



A New Constrain $\mathbb{K} - \bar{\mathbb{Q}}$ Group Operating on Neutrosophic Fuzzy Subgroup and Normal Subgroup

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Abstract:

This research presents an innovative concept known as the Neutrosophic $\mathbb{K} - \bar{\mathbb{Q}}$ group operating on fuzzy subsets, extending the traditional algebraic framework to include elements of fuzziness and neutrosophy. The study defines the structure and algebraic properties of Neutrosophic $\mathbb{K} - \bar{\mathbb{Q}}$ group operating fuzzy subgroups, highlighting how these subsets behave under group operations influenced by uncertainty and indeterminacy. Several foundational algebraic characteristics are discussed, including closure, associativity, identity, and inverses within this specialized fuzzy context. In addition, the notion of homomorphisms is incorporated by examining how fuzzy subgroups within the Neutrosophic $\mathbb{K} - \bar{\mathbb{Q}}$ group behave under structure-preserving mappings. Specifically, we analyze the homomorphic image and preimage of these subgroups, establishing essential results that contribute to understanding their structural consistency and transformation properties. Furthermore, the concept of the direct product of Neutrosophic $\mathbb{K} - \bar{\mathbb{Q}}$ group operating fuzzy subgroups is introduced. We demonstrate that the direct product maintains the integrity of the fuzzy subgroup structure, preserving its key features across multiple components. This idea is not only applied to the direct product of two such subgroups but is also extended to a finite number of them, reinforcing the robustness and generalizability of the model. These developments open new avenues in fuzzy algebra and Neutrosophic logic-based group theory.

Keywords: Fuzzy Set (FS), Neutrosophic $\kappa - \bar{Q}$ group operating on fuzzy subsets (Neu $\kappa - \bar{Q}$ GOFsb), Neutrosophic $\kappa - \bar{Q}$ group operating fuzzy subgroups (Neu $\kappa - \bar{Q}$ GOFSG), Neutrosophic $\kappa - \bar{Q}$ group operating fuzzy normal subgroups (Neu $\kappa - \bar{Q}$ GOFNSG).

1. Introduction

In 1994, Ajmal [1] introduced a novel notation for group homomorphisms, the correspondence theorem, and fuzzy quotient groups. Mordeson created Fuzzy Group Theory in 2005 [6]. Mukherjee [7] developed fuzzy normal subgroups and cosets in 1984. Gupta [5] established the new concept of Theory of T-norms and fuzzy inference methods in 1984. Das [4] introduced the concept of fuzzy groups and level subgroups in 1981. Khare S S[3] introduced the concept of Fuzzy Homomorphism and Algebraic Structures in 1993. Sherwood H [2] established the concept of fuzzy groups in 1979. Rosenfeld A [9] was first suggested by Fuzzy Groups in 1971. Ismai et.al.[8], introduced the concept of $\kappa - \bar{Q}$ -Fuzzy Orders Relative to $\kappa - \bar{Q}$ -Fuzzy Subgroups and Cyclic group on various fundamental aspects in 2020. In 2016, Abdul Salam[10] described a new notation for a group acting on a fuzzy algebraic structure. Sherwood H [2] established the concept of fuzzy groups in 1979. Dragan Pamucar et.al.[12] introduces the concept of Neutrosophic fuzzy set and its application in decision making in 2020. In 2018, Thiruveni[14], described the notation of Neutrosophic Q-Fuzzy Subgroups. Yager R R [15] Pioneered Fuzzy Sets and Possibility Theory in 1982. Tarnauceanu[13] introduced the concept of classifying fuzzy normal subgroups of finite groups in 2015. Zadeh [16] introduced fuzzy sets for the first time in 1965. Nagarajan [11] introduced the concept of a novel structure and constructed Q-fuzzy groups in 2009.

This research introduces the Neutrosophic $\kappa - \bar{Q}$ group operating on fuzzy subsets, extending classical algebraic structures to encompass fuzziness and indeterminacy. And defines the structure and algebraic properties of Neutrosophic $\kappa - \bar{Q}$ group operating fuzzy subgroups and normal subgroups, highlighting how these subsets behave under group operations influenced by uncertainty and indeterminacy. The study also explores homomorphisms, analyzing how fuzzy subgroups behave under structure-preserving mappings, including their images and preimages. Additionally, it presents the direct product of such fuzzy subgroups, proving that key structural properties remain intact across multiple components. Extending this concept to finitely many subgroups, the work enhances the generality and applicability of the model, offering valuable insights into fuzzy algebra and Neutrosophic group theory.

2. Preliminaries

Definition 2.1[9]: Assume that, \tilde{G} is any group. If, $\tilde{\alpha}(ab) \geq \{\tilde{\alpha}(a) \wedge \tilde{\alpha}(b)\}$ and $\tilde{\alpha}(a^{-1}) = \tilde{\alpha}(a)$, then a mapping $\tilde{\alpha}: \tilde{G} \rightarrow [0,1]$ is a fuzzy group.

Definition 2.2[11]: Let, \check{G} be a group and \bar{Q} be a set. \bar{Q} fuzzy set is a mapping $\acute{\alpha}: \check{G} \times \bar{Q} \rightarrow [0,1]$ in \check{G} . We define the set $U(\mu; t) = \{a \in \check{G} / \acute{\alpha}(a, \eta) \geq t, \eta \in \bar{Q}\}$ for each \bar{Q} -fuzzy set $\acute{\alpha}$ in \check{G} and $t \in [0,1]$.

Definition 2.3[8]: Let, \check{G} and \bar{Q} be any two nonempty sets, with $\kappa \in [0,1]$ and \bar{Q} -FSb of a set \check{G} . The fuzzy set $\acute{\alpha}^\kappa$ of \check{G} , also known as the, κ - \bar{Q} -FSb of \check{G} , is defined as $\acute{\alpha}^\kappa(\theta, \eta) = (\acute{\alpha}(\theta, \eta), \kappa)$.

Definition 2.4[12]: A neutrosophic fuzzy set, $\acute{\alpha}$ on the universe of discourse \mathbf{X} characterized by a truth membership function $T_{\acute{\alpha}}(a)$, an indeterminacy function $I_{\acute{\alpha}}(a)$ and a falsity membership function $F_{\acute{\alpha}}(a)$ is defined as $\acute{\alpha} = \{ \{a, T_{\acute{\alpha}}(a), I_{\acute{\alpha}}(a), F_{\acute{\alpha}}(a)\} : a \in \mathbf{X} \}$, where $T_{\acute{\alpha}}, I_{\acute{\alpha}}, F_{\acute{\alpha}}: \mathbf{X} \rightarrow [0,1]$ and $0 \leq T_{\acute{\alpha}}(a) + I_{\acute{\alpha}}(a) + \acute{\alpha}(a) \leq 3$.

3. A New Constrain $\mathbb{K} - \bar{Q}$ Group Operating on Neutrosophic Fuzzy Subgroup and Normal Subgroup

Definition 3.1: Let \check{G} and \bar{Q} be any two non-empty sets, $\mathbb{K} \in [0,1]$ and $q \in \bar{Q}$. Let $\tilde{\alpha}_0^{\mathbb{K} *}$ is fuzzy set on $\tilde{\alpha}_0^{\mathbb{K} *} : (\check{G} \times \check{S}) \times \bar{Q} \rightarrow [0,1]$, for each set $\tilde{\alpha}_0^{\mathbb{K} *}$ on \check{S} . Thus the fuzzy set $(\tilde{\alpha}_0^{\mathbb{K} *})$ on $(\check{G} \times \check{S}) \times \bar{Q}$ as $\{ (a, \check{s}), \tilde{\alpha}_0^{\mathbb{K} *} (a * \check{s}, q), \forall a \in \check{G}, \check{s} \in \check{S} \text{ and } q \in \bar{Q} \} \Rightarrow \tilde{\alpha}_0^{\mathbb{K} *} (a * \check{s}, q) = \tilde{\alpha}_0^{\mathbb{K} *} ((a * \check{s}, q), \mathbb{K})$ be an $\mathbb{K} - \bar{Q}$ group operating fuzzy set in \check{G} and \check{S} . Then a *Neu $\mathbb{K} - \bar{Q}$ GOFsb* $\omega_0^{\mathbb{K} *}$ is \check{G} and \bar{Q} defined by

$$\begin{aligned} \omega_0^{\mathbb{K} *} &= \{ (a * \check{s}, q), \tilde{\alpha}_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} (a * \check{s}, q) = \tilde{\alpha}_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} ((a * \check{s}, q), \mathbb{K}), T_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} (a * \check{s}, q) \\ &= T_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} ((a * \check{s}, q), \mathbb{K}), I_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} (a * \check{s}, q) = I_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} ((a * \check{s}, q), \mathbb{K}), F_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} (a * \check{s}, q) \\ &= F_{\omega_0^{\mathbb{K} *}}^{\mathbb{K} *} ((a * \check{s}, q), \mathbb{K}), \forall a \in \check{G}, \check{s} \in \check{S} \text{ and } q \in \bar{Q} \} \end{aligned}$$

Definition 3.2: Let \check{G} and \bar{Q} be any two non-empty sets, $\mathbb{K} \in [0,1]$ and $q \in \bar{Q}$. Let $\omega_0^{\mathbb{K} *} : (\check{G} \times \check{S}) \times \bar{Q} \rightarrow [0,1]$ be a *Neu $\mathbb{K} - \bar{Q}$ GOFsb* in \check{G} and \check{S} . Then $\omega_0^{\mathbb{K} *}$ is called *Neu $\mathbb{K} - \bar{Q}$ GOFSG* of \check{G} . If its following conditions

- (i) $T_{\omega_0}^{\mathbb{K}} * (a \dot{b} * \mathfrak{s}, q) \geq \{T_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge T_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$
 $I_{\omega_0}^{\mathbb{K}} * (a \dot{b} * \mathfrak{s}, q) \geq \{I_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge I_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$
 $F_{\omega_0}^{\mathbb{K}} * (a \dot{b} * \mathfrak{s}, q) \leq \{F_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \vee F_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$
- (ii) $T_{\omega_0}^{\mathbb{K}} * (a^{-1} * \mathfrak{s}, q) = T_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K})$
 $I_{\omega_0}^{\mathbb{K}} * (a^{-1} * \mathfrak{s}, q) = I_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K})$
 $F_{\omega_0}^{\mathbb{K}} * (a^{-1} * \mathfrak{s}, q) = F_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}), \forall a, b \in \check{G}, q \in \check{Q}, \mathfrak{s} \in \check{S} \text{ and } \mathbb{K} \in [0,1]$

Theorem 3.3: Let $\omega_0^{\mathbb{K}} *$ be *Neu* $\mathbb{K} - \check{Q}$ *GOF* *Sb* of \check{G} be a *Neu* $\mathbb{K} - \check{Q}$ *GOF* *SG* in \check{G} and \check{S} if and only

- if $T_{\omega_0}^{\mathbb{K}} * (a \dot{b}^{-1} * \mathfrak{s}, q) \geq \{T_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge T_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$,
- $I_{\omega_0}^{\mathbb{K}} * (a \dot{b}^{-1} * \mathfrak{s}, q) \geq \{I_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge I_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$ and
- $F_{\omega_0}^{\mathbb{K}} * (a \dot{b}^{-1} * \mathfrak{s}, q) \geq \{F_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge F_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$.

Proof:

Let $\omega_0^{\mathbb{K}} *$ be *Neu* $\mathbb{K} - \check{Q}$ *GOF* *SG* of \check{G} operating on \check{S} .

Now,

- (i) $T_{\omega_0}^{\mathbb{K}} * (a \dot{b} * \mathfrak{s}, q) \geq \{T_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge T_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$,
 $I_{\omega_0}^{\mathbb{K}} * (a \dot{b} * \mathfrak{s}, q) \geq \{I_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge I_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$ and
 $F_{\omega_0}^{\mathbb{K}} * (a \dot{b} * \mathfrak{s}, q) \leq \{F_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \vee F_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$
- (ii) $T_{\omega_0}^{\mathbb{K}} * (a^{-1} * \mathfrak{s}, q) = T_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K})$
 $I_{\omega_0}^{\mathbb{K}} * (a^{-1} * \mathfrak{s}, q) = I_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K})$ and
 $F_{\omega_0}^{\mathbb{K}} * (a^{-1} * \mathfrak{s}, q) = F_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}), \forall a, b \in \check{G}, q \in \check{Q}, \mathfrak{s} \in \check{S} \text{ and } \mathbb{K} \in [0,1]$.

And also,

- $\Leftrightarrow T_{\omega_0}^{\mathbb{K}} * (a \dot{b}^{-1} * \mathfrak{s}, q) \geq \{T_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge T_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$,
 $I_{\omega_0}^{\mathbb{K}} * (a \dot{b}^{-1} * \mathfrak{s}, q) \geq \{I_{\omega_0} * ((a * \mathfrak{s}, q), \mathbb{K}) \wedge I_{\omega_0} * ((b * \mathfrak{s}, q), \mathbb{K})\}$ and

$$F_{\omega_0^{\mathbb{K}}} * (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) \leq \{F_{\omega_0} * ((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \vee F_{\omega_0} * ((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\}$$

Theorem 3.4: Let $\{\omega_0^{\mathbb{K}} *_{1}, \omega_0^{\mathbb{K}} *_{2}, \omega_0^{\mathbb{K}} *_{3}, \dots, \omega_0^{\mathbb{K}} *_{n}\}$ be *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Then $\omega_0^{\mathbb{K}} * = \cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}$ is *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Proof:

Let $\{\omega_0^{\mathbb{K}} *_{1}, \omega_0^{\mathbb{K}} *_{2}, \omega_0^{\mathbb{K}} *_{3}, \dots, \omega_0^{\mathbb{K}} *_{n}\}$ be *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Let $\omega_0^{\mathbb{K}} * = \cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}, \forall \mathfrak{a}, \mathfrak{b} \in \check{G}, q \in \bar{Q}, \mathfrak{s} \in \check{S}$ and $\mathbb{K} \in [0,1]$.

Then,

$$\begin{aligned} \omega_0^{\mathbb{K}} * (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) &= \cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) \\ &= \{(\mathfrak{a} * \mathfrak{s}, q), (T_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q)), (I_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q)), (F_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q))\} \end{aligned}$$

Now,

$$\begin{aligned} T_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) &= \max T_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) \\ &\geq \max \{T_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a} * \mathfrak{s}, q) \wedge \max T_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{b} * \mathfrak{s}, q)\} \\ &\geq \max \{T_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \wedge T_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\} \\ &= \{\max T_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \wedge \max T_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\} \\ &= \{T_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a} * \mathfrak{s}, q) \wedge T_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{b} * \mathfrak{s}, q)\} \end{aligned}$$

$$\begin{aligned} I_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) &= \max I_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) \\ &\geq \max \{I_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a} * \mathfrak{s}, q) \wedge \max I_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{b} * \mathfrak{s}, q)\} \\ &\geq \max \{I_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \wedge I_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\} \\ &= \{\max I_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \wedge \max I_{\omega_0^{\mathbb{K}} *_{i}} ((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\} \\ &= \{I_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a} * \mathfrak{s}, q) \wedge I_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{b} * \mathfrak{s}, q)\} \end{aligned}$$

$$\begin{aligned} F_{\cup_{i=1}^n \omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) &= \min F_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) \\ &\leq \min \{F_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{a} * \mathfrak{s}, q) \vee \min F_{\omega_0^{\mathbb{K}} *_{i}} (\mathfrak{b} * \mathfrak{s}, q)\} \end{aligned}$$

$$\begin{aligned} &\leq \min\{F_{\omega_0^{*i}}((a * \xi, q), \mathbb{K}) \vee F_{\omega_0^{*i}}((b * \xi, q), \mathbb{K})\} \\ &= \{ \min F_{\omega_0^{*i}}((a * \xi, q), \mathbb{K}) \vee \min F_{\omega_0^{*i}}((b * \xi, q), \mathbb{K}) \} \\ &= \left\{ F_{\bigcup_{i=1}^n \omega_0^{*i}}(a * \xi, q) \vee F_{\bigcup_{i=1}^n \omega_0^{*i}}(b * \xi, q) \right\} \end{aligned}$$

Hence, $\omega_0^{*n} = \bigcup_{i=1}^n \omega_0^{*i}$ is a *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Proposition 3.5: Let $\{\omega_0^{*1}, \omega_0^{*2}, \omega_0^{*3}, \dots, \omega_0^{*n}\}$ be *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Then $\omega_0^{*n} = \bigcap_{i=1}^n \omega_0^{*i}$ is *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Proof:

Let $\{\omega_0^{*1}, \omega_0^{*2}, \omega_0^{*3}, \dots, \omega_0^{*n}\}$ be *Neu* $\mathbb{K} - \bar{Q}$ *GOFSG* of \check{G} operating on \check{S} .

Let $\omega_0^{*n} = \bigcap_{i=1}^n \omega_0^{*i}, \forall a, b \in \check{G}, q \in \bar{Q}, \xi \in \check{S}$ and $\mathbb{K} \in [0,1]$.

Then,

$$\begin{aligned} \omega_0^{*n}(ab^{-1} * \xi, q) &= \bigcap_{i=1}^n \omega_0^{*i}(ab^{-1} * \xi, q) \\ &= \left\{ (a * \xi, q), \left(T_{\bigcap_{i=1}^n \omega_0^{*i}}(ab^{-1} * \xi, q) \right), \left(I_{\bigcap_{i=1}^n \omega_0^{*i}}(ab^{-1} * \xi, q) \right), \left(F_{\bigcap_{i=1}^n \omega_0^{*i}}(ab^{-1} * \xi, q) \right) \right\} \end{aligned}$$

Now,

$$\begin{aligned} T_{\bigcap_{i=1}^n \omega_0^{*i}}(ab^{-1} * \xi, q) &= \max T_{\omega_0^{*i}}(ab^{-1} * \xi, q) \\ &\geq \max \left\{ T_{\omega_0^{*i}}(a * \xi, q) \wedge \max T_{\omega_0^{*i}}(b * \xi, q) \right\} \\ &\geq \max \left\{ T_{\omega_0^{*i}}((a * \xi, q), \mathbb{K}) \wedge T_{\omega_0^{*i}}((b * \xi, q), \mathbb{K}) \right\} \\ &= \left\{ \max T_{\omega_0^{*i}}((a * \xi, q), \mathbb{K}) \wedge \max T_{\omega_0^{*i}}((b * \xi, q), \mathbb{K}) \right\} \\ &= \left\{ T_{\bigcap_{i=1}^n \omega_0^{*i}}(a * \xi, q) \wedge T_{\bigcap_{i=1}^n \omega_0^{*i}}(b * \xi, q) \right\} \end{aligned}$$

$$\begin{aligned} I_{\bigcap_{i=1}^n \omega_0^{*i}}(ab^{-1} * \xi, q) &= \max I_{\omega_0^{*i}}(ab^{-1} * \xi, q) \\ &\geq \max \left\{ I_{\omega_0^{*i}}(a * \xi, q) \wedge \max I_{\omega_0^{*i}}(b * \xi, q) \right\} \\ &\geq \max \left\{ I_{\omega_0^{*i}}((a * \xi, q), \mathbb{K}) \wedge I_{\omega_0^{*i}}((b * \xi, q), \mathbb{K}) \right\} \\ &= \left\{ \max I_{\omega_0^{*i}}((a * \xi, q), \mathbb{K}) \wedge \max I_{\omega_0^{*i}}((b * \xi, q), \mathbb{K}) \right\} \end{aligned}$$

$$= \{I_{\cap_{i=1}^n \omega_0^{\mathbb{K} *}_i}(\mathfrak{a} * \mathfrak{s}, q) \wedge I_{\cap_{i=1}^n \omega_0^{\mathbb{K} *}_i}(\mathfrak{b} * \mathfrak{s}, q)\}$$

$$F_{\cap_{i=1}^n \omega_0^{\mathbb{K} *}_i}(\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) = \min F_{\omega_0^{\mathbb{K} *}_i}(\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q)$$

$$\leq \min \{F_{\omega_0^{\mathbb{K} *}_i}(\mathfrak{a} * \mathfrak{s}, q) \vee \min F_{\omega_0^{\mathbb{K} *}_i}(\mathfrak{b} * \mathfrak{s}, q)\}$$

$$\leq \min \{F_{\omega_0^{\mathbb{K} *}_i}((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \vee F_{\omega_0^{\mathbb{K} *}_i}((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\}$$

$$= \{\min F_{\omega_0^{\mathbb{K} *}_i}((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \vee \min F_{\omega_0^{\mathbb{K} *}_i}((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K})\}$$

$$= \{F_{\cap_{i=1}^n \omega_0^{\mathbb{K} *}_i}(\mathfrak{a} * \mathfrak{s}, q) \vee F_{\cap_{i=1}^n \omega_0^{\mathbb{K} *}_i}(\mathfrak{b} * \mathfrak{s}, q)\}$$

Hence, $\omega_0^{\mathbb{K} *} = \cap_{i=1}^n \omega_0^{\mathbb{K} *}_i$ is a *Neu K – Q GOFSG* of \check{G} operating on \check{S} .

Definition 3.6: Let $\omega_0^{\mathbb{K} *}$ be *Neu K – Q GOFSG* of \check{G} is said to be *Neu K – Q GOFNSG* of \check{G}

operating on \check{S} . If $T_{\omega_0^{\mathbb{K} *}}(\mathfrak{a}\mathfrak{b} * \mathfrak{s}, q) = T_{\omega_0^{\mathbb{K} *}}(\mathfrak{b}\mathfrak{a} * \mathfrak{s}, q) = T_{\omega_0^{\mathbb{K} *}}((\mathfrak{a}\mathfrak{b} * \mathfrak{s}, q), \mathbb{K}) = T_{\omega_0^{\mathbb{K} *}}((\mathfrak{b}\mathfrak{a} * \mathfrak{s}, q), \mathbb{K})$,

$$I_{\omega_0^{\mathbb{K} *}}(\mathfrak{a}\mathfrak{b} * \mathfrak{s}, q) = I_{\omega_0^{\mathbb{K} *}}(\mathfrak{b}\mathfrak{a} * \mathfrak{s}, q) = I_{\omega_0^{\mathbb{K} *}}((\mathfrak{a}\mathfrak{b} * \mathfrak{s}, q), \mathbb{K}) = I_{\omega_0^{\mathbb{K} *}}((\mathfrak{b}\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}),$$

$$F_{\omega_0^{\mathbb{K} *}}(\mathfrak{a}\mathfrak{b} * \mathfrak{s}, q) = F_{\omega_0^{\mathbb{K} *}}(\mathfrak{b}\mathfrak{a} * \mathfrak{s}, q) = F_{\omega_0^{\mathbb{K} *}}((\mathfrak{a}\mathfrak{b} * \mathfrak{s}, q), \mathbb{K}) = F_{\omega_0^{\mathbb{K} *}}((\mathfrak{b}\mathfrak{a} * \mathfrak{s}, q), \mathbb{K}) \text{ and}$$

$$T_{\omega_0^{\mathbb{K} *}}(\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1} * \mathfrak{s}, q) = T_{\omega_0^{\mathbb{K} *}}(\mathfrak{b} * \mathfrak{s}, q) = T_{\omega_0^{\mathbb{K} *}}((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K}) \quad ,$$

$$I_{\omega_0^{\mathbb{K} *}}(\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1} * \mathfrak{s}, q) = I_{\omega_0^{\mathbb{K} *}}(\mathfrak{b} * \mathfrak{s}, q) = I_{\omega_0^{\mathbb{K} *}}((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K}) \quad ,$$

$$F_{\omega_0^{\mathbb{K} *}}(\mathfrak{a}\mathfrak{b}\mathfrak{a}^{-1} * \mathfrak{s}, q) = F_{\omega_0^{\mathbb{K} *}}(\mathfrak{b} * \mathfrak{s}, q) = F_{\omega_0^{\mathbb{K} *}}((\mathfrak{b} * \mathfrak{s}, q), \mathbb{K}), \forall \mathfrak{a}, \mathfrak{b} \in \check{G}, q \in \check{Q}, \mathfrak{s} \in \check{S} \text{ and } \mathbb{K} \in [0,1].$$

Definition 3.7: Let $\omega_0^{\mathbb{K} *}$ be *Neu K – Q GOFsb* of \check{G} operating on \check{S} . Let $\alpha, \beta, \gamma \in [0,1]$ with

$\alpha + \beta + \gamma \leq 3$. Then (α, β, γ) is *K – Q group operating level subset* of $\omega_0^{\mathbb{K} *}$ is defined by

$$(\omega_0^{\mathbb{K} *})_{(\alpha, \beta, \gamma)} = \{(\mathfrak{a} * \mathfrak{s}, q), T_{\omega_0^{\mathbb{K} *}}(\mathfrak{a} * \mathfrak{s}, q) \geq \alpha, I_{\omega_0^{\mathbb{K} *}}(\mathfrak{a} * \mathfrak{s}, q) \geq \beta \text{ and } F_{\omega_0^{\mathbb{K} *}}(\mathfrak{a} * \mathfrak{s}, q) \leq \gamma; \mathfrak{a} \in \check{G}, q \in \check{Q}, \mathfrak{s} \in \check{S} \text{ and } \mathbb{K} \in [0,1]\}$$

Theorem 3.8: If $\omega_0^{\mathbb{K} *}$ is a *Neu K – Q GOFSG* of \check{G} operating on \check{S} and $(\alpha, \beta, \gamma) \in [0,1]$ then (α, β, γ)

is *K – Q group operating level subset* $(\omega_0^{\mathbb{K} *})_{(\alpha, \beta, \gamma)}$ of $\omega_0^{\mathbb{K} *}$ is a *Neu K – Q GOFSG* of \check{G} , where

$T_{\omega_0^{\mathbb{K}^*}}(e * \xi, q) \geq \alpha, I_{\omega_0^{\mathbb{K}^*}}(e * \xi, q) \geq \beta$ and $F_{\omega_0^{\mathbb{K}^*}}(e * \xi, q) \leq \gamma$, where e is identity element of \check{G} operating on \check{S} and $q \in \bar{Q}$ and $\mathbb{K} \in [0,1]$.

Proof:

Let $\omega_0^{\mathbb{K}^*}$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} operating on \check{S} .

Since $T_{\omega_0^{\mathbb{K}^*}}(e * \xi, q) \geq \alpha, I_{\omega_0^{\mathbb{K}^*}}(e * \xi, q) \geq \beta$ and $F_{\omega_0^{\mathbb{K}^*}}(e * \xi, q) \leq \gamma, e \in (\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)}$.

Therefore,

$$(\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)} \neq \{\}$$

Let $a, b \in (\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)}$ and $q \in \bar{Q}$ and $\mathbb{K} \in [0,1]$.

Then,

$$T_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \geq \alpha, I_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \geq \beta \text{ and } F_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \leq \gamma$$

$$T_{\omega_0^{\mathbb{K}^*}}(b * \xi, q) \geq \alpha, I_{\omega_0^{\mathbb{K}^*}}(b * \xi, q) \geq \beta \text{ and } F_{\omega_0^{\mathbb{K}^*}}(b * \xi, q) \leq \gamma$$

$$\Leftrightarrow \{T_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \wedge T_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)\} \geq \alpha \quad , \quad \{I_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \wedge I_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)\} \geq \beta \quad \text{and}$$

$$\{F_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \vee F_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)\} \leq \gamma$$

$$\Leftrightarrow T_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q) \geq \alpha, I_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q) \geq \beta \text{ and } F_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q) \leq \gamma$$

$$\Leftrightarrow (ab^{-1} * \xi, q) \in (\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)}$$

$$\Leftrightarrow (\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)} \text{ is a Neu } \mathbb{K} - \bar{Q} \text{ GOFSG of } \check{G} \text{ operating on } \check{S}.$$

Theorem 3.9: If $\omega_0^{\mathbb{K}^*}$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} then $\omega_0^{\mathbb{K}^*}$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} if and only if $(\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)}$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} for $\alpha, \beta, \gamma \in [0,1]$.

Proof:

Let $a, b \in (\omega_0^{\mathbb{K}^*})_{(\alpha, \beta, \gamma)}, q \in \bar{Q}, \xi \in \check{S}$ and $\mathbb{K} \in [0,1]$

Let $\omega_0^{\mathbb{K}^*}$ is Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} .

$$\Leftrightarrow T_{\omega_0}^{\mathbb{K}} * (a\bar{b}^{-1} * \varsigma, q) \geq \left\{ \left\{ T_{\omega_0} * ((a * \varsigma, q), \mathbb{K}) \wedge T_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \right\}, I_{\omega_0}^{\mathbb{K}} * (a\bar{b}^{-1} * \varsigma, q) \geq \right. \\ \left. \left\{ I_{\omega_0} * ((a * \varsigma, q), \mathbb{K}) \wedge I_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \right\} \right\} \\ , F_{\omega_0}^{\mathbb{K}} * (a\bar{b}^{-1} * \varsigma, q) \leq \left\{ \left\{ F_{\omega_0} * ((a * \varsigma, q), \mathbb{K}) \vee F_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \right\} \right\} \\ \Leftrightarrow T_{\omega_0}^{\mathbb{K}} * (a\bar{b}^{-1} * \varsigma, q) \geq \alpha, I_{\omega_0}^{\mathbb{K}} * (a\bar{b}^{-1} * \varsigma, q) \geq \beta, \quad F_{\omega_0}^{\mathbb{K}} * (a\bar{b}^{-1} * \varsigma, q) \leq \gamma \\ \Leftrightarrow (a\bar{b}^{-1} * \varsigma, q) \in (\omega_0^{\mathbb{K}} *)_{(\alpha, \beta, \gamma)} \text{ is a Neu } \mathbb{K} - \bar{Q} \text{ GOFSG of } \check{G} \text{ operating on } \check{S}.$$

Theorem 3.10: Let $\omega_0^{\mathbb{K}} *$ is a *Neu* $\mathbb{K} - \bar{Q}$ *GOFsb* of a group \check{G} operating on \check{S} . Then $\omega_0^{\mathbb{K}} *$ is a *Neu* $\mathbb{K} - \bar{Q}$ *GOFNSG* of \check{G} operating on \check{S} if and only if $(\omega_0^{\mathbb{K}} *)_{(\alpha, \beta, \gamma)}$ is a *Neu* $\mathbb{K} - \bar{Q}$ *GOFNSG* of \check{G} operating on \check{S} .

Proof:

Let $\omega_0^{\mathbb{K}} *$ be a *Neu* $\mathbb{K} - \bar{Q}$ *GOFNSG* of \check{G} operating on \check{S} .

$$T_{\omega_0}^{\mathbb{K}} * (a\bar{b}a^{-1} * \varsigma, q) = T_{\omega_0}^{\mathbb{K}} * (\bar{b} * \varsigma, q) = T_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \geq \alpha,$$

$$I_{\omega_0}^{\mathbb{K}} * (a\bar{b}a^{-1} * \varsigma, q) = I_{\omega_0}^{\mathbb{K}} * (\bar{b} * \varsigma, q) = I_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \geq \beta,$$

$$F_{\omega_0}^{\mathbb{K}} * (a\bar{b}a^{-1} * \varsigma, q) = F_{\omega_0}^{\mathbb{K}} * (\bar{b} * \varsigma, q) = F_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \leq \gamma, \forall a, \bar{b} \in \check{G} \text{ and } q \in \bar{Q} \text{ and } \mathbb{K} \in [0, 1].$$

Hence, $(\omega_0^{\mathbb{K}} *)_{(\alpha, \beta, \gamma)}$ is a *Neu* $\mathbb{K} - \bar{Q}$ *GOFNSG* of \check{G} operating on \check{S} .

4. A New Constrains of Homomorphism of Neutrosophic $\mathbb{K} - \bar{Q}$ Group Operating Fuzzy

Subgroup

Definition 4.1: Let \mathbb{X} and \bar{Q} be the non-empty sets. Let \check{G} and \check{G}' be any two groups. The function

$\tilde{\eta}_{\omega_0}^{\mathbb{K}} * : (\check{G} \times \bar{Q}) \times \check{S} \rightarrow (\check{G}' \times \bar{Q}) \times \check{S}$ is said to be $\mathbb{K} - \bar{Q}$ group operating homomorphism if

(i) $\tilde{\eta}_{\omega_0}^{\mathbb{K}} * : (\check{G} \times \bar{Q}) \times \check{S} \rightarrow (\check{G}' \times \bar{Q}) \times \check{S}$ is a $\mathbb{K} - \bar{Q}$ group operating homomorphism

(ii) $\tilde{\eta}_{\omega_0}^{\mathbb{K}} * (a\bar{b} * \varsigma, q) = \left\{ \tilde{\eta}_{\omega_0} * ((a * \varsigma, q), \mathbb{K}) \wedge \tilde{\eta}_{\omega_0} * ((\bar{b} * \varsigma, q), \mathbb{K}) \right\}, \forall a, \bar{b} \in \check{G}, \varsigma \in \check{S} \text{ and } q \in \bar{Q} \text{ and } \mathbb{K} \in [0, 1].$

Theorem 4.2: Let \check{G} and \check{G}' be any two groups and $\omega_0^{\mathbb{K}^*}$ be a $\mathbb{K} - \bar{Q}$ group operating homomorphism of \check{G} onto \check{G}' . If $\omega_0^{\mathbb{K}^*}$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G}' then $\tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (\omega_0^{\mathbb{K}^*})$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} .

Proof:

Let $\omega_0^{\mathbb{K}^*}$ be a $\mathbb{K} - \bar{Q}$ group operating fuzzy subgroup of \check{G}' operating on \check{S} .

By definition,

$$\tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (\omega_0^{\mathbb{K}^*}) = \left\{ \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (T_{\omega_0^{\mathbb{K}^*}}), \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (I_{\omega_0^{\mathbb{K}^*}}), \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (F_{\omega_0^{\mathbb{K}^*}}) \right\}$$

Let $a, b \in \check{G}, \check{s} \in \check{S}$ and $q \in \bar{Q}$ and $\mathbb{K} \in [0,1]$

We have,

$$\begin{aligned} \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (T_{\omega_0^{\mathbb{K}^*}})(ab^{-1} * \check{s}, q) &= T_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (ab^{-1} * \check{s}, q)) \\ &= T_{\omega_0^{\mathbb{K}^*}} * \{ \tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (a * \check{s}, q) \wedge \tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (b^{-1} * \check{s}, q) \} \end{aligned}$$

Since, $\tilde{\eta}_{\omega_0^{\mathbb{K}^*}}$ is a $\mathbb{K} - \bar{Q}$ group operating homomorphism.

$$\begin{aligned} &\geq \left\{ T_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (a * \check{s}, q)) \wedge T_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (b^{-1} * \check{s}, q)) \right\} \\ &\geq \left\{ T_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * ((a * \check{s}, q), \mathbb{K})) \wedge T_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * ((b^{-1} * \check{s}, q), \mathbb{K})) \right\} \\ &= \left\{ \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * ((T_{\omega_0^{\mathbb{K}^*}})(a * \check{s}, q)) \wedge \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * ((T_{\omega_0^{\mathbb{K}^*}})(b * \check{s}, q)) \right\}. \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * (I_{\omega_0^{\mathbb{K}^*}})(ab^{-1} * \check{s}, q) &= I_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (ab^{-1} * \check{s}, q)) \\ &= I_{\omega_0^{\mathbb{K}^*}} * \{ \tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (a * \check{s}, q) \wedge \tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (b^{-1} * \check{s}, q) \} \end{aligned}$$

Since, $\tilde{\eta}_{\omega_0^{\mathbb{K}^*}}$ is a $\mathbb{K} - \bar{Q}$ group operating homomorphism.

$$\begin{aligned} &\geq \left\{ I_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (a * \check{s}, q)) \wedge I_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * (b^{-1} * \check{s}, q)) \right\} \\ &\geq \left\{ I_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * ((a * \check{s}, q), \mathbb{K})) \wedge I_{\omega_0^{\mathbb{K}^*}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}^*}} * ((b^{-1} * \check{s}, q), \mathbb{K})) \right\} \\ &= \left\{ \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * ((I_{\omega_0^{\mathbb{K}^*}})(a * \check{s}, q)) \wedge \tilde{\eta}_{\omega_0^{\mathbb{K}^*}}^{-1} * ((I_{\omega_0^{\mathbb{K}^*}})(b * \check{s}, q)) \right\}. \end{aligned}$$

$$\begin{aligned} \tilde{\eta}_{\omega_0^{\mathbb{K}}}^{-1} * (F_{\omega_0^{\mathbb{K}}} * (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q)) &= F_{\omega_0^{\mathbb{K}}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}}} * (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q)) \\ &= F_{\omega_0^{\mathbb{K}}} * \{ \tilde{\eta}_{\omega_0^{\mathbb{K}}} * (\mathfrak{a} * \mathfrak{s}, q) \vee \tilde{\eta}_{\omega_0^{\mathbb{K}}} * (\mathfrak{b}^{-1} * \mathfrak{s}, q) \} \end{aligned}$$

Since, $\tilde{\eta}_{\omega_0^{\mathbb{K}}}$ is a $\mathbb{K} - \bar{Q}$ group operating homomorphism.

$$\begin{aligned} &\leq \{ F_{\omega_0^{\mathbb{K}}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}}} * (\mathfrak{a} * \mathfrak{s}, q)) \vee F_{\omega_0^{\mathbb{K}}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}}} * (\mathfrak{b}^{-1} * \mathfrak{s}, q)) \} \\ &\leq \{ F_{\omega_0^{\mathbb{K}}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}}} * ((\mathfrak{a} * \mathfrak{s}, q), \mathbb{K})) \vee F_{\omega_0^{\mathbb{K}}} * (\tilde{\eta}_{\omega_0^{\mathbb{K}}} * ((\mathfrak{b}^{-1} * \mathfrak{s}, q), \mathbb{K})) \} \\ &= \{ \tilde{\eta}_{\omega_0^{\mathbb{K}}}^{-1} * ((F_{\omega_0^{\mathbb{K}}} * (\mathfrak{a} * \mathfrak{s}, q)) \vee \tilde{\eta}_{\omega_0^{\mathbb{K}}}^{-1} * ((F_{\omega_0^{\mathbb{K}}} * (\mathfrak{b} * \mathfrak{s}, q))) \}. \end{aligned}$$

Hence, $\tilde{\eta}_{\omega_0^{\mathbb{K}}}^{-1} * (\omega_0^{\mathbb{K}})$ is a *Neu* $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} .

Note 4.3: Let $\mathfrak{a}, \mathfrak{b} \in I$ and i is any positive integer. If $\mathfrak{a} \leq \mathfrak{b}$ then operating fuzzy subgroup of \check{G} .

- (i) $(\mathfrak{a} * \mathfrak{s}, q)^i \leq (\mathfrak{b} * \mathfrak{s}, q)^i$
- (ii) $((\mathfrak{a} * \mathfrak{s}, q) \wedge (\mathfrak{b} * \mathfrak{s}, q))^i = ((\mathfrak{a} * \mathfrak{s}, q)^i \wedge (\mathfrak{b} * \mathfrak{s}, q)^i)$
- (iii) $((\mathfrak{a} * \mathfrak{s}, q) \vee (\mathfrak{b} * \mathfrak{s}, q))^i = ((\mathfrak{a} * \mathfrak{s}, q)^i \vee (\mathfrak{b} * \mathfrak{s}, q)^i)$

Theorem 4.4: Let $\omega_0^{\mathbb{K}}$ is a *Neu* $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} . Then

$$(\omega_0^{\mathbb{K}})^i = \left\{ (\mathfrak{a} * \mathfrak{s}, q), (T_{\omega_0^{\mathbb{K}}} * (\mathfrak{a} * \mathfrak{s}, q))^i, (I_{\omega_0^{\mathbb{K}}} * (\mathfrak{a} * \mathfrak{s}, q))^i, (F_{\omega_0^{\mathbb{K}}} * (\mathfrak{a} * \mathfrak{s}, q))^i, \forall \mathfrak{a} \in \check{G}, \mathfrak{s} \in \check{S} \text{ and } q \in \bar{Q} \text{ and } \mathbb{K} \in [0,1] \right\}$$

is a *Neu* $\mathbb{K} - \bar{Q}$ GOFSG of \check{G}^i .

Proof:

Let $(\check{G}, *)$ is a group then $(\check{G}^i, *)$ is also a group.

Let $\omega_0^{\mathbb{K}}$ is a *Neu* $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} .

Now, $\mathfrak{a} \in \check{G}, \mathfrak{s} \in \check{S}$ and $q \in \bar{Q}$ and $\mathbb{K} \in [0,1]$

We have,

$$T_{\omega_0^{\mathbb{K}}} *^i (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q) = T_{\omega_0^{\mathbb{K}}} * (\mathfrak{a}\mathfrak{b}^{-1} * \mathfrak{s}, q)^i$$

$$\begin{aligned}
 &\geq \{T_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \wedge T_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)\}^i \\
 &\geq \{T_{\omega_0^{\mathbb{K}^*}}(a * \xi, q)^i \wedge T_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)^i\} \\
 &= \{T_{\omega_0^{\mathbb{K}^*}}^i(a * \xi, q) \wedge T_{\omega_0^{\mathbb{K}^*}}^i(a * \xi, q)\} \\
 I_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q) &= I_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q)^i \\
 &\geq \{I_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \wedge I_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)\}^i \\
 &\geq \{I_{\omega_0^{\mathbb{K}^*}}(a * \xi, q)^i \wedge I_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)^i\} \\
 &= \{I_{\omega_0^{\mathbb{K}^*}}^i(a * \xi, q) \wedge I_{\omega_0^{\mathbb{K}^*}}^i(a * \xi, q)\} \\
 F_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q) &= F_{\omega_0^{\mathbb{K}^*}}(ab^{-1} * \xi, q)^i \\
 &\leq \{F_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \vee F_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)\}^i \\
 &\leq \{F_{\omega_0^{\mathbb{K}^*}}(a * \xi, q)^i \vee F_{\omega_0^{\mathbb{K}^*}}(b * \xi, q)^i\} \\
 &= \{F_{\omega_0^{\mathbb{K}^*}}^i(a * \xi, q) \vee F_{\omega_0^{\mathbb{K}^*}}^i(a * \xi, q)\} \\
 \therefore \omega_0^{\mathbb{K}^*} &\text{ is a Neu } \mathbb{K} - \bar{Q} \text{ GOFSG of } \bar{G}^i.
 \end{aligned}$$

5. Direct Product of Neutrosophic $\mathbb{K} - Q$ group operating Fuzzy Subgroup

Definition 5.1: Let $\omega_0^{\mathbb{K}^*}$ and $\mu_0^{\mathbb{K}^*}$ be a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \mathbb{X} and \hat{Y} respectively. Then the cartesian product of $\omega_0^{\mathbb{K}^*}$ and $\mu_0^{\mathbb{K}^*}$ denoted by $(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})$ is defined by

$$(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*}) = \left\{ \left\{ ((a, b) * \xi, q), T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})}((a, b) * \xi, q), I_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})}((a, b) * \xi, q), F_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})}((a, b) * \xi, q) \right\}; a \in \mathbb{X}, b \in \hat{Y}, q \in \bar{Q}, \xi \in \check{S} \text{ and } \mathbb{K} \in [0, 1] \right\}$$

Where,

$$T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})}((a, b) * \xi, q) = \{T_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \wedge T_{\mu_0^{\mathbb{K}^*}}(b * \xi, q)\}$$

$$I_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})}((a, b) * \xi, q) = \{I_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \wedge I_{\mu_0^{\mathbb{K}^*}}(b * \xi, q)\}$$

$$F_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})}((a, b) * \xi, q) = \{F_{\omega_0^{\mathbb{K}^*}}(a * \xi, q) \vee F_{\mu_0^{\mathbb{K}^*}}(b * \xi, q)\}.$$

Theorem 5.2: If $\omega_0^{\mathbb{K}^*}$ and $\mu_0^{\mathbb{K}^*}$ be a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} with \mathbb{X} and \hat{Y} respectively then $(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})$ is a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} .

Proof:

Let $\omega_0^{\mathbb{K}^*}$ and $\mu_0^{\mathbb{K}^*}$ be a Neu $\mathbb{K} - \bar{Q}$ GOFSG of \check{G} .

Now,

$$(\mathbf{a}_1, \mathbf{a}_2, \mathbf{b}_1, \mathbf{b}_2) \in (\mathbb{X} \times \hat{Y}), \mathfrak{s} \in \check{S}, q \in \bar{Q} \text{ and } \mathbb{K} \in [0,1]$$

$$\begin{aligned} \text{(i)} \quad & T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_1, \mathbf{b}_1) * \mathfrak{s}, q)((\mathbf{a}_2, \mathbf{b}_2) * \mathfrak{s}, q) \right) = T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left((\mathbf{a}_1 \mathbf{a}_2, \mathbf{b}_1 \mathbf{b}_2) * \mathfrak{s}, q \right) \\ & \geq \{ T_{\omega_0^{\mathbb{K}^*}} \left((\mathbf{a}_1 \mathbf{a}_2) * \mathfrak{s}, q \right) \wedge T_{\mu_0^{\mathbb{K}^*}} \left((\mathbf{b}_1 \mathbf{b}_2) * \mathfrak{s}, q \right) \} \\ & = \{ \{ T_{\omega_0^{\mathbb{K}^*}} \left(\mathbf{a}_1 * \mathfrak{s}, q \right) \wedge T_{\omega_0^{\mathbb{K}^*}} \left(\mathbf{a}_2 * \mathfrak{s}, q \right) \} \wedge \{ T_{\mu_0^{\mathbb{K}^*}} \left(\mathbf{b}_1 * \mathfrak{s}, q \right) \wedge T_{\mu_0^{\mathbb{K}^*}} \left(\mathbf{b}_2 * \mathfrak{s}, q \right) \} \} \\ & = \{ \{ T_{\omega_0^{\mathbb{K}^*}} \left((\mathbf{a}_1 * \mathfrak{s}, q), \mathbb{K} \right) \wedge T_{\omega_0^{\mathbb{K}^*}} \left((\mathbf{a}_2 * \mathfrak{s}, q), \mathbb{K} \right) \} \wedge \{ T_{\mu_0^{\mathbb{K}^*}} \left((\mathbf{b}_1 * \mathfrak{s}, q), \mathbb{K} \right) \wedge T_{\mu_0^{\mathbb{K}^*}} \left((\mathbf{b}_2 * \mathfrak{s}, q), \mathbb{K} \right) \} \} \\ & \geq \{ \{ T_{\omega_0^{\mathbb{K}^*}} \left((\mathbf{a}_1 * \mathfrak{s}, q), \mathbb{K} \right) \wedge T_{\mu_0^{\mathbb{K}^*}} \left((\mathbf{b}_1 * \mathfrak{s}, q), \mathbb{K} \right) \} \wedge \{ T_{\omega_0^{\mathbb{K}^*}} \left((\mathbf{a}_2 * \mathfrak{s}, q), \mathbb{K} \right) \wedge T_{\mu_0^{\mathbb{K}^*}} \left((\mathbf{b}_2 * \mathfrak{s}, q), \mathbb{K} \right) \} \} \\ & = \{ T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_1, \mathbf{b}_1) * \mathfrak{s}, q), \mathbb{K} \right) \wedge T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_2, \mathbf{b}_2) * \mathfrak{s}, q), \mathbb{K} \right) \} \\ & T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_1, \mathbf{b}_1) * \mathfrak{s}, q)((\mathbf{a}_2, \mathbf{b}_2) * \mathfrak{s}, q) \right) \\ & \geq \{ \{ T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_1, \mathbf{b}_1) * \mathfrak{s}, q), \mathbb{K} \right) \wedge T_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_2, \mathbf{b}_2) * \mathfrak{s}, q), \mathbb{K} \right) \} \} \end{aligned}$$

$$\begin{aligned} I_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left(((\mathbf{a}_1, \mathbf{b}_1) * \mathfrak{s}, q)((\mathbf{a}_2, \mathbf{b}_2) * \mathfrak{s}, q) \right) &= I_{(\omega_0^{\mathbb{K}^*} \times \mu_0^{\mathbb{K}^*})} \left((\mathbf{a}_1 \mathbf{a}_2, \mathbf{b}_1 \mathbf{b}_2) * \mathfrak{s}, q \right) \\ &\geq \{ I_{\omega_0^{\mathbb{K}^*}} \left((\mathbf{a}_1 \mathbf{a}_2) * \mathfrak{s}, q \right) \wedge I_{\mu_0^{\mathbb{K}^*}} \left((\mathbf{b}_1 \mathbf{b}_2) * \mathfrak{s}, q \right) \} \\ &= \{ \{ I_{\omega_0^{\mathbb{K}^*}} \left(\mathbf{a}_1 * \mathfrak{s}, q \right) \wedge I_{\omega_0^{\mathbb{K}^*}} \left(\mathbf{a}_2 * \mathfrak{s}, q \right) \} \wedge \{ I_{\mu_0^{\mathbb{K}^*}} \left(\mathbf{b}_1 * \mathfrak{s}, q \right) \wedge I_{\mu_0^{\mathbb{K}^*}} \left(\mathbf{b}_2 * \mathfrak{s}, q \right) \} \} \end{aligned}$$

$$\begin{aligned}
 &= \{ \{ I_{\omega_0} * ((a_1 * \varsigma, q), \mathbb{K}) \wedge I_{\omega_0} * ((a_2 * \varsigma, q), \mathbb{K}) \} \wedge \{ I_{\mu_0} * ((b_1 * \varsigma, q), \mathbb{K}) \wedge I_{\mu_0} * ((b_2 * \varsigma, q), \mathbb{K}) \} \} \\
 &\geq \{ \{ I_{\omega_0} * ((a_1 * \varsigma, q), \mathbb{K}) \wedge I_{\mu_0} * ((b_1 * \varsigma, q), \mathbb{K}) \} \wedge \{ I_{\omega_0} * ((a_2 * \varsigma, q), \mathbb{K}) \wedge I_{\mu_0} * ((b_2 * \varsigma, q), \mathbb{K}) \} \} \\
 &= \{ I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q), \mathbb{K}) \wedge I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_2, b_2) * \varsigma, q), \mathbb{K}) \} \\
 &I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q) ((a_2, b_2) * \varsigma, q)) \\
 &\geq \{ \{ I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q), \mathbb{K}) \wedge I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_2, b_2) * \varsigma, q), \mathbb{K}) \} \}
 \end{aligned}$$

$$\begin{aligned}
 &F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q) ((a_2, b_2) * \varsigma, q)) = F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * ((a_1 a_2, b_1 b_2) * \varsigma, q) \\
 &\leq \{ F_{\omega_0}^{\mathbb{K}} * ((a_1 a_2) * \varsigma, q) \vee F_{\mu_0}^{\mathbb{K}} * ((b_1 b_2) * \varsigma, q) \} \\
 &= \{ \{ F_{\omega_0}^{\mathbb{K}} * (a_1 * \varsigma, q) \vee F_{\omega_0}^{\mathbb{K}} * (a_2 * \varsigma, q) \} \vee \{ F_{\mu_0}^{\mathbb{K}} * (b_1 * \varsigma, q) \vee F_{\mu_0}^{\mathbb{K}} * (b_2 * \varsigma, q) \} \} \\
 &= \{ \{ F_{\omega_0} * ((a_1 * \varsigma, q), \mathbb{K}) \vee F_{\omega_0} * ((a_2 * \varsigma, q), \mathbb{K}) \} \vee \{ F_{\mu_0} * ((b_1 * \varsigma, q), \mathbb{K}) \vee I_{\mu_0} \\
 &\quad * ((b_2 * \varsigma, q), \mathbb{K}) \} \} \\
 &\leq \{ \{ F_{\omega_0} * ((a_1 * \varsigma, q), \mathbb{K}) \vee F_{\mu_0} * ((b_1 * \varsigma, q), \mathbb{K}) \} \vee \{ F_{\omega_0} * ((a_2 * \varsigma, q), \mathbb{K}) \vee I_{\mu_0} \\
 &\quad * ((b_2 * \varsigma, q), \mathbb{K}) \} \} \\
 &= \{ F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q), \mathbb{K}) \vee F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_2, b_2) * \varsigma, q), \mathbb{K}) \} \\
 &F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q) ((a_2, b_2) * \varsigma, q)) \\
 &\leq \{ \{ F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_1, b_1) * \varsigma, q), \mathbb{K}) \vee F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * (((a_2, b_2) * \varsigma, q), \mathbb{K}) \} \}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii) } &T_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * ((a, b)^{-1} * \varsigma, q) = T_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})} * ((a^{-1}, b^{-1}) * \varsigma, q) \\
 &= \{ T_{\omega_0}^{\mathbb{K}} * (a^{-1} * \varsigma, q) \wedge T_{\mu_0}^{\mathbb{K}} * (b^{-1} * \varsigma, q) \} \\
 &\geq \{ T_{\omega_0}^{\mathbb{K}} * (a * \varsigma, q) \wedge T_{\mu_0}^{\mathbb{K}} * (b * \varsigma, q) \}
 \end{aligned}$$

$$\begin{aligned}
 &= \{T_{\omega_0} * ((a * \xi, q), \mathbb{K}) \wedge T_{\mu_0} * ((b * \xi, q), \mathbb{K})\} \\
 T_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b)^{-1} * \xi, q) &\geq T_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b) * \xi, q) \\
 I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b)^{-1} * \xi, q) &= I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a^{-1}, b^{-1}) * \xi, q) \\
 &= \{I_{\omega_0^{\mathbb{K}}} * (a^{-1} * \xi, q) \wedge I_{\mu_0^{\mathbb{K}}} * (b^{-1} * \xi, q)\} \\
 &\geq \{I_{\omega_0^{\mathbb{K}}} * (a * \xi, q) \wedge I_{\mu_0^{\mathbb{K}}} * (b * \xi, q)\} \\
 &= \{I_{\omega_0} * ((a * \xi, q), \mathbb{K}) \wedge I_{\mu_0} * ((b * \xi, q), \mathbb{K})\} \\
 I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b)^{-1} * \xi, q) &\geq I_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b) * \xi, q) \\
 F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b)^{-1} * \xi, q) &= F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a^{-1}, b^{-1}) * \xi, q) \\
 &= \{F_{\omega_0^{\mathbb{K}}} * (a^{-1} * \xi, q) \vee F_{\mu_0^{\mathbb{K}}} * (b^{-1} * \xi, q)\} \\
 &\leq \{F_{\omega_0^{\mathbb{K}}} * (a * \xi, q) \vee F_{\mu_0^{\mathbb{K}}} * (b * \xi, q)\} \\
 &= \{F_{\omega_0} * ((a * \xi, q), \mathbb{K}) \vee F_{\mu_0} * ((b * \xi, q), \mathbb{K})\} \\
 F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b)^{-1} * \xi, q) &\leq F_{(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})}((a, b) * \xi, q)
 \end{aligned}$$

Hence, $(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})$ is a *Neu $\mathbb{K} - \bar{Q}$ GOFSG* of $(\omega_0^{\mathbb{K}} * \mu_0^{\mathbb{K}})$.

6. Conclusion

This paper introduces a novel notation for the Neutrosophic $\mathbb{K} - \bar{Q}$ group, which functions over fuzzy subsets, and explores its fundamental algebraic properties. A new structure, termed the Neutrosophic $\mathbb{K} - \bar{Q}$ group operating fuzzy subgroup, is defined, and several key algebraic characteristics of this structure are thoroughly examined. The study further investigates the behavior of this subgroup under homomorphisms by analyzing both the homomorphic image and preimage, establishing critical results in this context. Additionally, the paper proves that the direct product of two Neutrosophic $\mathbb{K} - \bar{Q}$ group operating fuzzy subgroups results in another Neutrosophic $\mathbb{K} - \bar{Q}$ group operating fuzzy subgroup. This concept is further generalized to accommodate the direct product of a finite collection of such subgroups. Overall, the findings contribute to a deeper understanding of fuzzy subgroup behavior in the framework of Neutrosophic algebraic systems and lay the groundwork for further theoretical development and practical applications in uncertain or imprecise environments.

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