A novel computational method for neutrosophic uncertainty related quadratic fractional programming problems

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Abstract: This study introduces a novel method for addressing the pentagonal quadratic fractional programming problem (PQFPP). We employ pentagonal neutrosophic numbers for the objective function's cost, resources, and technological coefficients. The paper transforms the PQFPP into a standard quadratic fractional programming (QFP) problem via the score function. By leveraging the Taylor series approach, the modified QFP is simplified to a single-objective linear programming (LP) task, amenable to resolution through conventional LP algorithms or software tools. A numerical example serves to demonstrate the efficacy of the suggested approach. Moreover, comparative analyses and benefits reveal that the newly developed techniques outperform existing solutions in current scholarly works.

Keywords: Quadratic fractional programming; Score function; Taylor series; Linear programming; Decision making; Optimal solution.

1. Introduction

The issue of fractional programming (FP) comes into play when the goal is to optimize the relationship between variables and constraints in decision-making scenarios. In problems of numerical optimization, FP can be considered an extension of linear fractional programming (LFP). In FP, the objective function is composed of a ratio between two generally nonlinear functions.

Fractional programming finds applications in diverse areas of decision-making, including but not limited to traffic management (cited from Dantzig et al., 1966), network flow optimization (referenced from Arisawa and Elmaghraby, 1972), and strategic games (based on Isbell and Marlow, 1956). Schaible (1976 and 1982) gives a review of various applications. Enormous approaches are introduced to solve LFP problems (Gupta and Chakraborty, 1998; Tantawy, 2007; 2008; Pandey and Punnen, 2007; Pop and Stancu, 2008; Kim and Mehrotra, 2021; Bennani et al., 2021; Park and Lim, 2021; Das and Mandal, 2017; Das, 2019; 2021; Farnam and Darehmiraki, 2021; Mekawy, 2022; Edalatpanah, 2023; Jiao and Shang, 2023).

In the domain of operations research, quadratic fractional programming (QFP) issues are widely applicable. These problems can be categorized by the uniformity of the constraints and the divisibility of the objective function, as outlined by Sharma and Singh in 2013. Ibaraki et al. (1976) introduced some models for solving QFP. Gupta and Puri's 1994 work focused on a specialized QFP scenario, aiming to minimize a certain quadratic fractional function under generalized constraints, further narrowed down to an extreme point of a convex polytope. Benson, in 2006, explored fractional programming problems that maximize a particular ratio of two convex functions, with at least one being a quadratic form, and detailed their mathematical attributes. In 2006, Mishra and Ghosh introduced an interactive fuzzy programming technique to find satisfactory solutions for a two-tier QFP problem involving dual decision-makers. Zhang and Hayashi, in 2011, dealt with a fractional programming problem that minimizes the ratio of two indefinite quadratic functions under dual quadratic constraints, converting the original problem into a univariate nonlinear equation.

Khurana and Arora, in 2011, put forth a methodology for tackling QFP problems incorporating uniform constraints. Sharma and Singh, in 2013, devised an iterative approach based on simplex techniques for solving QFP problems with factorization. Suleiman and Nawkhass, in 2013, advocated that a revised simplex method outperforms in addressing QFP issues and applied Wolfe's method to solve them. Lur and colleagues, in 2014, proposed a new continuous-time QFP (CQFP) model using the parametric and discretization methods, leading to an approximation algorithm with any desired accuracy.

Continuing, Singh and Haldar, in 2015, innovated a method for bi-level quadratic linear fractional programming issues by transforming the original problem. Youness et al., in 2016, introduced a parametric approach to solve nonlinear fractional optimization problems, relying on a two-dimensional algorithm. Sharma et al., in 2017, employed the ϵϵ-scalarization technique coupled with an integer feasible solution ranking to identify all non-dominated points for bi-objective quadratic fractional integer programming issues. Jain and colleagues, in 2018, offered an algorithmic solution for quadratic fractional integer programming problems involving bounded variables, using complete ranking and scanning.

Kassa and Tsegay, in 2018, presented an algorithm for a tri-level programming issue involving quadratic fractional objectives at each tier, using a fuzzy goal programming strategy. Sivri and colleagues, in 2018, proposed a computational technique that simplifies QFP into a linear programming task. Lara, in 2019, established optimality conditions for general QFP issues using a generalized asymptotic function for dealing with quasi-convexity. Gharanjik et al., in 2019, introduced a novel optimization schema for signal design problems involving max-min FQP issues, simplifying the original problem using a penalized version.

Lastly, Consolini et al., in 2020, rephrased an FQP issue into a Celis–Dennis–Tapia (CDT) problem, which served to outline a local search algorithm. Lachhwani, in 2020, recommended a holistic method for solving multi-level QFP problems based on fuzzy goal programming. For other recent research on the topic, refer to works by Taghi-Nezhad and Taleshian (2018), Badrloo and Husseinzadeh Kashan (2019), Yang and Xia (2020), Kausar et al. (2021), Jafari and Sheykhan (2021), Rani et al. (2021), Xiao et al. (2022), Ju et al. (2022), Zhou et al. (2022), and Berahas et al. (2023).

Real-world data is inaccurate and very difficult to be determined exactly. Therefore, a mathematical model of a problem does not generally have accurate output to fulfill sufficient efficiency. As a result, in optimization problems, an appropriate tool is required by which the uncertainty of data is overcome. Fuzzy set theory serves as a pivotal research methodology for addressing issues associated with vagueness and uncertainty, and it has found applications across diverse academic disciplines. Initially introduced by Zadeh in 1965 (Zadeh, 1965), fuzzy numbers are constrained to a single membership function. In practical scenarios, the attributes of data such as certainty, accuracy, and reliability are typically elusive. Given that fuzzy numbers' optimal solutions are bound by a restricted set of constraints, a new theoretical framework known as 'Neutrosophic sets' was introduced. This approach was first conceptualized by Smarandache in 1995 (Smarandache, 1998). After that this logic was developed and studied by several scholars (Rivieccio, 2008; Guo and Cheng, 2009; Ye, 2014; Smarandache, 2020; Edalatpanah, 2018; Garg, 2020; Abdel-Basset et al., 2020; Debnath, 2021; Mohanta and Toragay, 2023; Edalatpanah et al., 2023; Bhat, 2023, etc.). For researchers, it's crucial to extend traditional linear programming issues into their neutrosophic counterparts, incorporating three distinct membership functions: truth, indeterminacy, and falsity. This capacity to manage ambiguous and nebulous data can significantly enhance the adoption and utility of linear programming. Unlike fuzzy linear programming, which relies solely on a single membership function, neutrosophic linear programming offers a more nuanced approach by employing three types of membership functions. For an in-depth understanding, refer to works by Ye (2018), Abdel-Basset et al. (2019), Khatter (2020), Basumatary and Broumi (2020), Das and Dash (2020), Das et al. (2020a,b,c), Das and Edalatpanah (2022), Kumar et al. (2021), and Abdelfattah (2021).

For the best of our mind, it has been observed from the literature study that there was no study in a neutrosophic quadratic fractional programming problems. Taking this opportunity, we introduced a new method for solving PQFPP.

Contribution: One of the key strengths of the neutrosophic set lies in its ability to aid decision-makers through its incorporation of degrees of truth, falsity, and indeterminacy. In this context, the degree of indeterminacy is often viewed as an autonomous variable with a crucial role in decision processes. Given the inherent uncertainties in real-world scenarios, utilizing pentagonal neutrosophic linear fractional programming problems (PNQLFPP) offers a more realistic approach than traditional PQFPP. In this study, we introduce a PNQLFPP model, wherein all coefficients are treated as pentagonal neutrosophic numbers. We present a novel algorithm that leverages a recently-developed ranking function along with the Taylor series method to solve PNQLFPPs. As far as we are aware, this is the inaugural methodology for addressing PNQLFPPs using a ranking function. Consequently, a direct comparison with existing techniques is not applicable for validating our approach. We illustrate the utility and efficacy of our method through a diet planning example, thereby showcasing its real-world applicability.

Motivation:

Neutrosophic sets serve as a cornerstone in modeling uncertainty, a key element in the creation of applied mathematical models in science, engineering structures, and medical diagnostic problems. Given the absence of existing studies that tackle PNQLFPP, our work pioneers a new approach that employs a ranking function along with the Taylor series method for resolving issues related to PNQLFP. Queries such as the feasibility of incorporating this into operations research grounded in linear programming, or its real-world applicability, have yet to be answered. It is with this backdrop that we seek to advance the discourse through this paper.

Novelties:

In recent years, scholarly focus has shifted toward the enhancement and refinement of theories within the realm of neutrosophic studies, with ongoing efforts to broaden their utility across diverse neutrosophic subfields. Against this backdrop, our core objective in relation to PNQLFP problem theory is to validate the conceptual framework through several pivotal aspects:

- ✓ Unveiling an efficient ranking function.
- ✓ Incorporating the Taylor series method and elucidating its applications.
- ✓ Real-world applications of the PNQLFP problem.
- ✓ Benchmarking our findings against earlier established outcomes.

In addressing this research void, our paper debuts the concept of pentagonal neutrosophic quadratic fractional programming problems. We then transform this into a more straightforward problem via the scoring function associated with pentagonal neutrosophic numbers, ultimately reducing it to a linear programming (LP) issue through the application of the Taylor Series. The rest of the paper unfolds as follows: Section 2 provides essential background information. The formulation of the pentagonal neutrosophic quadratic fractional programming issue is detailed in Section 3. A methodology to arrive at an optimal solution is laid out in Section 4. Section 5 brings in a numerical example to elucidate the concept. Insights into the results and merits of our approach are discussed in Section 6, and Section 7 closes with concluding observations.

2. Foundational Concepts

3.

In this part, we outline fundamental ideas and findings concerning fuzzy numbers, pentagonal fuzzy numbers, the neutrosophic set, pentagonal neutrosophic numbers, and the arithmetic operations associated with them.

Definition 1. (Cited from Zadeh, 1965). A set A is termed as a fuzzy set within the realm of real numbers R if the range of its membership function falls between [0, 1].

Definition 2. (As per Abbasbandy and Hajjari, 2009). A number $\widetilde{A}_P = (r, s, t, u, v), r \le s \le t \le u \le v$, in the set of real numbers $\Re$, is identified as a pentagonal fuzzy number when its membership function is expressed as:

$$\mu_{\tilde{A}_P} = \begin{cases} 0, \quad x < r, \\ w_1\left(\frac{x-r}{s-r}\right), \text{for } r \le x \le s, \\ 1-(1-w_1)\left(\frac{x-s}{t-s}\right), \text{for } s \le x \le t \\ 1, \text{for } x = t, \\ 1-(1-w_2)\left(\frac{u-x}{u-t}\right), \text{for } t \le x \le u, \\ w_2\left(\frac{v-x}{v-u}\right), \text{for } u \le x \le v, \\ 0, \text{for } x > v. \end{cases}$$

Definition 3. (Based on Smarandache, 1998). A neutrosophic set, denoted as $\widetilde{B}^N$, within a non-empty set X is characterized as:

$$\widetilde{B}^N = \{\langle x;\ I_{\widetilde{B}^N}(x),\ J_{\widetilde{B}^N}(x), V_{\widetilde{B}^N}(x)\rangle : x \in X, I_{\widetilde{B}^N}(x),\ J_{\widetilde{B}^N}(x), V_{\widetilde{B}^N}(x) \in\ ]0_-, 1^+[\},$$

here, $I_{\widetilde{B}^N}(x), J_{\widetilde{B}^N}(x)$, and $V_{\widetilde{B}^N}(x)$ stand for the truth, indeterminacy, and falsity membership functions, respectively. The sum of them is unrestricted, falling within the range $0^- \le \mathrm{Sup}\{I_{\widetilde{B}^N}(x)\} + \mathrm{Sup}\{J_{\widetilde{B}^N}(x)\} + \mathrm{Sup}\{V_{\widetilde{B}^N}(x)\} \le 3^+$. Additionally, $]0^-, 1^+[$ represents a nonstandard unit interval.

Definition 4. Referring to the terms outlined in Definition 3, if these membership functions are confined to the interval [0,1] and their aggregate sum lies in the range [0,3], such a set is termed a Single-Valued Neutrosophic set.

Definition 5. Assume that $\tau_{\tilde{p}}, \varphi_{\tilde{p}}, \omega_{\tilde{p}} \in [0, 1]$, and $r, s, t, u, v \in \mathbb{R}$ satisfying $r \le s \le t \le\ u \le v$. A Single-Valued Pentagonal Neutrosophic Set (SVPN), denoted as $\tilde{p}^{PN} = \langle (r, s, t, u, v) : \tau_{\tilde{p}}, \varphi_{\tilde{p}}, \omega_{\tilde{p}} \rangle$, is a specialized neutrosophic set on $\Re$. In this set, the truth-membership, hesitant-membership, and falsity-membership functions are represented by:

$$\mu_{\tilde{p}^{PN}}(x)=\begin{cases} 0, & x<r;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \tau_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v. \end{cases}$$

$$\rho_{\tilde{p}^{PN}}(x)=\begin{cases} 0, & x<r;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \varphi_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v. \end{cases}$$

$$\sigma_{\tilde{p}^{PN}}(x)=\begin{cases} 0, & x<r;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(s-r)^2}(x-r)^2\right), & r\le x\le s;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(t-s)^2}(x-t)^2+1\right), & s\le x\le t;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(u-t)^2}(x-t)^2+1\right), & t\le x\le u;\\ \omega_{\tilde{p}^{PN}}\left(\frac{1}{2}\frac{1}{(v-u)^2}(x-v)^2\right), & u\le x\le v;\\ 0, & x>v. \end{cases}$$

Here, $\tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}$, and $\omega_{\tilde{p}^{PN}}$ indicate the peak truth, nadir-hesitant, and nadir-falsity membership degrees, correspondingly. The SVPN $\tilde{p}^{PN} = \langle (r,s,t,u,v): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ can depict a vaguely defined value approximating $[s,u]$.

Definition 6. Suppose $\tilde{p}^{PN} = \langle (r,s,t,u,v): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ and $\tilde{q}^{PN} = \langle (r',s',t',u',v'): \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{q}^{PN}} \rangle$ are two distinct single-valued PQFNs. The following describes the arithmetic procedures that apply to them:

1. $\tilde{p}^{PN} \oplus \tilde{q}^{PN} = \langle (r+r', s+s', t-t', u+u', v+v');\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle$

2. $\tilde{p}^{PN} \ominus \tilde{q}^{PN} = \langle \left(r-v', s-u', t+t', u-s', v-r' \right);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle,$

3. $\tilde{p}^{PN} \otimes \tilde{q}^{PN} = \frac{1}{5}\gamma_q \langle (r,s,t,u,v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_q = \frac{1}{3}(r' + s' + t' + u' + v')\left(2 + \tau_{\tilde{q}^{PN}} - \varphi_{\tilde{q}^{PN}}\right) \neq 0,$

4. $\tilde{p}^{N} \oslash \tilde{q}^{N} = \frac{5}{\gamma_q} \langle (r,s,t,u,v);\ \tau_{\tilde{p}^{PN}} \wedge \tau_{\tilde{q}^{PN}}, \varphi_{\tilde{p}^{PN}} \vee \varphi_{\tilde{q}^{PN}}, \omega_{\tilde{p}^{PN}} \vee \omega_{\tilde{q}^{PN}} \rangle, \gamma_q \neq 0,$

5. $k\,\tilde{p}^{PN} = f(x) = \begin{cases} \langle (kr,ks,kt,ku,kv): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle, k > 0, \\ \langle (kv,ku,kt,ks,kr);\ \tau_{\tau_{\tilde{p}^{PN}}},\ \varphi_{\tau_{\tilde{p}^{PN}}}, \omega_{\tau_{\tilde{p}^{PN}}} \rangle, k < 0, \end{cases}$

6. $\tilde{p}^{PN^{-1}} = \langle \left(\frac{1}{v}, \frac{1}{u}, \frac{1}{t}, \frac{1}{s}, \frac{1}{r} \right);\ \tau_{\tau_{\tilde{p}^{PN}}},\ \varphi_{\tau_{\tilde{p}^{PN}}}, \omega_{\tau_{\tilde{p}^{PN}}} \rangle, \tilde{p}^{PN} \neq 0.$

Definition 7. Assume $\tilde{p}^{PN} = \langle (r,s,t,u,v): \tau_{\tilde{p}^{PN}}, \varphi_{\tilde{p}^{PN}}, \omega_{\tilde{p}^{PN}} \rangle$ is a single-valued pentagonal neutrosophic number. In this context, the Accuracy and Score functions are delineated in the following manner:

$$AC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * [2 + \tau_{\tilde{p}^{PN}} - \varphi_{\tilde{p}^{PN}}].$$

$$SC(\tilde{p}^{PN}) = \left(\frac{1}{15}\right)(r + s + t + u + v) * \left[2 + \tau_{\tilde{p}^{PN}} - \varphi_{\tilde{p}^{PN}} - \omega_{\tilde{p}^{PN}}\right].$$

Definition 8. (As per Thamariselvi and Santhi, 2016). Referring to the terms outlined in Definition 7, the ordinal relationships between A and B, predicated on their Accuracy and Score functions, are specified as follows:

1. If $SC(\tilde{p}^{PN}) < SC(\tilde{q}^{PN})$, then $p < q$,
2. If $SC(\tilde{p}^{PN}) = SC(\tilde{q}^{PN})$, then $p = q$,
3. If $AC(\tilde{p}^{PN}) < AC(\tilde{q}^{PN})$, then $p < q$
4. If $AC(\tilde{p}^{PN}) > AC(\tilde{q}^{PN})$, then $p > q$,
5. If $AC(\tilde{p}^{PN}) = AC(\tilde{q}^{PN})$, then$p = q$.

Definition 9. (Based on Sivri et al., 2018). The initial pair of terms in the Taylor series, stemming from $f(X_1, X_2, \dots, X_2)$, when evaluated at a given point $Q = (q_1, q_2, \dots, q_n)$, are characterized as follows:

$$f(Q) + \nabla_{X_1} f(Q)(X_1 - q_1) + \nabla_{X_2} f(Q)(X_2 - q_2) + \cdots + \nabla_{X_n} f(Q)(X_n - q_n) = 0.$$

3. Problem statement

Quadratic fractional programming with pentagonal neutrosophic parameters can be formulated as

$$\text{(PQFP)} \quad \max(\text{or} \min) \ \tilde{Z}^{PN}(X) = \frac{f(\tilde{C}^{PN}, \tilde{E}^{PN})}{g(\tilde{D}^{PN}, \tilde{G}^{PN})} = \frac{(\tilde{C}^T)^{PN} X + \frac{1}{2} X^T \tilde{E}^{PN} X}{(\tilde{D}^T)^{PN} X + \frac{1}{2} X^T \tilde{G}^{PN} X}$$

Subject to

$$X \in \tilde{M}^{PN} = \left\{X \in \Re^n : \tilde{A}^{PN} \otimes X(\lesssim, \eqsim, \gtrsim)\tilde{b}^{PN}; X \geq 0\right\}$$

Where, $\tilde{b}^{PN} = (\tilde{b}_1^{PN}, \tilde{b}_2^{PN}, \dots, \tilde{b}_m^{PN}), \tilde{c}^{PN} = (\tilde{c}_1^{PN}, \tilde{c}_2^{PN}, \dots, \tilde{c}_n^{PN})$, $\tilde{d}^{PN} = (\tilde{d}_1^{PN}, \tilde{d}_2^{PN}, \dots, \tilde{d}_n^{PN})$, and are neutrosophic cost vector and neutrosophic right- hand side vector. $X = (X_1, X_2, \dots, X_n)$ is a vector of decision variables, and $\tilde{E}^{PN} = [\tilde{e}_{ij}^{PN}]_{n\times n}$, $\tilde{G}^{PN} = [\tilde{g}_{ij}^{PN}]_{n\times n}$ is a matrix of quadratic form which is symmetric and positive semi-definite, and $\tilde{A}^{PN} = [\tilde{a}_{ij}^{PN}]_{m\times n}$. It is assumed that all of $\tilde{A}^{PN}, \tilde{b}^{PN}, \tilde{C}^{PN}, \tilde{D}^{PN}, \tilde{E}^{PN} and\ \tilde{G}^{PN} \in F(R)$, where $F(R)$ denotes the set of all pentagonal neutrosophic parameters.

Definition 10. In the context of the PQFP problem, a feasible fuzzy solution, denoted as $\overline{X}$, is termed an optimal fuzzy solution if $\tilde{Z}^{PN}(X) \binom{\gtrsim}{\lesssim} \tilde{Z}^{PN}(\overline{X})$ for every individual x.

Utilizing the score function associated with the Pentagonal fuzzy number, the PQFP problem is reformulated as follows:

$$\text{(QFP)} \quad \max(\text{or } \min)\ Z(X) = \frac{C^T X + \frac{1}{2} X^T E X}{D^T X + \frac{1}{2} X^T G X}$$

Subject to

$$x \in M = \{X \in \Re^n : A\, X(\le, =, \ge) b; X \ge 0\}.$$

It assumed that $Z(X)$ is a function of class $C^{(1)}$.

4. Solution method

The steps of the solution procedure are:

Step 1: Consider the PQFP problem.

Step 2: Convert the PQFP into the QFP based on the score function.

Step 3: Choose an arbitrary initial non-zero feasible point, say x^*.

Step 4: Expand the Taylor series at x^* (Definition 9) to linearize the objective function.

Step 5: Solve the linear programming

$$\text{(LP)} \quad \max(\text{or } \min)\ Z^{Linear}(X) = H^T X$$

Subject to $x \in M$.

Assume X° represents the optimal solution for the Linear Programming (LP) problem.

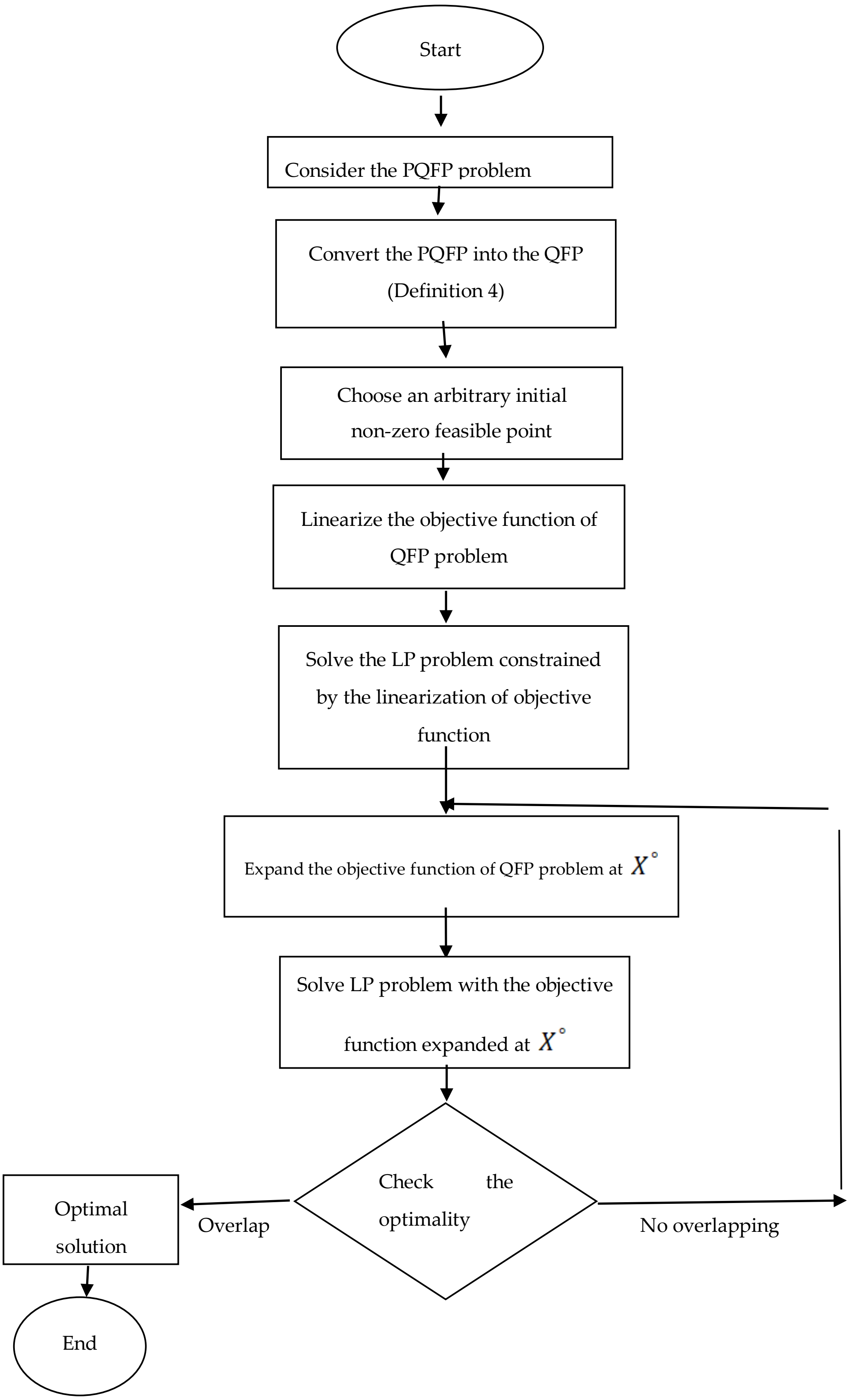

Fig.1. Solution method flow chart

Step 6: Expand the objective function of the QFP problem (Definition 9) at X°.

Step 7: Solve LP problem with expanded objective function resulting from step 6 as constrained. Let $\widehat{X}$ be the solution.

Step 8: Check the optimality

If the solutions X° and $\hat{X}$ overlap stop with the final optimal solution. Otherwise, assign X° to $\hat{X}$ and return to step 6.

Fig. 1 depicts the flowchart outlining the steps of the solution method.

5. Illustrative Example

This section is dedicated to the application of our proposed method. We examine the following PQFP issue:

$$\max \tilde{Z}^{PN}(x) = \frac{(x_1 x_2)\begin{pmatrix} 0 & \tilde{e}_{12}^{PN} \\ 0 & \tilde{e}_{22}^{PN} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(+)\tilde{c}_1^{PN}x_1(+)\tilde{c}_2^{PN}x_2(+)\tilde{c}_0^{PN}}{(x_1 x_2)\begin{pmatrix} \tilde{g}_{11}^{PN} & 0 \\ \tilde{g}_{21}^{PN} & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}(+)\tilde{d}_1^{PN}x_1(+)\tilde{d}_2^{PN}x_2(+)\tilde{d}_0^{PN}}$$

Subject to (1)

$\tilde{a}_{11}^{PN}x_1(+)\tilde{a}_{12}^{PN}x_2 \lesssim \tilde{b}_1^{PN},$

$\tilde{a}_{21}^{PN}x_1(+)\tilde{a}_2^{PN}x_2 \lesssim \tilde{b}_2^{PN},$

$x_1,\ x_2 \geq 0.$

Where,

$\tilde{e}_{12}^{PN} = \tilde{e}_{22}^{PN} = \tilde{g}_{11}^{PN} = \tilde{g}_{21}^{PN} = \langle(1,1,1,1,1);1,0,0\rangle;\ \tilde{c}_1^{PN} = \langle(1,1,1,1,1);1,0,0\rangle;$

$\tilde{c}_2^{PN} = \langle(2,3,4,5,6);1,0,0\rangle; \tilde{c}_0^{PN} = \langle(1,2,3,4,5);1,0,0\rangle = \tilde{d}_2^{PN}$

$\tilde{d}_1^{PN} = \langle(1,5,6,7,11);1,0,0\rangle; \tilde{d}_0^{PN} = \langle(2,6,8,10,14);1,0,0\rangle;$

$\tilde{a}_{11}^{PN} = \tilde{a}_{12}^{PN} = \tilde{a}_{21}^{PN}x_1 = \tilde{b}_1^{PN} = \langle(1,1,1,1,1);1,0,0\rangle;$

$\tilde{a}_2^{PN} = \langle(0, 1, 2, 3, 4); 1, 0, 0\rangle;\ \tilde{b}_2^{PN} = \langle(1, 4, 5{,}9, 16); 1, 0, 0\rangle$

Let us apply the steps of the solution method as

Step 2: Based on the ranking function of the pentagonal neutrosophic numbers, Problem (1) converts into

$$\max Z(x) = \frac{(x_1 x_2)\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + x_1 + 4x_2 + 3}{(x_1 x_2)\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 6x_1 + 4x_2 + 8}$$

Subject to (2)

$-x_1 + x_2 \leq 1,$

$x_1 + 2x_2 \leq 7,$

$x_1,\ x_2 \geq 0.$

Or equivalently,

$$\max Z(x) = \frac{x_2^2 + x_1 x_2 + x_1 + 4x_2 + 3}{x_1^2 + x_1 x_2 + 6x_1 + 4x_2 + 8}$$

Subject to (3)

$-x_1 + x_2 \leq 1,$

$x_1 + 2x_2 \leq 7,$

$x_1,\ x_2 \geq 0.$

Steps 3 and 4: Select $x^* = (1, 1)$ to be a random optimal point that is not zero. Utilize the Taylor series centered at this point to approximate the objective function in a linear form as follows:

$$\nabla_{x_1} Z = \frac{(x_1^2 + x_1 x_2 + 6x_1 + 4x_2 + 8)(x_2 + 1) - (x_2^2 + x_1 x_2 + x_1 + 4x_2 + 3)(x_2 + 4)}{(x_1^2 + x_1 x_2 + 6x_1 + 4x_2 + 8)^2}$$

$$\nabla_{x_2} Z = \frac{(x_1^2 + x_1 x_2 + 6x_1 + 4x_2 + 8)(2x_2 + x_1 + 4) - (x_2^2 + x_1 x_2 + x_1 + 4x_2 + 3)(2x_1 + x_2 + 6)}{(x_1^2 + x_1 x_2 + 6x_1 + 4x_2 + 8)^2}$$

Step 5: Construct the LP problem as

$$\max Z = -\frac{1}{8}x_1 + \frac{9}{40}x_2$$

Subject to (4)

$$-x_1 + x_2 \leq 1,$$

$$x_1 + 2x_2 \leq 7,$$

$$x_1,\ x_2 \geq 0.$$

The optimal solution is $X^{\circ} = (1.67, 2.67)$.

Steps 6 and 7: Elaborate on the objective function corresponding to the QFP issue (as per Definition 6) around the point X°. Subsequently, construct and resolve the ensuing LP problem as:

$$\max Z = -0.148x_1 + 0.188x_2$$

Subject to (5)

$$-x_1 + x_2 \leq 1,$$

$$x_1 + 2x_2 \leq 7,$$

$$x_1,\ x_2 \geq 0.$$

The optimal solution is $X^{\circ} = (1.67, 2.67) = \hat{X}$, with the optimum value $Z = 0.75$, and

$$\tilde{Z}^{NP} = \langle (1.61216, 1.91406, 2.21596, 2.51786, 2.8198); 1, 0, 0 \rangle.$$

6. Result Analysis

In this section, we analyze the efficacy of our proposed approach in comparison to the existing technique by Sivri et al. (2018) for addressing the PNQFP issue. Utilizing ranking functions and order definitions, we establish that our method yields more effective outcomes, as evidenced by the comparison:

$$Z_{\text{Proposed method}} = 0.75 > Z_{\text{Sivri et al. method (2018)}} = 0.74$$

Additionally, it's crucial to note that the literature currently lacks a method for addressing the PNQFP issue. Consequently, we juxtaposed our novel approach with prevalent techniques for solving the C-QFP problem.

Our findings indicate that the objective value yielded by our method surpasses those of existing approaches. This leads us to conclude that our method is notably more effective. Furthermore, the objective value generated by our method resides within the realm of neutrosophic values.

Benefits of the proposed approach are as follow:

i) The outcomes generated by our model outperform those of Sivri's. As evidenced in the results section, our objective function value stands at 0.75, compared to Sivri's 0.74. Given that the problem aims for maximization, our solution effectively achieves this goal.

ii) Crucially, in real-world scenarios, managers often grapple with options of agreement, uncertainty, and disagreement. Sivri's model restricts them to parameters set by decision-makers, a limitation we've addressed by incorporating a neutrosophic model in our approach.

iii) Our model is versatile enough to be applicable to both real-world and large-scale issues.

Summing up the discussion, our newly proposed algorithm presents an innovative avenue for tackling both uncertainty and indeterminacy in real-world situations.

7. Conclusions

In real-world settings, dealing with ambiguous, unclear, or incomplete information often necessitates the use of neutrosophic sets. This study focuses on a neutrosophic linear fractional programming issue involving pentagonal neutrosophic numbers and converts it into a QFP issue through a ranking function. Utilizing the Taylor series method, we further simplify the QFP issue into a linear programming (LP) problem solvable via standard LP algorithms or software. Scholars in this domain could find our approach useful for addressing both intricate and straightforward challenges. An example is included to validate the efficiency of our methodology. This new framework not only augments the realm of uncertain linear fractional programming but also introduces a novel, effective strategy for managing indeterminate optimization issues. Comparative evaluations with existing methodologies underscore the merits of our ranking approach. Future extensions could involve incorporating other specialized neutrosophic sets like pentagonal neutrosophic sets, neutrosophic rough sets, interval-valued neutrosophic sets, and plithogenic contexts.

Conflicts of Interest

Authors do not have any conflicts of interest.

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