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Taxi Geometry and Euclidean Geometry: Congruence and Similarity of Triangles

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Abstract: This article presents an instrument that describes the experience implemented with students of a graduate programme in the courses of Analytic Geometry and Linear Algebra. The purpose of the instrument, as a didactic resource, was to analyse how the connections between the classical cases of congruence and similarity of triangles in Euclidean Geometry and in Taxi Geometry can be explored, fostering the construction of geometric thinking through visual skills and enabling the exploration, experimentation, search for and conjecturing of geometric knowledge from the study of both geometries. The metric analysis of the productions shows that the three classical congruence criteria — SSS, SAS and ASA — cease to hold under the taxicab metric, and that equilaterality does not imply equiangularity in that geometry. The knowledge constructed through the comparison of the two geometries proved sufficient and pertinent for the correct solution of the problem proposed in the pedagogical exercise. In addition, a neutrosophic logic framework is applied in order to assess the degree of certainty, indeterminacy and contradiction of each category of bibliographic evidence on which the proposal rests.

Keywords: Taxi geometry; congruence; similarity; geometric thinking; representation and visualization; Neutrosophic logic; Indeterminacy.

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1. Introduction

Today we live amid constant change at the cultural and technological levels and in the way information is handled; hence the need for change in the current traditional educational system that, in most cases, is still found in our society.

This research seeks to make students participants in the construction of their own knowledge, so that in this way they become motivated by their own learning processes. Philip Davis and Reuben Hersh [1], authors of the text *The Mathematical Experience*, set out their views on what mathematics is, on the construction of mathematical knowledge and on the way in which it is taught. Davis and Hersh [1] further maintain that the lecture has no bearing on students' learning of content, and that genuine learning is achieved when each individual is confronted with problems for which they perceive the need to find a solution.

Drawing on Lakatos's work and his appraisal of the nature of mathematical knowledge, it can be asserted that teaching and learning, in order to be consistent with this epistemological view of mathematics, must not be limited to the formal process of chaining together axioms, definitions and theorems. For Lakatos [2], beyond this there is the history of the construction or discovery of mathematical knowledge, an informal mathematics created through a quasi-empirical process, which is unavoidable if one is to reach the knowledge accepted as such by the mathematical community. Following Lakatos's [2] line of research, the shift from a formalist or deductive approach to mathematics towards a more inductive and heuristic one is adopted here.

Similarly, the conception and teaching of geometry can be characterised by three fundamental aspects: geometry as reasoning, geometry as problem solving, and geometry as a means of establishing mathematical connections. The National Council of Teachers of Mathematics (NCTM), following an epistemology influenced by Lakatos and by constructivist psychology, promotes the use of a more active and participatory classroom methodology that allows students to construct their own knowledge. The premise adopted is that this type of methodology favours geometric learning and that, in addition, it contributes to the teaching and learning of the applications of geometry through the solution of problems found in the surrounding environment, which fosters the student's independence and autonomy.

In view of the above, a series of problems was proposed to the students in this study with the aim of analysing how the connections between the classical cases of congruence or similarity of triangles in Euclidean Geometry and Taxi Geometry can be explored, comparing these two geometries in order to pursue the construction of geometric thinking through the visual skills involved.

2. Rationale

In addressing the relevance of medieval pedagogy, Régine Pernoud, attentive to the possibilities of opening up the study of history through the journeys that may be undertaken, observes, according to Lauand [3, p. 19], that in both cases we encounter "the other", distant from us in time and space. This encounter with alternative logics of space and knowledge resonates with epistemological traditions that read indeterminacy and complexity as constitutive rather than accidental, from the dialogue between neutrosophic thought and Edgar Morin's complexity paradigm [24] to the recovery of indigenous and ancestral geometrical and philosophical traditions of Latin America within a neutrosophic framework [25].

In this respect, the Greeks, with Euclid, made a significant contribution to the mathematical organisation of their time: thanks to that humble and self-sacrificing work, consciously undertaken by those who had talent for a great deal more, mathematics was preserved in the West and was able to sustain itself until the tenth century, when it received fresh impetus from Gerbert and, from the following centuries onwards, developed ever further [3, p. 25].

From this starting point, it is appropriate to reflect on the historical development of education, and of geometry in particular, and on ways of broadening and articulating geometric knowledge in the school environment and in the initial and continuing education of teachers who teach mathematics. This analysis includes, for example, the "new" geometries, especially hyperbolic, spherical, Riemannian, differential, topological and fractal geometry, and Taxi Geometry, which emerged between the twentieth and twenty-first centuries.

Within the formalism of the statements of Euclid's *Elements*, it became necessary to obtain demonstrations in the Middle Ages. One example is the construction of an equilateral triangle with the instruments straightedge and compass, recalling that the straightedge was not graduated: it was used simply to draw straight lines, while the compass was used to transfer the measurements. A rigorous mathematical proof involving perfectly defined definitions and propositions was therefore required. By comparison, in present times we have Dynamic Geometry software, GeoGebra and others, which afford countless possibilities for visualising geometric inputs and the relationships established among them.

Speaking of the origins of geometry therefore opens up new insights into today's world, beyond those of Thales, for instance, who may be judged one of the precursors in bringing together geometry and visualisation. Moreover, according to Serres [4, p. 195], Thales's theorem compares a first triangle formed by a pyramid, its shadow cast on the sand and the solar ray with a second one constituted by any body accessible in its height, its shadow again and a similar ray of light: both triangles, having equal angles, are homothetic. In this sense, one may offer clues of visual skills at that time, showing that some people were able to exploit such skills in the construction of mathematical knowledge, on the basis of calculations involving heights inaccessible to humans with technological instruments not yet available at the time.

For this reason, it falls to today's geometry educators to conduct their research on visual skills that enable teachers and prospective mathematics teachers to develop their knowledge of geometries other than the Euclidean one. This research is therefore justified with regard to one of these, namely Taxi Geometry.

3. Theoretical Framework

The axiomatic method in mathematics draws its strength from Euclid, where a theorem, in order to be an accepted truth, needs to be anchored in other already demonstrated truths or in unquestionable truths: the postulates. Intuitive questions were thus quite present in the acceptance of the premises underlying the theory, unlike the formalist current, which did not grant the intuitive reality of mathematical objects the former value; for the formalists the postulates do not express obvious truths about the realities of pure intuition, and their concern is only with the formal logical procedure of reasoning based on postulates [5, p. 263]. For them, the axiomatic approach to a mathematical theory is a natural way of unravelling the network of relations among different facts and of exhibiting an essential logic of structure [5, p. 264].

This logic therefore predicts that, when a set of axioms is organised in order to characterise a given theory, certain undefined concepts are present, such as point and line, for example, in order to define others, such as the distance between two points. These discussions eventually led to thinking about geometric entities other than those arising from Euclidean Geometry, and this led to the construction of new axiomatic systems, that is, to the creation of new geometries such as Hyperbolic Geometry (Bolyai, Lobachevsky and Gauss), with models such as Klein's or Poincaré's for the latter. Elliptic or Riemannian Geometry also has its origin here, as does Topology, in the nineteenth century, on which rests the important four-colour theorem, proved recently with the evolution of computing; and, more recently, Taxi Geometry, which we shall set out in greater detail as we anchor the research reported in this article.

Let us consider Klein's hyperbolic model, in which the geometric space is constituted by the interior of any circle in the plane (Figure 1). Points are as in Euclidean Geometry and lines are the chords of the circumference of this circle, excluding the endpoints, which do not belong to the geometric space. Three straight lines pass through point P and one (α) does not.

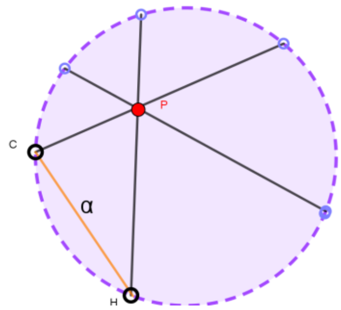

Figure 1. Klein's model of Hyperbolic Geometry

Source: prepared by the authors, designed in GeoGebra

It is observed that line α does not intersect the others, since the endpoints C and H do not belong to the geometric space (Klein's hyperbolic plane). This example therefore demonstrates that the parallel postulate cannot be deduced from the others of Euclidean Geometry. It is known that through a point not belonging to a line there passes a single line parallel to it that does not intersect it. That statement is not valid in this model.

Riemann studied geometry on the surface of a sphere, where points are considered as in Euclidean Geometry (A and P), while straight lines are the great circles (α, β, γ) of this surface (Figure 2). It is observed that, given the curve γ and a point P outside this curve — for example the north pole — there are infinitely many lines passing through this point, but they all intersect the line γ (α and β being two of them). Therefore, the fifth postulate of Euclidean Geometry is not valid for this geometry either.

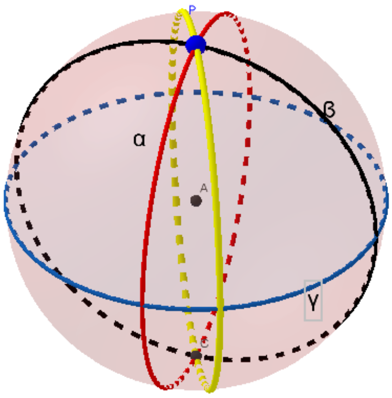

Figure 2. Riemann's model of Elliptic Geometry

Source: prepared by the authors, designed in GeoGebra

It is not the aim of this article to go deeply into these geometries; they are referred to only in order to establish a second relevant aspect that distinguishes them from Euclidean Geometry and that bears on what will be developed in the present research, namely the calculation of distances between points. According to Courant and Robbins [5], in Klein's model, if P and Q are two distinct points of a chord whose endpoints are O and S, then the non-Euclidean distance from P to Q is given by log (OSQP), where (OSQP) denotes the cross ratio.

This model characterises well the mathematical importance of axiomatic logic in the reconstruction of geometric concepts other than the Euclidean ones.

In turn, considering Riemann's model, the distance between two points on the spherical surface is given by the length of the great-circle arc passing through the two points (geodesic arc), which can readily be obtained with the tools of differential and integral calculus. This model is relevant to study and to understand because it is present in the real world, especially in navigation, since straight lines (geodesics) describe the shortest path between two points and can be associated with location in terms such as meridians, longitudes and latitudes.

On this brief basis of geometries other than the Euclidean one, we arrive at more modern times, in which one may think about how displacements occur in urbanised cities, organised in blocks, and ask: what is the geometry involved in these displacements and how are the distances between any two points calculated? Figure 3 illustrates the movement of a person or a car through such a city.

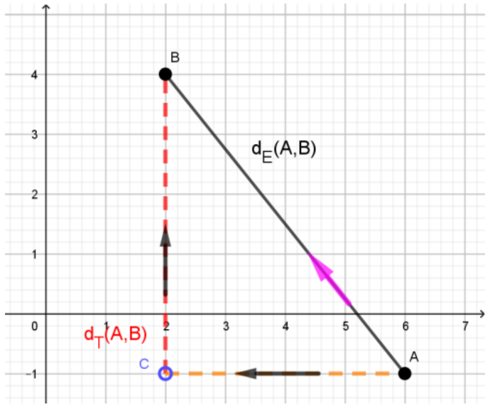

Figure 3. Displacements in an urbanised city

Source: prepared by the authors

The Euclidean displacement from point A to point B is represented by the solid dark line and its value is given by $d_E(A,B)$. The displacement in the urbanised city is given by $d_T(A,B)$, drawn in red, passing through C, that is, following an idealised triangular route. Considering the points given by their coordinates, that is, $A = (x_A, y_A)$ and $B = (x_B, y_B)$, it follows that:

$$d_E(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}; \quad d_T(A,B) = |x_A - x_B| + |y_A - y_B|$$

From this consideration of the ways of measuring, we shall analyse another modern non-Euclidean geometry, called Taxi Geometry, whose ideas go back, according to Reinhardt [6], to Hermann Minkowski's work on families of "metrics", that is, ways of measuring based on a set of axioms which characterise certain spaces as "metric spaces". According to the author, it was in 1952 that Karl Menger presented this geometry in an exhibition at the Museum of Science and Industry in Chicago under the name *TaxiCab Geometry*, which is where its designation actually comes from.

Ekici, Kocayusufoğlu and Akça [7] define the inner product and the norm of a vector in Taxi Geometry, discussing these definitions geometrically and how they may be explored. These authors use the trigonometric equality in this geometry to arrive at a comparison between the definitions in Euclidean Geometry and in Taxi Geometry [7, p. 305]:

$$\langle \alpha, \beta \rangle_E = \|\alpha\|_E \|\beta\|_E cos_E\, \theta \quad \text{and} \quad \langle \alpha, \beta \rangle_T = \|\alpha\|_T \|\beta\|_T cos_T\, \theta - R_T$$

Concerning the representations of conics in Taxi Geometry, Leivas [8] carried out research with seven students in an in-service course, the aim of which was to analyse how these students, all of them teachers, interpreted and represented ellipses, parabolas and hyperbolas using the metric of the legs, that is, how such loci appear in Taxi Geometry or Urban Geometry.

Leivas [9] also carried out research with students of a graduate programme in a course on Analytic Geometry and Linear Algebra, in which he used the Theory of Semiotic Representation Registers to solve a problem in a hypothetical city involving Taxi Geometry. The results showed that the students had some difficulty in visualising and representing the solution in this geometry compared with what would happen in Euclid's. In turn, the researchers concluded that Taxi Geometry is the one that yields an adequate real solution to the problem.

On the basis of these theoretical assumptions, it can be seen that Taxi Geometry offers possibilities for constructing geometric thinking that is relevant to the initial and continuing education of mathematics teachers, especially when addressing the visual aspects involved. With regard to visualisation, this is defined as a process of forming mental images with the purpose of constructing and communicating a certain mathematical concept, in order to help solve analytic or geometric problems [10, p. 112].

Moreover, these ideas put forward by Leivas [10] coincide with what Arcavi [11] says about visualisation. Arcavi [11, p. 217] states that it is the ability, the process and the product of creating, interpreting, using and reflecting upon figures, images and diagrams, in our minds, on paper or with technological tools, for the purpose of representing and communicating information, thinking about and developing ideas, and advancing understanding. From Arcavi's [11] perspective, visualisation is not only a tool for the understanding of mathematical concepts, but also for communicating mathematical ideas, criteria that reinforce what Leivas [10] proposes. These ideas of Arcavi [11], Arcavi and Hadas [12] and Leivas [10] are considered fundamental to the conduct of the present research.

In Euclidean Geometry, given the non-collinear points A, B and C, these determine a triangle and, from the measures of the sides AB, BC and AC, triangles can be classified as equilateral (three sides of the same measure), isosceles (two sides of the same measure and a different third one) and scalene (three sides of different measures). As for the interior angles, triangles are classified as acute (all angles smaller than 90º), obtuse (one obtuse angle) and right (one angle measuring 90º). In the last case, the sides forming the right angle are called legs and the side opposite the right angle is called the hypotenuse.

Another important definition in Euclidean Geometry is that of congruence between two triangles; for Durán C. [13, p. 159] it is a one-to-one correspondence between their vertices such that their corresponding angles and their corresponding sides are equal, the sign for congruence being $ABC \cong DEF$. The criteria for the congruence of triangles are the following:

- SAS: if two sides and the angle included between them of one triangle are equal to two sides and the angle included between them of another triangle, then both triangles are congruent.
- ASA: if two angles and their common side in one triangle are equal to two angles and their common side of another triangle, then the two are congruent.
- SSS: if the three sides of one triangle are equal to the three sides of another triangle, then both are congruent.

On the other hand, Durán [13, p. 155] states that a similarity between two triangles is a one-to-one correspondence between their vertices such that the corresponding angles are equal and their corresponding sides are proportional.

In this case of similar triangles one writes $ABC \sim DEF$, and certain correlations may occur, such as the fact that the relation is an equivalence relation (reflexive, symmetric and transitive). Durán C. [13] cites the similarity criteria:

- If one angle of a triangle is equal to one angle of another triangle, and the sides forming the first angle are proportional to the sides forming the other angle, then both triangles are similar.
- If two angles of a triangle are equal to two angles of another, then the two triangles are similar.
- If the three sides of one triangle are proportional to the three sides of another triangle, then the two are similar.

An equally important proposition in triangle relations is: "In a triangle, the greater side is opposite the greater angle, and vice versa"[1]. These relations are important for what we shall discuss in the sequence of the article, since we shall compare them with Taxi Geometry. The section that follows deals with the empirical framework of the research.

4. Methodology

This research is developed under a qualitative approach, within the interpretive tradition that goes back to Max Weber (1864–1920), a classical author of the social sciences and one of those considered to have given rise to this approach by introducing the term "to understand"; in that tradition it is stressed that, in addition to the description and measurement of social variables, the subjective meanings and the understanding of the context in which the phenomenon occurs must be

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[1] All the relations derive from Euclid and it is possible to demonstrate them.

taken into account. For this study, not only social variables at the macro level are considered, but also individual instances.

Three pairs of students took part, formed by affinity, identified hereafter by the pseudonyms Flatan, Carlia and Thanola; in what follows, "pair" and "working group" designate the same unit of analysis. All participants were informed of the purpose of the study and consented to the use of their written productions for research purposes. The stages carried out in the research are presented below.

First, the students were shown Figure 4 and were asked to observe it. This figure presents two triangles whose side measures are given in the taxicab metric.

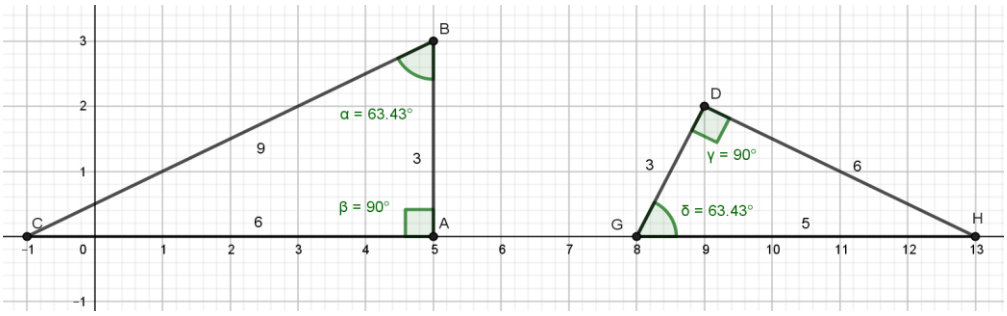


Figure 4. Triangles in Taxi Geometry

Source: prepared by the authors

Second, after the students observe Figure 4, they must complete Table 1.

Table 1. Triangles in Taxi Geometry

Triangles	Corresponding vertices			Corresponding angles			Corresponding sides		
Triangle ABC									
Triangle A'B'C'									

The students were then asked to form working teams in order to answer the following questions:

- Bearing in mind Euclidean Geometry, are the triangles presented in Figure 4 congruent?
- What happens in Taxi Geometry?
- Which metric relation holds for Euclidean Geometry and is not valid for Taxi Geometry?
- Bearing in mind the propositions of triangle congruence for Euclidean Geometry, indicate the case for which it does not hold in Taxi Geometry.

After the resolution of this first part, a further session with the students was proposed. This time no geometric figure was presented, since the aim was the visual representations; five further questions were formulated and a sheet of graph paper was provided so that they could produce the necessary visual representations. The questions presented were:

1. Represent three scalene triangles in Taxi Geometry and explain how you did it.
2. The same for three isosceles triangles.
3. Recall that, in Euclidean Geometry, the angles opposite sides of the same measure in an isosceles triangle also have the same measure. Use the protractor to measure the angles you constructed in Taxi Geometry. What conclusion do you reach?
4. Repeat for three equilateral triangles.
5. Recall that, in Euclidean Geometry, the interior angles of an equilateral triangle are all of the same size. Use the protractor to measure the angles you constructed in Taxi Geometry. What conclusion do you reach?

To begin the exploratory process in carrying out the activity, the students were told that it was not necessary to follow strictly the steps of the problem-solving methodology of Polya [14] and Onuchic *et al.* [15]; those stages are used in the analysis as an *a posteriori* interpretive category and not as a procedure prescribed to the participants. For this session, the first step is to hand out a sheet containing the wording of the questions, further blank sheets and a sheet of graph paper for the records and representations. The data collected were analysed by the researcher who applied the instrument and discussed at the end of the activity with all the participants. In this regard, Alves-Mazzotti and Gewandsznajder [16, p. 161] state that this should be done so that the participants may express their opinion on the relevance of the observations made by the researcher, to which we add the discussion about the validity or otherwise of the findings and, consequently, the formalisation of the content.

Of the set of questions in the second session, the analysis reported in this article focuses on questions 1, 4 and 5, concerning scalene and equilateral triangles. The records corresponding to questions 2 and 3, on isosceles triangles, were collected but are not analysed here and are left for further work.

4.1. Neutrosophic Assessment of Evidence Quality

As a complementary analytical framework, this study applies neutrosophic logic [17] in order to evaluate the epistemic state of each category of evidence on which the didactic proposal reported here rests. Neutrosophic logic extends classical logic by assigning to each proposition a triple (T, I, F), where $T \in [0,1]$ represents the degree of truth or certainty of the supporting evidence, $I \in [0,1]$ represents the degree of indeterminacy (incomplete documentation, methodological heterogeneity or gaps in the evidence base), and $F \in [0,1]$ represents the degree of falsity (contradictory or non-replicated findings). In the general single-valued setting the three components are independent, so that $0 \leq T + I + F \leq 3$ [17]; the separation of favourable evidence, contrary evidence and indeterminacy into coordinates that are not forced to sum to one has been formalised in Neutrosophic Paraconsistent Logic [19]. In the elicitation used here each category is assessed by distributing one unit of confidence among the three components, so that $T + I + F = 1$ in every row of Tables 5 and 6; this normalisation is a property of the elicitation protocol adopted and not of the logic. The framework has been applied to characterise epistemic uncertainty in heterogeneous evidence bases, from analyses of scientific production [18] and narrative reviews in the health sciences [20] to the multivariate assessment of educational service quality [21], to student loyalty and retention in higher education [27] and international student mobility [28], to strategic school management [26], and to decision structures induced from indeterminate soft data [22].

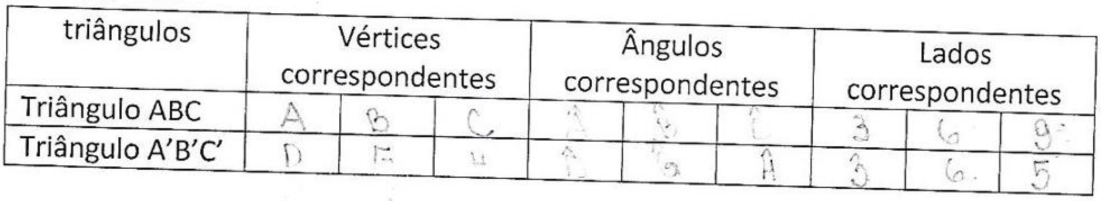
The assessment is applied to the bibliographic base of the study, that is, to the categories of evidence drawn from the sources consulted, and not to the productions of the participants, which are examined qualitatively in the preceding sections. Within that base the categories are of two distinct kinds, and the distinction is decisive for the interpretation of the triples: some rest on deductive mathematical results, whose degree of truth approaches unity and whose indeterminacy is limited to matters of formulation; others rest on didactic and epistemological literature, where convergence between independent traditions raises the degree of truth and incomplete documentation or limited scope raises the degree of indeterminacy. Each category was therefore evaluated according to three criteria: the convergence of the sources consulted, the nature of the warrant invoked (deductive or empirical), and the size and scope of the evidence base.

5. Results

The results are organised in two complementary planes. First, the geometric objects used in the instrument are analysed metrically, so that the didactic conclusions rest on demonstrated statements and on a single metric per statement. Second, the productions of the three pairs are examined in the light of that analysis.

5.1. Correspondence between the Elements of the Two Triangles

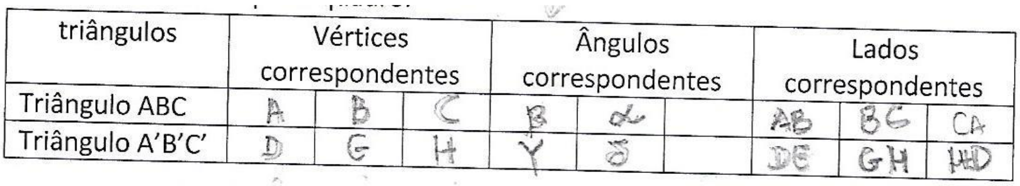
The records provided by the pairs with regard to the observation of the two triangles in Figure 4 and the completion of Table 1 are presented in Figures 5, 6 and 7 and are consolidated in Table 2.



triângulos	Vértices correspondentes			Ângulos correspondentes			Lados correspondentes		
Triângulo ABC	A	B	C	[illegible]	[illegible]	[illegible]	3	6	9
Triângulo A'B'C'	D	[illegible]	[illegible]	[illegible]	[illegible]	A	3	6	5

Figure 5. Production of the pair Flatan

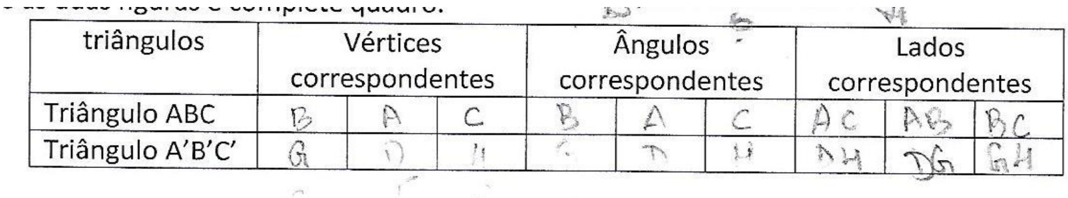
Source: participants' records



triângulos	Vértices correspondentes			Ângulos correspondentes			Lados correspondentes		
Triângulo ABC	A	B	C	β	α		AB	BG	CA
Triângulo A'B'C'	D	G	H	γ	δ		DE	GH	HD

Figure 6. Production of the pair Carlia

Source: participants' records



triângulos	Vértices correspondentes			Ângulos correspondentes			Lados correspondentes		
Triângulo ABC	B	A	C	B	A	C	AC	AB	BC
Triângulo A'B'C'	G	D	H	[illegible]	D	H	DH	DG	GH

Figure 7. Production of the pair Thanola

Source: participants' records

Table 2. Records of the three pairs for Table 1

Pair	Vertices	Angles	Sides
Flatan	A–D; B–G; C–H	α–δ; β–γ (Greek letters)	***AC ≅ DH; AB ≅ DG; BC ≅ GH***
Carlia	A–D; B–G; C–H	Â–D̂; B̂–Ĝ; Ĉ–Ĥ	***AC ≅ DH; AB ≅ DG; BC ≅ GH***
Thanola	A–D; B–G; C–H	Â–D̂; B̂–Ĝ; Ĉ–Ĥ	by the indicated measures: 3, 6, 9 and 3, 6, 5

Source: participants' records; transcription of Figures 5–7.

As regards the corresponding vertices, the three pairs identified them correctly in the two triangles. With respect to the angles there is divergence in the form of the record, possibly because they were represented by Greek letters: Flatan established the relation only between the data symbolised with Greek letters, whereas Carlia and Thanola recorded the correspondence by the letters of the three vertices, using the customary notation with a circumflex.

With respect to the correlation between the sides of the two triangles, Flatan and Carlia used the classical notation for segments, ***AC ≅ DH; AB ≅ DG; BC ≅ GH***, or in reverse order by the endpoints, whereas Thanola did so by the indicated measures. From this preliminary analysis it is concluded that there was no difficulty for the students, as expected, in identifying the correspondences between the elements of the two triangles.

5.2. Metric Analysis of the Triangles in Figure 4

The two triangles in Figure 4 are determined by the vertices C(−1, 0), A(5, 0), B(5, 3) and G(8, 0), D(9, 2), H(13, 0). The labels in the figure correspond to lengths in the taxicab metric. Table 3 gathers, for each triangle, the lengths of its sides in both metrics together with its interior angles, so that every subsequent statement can refer to a single metric.

Table 3. Dual-metric analysis of the triangles in Figure 4

Triangle	Vertices	Sides (Euclidean)	Sides (taxicab)	Interior angles
ABC	C(−1, 0); A(5, 0); B(5, 3)	AB = 3; CA = 6; CB = 6.708	AB = 3; CA = 6; $CB = 9$	90º; 63.43º; 26.57º
DGH	G(8, 0); D(9, 2); H(13, 0)	DG = 2.236; DH = 4.472; $GH = 5$	DG = 3; DH = 6; $GH = 5$	90º; 63.43º; 26.57º

Source: prepared by the authors; the labels in Figure 4 correspond to the taxicab metric.

Proposition 1. The triangles ABC and DGH in Figure 4 are similar in Euclidean Geometry, with similarity ratio $\frac{3}{\sqrt{5}}$ ≈ 1.342, and they are not congruent either in the Euclidean or in the taxicab metric.

Proof. From the vertices it follows that the interior angles of both triangles are 90º, 63.43º and 26.57º, so that, by the AA criterion, the triangles are similar. The correspondence that matches the angles is A ↔D, B ↔G, C ↔H, and for it $CB/GH = CA/DH = AB/DG = \frac{3}{\sqrt{5}}$ in the Euclidean metric. They are not congruent in that metric because the ratio differs from 1; and they are not congruent in the taxicab metric because the multisets of lengths are {3, 6, 9} and {3, 5, 6}, respectively. ■

Proposition 1 confirms the answer given by the three pairs to the first question: in Euclidean Geometry the two triangles are similar and not congruent. The reading according to which the figure would exemplify the classical ASA case does not hold, because the sides included between the two equal angles measure 3 units in the taxicab metric in both triangles, but 3 and √5 units in the Euclidean one: the criterion would be applied with sides from one metric and angles from another. As will be seen in Section 5.3, that same coincidence of lengths in the taxicab metric turns Figure 4 into a valid counterexample to the SAS and ASA criteria for that geometry.

5.3. Failure of the Euclidean Congruence Criteria under the Taxicab Metric

Lemma 1. Let $A, B, C \in \mathbb{R}^2$. Then $d_T(A, B) = d_T(A, C) + d_T(C, B)$ if and only if C belongs to the closed axis-parallel rectangle determined by A and B.

Proof. For the first coordinate, $|x_A - x_B| \leq |x_A - x_C| + |x_C - x_B|$, with equality if and only if x_C lies between x_A and x_B; the same holds for the second coordinate. Adding both inequalities yields $d_T(A, B) \leq d_T(A, C) + d_T(C, B)$, and equality is attained exactly when both coordinate inequalities are equalities, that is, when $min(x_A, x_B) \leq x_C \leq max(x_A, x_B)$ and $min(y_A, y_B) \leq y_C \leq max(y_A, y_B)$, which is precisely the membership of C in the rectangle indicated. ■

Corollary 1. Every right triangle whose legs are parallel to the coordinate axes is degenerate in the taxicab metric: the length of its hypotenuse equals the sum of the lengths of its legs. Consequently, the Pythagorean theorem does not hold in that metric.

Proof. If the legs are parallel to the axes, the vertex of the right angle belongs to the rectangle determined by the endpoints of the hypotenuse, and Lemma 1 gives the equality. For legs of lengths a and b one obtains a hypotenuse of length a + b, and $a^2 + b^2 \neq (a + b)^2$ unless a = 0 or b = 0. ■

Corollary 1 explains and validates Carlia's observation about the Pythagorean theorem and also accounts for the triples 2, 3, 5 and 3, 5, 8 obtained by Flatan's group (Figure 8): in all of them the greatest length is the sum of the other two. It should be noted that these triangles, although non-degenerate in the Euclidean metric, are degenerate in the taxicab one, and that triangle ABC in Figure 4 is in the same situation, since A belongs to the rectangle determined by C and B and $d_T(C, B) = 9 = 6 + 3$. Triangle DGH, by contrast, is not degenerate, since $3 + 6 = 9 > 5$.

In order to obtain non-degenerate counterexamples to the classical criteria it suffices to take a pair of triangles that are equilateral in the taxicab metric, such as those collected in Table 4.

Table 4. Two non-congruent triangles that are equilateral in the taxicab metric

Triangle	Vertices	Sides (taxicab)	Sides (Euclidean)	Interior angles
T_1	(0, 0); (0, 4); (2, 2)	4; 4; 4	4; 2.828; 2.828	90º; 45º; 45º
T_2	(0, 0); (1, 3); (3, 1)	4; 4; 4	3.162; 2.828; 3.162	53.13º; 63.43º; 63.43º

Source: prepared by the authors; neither is degenerate, since $4 + 4 > 4$.

Proposition 2. The SSS criterion does not hold in Taxi Geometry, and equilaterality does not imply equiangularity in that geometry.

Proof. Consider T_1, with vertices (0, 0), (0, 4) and (2, 2), and T_2, with vertices (0, 0), (1, 3) and (3, 1). All three sides of each measure 4 in the taxicab metric, and neither is degenerate, since $4 + 4 = 8 > 4$. However, their interior angles are 90º, 45º and 45º for T_1 and 53.13º, 63.43º and 63.43º for T_2, so that there is no one-to-one correspondence between their vertices preserving the angles: the triangles are not congruent despite having their three sides of equal measure. Since both are equilateral in the taxicab metric and their interior angles are not equal to one another, equilaterality does not imply equiangularity either. ■

Proposition 3. The SAS and ASA criteria do not hold in Taxi Geometry either.

Proof. The triangles in Figure 4 suffice. For the SAS criterion: the angle at A and the angle at D both measure 90º, and the sides forming them measure, in the taxicab metric, $AB = DG = 3$ and $AC = DH = 6$; however, the third sides measure $CB = 9$ and $GH = 5$, so that the triangles are not congruent. For the ASA criterion: the angles at A and at D measure 90º, the angles at B and at G measure 63.43º, and the sides included between them measure $AB = DG = 3$ in the taxicab metric; the triangles are still not congruent by Proposition 1. ■

Corollary 2. None of the three classical congruence criteria of Euclidean Geometry — SSS, SAS and ASA — transfers to Taxi Geometry.

Corollary 2 makes it possible to integrate into a single finding the three divergent answers that the pairs gave to the fourth question. Flatan pointed to the ASA case, Carlia to the SSS case and Thanola indicated "SAS or SAA$_o$ or ASA": each pair correctly identified a criterion that does indeed fail, and the divergence among them is not due to an error but to the fact that none of them yet had the general statement available. It should further be stressed that the criteria fail for different reasons: SSS fails because taxicab lengths do not determine the shape of the triangle, whereas SAS and ASA fail because Euclidean angle measure is not invariant under that metric.

5.4. The Notions of Triangle and Angle in Taxi Geometry

Taxi Geometry shares its points and its lines with the Euclidean plane and differs from it only in the metric: triangles exist in it and are in fact constructed by the participants throughout the study. Likewise, Ekici, Kocayusufoğlu and Akça [7] define an inner product and a cosine proper to this geometry, that is, precisely a notion of angle. The result sustained by the students' productions is therefore not the absence of these notions, but the loss of the relations that link them in Euclidean Geometry.

In precise terms, what is lost is the determination of angle measure by the lengths of the sides. In Euclidean Geometry the three lengths determine the three angles; in Taxi Geometry they do not, as Proposition 2 shows. Hence the classical relations between sides and angles — the equivalence between equilaterality and equiangularity, the equality of the base angles in an isosceles triangle, and the proposition according to which the greater side is opposite the greater angle — cease to hold. This is the formulation captured, in their own words, by the conclusions of Flatan's group reproduced in Figure 11.

5.5. Exploration Activity and Students' Productions

In the second stage of the research, an exploration activity is proposed to the students in which they must use visualisation processes, allowing the participants to choose the triangles and the steps in order to solve the problem posed to them. For this activity they were asked to represent three scalene triangles in Taxi Geometry and to explain how they did it; the following solution strategies were found among the working groups.

Flatan's working group. The members of this group began by discussing among themselves that they had to represent different scalene triangles (three different sides) in Taxi Geometry; they began by understanding the problem and then devising a plan, and started to draw three triangles with sides of different measures on the graph paper, calculating the measures according to Taxi Geometry.

For Flatan's group the following stages of work to solve the problem posed are evident:

- They begin by understanding the problem.
- They devise a plan.
- They carry out the plan.
- They review the plan by measuring the interior angles of the triangles; they mention that "*scalene triangles, like the Euclidean ones, are those that have all sides of different measures*".

Figure 8 shows the results of Flatan's group, in which they used graph paper to represent a variety of scalene triangles, including right, acute and obtuse ones. We observe the first representation in the upper left-hand corner, where they construct the triangle with sides 2, 3, 5 and also explain how the last measurement was found in Taxi Geometry: 1 unit to the right from the lower vertex, 1 unit up and 1 more to the right and 2 again to the top of the triangles, while the other two sides were marked horizontally with 2 units and vertically with 3 units, since they chose the more marked line of the paper for the unit, 1 cm at a scale of 1:100.

Upon constructing the right triangle 2, 3, 5, they immediately produced the 3, 5, 8 one and verbalised that the Pythagorean theorem of Euclidean Geometry was no longer valid in Taxi Geometry, making a third representation (see Figure 8). Corollary 1 formalises exactly this observation.

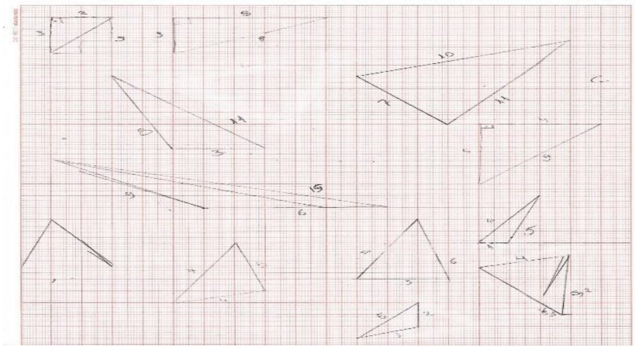


Figure 8. Triangles in Taxi Geometry obtained by Flatan's group

Source: participants' records

In representing the various triangles, new questions arose and the researcher observed and took note of these conclusions. One of them was the observation that, with the same measures, different triangles can be obtained, as in the initial and final cases 2, 3, 5. Thus, the visual aspects obtained concretely in this execution of the plan led the individuals to new mental constructions about the triangular geometric entity. This led them to continue working on the solution of the problem posed, in which the teacher intervened and commented in general terms that they should take the following elements into account in their constructions, posing new questions:

- Repeat for isosceles and equilateral triangles the following: recall that, in Euclidean Geometry, the angles opposite sides of the same measure in an isosceles triangle also have the same measure. Use the protractor to measure the angles you constructed in Taxi Geometry. What conclusion do you reach?
- Recall that, in Euclidean Geometry, the interior angles of an equilateral triangle are all of the same size. Use the protractor to measure the angles you constructed in Taxi Geometry. What conclusion do you reach?

The groups had didactic materials provided by the teacher and began to use the protractor, an instrument for measuring angles. With this, Flatan's working group produced the record shown in Figure 9.

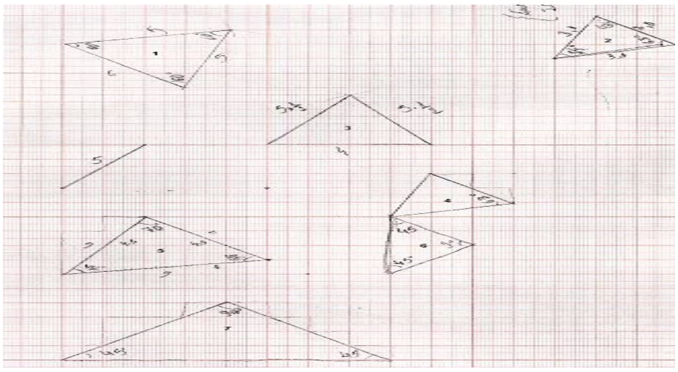


Figure 9. Triangles in Taxi Geometry obtained with the use of the protractor

Source: participants' records

In the course of this activity, some difficulties with the calculation of the measurements were observed, since the angle was now given and therefore side measurements were required that did not always yield a whole number, where the graph paper allowed a better approximation.

Carlia's working group devised the plan for its resolution as follows:

- First, we have to draw three scalene triangles (three sides with different measures in Taxi Geometry).
- 1st. Draw a Cartesian plane. 2nd. Choose the coordinates of the triangles ΔABC, ΔDEF and ΔGHI. 3rd. Determine the measure of the segments from the formula d_T so as to obtain three scalene triangles.

The pair presented their representations as in Figure 10, which already has the angles marked with the corresponding instrument. This group opted for graph paper and worked on millimetre paper.

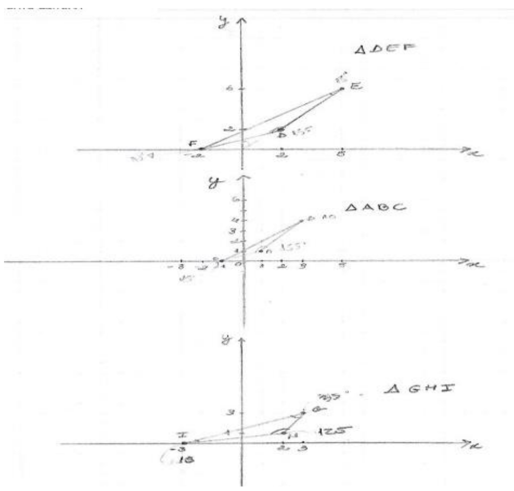
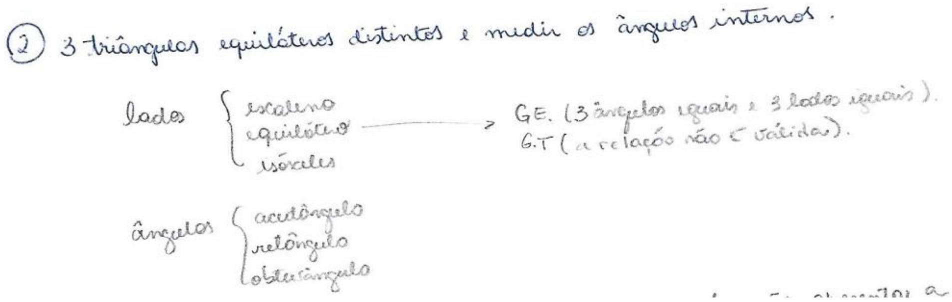


Figure 10. Triangles in Taxi Geometry obtained by Carlia

Source: participants' records

This working group was not conclusive, unlike Flatan's group, which clearly described its conclusions on the matter, as shown in Figure 11:



(2) 3 triângulos equiláteros distintos e medir os ângulos internos.

lados { escaleno, equilátero, isósceles → G.E. (3 ângulos iguais e 3 lados iguais). G.T (a relação não é válida).

ângulos { acutângulo, retângulo, obtusângulo

Figure 11. Conclusions given by Flatan's group

Source: participants' records

The group indicates: "*An equilateral triangle in Taxi Geometry does not present the Euclidean relation of 3 equal sides / 3 equal angles. In Taxi Geometry this relation is not valid*". This formulation coincides with the second part of Proposition 2.

The third working group, made up of Thanola, was more concise in its representations and observations, but its conclusion is important in indicating what guided them in solving the problem. This indication is presented below.

Thanola: "*We drew the Cartesian plane. We constructed three triangles (right, obtuse, acute). We calculated the measure of their sides by means of Taxi Geometry, through visualisation on graph paper*".

5.6. Evaluative Syntheses of the Working Groups

With regard to the problem-solving stages of Onuchic *et al.* [15], there is a differentiation from other authors because they include evaluation in one of them, which is relevant when it comes to the education of mathematics teachers, as in the present situation. From this point of view, as the final step of the activity reported here, we asked each working group to write an evaluative synthesis of it.

Flatan's group: "*With the previous activity we were able to construct some concepts about Taxi Geometry in relation to scalene and equilateral triangles, comparing them with the relations of Euclidean Geometry. The students took an active part in this construction of knowledge, since they were asked to think about these concepts, which made the activity very useful and interesting*".

Carlia's group: "*The activity was very interesting because I followed the steps of the problem-solving methodology using Taxi Geometry. This takes us out of thinking related only to Euclidean Geometry, solving the problems proposed and constructing new schemes*".

Thanola's group: "*The activity carried out through problem solving made it possible, from the very first moment, to understand the problem to be tackled, to reflect in order to devise a plan and also, with the construction and execution of the plan, to check whether what was achieved was consistent with the plan. The activity therefore allowed the student to arrive at the concept through construction, rather than through a concept given directly*".

Read in the light of the evaluation criterion of Onuchic *et al.* [15], the three syntheses agree in attributing the learning to the activity of construction and not to the exposition of the content, and two of the three explicitly mention the comparison between the two geometries as the mechanism that produced that learning. None of them, by contrast, formulates a judgement on the validity of the results obtained, which suggests that the evaluative stage was appropriated in its experiential rather than in its epistemic dimension.

As a result of this research practice, it can be concluded that exploration in solving problems posed to students in activities unknown to them from a given geometry — in this case Taxi Geometry — compared with a familiar one — in this case Euclidean Geometry — takes them out of their comfort zone and "forces" them to revisit concepts that until then they considered assimilated and unquestionable in the mathematical universe, not least because they did not know that whole universe.

Conservative and traditional teaching that provides concepts and demonstrations is certainly no longer recommended in mathematics education. It is firmly believed that the student must be included as the main actor of the process, the constructor of their own knowledge; in this sense, in Geometry, the development of geometric thinking can be fostered, according to our studies, either in Euclidean Geometry or in another, as in the present research in Taxi Geometry, and even if it is not possible to reach a certain depth of content, there is a stimulus for new discoveries and, with this, the development of advanced geometric thinking.

5.7. Neutrosophic Assessment of Evidence Categories

The assessment is applied on two planes. Table 5 evaluates the categories of evidence arising from the Results section itself, that is, what the study establishes; Table 6 evaluates the categories of evidence drawn from the literature, that is, what the study rests on. The distinction matters because the remedies each plane calls for are different.

Table 5. Neutrosophic Assessment of the Evidence Categories of the Results Section

Evidence Category (Results)	Basis in Section 5	T	I	F
Correct identification of the correspondences between the elements of the two triangles	5.1; Figures 5–7; Table 2	0.88	0.10	0.02
Euclidean similarity and non-congruence of the triangles in Figure 4	Proposition 1; Table 3	0.97	0.02	0.01
Degeneracy criterion in the taxicab metric and failure of the Pythagorean theorem	Lemma 1; Corollary 1; Figure 8	0.96	0.03	0.01
Failure of the SSS criterion and non-equivalence between equilaterality and equiangularity	Proposition 2; Table 4; Figure 11	0.95	0.04	0.01
Failure of the SAS and ASA criteria under the taxicab metric	Proposition 3; Figures 4 and 9	0.94	0.05	0.01

Evidence Category (Results)	Basis in Section 5	T	I	F
Integration of the three divergent answers of the pairs into a single statement	Corollary 2; Section 5.3	0.85	0.10	0.05
Visual construction of geometric thinking by the participants	5.5; Figures 8–10	0.62	0.33	0.05
Appropriation of the evaluative stage and transferability of the didactic outcome	5.6; syntheses of the three groups	0.50	0.43	0.07

Note: T = degree of truth/certainty; I = degree of indeterminacy; F = degree of falsity; the three degrees are elicited under the constraint $T + I + F = 1$ *(Section 4.1).*

Table 6 completes the assessment with the six categories of evidence drawn from the literature that sustains the study, together with the sources on which each category depends.

Table 6. Neutrosophic Assessment of the Bibliographic Evidence Categories

Evidence Category	Key Sources	T	I	F
Epistemological premises: quasi-empirical and heuristic construction of mathematical knowledge	[1], [2], [5]	0.80	0.15	0.05
Visualisation as a constitutive process of geometric thinking	[4], [10], [11], [12]	0.85	0.12	0.03
Formal metric foundations of non-Euclidean geometries (Klein, Riemann, taxicab metric and norm)	[5], [7]	0.95	0.04	0.01
Historical attribution of the origin of TaxiCab Geometry (Minkowski; Menger, 1952)	[6]	0.80	0.15	0.05
Non-transfer of the Euclidean congruence criteria to the taxicab metric	[7], [8], [9]	0.78	0.14	0.08
Problem solving as a didactic framework for the teaching of geometry	[14], [15], [16]	0.60	0.32	0.08

Note: T = degree of truth/certainty; I = degree of indeterminacy; F = degree of falsity; the three degrees are elicited under the constraint $T + I + F = 1$ *(Section 4.1).*

Table 5 shows a clear stratification of the results according to the nature of their warrant. The four categories with the highest degree of truth — from 0.94 to 0.97 — correspond to statements demonstrated in Section 5: Proposition 1, Lemma 1 with its Corollary 1, Proposition 2 and Proposition 3. Their certainty does not depend on the number of participants or on the variability of their productions, but solely on the validity of the proofs, so that the residual indeterminacy is limited to matters of formulation and the falsity is practically nil. It is this deductive base that allows the didactic conclusions of the study to rest on more than the agreement of three pairs.

The category concerning the integration of the divergent answers ($T = 0.85$, $F = 0.05$) occupies a characteristic intermediate position. Corollary 2 establishes deductively that the three classical criteria fail, with which the three answers of the pairs turn out to be partially correct and cease to be mutually contradictory; the residual falsity records that the pairs did not formulate that integration themselves, but that it is obtained in the analysis. It is worth noting the contrast with the assessment that would have corresponded to this same category before the metric analysis of Sections 5.2 and 5.3: three incompatible answers recorded without resolution would have yielded a high degree of falsity. The reduction of F is itself a result of the work reported here.

The assessment shows that the epistemic robustness of the study is not uniform across its evidence base, and that the variation follows the nature of the warrant rather than the number of sources. The highest degree of truth corresponds to the formal metric foundations of the non-Euclidean geometries ($T = 0.95$, $I = 0.04$, $F = 0.01$): the models of Klein and Riemann and the definitions of d_E, d_T and of the taxicab norm are deductive results, so their certainty is bounded only by the accuracy of their formulation and not by empirical variability. The category of visualisation as a constitutive process of geometric thinking follows ($T = 0.85$, $I = 0.12$), sustained by the convergence of two independent research traditions, those of Arcavi [11], [12] and Leivas [10], on a compatible characterisation of the construct.

The historical attribution of the origin of TaxiCab Geometry retains a moderate indeterminacy ($I = 0.15$). The claim that Minkowski addressed the families of metrics and that Menger presented the geometry exhibited in Chicago in 1952 rests on a single secondary source, now fully identified [6]: Reinhardt's article in The Mathematics Enthusiast, whose bibliographic data were incomplete in the first version of this manuscript. Completing the reference removes the documentary component of that indeterminacy, which in the first assessment reached 0.40; what remains records that the attribution still depends on one secondary source and not on primary documentation of Menger's exhibition.

The category concerning the non-transfer of the Euclidean congruence criteria to the taxicab metric attains a high degree of truth ($T = 0.78$) with a low residual falsity ($F = 0.08$). The taxicab norm and inner product defined in [7] establish that length and angle measure are governed by distinct structures in this geometry, and [8] and [9] document that classical Euclidean loci and relations do not carry over unchanged to the taxicab plane. The residual falsity reflects that the sources consulted approach the question from different angles — the algebraic structure of the norm in [7], the representation of conics in [8], the semiotic registers of a concrete problem in [9] — and that none of them offers a unified treatment of triangle congruence under this metric. Corollary 2 in Section 5.3 supplies that gap with a deductive warrant of its own.

The identification of the correspondences between the elements of the two triangles ($T = 0.88$) rests on convergent empirical evidence: the three pairs agreed, and their records are transcribed in Table 2, so that the claim is verifiable from Figures 5 to 7. Its moderate indeterminacy reflects only the divergence in the form of notation for the angles, which does not affect the content of the correspondence.

The last two categories concentrate the indeterminacy of the study and are didactic rather than mathematical in nature. The visual construction of geometric thinking ($T = 0.62$, $I = 0.33$) is well documented for the participating pairs through their own representations, but three pairs from a single course do not allow one to establish with what generality that process occurs. The appropriation of the evaluative stage and the transferability of the didactic outcome ($T = 0.50$, $I = 0.43$) is the weakest category: the three syntheses agree in attributing the learning to construction, but none of them issues a judgement on the validity of what was obtained, and the design provides for neither a control condition nor follow-up.

Read together, the two tables indicate where future work should concentrate. The deductive categories of Table 5 require no additional evidence, but at most verification of the proofs. The didactic categories, by contrast, share with the last row of Table 6 an indeterminacy of scope that is reduced only by widening the empirical base: replicating the sequence with more pairs, in more than one institution and with a record of subsequent performance. Keeping the two kinds of uncertainty apart is precisely what the tripartite representation allows and what a single quality score would conflate.

6. Conclusions

The research process on the study of Taxi Geometry and Euclidean Geometry with respect to the concepts of congruence and similarity of triangles made it possible to respond to the objective. The results obtained have generated statements that are essential in this work; they are:

The notions of congruence and similarity of triangles, defined in Euclidean Geometry through the equality or proportionality of sides and angles, admit formulation in Taxi Geometry, but they lose in it the criteria that characterise them: none of the three classical cases — SSS, SAS and ASA — transfers to that geometry, as Corollary 2 establishes.

The participating students manage to interpret, through their visual processes, that an equilateral triangle in Taxi Geometry does not present the Euclidean relation of three equal sides and three equal angles. The metric analysis of Section 5.3 confirms this conclusion and makes its scope precise: equilaterality does not imply equiangularity because taxicab lengths do not determine angle measure.

The notions of triangle and angle do exist in Taxi Geometry, which shares points and lines with the Euclidean plane and differs from it only in the metric; what is not preserved is the relation that links sides and angles in Euclidean Geometry. The students' productions evidence precisely that loss and not the absence of the objects.

This study makes it possible to delimit the scope of some postulates, theorems and definitions used in the congruence and similarity of triangles of Euclidean Geometry. An example of this was the students' observation that with the same measures different triangles can be obtained, an observation that Lemma 1 and Proposition 2 formalise.

Over time it has been shown that a traditional teaching for the construction of mathematical concepts persists. This research indicated that, through the exploration and solution of problems by means of a geometry unknown to them — in this case Taxi Geometry — and a known geometry — the Euclidean one — students begin to reason in a transformed manner that affects their way of thinking, becoming more active and more analytical individuals and participants in the construction of their own knowledge.

Another result achieved in this research is the way of presenting and addressing the problems proposed; to this end, a methodology was designed that makes it possible to give dynamism to the set of problems, structured according to the student's learning. Each of these is based on performance, motivation for learning, achievements and difficulties presented, which made possible the realisation and use of visual skills through problem solving, advancing in the learning of geometric concepts and in the development of mathematical thinking.

A didactic result worth highlighting is that the three pairs answered the initial question about Figure 4 correctly, and that each of them identified a congruence criterion that does indeed fail under the taxicab metric. The divergence among their answers expresses not an error but the absence of a general statement integrating them; obtaining that statement from their own productions is precisely the kind of closure that the problem-solving methodology pursues.

The incorporation of this teaching–learning instrument, made up of attractive and interesting situations as expressed by the participating students themselves, motivated them and thereby gave them the opportunity to promote their own learning.

The neutrosophic assessment in Tables 5 and 6 makes explicit the epistemic status both of what this work establishes and of what it rests on. The statements demonstrated in Section 5 attain degrees of truth between 0.94 and 0.97, and the integration of the pairs' divergent answers is resolved ($F = 0.05$) thanks to that deductive base; the metric foundations taken from the literature are equally robust ($T = 0.95$), as is the role attributed to visualisation ($T = 0.85$). Indeterminacy is concentrated, by contrast, in the transferability of the didactic outcome ($I = 0.43$), in the transfer of the problem-solving framework to this content ($I = 0.32$) and, to a lesser degree, in the documentary record of the historical origin of the geometry ($I = 0.15$, once the source is fully identified). Future work should reduce the first by replicating the sequence with a larger number of pairs and across more than one institution, and the second by widening the didactic sources; multi-criteria procedures with neutrosophic information [23] offer one route to aggregate the resulting evidence across categories. The incorporation of neutrosophic cognitive maps could further serve to model the coexistence of Euclidean and taxicab intuitions in the reasoning of the participants, a coexistence that this study documents but does not formalise [17], [18].

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