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An Introduction to Advanced Soft Set Variants: SuperHyperSoft Sets, IndetermSuperHyperSoft Sets, IndetermTreeSoft Sets, BiHyperSoft sets, GraphicSoft sets, and Beyond

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Abstract. This paper focuses on the study of Soft Sets, a concept that has led to the development of various extensions, including Double-Framed Soft Sets [5, 6, 10, 21], Hypersoft Sets [1], SuperHyperSoft Sets [36], ForestSoft Sets, TreeSoft Sets [3], IndetermSoft Sets [35], and IndetermHyperSoft Sets [35]. These extensions have been actively explored in recent research. Smarandache [https://fs.unm.edu/TSS/] introduced six new types of Soft Sets, such as: the HyperSoft Set (2018), IndetermSoft Set (2022), IndetermHyperSoft Set (2022), SuperHyperSoft Set, TreeSoft Set (2022) and ForestSoft Set (2024).

In this paper, we review several advanced soft set concepts, including IndetermSuperHyperSoft Sets, IndetermForest-Soft Sets, IndetermTreeSoft Sets, and other related structures. We hope that this work will inspire more researchers to explore Soft Sets and uncertainty modeling, further advancing studies in this field.

Keywords: Soft Set, Hypersoft set, Superhypersoft set, IndetermSoft Set, Treesoft set

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1. **Soft Set, Hypersoft Set, and SuperHypersoft Set**

A Soft Set offers a straightforward approach to parameterized decision modeling by associating attributes (or parameters) with subsets of a universal set, effectively addressing uncertainty in a structured manner [25, 26]. Several related mathematical frameworks, such as Fuzzy Sets [44], Neutrosophic Sets [14, 32, 41], and Rough Sets [30], have been introduced to handle different aspects of uncertainty. Building upon this foundation, the Hypersoft Set extends Soft Sets by incorporating multi-attribute decision modeling. Instead of assigning a single parameter to a subset of the universal set, a Hypersoft Set maps combinations of multiple attributes to subsets of the universal set, thereby improving its capability for handling complex decision-making scenarios [20, 33]. Further generalizing the concept, SuperHypersoft Sets expand the functionality of Hypersoft Sets by mapping power set combinations of multiple attribute values to subsets of a universal set. This higher-order approach allows for multidimensional decision-making and captures intricate interrelationships among attributes, making it a powerful tool for modeling complex systems [15, 16, 36, 42].

Definition 1.1 (Soft Set)**.** [25, 26] Let U be a universal set and A be a set of attributes. A soft set over U is a pair $(\mathcal{F}, S)$, where $S \subseteq A$ and $\mathcal{F} : S \to \mathcal{P}(U)$. Here, $\mathcal{P}(U)$ denotes the power set of U. Mathematically, a soft set is represented as:

$$(\mathcal{F}, S) = \{(\alpha, \mathcal{F}(\alpha)) \mid \alpha \in S, \mathcal{F}(\alpha) \in \mathcal{P}(U)\}.$$

Each $\alpha \in S$ is called a parameter, and $\mathcal{F}(\alpha)$ is the set of elements in U associated with α.

Definition 1.2 (Hypersoft Set)**.** [33] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be attribute domains. Define $\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m$, the Cartesian product of these domains. A hypersoft set over U is a pair $(G, \mathcal{C})$, where $G : \mathcal{C} \to \mathcal{P}(U)$. The hypersoft set is expressed as:

$$(G, \mathcal{C}) = \{(\gamma, G(\gamma)) \mid \gamma \in \mathcal{C}, G(\gamma) \in \mathcal{P}(U)\}.$$

For an m-tuple $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in \mathcal{C}$, where $\gamma_i \in \mathcal{A}_i$ for $i = 1, 2, \ldots, m$, $G(\gamma)$ represents the subset of U corresponding to the combination of attribute values $\gamma_1, \gamma_2, \ldots, \gamma_m$.

Definition 1.3 (SuperHyperSoft Set)**.** [36] Let U be a universal set, and let $\mathcal{P}(U)$ denote the power set of U. Consider n distinct attributes $a_1, a_2, \ldots, a_n$, where $n \geq 1$. Each attribute a_i is associated with a set of attribute values A_i, satisfying the property $A_i \cap A_j = \emptyset$ for all $i \neq j$.

Define $\mathcal{P}(A_i)$ as the power set of A_i for each $i = 1, 2, \ldots, n$. Then, the Cartesian product of the power sets of attribute values is given by:

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n).$$

A SuperHyperSoft Set over U is a pair $(F, \mathcal{C})$, where:

$$F : \mathcal{C} \to \mathcal{P}(U),$$

and F maps each element $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$ (with $\alpha_i \in \mathcal{P}(A_i)$) to a subset $F(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq U$. Mathematically, the SuperHyperSoft Set is represented as:

$$(F, \mathcal{C}) = \{(\gamma, F(\gamma)) \mid \gamma \in \mathcal{C}, F(\gamma) \in \mathcal{P}(U)\}.$$

Here, $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathcal{C}$, where $\alpha_i \in \mathcal{P}(A_i)$ for $i = 1, 2, \ldots, n$, and $F(\gamma)$ corresponds to the subset of U defined by the combined attribute values $\alpha_1, \alpha_2, \ldots, \alpha_n$.

2. Double-Framed Hypersoft Set

The Double-Framed Soft Set [5,6,21] and Double-Framed Hypersoft Set [10] are extended concepts of the Soft Set and Hypersoft Set, incorporating two frames for enhanced representation. Their definitions are provided below.

Definition 2.1 (Double Framed Soft Set)**.** [5, 6, 21] Let U be the universal set, and let A be a set of parameters. A *Double-Framed Soft Set* is a triple $\langle(\alpha, \beta); A\rangle$, where:

(1) $\alpha : A \to P(U)$ and $\beta : A \to P(U)$ are mappings from the parameter set A to the power set of U.

(2) $\alpha(x)$ represents the *positive frame* and $\beta(x)$ represents the *negative frame* for each parameter $x \in A$.

A Double-Framed Soft Set satisfies the condition:

$$\forall x, y \in A, \quad \alpha(x * y) \supseteq \alpha(x) \cap \alpha(y), \quad \beta(x * y) \subseteq \beta(x) \cup \beta(y),$$

where $*$ is a binary operation defined on A.

Definition 2.2 (Double-Framed Hypersoft Set (DFHSS))**.** [10] Let U be the universal set and $P(U)$ denote the power set of U. Let $\{a_1, a_2, \ldots, a_n\}$ represent n distinct attributes, where each attribute a_i is associated with a set of attribute values ϕ_i, satisfying the conditions:

$$\phi_i \cap \phi_j = \emptyset \quad \text{for } i \neq j, \quad i, j \in \{1, 2, \ldots, n\}.$$

A *Double-Framed Hypersoft Set (DFHSS)* is defined as a tuple:

$$(\pi_1, \pi_2; \phi_1 \times \phi_2 \times \cdots \times \phi_n),$$

where:

- $\phi_1 \times \phi_2 \times \cdots \times \phi_n$ is the Cartesian product of the attribute value sets.
- $\pi_1, \pi_2 : \phi_1 \times \phi_2 \times \cdots \times \phi_n \to P(U)$ are mappings that associate each tuple of attribute values with subsets of the universal set U.

Definition 2.3 (Double-Framed SuperHypersoft Set)**.** Let U be a universal set. Suppose there are n distinct attributes $a_1, a_2, \ldots, a_n$, each associated with a set of possible values A_i such that $A_i \cap A_j = \varnothing$ for all $i \neq j$. For each $i \in \{1, 2, \ldots, n\}$, let

$$\mathcal{P}(A_i)$$

denote the power set of A_i. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

which is the Cartesian product of these power sets.

A *Double-Framed SuperHypersoft Set (DFSHSS)* over U is then a triple

$$\bigl(\Theta_1, \Theta_2; \mathcal{C}\bigr),$$

where

$$\Theta_1 : \mathcal{C} \;\to\; \mathcal{P}(U), \quad \Theta_2 : \mathcal{C} \;\to\; \mathcal{P}(U).$$

That is, both Θ_1 and Θ_2 map each element of $\mathcal{C}$ (i.e., each n-tuple $\gamma = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in \mathcal{P}(A_i)$) to a subset of U.

Informally, $\Theta_1(\gamma)$ and $\Theta_2(\gamma)$ can be viewed as two distinct but related "frames" (e.g., a *positive* vs. *negative*, or *lower* vs. *upper* approximation) for the combined attribute values in γ.

3. Treesoft set and ForestSoft Set

A *TreeSoft Set* maps the power set of a hierarchical tree-like structure of attributes, Tree(A), to subsets of a universal set [40]. It is known that concepts such as TreeSoft Sets can generalize MultiSoft Sets and related frameworks.

Definition 3.1. [37] Let U be a universe of discourse, and let H be a non-empty subset of U, with $P(H)$ denoting the power set of H. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of attributes (parameters, factors, etc.), for some integer $n \geq 1$, where each attribute A_i (for $1 \leq i \leq n$) is considered a first-level attribute.

Each first-level attribute A_i consists of sub-attributes, defined as:

$$A_i = \{A_{i,1}, A_{i,2}, \ldots\},$$

where the elements $A_{i,j}$ (for $j = 1, 2, \ldots$) are second-level sub-attributes of A_i. Each second-level sub-attribute $A_{i,j}$ may further contain sub-sub-attributes, defined as:

$$A_{i,j} = \{A_{i,j,1}, A_{i,j,2}, \ldots\},$$

and so on, allowing for as many levels of refinement as needed. Thus, we can define sub-attributes of an m-th level with indices $A_{i_1,i_2,\ldots,i_m}$, where each i_k (for $k = 1, \ldots, m$) denotes the position at each level.

This hierarchical structure forms a tree-like graph, which we denote as Tree(A), with root A (level 0) and successive levels from 1 up to m, where m is the depth of the tree. The terminal nodes (nodes without descendants) are called *leaves* of the graph-tree.

A *TreeSoft Set* F is defined as a function:

$$F : P(\text{Tree}(A)) \to P(H),$$

where Tree(A) represents the set of all nodes and leaves (from level 1 to level m) of the graph-tree, and $P(\text{Tree}(A))$ denotes its power set.

A *ForestSoft Set* is formed by taking a collection of TreeSoft Sets and "gluing" (uniting) them together so as to obtain a single function whose domain is the union of all tree-nodes' power sets and whose values in $P(H)$ combine the images given by the individual TreeSoft Sets [31, 38].

Definition 3.2 (ForestSoft Set)**.** [38] Let U be a universe of discourse, $H \subseteq U$ be a non-empty subset, and $P(H)$ be the power set of H. Suppose we have a finite (or countable) collection of TreeSoft Sets

$$\{\, F_t : P(\text{Tree}(A^{(t)})) \to P(H) \}_{t \in T},$$

where each F_t is a TreeSoft Set corresponding to a tree Tree($A^{(t)}$) of attributes $A^{(t)}$.

We construct a *forest* by taking the (disjoint) union of all these trees:

$$\text{Forest}\big(\{A^{(t)}\}_{t \in T}\big) \;=\; \bigsqcup_{t \in T} \text{Tree}\big(A^{(t)}\big).$$

A *ForestSoft Set*, denoted by

$$\mathbf{F} : P\big(\text{Forest}(\{A^{(t)}\})\big) \;\longrightarrow\; P(H),$$

is defined as the *union* of all TreeSoft Set mappings F_t. Concretely, for any element $X \in P\big(\text{Forest}(\{A^{(t)}\})\big)$, we set

$$\mathbf{F}(X) \;=\; \bigcup_{\substack{t\in T \\ X\cap\,\text{Tree}(A^{(t)})\neq\varnothing}} F_t\big(X \cap \text{Tree}(A^{(t)})\big),$$

where we only apply F_t to that portion of X belonging to the tree $\text{Tree}(A^{(t)})$.

4. Double-Framed Treesoft Set

We now define the Double-Framed Treesoft Set, extending the idea of a Treesoft Set (which maps subsets of a hierarchical attribute tree to subsets of the universe) by introducing two frames [17].

Definition 4.1 (Double-Framed Treesoft Set)**.** [17] Let:

- U be a universal set.
- $\text{Tree}(A)$ be a hierarchical attribute tree constructed from an attribute set $A = \{A_1, A_2, \ldots, A_n\}$ (with possibly multiple levels of sub-attributes, sub-sub-attributes, etc.).
- $P(\text{Tree}(A))$ be the power set of all nodes (including leaves) in the tree $\text{Tree}(A)$.

A *Double-Framed Treesoft Set (DFTS)* is a triple

$$\big(\Phi_1, \Phi_2; \text{Tree}(A)\big),$$

where

$$\Phi_1 : P\big(\text{Tree}(A)\big) \;\to\; \mathcal{P}(U), \quad \Phi_2 : P\big(\text{Tree}(A)\big) \;\to\; \mathcal{P}(U).$$

For each subset of nodes $X \subseteq \text{Tree}(A)$, $\Phi_1(X)$ and $\Phi_2(X)$ represent two distinct frames (e.g., *positive* vs. *negative* or *lower* vs. *upper*) for the elements of U relevant to the portion of the tree in X.

5. New Concepts: Double-Framed Forestsoft Set

In this paper, we introduce a new concept, the Double-Framed Forestsoft Set, which is defined as follows. We anticipate that future research will further explore its applications and mathematical properties.

Definition 5.1 (Double-Framed Forestsoft Set)**.** Let U be a universal set, $H \subseteq U$ be a non-empty subset, and let

$$\text{Forest}\big(\{A^{(t)}\}_{t\in T}\big)$$

be a forest formed by the disjoint union of attribute trees $\text{Tree}(A^{(t)})$ for $t \in T$. Let

$$P\big(\text{Forest}(\{A^{(t)}\}_{t\in T})\big)$$

denote the power set of the forest. A *Double-Framed Forestsoft Set* (DFFS) is defined as a pair of mappings

$$\big(\Psi_1, \Psi_2\big) : P\big(\text{Forest}(\{A^{(t)}\}_{t\in T})\big) \to P(H),$$

such that for each $X \subseteq \text{Forest}(\{A^{(t)}\}_{t\in T})$,

$$\Psi_1(X) \quad \text{and} \quad \Psi_2(X)$$

represent two distinct frames (for example, a positive frame and a negative frame) associated with the subset X of the forest. These frames provide dual evaluations of the objects in H with respect to the hierarchical attributes in X.

Example 5.2 (Evaluating Environmental Impact)**.** Suppose

$$U = \{\text{Site1, Site2, Site3, Site4}\}$$

represents a set of environmental sites, and let $H = U$. Assume that the attributes are organized into trees reflecting factors such as *Pollution Level* and *Biodiversity*. For simplicity, consider a forest formed by two trees:

- $\text{Tree}(A^{(1)})$ representing *Pollution* with nodes {Low, Medium, High}.
- $\text{Tree}(A^{(2)})$ representing *Biodiversity* with nodes {Rich, Moderate, Poor}.

A traditional Forestsoft Set **F** might assign:

$$\mathbf{F}(X) = \{\text{Site2, Site3}\} \quad \text{for } X = \{\text{High Pollution}, \text{Poor Biodiversity}\}.$$

Now, a Double-Framed Forestsoft Set introduces two mappings:

$$\Psi_1, \Psi_2 : P\big(\text{Forest}(\{A^{(1)}, A^{(2)}\})\big) \to P(U).$$

For the same X, suppose:

$$\Psi_1(X) = \{\text{Site2}\} \quad \text{(a positive evaluation, e.g., based on recovery potential)},$$

$$\Psi_2(X) = \{\text{Site3}\} \quad \text{(a negative evaluation, e.g., based on risk assessment)}.$$

Thus, the Double-Framed Forestsoft Set provides a dual perspective for the environmental assessment of sites under the attributes in X.

Example 5.3 (Evaluating Agricultural Land)**.** Let

$$U = \{\text{Field1, Field2, Field3}\}$$

be a set of agricultural fields, and suppose $H = U$. Assume the forest is built from two trees representing:

- $\text{Tree}(A^{(1)})$: *Soil Quality* with nodes {Poor, Average, Excellent}.
- $\text{Tree}(A^{(2)})$: *Irrigation Availability* with nodes {Scarce, Moderate, Abundant}.

A Double-Framed Forestsoft Set assigns two evaluations to each subset $X \subseteq \text{Forest}(\{A^{(1)}, A^{(2)}\})$. For instance, for

$$X = \{\text{Excellent Soil, Abundant Irrigation}\},$$

one might have:

$$\Psi_1(X) = \{\text{Field1, Field3}\} \quad \text{(indicating fields highly favorable for production)},$$

$$\Psi_2(X) = \{\text{Field2}\} \quad \text{(indicating fields with potential risks, such as waterlogging)}.$$

This dual evaluation allows decision-makers to consider both the positive and negative aspects of the agricultural land.

6. Bipolar Hypersoft Set and Bipolar SuperHypersoft Set

A Bipolar Soft Set represents positive and negative memberships, ensuring consistency by mapping parameters to subsets of a universal set [2,8,9,19,22,43]. A Bipolar Hypersoft Set extends Bipolar Soft Sets by incorporating multi-attribute combinations for positive and negative memberships in decision-making frameworks [4,27–29].

Definition 6.1 (Bipolar Soft Set)**.** [8,9,43] A *Bipolar Soft Set* over a universal set U is a triple (F, G, A), where:

- $F : A \to P(U)$ is the *positive membership mapping*,
- $G : \neg A \to P(U)$ is the *negative membership mapping*,
- $A \subseteq E$, $\neg A = E \setminus A$, where E is a set of parameters.

The mappings satisfy the *consistency constraint*:

$$F(e) \cap G(\neg e) = \emptyset, \quad \forall e \in A.$$

A Bipolar Soft Set is represented as:

$$(F, G, A) = \{(e, F(e), G(\neg e)) \mid e \in A, F(e) \cap G(\neg e) = \emptyset\}.$$

Definition 6.2 (Bipolar Hypersoft Set)**.** [27–29] A *Bipolar Hypersoft Set (BHS-Set)* is a triple (F, G, A) over a universe of discourse U, where:

- $F : A \to \mathcal{P}(U)$ and $G : \neg A \to \mathcal{P}(U)$, with $\mathcal{P}(U)$ denoting the power set of U.
- The mappings satisfy the *consistency constraint*:

$$F(\alpha) \cap G(\neg\alpha) = \emptyset, \quad \forall \alpha \in A.$$

- $A = A_1 \times A_2 \times \cdots \times A_n$, where $A_i \subseteq E_i$ and $E = E_1 \times E_2 \times \cdots \times E_n$.
- $\neg A = \neg A_1 \times \neg A_2 \times \cdots \times \neg A_n$, where $\neg A_i = E_i \setminus A_i$.

The BHS-Set (F, G, A) is represented as:

$$(F, G, A) = \{(\alpha, F(\alpha), G(\neg\alpha)) \mid \alpha \in A \text{ and } F(\alpha) \cap G(\neg\alpha) = \emptyset\}.$$

The Bipolar SuperHypersoft Set, a generalization of the Bipolar Hypersoft Set, is defined as follows.

Definition 6.3 (Bipolar SuperHypersoft Set)**.** [13] Let U be a universe, and let $E_1, \dots, E_n$ be pairwise disjoint sets of parameter values. Define

$$\mathcal{A} \;=\; \mathcal{P}(E_1) \;\times\; \cdots \times\; \mathcal{P}(E_n).$$

Let $A \subseteq \mathcal{A}$, and denote by $\neg A$ its complement in $\mathcal{A}$. A *Bipolar SuperHypersoft Set (BSHS-Set)* is a triple (F, G, A) where

$$F : A \to \mathcal{P}(U), \quad G : \neg A \to \mathcal{P}(U),$$

and for every $\alpha \in A$, the following *consistency constraint* holds:

$$F(\alpha) \;\cap\; G(\neg\alpha) \;=\; \varnothing.$$

Equivalently, we may write

$$(F, G, A) \;=\; \big\{\, (\alpha, F(\alpha), G(\neg\alpha)) \;\big|\; \alpha \in A,\; F(\alpha) \cap G(\neg\alpha) = \varnothing \big\}.$$

7. **Forest SuperHypersoft Set**

A Forest Hypersoft Set models multi-level attributes as trees, mapping their combinations to subsets of a universal set for decision-making.

Definition 7.1 (Forest Hypersoft Set)**.** [31] Let U be a universal set, and let $H \subseteq U$ be a non-empty subset relevant to the decision or classification context. Suppose we have a finite set of root attributes $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$, with $n \geq 1$. Each attribute A_i can be expanded into a *tree* of sub-attributes:

$$A_i \;\to\; \begin{cases} A_{i1}, A_{i2}, \dots & \text{(level 1)} \\ A_{i1k}, A_{i2\ell}, \dots & \text{(level 2)} \\ \dots \end{cases}$$

Collecting all such trees for $A_1, A_2, \dots, A_n$ produces a *forest* of attributes, denoted

$$\text{Forest}(\mathcal{A}) \;=\; \{\text{Tree}(A_1),\, \text{Tree}(A_2), \dots, \text{Tree}(A_n)\}.$$

Leaf-Level Sub-Attributes. For each tree $\text{Tree}(A_i)$, let $\Gamma\big(\text{Tree}(A_i)\big)$ be the set of all possible *leaf-level sub-attributes* stemming from A_i. A single leaf-level sub-attribute might represent a path in the tree from A_i down to one of its final sub-sub-attributes. Then define

$$\Gamma\big(\text{Forest}(\mathcal{A})\big) \;=\; \bigcup_{i=1}^{n} \Gamma\big(\text{Tree}(A_i)\big),$$

so that any element $\alpha \in \Gamma(\text{Forest}(\mathcal{A}))$ is a final-level attribute value from one of the trees in the forest. Forest Combinations. Consider the family of all possible subsets of $\Gamma(\text{Forest}(\mathcal{A}))$:

$$\mathcal{C}_{\text{forest}} \;\subseteq\; \mathcal{P}\big(\Gamma(\text{Forest}(\mathcal{A}))\big).$$

Each $\gamma \in \mathcal{C}_{\text{forest}}$ is understood as a *forest-based combination* of leaf-level sub-attributes (potentially from different root attributes).

Mapping to the Universe. A *Forest Hypersoft Set* over (U, H) is given by a pair

$$(G,\ \mathcal{C}_{\text{forest}}),$$

where $G : \mathcal{C}_{\text{forest}} \to \mathcal{P}(H)$ satisfies the condition that for each $\gamma \in \mathcal{C}_{\text{forest}}$, $G(\gamma) \subseteq H$. Concretely,

$$(G,\ \mathcal{C}_{\text{forest}}) \ = \ \Big\{ \big(\gamma,\ G(\gamma)\big) \ \Big| \ \gamma \in \mathcal{C}_{\text{forest}},\ G(\gamma) \subseteq H \Big\}.$$

In words, $G(\gamma)$ is the subset of H that corresponds to the collective influence or membership of all final-level sub-attributes in γ.

Definition 7.2 (Forest SuperHypersoft Set (FSHS))**.** [12] Let:

- U be a universal set, and let $H \subseteq U$ be a non-empty subset.
- $\mathcal{A} = \{A_1, A_2, \ldots, A_n\}$ be $n \geq 1$ root attributes, each expanding into a tree of sub-attributes, collectively forming a forest Forest($\mathcal{A}$).
- $\Gamma(\text{Forest}(\mathcal{A}))$ denote the set of all final-level sub-attributes (the leaves) across all attribute trees.
- Define the *forest super-domain* as the power set of leaf-level sub-attributes:

$$\widetilde{\mathcal{C}}_{\text{forest}} \ = \ \mathcal{P}\big(\Gamma(\text{Forest}(\mathcal{A}))\big).$$

  An element $\beta \in \widetilde{\mathcal{C}}_{\text{forest}}$ is thus a subset of final-level sub-attributes (possibly from different root attributes).

A **Forest SuperHypersoft Set** (FSHS) over (U, H) is a pair

$$\big(G,\ \widetilde{\mathcal{C}}_{\text{forest}}\big),$$

where

$$G : \widetilde{\mathcal{C}}_{\text{forest}} \ \longrightarrow \ \mathcal{P}(H)$$

maps each set of leaf-level sub-attributes $\beta \subseteq \Gamma(\text{Forest}(\mathcal{A}))$ to a subset $G(\beta) \subseteq H$. In other words,

$$\big(G,\ \widetilde{\mathcal{C}}_{\text{forest}}\big) \ = \ \Big\{ \big(\beta,\ G(\beta)\big) \ \Big| \ \beta \in \widetilde{\mathcal{C}}_{\text{forest}},\ G(\beta) \subseteq H \Big\}.$$

8. **IndetermSoft set and IndetermHyperSoft Set**

An *IndetermSoft Set* models uncertainty by associating attribute values with subsets of a universal set where indeterminacy exists in attributes or images. And an *IndetermHyperSoft Set* generalizes IndetermSoft Sets to multiple distinct attributes, mapping combinations of indeterminate attribute values to subsets of a universal set.

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

where indeterminacy can occur in any A_i, $P(H)$, or $F(a_1, a_2, \ldots, a_n)$ [24, 34, 37, 39].

Definition 8.1 (IndetermSoft set)**.** [24, 34, 37, 39] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let A be the set of attribute values for an attribute a. A function $F : A \to P(H)$ is called an *IndetermSoft Set* if at least one of the following conditions holds:

(1) A has some indeterminacy.

(2) $P(H)$ has some indeterminacy.

(3) There exists at least one $v \in A$ such that $F(v)$ is indeterminate (unclear, uncertain, or not unique).

(4) Any two or all three of the above conditions.

An IndetermSoft Set is represented mathematically as:

$$F : A \to H(\cap, \cup, \oplus, \neg),$$

where $H(\cap, \cup, \oplus, \neg)$ represents a structure closed under the IndetermSoft operators.

Definition 8.2 (IndetermHyperSoft Set)**.** [24, 34, 37, 39] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the powerset of H. Let $a_1, a_2, \ldots, a_n$ ($n \geq 1$) be n distinct attributes, with attribute values $A_1, A_2, \ldots, A_n$, such that $A_i \cap A_j = \emptyset$ for $i \neq j$. The pair $(F, A_1 \times A_2 \times \cdots \times A_n)$, where

$$F : A_1 \times A_2 \times \cdots \times A_n \to P(H),$$

is called an *IndetermHyperSoft Set* if:

(1) Any A_i or $P(H)$ exhibits indeterminacy.

(2) For $(a_1, a_2, \ldots, a_n) \in A_1 \times A_2 \times \cdots \times A_n$, $F(a_1, a_2, \ldots, a_n)$ is indeterminate.

9. **IndetermTreeSoft Set and IndetermForestSoft Set**

The IndetermTreeSoft Set integrates the concepts of TreeSoft Sets and IndetermSoft Sets into a unified framework [18]. Its definition and related details are provided below.

Definition 9.1 (IndetermTreeSoft Set)**.** [18] Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and $P(H)$ the power set of H. Let $\text{Tree}(A)$ be the hierarchical tree of attributes (as defined in the TreeSoft Set, with levels of sub-attributes) and $P(\text{Tree}(A))$ its power set.

A mapping

$$F \; : \; P(\text{Tree}(A)) \; \longrightarrow \; P(H)$$

is called an *IndetermTreeSoft Set* if one or more of the following conditions of indeterminacy hold:

(1) The tree of attributes, $\text{Tree}(A)$, includes some indeterminate or uncertain nodes (e.g., an attribute node has unspecified or overlapping sub-attributes).

(2) The codomain $P(H)$ contains uncertain or partially unknown subsets (e.g., membership of some elements of H is not uniquely determined).

(3) For at least one $X \in P(\text{Tree}(A))$, the image $F(X) \subseteq H$ itself is indeterminate or ambiguous.

Mathematically, one may allow each value $F(X)$ to be an element of a more general structure $H(\cap, \cup, \oplus, \neg)$ under partial or fuzzy membership (if desired), provided it accommodates indeterminacy.

The IndetermForestSoft Set combines the concepts of IndetermSoft Sets and ForestSoft Sets. Its definitions and related details are provided below [18].

Definition 9.2 (IndetermForestSoft Set)**.** Let U be a universe of discourse, $H \subseteq U$ a non-empty subset, and let

$$\{F_t : P(\mathrm{Tree}(A^{(t)})) \to P(H)\}_{t\in T}$$

be a finite or countable collection of TreeSoft Sets, each possibly containing some indeterminacy as described in Definition 9.1. Let

$$\mathrm{Forest}(\{A^{(t)}\}) = \bigsqcup_{t\in T} \mathrm{Tree}(A^{(t)})$$

be the disjoint union of all attribute trees.

A mapping

$$\mathbf{F} : P(\mathrm{Forest}(\{A^{(t)}\})) \longrightarrow P(H)$$

is called an *IndetermForestSoft Set* if for each $X \in P(\mathrm{Forest}(\{A^{(t)}\}))$,

$$\mathbf{F}(X) = \bigcup_{\substack{t\in T \\ X\cap \mathrm{Tree}(A^{(t)}) \neq \varnothing}} F_t(X \cap \mathrm{Tree}(A^{(t)})),$$

and at least one of the following holds:

(1) There is indeterminacy in some $\mathrm{Tree}(A^{(t)})$ or in the way these trees are combined.

(2) The codomain $P(H)$ has elements that are uncertain or ambiguous.

(3) For some input X, the image $\mathbf{F}(X)$ is not uniquely defined or is subject to partial information.

In other words, an IndetermForestSoft Set unites multiple IndetermTreeSoft Sets (or TreeSoft Sets, some or all of which may have indeterminacy) into a single framework while preserving the possibility of ambiguity.

10. **IndetermSuperHyperSoft Set**

The IndetermSuperHyperSoft Set combines the concepts of SuperHyperSoft Sets and IndetermHyperSoft Sets. Its definition is provided below [18].

Definition 10.1 (IndetermSuperHyperSoft Set)**.** Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be n distinct attributes with corresponding sets of values A_i, each of which may exhibit partial or total indeterminacy. Define

$$\mathcal{C} = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

which forms the domain of a *SuperHyperSoft Set* (Definition [36]).

A mapping

$$F : \mathcal{C} \longrightarrow \mathcal{P}(U)$$

is called an *IndetermSuperHyperSoft Set* if one or more of the following indeterminacy conditions hold:

(1) Some $\mathcal{P}(A_i)$ is uncertain or incompletely specified (i.e., the attribute values or their subsets are not fully determined).

(2) The codomain $\mathcal{P}(U)$ contains partially unknown subsets or ambiguous memberships.

(3) For at least one $\gamma \in \mathcal{C}$, the set $F(\gamma) \subseteq U$ is not uniquely defined or is subject to ambiguity.

Additionally, higher-level nesting (if each A_i itself is replaced by $\tilde{\mathcal{P}}(A_i)$ for extended hyperstructures) may introduce further levels of indeterminacy, consistent with the definition of IndetermHyperSoft Sets.

11. Incomplete HyperSoft Set and Incomplete SuperHyperSoft Set

The incomplete soft set is known as a special case of the IndetermSoft set, and the following definition of the incomplete soft set has been introduced [23].

Definition 11.1. [23] An *incomplete soft set* is defined as a tuple:

$$S = (U, E, f),$$

where:

- U is a non-empty set of objects, referred to as the universe.
- E is a non-empty set of parameters.
- $f : E \to \mathcal{P}(U) \cup \{*\}$, a mapping from the parameter set E to the power set of U ($\mathcal{P}(U)$) or the special symbol $*$.

Here, $*$ indicates missing or undefined information for a given parameter $e \in E$. If $f(e) = *$, the information associated with the parameter e is incomplete. Otherwise, $f(e) \subseteq U$ as in a standard soft set.

The Incomplete HyperSoft Set is a generalized concept derived from the Incomplete Soft Set. Its formal definition is provided below.

Definition 11.2 (Incomplete HyperSoft Set)**.** [18] Let U be a universal set, and let $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_m$ be m (possibly indeterminate) attribute domains. Define the Cartesian product

$$\mathcal{C} \;=\; \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_m.$$

An *Incomplete HyperSoft Set* over U is defined as a tuple

$$\Big(U,\; \mathcal{A}_1,\; \mathcal{A}_2,\; \ldots,\; \mathcal{A}_m,\; G\Big),$$

where $G : \mathcal{C} \to \big(\mathcal{P}(U) \,\cup\, \{*\}\big)$. Concretely,

$$G(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if sufficient information is available,} \\ *, & \text{if the information is missing or undefined,} \end{cases}$$

for each $\gamma \in \mathcal{C}$. Here, $*$ denotes incomplete or missing data in analogy with the incomplete soft set.

In this definition:

(1) U is a non-empty universe of discourse.

(2) $\mathcal{A}_i$ are attribute domains, each of which may be partially incomplete or fully defined.

(3) $\mathcal{C} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_m$ represents all possible combinations of attribute values.

(4) $G(\gamma)$ lies in $\mathcal{P}(U) \cup \{*\}$. If $G(\gamma) = *$, the mapping is incomplete for the given γ.

The Incomplete SuperHyperSoft Set is a generalized concept derived from the Incomplete HyperSoft Set. Its formal definition is provided below.

Definition 11.3 (Incomplete SuperHyperSoft Set)**.** [18] Let U be a universal set, and let $\{a_1, a_2, \ldots, a_n\}$ be $n \geq 1$ distinct attributes, each with a set of possible values A_i. Define

$$\mathcal{C} \;=\; \mathcal{P}(A_1) \;\times\; \mathcal{P}(A_2) \;\times\; \cdots \;\times\; \mathcal{P}(A_n),$$

as in the definition of a *SuperHyperSoft Set*. An *Incomplete SuperHyperSoft Set* is a tuple

$$\Bigl(U,\; \{A_i\}_{i=1}^{n},\; F\Bigr),$$

where

$$F : \mathcal{C} \;\longrightarrow\; \bigl(\mathcal{P}(U) \,\cup\, \{*\}\bigr).$$

Specifically, for each $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_i \subseteq A_i$,

$$F(\gamma) \;=\; \begin{cases} S_\gamma \subseteq U, & \text{if the information is complete,} \\ *, & \text{if the information is incomplete or undefined.} \end{cases}$$

Here, $*$ indicates a missing or indeterminate value at the superhyper level (e.g., incomplete data about chosen subsets α_i or how they map into U).

12. New Concept: GraphicSoft Set

In this section, we introduce a new type of Soft Set. A GraphicSoft Set extends the concept of Soft Sets by incorporating a graph structure to represent relationships between attributes, mapping subsets of the graph to subsets of a universal set.

Definition 12.1 (Graph and Its Power Set)**.** (cf. [11]) Let $G = (V, E)$ be a finite graph, where V is the set of vertices (each representing an attribute) and $E \subseteq V \times V$ is the set of edges (representing relationships among these attributes). Define a *subgraph* H of G as a graph $H = (V_H, E_H)$ with $V_H \subseteq V$ and $E_H \subseteq E \cap (V_H \times V_H)$. The *power set of* G, denoted $\mathcal{P}(G)$, is defined as the collection of all subgraphs of G:

$$\mathcal{P}(G) = \{\, H \mid H \text{ is a subgraph of } G \,\}.$$

Definition 12.2 (GraphicSoft Set)**.** Let U be a universe of discourse, and let $G = (V, E)$ be a graph representing a set of attributes and their relationships. A *GraphicSoft Set* is defined as a mapping

$$F : \mathcal{P}(G) \to \mathcal{P}(U),$$

which assigns to each subgraph $H \in \mathcal{P}(G)$ a subset $F(H) \subseteq U$. Intuitively, $F(H)$ represents the set of objects in U that possess the combined attributes described by the subgraph H.

Example 12.3 (Attributes of Individuals)**.** Let

$$U = \{\text{Alice}, \text{Bob}, \text{Charlie}, \text{Diana}\}$$

be a set of individuals. Define a graph

$$G = (V, E)$$

with

$$V = \{\text{Smart}, \text{Friendly}, \text{Athletic}\}$$

and let the edges in E indicate potential relationships (e.g., "Smart" might be related to "Friendly"). Define a GraphicSoft Set $F : \mathcal{P}(G) \to \mathcal{P}(U)$ as follows:

- For the subgraph H_1 containing only the vertex {Smart}, set

$$F(H_1) = \{\text{Alice}, \text{Charlie}\}.$$

- For the subgraph H_2 with only {Friendly}, set

$$F(H_2) = \{\text{Bob}, \text{Diana}\}.$$

- For the subgraph H_3 with vertices $\{\text{Smart}, \text{Friendly}\}$ (and the edge connecting them), define

$$F(H_3) = F(\{\text{Smart}\}) \cap F(\{\text{Friendly}\}) = \{\text{Alice}\}.$$

- For the entire graph G itself, let

$$F(G) = \{\text{Alice}, \text{Bob}\},$$

  indicating that only Alice and Bob exhibit all the attributes in G.

Example 12.4 (Houses and Their Features)**.** Let

$$U = \{\text{House1}, \text{House2}, \text{House3}, \text{House4}\}$$

be a set of houses. Define a graph

$$G = (V, E)$$

with

$$V = \{\text{Red}, \text{Big}, \text{Modern}\}.$$

Assume the edges in E express compatibility (e.g., "Red" might be frequently seen with "Modern"). Define a GraphicSoft Set F as follows:

- For the subgraph H_1 with vertex {Red}, let

$$F(H_1) = \{\text{House1}, \text{House3}\}.$$

- For the subgraph H_2 with vertex {Big}, let

$$F(H_2) = \{\text{House2}, \text{House3}\}.$$

- For the subgraph H_3 with vertices {Modern}, let

$$F(H_3) = \{\text{House3}, \text{House4}\}.$$

- For the subgraph H_4 with vertices {Red, Big}, define

$$F(H_4) = F(\{\text{Red}\}) \cap F(\{\text{Big}\}) = \{\text{House3}\}.$$

- For the full graph G, let

$$F(G) = \{\text{House3}\},$$

meaning that only House3 exhibits all three features.

Theorem 12.5 (Monotonicity of GraphicSoft Sets)**.** *Assume $F : \mathcal{P}(G) \to \mathcal{P}(U)$ is defined such that for any subgraph $H \in \mathcal{P}(G)$,*

$$F(H) = \bigcap_{v \in V(H)} F(\{v\}),$$

where $F(\{v\})$ is the set of objects in U possessing attribute v. Then, if H_1 and H_2 are subgraphs of G with $H_1 \subseteq H_2$ (i.e., $V(H_1) \subseteq V(H_2)$), we have

$$F(H_2) \subseteq F(H_1).$$

Proof. Since $H_1 \subseteq H_2$, we have $V(H_1) \subseteq V(H_2)$. Then,

$$F(H_2) = \bigcap_{v \in V(H_2)} F(\{v\}) \quad \text{and} \quad F(H_1) = \bigcap_{v \in V(H_1)} F(\{v\}).$$

Because intersecting over a larger set of indices yields a smaller (or equal) set, it follows that

$$\bigcap_{v \in V(H_2)} F(\{v\}) \subseteq \bigcap_{v \in V(H_1)} F(\{v\}),$$

which implies $F(H_2) \subseteq F(H_1)$. □

13. **New Concept: ClusterSoft Set and Cluster-HyperSoft Set**

In this section, we introduce a new type of Soft Set. A ClusterSoft Set groups multiple Soft Sets, capturing relationships among clustered attributes and mapping them to subsets of a universal set for decision modeling.

Definition 13.1 (ClusterSoft Set)**.** Let $\{F_i\}_{i \in I}$ be a finite family of soft sets over a universe U, where each soft set F_i is a mapping

$$F_i : A_i \to \mathcal{P}(U)$$

for some set of attributes A_i. Suppose the index set I is partitioned into clusters $\{C_j\}_{j \in J}$ with each $C_j \subseteq I$ and $C_j \cap C_k = \emptyset$ for $j \neq k$. A *ClusterSoft Set* is defined as a mapping

$$G : \{C_j : j \in J\} \to \mathcal{P}(U)$$

given by

$$G(C_j) = \bigcup_{i \in C_j} F_i^*(A_i),$$

where $F_i^*(A_i)$ denotes the set of objects in U associated with soft set F_i (possibly after appropriate aggregation or normalization). The union is taken in the usual set-theoretic sense.

Example 13.2 (Houses Grouped by Features)**.** Let

$$U = \{\text{House1}, \text{House2}, \text{House3}, \text{House4}, \text{House5}\}$$

be a set of houses. Suppose we have two soft sets:

- $F_1 : \{\text{Color}\} \to \mathcal{P}(U)$ with $F_1(\text{Red}) = \{\text{House1}, \text{House2}\}$.
- $F_2 : \{\text{Size}\} \to \mathcal{P}(U)$ with $F_2(\text{Large}) = \{\text{House2}, \text{House3}, \text{House4}\}$.

Let the index set be $I = \{1, 2\}$ and define a single cluster $C_1 = \{1, 2\}$. Then the ClusterSoft Set G is given by

$$G(C_1) = F_1(\text{Red}) \cup F_2(\text{Large}) = \{\text{House1}, \text{House2}\} \cup \{\text{House2}, \text{House3}, \text{House4}\} = \{\text{House1}, \text{House2}, \text{House3}, \text{House4}\}.$$

Example 13.3 (Students Grouped by Skill Sets)**.** Let

$$U = \{\text{Student1}, \text{Student2}, \text{Student3}, \text{Student4}\}$$

be a set of students. Define two soft sets:

- $F_1 : \{\text{Mathematics}\} \to \mathcal{P}(U)$ with $F_1(\text{Strong}) = \{\text{Student1}, \text{Student3}\}$.
- $F_2 : \{\text{Language}\} \to \mathcal{P}(U)$ with $F_2(\text{Fluent}) = \{\text{Student2}, \text{Student3}, \text{Student4}\}$.

Let the index set be $I = \{1, 2\}$ and form a single cluster $C_1 = \{1, 2\}$. Then, the ClusterSoft Set G is defined as

$$G(C_1) = F_1(\text{Strong}) \cup F_2(\text{Fluent}) = \{\text{Student1}, \text{Student3}\}$$
$$\cup \{\text{Student2}, \text{Student3}, \text{Student4}\} = \{\text{Student1}, \text{Student2}, \text{Student3}, \text{Student4}\}.$$

Theorem 13.4 (Disjoint Union Property)**.** *Let $\{F_i\}_{i \in I}$ be a family of soft sets over U and let $C_1, C_2 \subseteq I$ be two disjoint clusters (i.e., $C_1 \cap C_2 = \emptyset$). Then, the ClusterSoft Set satisfies*

$$G(C_1 \cup C_2) = G(C_1) \cup G(C_2).$$

Proof. By definition,

$$G(C_1 \cup C_2) = \bigcup_{i \in C_1 \cup C_2} F_i^*(A_i).$$

Since C_1 and C_2 are disjoint, the union can be split as

$$\bigcup_{i \in C_1 \cup C_2} F_i^*(A_i) = \left(\bigcup_{i \in C_1} F_i^*(A_i) \right) \cup \left(\bigcup_{i \in C_2} F_i^*(A_i) \right).$$

This equals $G(C_1) \cup G(C_2)$, proving the assertion. □

Theorem 13.5 (Non-emptiness)**.** *Assume that for every $i \in I$, the soft set F_i is non-empty (i.e., $F_i(A_i) \neq \emptyset$). Then for any non-empty cluster $C \subseteq I$, the ClusterSoft Set $G(C)$ is non-empty.*

Proof. Since each $F_i(A_i)$ is non-empty for $i \in C$ and $C \neq \emptyset$, the union

$$G(C) = \bigcup_{i \in C} F_i^*(A_i)$$

is a union of non-empty sets. Therefore, $G(C) \neq \emptyset$. □

Definition 13.6 (Cluster-HyperSoft Set)**.** Let U be a universal set. Suppose that for each $i \in I$, there is a HyperSoft Set

$$G_i : \mathcal{C}_i \to \mathcal{P}(U),$$

where $\mathcal{C}_i \subseteq A_{i1} \times A_{i2} \times \cdots \times A_{im_i}$ is the Cartesian product of attribute domains corresponding to the i-th evaluation. Assume that the index set I is partitioned into disjoint clusters $\{C_k\}_{k \in K}$ (i.e., $C_k \subseteq I$ and $C_k \cap C_{k'} = \emptyset$ for $k \neq k'$).

Then, the *Cluster-HyperSoft Set* is defined as a mapping

$$H : \{C_k \mid k \in K\} \to \mathcal{P}(U)$$

given by

$$H(C_k) = \bigcup_{i \in C_k} G_i(\gamma_i),$$

where for each $i \in C_k$, a specific element $\gamma_i \in \mathcal{C}_i$ is chosen (or determined by an aggregation operator). In other words, the Cluster-HyperSoft Set aggregates the hypersoft evaluations of the soft sets within each cluster.

Example 13.7 (Cluster-HyperSoft Set: Product Quality Evaluation)**.** Let

$$U = \{\text{Product1, Product2, Product3, Product4}\}$$

be a set of products. Assume we have two hyper soft sets:

- $G_1 : \mathcal{C}_1 \to \mathcal{P}(U)$ where $\mathcal{C}_1 = \{(\text{High}, \text{Low}), (\text{High}, \text{Medium})\}$ represents quality and cost parameters. Let
$$G_1(\text{High}, \text{Low}) = \{\text{Product1}, \text{Product2}\}.$$
- $G_2 : \mathcal{C}_2 \to \mathcal{P}(U)$ where $\mathcal{C}_2 = \{(\text{Medium}, \text{High}), (\text{Medium}, \text{Medium})\}$. Let
$$G_2(\text{Medium}, \text{High}) = \{\text{Product3}\}.$$

Let the index set be $I = \{1, 2\}$ and form a single cluster $C_1 = \{1, 2\}$. Then, by choosing $\gamma_1 = (\text{High}, \text{Low})$ for G_1 and $\gamma_2 = (\text{Medium}, \text{High})$ for G_2, we define the Cluster-HyperSoft Set as:

$$H(C_1) = G_1(\text{High}, \text{Low}) \cup G_2(\text{Medium}, \text{High}) = \{\text{Product1}, \text{Product2}\} \cup \{\text{Product3}\} = \{\text{Product1}, \text{Product2}, \text{Product3}\}.$$

Theorem 13.8 (Monotonicity of Cluster-HyperSoft Set Aggregation)**.** *Let C_1 and C_2 be clusters with $C_1 \subseteq C_2 \subseteq I$. Then,*

$$H(C_1) \subseteq H(C_2).$$

Proof. Since $C_1 \subseteq C_2$, every index $i \in C_1$ is also contained in C_2. Therefore, the union over C_1 is a subset of the union over C_2:

$$H(C_1) = \bigcup_{i \in C_1} G_i(\gamma_i) \subseteq \bigcup_{i \in C_2} G_i(\gamma_i) = H(C_2).$$

Thus, the monotonicity property holds. □

14. **New Concept: CycleSoft Set and Cycle-Hypersoft set**

In this section, we introduce a new type of Soft Set. A CycleSoft Set extends Soft Sets by organizing parameters in a cycle graph, mapping cycle subgraphs to subsets of a universal set for structured decision-making.

Definition 14.1 (CycleSoft Set)**.** Let U be a universal set and let $C = (A, E_C)$ be a cycle graph, where A is a set of parameters arranged in a cycle and

$$E_C = \{(a_i, a_{i+1}) \mid a_i, a_{i+1} \in A\} \cup \{(a_n, a_1)\}$$

describes the cyclic adjacency among the parameters. Define the power set of C as

$$\mathcal{P}(C) = \{H \mid H \text{ is a subgraph of } C\}.$$

A *CycleSoft Set* is a mapping

$$F : \mathcal{P}(C) \to \mathcal{P}(U),$$

where for each subgraph $H \in \mathcal{P}(C)$, $F(H) \subseteq U$ represents the set of objects associated with the combination of parameters corresponding to H. A common aggregation is to define, for each H,

$$F(H)(x) = \bigcap_{a \in V(H)} f(a)(x),$$

with $f(a) : U \to [0, 1]$ (or characteristic functions in the crisp case).

Example 14.2 (CycleSoft Set: Social Event Preferences)**.** Let

$$U = \{\text{Event1, Event2, Event3, Event4}\}$$

be a set of social events. Consider parameters arranged in a cycle:

$$A = \{\text{Music, Food, Dancing, Art}\},$$

with the cycle graph

$$C = \big(A,\ \{(\text{Music}, \text{Food}),\ (\text{Food}, \text{Dancing}),\ (\text{Dancing}, \text{Art}),\ (\text{Art}, \text{Music})\}\big).$$

Define the basic assignments:

$$F(\{\text{Music}\}) = \{\text{Event1}, \text{Event2}\}, \quad F(\{\text{Food}\}) = \{\text{Event2}, \text{Event3}\},$$

$$F(\{\text{Dancing}\}) = \{\text{Event3}, \text{Event4}\}, \quad F(\{\text{Art}\}) = \{\text{Event1}, \text{Event4}\}.$$

For the subgraph $H = \{\text{Music}, \text{Food}\}$ (i.e., two adjacent parameters), we define

$$F(H) = F(\{\text{Music}\}) \cap F(\{\text{Food}\}) = \{\text{Event2}\}.$$

Thus, the CycleSoft Set captures the joint effect of consecutive attributes arranged in a cyclic order.

Theorem 14.3 (Monotonicity of CycleSoft Set)**.** *Let $H_1, H_2 \in \mathcal{P}(C)$ be subgraphs with $H_1 \subseteq H_2$ (i.e., $V(H_1) \subseteq V(H_2)$). If F is defined via the intersection operator as*

$$F(H)(x) = \bigcap_{a \in V(H)} f(a)(x),$$

then for all $x \in U$,

$$F(H_2)(x) \subseteq F(H_1)(x).$$

Proof. Since $H_1 \subseteq H_2$, every parameter in H_1 is contained in H_2. Hence,

$$F(H_1)(x) = \bigcap_{a \in V(H_1)} f(a)(x) \quad \text{and} \quad F(H_2)(x) = \bigcap_{a \in V(H_2)} f(a)(x).$$

Since the intersection over a larger set of indices yields a subset (or equal set) of the intersection over a smaller set, we have

$$F(H_2)(x) \subseteq F(H_1)(x),$$

for all $x \in U$. □

Definition 14.4 (Cycle-Hypersoft Set)**.** Let U be a universal set, and let $C = (A, E_C)$ be a cycle graph where $A = \{a_1, a_2, \ldots, a_n\}$ is a set of parameters arranged in a cyclic order and

$$E_C = \{(a_i, a_{i+1}) \mid 1 \leq i < n\} \cup \{(a_n, a_1)\}$$

describes the cyclic adjacency among the parameters. Define the hypersoft structure on C by considering the Cartesian product of the power sets of attribute values for each a_i; that is, let

$$\mathcal{H}(C) = \mathcal{P}(A_1) \times \mathcal{P}(A_2) \times \cdots \times \mathcal{P}(A_n),$$

where for each a_i, A_i is the set of possible attribute values. A *Cycle-Hypersoft Set* is a mapping

$$F : \mathcal{H}(C) \to \mathcal{P}(U),$$

which assigns to each n-tuple $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ (with $\alpha_i \in \mathcal{P}(A_i)$) a subset $F(\gamma) \subseteq U$. This mapping aggregates the information from all cyclically arranged attributes in a higher-dimensional manner.

Example 14.5 (Cycle-Hypersoft Set: Social Media Preferences)**.** Let

$$U = \{\text{User1}, \text{User2}, \text{User3}, \text{User4}\}$$

represent a set of social media users. Suppose the parameters are arranged in a cycle:

$$A = \{\text{Music}, \text{Movies}, \text{Sports}, \text{Art}\},$$

with corresponding attribute value sets:

$$A_{\text{Music}} = \{\text{Rock}, \text{Pop}\}, \quad A_{\text{Movies}} = \{\text{Action}, \text{Drama}\},$$

$$A_{\text{Sports}} = \{\text{Football}, \text{Basketball}\}, \quad A_{\text{Art}} = \{\text{Modern}, \text{Classic}\}.$$

Then, the domain of the Cycle-Hypersoft Set is

$$\mathcal{H}(C) = \mathcal{P}(A_{\text{Music}}) \times \mathcal{P}(A_{\text{Movies}}) \times \mathcal{P}(A_{\text{Sports}}) \times \mathcal{P}(A_{\text{Art}}).$$

For instance, one may define:

$$F\Big(\{\text{Rock}\}, \{\text{Action}\}, \{\text{Football}\}, \{\text{Modern}\}\Big) = \{\text{User1}, \text{User3}\},$$

and

$$F\Big(\{\text{Pop}\}, \{\text{Drama}\}, \{\text{Basketball}\}, \{\text{Classic}\}\Big) = \{\text{User2}, \text{User4}\}.$$

Thus, the Cycle-Hypersoft Set models the joint effect of cyclically arranged attributes on user preferences.

Theorem 14.6 (Monotonicity in Cycle-Hypersoft Set)**.** *Let $\gamma = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\gamma' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_n)$ be two elements of $\mathcal{H}(C)$ such that for each i, $\alpha_i \subseteq \alpha'_i$. Then,*

$$F(\gamma) \subseteq F(\gamma'),$$

provided that the aggregation operator used in F is monotonic with respect to set inclusion.

Proof. Since for each i we have $\alpha_i \subseteq \alpha'_i$, the combination of attribute values in γ is a subset of that in γ'. Assuming that the aggregation operator in F (for example, a union or intersection operator) is monotonic with respect to inclusion, it follows that

$$F(\gamma) \subseteq F(\gamma'),$$

for all $x \in U$. □□

15. New Concept: Bipartite-Soft Set (Bisoft Set)

In this section, we introduce a new type of Soft Set. A Bipartite-Soft Set (Bisoft Set) extends Soft Sets by structuring attributes in a bipartite graph, mapping subsets of partitions to subsets of a universal set.

Definition 15.1 (Bipartite-Soft Set (Bisoft Set))**.** Let U be a universal set and let X and Y be two disjoint sets of parameters (i.e., $X \cap Y = \emptyset$). Consider the bipartite graph $B = (X \cup Y, E_B)$ where $E_B \subseteq X \times Y$ represents the interrelationships between parameters in X and those in Y. Define the Cartesian product of the power sets as

$$\mathcal{C} = \mathcal{P}(X) \times \mathcal{P}(Y).$$

A *Bipartite-Soft Set* (or *Bisoft Set*) is defined as a mapping

$$F : \mathcal{C} \to \mathcal{P}(U),$$

such that for each pair $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y)$, $F(A, B) \subseteq U$ represents the set of objects associated with the combination of attributes in A (from the first partition) and B (from the second partition).

Example 15.2 (Bipartite-Soft Set: Employee Evaluation)**.** Let

$$U = \{\text{Employee1}, \text{Employee2}, \text{Employee3}, \text{Employee4}\}$$

be a set of employees. Suppose the evaluation criteria are divided into two disjoint groups:

$$X = \{\text{Technical Skill}, \text{Experience}\} \quad \text{and} \quad Y = \{\text{Communication}, \text{Teamwork}\}.$$

Define a Bipartite-Soft Set $F : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(U)$ by:

- $F(\{\text{Technical Skill}\}, \{\text{Communication}\}) = \{\text{Employee1}, \text{Employee3}\}$.
- $F(\{\text{Experience}\}, \{\text{Teamwork}\}) = \{\text{Employee2}, \text{Employee4}\}$.
- $F(\{\text{Technical Skill}, \text{Experience}\}, \{\text{Communication}, \text{Teamwork}\}) = \{\text{Employee3}\}$.

Thus, the Bipartite-Soft Set aggregates information from the two groups of attributes to provide an evaluation of employees.

Example 15.3 (Bipartite-Soft Set: Product Analysis)**.** Let

$$U = \{\text{Product1}, \text{Product2}, \text{Product3}\}$$

be a set of products. Suppose the attributes are divided into two categories:

$$X = \{\text{Design}, \text{Durability}\} \quad \text{and} \quad Y = \{\text{Price}, \text{Market Appeal}\}.$$

Define the Bipartite-Soft Set $F : \mathcal{P}(X) \times \mathcal{P}(Y) \to \mathcal{P}(U)$ as follows:

- $F(\{\text{Design}\}, \{\text{Price}\}) = \{\text{Product1}, \text{Product2}\}$.
- $F(\{\text{Durability}\}, \{\text{Market Appeal}\}) = \{\text{Product2}, \text{Product3}\}$.
- $F(\{\text{Design}, \text{Durability}\}, \{\text{Price}, \text{Market Appeal}\}) = \{\text{Product2}\}$.

This example demonstrates how the Bipartite-Soft Set can be used to integrate and analyze product attributes from two different domains.

Theorem 15.4 (Union Property for Bipartite-Soft Sets)**.** *Let* (A_1, B_1) *and* (A_2, B_2) *be two pairs of subsets where* $A_1, A_2 \subseteq X$ *and* $B_1, B_2 \subseteq Y$*. Suppose that the Bipartite-Soft Set* F *is defined such that for any* $(A, B) \in \mathcal{P}(X) \times \mathcal{P}(Y)$*,*

$$F(A, B) = F(A_1, B_1) \cup F(A_2, B_2),$$

whenever $A = A_1 \cup A_2$ *and* $B = B_1 \cup B_2$*. Then,*

$$F(A, B) = F(A_1, B_1) \cup F(A_2, B_2).$$

Proof. By the definition of the Bipartite-Soft Set, if $A = A_1 \cup A_2$ and $B = B_1 \cup B_2$, then

$$F(A, B) = \{x \in U \mid x \text{ is associated with either } (A_1, B_1) \text{ or } (A_2, B_2)\}.$$

Thus,

$$F(A, B) = F(A_1, B_1) \cup F(A_2, B_2),$$

which completes the proof. □

Theorem 15.5 (Monotonicity of Bipartite-Soft Set)**.** *Let* $A_1 \subseteq A_2 \subseteq X$ *and* $B_1 \subseteq B_2 \subseteq Y$*. Then, for every* $x \in U$*,*

$$F(A_1, B_1)(x) \subseteq F(A_2, B_2)(x),$$

assuming that the mapping F *is monotonic with respect to set inclusion.*

Proof. Since $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, the set of parameters considered in (A_1, B_1) is a subset of those in (A_2, B_2). Thus, the set of objects associated with (A_1, B_1) is contained in the set of objects associated with (A_2, B_2), i.e.,

$$F(A_1, B_1) \subseteq F(A_2, B_2).$$

This proves the monotonicity property. □

16. **New Concept: Bipartite-HyperSoft Set and Bipartite-SuperhyperSoft Set**

In this section, we introduce a new type of Soft Set. The Bipartite-HyperSoft Set and Bipartite-SuperhyperSoft Set are extensions of the Bipartite-Soft Set. Their definitions and related properties are provided below.

Definition 16.1 (Bipartite-HyperSoft Set)**.** Let U be a universal set. Let X and Y be two disjoint sets of parameters, and assume that each parameter in X is associated with a set of attribute values A and each parameter in Y is associated with a set of attribute values B. Denote by $\mathcal{P}(A)$ and $\mathcal{P}(B)$ the power sets of A and B, respectively. A *Bipartite-HyperSoft Set* is a mapping

$$F : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(U),$$

which assigns to each pair (S, T) with $S \subseteq A$ and $T \subseteq B$ a subset $F(S, T) \subseteq U$. This mapping captures the combined influence of attribute values from both disjoint groups.

Example 16.2 (Bipartite-HyperSoft Set: Product Evaluation)**.** Let

$$U = \{\text{Product1, Product2, Product3, Product4}\}.$$

Assume we evaluate products based on two criteria:

- Design attributes: Let $A = \{\text{Modern, Classic}\}$.
- Performance attributes: Let $B = \{\text{Efficient, Robust}\}$.

Thus, the domain of the mapping is $\mathcal{P}(A) \times \mathcal{P}(B)$. Define $F : \mathcal{P}(A) \times \mathcal{P}(B) \to \mathcal{P}(U)$ as follows:

- $F(\{\text{Modern}\}, \{\text{Efficient}\}) = \{\text{Product1}, \text{Product2}\}$.
- $F(\{\text{Classic}\}, \{\text{Robust}\}) = \{\text{Product3}\}$.
- $F(\{\text{Modern}, \text{Classic}\}, \{\text{Efficient}, \text{Robust}\}) = U$.

This Bipartite-HyperSoft Set captures the joint effects of design and performance attributes on the set of products.

Theorem 16.3 (Monotonicity of Bipartite-HyperSoft Set)**.** *Let (S_1, T_1) and (S_2, T_2) be two pairs in $\mathcal{P}(A) \times \mathcal{P}(B)$ with $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$. Then, for all $x \in U$,*

$$F(S_1, T_1)(x) \subseteq F(S_2, T_2)(x).$$

Proof. Since $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$, the combination of attribute values in (S_1, T_1) is a subset of that in (S_2, T_2). Under standard aggregation, the set $F(S_2, T_2)(x)$ includes all objects corresponding to the larger collection of attributes, hence

$$F(S_1, T_1)(x) \subseteq F(S_2, T_2)(x),$$

for all $x \in U$. □

Definition 16.4 (Bipartite-SuperhyperSoft Set)**.** Let U be a universal set. Let A and B be two disjoint sets of attribute values corresponding to two disjoint parameter groups. Consider the power sets $\mathcal{P}(A)$ and $\mathcal{P}(B)$. Then, define the double power sets $\mathcal{P}(\mathcal{P}(A))$ and $\mathcal{P}(\mathcal{P}(B))$. A *Bipartite-SuperhyperSoft Set* is a mapping

$$F : \mathcal{P}(\mathcal{P}(A)) \times \mathcal{P}(\mathcal{P}(B)) \to \mathcal{P}(U),$$

which assigns to each pair (S,T) with $S \subseteq \mathcal{P}(A)$ and $T \subseteq \mathcal{P}(B)$ a subset $F(S,T) \subseteq U$. This higher-order aggregation allows for modeling more complex interrelationships by considering sets of attribute value subsets.

Example 16.5 (Bipartite-SuperhyperSoft Set: Re-Conscientization in Education)**.** (cf. [7]) Let

$$U = \{\text{Program1, Program2, Program3, Program4}\},$$

be a set of educational programs offered by a university. In order to evaluate these programs in a manner that promotes re-conscientization—that is, a process of re-evaluating established educational paradigms and fostering both critical inquiry and ethical reflection—we introduce two disjoint sets of attribute values:

- $A = \{\text{Critical Inquiry, Philosophical Rigor}\}$, representing the intellectual and reflective dimensions.
- $B = \{\text{Ethical Reflection, Social Engagement}\}$, representing the ethical and societal dimensions.

To capture complex, higher-order relationships among these attributes, we consider the double power sets $\mathcal{P}(\mathcal{P}(A))$ and $\mathcal{P}(\mathcal{P}(B))$. We then define the mapping

$$F : \mathcal{P}(\mathcal{P}(A)) \times \mathcal{P}(\mathcal{P}(B)) \to \mathcal{P}(U)$$

as follows:

- $F\Big(\{\{\text{Critical Inquiry}\}\}, \{\{\text{Ethical Reflection}\}\}\Big) = \{\text{Program1}\}$.
  This indicates that Program1 is evaluated highly for its ability to promote both critical inquiry and ethical reflection, essential components of re-conscientization.
- $F\Big(\{\{\text{Philosophical Rigor}\}\}, \{\{\text{Social Engagement}\}\}\Big) = \{\text{Program2}\}$.
  Here, Program2 is recognized for its strong theoretical foundations paired with active social engagement, thereby challenging conventional educational norms.
- $F\Big(\{\{\text{Critical Inquiry}, \text{Philosophical Rigor}\}\}, \{\{\text{Ethical Reflection}, \text{Social Engagement}\}\}\Big) = \{\text{Program3, Program4}\}$.
  This captures programs that integrate both layers of intellectual depth and comprehensive ethical-social considerations, embodying a holistic approach to re-conscientization.

Thus, the Bipartite-SuperhyperSoft Set framework here models a multi-layered aggregation of attributes that reflect not only academic performance but also the transformative potential of education through re-conscientization.

Theorem 16.6 (Consistency Under Aggregation for Bipartite-SuperhyperSoft Set)**.** *Let (S_1, T_1) and (S_2, T_2) be elements of $\mathcal{P}(\mathcal{P}(A)) \times \mathcal{P}(\mathcal{P}(B))$ with $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$. Then,*

$$F(S_1, T_1) \subseteq F(S_2, T_2).$$

Proof. Since $S_1 \subseteq S_2$ and $T_1 \subseteq T_2$, the set of attribute value subsets considered in (S_1, T_1) is a subset of that in (S_2, T_2). Assuming the aggregation function F is monotonic with respect to set inclusion, it follows that

$$F(S_1, T_1) \subseteq F(S_2, T_2).$$

Thus, the aggregation is consistent under the inclusion of additional attribute value subsets. □

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Data Availability

This research is purely theoretical, involving no data collection or analysis. We encourage future researchers to pursue empirical investigations to further develop and validate the concepts introduced here.

Ethical Approval

As this research is entirely theoretical in nature and does not involve human participants or animal subjects, no ethical approval is required.

Conflicts of Interest

The authors confirm that there are no conflicts of interest related to the research or its publication.

Disclaimer

This work presents theoretical concepts that have not yet undergone practical testing or validation. Future researchers are encouraged to apply and assess these ideas in empirical contexts. While every effort has been made to ensure accuracy and appropriate referencing, unintentional errors or omissions may still exist. Readers are advised to verify referenced materials on their own. The views and conclusions expressed here are the authors' own and do not necessarily reflect those of their affiliated organizations.

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