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Finding Minimal Units In Several Two-Fold Fuzzy Finite Neutrosophic Rings

Raed Hatamleh*
Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid 21110, Jordan
raed@jadara.edu.jo

Ayman Hazaymeh
Department of Mathematics, Faculty of Science and Information Technology, Jadara University, P.O. Box 733, Irbid 21110, Jordan
aymanha@ jadara.edu.jo

Abstract:

In this paper, we have defined the concept of minimal units in finite two-fold fuzzy neutrosophic rings modulo integers as a generalization of classical elements of the group of units of the mentioned neutrosophic rings. We apply our results to characterize all minimal units in the following two-fold neutrosophic rings $\left(Z_n(I)\right)_{f_I}$ for $n \in \{2,3,4,5\}$.

Keywords: two-fold algebra, finite neutrosophic ring, modulo integer ring, minimal unit.

Introduction

The determination of the invertible elements in a ring has attracted the attention of many researchers, specifically in some modern rings such as neutrosophic rings, and n-cyclic refined neutrosophic rings [6-8]. Many interesting results have been proved about the problem of classifying the group of units in a number of special solutions, and also tables that calculate the value of these units and determine their exact number [9-10].

A unit in a ring R is the concept of being an invertible element concerning multiplication operation which make a group together.

Two-fold neutrosophic algebras are new algebraic structures presented by Smarandache [1] by combining neutrosophic values of truth, falsity, and indeterminacy with classical algebraic sets. These ideas were used by many authors to generalize other famous algebraic structures such as two-fold fuzzy number theoretical systems [2-3], two-fold modules and spaces [4], and two-fold fuzzy rings [5]. Also, they were used in the study of some special two-fold complex functions such as Gamma function [13], and in extending n-refined neutrosophic rings [11-15]. The complex fuzzy ring initiated on complex fuzzy space was given by Al-Husban [14-17]

In this work, we define the minimal two-fold neutrosophic finite ring modulo integers, and we determine all maximal units in these rings for the special values of n between 2 and 5. We have also classified all the units that have been calculated in tables showing their values as well as their number.

Main Discussion

Definition 2.1.

Let $Z_n = \{0,1,\dots,n-1\}$ be the ring of integers modulo n, and $Z_n(I) = \{a + bI \ ; a, b \in Z_n \ , I^2 = I\}$ be the corresponding neutrosophic ring. Assume that $f: Z_n \to [0,1]$ be a fuzzy mapping with $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$; $f_I: Z_n(I) \to [0,1]$ such that:

$$f_I(a + bI) = \min(f(a), f(b)).$$

The minimal two-fold neutrosophic ring$(Z_n(I))_{f_I}$ is defined as:

$$(Z_n(I))_{f_I} = \{(a + bI)_{f_I(c+dI)} \quad ; \quad a + bI, c + dI \in Z_n(I)\}.$$

Example 2.1.

Consider $(Z_5 = \{0,1,2,3,4\}, +,\cdot)$ the ring of integers modulo 5, take $f : Z_5 \to [0,1]$;

$$f(x) = \begin{cases} 1 & ; x = 1 \\ \frac{1}{2} & ; x \in \{2,3\} \\ \frac{1}{4} & ; x = 4 \\ 0 ; x = 0 \end{cases},$$

$Z_5(I) = \{0,1,2,3,4, I, 2I, 3I, 4I, 1 + I, 1 + 2I, 1 + 3I, 1 + 4I, 2 + I, 2 + 2I, 2 + 3I, 2 + 4I, 3 + I, 3 + 2I, 3 + 3I, 3 + 4I, 4 + I, 4 + 2I, 4 + 3I, 4 + 4I\}$.

$$f_I(3 + 4I) = \min(f(3), f(4)) = \tfrac{1}{4},\ f_I(1 + 2I) = \min(f(1), f(2)) = \tfrac{1}{2}\ .$$

By a similar approach, we can see:

$f_I(1+4I) = f_I(2+4I) = \frac{1}{4}$, $f_I(0) = f_I(I) = f_I(2I) = f_I(3I) = f_I(4I) = 0, f_I(1+3I) = \frac{1}{2}$,

$f_I(2+I) = f_I(3+I) = f_I(2) = f_I(3) = 1\backslash 2$, and so on .

Therefor, $(Z_5(I))_{f_I} = \{(a+bI)_0, (a+bI)_1, (a+bI)_{\frac{1}{2}}, (a+bI)_{\frac{1}{4}}; for\ \ all\ \ a, b \in Z_5\}$.

Definition 2.2.

An element $X = (a+bI)_{f_I(c+dI)}$ is called a minimal unit in the minimal two-fold neutrosophic ring $(Z_n(I))_{f_I}$ if and only if there exists:

$$Y = (m+nI)_{f_I(t+kI)} \in (Z_n(I))_{f_I}$$

such that:

$$X \circ Y = [(a+bI)(m+nI)]_{f_I[(c+dI)(t+kI)]} = 1_0.$$

Example 2.2.

Consider $Z_3 = \{0,1,2\}$ $, Z_3(I) = \{0,1,2, I, 2I, 1+I, 1+2I, 2+I, 2+2I\}$,

$$f\colon\ Z_3 \to [0,1]\ \ ;\ \ \begin{cases} f(0) = 0 \\ f(1) = 1 \\ f(2) = \dfrac{1}{2} \end{cases} \qquad , f_I\colon Z_n(I) \to [0,1]$$

such that:

$f_I(0) = f_I(I) = f_I(2I) = f_I(1) = 0, f_I(1+I) = 1, f_I(2) = 0, f_I(1+2I) = f_I(2+I) =$

$f_I(2+2I) = \frac{1}{2}$.

Hence

$\left(Z_3(I)\right)_{f_I} = \Big\{0_0, 0_1, 0_{\frac{1}{2}}, 1_0, 1_1, 1_{\frac{1}{2}}, I_0, I_1, I_{\frac{1}{2}}, (2I)_0, (2I)_1, (2I)_{\frac{1}{2}}, (1+I)_0, (1+I)_1, (1+I)_{\frac{1}{2}}, (2+$

$I)_0, (2+I)_1, (2+I)_{\frac{1}{2}}, (1+2I)_0, (1+2I)_1, (1+2I)_{\frac{1}{2}}, (2+2I)_0, (2+2I)_1, (2+2I)_{\frac{1}{2}}, 2_0, 2_1, 2_{\frac{1}{2}}\Big\}$.

Theorem 2.1.

An element $X = (a+bI)_{f_I(m+nI)} \in \left(Z_n(I)\right)_{f_I}$ is a minimal unit if and only if:

1] $\gcd(a, n) = \gcd(a+b, n) = 1$.

2] $\exists\ \ t+kI \in Z_n(I)$ such that: $\mathrm{m}t \equiv 0(mod\ \ n)$ or $(m+n)(t+k) - mt \equiv 0(mod\ \ n)$.

Proof.

X is a minimal unit if and only if there exists $Y = (c+dI)_{f_I(t+kI)}$ with $X \circ Y = 1_0$, so that:

$$(a+bI)(c+dI)=1 \Rightarrow \begin{cases} ac=1 \\ ad+bc+bd=0 \end{cases} \Rightarrow \begin{cases} ac=1 \\ (a+b)(c+d)-ac=0 \end{cases}$$

$$\Rightarrow \begin{cases} ac=1 \\ (a+b)(c+d)=ac=1 \end{cases} \Rightarrow \begin{cases} \gcd(a,n)=1 \\ \gcd(a+b,n)=1 \end{cases}.$$

On the other hand:

$f_I[(m+nI)(t+kI)]=0 \Rightarrow f_I[mt+I(mk+nt+nk)]=0 \Rightarrow f_I\big[mt+I[(m+n)(t+k)-mt]\big]=0$, thus:

$$f(mt)=0 \text{ or } f[(m+n)(t+k)-mt]=0,$$

hence

$$\begin{cases} mt=0\equiv 0(mod\ \ n) \\ or \\ (m+n)(t+k)-mt\equiv 0(mod\ \ n) \end{cases}.$$

Minimal units of $\left(Z_2(I)\right)_{f_I}$:

$Z_2=\{0,1\},\ \ Z_2(I)=\{0,1,I,1+I\}, \cup(Z_2)=\cup\left(Z_2(I)\right)=\{1\}$.

For any $f:\ \ Z_2\rightarrow[0,1]$,

we have:

$$f(0)=0, f(1)=1,$$

that is because (f) is a fuzzy mapping, hence: $f_I: Z_2(I)\rightarrow[0,1]$ with:

$$f_I(0)=f_I(1)=f_I(I)=0, f_I(1+I)=1.$$

The minimal units of $\left(Z_2(I)\right)_{f_I}$ are : $\{1_0,1_1\}$, that is because $0\cdot(1+I)=0$.

Remark 2.1.

For any two-fold minimal neutrosophic ring $\left(Z_n(I)\right)_{f_I}$ and for any $m+nI\in Z_n(I)$, there exists $0\in Z_n(I)$ such that: $0\cdot(m+nI)=0$.

So that the minimal units of $Z_n(I)$ are:

$$M_u\left(Z_n(I)\right)=\{(a+bI)_x\ \ ; a+bI\in\cup\left(Z_n(I)\right)\ \ , x\in f(Z_n)\}$$

Minimal units of $\boldsymbol{Z_3(I)}$, $\boldsymbol{Z_4(I)}$:

$$Z_3=\{0,1,2\}, Z_3(I)=\{0,1,2,I,2I,1+I,1+2I,2+I,2+2I\},$$

$$\cup(Z_3)=\{1,2\}, \cup\left(Z_3(I)\right)=\{1,2,1+I,2+2I\}.$$

For any fuzzy mapping $f: Z_3\rightarrow[0,1]$,

we have

$$f(0)=0, f(1)=1, f(2)\in]0,1[,$$

$Z_4=\{0,1,2,3\}, Z_4(I)=\{0,1,2,3,I,2I,3I,1+I,1+2I,1+3I,2+I,2+3I,2+2I,3+I,3+2I,3+3I\}$,

$$\cup(Z_4)=\{1,3\}, \cup\left(Z_4(I)\right)=\{1,3,1+2I,3+2I\}.$$

For any fuzzy mapping $f: Z_4(I)\to[0,1]$,

we have

$$f(0)=0, f(1)=1, f(2), f(3)\in]0,1[.$$

Minimal units of $\left(Z_3(I)\right)_{f_I}$:

	$M_u\left(Z_3(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	2_1
5	2_0
6	$2_{f(2)}$
7	$(1+I)_0$
8	$(1+I)_1$
9	$(1+I)_{f(2)}$
10	$(2+2I)_0$
11	$(2+2I)_1$
12	$(2+2I)_{f(2)}$

Minimal units of $\left(Z_4(I)\right)_{f_I}$ in the case of $f(2)=f(3)$:

	$M_u\left(Z_4(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$

4	3_0
5	3_1
6	$3_{f(2)}$
7	$(1+2I)_0$
8	$(1+2I)_1$
9	$(1+2I)_{f(2)}$
10	$(3+2I)_0$
11	$(3+2I)_1$
12	$(3+2I)_{f(2)}$

Minimal units of $\left(Z_4(I)\right)_{f_I}$ in the case of $f(2) \neq f(3)$:

	$M_u\left(Z_4(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	$1_{f(3)}$
5	3_0
6	3_1
7	$3_{f(2)}$
8	$3_{f(3)}$
9	$(3+2I)_0$
10	$(3+2I)_1$
11	$(3+2I)_{f(2)}$
12	$(3+2I)_{f(3)}$
13	$(1+2I)_0$
14	$(1+2I)_1$
15	$(1+2I)_{f(2)}$
16	$(1+2I)_{f(3)}$

Minimal units of $\left(Z_5(I)\right)_{f_I}$:

$Z_5 = \{0,1,2,3,4\}, Z_5(I) = \{0,1,2,3,4, I, 2I, 3I, 4I, 1+I, 1+2I, 1+3I, 1+4I, 2+I, 2+2I, 2+3I, 2+4I, 3+I, 3+2I, 3+3I, 3+4I, 4+I, 4+2I, 4+3I, 4+4I\}$,

$\cup(Z_5) = \{1,2,3,4\}, \cup\left(Z_5(I)\right) = \{1,2,3,4,1+I, 1+2I, 1+3I, 2+I, 2+2I, 2+4I, 3+I, 3+3I, 3+4I, 4+2I, 4+3I, 4+4I\}$.

Minimal units of $\left(Z_5(I)\right)_{f_I}$ in the case of $f(2) = f(3) = f(4)$:

	$M_u\left(Z_5(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	2_0
5	2_1
6	$2_{f(2)}$
7	3_0
8	3_1
9	$3_{f(2)}$
10	4_0
11	4_1
12	$4_{f(2)}$
13	$(1+I)_0$
14	$(1+I)_1$
15	$(1+I)_{f(2)}$
16	$(1+2I)_0$
17	$(1+2I)_1$

18	$(1+2I)_{f(2)}$
19	$(1+3I)_0$
20	$(1+3I)_1$
21	$(1+3I)_{f(2)}$
22	$(2+I)_0$
23	$(2+I)_1$
24	$(2+I)_{f(2)}$
25	$(3+I)_0$
26	$(3+I)_1$
27	$(3+I)_{f(2)}$
28	$(2+2I)_0$
29	$(2+2I)_1$
30	$(2+2I)_{f(2)}$
31	$(2+4I)_0$
32	$(2+4I)_1$
33	$(2+4I)_{f(2)}$
34	$(3+4I)_0$
35	$(3+4I)_1$
36	$(3+4I)_{f(2)}$
37	$(4+2I)_0$
38	$(4+2I)_1$
39	$(4+2I)_{f(2)}$
40	$(4+3I)_0$
41	$(4+3I)_1$
42	$(4+3I)_{f(2)}$
43	$(4+4I)_0$
44	$(4+4I)_1$
45	$(4+4I)_{f(2)}$

46	$(3+3I)_0$
47	$(3+3I)_1$
48	$(3+3I)_{f(2)}$

Minimal units of $\left(Z_5(I)\right)_{f_I}$ in the case of $f(2) \neq f(3) \neq f(4)$:

	$M_u\left(Z_5(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(2)}$
4	$1_{f(3)}$
5	$1_{f(4)}$
6	2_0
7	2_1
8	$2_{f(2)}$
9	$2_{f(3)}$
10	$2_{f(4)}$
11	3_0
12	3_1
13	$3_{f(2)}$
14	$3_{f(3)}$
15	$3_{f(4)}$
16	4_0
17	4_1
18	$4_{f(2)}$
19	$4_{f(3)}$
20	$4_{f(4)}$
21	$(1+I)_0$
22	$(1+I)_1$

23	$(1+I)_{f(2)}$
24	$(1+I)_{f(3)}$
25	$(1+I)_{f(4)}$
26	$(1+2I)_0$
27	$(1+2I)_1$
28	$(1+2I)_{f(2)}$
29	$(1+2I)_{f(3)}$
30	$(1+2I)_{f(4)}$
31	$(1+3I)_0$
32	$(1+3I)_1$
33	$(1+3I)_{f(2)}$
34	$(1+3I)_{f(3)}$
35	$(1+3I)_{f(4)}$
36	$(2+I)_0$
37	$(2+I)_1$
38	$(2+I)_{f(2)}$
39	$(2+I)_{f(3)}$
40	$(2+I)_{f(4)}$
41	$(2+2I)_0$
42	$(2+2I)_1$
43	$(2+2I)_{f(2)}$
44	$(2+2I)_{f(3)}$
45	$(2+2I)_{f(4)}$
46	$(2+4I)_0$
47	$(2+4I)_1$
48	$(2+4I)_{f(2)}$
49	$(2+4I)_{f(3)}$
50	$(2+4I)_{f(4)}$

51	$(3+I)_0$
52	$(3+I)_1$
53	$(3+I)_{f(2)}$
54	$(3+I)_{f(3)}$
55	$(3+I)_{f(4)}$
56	$(3+3I)_0$
57	$(3+3I)_1$
58	$(3+3I)_{f(2)}$
59	$(3+3I)_{f(3)}$
60	$(3+3I)_{f(4)}$
61	$(3+4I)_0$
62	$(3+4I)_1$
63	$(3+4I)_{f(2)}$
64	$(3+4I)_{f(3)}$
65	$(3+4I)_{f(4)}$
66	$(4+2I)_0$
67	$(4+2I)_1$
68	$(4+2I)_{f(2)}$
69	$(4+2I)_{f(3)}$
70	$(4+2I)_{f(4)}$
71	$(4+3I)_0$
72	$(4+3I)_1$
73	$(4+3I)_{f(2)}$
74	$(4+3I)_{f(3)}$
75	$(4+3I)_{f(4)}$
76	$(4+4I)_0$
77	$(4+4I)_1$
78	$(4+4I)_{f(2)}$

79	$(4+4I)_{f(3)}$
80	$(4+4I)_{f(4)}$

Minimal units of $\left(Z_5(I)\right)_{f_I}$ in the case of $f(a)=f(b)\neq f(c)$; where $a,b,c\in\{2,3,4\}$:

	$M_u\left(Z_5(I)\right)_{f_I}$
1	1_0
2	1_1
3	$1_{f(a)}$
4	$1_{f(c)}$
5	2_0
6	2_1
7	$2_{f(a)}$
8	$2_{f(c)}$
9	3_0
10	3_1
11	$3_{f(a)}$
12	$3_{f(c)}$
13	4_0
14	4_1
15	$4_{f(a)}$
16	$4_{f(c)}$
17	$(1+I)_0$
18	$(1+I)_1$
19	$(1+I)_{f(a)}$
20	$(1+I)_{f(c)}$
21	$(1+2I)_0$
22	$(1+2I)_1$

23	$(1+2I)_{f(a)}$
24	$(1+2I)_{f(c)}$
25	$(4+4I)_0$
26	$(4+4I)_1$
27	$(4+4I)_{f(a)}$
28	$(4+4I)_{f(c)}$
29	$(1+3I)_0$
30	$(1+3I)_1$
31	$(1+3I)_{f(a)}$
32	$(1+3I)_{f(c)}$
33	$(2+I)_0$
34	$(2+I)_1$
35	$(2+I)_{f(a)}$
36	$(2+I)_{f(c)}$
37	$(2+2I)_0$
38	$(2+2I)_1$
39	$(2+2I)_{f(a)}$
40	$(2+2I)_{f(c)}$
41	$(2+4I)_0$
42	$(2+4I)_1$
43	$(2+4I)_{f(a)}$
44	$(2+4I)_{f(c)}$
45	$(3+I)_0$
46	$(3+I)_1$
47	$(3+I)_{f(a)}$
48	$(3+I)_{f(c)}$
49	$(3+3I)_0$
50	$(3+3I)_1$

51	$(3+3I)_{f(a)}$
52	$(3+3I)_{f(c)}$
53	$(3+4I)_0$
54	$(3+4I)_1$
55	$(3+4I)_{f(a)}$
56	$(3+4I)_{f(c)}$
57	$(4+2I)_0$
58	$(4+2I)_1$
59	$(4+2I)_{f(a)}$
60	$(4+2I)_{f(c)}$
61	$(4+3I)_0$
62	$(4+3I)_1$
63	$(4+3I)_{f(a)}$
64	$(4+3I)_{f(c)}$

Conclusion

In this paper, we defined the concept of minimal units in finite two-fold finite neutrosophic rings modulo integers as a generalization of classical elements of the group of units of the mentioned neutrosophic rings. We applied our results to characterize all minimal units in the following two-fold neutrosophic rings $\left(Z_n(I)\right)_{f_I}$ for $n \in \{2,3,4,5\}$.

In the future, we aim to classify all minimal units for higher orders cases $n \geq 6$.

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