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Topology-Aware Neutrosophic Graph Structures for Modeling Innovation Performance of Agricultural Technology Enterprises in the Digital Economy Era

Xianzhang Meng[1]* ,Yuhao Lu[2]

[1]Jilin Business and Technology College, Changchun, Jilin, 130507, China

[2]Changchun University, Changchun, Jilin,130022, China

*Corresponding author, E-mail: mxzmxz7220@163.com

Abstract-With the proliferation of smart agricultural initiatives in the digital economy, a significant increase in uncertainties has arisen in how stakeholders interact across different dimensions. Existing neutrosophic models, including neutrosophic graphs, failed to model structural realities that govern these interactions. To address this challenge, this study presents a novel mathematical framework, topology-aware neutrosophic graphs, that integrates neutrosophic graph theories with domain-specific topological constraints for modeling enterprise connectivity. Our key contributions include: 1) providing formal definition of topology-aware neutrosophic graphs theory that integrate neutrosophic uncertainty with structural constraints; 2) presenting customized operations and analytical tools for exploring graph properties under topological constraints; and 3) demonstrating through a detailed case study how our framework reduces uncertainty and improves interpretability compared to custom approaches. We study the applicability of the proposed framework on a real case study, and the results show the ability to capture uncertainty more realistically while filtering implausible relationships. Based on comparative analysis against topology-agnostic models, we prove the benefits of our framework in reducing network ambiguity and enhancing interpretability.

Keywords: Neutrosophic Sets; Neutrosophic Graph Theory; Uncertainty Modeling; Graph Homomorphism; Agricultural Technology; Agri-Digital Transformation; AgriTech Enterprises.

1. Introduction

Agricultural enterprises encompass a broad range of entities involved in the production, processing, distribution, and commercialization of agricultural goods and services [1]. These enterprises form the backbone of national and global food systems, involving actors such as farmers, cooperatives, agri-processors, distributors, agri-tech companies, and policy regulators [2], [3]. Traditionally, these enterprises have operated in siloed and often manual environments with limited automation, data sharing, or systemic coordination.

The rapid advancement of digital technologies has initiated a paradigm shift across agricultural value chains [4], [5]. The integration of tools such as Internet of Things (IoT) devices, remote sensing, data analytics, cloud computing, blockchain, and artificial intelligence has led to the emergence of smart agricultural enterprises. In this context, "smart" refers to the ability of enterprises to use real-time data and digital intelligence for optimizing decisions, automating operations, enhancing collaboration, and increasing adaptability across interconnected agricultural systems, as shown in Figure 1.

Modeling uncertainty in such interconnected systems is crucial for ensuring robust decision-making, adaptability, and operational resiliency. Traditional mathematical and graph-based models often fall short in modeling the multifaceted nature of uncertainty that stems from incomplete, imprecise, or conflicting information inherent in the agricultural enterprise ecosystem. Motivated by that, we explore more expressive frameworks that can handle degrees of indeterminacy.

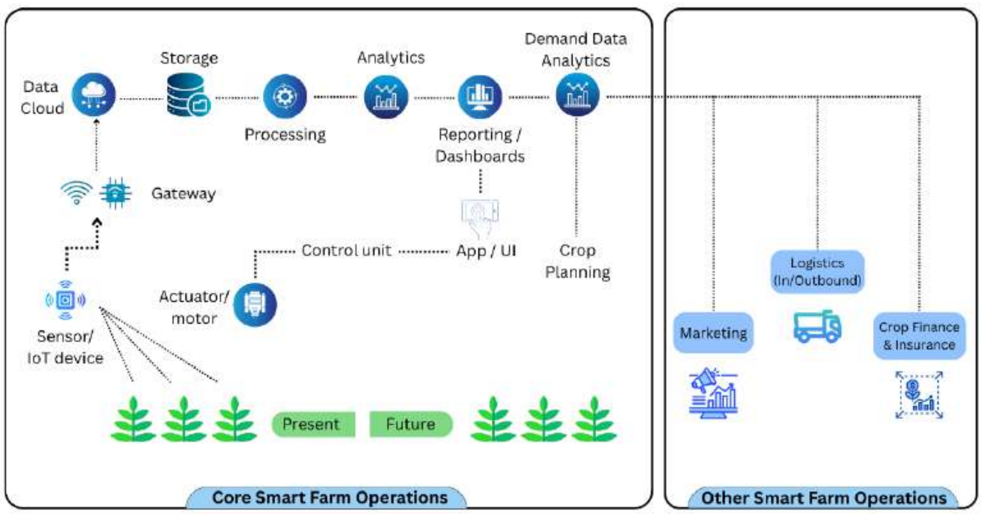

Figure 1. Visualization of core components and interactions in a smart agricultural enterprise.

Neutrosophic graph theory [6], [7] offers a powerful extension to classical and fuzzy graph models by incorporating neutrosophic sets, which allow each element to have levels of truth (T), indeterminacy (I), and falsity (F) [8], [9]. This makes it particularly suitable for representing the uncertainty-laden interactions within digital agricultural networks. Furthermore, incorporating topological awareness into neutrosophic graph structures enables an ironic representation of spatial, functional, and logical connectivity among enterprise components, which enhances the ability to analyze multifaceted interdependencies.

In this paper, we introduce a novel framework titled Topology-Aware Neutrosophic Graph Structures (TANGS), aimed at modeling and analyzing interactions in smart agricultural enterprises. The main contributions of this work are:

- Formalization of TANGS theory and related properties and operations, including union, complement, Cartesian product, and join, according to structural conditions.

- Application of the proposed TANGS to scenarios in smart agricultural enterprises, showcasing how it captures the uncertainty and topological intricacies of digital interactions.
- Quantitative and Qualitative analysis of topology-aware neutrosophic graphs representing real or hypothetical agri-enterprise systems to validate the expressive power of the framework.

The left part of this study is structured into 4 main sections. Section 2 discusses fundamental concepts and operations in neutrosophic graph theory. Section 3 introduces the topology-aware neutrosophic graph framework. Section 4 argues the application in the modeling of smart agricultural enterprises. Section 5 presents a discussion, and Section 7 concludes our research and charts out the potential directions for future work.

2. Preliminaries and Fundamental Definitions

Definition 1. Given X be a universe of discourse. A Neutrosophic Set A in X is characterized by membership functions, including truth-membership T_A, an indeterminacy-membership I_A, and a falsity-membership, F_A, such that:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where

$$T_A(x), I_A(x), F_A(x) \in [0^-, 1^+] \tag{1}$$

and $0^-$and 1^+denote possible non-standard values slightly less than 0 and slightly greater than 1, which allow paraconsistent and over-/under-defined modeling.

Definition 2. Given V be a non-empty set of vertices, a neutrosophic graph $\mathfrak{G} = \langle \Psi, \Omega\rangle$ is defined over V with neutrosophic vertex set $\Psi = \{\langle T^R_{\sigma\Psi}(\grave{v}_j), I^N_{\sigma\Psi}(\grave{v}_j), F^S_{\sigma\Psi}(\grave{v}_j)\rangle : v_i \in V\}$ and neutrosophic edge set $\Omega = \{\langle T^R_{\sigma\Omega}(\grave{v}_i, \grave{v}_j), I^N_{\sigma\Omega}(\grave{v}_i, \grave{v}_j), F^S_{\sigma\Omega}(\grave{v}_i, \grave{v}_j)\rangle : (\grave{v}_i, \grave{v}_j) \in V \times V\}$, such that $T^R\sigma_{\boldsymbol{\Psi}}, I^N\sigma_{\boldsymbol{\Psi}}, and\ F^S\sigma_{\boldsymbol{\Psi}}$ denote membership function mappings from V to the closed interval [0,1] , for each vertex $v_i \in V$.

The following conditions must hold for all $v_i \in V$, and $(\grave{v}_i, \grave{v}_j) \in V \times V$:

I. Vertex Constraint

$$0 \leq T^R_{\sigma\Psi}(v_i) + I^N_{\sigma\Psi}(v_i) + F^S_{\sigma\Psi}(v_i) \leq 3 \tag{2}$$

II. Edge Constraints

$$\begin{aligned} T^R_{\sigma\Omega}(\grave{v}_i\grave{v}_j) &\leq \min\left(T^R_{\sigma\boldsymbol{\Psi}}(\grave{v}_i), T^R_{\sigma\boldsymbol{\Psi}}(\grave{v}_j)\right), \\ I^N_{\sigma\Omega}(\grave{v}_i\grave{v}_j) &\leq \min\left(I^N_{\sigma\boldsymbol{\Psi}}(\grave{v}_i), I^N_{\sigma\boldsymbol{\Psi}}(\grave{v}_j)\right), \\ F^S_{\sigma\Omega}(\grave{v}_i\grave{v}_j) &\geq \max\left(F^S_{\sigma\boldsymbol{\Psi}}(\grave{v}_i), F^S_{\sigma\boldsymbol{\Psi}}(\grave{v}_j)\right) \end{aligned} \tag{3}$$

III. Edge Summation Constraint

(4)

$$0 \le T^{R}_{\sigma\Omega}(\grave{v}_i, \grave{v}_j) + I^{N}_{\sigma\Omega}(\grave{v}_i, \grave{v}_j) + F^{S}_{\sigma\Omega}(\grave{v}_i, \grave{v}_j) \le 3$$

Definition 3. *Given* $\mathfrak{G} = \langle \Psi, \Omega \rangle$ *as* neutrosophic graph, then, the order of this graph is computed as:

$$O(\mathfrak{G}) = \left(\sum_{\grave{v} \in V} T^{R}\sigma_{\Psi}(\grave{v}),\ \sum_{\grave{v} \in V} I^{N}\sigma_{\Psi}(\grave{v}),\ \sum_{\grave{v} \in V} F^{S}\sigma_{\Psi}(\grave{v}) \right) \tag{5}$$

while the degree of each vertex $\grave{v}$ *in G* is computed as:

$$\mathfrak{D}(\mathfrak{G}) = \left(\sum_{\grave{v} \in \Omega} T^{R}\sigma_{\Omega}(\grave{v}, \grave{v}),\ \sum_{\grave{v} \in \Omega} I^{N}\sigma_{\Omega}(\grave{v}, \grave{v}),\ \sum_{\grave{v} \in \Omega} F^{S}\sigma_{\Omega}(\grave{v}, \grave{v}) \right) \tag{6}$$

Example 1. *Given* $A = \{a, b, c, d\}$ *and* $B = \{(a, b), (a, c), (b, c), (a, d)\}$ on $\mathfrak{G} = \langle \Psi, \Omega \rangle$ where V *is* neutrosophic graph subset *of* $\boldsymbol{\Psi}$, *and* E *is a* neutrosophic graph subset *of* $E \subseteq V \times V$, as given:

$$V = \left(\frac{(0.32, 0.14, 0.45)}{a}, \frac{(0.64, 0.42, 0.31)}{b}, \frac{(0.87, 0.54, 0.31)}{c}, \frac{(0.41, 0.23, 0.28)}{d} \right)$$

$$E = \left(\frac{(0.33, 0.17, 0.26)}{a, b}, \frac{(0.71, 0.43, 0.38)}{a, c}, \frac{(0.82, 0.51, 0.16)}{b, c}, \frac{(0.53, 0.28, 0.23)}{a, d} \right).$$

Figure 2 illustrates the neutrosophic graph corresponding to Example 1, where each vertex and edge is characterized by a neutrosophic triplet. The set of vertices $V = \{a, b, c, d\}$ and the set of edges $E = \{ab, ac, bc, ad\}$ are enriched with their associated neutrosophic values, providing a more expressive structure for modeling uncertainty in complex networks.

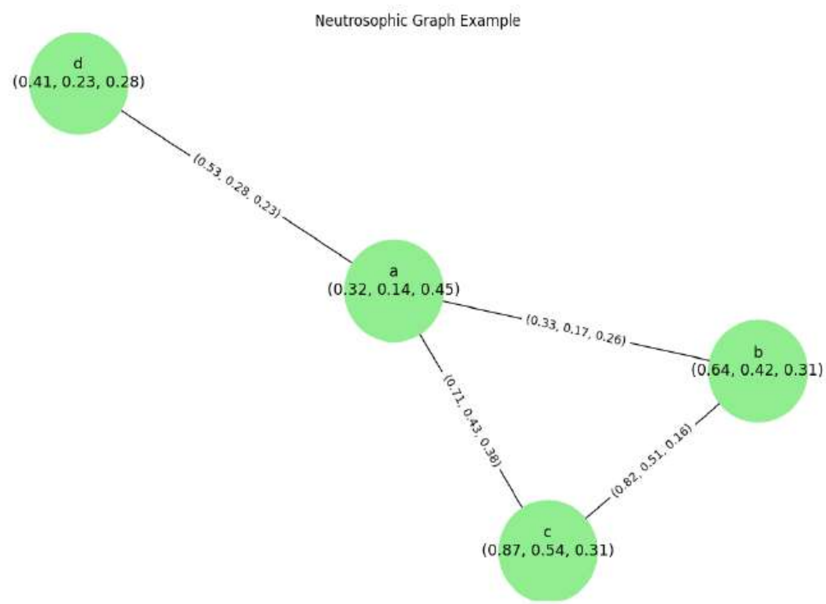

Figure 2. Neutrosophic Graph Representation of Example 1 with Neutrosophic Triplets Assigned to Nodes and Edges.

The order of the neutrosophic graph is expressed as follows $O(\mathfrak{G}) = (2.34, 2.08, 1.95)$, while the individual degree of each vertex is given as follows:

$$\mathfrak{D}(a) = (1.18, 1.05, 1.40), \mathfrak{D}(b) = (1.60, 0.90, 0.42),$$
$$\mathfrak{D}(c) = (0.91, 0.74, 0.82), \mathfrak{D}(d) = (0.23, 0.31, 0.68)$$

$$0 \leq T^R_{\sigma_\Omega}(\grave{v}_i\grave{v}_j) + I^N_{\sigma_\Omega}(\grave{v}_i\grave{v}_j) + F^S_{\sigma_\Omega}(\grave{v}_i\grave{v}_j) \leq 3 \tag{7}$$

Definition 4. Given two neutrosophic graphs $\mathfrak{G}_1 = \langle \Psi_1, \Omega_1 \rangle$, *and* $\mathfrak{G}_2 = \langle \Psi_2, \Omega_2 \rangle$ defined over vertex sets V_1 and V_2, respectively; then, the *Cartesian product of* $\mathfrak{G}_1$, *and* $\mathfrak{G}_2$ *defined as:*

$$\mathfrak{G}_1 \times \mathfrak{G}_2 = \langle \Omega_1 \times \Omega_2, \boldsymbol{\Psi}_1 \times \boldsymbol{\Psi}_2 \rangle, \tag{8}$$

with the following component-wise mappings:

I. Edge Neutrosophic Components (for all $(\grave{v}_i, \grave{v}_j) \in V \times V$)

$$\begin{aligned} T^R\sigma_{\Omega_1\times\Omega_2}(\grave{v}_1, \grave{v}_2) &= min\{T^R\sigma_{\Omega_1}(\grave{v}_1), T^R\sigma_{\Omega_2}(\grave{v}_2)\} \\ I^N\sigma_{\Omega_1\times\Omega_2}(\grave{v}_1, \grave{v}_2) &= min\{I^N\sigma_{\Omega_1}(\grave{v}_1), I^N\sigma_{\Omega_2}(\grave{v}_2)\} \\ F^S\sigma_{\Omega_1\times\Omega_2}(\grave{v}_1, \grave{v}_2) &= max\{F^S\sigma_{\Omega_1}(\grave{v}_1), F^S\sigma_{\Omega_2}(\grave{v}_2)\} \end{aligned} \tag{9}$$

II. *Vertex Neutrosophic Components (for all $(\grave{v}_i, \grave{v}_j) \in V \times V$)*

$$\begin{aligned} T^R\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}(\grave{v}_1, \grave{v}_2) &= min\{T^R\sigma_{\boldsymbol{\Psi}_1}(\grave{v}_1), T^R\sigma_{\boldsymbol{\Psi}_2}(\grave{v}_2)\} \\ I^N\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}(\grave{v}_1, \grave{v}_2) &= min\{I^N\sigma_{\boldsymbol{\Psi}_1}(\grave{v}_1), I^N\sigma_{\boldsymbol{\Psi}_2}(\grave{v}_2)\} \\ F^S\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}(\grave{v}_1, \grave{v}_2) &= max\{F^S\sigma_{\boldsymbol{\Psi}_1}(\grave{v}_1), F^S\sigma_{\boldsymbol{\Psi}_2}(\grave{v}_2)\} \end{aligned} \tag{10}$$

III. Edge Set Mapping (for all $(\grave{v}_i, \grave{v}_j) \in V \times V$)

$$\begin{aligned} T^R\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}\big((\grave{v}_1, \grave{t}), (\grave{v}_1, \grave{t})\big) &= min\{T^R\sigma_{\boldsymbol{\Psi}_1}(\grave{v}_1\grave{v}_1), T^R\grave{\sigma}_{\boldsymbol{\Psi}_2}(\grave{t})\} \\ I^N\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}\big((\grave{v}_1, \grave{t}), (\grave{v}_1, \grave{t})\big) &= min\{I^N\sigma_{\boldsymbol{\Psi}_1}(\grave{v}_1\grave{v}_1), I^N\grave{\sigma}_{\boldsymbol{\Psi}_2}(\grave{t})\} \\ F^S\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}\big((\grave{v}_1, \grave{t}), (\grave{v}_1, \grave{t})\big) &= max\{F^S\sigma_{\boldsymbol{\Psi}_1}(\grave{v}_1\grave{v}_1), F^S\grave{\sigma}_{\boldsymbol{\Psi}_2}(\grave{t})\} \end{aligned} \tag{11}$$

Example 2. Given neutrosophic graph $\mathfrak{G}_1$ with vertex set $V_1 = \left\{\underbrace{(0.7,0.2,0.1)}_{a}, \underbrace{(0.6,0.3,0.2)}_{b}\right\}$, and Edge set $E_1 = \underbrace{(0.5,0.2,0.3)}_{\text{(a,b)}}$; and neutrosophic graph $\mathfrak{G}_2$ with vertex set $V_2 = \left\{\underbrace{(0.9,0.1,0.0)}_{x}, \underbrace{(0.6,0.2,0.4)}_{y}, \underbrace{(0.5,0.3,0.5)}_{z}\right\}$, and Edge set $E_2 = \underbrace{(0.8,0.2,0.3)}_{\text{(x,y)}}, \underbrace{(0.4,0.3,0.6)}_{\text{(y,z)}}$. Then, the Vertex Set of cartesian product is formed as $\mathrm{G}_1 \times \mathrm{G}_2 = \{(\mathrm{a,x}), (\mathrm{a,y}), (\mathrm{a,z}), (\mathrm{b,x}), (\mathrm{b,y}), (\mathrm{b,z})\}$, as shown in Figure 3.

Definition 5. Given $\mathfrak{G}_1 \times \mathfrak{G}_2$ *as a* product neutrosophic graph. The degree of a vertex in $\mathrm{G}_1 \times \mathrm{G}_2$ could be computed as bellows. For any $(\grave{v}_1, \grave{v}_2) \in V_1 \times V_2$:

(12)

$$\mathfrak{D}_{\mathfrak{G}_1\times\mathfrak{G}_2}(\grave{v}_1,\grave{v}_2)=\begin{pmatrix}\sum_{(\grave{v}_1,\grave{v}_2)(\grave{v}_1,\grave{v}_2)\in E}T^R\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}\big((\grave{v}_1,\grave{v}_2),(\grave{v}_1,\grave{v}_2)\big),\\ \sum_{(\grave{v}_1,\grave{v}_2)(\grave{v}_1,\grave{v}_2)\in E}I^N\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}\big((\grave{v}_1,\grave{v}_2),(\grave{v}_1,\grave{v}_2)\big),\\ \sum_{(\grave{v}_1,\grave{v}_2)(\grave{v}_1,\grave{v}_2)\in E}F^S\sigma_{\boldsymbol{\Psi}_1\times\boldsymbol{\Psi}_2}\big((\grave{v}_1,\grave{v}_2),(\grave{v}_1,\grave{v}_2)\big).\end{pmatrix}$$

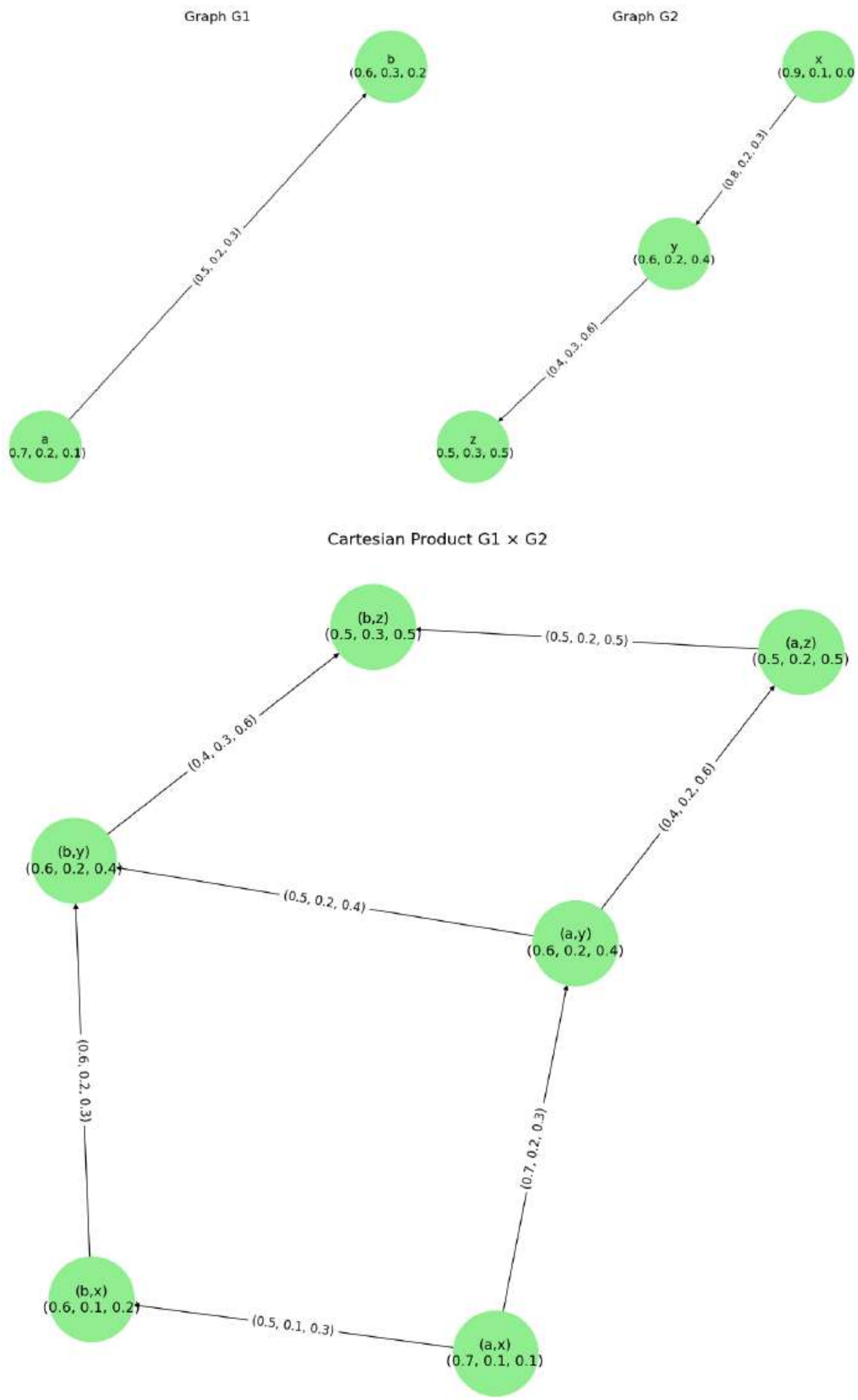

Figure 3. Cartesian product of neutrosophic graphs $\langle \mathfrak{G}_1 \times \mathfrak{G}_2 \rangle$ from Example 2.

Definition 6. Let $\mathfrak{G}_1 = \langle \Psi_1, \Omega_1 \rangle$, and $\mathfrak{G}_2 = \langle \Psi_2, \Omega_2 \rangle$ be two neutrosophic graphs, the composition of $\mathfrak{G}_1$ and $\mathfrak{G}_2$, denoted as $\mathfrak{G}_1 \circ \mathfrak{G}_2 = \langle \Omega_1 \circ \Omega_2, \Psi_1 \circ \Psi_2 \rangle$ is defined on the vertex set $V = V_1 \times V_2$ with neutrosophic mappings constructed as follows:

I. Edge-Based Composition (Pair of vertices $(v_1, v_2) \in V_1 \times V_2$):

$$
\begin{aligned}
T^R_{\sigma_{\Omega_1 \circ \Omega_2}}(v_1, v_2) &= \min\left\{T^R_{\sigma\_\Omega_1}(v_1), T^R_{\sigma_{\Omega_1}}(v_2)\right\},\\
I^N_{\sigma_{\Omega_1 \circ \Omega_2}}(v_1, v_2) &= \min\left\{I^N_{\sigma\_\Omega_1}(v_1), I^N_{\sigma_{\Omega_1}}(v_2)\right\},\\
F^S_{\sigma_{\Omega_1 \circ \Omega_2}}(v_1, v_2) &= \max\left\{F^S_{\sigma_{\Omega_1}}(v_1), F^S_{\sigma_{\Omega_1}}(v_2)\right\}.
\end{aligned} \tag{13}
$$

II. Vertex-Based Composition (for all $v_1 \in V_1$, and $v_2, u_2 \in V_2$, with $(v_2, u_2) \in E_2$):

$$
\begin{aligned}
T^R_{\sigma_{\Omega_1 \circ \Omega_2}}(v_1, v_2) &= \min\left\{T^R_{\sigma\_\Omega_1}(v_1), T^R_{\sigma_{\Omega_1}}(v_2)\right\}\\
I^N_{\sigma_{\Omega_1 \circ \Omega_2}}(v_1, v_2) &= \min\left\{I^N_{\sigma\_\Omega_1}(v_1), I^N_{\sigma_{\Omega_1}}(v_2)\right\}\\
F^S_{\sigma_{\Omega_1 \circ \Omega_2}}(v_1, v_2) &= \max\left\{F^S_{\sigma\_\Omega_1}(v_1), F^S_{\sigma_{\Omega_1}}(v_2)\right\}
\end{aligned} \tag{14}
$$

III. Edges from $\mathfrak{G}_1$ with fixed vertex $t \in V_2$:

For all $t \in V_2$, and $(v_1, u_1) \in E_1$:

$$
\begin{aligned}
T^R_{\sigma_{\Omega_1 \circ \Omega_2}}\big((v_1, t), (u_1, t)\big) &= \min\left\{T^R_{\sigma_{\Omega_1}}(v_1, u_1), T^R_{\sigma_{\Omega_2}}(t)\right\}\\
I^N_{\sigma_{\Omega_1 \circ \Omega_2}}\big((v_1, t), (u_1, t)\big) &= \min\left\{I^N_{\sigma_{\Omega_1}}(v_1, u_1), I^N_{\sigma_{\Omega_2}}(t)\right\}\\
F^S_{\sigma_{\Omega_1 \circ \Omega_2}}\big((v_1, t), (u_1, t)\big) &= \max\left\{F^S_{\sigma_{\Omega_1}}(v_1, u_1), F^S_{\sigma_{\Omega_2}}(t)\right\}
\end{aligned} \tag{15}
$$

IV. Full Composition Edges (for all $(v_1, v_2), (u_1, u_2) \in V_1 \times V_2$, with $(v_1, u_1) \in E_1$ and $v_2 \neq u_2$):

$$
\begin{aligned}
T^R_{\sigma_{\Omega_1 \circ \Omega_2}}\big((v_1, v_2), (u_1, u_2)\big) &= \min\left\{T^R_{\sigma_{\Omega_2}}(v_2), T^R_{\sigma_{\Omega_2}}(u_2), T^R_{\sigma_{\Omega_1}}(v_1, u_1)\right\},\\
I^N_{\sigma_{\Omega_1 \circ \Omega_2}}\big((v_1, v_2), (u_1, u_2)\big) &= \min\left\{I^N_{\sigma_{\Omega_2}}(v_2), I^N_{\sigma_{\Omega_2}}(u_2), I^N_{\sigma_{\Omega_1}}(v_1, u_1)\right\},\\
F^S_{\sigma_{\Omega_1 \circ \Omega_2}}\big((v_1, v_2), (u_1, u_2)\big) &= \max\left\{F^S_{\sigma_{\Omega_2}}(v_2), F^S_{\sigma_{\Omega_2}}(u_2), F^S_{\sigma_{\Omega_1}}(v_1, u_1)\right\}.
\end{aligned} \tag{16}
$$

Definition 7. Given $\mathfrak{G}_1 \circ \mathfrak{G}_2$ as a neutrosophic graph, then the degree of a vertex in $\mathfrak{G}_1 \circ \mathfrak{G}_2$ is defined as follows*: for any* $(\grave{v}_1, \grave{v}_2) \in V_1 \circ V_2$,

$$
\mathfrak{D}_{\mathfrak{G}_1 \circ \mathfrak{G}_2}(\grave{v}_1, \grave{v}_2) = \begin{pmatrix}
\sum_{(\grave{v}_1, \grave{v}_2)(\grave{v}_1, \grave{v}_2) \in E} T^R \sigma_{\boldsymbol{\Psi}_1 \circ \boldsymbol{\Psi}_2}\big((\grave{v}_1, \grave{v}_2), (\grave{v}_1, \grave{v}_2)\big),\\
\sum_{(\grave{v}_1, \grave{v}_2)(\grave{v}_1, \grave{v}_2) \in E} I^N \sigma_{\boldsymbol{\Psi}_1 \circ \boldsymbol{\Psi}_2}\big((\grave{v}_1, \grave{v}_2), (\grave{v}_1, \grave{v}_2)\big),\\
\sum_{(\grave{v}_1, \grave{v}_2)(\grave{v}_1, \grave{v}_2) \in E} F^S \sigma_{\boldsymbol{\Psi}_1 \circ \boldsymbol{\Psi}_2}\big((\grave{v}_1, \grave{v}_2), (\grave{v}_1, \grave{v}_2)\big).
\end{pmatrix} \tag{17}
$$

Example 3. Given neutrosophic graph $\mathfrak{G}_1$ with vertex set $V_1 = \left\{\underbrace{(0.7,0.2,0.1)}_{a}, \underbrace{(0.6,0.3,0.2)}_{b}\right\}$, and Edge set $E_1 = \underbrace{(0.5,0.2,0.3)}_{(a,b)}$; and neutrosophic graph $\mathfrak{G}_2$ with vertex set $V_2 = \left\{\underbrace{(0.9,0.1,0.0)}_{x}, \underbrace{(0.6,0.2,0.4)}_{y}\right\}$, and Edge set $E_2 = \left\{\underbrace{(0.8,0.1,0.4)}_{(x,y)}\right\}$, their composition, their composition $\mathfrak{G}_1 \circ \mathfrak{G}_2$ is shown in Figure 4.

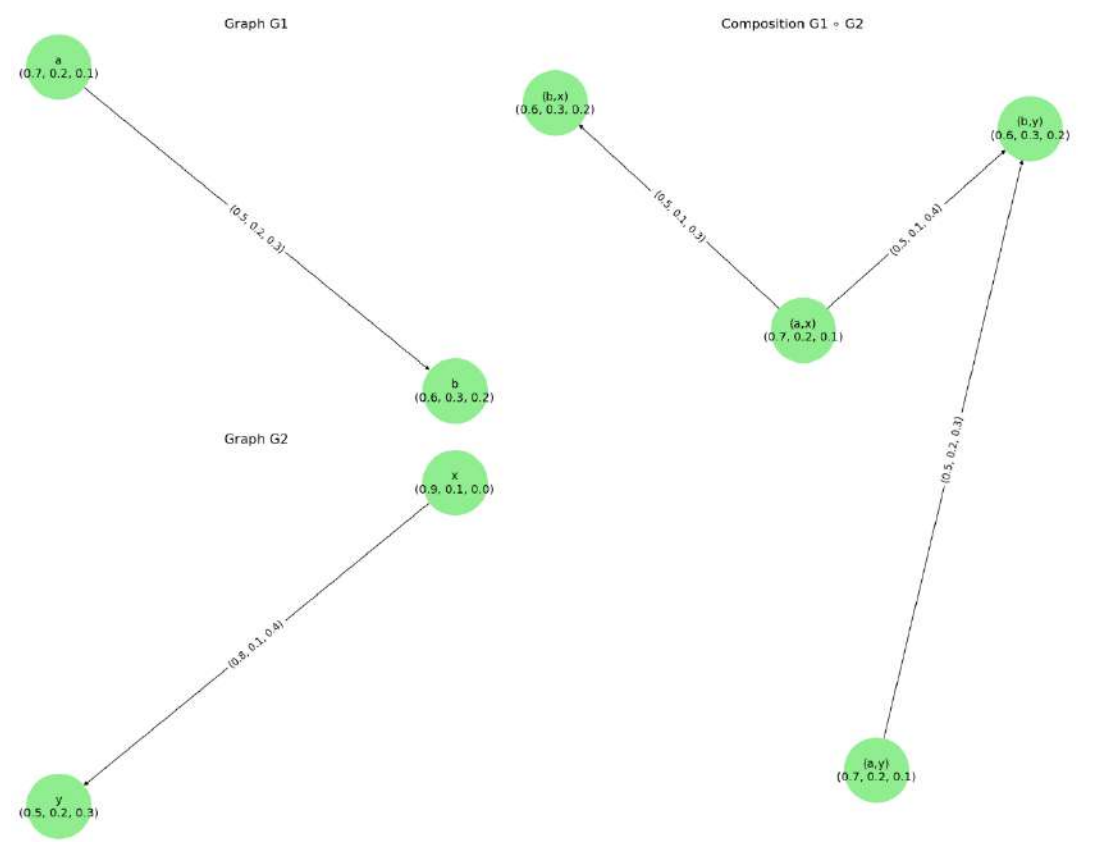

Figure 4. Composition of neutrosophic graphs $\mathfrak{G}_1 \circ \mathfrak{G}_2$ from Example 3.

Definition 8. Let $\mathfrak{G}_1 = \langle \Psi_1, \Omega_1 \rangle$, and $\mathfrak{G}_2 = \langle \Psi_2, \Omega_2 \rangle$ be two neutrosophic graphs, the union of $\mathfrak{G}_1$ and $\mathfrak{G}_2$, denoted as $\mathfrak{G}_1 \cup \mathfrak{G}_2 = \langle \Omega_1 \cup \Omega_2, \Psi_1 \cup \Psi_2 \rangle$ is defined on the vertex set $V = V_1 \times V_2$ as described below:

Vertex Membership Functions

I. If $v \in V_1 \setminus V_2$, then:

$$
\begin{aligned}
T^R_{\Omega_1 \cup \Omega_2}(v) &= T^R_{\sigma\Omega_1}(v) \\
I^N_{\Omega_1 \cup \Omega_2}(v) &= I^N_{\sigma\Omega_1}(v) \\
F^S_{\Omega_1 \cup \Omega_2}(v) &= F^S_{\sigma\Omega_1}(v)
\end{aligned} \tag{18}
$$

II. If $v \in V_2 \setminus V_1$, then:

$$\begin{aligned} T^R_{\sigma_{\Omega_1 \cup \Omega_2}}(v) &= T^R_{\sigma_{\Omega_2}}(v) \\ I^N_{\sigma_{\Omega_1 \cup \Omega_2}}(v) &= I^N_{\sigma_{\Omega_2}}(v) \\ F^S_{\sigma_{\Omega_1 \cup \Omega_2}}(v) &= F^S_{\sigma_{\Omega_2}}(v) \end{aligned} \tag{19}$$

III. If $v \in V_1 \cap V_2$, then:

$$\begin{aligned} T^R_{\sigma_{\Omega_1 \cup \Omega_2}}(v) &= \max\left\{T^R_{\sigma_{\Omega_1}}(v), T^R_{\sigma_{\Omega_2}}(v)\right\} \\ I^N_{\sigma_{\Omega_1 \cup \Omega_2}}(v) &= \max\left\{I^N_{\sigma_{\Omega_1}}(v), I^N_{\sigma_{\Omega_2}}(v)\right\} \\ F^S_{\sigma_{\Omega_1 \cup \Omega_2}}(v) &= \min\left\{F^S_{\sigma_{\Omega_1}}(v), F^S_{\sigma_{\Omega_2}}(v)\right\} \end{aligned} \tag{20}$$

Edge Membership Functions: Let $(v, \omega) \in E_1 \cup E_2$, then,

I. If $(v, \omega) \in E_1 \setminus E_2$, then:

$$\begin{aligned} T^R_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= T^R_{\sigma_{\Omega_1}}(v,\omega) \\ I^N_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= I^N_{\sigma_{\Omega_1}}(v,\omega) \\ F^S_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= F^S_{\sigma_{\Omega_1}}(v,\omega) \end{aligned} \tag{21}$$

II. If $(v, \omega) \in E_2 \setminus E_1$, then:

$$\begin{aligned} T^R_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= T^R_{\sigma_{\Omega_2}}(v,\omega) \\ I^N_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= I^N_{\sigma_{\Omega_2}}(v,\omega) \\ F^S_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= F^S_{\sigma_{\Omega_2}}(v,\omega) \end{aligned} \tag{22}$$

III. If $(v, \omega) \in E_1 \cap E_2$, then:

$$\begin{aligned} T^R_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= \max\left\{T^R_{\sigma_{\Omega_1}}(v,\omega), T^R_{\sigma_{\Omega_2}}(v,\omega)\right\} \\ I^N_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= \max\left\{I^N_{\sigma_{\Omega_1}}(v,\omega), I^N_{\sigma_{\Omega_2}}(v,\omega)\right\} \\ F^S_{\sigma_{\Omega_1 \cup \Omega_2}}(v,\omega) &= \min\left\{F^S_{\sigma_{\Omega_1}}(v,\omega), F^S_{\sigma_{\Omega_2}}(v,\omega)\right\} \end{aligned} \tag{23}$$

Example 4. Given neutrosophic graph $\mathfrak{G}_1$ and $\mathfrak{G}_2$ presented in Figure 5, with vertex set $V_1 = \left\{\underbrace{(0.7,0.2,0.1)}_{a}, \underbrace{(0.6,0.3,0.2)}_{b}\right\}$, and Edge set $E_1 = \underbrace{(0.5,0.2,0.3)}_{(a,b)}$; while neutrosophic graph $\mathfrak{G}_2$ with vertex set $V_2 = \left\{\underbrace{(0.8,0.1,0.4)}_{b}, \underbrace{(0.5,0.2,0.3)}_{c}\right\}$, and Edge set $E_2 = \left\{\underbrace{(0.6,0.1,0.2)}_{(b,c)}\right\}$, their union $\mathfrak{G}_1 \cup \mathfrak{G}_2$ is shown in Figure 5.

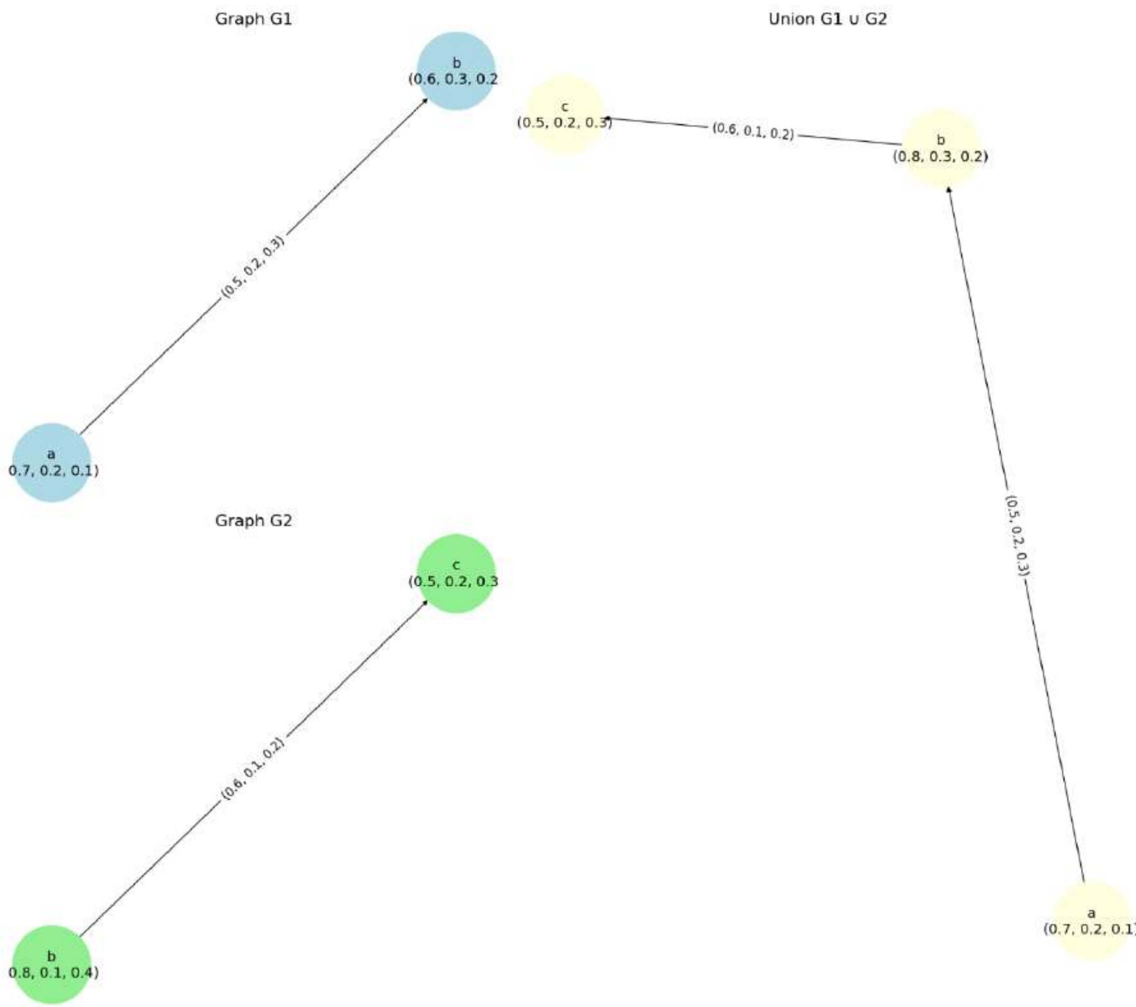

Figure 5. Union of neutrosophic graphs $\mathfrak{G}_1 \cup \mathfrak{G}_2$ from Example 4.

Definition 9 ([10]). Let $\mathfrak{G}_1 = \langle \Psi_1, \Omega_1 \rangle$, and $\mathfrak{G}_2 = \langle \Psi_2, \Omega_2 \rangle$ be two neutrosophic graphs such that their vertex sets are disjoint, i.e., $V_1 \cap V_2 = \emptyset$; Then, the join of $\mathfrak{G}_1$ and $\mathfrak{G}_2$, denoted by $\mathfrak{G}_1 + \mathfrak{G}_2 = \langle \Psi_1 + \Psi_2, \Omega_1 + \Omega_2 \rangle$, and is defined by the following neutrosophic membership functions:

I. Vertex Memberships

For every vertex $v \in V_1 \cup V_2$ The truth, indeterminacy, and falsity values in the join graph are inherited from the union:

$$\begin{aligned} T^R_{\sigma_{\Omega_1+\Omega_2}}(v) &= T^R_{\sigma_{\Omega_1 \cup \Omega_2}}(v) \\ I^N_{\sigma_{\Omega_1+\Omega_2}}(v) &= I^N_{\sigma_{\Omega_1 \cup \Omega_2}}(v) \\ F^S_{\sigma_{\Omega_1+\Omega_2}}(v) &= F^S_{\sigma_{\Omega_1 \cup \Omega_2}}(v) \end{aligned} \tag{24}$$

II. Edge Memberships

Let $(v,\omega) \in E_{1+2}$, the edge set of the join graph. Then:
a) If $(v,\omega) \in E_1 \cup E_2$ (original edges of $\mathfrak{G}_1$ or $\mathfrak{G}_2$):

$$\begin{aligned} T^{R}_{\sigma_{\Omega_1+\Omega_2}}(v,\omega) &= T^{R}_{\sigma_{\Omega_1\cup\,\Omega_2}}(v,\omega) \\ I^{N}_{\sigma_{\Omega_1+\Omega_2}}(v,\omega) &= I^{N}_{\sigma_{\Omega_1\cup\,\Omega_2}}(v,\omega) \\ F^{S}_{\sigma_{\Omega_1+\Omega_2}}(v,\omega) &= F^{S}_{\sigma_{\Omega_1\cup\,\Omega_2}}(v,\omega) \end{aligned} \tag{25}$$

b) If $(v,\omega) \in E$, where $v \in V_1, \omega \in V_2$ (i.e., a newly added edge connecting $\mathfrak{G}_1$ and $\mathfrak{G}_2$), then:

$$\begin{aligned} T^{R}_{\sigma_{\Omega_1+\Omega_2}}(v,\omega) &= \min\left\{T^{R}_{\sigma_{\Omega_1}}(v), T^{R}_{\sigma_{\Omega_2}}(\omega)\right\} \\ I^{N}_{\sigma_{\Omega_1+\Omega_2}}(v,\omega) &= \min\left\{I^{N}_{\sigma_{\Omega_1}}(v), I^{N}_{\sigma_{\Omega_2}}(\omega)\right\} \\ F^{S}_{\sigma_{\Omega_1+\Omega_2}}(v,\omega) &= \max\left\{F^{S}_{\sigma_{\Omega_1}}(v), F^{S}_{\sigma_{\Omega_2}}(\omega)\right\} \end{aligned} \tag{26}$$

Definition 10. Let $\mathfrak{G}_1 = \langle \Psi_1^{\mathcal{M}}, \Omega_1 \rangle$ and $\mathfrak{G}_2 = \langle \Psi_2^{\mathcal{M}}, \Omega_2 \rangle$ be two neutrosophic graphs such that vertex set $V = V_1 \times V_2$. Then, the mapping $\varphi: V_1 \to V_2$ is called a neutrosophic graph homomorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$symbolized as $\varphi: \mathfrak{G}_1 \to \mathfrak{G}_2$, when the following conditions hold:

I. Vertex Conditions

For all $v_1 \in V_1$:

$$\begin{aligned} T^{R}_{\sigma_{\Omega_1}}(v_1) &\le T^{R}_{\sigma_{\Omega_2}}\big(\varphi(v_1)\big), \\ I^{N}_{\sigma_{\Omega_1}}(v_1) &\le I^{N}_{\sigma_{\Omega_2}}\big(\varphi(v_1)\big), \\ F^{S}_{\sigma_{\Omega_1}}(v_1) &\ge F^{S}_{\sigma_{\Omega_2}}\big(\varphi(v_1)\big). \end{aligned} \tag{27}$$

II. Edge Conditions

For all $(v_1, u_1) \in E_1$:

$$\begin{aligned} T^{R}_{\sigma_{\Omega_1}}(v_1,u_1) &\le T^{R}_{\sigma_{\Omega_2}}\big(\varphi(v_1),\varphi(u_1)\big), \\ I^{N}_{\sigma_{\Omega_1}}(v_1,u_1) &\le I^{N}_{\sigma_{\Omega_2}}\big(\varphi(v_1),\varphi(u_1)\big), \\ F^{S}_{\sigma_{\Omega_1}}(v_1,u_1) &\ge F^{S}_{\sigma_{\Omega_2}}\big(\varphi(v_1),\varphi(u_1)\big). \end{aligned} \tag{28}$$

Weak Isomorphism: A bijective homomorphism $\varphi: V_1 \to V_2$ is called a weak isomorphism if:

$$\begin{aligned} T^{R}_{\sigma_{\Omega_1}}(v_1) &= T^{R}_{\Omega_2}\big(\varphi(v_1)\big), \\ I^{N}_{\sigma_{\Omega_1}}(v_1) &= I^{N}_{\Omega_2}\big(\varphi(v_1)\big), \ \text{ for all } v_1 \in V_1. \\ F^{S}_{\sigma_{\Omega_1}}(v_1) &= F^{S}_{\Omega_2}\big(\varphi(v_1)\big), \end{aligned} \tag{29}$$

Strong Co-Isomorphism: the objective homomorphism $\varphi: V_1 \to V_2$ is called a strong co-isomorphism if:

(30)

$$T^R_{\sigma_{\Omega_1}}(v_1,u_1) = T^R_{\sigma_{\Omega_2}}\big(\varphi(v_1),\varphi(u_1)\big)$$
$$I^N_{\sigma_{\Omega_1}}(v_1,u_1) = I^N_{\sigma_{\Omega_2}}\big(\varphi(v_1),\varphi(u_1)\big),\ \text{for all } (v_1,u_1) \in E_1$$
$$F^S_{\sigma_{\Omega_1}}(v_1,u_1) = F^S_{\sigma_{\Omega_2}}\big(\varphi(v_1),\varphi(u_1)\big)$$

Definition 11. Given $\mathfrak{G} = \langle \Psi, \Omega \rangle$ be a weak neutrosophic graph, where $\mathfrak{G}^* = \langle V, E \rangle$ is its underlying crisp graph, and $\overline{\mathfrak{G}} = \langle \overline{\Psi}, \bar{\Omega} \rangle$ be the complement of $\mathfrak{G}$, denoted as a complement of weak neutrosophic graph $\overline{\mathfrak{G}^*}$. Then, the vertex set remains unchanged. For every vertex $v \in V$, the truth, indeterminacy, and falsity values are preserved:

$$\begin{aligned} T^R_{\sigma_{\bar{\Omega}}}(v) &= T^R_{\Omega}(v) \\ I^N_{\sigma_{\bar{\Omega}}}(v) &= I^N_{\Omega}(v) \\ F^S_{\sigma_{\bar{\Omega}}}(v) &= F^S_{\Omega}(v) \end{aligned} \tag{31}$$

Also, for every pair $(v,u) \in V \times V$, the edge values in the complement graph $\bar{\Omega}$ are defined as follows:

$$\begin{aligned} T^R_{\sigma_{\bar{\Omega}}}(v,u) &= \begin{cases} 0, & \text{if } T^R_{\sigma_{\Omega}}(v,u) \neq 0 \\ \min\{T^R_{\sigma_{\Omega}}(v), T^R_{\sigma_{\Omega}}(u)\}, & \text{if } T^R_{\sigma_{\Omega}}(v,u) = 0 \end{cases} \\ I^N_{\sigma_{\bar{\Omega}}}(v,u) &= \begin{cases} 0, & \text{if } I^N_{\sigma_{\Omega}}(v,u) \neq 0 \\ \min\{I^N_{\sigma_{\Omega}}(v), I^N_{\sigma_{\Omega}}(u)\}, & \text{if } I^N_{\sigma_{\Omega}}(v,u) = 0 \end{cases} \\ F^S_{\sigma_{\bar{\Omega}}}(v,u) &= \begin{cases} 0, & \text{if } F^S_{\sigma_{\Omega}}(v,u) \neq 0 \\ \max\{F^S_{\sigma_{\Omega}}(v), F^S_{\sigma_{\Omega}}(u)\}, & \text{if } F^S_{\sigma_{\Omega}}(v,u) = 0 \end{cases} \end{aligned} \tag{32}$$

Definition 12: A neutrosophic graph $\mathfrak{G} = \langle \Psi, \Omega \rangle$ is said to be self-complementary if its complement $\overline{\mathfrak{G}}$ is neutrosophically isomorphic to $\mathfrak{G}$, that is, $\overline{\mathfrak{G}} \approx \mathfrak{G}$. Let V be the vertex set of $\mathfrak{G}$. Then, $\mathfrak{G}$ is self-complementary if and only if the following aggregate conditions hold over all unordered vertex pairs $(v,u) \in V \times V, v \neq u$:

$$\begin{aligned} \sum_{v\neq u} T^R_B(v,u) &= \sum_{v\neq u} \min\{T^R_A(v), T^R_A(u)\} \\ \sum_{v\neq u} I^N_B(v,u) &= \sum_{v\neq u} \min\{I^N_A(v), I^N_A(u)\} \\ \sum_{v\neq u} F^S_B(v,u) &= \sum_{v\neq u} \max\{F^S_A(v), F^S_A(u)\} \end{aligned} \tag{33}$$

These conditions reflect the complement symmetry of neutrosophic edge memberships. Alternatively, a neutrosophic graph $\mathfrak{G} = \langle \Psi, \Omega \rangle$ is self-complementary if for every pair of distinct vertices $v,u \in V$, the following relations hold for the edge $(v,u) \in E$: (34)

$$T^R_B(v,u) = \min\{T^R_A(v), T^R_A(u)\},$$

$$I^N_B(v,u) = \min\{I^N_A(v), I^N_A(u)\},$$

$$F_B^S(v,u) = \max\{F_A^S(v), F_A^S(u)\}$$

That is, each edge's neutrosophic truth, indeterminacy, and falsity values are directly derived from the vertex memberships, mimicking those that would appear in its complement.

3. Topology-Aware Neutrosophic Graph Structures

In this section, we introduce Topology-aware modeling approach to provides a dual-layer representation including semantic uncertainty, as well as structural Validity based on domain-specific topological consteraints. With the inclusion of topological constraints, we enable the graph structure to be sensitive to neighborhoods, connectivity classes, cluster membership, and path-based logic.

Definition 13. Let V denote a finite set of vertices and $E \subseteq V \times V$ a set of edges. A Topology-Aware Neutrosophic Graph, denoted by

$$\mathfrak{G}_T = \langle \Psi, \Omega, \tau \rangle,$$

is a triplet comprising the neutrosophic vertex set

$$\Psi = \left\{ \left\langle T_{\sigma\Psi}^R(v), I_{\sigma\Psi}^N(v), F_{\sigma\Psi}^S(v) \right\rangle : v \in V \right\}$$

satisfying: (35)

$$0 \le T_{\sigma\Psi}^R(v) + I_{\sigma\Psi}^N(v) + F_{\sigma\Psi}^S(v) \le 3$$

neutrosophic edge set

$$\Omega = \left\{ \left\langle T_{\sigma\Omega}^R(v,u), I_{\sigma\Omega}^N(v,u), F_{\sigma\Omega}^S(v,u) \right\rangle : (v,u) \in E' \subseteq E \right\} \quad (36)$$

where each edge (v,u) connects two vertices if and only if the topological constraint function $\tau(v,u) = \text{true}$, and the values of $T, I, F \in [0,1]$ satisfy:

$$T_{\sigma_R}^R(v,u) \le \min\left\{T_{\sigma\Psi}^R(v), T_{\sigma\Psi}^R(u)\right\}, \qquad I_{\sigma_R}^N(v,u) \le \min\left\{I_{\sigma\Psi}^N(v), I_{\sigma\Psi}^N(u)\right\},$$
$$F_{\sigma_R}^S(v,u) \ge \max\left\{F_{\sigma\Psi}^S(v), F_{\sigma\Psi}^S(u)\right\} \quad (37)$$

where $\tau: V \times V \to \{0,1\}$ is the topological constraint function, which determines whether an edge between two vertices v and u is structurally valid.

Definition 14. Topological Constraint Function**.** Let $\boldsymbol{V}$ be the vertex set of a neutrosophic graph. A topological constraint function is a mapping:

$$\tau: V \times V \to \{0,1\} \quad (38)$$

such that $\tau(v,u)=1$ if the pair (v,u) satisfies a specified topological condition, and 0 otherwise.

- **Distance Constraint:** For given metric $d: V\times V\to R^{+}$ and threshold δ,

$$\tau_d(v,u)=\begin{cases}1 & if\ d(v,u)\le\delta\\ 0 & otherwise\end{cases} \tag{39}$$

- **Role Constraint:** Given role mapping $\rho: V\to R$ and allowed relation set $H\subseteq R\times R$,

$$\tau_d(v,u)=\begin{cases}1 & \text{if}(\rho(v),\rho(u))\in H\\ 0 & otherwise\end{cases} \tag{40}$$

Definition 15. Topology-Aware Edge Filter**.** Given $\boldsymbol{\tau}$ and neutrosophic vertex set $\boldsymbol{A^M}$, the edge filter generates the set of permissible edges:

$$E'=\{\,(v,u)\in V\times V \mid v\ne u, \tau(v,u)=1\,\}$$

Each permissible edge (v,u) is assigned neutrosophic values:

$$\begin{aligned} T_{\sigma_R}^{R}(v,u) &= \min\{T_{\sigma_\Psi}^{R}(v), T_{\sigma_\Psi}^{R}(u)\}\\ I_{\sigma_R}^{N}(v,u) &= \min\{I_{\sigma_\Psi}^{N}(v), I_{\sigma_\Psi}^{N}(u)\}\\ F_{\sigma_R}^{S}(v,u) &= \max\{F_{\sigma_\Psi}^{S}(v), F_{\sigma_\Psi}^{S}(u)\}\end{aligned} \tag{41}$$

Theorem 1. Existence of Maximal Topology-Aware Neutrosophic Graph. Given a finite vertex set V and topological constraint function τ, there exists a unique maximal topology-aware neutrosophic graph $\mathfrak{G}_T^*$ containing all permissible edges defined by τ.

In practice, multiple constraints may apply simultaneously. Let $\tau_1,\tau_2,\dots,\tau_k$ be constraint functions. A combined constraint function τ_c is defined by:

$$\tau_c(v,u)=\prod_{i=1}^{k}\tau_i(v,u) \tag{42}$$

Thus, an edge is permitted only if all constraints are satisfied simultaneously.

Example 5: Given a small smart agricultural enterprise network (SAEN) with 3 entities: $v_1 = Farmer, v_2 = Supplier, v_3 = Digital\ Advisor$, where each node has associated neutrosophic membership values:

$$T^{R}_{\sigma\psi}(v_1) = 0.8, \quad I^{N}_{\sigma\psi}(v_1) = 0.1, \quad F^{S}_{\sigma\psi}(v_1) = 0.1$$
$$T^{R}_{\sigma\psi}(v_2) = 0.6, \quad I^{N}_{\sigma\psi}(v_2) = 0.3, \quad F^{S}_{\sigma\psi}(v_2) = 0.1$$
$$T^{R}_{\sigma\psi}(v_3) = 0.7, \quad I^{N}_{\sigma\psi}(v_3) = 0.2, \quad F^{S}_{\sigma\psi}(v_3) = 0.1$$

Let $\tau(v,u) = 1$ if $\text{role}(v) \to \text{role}(u) \in \mathcal{H}$, where $\mathcal{H} = \{$ (Farmer → Supplier), (Farmer → Digital Advisor)$\}$, then, $\tau(v_1,v_2) = 1, \tau(v_1,v_3) = 1$, and $\tau(v_2,v_3) = 0 \to$ edge not allowed.

Edge Construction Using Topological Filter:
For (v_1,v_2) :

$$T^{R}_{\sigma R}(v_1,v_2) = \min(0.8,0.6) = 0.6$$
$$I^{N}_{\sigma R}(v_1,v_2) = \min(0.1,0.3) = 0.1$$
$$F^{S}_{\sigma R}(v_1,v_2) = \max(0.1,0.1) = 0.1$$

For (v_1,v_3) :

$$T^{R}_{\sigma R}(v_1,v_3) = \min(0.8,0.7) = 0.7$$
$$I^{N}_{\sigma R}(v_1,v_3) = \min(0.1,0.2) = 0.1$$
$$F^{S}_{\sigma R}(v_1,v_3) = \max(0.1,0.1) = 0.1$$

For (v_2,v_3) :
Since $\tau(v_2,v_3) = 0$, the edge is filtered out → not included in E', then the resulting topology-aware neutrosophic graph is presented in Figure 6.

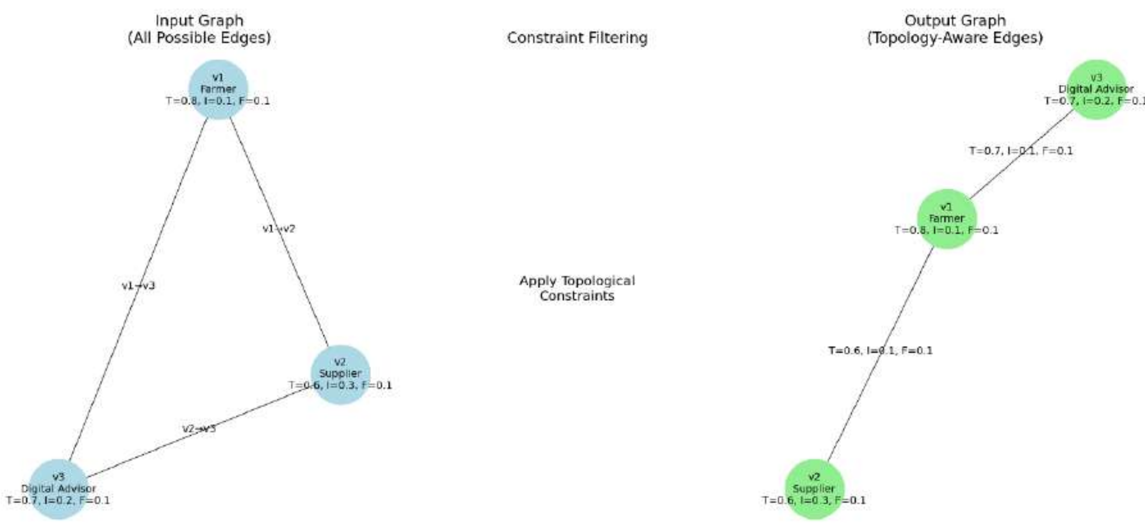

Figure 6. Process of applying topological constraints to a neutrosophic graph

Remark 1 (Flexible Constraint Modeling). One can model domain knowledge in smart agricultural enterprises by defining appropriate constraints reflecting spatial range, functional hierarchy, or dynamic interaction conditions.

In this following, we re-define operations on neutrosophic graphs — union, complement, join, and Cartesian product — by incorporating topological constraints, ensuring each operation respects the structural properties.

Definition 16. The union of two topology-aware neutrosophic graphs is $\mathfrak{G}_1$, and $\mathfrak{G}_2$ defined as, $\mathfrak{G}_1 \cup \mathfrak{G}_2 = \langle \Psi_1 \cup \Psi_2, \Omega_1 \cup \Omega_2, \tau \rangle$, where:

$$\tau(v,u) = \begin{cases} \tau_1(v,u), & v,u \in V_1 \\ \tau_2(v,u), & v,u \in V_2 \\ \text{undefined}, & \text{otherwise} \end{cases} \quad (43)$$

Vertex-level Memberships:
For any $v \in V_1 \cup V_2$:

$$T^R(v) = \begin{cases} T_1^R(v), & v \in V_1 \\ T_2^R(v), & v \in V_2 \end{cases}, I^N(v) = \begin{cases} I_1^N(v), & v \in V_1 \\ I_2^N(v), & v \in V_2 \end{cases}, F^S(v) = \begin{cases} F_1^S(v), & v \in V_1 \\ F_2^S(v), & v \in V_2 \end{cases} \quad (44)$$

Edge-level Memberships:
For any $(v,u) \in E_1 \cup E_2$:

$$T^R(v,u) = \begin{cases} T_1^R(v,u), & (v,u) \in E_1 \\ T_2^R(v,u), & (v,u) \in E_2 \end{cases} I^N(v,u) = \begin{cases} I_1^N(v,u), & (v,u) \in E_1 \\ I_2^N(v,u), & (v,u) \in E_2 \end{cases},$$
$$F^S(v,u) = \begin{cases} F_1^S(v,u), & (v,u) \in E_1 \\ F_2^S(v,u), & (v,u) \in E_2 \end{cases} \quad (45)$$

Definition 17. Topology-Aware Complement. Given $\mathfrak{G} = \langle \Psi, \Omega, \tau \rangle$, its complement $\overline{\mathfrak{G}}$ is:

$$\overline{\mathfrak{G}} = \langle \Omega, \overline{\Omega}, \tau \rangle$$

where for all $v, u \in V, v \neq u$, with $\tau(v,u) = 1$:

$$T_\Omega^R(v,u) = \begin{cases} 0, & T_\Omega^R(v,u) \neq 0 \\ \min\{T^R(v), T^R(u)\}, & T_\Omega^R(v,u) = 0 \end{cases}$$
$$I_\Omega^N(v,u) = \begin{cases} 0, & I_\Omega^N(v,u) \neq 0 \\ \min\{I^N(v), I^N(u)\}, & I_\Omega^N(v,u) = 0 \end{cases} \quad (46)$$
$$F_\Omega^S(v,u) = \begin{cases} 0, & F_\Omega^S(v,u) \neq 0 \\ \max\{F^S(v), F^S(u)\}, & F_\Omega^S(v,u) = 0 \end{cases}$$

Definition 18. Topology-Aware Join. The join operation creates edges between all vertices in V_1 and V_2 that satisfy the topological constraint $\tau(v,u) = 1$:

$$\mathfrak{G}_1 + \mathfrak{G}_2 = \langle \Psi_1 \cup \Psi_2, \Omega_1 \cup \Omega_2 \cup \Omega_{\text{join}}, \tau \rangle$$

Where For $v \in V_1, u \in V_2$ with $\tau(v,u) = 1$:

$$\begin{gathered} T^R(v,u) = \min\{T^R(v), T^R(u)\} \\ I^N(v,u) = \min\{I^N(v), I^N(u)\} \\ F^S(v,u) = \max\{F^S(v), F^S(u)\} \end{gathered} \tag{47}$$

Definition 19. Given two topology-aware neutrosophic graphs is $\mathfrak{G}_1$, and $\mathfrak{G}_2$, then, th Cartesian product graph $\mathfrak{G}_1 \times \mathfrak{G}_2$ with vertex set $V_1 \times V_2$. An edge exists between (v_1, v_2) and (u_1, u_2) if:

- $v_1 = u_1$ and $(v_2, u_2) \in E_2$ with $\tau_2(v_2, u_2) = 1$, or
- $v_2 = u_2$ and $(v_1, u_1) \in E_1$ with $\tau_1(v_1, u_1) = 1$.

In such cases:
If $v_2 = u_2$:

$$\begin{gathered} T^R\big((v_1,v_2),(u_1,v_2)\big) = \min\{T_1^R(v_1,u_1), T_2^R(v_2)\} \\ I^N\big((v_1,v_2),(u_1,v_2)\big) = \min\{I_1^N(v_1,u_1), I_2^N(v_2)\} \\ F^S\big((v_1,v_2),(u_1,v_2)\big) = \max\{F_1^S(v_1,u_1), F_2^S(v_2)\} \end{gathered} \tag{48}$$

If $v_1 = u_1$:

$$\begin{gathered} T^R\big((v_1,v_2),(v_1,u_2)\big) = \min\{T_2^R(v_2,u_2), T_1^R(v_1)\} \\ I^N\big((v_1,v_2),(v_1,u_2)\big) = \min\{I_2^N(v_2,u_2), I_1^N(v_1)\} \\ F^S\big((v_1,v_2),(v_1,u_2)\big) = \max\{F_2^S(v_2,u_2), F_1^S(v_1)\} \end{gathered} \tag{49}$$

Theorem 2 Edge Membership Boundaries under Join. For $v \in V_1, u \in V_2$ with $\tau(v,u) = 1$, the edge membership values in $\mathfrak{G}_1 + \mathfrak{G}_2$ satisfy:

$$\begin{gathered} T^R(v,u) \le \min\{T^R(v), T^R(u)\} \\ I^N(v,u) \le \min\{I^N(v), I^N(u)\} \\ F^S(v,u) \ge \max\{F^S(v), F^S(u)\} \end{gathered} \tag{50}$$

Proof:
By definition of the join operation (4.4.3), for each added edge (v,u) :

$$T^R(v,u) = \min\{T^R(v), T^R(u)\}$$

which implies:

$$T^R(v,u) \le T^R(v), T^R(v,u) \le T^R(u)$$

hence:

$$T^R(v,u) \le \min\{T^R(v), T^R(u)\}$$

$$I^N(v,u) = \min\{I^N(v), I^N(u)\} \le \min\{I^N(v), I^N(u)\}$$

$$F^S(v,u) = \max\{F^S(v), F^S(u)\} \ge \max\{F^S(v), F^S(u)\}$$

Corollary 1 (Complementarity Preservation). If $\mathfrak{G}$ is self-complementary under the topological constraint function τ, then the following equalities hold between the sums of edge membership values and the corresponding minimum or maximum of the vertex membership values:

$$\begin{gathered} \sum_{\substack{v\neq u,\\ \tau(v,u)=1}} T_R(v,u) = \sum_{\substack{v\neq u,\\ \tau(v,u)=1}} min\{T_R(v), T_R(u)\} \\ \sum_{\substack{v\neq u,\\ \tau(v,u)=1}} F_R(v,u) = \sum_{\substack{v\neq u,\\ \tau(v,u)=1}} min\{I_R(v), I_R(u)\} \\ \sum_{\substack{v\neq u,\\ \tau(v,u)=1}} F_R(v,u) = \sum_{\substack{v\neq u,\\ \tau(v,u)=1}} min\{F_R(v), F_R(u)\} \end{gathered} \tag{51}$$

4. Application to Smart Agricultural Enterprises

The agricultural sector has increasingly embraced digital transformation initiatives, driving the emergence of smart agricultural enterprises [11], [12]. These enterprises integrate heterogeneous actors such as farmers, suppliers, processors, distributors, and advisory services through digital platforms that facilitate real-time data exchange, predictive analytics, and collaborative decision-making [13]. To illustrate the practical application of our proposed framework, we construct a case study involving a SAEN. The network consists of twelve nodes, each representing a critical stakeholder or digital component in the agri-enterprise ecosystem, as shown in Table 1.

Table 1. summary of SAEN with 12 nodes categorized into five functional roles.

Role	Nodes	Description
Farmers	F1–F4	Producers managing cultivation and harvesting
Suppliers	S1–S2	Providers of agricultural inputs and services
Advisors	A1–A2	Experts/digital platforms providing guidance
Processors	P1–P2	Facilities converting raw goods into products
Distributors	D1–D2	Logistics agents connecting goods to markets

Indeed, each node is assigned a neutrosophic triplet based on four experts' assessments, as shown in Table 2, which summarizes the node-level statistics.

Table 2. Neutrosophic Vertex Membership Values from Four Experts

Node	Role	Expert 1	Expert 2	Expert 3	Expert 4
F1	Farmer	(0.75, 0.20, 0.10)	(0.72, 0.22, 0.12)	(0.78, 0.18, 0.14)	(0.74, 0.21, 0.11)
F2	Farmer	(0.60, 0.25, 0.15)	(0.58, 0.28, 0.14)	(0.62, 0.24, 0.16)	(0.61, 0.26, 0.15)
F3	Farmer	(0.70, 0.15, 0.20)	(0.72, 0.14, 0.18)	(0.69, 0.16, 0.19)	(0.71, 0.15, 0.20)
F4	Farmer	(0.65, 0.30, 0.10)	(0.66, 0.28, 0.12)	(0.67, 0.29, 0.11)	(0.64, 0.31, 0.13)
S1	Supplier	(0.85, 0.10, 0.05)	(0.82, 0.12, 0.06)	(0.86, 0.09, 0.05)	(0.84, 0.11, 0.05)
S2	Supplier	(0.80, 0.15, 0.10)	(0.78, 0.16, 0.09)	(0.79, 0.14, 0.11)	(0.81, 0.13, 0.10)
A1	Advisor	(0.70, 0.20, 0.15)	(0.68, 0.21, 0.14)	(0.71, 0.19, 0.16)	(0.72, 0.18, 0.13)
A2	Advisor	(0.75, 0.15, 0.10)	(0.73, 0.17, 0.11)	(0.74, 0.16, 0.12)	(0.76, 0.14, 0.11)
P1	Processor	(0.80, 0.10, 0.10)	(0.79, 0.11, 0.09)	(0.82, 0.09, 0.10)	(0.81, 0.10, 0.08)
P2	Processor	(0.78, 0.12, 0.15)	(0.76, 0.13, 0.14)	(0.77, 0.11, 0.16)	(0.79, 0.12, 0.13)
D1	Distributor	(0.65, 0.25, 0.15)	(0.66, 0.24, 0.16)	(0.64, 0.26, 0.15)	(0.65, 0.25, 0.14)
D2	Distributor	(0.68, 0.22, 0.20)	(0.69, 0.20, 0.19)	(0.67, 0.23, 0.21)	(0.68, 0.22, 0.20)

The connections of these nodes are governed by structural rules grounded in domain knowledge. First, role-based constraints, where Farmers may directly connect to suppliers, advisors, and processors. On the other hand, Distributors may connect only with processors, while advisors connect primarily to farmers and suppliers. Second, spatial clusters, which is summarized in Table 3.

Table 3. Cluster Membership

Cluster	Nodes
Cluster A	F1, F2, S1, A1, P1
Cluster B	F3, F4, S2, A2, P2
Cross-Cluster	D1, D2

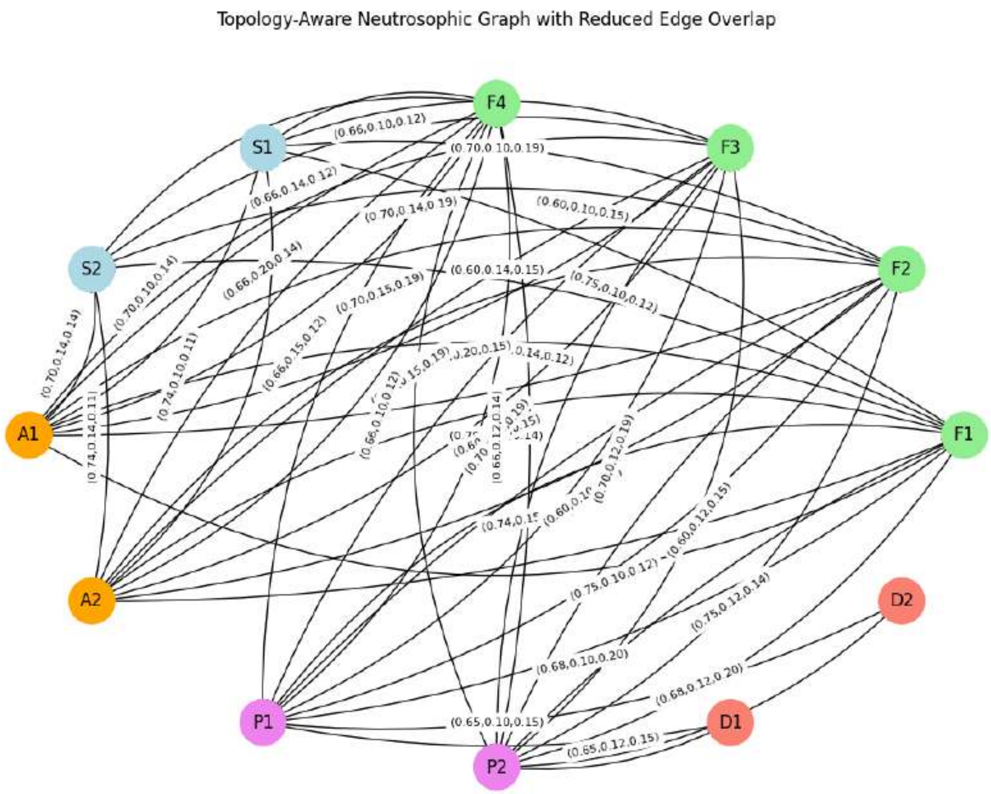

Figure 7. Topology-aware neutrosophic graph of the SAEN.

In addition, hierarchy constrained imposed that edges must respect the flow from upstream (producers/suppliers) to downstream (processors/distributors). Figure 7 illustrates the topology-aware neutrosophic graph of the SAEN. Nodes represent diverse roles, positioned evenly on a hendecagon to improve clarity. Directed edges, annotated with neutrosophic triplets, indicate role-constrained interactions whose uncertainty characteristics were derived from expert assessments. Table 4 show all edges in the SAEN, showing the source and destination nodes along with their associated neutrosophic triplets.

Table 4. Topology-Aware Neutrosophic Edge Information

Edge	Neutrosophic Triplet (T, I, F)	Edge	Neutrosophic Triplet (T, I, F)
(F1, S1)	(0.75, 0.10, 0.12)	(A1, F1)	(0.70, 0.20, 0.14)
(F1, S2)	(0.75, 0.14, 0.12)	(A1, F2)	(0.60, 0.20, 0.15)
(F1, A1)	(0.70, 0.20, 0.14)	(A1, F3)	(0.70, 0.15, 0.19)
(F1, A2)	(0.74, 0.15, 0.12)	(A1, F4)	(0.66, 0.20, 0.14)
(F1, P1)	(0.75, 0.10, 0.12)	(A1, S1)	(0.70, 0.10, 0.14)
(F1, P2)	(0.75, 0.12, 0.14)	(A1, S2)	(0.70, 0.14, 0.14)
(F2, S1)	(0.60, 0.10, 0.15)	(A2, F1)	(0.74, 0.15, 0.12)
(F2, S2)	(0.60, 0.14, 0.15)	(A2, F2)	(0.60, 0.15, 0.15)
(F2, A1)	(0.60, 0.20, 0.15)	(A2, F3)	(0.70, 0.15, 0.19)
(F2, A2)	(0.60, 0.15, 0.15)	(A2, F4)	(0.66, 0.15, 0.12)

(F2, P1)	(0.60, 0.10, 0.15)	(A2, S1)	(0.74, 0.10, 0.11)
(F2, P2)	(0.60, 0.12, 0.15)	(A2, S2)	(0.74, 0.14, 0.11)
(F3, S1)	(0.70, 0.10, 0.19)	(P1, F1)	(0.75, 0.10, 0.12)
(F3, S2)	(0.70, 0.14, 0.19)	(P1, F2)	(0.60, 0.10, 0.15)
(F3, A1)	(0.70, 0.15, 0.19)	(P1, F3)	(0.70, 0.10, 0.19)
(F3, A2)	(0.70, 0.15, 0.19)	(P1, F4)	(0.66, 0.10, 0.12)
(F3, P1)	(0.70, 0.10, 0.19)	(P1, D1)	(0.65, 0.10, 0.15)
(F3, P2)	(0.70, 0.12, 0.19)	(P1, D2)	(0.68, 0.10, 0.20)
(F4, S1)	(0.66, 0.10, 0.12)	(P2, F1)	(0.75, 0.12, 0.14)
(F4, S2)	(0.66, 0.14, 0.12)	(P2, F2)	(0.60, 0.12, 0.15)
(F4, A1)	(0.66, 0.20, 0.14)	(P2, F3)	(0.70, 0.12, 0.19)
(F4, A2)	(0.66, 0.15, 0.12)	(P2, F4)	(0.66, 0.12, 0.14)
(F4, P1)	(0.66, 0.10, 0.12)	(P2, D1)	(0.65, 0.12, 0.15)
(F4, P2)	(0.66, 0.12, 0.14)	(P2, D2)	(0.68, 0.12, 0.20)

Based on the above topology-aware neutrosophic graph structure, we recomputed the neutrosophic degrees of nodes based on both incoming and outgoing edges, as shown in Table 5.

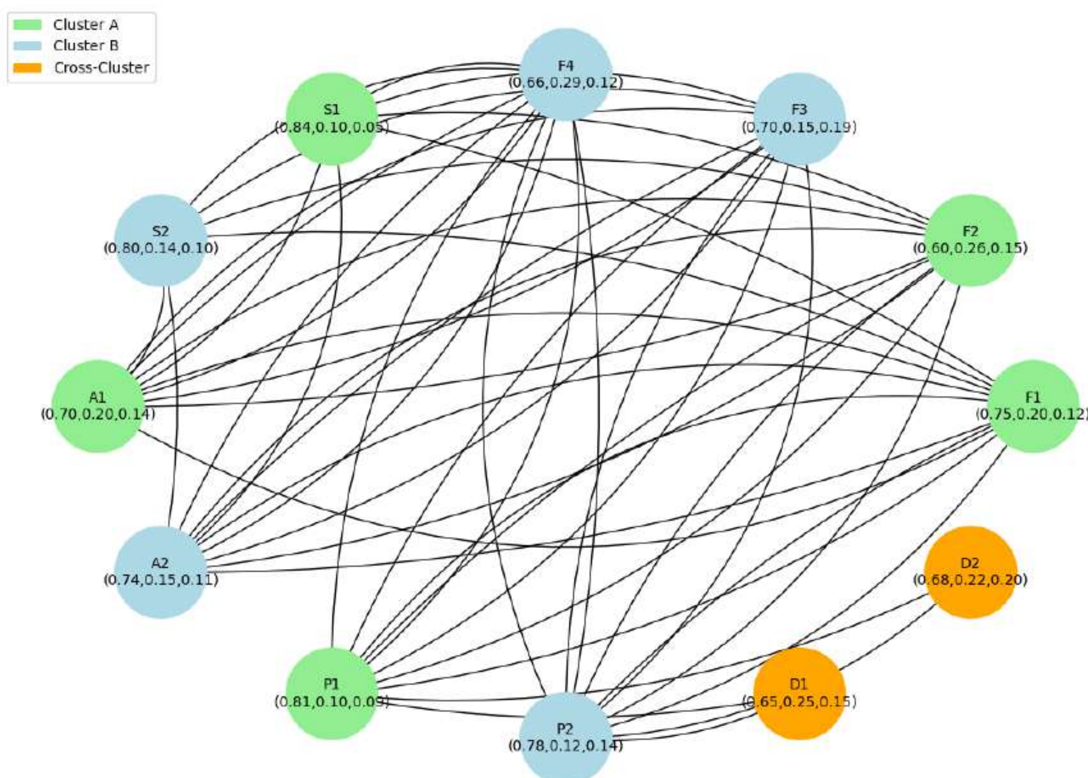

Figure 8. Visualization of cluster analysis of the SAEN.

The cluster analysis and visualization presented in Figure 8 highlight the structural and uncertainty-based characteristics of the SAEN. $Cluster\ A$, consisting primarily of upstream production and input nodes (farmers, suppliers, and processors), demonstrates relatively higher average truth degrees and lower indeterminacy, indicating stronger and more reliable interactions within tightly coupled local ecosystems. $Cluster\ B$, which mirrors structure of $Cluster\ A$ in a separate spatial region, exhibits similar patterns but with slight variations in uncertainty measures reflecting regional operational differences. As shown, cross-cluster nodes (D1 and D2) connect both clusters and show moderately elevated indeterminacy and falsity levels, consistent with the inherent risks and variability associated with downstream distribution and logistics.

Table 5. Topology-aware neutrosophic degrees of each node.

	Node	**Role**	$\sum_{\grave{v}\in\Omega} T^R\sigma_\Omega(\grave{v},\grave{v})$,	$\sum_{\grave{v}\in\Omega} I^N\sigma_\Omega(\grave{v},\grave{v})$	$\sum_{\grave{v}\in\Omega} F^S\sigma_\Omega(\grave{v},\grave{v})$
0	F1	Farmer	7.3800	1.39	1.285
1	F2	Farmer	6.0250	1.39	1.500
2	F3	Farmer	7.0450	1.29	1.925
3	F4	Farmer	6.5500	1.39	1.270
4	S1	Supplier	4.1575	0.63	0.830
5	S2	Supplier	4.1575	0.87	0.830
6	A1	Advisor	6.7300	1.72	1.555
7	A2	Advisor	6.9050	1.48	1.370
8	P1	Processor	6.7500	1.00	1.500
9	P2	Processor	6.7500	1.20	1.615
10	D1	Distributor	1.3000	0.22	0.300
11	D2	Distributor	1.3600	0.22	0.400

The results in Table 5 reveal that farming nodes representing farmer generally exhibit high truth degrees ($\sum_{\grave{v}\in\Omega} T^R\sigma_\Omega(\grave{v},\grave{v})$), which indicate strong and reliable connectivity within the SAEN. This aligns with their central role in initiating interactions with other stakeholders. Conversely, supplier and distributor nodes show lower indeterminacy ($\sum_{\grave{v}\in\Omega} I^S\sigma_\Omega(\grave{v},\grave{v})$) and falsity degrees ($\sum_{\grave{v}\in\Omega} F^S\sigma_\Omega(\grave{v},\grave{v})$), which indicate clearer and more definitive interactions, possibly due to more structured and contrectual relationships. The low falsity values across various roles reflect a well-integrated enterprise topology, with advisors and processors acting as bridges that reduce uncertainty across domains. The findings demonstrate the utility of TANGS in capturing both the structural and uncertain dynamics of systems.

5. Conclusion

In this article, we introduced a novel TANGS framework for modeling complex and uncertain interactions in SAEN. This framework creatively integrate domain-specific topological conditions—such as role-based, spatial, and hierarchical relationships— into neutrosophic uncertainty handling power to build a unified approach for providing more realistic and representation of enterprise dynamics. We also present a detailed case study to conduct proof of concept analysis, which demonstrated the effectiveness of topology-aware modeling for improving clarity, filters implausible connections, which indeed contribute informed decision-making. This work lays a foundation for future research on scalable, adaptive, and intelligent graph-based systems in precision agriculture and the broader digital economy.

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