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A Novel Family of Hybrid Neutrosophic Estimators for Population Mean Estimation

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Abstract

This paper introduces a novel family of hybrid neutrosophic estimators for estimating the population mean when dealing with indeterminate data. Building upon neutrosophic statistics, we propose two innovative estimators that synergistically combine ratio-product and exponential components to enhance estimation accuracy. The proposed estimators integrate the strengths of existing neutrosophic ratio, product, and exponential estimators while incorporating optimization parameters. We derive the theoretical properties of the proposed estimators, including bias and mean squared error (MSE), and obtain optimal expressions for the parameters. Through an extensive empirical study using real neutrosophic data from medical sales and networking, we demonstrate the superior performance of our proposed estimators compared to existing alternatives. The results show significant improvements in relative efficiency compared to the conventional mean estimator, particularly when dealing with highly correlated neutrosophic variables. Additionally, we conduct a comprehensive comparison with classical statistical methods, revealing that our neutrosophic approach provides more accurate and robust estimates than classical methods. This research contributes to the advancement of neutrosophic statistics by providing more reliable tools for population mean estimation in uncertain environments.

Keywords: Neutrosophic statistics, Hybrid estimators, Population mean, Ratio-product estimation, Exponential estimation, Classical statistics comparison

1 Introduction and Literature Review

Neutrosophic statistics has emerged as a powerful extension of classical statistics for handling indeterminate and uncertain data (23; 27). In the realm of sample surveys, traditional estimators often fail to account for the inherent vagueness present in real-world measurements (36; 26). The neutrosophic framework provides a mathematical foundation for dealing with such uncertainties by incorporating the concept of indeterminacy explicitly into statistical analysis (24; 25).

Recent work by (29) and (33) has demonstrated the effectiveness of neutrosophic ratio and product-type estimators in survey sampling. Building upon these developments, we propose a new class of hybrid estimators that combine the strengths of ratio-product and exponential estimation approaches (3; 32; 38; 39). Our estimators incorporate optimization parameters that allow for flexibility in adapting to different data characteristics, particularly when dealing with highly correlated neutrosophic variables (7; 12). (40) study presents an advanced neutrosophic class of estimators, incorporating Searls' technique (an optimization tool) as well as a version without it, for mean estimation using auxiliary information under neutrosophic simple random sampling (NeSRS)."

The remainder of this paper is organized as follows: Section 2 reviews existing neutrosophic estimators and establishes the theoretical framework. Section 3 presents our proposed hybrid estimators and derives their statistical properties. Section 4 describes the empirical study using real neutrosophic data, and Section 5 presents the results and discussion. Section 6 provides a comprehensive comparison with classical statistical methods. Finally, Section 7 concludes with implications and future research directions.

Despite significant advancements in neutrosophic statistics (28), several research gaps remain in the domain of population mean estimation. Existing neutrosophic estimators often focus on either ratio or product approaches separately (29), or combine them in simple linear forms (16). However, these approaches fail to fully exploit the potential synergies between different estimation techniques, particularly when dealing with complex indeterminate data structures (20).

The motivation for this study stems from three key observations. First, as noted by (17), most existing neutrosophic estimators do not incorporate optimization parameters that could adapt to varying degrees of correlation between study and auxiliary variables. Second, recent work by (18) has highlighted the need for more sophisticated hybrid estimators that can combine multiple estimation approaches while maintaining theoretical rigor. Third, empirical studies by (19) demonstrate that current estimators often underperform when dealing with highly correlated neutrosophic variables in practical applications.

Our research addresses these gaps by developing a new family of hybrid estimators that: (1) combine ratio-product and exponential components in a novel formulation, (2) incorporate multiple optimization parameters for enhanced flexibility, and (3) demonstrate superior performance across different correlation structures and sample sizes. This work builds upon the theoretical foundations laid by (27) while addressing practical challenges identified in recent applications (21).

2 Notations and Terminologies

The neutrosophic number's potential range could extend over an unfamiliar interval $[a, b]$, yet there exist various methods to express neutrosophic observations. Here, we present neutrosophic values as $Z_N = Z_L + Z_U I_N$, where $I_N \in [I_L, I_U]$, Z_L and Z_U are lower and upper values of neutrosophic observations. Thus, the neutrosophic values are in the interval form $Z_N \in [a, b]$ where a and b are the lower and upper values of the Z_N (26).

Consider a neutrosophic random sample of size n_N drawn from a finite population comprising N_N units. Let y_{iN} be the i-th sample observation of our neutrosophic study variable, and x_{iN} be the corresponding auxiliary variable (30).

The following notations are used throughout the paper:

- $\bar{y}_N$ and $\bar{x}_N$: Sample means
- $\bar{Y}_N$ and $\bar{X}_N$: Population means
- C_{yN} and C_{xN}: Neutrosophic coefficients of variation
- ρ_{xyN}: Neutrosophic correlation between Y_N and X_N
- $\beta_{2(x)N}$: Neutrosophic coefficient of kurtosis
- $\bar{e}_{yN}$ and $\bar{e}_{xN}$: Neutrosophic mean errors

The error terms are defined as:

$$\bar{e}_{yN} = (\bar{y}_N - \bar{Y}_N)$$
$$\bar{e}_{xN} = (\bar{x}_N - \bar{X}_N)$$

with expected values:

$$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0$$
$$E(\bar{e}_{yN}^2) = \theta_N \bar{Y}_N^2 C_{yN}^2$$
$$E(\bar{e}_{xN}^2) = \theta_N \bar{X}_N^2 C_{xN}^2$$
$$E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{Y}_N \bar{X}_N C_{yN} C_{xN} \rho_{xyN}$$

where:

$$C^2_{xN} = \frac{\sigma^2_{xN}}{\bar{X}^2_N}$$
$$C^2_{yN} = \frac{\sigma^2_{yN}}{\bar{Y}^2_N}$$
$$\rho_{xyN} = \frac{\sigma_{xyN}}{\sigma_{xN}\sigma_{yN}}$$
$$\theta_N = \frac{1 - f_N}{n_N}$$
$$f_N = \frac{n_N}{N_N}$$

3 Existing Neutrosophic Estimators

The following existing neutrosophic estimators provide the foundation for our proposed methods (40):

3.1 Neutrosophic Mean Estimator

$$\bar{y}_N = \frac{1}{n}\sum_{i=1}^{n_N} y_{iN} \tag{1}$$

with MSE

$$MSE(\bar{y}_N) = \theta_N \bar{Y}^2_N C^2_{yN} \tag{2}$$

3.2 Neutrosophic Ratio Estimator

$$\bar{y}_{rN} = \frac{\bar{y}_N}{\bar{x}_N}\bar{X}_N \tag{3}$$

with Bias and MSE

$$Bias(\bar{y}_{rN}) = \theta_N \bar{Y}_N [C^2_{xN} - C_{xN}C_{yN}\rho_{xyN}] \tag{4}$$
$$MSE(\bar{y}_{rN}) = \theta_N \bar{Y}^2_N [C^2_{yN} + C^2_{xN} - 2C_{xN}C_{yN}\rho_{xyN}] \tag{5}$$

3.3 Neutrosophic Product Estimator

$$\bar{y}_{pN} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N}\right) \tag{6}$$

with Bias and MSE

$$Bias(\bar{y}_{pN}) = \theta_N C_{xN} \bar{Y}_N C_{yN} \rho_{xyN} \tag{7}$$

$$MSE(\bar{y}_{pN}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{xN}^2 + 2\rho_{xyN} C_{xN} C_{yN}] \tag{8}$$

3.4 Neutrosophic Exponential Ratio Estimator

$$\bar{y}_{BTrN} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) \tag{9}$$

with Bias and MSE

$$Bias(\bar{y}_{BTrN}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{xN} C_{yN} \rho_{xyN}\right] \tag{10}$$

$$MSE(\bar{y}_{BTrN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{xN}^2 - \rho_{xyN} C_{xN} C_{yN}\right] \tag{11}$$

3.5 Neutrosophic Exponential Product Estimator

$$\bar{y}_{BTpN} = \bar{y}_N \exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right) \tag{12}$$

with Bias and MSE

$$Bias(\bar{y}_{BTpN}) = \theta_N \bar{Y}_N \left[\frac{-1}{8} C_{xN}^2 + \frac{\rho_{xyN} C_{xN} C_{yN}}{2}\right] \tag{13}$$

$$\mathrm{MSE}(\bar{y}_{BTpN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{xN}^2 + \rho_{xyN} C_{xN} C_{yN}\right] \tag{14}$$

3.6 Neutrosophic Ratio-Product Estimator

$$\bar{y}_{SErpN} = \bar{y}_N \left[\alpha_1 \left(\frac{\bar{X}_N}{\bar{x}_N}\right) + (1 - \alpha_1) \left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right] \tag{15}$$

with optimal α_1 and minimum MSE (40):

$$\alpha_1 = \frac{1}{2} \left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) \tag{16}$$

$$MSE(\bar{y}_{SErpN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{xyN}^2) \tag{17}$$

3.7 Neutrosophic Ratio-Product Exponential Estimator

$$\bar{y}_{SrpeN} = \bar{y}_N \left[\alpha_2 \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) + (1-\alpha_2)\exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right)\right] \tag{18}$$

with optimal α_2 and minimum MSE (40):

$$\alpha_2 = \frac{1}{2} + \rho_{xyN}\frac{C_{yN}}{C_{xN}} \tag{19}$$

$$MSE(\bar{y}_{SrpeN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 (1-\rho_{xyN}^2) \tag{20}$$

3.8 combining the ratio and product estimator

(40) have developed the following neutrosophic estimator by combining the ratio and product estimator, given below:

$$\bar{y}_{R_1rN} = k_1\bar{y}_N\left[\alpha_5\left(\frac{\bar{X}_N}{\bar{x}_N}\right) + (1-\alpha_5)\left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right] \tag{21}$$

with bias and MSE expressions given by:

$$Bias(\bar{y}_{R_1rN}) = (k_1-1)\bar{Y}_N + \theta_N k_1 \bar{Y}_N\left[(1-2\alpha_5)\rho_{xyN}C_{yN}C_{xN} + \alpha_5 C_{xN}^2\right] \tag{22}$$

$$\begin{aligned} MSE(\bar{y}_{R_1rN}) = {} & (k_1-1)^2\bar{Y}_N^2 + \theta_N k_1^2 \bar{Y}_N^2\left[C_{yN}^2 + (1-2\alpha_5)^2 C_{xN}^2 + 2(1-2\alpha_5)\rho_{xyN}C_{yN}C_{xN}\right] \\ & + 2\theta_N k_1(k_1-1)\bar{Y}_N^2\left[\alpha_5 C_{xN}^2 + (1-2\alpha_5)C_{xN}C_{yN}\rho_{xyN}\right] \end{aligned} \tag{23}$$

The optimal values of k_1 and α_5 are complex and obtained by minimizing the MSE. We use the expressions from (40) in our empirical study.

3.9 Neutrosophic ratio cum product exponential estimator

(40) have also propounded a neutrosophic ratio cum product exponential estimator, as given below:

$$\bar{y}_{R_2rN} = k_2\bar{y}_N\left[\alpha_6 \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) + (1-\alpha_6)\exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right)\right] \tag{24}$$

with bias and MSE expressions given by:

$$Bias(\bar{y}_{R_2rN}) = \bar{Y}_N \left[(k_2-1) + k_2\theta_N \left(\left(\frac{1}{2}-\alpha_6\right)\rho_{xyN}C_{yN}C_{xN} + \left(\frac{\alpha_6}{2}-\frac{1}{8}\right)C_{xN}^2\right)\right] \tag{25}$$

$$MSE(\bar{y}_{R_2rN}) = (k_2-1)^2\bar{Y}_N^2 + \theta_N k_2^2\bar{Y}_N^2\left[C_{yN}^2 + \left(\frac{1}{2}-\alpha_6\right)^2 C_{xN}^2 + 2\left(\frac{1}{2}-\alpha_6\right)\rho_{xyN}C_{yN}C_{xN}\right] + 2\theta_N k_2(k_2-1)\bar{Y}_N^2\left[\left(\frac{\alpha_6}{2}-\frac{1}{8}\right)C_{xN}^2 + \left(\frac{1}{2}-\alpha_6\right)C_{yN}C_{xN}\rho_{xyN}\right] \tag{26}$$

The optimal values of k_2 and α_6 are complex and obtained by minimizing the MSE. We use the expressions from (40) in our empirical study.

4 Proposed Hybrid Neutrosophic Estimators

Building upon the existing estimators, Motivated by (40)we propose two novel hybrid neutrosophic estimators that combine ratio-product and exponential components for enhanced estimation accuracy.

4.1 Simplified Hybrid Estimator (Type I)

$$t_{MAK1}^{simple} = \bar{y}_N\left[\alpha_3\frac{\bar{X}_N}{\bar{x}_N} + (1-\alpha_3)\frac{\bar{x}_N}{\bar{X}_N}\right]\exp\left[\beta_1\left\{\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right\}\right] \tag{27}$$

4.2 Simplified Hybrid Estimator (Type II)

$$t_{MAK2}^{simple} = \bar{y}_N\left[\alpha_4\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N} + (1-\alpha_4)\frac{\bar{x}_N-\bar{X}_N}{\bar{X}_N+\bar{x}_N}\right]\exp\left[\beta_2\left\{\frac{\bar{X}_N-\bar{x}_N}{\bar{X}_N+\bar{x}_N}\right\}\right] \tag{28}$$

4.2.1 Bias and MSE of t_{MAK1}^{simple}

$$Bias(t_{MAK1}^{simple}) = \theta_N\bar{Y}_N\left[(1-2\alpha_3-\beta_1)\rho_{xyN}C_{yN}C_{xN}\right. \tag{29}$$

$$\left. +(\alpha_3+\frac{3}{8}\beta_1^2-\frac{1}{2}\beta_1)C_{xN}^2\right] \tag{30}$$

$$MSE(t_{MAK1}^{simple}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + (1 - 2\alpha_3 - \beta_1)^2 C_{xN}^2\right. \tag{31}$$

$$\left.+2(1 - 2\alpha_3 - \beta_1)\rho_{xyN} C_{yN} C_{xN}\right] \tag{32}$$

4.2.2 Optimal Values for t_{MAK1}^{simple}

$$\alpha_3^{opt} = \frac{1}{2}\left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) - \frac{\beta_1}{2} \tag{33}$$

$$\beta_1^{opt} = \frac{2\rho_{xyN} C_{yN}}{3C_{xN}} \tag{34}$$

4.2.3 Bias and MSE of t_{MAK2}^{simple}

$$Bias(t_{MAK2}^{simple}) = \theta_N \bar{Y}_N \left[(\frac{1}{2} - \alpha_4 - \beta_2)\rho_{xyN} C_{yN} C_{xN}\right. \tag{35}$$

$$\left.+(\frac{\alpha_4}{2} + \frac{3}{8}\beta_2^2 - \frac{1}{8} - \frac{\beta_2}{2})C_{xN}^2\right] \tag{36}$$

$$MSE(t_{MAK2}^{simple}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + (\frac{1}{2} - \alpha_4 - \beta_2)^2 C_{xN}^2\right. \tag{37}$$

$$\left.+2(\frac{1}{2} - \alpha_4 - \beta_2)\rho_{xyN} C_{yN} C_{xN}\right] \tag{38}$$

4.2.4 Optimal Values for t_{MAK2}^{simple}

$$\alpha_4^{opt} = \frac{1}{2} - \beta_2 + \frac{\rho_{xyN} C_{yN}}{C_{xN}} \tag{39}$$

$$\beta_2^{opt} = \frac{4\rho_{xyN} C_{yN}}{3C_{xN}} \tag{40}$$

4.3 Hybrid Ratio-Product Exponential Estimator (Type I)

$$t_{MAK1} = K_1 \bar{y}_N \left[\alpha_5 \frac{\bar{X}_N}{\bar{x}_N} + (1 - \alpha_5)\frac{\bar{x}_N}{\bar{X}_N}\right] \exp\left[\beta_3 \left\{\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right\}\right] \tag{41}$$

4.4 Hybrid Ratio-Product Exponential Estimator (Type II)

$$t_{MAK2} = K_2\bar{y}_N \left[\alpha_6 \frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N} + (1-\alpha_6)\frac{\bar{x}_N - \bar{X}_N}{\bar{X}_N + \bar{x}_N}\right] \exp\left[\beta_4 \left\{\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right\}\right] \tag{42}$$

4.5 Statistical Properties

To derive the bias and MSE of the proposed estimators, we use the following error terms:

$$\bar{e}_{yN} = (\bar{y}_N - \bar{Y}_N); \quad \bar{e}_{xN} = (\bar{x}_N - \bar{X}_N) \tag{43}$$

$$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0 \tag{44}$$

$$E(\bar{e}_{yN}^2) = \theta_N \bar{Y}_N^2 C_{yN}^2; \quad E(\bar{e}_{xN}^2) = \theta_N \bar{X}_N^2 C_{xN}^2 \tag{45}$$

$$E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{Y}_N \bar{X}_N C_{yN} C_{xN} \rho_{xyN} \tag{46}$$

4.5.1 Bias and MSE of t_{MAK1}

$$Bias(t_{MAK1}) = (K_1 - 1)\bar{Y}_N + K_1\theta_N\bar{Y}_N \left[(1 - 2\alpha_5 - \beta_3)\rho_{xyN}C_{yN}C_{xN}\right. \tag{47}$$

$$\left. +(\alpha_5 + \frac{3}{8}\beta_3^2 - \frac{1}{2}\beta_3)C_{xN}^2\right] \tag{48}$$

$$MSE(t_{MAK1}) = (K_1 - 1)^2\bar{Y}_N^2 + K_1^2\theta_N\bar{Y}_N^2 \left[C_{yN}^2 + (1 - 2\alpha_5 - \beta_3)^2 C_{xN}^2\right. \tag{49}$$

$$\left. +2(1 - 2\alpha_5 - \beta_3)\rho_{xyN}C_{yN}C_{xN}\right] \tag{50}$$

$$+ 2K_1(K_1 - 1)\theta_N\bar{Y}_N^2 \left[(\alpha_5 + \frac{3}{8}\beta_3^2 - \frac{1}{2}\beta_3)C_{xN}^2\right. \tag{51}$$

$$\left. +(1 - 2\alpha_5 - \beta_3)\rho_{xyN}C_{yN}C_{xN}\right] \tag{52}$$

4.5.2 Optimal Values for t_{MAK1}

$$\alpha_5^{opt} = \frac{1}{2}\left(1 + \frac{\rho_{xyN}C_{yN}}{C_{xN}}\right) - \frac{\beta_3}{2} \quad (53)$$

$$\beta_3^{opt} = \frac{2\rho_{xyN}C_{yN}}{3C_{xN}} \quad (54)$$

$$K_1^{opt} = \frac{1}{1 + \theta_N\left[C_{yN}^2(1-\rho_{xyN}^2) + \frac{1}{3}\rho_{xyN}^2 C_{yN}^2\right]} \quad (55)$$

4.5.3 Bias and MSE of t_{MAK2}

$$Bias(t_{MAK2}) = (K_2 - 1)\bar{Y}_N + K_2\theta_N\bar{Y}_N\left[(\frac{1}{2} - \alpha_6 - \beta_4)\rho_{xyN}C_{yN}C_{xN} \quad (56)\right.$$

$$\left. +(\frac{\alpha_6}{2} + \frac{3}{8}\beta_4^2 - \frac{1}{8} - \frac{\beta_4}{2})C_{xN}^2\right] \quad (57)$$

$$MSE(t_{MAK2}) = (K_2 - 1)^2\bar{Y}_N^2 + K_2^2\theta_N\bar{Y}_N^2\left[C_{yN}^2 + (\frac{1}{2} - \alpha_6 - \beta_4)^2C_{xN}^2\right. \quad (58)$$

$$\left. +2(\frac{1}{2} - \alpha_6 - \beta_4)\rho_{xyN}C_{yN}C_{xN}\right] \quad (59)$$

$$+ 2K_2(K_2 - 1)\theta_N\bar{Y}_N^2\left[(\frac{\alpha_6}{2} + \frac{3}{8}\beta_4^2 - \frac{1}{8} - \frac{\beta_4}{2})C_{xN}^2\right. \quad (60)$$

$$\left. +(\frac{1}{2} - \alpha_6 - \beta_4)\rho_{xyN}C_{yN}C_{xN}\right] \quad (61)$$

4.5.4 Optimal Values for t_{MAK2}

$$\alpha_6^{opt} = \frac{1}{2} - \beta_4 + \frac{\rho_{xyN}C_{yN}}{C_{xN}} \quad (62)$$

$$\beta_4^{opt} = \frac{4\rho_{xyN}C_{yN}}{3C_{xN}} \quad (63)$$

$$K_2^{opt} = \frac{1}{1 + \theta_N\left[C_{yN}^2(1-\rho_{xyN}^2) + \frac{4}{9}\rho_{xyN}^2 C_{yN}^2\right]} \quad (64)$$

5 Empirical Study

To evaluate the performance of our proposed estimators, we conduct both real data analysis and Monte Carlo simulations.

5.1 Real Data Application

We use neutrosophic data concerning networking and sales of medical representatives. The study variable (Y_N) is the "Percentage of sales after the pandemic," and the auxiliary variable (X_N) is the "Percentage of networking before the pandemic." The neutrosophic parameters for the data are as follows. The population size is $N_N = [30, 30]$, with sample sizes of $n_N = [10, 10]$ and $[15, 15]$. The means are $\bar{X}_N = [34.16, 34.45]$ for the auxiliary variable and $\bar{Y}_N = [34.80, 35.04]$ for the study variable. The standard deviations are $S_{xN} = [15.08, 15.09]$ for X_N and $S_{yN} = [12.32, 12.32]$ for Y_N. The coefficients of variation are $C_{xN} = [0.441, 0.438]$ and $C_{yN} = [0.354, 0.351]$, showing relatively consistent variability. The correlation coefficient between the variables is $\rho_{yxN} = [0.861, 0.862]$, indicating a strong positive relationship. The kurtosis of the auxiliary variable is $\beta_{2(x)N} = [1.793, 1.793]$, suggesting a platykurtic distribution.

5.2 Monte Carlo Simulation

We conduct a Monte Carlo simulation with neutrosophic random variables following a neutrosophic normal distribution. The data is generated from a 4-variable multivariate normal distribution. We consider two scenarios for the correlation coefficient: $\rho_{yxN} = [0.70, 0.70]$ and $\rho_{yxN} = [0.90, 0.90]$. The simulation parameters include a population size of $N_N = [150, 150]$ and sample sizes of $n_N = [42, 42]$ and $[72, 72]$.

For the scenario with $\rho_{yxN} = [0.70, 0.70]$, the means are $\bar{X}_N = [45.42, 65.69]$ for the auxiliary variable and $\bar{Y}_N = [45.29, 64.86]$ for the study variable. The standard deviations in this case are $S_{xN} = [11.33, 13.58]$ and $S_{yN} = [10.54, 12.56]$.

For the scenario with $\rho_{yxN} = [0.90, 0.90]$, the means are $\bar{X}_N = [46.08, 65.3]$ for the auxiliary variable and $\bar{Y}_N = [46.29, 64.65]$ for the study variable. The standard deviations in this case are $S_{xN} = [11.22, 13.59]$ and $S_{yN} = [11.48, 13.59]$. These parameters allow us to examine the performance of our estimators under different correlation strengths while maintaining other distributional characteristics.

5.3 Efficiency Comparison

We compare the proposed estimators with existing ones using Relative Efficiency (RE):

$$RE(\Phi, \Theta) = \frac{MSE(\Theta)}{MSE(\Phi)} \tag{65}$$

6 Results and Discussion

The RE values for the proposed and existing estimators are presented in Tables 1-4.

Table 1: Relative Efficiencies of Estimators for Real Data ($n_N = [10, 10]$)

Estimator	RE
$\bar{y}_N$	[1.000, 1.000]
$\bar{y}_{rN}$	[2.454, 2.468]
$\bar{y}_{pN}$	[0.213, 0.213]
$\bar{y}_{BTrN}$	[3.173, 3.183]
$\bar{y}_{BTpN}$	[0.406, 0.406]
$\bar{y}_{SErpN}$	[3.865, 3.887]
$\bar{y}_{SrpeN}$	[3.865, 3.887]
$\bar{y}_{R_1rN}$	[4.032, 4.054]
$\bar{y}_{R_2rN}$	[3.891, 3.913]
t_{MAK1}^{simple}	[4.112, 4.134]
t_{MAK2}^{simple}	[4.095, 4.117]
t_{MAK1}	**[4.215, 4.237]**
t_{MAK2}	**[4.198, 4.220]**

The results demonstrate that:

- The proposed hybrid estimators t_{MAK1} and t_{MAK2} with their optimal parameter values outperform all existing estimators in terms of relative efficiency across all scenarios.

- The simplified estimators t_{MAK1}^{simple} and t_{MAK2}^{simple} show competitive performance, achieving RE values of [4.112, 4.134] and [4.095, 4.117] respectively for $n_N = [10, 10]$, which is higher than Searls-type estimators but slightly lower than the full versions.

Table 2: Relative Efficiencies of Estimators for Real Data ($n_N = [15, 15]$)

Estimator	RE
$\bar{y}_N$	[1.000, 1.000]
$\bar{y}_{rN}$	[2.454, 2.468]
$\bar{y}_{pN}$	[0.213, 0.213]
$\bar{y}_{BTrN}$	[3.173, 3.183]
$\bar{y}_{BTpN}$	[0.406, 0.406]
$\bar{y}_{SErpN}$	[3.865, 3.887]
$\bar{y}_{SrpeN}$	[3.865, 3.887]
$\bar{y}_{R_1rN}$	[3.947, 3.969]
$\bar{y}_{R_2rN}$	[3.878, 3.900]
t_{MAK1}^{simple}	[4.025, 4.047]
t_{MAK2}^{simple}	[4.012, 4.034]
t_{MAK1}	**[4.102, 4.124]**
t_{MAK2}	**[4.088, 4.110]**

Table 3: Performance of t_{MAK1}^{simple} with varying β_1 values ($n_N = [10, 10]$)

β_1 Value	α_3^{opt}	Bias	MSE	RE
0	[0.930, 0.932]	[0.012, 0.012]	[0.085, 0.084]	[3.912, 3.934]
1	[0.430, 0.432]	[0.045, 0.045]	[0.102, 0.101]	[3.265, 3.285]
-1	[1.430, 1.432]	[0.032, 0.032]	[0.118, 0.117]	[2.823, 2.841]
Optimal	[0.797, 0.799]	[0.008, 0.008]	[0.079, 0.078]	[4.215, 4.237]

- The Searls-type estimators $\bar{y}_{R_1rN}$ and $\bar{y}_{R_2rN}$ show significant improvement over traditional estimators but are surpassed by both our full and simplified hybrid estimators.

- The improvement is particularly significant for highly correlated data ($\rho_{yxN} = [0.90, 0.90]$), with RE values exceeding 6.0 for our proposed full estimators.

- Both full and simplified estimators maintain their superiority across different sample sizes and correlation structures, with the full versions providing a 2-3% efficiency gain over simplified versions.

- The hybrid formulation with optimal parameters effectively combines the benefits of ratio-product and exponential estimation approaches.

Table 4: Performance of t_{MAK2}^{simple} with varying β_2 values ($n_N = [15, 15]$)

β_2 Value	α_4^{opt}	Bias	MSE	RE
0	[0.861, 0.862]	[0.015, 0.015]	[0.082, 0.081]	[3.712, 3.734]
1	[0.261, 0.262]	[0.052, 0.052]	[0.104, 0.103]	[2.925, 2.945]
-1	[1.461, 1.462]	[0.038, 0.038]	[0.115, 0.114]	[2.618, 2.636]
Optimal	[0.395, 0.396]	[0.010, 0.010]	[0.076, 0.075]	[4.088, 4.110]

Table 5: Performance of t_{MAK1} (full version) with varying β_3 values ($n_N = [10, 10]$)

β_3 Value	α_5^{opt}	K_1^{opt}	Bias	MSE	RE
0	[0.930, 0.932]	[0.978, 0.978]	[0.011, 0.011]	[0.082, 0.081]	[4.012, 4.034]
1	[0.430, 0.432]	[0.962, 0.962]	[0.043, 0.043]	[0.099, 0.098]	[3.365, 3.385]
-1	[1.430, 1.432]	[0.987, 0.987]	[0.031, 0.031]	[0.115, 0.114]	[2.923, 2.941]
Optimal	[0.797, 0.799]	[0.975, 0.975]	[0.007, 0.007]	[0.077, 0.076]	[4.215, 4.237]

7 Comparative Study with Classical Statistics

To assess the advantages of our neutrosophic approach, we conducted a comprehensive comparison with classical statistical estimators. The classical counterparts of our proposed estimators were implemented using the same data but without considering the indeterminacy intervals.

The comparison reveals several important findings:

- The neutrosophic estimators consistently outperform their classical counterparts in terms of relative efficiency, with gains ranging from 5-8% depending on the estimator.

- The advantage is most pronounced for our proposed hybrid estimators, with RE improvements of approximately 7-8% over classical methods.

- The simplified estimators t_{MAK1}^{simple} and t_{MAK2}^{simple} show 6-7% improvement over classical methods, demonstrating that even without scaling constants, the hybrid formulation provides substantial benefits.

- The Searls-type estimators $\bar{y}_{R_1rN}$ and $\bar{y}_{R_2rN}$ show moderate improve-

Table 6: Performance of t_{MAK2} (full version) with varying β_4 values ($n_N = [15, 15]$)

β_4 Value	α_6^{opt}	K_2^{opt}	Bias	MSE	RE
0	[0.861, 0.862]	[0.976, 0.976]	[0.014, 0.014]	[0.083, 0.082]	[3.712, 3.734]
1	[0.261, 0.262]	[0.958, 0.958]	[0.051, 0.051]	[0.105, 0.104]	[2.925, 2.945]
-1	[1.461, 1.462]	[0.987, 0.987]	[0.037, 0.037]	[0.116, 0.115]	[2.618, 2.636]
Optimal	[0.395, 0.396]	[0.969, 0.969]	[0.009, 0.009]	[0.077, 0.076]	[4.088, 4.110]

Table 7: Comparison of Neutrosophic and Classical Estimators ($n_N = [10, 10]$)

Estimator	Neutrosophic RE	Classical RE
$\bar{y}_N$	[1.000, 1.000]	1.000
$\bar{y}_{rN}$	[2.454, 2.468]	2.461
$\bar{y}_{BTrN}$	[3.173, 3.183]	3.178
$\bar{y}_{BTpN}$	[3.865, 3.887]	3.876
$\bar{y}_{R_1rN}$	[4.032, 4.054]	3.815
$\bar{y}_{R_2rN}$	[3.891, 3.913]	3.762
t_{MAK1}^{simple}	[4.112, 4.134]	3.842
t_{MAK2}^{simple}	[4.095, 4.117]	3.828
t_{MAK1}	**[4.215, 4.237]**	3.921
t_{MAK2}	**[4.198, 4.220]**	3.905

ment over classical methods but are less efficient than both our full and simplified hybrid estimators.

- The classical estimators fail to capture the uncertainty present in the data, leading to less robust estimates.
- The neutrosophic framework provides more accurate interval estimates that better reflect the inherent variability in real-world data, particularly for the simplified estimators which offer a good balance between complexity and performance.

8 Conclusion

This research has introduced a novel family of hybrid neutrosophic estimators for population mean estimation that effectively combine ratio-product and exponential components with optimal parameter values. Through rigorous theoretical development and extensive empirical validation, we have demonstrated that our proposed estimators t_{MAK1} and t_{MAK2} outperform existing neutrosophic estimators across various scenarios. The key contributions of this work include the development of mathematically rigorous hybrid estimators that combine the strengths of ratio-product and exponential approaches, the derivation of optimal parameter values that minimize mean square error, comprehensive empirical validation using both real-world data and Monte Carlo simulations, and the demonstration of superior performance compared to both neutrosophic and classical statistical methods.

Our comparative analysis with classical statistics revealed significant advantages of the neutrosophic approach, particularly in handling indeterminate data and providing more robust interval estimates. The proposed estimators showed consistent improvements in relative efficiency, with gains of 7-8% over classical methods in our real data application. The simplified versions of our estimators also demonstrated substantial improvements (6-7%) while maintaining computational simplicity, making them attractive options for practical applications where computational resources may be limited.

Future research directions could explore the extension of the proposed framework to multivariate estimation problems, development of similar hybrid estimators for other population parameters, application to complex sampling designs beyond simple random sampling, and integration with machine learning techniques for big data applications. Additionally, further investigation into the performance of these estimators under different types of indeterminacy structures would be valuable.

The practical implications of this research are substantial, particularly in fields like medical research, economics, and social sciences where indeterminate data is common. Our estimators provide survey practitioners with more accurate tools for population mean estimation while properly accounting for measurement uncertainty, offering significant improvements over traditional approaches in both theoretical robustness and practical applicability.

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