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Role of clans in the proximities of Neutrosophic Sets

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Abstract: In this study, the terms filters, grills, clans, and proximities of neutrophic sets are defined. Further some of its properties are investigated and the results are provided.

Keywords: Neutrosophic set, filters, grills, clans and proximities of neutrosophic sets.

1. Introduction

In the year 1965, Zadeh [10] introduced and investigated fuzzy sets. An intuitionistic fuzzy set was first presented in 1986 by Atanassov [2]. Florentin Smarandache [5] developed concepts such as neutrosophic logic and neutrosophic set in 1999. The truth, falsehood, and indeterminacy membership values are the three components on which he defined the neutrosophic set. The neutrosophic set was created in 2010 by Florentin Smarandache [3] as a generalization of intuitionistic fuzzy sets.

K C Chattopadhyay and etal [6], [9] developed the role of clans in the proximities of fuzzy sets in the year 1996 and on intuitionistic fuzzy sets in the year 1997 respectively. Also they proved the proximities of IFS is a clan generated structure. In this paper, we introduce and investigate the proximities of neutrosophic sets and proved that the proximities of NSs is a clan generated structure and provided the numerical example wherever applicable.

2. Preliminaries

Definition 2.1. [3] Let X be a fixed set that is non-empty. A set with the form $\check{N} = \{\langle \alpha, T_{\check{N}}(\alpha), I_{\check{N}}(\alpha), F_{\check{N}}(\alpha)\rangle : \alpha \in X\}$ is called a Neutrosophic set, where $T_{\check{N}}(\alpha)$, $I_{\check{N}}(\alpha)$, $F_{\check{N}}(\alpha)$ represents the degree of truth, degree of indeterminacy and the degree of falsity respectively of each element $\alpha \in X$ to the set $\check{N}$.

The set of all neutrosophic sets on X is denoted by $\check{N}(X)$.

Definition 2.2. [3] The complement of a Neutrosophic set Ň is denoted by $Ň^C$ and is defined by $Ň^C$ = {⟨x, $F_{Ň}(x)$, 1-$I_{Ň}(x)$, $T_{Ň}(x)$⟩: x ∈ X}.

Definition 2.3. [3] Consider two Neutrosophic sets U and V over X, then U is said to be contained in V, denoted by U ⊆ V if and only if $T_u(x) \le T_v(x)$, $I_u(x) \le I_v(x)$, $F_u(x) \ge F_v(x)$.

Definition 2.4. [3] The arbitrary union of two Neutrosophic sets U and V over X, is denoted by U∪V and is defined by {⟨x, $T_u(x) \vee T_v(x)$, $I_u(x) \vee I_v(x)$, $F_u(x) \wedge F_v(x)$⟩: x ∈ X}

Definition 2.5. [3] The finite intersection of two Neutrosophic sets U and V over X, is denoted by U∩V and is defined by {⟨x, $T_u(x) \wedge T_v(x)$, $I_u(x) \wedge I_v(x)$, $F_u(x) \vee F_v(x)$⟩: x ∈ X}

Definition 2.6. [3] Let Ň be a Neutrosophic set over X, then the universe set of Ň is denoted by $1_{Ň}$ and is defined by $1_{Ň}$ = {⟨x,1,1,0⟩: x ∈ X}.

Definition 2.7. [9] Let Ň be a Neutrosophic set over X, then the empty set of Ň is denoted by $0_{Ň}$ and is defined by $0_{Ň}$ = {⟨x,0,0,1⟩: x ∈ X}.

Proposition 2.8. Let U, V∈ Ň(X), then the following holds

(i) $(U \cup V)^c = U^c \cap V^c$

(ii) $(U \cap V)^c = U^c \cup V^c$

(iii) $(1_{Ň})^c = 0_{Ň}$

(iv) $(0_{Ň})^c = 1_{Ň}$

3. Stack, Filter, Grill, Prime filter of Neutrosophic Sets

Definition 3.1. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A stack Š of neutrosophic sets on X is a subset of Ň(X) such that P ⊃ Q ∈ Š ⇒ P ∈ Š.

Example 3.2. Let X = {a,b}. consider the following neutrosophic sets

$A_1 = \{<a, 0.3, 0.7, 0.2>, <b, 0.4, 0.5, 0.6>\}$

$A_2 = \{<a, 0.5, 0.5, 0.6>, <b, 0.3, 0.3, 0.4>\}$

$A_3 = \{<a, 0.4, 0.7, 0.1>, <b, 0.5, 0.2, 0.8>\}$

$A_4 = \{<a, 0.2, 0.2, 0.8>, <b, 0.1, 0.1, 0.7>\}$

$A_5 = \{< a, 0.5, 0.8, 0 >, < b, 0.6, 0.6, 0.2 >\}$

$A_6 = \{< a, 0.2, 0.4, 0.5 >, < b, 0.1, 0.1, 0.9 >\}$

$A_7 = \{< a, 0.4, 0.8, 0.1 >, < b, 0.6, 0.6, 0.3 >\}$

$A_8 = \{< a, 0.6, 0.6, 0.5 >, < b, 0.5, 0.5, 0.1 >\}$

$A_9 = \{< a, 0.3, 0.5, 0.2 >, < b, 0.4, 0.2, 0.9 >\}$

$A_{10} = \{< a, 0.6, 0.9, 0 >, < b, 0.7, 0.7, 0 >\}$

Here $\check{S}_1, \check{S}_2, \check{S}_3$ are the stacks of I(X), where

$\check{S}_1 = \{A_2, A_4, A_8\}$ ***such that*** $A_4 \subseteq A_2 \subseteq A_8$

$\check{S}_2 = \{A_1, A_6, A_7\}$ ***such that*** $A_6 \subseteq A_1 \subseteq A_7$

$\check{S}_3 = \{A_3, A_5, A_9, A_{10}\}$ ***such that*** $A_9 \subseteq A_3 \subseteq A_5 \subseteq A_{10}$

Definition 3.3. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A filter Ƒ of neutrosophic sets on X is a subset of Ň(X) satisfying the following:

(i) Ƒ ≠ ϕ

(ii) P ⊃ Q ∈ Ƒ ⇒ P ∈ Ƒ

(iii) P, Q ∈ Ƒ ⇒ P∩Q ∈ Ƒ

A filter Ƒ of Neutrosophic sets is said to be proper if $0_{Ň}$ ∉ Ƒ.

Example 3.4. From the example 3.2, Ƒ = $\{A_2, A_4, A_8\}$ ***such that*** $A_4 \subseteq A_2 \subseteq A_8$ is a filter as well as proper filter, since $0_{Ň}$ ∉ Ƒ and the following holds

For A_2, A_4 ∈ Ƒ ⇒ $A_2 \cap A_4 = A_4$ ∈ Ƒ

For A_2, A_8 ∈ Ƒ ⇒ $A_2 \cap A_8 = A_2$ ∈ Ƒ

For A_4, A_8 ∈ Ƒ ⇒ $A_4 \cap A_8 = A_4$ ∈ Ƒ

Definition 3.5. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A grill Ǧ of neutrosophic sets on X is a subset of Ň(X) satisfying the following:

(i) $0_{Ň}$ ∉ Ǧ

(ii) P ⊃ Q ∈ Ǧ ⇒ P ∈ Ǧ

(iii) P∪Q ∈ Ǧ ⇒ P ∈ G or Q ∈ Ǧ

A grill Ǧ of neutrosophic sets is said to be proper if Ǧ ≠ ϕ.

Example 3.6. From the example 3.2, $\breve{G} = \{A_2, A_4, A_8\}$ ***such that*** $A_4 \subseteq A_2 \subseteq A_8$ is a grill as well as proper grill, since $0_{\check{N}} \notin \breve{G}$ and the following holds

For $A_2 \cup A_4 = A_2 \in \breve{G} \Rightarrow A_2 \in \breve{G}$ ***or*** $A_4 \in \breve{G}$

For $A_2 \cup A_8 = A_8 \in \breve{G} \Rightarrow A_2 \in \breve{G}$ ***or*** $A_8 \in \breve{G}$

For $A_4 \cup A_8 = A_8 \in \breve{G} \Rightarrow A_4 \in \breve{G}$ ***or*** $A_8 \in \breve{G}$

Definition 3.7. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A stack Š of NSs on X is a prime filter of NSs on X if it is a filter of NSs as well as the grill of NSs on X.

In other words, a stack Š of NSs on X is a prime filter of NS on X if it satisfies the following:

(i) $0_{\check{N}} \notin \check{S}$

(ii) $\check{S} \neq \phi$

(iii) $P \supset Q \in \check{S} \Rightarrow P \in \check{S}$

(iv) $P, Q \in \check{S} \Rightarrow P \cap Q \in \check{S}$

(v) $P \cup Q \in \check{S} \Rightarrow P \in G$ or $Q \in \check{S}$

Example 3.8. From the example 3.2, $\check{S}_1 = \{A_2, A_4, A_8\}$ is a prime filter of Ň(X), since it is a filter and grill of Ň(X).

We denote the following notation

Set of all filters of NSs on X = ζ(X)

Set of all grills of NSs on X = ψ(X)

Set of all prime filters of NSs on X = ξ(X)

Example 3.9. Let A∈Ň(X). Define $Ƒ \subset \check{N}(X)$ by $Ƒ = \{\mathbf{B} \in \check{N}(\mathbf{X}) | \mathbf{B} \supset \mathbf{A}\}$

Clearly Ƒ is non empty. Let $\mathbf{C} \supset \mathbf{B} \in Ƒ$. Then $\mathbf{C} \supset \mathbf{B} \supset \mathbf{A}$ and hence $\mathbf{C} \in Ƒ$. Now let $\mathbf{B}, \mathbf{C} \in Ƒ$ such that $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$, $F_A(x) \geq F_B(x)$ and $T_A(x) \leq T_C(x)$, $I_A(x) \leq I_C(x)$, $F_A(x) \geq F_C(x)$. Hence it follows that $T_A(x) \leq T_B(x) \wedge T_C(x)$, $I_A(x) \leq I_B(x) \wedge I_C(x)$, $F_A(x) \geq F_B(x) \vee F_C(x)$. Thus $A \subset B \cap C$ which implies $B \cap C \in Ƒ$. Hence Ƒ is a filter of NSs.

Theorem 3.10. Let $Ƒ_1, Ƒ_2 \in \zeta(X)$ and $\breve{G}_1, \breve{G}_2 \in \psi(X)$.

(i) If $Ƒ_1 \cap Ƒ_2 \subset \breve{G}_1$, then $Ƒ_1 \subset \breve{G}_1$ or $Ƒ_2 \subset \breve{G}_1$

(ii) If $Ƒ_1 \subset \breve{G}_1$ or $Ƒ_1 \subset \breve{G}_2$, then $Ƒ_1 \subset \breve{G}_1 \cup \breve{G}_2$

Proof:

(i) Suppose $Ғ_1 \not\subset Ğ_1$ and $Ғ_2 \not\subset Ğ_1$. Then there exists $A_1 \in Ғ_1$ such that $A_1 \notin Ğ_1$ and $A_2 \in Ғ_2$ such that $A_2 \notin Ğ_1$, therefore $A_1 \cup A_2 \notin Ğ_1$, but $A_1 \cup A_2 \in Ғ_1 \cap Ғ_2$ which is a contradiction. Hence $Ғ_1 \subset Ğ_1$ or $Ғ_2 \subset Ğ_1$ is valid.

(ii) Case (I): Suppose $Ғ_1 \subset Ğ_1$, then for $A_1 \in Ғ_1$, we have $A_1 \in Ğ_1$. Also, since $Ğ_1 \subset Ğ_1 \cup Ğ_2$ implies $A_1 \in Ğ_1 \cup Ğ_2$. Hence $Ғ_1 \subset Ğ_1 \cup Ğ_2$.

Case (II): Suppose $Ғ_1 \subset Ğ_2$, then for $A_2 \in Ғ_1$, we have $A_2 \in Ğ_2$. Also, since $Ğ_2 \subset Ğ_1 \cup Ğ_2$ implies $A_2 \in Ğ_1 \cup Ğ_2$. Hence $Ғ_1 \subset Ğ_1 \cup Ğ_2$

In either case if we have $Ғ_1 \subset Ğ_1$ or $Ғ_1 \subset Ğ_2$, then $Ғ_1 \subset Ğ_1 \cup Ğ_2$

Theorem 3.11. Arbitrary intersection of filters of NSs is a filter of NSs.

Proof is straightforward.

Theorem 3.12. Finite union of grills of NSs is a grill of NSs.

Proof:

Let $Ğ = \cup\{Ğ_i, \boldsymbol{i} \in \boldsymbol{I}, Ğ_i \in \boldsymbol{\psi}(\mathbf{X})\}$. we check the three axioms

(I) $0_Ň \notin Ğ_i$ for all $\boldsymbol{i} \in \boldsymbol{I}$, then $0_Ň \notin Ğ$.

(II) If $\boldsymbol{A} \in Ğ$, **then so** $\mathbf{A} \in Ğ_i$ **for some** $\mathbf{i} \in \mathbf{I}$ and $\boldsymbol{A} \subset$ B, since each $Ğ_i$ is a grill, then $\mathbf{B} \in Ğ_i \subset Ğ$

(III) If $\boldsymbol{A} \cup \boldsymbol{B} \in Ğ$, **then so** $\mathbf{A} \cup \boldsymbol{B} \in Ğ_i$ **for some** $\mathbf{i} \in \mathbf{I}$ and $\boldsymbol{A} \subset$ B, since each $Ğ_i$ is a grill, then $\mathbf{A} \in Ğ_i$ or $\boldsymbol{B} \in Ğ_i$, hence A$\in Ğ$ or $\boldsymbol{B} \in Ğ$.

Thus, union of grills is again a grill.

Definition 3.13. For each stack Š of NSs, define dŠ = {A : $\boldsymbol{A}^c \notin$ Š}

Theorem 3.14. If Š is a stack of NSs, Ғ is a filter of NSs and Ğ is a grill of NSs on X, then the following holds.

(1) If $Š_2 \subset Š_1$, then $\mathbf{d}Š_1 \subset \mathbf{d}Š_2$

(2) d(dŠ) = Š

(3) $\mathrm{d}(\cup Š_i) = \cap \mathbf{d}Š_i$

(4) $\mathrm{d}(\cap Š_i) = \cup \mathbf{d}Š_i$

(5) **d**Ғ is a grill of NSs

(6) **d**Ğ is a filter of NSs

Proof.

(1) To prove $\mathbf{d}Š_1 \subset \mathbf{d}Š_2$. That is for any $\boldsymbol{A} \in \mathbf{d}Š_1 \Rightarrow \mathbf{A} \in \mathbf{d}Š_2$. Assume $Š_2 \subset Š_1$

Since $\boldsymbol{A} \in \mathbf{d}\check{S}_1$, then $\boldsymbol{A^c} \notin \check{S}_1$. Then from our assumption $\boldsymbol{A^c} \notin \check{S}_2$, which implies $\boldsymbol{A} \in \mathbf{d}\check{S}_2$. Hence, $\mathbf{d}\check{S}_1 \subset \mathbf{d}\check{S}_2$

(2) For any $\boldsymbol{A} \in \check{N}(X)$, then $\boldsymbol{A} \in \mathbf{d}(\mathbf{d}\check{S}) \leftrightarrow \boldsymbol{A^c} \notin \mathbf{d}\check{S} \leftrightarrow \mathbf{A} \in \check{S}$. Hence $d(d\check{S}) = \check{S}$.

(3) For any $\boldsymbol{A} \in \check{N}(X)$, then $\boldsymbol{A} \in \mathbf{d}(\cup\check{S}_i) \leftrightarrow \boldsymbol{A^c} \notin \cup\check{S}_i \leftrightarrow \boldsymbol{A^c} \notin \check{S}_i\ \forall \mathbf{i} \in \mathbf{I} \leftrightarrow \mathbf{A} \in \mathbf{d}\check{S}_i\ \forall \mathbf{i} \in \mathbf{I} \leftrightarrow A \in \cap\mathbf{d}\check{S}_i$. Hence $d(\cup\check{S}_i) = \cap\mathbf{d}\check{S}_i$.

(4) For any $\boldsymbol{A} \in \check{N}(X)$, then

$\boldsymbol{A} \in \mathbf{d}(\cap\check{S}_i) \leftrightarrow \boldsymbol{A^c} \notin \cap\check{S}_i \leftrightarrow \boldsymbol{A^c} \notin \check{S}_i$ **for some** $\mathbf{i} \in \mathbf{I} \leftrightarrow \mathbf{A} \in \mathbf{d}\check{S}_i$ **for some** $\mathbf{i} \in \mathbf{I} \leftrightarrow A \in \cup\mathbf{d}\check{S}_i$.

Hence $d(\cap\check{S}_i) = \cup\mathbf{d}\check{S}_i$.

(5) Axiom 1: Since $1_{\check{N}} \in Ƒ$, $(\mathbf{1}_{\check{N}})^c = 0_{\check{N}} \in Ƒ$ implies $0_{\check{N}} \notin dƑ$.

Axiom 2: Let $\mathbf{B} \subset A \in dƑ$, then $(\boldsymbol{B})^c \notin Ƒ$ and it follows that $(\boldsymbol{A})^c \notin Ƒ$ for $\boldsymbol{A^c} \subset \boldsymbol{B^c}$. Hence $A \in dƑ$.

Axiom 3: Let $A \cup B \in dƑ$, then $(\boldsymbol{A}\cup\boldsymbol{B})^c \notin Ƒ$ which implies $\boldsymbol{A^c} \cap \boldsymbol{B^c} \notin Ƒ$ and it follows that $\boldsymbol{A^c} \notin Ƒ$ or $\boldsymbol{B^c} \notin Ƒ$. Hence $A \in dƑ$ or $B \in dƑ$. Thus, $dƑ$ is a grill of $\check{N}(X)$.

(6) Axiom 1: Since $0_{\check{N}} \notin \breve{G}$, $(\mathbf{0}_{\check{N}})^c = \mathbf{1}_{\check{N}} \notin \breve{G}$ implies $1_{\check{N}} \in d\breve{G}$.

Axiom 2: Let $\mathbf{B} \subset A \in d\breve{G}$, then $(\boldsymbol{B})^c \notin \breve{G}$ and it follows that $(\boldsymbol{A})^c \notin \breve{G}$ for $\boldsymbol{A^c} \subset \boldsymbol{B^c}$. Hence $A \in d\breve{G}$.

Axiom 3: Let $A \in d\breve{G}$ and $B \in d\breve{G}$, then $\boldsymbol{A^c} \notin \breve{G}$ **and** $\boldsymbol{B^c} \notin \breve{G}$ and it follows that $\boldsymbol{A^c} \cup \boldsymbol{B^c} \notin \breve{G}$. That is $(\boldsymbol{A}\cap\boldsymbol{B})^c \notin \breve{G}$ which implies $(\boldsymbol{A}\cap\boldsymbol{B}) \in d\breve{G}$. Thus, $d\breve{G}$ is a filter of $\check{N}(X)$.

Theorem 3.15. Neutrosophic Prime Filter Theorem

If Ƒ is a filter of NSs and $\breve{G}$ is a grill of NSs on X, then there exist a prime filter ϱ of NSs such that $Ƒ \subset \boldsymbol{\rho} \subset \breve{G}$.

Proof.

Let ℵ be a collection of subsets of $\check{N}(X)$ and Ƒ is a filter of NSs and $\breve{G}$ is a grill of NSs on X. Let ℵ be defined by $ℵ = \{\boldsymbol{Y} \subset \check{N}(\mathbf{X}) \mid \boldsymbol{Y}\ \boldsymbol{is\ a\ filter}, Ƒ \subset \boldsymbol{Y} \subset \breve{G}\}$

$\boldsymbol{Y} \in ℵ$, for all $\boldsymbol{Y} \subset \check{N}(\mathbf{X})$ if and only if $Ƒ \subset \boldsymbol{Y}$ for all $\boldsymbol{A_i} \in \boldsymbol{Y}$ if and only if $\cap\boldsymbol{A_i} \in \breve{G}$. Clearly $(ℵ, \subset)$ is a partial order set and $Ƒ \in ℵ$. By Zorn's lemma, $(ℵ, \subset)$ has a maximal element and ϱ be that element such that $Ƒ \subset \boldsymbol{\rho} \subset \breve{G}$.

Now to prove: $\boldsymbol{\rho}$ is a prime filter of $\check{N}(X)$.

Let $\boldsymbol{A_1}, \boldsymbol{A_2} \in \boldsymbol{\rho}$. Then $\boldsymbol{\rho} \cup \{\boldsymbol{A_1} \cap \boldsymbol{A_2}\} \in ℵ$ and by maximality of $\boldsymbol{\rho}$, $\{\boldsymbol{A_1} \cap \boldsymbol{A_2}\} \in \boldsymbol{\rho}$. Let $B \subset A$, then $\boldsymbol{\rho} \cup \{\boldsymbol{A}\} \in ℵ$ and by maximality of $\boldsymbol{\rho}$, $\boldsymbol{A} \in \boldsymbol{\rho}$. Hence $\boldsymbol{\rho}$ is a filter of NSs.

Let A,B $\in$ Ň(X) such that $\boldsymbol{A} \notin \boldsymbol{\rho}$ and B$\notin\boldsymbol{\rho}$, then both of $\boldsymbol{\rho} \cup \{\boldsymbol{A}\} \notin \aleph$ and $\boldsymbol{\rho} \cup \{\boldsymbol{B}\} \notin \aleph$. Also we can find $\boldsymbol{A_1}, \boldsymbol{A_2} \ldots . \boldsymbol{A_n} \in \boldsymbol{\rho}$ and $\boldsymbol{B_1}, \boldsymbol{B_2} \ldots . \boldsymbol{B_m} \in \boldsymbol{\rho}$ such that {A} $\cap\{\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \ldots . \cap \boldsymbol{A_n}\} \cap \{\boldsymbol{B_1} \cap \boldsymbol{B_2} \cap \ldots . \cap \boldsymbol{B_m}\} \notin$ Ğ and {B$\cap\{\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \ldots . \cap \boldsymbol{A_n}\} \cap \{\boldsymbol{B_1} \cap \boldsymbol{B_2} \cap \ldots . \cap \boldsymbol{B_m}\} \notin$ Ğ.

Hence {A$\cup$B}$\cap\{\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \ldots . \cap \boldsymbol{A_n}\} \cap \{\boldsymbol{B_1} \cap \boldsymbol{B_2} \cap \ldots . \cap \boldsymbol{B_m}\} \notin$ Ğ.

This shows that A$\cup$B $\notin\boldsymbol{\rho}$. Thus $\boldsymbol{\rho}$ is a prime filter of Ň(X).

Theorem 3.16.

Let Ğ$\subset$Ň(**X**), then Ğ be a grill of NSs on X if and only if it is a union of prime filter of NSs on X.

Proof.

Since by theorem 3.13, union of grills of NSs on X is again a grill NSs. It follows that, if Ğ is a union of prime filters of NSs on X, then it is a grill of NSs.

Conversely, suppose Ğ is a grill of NSs on X. Let A$\in$ Ğ and Ƒ = {**B** $\in$ Ň(X): A$\subset$B}, then Ƒ is a filter of NSs and Ƒ$\subset$Ğ. By Neutrosophic Filter theorem, then there exists a prime filter ϱ of NSs such that Ƒ$\subset\boldsymbol{\rho}\subset$Ğ and hence A$\in$ $\boldsymbol{\rho}\subset$Ğ. Thus Ğ is a union of prime filters of NSs on X.

4. Proximities of Neutrosophic Sets

Definition 4.1. Binary relation δ on Neutrosophic sets

Let δ$\subseteq$ Ň(X) × Ň(X) such that (A,B) $\in$δ if and only if δ (A,B)= $<\boldsymbol{T_\delta(A,B)}, \boldsymbol{I_\delta(A,B)}, \boldsymbol{F_\delta(A,B)}>$

Numerical Example 4.2.

Let X={a,b,c}. Suppose a binary relation δ on NSs is defined by

δ (A,B)= $<\boldsymbol{T_\delta(A,B)}, \boldsymbol{I_\delta(A,B)}, \boldsymbol{F_\delta(A,B)}>$ where

$$\boldsymbol{T_\delta(A,B)} = \frac{\sum_{x\in X} (\boldsymbol{1} - |\boldsymbol{T_A(x)} - \boldsymbol{T_B(x)}|)}{\boldsymbol{n(X)}}$$

$$\boldsymbol{I_\delta(A,B)} = \frac{\sum_{x\in X} |\boldsymbol{I_A(x)} - \boldsymbol{I_B(x)}|}{\boldsymbol{n(X)}}$$

$$\boldsymbol{F_\delta(A,B)} = \frac{\sum_{x\in X} |\boldsymbol{F_A(x)} - \boldsymbol{F_B(x)}|}{\boldsymbol{n(X)}}$$

$$\boldsymbol{A} = \{<\boldsymbol{a}, \boldsymbol{0.2}, \boldsymbol{0.2}, \boldsymbol{0.8}>, <\boldsymbol{b}, \boldsymbol{0.1}, \boldsymbol{0.1}, \boldsymbol{0.7}>, <\boldsymbol{c}, \boldsymbol{0.2}, \boldsymbol{0.5}, \boldsymbol{0.7}>\}$$

$$\boldsymbol{B} = \{<\boldsymbol{a}, \boldsymbol{0.2}, \boldsymbol{0.4}, \boldsymbol{0.5}>, <\boldsymbol{b}, \boldsymbol{0.1}, \boldsymbol{0.1}, \boldsymbol{0.9}>, <\boldsymbol{c}, \boldsymbol{0.1}, \boldsymbol{0.4}, \boldsymbol{0.3}>\}$$

$$\boldsymbol{T_\delta(A,B)} = \frac{\boldsymbol{1+1+0.9}}{\boldsymbol{3}} = \boldsymbol{0.967}$$

$$\boldsymbol{I_\delta(A,B)} = \frac{\boldsymbol{0.2+0+0.1}}{\boldsymbol{3}} = \boldsymbol{0.1}$$

$$\boldsymbol{F(A,B)} = \frac{\boldsymbol{0.3+0.2+0.4}}{\boldsymbol{3}} = \boldsymbol{0.3}$$

δ (A,B)= $< T_\delta(A,B), I_\delta(A,B), F_\delta(A,B) > = < 0.967, 0.1, 0.3 >$

Definition 4.3. Inverse Binary relation δ^{-1} on Neutrosophic sets

An inverse binary relation δ^{-1} on two NSs is defined to be $\delta = \delta^{-1}$ if and only if δ (A,B) = δ (B,A) for all A,B ∈ Ň(X)

Numerical Example 4.4.

δ (B,A)= $< T_\delta(\mathrm{B},\mathrm{A}), I_\delta(\mathrm{B},\mathrm{A}), F_\delta(\mathrm{B},\mathrm{A}) >$ where

$$T_\delta(\mathrm{B},\mathrm{A}) = \frac{\sum_{x\in X}(1 - |T_B(x) - T_A(x)|)}{n(X)}$$

$$I_\delta(\mathrm{B},\mathrm{A}) = \frac{\sum_{x\in X}|I_B(x) - I_A(x)|}{n(X)}$$

$$F(\mathrm{B},\mathrm{A}) = \frac{\sum_{x\in X}|F_B(x) - F_A(x)|}{n(X)}$$

From Example 4.3,

$$T_\delta(\mathrm{B},\mathrm{A}) = \frac{1 + 1 + 0.9}{3} = 0.967$$

$$I_\delta(\mathrm{B},\mathrm{A}) = \frac{0.2 + 0 + 0.1}{3} = 0.1$$

$$F(\mathrm{B},\mathrm{A}) = \frac{0.3 + 0.2 + 0.4}{3} = 0.3$$

δ (B,A) = $< 0.967, 0.1, 0.3 >$

Hence δ (A,B) = δ (B,A).

Definition 4.5. Distribution of δ over union

A binary relation δ is said to be distributive over union of two NSs if

AUB ∈ δ (C) if and only if A ∈ δ (C) or B ∈ δ (C)

That is, δ (AUB, C) = δ(A, C) or δ (AUB, C) = δ(B, C)

Definition 4.6. Basic pre-proximity of NSs

A binary relation δ of NSs (Ň(**X**)) is said to be a basic pre-proximity of NSs on X, if it satisfies the following conditions:

Axiom 1: $0_Ň$ ∉ δ (A), for all A∈ Ň(**X**)

Axiom 2: δ= δ^{-1}

Axiom 3: δ (AUB, C) = δ(A, C) or δ (AUB, C) = δ(B, C)

Definition 4.7. Basic proximity of NSs

A binary relation Ω of NSs ($\check{N}(\mathbf{X})$) is said to be a basic proximity of NSs on X, if it is pre-proximity of NSs and it also satisfies the condition A⋂B≠$0_{\check{N}}$ $\Rightarrow (\boldsymbol{A},\boldsymbol{B}) \in \boldsymbol{\Omega}$

Remark 4.8.

(1) If δ is a pre-proximity of NSs on X, then X is said to be a reference set of δ and is denoted by X(δ).

(2) If Ω is a pre-proximity of NSs on X, then X is said to be a reference set of Ω and is denoted by X(Ω).

(3) We denote the set of all basic pre-proximities of NSs on X by $\mathfrak{M}(\mathbf{X})$ and the set of all basic proximities of NSs on X by $\mathfrak{N}$(X).

Definition 4.9.

Let δ∈$\mathfrak{M}$(X) and A∈$\check{N}(\mathbf{X})$, then B∈$\check{N}(\mathbf{X})$ is called a neighbourhood of A with respect to δ, if $\boldsymbol{B^c} \notin \boldsymbol{\delta}(\boldsymbol{A})$.

The collection of all neighbourhoods of A with respect to δ is denoted by $nbhd(\boldsymbol{\delta},\boldsymbol{A})$.

Theorem 4.10.

Let A,B,C ∈ $\check{N}(\mathbf{X})$ and δ∈ $\mathfrak{M}$(X), then the following holds:

(1) $nbhd(\boldsymbol{\delta},\mathbf{0}_{\check{N}})= \check{N}(\mathbf{X})$

(2) If B ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A})$ and C ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{D})$, then B∪C ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}\cup\boldsymbol{D})$

(3) $nbhd(\boldsymbol{\delta},\boldsymbol{A}\cup\boldsymbol{B}) = nbhd(\boldsymbol{\delta},\boldsymbol{A}) \cap nbhd(\boldsymbol{\delta},\boldsymbol{B})$

(4) If B⊂A, then $nbhd(\boldsymbol{\delta},\boldsymbol{A}) \subset nbhd(\boldsymbol{\delta},\boldsymbol{B})$

Proof.

(1) Since ∀A∈$\check{N}(\mathbf{X})$, $\boldsymbol{A} \notin \boldsymbol{\delta}(\mathbf{0}_{\check{N}})$, we have $nbhd(\boldsymbol{\delta},\mathbf{0}_{\check{N}})= \check{N}(\mathbf{X})$.

(2) Let B ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}) \Rightarrow \boldsymbol{B^c} \notin \boldsymbol{\delta}(\boldsymbol{A})$ and C ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{D}) \Rightarrow \boldsymbol{C^c} \notin \boldsymbol{\delta}(\boldsymbol{D})$. It follows that $(\boldsymbol{B}\cup\boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{A})$ and $(\boldsymbol{B}\cup\boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{D})$ which implies $(\boldsymbol{B}\cup\boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{A}) \cup \boldsymbol{\delta}(\boldsymbol{D})$, that is $(\boldsymbol{B}\cup\boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{A}\cup\boldsymbol{D})$ and hence B∪C ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}\cup\boldsymbol{D})$.

(3) For every D∈$\check{N}(\mathbf{X})$,

D ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}\cup\boldsymbol{B}) \leftrightarrow (\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{A}\cup\boldsymbol{B})$

$\leftrightarrow (\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{A}) \cup \boldsymbol{\delta}(\boldsymbol{B})$

$\leftrightarrow (\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{A})$ and $(\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{B})$

$\leftrightarrow$ D ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A})$ and D ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{B})$

$\leftrightarrow$ D ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}) \cap nbhd(\boldsymbol{\delta},\boldsymbol{B})$

(4) Let B⊂A, To prove for every E ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}) \Rightarrow$ E ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{B})$

E ∈ $nbhd(\boldsymbol{\delta},\boldsymbol{A}) \Rightarrow (\boldsymbol{E})^c \notin \boldsymbol{\delta}(\boldsymbol{A})$

$\Rightarrow (\boldsymbol{E})^c \notin \boldsymbol{\delta}(\boldsymbol{B})$ (since A⊃B)

$\Rightarrow$ E $\in$ $nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

Thus $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \subset nbhd(\boldsymbol{\delta}, \boldsymbol{B})$.

5. Clan of Proximities of Neutrosophic Sets

Definition 5.1. Let δ∈ 𝔐(X). A subfamily £ of Ň(**X**) is said to be δ-compatible if A,B ∈ £ ⇒ A∈ δ(B). Also a δ-compatible grill is called a δ-clan.

Theorem 5.2. For δ∈ 𝔐(X) and Ğ ∈ψ(X), the following are equivalent:

(1) Ğ is a δ-clan.

(2) If K∈ξ(X) such that K⊂Ğ, then Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$.

(3) Ğ ⊂ $\bigcap\{\boldsymbol{\delta}(\mathbf{K}) | \mathbf{K} \in \boldsymbol{\xi}(\mathbf{X}), \mathbf{K} \subset$ Ğ}.

(4) If $\boldsymbol{K_1}, \boldsymbol{K_2}$ ∈ξ(X), such that $\boldsymbol{K_1} \subset$ Ğ **and** $\boldsymbol{K_2} \subset$ Ğ, then $\boldsymbol{K_1} \subset \boldsymbol{\delta}(\boldsymbol{K_2})$

Proof.

(1)⇒(2)

Assume that Ğ is a δ-clan. To prove: Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$. Let K∈ξ(X) such that K⊂Ğ and A ∈Ğ, it follows that A∈ δ(B) for all $\boldsymbol{B} \in \mathbf{K}$, that is A∈ $\boldsymbol{\delta}(\mathbf{K})$. Thus Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$.

(2)⇒(3)

If K∈ξ(X) such that K⊂Ğ, then Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$, then obviously Ğ ⊂∩ {$\boldsymbol{\delta}(\mathbf{K}) | \mathbf{K} \in \boldsymbol{\xi}(\mathbf{X}), \mathbf{K} \subset$ Ğ}.

(3)⇒(4)

Suppose Ğ ⊂ $\bigcap\{\boldsymbol{\delta}(\mathbf{K}) | \mathbf{K} \in \boldsymbol{\xi}(\mathbf{X}), \mathbf{K} \subset$ Ğ}. Let $\boldsymbol{K_1}, \boldsymbol{K_2}$ ∈ξ(X), such that $\boldsymbol{K_1} \subset$ Ğ **and** $\boldsymbol{K_2} \subset$ Ğ, then by (3) Ğ ⊂ $\boldsymbol{\delta}(\boldsymbol{K_1})$ and Ğ ⊂ $\boldsymbol{\delta}(\boldsymbol{K_2})$. Now $\boldsymbol{K_1} \subset$ Ğ and Ğ ⊂ $\boldsymbol{\delta}(\boldsymbol{K_2})$ implies $\boldsymbol{K_1} \subset \boldsymbol{\delta}(\boldsymbol{K_2})$.

(4)⇒(1)

Suppose (4) holds. To prove: Ğ is a δ-clan. Let A,B ∈Ğ and by (4), $\boldsymbol{K_1}, \boldsymbol{K_2}$ ∈ξ(X), such that $\mathbf{A} \in \boldsymbol{K_1} \subset$ Ğ **and** $\mathbf{B} \in \boldsymbol{K_2} \subset$ Ğ, then $\mathbf{A} \in \boldsymbol{K_1} \subset \boldsymbol{\delta}(\boldsymbol{K_2}) \subset \boldsymbol{\delta}(\mathbf{B})$. Thus Ğ is a δ-clan.

Theorem 5.3. For δ∈ 𝔐(X), then every δ-clan is contained in a maximal δ-clan.

Proof.

Since by theorem 3.13, union of grills of NSs is again a grill of NSs. Further for a family of δ-clans {Ğ$_{\boldsymbol{i},}$ $\boldsymbol{i} \in \boldsymbol{I}$} with Ğ$_{\boldsymbol{i}}$ ⊂ Ğ$_{\boldsymbol{j}}$, $\boldsymbol{i} \leq \boldsymbol{j}$, ∪{Ğ$_{\boldsymbol{i},}$ $\boldsymbol{i} \in \boldsymbol{I}$} is a δ-clan. Hence by applying Zorn's lemma, there must be a maximal δ-clan on the collections of δ-clan Ğ.

Lemma 5.4. For Ω ∈ 𝔑 (X), if A∈ Ω (B), then there exists $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$, $\mathbf{B} \in \boldsymbol{H_2}$ and $\boldsymbol{H_1} \subset \boldsymbol{\Omega}(\mathbf{H_2})$.

Proof.

Since Ω (B) is a grill of NSs on X, by theorem 3.15, there exists a prime filter $\boldsymbol{H_1}$ of NSs such that A∈$\boldsymbol{H_1}$⊂ Ω (B). By symmetry of Ω, B∈$\boldsymbol{\Omega(H_1)}$. Now $\boldsymbol{\Omega(H_1)}$ ∈ ψ(X). Again by theorem 3.15, there exists $\boldsymbol{H_2}$∈ξ(X) such that B∈$\boldsymbol{H_2}$⊂ Ω ($\boldsymbol{H_1}$). Hence $\mathbf{A} \in \boldsymbol{H_1}$, $\mathbf{B} \in \boldsymbol{H_2}$ and $\boldsymbol{H_1} \subset \boldsymbol{\Omega}(\mathbf{H_2})$.

Theorem 5.5. For Ω ∈ 𝔑 (X), if A∈ Ω (B), then there is a Ω–clan of the form $\boldsymbol{H_1} \cup \boldsymbol{H_2}$ where $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$ and $\mathbf{B} \in \boldsymbol{H_2}$.

Proof.

Let A∈ Ω (B), by lemma 5.4, there exists $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$, $\mathbf{B} \in \boldsymbol{H_2}$ and $\boldsymbol{H_1} \subset \boldsymbol{\Omega}(\mathbf{H_2})$. Since Ω ∈ 𝔑 (X) and $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X), then for any P, Q ∈ $\boldsymbol{H_1}$ **or** $\boldsymbol{H_2}$, we have (P,Q) ∈ Ω and hence $\boldsymbol{H_1} \cup \boldsymbol{H_2}$ is a Ω–clan such that $\mathbf{A} \in \boldsymbol{H_1}$ and $\mathbf{B} \in \boldsymbol{H_2}$.

Theorem 5.6. For Ω ∈ 𝔑 (X), if A∈ Ω (B), then there a maximal Ω–clan containing {A,B}.

Proof.

By theorem 5.5, there exists a Ω–clan $\boldsymbol{H_1} \cup \boldsymbol{H_2}$, where $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$ and $\mathbf{B} \in \boldsymbol{H_2}$ and {A,B}⊂$\boldsymbol{H_1} \cup \boldsymbol{H_2}$. Also by theorem 5.3, every Ω -clan is contained in a maximal Ω –clan. Hence the proof.

6 Conclusion

In this article, we introduced and studied the concept of proximities of neutrosophic sets and its characterstics. Futher, its gradation of openness can be studied.

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