Neutrosophic Sets and Systems, Vol. 85, 2025
University of New Mexico

Neutrosophic Divisor Cordial Labeling Graphs

Tephilla Joice P [1], A.Rajkumar[2],

[1] **Research scholar ,Department of Mathematics, Hindustan Institute of Technology & Science, Chennai-603 103, India; tephilla.maths@kcgcollege.com**

[2] **Department of Mathematics, Hindustan Institute of Technology & Science, Chennai-603 103, India; arajkumar@hindustanuniv.ac.in**

* Correspondence: arajkumar@hindustanuniv.ac.in

Abstract: In this paper we introduced a novel concept – Neutrosophic Divisor Cordial Labeling a have proved that graphs such as wheels, helms and closed helm graph satisfy this new labeling. This paper builds upon our previous work in Neutrosophic Cordial Labeling where we demonstrated that graphs such as cycle related graph satisfy the labeling. A main focus of this research is the investigation of the relationship between cordial and neutrosophic cordial labeling, highlighting similarities, differences between the two. Our new finding on Neutrosophic divisor cordial lebeling contribute to the advancement of graph labeling theory

Keywords: Neutrosophic, Labeling, Cordial Labeling, Divisor Cordial Labeling, Graphs, Helm, Closed Helm

1. Introduction

Graph labeling is the investigation of assignment of labels depicted by the integer to vertices and edges, has been a very important area of research in graph theory[7,13], providing valuable tools for the analysis and classification of the properties of graphs in theoretical and practical applications. Among the other labeling methods, Cordial Labeling[5,6] has been an especially fascinating method, distinguished by its capacity for producing balanced and harmonious sets of labels among a graph's vertices and edges. The advent of Cordial Labeling has spawned many extensions and variations, each seeking to discover new fronts of graph theory and its uses[10,12].

The emergence in recent times of Neutrosophic Logic[17], an extension of classical logic[3,8,12], has created a flexible and comprehensive system for managing indeterminate and inconsistent data[2,19]. In extending the elements of Neutrosophic Logic to Cordial Labeling, our prior work developed the model of Neutrosophic Cordial Labeling[21]. With this new strategy, one may enjoy a deeper and richer analysis of graph labeling, especially in scenarios where traditional solutions are ineffective. In our previous research, we effectively used Neutrosophic Cordial Labeling to several graph configurations, such as cycles, wheels, helms, and closed helms, showing the strength and flexibility of this new labeling method[1,9,6].

From these preliminary results, the current work investigates the application of Neutrosophic Cordial Labeling further in more complex graph structures like Helm and Closed Helm Graphs. With the study of these graphs, we intend to achieve better understanding of their properties and relation towards each other, extending the theoretical framework of Neutrosophic Cordial Labeling. Among the main motivations of this work is the comparison and contrasting of Cordial Labeling with its Neutrosophic counterpart, elucidating the commonalities and differences as well as the possible synergies between the two. This comparative study not only refines our insight into graph labeling techniques but also provides new paths for their extension to other graph-theoretic contexts[14,15].

Some recent research has led to the further development of graph labeling in fuzzy and neutrosophic settings. Bathusha et al. in 2024 proposed the interval-valued complex neutrosophic graphs, discussed their energy functions, and explored possible applications[4]. In contrast, Fujita and Smarandache in 2025 delved into sophisticated concepts like smart, zero divisor, layered, weak, semi, and chemical graphs, broadening the theoretical horizon of neutrosophic graphs[8]. Their later work on superhypertree-width and neutrosophictree-width offers insightful knowledge on neutrosophic graph structure classification and complexity[9].

Fuzzy and bipolar fuzzy graph models are also being considered with considerable interest. Studies on detour -interior and -boundary nodes in bipolar fuzzy graphs have progressed network reliability and optimization applications. Likewise, investigations on some graph indices under bipolar fuzzy settings and graph complete degree with bipolar fuzzy data have shed new light on connectivity and strength[23,24]. Additional contributions are research on estimating most affected cycles and busiest network paths using graph complexity functions in fuzzy environments, and empirical studies on operations of bipolar fuzzy graphs and their degrees. These contributions, as a whole, add to the expanding domain of graph theory in uncertain settings.

Aside from developing our knowledge of Neutrosophic Cordial Labeling, this paper presents an additional extension: Neutrosophic Divisor Cordial Labeling. This new concept is based on the theory of Divisor Cordial Labeling, initially conceived by Varatharajan et al. in 2011[22]. Incorporating Neutrosophic Logic into Divisor Cordial Labeling, our goal is to develop a more flexible and integrated labeling model with a broader focus on graph patterns. Here, we present strict proofs that wheels, helms, and closed helms graphs are Neutrosophic Divisor Cordial Labeling, thus widening the scope of such labeling methods[20].

With this work, we aim to further enhance the development of graph labeling theory, especially with regard to Neutrosophic Cordial and Neutrosophic Divisor Cordial Labeling. Through the examination of these notions and their interactions, we hope to improve the theoretical basis as well as real-world applicability of graph labeling methods in various fields[16,17]. Not only does this research advance our comprehension of current labeling methods, but it also opens doors to future investigations into more intricate and complex graph structures.

The results of this research can potentially affect many fields, such as network security, bioinformatics, and social network analysis, where graph models are fundamental. By further developing neutrosophic graph labeling methods, we seek to close the gap between theoretical breakthroughs and real-world applications, providing new opportunities for solving uncertainty and complexity problems in real life.

1.1 Organisation of the Manuscript

The manuscript is divided into six sections. The first part introduces the basic notions, setting the background for the research. The second part defines the preliminaries required for the understanding of the suggested labeling methods. The third part offers the results of the research, namely examining Helm and Closed Helm Graphs under Neutrosophic Divisor Cordial Labeling. The fourth part discusses possible directions for future research and extensions of this research. The fifth part describes the restrictions of the approach presented, such as computational problems when implementing this labeling in varying graph topologies. Last, the sixth part summarizes the paper and provides main contributions and takeaways.

2. Preliminaries

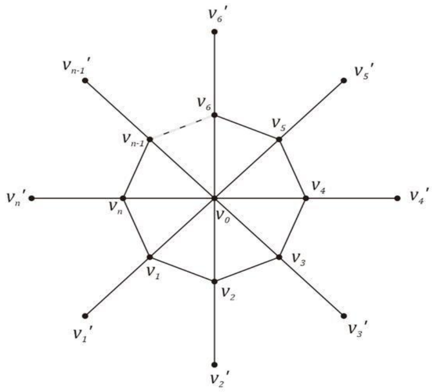
2.1 Graph[1]

A Graph is a finite set of vertices and finite set of edges G = (V(G), E(G)). Elements of V(G) and E(G) are called vertices and edges respectively

2.2 Helm Graph[21]

The helm graph H_n is constructed by taking an n-wheel graph and attaching a pendant edge to each vertex of the cycle

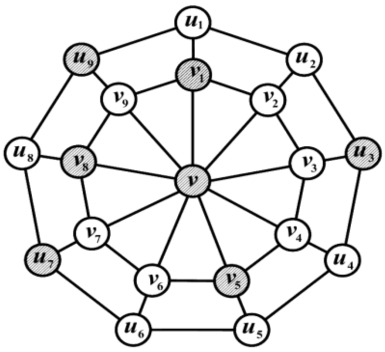
Example:



2.3 Closed Helm Graph[21]

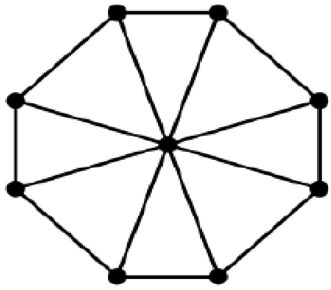
A closed helm CH_n is derived from Helm Graph H_n by adding edges between the pendant vertices, thereby forming a closed structure

Example: CH_9



2.4 Wheel Graph[21]

A wheel graph is formed by linking a single central vertex to all vertices of a cycle graph. The resulting structure resembles a wheel, with the cycle forming the rim and the central vertex as the hub.



2.5 Labeling of Graph[1,5]

Labeling of Graph refers to the assignment of integers to the vertices, edges, or both, of a graph, according to specific conditions or rules.

2.6 Cordial Labeling Graph[5]

Cordial labeling of a graph G involves an injectionf : V(G) → {0, 1} where each edge uv in G is assigned the label of ∣f(u)-f(v)∣. This labeling should meet the criteria that

$$|v_f(0) - v_f(1)| \leq 1 \text{and } |e_f(0) - e_f(1)| \leq 1|.$$

Here $v_f(i)$ indicates the count of vertices labeled i for i = 0 and 1, where $e_f(i)$ refers to the count of edges labeled i. A graph is considered *cordial* if it can be labeled in this manner.

2.7 Neutrosophic Logic[17]

Neutrosophic logic extends classical logic by addressing propositions with truth-values that may simultaneously exhibit degrees of truth, falsity, and indeterminacy. Each truth-value in this logic model is defined by three components: truth (T), indeterminacy (I), and falsity (F). These components are assigned real numbers within the interval [0,1]

2.8 Divisor Cordial Labeling[22]

A type of graph labeling that assign labels to the vertices or edges of a graph based on divisibility properties, aiming to create a balanced distribution. In divisor cordial labeling, each vertex v of a graph G is labeled with an integer f(v) such that each edge uv incident with vertices u and v is labeled as f(uv)=1 if f(u)divides f(v) or f(v) divides f(u); otherwise, f(uv)=0. The labeling is considered *divisor cordial* if $|e_f(1) - e_f(0)| \leq 1$

In our previous work we have introduced Neutrosophic Cordial Labeling.

2.9 Neutrosophic Cordial Labeling[21]

A neutrosophic graph $G = (V, \sigma, \mu)$ is said to be an neutrosophic cordial labeling graph if T_1, I_1, F_1 of vertices are assigned zero or one and T_2, I_2 *and* F_2 are assigned zero or one such that difference between the vertices assigned the value zero and one doesn't exceed one and similarly for edges. Edges are assigned values based on following criteria

$$T_2(v_i, v_j) = \begin{Bmatrix} \max\{T_1(v_i), T_1(v_j)\}\ if\ v_i > v_j\ or\ v_j > v_i \\ 0\ if\ v_i = v_j \end{Bmatrix}$$

$$I_2(v_i, v_j) = \begin{Bmatrix} \max\{I_1(v_i), I_1(v_j)\}\ if\ v_i > v_j\ or\ v_j > v_i \\ 0\ if\ v_i = v_j \end{Bmatrix}$$

$$F_2(v_i, v_j) = \begin{Bmatrix} \max\{F_1(v_i), F_1(v_j)\}\ if\ v_i > v_j\ or\ v_j > v_i \\ 0\ if\ v_i = v_j \end{Bmatrix}$$

and

$0 \leq T_2(v_i, v_j) + I_2(v_i, v_j) + F_2(v_i, v_j) \leq 3$

In this paper, we present a novel labeling approach called Neutrosophic Divisor Cordial Labeling.

2.10 Neutrosophic Divisor Cordial Labeling

A neutrosophic graph $G = (V, \sigma, \mu)$ is said to be an neutrosophic divisor cordial labeling graph if T_1, $I_1 and\ F_1$ are assigned values between 0 and 1 and edges are labeled 1 if $T_1(v_i)/T_1(v_j)$ or $T_1(v_j)/T_1(v_i)$, $I_1(v_i)/I_1(v_j)$ or $I_1(v_j)/I_1(v_i)$ and $F_1(v_i)/F_1(v_j)$ or $F_1(v_j)/F_1(v_i)$ otherwise 0 and it satisfies $|T_2(0) - T_2(1)| \leq 1, |I_2(0) - I_2(1)| \leq 1 and\ |F_2(0) - F_2(1)| \leq 1$ for edges

This paper introduces and analyzes Neutrosophic Divisor Cordial Labeling, providing theoretical results and applications for various graph structures.

3. Results and Discussion

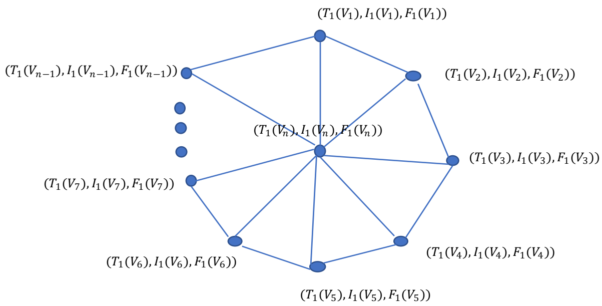
In this section, we have proved graphs such as Wheel, Helm and Closed Helm admits the above labeling technique.

Theorem 1

Wheel graph accepts Neutrosophic Divisor Cordial Labeling.

Proof

Let $G = (V, \sigma, \mu)$ be a neutrosophic graph where $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$, $V = \{v_1 + v_2 + v_3 + \cdots . + v_n\}$ such that $T_1: V \rightarrow \{0,1\}$, $I_1: V \rightarrow \{0,1\} and\ F_1: V \rightarrow \{0,1\}$ denote the degree of truth – membership function, indeterminacy – membership function and falsity – membership function of the vertex $v_i \in V$ and $T_2: V \rightarrow [0,1], I_2: V \rightarrow [0,1] and\ F_2: V \rightarrow [0,1]$ where $T_2(v_i, v_j), I_2(v_i, v_j), F_2(v_i, v_j)$ denote the degree of truth – membership function, indeterminacy – membership function and falsity – membership function of the edge (v_i, v_j) respectively such that for every (v_i, v_j).



Then,

$$|T_1(v_i)| = |I_1(v_i)| = |F_1(v_i)| = n$$
$$|T_2(v_i, v_j)| = |I_2(v_i, v_j)| = |F_2(v_i, v_j)| = 2n - 2$$

For the wheel graph, $T_1(v_i)$ - Truth memebership function are labeled using the function $(2m-1)2^{k_m}$ where $m \geq 1, k_m \geq 0$ where $(2m-1)2^{k_m} \leq n$.

Case I

If the value of $(2m-1)2^{k_m} \leq 9$, then the function $(2m-1)2^{k_m}$ is multiplied by 0.1. In other words, if $1 \leq (2m-1)2^{k_m} \leq 9, then\ (2m-1)2^{k_m} \times (0.1)$

Case II

If the value of $(2m-1)2^{k_m} \leq 99$, then the function $(2m-1)2^{k_m}$ is multiplied by 0.01. In other words, if $10 \leq (2m-1)2^{k_m} \leq 99, then\ (2m-1)2^{k_m} \times (0.1)^2$

Case III

If the value of $(2m-1)2^{k_m} \leq 999$, then the function $(2m-1)2^{k_m}$ is multiplied by 0.001. In other words, if $100 \leq (2m-1)2^{k_m} \leq 999, then\ (2m-1)2^{k_m} \times (0.1)^3$

and so on…

In general,

$$T_1(v_i) = \left\{ \begin{array}{c} (2m-1)2^{k_m} \times (0.1)^n \\ where\ m \geq 1, k_m \geq 0\ and \\ n = 1\ if\ 0 \leq (2m-1)2^{k_m} \leq 9 \\ n = 2\ if\ 10 \leq (2m-1)2^{k_m} \leq 99 \\ n = 3\ if\ 100 \leq (2m-1)2^{k_m} \leq 999 \\ \ldots \end{array} \right\}$$

and $(2m-1)2^{k_m} \leq n, the\ total\ number\ of\ vertices$

Indeterminacy – membership function is labeled using the following function

Case I

If T : odd, then

$$I_1(v_i) = \left(\frac{1-T}{2}\right)$$

Case II

If T :even, then

$$I_1(v_i) = \left(\frac{1-T}{3}\right)$$

In General,

$$I_1(v_i) = \left\{ \begin{array}{c} \left(\frac{1-T}{2}\right)\ if\ T\ is\ odd \\ \left(\frac{1-T}{3}\right)\ if\ T\ is\ even \end{array} \right\}$$

Falsity membership function is labelled using

$$F_1(v_i) = [1-(T+I)]$$

$T_1, I_1\ and\ F_1$ are labelled in the clockwise direction, $T_1(v_n) = 0.2, I_1(v_n) = F_1(v_n) = 0.1$ and the remaining vertices are labelled using the following order. If n is even, in addition to the above labeling interchange the values of $[T_1(v_1), I_1(v_1), F_1(v_1)]$ and $[T_1(v_2), I_1(v_2), F_1(v_2)]$

$$T_1(v_i) = \left\{ \begin{matrix} (2m-1)2^{k_m} \times (0.1)^n \\ where\ m \geq 1, k_m \geq 0\ and \\ n = 1\ if\ 0 \leq (2m-1)2^{k_m} \leq 9 \\ n = 2\ if\ 10 \leq (2m-1)2^{k_m} \leq 99 \\ n = 3\ if\ 100 \leq (2m-1)2^{k_m} \leq 999 \\ \ldots \end{matrix} \right\}$$

$$I_1(v_i) = \left\{ \begin{matrix} \left(\frac{1-T}{2}\right)\ if\ T\ is\ odd \\ \left(\frac{1-T}{3}\right)\ if\ T\ is\ even \end{matrix} \right\}$$

$$F_1(v_i) = [1 - (T + I)]$$

T_2, I_2 and F_2 are labelled 1 if two function divide each other otherwise it is labelled as zero.

$$T_2(v_i, v_j) = \left\{ \begin{matrix} 1\ if \frac{T_1(v_i)}{T_1(v_j)}\ or \frac{T_1(v_j)}{T_1(v_i)} \\ 0\ otherwise \end{matrix} \right\}$$

$$I_2(v_i, v_j) = \left\{ \begin{matrix} 1\ if \frac{I_1(v_i)}{I_1(v_j)}\ or\ \frac{I_1(v_j)}{I_1(v_i)} \\ 0\ otherwise \end{matrix} \right\}$$

$$F_2(v_i, v_j) = \left\{ \begin{matrix} 1\ if \frac{F_1(v_i)}{F_1(v_j)}\ or\ \frac{F_1(v_j)}{F_1(v_i)} \\ 0\ otherwise \end{matrix} \right\}$$

Then for truth membership function

$$T_2\ [1] = n - 1$$

In other words, for $T_2(v_i)$ the number of vertices labeled one are n-1

$$T_2\ [0] = n - 1$$

In other words, for $T_2(v_i)$ the number of vertices labeled Zero are n-1

$$\Rightarrow |T_2(0) - T_2(1)| \leq 1$$

In other words, for $T_2(v_i)$ the difference of number of vertices labeled one and zero is less than or equal to 1.

Similarly for Indeterminacy membership function

$$I_2\ [1] = n - 1$$

In other words, for $I_2(v_i)$ the number of vertices labeled one are n-1

$$I_2\ [0] = n - 1$$

In other words, for $I_2(v_i)$ the number of vertices labeled Zero are n-1

$$\Rightarrow |I_2(0) - I_2(1)| \leq 1$$

In other words, for $I_2(v_i)$ the difference of number of vertices labeled one and zero is less than or equal to 1.

For Falsity – membership function

$$F_2\,[1] = n - 1$$

In other words, for $F_2(v_i)$ the number of vertices labeled one are n-1

$$F_2\,[0] = n - 1$$

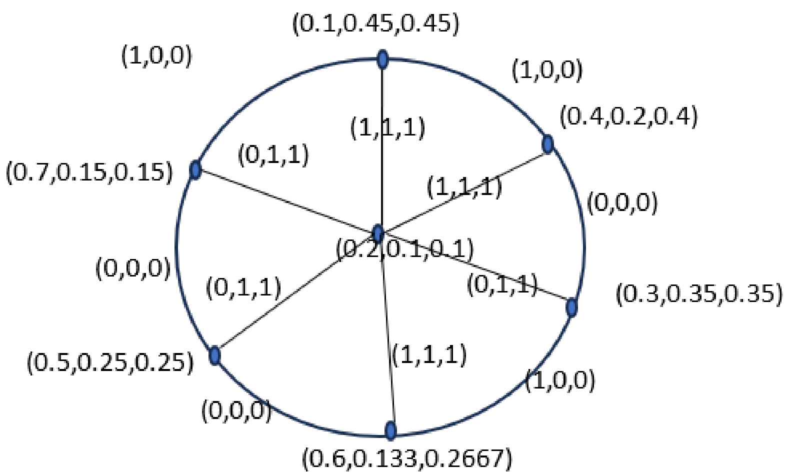
In other words, for $F_2(v_i)$ the number of vertices labeled Zero are n-1

$$\Rightarrow |F_2(0) - F_2(1)| \leq 1$$

In other words, for $F_2(v_i)$ the difference of number of vertices labeled one and zero is less than or equal to 1.

Thus G = W_n for n = 2m + 1 $: m \geq 2$ admits Neutrosophic Divisor Cordial Labeling

Neutrosophic Divisor Cordial Labeling for the Graph W_7 is given below



Total Number of Vertices = 7

Total Number of Edges = 12

Truth Membership Function

No of Edges Labeled one = 6

No of Edges Labeled zero = 6

$$\Rightarrow |T_2(0) - T_2(1)| = |6 - 6| \leq 1$$

Indeterminacy Membership Function

No of Edges Labeled one = 6

No of Edges Labeled zero = 6

$$\Rightarrow |I_2(0) - I_2(1)| = |6 - 6| \leq 1$$

Falsity Membership Function

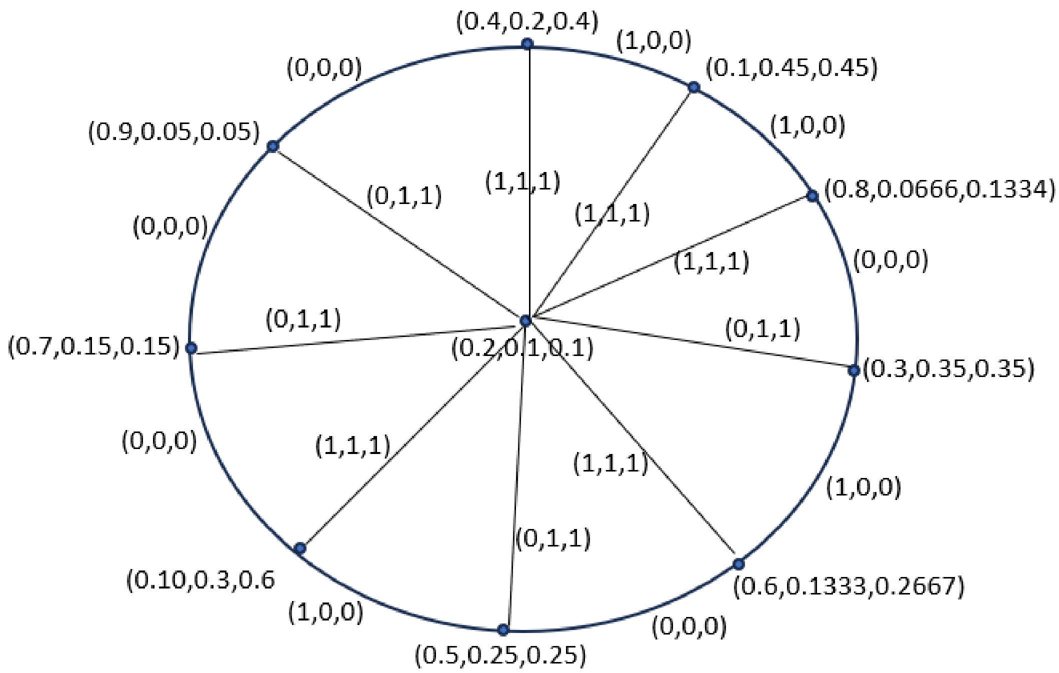
No of Edges Labeled one = 6

No of Edges Labeled zero = 6

$$\Rightarrow |F_2(0) - F_2(1)| = |6 - 6| \leq 1$$

W_7 admits Neutrosophic Divisor Cordial Labeling Graph.

Similarly, Neutrosophic Divisor Cordial Labeling for the Graph W_{10} is given below



Theorem 2

Helm Graph admits Neutrosophic Divisor Cordial Labeling.

Proof

Let $G = (V, \sigma, \mu)$ be a neutrosophic graph where $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$. The helm graph H_n is formed by starting with an n-wheel graph and attaching a pendant edge to each vertex of the cycle. It contains three types of vertices an apex of degree n, n vertices of degree 4 and n pendant vertices. Denote the apex vertex to be v. Vertices of degree four are denoted as $v_1, v_2, v_3, \ldots . v_n$. Denote the pendant vertices of Helm H_n to be $u_1, u_2, u_3, \ldots . u_n$. Denote the edges adjacent to the apex vertices to be $e_i{'s}$ where $1 \le i \le n$. Denote $g_1, g_2, g_3, \ldots . g_n$ to be the edges of the wheel's rim and the edges of the pendant vertices incident to rim of wheel are denoted as $h_1, h_2, h_3, \ldots . h_n$.

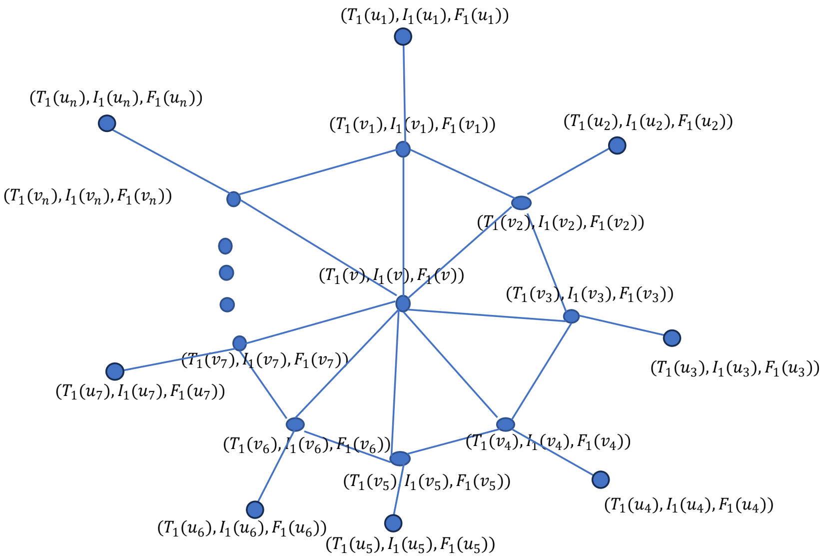
Vertex $V = \{v, v_1 + v_2 + v_3 + \cdots . + v_n, u_1, u_2, u_3, \ldots . u_n\}$ such that $T_1, I_1 and\ F_1$ denote the T, I, F of the vertex $v, v_i, u_i \in V$ and are assigned values between zero and one and $T_2, I_2\ and\ F_2$ denote the T, I, F of the edges between the vertices and are assigned the values "one" or "zero".

$$T_2(v, v_i)\ or\ T_2(e_i), T_2(v_i, v_j)\ or\ T_2(g_i), T_2(v_i, u_j)\ or\ T_2(h_i)$$

$$I_2(v, v_i)\ or\ I_2(e_i), I_2(v_i, v_j)\ or\ I_2(g_i), I_2(v_i, u_j)\ or\ I_2(h_i)\ \ and$$

$$F_2(v, v_i)\ or\ F_2(e_i), F_2(v_i, v_j)\ or\ F_2(g_i), F_2(v_i, u_j) or\ F_2(h_i)$$

represents the edges of T, I and F for helm graph



Then,

Total number of vertices is 2n+1 and for edges is 3n.

Vertices are labelled in the clockwise direction where $T_1(v_n) = 0.2, I_1(v_n) = F_1(v_n) = 0.1$ and the remaining vertices are labelled using the following order. If n is odd, in addition to the above labeling interchange the values of $[T_1(v_1), I_1(v_1), F_1(v_1)]$ and $[T_1(v_2), I_1(v_2), F_1(v_2)]$

For the vertices of degree 4 or vertices of wheels rim, $T_1(v_i)$ is labeled using the following function.

$$T_1(v_i) = \left\{ \begin{array}{c} (2m-1)2^{k_m} \times (0.1)^n \\ where\ m \geq 1, k_m \geq 0\ and \\ n = 1\ if\ 0 \leq (2m-1)2^{k_m} \leq 9 \\ n = 2\ if\ 10 \leq (2m-1)2^{k_m} \leq 99 \\ n = 3\ if\ 100 \leq (2m-1)2^{k_m} \leq 999 \\ \dots \end{array} \right\}$$

and $(2m-1)2^{k_m} \leq n, the\ total\ number\ of\ vertices$

$$I_1(v_i) = \left\{ \begin{array}{l} \left(\frac{1-T}{2}\right)\ if\ T\ is\ odd \\ \left(\frac{1-T}{3}\right)\ if\ T\ is\ even \end{array} \right\}$$

$$F_1(v_i) = [1 - (T(v_i) + I(v_i))]$$

For the pendant vertices, the vertices are labeled in clockwise direction.

$$T_1(u_i) = \begin{cases} \frac{T_1(v_i)}{2} & if\ i\ is\ odd \\ \frac{T_1(v_i)}{3} + 0.05 & if\ i\ is\ even \end{cases}$$

$$I_1(u_i) = \begin{cases} \frac{I_1(v_i)}{2} & if\ i\ is\ odd \\ \frac{I_1(v_i)}{3} & if\ i\ is\ even \end{cases}$$

$$F_1(u_i) = \begin{cases} \frac{F_1(v_i)}{2} & if\ i\ is\ odd \\ \frac{F_1(v_i)}{3} & if\ i\ is\ even \end{cases}$$

Edges between apex vertex and inner rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(e_i) = T_2(v, v_i) = \left\{ \begin{array}{c} 1\ if \frac{T_1(v)}{T_1(v_i)}\ or \frac{T_1(v_i)}{T_1(v)} \\ 0\ otherwise \end{array} \right\}$$

$$I_2(e_i) = I_2(v, v_i) = \left\{ \begin{array}{l} 1\ if \frac{I_1(v)}{I_1(v_i)}\ or\ \frac{I_1(v_i)}{I_1(v)} \\ 0\ otherwise \end{array} \right\}$$

$$F_2(e_i) = F_2(v, v_i) = \left\{ \begin{array}{c} 1\ if \frac{F_1(v)}{F_1(v_i)}\ or\ \frac{F_1(v_i)}{F_1(v)} \\ 0\ otherwise \end{array} \right\}$$

Edges between the vertices of inner rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(g_i) = T_2(v_i, v_j) = \begin{Bmatrix} 1\ if\ \dfrac{T_1(v_i)}{T_1(v_j)}\ or\ \dfrac{T_1(v_j)}{T_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$I_2(g_i) = I_2(v_i, v_j) = \begin{Bmatrix} 1\ if\ \dfrac{I_1(v_i)}{I_1(v_j)}\ or\ \dfrac{I_1(v_j)}{I_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$F_2(g_i) = F_2(v_i, v_j) = \begin{Bmatrix} 1\ if\ \dfrac{F_1(v_i)}{F_1(v_j)}\ or\ \dfrac{F_1(v_j)}{F_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

Edges between inner rim and outer rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(h_i) = T_2(v_i, u_j) = \begin{Bmatrix} 1\ if\ \dfrac{T_1(v_i)}{T_1(u_j)}\ or\ \dfrac{T_1(u_j)}{T_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$I_2(h_i) = I_2(v_i, u_j) = \begin{Bmatrix} 1\ if\ \dfrac{I_1(v_i)}{I_1(u_j)}\ or\ \dfrac{I_1(u_j)}{I_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$F_2(h_i) = F_2(v_i, u_j) = \begin{Bmatrix} 1\ if\ \dfrac{F_1(v_i)}{F_1(u_j)}\ or\ \dfrac{F_1(u_j)}{F_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

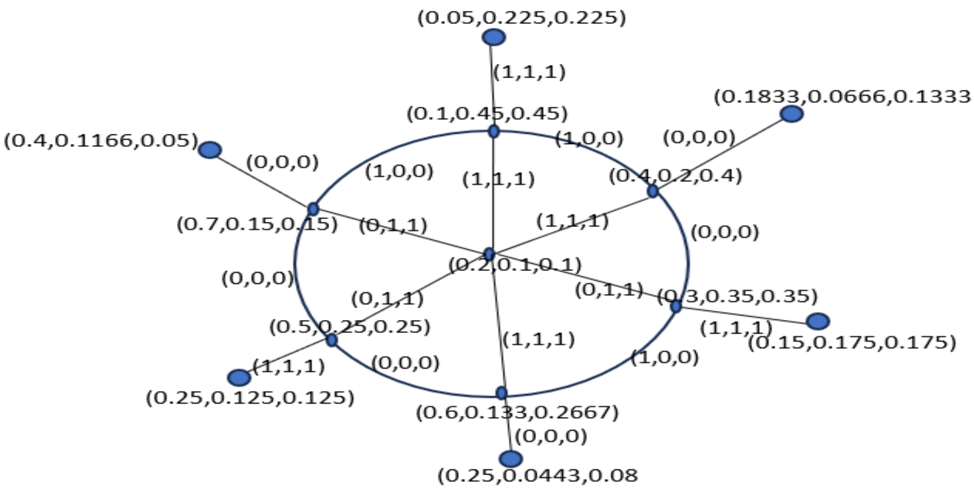
Case 1:

For $H_n: if\ n\ is\ even$

For truth membership function, $T_2\ [1] = \frac{3n}{2}$, $T_2\ [0] = \frac{3n}{2}$ and $|T_2(0) - T_2(1)| \leq 1$. Similarly for Indeterminacy membership function $I_2\ [1] = \frac{3n}{2}$, $I_2\ [0] = \frac{3n}{2}$ and $|I_2(0) - I_2(1)| \leq 1$ and for Falsity – membership function $F_2\ [1] = \frac{3n}{2}$, $F_2\ [0] = \frac{3n}{2}$ and $|F_2(0) - F_2(1)| \leq 1$

Thus $H_n: if\ n\ is\ even$ satisfies the above labeling.

Helm Graph: H6



T(1) = T(0) = I(1) = I(0) = F(1) = F(0) = 9. Thus satisfies the property of Neutrosophic Divisor Cordial Labeling.

Case II:

For $H_n: if\ n\ is\ odd$

For truth membership function, $T_2\,[1] = \frac{3n+1}{2}$, $T_2\,[0] = \frac{3n-1}{2}$ and $|T_2(0) - T_2(1)| \leq 1$. Similarly for Indeterminacy membership function, $I_2\,[1] = \frac{3n+1}{2}$, $I_2\,[0] = \frac{3n-1}{2}$ and $|I_2(0) - I_2(1)| \leq 1$ and for Falsity – membership function $F_2\,[1] = \frac{3n+1}{2}$, $F_2\,[0] = \frac{3n-1}{2}$ and $|F_2(0) - F_2(1)| \leq 1$

Thus G = W_n admits Neutrosophic Divisor Cordial Labeling.

Theorem 3

Closed Helm Graph admits Neutrosophic Divisor Cordial Labeling.

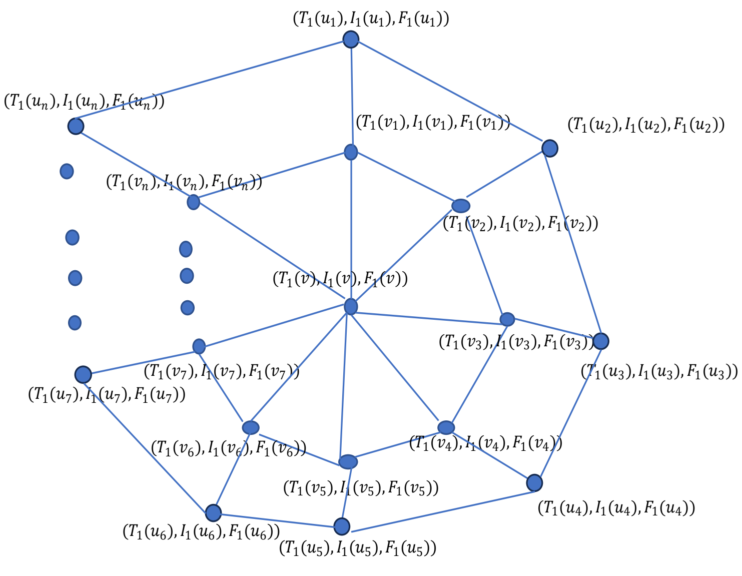
Proof:

A closed helm CH_n is the graph created by taking a helm graph $H_n$and adding edges between the pendant vertices. It contains apex vertex, inner rim and outer rim vertices. An Apex of degree n, n number of inner rim vertices of degree 4 and n number of outer rim vertices of degree n-1. The apex vertex is denoted as v, n vertices of degree 4 is denoted as $v_i: 1 \leq i \leq n$ and outer rim vertices is denoted as $u_i: 1 \leq i \leq n$. Edges adjacent to the apex vertex is denoted as $e_i: 1 \leq i \leq n$, the edges of the wheels inner rim is denoted as $g_i: 1 \le i \le n$ and the edges incident with the outer rim of the wheel is denoted as $h_i: 1 \le i \le n$ and the edges of the outer rim of the wheel is denoted as $w_i: 1 \le i \le n$

Let G = (V, σ, μ) be a neutrosophic graph where $\sigma = (T_1, I_1, F_1)$ and $\mu = (T_2, I_2, F_2)$, $V = [v, \{v_i : i \ge 1\}, \{u_i : i \ge 1\}]$ such that $T_1: V \rightarrow \{0,1\}, I_1: V \rightarrow \{0, 1\}$ and $F_1: V \rightarrow \{0, 1\}$ and $T_2: V \rightarrow \{0,1\}, I_2: V \rightarrow \{0,1\} and$ $F_2: V \rightarrow \{0,1\}$ where $T_2(e_i), I_2(e_i), F_2(e_i)$, $T_2(g_i), I_2(g_i), F_2(g_i)$, $T_2(h_i), I_2(h_i), F_2(h_i)$, $T_2(w_i), I_2(w_i), F_2(w_i)$ denote the degree of T, I and F of the edge.

Vertices are denoted as follows

$T_1(v_i)$ denote the truth – membership function and are labeled in the clockwise direction, $I_1(v_i)$ denote an indeterminacy – membership function and are labeled in the anticlockwise direction and $F_1(v_i)$ denote the falsity – membership function and are labeled in the clockwise direction. Similarly, $T_1(u_i)$ denote the truth – membership function and are labeled in the clockwise direction, $I_1(u_i)$ denote an indeterminacy – membership function and are labeled in the clockwise direction and $F_1(u_i)$ denote the falsity – membership function and are labeled in the anticlockwise direction.



Then,

Total number of vertices is 2n+1 and for edges 4n.

In which T_1, I_1 and F_1 of the Apex vertices are labeled as 1.

Truth membership function of the inner and outer rim vertices $v_1, v_2, v_3, \dots, v_n, u_1, u_2, \dots u_n$ are labeled in the clockwise direction using the function $(2m-1)2^{k_m}$.

Indeterminancy membership function and falsity membership function of the inner rim are labeled using the below function

$$I_1(v_i) = \begin{Bmatrix} \left(\frac{1-T}{3}\right) \ if\ i\ is\ odd \\ \left(\frac{1-T}{2}\right) \ if\ i\ is\ even \end{Bmatrix}$$

$$F_1(v_i) = [1 - (T(v_i) + I(v_i))]$$

Indeterminancy membership function and falsity membership function of the outer rim are labeled using the below function

$$I_1(u_i) = \frac{I_1(v_i)}{2}$$

$$F_1(u_i) = \frac{F_1(v_i)}{2}$$

Edges between apex vertex and inner rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(e_i) = T_2(v, v_i) = \begin{Bmatrix} 1\ if \frac{T_1(v)}{T_1(v_i)}\ or \frac{T_1(v_i)}{T_1(v)} \\ 0\ otherwise \end{Bmatrix}$$

$$I_2(e_i) = I_2(v, v_i) = \begin{Bmatrix} 1\ if \frac{I_1(v)}{I_1(v_i)}\ or\ \frac{I_1(v_i)}{I_1(v)} \\ 0\ otherwise \end{Bmatrix}$$

$$F_2(e_i) = F_2(v, v_i) = \begin{Bmatrix} 1\ if \frac{F_1(v)}{F_1(v_i)}\ or\ \frac{F_1(v_i)}{F_1(v)} \\ 0\ otherwise \end{Bmatrix}$$

Edges between the vertices of inner rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(g_i) = T_2(v_i, v_j) = \begin{Bmatrix} 1\ if \frac{T_1(v_i)}{T_1(v_j)}\ or \frac{T_1(v_j)}{T_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$I_2(g_i) = I_2(v_i, v_j) = \begin{Bmatrix} 1\ if \frac{I_1(v_i)}{I_1(v_j)}\ or\ \frac{I_1(v_j)}{I_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$F_2(g_i) = F_2(v_i, v_j) = \begin{Bmatrix} 1\ if \frac{F_1(v_i)}{F_1(v_j)}\ or\ \frac{F_1(v_j)}{F_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

Edges between inner rim and outer rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(h_i) = T_2(v_i, u_j) = \begin{Bmatrix} 1\ if \dfrac{T_1(v_i)}{T_1(u_j)}\ or \dfrac{T_1(u_j)}{T_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$I_2(h_i) = I_2(v_i, u_j) = \begin{Bmatrix} 1\ if \dfrac{I_1(v_i)}{I_1(u_j)}\ or\ \dfrac{I_1(u_j)}{I_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$F_2(h_i) = F_2(v_i, u_j) = \begin{Bmatrix} 1\ if \dfrac{F_1(v_i)}{F_1(u_j)}\ or\ \dfrac{F_1(u_j)}{F_1(v_i)} \\ 0\ otherwise \end{Bmatrix}$$

Edges between vertices of outer rim are labeled 1 and 0 if it satisfies the following condition

$$T_2(w_i) = T_2(u_i, u_j) = \begin{Bmatrix} 1\ if \dfrac{T_1(u_i)}{T_1(u_j)}\ or \dfrac{T_1(u_j)}{T_1(u_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$I_2(w_i) = I_2(u_i, u_j) = \begin{Bmatrix} 1\ if \dfrac{I_1(u_i)}{I_1(u_j)}\ or\ \dfrac{I_1(u_j)}{I_1(u_i)} \\ 0\ otherwise \end{Bmatrix}$$

$$F_2(w_i) = F_2(u_i, u_j) = \begin{Bmatrix} 1\ if \dfrac{F_1(u_i)}{F_1(u_j)}\ or\ \dfrac{F_1(u_j)}{F_1(u_i)} \\ 0\ otherwise \end{Bmatrix}$$

Then for $T_2(v_i)$, $I_2(v_i)$ and $F_2(v_i)$, the number of edges labeled zero are $\frac{4n}{2}$

$$T_2(0) = I_2(0) = F_2(0) = \frac{4n}{2}$$

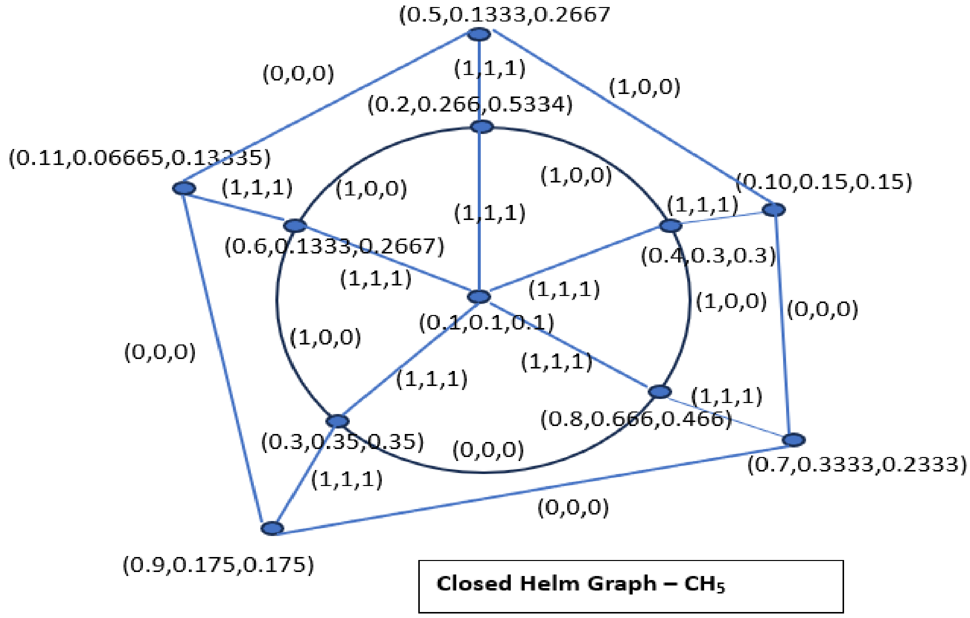
Similarly $T_2(v_i)$, $I_2(v_i)$ and $F_2(v_i)$, the number of edges labeled one are $\frac{4n}{2}$

$$T_2(1) = I_2(1) = F_2(1) = \frac{4n}{2}$$

Difference between the edges labeled one and zero for $T_2, I_2\ and\ F_2$ is less than or equal to zero, thus satisfying the condition of Neutrosophic Divisor Cordial Labeling.

Hence closed helm graph admits Neutrosophic Divisor Cordial Labeling

For a Closed Helm Graph CH_5



Closed Helm Graph – CH_5

Total Number of Vertices = 11
Total Number of Edges = 20
T(1) = T(0) = 10, I(1) = I(0) = 10, F(1) = F(0) = 10
Thus satisfies the property of Neutrosophic Divisor Cordial Labeling.
Hence, CH_5 admits Neutrosophic Divisor Cordial Labeling.

4.Future Work

In this study, we have introduced and analyzed Neutrosophic Divisor Cordial Labeling, extending the principles of neutrosophic logic to divisor cordial labeling. As a continuation of this research, we aim to explore the application of neutrosophic labeling to other labeling techniques and investigate its impact on various graph structures. Additionally, we seek to examine the practical applications of labeling graphs in decision-making systems, particularly in risk assessment, multi-criteria decision analysis, and optimization problems in complex networks, where uncertainty and indeterminate information are prevalent.

5.Limitation

Checking whether different types of graphs satisfy this labeling is challenging and requires a lot of calculations. The complexity increases as the graph structures become more complicated.

6.Conclusion

In this paper, we introduced the concept of Neutrosophic Divisor Cordial Labeling and demonstrated its applicability to various graph structures, including wheels, helms, and closed helms. By extending the principles of Neutrosophic Cordial Labeling, our study provides a deeper understanding of how neutrosophic logic can be integrated into graph labeling.

References:

1. Ancy Dsouza, Dr Kumudakshi.S, *A survey on Magic Labeling of digraphs*, IOSR Journal of Engineering (IOSR JEN), pp. 39-42
2. Ansari, A. Q. Biswas, R., and Aggarwal, S. , *Neutrosophication of Fuzzy Models*, IEEE Workshop On Computational Intelligence: Theories, Applications and Future Directions (hostedby IIT Kanpur), 2013.
3. Ansari, A. Q, Biswas R. and Aggarwal, S., *Extension to fuzzy logic representation: Moving towards neutrosophic logic* - A new laboratoryrat, Fuzzy Systems (FUZZ), 2013 IEEE International Conference, 1-8
4. Bathusha, S. S., Jayakumar, S., & Raj, S. A. K. (2024). The Energy of Interval-Valued Complex Neutrosophic Graph Structures: Framework, Application and Future Research Directions. Neutrosophic Systems With Applications, 13, 67-101.
5. Cahit I, *Cordial Graphs: A weaker version of graceful and harmonious graphs*, Arcs. Combinatorial 23(1987), 201-207
6. Cahit I., "*On cordial and 3-equitable labellings of graphs*", Util.Math., Vol. 37,pp. 189-198, (1990)
7. Farida A, R P Indah and N A Sudibyo, *Magic covering and edge magic labelling and its application,* Journal of Physics Conference Series, 2020
8. Fujita, T., & Smarandache, F. (2025). A Reconsideration of Advanced Concepts in Neutrosophic Graphs: Smart, Zero Divisor, Layered, Weak, Semi, and Chemical Graphs. Neutrosophic Systems With Applications, 25(2), 39-79
9. Fujita, T., & Smarandache, F. (2025). Some Graph Parameters for Superhypertree-width and Neutrosophictree-width. *Neutrosophic Systems With Applications*, *25*(1), 53-77
10. Gallian J.A, *A Dynamic Survey of Graph Labeling*, Electronic Journey of Combinatorics., 25th Edition, DS6(2022)
11. Gani A N, M. Akram, D.R. Subhashini*: Novel Properties of Fuzzy Labelling Graphs*, J. Math., (2014), 375135, 1-6
12. Gomathi M, Keerthika M, *Neutrosophic Labeling Graph, Neutrosophic Sets and Systems* : An International Journal of Information Science and Engineering, 2019, pp 261 – 272
13. Lakshmi prasanna et al., *Applications of Graph labelling in major areas of computer science*, International Journal of Research in Computer and Communication Technology Vol 3, Issue 8, pp. 819-823,August – 2014

14. Rajab Ali Borzooei, Hossein Rashmanlou, Sovan Samanta and Madhumangal Pal, *A study on fuzzy labeling graph*, Journal of intelligent and fuzzy systems, 2016,30, pp 3349-3355
15. Rosenfeld A: Fuzzy *sets and their applications to cognitive and decision processes*, Academic Press, 1st Edition, Proceedings of the U.S – Japan Seminar on Fuzzy Sets and their applications, held at the university of California, Berkeley, California, July 1 – 4, 1974
16. Rosa A: *On certain valuations of the vertices of a graph*, In: Proceeding of the International Symposium on Theory of Graphs, Rome, Italy (1966), 349 – 355
17. Smarandache F, *Neutrosophy and Neutrosophic Logic*, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability and Statistics University of New Mexico, Gallup, NM 87301, USA (2002)
18. Smarandache F, W. B. Vasantha Kandasamy, Ilanthenral Kandasamy, *Neutrosophic Graphs: A new Dimension to Graph Theory*, 2015, ISBN – 12, 978 – 1 -59973-362-3
19. Sridevi. R, Nagarajan. K, Nellaimurugan. A, Navaneethakrishnan. S *Fibonacci divisor Cordial graphs* International Journal of Mathematics and Soft Computing Vol.3, No.3, pp.33 – 39, (2013).
20. Tephilla Joice P and Sharon Philomena V *Fibonacci Divisor Cordial Labeling for Helm and Closed Helm* Bulletin of Pure and Applied Sciences Section - E - Mathematics & Statistics, 38E, pp 99–104 (2019)
21. Tephilla Joice P and Rajkumar A *Neutrosophic Cordial Labeling on Helm and Closed Helm Graph* International Journal of Neutrosophic Science, Vol 25, Issue 5, pp: 453-471(2025)
22. R. Varatharajan, S. Navaneethakrishnan and K. Nagarajan, *Divisor cordial graph*, International Journal of Mathematical Comb.,vol .4, pp.15 – 25,(2011).
23. Zadeh L.A, *Fuzzy Sets*, Information and Control, Elsevier, Vol. 8 pp 338-353(1965)
24. Zimmermann, *Fuzzy Set Theory-and Its Applications*, Fourth Edition, Springer Science+Business Media, LLC, ISBN 978-94-010-3870-6

Received: Nov. 8, 2024. Accepted: May 2, 2025