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Total Order Based Similarity Measures for Single Valued and Interval Valued Neutrosophic Triplets

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Abstract. Smarandeche (1995) introduced neutrosophic sets to address difficulties involving imprecise, indeterminate, and inconsistent information, as a generalization of Zadeh's fuzzy set and Atanassov's intuitionistic fuzzy set. It paved the way for that Smarandeche's single valued neutrosophic triplets (SVNT) and interval valued neutrosophic triplets IVNT for modelling real time applications based on such information. In this paper, we introduce the $\mathfrak{S}_1$-similarity measure for SVNT and the $\mathfrak{S}_2$-similarity measure for IVNT, through which $\mathfrak{S}_1$-ordering algorithm for SVNT and the $\mathfrak{S}_2$-ordering algorithm for IVNT are obtained, respectively. Further, we demonstrate that the $\mathfrak{S}_1$-ordering algorithm and the $\mathfrak{S}_2$-ordering algorithm inherit a total order on the set of all SVNTs and IVNTs, respectively. Finally, we present numerical illustrations and a comparative analysis to demonstrate that the proposed similarity measures outperform previous methods.

Keywords: Neutrosophic sets ; Similarity measure ; Total ordering ; MCDM.

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1. **Introduction**

In decision-making processes, we must always make the best choices while considering the inherent unpredictability of real-world events. To address this ambiguity, L.A. Zadeh [8] introduced the concept of fuzzy sets in 1965. Initially met with skepticism, fuzzy set theory posits that available data is not always precise but instead contains a degree of uncertainty. Analyzing this vagueness can lead to significant advancements when applied to multi-criteria decision-making (MCDM) problems. Over the years, extensive research has led to the generalization of fuzzy sets, giving rise to various extensions, including intuitionistic fuzzy sets [10], [11], [12], [14], [15], neutrosophic sets [1], [2], [3], [4], [24], [25], picture fuzzy sets [15],ternary fuzzy set [16], bi-polar fuzzy sets [17], hesitant fuzzy sets [18], Pythagorean fuzzy sets [23], Spherical fuzzy set [19], circular intuitionistic fuzzy sets [20], linear fuzzy number [26]. These extensions have found widespread applications in diverse domains. Among these, the theory of neutrosophic sets [1], [2] has emerged as one of the fastest-growing fields due to its effectiveness in real-world applications such as supply chain management and medical diagnosis [27], [28], [29], [30], [31], [32]. Neutrosophic sets have also been extended to other domains, including goal programming, graph theory, and optimization [33], [34], [35], [36], [37], [38]. Unlike fuzzy sets and intuitionistic fuzzy sets, neutrosophic sets consider three independent degrees: membership, non-membership, and indeterminacy, making them more robust in handling uncertainty [8], [9]. Every fuzzy and intuitionistic fuzzy set can be viewed as a special case of a neutrosophic set [1]. The field of neutrosophic sets has seen significant advancements, particularly in graph theory, multi-criteria decision-making, and economic assessments. For instance, Vetrivel et al. [39] provide a comprehensive review of the transition from fuzzy graphs to neutrosophic graph extensions, highlighting their enhanced capability in handling uncertainty. Naveed and Ali [40] propose a multi-criteria decision-making approach using correlation coefficients for multi-polar interval-valued neutrosophic soft sets, demonstrating its applicability through mathematical examples. Furthermore, Elsayed and Mohamed [41] conduct a comparative analysis of various MCDM techniques within neutrosophic environments, assessing their effectiveness in economic condition evaluations. In the literature, various total ordering algorithms for neutrosophic triplets have been proposed to address MCDM problems. In particular, total ordering methods for single-valued, interval-valued, and n-valued neutrosophic triplets [21], [22] have been widely adopted. One of the key research areas in neutrosophic set theory is the development of effective similarity measures [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52]. In pattern recognition problems, similarity measures play a crucial role in analyzing uncertainty. Smarandache [42] introduced various similarity measures for neutrosophic sets based on extended Hausdorff distance, set-theoretic approaches, and Type-1, Type-2, and Type-3 geometric distance models.

In this paper, we propose a novel similarity measure for single-valued neutrosophic triplets (SVNTs), which is later extended to interval-valued neutrosophic triplets (IVNTs). By defining similarity measures between any two SVNTs, we develop a corresponding similarity measure for single-valued neutrosophic sets (SVNSs) and subsequently extend this framework to interval-valued neutrosophic sets (IVNSs). Additionally, we introduce a total ordering algorithm based on the proposed similarity measures, facilitating better decision-making processes.

A comparative analysis with existing similarity measures in neutrosophic set theory is conducted to evaluate the efficacy of our approach.

1.1. *Addressing Research Gaps*

Despite extensive research on similarity measures for fuzzy and neutrosophic sets, several limitations persist in existing methods. This work identifies and addresses the following research gaps:

Lack of Total Ordering in Existing Measures:

Many existing similarity measures, such as those proposed by Broumi et al. [42], Ridvan Sahin et al. [43], Yanqiu Zeng et al. [44], Rakhal Das et al. [45], Kalyan Mondal et al. [46], Mahima Poonia et al. [47], and Sujit et al. [7], suffer from the issue of **partial ordering**. This means that multiple alternatives may receive the same similarity score, leading to ambiguity in decision-making. Such ambiguity is particularly problematic in scenarios that require a clear ranking, such as medical diagnosis and investment decision-making.

- **Impact on Decision-Making:** Without strict total ordering, decision-makers face uncertainty in selecting the best alternative, potentially leading to suboptimal or inconsistent choices.
- **Proposed Solution:** The proposed $\mathfrak{S}$-similarity measure ensures a strict ranking of alternatives, eliminating ambiguity and providing a **clear preference structure**.

Inconsistencies in Decision Outcomes:

Several widely used similarity measures, such as the Type-3 Geometric Distance Model proposed by Broumi et al. [42], have demonstrated **inconsistencies in ranking alternatives** when tested on benchmark problems. These inconsistencies arise due to:

- **Sensitivity to Parameter Variations:** Many existing methods rely on parameter-dependent formulations that may lead to fluctuating rankings when applied to different datasets.
- **Contradictions Across Comparative Studies:** A method that performs well in one domain (e.g., medical diagnosis) may fail in another (e.g., financial decision-making), indicating instability in similarity computations.
- **Proposed Solution:** The proposed $\mathfrak{S}$-similarity measure exhibits **greater stability across multiple datasets**, ensuring that rankings remain consistent regardless of parameter variations. This enhances reliability in real-world applications.

By addressing these gaps, this study contributes a **robust, interpretable, and computationally efficient similarity measure** that enhances decision-making across multiple domains.

1.2. *Outline of the Work*

The remainder of this paper is structured as follows. Section 2 provides an overview of essential concepts related to single-valued neutrosophic sets (SVNSs), interval-valued neutrosophic sets (IVNSs), and similarity measures. In Section 3, we introduce a similarity measure for single-valued neutrosophic triplets (SVNTs) and develop a total ordering method for SVNTs using the proposed similarity measure on $\mathbb{N}_{\mathbb{S}}'$. Section 4 extends this approach to interval-valued neutrosophic triplets (IVNTs), presenting a total ordering method based on similarity measures on $\mathbb{N}_{\mathbb{IV}}'$. To illustrate the applicability of the proposed similarity measure $\mathfrak{S}_1$, a numerical example involving medical diagnostics is presented in Section 5. Section 6 provides a comparative analysis between the proposed similarity measures and existing approaches, highlighting the advantages of the $\mathfrak{S}$-similarity measure. Section 7 discusses the limitations of the proposed work and outlines potential directions for future research. Finally, Section 8 presents concluding remarks.

2. **Preliminaries**

To make this study self-contained, we briefly explain a few outcomes and findings to be used later in this work.

Definition 2.1. [1] A neutrosophic set $\mathbb{NS}$ on universe of discourse $\mathbb{U}$ is defined as

$$\mathbb{NS} = \{(x : \mu(x), \iota(x), \nu(x)) \mid \mu(x), \iota(x), \nu(x) \in [0,1],\ 0 \leq \mu(x) + \iota(x) + \nu(x) \leq 3\}$$

where $\mu : \mathbb{U} \longrightarrow [0,1], \nu : \mathbb{U} \longrightarrow [0,1], \iota : \mathbb{U} \longrightarrow [0,1]$ and $\mu(x), \iota(x)$ and $\nu(x)$ stand for degrees of truthfulness, indeterminacy and falsity respectively.

Let $\mathbb{N}_{\mathbb{S}} = \{(T, I, F) \mid T, I, F \in [0,1],\ 0 \leq T+I+F \leq 3\}$ denote the collection of single valued neutrosophic triplet (SVNT) numbers where T, I and F stand for degrees of truthfulness, indeterminacy and falsity respectively.

Let $\mathbb{N}_{\mathbb{IV}} = \{([T^l, T^u], [I^l, I^u], [F^l, F^u]) \mid T^l, T^u, I^l, I^u, F^l, F^u \in [0,1],\ 0 \leq T^u + I^u + F^u \leq 3\}$ denote the collection of interval valued neutrosophic triplet (IVNT) numbers [2] with the constraints $T^l < T^u,\ I^l < I^u$ and $F^l < F^u$.

Definition 2.2. [3] Let $A, B, C \in \mathbb{N}_{\mathbb{S}}$ such that $A = (T_a, I_a, F_a),\ B = (T_b, I_b, F_b)$ and $C = (T_c, I_c, F_c)$, Then $A \subseteq B \subseteq C$ if $T_a \leq T_b \leq T_c$, $I_a \geq I_b \geq I_c$ and $F_a \geq F_b \geq F_c$.

Definition 2.3. [2] Let $A, B, C \in \mathbb{N}_{\mathbb{IV}}$ such that $A = ([T_a^l, T_a^u], [I_a^l, I_a^u], [F_a^l, F_a^u])$, $B = ([T_b^l, T_b^u], [I_b^l, I_b^u], [F_b^l, F_b^u])$ and $C = ([T_c^l, T_c^u], [I_c^l, I_c^u], [F_c^l, F_c^u])$. Then $A \subseteq B \subseteq C$ if

$$T_a^l \leq T_b^l \leq T_c^l \text{ and } T_a^u \leq T_b^u \leq T_c^u,$$
$$I_a^l \geq I_b^l \geq I_c^l \text{ and } I_a^u \geq I_b^u \geq I_c^u,$$
$$F_a^l \geq F_b^l \geq F_c^l \text{ and } F_a^u \geq F_b^u \geq F_c^u.$$

Definition 2.4. [4] A real-valued mapping $\mathfrak{S} : \mathbb{NS} \times \mathbb{NS} \longrightarrow [0, 1]$ is a similarity measure on $\mathbb{NS}$ if the following requirements are met for every $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{NS}$:

($\mathfrak{S}(1)$) $0 \leq \mathfrak{S}(\mathbf{A}, \mathbf{B}) \leq 1$,
($\mathfrak{S}(2)$) $\mathfrak{S}(\mathbf{A}, \mathbf{B}) = 1 \Longleftrightarrow \mathbf{A} = \mathbf{B}$,
($\mathfrak{S}(3)$) $\mathfrak{S}(\mathbf{A}, \mathbf{B}) = \mathfrak{S}(\mathbf{B}, \mathbf{A})$,
($\mathfrak{S}(4)$) If $\mathbf{A} \subseteq \mathbf{B} \subseteq \mathbf{C}$ then $\mathfrak{S}(\mathbf{A}, \mathbf{B}) \leq \mathfrak{S}(\mathbf{A}, \mathbf{C})$ and $\mathfrak{S}(\mathbf{B}, \mathbf{C}) \leq \mathfrak{S}(\mathbf{A}, \mathbf{C})$.

3. Similarity Measure on Single Valued Neutrosophic Triplets

Let $\mathbb{N}_{\mathbb{S}} = \{(T, I, F) \mid T, I, F \in [0, 1],\ 0 \leq T + I + F \leq 3\}$ be collection of single valued neutrosophic triplets.

Definition 3.1. Let $A, B \in \mathbb{N}_{\mathbb{S}}$ defined on singleton set $\mathbb{U}$, such that $A = (T_a, I_a, F_a)$ and $B = (T_b, I_b, F_b)$. Let $r \geq 1$ be fixed. The real valued function $\mathfrak{S}_1$ on $\mathbb{N}_{\mathbb{S}} \times \mathbb{N}_{\mathbb{S}}$ is defined as

$$\mathfrak{S}_1(A, B) = \frac{3^{\frac{1}{r}}}{3^{\frac{1}{r}} + \left(|T_a - T_b|^r + |I_a - I_b|^r + |F_a - F_b|^r\right)^{\frac{1}{r}}}.$$

Theorem 3.2. *The function $\mathfrak{S}_1$ is a similarity measure on $\mathbb{N}_{\mathbb{S}}$.*

Proof. Let $A, B, C \in \mathbb{N}_{\mathbb{S}}$ such that $A = (T_a, I_a, F_a)$, $B = (T_b, I_b, F_b)$ and $C = (T_c, I_c, F_c)$. Since $\mathfrak{S}_1$ is a function of all positive and absolute values, $0 \leq \mathfrak{S}_1(A, B)$. Since $0 \leq \left(|T_a - T_b|^r + |I_a - I_b|^r + |F_a - F_b|^r\right)^{\frac{1}{r}}$ and $r \geq 1$, $3^{\frac{1}{r}} \leq 3^{\frac{1}{r}} + \left(|T_a - T_b|^r + |I_a - I_b|^r + |F_a - F_b|^r\right)^{\frac{1}{r}}$. Hence $\mathfrak{S}_1(A, B) \leq 1$. Also, it is clear that $\mathfrak{S}_1(A, B) = 1$ iff $\left(|T_a - T_b|^r + |I_a - I_b|^r + |F_a - F_b|^r\right)^{\frac{1}{r}} = 0$, equivalently, $A = B$. Furthermore, it is easy to confirm that the function $\mathfrak{S}_1$ is symmetric in nature. In order to check condition $\mathfrak{S}_1(4)$ of definition 2.4, assume that $A \subseteq B \subseteq C$, then by definition 2.2, $T_a \leq T_b \leq T_c, I_a \geq I_b \geq I_c, F_a \geq F_b \geq F_c$. Therefore, $|T_a - T_b| \leq |T_a - T_c|, |I_a - I_b| \leq |I_a - I_c|$ and $|F_a - F_b| \leq |F_a - F_c|$. Since, $r \geq 1$,

$$3^{\frac{1}{r}} + \left(|T_a - T_b|^r + |I_a - I_b|^r + |F_a - F_b|^r\right)^{\frac{1}{r}} \leq 3^{\frac{1}{r}} + (|T_a - T_c|^r + |I_a - I_c|^r + |F_a - F_c|^r)^{\frac{1}{r}},$$

which implies $\mathfrak{S}_1(A, C) \leq \mathfrak{S}_1(A, B)$. Similarly, it can be verified that $\mathfrak{S}_1(A, C) \leq \mathfrak{S}_1(B, C)$. Hence, the proposed real valued function $\mathfrak{S}_1$ defined on $\mathbb{N}_{\mathbb{S}} \times \mathbb{N}_{\mathbb{S}}$ is a similarity measure on $\mathbb{N}_{\mathbb{S}}$.

□

Definition 3.3. The similarity measure betwen two single valued neutrosophic sets, A, B defined on set $\mathbb{U} = \{u_1, u_2, \ldots, u_n\}$ is

$$\mathfrak{S}(A,B) = \frac{\sum\limits_{j=1}^{n} \mathfrak{S}_{\mathfrak{1}}(A(u_j), B(u_j))}{n}$$

where $\mathfrak{S}(A(u_j), B(u_j))$ is a simlarity measure between SVNTs, $A(u_j)$ and $B(u_j)$.

Definition 3.4. The weighted similarity measure betwen two SVNTs, $A, B \in \mathbb{N}_{\mathbb{S}}$ defined on set $\mathbb{U} = \{u_1, u_2, \ldots, u_n\}$ is

$$\mathfrak{S}_{\mathfrak{w}}(A,B) = \sum_{j=1}^{n} \mathfrak{w}_j (\mathfrak{S}_{\mathfrak{1}}(A(u_j), B(u_j)))$$

where $\mathfrak{w}_j$ is the weight vector with $0 \leq \mathfrak{w}_j \leq 1$, $\sum\limits_{j=1}^{n} \mathfrak{w}_j = 1$ and $\mathfrak{S}_{\mathfrak{1}}(A(u_j), B(u_j))$ is similarity measure between SVNTs $A(u_j)$ and $B(u_j)$.

Definition 3.5. Let $\mathbb{N}_{\mathbb{S}}' = \{A_m = (T_m, I_m, F_m) \mid m = 1, 2, \cdots, n\}$ be a finite subset of $\mathbb{N}_{\mathbb{S}}$. Then $T_S, T_G, I_S, I_G, F_S, F_G \in [0,1]$ are defined as

$$T_S := \min_{1\leq j\leq n} T_j, T_G := \max_{1\leq j\leq n} T_j, I_S := \min_{1\leq j\leq n} I_j, I_G := \max_{1\leq j\leq n} I_j, F_S := \min_{1\leq j\leq n} F_j, F_G := \max_{1\leq j\leq n} F_j$$

respectively and M_T, M_I, M_F are defined as

$$M_T = (T_G, I_S, F_S), M_I = (T_S, I_G, F_S), M_F = (T_S, I_S, F_G).$$

Definition 3.6. Let $\mathbb{N}_{\mathbb{S}}' = \{A_1, A_2, A_3, \ldots, A_n\}$, where $A_m = \{A_m(u_1), A_m(u_2), \ldots, A_m(u_k)\}$ is a neutrosophic set defined on the universe $\mathbb{U} = \{u_1, u_2, \ldots, u_k\}$. Each element $A_m(u_i)$ is represented as:

$$A_m(u_i) = (T_m^i, I_m^i, F_m^i), \quad 1 \leq i \leq k.$$

Then, for $1 \leq i \leq k$, the following values are defined:

$$T_S^i := \min_{1\leq j\leq n} T_j^i, \quad T_G^i := \max_{1\leq j\leq n} T_j^i,$$

$$I_S^i := \min_{1\leq j\leq n} I_j^i, \quad I_G^i := \max_{1\leq j\leq n} I_j^i,$$

$$F_S^i := \min_{1\leq j\leq n} F_j^i, \quad F_G^i := \max_{1\leq j\leq n} F_j^i.$$

Further, the sets M_T, M_I, and M_F are defined as:

$$M_T = \{(T_G^1, I_S^1, F_S^1), (T_G^2, I_S^2, F_S^2), \ldots, (T_G^k, I_S^k, F_S^k)\},$$

$$M_I = \{(T_S^1, I_G^1, F_S^1), (T_S^2, I_G^2, F_S^2), \ldots, (T_S^k, I_G^k, F_S^k)\},$$

$$M_F = \{(T_S^1, I_S^1, F_G^1), (T_S^2, I_S^2, F_G^2), \ldots, (T_S^k, I_S^k, F_G^k)\}.$$

Remark 3.7. In the above definition M_T represents that it is better than all member in $\mathbb{N}_{\mathbb{S}}'$ with respect to truthfulness, M_F represents it is worse than anything else in $\mathbb{N}_{\mathbb{S}}'$ with respect to falsity and M_I represents it is worse than anything else in $\mathbb{N}_{\mathbb{S}}'$ with respect to indeterminacy.

3.1. *A Total Ordering on SVNT Via Similarity measure on* $\mathbb{N}_{\mathbb{S}}'$

In this section, we demonstrate the similarity measure $\mathfrak{S}_1$ preserve total ordering in $\mathbb{N}_{\mathbb{S}}'$.

3.1.1. $\mathfrak{S}_1-$*Ordering Algorithm for* $\mathbb{N}_{\mathbb{S}}'$

Let $A_p, A_q \in \mathbb{N}_{\mathbb{S}}'$ such that $A_p = (T_p, I_p, F_p)$ and $A_q = (T_q, I_q, F_q)$ and M_T, M_I and M_F are considered as definition 3.5.

Step 1

$A_p < A_q$ **if** $\mathfrak{S}_1(A_p, M_T) < \mathfrak{S}_1(A_q, M_T)$
$A_p > A_q$ **if** $\mathfrak{S}_1(A_p, M_T) > \mathfrak{S}_1(A_q, M_T)$
Go step 2 if $\mathfrak{S}_1(A_p, M_T) = \mathfrak{S}_1(A_q, M_T)$

Step 2

$A_p < A_q$ **if** $\mathfrak{S}_1(A_p, M_F) > \mathfrak{S}_1(A_q, M_F)$
$A_p > A_q$ **if** $\mathfrak{S}_1(A_p, M_F) < \mathfrak{S}_1(A_q, M_F)$
Go step 3 if $\mathfrak{S}_1(A_p, M_F) = \mathfrak{S}_1(A_q, M_F)$

Step 3

$A_p < A_q$ **if** $\mathfrak{S}_1(A_p, M_I) > \mathfrak{S}_1(A_q, M_I)$
$A_p > A_q$ **if** $\mathfrak{S}_1(A_p, M_I) < \mathfrak{S}_1(A_q, M_I)$
We can conclude that, $A_p = A_q$ **if** $\mathfrak{S}_1(A_p, M_I) = \mathfrak{S}_1(A_q, M_I)$

Theorem 3.8. *The proposed $\mathfrak{S}_1$-ordering algorithm for the set $\mathbb{N}_{\mathbb{S}}'$ establishes a total order on the collection of all single-valued neutrosophic triplets when the parameter $r = 1$.*

Proof. Let $A_p, A_q \in \mathbb{N}_{\mathbb{S}}'$ such that $A_p = (T_p, I_p, F_p)$ and $A_q = (T_q, I_q, F_q)$. We show that, either $A_p < A_q$ or $A_p > A_q$ or $A_p = A_q$. Here, M_T, M_I and M_F are considered as definition 3.5. First we find similarity measure between A_p, M_T and A_q, M_T. Suppose $\mathfrak{S}_1(A_p, M_T) < \mathfrak{S}_1(A_q, M_T)$ (or $\mathfrak{S}_1(A_p, M_T) > \mathfrak{S}_1(A_q, M_T)$), then we have $A_p < A_q$ (or $A_p > A_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_1(A_p, M_T) = \mathfrak{S}_1(A_q, M_T)$, that is,

$$\frac{3}{3 + |T_p - T_G| + |I_p - I_S| + |F_p - F_S|} = \frac{3}{3 + |T_q - T_G| + |I_q - I_S| + |F_q - F_S|}$$
$$I_p + F_p - T_p = I_q + F_q - T_q. \tag{1}$$

Then we have to go step 2, now find similarity measure between A_p, M_F and A_q, M_F. Suppose $\mathfrak{S}_1(A_p, M_F) > \mathfrak{S}_1(A_q, M_F)$ (or $\mathfrak{S}_1(A_p, M_F) < \mathfrak{S}_1(A_q, M_F)$) then we have $A_p < A_q$ (or $A_p > A_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_1(A_p, M_F) = \mathfrak{S}_1(A_q, M_F)$, that is,

$$T_p + I_p - F_p = T_q + I_q - F_q. \tag{2}$$

Then we have to go step 3, now find similarity measure between A_p, M_I and A_q, M_I. Suppose $\mathfrak{S}_1(A_p, M_I) > \mathfrak{S}_1(A_q, M_I)$ (or $\mathfrak{S}_1(A_p, M_I) < \mathfrak{S}_1(A_q, M_I)$) then we have $A_p < A_q$ (or $A_p > A_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_1(A_p, M_I) = \mathfrak{S}_1(A_q, M_I)$, that is,

$$T_p + F_p - I_p = T_q + F_q - I_q. \tag{3}$$

We are currently solving equations 1, 2 and 3. By adding equations 1 and 2, we get $I_p = I_q$. and adding equations 2 and 3, we get $T_p = T_q$ and substitute obtained results in any one of these three equation, we get $F_p = F_q$. Thus,

$$A_p = (T_p, I_p, F_p) = (T_q, I_q, F_q) = A_q.$$

Consequently, every pair of members in $\mathbb{N}_{\mathbb{S}}'$ is either greater than or equal to the other. Therefore, any two members in $\mathbb{N}_{\mathbb{S}}'$ are comparable. Thus, it leads to total ordering on $\mathbb{N}_{\mathbb{S}}'$. □

Theorem 3.9. *Let $\mathbb{N}_{\mathbb{S}}'' = \{A_m = (T_m, I_m, F_m) \in \mathbb{N}_{\mathbb{S}}' \mid m = 1, 2, \ldots, n, T_i \leq T_j$ and $I_i \geq I_j \ \forall\ i, j \in \{1, 2, \ldots, n\}\}$. The proposed $\mathfrak{S}_1$-ordering algorithm establishes a total order on the set $\mathbb{N}_{\mathbb{S}}''$ when the parameter $r = 2$ in $\mathfrak{S}_1$.*

Proof. Let $A_p, A_q \in \mathbb{N}_{\mathbb{S}}''$ such that $A_p = (T_p, I_p, F_p)$ and $A_q = (T_q, I_q, F_q)$. We show that, either $A_p < A_q$ or $A_p > A_q$ or $A_p = A_q$. Here, M_T, M_I and M_F are considered as definition 3.5. First we find similarity measure between A_p, M_T and A_q, M_T. Suppose $\mathfrak{S}_1(A_p, M_T) < \mathfrak{S}_1(A_q, M_T)$ (or $\mathfrak{S}_1(A_p, M_T) > \mathfrak{S}_1(A_q, M_T)$), then we have $A_p < A_q$ (or $A_p > A_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_1(A_p, M_T) = \mathfrak{S}_1(A_q, M_T)$, that is,

$$\begin{gathered}
\frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}} + (|T_p - T_G|^2 + |I_p - I_S|^2 + |F_p - F_S|^2)^{\frac{1}{2}}} = \frac{3^{\frac{1}{2}}}{3^{\frac{1}{2}} + (|T_q - T_G|^2 + |I_q - I_S|^2 + |F_q - F_S|^2)^{\frac{1}{2}}} \\
|T_p - T_G|^2 + |I_p - I_S|^2 + |F_p - F_S|^2 = |T_q - T_G|^2 + |I_q - I_S|^2 + |F_q - F_S|^2 \\
T_p^2 + I_p^2 + F_p^2 - 2(T_pT_G + I_pI_S + F_pF_S) = T_q^2 + I_q^2 + F_q^2 - 2(T_qT_G + I_qI_S + F_qF_S).
\end{gathered} \tag{4}$$

Then we have to go step 2, now we find similarity measure between A_p, M_F and A_q, M_F. Suppose $\mathfrak{S}_1(A_p, M_F) > \mathfrak{S}_1(A_q, M_F)$ (or $\mathfrak{S}_1(A_p, M_F) < \mathfrak{S}_1(A_q, M_F)$), then we have $A_p < A_q$ (or $A_p > A_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_1(A_p, M_F) = \mathfrak{S}_1(A_q, M_F)$, that is,

$$T_p^2 + I_p^2 + F_p^2 - 2(T_pT_S + I_pI_S + F_pF_G) = T_q^2 + I_q^2 + F_q^2 - 2(T_qT_S + I_qI_S + F_qF_G). \tag{5}$$

Then we have to go step 3, now we find similarity measure between A_p, M_I and A_q, M_I. Suppose $\mathfrak{S}_1(A_p, M_I) > \mathfrak{S}_1(A_q, M_I)$ (or $\mathfrak{S}_1(A_p, M_I) < \mathfrak{S}_1(A_q, M_I)$), then we have $A_p < A_q$ (or $A_p > A_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_1(A_p, M_I) = \mathfrak{S}_1(A_q, M_I)$, that is,

$$T_p^2 + I_p^2 + F_p^2 - 2(T_pT_S + I_pI_G + F_pF_S) = T_q^2 + I_q^2 + F_q^2 - 2(T_qT_S + I_qI_G + F_qF_S). \tag{6}$$

We are currently solving equations 4, 5 and 6. By subtracting equation 4 from 5, we get

$$\begin{gathered}T_p(T_G - T_S) + F_p(F_S - F_G) = T_q(T_G - T_S) + F_q(F_S - F_G) \\ (T_p - T_q)(T_G - T_S) = (F_p - F_q)(F_G - F_S)\end{gathered} \tag{7}$$

and subtracting equation 5 from 6, we get

$$(F_p - F_q)(F_G - F_S) = (I_p - I_q)(I_G - I_S) \tag{8}$$

and subtracting equation 6 from 4, we get

$$(I_p - I_q)(I_G - I_S) = (T_p - T_q)(T_G - T_S). \tag{9}$$

From equations 7, 8 and 9, we can obtain the relations

$$(T_p - T_q)(T_G - T_S) = (F_p - F_q)(F_G - F_S) = (I_p - I_q)(I_G - I_S) \tag{10}$$

Since, $T_G - T_S \geq 0, F_G - F_S \geq 0$ and $I_G - I_S \geq 0$, we can conclude that $T_p - T_q, F_p - F_q$ and $I_p - I_q$ are all have like sign. That is

$$T_p \geq T_q, F_p \geq F_q, I_p \geq I_q \quad \text{or} \quad T_p \leq T_q, F_p \leq F_q, I_p \leq I_q \tag{11}$$

Here $A_p, A_q \in \mathbb{N}_{\mathbb{S}}''$ and by the dentition of $\mathbb{N}_{\mathbb{S}}''$, we get

$$T_p \leq T_q \quad \text{and} \quad I_p \geq I_q \tag{12}$$

From the analysis of equations 11 and 12, we identify two possibilities: either $T_p = T_q$ or $I_p = I_q$. In the first case, assuming $T_p = T_q$, we substitute this condition into the equations 7 and 9. This yields equations $(F_p - F_q)(F_G - F_S) = 0$ and $(I_p - I_q)(I_G - I_S) = 0$. From these equations, we conclude that either $F_p = F_q$ or $F_G = F_S$ and either $I_p = I_q$ or $I_G = I_S$. Given the constraints $F_S \leq F_p \leq F_q \leq F_G$ and $I_S \leq I_p \leq I_q \leq I_G$, it follows that $F_p = F_q$ and $I_p = I_q$. Consequently, we can express this as

$$A_p = (T_p, I_p, F_p) = (T_q, I_q, F_q) = A_q.$$

In the second case, where $I_p = I_q$, we apply similar reasoning and demonstrate that

$$A_p = (T_p, I_p, F_p) = (T_q, I_q, F_q) = A_q.$$

The results indicate that every pair of members in $\mathbb{N}_{\mathbb{S}}''$ is either greater than or equal to the other, establishing that any two members in $\mathbb{N}_{\mathbb{S}}''$ are comparable. This leads to a total ordering on $\mathbb{N}_{\mathbb{S}}''$. □

Corollary 3.10. *If the parameter $r = 2$ in $\mathfrak{S}_1$, then the proposed $\mathfrak{S}_1$-ordering algorithm establishes a total order on the following sets:*

$\mathbb{N}_{\mathbb{S}_1''} = \{A_m = (T_m, I_m, F_m) \mid m = 1, 2, \ldots, n \text{ and } T_i \leq T_j \text{ and } F_i \geq F_j \ \forall\, i, j \in \{1, 2, \ldots, n\}\}$

$\mathbb{N}_{\mathbb{S}_2''} = \{A_m = (T_m, I_m, F_m) \mid m = 1, 2, \ldots, n \text{ and } F_i \leq F_j \text{ and } T_i \geq T_j \ \forall\, i, j \in \{1, 2, \ldots, n\}\}$

$\mathbb{N}_{\mathbb{S}_3''} = \{A_m = (T_m, I_m, F_m) \mid m = 1, 2, \ldots, n \text{ and } F_i \leq F_j \text{ and } I_i \geq I_j \ \forall\, i, j \in \{1, 2, \ldots, n\}\}$

$\mathbb{N}_{\mathbb{S}_4''} = \{A_m = (T_m, I_m, F_m) \mid m = 1, 2, \ldots, n \text{ and } I_i \leq I_j \text{ and } T_i \geq T_j \ \forall\, i, j \in \{1, 2, \ldots, n\}\}$

$\mathbb{N}_{\mathbb{S}_5''} = \{A_m = (T_m, I_m, F_m) \mid m = 1, 2, \ldots, n \text{ and } I_i \leq I_j \text{ and } F_i \geq F_j \ \forall\, i, j \in \{1, 2, \ldots, n\}\}$

4. Similarity Measure on Interval valued Neutrosophic Triplets

Let $\mathbb{N}_{\mathbb{IV}} = \{([T^l, T^u], [I^l, I^u], [F^l, F^u]) \mid T^l, T^u, I^l, I^u, F^l, F^u \in [0, 1],\ 0 \leq T^u + I^u + F^u \leq 3\}$ be the collection of interval valued neutrosophic triplets.

Definition 4.1. Let $A, B \in \mathbb{N}_{\mathbb{IV}}$ defined on singleton set $\mathbb{U}$, such that $A = ([T^l_a, T^u_a], [I^l_a, I^u_a], [F^l_a, F^u_a])$ and $B = ([T^l_b, T^u_b], [I^l_b, I^u_b], [F^l_b, F^u_b])$. Let $r \geq 1$ be fixed. The real valued function $\mathfrak{S}_2$ on $\mathbb{N}_{\mathbb{IV}} \times \mathbb{N}_{\mathbb{IV}}$ is defined as

$$\mathfrak{S}_2(A, B) = \frac{6^{\frac{1}{r}}}{6^{\frac{1}{r}} + \left(|T^l_a - T^l_b|^r + |I^l_a - I^l_b|^r + |F^l_a - F^l_b|^r + |T^u_a - T^u_b|^r + |I^u_a - I^u_b|^r + |F^u_a - F^u_b|^r\right)^{\frac{1}{r}}}.$$

Theorem 4.2. *The function $\mathfrak{S}_2$ is a similarity measure on $\mathbb{N}_{\mathbb{IV}}$.*

Proof. Let $A, B, C \in \mathbb{N}_{\mathbb{IV}}$ such that $A = ([T^l_a, T^u_a], [I^l_a, I^u_a], [F^l_a, F^u_a])$, $B = ([T^l_b, T^u_b], [I^l_b, I^u_b], [F^l_b, F^u_b])$ and $C = ([T^l_c, T^u_c], [I^l_c, I^u_c], [F^l_c, F^u_c])$. Since $\mathfrak{S}_2$ is a function of all positive and absolute values, $0 \leq \mathfrak{S}_2(A, B)$. Since $0 \leq \left(|T^l_a - T^l_b|^r + |I^l_a - I^l_b|^r + |F^l_a - F^l_b|^r + |T^u_a - T^u_b|^r + |I^u_a - I^u_b|^r + |F^u_a - F^u_b|^r\right)^{\frac{1}{r}}$ and $r \geq 1$, $6^{\frac{1}{r}} \leq 6^{\frac{1}{r}} + \left(|T^l_a - T^l_b|^r + |I^l_a - I^l_b|^r + |F^l_a - F^l_b|^r + |T^u_a - T^u_b|^r + |I^u_a - I^u_b|^r + |F^u_a - F^u_b|^r\right)^{\frac{1}{r}}$. Hence $\mathfrak{S}_2(A, B) \leq 1$. Also, it is clear that $\mathfrak{S}_2(A, B) = 1$ iff $\left(|T^l_a - T^l_b|^r + |I^l_a - I^l_b|^r + |F^l_a - F^l_b|^r + |T^u_a - T^u_b|^r + |I^u_a - I^u_b|^r + |F^u_a - F^u_b|^r\right)^{\frac{1}{r}} = 0$, equivalently, $A = B$. Furthermore, it is easy to confirm that the function $\mathfrak{S}_2$ is symmetric in nature. In order to check condition $\mathfrak{S}_2(4)$ of definition 2.4, assume that $A \subseteq B \subseteq C$, then by definition 2.3, $T^l_a \leq T^l_b \leq T^l_c, I^l_a \geq I^l_b \geq I^l_c, F^l_a \geq F^l_b \geq F^l_c, T^u_a \leq T^u_b \leq T^u_c, I^u_a \geq I^u_b \geq I^u_c, F^u_a \geq F^u_b \geq F^u_c$. Therefore, $|T^l_a - T^l_b| \leq |T^l_a - T^l_c|, |I^l_a - I^l_b| \leq |I^l_a - I^l_c|, |F^l_a - F^l_b| \leq |F^l_a - F^l_c|$ and $|T^u_a - T^u_b| \leq |T^u_a - T^u_c|, |I^u_a - I^u_b| \leq |I^u_a - I^u_c|$ and $|F^u_a - F^u_b| \leq |F^u_a - F^u_c|$. Since, $r \geq 1$,

$$\begin{aligned}&6^{\frac{1}{r}} + \left(|T^l_a - T^l_b|^r + |I^l_a - I^l_b|^r + |F^l_a - F^l_b|^r + |T^u_a - T^u_b|^r + |I^u_a - I^u_b|^r + |F^u_a - F^u_b|^r\right)^{\frac{1}{r}}\\ &\quad \leq 6^{\frac{1}{r}} + \left(|T^l_b - T^l_c|^r + |I^l_b - I^l_c|^r + |F^l_b - F^l_c|^r + |T^u_b - T^u_c|^r + |I^u_b - I^u_c|^r + |F^u_b - F^u_c|^r\right)^{\frac{1}{r}},\end{aligned}$$

which implies $\mathfrak{S}_2(A, C) \leq \mathfrak{S}_2(A, B)$. Similarly, it can be verified that $\mathfrak{S}_2(A, C) \leq \mathfrak{S}_2(B, C)$. Hence, the proposed real valued function $\mathfrak{S}_2$ defined on $\mathbb{N}_{\mathbb{IV}} \times \mathbb{N}_{\mathbb{IV}}$ is similarity measure on $\mathbb{N}_{\mathbb{IV}}$. □

Definition 4.3. The similarity measure betwen two IVNTs $A, B \in \mathbb{N}_{\mathbb{IV}}$ defined on set $\mathbb{U} = \{u_1, u_2, \ldots, u_n\}$ is

$$\mathfrak{S}_2(A,B) = \frac{\sum\limits_{j=1}^{n} \mathfrak{S}_2(A(u_j), B(u_j))}{n}$$

where $\mathfrak{S}_2(A(u_j), B(u_j))$ is simlarity measure between IVNTs, $A(u_j)$ and $B(u_j)$.

Definition 4.4. The weighted similarity measure betwen two IVNTs $A, B \in \mathbb{N}_{\mathbb{IV}}$ defined on set $\mathbb{U} = \{u_1, u_2, \ldots, u_n\}$ is

$$\mathfrak{S}_{\mathfrak{v}}(A,B) = \sum_{j=1}^{n} \mathfrak{v}_j(\mathfrak{S}_2(A(u_j), B(u_j)))$$

where $\mathfrak{v}_j$ is the weight vector with $0 \leq \mathfrak{v}_j \leq 1$, $\sum\limits_{j=1}^{n} \mathfrak{v}_j = 1$ and $\mathfrak{S}_2(A(u_j), B(u_j))$ is simlarity measure between IVNTs, $A(u_j)$ and $B(u_j)$.

Definition 4.5. Let $\mathbb{N}_{\mathbb{IV}}{}' = \{B_m = ([T_m^l, T_m^u], [I_m^l, I_m^u], [F_m^l, F_m^u]) \mid m = 1, 2, \cdots, n\}$ be a finite subset of $\mathbb{N}_{\mathbb{I}}\mathbb{V}$. Then $T_S^l, T_G^l, I_S^l, I_G^l, F_S^l, F_G^l, T_S^u, T_G^u, I_S^u, I_G^u, F_S^u, F_G^u \in [0,1]$ are defined as

$T_S^l := \min\limits_{1\leq j\leq n} T_j^l, T_G^l := \max\limits_{1\leq j\leq n} T_j^l, I_S^l := \min\limits_{1\leq j\leq n} I_j^l, I_G^l := \max\limits_{1\leq j\leq n} I_j^l, F_S^l := \min\limits_{1\leq j\leq n} F_j^l, F_G^l := \max\limits_{1\leq j\leq n} F_j^l$

$T_S^u := \min\limits_{1\leq j\leq n} T_j^u, T_G^u := \max\limits_{1\leq j\leq n} T_j^u, I_S^u := \min\limits_{1\leq j\leq n} I_j^u, I_G^u := \max\limits_{1\leq j\leq n} I_j^u, F_S^u := \min\limits_{1\leq j\leq n} F_j^u, F_G^u := \max\limits_{1\leq j\leq n} F_j^u$

respectively and $M_T, M_I, M_F, M'_T, M'_I, M'_F$ are defined as

$$M_T = ([T_G^l, T_G^u], [I_S^l, I_S^u], [F_S^l, F_S^u]),\ M'_T = ([T_S^l, T_G^u], [I_S^l, I_S^u], [F_S^l, F_S^u]),$$
$$M_I = ([T_S^l, T_S^u], [I_G^l, I_G^u], [F_S^l, F_S^u]),\ M'_I = ([T_S^l, T_S^u], [I_S^l, I_G^u], [F_S^l, F_S^u]),$$
$$M_F = ([T_S^l, T_S^u], [I_S^l, I_S^u], [F_G^l, F_G^u]),\ M'_F = ([T_S^l, T_S^u], [I_S^l, I_S^u], [F_S^l, F_G^u]).$$

Remark 4.6. In the above definition, M_T represents that it is better than all member in $\mathbb{N}_{\mathbb{IV}}{}'$, M_F represents it is worse than anything else in $\mathbb{N}_{\mathbb{IV}}{}'$ with respect to falsity and M_I represents it is worse than anything else in $\mathbb{N}_{\mathbb{IV}}{}'$ with respect to indeterminacy.

4.1. *A Total Ordering on IVNT Via Similarity measure on* $\mathbb{N}_{\mathbb{IV}}{}'$

In this section, we demonstrate the similarity measure $\mathfrak{S}_2$ preserve total ordering in $\mathbb{N}_{\mathbb{IV}}{}'$.

4.1.1. $\mathfrak{S}_2-$*Ordering Algorithm for* $\mathbb{N}_{\mathbb{IV}}{}'$

Let $B_p, B_q \in \mathbb{N}_{\mathbb{IV}}{}'$ such that $B_p = ([T_p^l, T_p^u], [I_p^l, I_p^u], [F_p^l, F_p^u])$ and $B_q = ([T_q^l, T_q^u], [I_q^l, I_q^u], [F_q^l, F_q^u])$ and $M_T, M_I, M_F, M'_T, M'_I$ and M'_F are considered as definition 4.5.

Step 1

$B_p < B_q$ **if** $\mathfrak{S}_2(B_p, M_T) < \mathfrak{S}_2(B_q, M_T)$
$B_p > B_q$ **if** $\mathfrak{S}_2(B_p, M_T) > \mathfrak{S}_2(B_q, M_T)$
Go step 2 if $\mathfrak{S}_2(B_p, M_T) = \mathfrak{S}_2(B_q, M_T)$

Step 2	$B_p < B_q$ **if** $\mathfrak{S}_2(B_p, M_F) > \mathfrak{S}_2(B_q, M_F)$ $B_p > B_q$ **if** $\mathfrak{S}_2(B_p, M_F) < \mathfrak{S}_2(B_q, M_F)$ **Go step 3 if** $\mathfrak{S}_2(B_p, M_F) = \mathfrak{S}_2(B_q, M_F)$
Step 3	$B_p < B_q$ **if** $\mathfrak{S}_2(B_p, M_I) > \mathfrak{S}_2(B_q, M_I)$ $B_p > B_q$ **if** $\mathfrak{S}_2(B_p, M_I) < \mathfrak{S}_2(B_q, M_I)$ **Go step 4,** $B_p = B_q$ **if** $\mathfrak{S}_2(B_p, M_I) = \mathfrak{S}_2(B_q, M_I)$
Step 3	$B_p < B_q$ **if** $\mathfrak{S}_2(B_p, M'_T) > \mathfrak{S}_2(B_q, M'_T)$ $B_p > B_q$ **if** $\mathfrak{S}_2(B_p, M'_T) < \mathfrak{S}_2(B_q, M'_T)$ **Go step 5 if** $\mathfrak{S}_2(B_p, M'_T) = \mathfrak{S}_2(B_q, M'_T)$
Step 4	$B_p < B_q$ **if** $\mathfrak{S}_2(B_p, M'_F) < \mathfrak{S}_2(B_q, M'_F)$ $B_p > B_q$ **if** $\mathfrak{S}_2(B_p, M'_F) > \mathfrak{S}_2(B_q, M'_F)$ **Go step 6 if** $\mathfrak{S}_2(A_p, M'_F) = \mathfrak{S}_2(B_q, M'_F)$
Step 6	$B_p < B_q$ **if** $\mathfrak{S}_2(B_p, M'_I) < \mathfrak{S}_2(B_q, M'_I)$ $B_p > B_q$ **if** $\mathfrak{S}_2(B_p, M'_I) > \mathfrak{S}_2(B_q, M'_I)$ **We can conclude that,** $B_p = B_q$ **if** $\mathfrak{S}_2(B_p, M'_I) = \mathfrak{S}_2(B_q, M'_I)$

Theorem 4.7. *The proposed $\mathfrak{S}_2$-ordering algorithm for the set $\mathbb{N}_{\mathbb{IV}}{}'$ establishes a total order on the collection of all interval valued neutrosophic triplets when the parameter $r = 1$.*

Proof. Let $B_p, B_q \in \mathbb{N}_{\mathbb{IV}}{}'$ such that $B_p = ([T^l_p, T^u_p], [I^l_p, I^u_p], [F^l_p, F^u_p])$ and $B_q = ([T^l_q, T^u_q], [I^l_q, I^u_q], [F^l_q, F^u_q])$. We show that, either $B_p < B_q$ or $B_p > B_q$ or $B_p = B_q$. Here, $M_T, M_I, M_F, M'_T, M'_I$ and M'_F are considered as definition 4.5. First we find similarity measure between B_p, M_T and B_q, M_T. Suppose $\mathfrak{S}_2(B_p, M_T) < \mathfrak{S}_2(B_q, M_T)$ (or $\mathfrak{S}_2(B_p, M_T) > \mathfrak{S}_2(B_q, M_T)$), then we have $B_p < B_q$ (or $B_p > B_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_2(B_p, M_T) = \mathfrak{S}_2(B_q, M_T)$, that is,

$$\frac{6}{6 + |T^l_p - T^l_G| + |I^l_p - I^l_S| + |F^l_p - F^l_S| + |T^u_p - T^u_G| + |I^u_p - I_S| + |F^u_p - F^u_S|}$$
$$= \frac{6}{6 + |T^l_q - T^l_G| + |I^l_q - I^l_S| + |F^l_q - F^l_S| + |T^u_q - T^u_G| + |I^u_q - I_S| + |F^u_q - F^u_S|} \quad (13)$$
$$I^l_p + I^u_p + F^l_p + F^u_p - T^l_p - T^u_p = I^l_q + I^u_q + F^l_q + F^u_q - T^l_q - T^u_q.$$

Then we have to go step 2, now find similarity measure between B_p, M_F and B_q, M_F. Suppose $\mathfrak{S}_2(B_p, M_F) > \mathfrak{S}_2(B_q, M_F)$ (or $\mathfrak{S}_2(B_p, M_F) < \mathfrak{S}_2(B_q, M_F)$), then we have $B_p < B_q$ (or $B_p > B_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_2(B_p, M_F) = \mathfrak{S}_2(B_q, M_F)$, that is,

$$T^l_p + T^u_p + I^l_p + I^u_p - F^l_p - F^u_p = T^l_q + T^u_q + I^l_q + I^u_q - F^l_q - F^u_q. \quad (14)$$

Then we have to go step 3, now find similarity measure between B_p, M_I and B_q, M_I. Suppose $\mathfrak{S}_2(B_p, M_I) > \mathfrak{S}_2(B_q, M_I)$ (or $\mathfrak{S}_2(B_p, M_I) < \mathfrak{S}_2(B_q, M_I)$), then we have $B_p < B_q$ (or $B_p >$

B_q). In such cases, the task is completed. Suppose, $\mathfrak{S}_2(B_p, M_I) = \mathfrak{S}_2(B_q, M_I)$, that is,

$$T_p^l + T_p^u + F_p^l + F_p^u - I_p^l - I_p^u = T_q^l + T_q^u + F_q^l + F_q^u - I_q^l - I_q^u. \tag{15}$$

Then we have to go step 4, now find similarity measure between B_p, M_T' and B_q, M_T'. Suppose $\mathfrak{S}_2(B_p, M_T') > \mathfrak{S}_2(B_q, M_T')$ (or $\mathfrak{S}_2(B_p, M_T') < \mathfrak{S}_2(B_q, M_T')$), then we have $B_p < B_q$ (or $B_p > B_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_2(B_p, M_T') = \mathfrak{S}_2(B_q, M_T')$, that is,

$$T_p^l + I_p^l + I_p^u + F_p^l + F_p^u - T_p^u = T_q^l + I_q^l + I_q^u + F_q^l + F_q^u - T_q^u. \tag{16}$$

Then we have to go step 5, now find similarity measure between B_p, M_F' and B_q, M_F'. Suppose $\mathfrak{S}_2(B_p, M_F') < \mathfrak{S}_2(B_q, M_F')$ (or $\mathfrak{S}_2(B_p, M_F') > \mathfrak{S}_2(B_q, M_F')$), then we have $B_p < B_q$ (or $B_p > B_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_2(B_p, M_F') = \mathfrak{S}_2(B_q, M_F')$, that is,

$$T_p^l + T_p^u + F_p^l + I_p^l + I_p^u - F_p^u = T_q^l + T_q^u + F_q^l + I_q^l + I_q^u - F_q^u. \tag{17}$$

Then we have to go step 6, now find similarity measure between B_p, M_I' and B_q, M_I'. Suppose $\mathfrak{S}_2(B_p, M_I') < \mathfrak{S}_2(B_q, M_I')$ (or $\mathfrak{S}_2(B_p, M_I') > \mathfrak{S}_2(B_q, M_I')$), then we have $B_p < B_q$ (or $B_p > B_q$). In such cases, the task is completed. Suppose, $\mathfrak{S}_2(B_p, M_I') = \mathfrak{S}_2(B_q, M_I')$, that is,

$$T_p^l + T_p^u + F_p^l + F_p^u + I_p^l - I_p^u = T_q^l + T_q^u + F_q^l + F_q^u + I_q^l - I_q^u. \tag{18}$$

We are currently solving equations 13, 14 and 15. By adding equations 13 and 14, we get

$$I_p^l + I_p^u = I_q^l + I_q^u. \tag{19}$$

By adding equations 14 and 15, we get

$$T_p^l + T_p^u = T_q^l + T_q^u. \tag{20}$$

By adding equations 14 and 15, we get

$$F_p^l + F_p^u = F_q^l + F_q^u. \tag{21}$$

By substituting equations 19 and 21 in 16, we get

$$T_p^l - T_p^u = T_q^l - T_q^u. \tag{22}$$

By substituting equations 19 and 20 in 17, we get

$$F_p^l - F_p^u = F_q^l - F_q^u. \tag{23}$$

By substituting equations 20 and 21 in 18, we get

$$I_p^l - I_p^u = I_q^l - I_q^u. \tag{24}$$

When we are solving the system of equations 20 & 22, 21 & 23 and 19 & 24, we get

$$T_p = T_q, F_p = F_q \text{ and } I_p = I_q$$

respectively. Thus,

$$B_p = ([T_p^l, T_p^u], [I_p^l, I_p^u], [F_p^l, F_p^u]) = ([T_q^l, T_q^u], [I_q^l, I_q^u], [F_q^l, F_q^u]) = B_q.$$

Consequently, every pair of members in $\mathbb{N}_{\mathbb{IV}}{}'$ is either greater than or equal to the other. Therefore, any two members in $\mathbb{N}_{\mathbb{IV}}{}'$ are comparable. Thus, it leads to total ordering on $\mathbb{N}_{\mathbb{IV}}{}'$.

□

5. Numerical Illustration

5.1. *Application of Proposed Similarity Measure $\mathfrak{S}_1$ in Medical Diagnosis*

To illustrate the effectiveness of the proposed similarity measure $\mathfrak{S}_1$, we consider multiple medical diagnostic scenarios where different patients exhibit varying symptoms. This allows us to analyze how well the measure differentiates between similar and dissimilar cases.

Case 1: Initial Diagnosis for a Single Patient

Example 5.1. Consider a patient P_1 exhibiting a set of symptoms (S) and Diagnosis (D) :

$$S = \{S_1(\text{Fever}), S_2(\text{Headache}), S_3(\text{Stomach pain}), S_4(\text{Cough}), S_5(\text{Chest pain})\}$$

$$D = \{D_1(\text{Viral fever}), D_2(\text{Malaria}), D_3(\text{Typhoid}), D_4(\text{Gastritis}), D_5(\text{Stenocardia})\}.$$

The patient's symptom profile, evaluated under the SVNT framework, is given as : $P_1 = \{S_1(0.8, 0.2, 0.1), S_2(0.6, 0.3, 0.1), S_3(0.2, 0.1, 0.8), S_4(0.6, 0.5, 0.1), S_5(0.1, 0.4, 0.6)\}$. The corresponding diagnostic profiles for each disease, represented in SVNT form, are presented in Table 1.

Table 1. SVNT representation of diagnostic information for different diseases.

	S_1	S_2	S_3	S_4	S_5
D_1	(0.4, 0.6, 0.0)	(0.3, 0.2, 0.5)	(0.1, 0.3, 0.7)	(0.4, 0.3, 0.3)	(0.1, 0.2, 0.7)
D_2	(0.7, 0.3, 0.0)	(0.2, 0.2, 0.6)	(0.0, 0.1, 0.9)	(0.7, 0.3, 0.0)	(0.1, 0.1, 0.8)
D_3	(0.3, 0.4, 0.3)	(0.6, 0.3, 0.1)	(0.2, 0.1, 0.7)	(0.2, 0.2, 0.6)	(0.1, 0.0, 0.9)
D_4	(0.1, 0.2, 0.7)	(0.2, 0.4, 0.4)	(0.8, 0.2, 0.0)	(0.2, 0.1, 0.7)	(0.2, 0.1, 0.7)
D_5	(0.1, 0.1, 0.8)	(0.0, 0.2, 0.8)	(0.2, 0.0, 0.8)	(0.2, 0.0, 0.8)	(0.8, 0.1, 0.1)

We apply the similarity measure $\mathfrak{S}_1$ to evaluate the relationship between the patient P_1 and each symptom S_i associated with the diagnoses D_i $(i = 1, 2, \ldots, 5)$ for $r = 1$. This computation is conducted following the formal definition of similarity measures, as stated in Definition 3.1. The obtained similarity values provide a quantitative representation of the degree of resemblance between P_1 and the given symptoms in relation to different diagnoses.

These values are systematically presented in Table 2, allowing for a clear comparison and assessment of symptom-based similarity within the diagnostic framework.

Building upon these results, we further compute the similarity between P_1 and each diagnosis D_i $(i = 1, 2, \ldots, 5)$ for $r = 1$, utilizing the data from Table 2 and the similarity measure definition provided in 3.3. This step integrates the symptom-wise similarity values to derive an overall similarity measure for each diagnosis, enabling a comprehensive evaluation of the patient's association with different medical conditions. The computed similarity values are presented in Table 3, offering insights into the most relevant diagnoses for P_1.

TABLE 2. Similarity measure between P_1 and each symptoms with respect to diagnosis .

		S_1	S_2	S_3	S_4	S_5
	D_1	0.7692	0.7895	0.8824	0.8333	0.9091
	D_2	0.9091	0.7500	0.9091	0.8824	0.8571
$\mathfrak{S}_1$	D_3	0.7692	1.000	0.9677	0.7143	0.8108
	D_4	0.6977	0.7895	0.6667	0.6818	0.8571
	D_5	0.6667	0.6818	0.9677	0.6522	0.6667

TABLE 3. Similarity measure between P_1 and each diagnosis.

	D_1	D_2	D_3	D_4	D_5
$\mathfrak{S}_1$	0.8367	**0.8615**	0.8524	0.7386	0.7271

Since D_2 (Malaria) has the highest similarity score, the patient is diagnosed with Malaria.

Case 2: Comparative Diagnosis for Multiple Patients

Example 5.2. To further evaluate the robustness of $\mathfrak{S}_1$, we introduce two additional patients, P_2 and P_3, with different symptom distributions.

Patient P_2**:** $\{S_1(0.5, 0.3, 0.2), S_2(0.4, 0.5, 0.1), S_3(0.3, 0.2, 0.5), S_4(0.7, 0.3, 0.0), S_5(0.2, 0.4, 0.4)\}$.

Patient P_3**:** $\{S_1(0.2, 0.5, 0.3), S_2(0.1, 0.6, 0.3), S_3(0.8, 0.1, 0.1), S_4(0.2, 0.3, 0.5), S_5(0.7, 0.2, 0.1)\}$.

By applying the similarity measure $\mathfrak{S}_1$, we obtain the similarity values between P_2, P_3 and each diagnosis D_i $(i = 1, 2, \ldots, 5)$ for $r = 1$ by similarity measures definitions 3.1 and 3.3 and presented in Table 4. For P_2, the highest similarity is with D_1 (Viral Fever), while for P_3, the highest similarity is with D_5 (Stenocardia), indicating different diagnoses due to varying symptoms.

Thus, The measure effectively assigns diagnoses based on the highest similarity values, ensuring reliable decision-making. Different patients with different symptom intensities receive

TABLE 4. Similarity measure for additional patients.

		D_1	D_2	D_3	D_4	D_5
$\mathfrak{S}_1$	P_2	**0.8752**	0.8641	0.8327	0.7413	0.7295
	P_3	0.7221	0.7345	0.7956	0.8421	**0.8613**

appropriate diagnoses. The measure distinguishes between closely related conditions (e.g., Viral Fever vs. Malaria vs. Typhoid) while still assigning accurate diagnoses. The framework supports real-world medical decision-making where patients exhibit uncertain and varying symptom levels. Thus, the proposed similarity measure $\mathfrak{S}_1$ proves to be highly effective in medical diagnostics, offering a systematic approach for decision-making in uncertain and vague environments.

6. Comparative analysis with other similarity measure existing in the literature

6.1. *Comparative Study -I for $\mathfrak{S}-$ Similarity Measure on single valued Neutrosophic Sets*

In this section, we are comparing our $\mathfrak{S}-$similarity measure with existing similarity measure on interval valued neutrosophic sets in [7]. Now let us consider the following MCDM similarity measure problem (adapted from Sujit Das, [7]). The investor has chosen five companies and will select one based on certain criteria and the set of alternative companies $A = \{I_1$ (Automobile company), I_2 (Food manufacturing company), I_3 (Availability of labor,) I_4(Oil company), I_5 (pharmaceutical company)$\}$ and a set of decision criteria $C = \{C_1$ (Risk), C_2 (Availability of raw materials), C_3 (Security), C_4 (Market demand), C_5 (Production quantity)$\}$. The company information I_i $(i = 1, 2, \ldots, 5)$ shown in Table 5 with respect to decision criteria C_i $(i = 1, 2, \ldots, 5)$. In [7], SVNT are represented as (μ, T, I, F) and we indicate them as (T, I, F). Thus, the table given in [7] and similarity measure technique have been adjusted to use $\mu = 1$. The Ideal alternating company $A^* = \{(0.9, 0.1, 0.1),\ (0.3, 0.36, 0.2),\ (1.0, 0.0, 0.0),\ (0.72, 0.7, 0.3),\ (0.7, 0.2, 0.3)\}$ is assumed from [7]. By existing methods in [7], the S_1 and S_2 similarity measures between

TABLE 5. The company information I_i with respect to decision attribute C_i.

	C_1	C_2	C_3	C_4	C_5
I_1	(0.7, 0.4, 0.3)	(0.27, 0.4, 0.7)	(0.5, 0.1, 0.2)	(0.5, 0.9, 0.4)	(0.1, 0.3, 0.8)
I_2	(0.5, 0.8, 0.2)	(0.15, 0.36, 0.78)	(0.9, 0.0, 0.2)	(0.6, 0.96, 0.45)	(0.3, 0.5, 0.75)
I_3	(0.9, 0.1, 0.1)	(0.3, 0.6, 0.9)	(0.25, 0.4, 0.5)	(0.72, 0.85, 0.3)	(0.3, 0.45, 0.87)
I_4	(0.8, 0.6, 0.3)	(0.1, 0.8, 0.2)	(1.0, 0.5, 0.0)	(0.57, 0.8, 0.35)	(0.1, 0.8, 0.6)
I_5	(0.65, 0.2, 0.8)	(0.2 0.45 0.65)	(0.7 0.4, 0.6)	(0.4, 0.7, 0.6)	(0.7, 0.2, 0.3)

I_i $(i = 1, 2, 3, 4, 5)$ and A^* are presented in table 6. From table 6, we can order the companies

TABLE 6. Similarity measure between A^* and I_i $(i = 1, 2, \ldots, 5)$.

	I_1	I_2	I_3	I_4	I_5
S_1	0.82	0.80	0.80	0.81	0.82
S_2	0.71	0.70	0.67	0.69	0.71

as $I_1 = I_5 > I_4 > I_3 = I_4$ by S_1 similarity measure and $I_1 = I_5 > I_2 > I_4 > I_3$ by S_2 similarity measure. It is evident that an investor may be less confident in his ability to choose the company to invest in. this is due to the lack of total ordering.

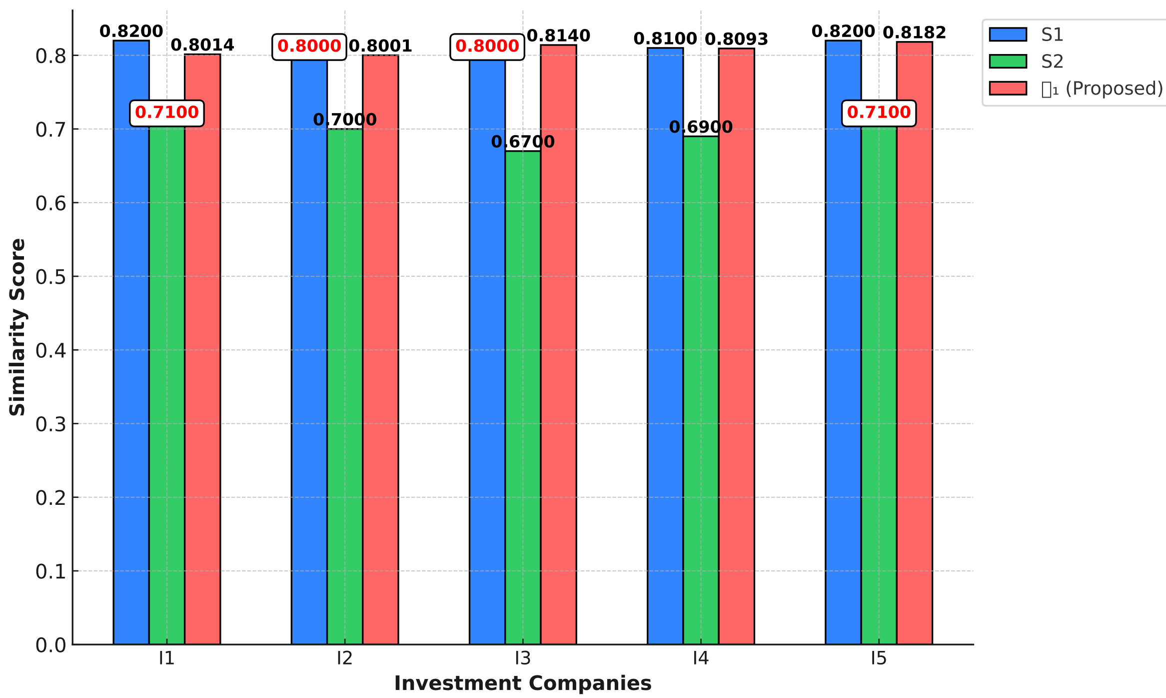

FIGURE 1. Comparison of $\mathfrak{S}_1$ Similarity Measure with Selected Methods

As per our proposed $\mathfrak{S}_1$ ordering algorithm (Algorithm 3.1.1), we initially determine M_T, M_I, M_F using Definition 3.6. The computed values are:

$$M_T = \{(0.9, 0.1, 0.1),\ (0.3, 0.36, 0.2),\ (1.0, 0.0, 0.0),\ (0.72, 0.7, 0.3),\ (0.7, 0.2, 0.3)\},$$

$$M_I = \{(0.5, 0.8, 0.1),\ (0.1, 0.8, 0.2),\ (0.25, 0.5, 0.0),\ (0.4, 0.96, 0.3),\ (0.1, 0.8, 0.3)\},$$

$$M_F = \{(0.5, 0.1, 0.8),\ (0.1, 0.36, 0.9),\ (0.25, 0.0, 0.6),\ (0.4, 0.7, 0.6),\ (0.1, 0.2, 0.87)\}.$$

Step 1: Computation of $\mathfrak{S}_1$-Similarity Measure The $\mathfrak{S}_1$-similarity measure is computed between M_T and each company I_i $(i = 1, 2, 3, 4, 5)$ using Definitions 3.1 and 3.3. The computed values are presented in Table 7.

Step 2: Ranking of Companies From Table 7, the companies are ranked as follows:

$$I_5 > I_3 > I_4 > I_1 > I_2. \tag{25}$$

Since all companies have distinct similarity scores, this ranking is sufficient, and further similarity computations are not required.

Step 3: Tie-Breaking Strategy If two or more companies have identical similarity scores, additional computations are performed as follows:

(1) **First-Level Tie Resolution:**
- Compute the $\mathfrak{S}_1$-similarity measure between M_I and I_i $(i = 1, 2, 3, 4, 5)$.
- If a strict ranking is achieved, no further computations are needed.

(2) **Second-Level Tie Resolution (If Necessary):**
- If the tie persists, compute the $\mathfrak{S}_1$-similarity measure between M_F and I_i $(i = 1, 2, 3, 4, 5)$.
- This final step ensures a complete ordering.

Step 4: Ensuring a Complete Order By Theorem 3.8, the above procedure guarantees a total ordering of the companies, ensuring a definitive ranking based on similarity measures.

Table 7. $\mathfrak{S}-$Similarity measure between M_T and I_i $(i = 1, 2, \ldots, 5)$.

	I_1	I_2	I_3	I_4	I_5
$\mathfrak{S}_1$	0.8014	0.8001	0.8140	0.8093	**0.8182**

The aforementioned considerations confirm that our proposed SS total ordering method demonstrates superior performance compared to existing similarity methods in [7]. This advantage is clearly illustrated in Figure 1, showcasing its effectiveness and reliability in various scenarios.

6.2. *Comparative Study -II for $\mathfrak{S}-$Similarity Measure on Single valued Neutrosophic Sets*

In this section, we are comparing our $\mathfrak{S}-$similarity measure with existing similarity measure on single valued neutrosophic sets in [42]. To compare both these method we apply various similarity measures given by [42] to our example 5.1 from our proposed numerical illustration section. For the same problem given in example 5.1 we get the following results by applying similarity measures. Here $\mathfrak{S}(D_1)$ denotes the respective similarity measure between P_1 and

D_1. From the table 8, we can say that our proposed similarity measure method co-inside

Table 8. Solution for example 5.1 by various similarity measures

Similarity Measure Based on	D_1	D_2	D_3	D_4	D_5	Result
Extended Hausdorff Distance	**0.92**	0.9	0.9	0.84	0.86	D_1(Viral fever)
Set - Theoretic Approach	0.63	0.675	**0.741**	0.308	0.317	D_3(Typhoid)
Type1 Geometric Distance Model	0.722	**0.8**	0.763	0.42	0.333	D_2(Malaria)
Type 2 Geometric Distance Model	0.9	**0.92**	0.91	0.78	0.76	D_2(Malaria)
Type 3 Geometric Distance Model	**0.71**	**0.71**	0.66	0.58	0.58	D_1(Viral fever) or D_2(Malaria)

with the similarity measure based on Type 1, Type 2 geometric distance model method and differ with other methods for this particular problem. When we analyze these results we clearly see that in Type 3 Geometric Distance Model we got the result as either the patient is diagnosed by malaria or viral fever which is a shortcoming because we are not able to give precise alternative as a solution to this problem and in extended Hausdorff Distance method we can order in well manner. This ambiguity occurs because of the lack of total ordering technique embedded in the similarity measure. For example consider the following three SVNT $A = (0.5, 0.5, 0.5), B = (1, 0.6, 0.3), C = (1, 0.3, 0.9)$ by applying all these methods we get $S(A, B) = S(A, C)$ but when we apply our method we get $S(A, B) > S(A, C)$. Therefore in some circumstances our method will works better than the other methods.

Upon analyzing the results, we observe that the **Type 3 Geometric Distance Model** struggles with ambiguity, as it diagnoses the patient with either malaria or viral fever, failing to provide a precise alternative. Similarly, the **Extended Hausdorff Distance Method** lacks a well-structured ordering mechanism, leading to inconsistencies in ranking. These shortcomings arise due to the absence of a robust total ordering technique in the similarity measure. In contrast, our proposed $\mathfrak{S}_1$ **similarity measure** incorporates an inherent total ordering property, ensuring a well-defined and consistent ranking of outcomes. This capability eliminates ambiguity, enhances decision-making accuracy, and provides clearer diagnostic conclusions, as illustrated in Figure 2.

For instance, consider the following three SVNTs, $A = (0.5, 0.5, 0.5)$, $B = (1, 0.6, 0.3)$, $C = (1, 0.3, 0.9)$. When applying the existing methods, we obtain $S(A, B) =$

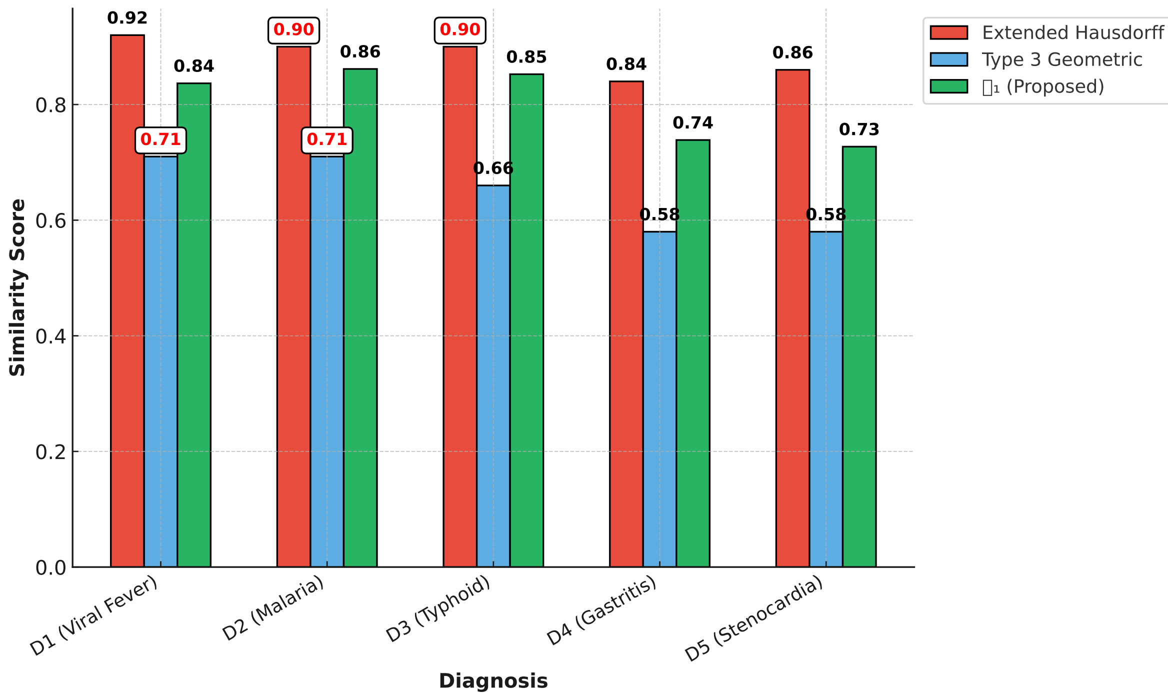

FIGURE 2. Comparison of $\mathfrak{S}_1$ Similarity Measure with Selected Methods

$S(A,C)$, meaning they fail to distinguish between the two cases. However, with our proposed method, we get $S(A,B) > S(A,C)$, demonstrating its ability to establish a meaningful and precise ordering. This result further highlights the superiority of our approach in addressing the limitations of existing similarity measures.

6.3. *Comparative Study -III with $\mathfrak{S}-$ Similarity Measure on Interval valued Neutrosophic Sets*

To demonstrate the practicality of proposed similarity measures $\mathfrak{S}_2$, we first provide a numerical example to choose a company for internet of things industry (adapted from X Peng [6]) and we are comparing our $\mathfrak{S}-$similarity measure with existing similarity measure on interval valued neutrosophic sets in [6]. Let us consider the following for choosing a company for internet of things industry: a set of alternative companies $A = \{A_1, A_2, A_3, A_4, A_5\}$ and a set of decision attribute $C = \{C_1$ (Connectivity), C_2 (Value), C_3 (Security), C_4 (Telepresence), C_5 (Intelligence)$\}$. Assuming a ideal alternative company A^* as $([1,1],[0,0],[0,0])$. The company information A_i $(i = 1,2,\ldots,5)$ shown in Table 9 with respect to decision attribute C_i $(i = 1,2,\ldots,5)$.

We can find similarity measure between A^* and A_i $(i = 1,2,\ldots,5)$ for $r = 1$ by similarity measures definitions 4.1 and 4.3 and which is presented in Table 10. From the similarity measure $\mathfrak{S}_2$ values in Table 10, we can see that the company A_1 is preferable for internet of

TABLE 9. For the IVNT decision, The company information A_i with respect to decision attribute C_i.

	C_1	C_2	C_3
A_1	([0.8, 0.9], [0.1, 0.2], [0.1, 0.2])	([0.8, 0.9], [0.1, 0.2], [0.2, 0.3])	([0.8, 0.9], [0.2, 0.3], [0.1, 0.2])
A_2	([0.8, 0.8], [0.1, 0.2], [0.1, 0.2])	([0.8, 0.9], [0.2, 0.3], [0.3, 0.3])	([0.7, 0.8], [0.2, 0.3], [0.1, 0.3])
A_3	([0.7, 0.8], [0.1, 0.2], [0.1, 0.2])	([0.7, 0.8], [0.2, 0.3], [0.3, 0.4])	([0.6, 0.8], [0.2, 0.3], [0.1, 0.3])
A_4	([0.5, 0.6], [0.1, 0.3], [0.2, 0.3])	([0.6, 0.7], [0.2, 0.3], [0.3, 0.4])	([0.6, 0.8], [0.2, 0.3], [0.2, 0.3])
A_5	([0.3, 0.4], [0.2, 0.3], [0.2, 0.3])	([0.5, 0.6], [0.3, 0.4], [0.4, 0.5])	([0.5, 0.7], [0.4, 0.5], [0.3, 0.4])

	C_4	C_5
A_1	([0.8, 0.9], [0.1, 0.2], [0.1, 0.2])	(([0.7, 0.8], [0.1, 0.2], [0.1, 0.2])
A_2	([0.8, 0.9], [0.2, 0.2], [0.2, 0.2])	([0.7, 0.8], [0.2, 0.3], [0.1, 0.2])
A_3	([0.8, 0.9], [0.2, 0.3], [0.3, 0.4])	([0.6, 0.7], [0.2, 0.3], [0.2, 0.3])
A_4	([0.7, 0.8], [0.2, 0.3], [0.3, 0.4])	([0.5, 0.6], [0.2, 0.3], [0.2, 0.3])
A_5	([0.6, 0.7], [0.2, 0.3], [0.4, 0.4])	([0.4, 0.6], [0.5, 0.6], [0.2, 0.4])

TABLE 10. Similarity measure between A^* and A_i $(i = 1, 2, \ldots, 5)$.

	A_1	A_2	A_3	A_4	A_5
$\mathfrak{S}_2$	**0.8549**	0.8292	0.8007	0.7714	0.7166

things industry. Our $\mathfrak{S}_2$ similarity results are the same as shown by X Peng [6]. Thus, our proposed similarity measure is effectively applied in decision making.

6.4. *Advantages of the Proposed $\mathfrak{S}$-Similarity Measure*

This section highlights the advantages of the proposed $\mathfrak{S}$-similarity measure over existing methods.

(1) **Elimination of Ambiguity through Total Ordering**

Several existing similarity measures, including the Type 3 Geometric Distance Model and methods presented in [42], fail to provide a definitive ranking of alternatives, leading to uncertainty in decision-making. The proposed $\mathfrak{S}$-similarity measure incorporates a *total ordering mechanism*, ensuring a precise differentiation among alternatives.

(2) **Enhanced Decision-Making Accuracy**

- In *Comparative Study - I*, traditional methods such as the Type 3 Geometric Distance Model [42] produced inconsistent results, often identifying multiple possible diagnoses without a clear preference. The proposed $\mathfrak{S}$-similarity measure eliminates this issue by **providing a unique ranking**, thereby enhancing medical diagnosis reliability.

- In *Comparative Study - II*, decision-makers using similarity measures from [7] encountered difficulties in ranking investment options due to tied similarity scores. The proposed measure successfully resolves these ties, leading to a **more confident and accurate decision-making process**.

(3) **Robustness Across Different Neutrosophic Set Types**

Unlike existing approaches that are often restricted to either *single-valued neutrosophic sets (SVNSs) or interval-valued neutrosophic sets (IVNSs)*, the proposed $\mathfrak{S}$-similarity measure demonstrates adaptability across both representations, making it suitable for a broader range of applications.

7. Limitations and Future Research Directions

While the proposed similarity measures $\mathfrak{S}_1$ and $\mathfrak{S}_2$ effectively establish a total ordering of the members of SVNTs and IVNTs when $r = 1$, their applicability is currently limited when $r = 2$. In this case, they provide a total ordering only for a specific subclass of SVNTs, as demonstrated in Theorem 3.9, rather than the entire class. This limitation indicates the need for further refinement to extend their applicability to the general class of SVNTs and IVNTs.

As a future direction, we aim to extend our work by developing similarity measures that can handle the entire class of SVNTs and IVNTs without any restrictions on r. This enhancement will ensure a more comprehensive and robust approach to total ordering within these frameworks.

To address this limitation and further enhance the effectiveness of the proposed similarity measures, several promising research directions can be pursued:

- **Extending the Framework**: Expanding the $\mathfrak{S}$-similarity measures to other neutrosophic environments, such as hesitant neutrosophic sets, refined neutrosophic sets, and multi-valued neutrosophic triplets, to increase their versatility.
- **Generalization for Total Ordering**: Developing an extended version of the similarity measures that ensures total ordering for the entire class of SVNTs and IVNTs, without any restrictions on r.
- **Algorithmic Enhancements**: Refining the $\mathfrak{S}_1$- and $\mathfrak{S}_2$-ordering algorithms to improve computational efficiency and adaptability for large-scale decision-making problems.

- **Hybrid Decision-Making Models**: Integrating the proposed measures into machine learning-based classification and clustering frameworks to enhance intelligent decision-support systems.
- **Real-World Applications**: Applying the proposed approach to practical domains such as medical diagnosis, risk assessment, supply chain optimization, and financial decision-making to evaluate its effectiveness in complex decision environments.

By addressing these future directions, the proposed $\mathfrak{S}$-similarity measures can be further refined and extended, contributing to more advanced decision-making methodologies in neutrosophic environments. The incorporation of total ordering in similarity measures offers a more structured and systematic approach to multi-criteria decision analysis, making it a valuable tool for researchers and practitioners alike.

8. Conclusions

In the rapidly advancing field of decision-making and similarity assessment, the ability to establish precise and reliable similarity measures is critical for selecting optimal alternatives based on given attributes. This research introduced a novel $\mathfrak{S}$-similarity measure for ingle-Valued Neutrosophic Triplets (SVNTs)and Interval-Valued Neutrosophic Triplets (IVNTs), along with their generalized forms. Unlike existing similarity measures, the proposed $\mathfrak{S}$-similarity measures inherently possess total ordering, which is crucial for accurate decision-making and ranking of alternatives. Conventional similarity measures often lack this feature, leading to ambiguity, a limitation effectively addressed by the proposed $\mathfrak{S}_1$-ordering algorithm for SVNTs and the $\mathfrak{S}_2$-ordering algorithm for IVNTs.

Key Contributions:

- Novel $\mathfrak{S}$-similarity measures for SVNTs and IVNTs with total ordering.
- Development of the $\mathfrak{S}_1$- and $\mathfrak{S}_2$-ordering algorithms, ensuring precise decision-making.
- Application of the proposed similarity measures in Multi-Criteria Decision-Making (MCDM) problems adapted from Ye [5], X. Peng [6], and Sujit Das [7].
- Validation through comparative analysis, demonstrating that the proposed measures are computationally efficient and yield results consistent with existing methods.

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