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New Fixed Point Results in Neutrosophic Metric Spaces

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Abstract: In this manuscript, we give the generalization of banach's, Kannan's and Chatterjee's fixed Point theorems in neutrosophic metric spaces by using new (TS-IFα) contractive mappings. Also, we establish common fixed point results in neutrosophic metric space by using Occasionally weakly compatible maps for integral type inequalities.

Keywords: Fuzzy metric space; Neutrosophic metric space; Banach; Kannan; Chatterjee; Fixed point theorems.

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1. Introduction

Fixed point theory is an important tool to find the existence and uniqueness of solution of integral and differential equations. Researchers in [1-9] worked on several generalized fixed point results. After being given the notion of fuzzy sets by L. A. Zadeh [10], a large number of researchers provide many generalizations. In this continuation, Kramosil and Michalek [11] originated the approach of fuzzy metric spaces, George and Veeramani [12] diversify the approach of fuzzy metric spaces. Garbiec [13] tossed the fuzzy interpretation of Banach contraction principle in fuzzy metric spaces.

The approach of intuitionistic fuzzy metric spaces (IFMS) was tossed by Park in [14]. Kirişci and Simsek [5] tossed the approach of neutrosophic metric space (NMS). Simsek and Kirişci [6] and Sowndrarajan et al. [1] proved some fixed point (FP) results in the setting of NMS. Kiran and Khatoon [15] proved Kannan's and Chatterjee's FP results in the sense of IFMS by using (TS-IFα) mappings for $\alpha: \Omega \to [0,1)$. Patel et al. [9] prove common fixed point (FP) results in the sense of

Occasionally weakly compatible (OWC) maps on IFMS for integral type inequality. Authors in [16-20] worked on different generalizations of NMSs and proved several fixed point results.
In this manuscript, we give the generalization of banach's, Kannan's and Chatterjee's FP theorems in NMS by using new (TS-IFα) mappings and for $\alpha: \Omega \times \Omega \to [0,1)$. Also, we establish common fixed point results in NMS by using Occasionally weakly compatible maps for integral type inequality. Our results are more generalized in the existing literature.

Theorem 1.1 [8] Let $(\Omega, \psi, \phi, *, \circ)$ be complete IFMS and a mapping $h: \Omega \to \Omega$ is named an intuitionistic fuzzy contraction depends on α (IFC_α) if there exists a mapping $\alpha: \Omega \to [0,1)$ where $\alpha(h\varkappa) \leq \alpha(\varkappa)$ such that

$$\frac{1}{\psi(h\varkappa, h\mathrm{y}, t)} - 1 \leq \alpha(\varkappa)\left(\frac{1}{\psi(\varkappa, \mathrm{y}, t)}\right)$$

and

$$\phi(h\varkappa, h\mathrm{y}, t) \leq \alpha(\varkappa)\phi(\varkappa, \mathrm{y}, t)$$

$\forall\, \varkappa, \mathrm{y} \in \Omega$ and $t > 0$. Then h has a unique FP.

Theorem 1.2 [7] Let $(\Omega, \psi, \phi, *, \circ)$ be IFMS and a mapping $h: \Omega \to \Omega$ is named to be $(DC - IF)$ contraction mapping if $\exists\, \delta \in (0,1)$ such that

$$\delta\psi(h\varkappa, h\mathrm{y}, t) \geq \psi(\varkappa, \mathrm{y}, t), \qquad \frac{1}{\delta}\phi(h\varkappa, h\mathrm{y}, t) \leq \phi(\varkappa, \mathrm{y}, t)$$

$\forall\, \varkappa, \mathrm{y} \in \Omega$ and $t > 0$. Then h has a unique FP.

Definition 1.1 [5] Suppose $\Omega \neq \emptyset$, assume a six tuple $(\Omega, \psi, \phi, \theta, *, \circ)$ where $*$ is a continuous t-norm (CTN), $\circ$ is a continuous t-conorm (CTCN), ψ, ϕ and θ NS on $\Omega \times \Omega \times (0, \infty)$. If $(\Omega, \psi, \phi, \theta, *, \circ)$ meet the below circumstances for all $\varkappa, \mathrm{y}, v \in \Omega$ and $t, s > 0$:

(NS1) $\psi(\varkappa, \mathrm{y}, t) + \phi(\varkappa, \mathrm{y}, t) + \theta(\varkappa, \mathrm{y}, t) \leq 3,$

(NS2) $0 \leq \psi(\varkappa, \mathrm{y}, t) \leq 1,$

(NS3) $\psi(\varkappa, \mathrm{y}, t) = 1 \iff \varkappa = \mathrm{y},$

(NS4) $\psi(\varkappa, \mathrm{y}, t) = \psi(\mathrm{y}, \varkappa, t),$

(NS5) $\psi\big(\varkappa, v(t+s)\big) \geq \psi(\varkappa, \mathrm{y}, t)^{*}\psi(\mathrm{y}, v, s)$,

(NS6) $\psi(\varkappa, \mathrm{y}, \cdot): [0, \infty) \to [0,1]$ is a continuous,

(NS7) $\lim_{t\to\infty} \psi(\varkappa, \mathrm{y}, t) = 1$,

(NS8) $0 \leq \phi(\varkappa, \mathrm{y}, t) \leq 1$,

(NS9) $\phi(\varkappa, \mathrm{y}, t) = 0 \Leftrightarrow \varkappa = \mathrm{y}$,

(NS10) $\phi(\varkappa, \mathrm{y}, t) = \phi(\mathrm{y}, \varkappa, t)$,

(NS11) $\phi\big(\varkappa, v, (t+s)\big) \leq \phi(\varkappa, \mathrm{y}, t) \circ \phi(\mathrm{y}, v, s)$,

(NS12) $\phi(\varkappa, \mathrm{y}, \cdot): [0, \infty) \to [0,1]$ is a continuous,

(NS13) $\lim_{t\to\infty} \phi(\varkappa, \mathrm{y}, t) = 0$,

(NS14) $0 \leq \theta(\varkappa, \mathrm{y}, t) \leq 1$,

(NS15) $\theta(\varkappa, \mathrm{y}, t) = 0 \Leftrightarrow \varkappa = \mathrm{y}$,

(NS16) $\theta(\varkappa, \mathrm{y}, t) = \theta(\mathrm{y}, \varkappa, t)$,

(NS17) $\theta\big(\varkappa, v, (t+s)\big) \leq \theta(\varkappa, \mathrm{y}, t) \circ \theta(\mathrm{y}, v, s)$,

(NS18) $\theta(\varkappa, \mathrm{y}, \cdot): [0, \infty) \to [0,1]$ is a continuous,

(NS19) $\lim_{t\to\infty} \theta(\varkappa, \mathrm{y}, t) = 0$,

(NS20) If $t \leq 0$ then $\psi(\varkappa, \mathrm{y}, t) = 0, \phi(\varkappa, \mathrm{y}, t) = 1, \theta(\varkappa, \mathrm{y}, t) = 1$.

Then $(\Omega, \psi, \phi, \theta)$ Neutrosophic metric on Ω and $(\Omega, \psi, \phi, \theta, *, \circ)$ is an NMS.

Definition 1.2 [6] Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be an NMS. Then

(i) a sequence $\{\varkappa_n\}$ in Ω is named to be G-Cauchy sequence (GCS) if and only if for all

$t > 0$ and $\mathrm{m} > 0$,

$\lim_{n\to\infty} \psi(\varkappa_n, \varkappa_{n+m}, t) = 1$, $\lim_{n\to\infty} \phi(\varkappa_n, \varkappa_{n+m}, t) = 0$ and $\lim_{n\to\infty} \theta(\varkappa_n, \varkappa_{n+m}, t) = 0$

.

(ii) a sequence $\{\varkappa_n\}$ in Ω is named to be G-convergent (GC) to $\varkappa$ in Ω, if and only if for all $t > 0$,

$$\lim_{n\to\infty} \psi(\varkappa_n, \varkappa, t) = 1, \lim_{n\to\infty} \phi(\varkappa_n, \varkappa, t) = 0 \text{ and } \lim_{n\to\infty} \theta(\varkappa_n, \varkappa, t) = 0.$$

(iii) an ENMS is named to be complete iff each GCS is convergent.

Definition 1.3 [2] A self mappings pair (h, g) of IFMS is named to be weakly compatible if they commute at the coincident points i.e. $hu = gu$ for some $u \in \Omega$, then $hgu = ghu$.

Definition 1.4 [2] Let $(\Omega, \psi, \phi, *, \circ)$ be an IFMS. $h, g: \Omega \to \Omega$. A point $\varkappa \in \Omega$ is called coincident point of h and g if and only if $h\varkappa = g\varkappa$.

Definition 1.5 [3] A self mappings pair (h, g) of IFMS is named to be OWC iff there is a point $\varkappa \in \Omega$ is coincident point if it commutes at the coincident points h and g at which h and g commute.

Lemma 1.1 [3] Let $(\Omega, \psi, \phi, *, \circ)$ be an IFMS. $h, g: \Omega \to \Omega$ and h, g have unique coincident point, $w = h\varkappa = g\varkappa$, then h and g has a unique common FP w.

Lemma 1.2 [4] Let $(\Omega, \psi, \phi, *, \circ)$ be an IFMS and $\forall\ \varkappa, y \in \Omega, t > 0$ and $k \in (0,1)$ such that $\psi(\varkappa, y, kt) \geq \psi(\varkappa, y, t)$ and $\phi(\varkappa, y, kt) \leq \phi(\varkappa, y, t)$ then $\varkappa = y$.

2. Main Section-I

In this section, we give some theorems by using new (TS-IFα) mapping, also examine a result with example.

Theorem 2.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a G-complete NMS and $h: \Omega \to \Omega$ be (TS-IFα) mapping, i.e., if $\exists\ \alpha: \Omega \times \Omega \to [0,1)$ where

$\alpha(h\varkappa, h\mathrm{y}) \leq \alpha(\varkappa, \mathrm{y})$

such that

$\alpha(\varkappa, \mathrm{y})\psi(h(\varkappa), h(\mathrm{y}), t) \geq \psi(\varkappa, \mathrm{y}, t),$

$$\frac{1}{\alpha(\varkappa, \mathrm{y})}\phi(h(\varkappa), h(\mathrm{y}), t) \leq \phi(\varkappa, \mathrm{y}, t)$$

and

$$\frac{1}{\alpha(\varkappa, \mathrm{y})}\theta(h(\varkappa), h(\mathrm{y}), t) \leq \theta(\varkappa, \mathrm{y}, t)$$

$\forall\, t > 0.$ Then h has a unique fixed point (FP).

Proof: Let $\varkappa_0 \in \Omega$ be a random point. We build a sequence $\varkappa_m \in \Omega$ by

$$\varkappa_m = h^m \varkappa_0 = h\varkappa_{m-1}$$

For $m \in \mathbb{N}$. Now $\forall\, t > 0$ we obtain

$$\psi(\varkappa_{m+1}, \varkappa_m, t) \geq \frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m}\psi(\varkappa_1, \varkappa_0, t)$$

Now for $m, n \in \mathbb{N}$ such that $n \geq m,$ we deduce

$$\psi(\varkappa_m, \varkappa_{n+p}, t) \geq \frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m}\psi\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)$$

Hence, we get

$$1 \leq \lim_{m\to\infty}\left(\frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m}\psi\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)\right) \leq \lim_{m\to\infty}\psi(\varkappa_m, \varkappa_{n+p}, t) \leq 1$$

That is

$\lim_{m\to\infty}\psi(\varkappa_m, \varkappa_{n+p}, t) = 1.$

Again for $m \in \mathbb{N}$. Now $\forall\, t > 0$ we obtain

$$\phi(\varkappa_{m+1}, \varkappa_m, t) \leq \left(\alpha(\varkappa_1, \varkappa_0)\right)^m\phi(\varkappa_1, \varkappa_0, t)$$

Now for $m, n \in \mathbb{N}$ such that $n \geq m,$ we deduce

$$\phi(\varkappa_m, \varkappa_{n+p}, t) \leq \frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m} \phi\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)$$

Hence, we get

$$0 \geq \lim_{m\to\infty}\left(\frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m} \phi\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)\right) \geq \lim_{m\to\infty} \phi(\varkappa_m, \varkappa_{n+p}, t) \geq 0$$

This implies

$$\lim_{m\to\infty} \phi(\varkappa_m, \varkappa_{n+p}, t) = 0$$

Similarly, for $m \in \mathbb{N}$. Now $\forall\, t > 0$ we obtain

$$\theta(\varkappa_{m+1}, \varkappa_m, t) \leq \left(\alpha(\varkappa_1, \varkappa_0)\right)^m \theta(\varkappa_1, \varkappa_0, t)$$

Now for $m, n \in \mathbb{N}$ such that $n \geq m$, we deduce

$$\theta(\varkappa_m, \varkappa_{n+p}, t) \leq \frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m} \theta\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)$$

Hence, we get

$$0 \geq \lim_{m\to\infty}\left(\frac{1}{\left(\alpha(\varkappa_1, \varkappa_0)\right)^m} \theta\left(\varkappa_1, \varkappa_0, \frac{t}{p}\right)\right) \geq \lim_{m\to\infty} \theta(\varkappa_m, \varkappa_{n+p}, t) \geq 0$$

This implies

$$\lim_{m\to\infty} \theta(\varkappa_m, \varkappa_{n+p}, t) = 0.$$

Hence, $\{\varkappa_m\} \in \Omega$ GSC sequence. Due to the completeness of Ω there exists $v \in \Omega$ such that

$\varkappa_m \to v$ as $m \to \infty$. We have

$$\lim_{m\to\infty} \psi(\varkappa_m, v, t) = 1,$$

$$\lim_{m\to\infty} \phi(\varkappa_m, v, t) = 0$$

And

$$\lim_{m\to\infty} \theta(\varkappa_m, v, t) = 0.$$

Now we examine that v is a FP of h. Therefore, h is (TS-IFα) $\forall\, m \in \mathbb{N}$ we obtain

$$\psi(hv, h\varkappa_m, t) \geq \frac{1}{\alpha(v, \varkappa_m)} \psi(v, \varkappa_m, t)$$

$$\lim_{m\to\infty}\psi(hv,h\varkappa_m,t)\geq\lim_{m\to\infty}\frac{1}{\alpha(v,\varkappa_m)}\psi(v,\varkappa_m,t)=\frac{1}{\alpha(v,v)}>1$$

This implies

$$1<\lim_{m\to\infty}\psi(hv,h\varkappa_m,t)\leq 1$$

Hence

$$\lim_{m\to\infty}\psi(hv,h\varkappa_m,t)=1$$

Again, we have

$$\phi(hv,h\varkappa_m,t)\leq\alpha(v,\varkappa_m)\phi(v,\varkappa_m,t)$$

Now for $t>0,\forall\, m\in\mathbb{N}$, we get

$$\lim_{m\to\infty}\phi(hv,h\varkappa_m,t)=0$$

Similarly, we have

$$\theta(hv,h\varkappa_m,t)\leq\alpha(v,\varkappa_m)\theta(v,\varkappa_m,t)$$

Now for $t>0,\forall\, m\in\mathbb{N}$, we get

$$\lim_{m\to\infty}\theta(hv,h\varkappa_m,t)=0.$$

Hence, $h\varkappa_m\to hv$, this implies $v=hv$. Now we examine the uniqueness of FP. Let v_1 be another FP. We have

$$\psi(v,v_1,t)=\psi(hv,hv_1,t)\geq\frac{1}{\left(\alpha(v,v_1)\right)^m}\psi(v,v_1,t)\to 1 \text{ as } m\to\infty$$

This implies

$$1<\lim_{m\to\infty}\frac{1}{\left(\alpha(v,v_1)\right)^m}\psi(v,v_1,t)\leq\psi(v,v_1,t)\leq 1$$

Hence,

$$\psi(v,v_1,t)=1$$

Again, we have

$$\phi(v,v_1,t)=\phi(hv,hv_1,t)\leq\left(\alpha(v,v_1)\right)^m\phi(v,v_1,t)\to 0 \text{ as } m\to\infty$$

This implies

$$0\leq\phi(v,v_1,t)\leq\lim_{m\to\infty}\left(\alpha(v,v_1)\right)^m\phi(v,v_1,t)<0$$

Hence

$\phi(v, v_1, t) = 0$

Similarly, we have

$$\theta(v, v_1, t) = \theta(hv, hv_1, t) \leq (\alpha(v, v_1))^m \theta(v, v_1, t) \to 0 \text{ as } m \to \infty$$

This implies

$$0 \leq \theta(v, v_1, t) \leq \lim_{m\to\infty} (\alpha(v, v_1))^m \theta(v, v_1, t) < 0$$

Hence

$\theta(v, v_1, t) = 0.$

This examine that $v = v_1$.

Example 2.1 Let $\Omega = [0,1]$. Define $\alpha: \Omega \times \Omega \to [0,1)$ by

$$\alpha(\varkappa) = \begin{cases} 0 & \text{if } \varkappa = 0 \text{ or } y = 0, \\ \dfrac{1}{(\max\{\varkappa, y\})^2} & \text{if otherwise} \end{cases} \quad \forall \varkappa, y \in \Omega$$

and define a G-complete NMS in [1] by

$$\psi(\varkappa, y, t) = \frac{t}{t + |\varkappa - y|}, \phi(\varkappa, y, t) = \frac{|\varkappa - y|}{t + |\varkappa - y|} \text{ and } \theta(\varkappa, y, t) = \frac{|\varkappa - y|}{t}$$

Also define $h: \Omega \to \Omega$ by

$$h(\varkappa) = \begin{cases} 0 & \text{if } \varkappa = 0, \\ \dfrac{1}{\varkappa} & \text{if } \varkappa \in (0,1] \end{cases}$$

Then the mapping is a new (TS-IFα) contractive mapping.

We have for cases:

(i) If $\varkappa = y = 0$, then $h\varkappa = hy = 0$;

(ii) If $\varkappa = 0$ and $y \in (0,1]$, then $h\varkappa = 0$ and $hy = \frac{1}{y}$;

(iii) If $y = 0$ and $\varkappa \in (0,1]$, then $hy = 0$ and $h\varkappa = \frac{1}{\varkappa}$;

(iv) If $\varkappa, y \in (0,1]$, then $h\varkappa = \frac{1}{\varkappa}$ and $hy = \frac{1}{y}$;

Then all the circumstances of theorem 2.1 are fulfilled and 0 is a unique FP of h.

Theorem 2.2 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a G-complete NMS and $h: \Omega \to \Omega$ be a contractive mapping such that ∃ $\alpha: \Omega \times \Omega \to \left(0, \frac{1}{2}\right)$, where

$$\alpha(hx, hy) \leq \alpha(x, y)$$

Such that

$$\alpha(x, y)\psi(h(x), h(y), t) \geq [\psi(x, hx, t) + \psi(y, hy, t)],$$

$$\frac{1}{\alpha(x, y)}\phi(h(x), h(y), t) \leq [\phi(x, hx, t) + \phi(y, hy, t)]$$

And

$$\frac{1}{\alpha(x, y)}\theta(h(x), h(y), t) \leq [\theta(x, hx, t) + \theta(y, hy, t)]$$

$\forall\, t > 0.$ Then h has a unique FP.

Theorem 2.3 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a G-complete NMS and $h: \Omega \to \Omega$ be a contractive mapping such that ∃ $\alpha: \Omega \times \Omega \to \left(0, \frac{1}{2}\right)$, where

$$\alpha(hx, hy) \leq \alpha(x, y)$$

Such that

$$\alpha(x, y)\psi(h(x), h(y), t) \geq [\psi(x, hy, t) + \psi(y, hx, t)],$$

$$\frac{1}{\alpha(x, y)}\phi(h(x), h(y), t) \leq [\phi(x, hy, t) + \phi(y, hx, t)]$$

And

$$\frac{1}{\alpha(x, y)}\theta(h(x), h(y), t) \leq [\theta(x, hy, t) + \theta(y, hx, t)]$$

$\forall\, t > 0.$ Then h has a unique FP.

Theorem 2.4 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a G-complete NMS and $h: \Omega \to \Omega$ be a contractive mapping such that ∃ $\alpha: \Omega \times \Omega \to [0,1)$, where

$$\alpha(hx, hy) \leq \alpha(x, y)$$

Such that

$$\frac{1}{\psi(hx, hy, t)} - 1 \leq \alpha(x, y)\left(\frac{1}{\psi(x, y, t)}\right),$$

$\phi(h\varkappa, hy, t) \leq \alpha(\varkappa, y)\phi(\varkappa, y, t)$

and

$\theta(h\varkappa, hy, t) \leq \alpha(\varkappa, y)\theta(\varkappa, y, t)$

$\forall\ \varkappa, y \in \Omega$ and $t > 0$. Then h has a unique FP.

Theorem 2.2 and 2.3 are the generalizations of Kannan's and Chatterjee's FP theorems in NMS. Theorem 2.4 is neutrosophic contraction mapping theorem. We can prove easily by using theorem 2.1.

3. Main Section-II

Definition 3.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a NMS. $h, g: \Omega \to \Omega$. A point $\varkappa \in \Omega$ is called coincident point of h and g if and only if $h\varkappa = g\varkappa$.

Definition 3.2 A self mappings pair (h, g) of a NMS is named to be weakly compatible if they commute at the coincident points i.e. $hu = gu$ for some $u \in \Omega$, then $hgu = ghu$.

Definition 3.3 A self mappings pair (h, g) of NMS is named to be OWC iff there is a point $\varkappa \in \Omega$ is coincident point if it commutes at the coincident points h and g at which h and g commute.

Lemma 3.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be a NMS. $h, g: \Omega \to \Omega$ and h, g have unique coincident point, $w = h\varkappa = g\varkappa$, then h and g has a unique common FP w.

Proof easily follows from [9] and [3].

Lemma 3.2 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be an NMS and $\forall\ \varkappa, y \in \Omega, t > 0$ and $k \in (0,1)$ such that $\psi(\varkappa, y, kt) \geq \psi(\varkappa, y, t), \phi(\varkappa, y, kt) \leq \phi(\varkappa, y, t)$ and $\theta(\varkappa, y, kt) \leq \theta(\varkappa, y, t)$ then $\varkappa = y$.

Proof easily follows from [4] and [9].

Theorem 3.1 Let $(\Omega, \psi, \phi, \theta, *, \circ)$ be an NMS with CTM $*$ and CTCN $\circ$. Let $A, B, C, D: \Omega \to \Omega$ and the pairs (A, C) and (B, D) are OWC. If $\exists\ k \in (0,1)$ such that

$$\int_0^{\psi(A\varkappa, By, kt)} f(t)dt \geq \int_0^{\min\left\{\begin{array}{c}\psi(C\varkappa, Dy, t), \psi(By, C\varkappa, t), \psi(C\varkappa, A\varkappa, t), \psi(By, Dy, t), \\ \psi(A\varkappa, Dy, t), \left(\frac{\psi(C\varkappa, A\varkappa, t)}{\psi(By, Dy, t)}\right)\end{array}\right\}} f(t)dt,$$

$$\int_0^{\phi(A\varkappa,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{matrix}\phi(C\varkappa,Dy,t),\phi(By,C\varkappa,t),\phi(C\varkappa,A\varkappa,t),\phi(By,Dy,t),\\ \phi(A\varkappa,Dy,t).\left(\frac{\phi(C\varkappa,A\varkappa,t)}{\phi(By,Dy,t)}\right)\end{matrix}\right\}} f(t)dt$$

And

$$\int_0^{\theta(A\varkappa,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{matrix}\theta(C\varkappa,Dy,t),\theta(By,C\varkappa,t),\theta(C\varkappa,A\varkappa,t),\theta(By,Dy,t),\\ \theta(A\varkappa,Dy,t).\left(\frac{\theta(C\varkappa,A\varkappa,t)}{\theta(By,Dy,t)}\right)\end{matrix}\right\}} f(t)dt$$

$\forall\ \varkappa, y \in \Omega$ and $t > 0$. *Then, there exists a unique FP of* A, B, C and D.

Proof: *Because pairs* (A, C) and (B, D) are OWC, so there exist $\varkappa, y \in \Omega$ such that $A\varkappa = C\varkappa$ and $By = Dy$. We claim that $A\varkappa = By$, we have

$$\int_0^{\psi(A\varkappa,By,kt)} f(t)dt \geq \int_0^{min\left\{\begin{matrix}\psi(A\varkappa,By,t),\psi(By,A\varkappa,t),\psi(A\varkappa,A\varkappa,t),\psi(By,By,t),\\ \psi(A\varkappa,By,t).\left(\frac{\psi(A\varkappa,A\varkappa,t)}{\psi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\int_0^{\psi(By,By,kt)} f(t)dt \geq \int_0^{min\left\{\begin{matrix}\psi(By,By,t),\psi(By,By,t),\psi(By,By,t),\psi(By,By,t),\\ \psi(By,By,t).\left(\frac{\psi(By,By,t)}{\psi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\geq \int_0^{\psi(By,By,t)} f(t)dt,$$

$$\int_0^{\phi(A\varkappa,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{matrix}\phi(A\varkappa,By,t),\phi(By,A\varkappa,t),\phi(A\varkappa,A\varkappa,t),\phi(By,By,t),\\ \phi(A\varkappa,By,t).\left(\frac{\phi(A\varkappa,A\varkappa,t)}{\phi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\int_0^{\phi(By,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{matrix}\phi(By,By,t),\phi(By,By,t),\phi(By,By,t),\phi(By,By,t),\\ \phi(By,By,t).\left(\frac{\phi(By,By,t)}{\phi(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\leq \int_0^{\phi(By,By,t)} f(t)dt$$

And

$$\int_0^{\theta(A\varkappa,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{matrix}\theta(A\varkappa,By,t),\theta(By,A\varkappa,t),\theta(A\varkappa,A\varkappa,t),\theta(By,By,t),\\ \theta(A\varkappa,By,t).\left(\frac{\theta(A\varkappa,A\varkappa,t)}{\theta(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\int_0^{\theta(By,By,kt)} f(t)dt \leq \int_0^{max\left\{\begin{matrix}\theta(By,By,t),\theta(By,By,t),\theta(By,By,t),\theta(By,By,t),\\ \theta(By,By,t).\left(\frac{\theta(By,By,t)}{\theta(By,By,t)}\right)\end{matrix}\right\}} f(t)dt$$

$$\leq \int_0^{\theta(By,By,t)} f(t)dt$$

Hence, from lemma 3.2, $A\varkappa = By$ i.e $A\varkappa = C\varkappa = By = Dy$. *Assume* v *be another point such that* $Av = Cv$ *then, we obtain* $Av = Cv = By = Dy$ so $A\varkappa = Av$ and $w = A\varkappa = C\varkappa$ *is the unique point of* A and C. *From lemma 3.1, only* w *is common FP of* A *and* C. *Likewise, there is* $v \in \Omega$ *a unique point such that* $v = Bv = Dv$. *Now we examine that* $v = w$.

$$\int_0^{\psi(w,v,kt)} f(t)dt = \int_0^{\psi(Aw,Bv,kt)} f(t)dt$$

$$\geq \int_0^{min\left\{\substack{\psi(Cw,Dv,t),\psi(Bv,Cw,t),\psi(Cw,Aw,t),\psi(Bv,Dv,t), \\ \psi(Aw,Dv,t).\left(\frac{\psi(Cw,Aw,t)}{\psi(Bv,Dv,t)}\right)}\right\}} f(t)dt$$

$$= \int_0^{min\left\{\substack{\psi(w,v,t),\psi(v,w,t),\psi(w,w,t),\psi(v,v,t), \\ \psi(w,v,t).\left(\frac{\psi(w,w,t)}{\psi(v,v,t)}\right)}\right\}} f(t)dt$$

$$= \int_0^{min\{\psi(w,v,t),\psi(v,w,t),1,1,\psi(w,v,t)\}} f(t)dt = \int_0^{\psi(w,v,t)} f(t)dt,$$

$$\int_0^{\phi(w,v,kt)} f(t)dt = \int_0^{\phi(Aw,Bv,kt)} f(t)dt$$

$$\leq \int_0^{max\left\{\substack{\phi(Cw,Dv,t),\phi(Bv,Cw,t),\phi(Cw,Aw,t),\phi(Bv,Dv,t), \\ \phi(Aw,Dv,t).\left(\frac{\phi(Cw,Aw,t)}{\phi(Bv,Dv,t)}\right)}\right\}} f(t)dt$$

$$= \int_0^{max\left\{\substack{\phi(w,v,t),\phi(v,w,t),\phi(w,w,t),\phi(v,v,t), \\ \phi(w,v,t).\left(\frac{\phi(w,w,t)}{\phi(v,v,t)}\right)}\right\}} f(t)dt$$

$$= \int_0^{max\{\phi(w,v,t),\phi(v,w,t),1,1,\phi(w,v,t)\}} f(t)dt = \int_0^{\phi(w,v,t)} f(t)dt$$

And

$$\int_0^{\theta(w,v,kt)} f(t)dt = \int_0^{\theta(Aw,Bv,kt)} f(t)dt$$

$$\leq \int_0^{max\left\{\substack{\theta(Cw,Dv,t),\theta(Bv,Cw,t),\theta(Cw,Aw,t),\theta(Bv,Dv,t), \\ \theta(Aw,Dv,t).\left(\frac{\theta(Cw,Aw,t)}{\theta(Bv,Dv,t)}\right)}\right\}} f(t)dt$$

$$= \int_0^{\max\left\{\begin{array}{c}\Theta(w,v,t),\Theta(v,w,t),\Theta(w,w,t),\Theta(v,v,t),\\ \Theta(w,v,t).\left(\frac{\Theta(w,w,t)}{\Theta(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\{\Theta(w,v,t),\Theta(v,w,t),1,1,\Theta(w,v,t)\}} f(t)dt = \int_0^{\Theta(w,v,t)} f(t)dt$$

Hence, from lemma 3.2, $w = v$. *That is,* v *is a common FP of* A, B, C *and* D. *Now assume another common FP* u *of* A, B, C *and* D *for examining the uniqueness. Then*

$$\int_0^{\psi(u,v,kt)} f(t)dt = \int_0^{\psi(Au,Bv,kt)} f(t)dt$$

$$\geq \int_0^{\min\left\{\begin{array}{c}\psi(Cu,Dv,t),\psi(Bv,Cu,t),\psi(Cu,Au,t),\psi(Bv,Dv,t),\\ \psi(Au,Dv,t).\left(\frac{\psi(Cu,Au,t)}{\psi(Bv,Dv,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\min\left\{\begin{array}{c}\psi(u,v,t),\psi(v,u,t),\psi(u,u,t),\psi(v,v,t),\\ \psi(u,v,t).\left(\frac{\psi(u,u,t)}{\psi(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\min\{\psi(u,v,t),\psi(v,u,t),1,1,\psi(u,v,t)\}} f(t)dt = \int_0^{\psi(u,v,t)} f(t)dt,$$

$$\int_0^{\phi(u,v,kt)} f(t)dt = \int_0^{\phi(Au,Bv,kt)} f(t)dt$$

$$\leq \int_0^{\max\left\{\begin{array}{c}\phi(Cu,Dv,t),\phi(Bv,Cu,t),\phi(Cu,Au,t),\phi(Bv,Dv,t),\\ \phi(Au,Dv,t).\left(\frac{\phi(Cu,Au,t)}{\phi(Bv,Dv,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\left\{\begin{array}{c}\phi(u,v,t),\phi(v,u,t),\phi(u,u,t),\phi(v,v,t),\\ \phi(u,v,t).\left(\frac{\phi(u,u,t)}{\phi(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\{\phi(u,v,t),\phi(v,u,t),1,1,\phi(u,v,t)\}} f(t)dt = \int_0^{\phi(u,v,t)} f(t)dt$$

And

$$\int_0^{\Theta(u,v,kt)} f(t)dt = \int_0^{\Theta(Au,Bv,kt)} f(t)dt$$

$$\leq \int_0^{\max\left\{\begin{array}{c}\Theta(Cu,Dv,t),\Theta(Bv,Cu,t),\Theta(Cu,Au,t),\Theta(Bv,Dv,t),\\ \Theta(Au,Dv,t).\left(\frac{\Theta(Cu,Au,t)}{\Theta(Bv,Dv,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\left\{\begin{array}{c}\theta(u,v,t),\theta(v,u,t),\theta(u,u,t),\theta(v,v,t),\\ \theta(u,v,t).\left(\frac{\theta(u,u,t)}{\theta(v,v,t)}\right)\end{array}\right\}} f(t)dt$$

$$= \int_0^{\max\{\theta(u,v,t),\theta(v,u,t),1,1,\theta(u,v,t)\}} f(t)dt = \int_0^{\theta(u,v,t)} f(t)dt$$

Hence, from lemma 3.2, $v = w$. *Hence* v *is a unique common FP.*

Corollary 3.1 Let $(\Omega,\psi,\phi,\theta,*,\circ)$ be a NMS with CTM $a * \mathrm{b} = \min\{\mathrm{a},\mathrm{b}\}$ and CTCN $\mathrm{a} \circ \mathrm{b} = \max\{\mathrm{a},\mathrm{b}\}$. Let $A,B,C,D:\Omega \to \Omega$ and the pairs (A,C) and (B,D) are OWC. If $\exists\, k \in (0,1)$ such that

$$\int_0^{\psi(Aϰ,By,kt)} f(t)dt \geq \int_0^{\min\left\{\begin{array}{c}\psi(Cϰ,Dy,t)*\psi(By,Cϰ,t)*\psi(Cϰ,Aϰ,t)*\psi(By,Dy,t)*\\ \psi(Aϰ,Dy,t).\left(\frac{1+\psi(Cϰ,Aϰ,t)}{1+\psi(By,Dy,t)}\right)\end{array}\right\}} f(t)dt,$$

$$\int_0^{\phi(Aϰ,By,kt)} f(t)dt \leq \int_0^{\max\left\{\begin{array}{c}\phi(Cϰ,Dy,t)\circ\phi(By,Cϰ,t)\circ\phi(Cϰ,Aϰ,t)\circ\phi(By,Dy,t)\circ\\ \phi(Aϰ,Dy,t).\left(\frac{1+\phi(Cϰ,Aϰ,t)}{1+\phi(By,Dy,t)}\right)\end{array}\right\}} f(t)dt$$

And

$$\int_0^{\theta(Aϰ,By,kt)} f(t)dt \leq \int_0^{\max\left\{\begin{array}{c}\theta(Cϰ,Dy,t)\circ\theta(By,Cϰ,t)\circ\theta(Cϰ,Aϰ,t)\circ\theta(By,Dy,t)\circ\\ \theta(Aϰ,Dy,t).\left(\frac{1+\theta(Cϰ,Aϰ,t)}{1+\theta(By,Dy,t)}\right)\end{array}\right\}} f(t)dt$$

$\forall\, ϰ, \mathrm{y} \in \Omega$ *and* $t > 0$. *Then, there exists a unique FP of* A, B, C *and* D.

Proof: *Easily can prove on the lines of theorem 3.1.*

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