Plithogenic Quasigroups with Application to Decision Making

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Abstract: In this paper, we extend the concept of neutrosophic sets by introducing plithogenic quasigroups by associating a quasigroup $(Q, \cdot)$ with a plithogenic structure: attributes V, dominant vD, contradiction degrees ci ∈ [0, 1], and neutrosophic degrees d(x, vi). The plithogenic product ·P aggregates componentwise via (1 − ci)tN + ci sN with tN = min, sN = max. We prove that (P(Q), ·P) is a quasigroup and its parastrophes; the Latin square property lifts, guaranteeing unique solvability of a ·P x = b. Solvability holds iff each component bi lies in the ci-weighted interval between ai and xi: ci = 1 forces bi ≥ ai while ci = 0 forces bi ≤ ai. We characterize isotopies, homomorphisms, and nuclei, showing 0 < ci < 1 yields trivial nuclei and contradicts associativity. We applied plithogenic quasigroup properties to portfolio selection with V = {Risk, Return, Liquidity}, setting cRisk = 0.9 blocks buy signals whereas cRisk = 0.1 enables solvability, where ci functions as a parameter connecting algebraic feasibility to risk policy in decision making.

Keywords: Plithogenic set, quasigroup, neutrosophic set, contradiction degree, non-associative algebra, Latin square.

1. Introduction

Quasigroups are algebraic structures that generalize groups by dropping the associativity requirement. The study of quasigroups and loops has a history spanning more than two centuries. Its origins can be traced to Euler's 1782 conjecture on the non-existence of certain pairs of mutually orthogonal Latin squares.

Algebraic structures that model uncertainty and contradiction have become central to decision science, particularly in multi-criteria problems where attributes conflict. Neutrosophic sets, introduced by Smarandache [29], capture truth, indeterminacy, and falsity degrees, while plithogenic sets [28] refine this by assigning contradiction degrees c(vi, vD) ∈ [0, 1] between each attribute vi and a dominant attribute vD. These degrees govern how strongly an attribute opposes the dominant one during aggregation.

While plithogenic aggregation has been studied for set-theoretic and matrix operations, its extension to non-associative algebraic systems remains unexplored. Quasigroups are precisely the non-associative generalization of groups: they guarantee unique solvability of $a \cdot x = b$ and $y \cdot a = b$ but need not satisfy $(xy)z = x(yz)$. This property makes them ideal for modeling operations where the order of application matters — for example, sequential portfolio adjustments, layered encryption, or step-wise medical treatments.

We introduce plithogenic quasigroups $(P(Q), \cdot_P)$ by equipping a base quasigroup $(Q, \cdot)$ with an attribute set V, contradiction degrees c_i, and componentwise aggregation using neutrosophic t-norm $t_N = \min$ and s-norm $s_N = \max$. The plithogenic product is defined as

$$d(x \cdot_P y, v_i) = (1 - c_i) \min\{d(x, v_i), d(y, v_i)\} + c_i \max\{d(x, v_i), d(y, v_i)\},$$

so that $c_i = 0$ yields pure consensus while $c_i = 1$ yields pure dominance of the higher degree.

The theory of neutrosophic soft sets has developed significantly, studied by many authors since it was introduced by Maji [16]. Broumi and Smarandache studied its interval-valued and intuitionistic [10] neutrosophic soft sets. Oyem et al. [23] introduced the Neutrosophic quasigroup, which extends neutrosophic sets' properties to non-associative algebraic structures known as quasigroups. Its algebraic properties have been studied by Broumi et al. [11, 12], who provided new operations and examined distributivity and associativity. Its applications to incomplete data were considered by Saha et al. [25].

Also, fuzzy soft sets and their decision-making uses laid groundwork for algebraic treatments in soft and fuzzy settings. Recent developments include Soft Neutrosophic versions of algebraic structures [4, 5, 9, 26, 6] and Neutrosophic Soft versions [17, 7, 32, 1], indicating that soft set theory offers a natural generalization of neutrosophic sets to algebra.

Oyem et al. [20, 21, 22, 24] introduced the notion of order for finite soft quasigroups and applied it to egalitarianism. They established several algebraic relationships between the order of a quasigroup and the order of its associated soft quasigroups, and, in other related work, investigated various algebraic properties of soft quasigroups and soft parastrophes, showing that certain operations are not distributive over one another.

2. Preliminaries

This section provides a brief review of the basic concepts and known results on quasigroups and soft sets.

2.1 Quasigroups and Parastrophy

Definition 2.1 (Quasigroup). Let Q be a nonempty set and let (·) be a binary operation on Q. The pair (Q, ·) is called a groupoid or magma.

If for all a, b ∈ Q the equations a · x = b and y · a = b have unique solutions x, y ∈ Q, then (Q, ·) is a quasigroup.

A quasigroup (Q, ·) for which there exists a unique element e ∈ Q such that x · e = e · x = x for all x ∈ Q is called a loop. The element e is the identity element of the loop.

Throughout this paper we write xy for x · y, and we adopt the convention that · has lower precedence than juxtaposition. For example, x · yz means x(yz).

Definition 2.2. For a fixed element x ∈ Q, the left translation Lx and right translation Rx are the maps Q → Q defined by yLx = x · y, yRx = y · x.

A groupoid is a quasigroup if and only if Lx and Rx are permutations of Q for all x ∈ Q. Since Lx and Rx are bijective, their inverses exist. We define the left and right divisions by $x \backslash y = yLx^{-1}$, $x/y = xRy^{-1}$, so that $x \backslash y = z \Leftrightarrow x \cdot z = y$, and $x/y = z \Leftrightarrow z \cdot y = x$.

In the language of universal algebra, a quasigroup can be presented as an algebra (Q, ·, \, /) of type (2, 2, 2). For more on quasigroups, readers can check [3, 30].

Definition 2.3 (Subquasigroup) (Pflugfelder [31]). Let (Q, ·) be a groupoid, and let H ⊆ Q with H ≠ ∅. If (H, ·) is itself a groupoid under the same operation, then H is called a subgroupoid of Q.

If (Q, ·) is a quasigroup and (H, ·) is a quasigroup, then H is called a subquasigroup of Q. In either case we write H ≤ Q.

Definition 2.4 (Plithogenic Set [28, 27, 2, 15, 29]). Let U be a universe of discourse and let P ⊆ U. A plithogenic set P is defined as the quintuple P = (P, V, vD, d, c) where:

1. P is the set of elements;

2. V = {v1, v2, …, vm} is a set of attributes characterizing the elements of P;

3. vD ∈ V is the dominant attribute, used as a reference for measuring contradiction;

4. d : P × V → $[0, 1]^3$ is the degree of appurtenance function, such that for each x ∈ P and vi ∈ V, d(x, vi) = (ti(x), ii(x), fi(x)) denotes the neutrosophic degrees of truth, indeterminacy, and falsity of element x with respect to attribute vi;

5. $c : V \times V \to [0, 1]$ is the contradiction degree function, where $c(v_i, v_j)$ measures the degree to which attribute v_i contradicts attribute v_j. In particular, we set $c_i = c(v_i, v_D)$ as the contradiction degree of v_i relative to the dominant attribute v_D, with $c(v_D, v_D) = 0$.

For any $x, y \in P$, the plithogenic aggregation with respect to attribute v_i is given by

$$d_{agg}(x, y, v_i) = (1 - c_i)\, t_N\{d(x, v_i), d(y, v_i)\} + c_i\, s_N\{d(x, v_i), d(y, v_i)\},$$

where t_N and s_N are a neutrosophic t-norm and s-norm, respectively.

3. Plithogenic Quasigroups

Definition 3.1 (Plithogenic Quasigroup). Let $(Q, \cdot)$ be a quasigroup, $V = \{v_1, \ldots, v_m\}$ an attribute set with dominant attribute $v_D \in V$, and $c_i = c(v_i, v_D) \in [0, 1]$ the contradiction degree of v_i relative to v_D. Choose neutrosophic t-norm $t_N = \min$ and s-norm $s_N = \max$. For each $x \in Q$, let $d(x, v_i) = (t_i, i_i, f_i) \in [0, 1]^3$ denote the neutrosophic degrees of truth, indeterminacy, and falsity for attribute v_i.

A plithogenic quasigroup over Q is then the structure $P(Q) = (Q, V, v_D, d, c, \cdot_P)$, where the plithogenic product $(\cdot_P)$ is defined for $x, y \in P(Q)$ by:

1. Base: $x \cdot_P y = x \cdot y$ in $(Q, \cdot)$;

2. Degrees: for each $v_i \in V$, $d(x \cdot_P y, v_i) = (1 - c_i)\, t_N\{d(x, v_i), d(y, v_i)\} + c_i\, s_N\{d(x, v_i), d(y, v_i)\}$.

Theorem 3.1 (Closure of Latin Square Property). If $(Q, \cdot)$ is a quasigroup, then $(P(Q), \cdot_P)$ is a quasigroup. That is, for any $a, b \in P(Q)$, the equations $a \cdot_P x = b$ and $y \cdot_P a = b$ have unique solutions $x, y \in P(Q)$ whenever they are solvable.

Proof. Since $(Q, \cdot)$ is a quasigroup, there exists a unique $x_0 \in Q$ such that $a \cdot x_0 = b$ in the base. Construct $x \in P(Q)$ with base element x_0.

For each $v_i \in V$, let $a_i = d(a, v_i)$, $b_i = d(b, v_i)$, and set $x_i = d(x, v_i)$. By definition of $\cdot_P$, $b_i = (1 - c_i)\, t_N\{a_i, x_i\} + c_i\, s_N\{a_i, x_i\}$. For $t_N = \min$, $s_N = \max$, this equation has a unique solution x_i provided it exists. The argument for $y \cdot_P a = b$ is symmetric. Hence $(P(Q), \cdot_P)$ satisfies the Latin square property whenever solutions exist. □

Corollary 3.1 (Solvability Condition). Let $a, b \in P(Q)$ with $a_i = d(a, v_i)$ and $b_i = d(b, v_i)$ for $v_i \in V$. Equation $a \cdot_P x = b$ is solvable in $P(Q)$ if and only if for every $v_i \in V$ and each component $* \in \{t, i, f\}$,

$$x^*_i = (b^*_i - (1 - c_i)a^*_i) / c_i \quad \in [0,1], \quad \text{when } b^*_i \geq a^*_i;$$

$$x^*_i = (b^*_i - c_i a^*_i) / (1 - c_i) \quad \in [0,1], \quad \text{when } b^*_i < a^*_i;$$

with the convention that 0/0 is interpreted as any value in [0, 1] when $c_i = 0$ or $c_i = 1$. If any component $x^*_i \notin [0, 1]$, the equation has no solution in P(Q).

Example 3.1. Let Q = {e, a, b} with operation given by its Latin square:

·	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Take attributes V = {Growth, Safety} with v_D = Growth, $c_{Growth} = 0$, $c_{Safety} = 0.4$. Define elements:

x1 = [e] : G(0.90, 0.10, 0.00), S(0.80, 0.10, 0.10),

x2 = [a] : G(0.60, 0.30, 0.10), S(0.90, 0.10, 0.00),

x3 = [b] : G(0.30, 0.40, 0.30), S(0.50, 0.30, 0.20).

The plithogenic product yields the following 3×3 Latin square (Table 1):

Table 1: Plithogenic 3×3 Latin square for (P(Q), ·P)

·P	x1	x2	x3
x1	e: G(0.90,0.10,0.00) S(0.80,0.10,0.10)	a: G(0.60,0.30,0.10) S(0.86,0.10,0.04)*	b: G(0.30,0.40,0.30) S(0.62,0.22,0.16)
x2	a: G(0.60,0.30,0.10) S(0.86,0.10,0.04)	b: G(0.30,0.40,0.30) S(0.62,0.22,0.16)	e: G(0.90,0.10,0.00) S(0.80,0.10,0.10)
x3	b: G(0.30,0.40,0.30) S(0.62,0.22,0.16)	e: G(0.90,0.10,0.00) S(0.80,0.10,0.10)	a: G(0.60,0.30,0.10) S(0.86,0.10,0.04)

Tables 2 and 3 verify uniqueness of solutions for right and left division:

Table 2: Right division test: x1 ·P x = x2

Test x	Base	Growth	Safety	= x2
x1	e	(0.90, 0.10, 0.00)	(0.80, 0.10, 0.10)	No

Test x	Base	Growth	Safety	= x2
x2	a	(0.60, 0.30, 0.10)	(0.86, 0.10, 0.04)	Yes
x3	b	(0.30, 0.40, 0.30)	(0.62, 0.22, 0.16)	No

Table 3: Left division test: y ·P x1 = x2

Test y	Base	Growth	Safety	= x2
y = x1	e	(0.90, 0.10, 0.00)	(0.80, 0.10, 0.10)	No
y = x2	a	(0.60, 0.30, 0.10)	(0.86, 0.10, 0.04)	Yes
y = x3	b	(0.30, 0.40, 0.30)	(0.62, 0.22, 0.16)	No

Remark 3.1 (Contradiction Degree as Risk Dial)

ci determines how we can combine the attribute vi. If ci = 0: strict AND, both inputs must be high. If ci = 1: strict OR, the higher input prevails. If 0 < ci < 1: smooth interpolation between the two. A larger ci also reduces solvability if bi < ai by giving weak inputs veto power, making ci a direct measure of risk tolerance.

Corollary 3.1 shows that ci also governs solvability: large ci restricts solutions when bi < ai, acting as a veto. Thus ci adjusts how risk preferences enter the algebraic structure.

Definition 3.2 (Plithogenic Quasigroup Homomorphism). Let P(Q1) = (Q1, V, vD, d1, c, ·P1) and P(Q2) = (Q2, V, vD, d2, c, ·P2) be two plithogenic quasigroups over base quasigroups (Q1, ·1) and (Q2, ·2) with the same attribute set V, dominant attribute vD, and contradiction degrees c.

A map φ : P(Q1) → P(Q2) is a plithogenic quasigroup homomorphism if:

1. Base homomorphism: for all x, y ∈ P(Q1), φ(x ·P1 y) = φ(x) ·P2 φ(y) on the base elements. That is, the restriction φ0 : Q1 → Q2 given by φ0(x) = φ(x) is a quasigroup homomorphism.

2. Degree preservation: for all x ∈ P(Q1) and all vi ∈ V, d2(φ(x), vi) = d1(x, vi). That is, φ preserves the neutrosophic degrees (t, i, f) for every attribute.

If φ0 is bijective, we call φ a plithogenic quasigroup isomorphism, and write P(Q1) ≅ P(Q2).

Corollary 3.2. If $\varphi : P(Q1) \to P(Q2)$ is a plithogenic quasigroup homomorphism, then for any a, b ∈ P(Q1), the unique solution x to a ·P1 x = b is mapped to the unique solution to φ(a) ·P2 y = φ(b), namely y = φ(x).

Definition 3.3 (Kernel and Image of a Plithogenic Quasigroup Homomorphism). Let $\varphi : P(Q1) \to P(Q2)$ be a plithogenic quasigroup homomorphism as in Definition 3.2, and let $\varphi 0 : Q1 \to Q2$ be the induced base homomorphism.

1. The kernel of φ is the substructure of P(Q1) defined by ker(φ) = {x ∈ P(Q1) : φ(x) = e2}, where e2 ∈ P(Q2) is a fixed element whose base is the identity of (Q2, ·2) and d2(e2, vi) = (1, 0, 0) for all vi ∈ V. If (Q2, ·2) has no identity, fix any distinguished element e2 ∈ P(Q2).

2. The image of φ is the substructure of P(Q2) given by Im(φ) = {φ(x) ∈ P(Q2) : x ∈ P(Q1)}. Then Im(φ) is a plithogenic subquasigroup of P(Q2).

Proposition 3.1 (First Isomorphism Theorem for Plithogenic Quasigroups). Let $\varphi : P(Q1) \to P(Q2)$ be a surjective plithogenic quasigroup homomorphism. Then there exists a plithogenic quasigroup congruence ∼φ on P(Q1) defined by $x \sim\varphi\, y \Leftrightarrow \varphi(x) = \varphi(y)$, and we have a plithogenic quasigroup isomorphism $P(Q1)/\sim\varphi \cong P(Q2)$. Moreover, x ∼φ y if and only if $x \cdot P1\, y^{-1} \in \ker(\varphi)$ when (Q1, ·1) is a loop, and d1(x, vi) = d1(y, vi) for all vi ∈ V.

Remark 3.2. Unlike groups, ker(φ) in a quasigroup is not automatically a subquasigroup, because quasigroups need not have identities or inverses. However, ker(φ) is a normal subquasigroup in the sense of quasigroup theory if (Q1, ·1) is a loop. The degree preservation condition ensures that ker(φ) consists of elements that are neutrosophically indistinguishable from the distinguished element e2.

Example 3.2 (Plithogenic Quasigroup Homomorphism). Let P(Q1) be the plithogenic quasigroup from Example 3.1, with base Q1 = {e, a, b} and elements x1, x2, x3 as defined there.

Define a second base quasigroup (Q2, ◦) by Q2 = {ē, ā, b̄} with an isomorphic Cayley table, via φ0(e) = ē, φ0(a) = ā, φ0(b) = b̄.

Construct P(Q2) using the same attributes V = {Growth, Safety} and contradiction degrees cGrowth = 0, cSafety = 0.4. Define elements y1 = [ē], y2 = [ā], y3 = [b̄] with the same degree triples as x1, x2, x3, respectively.

Define $\varphi : P(Q1) \to P(Q2)$ by φ(xi) = yi for i = 1, 2, 3. Then φ is a plithogenic quasigroup homomorphism because:

1. Base homomorphism: φ_0 is an isomorphism of quasigroups, so $\varphi(x_i \cdot_{P1} x_j) = \varphi(x_i) \cdot_{P2} \varphi(x_j)$ for all i, j. For instance, $\varphi(x_1 \cdot_{P1} x_2) = \varphi(x_2) = y_2 = y_1 \cdot_{P2} y_2 = \varphi(x_1) \cdot_{P2} \varphi(x_2)$.

2. Degree preservation: by construction, $d_1(x_i, v) = d_2(y_i, v)$ for all i and $v \in V$. For example, $d_1(x_1, \text{Growth}) = (0.90, 0.10, 0.00) = d_2(y_1, \text{Growth})$.

Hence $P(Q_1) \cong P(Q_2)$ as plithogenic quasigroups. Moreover, $\ker(\varphi) = \{x_1\}$, because $\varphi(x_1) = y_1 = e_2$, and $\mathrm{Im}(\varphi) = P(Q_2)$.

Example 3.3 (Non-isomorphic Plithogenic Homomorphism). Let $P(Q_1)$ be as in Example 3.1 with x_1, x_2, x_3. Consider a base quasigroup $(Q_2, *)$ with $Q_2 = \{ê, â\}$ and its 2×2 Cayley table, and construct $P(Q_2)$ with the same V = {Growth, Safety}, $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$.

Suppose we try to define a base homomorphism $\varphi_0 : Q_1 \to Q_2$ by $\varphi_0(e) = ê$, $\varphi_0(a) = â$, $\varphi_0(b) = â$, so that a and b collapse to the same base element. For φ to lift to a plithogenic quasigroup homomorphism, we must have $d_1(x, v_i) = d_2(\varphi(x), v_i)$ for all x and v_i.

But in $P(Q_1)$ we have $d_1(x_2, \text{Growth}) = (0.60, 0.30, 0.10)$ and $d_1(x_3, \text{Growth}) = (0.30, 0.40, 0.30)$. Since $\varphi(x_2) = \varphi(x_3) = z$ for some $z \in P(Q_2)$ with base â, degree preservation forces $d_2(z, \text{Growth}) = (0.60, 0.30, 0.10) = (0.30, 0.40, 0.30)$, which is impossible. Therefore no plithogenic homomorphism exists that collapses x_2 and x_3.

If instead we try to collapse x_1 and x_3 by setting $\varphi_0(e) = \varphi_0(b) = ê$, we need $d_1(x_1, \text{Safety}) = (0.80, 0.10, 0.10) = d_1(x_3, \text{Safety}) = (0.50, 0.30, 0.20)$, again impossible.

Hence the only plithogenic homomorphisms from $P(Q_1)$ are injective on elements with distinct degrees. In this example, any φ must send x_1, x_2, x_3 to three distinct elements of $P(Q_2)$ with matching degrees. If $P(Q_2)$ has only two elements, no plithogenic homomorphism exists.

Remark 3.3 (Degrees Restrict Homomorphisms). In classical quasigroups, any surjective base homomorphism works. For plithogenic quasigroups, degree preservation is a strong constraint: you cannot identify two elements unless all their (t, i, f) vectors agree for every attribute. Thus c_i and d together turn the algebra into a labeled structure where homomorphisms must respect the data. This models the real requirement that you cannot merge two portfolios with different risk scores.

Definition 3.4 (Plithogenic Quasigroup Parastrophes). Let $P(Q) = (Q, V, v_D, d, c, \cdot_P)$ be a plithogenic quasigroup over base quasigroup $(Q, \cdot)$. A quasigroup $(Q, \cdot)$ has exactly six parastrophes determined by permuting the ternary relation $E(x, y, z) \Leftrightarrow x \cdot y = z$: the main operation ·, right division /, left division \, the opposite ∘, opposite right division //, and opposite left division \\.

The plithogenic quasigroup parastrophes of P(Q) are the six operations ·P, /P, \P, ∘P, //P, \\P on P(Q), defined by (1) Base parastrophes: for each operation, the base elements satisfy the corresponding parastrophe of (Q, ·); and (2) Degree rule: for all vi ∈ V, the neutrosophic degrees are computed by the same aggregation formula, subject to the relation being consistent with the base parastrophe.

Theorem 3.2 (Closure of Parastrophes). If (P(Q), ·P) is a plithogenic quasigroup, then (P(Q), *P) is a plithogenic quasigroup for each parastrophe *P ∈ {/P, \P, ∘P, //P, \\P}.

Proof. Each base parastrophe (Q, *) is a quasigroup. The degree rule for *P is identical to ·P, so solvability follows from Corollary 3.1 with x, y, z permuted. Uniqueness is inherited from the base. Hence (P(Q), *P) satisfies the Latin square property. □

Theorem 3.3 (Isotopy Invariance of Plithogenic Structure). Let (Q1, ·1) and (Q2, ·2) be isotopic quasigroups via (α, β, γ), i.e. α(x)·2β(y) = γ(x·1y) for bijections α, β, γ : Q1 → Q2. Let P(Q1) = (Q1, V, vD, d1, c, ·P1) be a plithogenic quasigroup. Define P(Q2) with degrees d2(α(x), vi) = d1(x, vi) for all x ∈ Q1, vi ∈ V. Then P(Q2) = (Q2, V, vD, d2, c, ·P2) is a plithogenic quasigroup, and the triple (α, β, γ) extends to a plithogenic isotopy P(Q1) ≃ P(Q2).

Proof. The base is an isotopy by hypothesis. Degree preservation holds by construction. Since the aggregation rule uses only d and c, and c is fixed, γ(x·P1y) = α(x)·P2β(y). The Latin square property follows from Theorem 3.1. □

Theorem 3.4 (Lagrange-type Property for Finite Plithogenic Loops). Let P(Q) be a plithogenic quasigroup where (Q, ·) is a finite loop of order n. Let P(H) be a plithogenic subloop, i.e. H ≤ Q and P(H) uses restrictions of d, c. Then |H| divides n, and for each vi ∈ V, the multiset of degrees {d(x, vi) : x ∈ H} is determined by |H| and d.

Remark 3.4. Unlike groups, P(H) need not partition P(Q). The theorem only restricts |H|, not the degree distribution. This is why portfolio constraints must be checked via Corollary 3.1, not just by size.

Theorem 3.5 (Automorphism Group Structure). Let Aut(P(Q)) = {φ : P(Q) → P(Q) plithogenic isomorphism}. Then Aut(P(Q)) ≅ {σ ∈ Aut(Q, ·) : d(σ(x), vi) = d(x, vi) ∀x ∈ Q, vi ∈ V}. That is, automorphisms are exactly base automorphisms that fix all neutrosophic degree vectors.

Proof. By Definition 3.2, any φ must preserve base and degrees. Conversely, any base automorphism σ preserving d extends uniquely to φ(x) = σ(x) with the same degrees, and φ respects ·P because the aggregation rule depends only on d, c. □

Theorem 3.6 (Cancellation and Idempotence Criteria). Let P(Q) be a plithogenic quasigroup. Let a ∈ P(Q); then (1) a·P x = a·P y ⇒ x = y and x·P a = y·P a ⇒ x = y (cancellation); and (2) a·P a = a iff a·a = a in (Q, ·), since d(a·P a, vi) = d(a, vi) by idempotence of tN, sN. Hence, if ci = 0 for all i, then ·P is idempotent iff · is idempotent; if ci = 1 for all i, the same holds.

Proof. (1) follows from the Latin square property in Theorem 3.1. For (2), compute degrees: (1−ci)tN(t,t) + ci sN(t,t) = (1−ci)t + ci t = t, similarly for i, f. Therefore d(a·P a) = d(a), so idempotence depends only on the base value. □

4. Application: Using Plithogenic Quasigroups as a Tool for Decision Making

We now use plithogenic quasigroup properties to model a portfolio manager choosing between three stocks under conflicting criteria. The plithogenic quasigroup structure captures both the algebraic flexibility of stocks and the neutrosophic uncertainty of each attribute.

4.1 Setup: Stocks as Elements of P(Q)

Let Q = {e, a, b} represent three candidate stocks: e = A, a = M, b = T. (Q, ·) is the quasigroup where x·y denotes switching from x to y, given by the Latin square in Example 3.1. This encodes the decision action: selecting a stock and then switching to another.

Let attributes V = {Growth, Safety, Liquidity} with dominant attribute vD = Growth. Contradiction degrees reflect analyst policy: cGrowth = 0, cSafety = 0.4, cLiquidity = 0.2. c = 0 for Growth means we require consensus: a portfolio switch only has high Growth if both stocks rate high. c = 0.4 for Safety means we tolerate conflict: the safer stock can dominate.

Expert evaluations give neutrosophic degrees d(x, vi) = (t, i, f):

Stock	Growth	Safety	Liquidity
x1 = A	(0.90, 0.10, 0.00)	(0.80, 0.10, 0.10)	(0.95, 0.05, 0.00)
x2 = M	(0.60, 0.30, 0.10)	(0.90, 0.10, 0.00)	(0.85, 0.10, 0.05)
x3 = T	(0.30, 0.40, 0.30)	(0.50, 0.30, 0.20)	(0.70, 0.20, 0.10)

This defines the plithogenic quasigroup P(Q) = (Q, V, vD, d, c, ·P).

4.2 Plithogenic Product as Strategy Composition for Decision Rule

The product $x_i \cdot_P x_j$ models the strategy "hold x_i, then switch to x_j." By Theorem 3.1, the result is a unique portfolio with aggregated degrees.

Example: the manager holds A and considers switching to M. Growth (min, since $c_G = 0$): $\min\{(0.9,0.1,0.0),(0.6,0.3,0.1)\} = (0.6, 0.3, 0.1)$. Safety ($0.6\cdot\min + 0.4\cdot\max$): $x_1 \cdot_P x_2 = M$ with Safety degrees (0.86, 0.10, 0.04). Liquidity ($0.8\cdot\min + 0.2\cdot\max$): (0.87, 0.09, 0.04).

Interpretation: Growth reduces to M's level because $c_G = 0$ enforces consensus. Safety increases because $c_S = 0.4$ lets M's higher Safety dominate.

4.3 Solving for a Target Portfolio

Suppose the investor requires a target profile b = Target with $d(b, G) = (0.60, 0.30, 0.10)$, $d(b, S) = (0.86, 0.10, 0.04)$. If the current holding is $x_1 = A$, we can solve for $x_1 \cdot_P x = b$ using Corollary 3.1. Table 2 shows the uniqueness of the solution $x = x_2 = M$. So M is the unique switch that meets the target criteria.

If no stock solves the equation, then the target cannot be determined under the current parameters. The manager must either change c_i or adjust the target, which is exactly Remark 3.1: c_i is the risk dial.

Conclusion. In this work, we were able to combine plithogenic set properties and quasigroup properties to provide a framework where the policy c_i, uncertainty (t, i, f), and algebra ($\cdot$, /, \) are inseparable primitives for making decisions that involve uncertainty.

5. Future Work

With the extension of the theory of neutrosophic sets to plithogenic quasigroups, several research directions open up to other algebraic structures. Future work can aim to unify plithogeny with the major non-associative algebraic theories.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: Mar. 1, 2026. Accepted: Sept. 5, 2026.