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Neutrosophic Pythagorean Cubic Soft Sets

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Abstract: In this paper we introduce the notion of Neutrosophic Pythagorean Cubic Soft Sets (NPCSS). Here we define Internal Neutrosophic Pythagorean Cubic Soft Sets (INPCSS), External Neutrosophic Pythagorean Cubic Soft Sets (ENPCSS) and also propose the new idea of $\frac{1}{3}$ INPCSS (or $\frac{2}{3}$ ENPCSS), and $\frac{2}{3}$ INPCSS ($\frac{1}{3}$ ENPCSS). Further Neutrosophic Pythagorean Cubic Soft Sets as P-order, P-union, P-intersection as well as R-order, R-union, R-intersection are introduced for Neutrosophic Pythagorean Cubic Soft Sets which act as a tool to study some of their properties of newly introduced sets. And also, in this paper we introduce the concepts of P−OR and P−AND for Neutrosophic Pythagorean Cubic Soft Sets and explore their fundamental properties. We establish that the P−union and P−intersection of two Internal Neutrosophic soft cubic sets also result in Internal Neutrosophic Pythagorean Cubic Soft Sets. Additionally, we examine the conditions under which the P−union and P−intersection of two T−External (or I−External, F−External) Neutrosophic soft cubic sets remain T−External (or I−External, F−External) within the framework of Neutrosophic Pythagorean Cubic Soft Sets.

Keywords: NCPSS, INPCSS, ENPCSS, OR, AND.

1. Introduction

The concept of a cubic set, which merges fuzzy sets and interval-valued fuzzy sets, was first introduced by Y. B. Jun et al. [3,4]. Later, in 2012, Saleem Abdullah et al. [23] extended this idea by incorporating cubic sets into soft set theory. In 2014, Muhiuddin et al. [8] further expanded on this framework by defining Internal and External Cubic Soft Sets, along with various operations such as P-cubic soft subsets, R-cubic soft subsets, P-union, P-intersection, R-union, and R-intersection of Cubic Soft Sets.

Soft set theory, introduced by Molodtsov, is a mathematical tool for handling uncertainty. It has been widely applied in decision-making and medical sciences. Cagman et al. extended it to soft topology, while Shabir and Naz introduced key concepts like soft interior and soft closure, paving the way for its integration into traditional topology. Maji et al. defined fundamental operations such as union, intersection, and complement, making soft sets valuable in various fields and also introduced Intuitionistic Fuzzy and Neutrosophic soft sets, which were further developed by Deli and Broumi for decision-making applications.

The algebraic properties of soft sets were studied by Ali et al., and their connections to BCK/BCI-algebras were explored by Iseki, Jun, and others. Soft set theory has since evolved across multiple domains.

Cubic set theory, introduced by Jun et al., combines interval-valued fuzzy sets with fuzzy sets to represent both certainty and uncertainty. It was extended to Neutrosophic cubic sets, with defined operations like R-union and P-intersection. Jun and Lee applied cubic sets to BCK/BCI-algebras, while Abdullah introduced Cubic Soft Sets, further expanding their mathematical applications. Recent studies have focused on merging soft sets and cubic sets.

A significant development came in 2016 when Ali et al. [9] advanced the concept by introducing Neutrosophic cubic sets, which broadened the scope of cubic sets. They also introduced Internal and External Neutrosophic cubic sets, formulated the Hamming distance between them, and proposed a decision-making approach based on similarity measures in pattern recognition. Around the same time, Jun et al. [3,4] examined different types of Neutrosophic cubic sets, such as truth-Internal, indeterminacy-Internal, falsity-Internal, truth-External, indeterminacy-External, and falsity-External, and explored their properties.

This chapter presents a novel approach by merging Neutrosophic sets and interval Neutrosophic sets with soft sets, leading to the introduction of Neutrosophic Pythagorean Cubic Soft Sets. Various properties of these sets are analyzed in detail.

2. **Preliminary:**

Definition 2.1 *[1]*

A fuzzy set F on a nonempty universe of discourse U is defined as
$F = \{\langle x, \mu_A(x)\rangle,\ x \in U\}$, where $\mu: U \to [0, 1]$.

Definition 2.2 *[2]*

Let U be a nonempty universe. An intuitionistic fuzzy set (IFS) IF is an ordered pair IF $= \{\langle x, \mu_A(x), \nu_A(x)\rangle,\ x \in U\}$, where the functions $\mu: U \to [0, 1]$ and $\nu: U \to [0, 1]$ denote the degree of membership and the degree of nonmembership respectively and $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Definition 2.3 *[3]*

A Neutrosophic set A on the universe of discourse U is defined as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle\}, x \in X$ where $T, I, F: X \to]^-0, 1^+[$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

Definition 2.4 *[4]*

If $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle/x \in X\}$ and $B = \{\langle x, T_B(x), I_B(x), F_B(x)\rangle\ /x \in X\}$ are any two Neutrosophic sets of U, then

(i) $A \subseteq B \Leftrightarrow T_A(x) \leq T_B(x)$; $I_A(x) \leq I_B(x)$ and $F_A(x) \geq F_B(x)$;

(ii) $A = B \Leftrightarrow T_A(x) = T_B(x)$; $I_A(x) = I_B(x)$ and $F_A(x) = F_B(x)\ \forall\ x \in X$;

(iii) $A \cap B = \{\langle x, T_{(A\cap B)}(x), I_{(A\cap B)}(x), F_{(A\cap B)}(x)\rangle / x \in X\}$

where $T_{A\cap B}(x) = \min\{T_A(x), T_B(x)\}$, $I_{A\cap B}(x) = \min\{I_A(x), I_B(x)\}$,

$F_{A\cap B}(x) = \max\{F_A(x), F_B(x)\}$;

(iv) $A \cup B = \{\langle x, T_{(A\cup B)}(x), I_{(A\cup B)}(x), F_{(A\cup B)}(x)\rangle / x \in X\}$

where $T_{A\cup B}(x) = \max\{T_A(x), T_B(x)\}$, $I_{A\cup B}(x) = \max\{I_A(x), I_B(x)\}$,

$F_{A\cup B}(x) = \min\{F_A(x), F_B(x)\}$.

Definition 2.5 *[5]*

0_N and 1_N in X as follows:

0_N may be defined as:

(01) $0_N = \{\langle x, 0, 0, 1\rangle : x \in X\}$

(02) $0_N = \{\langle x, 0, 1, 1\rangle : x \in X\}$

(03) $0_N = \{\langle x, 0, 1, 0\rangle : x \in X\}$

(04) $0_N = \{\langle x, 0, 0, 0\rangle : x \in X\}$

1_N may be defined as:

(11) $1_N = \{\langle x, 1, 0, 0\rangle : x \in X\}$

(12) $1_N = \{\langle x, 1, 0, 1\rangle : x \in X\}$

(13) $1_N = \{\langle x, 1, 1, 0\rangle : x \in X\}$

(14) $1_N = \{\langle x, 1, 1, 1\rangle : x \in X\}$

Definition 2.6 *[6]*

Let U be the initial universal set and E be a set of parameters. Let P(U) denote the power set of U. Consider a non-empty set A, $A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F: A \rightarrow P(U)$.

Definition 2.7 *[7]*

For any two soft sets (F, A) and (G, B) over a common universe U, we say that (F, A) is a subset of (G, B) if

(i) $A \subseteq B$

(ii) $\forall$ e $\in$ A, F(e) and G(e) are identical approximations.

We write $(F, A) \subseteq (G, B)$.

And when (F, A) is soft super set of (G, B), and (G, B) is a soft subset of (F, A). We denote it by $(F, A) \supseteq (G, B)$.

Definition 2.8 *[7]*

Two soft sets (F, A) and (G, B) over a common universe U are said to be soft equal, if (F, A) is a soft subset of (G, B) and (G, B) is a soft subset of (F, A).

Definition 2.9 *[8]*

Let U be an initial universe set and E be a set of parameters. Consider $A \subset E$. Let P(U) denotes the set of all Neutrosophic sets of U. The collection (F, A) is termed to be the soft Neutrosophic set over U, where F is a mapping given by $F: A \to P(U)$.

Definition 2.10 *[9,10]*

A d−algebra is a nonempty set X with a constant 0 and a binary operation * satisfying the following axioms.

(i) $x * x = 0$

(ii) $0 * x = 0$

(iii) $x * y = 0$ and $y * x = 0$ imply $x = y$, for all $x, y \in X$.

A BCK-algebra is a d−algebra satisfying additional axioms:

(iv) $((x * y) * (x * z)) * (z * y) = 0$

(v) $(x * (x * y)) * y = 0$ for all $x, y, z \in X$.

If X we can define a binary relation "≤" by $x \le y$ if and only if $x * y = 0$. A mapping $f: X \to Y$ of d−algebra is a called a d−homomorphism if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$.

Definition 2.11 *[10]*

Let X be a d−algebra and let $\varphi \neq I \subseteq X$. I is called a d−subalgebra of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called BCK-ideal of X if it satisfies

(a) $0 \in I$

(b) $x * y \in I$ and $y \in I$ imply $x \in I$

I is called d−ideal of X if it satisfies (b) and

(c) $x \in I$ and $y \in X$ imply $x * y \in I$, ie., $I * X \subseteq I$.

(d) $(x * z) * (y * z) * (x * y) = 0$.

Definition 2.12 *[11]*

Let X be a d−algebra and $0 \in I \subseteq X$. I is called a quick ideal of X if for any $x, y \in X$ with $x * y \neq 0$, $x * y \in I$ imply $x, y \in I$.

3. NEUTROSOPHIC PYTHAGOREAN CUBIC SOFT SETS

In this section we introduce the notion of Neutrosophic Pythagorean Cubic Soft Sets (NPCSS). Here we define Internal Neutrosophic Pythagorean Cubic Soft Sets (INPCSS), External

Neutrosophic Pythagorean Cubic Soft Sets (ENPCSS) and also propose the new idea of $\frac{1}{3}$ INPCSS (or $\frac{2}{3}$ ENPCSS), and $\frac{2}{3}$ INPCSS ($\frac{1}{3}$ ENPCSS). Further Neutrosophic Pythagorean Cubic Soft Sets as P-order, P-union, P-intersection as well as R-order, R-union, R-intersection are introduced for Neutrosophic Pythagorean Cubic Soft Sets which acts as a tool to study some of their properties of newly introduced sets.

Definition 3.1 Let K be an initial universe set. Let NPC(K) denote the set of all Neutrosophic Pythagorean cubic sets and E be the set of parameters. Let D $\subset$ E, then

$(Q, D) = \{Q(ei)= \{\langle u, A_{e_i}(u), \lambda e_i(u)\rangle : u \in U\} e_i \in D\}$

Where $A_{ei}(u)=\{\langle u, A^T{}_{ei}(u), A^I{}_{ei}(u)\ A^F{}_{ei}(u)\rangle / u \in U\}$ is an Interval Neutrosophic Pythagorean Set,

$\lambda e_i(u) =\{\langle u, \lambda^T{}_{ei}(u), \lambda^I{}_{ei}(u)\ \lambda^F{}_{ei}(u)\rangle / u \in U\}$ is a Neutrosophic Pythagorean set.

The pair (Q, D) is termed to be the Neutrosophic Pythagorean Cubic Soft Set over U where Q is a mapping given by Q: D → NPC(K). The sets of all Neutrosophic Pythagorean cubic soft sets over U will be denoted by C_N^U.

Example 3.2 Let K = {u1, u2, u3, u4} be the set of Hand ball players under consideration and E = {e1, e2, e3, e4} be the set of parameters, where e1, e2, e3, e4 represent fitness, good current form, good domestic hand ball record and good moral character respectively. Let D = {e1, e2, e3} $\subseteq$ E. then, the Neutrosophic Pythagorean Cubic Soft Set,

$(Q, D) = \{Q(ei) = \{\langle u, A_{e_i}(u), \lambda e_i(u)\rangle : u \in U\}\ e_i \in D\}$, i = 1, 2, 3 in U is

Table 3.1 tabular representation of (Q,D)

U	e_1	e_2	e_3
u_1	([0.2,0.4],[0.4,0.7],[0.3,0.6])(0.3, 0.5, 0.55)	([0.7,0.4],[0.7,0.5],[0.5,0.3])(0.7,0.6, 0.5)	([0.6,0.3],[0.5,0.4],[0.7,0.4]) (0.5, 0.45, 0.55)
u_2	([0.6,0.3],[0.7,0.4],[0.5,0.2])(0.5, 0.6, 0.3)	([0.5,0.3],[0.6,0.4],[0.7,0.3])(0.4,0.5, 0.4)	([0.7,0.4],[0.5,0.3],[0.8,0.5]) (0.5, 0.4, 0.7)
u_3	([0.5,0.7],[0.2,0.4], [0.4,0.6]) (0.6, 0.3, 0.5)	([0.4,0.2],[0.9,0.6],[0.8,0.5])(0.35,0.7, 0.6)	([0.6,0.3],[0.7,0.4],[0.8,0.5]) (0.7, 0.3, 0.4)
u_4	([0.6,0.8],[0.3,0.6],[0.2,0.5]) (0.7, 0.4, 0.4)	([0.6, 0.3],[0.7, 0.4],[0.5,0.2])(0.3, 0.5, 0.35)	([0.5,0.1],[0.8,0.4],[0.6,0.4]) (0.2, 0.7 0.5)
u_5	([0.6,0.2],[0.5,0.3],[0.5,0.3])(0.4,0.45,0.35)	([0.5,0.1],[0.6,0.4],[0.6,0.4])(0.45, 0.55, 0.55)	([0.7,0.3],[0.9,0.6],[0.6,0.3]) (0.55, 0.65, 0.45)

Definition 3.3 The complement of a Neutrosophic Pythagorean Cubic Soft Set

(Q, D) = {Q(ei) = {⟨u, A_{e_i} (u), λ_{e_i}(u)⟩: u ∈ U}e_i ∈ D} is denoted by $(Q, D)^c$ and defined as $(Q, D)^c = \{(Q, D)^c = (Q^c, D)\}$,

where $Q^c : D \rightarrow$ NPC(U) and

$(Q, D)^c$ ={(Q(ei))^c = {⟨u, $A^c{}_{e_i}$ (u), $\lambda^c{}_{e_i}$(u)⟩ : u ∈ U} e_i ∈ D}.$(Q, D)^c = \{\langle u, ([1-A^{+T}{}_{ei}(u), 1- A^{-T}{}_{ei}(u)], [1 - A^{+I}{}_{ei}(u), 1 - A^{+I}{}_{ei}(u)], [1 - A^{+F}{}_{ei}(u), 1 - A^{+F}{}_{ei}(u)]), (1 - \lambda^T{}_{ei}(u), 1 - \lambda^I{}_{ei}(u), 1 - \lambda^F{}_{ei}(u)) \rangle : u \in U\} e_i \in D$.

Example 3.4 Let K = {u1, u2} be the initial universe and E = {e1, e2} parameters set. Let (Q, D) be a Neutrosophic Pythagorean Cubic Soft Set over K and defined as
(Q, D) ={Q(ei) = {{⟨u, A_{e_i} (u), λ_{e_i}(u)⟩: u ∈ U} e_i ∈ D}. Then

U	Q (e1)	Q (e2)
u1	([0.2,0.4],[0.9,0.6],[0.7,0.3])(0.5,0.8.0.4)	([0.2,0.4],[0.7,0.6],[0.5,0.1])(0.4,0.2,0.6)
u2	([0.4,0.6],[0.8,0.5],[0.2,0.1])(0.8,0.3,0.4)	([0.6,0.7],[0.3,0.8],[0.9,0.5])(0.2,0.6,0.7)

$(P, M)^c$={$(P(e_i))^c = \{\langle u, A_{e_i}{}^c(u), \lambda_{e_i}{}^c(u) \rangle : u \in U\} e_i \in D\}$ is defined as

U	Q^c (e1)	Q^c (e2)
u1	([0.6,0.8],[0.4,0.1],[0.7,0.3])(0.5,0.2.0.6)	([0.6,0.8],[0.4,0.3],[0.9,0.5])(0.6,0.8,0.4)
u2	([0.4,0.6],[0.5,0.2],[0.9,0.8])(0.2,0.7,0.6)	([0.3,0.4],[0.2,0.7],[0.5,0.1])(0.8,0.4,0.3)

Definition 3.5 Let K be an initial universal set. A Neutrosophic Pythagorean Cubic Soft Set (Q, D) in K is said to be

- Truth-Internal (briefly, T-Internal) if the following inequality is valid
  (∀ u ∈ U, e_i ∈ D) $(A^{-T}{}_{ei}(u) \leq \lambda^T{}_{ei}(u) \leq A^{+T}{}_{ei}(u))$, (2.1)
- Indeterminacy-Internal (briefly, I–Internal) if the following inequality is valid
  (∀ u ∈ U, e_i ∈ D) $(A^{-I}{}_{ei}(u) \leq \lambda^I{}_{ei}(u) \leq A^{+I}{}_{ei}(u))$, (2.2)
- Falsity-Internal (briefly, F−Internal) if the following inequality is valid
  (∀ u ∈ U, e_i ∈ D) $(A^{-F}{}_{ei}(u) \leq \lambda^F{}_{ei}(u) \leq A^{+F}{}_{ei}(u))$, (2.3)

If a Neutrosophic Pythagorean cubic set in K satisfies (2.1), (2.2), (2.3) we say that (Q, D) is an Internal Neutrosophic Pythagorean cubic soft (INPCSS) in K.

Table 3.2: The tabular representation of INPCSS (Q, D)

U	e1	e2	e3
u1	([0.3,0.5],[0.2,0.4],[0.1,0.5])(0.4, 0.35,0.3)	([0.1,0.3],[0.3,0.6],[0.2,0.5])(0.2, 0.4,0.45)	([0.2,0.5],[0.2,0.4],[0.4,0.6]) (0.4,0.3,0.5)
u2	([0.8,0.3],[0.1,0.2],[0.9,0.4])(0.2, 0.15,0.3)	([0.3,0.6],[0.2,0.5],[0.4,0.7])(0.4, 0.35,0.5)	([0.4,0.6],[0.3,0.5],[0.4,0.6]) (0.5,0.4,0.35)
u3	([0.2,0.7],[0.1,0.4],[0.3,0.5])(0.3, 0.25,0.4)	([0.4,0.6],[0.1,0.3],[0.3,0.5])(0.5, 0.25,0.35)	([0.5,0.7],[0.9,0.3],[0.1,0.3]) (0.55,0.2,0.4)

u4	([0.6,0.4],[0.2,0.3],[0.2,0.7])(0.2,0.2,0.4)	([0.7,0.2],[0.2,0.5],[0.1,0.4])(0.55,0.3,0.3)	([0.1,0.5],[0.2,0.4],[0.5,0.7])(0.45,0.35,0.55)
u5	([0.2,0.6],[0.9,0.2],[0.1,0.3])(0.55,0.1,0.25)	([0.3,0.7],[0.4,0.6],[0.4,0.6])(0.5,0.55,0.45)	([0.2,0.5],[0.1,0.6],[0.3,0.6])(0.3,0.4,0.5)

Definition 3.6 Let K be an initial universal set. A Neutrosophic Pythagorean Cubic Soft Set (Q, D) in K is said to be

- Truth-External (briefly, T− External) if the following inequality is valid
  $(\forall\ u \in U, e_i \in D)\ \lambda^{T}{}_{ei}(u) \notin [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)],$ (2.4)
- Indeterminacy-External (briefly, I− External) if the following inequality is valid
  $(\forall\ u \in U, e_i \in D)\ \lambda^{I}{}_{ei}(u) \notin [(A^{-I}{}_{ei}(u),\ A^{+I}{}_{ei}(u)]$ (2.5)
- Falsity-External (briefly, F− External) if the following inequality is valid
  $(\forall\ u \in U, e_i \in D)\ \lambda^{F}{}_{ei}(u) \notin [(A^{-F}{}_{ei}(u),\ A^{+F}{}_{ei}(u)],$ (2.6)

If a Neutrosophic Pythagorean Cubic Soft Set (Q, D) in U satisfies (2.4), (2.5) and (2.6) we say that (Q, D) is an External Neutrosophic Pythagorean Cubic Soft Set (ENPSS) in U.

Example 3.7 Let K = {g1, g2, g3, g4} be the set of hand ball players under consideration and E = {e1, e2, e3, e4} be the set of parameters, e1, e2, e3, e4 represent fitness, good current form, good domestic hand ball record and good moral characters respectively. Let D = {e1, e2, e3, e4} ∈ E. Then, the Neutrosophic Pythagorean Cubic Soft Set (Q, D) = {Q(ei) = {⟨g, A_{e_i} (g), λ_{e_i}(g)⟩: g ∈ U} e_i ∈ D}, i =1,2,3 in U is an External Neutrosophic Pythagorean Cubic Soft Set (ENPCSS) in U.

Table 3.3: The tabular representation of ENPCSS (Q, D)

U	e1	e2	e3
p1	([0.3,0.4],[0.3,0.4],[0.1,0.4])(0.5,0.4,0.5)	([0.1,0.3],[0.3,0.5],[0.3,.6])(0.9,0.3,0.35)	([0.3,0.4],[0.3,0.4],[0.4,0.5])(0.5,0.4,0.3)
p2	([0.9,0.3],[0.1,0.3],[0.9,0.4])(0.4,0.3,0.4)	([0.3,0.5],[0.3,0.4],[0.4,0.7])(0.3,0.1,0.3)	([0.4,0.5],[0.3,0.4],[0.3,0.5])(0.3,0.5,0.1)
p3	([0.3,0.4],[0.1,0.4],[0.3,0.4])(0.5,0.4,0.1)	([0.4,0.5],[0.1,0.3],[0.3,0.4])(0.3,0.4,0.25)	([0.4,0.7],[0.9,0.3],[0.1,0.4])(0.4,0.4,0.4)
p4	([0.1,0.4],[0.9,0.3],[0.3,0.7])(0.4,0.4,0.1)	([0.4,0.7],[0.3,0.4],[0.1,0.4])(0.4,0.1,0.4)	([0.1,0.4],[0.3,0.4],[0.3,0.7])(0.5,0.35,0.25)
p5	([0.4,0.5],[0.9,0.3],[0.1,0.3])(0.25,0.3,0.4)	([0.3,0.7],[0.4,0.5],[0.4,0.5])(0.3,0.25,0.3)	([0.3,0.4],[0.1,0.5],[0.3,0.5])(0.1,0.7,0.3)

Theorem 3.8 Let (Q, D) = {Q (e_i) = {⟨u, A_{e_i} (u), λ_{e_i}(u)⟩: u ∈ U}e_i ∈ D} be a Neutrosophic Pythagorean Cubic Soft Set in U which is not an ENPCSS. Then, there exists at least one e_i ∈ D for which there exist some u ∈ U such that

$\lambda_{ei}{}^{T}(u) \in [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)],$

$\lambda^{I}{}_{ei}(u) \in [(A^{-I}{}_{ei}(u),\ A^{+I}{}_{ei}(u)],$

$\lambda^{F}{}_{ei}(u) \in [(A^{-F}{}_{ei}(u),\ A^{+F}{}_{ei}(u)]$

Proof. By the definition of an External Neutrosophic Pythagorean Cubic Soft Set (ENPCSS) we know that

$\lambda^{T}{}_{ei}(u) \notin [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)],\ \lambda^{I}{}_{ei}(u) \notin [(A^{-I}{}_{ei}(u),\ A^{+I}{}_{ei}(u)],$

$\lambda^{F}{}_{ei}(u) \notin [(A^{-F}{}_{ei}(u),\ A^{+F}{}_{ei}(u)],$

for all u ∈ U, corresponding to each e_i∈ D. But given that (Q, D) is not ENPCSS so far at least one e_i∈ D there exist some u ∈ U such that

$A^{-T}{}_{ei}(u) \le \lambda^{T}{}_{ei}(u) \le A^{+T}{}_{ei}(u),$

$A^{-I}{}_{ei}(u) \le \lambda^{I}{}_{ei}(u) \le A^{+I}{}_{ei}(u),$

$A^{-F}{}_{ei}(u) \le \lambda^{F}{}_{ei}(u) \le A^{+F}{}_{ei}(u).$

Hence the result.

Theorem 3.9 Let (Q, D) be a Neutrosophic Pythagorean Cubic Soft Set in U. If (Q, D) is both T-Internal and T-External in U, then (∀ u ∈ U, e_i ∈ D)

$$\lambda^{T}{}_{ei}(u) \in \{A^{-T}{}_{ei}(u)/u \in U,\ e_i \in D\} \cup \{A^{+T}{}_{ei}(u)/\ u \in U,\ e_i \in D\} \quad (2.7)$$

Proof. Consider the definition 3.5 and 3.6 which implies that $A^{-T}{}_{ei}(u) \le \lambda^{T}{}_{ei}(u) \le A^{+T}{}_{ei}(u)$ and $\lambda^{T}{}_{ei}(u) \notin [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)]$ for all u ∈ U, e_i∈ D. Then it follows that $\lambda^{T}{}_{ei}(u) = A^{-T}{}_{ei}(u)$ or $\lambda^{T}{}_{ei}(u) = A^{+T}{}_{ei}(u)$, and hence $\lambda^{T}{}_{ei} \in \{A^{-T}{}_{ei}/u \in U, e_i \in D\}$. Hence the result

Similarly, the following Theorems hold for the indeterminate and falsity values.

Theorem 3.10 Let (Q, D) be a Neutrosophic Pythagorean Cubic Soft Set in a non-empty set U. if (Q, D) is both I- Internal and I- External, then

(∀ u ∈ U, e_i ∈ D), $(\lambda^{I}{}_{ei}(u) \in [(A^{-I}{}_{ei}(u)/\ u \in U, e_i \in D\} \cup \{A^{+I}{}_{ei}(u)/\ u \in U, e_i \in D\})$.

Theorem 3.11 Let (Q, D) be a Neutrosophic Pythagorean Cubic Soft Set in a non-empty set U. If (Q, D) is both F- Internal and F- External, then

(∀ u ∈ U, e_i∈ D), $(\lambda^{F}{}_{ei}(u) \in [(A^{-F}{}_{ei}(u)/\ u \in U, e_i \in D\} \cup \{A^{+F}{}_{ei}(u)/\ u \in U, e_i \in D\})$.

Definition 3.12 Let $\mathfrak{L}$ = (Q, D) ∈ C_N^U. If

$A^{-T}{}_{ei}(u) \le \lambda^{T}{}_{ei}(u) \le A^{+T}{}_{ei}(u),\ (A^{-I}{}_{ei}(u) \le \lambda^{I}{}_{ei}(u) \le A^{+I}{}_{ei}(u)$ and $\lambda^{F}{}_{ei}(u) \notin [(A^{-F}{}_{ei}(u),\ A^{+F}{}_{ei}(u)]$ or

$A^{-T}{}_{ei}(u) \le \lambda^{T}{}_{ei}(u) \le A^{+T}{}_{ei}(u),\ (A^{-F}{}_{ei}(u) \le \lambda^{F}{}_{ei}(u) \le A^{+F}{}_{ei}(u)$ and $\lambda^{I}{}_{ei}(u) \notin [(A^{-I}{}_{ei}(u),\ A^{+I}{}_{ei}(u)]$ or

$(A^{-F}{}_{ei}(u) \le \lambda^{F}{}_{ei}(u) \le A^{+F}{}_{ei}(u),\ (A^{-I}{}_{ei}(u) \le \lambda^{I}{}_{ei}(u) \le A^{+I}{}_{ei}(u)$ and $\lambda^{T}{}_{ei}(u) \notin [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)]$ for all u ∈ U corresponding to each e_i ∈ D. Then $\mathfrak{L}$ is called an $\frac{2}{3}$ Internal Neutrosophic Pythagorean Cubic Soft Set ($\frac{2}{3}$INPCSS) or $\frac{1}{3}$ ENPCSS.

Example 3.13 Let $\mathfrak{L}$ = (Q, D) ∈ C_N^U. If

(Q, D) = P (e) = {⟨u, ([0.8, 0.5], [0.5, 0.3], [0.7, 0.5]), (0.7, 0.6, 0.6) ⟩ for all u ∈ U corresponding to each e_i∈ D. Then $\mathfrak{L}$= (Q, D) is a $\frac{2}{3}$ INPCSS.

Definition 3.14 Let $\mathfrak{L}$ = (Q, D) ∈ C_N^U. If

$A^{-T}{}_{ei}(u) \le \lambda^{T}{}_{ei}(u) \le A^{+T}{}_{ei}(u),\ \lambda^{I}{}_{ei}(u) \notin [(A^{-I}{}_{ei}(u),\ A^{+I}{}_{ei}(u)]$ and $\lambda^{F}{}_{ei}(u) \notin [(A^{-F}{}_{ei}(u),\ A^{+F}{}_{ei}(u)]$ or

$(A^{-F}{}_{ei}(u) \le \lambda^{F}{}_{ei}(u) \le A^{+F}{}_{ei}(u), \lambda^{T}{}_{ei}(u) \notin [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)]$ and $\lambda^{I}{}_{ei}(u) \notin [(A^{-I}{}_{ei}(u),\ A^{+I}{}_{ei}(u)]$ or

$(A^{-I}{}_{ei}(u) \le \lambda^{I}{}_{ei}(u) \le A^{+I}{}_{ei}(u),\ \lambda^{F}{}_{ei}(u) \notin [(A^{-F}{}_{ei}(u),\ A^{+F}{}_{ei}(u)]$ and $\lambda^{T}{}_{ei}(u) \notin [(A^{-T}{}_{ei}(u),\ A^{+T}{}_{ei}(u)]$

for all u ∈ U corresponding to each e_i ∈ D. Then $\mathfrak{L}$ is called an $\frac{2}{3}$ External Neutrosophic Pythagorean Cubic Soft Set ($\frac{2}{3}$ ENPCSS) or $\frac{1}{3}$ INPCSS.

Example 3.15 Let $\mathfrak{L}$= (Q, D) ∈ C_N^U. If

(Q, D) = P (e) = {⟨u, ([0.8, 0.5], [0.5, 0.3], [0.7, 0.5]), (0.7, 0.6, 0.4) ⟩

all u ∈ U corresponding to each e_i ∈ D. Then $\mathfrak{L}$= (Q, D) is a $\frac{1}{3}$ INPCSS.

Definition 3.16 Let (Q, D) = {Q(e_i) = {⟨u, A_{e_i} (u), λe_i(u)⟩: u ∈ U} e_i ∈ D} and (J, H) = {J(e_i) = {⟨u, B_{e_i} (u), μ_{ei} (u)⟩: u ∈ U}e_i∈ H} be two Neutrosophic Pythagorean Cubic Soft Set in U. Let D and H be any two subsets of E (set of parameters), then we have the following

1. (Q, D) = (J, H) if and only if the following conditions are satisfied

   (a) D = H and

   (b) Q(e_i) = H(e_i) for all e_i∈ M if and only if Ae_i (u) = Be_i (u) and λe_i (u) = μe_i (u) for all u ∈ U corresponding to eache_i∈ M.

2. If (Q, D) and (J, H) are two Neutrosophic Pythagorean Cubic Soft Sets then we define and denote P− order as (Q, D) ⊆P (J, H) if and only if the following conditions are satisfied

   (c) D ⊆ H and

   (d) P(e_i) ≤P Q(e_i) for all e_i ∈ D if and only if Ae_i (u) ⊆ Be_i (u) and λe_i (u) ≤μ e_i (u) for all u ∈ U corresponding to each e_i∈ M.

3. If (Q, D) and (J, H) are two Neutrosophic Pythagorean Cubic Soft Sets then we define and denote P− order as (Q, H) ⊆R (J, H) if and only if the following conditions are satisfied

   (e) D ⊆ H and

   (f) P(e_i) ≤R Q(e_i) for all e_i ∈ D if and only if Ae_i(u) ⊆ Be_i(u) and λe_i (u) ≥ μe_i (u) for all u ∈ U corresponding to each e_i ∈ D. We now define the P–union, P–intersection, R–union and R–intersection of Neutrosophic Pythagorean Cubic Soft Sets as follows:

Definition 3.17 Let (Q, D) and (J, H) be two Neutrosophic Pythagorean Cubic Soft Sets (NPCSS) in U, where D and H are any two subsets of the parameters set E. Then we define P–union as (Q, D) $\cup_P$ (J, H) = (H, C) where C = D ∪ H

$$H(e_i) = \begin{cases} Q(e_i) & \text{if } e_i \in D - H \\ J(e_i) & \text{if } e_i \in H - D \\ Q(e_i) \vee_P J(e_i) & \text{if } e_i \in D \cap H \end{cases}$$

Where $Q(e_i) \vee_P J(e_i)$ is defined as $Q(e_i) \vee_P J(e_i) = \{\langle u, \max\{A_{e_i}(u), B_{e_i}(u)\}, \lambda_{e_i}(u) \vee \mu_{e_i}(u) \rangle : u \in U\}$ $e_i \in D \cap H$ Where $A_{e_i}(u), B_{e_i}(u)$ represent Interval Neutrosophic Pythagorean sets and $\lambda_{e_i}(u) \vee \mu_{e_i}(u)$ represent Neutrosophic Pythagorean sets. Hence

$Q^T(e_i) \vee_P J^T(e_i) = \{\langle u, \max\{A_{e_i}{}^T(u), B_{e_i}{}^T(u)\}, (\lambda_{e_i}{}^T \vee \mu_{e_i}{}^T(u) \rangle : u \in U\}$ $e_i \in D \cap H$,

$Q^I(e_i) \vee_P J^I(e_i) = \{\langle u, \max\{A_{e_i}{}^I(u), B_{e_i}{}^I(u)\}, (\lambda_{e_i}{}^I \vee \mu_{e_i}{}^I(u) \rangle : u \in U\}$ $e_i \in D \cap H$,

$Q^F(e_i) \vee_P J^F(e_i) = \{\langle u, \max\{A_{e_i}{}^F(u), B_{e_i}{}^F(u)\}, (\lambda_{e_i}{}^F \vee \mu_{e_i}{}^F(u) \rangle : u \in U\}$ $e_i \in D \cap H$.

Definition 3.18 Let (Q,D) and (J,H) be two Neutrosophic Pythagorean soft cubic sets(NPCSS) in U where D and H are any subsets of the parameter set E. Then we define P- intersection as (Q,D)$\cap_P$(J,H) = (H,C) where C = D∩H, $H(e_i) = Q(e_i) \wedge_P J(e_i)$ and $e_i \in D \cap H$. Here $Q(e_i) \wedge_P J(e_i)$ is defined as

$Q(e_i) \wedge_P J(e_i) = H(e_i) = \{\langle u, \min\{A_{e_i}(u), B_{e_i}(u)\}, \lambda_{e_i}(u) \wedge \mu_{e_i}(u) \rangle : u \in U\}$ $e_i \in D \cap H$ where $A_{e_i}(u), B_{e_i}(u)$ represent Interval Neutrosophic Pythagorean Sets and $\lambda_{e_i}(u) \vee \mu_{e_i}(u)$ represent Neutrosophic Pythagorean sets. Hence

$Q^T(e_i) \wedge_P J^T(e_i) = \{\langle u, \min\{A_{e_i}{}^T(u), B_{e_i}{}^T(u)\}, (\lambda_{e_i}{}^T \wedge \mu_{e_i}{}^T(u) \rangle : u \in U\}$ $e_i \in D \cap H$,

$Q^I(e_i) \wedge_P J^I(e_i) = \{\langle u, \min\{A_{e_i}{}^I(u), B_{e_i}{}^I(u)\}, (\lambda_{e_i}{}^I \wedge \mu_{e_i}{}^I(u) \rangle : u \in U\}$ $e_i \in D \cap H$,

$Q^F(e_i) \wedge_P J^F(e_i) = \{\langle u, \min\{A_{e_i}{}^F(u), B_{e_i}{}^F(u)\}, (\lambda_{e_i}{}^F \wedge \mu_{e_i}{}^F(u) \rangle : u \in U\}$ $e_i \in D \cap H$.

Example 3.19 Let U = {u1, u2, u3, u4} be the initial universe, D =H= {e1, e2} are any subset of parameter's set E = {e1, e2, e3}. Let (Q, D) be NPCSS defined as

$$(Q, D) = \{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u) \rangle : u \in U\}\ e_i \in D\} \text{ is}$$

Table 3.4: The tabular representation of (Q, D)

U	Q(e1)	Q (e2)
u1	([0.4,0.5],[0.2,0.7],[0.2,0.3])(0.3,0.1,0.4)	([0.1,0.5],[0.8,0.2],[0.1,0.7])(0.6,0.4,0.5)
u2	([0.1,0.4],[0.3,0.6],[0.4,0.5])(0.2,0.4,0.3)	([0.3,0.4],[0.2,0.4],[0.1,0.3])(0.5,0.3,0.4)
u3	([0.2,0.3],[0.6,0.8],[0.8,0.1])(0.4,0.5,0.3)	([0.1,0.2],[0.8,0.2],[0.3,0.4])(0.4,0.2,0.3)
u4	([0.8,0.6],[0.1,0.3],[0.5,0.6])(0.5,0.2,0.5)	([0.4,0.5],[0.3,0.4],[0.2,0.3])(0.7,0.4,0.4)
u5	([0.3,0.4],[0.2,0.4],[0.1,0.3])(0.6,0.3,0.1)	([0.2,0.5],[0.1,0.2],[0.4,0.5])(0.3,0.2,0.2)

Let (J, H) be NPCSS defines as

$(J, H) = \{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u) \rangle : u \in U\}\ e_i \in H\}$ is

Table 3.5: The tabular representation of (J, H)

U	J(e1)	J(e2)
u1	([0.2,0.3],[0.6,0.1],[0.1,0.1])(0.4,0.5,0.1)	([0.3,0.5],[0.6,0.7],[0.1,0.3])(0.7,0.5,0.4)
u2	([0.5,0.7],[0.2,0.3],[0.1,0.3])(0.2,0.4,0.5)	([0.1,0.4],[0.3,0.6],[0.4,0.5])(0.4,0.2,0.3)
u3	([0.2,0.5],[0.3,0.6],[0.2,0.3])(0.2,0.3,0.5)	([0.3,0.6],[0.1,0.2],[0.1,0.3])(0.5,0.2,0.1)
u4	([0.5,0.6],[0.2,0.3],[0.1,0.3])(0.5,0.4,0.4)	([0.2,0.3],[0.6,0.1],[0.1,0.1])(0.3,0.5,0.5)
u5	([0.1,0.5],[0.1,0.3],[0.2,0.4])(0.2,0.1,0.2)	([0.4,0.5],[0.5,0.6],[0.2,0.3])(0.5,0.6,0.4)

Let P−union is denoted by (P, M)$\cup_P$ (Q, N) and defined as Let P−intersection is denoted

Table 3.6: The tabular representation of (Q, D) $\cup_P$ (J, H)

U	Q $\cup_P$ J(e1)	Q $\cup_P$ J(e2)
u1	([0.4,0.5],[0.6,0.7],[0.2,0.3])(0.4,0.5,0.4)	([0.3,0.5],[0.8,0.7],[0.1,0.7])(0.7,0.5,05)
u2	([0.5,0.7],[0.3,0.8],[0.4,0.5])(0.2,0.4,0.5)	([0.3,0.4],[0.3,0.6],[0.4,0.5])(0.5,0.3,0.4)
u3	([0.2,0.5],[0.6,0.8],[0.8,0.3])(0.5,0.4,0.5)	([0.3,0.6],[0.8,0.2],[0.3,0.4])(0.5,0.2,0.3)
u4	([0.8,0.6],[0.2,0.3],[0.5,0.6])(0.5,0.4,0.5)	([0.4,0.5],[0.6,0.4],[0.2,0.3])(0.7,0.5,0.5)
u5	([0.3,0.5],[0.2,0.4],[0.2,0.4])(0.6,0.3,0.2)	([0.4,0.5],[0.5,0.6],[0.4,0.5])(0.5,0.6,0.4)

by (P, M)$\cap_P$(Q, N) and defined as

Table 3.7: The tabular representation of (Q, D)$\cap_P$ (J, H)

U	Q $\cap_P$ J(e1)	Q $\cap_P$ J(e2)
u1	([0.2,0.3],[0.2,0.1],[0.1,0.1])(0.3,0.1,0.1)	([0.1,0.5],[0.6,0.2],[0.1,0.3])(0.6,0.4,0.4)
u2	([0.1,0.4],[0.2,0.3],[0.1,0.3])(0.2,0.4,0.3)	([0.1,0.4],[0.2,0.4],[0.1,0.3])(0.4,0.2,0.3)
u3	([0.2,0.3],[0.3,0.6],[0.2,0.1])(0.2,0.3,0.3)	([0.1,0.2],[0.1,0.2],[0.1,0.3])(0.4,0.2,0.1)
u4	([0.5,0.6],[0.1,0.3],[0.1,0.3])(0.5,0.2,0.4)	([0.2,0.3],[0.3,0.1],[0.1,0.1])(0.3,0.4,0.4)
u5	([0.1,0.4],[0.1,0.3],[0.1,0.3])(0.2,0.1,0.1)	([0.2,0.5],[0.1,0.2],[0.2,0.3])(0.3,0.2,0.2)

Definition 3.20 Let (Q, D) and (J, H) be two Neutrosophic Pythagorean Cubic Soft Sets (NPCSS) in U where D and H are any subsets of the parameter set E. Then we define R – union of Neutrosophic Pythagorean Cubic Soft Set as (Q, D) $\cup_R$ (J, H) = (H, C) where C= D∪H,

$$H(e_i) = \begin{cases} Q(e_i) & \text{if } e_i \in D - H \\ J(e_i) & \text{if } e_i \in H - D \\ Q(e_i) \vee_R J(e_i) & \text{if } e_i \in D \cap H \end{cases}$$

Where $Q(e_i) \vee_R J(e_i)$ is defined as

$Q(e_i) \vee_R J(e_i) = \{\langle u, \max\{ A_{e_i}(u), B_{e_i}(u)\}, \lambda_{e_i}(u) \wedge \mu_{e_i}(u) \rangle : u \in U\}$ $e_i \in D \cap H$

where $A_{e_i}(u)$, $B_{e_i}(u)$ represent interval Neutrosophic Pythagorean sets and $\lambda_{e_i}(u)$, $\mu_{e_i}(u)$ represent Neutrosophic Pythagorean sets. Hence

$Q^T(e_i) \vee_R J^T(e_i) = \{\langle u, \max\{A_{e_i}{}^T(u), B_{e_i}{}^T(u)\}, (\lambda_{e_i}{}^T \wedge \mu_{e_i}{}^T)(u) \rangle : u \in U\}$ $e_i \in D \cap H$,
$Q^I(e_i) \vee_R J^I(e_i) = \{\langle u, \max\{A_{e_i}{}^I(u), B_{e_i}{}^I(u)\}, (\lambda_{e_i}{}^I \wedge \mu_{e_i}{}^I)(u) \rangle : u \in U\}$ $e_i \in D \cap H$,
$Q^F(e_i) \vee_R J^F(e_i) = \{\langle u, \max\{A_{e_i}{}^F(u), B_{e_i}{}^F(u)\}, (\lambda_{e_i}{}^F \wedge \mu_{e_i}{}^F)(u) \rangle : u \in U\}$ $e_i \in D \cap H$.

Definition 3.21 Let (Q, D) and (J, H) be two Neutrosophic Pythagorean Cubic Soft Sets (NPCSS) in U where D and H are any subsets of parameter set E. Then we define R- intersection of Neutrosophic Pythagorean Cubic Soft Set as $(Q, D) \cap_R (J, H) = (H, C)$ where $C = D \cap H$, $H(e_i) = Q(e_i) \wedge_R J(e_i)$, $H(e_i) = Q(e_i) \wedge_R J(e_i)$ and $e_i \in D \cap H$. Here $Q(e_i) \wedge_R J(e_i)$ is defined as
$Q(e_i) \wedge_R J(e_i) = H(e_i) = \{\langle u, \min\{A_{e_i}(u), B_{e_i}(u)\}, \lambda_{e_i}(u) \wedge \mu_{e_i}(u) \rangle : u \in U\}$ $e_i \in D \cap H$
where $A_{e_i}(u)$, $B_{e_i}(u)$ represent interval Neutrosophic Pythagorean Cubic Sets and $\lambda_{e_i}(u)$,$\mu_{e_i}(u)$ Neutrosophic Pythagorean Cubic Set. Hence
$Q^T(e_i) \wedge_R J^T(e_i) = \{\langle u, \min\{A_{e_i}{}^T(u), B_{e_i}{}^T(u)\}, (\lambda_{e_i}{}^T \vee \mu_{e_i}{}^T)(u) \rangle : u \in U\}$ $e_i \in D \cap H$,
$Q^I(e_i) \wedge_R J^I(e_i) = \{\langle u, \min\{A_{e_i}{}^I(u), B_{e_i}{}^I(u)\}, (\lambda_{e_i}{}^I \vee \mu_{e_i}{}^I)(u) \rangle : u \in U\}$ $e_i \in D \cap H$,
$Q^F(e_i) \wedge_R J^F(e_i) = \{\langle u, \min\{A_{e_i}{}^F(u), B_{e_i}{}^F(u)\}, (\lambda_{e_i}{}^F \vee \mu_{e_i}{}^F)(u) \rangle : u \in U\}$ $e_i \in D \cap H$.
Let (Q, D) of NPCSS defined as
$(Q, D) = \{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u) \rangle : u \in U\}$ $e_i \in H\}$ is

Table 3.8: The tabular representation of (Q, D)

U	Q (e1)	Q (e2)
u1	([0.1,0.2],[0.2,0.4],[0.2,0.4])(0.25, 0.25, 0.3)	([0.2,0.6],[0.2,0.4],[0.9,0.4])(0.85, 0.85, 0.5)
u2	([0.3,0.6],[0.9,0.4],[0.1,0.3])(0.65, 0.4, 0.2)	([0.4,0.7],[0.4,0.5],[0.1,0.4])(0.35, 0.2, 0.3)
u3	([0.5,0.8],[0.25,0.1],[0.2,0.3])(0.4, 0.5, 0.45)	([0.3,0.8],[0.3,0.6],[0.2,0.4]) (0.15, 0.2, 0 . 55)

Let (Q, D) of NPCSS defined as
$(Q, D) = \{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u) \rangle : u \in U\}$ $e_i \in H\}$ is

Table 3.9: The tabular representation of (J, H)

U	Q(e1)	Q(e2)
u1	([0.2,0.4],[0.3,0.4],[0.3,0.4])(0.15, 0.2, 0.5)	([0.9,0.7],[0.4,0.6],[0.2,0.4])(0.3, 0.5, 0.7)
u2	([0.1,0.4],[0.4,0.6],[0.2,0.4])(0.2, 0.3, 0.5)	([0.3,0.6],[0.3,0.5],[0.4,0.7])(0.4, 0.2, 0.5)
u3	([0.5,0.7],[0.3,0.6],[0.6,0.8])(0.8, 0.2, 0.3)	([0.6,0.8],[0.3,0.4],[0.6,0.8])(0.4, 0.35, 0.1)

Let R−union is denoted by $(Q, D) \cup_R (J, H)$ and defined as

Table 3.10: The tabular representation of $(Q, D) \cup_R (J, H)$

U	$Q \cup_R J(e_1)$	$Q \cup_R J(e_2)$
u_1	([0.2,0.4],[0.3,0.4],[0.3,0.4])(0.25, 0.2, 0.4)	([0.9,0.7],[0.4,0.6],[0.9,0.4])(0.85, 0.85, 0.7)
u_2	([0.3,0.6],[0.9,0.6],[0.2,0.4])(0.65, 0.4, 0.5)	([0.4,0.7],[0.4,0.5],[0.4,0.7])(0.35, 0.2, 0.5)
u_3	([0.5,0.8],[0.3,0.6],[0.6,0.8])(0.8, 0.5, 0.45)	([0.6,0.8],[0.3,0.6],[0.6,0.8])(0.4, 0.35, 0.55)

Let R−intersection is denoted by $(Q, D) \cap_R (J, H)$ and defined as

Table 3.11: The tabular representation of $(Q, D) \cap_R (J, H)$

U	(Q ∩R J) (e1)	(Q ∩R J) (e2)
u1	([0.1,0.2],[0.2,0.4],[0.2,0.4])(0.15, 0.25, 0.3)	([0.2,0.6],[0.2,0.4],[0.2,0.4])(0.3, 0.5, 0.5)
u2	([0.1,0.4],[0.4,0.4],[0.1,0.3])(0.2, 0.3, 0.2)	([0.3,0.6],[0.3,0.5],[0.1,0.4])(0.4, 0.2, 0.3)
u3	([0.5,0.7],[0.25,0.1],[0.2,0.3])(0.4, 0.2, 0.3)	([0.3,0.8],[0.3,0.4],[0.2,0.4])(0.15, 0.2, 0.1)

4. P−UNION AND P−INTERSECTION OF NEUTROSOPHIC PYTHAGOREAN CUBIC SOFT SET

In this section we introduce the concepts of P−OR and P−AND for Neutrosophic Pythagorean Cubic Soft Sets and explore their fundamental properties. We establish that the P−union and P−intersection of two Internal Neutrosophic soft cubic sets also result in Internal Neutrosophic Pythagorean Cubic Soft Sets. Additionally, we examine the conditions under which the P−union and P−intersection of two T−External (or I−External, F−External) Neutrosophic soft cubic sets remain T−External (or I−External, F−External) within the framework of Neutrosophic Pythagorean Cubic Soft Sets.

Definition 4.1 Let (Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$
(J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be Neutrosophic Pythagorean Cubic Soft Set (NPCSS) in U. Then

1. P − OR is denoted by (Q,D) $\vee_P$ (J, H) and defined as (Q,D) $\wedge_P$ (J, H) = $(\mathfrak{H}, D \times H)$ where $\mathfrak{H}(\alpha_i, \beta_i)$= $Q(\alpha_i) \cup_P J(\beta_i)$ for all $(\alpha_i, \beta_i) \in D \times H$.
2. P − AND is denoted by (Q,D) $\wedge_P$ (J, H) and defined as (Q,D) $\wedge_P$ (J, H) = $(\mathfrak{H}, D \times H)$ where $\mathfrak{H}(\alpha_i, \beta_i)$= $Q(\alpha_i) \cap_P J(\beta_i)$ for all $(\alpha_i, \beta_i) \in D \times H$.

Example 4.2 Let U = {u1, u2, u3} be the initial universe and the parameter set E= {e1, e2}. Let (Q, D) and (J, H) be two NPCSS over U is defined as
(Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

Table 4.1: The tabular representation of (Q, D)

U	Q (e1)	Q (e2)
u1	([0.4,0.5],[0.5,0.6],[0.4,0.5])(0.3, 0.4, 0.5)	([0.2,0.5],[0.1,0.6],[0.1,0.3])(0.2, 0.3, 0.3)
u2	([0.3,0.4],[0.6,0.7],[0.1,0.2])(0.4, 0.5, 0.5)	([0.2,0.4],[0.5,0.7],[0.1,0.5])(0.3, 0.6, 0.4)
u3	([0.1,0.2],[0.1,0.2],[0.2,0.4])(0.2, 0.3, 0.6)	([0.3,0.6],[0.1,0.4],[0.2,0.5]) (0.4, 0.5, 0.5)

and (J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$

Table 4.2: The tabular representation of (J, H)

U	J(e1)	J(e2)

u_1	([0.6,0.8],[0.2,0.4],[0.2,0.3])(0.6, 0.3, 0.5)	([0.3,0.6],[0.9,0.2],[0.2,0.1])(0.4, 0.1, 0.1)
u_2	([0.4,0.5],[0.2,0.6],[0.8,0.1])(0.5, 0.3, 0.1)	([0.3,0.5],[0.3,0.6],[0.1,0.4])(0.5, 0.4, 0.3)
u_3	([0.2,0.3],[0.9,0.1],[0.1,0.3])(0.4, 0.2, 0.4)	([0.4,0.7],[0.5,0.3],[0.2,0.3]) (0.6, 0.4, 0.3)

Table 4.3: The tabular representation of (Q, D) $\vee_P$ (J, H)

U	H (e1, e1)	H (e1, e2)	H (e2, e1)	H (e2, e2)
U	P (e1) ∪ Q(e1)	P (e1) ∪ Q(e2)	P (e2) ∪ Q(e1)	P (e2) ∪ Q(e2)
u1	([0.6,0.8],[0.5,0.6], [0.4,0.5])(0.6, 0.4, 0.5)	([0.5,0.6],[0.6,0.7],[0.5,0.6]) (0.5, 0.5, 0.6)	([0.7,0.9][0.3,0.5] [0.3,0.4])(0.7,0.4,0.5)	([0.4,0.7][0.2,0.7] [0.2,0.4])(0.5,0.4,0.4)
u2	([0.5,0.6],[0.7,0.8], [0.2,0.3])(0.6, 0.6, 0.6)	([0.4,0.6],[0.7,0.8],[0.2,0.5]) (0.6, 0.6, 0.6)	([0.5,0.6][0.6,0.8] [0.2,0.6])(0.6,0.7,0.5)	([0.4,0.6][0.6,0.8] [0.2,0.6])(0.6,0.7,0.5)
u3	([0.3,0.4],[0.2,0.3] ,[0.3,0.5])(0.5, 0.4, 0.6)	([0.5,0.8],[0.2,0.3], [0.3,0.5]) (0.7, 0.4, 0.6)	([0.4,0.7][0.2,0.5] [0.3,0.6])(0.5,0.6,0.6)	([0.5,0.8][0.2,0.5] [0.3,0.6])(0.7,0.6,0.6)

Table 4.4: The tabular representation of (Q, D) $\wedge_P$ (J, H)

U	H (e1, e1)	H (e1, e2)	H (e2, e1)	H (e2, e2)
U	P (e1) ∩ Q(e1)	P (e1) ∩ Q(e2)	P (e2) ∩ Q(e1)	P (e2) ∩ Q(e2)
u1	([0.5,0.6],[0.3,0.5], [0.3,0.4]) (0.4,0.4,0.5)	([0.4,0.7],[0.1,0.3],[0.1,0.2]) (0.4,0.2,0.2)	([0.3,0.6],[0.2,0.7], [0.2,0.4]) (0.3,0.4,0.4)	([0.3,0.6],[0.1,0.3], [0.1,0.2]) (0.3,0.2,0.2)
u 2	([0.4,0.5],[0.3,0.7],[0.1,0.2]) (0.5,0.4,0.2)	([0.4,0.5],[0.4,0.7],[0.2,0.3]) (0.5,0.5,0.4)	([0.3,0.5],[0.3,0.7],[0.1,0.2]) (0.4,0.4,0.2)	([0.3,0.5],[0.4,0.7],[0.2,0.5]) (0.4,0.5,0.4)
u3	([0.2,0.3],[0.1,0.2],[0.2,0.4]) (0.3,0.3,0.5)	([0.2,0.3],[0.1,0.4],[0.1,0.4]) (0.3,0.3,0.4)	([0.3,0.4],[0.1,0.2],[0.2,0.4]) (0.5,0.3,0.5)	([0.4,0.7],[0.1,0.4],[0.1,0.4]) (0.5,0.3,0.4)

Proposition 4.3 Let U be the initial universe and MNL and S be subsets of the parametric set E Then for any Neutrosophic Pythagorean Cubic Soft Sets A = (P, M), B = (Q, N), C = (E, L), D = (T, S) the following properties hold

1. If $A \subseteq_P B$ and $B \subseteq_P C$ then $A \subseteq_P C$.
2. If $A \subseteq_P B$, then $B^C \subseteq_P A^C$.
3. If $A \subseteq_P B$ and $A \subseteq_P C$ then $A \subseteq_P B\ A \cap_P C$.
4. If $A \subseteq_P B$ and $C \subseteq_P B$ then $A \cup_P C \subseteq_P B$.
5. If $A \subseteq_P B$ and $C \subseteq_P D$ then $A \cup_P C \subseteq_P B \cup_P D$.

Proof.

(1) To Prove $A \subseteq_P B$ and $B \subseteq C$. Then $A \subseteq_P C$. Consider

$A \subseteq_P B \Rightarrow (P, M) \subseteq_P (Q, N) \Rightarrow P(e_i) \leq Q(e_i)\ \forall\ e_i \in M,\ M \subseteq N$

$B \subseteq_P C \Rightarrow (Q, N) \subseteq_P (H, L) \Rightarrow Q(e_i) \leq H(e_i)\ \forall\ e_i \in N,\ N \subseteq L\ e_i \in M \Rightarrow e_i \in N$, since $M \subseteq N$

$$Q(e_i) \leq H(e_i)\ \forall\ e_i \in M$$

$$P(e_i) \leq Q(e_i) \leq H(e_i)$$

$$\Rightarrow P(e_i) \leq H(e_i)\ \forall\ e_i \in M \subseteq N \subseteq L$$

$$\Rightarrow (P, M) \subseteq_P (H, L)$$

$$\Rightarrow A \subseteq_P B$$

(2) To prove $A \subseteq_P B$. Then $B^C \subseteq_P A^C$.

If $M=N \Rightarrow M \subseteq N$ and $N \subseteq M$.

$A \subseteq_P B \Rightarrow M \subseteq N$ and $P(e_i) \leq Q(e_i)$ by definition

$P(e_i) \leq Q(e_i) \Rightarrow A(e_i) \subseteq B(e_i)$ and $\lambda(e_i) \subseteq \mu(e_i)\ \forall\ u \in U$

$A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u),\ A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u),$

$A_{e_i}{}^{-I}(u) \geq B_{e_i}{}^{-I}(u),\ A_{e_i}{}^{+I}(u) \leq B_{e_i}{}^{+I}(u),$

$A_{e_i}{}^{-F}(u) \geq B_{e_i}{}^{-F}(u),\ A_{e_i}{}^{+F}(u) \leq B_{e_i}{}^{+F}(u)$

and $\lambda_{e_i}{}^{T}(u) \leq \mu_{e_i}{}^{T}(u),\ \lambda_{e_i}{}^{I}(u) \geq \mu_{e_i}{}^{I}(u),\ \lambda_{e_i}{}^{F}(u) \geq \mu_{e_i}{}^{F}(u).$

$A^C = \{\langle u, A^C{}_{e_i}(u), \lambda^C{}_{e_i}(u)\rangle : \forall\ u \in U, e_i \in M\}$

Where $A^C{}_{e_i}(u)=\{\langle u, [1 - A_{e_i}{}^{+T}(u), A_{e_i}{}^{-T}(u)], [1 - A_{e_i}{}^{+I}(u), A_{e_i}{}^{-I}(u)], [1 - A_{e_i}{}^{+F}(u), A_{e_i}{}^{-F}(u)]\rangle : u \in U\}$ $\lambda^C{}_{e_i}(u)= \{\langle u, [1 - \lambda_{e_i}{}^{T}(u), 1 - \lambda_{e_i}{}^{I}(u), 1 - \lambda_{e_i}{}^{F}(u)]\rangle : u \in U\}$.

and $B^C = \{\langle u, B^C{}_{e_i}(u), B^C{}_{e_i}(u)\rangle : \forall\ u \in U, e_i \in N\}$

Where

$B^C{}_{e_i}(u)= \{\langle u, [1 - B_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)], [1 - B_{e_i}{}^{+I}(u), B_{e_i}{}^{-I}(u)], [1 - B_{e_i}{}^{+F}(u), B_{e_i}{}^{-F}(u)]\rangle : u \in U\}$

$\mu^C{}_{e_i}(u)= \{\langle u, [1 - \mu_{e_i}{}^{T}(u), 1 - \mu_{e_i}{}^{I}(u), 1 - \mu_{e_i}{}^{F}(u)]\rangle : u \in U\}$.

Now

$A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) \Rightarrow -A_{e_i}{}^{-T}(u) \geq -B_{e_i}{}^{-T}(u).$

Adding 1 on both sides, we get

$1 - A_{e_i}{}^{-T}(u) \geq 1 - B_{e_i}{}^{-T}(u).$

Similarly,

$$1 - A_{e_i}{}^{+T}(u) \geq 1 - B_{e_i}{}^{+T}(u),$$
$$1 - A_{e_i}{}^{+I}(u) \geq 1 - B_{e_i}{}^{-I}(u),$$
$$1 - A_{e_i}{}^{+I}(u) \geq 1 - B_{e_i}{}^{+I}(u),$$
$$1 - A_{e_i}{}^{-F}(u) \geq 1 - B_{e_i}{}^{-F}(u),$$
$$1 - A_{e_i}{}^{+F}(u) \geq 1 - B_{e_i}{}^{+F}(u) \qquad (2.8)$$

And

$$1 - \lambda_{e_i}{}^{T}(u) \geq 1 - \mu_{e_i}{}^{T}(u),$$
$$1 - \lambda_{e_i}{}^{I}(u) \geq 1 - \mu_{e_i}{}^{I}(u),$$
$$1 - \lambda_{e_i}{}^{F}(u) \geq 1 - \mu_{e_i}{}^{F}(u). \qquad (2.9)$$

From (2.8) and (2.9),

$$B^C \subseteq A^C \quad \text{since } N \subseteq M$$

Hence the proof.

Similarly, we can prove for (3), (4) and (5).

Theorem 4.4 Let (P, M) be a Neutrosophic Pythagorean Cubic Soft Set over U.

1. If (P, M) is an Internal Neutrosophic Pythagorean Cubic Soft Set, then $(P, M)^C$ is also an Internal Neutrosophic Pythagorean Cubic Soft Set (INPCSS).
2. If (P, M) is an External Neutrosophic Pythagorean Cubic Soft Set, then $(P, M)^C$ is also an External Neutrosophic Pythagorean Cubic Soft Set (ENPCSS).

Proof.

1. Given (P, M) = $\{P(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ is an INPCSS this implies

$A_{e_i}{}^{-T}(u) \leq \lambda_{e_i}{}^{T}(u) \leq A_{e_i}{}^{+T}(u)$, $A_{e_i}{}^{-I}(u) \leq \lambda_{e_i}{}^{I}(u) \leq A_{e_i}{}^{+I}(u)$,

$A_{e_i}{}^{-F}(u) \leq \lambda_{e_i}{}^{F}(u) \leq A_{e_i}{}^{+F}(u)$, for all $e_i \in M$ and for all $u \in U$.This implies

$1 - A_{e_i}{}^{+T}(u) \leq 1 - \lambda_{e_i}{}^{T}(u) \leq 1 - A_{e_i}{}^{-T}(u)$, $1 - A_{e_i}{}^{+I}(u) \leq 1 - \lambda_{e_i}{}^{I}(u) \leq 1 - A_{e_i}{}^{-I}(u)$, $1 - A_{e_i}{}^{+F}(u) \leq 1 - \lambda_{e_i}{}^{F}(u) \leq 1 - A_{e_i}{}^{-F}(u)$

for all $e_i \in M$ and for all $u \in U$.

Hence, $(P, M)^C$ is an INPCSS.

2. Given (P, M) = $\{P(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ is an ENPCSS this implies

$\lambda_{e_i}{}^{T}(u) \notin (A_{e_i}{}^{-T}(u), A_{e_i}{}^{+T}(u))$, $\lambda_{e_i}{}^{I}(u) \notin (A_{e_i}{}^{-I}(u), A_{e_i}{}^{+I}(u))$, $\lambda_{e_i}{}^{F}(u) \notin (A_{e_i}{}^{-F}(u), A_{e_i}{}^{+F}(u))$, for all $e_i \in M$ and for all $u \in U$. Since,

$\lambda_{e_i}{}^{T}(u) \notin (A_{e_i}{}^{-T}(u), A_{e_i}{}^{+T}(u))$ & $0 \leq A_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{+T}(u) \leq 1$,

$\lambda_{e_i}{}^{I}(u) \notin (A_{e_i}{}^{-I}(u), A_{e_i}{}^{+I}(u))$ & $0 \leq A_{e_i}{}^{-I}(u) \leq A_{e_i}{}^{+I}(u) \leq 1$,

$\lambda_{e_i}{}^{F}(u) \notin (A_{e_i}{}^{-F}(u), A_{e_i}{}^{+F}(u))$ & $0 \leq A_{e_i}{}^{-F}(u) \leq A_{e_i}{}^{+F}(u) \leq 1$.

So, we have

$\lambda_{e_i}{}^{T}(u) \leq A_{e_i}{}^{-T}(u)$ or $A_{e_i}{}^{+T}(u) \leq \lambda_{e_i}{}^{T}(u)$,

$\lambda_{e_i}{}^{I}(u) \leq A_{e_i}{}^{-I}(u)$ or $A_{e_i}{}^{+I}(u) \leq \lambda_{e_i}{}^{I}(u)$,

$\lambda_{e_i}{}^{F}(u) \leq A_{e_i}{}^{-F}(u)$ or $A_{e_i}{}^{+F}(u) \leq \lambda_{e_i}{}^{F}(u)$,

This implies

$1 - \lambda_{e_i}{}^{T}(u) \geq 1 - A_{e_i}{}^{-T}(u)$ or $1 - A_{e_i}{}^{+T}(u) \geq 1 - \lambda_{e_i}{}^{T}(u)$,

$1 - \lambda_{e_i}{}^{I}(u) \geq 1 - A_{e_i}{}^{-I}(u)$ or $1 - A_{e_i}{}^{+I}(u) \geq 1 - \lambda_{e_i}{}^{I}(u)$,

$1 - \lambda_{e_i}{}^{F}(u) \geq 1 - A_{e_i}{}^{-F}(u)$ or $1 - A_{e_i}{}^{+F}(u) \geq 1 - \lambda_{e_i}{}^{F}(u)$,

for all $e_i \in M$ and for all $u \in U$.

Thus $1 - \lambda_{e_i}{}^{T}(u) \notin (1 - A_{e_i}{}^{-T}(u), 1 - A_{e_i}{}^{+T}(u))$, $1 - \lambda_{e_i}{}^{I}(u) \notin (1 - A_{e_i}{}^{-I}(u), 1 - A_{e_i}{}^{+I}(u))$, $1 - \lambda_{e_i}{}^{F}(u) \notin (1 - A_{e_i}{}^{-F}(u), 1 - A_{e_i}{}^{+F}(u))$.

Hence (P,M) is an ENPCSS.

Theorem 4.5 Let (Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

(J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be two Internal Neutrosophic Pythagorean Cubic Soft Sets. Then

1. (Q, D) $\cup_P$(J, H) is an INPCSS
2. (Q, D) $\cap_P$ (J, H) is an ENPCSS

Proof.

1. Consider (Q, D) and (J, H) to be two Internal Neutrosophic Pythagorean Cubic Soft Sets, then we have for (Q, D) as ${A_{e_i}}^{-T}(u) \leq {\lambda_{e_i}}^{T}(u) \leq {A_{e_i}}^{+T}(u)$, ${A_{e_i}}^{-I}(u) \leq {\lambda_{e_i}}^{I}(u) \leq {A_{e_i}}^{+I}(u)$, ${A_{e_i}}^{-F}(u) \leq {\lambda_{e_i}}^{F}(u) \leq {A_{e_i}}^{+F}(u)$, for all $e_i \in M$ and for all $u \in U$. Also for (D, H) we have ${B_{e_i}}^{-T}(u) \leq {\mu_{e_i}}^{T}(u) \leq {B_{e_i}}^{+T}(u)$, ${B_{e_i}}^{-I}(u) \leq {\mu_{e_i}}^{I}(u) \leq {B_{e_i}}^{+I}(u)$, ${B_{e_i}}^{-F}(u) \leq {\mu_{e_i}}^{F}(u) \leq {B_{e_i}}^{+F}(u)$, for all $e_i \in M$ and for all $u \in U$. Then we have

$$\max\{{A_{e_i}}^{-T}(u), {B_{e_i}}^{-T}(u)\} \leq ({\lambda_{e_i}}^{T} \vee {\mu_{e_i}}^{T})(u) \leq \max\{{A_{e_i}}^{+T}(u), {B_{e_i}}^{+T}(u)\}$$
$$\max\{{A_{e_i}}^{-I}(u), {B_{e_i}}^{-I}(u)\} \leq ({\lambda_{e_i}}^{I} \vee {\mu_{e_i}}^{I})(u) \leq \max\{{A_{e_i}}^{+I}(u), {B_{e_i}}^{+I}(u)\}$$
$$\max\{{A_{e_i}}^{-F}(u), {B_{e_i}}^{-F}(u)\} \leq ({\lambda_{e_i}}^{F} \vee {\mu_{e_i}}^{F})(u) \leq \max\{{A_{e_i}}^{+F}(u), {B_{e_i}}^{+F}(u)\}$$

$$H(e_i) = \begin{cases} Q(e_i) & \text{if } e_i \in D - H \\ J(e_i) & \text{if } e_i \in H - D \\ Q(e_i) \vee_P J(e_i) & \text{if } e_i \in D \cap H \end{cases}$$

If $e_i \in D \cap H$, then $Q(e_i) \vee_P J(e_i)$ is defined as

$Q(e_i) \vee_P J(e_i) = H(e_i) = \{\langle u, \max\{A_{e_i}(u), B_{e_i}(u)\}, (\lambda_{e_i} \vee \mu_{e_i})(u)\rangle : u \in U, e_i \in D \cap H\}$

Where

$$Q^T(e_i) \vee_P J^T(e_i) = \left\{\left\langle u, \max\{{A^T}_{e_i}(u), {B^T}_{e_i}(u)\}, ({\lambda^T}_{e_i} \vee {\mu^T}_{e_i})(u)\right\rangle : u \in U, e_i \in D \cap H\right\}$$

$$Q^I(e_i) \vee_P J^I(e_i) = \left\{\left\langle u, \max\{{A^I}_{e_i}(u), {B^I}_{e_i}(u)\}, ({\lambda^I}_{e_i} \vee {\mu^I}_{e_i})(u)\right\rangle : u \in U, e_i \in D \cap H\right\}$$

$$Q^F(e_i) \vee_P J^F(e_i) = \left\{\left\langle u, \max\{{A^F}_{e_i}(u), {B^F}_{e_i}(u)\}, ({\lambda^F}_{e_i} \vee {\mu^F}_{e_i})(u)\right\rangle : u \in U, e_i \in D \cap H\right\}$$

Thus (Q, D) $\cup_P$(J, H) is an INPCSS if $e_i \in D \cap H$. If $e_i \in D - H$ or $e_i \in H - D$ then the result is trivial.

Hence (Q, D) $\cup_P$(J, H) is an INPCSS in all cases.

2. Since (Q, D) $\cap_P$ (J, H) = (H, C) where $D \cap H = C$ and $H(e_i) = Q(e_i) \wedge_P J(e_i)$. If $e_i \in D \cap H$ then $Q(e_i) \wedge_P J(e_i)$ is defined as

   $H(e_i) = Q(e_i) \wedge_P J(e_i) = \{\langle u, \min\{A_{e_i}(u), B_{e_i}(u)\}, (\lambda_{e_i} \wedge \mu_{e_i})(u)\rangle : u \in U, e_i \in D \cap H\}$.

Also given that (Q, D) and (J, H) are INPCSS. ${A_{e_i}}^{-T}(u) \leq {\lambda_{e_i}}^{T}(u) \leq {A_{e_i}}^{+T}(u)$, ${A_{e_i}}^{-I}(u) \leq {\lambda_{e_i}}^{I}(u) \leq {A_{e_i}}^{+I}(u)$, ${A_{e_i}}^{-F}(u) \leq {\lambda_{e_i}}^{F}(u) \leq {A_{e_i}}^{+F}(u)$, for all $e_i \in M$ and for all $u \in U$. For (D, H) we have ${B_{e_i}}^{-T}(u) \leq {\mu_{e_i}}^{T}(u) \leq {B_{e_i}}^{+T}(u)$, ${B_{e_i}}^{-I}(u) \leq {\mu_{e_i}}^{I}(u) \leq {B_{e_i}}^{+I}(u)$, ${B_{e_i}}^{-F}(u) \leq {\mu_{e_i}}^{F}(u) \leq {B_{e_i}}^{+F}(u)$, for all $e_i \in D$ and for all $u \in U$.

$$\min\{{A_{e_i}}^{-T}(u), {B_{e_i}}^{-T}(u)\} \leq ({\lambda_{e_i}}^{T} \wedge {\mu_{e_i}}^{T})(u) \leq \min\{{A_{e_i}}^{+T}(u), {B_{e_i}}^{+T}(u)\}$$
$$\min\{{A_{e_i}}^{-I}(u), {B_{e_i}}^{-I}(u)\} \leq ({\lambda_{e_i}}^{I} \wedge {\mu_{e_i}}^{I})(u) \leq \min\{{A_{e_i}}^{+I}(u), {B_{e_i}}^{+I}(u)\}$$
$$\min\{{A_{e_i}}^{-F}(u), {B_{e_i}}^{-F}(u)\} \leq ({\lambda_{e_i}}^{F} \wedge {\mu_{e_i}}^{F})(u) \leq \min\{{A_{e_i}}^{+F}(u), {B_{e_i}}^{+F}(u)\}$$

for all $e_i \in D \cap H$ and for all $u \in U$. Hence (Q, D)$\cap_P$(J, H) is an INPCSS.

Definition 4.6 Given two Neutrosophic Pythagorean Cubic Soft Sets (NPCSS)

(Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

and (J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ If we interchange λ and μ then the new Neutrosophic Pythagorean Cubic Soft Set (NPCSS) are denoted and defined as

$(Q, D)^* = \{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

and $(J, H)^* = \{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ respectively.

Theorem 4.8 For two ENPCSSs (Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ in U. If

$(Q, D)^* = \{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

and $(J, H)^* = \{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ are INPCSS in U then (Q, D)$\cup_P$(J, H) is an INPCSS in U.

Proof. By similar way to theorem 4.7 we can obtain the result.

Theorem 4.9 Let (Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$, be ENPCSS in U such that

$(Q, D)^* = \{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

and $(J, H)^* = \{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$

be ENPCSS in U. Then P-union of (Q, D) and (J, H) is an ENPCSS in U.

Proof. Since (Q, D), (J, H), $(Q, D)^*$ and $(J, H)^*$ are ENPCSS so by definition of an External soft cubic set for (Q, D), (J, H), $(Q, D)^*$ and $(J, H)^*$ we have

$\lambda_{e_i}{}^{T} \notin \left(A_{e_i}{}^{-T}(u), A_{e_i}{}^{+T}(u)\right), \lambda_{e_i}{}^{I} \notin \left(A_{e_i}{}^{-I}(u), A_{e_i}{}^{+I}(u)\right), \lambda_{e_i}{}^{F} \notin \left(A_{e_i}{}^{-F}(u), A_{e_i}{}^{+F}(u)\right)$, for all $e_i \in H$ and for all $u \in U$.

$\mu_{e_i}{}^{T} \notin \left(B_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\right), \mu_{e_i}{}^{I} \notin \left(B_{e_i}{}^{-I}(u), B_{e_i}{}^{+I}(u)\right), \mu_{e_i}{}^{F} \notin \left(B_{e_i}{}^{-F}(u), B_{e_i}{}^{+F}(u)\right)$, for all $e_i \in D$ and for all $u \in U$.

$\mu_{e_i}{}^{T} \notin \left(A_{e_i}{}^{-T}(u), A_{e_i}{}^{+T}(u)\right), \mu_{e_i}{}^{I} \notin \left(A_{e_i}{}^{-I}(u), A_{e_i}{}^{+I}(u)\right), \mu_{e_i}{}^{F} \notin \left(A_{e_i}{}^{-F}(u), A_{e_i}{}^{+F}(u)\right)$, for all $e_i \in D$ and for all $u \in U$.

$\lambda_{e_i}{}^{T} \notin \left(B_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\right), \lambda_{e_i}{}^{I} \notin \left(B_{e_i}{}^{-I}(u), B_{e_i}{}^{+I}(u)\right), \lambda_{e_i}{}^{F} \notin \left(B_{e_i}{}^{-F}(u), B_{e_i}{}^{+F}(u)\right)$, for all $e_i \in H$ and for all $u \in U$ respectively.

Thus, we have

$(\lambda_{e_i}{}^{T} \vee \mu_{e_i}{}^{T})(u) \notin \{\max\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{-T}(u)\}, \max\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{+T}(u)\}\}$,

$(\lambda_{e_i}{}^{I} \vee \mu_{e_i}{}^{I})(u) \notin \{\max\{A_{e_i}{}^{-I}(u), B_{e_i}{}^{-I}(u)\}, \max\{A_{e_i}{}^{+I}(u), B_{e_i}{}^{+I}(u)\}\}$,

$(\lambda_{e_i}{}^{F} \vee \mu_{e_i}{}^{F})(u) \notin \{\max\{A_{e_i}{}^{-F}(u), B_{e_i}{}^{-F}(u)\}, \max\{A_{e_i}{}^{+F}(u), B_{e_i}{}^{+F}(u)\}\}$,

For all $e_i \in D \cap H$ and for all $u \in U$.

Thus we have $(\lambda_{e_i} \vee \mu_{e_i})(u) \notin \max\{A_{e_i}(u), B_{e_i}(u)\}$ for all $e_i \in D \cap H$ and for all $u \in U$. also since (Q,D) $\cup_P$ (J,H)=(H,C) where D∪H = C and

$$H(e_i) = \begin{cases} Q(e_i) & \text{if } e_i \in D - H \\ J(e_i) & \text{if } e_i \in H - D \\ Q(e_i) \vee_P J(e_i) & \text{if } e_i \in D \cap H \end{cases}$$

if $e_i \in D \cap H$, then $Q(e_i) \vee_P J(e_i)$ is defined as

$Q(e_i) \vee_P J(e_i) = H(e_i)=\{\langle x, \max\{A_{e_i}(u), B_{e_i}(u)\}, (\lambda_{e_i} \vee \mu_{e_i})(u)\rangle : u \in U, e_i \in D \cap H\}$

Where

$$Q^T(e_i) \vee_P J^T(e_i) = \left\{\left\langle x, \max\{A^T{}_{e_i}(u), B^T{}_{e_i}(u)\}, \left(\lambda^T{}_{e_i} \vee \mu^T{}_{e_i}\right)(u)\right\rangle : u \in U, e_i \in D \cap H\right\},$$

$$Q^I(e_i) \vee_P J^I(e_i) = \left\{\left\langle x, \max\{A^I{}_{e_i}(u), B^I{}_{e_i}(u)\}, \left(\lambda^I{}_{e_i} \vee \mu^I{}_{e_i}\right)(u)\right\rangle : u \in U, e_i \in D \cap H\right\},$$

$$Q^F(e_i) \vee_P J^F(e_i) = \left\{\left\langle x, \max\{A^F{}_{e_i}(u), B^F{}_{e_i}(u)\}, \left(\lambda^F{}_{e_i} \vee \mu^F{}_{e_i}\right)(u)\right\rangle : u \in U, e_i \in D \cap H\right\}.$$

By definition of an External soft cubic set (Q, D)$\cup_P$ (J, H) is an ENPCSS in U.

Example 4.10 Let (Q, D) and (J, H) be Neutrosophic Pythagorean Cubic Soft Sets in U, where

$(P, M) = P(e_1) = \{\langle x, ([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), (0.8, 0.3, 0.4)\rangle, e_1 \in D\}$,

$(Q, N) = Q(e_1) = \{\langle x, ([0.7, 0.9], [0.6, 0.8], [0.4, 0.7]), (0.4, 0.7, 0.3)\rangle, e_1 \in H\}$

for all $u \in U$.

Then (Q, D) and (J, H) are T- External Neutrosophic Pythagorean Cubic Soft Sets in U and

(Q, D) $\cap_P$ (J, H) = (Q, D) $\cap$(J, H) =Q$\cap$ Q(e1) = $\{\langle x, ([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), (0.8, 0.3, 0.4)\rangle, e_1 \in D\cap H\}$ for all $u \in U$. (Q, D) $\cap_P$ (J, H) is not a T- External Neutrosophic cubic set, since

$(\lambda_{e_i}{}^T \vee \mu_{e_i}{}^T)(u) = 0.4 \in (0.3,0.5) = ((A^T{}_{e_1} \cap B^T{}_{e_1})^-(u), (A^T{}_{e_1} \cap B^T{}_{e_1})^+(u))$

From the above example it is clear that P – intersection of T – External Neutrosophic Pythagorean Cubic Soft Sets may not be an T -External Neutrosophic Pythagorean Cubic Soft Set.

Now we provide a condition for the P -intersection of T - External (resp. I – External and F -External) Neutrosophic Pythagorean Cubic Soft Sets to be T – External (resp. I- External and F- External) Neutrosophic Pythagorean Cubic Soft Set in the following theorems.

Theorem 4.11 Let (Q, D) = $\{Q(e_i) = \{\langle x, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) = $\{J(e_i) = \{\langle x, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be T- ENPCSS in U such that

$$(\lambda_{e_i}{}^T \vee \mu_{e_i}{}^T)(u) \in \left\{\begin{matrix} \max\{\min\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \min\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}\} \\ \min\{\max\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \max\{A_{e_i}{}^{-T}(u) B_{e_i}{}^{+T}(u)\}\} \end{matrix}\right\} \quad (2.10)$$

For all $e_i \in D$ and for all $e_i \in H$ and for all $u \in U$. Then (Q, D) $\cap_P$ (J, H) is also an T -ENPCSS.

Proof. Consider (Q,D) $\cap_P$ (J,H) = (H, C) where $D \cap H = C$ where

$H(e_i) = Q(e_i) \wedge_P J(e_i)$ is defined as

$$Q(e_i) \wedge_P J(e_i) = H(e_i) = \left\{\left\langle x, \min\{A_{e_i}(u), B_{e_i}(u)\}, \left(\lambda_{e_i}(u) \wedge \mu_{e_i}(u)\right)(u)\right\rangle : u \in U, e_i \in D \cap H\right\},$$

For each $e \in D \cap H$. Take $\alpha_{e_i}{}^T = \min\{\max\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \max\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}$ and $\beta_{e_i}{}^T = \max\{\min\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}\}, \min\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}\}$.

Then $\alpha_{e_i}{}^T$ is one of $A_{e_i}{}^{-T}(u), B_{e_i}{}^{-T}(u), A_{e_i}{}^{+T}(u)$ and $, B_{e_i}{}^{+T}(u)$.

Now we consider $\alpha_{e_i}{}^{T} = A_{e_i}{}^{-T}(u)$ or $A_{e_i}{}^{-T}(u)$ only, as the remaining cases are similar to this one.

If $\alpha_{e_i}{}^{T} = A_{e_i}{}^{-T}(u)$ then $\leq B_{e_i}{}^{+T}(u) \leq A_{e_i}{}^{+T}(u)$ and so $\beta_{e_i}{}^{T} = B_{e_i}{}^{+T}(u)$. Thus $B_{e_i}{}^{-T}(u) = (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{-}(u) \leq (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{+}(u) = B_{e_i}{}^{+T}(u) = \beta_{e_i}{}^{T} < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u)$.

Hence $(\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \notin (((A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{-}(u), (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{+}(u))$.

If $\alpha_{e_i}{}^{T} = A_{e_i}{}^{+T}(u)$, then

$$B_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u) \text{ and so } \beta_{e_i}{}^{T} = \max\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{-T}(u)\}.$$

Assume that $\beta_{e_i}{}^{T} = A_{e_i}{}^{-T}(u)$ then $B_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{-T}(u) < A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$.

So from this we can write $B_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$ or $B_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) = A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$.

For this case $B_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$ which is a contradiction to the fact that (Q,D) and (J,H) are T -ENPCSS.

For the case $B_{e_i}{}^{-T}(u) \leq A_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$ we have $(\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \notin ((A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{-}(u), (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{+}(u))$ because

$$(\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) = A_{e_i}{}^{+T}(u) = (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{+}(u).$$

Again assume that $\beta_{e_i}{}^{T} = B_{e_i}{}^{T}(u)$, then

$$A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u).$$

From this we have $A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$ or

$$A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u).$$

For this case $A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \leq A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$

It is contradiction to the fact that (Q, D) and (J, H) are T – ENPCSS. And if we take the case $A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) < (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) = A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$, we get $(\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) \notin ((A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{-}(u), (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{+}(u))$ because

$$(\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u) = A_{e_i}{}^{+T}(u) = (A_{e_i}{}^{T} \cap B_{e_i}{}^{T})^{+}(u).$$

Hence in all the cases (Q, D) $\cap_P$ (J, H) is a T – ENPCSS in U.

Theorem 4.12 Let (Q, D) = $\{Q(e_i) = \{\langle x, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) =$\{J(e_i) = \{\langle x, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be I- ENPCSSs in U such that

$$(\lambda_{e_i}{}^{I} \wedge \mu_{e_i}{}^{I})(u) \in \begin{Bmatrix} \min\{\max\{A_{e_i}{}^{+I}(u), B_{e_i}{}^{-I}(u)\}, \min\{A_{e_i}{}^{-I}(u), B_{e_i}{}^{+I}(u)\}\} \\ \max\{\min\{A_{e_i}{}^{+I}(u), B_{e_i}{}^{-I}(u)\}, \max\{A_{e_i}{}^{-I}(u)B_{e_i}{}^{+I}(u)\}\} \end{Bmatrix} \tag{2.11}$$

for all $e_i \in D$ and for all $e_i \in H$ and for all $u \in U$. Then (Q, D) $\cap_P$ (J, H) is also an I -ENPCSS.

Proof. Similar to the theorem 4.11.

Theorem 4.13 Let (Q, D) =$\{Q(e_i) = \{\langle x, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) =$\{J(e_i) = \{\langle x, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be F- ENPCSSs in U such that

$$(\lambda_{e_i}{}^{F} \wedge \mu_{e_i}{}^{F})(u) \in \begin{Bmatrix} \min\{\max\{A_{e_i}{}^{+F}(u), B_{e_i}{}^{-F}(u)\}, \min\{A_{e_i}{}^{-F}(u), B_{e_i}{}^{+F}(u)\}\} \\ \max\{\min\{A_{e_i}{}^{+F}(u), B_{e_i}{}^{-F}(u)\}, \max\{A_{e_i}{}^{-F}(u)B_{e_i}{}^{+F}(u)\}\} \end{Bmatrix} \tag{2.12}$$

for all $e_i \in D$ and for all $e_i \in H$ and for all $u \in U$. Then (Q, D) $\cap_P$ (J, H) is also an F -ENPCSS.

Proof. Similar to the theorem 4.11.

Corollary 4.14 Let (Q, D) = $\{Q(e_i) = \{\langle x, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) = $\{J(e_i) = \{\langle x, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be ENPCSSs in U. Then P – intersection (Q, D) $\cap_P$ (J, H) is also an ENPCSS in U when the conditions (2.10), (2.11) and (2.12) are valid.

Theorem 4.15 If Neutrosophic Pythagorean Cubic Soft Set

(Q, D) = $\{Q(e_i) = \{\langle x, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and

(J, H) = $\{J(e_i) = \{\langle x, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ in U satisfy the following condition

$$\min\{\max\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \max\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}\} = (\lambda_{e_i}{}^{T} \wedge \mu_{e_i}{}^{T})(u)$$
$$= \max\{\min\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \min\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}\} \quad (2.13)$$

Then the (Q, D) $\cap_P$ (J, H) is both a T- Internal Neutrosophic Pythagorean Cubic Soft Set and T – External soft Neutrosophic Pythagorean cubic set in U.

Proof. Consider (Q,D) $\cap_P$ (J, H) = (H,C) where D∩ H =C, where $H(e_i) = Q(e_i) \wedge_P J(e_i)$ is defined as

$$Q(e_i) \wedge_P J(e_i) = H(e_i) = \left\{\left\langle x, \min\{A_{e_i}(u), B_{e_i}(u)\}, \left(\lambda_{e_i}(u) \wedge \mu_{e_i}(u)\right)(u)\right\rangle : u \in U, e_i \in D \cap H\right\}$$

Where $F^T(e_i) \wedge_P G^T(e_i) = H(e_i)$

$$= \{Q(e_i)\left\{\left\langle u, \min\{A^T{}_{e_i}(u), B^T{}_{e_i}(u)\}, \left(\lambda^T{}_{e_i} \wedge \mu^T{}_{e_i}\right)(u))\right\rangle : u \in U\} e_i \in D \cap H\right\}$$

for each $e_i \in D \cap H$. Take $\alpha_{e_i}{}^T = \min\{\max\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \max\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}\}$ and $\beta_{e_i}{}^T = \max\{\min\{A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)\}, \min\{A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u)\}\}$. Then $\alpha_{e_i}{}^T$ is one of the $A_{e_i}{}^{-T}(u), B_{e_i}{}^{+T}(u), A_{e_i}{}^{+T}(u), B_{e_i}{}^{-T}(u)$. Now we consider $\alpha_{e_i}{}^T = A_{e_i}{}^{-T}(u)$ or $A_{e_i}{}^{+T}(u)$ only, as the remaining cases are similar to this one. If $\alpha_{e_i}{}^T = A_{e_i}{}^{-T}(u)$.Then $B_{e_i}{}^{-T}(u) \le B_{e_i}{}^{+T}(u) \le A_{e_i}{}^{-T}(u)$, $A_{e_i}{}^{+T}(u)$, and so $\beta_{e_i}{}^T = B^T{}_{e_i}(u)$. This implies

$$A_{e_i}{}^{-T}(u) = \alpha_{e_i}{}^T = \left(\lambda^T{}_{e_i} \wedge \mu^T{}_{e_i}\right)(u) = \beta_{e_i}{}^T = B_{e_i}{}^{+T}(u).$$

Thus $B_{e_i}{}^{-T}(u) \le B_{e_i}{}^{+T}(u) = \left(\lambda^T{}_{e_i} \wedge \mu^T{}_{e_i}\right)(u) = A_{e_i}{}^{-T}(u) \le A_{e_i}{}^{+T}(u)$,

which implies that

$$\left(\lambda^T{}_{e_i} \wedge \mu^T{}_{e_i}\right)(u) = B_{e_i}{}^{+T}(u) = (A_{e_i}{}^T \cap B_{e_i}{}^T)^+(u).$$

Hence $(\lambda_{e_i}{}^T \wedge \mu_{e_i}{}^T)(u) \notin ((A_{e_i}{}^T \cap B_{e_i}{}^T)^-(u), (A_{e_i}{}^T \cap B_{e_i}{}^T)^+(u))$ and $(A_{e_i}{}^T \cap B_{e_i}{}^T)^-(u) \le (\lambda_{e_i}{}^T \wedge \mu_{e_i}{}^T)(u) \le (A_{e_i}{}^T \cap B_{e_i}{}^T)^+(u)$. If $\alpha_{e_i}{}^T = A_{e_i}{}^{+T}(u)$ Then $B_{e_i}{}^{-T}(u) \le A_{e_i}{}^{+T}(u) \le B_{e_i}{}^{+T}(u)$, and so

$$\left(\lambda^T{}_{e_i} \wedge \mu^T{}_{e_i}\right)(u) = A_{e_i}{}^{+T}(u) = (A_{e_i}{}^T \cap B_{e_i}{}^T)^+(u).$$

Hence $(\lambda_{e_i}{}^T \wedge \mu_{e_i}{}^T)(u) \notin ((A_{e_i}{}^T \cap B_{e_i}{}^T)^-(u), (A_{e_i}{}^T \cap B_{e_i}{}^T)^+(u))$ and

$({A_{e_i}}^T \cap {B_{e_i}}^T)^-(u) \leq ({\lambda_{e_i}}^T \wedge {\mu_{e_i}}^T)(u) \leq ({A_{e_i}}^T \cap {B_{e_i}}^T)^+(u)$. Consequently, we note that (Q,D) $\cap_P$ (J,H) is both T – Internal Neutrosophic Pythagorean Cubic Soft Set and T – External soft Neutrosophic Cubic Set in U.

Similarly, we have the following theorems

Theorem 4.16 If Neutrosophic Pythagorean Cubic Soft Set

(Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and

(J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ in U satisfy the following condition

$$\min\{\max\{{A_{e_i}}^{+I}(u), {B_{e_i}}^{-I}(u)\}, \max\{{A_{e_i}}^{-I}(u), {B_{e_i}}^{+I}(u)\}\} = \left(\lambda^I{}_{e_i} \wedge \mu^I{}_{e_i}\right)(u)$$

$$= \max\{\min\{{A_{e_i}}^{+I}(u), {B_{e_i}}^{-I}(u)\}, \min\{{A_{e_i}}^{-I}(u) {B_{e_i}}^{+I}(u)\}\} \quad (4.14)$$

Then the (Q, D) $\cap_P$ (J, H) is both an I– Internal Neutrosophic Pythagorean Cubic Soft Set and I – External soft Neutrosophic Cubic Set in U.

Theorem 4.17 If Neutrosophic Pythagorean Cubic Soft Set

(Q, D) =$\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$

(J, H) = $\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ in U satisfy the following condition

$$\min\{\max\{{A_{e_i}}^{+F}(u), {B_{e_i}}^{-F}(u)\}, \max\{{A_{e_i}}^{-F}(u), {B_{e_i}}^{+F}(u)\}\} = \left(\lambda^F{}_{e_i} \wedge \mu^F{}_{e_i}\right)(u)$$

$$= \max\{\min\{{A_{e_i}}^{+F}(u), {B_{e_i}}^{-F}(u)\}, \min\{{A_{e_i}}^{-F}(u) {B_{e_i}}^{+F}(u)\}\} \quad (2.14)$$

Then the (Q, D) $\cap_P$ (J, H) is both an F – Internal Neutrosophic Pythagorean Cubic Soft set and F – External soft Neutrosophic Cubic Set in U.

Corollary 4.18 Let (Q, D) = $\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and

(J, H) =$\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ (2.15)

be NSPCSSs in U. Then P – intersection (Q, D) $\cap_P$ (J, H) is also an ENPCSS and an INPCSS in U when the conditions (2.13), (2.14) and (2.15) are valid.

The following example shows that the P-union of T – External Neutrosophic Pythagorean Cubic Soft Sets may not be an T – External Neutrosophic Pythagorean Cubic Soft Set.

Example 4.19 Let (Q, D) and (J, H) be Neutrosophic Pythagorean Cubic Soft Sets in U, where

(Q, D) = Q (e_i) = $\{\langle u, ([0.3, 0.5], [0.2, 0.5], [0.5, 0.7]), (0.8, 0.3, 0.6)\rangle \; e_i \in D\}$

(J, H) = J (e_i) = $\{\langle u, ([0.7, 0.9]\, [0.6, 0.8]\, [0.4, 0.7]), (0.4, 0.7, 0.5)\rangle \; e_i \in H\}$

Then (Q, D) and (J, H) are T -External Neutrosophic Pythagorean Cubic Soft Sets in U and (Q, D) ∪ (J, H) =Q∪ J(e_i) = $\{\langle u, ([0.7, 0.9][0.6, 0.8][0.5, 0.7]), (0.8, 0.3, 0.6)\rangle\}$

(Q, D) $\cup_P$ (J, H) is not an T – External Neutrosophic Pythagorean Cubic Set in U since $({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T) = 0.8 \in (0.7, 0.9) = \notin (({A_{e_i}}^T \cap {B_{e_i}}^T)^-(u), ({A_{e_i}}^T \cap {B_{e_i}}^T)^+(u))$,

We consider a condition for the P -union of T – External (resp. I -External and F -External) Neutrosophic Pythagorean Cubic Soft Sets to be T – External (resp. I-External and F -External) Neutrosophic Pythagorean Cubic Soft Set.

Theorem 4.20 Let (Q, D) $=\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\}\ e_i \in D\}$ and (J, H) $=\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be T- ENPCSS in U such that

$$({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) \in \begin{cases} \max\{\min\{{A_{e_i}}^{+T}(u), {B_{e_i}}^{-T}(u)\}, \min\{{A_{e_i}}^{-T}(u), {B_{e_i}}^{+T}(u)\}\} \\ \min\{\max\{{A_{e_i}}^{+T}(u), {B_{e_i}}^{-T}(u)\}, \max\{{A_{e_i}}^{-T}(u){B_{e_i}}^{+T}(u)\}\} \end{cases} \quad (2.16)$$

for all $e_i \in D$ and for all $e_i \in H$ and for all $u \in U$. Then (Q, D) $\cup_P$ (J, H) is also an T - ENPCSS.

Proof. Consider (Q, D) $\cup_P$ (J, H) = (H, C) where $D \cup H = C$ and

$$H(e_i) = \begin{cases} Q(e_i) & \text{if } e_i \in D - H \\ H(e_i) & \text{if } e_i \in H - D \\ Q(e_i) \vee_P H(e_i) & \text{if } e_i \in D \cap H \end{cases}$$

Where $H(e_i) = Q(e_i) \vee_P J(e_i)$ is defined as

$$Q(e_i) \vee_P J(e_i) = H(e_i) = \{\langle u, \max\{A_{e_i}(u), B_{e_i}(u)\}, (\lambda_{e_i} \vee \mu_{e_i})(u)\rangle : u \in U, e_i \in D \cap H\}$$

$$Q^T(e_i) \vee_P J^T(e_i) = \left\{\left\langle x, \max\{{A^T}_{e_i}(u), {B^T}_{e_i}(u)\}, \left({\lambda^T}_{e_i} \vee {\mu^T}_{e_i}\right)(u)\right\rangle : u \in U, e_i \in D \cap H\right\}$$

If $e_i \in D \cap H$, ${\alpha_{e_i}}^T = \min\{\max\{{A_{e_i}}^{+T}(u), {B_{e_i}}^{-T}(u)\}, \max\{{A_{e_i}}^{-T}(u), {B_{e_i}}^{+T}(u)\}\}$ and ${\beta_{e_i}}^T = \max\{\min\{{A_{e_i}}^{+T}(u), {B_{e_i}}^{-T}(u)\}, \min\{{A_{e_i}}^{-T}(u), {B_{e_i}}^{+T}(u)\}\}$.

Then ${\alpha_{e_i}}^T$ is one of ${A_{e_i}}^{-T}(u), {B_{e_i}}^{-T}(u), {A_{e_i}}^{+T}(u), {B_{e_i}}^{+T}(u)$.

Now we consider ${\alpha_{e_i}}^T = {A_{e_i}}^{-T}(u)$ or ${A_{e_i}}^{+T}(u)$only, as the remaining cases are similar to this one.

If ${\alpha_{e_i}}^T = {A_{e_i}}^{-T}(u)$, then ${B_{e_i}}^{-T}(u) \le {B_{e_i}}^{+T}(u) \le {A_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u)$ and so ${\beta_{e_i}}^T = {B_{e_i}}^{+T}(u)$. Thus $({A_{e_i}}^T \cup {B_{e_i}}^T)^-(u) = {A_{e_i}}^{-T}(u) = {\alpha_{e_i}}^T(u) > ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u)$.

Hence $({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) \notin (({A_{e_i}}^T \cup {B_{e_i}}^T)^-(u), ({A_{e_i}}^T \cup {B_{e_i}}^T)^+(u))$.

If ${\alpha_{e_i}}^T = {A_{e_i}}^{-T}(u)$, then ${B_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u) \le {A_{e_i}}^{+T}(u)$ and so ${\beta_{e_i}}^T = \max\{{A_{e_i}}^{-T}(u), {B_{e_i}}^{-T}(u)\}$.

Assume that ${\beta_{e_i}}^T = {A_{e_i}}^{-T}(u)$, then

$${B_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u) \le ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) \le {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u).$$

So from this we can write ${B_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u) \le ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) \le {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$ or ${B_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u) = ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) \le {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$.

For the case ${B_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u) < ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) < {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$.

It is contradiction to the fact that (Q,D) and (J,H) are T -ENPCSS.

For the case ${B_{e_i}}^{-T}(u) \le {A_{e_i}}^{+T}(u) < ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) < {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$ we have $({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) \notin (({A_{e_i}}^T \cup {B_{e_i}}^T)^-(u), ({A_{e_i}}^T \cup {B_{e_i}}^T)^+(u))$because $({A_{e_i}}^T \cup {B_{e_i}}^T)^-(u) = {A_{e_i}}^{-T}(u) = ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u)$.

Again assume that ${\beta_{e_i}}^T = {B_{e_i}}^{-T}(u)$, then

${A_{e_i}}^{-T}(u) \le {B_{e_i}}^{-T}(u) < ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) < {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$.

So from this we can write

${A_{e_i}}^{-T}(u) \le {B_{e_i}}^{-T}(u) < ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) < {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$ or

${A_{e_i}}^{-T}(u) \le {B_{e_i}}^{-T}(u) = ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) < {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$.

For this case we write ${A_{e_i}}^{-T}(u) \le {B_{e_i}}^{-T}(u) < ({\lambda_{e_i}}^T \vee {\mu_{e_i}}^T)(u) < {A_{e_i}}^{+T}(u) \le {B_{e_i}}^{+T}(u)$.

It is contradiction to the fact that (Q,D) and (J,H) are T -ENPCSS. And if we take the case $A_{e_i}{}^{-T}(u) \leq B_{e_i}{}^{-T}(u) = (\lambda_{e_i}{}^{T} \vee \mu_{e_i}{}^{T})(u) < A_{e_i}{}^{+T}(u) \leq B_{e_i}{}^{+T}(u)$, we get have $(\lambda_{e_i}{}^{T} \vee \mu_{e_i}{}^{T})(u) \notin ((A_{e_i}{}^{T} \cup B_{e_i}{}^{T})^{-}(u), (A_{e_i}{}^{T} \cup B_{e_i}{}^{T})^{+}(u))$ because $(A_{e_i}{}^{T} \cup B_{e_i}{}^{T})^{-}(u) = B_{e_i}{}^{-T}(u) = (\lambda_{e_i}{}^{T} \vee \mu_{e_i}{}^{T})(u)$. If $e_i \in D - H$ or $e_i \in H - D$, then we have trivial results. Hence (Q, D) $\cup_P$ (J, H) is an T -ENPCSS in U. similarly we have the following theorems.

Theorem 4.21 Let (Q, D) $=\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) $=\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be I- ENPCSs in U such that

$$(\lambda_{e_i}{}^{I} \vee \mu_{e_i}{}^{I})(u) \in \begin{Bmatrix} \max\{\min\{A_{e_i}{}^{+I}(u), B_{e_i}{}^{-I}(u)\}, \min\{A_{e_i}{}^{-I}(u), B_{e_i}{}^{+I}(u)\}\} \\ \min\{\max\{A_{e_i}{}^{+I}(u), B_{e_i}{}^{-I}(u)\}, \max\{A_{e_i}{}^{-I}(u) B_{e_i}{}^{+I}(u)\}\} \end{Bmatrix} \quad (2.17)$$

for all $e_i \in D$ and for all $e_i \in H$ and for all $u \in U$. Then (Q, D) $\cup_P$ (J, H) is also an I -ENPCSS

Theorem 4.22 Let (Q, D) $=\{Q(e_i) = \{\langle u, A_{e_i}(u), \lambda_{e_i}(u)\rangle : u \in U\} e_i \in D\}$ and (J, H) $=\{J(e_i) = \{\langle u, B_{e_i}(u), \mu_{e_i}(u)\rangle : u \in U\} e_i \in H\}$ be F- ENPCSS in U such that

$$(\lambda_{e_i}{}^{F} \vee \mu_{e_i}{}^{F})(u) \in \begin{Bmatrix} \max\{\min\{A_{e_i}{}^{+F}(u), B_{e_i}{}^{-F}(u)\}, \min\{A_{e_i}{}^{-F}(u), B_{e_i}{}^{+F}(u)\}\} \\ \min\{\max\{A_{e_i}{}^{+F}(u), B_{e_i}{}^{-F}(u)\}, \max\{A_{e_i}{}^{-F}(u) B_{e_i}{}^{+F}(u)\}\} \end{Bmatrix} \quad (2.18)$$

for all $e_i \in D$ and for all $e_i \in H$ and for all $u \in U$. Then (Q, D) $\cup_P$ (J, H) is also an F-ENPCSS

5. Conclusion

This research paper introduced and explored the concept of Neutrosophic Pythagorean cubic soft sets, including definitions of Neutrosophic Pythagorean Cubic soft sets, Internal Neutrosophic Pythagorean Cubic Soft Sets. The investigation of P-union and P-intersection operations, along with their properties, was conducted. In future, R-union and R-intersection operations will introduced, enabling the examination of combined properties. These novel concepts and operations contributed to the advancement of neutrosophic set theory, potentially benefiting applications in decision-making, uncertainty modelling, and more.

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