$) (i) Let $\Lambda_1 \in N_{e.}$bO($\chi_{\text{п}_{e.}}$) then by claim 3.15 $\Lambda_1 \in N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$). Hence $N_{e.}$bO($\chi_{\text{п}_{e.}}$) $\subseteq N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$) (ii). Therefore from (i) and (ii) $N_{e.}$bO($\chi_{\text{п}_{e.}}$) = $N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$).

5. Theoretical implications and applications of Neutrosophic γ -generalized α -closed sets.

In this section, we explore various theoretical applications of Neutrosophic γ -generalized α -closed sets by introducing new baces and deriving several noteworthy claim s.

Definition 5.1:

If each $N_{e.}(bG\alpha)CS$ is a $N_{e.}$ Closed set space ($\chi_{\text{п}_{e.}}$,τ), then the bace is referred to as a $N_{e.}b_{g\alpha c}T_{1/2}$ base.

Definition 5.2:

A neutrosophic topological space $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ), is said to have a $b_{g\alpha b}T_{1/2}$ ($N_{e.}b_{g\alpha b}T_{1/2}$) -bace if every $b_{g\alpha b}T_{1/2}$ ($N_{e.}b_{g\alpha b}T_{1/2}$) -closed set $\in \chi_{\text{п}_{e.}}$ is also a $N_{e.}(b)$-closed set.

Illustration 5.3:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right) \rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a N_e.T on $\chi_{\text{п}_{e.}}$.

Here the $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a Neutrosophic $b_{g\alpha c}T_{1/2}$bace.

Definition 5.4:

A $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) stands a Neutrosophic $b_{g\alpha P}T_{1/2}$ ($N_{e.}b_{g\alpha P}T_{1/2}$) bace if each $N_{e.}$(γGα) closed set is a $N_{e.}$(P)closed set $\in \chi_{\text{п}_{e.}}$.

Illustration 5.5:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right) \rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a N_e.T on $\chi_{\text{п}_{e.}}$.

. Here $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace.

Claim 5.6:

Every $N_{e.}b_{g\alpha c}T_{1/2}$ bace is a $N_{e.}b_{g\alpha b}T_{1/2}$ space but not conversely in general.

Proof:

Let $\chi_{\text{п}_{e.}}$ is a $N_{e.}$ γGαcT1/2 bace. Let A is a $N_{e.}$(γGα)CS $\in \chi_{\text{п}_{e.}}$. By hypothesis, Λ_1 is a $N_{e.}$CS $\in \chi_{\text{п}_{e.}}$. Since every $N_{e.}$CS is a $N_{e.}$(γ)CS, Λ_1 is a $N_{e.}$(γ)CS $\in \chi_{\text{п}_{e.}}$. Hence $\chi_{\text{п}_{e.}}$ is a $N_{e.}b_{g\alpha b}T_{1/2}$bace.

Illustration 5.7:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right) \rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a N_e.T on $\chi_{\text{п}_{e.}}$.

. Here $N_{e.}$TS ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace ,as every $N_{e.}$(γGα)CS is a $N_{e.}$(γ)CS in ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$), but not an $N_{e.}b_{g\alpha c}T_{1/2}$bace, as $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ remains a Neutrosophic $(bG\alpha)$ closed set but not an Neutrosophic $(bG\alpha)$ closed set ∈ ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$).

Claim 5.8:

Every $N_{e.}b_{g\alpha P}T_{1/2}$ bace is a $N_{e.}b_{g\alpha b}T_{1/2}$bace but not conversely in general.

Proof:

Let $\chi_{\text{n}_{e.}}$ is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace and let Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{e.}}$. By hypothesis, Λ_1 is a $N_{e.}$(P)CS in $\chi_{\text{n}_{e.}}$. Since every $N_{e.}$(P)CS is a $N_{e.}$(γ)CS, Λ_1 is a $N_{e.}$(γ)CS in $\chi_{\text{n}_{e.}}$. Hence $\chi_{\text{n}_{e.}}$ is a $N_{e.}b_{g\alpha b}T_{1/2}$ bace.

Illustration 5.9:

Let $\chi_{\text{n}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$. Then $\tau_{\text{n}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a $N_{e.}$T on $\chi_{\text{n}_{e.}}$.

. Here $N_{e.}$TS ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace ,as every $N_{e.}$(γGα)CS is a $N_{e.}$(γ)CS in ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$), but not an $N_{e.}b_{g\alpha P}T_{1/2}$ bace, as $\Lambda_1 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$ is a $N_{e.}(bG\alpha)CS$ but not an $N_{e.}(P)CS$ in ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$).

Claim 5.10:

Let ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha b}T_{1/2}$bace. Then

(i) Any union of $N_{e.}$(γGα)CSs is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{e.}}$.

(ii) Any intersection of $N_{e.}$(γGα)OSs is a $N_{e.}$(γGα)OS in $\chi_{\text{n}_{e.}}$.

Proof:

(i) Let {Ai}denote a family of $N_{e.}$(γGα) closed sets within the space $\chi_{\text{n}_{e.}}$. Since ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}$ γGαbT1/2 bace, every $N_{e.}$(γGα)CS is a $N_{e.}$(γ)CS and hence each Ai , i J is a $N_{e.}$(γ)CS in ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$). But any union of Neutrosophic −γ closed set stands a $N_{e.}$(γ)CS, Subsequently every one $N_{e.}$(γ) closed set stands a $N_{e.}$(γGα)CS, ∪Ai stands an $N_{e.}$(γGα)CS in $\chi_{\text{n}_{e.}}$.

(ii) can be proved by taking complement in (i).

Claim 5.11: Let Λ_1 be a set that qualifies as both a $N_{e.}$OS and a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{e.}}$ Within the space $\chi_{\text{n}_{e.}}$ If the space $\chi_{\text{n}_{e.}}$ satisfies the conditions of a $N_{e.}$ γGαcT1/2 , then

(i) Λ_1 must be a $N_{e.}$(R) open set in $\chi_{\text{n}_{e.}}$,

(ii) Λ_1 must be a $N_{e.}$(R) closed set $\chi_{\text{n}_{e.}}$,

(iii) Λ_1 must be a $N_{e.}$Q set in $\chi_{\text{n}_{e.}}$.

Proof: Let Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{e.}}$, then by Definition 4.1, Λ_1 is a $N_{e.}$CS in $\chi_{\text{n}_{e.}}$. Now (i) $N_{e.}$int($N_{e.}$cl(Λ_1)) = $N_{e.}$int(Λ_1) = Λ_1 and therefore Λ_1 is a $N_{e.}(R)OS$ in $\chi_{\text{n}_{e.}}$, (ii) $N_{e.}$cl($N_{e.}$int(Λ_1)) = $N_{e.}$cl(Λ_1) = Λ_1 and therefore Λ_1 is a $N_{e.}$(R)CS in $\chi_{\text{n}_{e.}}$ and (iii) from (i) and (ii) $N_{e.}$int($N_{e.}$cl(Λ_1)) = $N_{e.}$cl($N_{e.}$int(Λ_1)). Hence Λ_1 is a $N_{e.}$Q-set in $\chi_{\text{n}_{e.}}$.

Claim 5.12:

Let ($\chi_{\text{n}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha b}T_{1/2}$bace, then the following conditions are equivalent:

(i) Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\text{n}_{e.}}$,

(ii) $\Lambda_1 \subseteq N_{e.}$cl($N_{e.}$int($N_{e.}$cl($\Lambda_1$))),

(iii) $N_{e.}$cl(Λ_1) ∈ $N_{e.}$RC($\chi_{\text{n}_{e.}}$).

Proof:

$(i) \rightarrow (ii)$ Let Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\text{n}_{\text{e.}}}$. Then since $\chi_{\text{n}_{\text{e.}}}$ is a $N_{e.}b_{g\alpha b}T_{1/2}$bace, Λ_1 is a $N_{e.}$(γ)OS in $\chi_{\text{n}_{\text{e.}}}$. Therefore $\Lambda_1 \subseteq N_{e.}$cl($N_{e.}$int($N_{e.}$cl($\Lambda_1$))).

$(ii) \rightarrow (iii)$ Let $\Lambda_1 \subseteq N_{e.}$ cl($N_{e.}$ int($N_{e.}$ cl(Λ_1))). Then $N_{e.}$ cl(Λ_1)$\subseteq N_{e.}$ cl($N_{e.}$ cl($N_{e.}$ int($N_{e.}$ cl(Λ_1)))) = $N_{e.}$cl($N_{e.}$int($N_{e.}$cl(Λ_1)))$\subseteq N_{e.}$cl(Λ_1).Therefore $N_{e.}$cl(Λ_1) = $N_{e.}$cl($N_{e.}$int($N_{e.}$cl(Λ_1))). Hence
$N_{e.}$cl(Λ_1) $N_{e.}$RC($\chi_{\text{n}_{\text{e.}}}$).

$(iii) \rightarrow (i)$ Since cl(Λ_1) is a $N_{e.}$(R)CS in $\chi_{\text{n}_{\text{e.}}}$, $N_{e.}$cl(Λ_1)=$N_{e.}$cl($N_{e.}$int($N_{e.}$cl(Λ_1))) and since $\Lambda_1 \subseteq N_{e.}$cl($\Lambda_1$), $\Lambda_1 \subseteq N_{e.}$cl($N_{e.}$int($N_{e.}$cl($\Lambda_1$))). Therefore Λ_1 is a N(γ)OS. Hence Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\text{n}_{\text{e.}}}$

Claim 5.13:

Let ($\chi_{\text{n}_{\text{e.}}}$, $N_{e.}$τ) is a $N_{e.}$ γGαbT1/2 space, then the following conditions are equivalent:

(i) Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\text{n}_{\text{e.}}}$,

(ii) $N_{e.}$int($N_{e.}$cl($N_{e.}$int(Λ_1))) $\subseteq \Lambda_1$,

(iii) $N_{e.}$int(Λ_1) $\in N_{e.}$RO($\chi_{\text{n}_{\text{e.}}}$).

Proof: This claim can be easily proved by taking complement in claim 4.16

6. Limitations of the Study

While the study successfully introduces and generalizes the concept of Neutrosophic $\boldsymbol{\gamma}$ -generalized $\boldsymbol{\alpha}$ -closed sets, it is not without limitations. Firstly, the research is entirely theoretical and lacks practical applications or real-world data validation. The examples used are limited to small, finite Neutrosophic spaces, which may not reflect the behavior of these sets in large or complex topological systems. Secondly, no algorithmic or computational methods are developed to detect or implement these sets in applied settings. Thirdly, the study does not address the dynamic behavior of these sets under changes in the underlying topological space. Lastly, potential applications in decision-making, data analysis, or artificial intelligence are not explored, leaving the practical relevance of the proposed sets for future investigation.

7. Future Work

The proposed class of Neutrosophic $\boldsymbol{\gamma}$ -generalized α-closed sets (Ne.(γG$\boldsymbol{\alpha}$)CS opens multiple avenues for future investigation. One notable avenue is the creation of computational algorithms to detect and analyze (Ne.(γG$\boldsymbol{\alpha}$)CS in large Neutrosophic topological spaces, making the concept applicable to practical decision-making and uncertainty modeling.. Future work may also investigate dynamic Neutrosophic systems where the topology evolves over time, requiring adaptive closure properties. In addition, exploring the application of (Ne.(γG$\boldsymbol{\alpha}$)CS) in fields such as digital topology, image processing, data clustering, and granular computing could provide real-world relevance. Another direction involves studying dual concepts like Neutrosophic $\boldsymbol{\gamma}$ -generalized α-interior sets and their topological implications. Overall, the foundational structure developed in this study paves the way for further theoretical expansion and interdisciplinary applications in systems that involve incomplete, imprecise, or inconsistent information.

The comparative analysis table.1 evaluates the proposed Neutrosophic γ -generalized α-closed sets (Ne.(γ Gα)CS) alongside traditional Neutrosophic closed set types—namely α-closed, semi-closed, pre-closed, and γ -closed sets. Each class is compared based on criteria such as openness foundation, closure operator used, scope of generalization, and inclusion relationships. Traditional set types depend on specific types of open sets (α, semi, pre, γ) and corresponding closures, often with narrow generalization and limited structural relationships. In contrast, (Ne.(γGα)CS) utilizes β -closure and α-open sets, providing a unified and more flexible framework. The table confirms that (Ne.(γGα)CS) includes all traditional types as special cases, while none of the others offer similar inclusiveness. Reverse implications do not generally hold for (Ne.(γGα)CS), which is supported through counterexamples in the paper. The proposed class also demonstrates improved behavior under operations like union and intersection, which is often not preserved in other types. Furthermore, it better captures uncertainty and hybrid behavior due to its broader formulation. This enhanced expressiveness makes (Ne.(γGα)CS) more applicable to advanced modeling in uncertain topological environments. The comparison validates the generality, strength, and necessity of the proposed class within Neutrosophic topology.

Table 1: Comparison between the proposed Neutrosophic γ-generalized α-closed sets and traditional Neutrosophic closed set types

Feature	**α - Closed Sets**	**Semi-Closed Sets**	**Pre-Closed Sets**	**γ - Closed Sets**	**Proposed Ne.(γ G α)CS**
Openness Basis	α -open sets	Semi-open sets	Pre-open sets	γ -open sets	α -open sets
Closure Type Used	α -closure or identity	Semi-closure	Pre-closure	γ -closure	β -closure (broader)
Defined via	Inclusion via α -open set	Superset's semi-open relation	Pre-open neighborhood inclusion	γ -open neighborhood containment	β -closure inclusion inside α-open sets
Scope of Generalization	Narrow	Moderate	Moderate	Broader than α	Broadest – generalizes all
Inclusion of Other Sets	Does not include others	Does not include others	Does not include others	Partial inclusion of α and semi	Includes α, semi, pre, and γ as special cases
Reverse Implication	May hold in special cases	Not always true	Often fails	Rarely holds	Proven false via counterexamples
Support for Hybrid Behavior	Limited	Limited	Limited	Partial	High – designed for uncertain overlap
Behavior under Union/Intersection	Not always closed	Not preserved	Not preserved	Sometimes preserved	Analyzed in claims; flexible

Expressiveness under Uncertainty	Low	Moderate	Moderate	Moderate	High – handles mixed/indeterminate membership
Application Readiness	Theoretical	Theoretical	Theoretical	Theoretical	Theoretical; open for future applications

8. Conclusion

In this articles, introduced and examined a new class of sets in Neutrosophic topology, namely Neutrosophic γ -generalized α-closed sets (γ GS-closed sets) and their counterparts, Neutrosophic γ - generalized α-open sets (γ GS-open sets). These sets represent a meaningful generalization of existing Neutrosophic closed and open set concepts, enriching the structural framework of Neutrosophic topological spaces. We have discussed several foundational properties of these sets and explored their relationships with previously established classes of Neutrosophic sets, highlighting their uniqueness and broader applicability.

The results obtained in this work not only contribute to the theoretical development of Neutrosophic topology but also pave the way for further generalizations and refinements. Future research could focus on extending these sets under different topological operators, examining their behavior in product spaces, and exploring their role in Neutrosophic continuity, compactness, and separation axioms. Additionally, potential applications in fields dealing with uncertainty, such as decision-making, data analysis, and artificial intelligence, can be explored by leveraging the flexible nature of γ GS-closed and γ GS-open sets. This work thus lays a solid foundation for advancing both theoretical investigations and practical applications within the broader domain of Neutrosophic mathematics.

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Data Availability

The present work is wholly theoretical, with no inclusion of data gathering or analytical evaluation. Prospective researchers are invited to conduct empirical research to further investigate and substantiate the ideas outlined in this paper.

Ethical Approval

Ethical approval is not required, as this purely theoretical research involves no animal subjects or human participants.

Conflicts of Interest

It is affirmed by the authors that this research and its publication involve no conflicts of interest.

Disclaimer

This work proposes untested theoretical concepts, inviting future empirical validation. While accuracy and proper citation have been prioritized, inadvertent errors may exist, and readers should verify sources. The views expressed are solely the authors' and not necessarily those of their institutions.

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