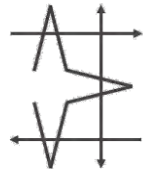


Neutrosophic Sets and Systems

<A> <neutA> <antiA>

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Information for Authors and Subscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea $\langle A \rangle$ together with its opposite or negation $\langle \text{anti}A \rangle$ and with their spectrum of neutralities $\langle \text{neut}A \rangle$ in between them (i.e. notions or ideas supporting neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$). The $\langle \text{neut}A \rangle$ and $\langle \text{anti}A \rangle$ ideas together are referred to as $\langle \text{non}A \rangle$.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on $\langle A \rangle$ and $\langle \text{anti}A \rangle$ only).

According to this theory every idea $\langle A \rangle$ tends to be neutralized and balanced by $\langle \text{anti}A \rangle$ and $\langle \text{non}A \rangle$ ideas - as a state of equilibrium.

In a classical way $\langle A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{anti}A \rangle$ are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that $\langle A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{anti}A \rangle$ (and $\langle \text{non}A \rangle$ of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]0, 1[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the $\langle \text{neut}A \rangle$, which means neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$.

$\langle \text{neut}A \rangle$, which of course depends on $\langle A \rangle$, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and "**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.

FOLDOC Dictionary of Computing (1, 2), Webster

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Dictionary (2), Chinese Youdao Dictionary (2) etc. have included these scientific neologisms.



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Internal Neutrosophic Spiral Dynamics: Connecting Human Neutrosophic Evolution with the Law of Internal Breakdown of Dynamic Systems

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Abstract

These two bellow ideas [1 & 2] seem to fit together remarkably well and may actually be viewed as **two complementary laws of neutrosophic dynamics**:

1. **Law of Internal Breakdown of a Neutrosophic Dynamic System** (2017)
"It is easier to break a system from inside than from outside."
2. **Law of Spiral Neutrosophic Human Evolution** (2019)
Human beings evolve through successive neutrosophic cycles, combining progress, regress, uncertainty, and transcendence, moving in a spiral rather than a straight line.

The strongest conceptual comparisons of them are likely with:

- Ludwig von Bertalanffy (General System Theory),
- Ilya Prigogine (self-organization),
- Edward Lorenz (chaos and attractors),
- Pierre Teilhard de Chardin (spiral/spiritual evolution),
- Arnold J. Toynbee (civilizational cycles),

because they provide the closest non-neutrosophic parallels to our proposed Internal Neutrosophic Spiral Dynamics framework.

1. Unified Interpretation

The first theory describes the **destructive dynamics** of a system.

The second theory describes the **constructive dynamics** of a human being.

Together they suggest:

Every neutrosophic system evolves through a spiral trajectory, but the direction of the spiral depends on the balance between internal constructive and internal destructive forces.

Symbolically:

$$Evolution = Internal Construction - Internal Destruction$$

or in neutrosophic form:

$$E = (T, I, F)$$

where

- T = constructive/evolutionary tendencies,
- I = indeterminate tendencies,
- F = destructive tendencies.

2. A New Principle

One may formulate:

Principle of Internal Neutrosophic Dominance

For any complex neutrosophic dynamic system,

$$| Internal Influence | > | External Influence |$$

over sufficiently long periods.

Meaning:

- civilizations are mainly destroyed from within;
- organizations collapse from within;
- individuals evolve from within;
- spiritual growth starts from within.

This connects both theories naturally.

3. Spiral Evolution versus Spiral Devolution

Our spiral model may be extended.

Instead of a single spiral, we may have:

Ascending Spiral



where

$$T > F.$$

The human approaches higher consciousness.

Descending Spiral

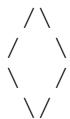


where

$$F > T.$$

The human deteriorates morally, intellectually, spiritually.

Oscillating Spiral



where T and F alternate.

Most people probably belong here.

4. Comparison with Other Theories:

Darwinian Evolution

Darwin:

$$Evolution = External Adaptation$$

through natural selection.

Our model:

$$\textit{Evolution} = \textit{External Factors} + \textit{Internal Consciousness} + \textit{Indeterminacy}$$

The neutrosophic spiral is broader because it includes:

- biology,
- psychology,
- spirituality,
- uncertainty.

Hegel's Dialectics

Hegel:

$$\textit{Thesis} \rightarrow \textit{Antithesis} \rightarrow \textit{Synthesis}$$

Our spiral:

$$(T, I, F) \rightarrow (T', I', F')$$

with no final synthesis required.

Contradictions remain partially unresolved.

This is closer to Neutrosophy.

Teilhard de Chardin

Teilhard viewed evolution as movement toward an Omega Point.

Our theory differs because:

- evolution is not necessarily monotonic;
- regressions occur;
- indeterminacy is fundamental;
- multiple spirals may coexist.

Systems Theory

Classical systems theory studies feedback loops.

Our 2017 law introduces:

Internal Critical Threshold

$$m_S = \tau_{DN}$$

Beyond this threshold the system breaks from inside.

Combining with spiral evolution yields:

Evolutionary Threshold

There may exist a critical level

$$\tau_E$$

beyond which a human jumps to a higher spiral.

5. Possible New Theorem

The Spiral Stability Theorem

Let

$$S(t)$$

be a neutrosophic dynamic system.

Let

$$T(t), I(t), F(t)$$

represent constructive, indeterminate, and destructive components.

Then:

- if

$$\int T(t) dt > \int F(t) dt$$

the system follows an ascending spiral;

- if

$$\int T(t) dt < \int F(t) dt$$

the system follows a descending spiral;

- if approximately equal,

$$\int T(t) dt \approx \int F(t) dt$$

the system oscillates around a neutral spiral.

This could become a formal theorem in a future paper.

6. Applications

A. Civilization Dynamics

Roman Empire.

External enemies were weaker than internal corruption.

Thus:

$$F_{inside} > F_{outside}$$

which agrees with the 2017 law.

Civilizations therefore evolve through spirals:

growth $\rightarrow$ stagnation $\rightarrow$ decay $\rightarrow$ rebirth.

B. Education

A student evolves not mainly because of teachers.

The dominant factor is internal motivation.

Hence:

$$Growth_{inside} > Growth_{outside}$$

The same principle appears in reverse for failure.

C. Artificial Intelligence

An AI system can be attacked externally.

Yet:

- corrupted training data,
- malicious insiders,
- poisoned models

often cause greater damage.

This is a direct analogue of the internal-breakdown law.

AI development itself follows spiral evolution:

learning,
forgetting,
re-learning,
self-correction.

D. Organizations

Companies usually fail because of:

- bad management,
- internal conflicts,
- corruption,

rather than competitors.

Thus:

$$Failure_{inside} > Failure_{outside}$$

E. Spiritual Development

This may be the deepest application.

The 2019 theory says:

"The Divine is in the Man."

The 2017 theory implies:

The greatest obstacle is also in the man.

Hence:

$$\textit{Human} = \textit{Potential Divinity} + \textit{Potential Self Destruction} + \textit{Indeterminacy}$$

which is a perfectly neutrosophic formulation.

7. A New Unified Model

I would propose naming the synthesis:

Internal Neutrosophic Spiral Dynamics (INSD)

Definition.

A neutrosophic dynamic system evolves along an ascending or descending spiral according to the competition between its internal constructive, destructive, and indeterminate forces.

Symbolically:

$$\textit{INSD} = (T, I, F)_{\textit{internal}} \rightarrow \textit{Spiral Trajectory}$$

The 2017 paper provides the **mechanism of collapse**.

The 2019 paper provides the **mechanism of evolution**.

Together they form a unified theory of:

- personal growth,
- social development,
- organizational behavior,
- civilization cycles,
- spiritual evolution,
- artificial intelligence evolution.

The connection within the Internal Neutrosophic Spiral Dynamics is strong between the Human Neutrosophic Evolution and the Law of Internal Breakdown of Dynamic Systems.

These two ideas seem to fit together remarkably well and may actually be viewed as **two complementary laws of neutrosophic dynamics**:

1. **Law of Internal Breakdown of a Neutrosophic Dynamic System** (2017)
"It is easier to break a system from inside than from outside."
2. **Law of Spiral Neutrosophic Human Evolution** (2019)
Human beings evolve through successive neutrosophic cycles, combining progress, regress, uncertainty, and transcendence, moving in a spiral rather than a straight line.

1. Unified Interpretation

The first theory describes the **destructive dynamics** of a system.

The second theory describes the **constructive dynamics** of a human being.

Together they suggest:

Every neutrosophic system evolves through a spiral trajectory, but the direction of the spiral depends on the balance between internal constructive and internal destructive forces.

Symbolically:

$$Evolution = Internal Construction - Internal Destruction$$

or in neutrosophic form:

$$E = (T, I, F)$$

where

- T = constructive/evolutionary tendencies,
- I = indeterminate tendencies,
- F = destructive tendencies.

2. A New Principle

One may formulate:

Principle of Internal Neutrosophic Dominance

For any complex neutrosophic dynamic system,

$$| Internal Influence | > | External Influence |$$

over sufficiently long periods.

Meaning:

- civilizations are mainly destroyed from within;
- organizations collapse from within;
- individuals evolve from within;
- spiritual growth starts from within.

This connects both theories naturally.

3. Spiral Evolution versus Spiral Devolution

Our spiral model may be extended.

Instead of a single spiral, we may have:

Ascending Spiral



where

$$T > F.$$

The human approaches higher consciousness.

Descending Spiral



where

$$F > T.$$

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Together they form a unified theory of:

- personal growth,
- social development,

- organizational behavior,
- civilization cycles,
- spiritual evolution,
- artificial intelligence evolution.

We think that the paper can be positioned as a **new neutrosophic theory of evolution and collapse**, where the two Vatuui–Smarandache papers become the foundational pillars.

Internal Neutrosophic Spiral Dynamics:

Connecting Human Neutrosophic Evolution with the Internal Breakdown Law of Dynamic Systems

Abstract

This paper unifies two previously introduced concepts: (1) the principle that a neutrosophic dynamic system is easier to break from inside than from outside, and (2) the spiral model of human neutrosophic evolution. We show that both phenomena arise from the interaction of truth-components, indeterminacy-components, and falsity-components acting within a dynamic system. A new framework, called Internal Neutrosophic Spiral Dynamics (INSD), is proposed. The theory explains growth, decline, oscillation, self-organization, self-destruction, and transformation in individuals, organizations, societies, civilizations, and intelligent systems.

1. Introduction

- Neutrosophy and dynamic systems.
- Internal versus external influences.
- Spiral models of development.
- Motivation for unification.

Citations:

- Smarandache (1998, 1999)
- Smarandache & Vatuui (2017)
- Smarandache & Vatuui (2019)

2. Easier to Break from Inside than from Outside

Review of:

Definition 2.1. Internal Breakdown Principle

For a neutrosophic dynamic system S ,

$$D_{int} > D_{ext}$$

where

- D_{int} = cumulative internal destructive force,
- D_{ext} = cumulative external destructive force.

Examples:

- Roman Empire.
- Companies.
- Political systems.
- Biological organisms.

3. Human Neutrosophic Evolution in Spiral

Review of:

Definition 3.1

A human state at time t is

$$H(t) = (T(t), I(t), F(t)).$$

Evolution proceeds through successive cycles:

$$H_1 \rightarrow H_2 \rightarrow \cdots \rightarrow H_n$$

forming a spiral trajectory.

Add a figure:

Neutrosophic Spiral of Human Development

with ascending, descending, and oscillatory branches.

4. Internal Neutrosophic Spiral Dynamics

This is the new contribution.

Definition 4.1

Let

$$S(t) = (T(t), I(t), F(t)).$$

The evolution operator is

$$\mathcal{E}(t) = T(t) - F(t).$$

while indeterminacy modulates the trajectory.

Definition 4.2

The Internal Neutrosophic Potential

$$P_{INS D} = \alpha T + \beta I - \gamma F$$

where

$$\alpha, \beta, \gamma > 0.$$

5. Fundamental Theorems

Theorem 5.1 (Ascending Spiral Theorem)

If

$$\int_{t_0}^{t_1} T(t) dt > \int_{t_0}^{t_1} F(t) dt$$

then the system tends toward an ascending spiral.

Theorem 5.2 (Descending Spiral Theorem)

If

$$\int_{t_0}^{t_1} T(t) dt < \int_{t_0}^{t_1} F(t) dt$$

then the system tends toward a descending spiral.

Theorem 5.3 (Oscillatory Spiral Theorem)

If

$$\int T(t) dt \approx \int F(t) dt$$

then the system oscillates around a neutral equilibrium spiral.

Theorem 5.4 (Internal Dominance Theorem)

For sufficiently complex systems,

$$|F_{int}| > |F_{ext}|$$

implies that internal causes dominate collapse.

This theorem directly links the 2017 paper with the spiral model.

6. Comparison with Other Theories

Darwin

$$Evolution = Adaptation$$

versus

$$Evolution = (T, I, F)$$

Hegel

$$Thesis \rightarrow Antithesis \rightarrow Synthesis$$

versus neutrosophic spiral evolution.

Teilhard de Chardin

Omega-point evolution versus neutrosophic multi-directional evolution.

Prigogine

Self-organization and dissipative structures.

General System Theory

Bertalanffy and internal dynamics.

7. Refined Neutrosophic Version

A particularly interesting extension.

Let

$$\begin{aligned} T &= (T_1, T_2, \dots, T_p) \\ I &= (I_1, I_2, \dots, I_r) \\ F &= (F_1, F_2, \dots, F_s) \end{aligned}$$

Then

$$P_{INS D} = \sum_{i=1}^p \alpha_i T_i + \sum_{j=1}^r \beta_j I_j - \sum_{k=1}^s \gamma_k F_k.$$

This connects naturally with Our recent work on:

- Refined Neutrosophic Attractors,
- Refined Neutrosophic Repellers,
- Refined Neutrosophic Neutrals,
- Refined Neutrosophic Lattices.

8. Applications

Human Development

- education,
- psychology,
- spirituality.

Organizations

- management,
- leadership,
- governance.

Artificial Intelligence

- self-learning systems,
- model drift,
- internal bias formation.

Civilizations

- growth,
- stagnation,
- collapse,
- renewal.

Neutrosophic Decision-Making

- dynamic MCDM systems,
- adaptive planning.

9. Future Research

- Refined Internal Neutrosophic Spiral Dynamics.
- Neutrosophic Attractor–Repeller Spirals.
- Neutrosophic Evolution Cubes.
- Neutrosophic Spiral Manifolds.
- Neutrosophic Evolution Equations.
- Neutrosophic Spiral Differential Systems.

10. Conclusion

The 2017 paper provides a neutrosophic law of internal collapse, while the 2019 paper provides a neutrosophic law of internal evolution. Their synthesis leads to a general framework—Internal Neutrosophic Spiral Dynamics—capable of describing constructive, destructive, and indeterminate trajectories in complex systems. This framework extends classical evolution theories, systems theory, and dynamical systems theory within the neutrosophic paradigm.

This is one of the first papers explicitly connecting **neutrosophic dynamic systems, spiral evolution, attractors/repellers, and internal system collapse** into a single mathematical framework.

A. Primary References

We organize the bibliography into six groups, including the foundational papers.

1. Florentin Smarandache, A. R. Vatuiu,
Easier to Break from Inside than from Outside a Neutrosophic Dynamic System,
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Available at:
<https://fs.unm.edu/EasierMaiUsor.pdf>
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Human Neutrosophic Evolution in Spiral or The Divine is in the Man,

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3. Florentin Smarandache,
Neutrosophy: Neutrosophic Probability, Set, and Logic,
American Research Press, 1998.
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American Research Press, 1999.

B. Neutrosophic Dynamic Systems

5. Florentin Smarandache,
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6. Florentin Smarandache,
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7. Florentin Smarandache,
Plithogeny, Plithogenic Set, Logic, Probability and Statistics.
8. W. B. Vasantha Kandasamy and Florentin Smarandache,
Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps.
9. Florentin Smarandache,
Neutrosophic Triplet Structures.

C. Dynamical Systems and Attractors

10. Henri Poincaré,
Science and Method.
11. Stephen Smale,
"Differentiable Dynamical Systems,"
Bulletin of the American Mathematical Society, 1967.
12. Edward Lorenz,
The Essence of Chaos.
13. Nonlinear Dynamics and Chaos by Steven Strogatz.
14. Ilya Prigogine,
Order out of Chaos.
15. Hermann Haken,
Synergetics: An Introduction.

D. Spiral Evolution and Development

16. Georg Wilhelm Friedrich Hegel,
Phenomenology of Spirit.
17. Pierre Teilhard de Chardin,
The Phenomenon of Man.

18. Arnold J. Toynbee,
A Study of History.
19. Oswald Spengler,
The Decline of the West.
20. Ken Wilber,
A Theory of Everything.

E. Internal versus External Causes

21. Peter Senge,
The Fifth Discipline.
22. Russell Ackoff,
Systems Thinking for Curious Managers.
23. Ludwig von Bertalanffy,
General System Theory.
24. W. Edwards Deming,
The New Economics.
25. Nassim Nicholas Taleb,
Antifragile.

These references support the thesis that many failures originate internally rather than externally.

F. Connections with our Recent Work

Since the paper would fit naturally into our current research direction, we would also cite:

26. Florentin Smarandache,
Refined Neutrosophic Attractors, Repellers and Neutrals.
27. Florentin Smarandache,
Refined Neutrosophic Lattices.
28. Florentin Smarandache,
Neutrosophic Dynamic Systems.
29. Florentin Smarandache,
Neutrosophic Cubes and Geometric Representations.
30. Florentin Smarandache,
Neutrosophic Over/Under/Off-Structures.

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An Approach to Bilevel Multi-Objective Linear Fractional Probabilistic Optimization in Neutrosophic Environments with Applications

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Abstract: In real-life organization sectors, decision-makers (DMs) always encounter several complexities like hierarchical decision-process with multiple conflict ratio goals, constraints associated with probability and uncertainty parameter with hesitance scope. Despite the important role of optimization method for real-life problem, a very few success has been achieved to deal with this applicable complex problem. Thus, the current study has focused on investigating an efficient method consisting of a goal-programming (GP), chance-constrained optimization (CCO) and intuitionistic fuzzy mathematical (IFM) approach for solving hierarchical multi-objective linear fractional probabilistic optimization (HMOLFPO) problems under neutrosophic environment. Here each hierarchical decision-making optimize numerous conflict affair-ratio objective functions under chance constraints with uncertain coefficients and quantity defined as neutrosophic linear numbers of the form $\gamma + \omega I$ where $\gamma, \omega \in \mathbb{R}$ and I represents indeterminacy. In the proposed method, first the crisp linear model of neutrosophic HMOLFPO problem is formulated using the concept of interval ratio, CCO and linearized strategy. Then, the membership and non-memberships are investigated to measure the degree of satisfaction and dissatisfaction for each imprecise goal of DMs. Finally, GP with an IFM method is developed to generate a non-dominant solution for all DMs at both levels. Unlike existing methods, these strategy helps to avoid a decision-locks between upper and lower DMs. The applicability and efficiency of the suggested method are illustrated by solving numerical examples and industrial production real-life problem.

Keywords: Multi-Objective Fractional Problem · Chance-Constrained Optimization · Bi-level Decision-Making Problem · Neutrosophic Linear Number · Intuitionistic Fuzzy Goal Programming.

1 Introduction

Decentralized bi-level decision-making (DBLDM) problems, stated as mathematical programming with a sole DM at the leader-level and numerous DMs at the follower-level, are frequently found in the hierarchical structures of real-life applications [36, 37]. In DBLDM problem, the ULDM (standing for Upper-Level Decision-Maker) decides his goals and/or decision-control choices first, then each LLDM (standing for Lower-Level Decision-Maker) react by selecting their optimal decision variables that optimize the LLDM problem. That means, even though the ULDM independently maximizes its own benefits, the LLDM's response could influence the decision. As a result, most real-world decision scenarios involve the issue of distributing decision authority appropriately, which commonly leads to decision deadlock [8, 25, 39, 41].

In most real-world decision-making problems, the values to be allocated to the parameters cannot be exactly determined due to ambiguous and insufficient information. In order to solve this issue, fuzzy sets (FSs) have been introduced by Bellman and Zadeh [10] to simplify the task of demonstrating the imprecise value of parameters with the aid of fuzzy numbers. Since fuzzy numbers only take into account the truth-membership function, one of their main drawbacks is their inability to effectively capture ambiguous and inconsistent data with hesitation aspects [11, 15, 35, 38]. An extension of FSs that takes into account the membership for truth and falsity function is called an intuitionistic fuzzy set (IFS) introduced by Atanassov [7] to overcome these drawbacks. In the last decade, a large number of studies and applications on the domain of fuzzy and intuitionistic fuzzy bi-level multi-objective decision-making problems have been presented. Among these studies, we can mention the works of [5, 13, 20–22, 25, 36, 39, 46, 52, 56].

Even with IFS and FS-based optimization methods, the ability to cover imprecise and inconsistent information in various areas is still trailing behind the more realistic scenarios of decision-making, where neutral ideas which neither affirmation nor negation cannot be addressed [3, 37]. As a result, handling the unpredictable conditions of ambiguous information and experience becomes difficult. To address this difficulty, Smarandache [55] proposed the neutrosophic set (NS). This novel set type was designed to handle decision-making scenarios including incomplete, inconsistent, and ambiguous data. In this case, indeterminacy is regarded as an explicitly element that significantly influences decision-making. Many researchers have used the basic concept of NS, such as [23, 28, 29, 37] in decision-making processes. NS uses membership functions for truth, falsehood, and indeterminacy to aid in human-like decision-making.

Significant progress in the research of neutrosophic optimization environments and related practical concerns has been made in recent years. For instance, a neutrosophic-GP (NGP) technique was devised by Ahmad et.al. [4] for the multiple objectives in a hierarchical mathematical problem with a rough parameter in objective functions. GP was used by Pramenilk and Dey [43] to solve the multilevel mathematical problem utilizing a neutrosophic number. To address transportation application concerns, Hosseinzadeh and Tayyebi [23] searched for a compromise optimal solution to the multi-objective neutrosophic linear optimization (MONLO) problem. Buvaneshwari and Anuradha [12] created a novel method based on score and accuracy ranking approaches to solve a bi-objective interval-valued neutrosophic assignment problem.

A NGP approach was presented by Maiti et al. [33] to solve the hierarchical with MONLO problem with coefficients provided in the $c + Id$ form, where I denotes indeterminacy and $c, d \in \mathfrak{R}$. Emam et al. [18,19] addressed various neutrosophic optimization problem by considering trapezoidal neutrosophic parameter and quadratic objective type.

In a real-life decision-making environment, despite having vast experience, the DMs cannot always articulate the goals precisely due to the uncertain behavior of the input parameter and the probabilistic nature of the constraints in the problem's model formulation process [14]. For example, the variability of resources—such as time, power, financial resources, and machinery—typically affected from irregular fluctuations or stochastic factors. These unpredictable oscillations are not always foreseeable or preventable; however, they generally adhere to a random process, making the representation of known random variables a logical approach. The discipline of mathematics that examines the theory, and methodologies related to optimization problems that incorporate random variables is known as chance constraint optimization. A limited number of methods have been explored in the literature to address real-world application problems characterized by chance constraints and imprecise input data. A concise overview of existing studies aimed at resolving HMOLFPO problems within FS, IFS, or NS-based optimization methods, along with the contributions of the current study, is presented in Table 1. To the best of our knowledge, there has been no algorithm developed in the literature specifically for addressing HMOLFPO problems under neutrosophic environment.

This study aims to present an efficient methodology for the HMOLFPO problems under neutrosophic environment, characterized by neutrosophic linear numbers for all input parameters and random variables for some right-hand resources. The DMs at hierarchical levels need optimize multiple linear-fractional objective functions. The proposed solution method transforms the introduced problem into a crisp bi-level multi-objective linear optimization problem using interval ratio ranking, stochastic optimization, and modified general linear-transformation techniques, thereby streamlining the process. The degree of achievable for the intuitionistic fuzzy aspiration level of the objective function for each DM, along with the decision-control variable under the upper-level's supervision, is quantified using membership and non-membership functions. Finally, we employ an interactive GP method on the intuitionistic fuzzy decision set to derive a preferred optimal solution.

The paper is structured as follows: Section 2 introduces the theory of the neutrosophic linear number concept. In Section 3, we present the mathematical formulation of the HMOLFPO problem within a neutrosophic optimization environment, along with its crisp and linear model. Section 4 details our proposed method for addressing the newly introduced HMOLFPO problems under neutrosophic environment, using the interactive goal programming method and intuitionistic fuzzy decision sets. The algorithm for the proposed solution method is outlined in Section 5. Finally, Section 6 demonstrates the applicability of the proposed methodology through two numerical examples and production planning problems. Sect. 7 discusses conclusions and further scope based on the work conducted thus far.

2 Basic concepts and preliminaries

This section briefly discusses essential concepts related to neutrosophic sets, neutrosophic linear numbers, and their arithmetic operations needed for this task.

Definition 2.1. [12,37] Let X be a non-empty general set and $x \in X$. Then a neutrosophic set, $\tilde{A}^N$ is defined as $\tilde{A}^N = \{(x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)) | x \in X\}$ where the membership function $T_{\tilde{A}^N} | X \rightarrow [0, 1]$, $I_{\tilde{A}^N} | X \rightarrow [0, 1]$ and $F_{\tilde{A}^N} | X \rightarrow [0, 1]$ are illustrates the degree of truth, indeterminacy and falsity of member/ element $x \in X$ in $\tilde{A}^N$ so that $0 \leq T_{\tilde{A}^N}(x) + I_{\tilde{A}^N}(x) + F_{\tilde{A}^N}(x) \leq 3$, unlike IFS where restricted as $T_{\tilde{A}^N}(x) + I_{\tilde{A}^N}(x) + F_{\tilde{A}^N}(x) = 1$

Definition 2.2. [33] A neutrosophic linear number (NLN) is denoted as $N = \gamma + \omega I$ where γ and ω are real/ complex numbers and $I \in [I^L, I^R]$ represents indeterminacy. Here, γ and ω are called determinate and indeterminate part of N, respectively. Thus, the NLN, N can be written as:

$$N = \gamma + \omega I = \gamma + \omega [I^L, I^R] = [\gamma + \omega I^L, \gamma + \omega I^R] = [N^L, N^R] (\text{say}).$$

Here, N is transferred into an interval-number $[N^L, N^R]$. Suppose a NLN considered as $N = 3 + 5I$ where 3 is determinate and 5I is indeterminate part. Here, let it be $I \in [0.3, 0.7]$, so N transforms into an interval-number of the form $[4.5, 6.5]$.

Now, the arithmetic operation on NLN discussed as follows: [33,43] Suppose that two NLNs given as $N_1 = \gamma_1 + \omega_1 I_1 = [\gamma_1 + \omega_1 I_1^L, \gamma_1 + \omega_1 I_1^R] = [N_1^L, N_1^R]$ and $N_2 = \gamma_2 + \omega_2 I_2 = [\gamma_2 + \omega_2 I_2^L, \gamma_2 + \omega_2 I_2^R] = [N_2^L, N_2^R]$ where, $I_1 \in [I_1^L, I_1^R]$ and $I_2 \in [I_2^L, I_2^R]$, then

3 Mathematical Development of the Problem

In this section, the mathematical formulation of the introduced neutrosophic bi-level multi-objective linear fractional chance-constraint optimization (NBL-MOLFCCo) problems and its corresponding crisp parametric, deterministic constraints and linear-models are developed.

3.1 Problem Definition

Assume that a NBL-MOLFCCo problem is constructed having two stage/ hierarchical level decision-makers (DMs) called as ULDM and LLDM, where the ULDM having a single decision-maker and denoted as DM_1^1 that controlling a decision vector $\mathbf{x}_1 = (x_{11}, x_{12}, \dots, x_{1n_1})$ and LLDM having multiple decision-makers and denoted as DM_l^2 where, $l = 2, 3, \dots, p$ which control over a different decision vector $\mathbf{x}_l = (x_{l1}, x_{l2}, \dots, x_{ln_l})' \in R^{n_l}, \forall l = 2, 3, \dots, p$. Also assume that each decision-maker $DM_l, l = 1, 2, \dots, p$ has several affair ratio objective functions with neutrosophic linear coefficients of the form $\gamma + \omega I$ that are independently optimized under the given feasible sets. Thus, the neutrosophic affair fractional objective functions under $DM_l, l = 1, 2, \dots, p$ are:

$$\tilde{f}_{lk}(\mathbf{x}) = \tilde{f}_{lk}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p) : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \dots \times \mathbb{R}^{n_p} \rightarrow \mathbb{R}^{l_l} l = 1, \dots, p, k = 1, 2, \dots, t_l.$$

Therefore, the mathematical formulation of the neutrosophic bi-level multi-objective linear fractional chance-constraint optimization (NBL-MOLFCCO) problem is defined as follows:

ULDM

$$\max_{\mathbf{x}_1} \tilde{f}_{1k}(\mathbf{x}) = \max_{\mathbf{x}_1} \frac{\tilde{N}_{1k}(\mathbf{x})}{\tilde{D}_{1k}(\mathbf{x})} = \frac{(c_{1k}^1 + \bar{c}_{1k}^1 I_{1k}^1)x_1 + (c_{1k}^2 + \bar{c}_{1k}^2 I_{1k}^2)x_2 + \dots + (c_{1k}^p + \bar{c}_{1k}^p I_{1k}^p)x_p + (\alpha_{1k} + \bar{\alpha}_{1k} I_{1k})}{(d_{1k}^1 + \bar{d}_{1k}^1 J_{1k}^1)x_1 + (d_{1k}^2 + \bar{d}_{1k}^2 J_{1k}^2)x_2 + \dots + (d_{1k}^p + \bar{d}_{1k}^p J_{1k}^p)x_p + (\beta_{1k} + \bar{\beta}_{1k} J_{1k})},$$

where, $x_2, x_3, \dots, x_p$ solves

LLDM

$$\left\{ \begin{array}{l} \max_{\mathbf{x}_2} \tilde{f}_{2k}(\mathbf{x}) = \max_{\mathbf{x}_2} \frac{\tilde{N}_{2k}(\mathbf{x})}{\tilde{D}_{2k}(\mathbf{x})} = \frac{(c_{2k}^1 + \bar{c}_{2k}^1 I_{2k}^1)x_1 + (c_{2k}^2 + \bar{c}_{2k}^2 I_{2k}^2)x_2 + \dots + (c_{2k}^p + \bar{c}_{2k}^p I_{2k}^p)x_p + (\alpha_{2k} + \bar{\alpha}_{2k} I_{2k})}{(d_{2k}^1 + \bar{d}_{2k}^1 J_{2k}^1)x_1 + (d_{2k}^2 + \bar{d}_{2k}^2 J_{2k}^2)x_2 + \dots + (d_{2k}^p + \bar{d}_{2k}^p J_{2k}^p)x_p + (\beta_{2k} + \bar{\beta}_{2k} J_{2k})}, \\ \dots \\ \dots \\ \max_{\mathbf{x}_p} \tilde{f}_{pk}(\mathbf{x}) = \max_{\mathbf{x}_p} \frac{\tilde{N}_{pk}(\mathbf{x})}{\tilde{D}_{pk}(\mathbf{x})} = \frac{(c_{pk}^1 + \bar{c}_{pk}^1 I_{pk}^1)x_1 + (c_{pk}^2 + \bar{c}_{pk}^2 I_{pk}^2)x_2 + \dots + (c_{pk}^p + \bar{c}_{pk}^p I_{pk}^p)x_p + (\alpha_{pk} + \bar{\alpha}_{pk} I_{pk})}{(d_{pk}^1 + \bar{d}_{pk}^1 J_{pk}^1)x_1 + (d_{pk}^2 + \bar{d}_{pk}^2 J_{pk}^2)x_2 + \dots + (d_{pk}^p + \bar{d}_{pk}^p J_{pk}^p)x_p + (\beta_{pk} + \bar{\beta}_{pk} J_{pk})}, \end{array} \right. \quad (1)$$

Subject to:

$$\mathbf{x} \in \tilde{X}_N = \left\{ \begin{array}{l} \sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti}) x_i \begin{pmatrix} \leq \\ = \\ \geq \end{pmatrix} b_t + \bar{b}_t I_t, \quad t = 1, 2, \dots, m \\ Pr \left\{ \sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti}) x_i \begin{pmatrix} \leq \\ \geq \end{pmatrix} b_t \right\} \geq \alpha_t, \quad t = m+1, \dots, s \\ x_i \geq 0, \quad i = 1, 2, \dots, n \end{array} \right.$$

Where,

- $\tilde{f}_{lk}(\mathbf{x}) = \frac{\tilde{N}_{lk}(\mathbf{x})}{\tilde{D}_{lk}(\mathbf{x})}$, $k = 1, 2, \dots, t_l$ indicates the neutrosophic affair ratio objective function under DM_l .
- $(c_{lk}^j + \bar{c}_{lk}^j I_{lk}^j)$ and $(d_{lk}^j + \bar{d}_{lk}^j J_{lk}^j)$, $j = 1, 2, \dots, p$ are n-dimension row vectors of neutrosophic linear coefficients of $\tilde{N}_{lk}$ (numerator) and $\tilde{D}_{lk}$ (denominator) of $\tilde{f}_{lk}(\mathbf{x}) = \frac{\tilde{N}_{lk}}{\tilde{D}_{lk}}$ respectively.
- Assume that $\tilde{D}_{lk}(\mathbf{x}) = (d_{lk}^1 + \bar{d}_{lk}^1 J_{lk}^1)x_1 + \dots + (d_{lk}^p + \bar{d}_{lk}^p J_{lk}^p)x_p + (\beta_{lk} + \bar{\beta}_{lk} J_{lk}) > 0, \forall \mathbf{x} \in \tilde{X}_N$.
- $b_t, t = m+1, \dots, s$ are autonomous random variables with follows a Cauchy distribution having mean θ and standard deviation σ .
- $\alpha_i \in (0, 1)$ is act as the admissible risk of constraint violation. "Pr{·}" represent probability.
- The scalars $\alpha_{lk} + \bar{\alpha}_{lk} I_{lk}$ and $\beta_{lk} + \bar{\beta}_{lk} J_{lk}$ are neutrosophic linear numbers.
- Each decision vectors monitored by DM_l are $\mathbf{x}_l \in \mathfrak{R}^{n_l}$, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_p) \in \mathfrak{R}^n$ with $n = \sum_{l=1}^p n_l$.
- $\tilde{X}_N \neq \emptyset$ is a compact and convex neutrosophic feasible region for the NBL-MOLFCCO problem (1).
- $a_{ti} + \bar{a}_{ti} I_{ti}$ acts as the neutrosophic linear coefficient of the ith decision variable and its right-side quantity is denoted as $b_t + \bar{b}_t I_t, t = 1, 2, \dots, m$.
- $I_{lk}^j \in [I_{lk}^{Lj}, I_{lk}^{Rj}]$, $J_{lk}^j \in [J_{lk}^{Lj}, J_{lk}^{Rj}]$, $I_{lk} \in [I_{lk}^L, I_{lk}^R]$, $J_{lk} \in [J_{lk}^L, J_{lk}^R]$, $I_t \in [I_t^L, I_t^R]$, $I \in [I^L, I^R]$ and $I_{lk}^{Lj}, I_{lk}^{Rj}, J_{lk}^{Lj}, J_{lk}^{Rj}, I_{lk}^L, I_{lk}^R, I_t^L, I_t^R, I^L, I^R$ are real numbers ($\mathfrak{R}$).

3.2 Formulation of Crisp Model of NBL-MOLFCCO Problem

Following the discussion given in section 2, we reduce each neutrosophic linear number (NLN) into an interval number. Thus, all uncertain coefficients and quantity in NBL-MOLFCCO problem (1) reduced to:

$$\begin{aligned}
 c_{lk}^j + \bar{c}_{lk}^j I_{lk}^j &= c_{lk}^j + \bar{c}_{lk}^j [I_{lk}^{Lj}, I_{lk}^{Rj}] = [c_{lk}^j + \bar{c}_{lk}^j I_{lk}^{Lj}, c_{lk}^j + \bar{c}_{lk}^j I_{lk}^{Rj}] = [m_{lk}, n_{lk}] \text{ (say)} \\
 d_{lk}^j + \bar{d}_{lk}^j J_{lk}^j &= d_{lk}^j + \bar{d}_{lk}^j [J_{lk}^{Lj}, J_{lk}^{Rj}] = [d_{lk}^j + \bar{d}_{lk}^j J_{lk}^{Lj}, d_{lk}^j + \bar{d}_{lk}^j J_{lk}^{Rj}] = [e_{lk}, f_{lk}] \text{ (say)} \\
 \alpha_{lk} + \bar{\alpha}_{lk} I_{lk} &= \alpha_{lk} + \bar{\alpha}_{lk} [I_{lk}^L, I_{lk}^R] = [\alpha_{lk} + \bar{\alpha}_{lk} I_{lk}^L, \alpha_{lk} + \bar{\alpha}_{lk} I_{lk}^R] = [\alpha_{lk}^L, \alpha_{lk}^R] \text{ (say)} \\
 \beta_{lk} + \bar{\beta}_{lk} J_{lk} &= \beta_{lk} + \bar{\beta}_{lk} [J_{lk}^L, J_{lk}^R] = [\beta_{lk} + \bar{\beta}_{lk} J_{lk}^L, \beta_{lk} + \bar{\beta}_{lk} J_{lk}^R] = [\beta_{lk}^L, \beta_{lk}^R] \text{ (say)} \\
 a_{ti} + \bar{a}_{ti} I_{ti} &= a_{ti} + \bar{a}_{ti} [I_{ti}^L, I_{ti}^R] = [a_{ti} + \bar{a}_{ti} I_{ti}^L, a_{ti} + \bar{a}_{ti} I_{ti}^R] = [a_{ti}^L, a_{ti}^R] \text{ (say)} \\
 b_t + \bar{b}_t I_t &= b_t + \bar{b}_t [I_t^L, I_t^R] = [b_t + \bar{b}_t I_t^L, b_t + \bar{b}_t I_t^R] = [b_t^L, b_t^R] \text{ (say)}
 \end{aligned} \tag{2}$$

Based on ratio optimization concept [36,41] and interval number given in Eqs. (2), the neutrosophic objective function $\tilde{f}_{lk}(\mathbf{x}) = \frac{\tilde{N}_{lk}}{D_{lk}}$ of minimization and maximization-form problems are transformed into crisp form $f_{lk}(\mathbf{x}) = \frac{N_{lk}^L(\mathbf{x})}{D_{lk}^R(\mathbf{x})}$ and $f_{lk}(\mathbf{x}) = \frac{N_{lk}^R(\mathbf{x})}{D_{lk}^L(\mathbf{x})}$ respectively, where defined as:

$$f_{lk}(\mathbf{x}) = \frac{N_{lk}^L(\mathbf{x})}{D_{lk}^R(\mathbf{x})} = \frac{m_{lk}^1 x_1 + m_{lk}^2 x_2 + \dots + m_{lk}^p x_p + \alpha_{lk}^L}{n_{lk}^1 x_1 + n_{lk}^2 x_2 + \dots + n_{lk}^p x_p + \beta_{lk}^R}, \quad l = 1, 2, \dots, p, k = 1, 2, \dots, t_l \tag{3}$$

$$f_{lk}(\mathbf{x}) = \frac{N_{lk}^R(\mathbf{x})}{D_{lk}^L(\mathbf{x})} = \frac{n_{lk}^1 x_1 + n_{lk}^2 x_2 + \dots + n_{lk}^p x_p + \alpha_{lk}^R}{m_{lk}^1 x_1 + m_{lk}^2 x_2 + \dots + m_{lk}^p x_p + \beta_{lk}^L}, \quad l = 1, 2, \dots, p, k = 1, 2, \dots, t_l \tag{4}$$

Suppose an optimization problem have a constraint in the form of $\sum_{i=1}^n [a_{ti}^L, a_{ti}^R] x_i \geq [b_t^L, b_t^R]$, then the minimum and maximum value range are obtained from the inequality of $\sum_{i=1}^n a_{ti}^L x_i \geq b_t^R$ and $\sum_{i=1}^n a_{ti}^R x_i \geq b_t^L$ respectively.

Considering Proposition 3.1 on the constraint sets $\tilde{X}_N$ of NBL-MOLFCCO problem (1) and to develop the maximum relaxed feasible region, the corresponding crisp constraints of neutrosophic linear constraints in the problem (1) are formulated as follows:

$$\begin{aligned}
 \sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti}) x_i &\leq [b_t + \bar{b}_t I_t] \Rightarrow \sum_{i=1}^n a_{ti}^L x_i \leq b_t^R, \quad t = 1, 2, \dots, m_1 \\
 \sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti}) x_i &\geq [b_t + \bar{b}_t I_t] \Rightarrow \sum_{i=1}^n a_{ti}^R x_i \geq b_t^L, \quad t = m_1 + 1, \dots, m_2 \\
 \sum_{i=1}^n (a_{ti} + \bar{a}_{ti} I_{ti}) x_i &= [b_t + \bar{b}_t I_t] \Rightarrow \sum_{i=1}^n a_{ti}^L x_i \leq b_t^R \text{ \& } \sum_{i=1}^n a_{ti}^R x_i \geq b_t^L, \quad t = m_2 + 1, \dots, m
 \end{aligned} \tag{5}$$

Therefore, following the above discussion the equivalent crisp model of the NBL-MOLFCCO problem (1) is developed as:

$$\text{ULDM} \left\{ \max f_{1k}(\mathbf{x}) = \max \frac{N_{1k}^R(\mathbf{x})}{D_{1k}^L(\mathbf{x})} = \frac{n_{1k}^1 x_1 + n_{1k}^2 x_2 + \dots + n_{1k}^p x_p + \alpha_{1k}^R}{m_{1k}^1 x_1 + m_{1k}^2 x_2 + \dots + m_{1k}^p x_p + \beta_{1k}^L}, \quad k = 1, 2, \dots, t_1 \right. \tag{6}$$

where, $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p$ solves

$$\text{LLDM} \left\{ \max f_{2k}(\mathbf{x}) = \max \frac{N_{2k}^R(\mathbf{x})}{D_{2k}^L(\mathbf{x})} = \frac{n_{2k}^1 x_1 + n_{2k}^2 x_2 + \dots + n_{2k}^p x_p + \alpha_{2k}^R}{m_{2k}^1 x_1 + m_{2k}^2 x_2 + \dots + m_{2k}^p x_p + \beta_{2k}^L}, \quad k = 1, 2, \dots, t_2 \right. \tag{7}$$

Subject to:

$$\begin{aligned}
 & \sum_{i=1}^n a_{ti}^L x_i \leq b_t^R, \quad t = 1, 2, \dots, m_1 \\
 & \sum_{i=1}^n a_{ti}^R x_i \geq b_t^L, \quad t = m_1 + 1, \dots, m_2 \\
 & \sum_{i=1}^n a_{ti}^L x_i \leq b_t^R \text{ and } \sum_{i=1}^n a_{ti}^R x_i \geq b_t^L, \quad t = m_2 + 1, \dots, m \\
 & Pr \left[\sum_{i=1}^n a_{ti}^L x_i \leq b_t \right] \geq \alpha_t, \quad t = m + 1, \dots, s_0 \\
 & Pr \left[\sum_{i=1}^n a_{ti}^R x_i \geq b_t \right] \geq \alpha_t, \quad t = s_0 + 1, \dots, s \\
 & x_i \geq 0, \quad i = 1, 2, \dots, n
 \end{aligned} \tag{8}$$

3.3 Deterministic Constraint Formulation of NBL-MOLFCCO Problem

Given that the right-hand-side (RHS) variable $b_t, t = m + 1, \dots, s$ in problem (3) be an autonomous Cauchy-random variable having arithmetic mean θ_t and standard deviation σ_t . Since $Pr\{g_t(\mathbf{x}) \leq b_t\} \geq \alpha$ is equivalent to $Pr\{g_t(\mathbf{x}) \geq b_t\} \leq 1 - \alpha$, here we consider the first type of chance constraints. The probability density function, $\Phi(g_t(\mathbf{x}))$ of the Cauchy-random variable b_t is defined as:

$$\Phi(g_t(\mathbf{x})) = Pr[g_t(\mathbf{x}) \leq b_t] = \int_{g_t(\mathbf{x})}^{\infty} \frac{\sigma_t}{\pi(\sigma_t^2 + (b_t - \theta_t)^2)} db_t \quad t = m + 1, \dots, s_0. \tag{9}$$

Theorem 3.1. If the RHS b_t is an autonomous Cauchy random variable having arithmetic mean θ_t and standard-deviation σ_t , then the chance constraint, $Pr[\sum_{i=1}^n a_{ti}^L x_i \leq b_t] \geq \alpha_t, t = m + 1, \dots, s_0$ and $Pr[\sum_{i=1}^n a_{ti}^R x_i \geq b_t] \geq \alpha_t, t = s_0 + 1, \dots, s$ of problem (3) is equivalent to

$$\sum_{i=1}^n a_{ti}^L x_i \leq \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right) \pi \quad \text{and} \quad \sum_{i=1}^n a_{ti}^R x_i \geq \theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right) \pi, \quad \text{respectively.}$$

Proof. Let $g_t(\mathbf{x}) = \sum_{i=1}^n a_{ti}^L x_i$ for each $t = m + 1, \dots, r_0$, then using Eqs.(4) the chance constraint in problem (3) can be written as:

$$\Phi(g_t(\mathbf{x})) = Pr[g_t(\mathbf{x}) \leq b_t] = \int_{g_t(\mathbf{x})}^{\infty} \frac{\sigma_t}{\pi(\sigma_t^2 + (b_t - \theta_t)^2)} db_t \geq \alpha_t \quad \text{for } t = m + 1, \dots, s_0.$$

By determining the integration, we obtain:

$$\frac{1}{\pi} \left[\tan^{-1} \left(\frac{b_t - \theta_t}{\sigma_t} \right) \right]_{g_t(\mathbf{x})}^{\infty} \geq \alpha_t \quad \text{for } t = m + 1, \dots, s_0.$$

Substituting the tolerance limit of integration, we obtain the following results:

$$\frac{\pi}{2} - \tan^{-1} \left(\frac{g_t(\mathbf{x}) - \theta_t}{\sigma_t} \right) \geq \pi \alpha_t \quad \text{for } t = m + 1, \dots, s_0.$$

It can be further simplified as:

$$\tan^{-1} \left(\frac{g_t(\mathbf{x}) - \theta_t}{\sigma_t} \right) \leq \frac{\pi}{2} - \pi \alpha_t \quad \text{for } t = m + 1, \dots, s_0. \tag{5}$$

By applying tan (tangent) function on both sides of Eqs. (5) and then solving for $g_t(\mathbf{x})$, we obtain:

$$g_t(\mathbf{x}) \leq \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right) \pi \quad \text{for } t = m + 1, \dots, s_0.$$

Now, replacing $g_t(\mathbf{x})$ by $\sum_{i=1}^n a_{ti}^L x_i$, we obtain the following equivalent deterministic-constraint:

$$\sum_{i=1}^n a_{ti}^L x_i \leq \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right) \pi, \quad \text{for each } t = m+1, \dots, s_0.$$

Since $Pr[g_t(\mathbf{x}) \geq b_t] = 1 - Pr[g_t(\mathbf{x}) \leq b_t]$, the analogous deterministic constraint of the chance constraint $Pr[\sum_{i=1}^n a_{ti}^R x_i \geq b_t] \geq \alpha_t, t = s_0 + 1, \dots, s$ of problem (3) in similar fashion formulated as:

$$\sum_{i=1}^n a_{ti}^R x_i \geq \theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right) \pi \quad \text{for } t = s_0 + 1, \dots, s.$$

□

Therefore, following Theorem 3.1 the equivalent deterministic constraint model of problem (3) is formulated as follows:

$$\text{ULDM} \left\{ \max f_{1k}(\mathbf{x}) = \max \frac{N_{1k}^R(\mathbf{x})}{D_{1k}^L(\mathbf{x})} = \frac{\mathbf{n}_{1k}^1 \mathbf{x}_1 + \mathbf{n}_{1k}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{1k}^p \mathbf{x}_p + \alpha_{1k}^R}{\mathbf{m}_{1k}^1 \mathbf{x}_1 + \mathbf{m}_{1k}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{1k}^p \mathbf{x}_p + \beta_{1k}^L}, \quad k = 1, 2, \dots, t_1 \right. \quad (10)$$

where, $\mathbf{x}_2, \mathbf{x}_3, \dots, \mathbf{x}_p$ solves

$$\text{LLDM} \left\{ \begin{array}{l} \max f_{2k}(\mathbf{x}) = \max \frac{N_{2k}^R(\mathbf{x})}{D_{2k}^L(\mathbf{x})} = \frac{\mathbf{n}_{2k}^1 \mathbf{x}_1 + \mathbf{n}_{2k}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{2k}^p \mathbf{x}_p + \alpha_{2k}^R}{\mathbf{m}_{2k}^1 \mathbf{x}_1 + \mathbf{m}_{2k}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{2k}^p \mathbf{x}_p + \beta_{2k}^L}, \quad k = 1, 2, \dots, t_2 \\ \dots \\ \max f_{pk}(\mathbf{x}) = \max \frac{N_{pk}^R(\mathbf{x})}{D_{pk}^L(\mathbf{x})} = \frac{\mathbf{n}_{pk}^1 \mathbf{x}_1 + \mathbf{n}_{pk}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{pk}^p \mathbf{x}_p + \alpha_{pk}^R}{\mathbf{m}_{pk}^1 \mathbf{x}_1 + \mathbf{m}_{pk}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{pk}^p \mathbf{x}_p + \beta_{pk}^L}, \quad k = 1, 2, \dots, t_p \end{array} \right. \quad (11)$$

Subject to:

$$\begin{aligned} \sum_{i=1}^n a_{ti}^L x_i &\leq b_t^R, \quad t = 1, 2, \dots, m_1 \\ \sum_{i=1}^n a_{ti}^R x_i &\geq b_t^L, \quad t = m_1 + 1, \dots, m_2 \\ \sum_{i=1}^n a_{ti}^L x_i &\leq b_t^R \text{ and } \sum_{i=1}^n a_{ti}^R x_i \geq b_t^L, \quad t = m_2 + 1, \dots, m \\ \sum_{i=1}^n a_{ti}^L x_i &\leq \theta_t + \sigma_t \tan\left(\frac{1}{2} - \alpha_t\right) \pi, \quad t = m+1, \dots, s_0 \\ \sum_{i=1}^n a_{ti}^R x_i &\geq \theta_t + \sigma_t \tan\left(\alpha_t - \frac{1}{2}\right) \pi, \quad t = s_0 + 1, \dots, s \\ x_i &\geq 0, \quad i = 1, 2, \dots, n \end{aligned} \quad (12)$$

Definition 3.1. The DM_l –MOLFO problem (7) is said to be a concave-convex fractional decision-making problem if the numerator, $N_{lk}(\mathbf{x})$ is concave function, the denominator, $D_{lk}(\mathbf{x})$ is convex function with $D_{lk}(\mathbf{x}) > 0, \forall \mathbf{x} \in X$ for each l, k and region X is convex feasible set.

Definition 3.2. For any decision-control vector $\mathbf{x}_1^* \in \{\mathbf{x}_1^* | \mathbf{x} = (\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_p^*) \in X\}$ given by $DM_1, \mathbf{x}_1^*, l = 2, 3, \dots, p$ be a Pareto-optimal (efficient) solution to the LLDM (i.e., DM_l –MOLFO, $l = 2, 3, \dots, p$) problem, if there does not exist a new-vector $\mathbf{z}_l \in X$, such that $f_{lk}(\mathbf{x}_l^*) \leq f_{lk}(\mathbf{z}_l), \forall k = 1, 2, \dots, t_l$ and $f_{lk}(\mathbf{x}_l^*) < f_{lk}(\mathbf{z}_l)$ for some k . Hence, $\mathbf{x}^* = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_p^*\}$ is called a feasible-solution to problem (3).

Definition 3.3. Let $\mathbf{x}^* = \{\mathbf{x}_1^*, \mathbf{x}_2^*, \dots, \mathbf{x}_p^*\}$ be a feasible-solution to problem (3). Then, $\mathbf{x}^*$ be an efficient solution to problem (3) if there doesn't exists a new feasible-vector $\mathbf{x} \in X$ such that $f_{lk}(\mathbf{x}^*) \leq f_{lk}(\mathbf{x}), \forall k = 1, 2, \dots, t_1$ and $f_{lk}(\mathbf{x}^*) < f_{lk}(\mathbf{x}),$ for some k .

Assume that in the problem (7), if $D_{lk}^L(\mathbf{x})$ is concave with $D_{lk}^L(\mathbf{x}) > 0, \forall \mathbf{x} \in X$ and $N_{lk}^R(\mathbf{x})$ is concave with $N_{lk}^R(\mathbf{x}) < 0, \forall \mathbf{x} \in X$, then the negative of $N_{lk}(\mathbf{x})$ is positive and convex linear-functional for each feasible decision-vector $\mathbf{x}$ and $\max_{\mathbf{x} \in X} \frac{N_{lk}^R(\mathbf{x})}{D_{lk}^L(\mathbf{x})} \iff \min_{\mathbf{x} \in X} \frac{-N_{lk}^R(\mathbf{x})}{D_{lk}^L(\mathbf{x})} \iff \max_{\mathbf{x} \in X} \frac{D_{lk}^L(\mathbf{x})}{-N_{lk}^R(\mathbf{x})}$. Therefore, this way it converted into concave-convex fractional decision-making problem [35][41].

Suppose that T be represent an index set such that $T = \{k = 1, 2, \dots, t_l, |N_{lk}^R(\mathbf{x}) \geq 0 \text{ for some } \mathbf{x} \in X\}$, $T^c = \{k = 1, 2, \dots, t_l, |N_{lk}^R(\mathbf{x}) < 0, \forall \mathbf{x} \in X\}$, and $T \cup T^c = \{1, 2, \dots, t_l\}$ for each $l = 1, 2, \dots, p$. Let us choose a new vector of n -variables $\mathbf{w}$ that related with each component of $\mathbf{x} \in \mathfrak{R}_+^n$ as:

$$\mathbf{w} = r\mathbf{x}, \text{ where } r = \min \left\{ \frac{1}{\mathbf{m}_{lk}^1 \mathbf{x}_1 + \mathbf{m}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{lk}^p \mathbf{x}_p + \beta_{lk}^L}, k \in T \right\} > 0 \text{ and}$$

$$\mathbf{w} = r\mathbf{x}, \text{ where } r = \min \left\{ \frac{-1}{\mathbf{n}_{lk}^1 \mathbf{x}_1 + \mathbf{n}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{lk}^p \mathbf{x}_p + \alpha_{lk}^R}, k \in T^c \right\} > 0 \text{ for each } l.$$

This is equivalent to:

$$\mathbf{m}_{lk}^1 \mathbf{x}_1 + \mathbf{m}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{lk}^p \mathbf{x}_p + \beta_{lk}^L \geq r \Rightarrow r(\mathbf{m}_{lk}^1 \mathbf{x}_1 + \mathbf{m}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{lk}^p \mathbf{x}_p + \beta_{lk}^L) \leq 1 \text{ for } k \in T \text{ and}$$

$$\frac{-1}{\mathbf{n}_{lk}^1 \mathbf{x}_1 + \mathbf{n}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{lk}^p \mathbf{x}_p + \alpha_{lk}^R} \geq r \Rightarrow -r(\mathbf{n}_{lk}^1 \mathbf{x}_1 + \mathbf{n}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{lk}^p \mathbf{x}_p + \alpha_{lk}^R) \leq 1 \text{ for } k \in T^c.$$

Using this new vector $\mathbf{w} \in \mathfrak{R}_+^n$ with relation operator $\mathbf{w} = r\mathbf{x}$, where, $r > 0, \mathbf{x} \in \mathfrak{R}_+^n$, we obtain:

$$\mathbf{x}_l = \frac{\mathbf{w}_l}{r} \iff x_{lj} = \frac{w_{lj}}{r}, j = 1, 2, \dots, n_l \iff x_i = \frac{w_i}{r}, i = 1, 2, \dots, n.$$

Thus, each affian fractional objective function $f_{lk}(\mathbf{x}) = \frac{N_{lk}^R(\mathbf{x})}{D_{lk}^L(\mathbf{x})} = \frac{\mathbf{n}_{lk}^1 \mathbf{x}_1 + \mathbf{n}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{n}_{lk}^p \mathbf{x}_p + \alpha_{lk}^R}{\mathbf{m}_{lk}^1 \mathbf{x}_1 + \mathbf{m}_{lk}^2 \mathbf{x}_2 + \dots + \mathbf{m}_{lk}^p \mathbf{x}_p + \beta_{lk}^L}$ in the DM_l -MOLFO problem (7) is converted into an equivalent linear-objective function:

$$f_{lk}(\mathbf{w}, r) = (\mathbf{n}_{lk}^1 \frac{\mathbf{w}_1}{r} + \mathbf{n}_{lk}^2 \frac{\mathbf{w}_2}{r} + \dots + \mathbf{n}_{lk}^p \frac{\mathbf{w}_p}{r} + \alpha_{lk}^R)r = \mathbf{n}_{lk}^1 \mathbf{w}_1 + \mathbf{n}_{lk}^2 \mathbf{w}_2 + \dots + \mathbf{n}_{lk}^p \mathbf{w}_p + \alpha_{lk}^R r = rN_{lk}^R(\frac{\mathbf{w}}{r}), \quad k \in T$$

$$f_{lk}(\mathbf{w}, r) = (\mathbf{m}_{lk}^1 \frac{\mathbf{w}_1}{r} + \mathbf{m}_{lk}^2 \frac{\mathbf{w}_2}{r} + \dots + \mathbf{m}_{lk}^p \frac{\mathbf{w}_p}{r} + \beta_{lk}^L)r = \mathbf{m}_{lk}^1 \mathbf{w}_1 + \mathbf{m}_{lk}^2 \mathbf{w}_2 + \dots + \mathbf{m}_{lk}^p \mathbf{w}_p + \beta_{lk}^L r = rD_{lk}^L(\frac{\mathbf{w}}{r}), \quad k \in T^c.$$

Therefore, using these linear-transformation strategies, the bi-level multi-objective linear optimization (BL-MOLO) model (8) of the crisp problem (6) is developed as:

$$\text{ULDM} \begin{cases} \max_{w_1} (f_{1k}(\mathbf{w}, r) = rN_{1k}^R(\mathbf{w}) \text{ for } k \in T; f_{1k}(\mathbf{w}, r) = rD_{1k}^L(\mathbf{w}) \text{ for } k \in T^c) & [DM_1], \\ \text{where, } \mathbf{w}_2, \mathbf{w}_3, \dots, \mathbf{w}_p \text{ solves;} \end{cases} \quad (13)$$

$$\text{LLDM} \begin{cases} \max_{w_2} (f_{2k}(\mathbf{w}, r) = rN_{2k}^R(\mathbf{w}) \text{ for } k \in T; f_{2k}(\mathbf{w}, r) = rD_{2k}^L(\mathbf{w}) \text{ for } k \in T^c) & [DM_2], \\ \vdots \\ \max_{w_p} (f_{pk}(\mathbf{w}, r) = rN_{pk}^R(\mathbf{w}), \text{ for } k \in T; f_{pk}(\mathbf{w}, r) = rD_{pk}^L(\mathbf{w}) \text{ for } k \in T^c) & [DM_p], \end{cases} \quad (14)$$

Subject to:

$$\begin{aligned}
& \sum_{i=1}^n a_{ti}^L w_i - r b_t^L \leq 0, \quad t = 1, 2, \dots, m_1, m_2 + 1, \dots, m \\
& \sum_{i=1}^n a_{ti}^R w_i - r b_t^L \geq 0, \quad t = m_1 + 1, \dots, m_2, m_2 + 1, \dots, m \\
& r D_{lk}(\mathbf{w}) \leq 1, \quad \text{for } k \in T; k = 1, 2, \dots, t_l; l = 1, 2, \dots, p \\
& -r N_{lk}(\mathbf{w}) \leq 1, \quad \text{for } k \in T^c, k = 1, 2, \dots, t_l; l = 1, 2, \dots, p \\
& \sum_{i=1}^n a_{ti}^L w_i - r \left(\theta_t + \sigma_t \tan \left(\frac{1}{2} - \alpha_t \right) \pi \right) \leq 0, \quad t = m + 1, \dots, s_0 \\
& \sum_{i=1}^n a_{ti}^R w_i - r \left(\theta_t + \sigma_t \tan \left(\alpha_t - \frac{1}{2} \right) \pi \right) \geq 0, \quad t = s_0 + 1, \dots, s \\
& \mathbf{w} \in \mathbb{R}^n, \mathbf{w} \geq 0, r > 0.
\end{aligned} \tag{15}$$

Where, it assumed that $S^{wr} \neq \emptyset$ and a bounded constraint region of BL-MOLO (8) problem.

4 Proposed Solution Method

In this section, we introduced a new hybrid method that combined intuitionistic fuzzy optimization and goal programming is called IFGP (intuitionistic fuzzy goal programming) method.

4.1 Intuitionistic Fuzzy Optimization (IFO)

In order to address real-world decision-making scenarios and solve the resulting BL-MOLO (8) problem, IFO play a crucial role and offer numerous advantageous features. Hence, to solve the problem (8) by considering the real-life decision scenarios, we need to develop the equivalent IFO model of problem (8) by converting each linear-objective function $f_{lk}(\mathbf{w}, r), k = 1, 2, \dots, t_l, l = 1, 2, \dots, p$ into an intuitionistic-fuzzy-goal (IFG) by injecting an aspiration level. To find aspiration levels or goals for objective functions, we solve a sole-objective function under the relaxed feasible sets.

Let $f_{lk}^I = f_{lk}(\mathbf{w}_{lk}^*, r^*) = \max_{(\mathbf{w}, r) \in S^{wr}} f_{lk}(\mathbf{w}, r)$ is an Ideal (global optimal) value at the Ideal (best optimal) solution $(\mathbf{w}_{lk}^*, r^*), l = 1, 2, \dots, p$ obtained from solving a single-objective function under feasible convex region S^{wr} . Thus, the IFO model of problem (8) is formulated according to:

$$\text{Determine } (\mathbf{w}, r) \in S^{wr} \tag{16}$$

$$\text{Subject to: } f_{lk}(\mathbf{w}, r) \gtrsim f_{lk}^I, \quad l = 1, 2, \dots, p, k = 1, 2, \dots, t_l \tag{17}$$

Where, $\gtrsim$ is an intuitionistic fuzzy inequality which means that "significantly greater than or equal to".

4.1.1 Membership and Non-membership functions

We will use the membership function $\mu_{lk}(f_{lk}(\mathbf{w}, r))$ to gauge acceptance and satisfaction and the non-membership function $\nu_{lk}(f_{lk}(\mathbf{w}, r))$ to assess rejection and dissatisfaction for each objective function $f_{lk}(\mathbf{w}, r) \gtrsim f_{lk}^I$. To establish these functions, we must first define relaxation bounds and the aspired level. This requires calculating a payoff matrix following Eqs. (10), based on individual ideal solutions $(\mathbf{w}_{lk}^*, r^*)$, obtained by solving each lk th single-objective maximization problem, where $f_{lk}(\mathbf{w}_{lk}^*, r^*) = \max_{(\mathbf{w}, r) \in S^{wr}} f_{lk}(\mathbf{w}, r)$.

$\max_{(\mathbf{w}, r) \in S^{wr}} f_{lk}(\mathbf{w}, r)$	$f_{l1}(\mathbf{w}, r)$	$f_{l2}(\mathbf{w}, r)$	$\dots$	$f_{lt_l}(\mathbf{w}, r)$
$(\mathbf{w}_{l1}^*, r^*)$	$f_{l1}(\mathbf{w}_{l1}^*, r^*)$	$f_{l2}(\mathbf{w}_{l1}^*, r^*)$	$\dots$	$f_{lt_l}(\mathbf{w}_{l1}^*, r^*)$
$(\mathbf{w}_{l2}^*, r^*)$	$f_{l1}(\mathbf{w}_{l2}^*, r^*)$	$f_{l2}(\mathbf{w}_{l2}^*, r^*)$	$\dots$	$f_{lt_l}(\mathbf{w}_{l2}^*, r^*)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(\mathbf{w}_{lt_l}^*, r^*)$	$f_{l1}(\mathbf{w}_{lt_l}^*, r^*)$	$f_{l2}(\mathbf{w}_{lt_l}^*, r^*)$	$\dots$	$f_{lt_l}(\mathbf{w}_{lt_l}^*, r^*)$

(18)

The value of $U_{lk}^\mu = \max\{f_{lk}(\mathbf{w}_{l1}^*, r^*), f_{lk}(\mathbf{w}_{l2}^*, r^*), \dots, f_{lk}(\mathbf{w}_{lt_l}^*, r^*)\}$ gives aspired-level or upper relaxation limit, where as, the value of $L_{lk}^\mu = \min\{f_{lk}(\mathbf{w}_{l1}^*, r^*), f_{lk}(\mathbf{w}_{l2}^*, r^*), \dots, f_{lk}(\mathbf{w}_{lt_l}^*, r^*)\}$ gives lower acceptable relaxation limit for $\mu_{lk}(f_{lk}(\mathbf{w}, r))$. Similar, the upper and lower-relaxation limit for $\nu_{lk}(f_{lk}(\mathbf{w}, r))$ are determined by $U_{lk}^\nu = U_{lk} - \epsilon_{lk}(U_{lk} - L_{lk})$ and $L_{lk}^\nu = L_{lk}$, where $\epsilon_{lk} \in (0, 1)$ respectively. The arithmetic weighted value of $\epsilon_{lk} \in (0, 1)$ first set by DMs.

The geometrical definition of membership, $\mu_{lk}(f_{lk}(\mathbf{w}, r))$ and non membership, $\nu_{lk}(f_{lk}(\mathbf{w}, r))$ functions for lk th, $k = 1, 2, \dots, t_l$ objective functions of DM_l – MOLO problem are shown in the figure:1-(1) and formulated mathematically as:

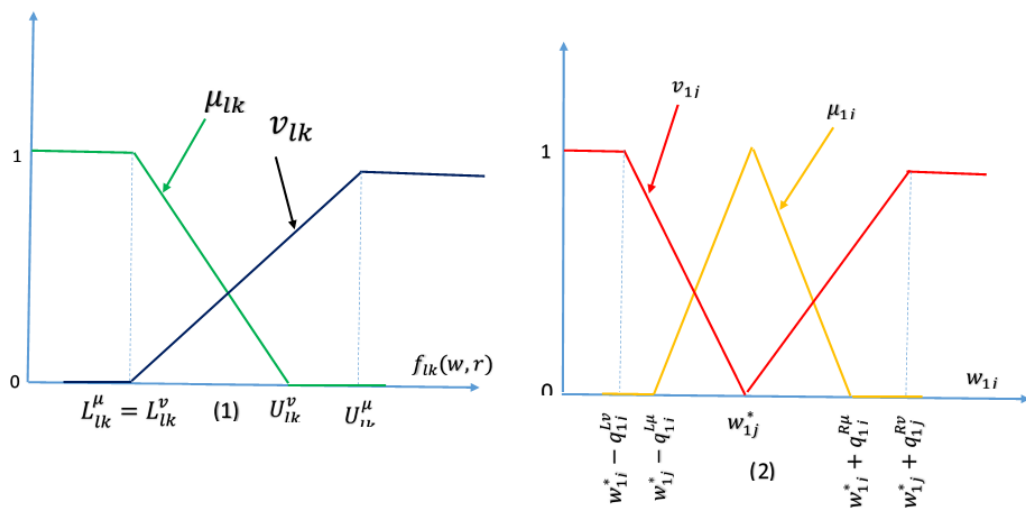


Figure 1: Membership and non membership function for each $f_{lk}(\mathbf{w}, r)$, (a) and w_{1j} , (b)

$$\mu_{lk}(f_{lk}(\mathbf{w}, r)) = \begin{cases} 1, & f_{lk}(\mathbf{w}, r) \geq U_{lk}^\mu \\ \frac{f_{lk}(\mathbf{w}, r) - L_{lk}^\mu}{U_{lk}^\mu - L_{lk}^\mu}, & L_{lk}^\mu \leq f_{lk}(\mathbf{w}, r) \leq U_{lk}^\mu \\ 0, & f_{lk}(\mathbf{w}, r) \leq L_{lk}^\mu \end{cases} \quad (19)$$

$$\nu_{lk}(f_{lk}(\mathbf{w}, r)) = \begin{cases} 0, & f_{lk}(\mathbf{w}, r) \geq U_{lk}^\nu \\ \frac{U_{lk}^\nu - f_{lk}(\mathbf{w}, r)}{U_{lk}^\nu - L_{lk}^\nu}, & L_{lk}^\nu \leq f_{lk}(\mathbf{w}, r) \leq U_{lk}^\nu \\ 1, & f_{lk}(\mathbf{w}, r) \leq L_{lk}^\nu \end{cases} \quad (20)$$

4.2 Goal Programming (GP) method

GP is an optimization technique designed to handle decision-making situation where a number of conditions characterized as goals are to be met as closely as possible. In an intuitionistic fuzzy decision environment, the goal of each decision-maker is to achieve the highest level of achievement. Therefore, each decision-maker should try to maximize and minimize his/her linear membership function (MF) and non-membership function (NMF) in order to attain their aspired-level 1(unity) and 0(zero), respectively by minimize their unwanted deviation variables with the help of interactive GP method. Therefore, the concept of GP model can be employed to formulate the flexible membership and non membership goals with the aspired level 1 and 0 for MF ($\mu_{lk}(f_{lk}(\mathbf{w}, r))$) and NMF ($\nu_{lk}(f_{lk}(\mathbf{w}, r))$) of objective functions defined in the Eqs. (11) accordingly:

$$\begin{aligned}\mu_{lk}(f_{lk}(\mathbf{w}, r)) + D_{lk}^{-\mu} - D_{lk}^{+\mu} &= 1, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l. \\ \nu_{lk}(f_{lk}(\mathbf{w}, r)) + D_{lk}^{-\nu} - D_{lk}^{+\nu} &= 0, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l.\end{aligned}\quad (21)$$

Here, the variables $D_{lk}^{-\mu}, d_{1j}^{-\mu}, D_{lk}^{-\nu}, d_{1j}^{-\nu}$ are used to represents negative-deviation and $D_{lk}^{+\mu}, d_{1j}^{+\mu}, D_{lk}^{+\nu}, d_{1j}^{+\nu}$ are positive-deviation from aspiration levels. Since the over-deviation, $D_{lk}^{+\mu}, d_{1j}^{+\mu}$ and under-deviation, $D_{lk}^{-\nu}, d_{1j}^{-\nu}$ from an IFGs are illustrates highest and lowest-achievement of MF and NMFs respectively [35,39], those variables have been fully-accepted and hence, it removed from the Eqs. (12). Therefore, the Eqs. (12) is reduced to:

$$\begin{aligned}\mu_{lk}(f_{lk}(\mathbf{w}, r)) + D_{lk}^{-\mu} &\geq 1, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l. \\ \nu_{lk}(f_{lk}(\mathbf{w}, r)) - D_{lk}^{+\nu} &\leq 0, \quad \forall l = 1, 2, \dots, p; k = 1, 2, \dots, t_l.\end{aligned}\quad (22)$$

Now, using the summarized goal programming in the Eqs. (13), we introduced the IFO plus GP method called as IFGP method. Therefore, the proposed IFGP model for relaxed upper-level decision-maker (DM_1) of problem (1) is formulated as:

IFGP Model-I for ULDM problem:

$$\begin{aligned}\min \sum_{k=1}^{t_1} (W_{1k}^{\mu} D_{1k}^{-\mu} + W_{1k}^{\nu} D_{1k}^{+\nu}) \\ \text{Subject to: } \begin{cases} \mu_{1k}(f_{1k}(\mathbf{w}, r)) + D_{1k}^{-\mu} \geq 1; \quad \nu_{1k}(f_{1k}(\mathbf{w}, r)) - D_{1k}^{+\nu} \leq 0, \quad \forall k = 1, 2, \dots, t_1. \\ (\mathbf{w}, r) \in S^{wr} \text{ (system constraint from Eqs.(8)); } \quad D_{1k}^{-\mu}, D_{1k}^{+\nu} \geq 0. \end{cases}\end{aligned}\quad (23)$$

4.3 Negotiation between Bilevel Decision-Makers

The benefit of DM_l for $l = 2, 3, \dots, p$ (follower-level decision-maker) depends on an imprecise-tolerance of the decision control-variable of a DM_1 . The decision-control variable at the ULDM is set within a tolerance range in order to avoid a decision lock for the follower-level, i.e,

$$w_{1j} \approx w_{1j}^*, \quad j = 1, 2, \dots, n_1.$$

Thus, the decision-control variable range for $w_{1j}, j = 1, 2, \dots, n_1$ should be "nearly w_{1j}^* , with maximum acceptable right relaxation values $q_{1j}^{R\mu} > 0$ and maximum acceptable left relaxation values $q_{1j}^{L\mu} > 0$. This values are specified by the ULDM. Those parameters $q_{1j}^{R\mu}$ and $q_{1j}^{L\mu}$ haven't necessarily the same values.

The level of optimality for the decision-control variable monitored by DM_1 is fully satisfactory at w_{1j}^* , monotonically increasing in the $[w_{1j}^* - q_{1j}^{L\mu}, w_{1j}^*]$, monotonically decreasing in the $[w_{1j}^*, w_{1j}^* + q_{1j}^{R\mu}]$, otherwise is zero-satisfactory by ULDM. But due to inaccurate decisions, zero-acceptance by ULDM does not imply complete-rejection, thus the DM may interpret rejection in an optimistic manner. As a result, regarding rejection over the interval $[w_{1j}^* - q_{1j}^{L\mu}, w_{1j}^* - q_{1j}^{L\mu}]$ and $[w_{1j}^* + q_{1j}^{R\mu}, w_{1j}^* + q_{1j}^{R\mu}]$ where, $q_{1j}^{L\mu} \geq q_{1j}^{L\mu} > 0$, and/or $0 < q_{1j}^{R\mu} \leq q_{1j}^{R\mu}$, the DM is open-minded.

With the help of this discussion, we can built the membership, $\mu_{1j}(w_{1j})$ and non membership functions, $\nu_{1j}(w_{1j})$ for the decision-control variable w_{1j} monitored by DM_1 in the intuitionistic fuzzy environment, as illustrated in Figure1- (2). These functions represent the acceptance and rejection levels for the variable w_{1j} by ULDM and defined mathematically as:

$$\mu_{1j}(w_{1j}) = \begin{cases} \frac{w_{1j} - (w_{1j}^* - q_{1j}^{L\mu})}{q_{1j}^{L\mu}}, & \text{if } w_{1j}^* - q_{1j}^{L\mu} \leq w_{1j} \leq w_{1j}^*, \\ \frac{w_{1j}^* + q_{1j}^{R\mu} - w_{1j}}{q_{1j}^{R\mu}}, & \text{if } w_{1j}^* \leq w_{1j} \leq w_{1j}^* + q_{1j}^{R\mu}, \end{cases} \quad j = 1, 2, \dots, n_1. \quad (24)$$

$$\nu_{1j}(w_{1j}) = \begin{cases} \frac{w_{1j}^* - w_{1j}}{q_{1j}^{L\nu}}, & \text{if } w_{1j}^* - q_{1j}^{L\nu} \leq w_{1j} \leq w_{1j}^*, \\ \frac{w_{1j}^* - w_{1j}}{q_{1j}^{R\nu}}, & \text{if } w_{1j}^* \leq w_{1j} \leq w_{1j}^* + q_{1j}^{R\nu}, \\ 1, & \text{otherwise.} \end{cases} \quad j = 1, 2, \dots, n_1. \quad (25)$$

To formulate the next IFGP model- II (17), the vector of decision-variables $w_{1j}, j = 1, 2, \dots, n_1$ monitored by ULDM used as a binding additional constraints for follower-level DM_l – MOLDM, Eqs.(8) where, $l = 2, 3, \dots, p$ problem. Therefore, the IFGP model- II for NBL-MOLFCCo problem (1) given as follows:

IFGP Model- II for LLDM problem:

$$\begin{aligned} \min \quad & \sum_{l=1}^p \sum_{k=1}^{t_l} (W_{lk}^{\mu} D_{lk}^{-\mu} + W_{lk}^{\nu} D_{lk}^{+\nu}) + \sum_{j=1}^{n_1} [w_{1j}^{L\mu} d_{1j}^{-L\mu} + w_{1j}^{R\mu} d_{1j}^{-R\mu} + w_{1j}^{L\nu} d_{1j}^{+L\nu} + w_{1j}^{R\nu} d_{1j}^{+R\nu}] \\ \text{Subject to:} \quad & \begin{cases} \mu_{1k}(f_{1k}(\mathbf{w}, r)) + D_{1k}^{-\mu} \geq 1; \quad \nu_{1k}(f_{1k}(\mathbf{w}, r)) - D_{1k}^{+\nu} \leq 0, \quad \forall k = 1, 2, \dots, t_1, \\ \mu_{1j}^L(w_{1j}) + d_{1j}^{-L\mu} \geq 1; \quad \nu_{1j}^L(w_{1j}) - d_{1j}^{+L\nu} \leq 0, \quad \forall j = 1, 2, \dots, n_1, \\ \mu_{1j}^R(w_{1j}) + d_{1j}^{-R\mu} \geq 1; \quad \nu_{1j}^R(w_{1j}) - d_{1j}^{+R\nu} \leq 0, \quad \forall j = 1, 2, \dots, n_1, \\ (\mathbf{w}, r) \in S^{wr} \text{ (see Eqs. 8); } \quad D_{1k}^{-\mu}, D_{1k}^{+\nu}, d_{1j}^{-L\mu}, d_{1j}^{+L\nu}, d_{1j}^{-R\mu}, d_{1j}^{+R\nu} \geq 0. \end{cases} \end{aligned} \quad (26)$$

Here, $W_{lk}^{\mu}, W_{lk}^{\nu}, w_{1j}^{L\mu}, w_{1j}^{R\mu}, w_{1j}^{L\nu}$, and $w_{1j}^{R\nu}$ are numerical weights indicates the relative importance of IFG (9). These parametric values are calculated using the weighting scheme introduced by [35] as follows:

$$W_{lk}^{\mu} = \frac{1}{U_{lk}^{\mu} - L_{lk}^{\mu}}, \quad \text{and} \quad W_{lk}^{\nu} = \frac{1}{U_{lk}^{\nu} - L_{lk}^{\nu}} \quad (27)$$

$$w_{1j}^{L\mu} = \frac{1}{q_{1j}^{L\mu}}; \quad w_{1j}^{R\mu} = \frac{1}{q_{1j}^{R\mu}}, \quad w_{1j}^{L\nu} = \frac{1}{q_{1j}^{L\nu}} \quad \text{and} \quad w_{1j}^{R\nu} = \frac{1}{q_{1j}^{R\nu}}. \quad (28)$$

Definition 4.1. Let w^* with corresponding r^* be an intuitionistic fuzzy (IF)-efficient solution to proposed IFGP model-II (17). Then there does not exist another vector $(\mathbf{w}, r) \in S^{wr}$ such that

$$\mu_{lk}(f_{lk}(\mathbf{w}, r)) \geq \mu_{lk}(f_{lk}(\mathbf{w}^*, r^*)) \wedge \nu_{lk}(f_{lk}(\mathbf{w}, r)) \leq \nu_{lk}(f_{lk}(\mathbf{w}^*, r^*)), \quad \forall l, k, \text{ and}$$

$$\mu_{lk}(f_{lk}(\mathbf{w}, r)) > \mu_{lk}(f_{lk}(\mathbf{w}^*, r^*)) \wedge \nu_{lk}(f_{lk}(\mathbf{w}, r)) < \nu_{lk}(f_{lk}(\mathbf{w}^*, r^*)) \text{ for some } l, k.$$

Theorem 4.1. A unique IF-efficient solution to proposed IFGP model-II (17) is also known as a Pareto-optimal (efficient) solution to the NBL-MOLFCCO problem (1).

Proof. Suppose that $(\mathbf{w}^*, r^*)$ is unique IF-efficient solution to IFGP model-II (17). Thus, we have:

$$\mu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) > \mu_{1k}(f_{1k}(\mathbf{w}, r)) \wedge \nu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) < \nu_{1k}(f_{1k}(\mathbf{w}, r)), \quad \forall k = 1, 2, \dots, t_1.$$

Assume that $(\mathbf{w}^*, r) = (\mathbf{w}_1^*, \mathbf{w}_2^*, \dots, \mathbf{w}_p^*, r^*)$ is not efficient solution to BL-MOLO (3) problem. Then there exist a new vector $(\mathbf{w}, r) \in S^{wr}$ such that

$$f_{1k}(\mathbf{w}^*, r^*) \leq f_{1k}(\mathbf{w}, r), \quad \forall k = 1, 2, \dots, t_1 \quad \text{and} \quad f_{1k}(\mathbf{w}^*, r^*) < f_{1k}(\mathbf{w}, r) \text{ for some } k.$$

Following that, we obtain:

$$f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^\mu \leq f_{1k}(\mathbf{w}, r) - L_{1k}^\mu, \quad \forall k \quad \text{and} \quad f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^\mu < f_{1k}(\mathbf{w}, r) - L_{1k}^\mu \text{ for some } k.$$

$$U_{1k}^\nu - f_{1k}(\mathbf{w}^*, r^*) \geq U_{1k}^\nu - f_{1k}(\mathbf{w}, r), \quad \forall k \quad \text{and} \quad U_{1k}^\nu - f_{1k}(\mathbf{w}^*, r^*) > U_{1k}^\nu - f_{1k}(\mathbf{w}, r), \text{ for some } k.$$

Divide the above inequality using positive quantity $U_{1k}^\mu - L_{1k}^\mu$ and $U_{1k}^\nu - L_{1k}^\nu$ respectively, to obtain:

$$\frac{f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^\mu}{U_{1k}^\mu - L_{1k}^\mu} \leq \frac{f_{1k}(\mathbf{w}, r) - L_{1k}^\mu}{U_{1k}^\mu - L_{1k}^\mu}, \quad \forall k \quad \text{and} \quad \frac{f_{1k}(\mathbf{w}^*, r^*) - L_{1k}^\mu}{U_{1k}^\mu - L_{1k}^\mu} < \frac{f_{1k}(\mathbf{w}, r) - L_{1k}^\mu}{U_{1k}^\mu - L_{1k}^\mu}, \text{ for some } k.$$

$$\frac{U_{1k}^\nu - f_{1k}(\mathbf{w}^*, r^*)}{U_{1k}^\nu - L_{1k}^\nu} \geq \frac{U_{1k}^\nu - f_{1k}(\mathbf{w}, r)}{U_{1k}^\nu - L_{1k}^\nu}, \quad \forall k \quad \text{and} \quad \frac{U_{1k}^\nu - f_{1k}(\mathbf{w}^*, r^*)}{U_{1k}^\nu - L_{1k}^\nu} > \frac{U_{1k}^\nu - f_{1k}(\mathbf{w}, r)}{U_{1k}^\nu - L_{1k}^\nu}, \text{ for some } k.$$

This implies that

$$\mu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) \leq \mu_{1k}(f_{1k}(\mathbf{w}, r)), \quad \forall k \quad \text{and} \quad \mu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) < \mu_{1k}(f_{1k}(\mathbf{w}, r)), \text{ for some } k.$$

$$\nu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) \geq \nu_{1k}(f_{1k}(\mathbf{w}, r)), \quad \forall k \quad \text{and} \quad \nu_{1k}(f_{1k}(\mathbf{w}^*, r^*)) > \nu_{1k}(f_{1k}(\mathbf{w}, r)), \text{ for some } k.$$

This implies that $(\mathbf{w}, r)$ is also an IF-efficient solution to (17) problem. Thus, we have reached at contradiction to $(\mathbf{w}^*, r^*)$ is unique IF-efficient solution to (17) problem. Therefore, the $\mathbf{x}^* = \frac{\mathbf{w}^*}{r^*}$ is Pareto-optimal (efficient) solution to NBL-MOLFCCO (1) problem. $\square$

5 Algorithm of proposed methodology for NBL-MOLFCCo problem

The algorithm of the proposed efficient method, which combined the benefits of the interactive goal programming approach and the intuitionistic fuzzy decision set for solving the introduced NBL-MOLFCCo (1) problem, is developed as follows:

1. Based on discussion given in subsection 3.2, the crisp-parametric model of NBL-MOLFCCo problem (1) is formulated according to problem (3).
2. Following the chance-constraint optimization strategy discussed in subsection 3.3, the deterministic constraint of problem (3) is developed as illustrated in Eqs. 6.
3. Using introduced linearization techniques, convert a problem (3) into crisp BL-MOLO problem (8).
4. Solve each linear objective function $f_{lk}(\mathbf{w}, r)$ independently under the convex constraint set S^{wr} and then construct a payoff-matrix (10) to obtain the upper U_{lk}^μ, U_{lk}^ν and lower L_{lk}^μ, L_{lk}^ν relaxation limits for each $f_{lk}(\mathbf{w}, r), l = 1, 2, \dots, p, k = 1, 2, \dots, t_l$, based on the obtained individual ideal solution.
5. Calculate the weights W_{lk}^μ and W_{lk}^ν as defined in Eqs. (18) for each IFG of objective functions $f_{lk}(\mathbf{w}, r), k = 1, 2, \dots, t_l, l = 1, 2, \dots, p$
6. Construct both the membership function $\mu_{lk}(f_{lk}(\mathbf{w}, r))$ and non membership function $\nu_{lk}(f_{lk}(\mathbf{w}, r))$ based on Eqs. (11) for each of the objective $f_{lk}(\mathbf{w}, r)$ function in the BL-MOLO problem (8).
7. Formulate the proposed IFGP Model- I (14) for ULDM problem and solve it to generate decision-control solution of DM_1 , set as: $\mathbf{w}_1^* = (w_{11}^*, w_{12}^*, \dots, w_{1n_1}^*)$
8. The leader-level decision-maker (DM_1) set the maximum acceptance for membership and minimum rejection for non membership as right $q_{1j}^{R\nu}, q_{1j}^{R\mu}$, and left $q_{1j}^{L\nu}, q_{1j}^{L\mu}$, relaxation limit on its decision-control variables $\mathbf{w}_1 = (w_{11}, w_{12}, \dots, w_{1n_1})$.
9. Build the linear-membership $\mu_{1j}(w_{1j})$ see Eqs (15) and linear non membership $\nu_{1j}(w_{1j})$ see Eqs.(16) functions for binding-constraints $w_{1j} \approx w_{1j}^*, j = 1, 2, \dots, n_1$.
10. Assign the weight of importance relation for imprecise aspiration-level of decision-control variables using Eqs.(19).
11. Formulate the next proposed IFGP Model- II (17) for NBL-MOLFCCo problem (1) and solve it to get a preferable compromise IF-efficient solution.
12. If all decision-makers ($DM_l, l = 1, 2, \dots, p$) are satisfied for the preferable IF-efficient solution in Step 11, stop and the Pareto-optimal solution to NBL-MOLFCCo problem (1) is obtained by backward substitution $\mathbf{x}^* = (x_1^*, x_2^*, \dots, x_p^*) = (\frac{w_1^*}{r}, \frac{w_2^*}{r}, \dots, \frac{w_p^*}{r})$. Else, modify the upper and lower-relaxation limit for objective function, $f_{lk}(\mathbf{w}, r), k = 1, 2, \dots, t_l, l = 1, 2, \dots, p$ and/ or decision-control variable $w_{1j}, j = 1, 2, \dots, n_1$, then go to Step 5.

We developed the distance-metric function, $\mathcal{D}(\mathbf{x})$, to analysis the performance of different algorithms available in the literature study for solving the NBL-MOLFCCo problem (1).

Definition 5.1. Let f_{lk}^I is Ideal value of crisp linear-fractional objective function $f_{lk}(\mathbf{x})$, $l = 1, \dots, p$, $k = 1, 2, \dots, t_l$ obtained by solving $f_{lk}^I = \max_{\mathbf{x} \in X} f_{lk}(\mathbf{x})$ and f_{lk}^* is the objective/ optimal values at the IF-efficient solution $\mathbf{x}^* \in X$ obtained using available approaches in the literature. Then the distance-metric value $\mathcal{D}(\mathbf{x})$ between f_{lk}^I and $f_{lk}^* = f_{lk}^*(\mathbf{x}^*)$ can be calculated as follows [35,39]:

$$\mathcal{D}(\mathbf{x}^*) = \frac{\sqrt{\sum_{l=1}^p \sum_{k=1}^{t_l} (f_{lk}^I - f_{lk}^*)^2}}{2 \times \sum_{l=1}^p t_l} \quad (29)$$

Where t_l , $l = 1, 2, \dots, p$ is presents the total number of neutrosophic linear-fractional objective function under l^{th} DM in the NBL-MOLFCCo problem (1). The smaller value of $\mathcal{D}(\mathbf{x}^*)$ delivered the most preferable IF-efficient solution is $\mathbf{x}^*$ among all existing solutions.

6 Numerical Illustrations

To illustrate and assess the suggested efficient method, two distinct numerical-examples and real-life application problem are examined in this section, where the example 6.2 is directly lifted from Baky [8] and Saraj and Safaraei [51] papers.

6.1 Numerical Example 1:

Consider NBL-MOLFCCO problem with neutrosophic linear parameters and right-hand-side quantity follow Cauchy random distribution having mean (θ) and standard division (σ) mathematically formulated as:

$$\begin{aligned}
& \text{ULDM} \left\{ \begin{aligned} & \max_{x_1} (f_{11}(x), f_{12}(x)), \\ & \text{where,} \\ & f_{11}(x) = \frac{(2+3[0.2,0.4])x_1 + (-4+0.5[0.2,0.5])x_2 + (3+3[0.15,0.25])x_3 + (1+0.3[0,1])}{(0.3+2[0.3,0.5])x_1 + (3+1.5[0.1,0.5])x_2 + (1+5[0.1,0.5])x_3}, \\ & f_{12}(x) = \frac{3[0.1,0.6]x_1 + (0.3+4[0.3,0.5])x_2 + (-3.3+2[0.4,0.6])x_3 + (3+3[0.15,0.25])}{3[0.3,0.7]x_1 + (2+1.7[0.2,0.5])x_2 + (1+1.1[0.4,1])x_3}. \end{aligned} \right. \\
& \text{LLDM} \left\{ \begin{aligned} & \text{Use } x_1 \text{ and } x_2, x_3 \text{ solves,} \\ & \max_{x_2} (f_{21}(x), f_{22}(x)), \\ & \text{where,} \\ & f_{21}(x) = \frac{(0.3+2[0.5,0.9])x_1 + (1.19+2[0.1,0.3])x_2 + (0.3+2.9[0.5,1])}{(0.4+3[0.4,0.6])x_1 + (1.4+1[0.5,0.7])x_2 + (0.1+2[0.6,0.8])x_3}, \\ & f_{22}(x) = \frac{(3.9+6[0.2,0.5])x_1 + (0.5+1.06[0.5,1])x_2 - (0.2+1.9[0.5,1])x_3}{(1.3+2[0.2,0.4])x_1 + (0.7+2[0.8,0.9])x_2 + (0.8+2[0.6,0.8])x_3}. \\ & \max_{x_3} (f_{31}(x), f_{32}(x)), \\ & \text{where,} \\ & f_{31}(x) = \frac{(4.3+5[0.2,0.6])x_1 - (2+3[0.2,0.4])x_2 + (2+3[0.2,0.4])x_3 + (0.36+3[0.2,0.4])}{(0.5+3[0.12,0.2])x_1 + (1.45+4[0.5,0.7])x_2}, \\ & f_{32}(x) = \frac{(3.2+3[0.5,1])x_1 - (1.3+2[0.2,0.5])x_3}{(0.3+2[0.3,0.4])x_1 + (2.1+1.5[0.3,0.5])x_2 + (2.1+2.4[0.1,0.3])x_3}. \end{aligned} \right. \quad (30)
\end{aligned}$$

Subject to:

$$\left\{ \begin{aligned} & (1.5 + 1[0.2, 0.5])x_1 + (1.2 + 0.9[0.1, 0.2])x_2 \geq (2.1 + 2[0.6, 0.7]), \\ & Pr[(1.5 + 2[0.6, 0.8])x_1 + (0.44 + 2[0.5, 0.7])x_2 + (0.7 + 2[0.8, 0.9])x_3 \leq b_1] \geq 0.95, \\ & Pr[(0.24 + 2[0.6, 0.8])x_1 - (1.5 + 2[0.3, 0.5])x_2 + (2 + 0.6[0.6, 0.9])x_3 \leq b_2] \geq 0.95, \\ & x_1, x_2, x_3 \geq 0. \end{aligned} \right.$$

where, the random variables b_1, b_2 having mean $\theta_1 = 11.002, \theta_2 = 5.767$ and standard division $\sigma_1 = 7.4$, and $\sigma_2 = 3.9$.

Based on interval-ratio and chance-constraint optimization method developed in the proposed methodology, the analogous crisp-parametric and deterministic constraint model of problem (21) formulated as:

$$\begin{aligned}
& \text{ULDM} \left\{ \begin{aligned} & \max_{x_1} \{f_{11}(x) = \frac{3.2x_1 - 3.75x_2 + 3.75x_3 + 1.3}{0.9x_1 + 3.15x_2 + 1.5x_3}, \quad f_{12}(x) = \frac{1.8x_1 + 2.3x_2 - 2.1x_3 + 3.75}{0.9x_1 + 2.34x_2 + 1.44x_3}\} \\ & \text{Use } x_1 \text{ and } x_2, x_3 \text{ solves:} \end{aligned} \right. \\
& \text{LLDM} \left\{ \begin{aligned} & \max_{x_2} \{f_{21}(x) = \frac{2.1x_1 + 1.79x_2 + 3.2}{1.6x_1 + 1.9x_2 + 1.3x_3}, \quad f_{22}(x) = \frac{6.9x_1 + 1.56x_2 - 2.1x_3}{1.7x_1 + 2.3x_2 + 2x_3}\} \\ & \max_{x_3} \{f_{31}(x) = \frac{7.3x_1 - 3.2x_2 + 3.2x_3 + 1.56}{0.86x_1 + 3.45x_2}, \quad f_{32}(x) = \frac{6.2x_1 - 2.3x_3}{0.9x_1 + 2.55x_2 + 2.34x_3}\} \end{aligned} \right. \quad (31) \\
& \text{Subject to: } S = \left\{ \begin{aligned} & 2x_1 + 1.38x_2 \geq 3.3, \quad 2.7x_1 + 1.44x_2 + 2.3x_3 \leq 10.95, \\ & 1.44x_1 - 2.1x_2 + 2.36x_3 \leq 5.74, \quad x_1, x_2, x_3 \geq 0. \end{aligned} \right.
\end{aligned}$$

The individual maximum i.e., solve $\max_{x \in S} f_{lk}(x)$ and minimum i.e., solve $\min_{x \in S} f_{lk}(x)$ value of NBL-MOLFCCO (22) model presented in the Table 2.

Table 1: Individual min. and max. value for model (22)

	f_{11}	f_{12}	f_{21}	f_{22}	f_{31}	f_{32}
$\max f_{lk}(x)$	4.431	4.526	2.525	4.059	12.802	6.889
$\min f_{lk}(x)$	-1.136	0.016	0.067	-0.26	-0.868	-0.547

Following the introduced linearized techniques, the NBL-MOLFCCO (22) model can be reformulated according to:

$$\text{ULDM} \left\{ \begin{array}{l} \max\{f_{11}(\mathbf{w}, r) = 3.2w_1 - 3.75w_2 + 3.75w_3 + 1.3r, \quad f_{12}(\mathbf{w}, r) = 1.8w_1 + 2.3w_2 - 2.1w_3 + 3.75r\} \\ \text{Use } w_1 \text{ and } w_2, w_3 \text{ solves:} \\ \max\{f_{21}(\mathbf{w}, r) = 2.1w_1 + 1.79w_2 + 3.2r, \quad f_{22}(\mathbf{w}, r) = 6.9w_1 + 1.56w_2 - 2.1w_3\} \\ \max\{f_{31}(\mathbf{w}, r) = 7.3w_1 - 3.2w_2 + 3.2w_3 + 1.56r, \quad f_{32}(\mathbf{w}, r) = 6.2w_1 - 2.3w_3\} \\ \text{Subject to: } S^{wr} = \left\{ \begin{array}{l} 0.9w_1 + 3.15w_2 + 1.5w_3 \leq 1, \quad 0.9w_1 + 2.34w_2 + 1.44w_3 \leq 1, \\ 1.6w_1 + 1.9w_2 + 1.3w_3 \leq 1, \quad 1.7w_1 + 2.3w_2 + 2w_3 \leq 1, \\ 0.86w_1 + 3.45w_2 \leq 1, \quad 0.9w_1 + 2.55w_2 + 2.34w_3 \leq 1, \\ 2w_1 + 1.38w_2 - 3.3r \geq 0, \quad 2.7w_1 + 1.44w_2 + 2.3w_3 - 10.95r \leq 0, \\ 1.44w_1 - 2.1w_2 + 2.36w_3 - 5.74r \leq 0, \quad w_1, w_2, w_3 \geq 0, r > 0. \end{array} \right. \end{array} \right. \quad (32)$$

Formulate the IFGP Model for DM_1 -MOLDM (23) problem and solve it.

IFGP Model- I for DM_1 -MOLDM problem:

$$\begin{aligned} & \min 0.295D_{11}^{-\mu} + 0.299D_{11}^{+\mu} + 0.421D_{12}^{-\mu} + 0.434D_{11}^{+\mu} \\ & \text{Subject to: } \left\{ \begin{array}{l} 3.2w_1 - 3.75w_2 + 3.75w_3 + 1.3r + 3.383D_{11}^{-\mu} \geq 2.346, \\ -3.2w_1 + 3.75w_2 - 3.75w_3 - 1.3r - 3.337D_{11}^{+\mu} \leq -2.3, \\ 1.8w_1 + 2.3w_2 - 2.1w_3 + 3.75r + 2.3798D_{12}^{-\mu} \geq 2.396, \\ -1.8w_1 - 2.3w_2 + 2.1w_3 - 3.75r - 2.3038D_{12}^{+\mu} \leq -2.32, \\ S^{wr} \text{ (Use the system constraint from Eqs. (23)), } D_{11}^{-\mu}, D_{11}^{+\mu}, D_{12}^{-\mu}, D_{12}^{+\mu} \geq 0. \end{array} \right. \end{aligned} \quad (33)$$

Thus, the optimal solution of Model (24) is $w_1^* = 0.5882353$, $w_2^* = 0$, $w_3^* = 0$, $r = 0.3565062$.

For the sake of avoiding decision-lock, the control variable of upper-level w_1 set as binding constraint for the next-level with given acceptable tolerance $q_{11}^{R\mu} = q_{11}^{L\mu} = 0.025$ and rejected tolerance $q_{11}^{L\nu} = q_{11}^{R\nu} = 0.05$. Therefore, the suggested IFGP Model- II for NBL-MOLFCCCO (21) problem formulated and solved as:

$$\begin{aligned} & \min 0.295D_{11}^{-\mu} + 0.299D_{11}^{+\mu} + 0.421D_{12}^{-\mu} + 0.434D_{11}^{+\mu} + 0.433D_{21}^{-\mu} + 0.446D_{21}^{+\mu} \\ & + 0.233D_{22}^{-\mu} + 0.236D_{22}^{+\mu} + 0.175D_{31}^{-\mu} + 0.195D_{31}^{+\mu} + 0.238D_{32}^{-\mu} + 0.241D_{32}^{+\mu} \\ & + 0.404D_{11}^{-\mu} + 0.404D_{11}^{+\mu} + 0.204D_{11}^{-\mu} + 0.204D_{11}^{+\mu} \end{aligned}$$

Subject to:

$$\begin{aligned} & 2.1w_1 + 1.79w_2 + 3.2r + 2.31D_{21}^{-\mu} \geq 2.376, \\ & -2.1w_1 - 1.79w_2 - 3.2r - 2.24D_{21}^{+\mu} \leq -2.31, \\ & 6.9w_1 + 1.56w_2 - 2.1w_3 + 3.831D_{22}^{-\mu} \geq 4.058, \\ & -6.9w_1 - 1.56w_2 + 2.1w_3 - 3.783D_{22}^{+\mu} \leq -4.01, \\ & 7.3w_1 - 3.2w_2 + 3.2w_3 + 1.56r + 5.718D_{31}^{-\mu} \geq 4.85, \\ & -7.3w_1 + 3.2w_2 - 3.2w_3 - 1.56r - 5.118D_{31}^{+\mu} \leq -4.25, \\ & 6.2w_1 - 2.3w_3 + 4.193D_{32}^{-\mu} \geq 3.647, \\ & -6.2w_1 + 2.3w_3 - 4.146D_{32}^{+\mu} \leq -3.6, \\ & S^{wr} \text{ (Use the system constraint from Eqs.(24))}, \\ & D_{11}^{-\mu}, D_{11}^{+\mu}, D_{12}^{-\mu}, D_{12}^{+\mu}, D_{21}^{-\mu}, D_{21}^{+\mu}, D_{22}^{-\mu}, D_{22}^{+\mu}, \\ & D_{31}^{-\mu}, D_{31}^{+\mu}, D_{32}^{-\mu}, D_{32}^{+\mu}, d_{11}^{-L\mu}, d_{11}^{-R\mu}, d_{11}^{+L\mu}, d_{11}^{+R\mu} \geq 0. \end{aligned} \tag{34}$$

Using Lingo-19 software, the compromise IF-efficient solution of the BL-MOLO problem is generated as: $w_1 = 0.5882353$, $w_2 = 0$, $w_3 = 0$, $r = 0.3565062$ and its corresponding efficient solution of NBL-MOLFCC0 (21) problem using backward technique obtained as: $\mathbf{x}^* = (x_1 = 1.650, x_2 = 0, x_3 = 0)$. The objective values, degree of satisfaction and dissatisfaction at these efficient solution are illustrated in Table-3.

Table 2: Objective, satisfaction and dissatisfaction value at $\mathbf{x}^*$ for model (22)

	f_{11}	f_{12}	f_{21}	f_{22}	f_{31}	f_{32}
Objective value ($f_{lk}(\mathbf{x}^*)$)	4.431	4.525	2.525	4.059	9.589	6.889
Satisfaction level ($\mu_{lk}(\mathbf{x}^*)$)	1	1	1	1	0.94	1
Dissatisfaction level ($\nu_{lk}(\mathbf{x}^*)$)	0	0	0	0	0.04	0

6.2 Numerical Example 2:

The proposed method cannot be directly compared with existing methodologies, as the introduced NBL-MOLFCC0 problem is a novel formulation not previously addressed in the literature. While the NBL-MOLFCC0 problem represents a general framework applicable to real-world decision-making scenarios involving uncertain parameters and chance constraints and can be reduced to fuzzy, intuitionistic fuzzy, or crisp-type problems, a direct comparison is not feasible due to its unique characteristics. Therefore, we will use a crisp-type problem adapted from Baky [8] and Saraj and Safarai [51] as a benchmark to evaluate the efficiency of the proposed method.

$$\begin{aligned}
& \text{ULDM} \left\{ \begin{array}{l} \max(f_{11}(\mathbf{x}) = \frac{-x_1-4x_2+x_3+1}{2x_1+3x_2+x_3+2}; \quad f_{12}(\mathbf{x}) = \frac{-2x_1+x_2+3x_3+4}{2x_1-x_2+x_3+5}) \quad [DM_1] \\ \text{Use } x_1, \text{ where } x_2 \text{ and } x_3 \text{ solves;} \end{array} \right. \\
& \text{LLDM} \left\{ \begin{array}{l} \max(f_{21}(\mathbf{x}) = \frac{3x_1-2x_2+2x_3}{x_1+x_2+x_3+3}; \quad f_{22}(\mathbf{x}) = \frac{-7x_1-2x_2+x_3+1}{5x_1+2x_2+x_3+1}) \quad [DM_2] \\ \max(f_{31}(\mathbf{x}) = \frac{x_1+x_2+x_3-4}{x_1-2x_2+10x_3+6}; \quad f_{32}(\mathbf{x}) = \frac{2x_1-x_2+x_3+4}{-x_1+x_2+x_3+10}) \quad [DM_3] \end{array} \right.
\end{aligned} \tag{35}$$

Since the given problem (26) formulated as crisp-parametric and deterministic-constraint model, directly go to the linearization procedure to obtain the following BL-MOLO problem:

$$\text{ULDM} \left\{ \begin{array}{l} \max(f_{11}(\mathbf{w}, r) = -w_1 - 4w_2 + w_3 + r; \quad f_{12}(\mathbf{w}, r) = -2w_1 + w_2 + 3w_3 + 4r) \\ \text{Use } w_1, \text{ where } w_2 \text{ and } w_3 \text{ solves} \\ \text{LLDM} \left\{ \begin{array}{l} \max(f_{21}(\mathbf{w}, r) = 3w_1 - 2w_2 + 2w_3; \quad f_{22}(\mathbf{w}, r) = -7w_1 - 2w_2 + w_3 + r) \\ \max(f_{31}(\mathbf{w}, r) = w_1 + w_2 + w_3 - 4r; \quad f_{32}(\mathbf{w}, r) = 2w_1 - w_2 + w_3 + 4r) \end{array} \right. \end{array} \right. \tag{36}$$

Subject to:

$$\begin{aligned}
& 2w_1 + 3w_2 + w_3 \leq 1, \\
& 2w_1 - w_2 + w_3 \leq 1, \\
& w_1 + w_2 + w_3 + 3r \leq 1, \\
& 5w_1 + 2w_2 + w_3 + r \leq 1, \\
& w_1 - 2w_2 + 10w_3 + 6r \leq 1, \\
& -w_1 + w_2 + w_3 + 10r \leq 1, \\
& w_1, w_2, w_3 \geq 0, r > 0.
\end{aligned} \tag{37}$$

Following the next steps of our proposed method, we generated different results as given in Table 4. Based on information given in Table 4, we formulate IFGP Model-I for the upper-level decision-making problem as shown below:

Table 3: The numerical values $U_{lk}^\mu, U_{lk}^\nu, L_{lk}^\mu = L_{lk}^\nu, W_{lk}^\mu$ and W_{lk}^ν for each $f_{lk}(\mathbf{x})$

	$f_{11}(\omega, r)$	$f_{12}(\omega, r)$	$f_{21}(\omega, r)$	$f_{22}(\omega, r)$	$f_{31}(\omega, r)$	$f_{32}(\omega, r)$
$U_{lk}^\mu = \max_{(\omega, r) \in S^{wr}} f_{lk}(\omega, r)$	0.125	0.5606	0.6164	0.125	0.0204	0.8258
$L_{lk}^\mu = L_{lk}^\nu = \min_{(\omega, r) \in S^{wr}} f_{lk}(\omega, r)$	-0.50	-0.1818	-1.1818	-0.33	0.003	-0.56
U_{lk}^ν	0.1	0.5	0.6	0.1	0.8	-0.5
$W_{lk}^\mu = \frac{1}{U_{lk}^\mu - L_{lk}^\mu}$	1.6	1.78	1.25	0.765	2.85	1.215
$W_{lk}^\nu = \frac{1}{U_{lk}^\nu - L_{lk}^\nu}$	1.61	1.786	1.28	0.768	2.857	1.25

IFGP Model- I for DM₁-MOLDM problem

$$\begin{aligned}
& \min 1.6D_{11}^{-\mu} + 1.61D_{11}^{+\nu} + 1.78D_{12}^{-\mu} + 1.786D_{12}^{+\nu} \\
& \text{Subject to:} \\
& -w_1 - 4w_2 + w_3 + r + 0.625D_{11}^{-\mu} \geq 0.125, \\
& w_1 + 4w_2 - w_3 - r - 0.62D_{11}^{+\nu} \leq -0.1, \\
& -2w_1 + w_2 + 3w_3 + 4r + 0.5606D_{12}^{-\mu} \geq 0.5606, \\
& 2w_1 - w_2 - 3w_3 - 4r - 0.56D_{12}^{+\nu} \leq -0.5, \\
& S^{wr} \text{ (Use the system constraint from Eqs. (27)).}
\end{aligned} \tag{38}$$

Solving the above problem (28) via LINGO-19 software, we obtained the solution ($w_1^* = 0, w_2^* = 0.0192, w_3^* = 0.0577, r = 0.0769$) to the ULDM problem. Thus, the decision control variable of *ULDM* is $w_1 = 0$.

Now, the ULDM set acceptable and rejected relaxation on its decision-control variable $w_1 = 0$ to avoid decision-lock for follower-level DMs, as: $q_{11}^{R\mu} = 0.155$ and $q_{11}^{R\nu} = 1$. The degree of membership and nonmembership for IFG of the decision control variable w_1 are measured by $\mu_{11}(w_{11}) = \frac{0.155-w_1}{0.155}$ and $\nu_{11}(w_{11}) = \frac{w_1}{1}$, respectively. Also, the weight for IFG of the decision control variable w_1 using the formula set in Eqs. (19), are: $w_1^{R\mu} = 6.45, w_1^{R\nu} = 1$. Therefore, the final proposed IFGP Model- II for the given problem (26) formulated as:

IFGP Model- II for problem (26)

$$\begin{aligned}
& \min[1.6D_{11}^{-\mu} + 1.61D_{11}^{+\nu} + 1.78D_{12}^{-\mu} + 1.786D_{12}^{+\nu} + 1.25D_{21}^{-\mu} + 1.28D_{21}^{+\nu} \\
& \quad + 0.765D_{22}^{-\mu} + 0.768D_{22}^{+\nu} + 2.85D_{31}^{-\mu} + 2.857D_{31}^{+\nu} + 1.215D_{32}^{-\mu} + 1.25D_{32}^{+\nu} + 6.45d_{11}^{-\mu} + d_{11}^{+\nu}] \\
& \text{Subject to:} \\
& 3w_1 - 2w_2 + 2w_3 + 0.7982D_{21}^{-\mu} \geq 0.6164, \\
& -3w_1 + 2w_2 - 2w_3 - 0.7818D_{21}^{+\nu} \leq -0.6, \\
& -7w_1 - 2w_2 + w_3 + r + 1.3068D_{22}^{-\mu} \geq 0.125, \\
& 7w_1 + 2w_2 - w_3 - r - 1.3018D_{22}^{+\nu} \leq -0.1, \\
& w_1 + w_2 + w_3 - 4r + 0.3504D_{31}^{-\mu} \geq 0.0204, \\
& -w_1 - w_2 - w_3 + 4r - 0.35D_{31}^{+\nu} \leq 0, \\
& 2w_1 - w_2 + w_3 + 4r + 0.8228D_{32}^{-\mu} \geq 0.8258, \\
& -2w_1 + w_2 - w_3 - 4r - 0.797D_{32}^{+\nu} \leq -0.8, \\
& -w_1 + 0.155d_{11}^{-\mu} \geq 0, \quad w_1 - d_{11}^{+\nu} \leq 0, \\
& S^{wr} \text{ (Use system constraint from Eqs. (27))}, \\
& -w_1 - 4w_2 + w_3 + r + 0.625D_{11}^{-\mu} \geq 0.125, \\
& w_1 + 4w_2 - w_3 - r - 0.62D_{11}^{+\nu} \leq -0.1, \\
& -2w_1 + w_2 + 3w_3 + 4r + 0.5606D_{12}^{-\mu} \geq 0.5606, \\
& 2w_1 - w_2 - 3w_3 - 4r - 0.56D_{12}^{+\nu} \leq -0.5, \\
& D_{lk}^{-\mu}, D_{lk}^{+\nu} \geq 0, \forall l, k, \quad d_{11}^{-\mu}, d_{11}^{+\nu} \geq 0, r > 0.
\end{aligned} \tag{39}$$

Finally, solving the above problem (29) using LINGO-19 software, we generate an IF-efficient solution to the given problem (26) as $w_1^* = 0.00348983, w_2^* = 0, w_3^* = 0.06365932, r = 0.06016949$. Therefore, the preferable solution obtained with backward substitution are $x_1^* = 0.058, x_2^* = 0, x_3^* = 1.058$.

In literature review, Saraj and Safarai [51] proposed a Taylor series method using fuzzy decision sets, while Baky [8] presented the FGP approach for the same numerical example (2(26)), yielding the result $\hat{x} = (0, 0, 1)$. To compare the priority of the proposed solution $\mathbf{x}^* = (0.058, 0, 1.058)$ with the existing solution $\hat{x} = (0, 0, 1)$, we employed a distance-metric function. As shown in Table 5, the distance-metric value for $\mathbf{x}^* = (0.058, 0, 1.058)$ is less than that for $\hat{x} = (0, 0, 1)$ from Saraj and Safarai [51] and Baky [8]. Thus, the proposed IFGP approach yields the most satisfactory Pareto-optimal solution.

6.3 Application: Industrial production planning problem

This subsection explores the applicability of the proposed efficient method to the real-world NBL-MOLFCC problem. We focus on six machine types—lathes, drill presses, grinders, jigsaws, band saws, and milling machines—used in industrial production planning [16, 37]. These machines will utilize their weekly capacities to produce three items. Table 6 presents each machine's total neutrosophic linear available time, along with parameters b_1, b_2 and b_3 , which follow a Cauchy random distribution with known mean $\theta_1 = 100, \theta_2 = 90.6, \theta_3 = 150.27$ and standard deviation $\sigma_1 = 1443.6, \sigma_2 = 1000.29, \sigma_3 = 1328.86$.

Table 4: Comparison of IF-efficient solution for numerical example 2(26) obtained by IFGP method and the fuzzy-based approach by Saraj and Safaraei (2015), and Baky (2009).

Methods	$\mathcal{D}(\mathbf{x})$	$f_{01}(\mathbf{x})$	$f_{02}(\mathbf{x})$	$f_{11}(\mathbf{x})$	$f_{12}(\mathbf{x})$	$f_{21}(\mathbf{x})$	$f_{22}(\mathbf{x})$
$\max f_{lk}(\mathbf{x})$		0	0.67	1.25	1.47	1	0.02
1.25							
Proposed	0.10443	0.63	1.15	0.56	0.9	1	-0.17
0.47							
(Saraj and Safaraei, 2015; Baky, 2009)	0.10644	0.67	1.17	0.51	-0.19	0.45	

Table 5: Portfolio of available capacities with NLN parameters

Machine	Machine time	Product-1 (w_1)	Product-2 (w_2)	Product-3 (w_3)
Milling	b_1	$(2 + 20[0.5, 0.9])$	$(15 + 4[0.5, 0.9])$	0
Lathe	b_2	$(2 + 2[0.5, 0.8])$	$(0 + 18[0.5, 0.7])$	$(6 + 4[0.5, 0.9])$
Grinder	b_3	$(5 + 10[0.5, 1])$	$(7 + 12[0.5, 0.9])$	$(13.5 + 5[0.3, 0.6])$
Jigsaw	$(1300 + 50.2[0.3, 0.5])$	$(4 + 4[0.5, 0.8])$	0	$(14 + 4[0.5, 0.9])$
Drill press	$(890.925 + 18[0.3, 0.5])$	0	$(6 + 12[0.5, 0.7])$	$(9 + 16[0.5, 0.8])$
Band saw	$(1070 + 10[0.2, 0.5])$	$(5.5 + 8[0.5, 0.9])$	$(5.5 + 8[0.5, 0.9])$	$(2 + 4[0.5, 1])$
Profit		$(40 + 20[0.1, 0.5])$	$(80 + 40[0.1, 0.5])$	$(16 + 2[0.4, 0.8])$
Product liability		$(0.9[0.6, 0.8])$	$(0.8 + 0.05[0.5, 1])$	$(0.7 + 0.2[0.1, 0.4])$
Quality		$(90 + 4[0.2, 0.5])$	$(70 + 10[0.3, 0.5])$	$(45 + 10[0.2, 0.5])$
Worker satisfaction		$(20 + 10[0.2, 0.5])$	$(60 + 80[0.1, 0.5])$	$(65 + 20[0.3, 0.5])$

Let w_1, w_2, w_3 stand for the three products. Then, with the data in Table-6 and the formulation of the crisp model that was covered in the preceding subsection, the crisp model is provided as follows:

$$\begin{aligned} \text{ULDM} & \begin{cases} \max_{w_1, w_2} (z_1(w) = 50w_1 + 100w_2 + 17.6w_3 \rightarrow \text{profit}, \\ z_2(w) = 0.72w_1 + 0.85w_2 + 0.78w_3 \rightarrow \text{product liability}), \end{cases} \\ \text{LLDM} & \begin{cases} \max_{w_3} (z_{21}(w) = 92w_1 + 75w_2 + 50w_3 \rightarrow \text{Quality}, \\ z_{22}(w) = 25w_1 + 100w_2 + 75w_3 \rightarrow \text{Worker's Satisfaction}), \end{cases} \end{aligned}$$

Subject to:

$$\begin{cases} 12w_1 + 17w_2 \leq 1411 \rightarrow \text{Milling Machine}, \\ 3w_1 + 9w_2 + 8w_3 \leq 998.025 \rightarrow \text{Lathe}, \\ 10w_1 + 13w_2 + 15w_3 \leq 1753 \rightarrow \text{Grinder}, \\ 6w_1 + 16w_3 \leq 1325.1 \rightarrow \text{Jigsaw}, \\ 12w_2 + 17w_3 \leq 899.925 \rightarrow \text{Drill Press}, \\ 9.5w_1 + 9.5w_2 + 4w_3 \leq 1075 \rightarrow \text{Band Saw}, \\ w_1, w_2, w_3 \geq 0. \end{cases} \quad (40)$$

Now, according to the proposed method, the IFGP model of ULDM is formulated as:

$$\begin{aligned}
 & \min 0.00315D_{11}^{-\mu} + 0.0044D_{11}^{+\nu} + 0.022D_{12}^{-\mu} + 0.033D_{12}^{+\nu} \\
 & \text{Subject to:} \\
 & 50w_1 + 100w_2 + 17.5w_3 + 317.15D_{11}^{-\mu} \geq 7767.995, \\
 & -50w_1 - 100w_2 - 17.5w_3 - 225D_{11}^{+\nu} \leq -5767.995, \\
 & 0.72w_1 + 0.85w_2 + 0.78w_3 + 45D_{12}^{-\mu} \geq 105.824, \\
 & -0.72w_1 - 0.85w_2 - 0.78w_3 - 30D_{12}^{+\nu} \leq -90.824, \\
 & \text{Include system of constraint from Eqs. (30).}
 \end{aligned} \tag{41}$$

The optimal solution of the IFGP (31) model of DM1 is $w^* = (w_1 = 64.60414, w_2 = 37.3694, w_3 = 26.5627)$.

Consider the decision control-variables of DM1, $w_1 = 64.604$, and $w_2 = 37.36$ as binding constraints for the subsequent LLDMs. Based on suggested solution method algorithm, the IFGP model of a neutrosophic bi-level multi-objective production planning problem is formulated as follows:

$$\begin{aligned}
 & \min 0.00315D_{11}^{-\mu} + 0.0044D_{11}^{+\nu} + 0.022D_{12}^{-\mu} + 0.033D_{12}^{+\nu} + 0.00033D_{21}^{-\mu} + 0.0005D_{21}^{+\nu} \\
 & \quad + 0.00033D_{22}^{-\mu} + 0.0005D_{22}^{+\nu} + 100d_{12}^{-L\mu} + 100d_{12}^{-R\mu} + 50d_{12}^{+L\nu} + 50d_{12}^{+R\nu} \\
 & \quad + 50d_{11}^{-L\mu} + 50d_{11}^{-R\mu} + 20d_{11}^{+L\nu} + 20d_{11}^{+R\nu} \\
 & \text{Subject to:} \\
 & 92w_1 + 75w_2 + 50w_3 + 3000D_{21}^{-\mu} \geq 10950.67, \\
 & -92w_1 - 75w_2 - 50w_3 - 2000D_{21}^{+\nu} \leq -9950.67, \\
 & 25w_1 + 100w_2 + 75w_3 + 3000D_{22}^{-\mu} \geq 7783.33, \\
 & -25w_1 - 100w_2 - 75w_3 - 2000D_{22}^{+\nu} \leq -6783.33, \\
 & w_1 - 0.02d_{11}^{-L\mu} \geq 64.604, \quad -w_1 + 0.02d_{11}^{-R\mu} \geq -64.604, \\
 & -w_1 - 0.05d_{11}^{+L\nu} \leq -64.604, \quad w_1 - 0.05d_{11}^{+R\nu} \leq 64.604, \\
 & w_2 + 0.01d_{12}^{-L\mu} \geq 37.36, \quad -w_2 + 0.01d_{12}^{-R\mu} \geq -37.36, \\
 & -w_2 - 0.02d_{12}^{+L\nu} \leq -37.36, \quad w_2 - 0.02d_{12}^{+R\nu} \leq 37.36, \\
 & \text{Include system of constraint from Eqs. (31).}
 \end{aligned} \tag{42}$$

Using LINGO-19 software, we obtained an optimal solution to the above model: $w_1 = 64.604, w_2 = 37.36, w_3 = 26.5691$.

The optimal solutions obtained from the proposed approach for the real-life production planning problem (31) match those achieved by current methods, such as TOPSIS given by El Sayed and Baky [16] and new method suggested by Moges and Wordofa [37].

7 Conclusion

This study introduces an efficient methodology to address a contemporary NBL-MOLFCCCO problem within an uncertain optimization, wherein each input data is characterized as a neutrosophic linear pa-

parameter, and certain right-hand-side quantities adhere to a Cauchy random distribution defined by their mean and standard deviation. The primary objective of this endeavor is to devise a solution method that is both practical and accessible for implementation, while adequately reflecting real-world decision-making contexts. In these scenarios, multiple DMs collaboratively within a hierarchical structure and engaging in various decision-making processes such as agreement, disagreement, and uncertainty. In this framework, all unknown coefficients pertaining to constraints, objective functions, and resources are treated as neutrosophic linear numbers, alongside chance constraints, thereby accommodating real-world complexities. The standard formulations are subsequently derived utilizing modified interval and chance-constraint methodologies. Thereafter, through a linearization approach, the resultant model is reformulated into a bi-level multi-objective linear mathematical model. Ultimately, the formulated model is resolved through the application of the proposed method. The main advantage of the proposed algorithm is its ability to eliminate the frequent rejection of solutions by the ULDM, thereby avoiding the need for reassessment of the problem through a redefinition of the elicited membership and non-membership functions necessary for achieving an adequate decision. To illustrate the efficacy and applicability of this method, two numerical examples along with a real-world application problem have been presented. The distinctiveness of this method is evident in its adeptness at managing uncertain data while maintaining a straightforward execution.

The proposed methodology will undergo further refinement for its implementation in a real-world context, specifically addressing multi-objective linear fractional hierarchical decision-making within an uncertain environment.

Author contributions

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Conflict of Interest

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NeutroSemigroups Generated by Upside-Down Logic

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Abstract. Current models of NeutroAlgebras generally depend on static assignments to construct partial axioms. This paper introduces a deterministic method to generate NeutroAlgebraic structures over plithogenic spaces. We define a dual-threshold mode selector that combines the plithogenic intersection with the Three-Mode Upside-Down operator, strictly parameterized by the contradiction degree function. By introducing a novel binary operation $\star$, we prove that the classical associative law fails within specific topological sub-regions due to contradiction-triggered truth reversals. Since the associative property holds partially, fails partially, and remains indeterminate elsewhere, the space equipped with $\star$ forms a NeutroSemigroup. This approach provides a structural generation of NeutroAxioms rather than relying on arbitrary assignments. The theoretical results are verified through an illustration in dynamic threat evaluation. We conclude by posing open problems regarding the existence of NeutroIdentity and NeutroInverse elements under the defined operation.

Keywords: NeutroSemigroup; Upside-Down Logic; Plithogenic Set; Contradiction Degree; NeutroAxiom; NeutroAlgebra

1. Introduction

Let X be a plithogenic space where elements possess multiple attribute values. The standard plithogenic intersection $\wedge_p$ aggregates these attributes using linear combinations of fuzzy t-norms and t-conorms. Smarandache defined this intersection to be strictly parameterized by a contradiction degree function $c(a, b)$, which measures the dissimilarity between attribute values [1,2]. This foundational framework effectively models multi-dimensional indeterminacy but assumes the universal validity of classical algebraic axioms.

To formalize the failure of classical axioms in indeterminate environments, Smarandache introduced the theory of NeutroAlgebras. He defined NeutroAlgebras as generalized algebraic structures where operations or axioms evaluate to partially true, partially indeterminate, and partially false [3,4]. Current literature primarily establishes these NeutroAlgebraic spaces by utilizing static, finite Cayley table assignments to demonstrate the partiality of axioms [5].

Independent of algebraic spaces, logical evaluations exhibit context-dependent truth reversals. To model these contextual transitions, Fujita [6] defined the Three-Mode Upside-Down operator on plithogenic neutrosophic sets. This operator utilizes Smarandache's Upside-Down logic [7] to exchange the degrees of truth and falsity under high contradiction. However, Fujita restricted the application of this mechanism entirely to propositional truth valuations.

This paper introduces a deterministic method to analytically generate NeutroAlgebraic structures, advancing the previously mentioned frameworks. Instead of assigning NeutroAlgebraic partiality statically, and rather than restricting the Upside-Down operator to propositional logic, we embed this reversal mechanism directly into the structural axioms of an algebraic space. We define a deterministic dual-threshold mode selector $M(a, b)$ that integrates the plithogenic intersection with the Three-Mode Upside-Down operator. This piecewise function links the contradiction degree directly to the failure of the classical associative law, generating NeutroAxioms from topological boundaries. By introducing a novel binary operation $\star$ based on $M(a, b)$, we prove that contradiction-triggered truth reversals force the associative property to hold partially, fail partially, and remain indeterminate elsewhere. Consequently, the plithogenic space equipped with $\star$ forms a NeutroSemigroup. Finally, the theoretical failure of associativity is verified in a dynamic threat evaluation application.

2. Preliminaries

In this section, we review the foundational concepts of plithogenic sets, contradiction degree functions, and NeutroAlgebraic structures required for the subsequent theorems.

Definition 2.1 ([1]). Let U be a universe of discourse and $P \subseteq U$ be a non-empty set. A plithogenic set X is defined as a quintuple $X = (P, v, P_v, d, c)$, where v is a designated attribute, P_v is the domain of admissible values for the attribute v , $d : P \times P_v \rightarrow [0, 1]^s$ is the degree of appurtenance function (DAF), and $c : P_v \times P_v \rightarrow [0, 1]^t$ is the degree of contradiction function (DCF), with $s, t \in \mathbb{N}^+$. For any $x \in P$ and $a \in P_v$, $d(x, a)$ represents the appurtenance degree of x to the set P with respect to the attribute value a .

Definition 2.2 ([1]). The contradiction degree function $c : P_v \times P_v \rightarrow [0, 1]^t$ quantifies the dissimilarity between any two attribute values $a, b \in P_v$. It satisfies the following conditions:

- (1) Reflexivity: $c(a, a) = 0$ for all $a \in P_v$.
- (2) Symmetry: $c(a, b) = c(b, a)$ for all $a, b \in P_v$.

Definition 2.3 ([2]). Let $a, b \in P_v$ be two attribute values. Let $\wedge_F$ and $\vee_F$ denote a classical fuzzy t-norm and t-conorm, respectively. The plithogenic intersection $\wedge_p$ between the appurtenance degrees of a and b is defined as a linear combination modulated by the contradiction degree $c(a, b)$:

$$a \wedge_p b = (1 - c(a, b)) \cdot (a \wedge_F b) + c(a, b) \cdot (a \vee_F b). \quad (1)$$

To clarify the operational mechanics of the plithogenic intersection and its notation under a fixed contradiction degree, consider the following example:

Example 2.4. Let $a, b \in P_v$ be two attribute values with a contradiction degree $c(a, b) = 0.2$. Assume their truth components under a specific evaluation are $T_a = 0.8$ and $T_b = 0.5$. Utilizing the standard fuzzy operators $\wedge_F(u, v) = \min(u, v)$ and $\vee_F(u, v) = \max(u, v)$, the truth component of the plithogenic intersection computes as:

$$T_{a \wedge_p b} = (1 - 0.2) \cdot \min(0.8, 0.5) + 0.2 \cdot \max(0.8, 0.5) = 0.8(0.5) + 0.2(0.8) = 0.56.$$

This illustrates the algebraic interpolation between the t-norm and t-conorm bounds dictated by $c(a, b)$.

Definition 2.5 ([3]). Let X be a non-empty space equipped with a well-defined binary operation. An algebraic axiom defined on X is called a NeutroAxiom if it evaluates to true (T) for some elements in X , indeterminate (I) for other elements, and false (F) for the remaining elements, such that $(T, I, F) \notin \{(1, 0, 0), (0, 0, 1)\}$.

Definition 2.6 ([3]). Let X be a non-empty set endowed with a well-defined binary operation. If the associativity axiom on X operates as a NeutroAxiom, the algebraic structure is called a NeutroSemigroup.

Definition 2.7 ([8]). A fuzzy t-norm $t : [0, 1]^2 \rightarrow [0, 1]$ is a commutative, associative, and non-decreasing binary operation with the identity element 1. Dually, a fuzzy t-conorm $s : [0, 1]^2 \rightarrow [0, 1]$ satisfies the same algebraic properties with the identity element 0. These operators are used to model the generalized intersection and union, respectively.

3. The Operator and Deterministic Mode Selection

In this section, we formalize the modes of the Three-Mode Upside-Down operator and establish a deterministic selection mechanism. By coupling the contradiction degree directly to a dual-threshold function, we eliminate randomness in mode activation, ensuring that the resulting algebraic operations are strictly well-defined prior to evaluating the semigroup axioms.

Definition 3.1 ([6]). Let $x = (T, I, F) \in [0, 1]^3$ be a neutrosophic triplet representing an appurtenance degree. Let $s : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a standard fuzzy t-conorm. The three

fundamental transformation modes operating on the plithogenic neutrosophic components are defined as mappings from $[0, 1]^3$ to $[0, 1]^3$:

$$Keep(T, I, F) = (T, I, F), \quad (2)$$

$$SwapTF(T, I, F) = (F, I, T), \quad (3)$$

$$EnrichI(T, I, F) = (0, s(I, s(T, F)), 0). \quad (4)$$

Definition 3.2. Let P_v be the domain of admissible attribute values and let $c : P_v \times P_v \rightarrow [0, 1]$ be the contradiction degree function. We introduce two fixed activation thresholds $\tau_1, \tau_2 \in [0, 1]$ such that $0 \leq \tau_1 < \tau_2 \leq 1$. The deterministic mode selector $M : P_v \times P_v \rightarrow \{Keep, EnrichI, SwapTF\}$ is defined as a piecewise function evaluated for any pair of attribute values $a, b \in P_v$:

$$M(a, b) = \begin{cases} Keep, & \text{if } 0 \leq c(a, b) < \tau_1, \\ EnrichI, & \text{if } \tau_1 \leq c(a, b) < \tau_2, \\ SwapTF, & \text{if } \tau_2 \leq c(a, b) \leq 1. \end{cases} \quad (5)$$

The piecewise construction of the mode selector M partitions the plithogenic space into three mutually exclusive regions based on the severity of the contradiction. Rather than leaving the mode selection to an external or random variable, this function strictly forces the system to either preserve the components, absorb the polar supports into indeterminacy, or completely invert the truth values, bounded entirely by the thresholds τ_1 and τ_2 .

Remark 3.3. The activation thresholds τ_1 and τ_2 utilized in the deterministic mode selector are exogenous parameters rather than algebraic constants. Their numerical values are dictated by the contextual requirements of the given application. Also, the breakdown of the associative law is invariant with respect to these values. Provided the condition $0 \leq \tau_1 < \tau_2 \leq 1$ holds, the generation of the NeutroAxiom and the resulting NeutroSemigroup structure remain independent of the threshold selection.

Definition 3.4. Let X be a plithogenic space. For any two attribute values $a, b \in P_v$ with their respective neutrosophic appurtenance degrees $x = (T_x, I_x, F_x)$ and $y = (T_y, I_y, F_y)$, we define the new binary operation $\star$ as the composition of the deterministic mode selector M and the standard plithogenic intersection $\wedge_p$:

$$x \star y = M(a, b)(x \wedge_p y). \quad (6)$$

Consequently, the operation $\star$ computes the plithogenic intersection of x and y , and subsequently maps the resulting neutrosophic components according to the exact mode dictated by the contradiction degree $c(a, b)$.

Example 3.5. To illustrate the mechanism of the mode selector M , assume the activation thresholds are fixed at $\tau_1 = 0.5$ and $\tau_2 = 0.8$. Let the standard plithogenic intersection of two elements x and y yield a resulting neutrosophic triplet $x \wedge_p y = (0.7, 0.2, 0.1)$. Assume the standard bounded sum $s(u, v) = \min\{1, u + v\}$ is utilized as the t-conorm. The binary operation $x \star y$ maps this triplet strictly based on the regional partition of the contradiction degree $c(a, b)$:

- (1) Assume $c(a, b) = 0.3$. The condition $c(a, b) < \tau_1$ holds. The selector assigns *Keep*, yielding $x \star y = (0.7, 0.2, 0.1)$.
- (2) Assume $c(a, b) = 0.6$. The condition $\tau_1 \leq c(a, b) < \tau_2$ holds. The selector assigns *EnrichI*. The polar components (truth and falsity) are mapped to 0, and the indeterminacy is enriched via the t-conorm s : $s(0.2, s(0.7, 0.1)) = \min\{1, 0.2 + 0.8\} = 1.0$. Thus, $x \star y = (0, 1.0, 0)$.
- (3) Assume $c(a, b) = 0.9$. The condition $c(a, b) \geq \tau_2$ holds. The selector assigns *SwapTF*, interchanging the truth and falsity components. This yields $x \star y = (0.1, 0.2, 0.7)$.

This demonstrates how the continuous spectrum of the contradiction degree discretizes the algebraic output into distinct topological regions, ensuring the operation $\star$ remains strictly deterministic.

4. Main Results: The NeutroSemigroup Structure

In this section, we investigate the algebraic properties of the binary operation $\star$ defined in Section 3. First, we demonstrate the closure and well-definedness of the operation under the proposed dual-threshold mechanism. Then, we evaluate the associativity axiom, proving that the structural inversion triggered by the mode selector forces the associative law to break down in specific sub-regions. Finally, we establish that the plithogenic space equipped with the operation $\star$ constitutes a NeutroSemigroup.

Lemma 4.1. *The binary operation $\star$ is closed and well-defined on the plithogenic space X .*

Proof. Let $(x_1, y_1), (x_2, y_2) \in X \times X$ such that $x_1 = x_2$ and $y_1 = y_2$. Let a_1, b_1 be the attributes corresponding to x_1, y_1 , and a_2, b_2 be the attributes corresponding to x_2, y_2 . The equality of elements strictly implies $a_1 = a_2$ and $b_1 = b_2$.

Since the contradiction degree function $c : P_v \times P_v \rightarrow [0, 1]$ is a well-defined mapping, we have $c(a_1, b_1) = c(a_2, b_2)$. The thresholds τ_1 and τ_2 partition the codomain of c into disjoint intervals. Consequently, $c(a_1, b_1)$ and $c(a_2, b_2)$ map to the identical interval, yielding $M(a_1, b_1) = M(a_2, b_2) = M_0$, where $M_0 \in \{\textit{Keep}, \textit{EnrichI}, \textit{SwapTF}\}$.

The standard plithogenic intersection $\wedge_p$ is well-defined on $[0, 1]^3$, implying $x_1 \wedge_p y_1 = x_2 \wedge_p y_2$. Applying the mode operator M_0 , we obtain:

$$x_1 \star y_1 = M_0(x_1 \wedge_p y_1) = M_0(x_2 \wedge_p y_2) = x_2 \star y_2.$$

Furthermore, $\wedge_p$ is closed on $[0, 1]^3$, and the transformation modes defined in M map $[0, 1]^3$ into $[0, 1]^3$. Thus, $x_1 \star y_1 \in X$. This produces a unique and well-defined neutrosophic element in X . $\square$

Theorem 4.2. *The associativity of the binary operation $\star$ operates as a NeutroAxiom in the plithogenic space X .*

Proof. Let $x, y, z \in X$ corresponding to the attributes $a, b, d \in P_v$. We evaluate $(x \star y) \star z = x \star (y \star z)$.

Case 1 (Degree of Truth T): Assume $c(a, b) < \tau_1$, $c(b, d) < \tau_1$, and $c(a, d) < \tau_1$. The mode selector yields *Keep* for all operations. The equation reduces to $(x \wedge_p y) \wedge_p z = x \wedge_p (y \wedge_p z)$. The standard plithogenic intersection preserves associativity on $[0, 1]^3$. The equality holds.

Case 2 (Degree of Falsehood F): Assume $c(a, b) \geq \tau_2$ and $c(b, d) < \tau_1$. Let $c(a \wedge_p b, d) < \tau_1$ and $c(a, b \wedge_p d) \geq \tau_2$. The left-hand side evaluates as: $(x \star y) \star z = \text{SwapTF}(x \wedge_p y) \star z = (F_{xy}, I_{xy}, T_{xy}) \wedge_p (T_z, I_z, F_z)$. Let T_L denote the truth component of this expression. Letting $c_L = c(a \wedge_p b, d)$, the standard aggregation yields:

$$T_L = (1 - c_L) \cdot \min(F_{xy}, T_z) + c_L \cdot \max(F_{xy}, T_z). \quad (7)$$

The right-hand side evaluates as: $x \star (y \star z) = x \star (y \wedge_p z) = \text{SwapTF}(x \wedge_p (y \wedge_p z))$. Let T_R denote the truth component of the right-hand side. By definition, T_R equals the falsity component of the total intersection, denoted F_{xyz} .

Assume $(x \star y) \star z = x \star (y \star z)$ holds universally in this sub-space. This requires $T_L = T_R$ for all elements. The term $T_R = F_{xyz}$ is invariant with respect to the variable T_z . Conversely, T_L depends on T_z . Evaluating the condition $T_L = T_R$ at the boundary $T_z = 0$ yields $T_L = c_L \cdot F_{xy}$. Evaluating at $T_z = 1$ yields $T_L = (1 - c_L) \cdot F_{xy} + c_L$. Since T_R is invariant, we equate both evaluations:

$$c_L \cdot F_{xy} = (1 - c_L) \cdot F_{xy} + c_L. \quad (8)$$

For arbitrary $F_{xy} \in (0, 1)$, this algebraic equality holds if and only if $c_L = 1$. This strictly contradicts the initial premise $c_L < \tau_1 \leq 1$. The assumption fails. Thus, $T_L \neq T_R$.

The associativity equation evaluates to true in Case 1 and false in Case 2. It constitutes a NeutroAxiom. $\square$

Lemma 4.3. *The plithogenic space $(X, \star)$ lacks a universal identity element.*

Proof. Assume there exists a universal identity $e = (T_e, I_e, F_e) \in X$ such that $x \star e = x$ for all $x \in X$. This requires the mode selector to yield *Keep*, implying $c(a_x, a_e) < \tau_1$ for all attributes. The equation reduces to $x \wedge_p e = x$. Evaluating the truth component yields:

$$(1 - c) \cdot \min(T_x, T_e) + c \cdot \max(T_x, T_e) = T_x, \quad (9)$$

where $c = c(a_x, a_e)$. For this equality to hold for arbitrary $T_x \in [0, 1]$ under a fixed $c \in (0, 1)$, T_e must simultaneously satisfy $\min(T_x, T_e) = T_x \implies T_e = 1$ and $\max(T_x, T_e) = T_x \implies T_e = 0$. This is a contradiction. Thus, e does not exist. $\square$

Remark 4.4. By Lemma 4.3, the absence of a universal identity element prevents the extension of the algebraic structure into a NeutroGroup. Consequently, the structural generation strictly terminalizes at a NeutroSemigroup. Formulating a NeutroGroup under $\star$ requires restricting the space to localized topological sub-regions where partial identities hold.

Corollary 4.5. *The plithogenic space X equipped with the binary operation $\star$ constitutes a NeutroSemigroup.*

Proof. By Lemma 4.1, the space X is endowed with a well-defined binary operation $\star$. By Theorem 4.2, the associativity of the operation $\star$ acts as a NeutroAxiom. These two conditions satisfy the requirements of a NeutroSemigroup as outlined in Definition 2.6 $\square$

5. Illustrative Applications

To demonstrate the operational mechanics of the deterministic mode selector $M(a, b)$ across different parameter thresholds, we present two distinct applied scenarios. The first application examines the failure of associativity triggered by a strict truth reversal in dynamic threat evaluation. The second application illustrates a structural absorption into indeterminacy and its subsequent algebraic collapse in financial fraud detection.

5.1. Dynamic Threat Evaluation

To demonstrate the breakdown of associativity under the operation $\star$ in an applied context, we adapt the security patch deployment and threat evaluation model introduced by Fujita [6]. In this framework, the evaluation of a software security patch relies on multi-attribute threat intelligence reports.

Let X be the plithogenic space of threat evaluations. We define the classical fuzzy t-norm as $\wedge_F(u, v) = \min(u, v)$ and the t-conorm as $\vee_F(u, v) = \max(u, v)$. In accordance with Remark 3.3, the exogenous mode selector thresholds are fixed at $\tau_1 = 0.4$ and $\tau_2 = 0.7$ to reflect the true nature of specific security protocols. Consider three threat intelligence sources

corresponding to the attribute values a (Heuristic Analysis), b (Signature Matching), and d (Network Behavior). The neutrosophic values assigned to the patch by these sources are given as:

$$\begin{aligned}x &= (0.7, 0.2, 0.1) \quad \text{for attribute } a, \\y &= (0.6, 0.3, 0.2) \quad \text{for attribute } b, \\z &= (0.8, 0.1, 0.2) \quad \text{for attribute } d.\end{aligned}$$

Assume the contradiction degrees between the intelligence sources are empirically established as follows: $c(a, b) = 0.8$, $c(b, d) = 0.2$, $c(a \wedge_p b, d) = 0.2$, and $c(a, b \wedge_p d) = 0.9$.

We evaluate the left-hand side of the associativity equation: $(x \star y) \star z$. For the inner operation $x \star y$, the contradiction degree is $c(a, b) = 0.8 \geq \tau_2$. The mode selector yields $M(a, b) = \text{SwapTF}$. The standard plithogenic intersection $x \wedge_p y$ is computed as:

$$\begin{aligned}x \wedge_F y &= (0.6, 0.2, 0.1) \\x \vee_F y &= (0.7, 0.3, 0.2) \\x \wedge_p y &= (1 - 0.8)(0.6, 0.2, 0.1) + 0.8(0.7, 0.3, 0.2) \\&= (0.12, 0.04, 0.02) + (0.56, 0.24, 0.16) \\&= (0.68, 0.28, 0.18).\end{aligned}\tag{10}$$

Applying the selected mode:

$$x \star y = \text{SwapTF}(0.68, 0.28, 0.18) = (0.18, 0.28, 0.68).\tag{11}$$

Let $w = x \star y$. The subsequent operation is $w \star z$. The contradiction degree is $c(a \wedge_p b, d) = 0.2 < \tau_1$. The mode selector yields $M(a \wedge_p b, d) = \text{Keep}$.

$$\begin{aligned}w \wedge_F z &= (\min(0.18, 0.8), \min(0.28, 0.1), \min(0.68, 0.2)) = (0.18, 0.10, 0.20) \\w \vee_F z &= (\max(0.18, 0.8), \max(0.28, 0.1), \max(0.68, 0.2)) = (0.80, 0.28, 0.68) \\w \wedge_p z &= (1 - 0.2)(0.18, 0.10, 0.20) + 0.2(0.80, 0.28, 0.68) \\&= (0.144, 0.080, 0.160) + (0.160, 0.056, 0.136) \\&= (0.304, 0.136, 0.296).\end{aligned}\tag{12}$$

Since the mode is *Keep*, the left-hand side evaluates to $(0.304, 0.136, 0.296)$.

We now evaluate the right-hand side of the equation: $x \star (y \star z)$. For the inner operation $y \star z$, the contradiction degree is $c(b, d) = 0.2 < \tau_1$. The mode selector yields $M(b, d) = \text{Keep}$.

$$\begin{aligned}
 y \wedge_F z &= (0.6, 0.1, 0.2) \\
 y \vee_F z &= (0.8, 0.3, 0.2) \\
 y \wedge_p z &= 0.8(0.6, 0.1, 0.2) + 0.2(0.8, 0.3, 0.2) \\
 &= (0.48, 0.08, 0.16) + (0.16, 0.06, 0.04) \\
 &= (0.64, 0.14, 0.20).
 \end{aligned} \tag{13}$$

Applying the selected mode yields $y \star z = \text{Keep}(0.64, 0.14, 0.20) = (0.64, 0.14, 0.20)$. Let $u = y \star z$. The subsequent operation is $x \star u$. The contradiction degree is $c(a, b \wedge_p d) = 0.9 \geq \tau_2$. The mode selector yields $M(a, b \wedge_p d) = \text{SwapTF}$.

$$\begin{aligned}
 x \wedge_F u &= (\min(0.7, 0.64), \min(0.2, 0.14), \min(0.1, 0.20)) = (0.64, 0.14, 0.10) \\
 x \vee_F u &= (\max(0.7, 0.64), \max(0.2, 0.14), \max(0.1, 0.20)) = (0.70, 0.20, 0.20) \\
 x \wedge_p u &= (1 - 0.9)(0.64, 0.14, 0.10) + 0.9(0.70, 0.20, 0.20) \\
 &= (0.064, 0.014, 0.010) + (0.630, 0.180, 0.180) \\
 &= (0.694, 0.194, 0.190).
 \end{aligned} \tag{14}$$

Applying the selected mode:

$$x \star u = \text{SwapTF}(0.694, 0.194, 0.190) = (0.190, 0.194, 0.694). \tag{15}$$

Since $(0.304, 0.136, 0.296) \neq (0.190, 0.194, 0.694)$, the associative property fails under the defined threshold parameters, verifying the theoretical results of Theorem 4.2.

5.2. Algebraic Collapse in Transaction Legitimacy

The deterministic generation of NeutroAlgebraic partiality extends into spaces governed by algorithmic legitimacy. Consider a transaction evaluation space where a request is verified against multiple attributes. Let a denote cryptographic authentication and b denote geospatial consistency. Assume an evaluation yields $x_a = (0.9, 0.1, 0.1)$, indicating a correct password, and $x_b = (0.2, 0.1, 0.8)$, indicating an anomalous location.

The contradiction degree between these evaluations evaluates to $c(a, b) = 0.5$. Under the thresholds $\tau_1 = 0.4$ and $\tau_2 = 0.7$, the condition $\tau_1 \leq c(a, b) < \tau_2$ holds. The deterministic selector $M(a, b)$ triggers the *EnrichI* mode. Algebraically, the polar components (Truth and Falsity) are mapped strictly to 0. The structural weight of the operation is absorbed into the indeterminacy component by the t-conorm. The transaction is neither approved nor rejected; it is structurally locked in an indeterminate state, demanding external manual resolution.

Suppose a third attribute d , representing physical impossibility (e.g., an ATM withdrawal on a different continent), enters the evaluation with $x_d = (0.1, 0.1, 0.9)$. The contradiction degree escalates to $c(a \wedge_p b, d) = 0.8 \geq \tau_2$. The threshold is breached. The selector M shifts to the *SwapTF* mode. The operation $\star$ applies the truth reversal, forcing any residual truth into absolute falsity.

The associative equation $(x_a \star x_b) \star x_d = x_a \star (x_b \star x_d)$ fails because the intermediate *EnrichI* collapse is irreversible. The topological transition between the sub-regions $[0, \tau_1)$, $[\tau_1, \tau_2)$, and $[\tau_2, 1]$ acts as a one-way trapdoor. The failure of associativity is not an operational defect; it guarantees that chronological re-evaluations cannot bypass the threshold-triggered lockdowns.

6. Conclusions and Open Problems

In this paper, we constructed an analytical generator for NeutroAlgebraic structures by synthesizing the plithogenic intersection and the Three-Mode Upside-Down operator. We defined a dual-threshold mode selector $M(a, b)$ modulated by the contradiction degree function. Based on this selector, we introduced the binary operation $\star$ and proved that the plithogenic space equipped with $\star$ forms a NeutroSemigroup. We demonstrated that the classical associative law fails to hold universally, as the contradiction degree dictates the sub-regions where associativity evaluates to true or false.

The transition from classical partial algebras to NeutroAlgebraic structures under threshold-based mode selectors presents the following open algebraic problems:

- By Lemma 4.3, the space lacks a universal identity element. A possible approach to construct partial identities (NeutroIdentities) is to restrict the space to localized topological sub-regions where the contradiction degree satisfies $c(a, b) < \tau_1$. Within these boundaries, the mode selector continuously yields *Keep*, allowing localized identities to emerge analytically. Establishing the formal conditions for these NeutroIdentities will determine whether the current NeutroSemigroup can be extended into a NeutroGroup.
- The algebraic classification of the operation $\star$ under asymmetric t-norms and t-conorms.

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The domain of a Neutrosophic metric spaces of Tribonacci matrix defined by Triple sequence spaces

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Abstract. In this paper, we introduce Tribonacci triple sequence spaces by using Neutrosophic metric spaces. We introduce Neutrosophic metric spaces in Tribonacci triple sequence spaces and prove some topological and algebraic properties based on the results with respect to these spaces.

Keywords: Tribonacci matrix and sequence, t-norm, t-conorm, Neutrosophic metric space, Tribonacci analytic and Tribonacci entire, triple sequence spaces.

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1. Introduction

The theory of fuzzy sets was firstly given by Zadeh in 1965 [1]. Intuitionistic fuzzy sets for such cases were initiated by Atanassov [2], Atanassov et al. [3] used this concept in decision-making problems. More over, fuzzy metric space has used by practical researches as for example decision-making fixed point theory medical imaging.

Triangular norms space (t-norms) (TNS) were given by Menger [4] TNS are used to generalize with the probability distribution of triangle inequality in metric space terms. Triangular conorms space (t-conorms) (TCS) known as dual operations of TNS, TNS and TCS important for fuzzy operations.

The concept neutrosophy implies impartial knowledge of thought and then neutral describes the basic difference between neutral fuzzy, intuitive fuzzy set and logic. The neutrosophic set (NS) was investigated by F.Smarandache [5], who defined the degree of indeterminacy as an independent component. In [6], neutrosophic logic was firstly examined. It is a logic where each proposition is determined to have a degree truth (T), falsity (F) and indeterminacy (I). A neutrosophic set (NS) is determined as a set where every component of the universe has a degree of T, F and I. Neutrosophic soft linear spaces (NSLS) were considered by Bera and Mahapatra [7]. Subsequently, in [8], the concept of neutrosophic soft normed linear space (NSNLS) was defined and the features of (NSNLS) were examined. Significant results on this topic can be found in [9]-[13]. Kirisci and Simsek [14] defined a new concept known as Neutrosophic Metric Space (NMS) with continuous t-norms and continuous t-conorms.

Neutrosophic Normed Space (NNS) and statistical convergence in NNS have been investigated by Kirisci and Simsek [15]. Neutrosophic set and Neutrosophic logic have been used in applied science and theoretical science, such as decision making, robotics and summability theory.

A triple sequence (real or complex) can be defined as a function $X : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{R}(\mathbb{C})$ where $\mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequences were introduced and investigated initially by Esi et al [16,17], Subramanian et al [18-20]. Hazarika et al [21] and many others.

Let $x = (x_{mnk})$ be a triple sequence of real or complex numbers. Then the series $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} x_{mnk}$ is called a triple series. The triple series is said to be convergent if and only if the triple sequence (S_{mnk}) is convergent, where $(S_{mnk}) = \sum_{i,j,q=1}^{(m,n,k)} x_{ijq}$ ($(m,n,k) = 1,2,3,\dots$).

Consider a triple sequence $x = (x_{mnk})$ is said to be analytic $\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty$. The vector space of all triple analytic sequences are usually denoted by Λ^3 . A triple sequence (x_{mnk}) is called triple entire sequence if $|x_{mnk}|^{\frac{1}{m+n+k}} \rightarrow 0$ as $m, n, k \rightarrow \infty$. The vector space of all triple entire sequences are usually denoted by Γ^3 . Let the set of sequences with this property be denoted by Λ^3 and Γ^3 is a metric space with the metric $d(x, y) = \sup_{m,n,k} \{|x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k = 1, 2, 3, \dots\}$ for all $x = (x_{mnk})$ and $y = (y_{mnk})$ in Γ^3 . Let $\phi = \{\text{finite sequences}\}$.

2. Preliminaries

Let λ and μ be two sequence spaces and $A = (a_{kl}^{mn})$ is an infinite four dimensional matrix of real or complex numbers a_{kl}^{mn} , where $k, l, m, n \in \mathbb{N}$. Then we say that A defines a matrix mapping from λ to μ and we denote it by writing $A : \lambda \rightarrow \mu$. If for every sequence $x = (x_{mnk}) \in \lambda$ the sequence $Ax = \{A(Ax)_{mnk}\} \in \mu$, then A transform for all $x \in \mu$ where $(Ax)_{mnk} = \sum_m \sum_n \sum_k a_{kl}^{mn} x_{mnk}$ ($k, l \in \mathbb{N}$).

By $(\lambda : \mu)$, we denote the class of matrices A such that $A : \lambda \rightarrow \mu$. Thus $a \in (\lambda, \mu)$ if and only if $(Ax) \in \mu$ forever $x \in \lambda$. The approach of constructing the new sequence spaces by means of the matrix domain of a particular limitation method.

First, we give some background about Tribonacci numbers. The studies on Tribonacci numbers were first initiated by a 14-year old student Mark Feinberg in 1963. In 1963, Mark Feinberg [22] defined the sequence $(t_n)_{n \in \mathbb{N}}$ of Tribonacci numbers given by the third recurrence relation $t_n = t_{n-1} + t_{n-2} + t_{n-3}, n \geq 3$ with $t_0 = t_1 = 1, t_2 = 2$ and $t_3 = 4$.

Thus, the first few numbers of Tribonacci sequence are 1,1,2,4,7,13,24,44,81,... .

Some basic properties of Tribonacci sequence are $\lim_{n \rightarrow \infty} \frac{t_n}{t_{n-1}} = 0.54368901...$

$$\sum_{k=0}^n t_k = \frac{t_{n+2} + t_{n-1}}{2}, n \geq 0 \quad \lim_{n \rightarrow \infty} \frac{t_{n+1}}{t_n} = 1.83929(\text{approx}).$$

Throughout this paper, we use the lower triangular Tribonacci matrix $T = (t_{ik})$, defined as

$$\text{follows } t_{ik} = \begin{cases} \frac{2tk}{t_{n+2} + t_{n-1}} & \text{if } 0 \leq k \leq n \\ 0 & k \geq n \end{cases}$$

Equivalently,

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1/4 & 1/2 & 0 & 0 & 0 & \cdots \\ 1/4 & 1/4 & 1/2 & 0 & 0 & \cdots \\ 1/8 & 1/8 & 1/4 & 1/2 & 0 & \cdots \\ 1/15 & 1/15 & 2/15 & 4/15 & 7/15 & 0 \\ 1/28 & 1/28 & 1/14 & 1/7 & 1/14 & 13/28 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Now we define three dimensional infinite matrix $T = (t_{uvw}^{mnk})$ is given by

$$t_{uvw}^{mnk} = \begin{cases} \frac{3t_{mnk}}{t_{u+2,v+2,w+2} + t_{uvw} - 1} & \text{if } 0 \leq m \leq u, 0 \leq n \leq v, 0 \leq k \leq w \\ 0 & (m > u, n > v, k > w) \end{cases}$$

$$\text{Equivalently, } T = (t_{uvw}^{mnk}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1/2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1/4 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1/2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 7/15 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 13/28 & 0 & \cdots \end{bmatrix}$$

Lemma 2.1. A matrix $A = (a_{uvw}^{mnk})$; m,n,k and $u,v,w \in \mathbb{N}$ is said to be regular if and only if the following conditions hold:

- (1) There exists $M > 0$ such that for every $u,v,w \in \mathbb{N}$, the equivalently

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} |a_{uvw}^{mnk}| \leq M \text{ holds.}$$

- (2) $\lim_{u,v,w \rightarrow \infty} a_{uvw}^{mnk} = 0$ for every $m,n,k \in \mathbb{N}$.

- (3) $\lim_{u,v,w \rightarrow \infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} a_{uvw}^{mnk} = 1$.

Clearly, Lemma 2.1 implies that the matrix T is regular. Define the sequence $y = (y_{uvw})$ which will be frequently used as the T by transform of the sequence $x = (x_{mnk})$

$$y_{uvw} = T_{uvw}x = \sum_{m=0}^u \sum_{n=0}^v \sum_{k=0}^w \left(\frac{3t_{mnk}}{t_{u+2,v+2,w+2} + t_{uvw-1}} \right) x_{mnk}; u, v, w \in \mathbb{N} \quad (2.1)$$

We also use the convention that any term with the negative subscripts example x_{-1-1-1} or t_{-1-1-1} , shall be considered as naught.

Definition 2.2. [23] A binary operation is $\star : [0, 1]^2 \times [0, 1] \rightarrow [0, 1]$. The operation $\star$ satisfying the following conditions, then it is called continuous TN, for each $s, t, u, v \in [0, 1]$,

- (1) $s \star 1 = s$.
- (2) If $s \leq u$ and $t \leq v$, then $s \star t \leq u \star v$.
- (3) $\star$ is continuous.
- (4) $\star$ is commutative and associative.

Definition 2.3. [23] A binary operation $\circ$ is satisfying the following conditions, then it is called continuous TC

- (1) $s \circ 0 = s$.
- (2) If $s \leq u$ and $t \leq v$, then $s \circ t \leq u \circ v$.
- (3) $\circ$ is continuous.
- (4) $\circ$ is commutative and associative.

Remark 2.4. [23] If $0 < \nu_j < 1$, $j = 1$ to 7 , $\star$ and $\circ$ are continuous t-norm, continuous t-conorm respectively. Then

- (1) If we take $0 < \nu_1, \nu_2 < 1$ for $\nu_1 > \nu_2$ then there exists $0 \leq \nu_3, \nu_4 < 1$ such that $\nu_1 \star \nu_3 \geq \nu_2$; $\nu_1 \geq \nu_4 \circ \nu_2$.
Further if we choose $\nu_5 \in (0, 1)$ then there exist $\nu_6, \nu_7 \in (0, 1)$ such that $\nu_6 \star \nu_6 \geq \nu_5$ and $\nu_7 \circ \nu_7 \circ \nu_5$.

Neutrosophic Metric Spaces:

Definition 2.5. [23] Take V to be an arbitrary set

$$N = \{ \langle a, G(a), B(a), Y(a) \rangle : a \in V \}$$

be a NS such that $N : V \times V \times \mathbb{R}^+ \rightarrow [0, 1]$. Let $\star$ and $\circ$ show the continuous TN and continuous TC respectively. The four-tuple $(V, N, \star, \circ)$ is called a Neutrosophic metric space (NMS) when the following conditions are satisfied $\forall a, b, c \in V$

- (1) $0 \leq G(a, b, \lambda) \leq 1, 0 \leq B(a, b, \lambda) \leq 1, 0 \leq Y(a, b, \lambda) \leq 1; \forall \lambda \in \mathbb{R}^+$.
- (2) $G(a, b, \lambda) + B(a, b, \lambda) + Y(a, b, \lambda) \leq 3, (\forall \lambda \in \mathbb{R}^+)$

- (3) $G(a, b, \lambda) = 1(\text{for } \lambda > 0) \Leftrightarrow a = b$
- (4) $G(a, b, \lambda) = G(b, a, \lambda)(\text{for } \lambda > 0)$.
- (5) $G(a, b, \lambda) \star G(b, c, \mu) \leq G(a, c, \lambda + \mu)(\forall \lambda, \mu > 0)$
- (6) $G(a, b, \circ) : [0, \infty) \rightarrow [0, 1]$ is continuous.
- (7) $\lim_{\lambda \rightarrow \infty} G(a, b, \lambda) = 1(\forall \lambda > 0)$
- (8) $B(a, b, \lambda) = 0(\text{for } \lambda > 0) \Leftrightarrow a = b$.
- (9) $B(a, b, \lambda) = B(b, a, \lambda)(\text{for } \lambda > 0)$
- (10) $B(a, b, \lambda) \circ B(b, c, \mu) \geq B(a, c, \lambda + \mu)(\forall \lambda, \mu > 0)$
- (11) $B(a, b, \circ) : [0, \infty) \rightarrow [0, 1]$ is continuous.
- (12) $\lim_{\lambda \rightarrow \infty} B(a, b, \lambda) = 0(\forall \lambda > 0)$
- (13) $Y(a, b, \lambda) = 0(\forall \lambda > 0) \Leftrightarrow a = b$
- (14) $Y(a, b, \lambda) = Y(b, a, \lambda)(\forall \lambda > 0)$.
- (15) $Y(a, b, \lambda).Y(b, c, \mu) \geq Y(a, c, \lambda + \mu)(\forall \lambda, \mu > 0)$.
- (16) $Y(a, b, \circ) : [0, \infty) \rightarrow [0, 1]$ is continuous.
- (17) $\lim_{\lambda \rightarrow \infty} Y(a, b, \lambda) = 0(\text{for } \lambda > 0)$
- (18) If $\lambda \leq 0$ then $G(a, b, \lambda) = 0, B(a, b, \lambda) = 1$ and $Y(a, b, \lambda) = 1$

Then $N = (G \circ B \circ Y)$ is called Neutrosophic Metric (NM) on V . The functions $G(a, b, \lambda), B(a, b, \lambda)$ and $Y(a, b, \lambda)$ denote the degree of nearness, the degree of neutralness and the degree of non-neutralness between a and b with respect to λ respectively.

Example 2.6. [23] Let (V, d) be an NMS, the binary operations $\star$ and $\circ$ as default (min) TN $a \star b = \min\{a, b\}$ and default (max) TC $a \circ b = \max\{a, b\}$.

$$G(a, b, \lambda) = \frac{\lambda}{\lambda + d(a, b)};$$

$$B(a, b, \lambda) = \frac{d}{\lambda + d(a, b)};$$

$$Y(a, b, \lambda) = \frac{d(a, b)}{\lambda}; \forall a, b \in V \text{ and } \lambda > 0. \text{ Then } (V, N, \star, \circ) \text{ is NMS such that } N : V \times V \times \mathbb{R}^+ \rightarrow [0, 1].$$

Example 2.7. [23] Let (V, d) be an NMS, the binary operations $\star$ and $\circ$ as TN $a \star b = \max\{0, a + b - 1\}$ and TC $a \circ b = a + b - ab \forall a, b \in V$ and $\lambda > 0$.

$$G(a, b, \lambda) = \begin{cases} \frac{a}{b} & (a \leq b) \\ \frac{b}{a} & (b \leq a) \end{cases};$$

$$B(a, b, \lambda) = \begin{cases} \frac{b-a}{Y} & (ax < b) \\ \frac{a-b}{x} & (b \leq a) \end{cases};$$

$$Y(a, b, \lambda) = \begin{cases} b - a & (a \leq b) \\ a - b & (b \leq a) \end{cases};$$

Then $(V, N, \star, \circ)$ is NMS such that $N : V \times V \times \mathbb{R}^+ \rightarrow [0, 1]$.

Remark 2.8. [23] $N = \{ \langle a, G(a), B(a), Y(a) \rangle : a \in V \}$ defined.

case(i) : Suppose Example 2.6 is not an NM with $TN \ a \star b = \max\{0, a + b - 1\}$ and $TC \ a \circ b = a + b - ab$. Case(ii): Suppose Example 2.7 is not an NM, then $N = \{ \langle a, G(a), B(a), Y(a) \rangle : a \in V \}$ with $TN \ a \star b = \min\{a, b\}$ and $TC \ a \circ b = \max\{a, b\}$.

Definition 2.9. Let V be an NMS the triple sequence $x = (x_{mnk}) \in V, 0 < \epsilon < 1$ and $\lambda > 0$. Then the triple sequence $x = (x_{mnk})$ converges to L if and only if there exists $N \in \mathbb{N}^3$ such that $G(x_{mnk}, x, \lambda) > 1 - \epsilon$; $B(x_{mnk}, x, \lambda) < \epsilon$ and $Y(x_{mnk}, x, \lambda) < \epsilon$.

That is

$$\lim_{m,n,k \rightarrow \infty} G(x_{mnk}, x, \lambda) = 1 ;$$

$$\lim_{m,n,k \rightarrow \infty} B(x_{mnk}, x, \lambda) = 0 ;$$

$$\lim_{m,n,k \rightarrow \infty} Y(x_{mnk}, x, \lambda) = o \text{ as } \lambda > 0 ;$$

In that case, the triple sequence $x = (x_{mnk})$ is called a convergent sequence in V . The convergent in NMS is denoted by $N - \lim_{m,n,k \rightarrow \infty} x_{mnk} = L$.

Definition 2.10. Take a $(V, N, \star, \circ)$ to be NMS. A triple sequence $x = (x_{mnk})$ in V is called Cauchy if for each $\epsilon > 0$ and each $\lambda > 0$ there exists $N \in \mathbb{N}^3$ such that $G(x_{mnk}, x_{pqr}, \lambda) > 1 - \epsilon$;

$$B(x_{mnk}, x_{pqr}, \lambda) < \epsilon;$$

$$Y(x_{mnk}, x_{pqr}, \lambda) < \epsilon; \forall m, n, k, p, q, r \geq N.$$

Hence $(V, N, \star, \circ)$ is called complete if every Cauchy sequence is convergent with respect to τ_N .

Definition 2.11. Let $(V, N, \star, \circ)$ be an NMS. A subset A of V is called Neutrosophic bounded (NB) if there exist $\lambda > 0$ and $\epsilon \in (0, 1)$ such that $G(a, b, \lambda) > 1 - \epsilon$, $B(a, b, \lambda) < \epsilon$ and $Y(a, b, \lambda) < \epsilon \forall a, b \in A$.

Definition 2.12. Give $(V, N, \star, \circ)$ be a NMS, $0 < \epsilon < 1$, $\lambda > 0$ and $a \in V$. The set $O(a, \epsilon, \lambda) = \{b \in V : G(a, b, \lambda) > 1 - \epsilon, B(a, b, \lambda) < \epsilon \text{ and } Y(a, b, \lambda) < \epsilon\}$ is said to be the open ball (OB) with center a and radius ϵ with respect to λ .

By using this T-transformation and the notion of Neutrosophic, we can define some sequence spaces, namely $L_{\Gamma^3(P,Q,R)}(T)$ and $L_{\Lambda^3(P,Q,R)}(T)$.

3. Main Results

In this section, we introduce the following sequence spaces.

$$\begin{aligned} L_{\Gamma^3(P,Q,R)}(T) &= \lim_{m,n,k \rightarrow \infty} \{x_{\frac{1}{m+n+k}} \in w^3 : (T_{uvw}(x))\} = 0 \\ &= \lim_{m,n,k \rightarrow \infty} \{x_{\frac{1}{m+n+k}} \in w^3 : \forall \epsilon > 0, \forall \lambda, \text{ there exist } (p, q, r) \in \mathbb{N} : \forall u \geq p, v \geq q, w \geq r, \\ G(T(x_{mnk}), x \circ \lambda) &> 1 - \epsilon, \\ B(T(x_{mnk}), x \circ \lambda) &< \epsilon \text{ and} \end{aligned}$$

$$\begin{aligned}
& Y(T(x_{mnk}), x \circ \lambda) < \epsilon \} \quad (3.1) \\
& = \lim_{m,n,k \rightarrow \infty} \{x = x_{mnk}^{\frac{1}{m+n+k}} \{ (u, v, w \in \mathbb{N}) \in w^3 : (u, v, w) \in \mathbb{N}, \forall \epsilon > 0 \\
& G(T(x_{mnk})x, \lambda) \leq 1 - \epsilon; \\
& B(T(x_{mnk})x, \lambda) \geq \epsilon \text{ and} \\
& Y(T(x_{mnk})x, \lambda) \geq \epsilon \} = 0\} \\
& L_{\Lambda^3(P,Q,R)}(T) = \sup_{(m,n,k)} \{x_{mnk}^{\frac{1}{m+n+k}} \in w^3 : (T_{uvw}(x)) \leq M\} \\
& \sup_{(m,n,k)} \{x_{mnk}^{\frac{1}{m+n+k}} \in w^3 : \forall \epsilon \in (0, 1), \forall \lambda, \text{ there exist } (p, q, r) \in \mathbb{N} : \forall u \geq p, v \geq q, w \geq r, \\
& G(T(x_{mnk}), x, \lambda) \geq 1 - \epsilon, \\
& B(T(x_{mnk}), x, \lambda) \leq \epsilon \text{ and} \\
& Y(T(x_{mnk}), x, \lambda) \leq \epsilon \} \leq M. \} \quad (3.2) \\
& = \sup_{m,n,k} \{x_{mnk}^{\frac{1}{m+n+k}} \in w^3 : \{(uvw) \in \mathbb{N} : \forall \epsilon > 0, \\
& G(T(x_{mnk}), x, \lambda) \geq 1 - \epsilon, \\
& B(T(x_{mnk}), x, \lambda) \geq \epsilon \text{ and} \\
& Y(T(x_{mnk}), x, \lambda) \geq \epsilon \} < \infty\}
\end{aligned}$$

Definition 3.1. Give $(V, N, \star, \circ)$ be an NMS, $0 < \epsilon < 1, \lambda > 0$ and $a \in V$.

The set $cl(a, \epsilon, \lambda) = \{b \in V : G(T(a), T(b), \lambda) < 1 - \epsilon; B(T(a), T(b), \lambda) > \epsilon$

and $Y(T(a), T(b), \lambda) > \epsilon$ is finite $\}$ is said to be a closed ball (CB) with center a radius ϵ with respect to λ , then

If $(X_{mnk}) \in L_{(P,Q,R)}(T)$, then (x_{mnk}) converges to some $\beta \in Y$, denoted by $(x_{mnk})^{(PQR)} \rightarrow \beta$.

Theorem 3.2. The inclusion relation $L_{\Gamma^3(PQR)}(T) \subset L_{(PQR)}(T) \subset L_{\Lambda^3(P,Q,R)}(T)$ holds.

Proof. It can be easily seen that $L_{\Gamma^3(PQR)}(T) \subset L_{(PQR)}(T)$. Now to prove that $L_{(PQR)}(T) \subset L_{\Lambda^3(P,Q,R)}(T)$

Let $x = (x_{mnk})^{\frac{1}{m+n+k}} \in L_{(P,Q,R)}(T)$.

Then there exist β such that $N_{(P,Q,R)}(T) - \lim_{m,n,k \rightarrow \infty} (x_{mnk}^{\frac{1}{m+n+k}}) = \beta$. Thus for every $0 < \epsilon < 1$ and $\lambda > 0$, the set A is finite.

$A = \{(u, v, w) \in \mathbb{N} :$

$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \frac{\beta}{2}, \frac{\lambda}{2}) > 1 - \epsilon; B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \frac{\beta}{2}, \frac{\lambda}{2}) < \epsilon,$

$Y(T_{uvw}(x_{mnk}), x, \frac{\beta}{2}, \frac{\lambda}{2}) < \epsilon\}$

Let $G(\beta, \frac{\lambda}{2}) = e; B(\frac{\beta}{2}, \frac{\lambda}{2}) = f$, and $Y(\beta, \frac{\lambda}{2}) = g$ for all $\lambda > 0$.

Since $(e, f, g) \in (0, 1)$ and $0 < \epsilon < 1$, there exist $s_1, s_2, s_3 \in (0, 1)$ such that

$(1 - \epsilon) \star e > 1 - s_1, \epsilon \circ f < s_2$ and $\epsilon \circ g < s_3$, we have

$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \lambda) = G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) \geq G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \frac{\beta}{2}, \frac{\lambda}{2}) \star G(\frac{\beta}{2}, \frac{\lambda}{2})$

$\geq (1 - \epsilon) \star e$

$> 1 - s_1.$

$$\begin{aligned}
& B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \lambda) = B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) \\
& \leq B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \frac{\beta}{2}, \frac{\beta}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}) \\
& \leq B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \frac{\beta}{2}, \frac{\lambda}{2}) \star B(\frac{\beta}{2}, \frac{\lambda}{2}) < \epsilon \circ f < s_2. \\
& \text{and } Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \lambda) = Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) \\
& \leq Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), \frac{\beta}{2}, \frac{\beta}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}) \\
& \leq Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \frac{\beta}{2}, \frac{\beta}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}) \star Y(\frac{\beta}{2}, \frac{\lambda}{2}) < \epsilon \star g < s_3.
\end{aligned}$$

Taking $S = \max\{s_1, s_2, s_3\}$, we have the set $\{(u, v, w) \in \mathbb{N} : \exists s \in (0, 1) :$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \lambda) > 1 - s$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \lambda) < s \text{ and}$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \lambda) < s.$$

$$\implies x = (x_{mnk}) \in L_{\Lambda^3(P, Q, R)}(T)$$

$$\implies L_{(P, Q, R)}(T) \subset L_{\Lambda^3(P, Q, R)}(T).$$

The converse is not valid. We present the following examples. $\square$

Example 3.3. Let $(\mathbb{R}, |., .|)$ be an NMS such that

$$d(xy) = \sup_{m, n, k} \{|x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k = 1, 2, 3, \dots\}$$

Let $x \star y = \min\{x, y\}$ and $x \circ y = \max\{x, y\}, \forall x, y \in (0, 1)$.

Now define metric (P, Q, R) on $\eta^2 \times (0, \infty)$ as follows :

$$G(x, y, \lambda) = \frac{\lambda}{\lambda + d(x, y)};$$

$$B(x, y, \lambda) = \frac{d(x, y)}{\lambda + d(x, y)};$$

$$Y(x, y, \lambda) = \frac{d(x, y)}{\lambda}.$$

Then $(\mathbb{R}, N, \star, \circ)$ is an NMS.

Consider the sequence $(x_{mnk})^{\frac{1}{m+n+k}} = \{1\}$.

Hence $(x_{mnk}) \in L_{(P, Q, R)}(T)$ and $(x_{mnk}^{\frac{1}{m+n+k}}) \rightarrow \{1\}$ but $(x_{mnk}^{\frac{1}{m+n+k}}) \notin L_{\Lambda^3(P, Q, R)}(T)$.

Example 3.4. Let $(\mathbb{R}, |., .|)$ be an NMS and (P, Q, R) be the NMS as defined in the above example. Consider the sequence $(x_{mnk}) = (-1)^{m+n+k}$. Then $(x_{mnk}) \in L_{\Lambda^3(P, Q, R)}(T)$, but $(x_{mnk}) \notin L_{(P, Q, R)}(T)$.

Theorem 3.5. The space $L_{\Gamma^3(P, Q, R)}(T)$ is linear space.

Proof. Let $x = (x_{mnk}), y = (y_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$. Then $\exists \beta_1, \beta_2 \in Y$ such that (y_{mnk}) and (z_{mnk}) $\mathcal{N}$ -convergence to β_1 and β_2 respectively. We shall prove that for any scalars ξ_1 and ξ_2 , the sequence $\xi_1 x_{mnk} + \xi_2 y_{mnk}$, $\mathcal{N}$ converges to $\eta_1 \beta_1 + \eta_2 \beta_2$. For $\lambda > 0$ and $0 < \epsilon < 1$. Consider the following finite sets ξ_1 and ξ_2 :

$$\xi_1 = \lim_{m, n, k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) \leq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) \geq \epsilon$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) \geq \epsilon\} = 0$$

$$\xi_1^c = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) < \epsilon$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) < \epsilon\} = 0$$

$$\xi_2 = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) \leq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) \geq \epsilon$$

$$Y(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) \geq \epsilon\} = 0$$

$$\xi_2^c = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) < \epsilon$$

$$Y(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) < \epsilon\} = 0$$

Define the set $\xi_3 = \xi_1 \cup \xi_2$ so that ξ_3 is finite. It follows that $\xi_3^c \neq \phi$. Now we prove that for each $x = (x_{mnk}), y = (y_{mnk}) \in L_{\Gamma^3(P,Q,R)}(T)$,

$$\xi_3^c \subset \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G((\xi_1 T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2 T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1, \beta_1, \xi_2, \beta_2), \lambda) > 1 - \epsilon \text{ or}$$

$$B((\xi_1 T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2 T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1, \beta_1, \xi_2, \beta_2), \lambda) < \epsilon$$

$$Y((\xi_1 T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2 T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1, \beta_1, \xi_2, \beta_2), \lambda) < \epsilon\} = 0$$

Let $a, b, c \in \xi_3^c$. In this case, $\lim_{m,n,k \rightarrow \infty}$

$$\{G(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) > 1 - \epsilon \text{ or}$$

$$B(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) < \epsilon$$

$$Y(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) < \epsilon = 0 \text{ and}$$

$$= \lim_{m,n,k \rightarrow \infty} \{G(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) > 1 - \epsilon \text{ or}$$

$$B(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) < \epsilon$$

$$Y(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|}) < \epsilon\} = 0$$

$$\lim_{m,n,k \rightarrow \infty} G((\xi_1 T_{abc}(x), x, \xi_2 T_{abc}(y), y), (\xi_1 \circ \beta_1, \xi_1 \circ \beta_2), \lambda)$$

$$\begin{aligned}
&\geq \lim_{m,n,k \rightarrow \infty} \{G(\xi_1 T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_1, \beta_1, \frac{\lambda}{2}) \star G(\xi_2 T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y, \xi_2, \beta_2, \frac{\lambda}{2})\} \\
&= \lim_{m,n,k \rightarrow \infty} \{G(T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2|\xi_1|}) \star G(T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta_2, \frac{\lambda}{2|\xi_2|})\} \\
&> \lim_{m,n,k \rightarrow \infty} \{(1 - \epsilon) \star (1 - \epsilon)\} \\
&= \lim_{m,n,k \rightarrow \infty} \{1 - \epsilon\} = 0 \\
&\implies \lim_{m,n,k \rightarrow \infty} \{G((\xi_1, T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2, T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1, \beta_1, \xi_2, \beta_2), \lambda) \\
&> 1 - \epsilon\} = 0
\end{aligned}$$

In a similar way,

$$\begin{aligned}
&\lim_{m,n,k \rightarrow \infty} \{B((\xi_1, T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2, T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1 \circ \beta_1, \xi_2 \circ \beta_2), \lambda) < \epsilon\} = 0 \text{ and} \\
&\lim_{m,n,k \rightarrow \infty} \{Y((\xi_1, T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2, T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1 \circ \beta_1, \xi_2 \circ \beta_2), \lambda) < \epsilon\} = 0
\end{aligned}$$

We have, $\xi_3^c \subset \lim_{m,n,k \rightarrow \infty} (a, b, c) \in \mathbb{N}$:

$$\begin{aligned}
&G((\xi_1, T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2, T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1 \circ \beta_1, \xi_2 \circ \beta_2), \lambda) \\
&> 1 - \epsilon\}, \\
&\{B((\xi_1, T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2, T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1 \circ \beta_1, \xi_2 \circ \beta_2), \lambda) < \epsilon\}, \text{ or} \\
&\{Y((\xi_1, T_{abc}(x_{mnk}^{\frac{1}{m+n+k}}), x, \xi_2, T_{abc}(y_{mnk}^{\frac{1}{m+n+k}}), y), (\xi_1 \circ \beta_1, \xi_2 \circ \beta_2), \lambda) < \epsilon\} = 0 \\
&\implies \text{the sequence } (\xi_1 x_{mnk} + \xi_2 y_{mnk}) \text{ } \mathcal{N}\text{-converges to } (\xi_1 \beta_1 + \xi_2 \beta_2).
\end{aligned}$$

Hence $(\xi_1 x_{mnk} + \xi_2 y_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$.

Hence $L_{\Gamma^3(P, Q, R)}(T)$ is a linear space. $\square$

Theorem 3.6. Every open ball with the center at v and the radius $r > 0$ with respect to the parameter if fuzziness $0 < \epsilon < 1$, i.e, $O_x(r, \epsilon)(T)$ is an open set in $L_{\Gamma^3(P, Q, R)}(T)$.

Proof. Consider the open ball with center at x and radius $r > 0$ with the parameter of neutrosophic $0 < \epsilon < 1$,

$$\begin{aligned}
O_x(r, \epsilon)(T) &= \lim_{m,n,k \rightarrow \infty} \{x = (x_{uvw}) \in w : \{(u, v, w) \in \mathbb{N} : \\
&G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) \leq 1 - \epsilon \text{ or} \\
&B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) \geq \epsilon \\
&R(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) \geq \epsilon\} \text{ is finite} \} = 0 \\
\text{Then } O_x^c(r, \epsilon)(T) &= \lim_{m,n,k \rightarrow \infty} \{x = (x_{uvw}) \in w : \{(u, v, w) \in \mathbb{N} : \\
&G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) > 1 - \epsilon \text{ or} \\
&B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) < \epsilon, \\
&Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) < \epsilon\} = 0\}
\end{aligned}$$

Let $y = (y_{uvw}) \in O_x(r, \epsilon)(T)$. Then the set $= \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$$\begin{aligned}
&G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) > 1 - \epsilon \text{ or} \\
&B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) < \epsilon, \\
&Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r) < \epsilon\} = 0 \exists r_0 \in (0, r) \text{ such that}
\end{aligned}$$

$\lim_{m,n,k \rightarrow \infty} \{G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r_0) > 1 - \epsilon \text{ or}$
 $B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r_0) < \epsilon,$
 $Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r_0)) < \epsilon\} = 0$
 Put $\epsilon_0 = G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r_0) > 1 - \epsilon = \epsilon_0$.
 then $\exists s \in (0, 1)$ such that $\epsilon_0 > 1 - s > 1 - \epsilon$.

For $\epsilon_0 > 1 - s$, we can have $\epsilon_1, \epsilon_2, \epsilon_3 \in (0, 1)$ such that $\epsilon_0 \star \epsilon_1 > 1 - s$;
 $(1 - \epsilon_0) \circ (1 - \epsilon_2) < s$ and $(1 - \epsilon_0) \circ (1 - \epsilon_3) < s$.

Let $\epsilon_4 = \max\{\epsilon_1, \epsilon_2, \epsilon_3\}$.

Now consider the open ball $O_x^c(r, r_0, 1 - \epsilon_4)(T)$.

Now to show that $O_x^c(r, r_0, 1 - \epsilon_4)(T) \subset O_x^c(r, \epsilon)(T)$.

Let $z = (z_{uvw}) \in O_z^c(r, r_0, 1 - \epsilon_4)(T)$.

Then $= \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0) > \epsilon_4 \text{ or}$
 $B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0) < 1 - \epsilon_4;$
 $Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0) < 1 - \epsilon_4\} = 0$

Therefore $\lim_{m,n,k \rightarrow \infty} \{G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, r)$
 $\geq \lim_{m,n,k \rightarrow \infty} G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r_0) \quad \star$
 $G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0)\}$
 $\geq \epsilon_0 \star \epsilon_4 \geq \epsilon_0 \star \epsilon_1$
 $> (1 - s) > (1 - \epsilon) = 0$

$\implies \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r) > 1 - \epsilon\} = 0$

In similarly, $\lim_{m,n,k \rightarrow \infty} B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r) < 0$

$= \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} : B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r) < \epsilon\} = 0$

$= \lim_{m,n,k \rightarrow \infty} Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r)$

$= \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} : Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r) < \epsilon\} = 0$

Therefore the set $= \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0) > 1 - \epsilon \text{ or}$

$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0) < \epsilon$

$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, (T_{uvw}(z_{mnk}^{\frac{1}{m+n+k}}), z, r, r_0) < \epsilon\} = 0,$

$\implies x = (x_{mnk}) \in O_x^c(r, \epsilon)(T)$.

$\implies O_x^c(r, r_0, 1 - \epsilon_4)(T) \subset O_y^c(r, \epsilon)(T)$. $\square$

Remark 3.7. The space $L_{\Gamma^3(P,Q,R)}(T)$ is NMS with respect to the Neutrosophic metric (P, Q, R) .

Theorem 3.8. *The topology $\tau_{(P,Q,R)}(T)$ on the space $L_{\Gamma^3(P,Q,R)}(T)$ is first countable.*

Proof. For each $x = (x_{mnk}) \in L_{\Gamma^3(P,Q,R)}(T)$.

Consider the set $B = \{O_x(\frac{1}{efg}, \frac{1}{efg}) : e, f, g = 1, 2, 3, \dots\}$ which is a countable local base at x .

Therefore, the topology $\tau_{(P,Q,R)}(T)$ on the space $L_{\Gamma^3(P,Q,R)}(T)$ is first countable. $\square$

4. On the Tribonacci triple sequence T_{uvw}

Definition 4.1. Give $(V, N, \star, \circ)$ be a NMS, A tripe sequence $x = (x_{mnk}) \in V$ is said to be Tribonacci convergent to $\beta \in \eta$ if for every $\epsilon > 0$ the set B_1 is finite, where

$$B_1 = \{(u, v, w) \in \mathbb{N} : |T_{uvw}(x_{mnk}), x, \beta| \geq \epsilon\}$$

Definition 4.2. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in V$ is said to be Neutrosophic Tribonacci convergent to $\beta \in \eta$ with respect to Neutrosophic metric (P, Q, R) denoted by $(x_{mnk}) \rightarrow \beta$ is for every $\epsilon \in (0, 1)$ and $\lambda > 0$, the set T_1 is finite, where $T_1 = \{(u, v, w) \in \mathbb{N} :$

$$G(T_{uvw}(x_{mnk}), x, \beta, \lambda) \leq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}), x, \beta, \lambda) \geq \epsilon,$$

$$YS(T_{uvw}(x_{mnk}), x, \beta, \lambda) \geq \epsilon\}$$

We write $N_{(P,Q,R)}(T) - \lim_{m,n,k \rightarrow \infty} (x_{mnk}) = \beta$.

Definition 4.3. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{m,n,k}) \in V$ is said to be Tribonacci Cauchy if for every $\epsilon > 0$, then $\exists(m, n, k) = (mnk(\epsilon)) \in \mathbb{N}$ such that the set B_2 is finite where

$$B_2 = \{(u, v, w) \in \mathbb{N} : |T_{uvw}(x_{mnk}), T_{uvw}(x_{pqr})| \geq \epsilon.$$

Definition 4.4. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{m,n,k}) \in V$ is said to be Neutrosophic Tribonacci Cauchy with respect to Neutrosophic metric (P, Q, R) is for every $\epsilon \in (0, 1)$ and $\lambda > 0$ there exist $(P, Q, R) \in \mathbb{N}$ such that the set T_2 is finite, where $T_2 = \{(u, v, w) \in \mathbb{N} : G(T_{uvw}(x_{mnk}), T_{uvw}(x_{pqr})\lambda) \leq 1 - \epsilon.$ or

$$B(T_{uvw}(x_{mnk}), T_{uvw}(x_{pqr})\lambda) \geq \epsilon.$$

$$Y(T_{uvw}(x_{mnk}), T_{uvw}(x_{pqr})\lambda) \geq \epsilon.$$

Definition 4.5. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{m,n,k}) \in V$ is said to be Tribonacci bounded if there exists $M > 0$ such that the set

$$B_3 = \{(u, v, w) \in \mathbb{N} : |T_{uvw}(x_{mnk})| > M\}$$

Definition 4.6. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{m,n,k}) \in V$ and a subset D of V is said to be Neutrosophic Tribonacci bounded with respect to Neutrosophic metric

(P, Q, R) if $\forall x \in D \exists 0 < \epsilon < 1$ and $\lambda > 0$ such that the set $\{(u, v, w) \in \mathbb{N} :$

$$G(T_{uvw}(x_{mnk}), \lambda) \leq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}), \lambda) \geq \epsilon,$$

$$Y(T_{uvw}(x_{mnk}), \lambda) \geq \epsilon\} \quad (4.6.1)$$

Theorem 4.7. *Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in V$ is neutrosophic Tribonacci entire, then the $N_{(P,Q,R)}(T)$ -limit is unique.*

Proof. Let $x = (x_{mnk}^{\frac{1}{m+n+k}}) \in V$ such that $(x_{mnk}^{\frac{1}{m+n+k}})$ is Neutrosophic Tribonacci entire. Let $N_{(P,Q,R)}(T) - \lim_{m,n,k \rightarrow \infty} (x_{mnk}^{\frac{1}{m+n+k}}) = \beta_1$ and $N_{(P,Q,R)}(T) - \lim_{m,n,k \rightarrow \infty} (x_{mnk}^{\frac{1}{m+n+k}}) = \beta_2$. To prove that $\beta_1 = \beta_2$. Now, for given $\epsilon \in (0, 1)$, $\exists \epsilon \in (0, 1)$ such that

$(1 - \epsilon_1) \star (1 - \epsilon_1) > 1 - \epsilon$ and $\epsilon_1 \circ \epsilon_2 < \epsilon$. Therefore the sets ξ_1 and ξ_2 are finite, where

$$\xi_1 = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) \leq 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) \geq \epsilon_1$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) \geq \epsilon_1\} = 0$$

$$\xi_2 = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2}) \leq 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2}) \geq \epsilon_1$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2}) \geq \epsilon_1\} = 0$$

$$\xi_1^c = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) > 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) < \epsilon_1$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) < \epsilon_1\} = 0$$

$$\xi_2^c = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2}) > 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2}) < \epsilon_1$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2}) < \epsilon_1\} = 0$$

Then $\xi_1^c \cap \xi_2^c \neq \phi$. Taking $(u, v, w) \in (\xi_1^c \cap \xi_2^c)$, we have

$$\lim_{m,n,k \rightarrow \infty} (G(\beta_1, \beta_2, \lambda)) \quad \geq$$

$$\lim_{m,n,k \rightarrow \infty} \{G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) \star G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2})$$

$$> \{(1 - \epsilon_1) \star (1 - \epsilon_1)$$

$$> (1 - \epsilon_1) = 1 - \epsilon$$

$$= \lim_{m,n,k \rightarrow \infty} B(\beta_1, \beta_2, \lambda)$$

$$\leq \lim_{m,n,k} \{B(T(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) \circ B(T(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta_2, \frac{\lambda}{2})\}$$

$$\leq \{\epsilon_1 \circ \epsilon_2\} \leq \epsilon_1 = \epsilon_1 \text{ and}$$

$$\lim_{m,n,k \rightarrow \infty} B(\beta_1, \beta_2, \lambda)$$

$$\leq \lim_{m,n,k \rightarrow \infty} \{Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), x, \beta_1, \frac{\lambda}{2}) \circ Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), x, \beta_2, \frac{\lambda}{2})\}$$

$$< \lim_{m,n,k \rightarrow \infty} (\epsilon_1 \circ \epsilon_2) < \epsilon_1 = \epsilon$$

Since $\epsilon \in (0, 1)$ is arbitrary, therefore $G(\beta_1, \beta_2, \lambda) = 1, B(\beta_1, \beta_2, \lambda) = 0$ and

$$Y(\beta_1, \beta_2, \lambda) = 0 \text{ for all } \lambda > 0.$$

Hence $\beta_1 = \beta_2$.

Thus $N_{(P,Q,R)}(T)$ - limit is unique. $\square$

Theorem 4.8. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in V$ is Neutrosophic Tribonacci entire of every subset D of $L_{\Gamma^3(P,Q,R)}(T)$ is Neutrosophic Tribonacci bounded.

Proof. The proof of this theorem follows from Theorem 3.2 and can be verified on similar. $\square$

Theorem 4.9. Let $(V, N, \star, \circ)$ be an NMS, $L_{\Gamma^3(P,Q,R)}(T)$ and $\tau_{(P,Q,R)}(T)$ be the topology on $L_{\Gamma^3(P,Q,R)}(T)$. Let $x = (x_{\frac{1}{mnk}^{m+n+k}})^{efg}$ be a triple sequence of points in $L_{\Gamma^3(P,Q,R)}(T)$. Then $(x_{mnk} - x) \rightarrow 0$ as $m, n, k \rightarrow \infty$ if and only if

$$G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), \lambda) \rightarrow 1,$$

$$B(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), \lambda) \rightarrow 0 \text{ and}$$

$$Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), \lambda) \rightarrow 0 \text{ as } m, n, k \rightarrow \infty$$

Proof. Let $(x_{mnk} - x) \rightarrow 0$ as $m, n, k \rightarrow \infty$. Fix $r > 0$. Then, for $0 < \epsilon < 1$, there exists $(e, f, g) \in \mathbb{N}$ such that $(x_{mnk}) \in O_x(r, \epsilon)(T)$ for all $m \geq e, n \geq f, k \geq g$. Then the set A is finite, where

$$A = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) \leq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) \geq \epsilon,$$

$$Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) \geq \epsilon\} = 0.$$

$$\text{ie. } A^c = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) < \epsilon,$$

$$Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) < \epsilon\} = 0$$

For $\{(u, v, w) \in \mathbb{N} : \} \subseteq A^c$,

$$\text{ie. } \lim_{m,n,k \rightarrow \infty} \{G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(x), r) > 1 - \epsilon$$

$$\begin{aligned} \implies & 1 - G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) > \epsilon \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) < \epsilon \text{ and} \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) < \epsilon \} = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} \implies & 1 - G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \\ \implies & G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 1 \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \end{aligned}$$

Conversely, suppose that for each $(m, n, k) > 0$,

$$\begin{aligned} & G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 1 \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ and} \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ as } m, n, k \rightarrow \infty. \end{aligned}$$

Then for each $\epsilon \in (0, 1)$ there exists $(e, f, g) \in \mathbb{N}$: such that

$$\begin{aligned} \implies & 1 - G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ and} \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \end{aligned}$$

for all $m \geq e, n \geq f, k \geq g$

$$\begin{aligned} & G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) > 1 - \epsilon \\ \text{ie. } & 1 - G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) > \epsilon \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) < \epsilon \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) < \epsilon \end{aligned}$$

Consider that the Neutrosophic $\mathcal{N}$ generated by the set $\{a < e, b < f, c \leq g\}$ implies the collection of sets generated by the set

$$\{(m, n, k) \in \mathbb{N} : u \geq e, v \geq f, w \geq g. \text{ Thus } = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$\begin{aligned} & G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) > 1 - \epsilon \text{ or} \\ & B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) < \epsilon \\ & Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) < \epsilon \} = 0 \end{aligned}$$

$$\implies (x_{mnk} \in O_x(r, \epsilon)(T) \text{ for all } m \geq e, n \geq f, k \geq g$$

Hence $(x_{mnk} - x) \rightarrow 0$ as $m, n, k \rightarrow \infty$.

ie. $(x_{mnk}) \rightarrow x$ as $m, n, k \rightarrow \infty$ $\square$

Theorem 4.10. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in V$ is Neutrosophic Tribonacci of every closed ball with the center at x and the radius $r > 0$ with respect to the parameter of fuzziness $0 < \epsilon < 1$, ie. $O_x[r, \epsilon](T)$ is a closed set in $L_{\Gamma^3(P, Q, R)}(T)$.

Proof. Let $x = (x_{mnk}) \in V$ be such that $x \in \overline{O_x[r, \epsilon](T)}$. Then there exists a triple sequence $(x_{mnk}) = (x_{mnk})^{efg} \in O_x[r, \epsilon](T)$ such that $(x_{mnk} - x) \rightarrow 0$ as $m, n, k \rightarrow \infty$. This implies the set

$$X = \lim_{m, n, k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \geq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \leq \epsilon$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \leq \epsilon\} = 0$$

Since $(x_{mnk} - x) \rightarrow 0$ as $m, n, k \rightarrow \infty$

ie. $(x_{mnk}) \rightarrow x$ as $m, n, k \rightarrow \infty$ by Theorem 4.9,

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 1$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), r) \rightarrow 0 \text{ as } m, n, k \rightarrow \infty \text{ for all } (a, b, c) > 0 \text{ as } \lambda \rightarrow \infty.$$

Hence for $(m, n, k) \in x$.

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(x), \lambda + r)$$

$$\geq \lim_{m, n, k \rightarrow \infty, u, v, w \rightarrow \infty} \{G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), \lambda) \star G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r)\}$$

$$\geq 1 \star (1 - \epsilon) = 1 - \epsilon \text{ as } m, n, k \rightarrow \infty.$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), \lambda + r)$$

$$\leq \lim_{m, n, k \rightarrow \infty, u, v, w \rightarrow \infty} \{B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), \lambda) \circ B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r)\}$$

$$\leq 0 \circ \epsilon = \epsilon \text{ and}$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), \lambda + r)$$

$$\leq \lim_{m, n, k \rightarrow \infty, u, v, w \rightarrow \infty} \{Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), \lambda) \circ Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r)\}$$

$$\leq 0 \circ \epsilon = \epsilon$$

In particular, for $(e, f, g) \in \mathbb{N}$, take $\lambda = \frac{1}{efg}$. Then

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r) =$$

$$\lim_{mnk \rightarrow \infty, e, f, g \rightarrow \infty} \{G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r + \frac{1}{efg}) \geq 1 - \epsilon\}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r) =$$

$$\lim_{mnk \rightarrow \infty, e, f, g \rightarrow \infty} \{B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r + \frac{1}{efg}) \leq \epsilon\} \text{ and}$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r) =$$

$$\lim_{mnk \rightarrow \infty, e, f, g \rightarrow \infty} \{Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r + \frac{1}{efg}) \leq \epsilon\} \text{ and}$$

$$\implies \lim_{m, n, k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} : G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r) \geq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r) \leq \epsilon$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y), r) \leq \epsilon\} = 0.$$

$$y \in O_x[r, \epsilon](T).$$

Therefore $O_x[r, \epsilon](T)$ is a closed set. $\square$

Theorem 4.11. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$. Then for some $\beta \in \eta$, $(x_{mnk}) \rightarrow \beta$ if and only if for every $\epsilon \in (0, 1)$ and $\lambda > 0$ there exists positive integers $N = N(x, \epsilon, \lambda)$ such that

$$\lim_{m,n,k \rightarrow \infty} \{N \in \mathbb{N} :$$

$$G(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) > 1 - \epsilon$$

$$B(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon$$

$$Y(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon\} = 0$$

Proof. Suppose $(x_{mnk}) \rightarrow \beta$ for some $\beta \in Y$. For given $\epsilon \in (0, 1)$ such that $(1 - \epsilon) \star (1 - \epsilon) > 1 - r$ and $\epsilon \circ \epsilon < r$. Since $(x_{mnk}) \rightarrow \beta$, for all $\lambda > 0$,

$$S = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) \leq 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon\} = 0$$

Conversely, let us choose $N \in S^c$. Then,

$$G(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) > 1 - \epsilon$$

$$B(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon$$

$$Y(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon \text{ as } m, n, k \rightarrow \infty.$$

To prove that there exists a positive integer $N = N(x, \epsilon, \lambda)$ such that

$$\lim_{m,n,k \rightarrow \infty} \{(e, f, g) \in \mathbb{N} :$$

$$G(T_{efg}(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \leq 1 - r \text{ or}$$

$$B(T_{efg}(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \geq r$$

$$Y(T_{efg}(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \geq r\} = 0.$$

We have $x = (x_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$ define,

$$\xi = \lim_{m,n,k \rightarrow \infty} \{(e, f, g) \in \mathbb{N} :$$

$$G(T_{efg}(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \leq 1 - r \text{ or}$$

$$B(T_{efg}(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \geq r$$

$$Y(T_{efg}(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \geq r\} = 0.$$

We shall prove that $\xi \subset \eta$. Let on the contrary $\xi \not\subset \eta$. ie, $\exists m \in \xi$ such that $m \notin \eta$. Then

$$G(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \leq 1 - r \text{ or}$$

$$G(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \frac{\lambda}{2}) > 1 - \epsilon \text{ as } m, n, k \rightarrow \infty$$

In particular, $G(T_N(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) > 1 - \epsilon$ as $m, n, k \rightarrow \infty$.

Therefore, we have $1 - r \geq G(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda)$
 $\geq G(T_m(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) \star G(T_m(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2})$
 $\geq (1 - \epsilon) \star (1 - \epsilon)$
 $> 1 - \epsilon = 1 - r.$

which is a contradiction. Similarly,

$B(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \geq r$ or
 $B(T_m(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon$ as $m, n, k \rightarrow \infty$.

In particular $B(T_N(x), \beta, \frac{\lambda}{2}) < \epsilon$.

Therefore we have

$r \leq B(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda)$
 $\leq B(T_m(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) \circ B(T_N(x), x, \beta, \frac{\lambda}{2})$
 $\leq \epsilon \circ \epsilon < \epsilon = r$

Similarly, in the other way,

$Y(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda) \geq r$
 $Y(T_m(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) < \epsilon$ as $m, n, k \rightarrow \infty$.

In particular $Y(T_N(x), x, \beta, \frac{\lambda}{2}) < \epsilon$.

$r \leq Y(T_m(x_{mnk}^{\frac{1}{m+n+k}}), T_N(x), \lambda)$
 $Y(T_m(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \frac{\lambda}{2}) \circ Y(T_N(x), x, \beta, \frac{\lambda}{2})$
 $\leq \epsilon \circ \epsilon < \epsilon = r$

which is a contradiction.

Hence $\xi \not\subseteq \eta$ since $\eta \in \mathbb{N} \implies \xi \in N$. $\square$

Definition 4.12. Consider $D \subseteq L_{\Gamma^3(P,Q,R)}(T)$. Then D is compact if and only if every open cover of D by the open set of $\tau_{(P,Q,R)}(T)$ has a finite subcover.

Theorem 4.13. Every finite subset D of $L_{\Gamma^3(P,Q,R)}(T)$ is compact.

Proof. Let

$$D = \begin{pmatrix} x_{111}^{\frac{1}{3}} & x_{112}^{\frac{1}{4}} & \cdots & x_{11n}^{\frac{1}{2+n}} \\ x_{221}^{\frac{1}{4}} & x_{222}^{\frac{1}{6}} & \cdots & x_{22n}^{\frac{1}{4+n}} \\ \vdots & \vdots & \vdots & \vdots \\ x_{m11}^{\frac{1}{2+m}} & x_{m22}^{\frac{1}{4+m}} & \cdots & x_{mnk}^{\frac{1}{m+n+k}} \end{pmatrix}$$

be the finite subsets of $L_{\Gamma^3(P,Q,R)}(T)$. For $r > 0$ and $0 < \epsilon < 1$.

Let us assume that $\{O_x(r, \epsilon)(T) : x \in D\}$ is an open cover of D. Then $D \subseteq \bigcup_{x \in D} O_x(r, \epsilon)(T)$.

Now for all $(x_{mnk}^{\frac{1}{m+n+k}}) \in D, m, n, k = 1, 2, 3, \dots (a, b, c)$, we have $(x_{mnk}^{\frac{1}{m+n+k}}) \in U_{x_{mnk} \in D} O_{mnk}(r, \epsilon)(T)$.

$$\implies x = (x_{mnk}^{\frac{1}{m+n+k}}) \in O_{x_{mnk}}(r, \epsilon)(T)$$

for some $(m, n, k) \in (1, 2, 3, \dots (a, b, c))$

Then $\{O_{x_{mnk}(r, \epsilon)(T): m, n, k=1, 2, 3, \dots (abc)}\}$

is a finite subcover of D.

Hence D is compact. $\square$

Theorem 4.14. Let $D \subseteq L_{\Gamma^3(P, Q, R)}(T)$. Then D is compact iff every triple sequence in D has an entire subsequence.

Proof. Suppose that D is compact subset of $L_{\Gamma^3(P, Q, R)}(T)$.

Let $x = (x_{mnk}^{\frac{1}{m+n+k}})$ be a triple sequence in D. For given $0 < t < 1$ and $r > 0$, let $\{O_x(\frac{r}{3}, \epsilon)(T) : x = (x_{mnk}) \in D\}$. be an open cover of S.

$$\implies x = (x_{mnk}^{\frac{1}{m+n+k}}) \in \cup_{x \in D} O_x(\frac{r}{3}, \epsilon)(T).$$

Then there exists some $x = (x_{mnk}) \in D$ such that $x = (x_{mnk}^{\frac{1}{m+n+k}}) \in O_x(\frac{r}{3}, \epsilon)(T)$. We have

$$\eta_1 = \lim_{m, n, k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}, T_{uvw}(x), \frac{r}{3}) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}, T_{uvw}(x), \frac{r}{3}) < \epsilon$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}, T_{uvw}(x), \frac{r}{3}) < \epsilon\} = 0$$

Since D is compact, there exist a finite subcover $\{O_{x_{mnk}}(\frac{r}{3}, \epsilon)(T) : (x_{mnk}) \in D \text{ and } m, n, k=1, 2, \dots (a, b, c)\}$.

of D such that $D \subseteq \cup_{m, n, k=1}^{a, b, c} O_{x_{mnk}}(\frac{r}{3}, \epsilon)(T)$.

Let $x = (x_{m_p n_p k_p})$ be a subsequence of (x_{mnk}) . Then

$$x = (x_{m_p n_p k_p}) \in \cup_{m, n, k=1}^{a, b, c} O_{x_{mnk}}(\frac{r}{3}, \epsilon)(T)$$

$$x = (x_{m_p n_p k_p}) \in O_{x_{mnk}}(\frac{r}{3}, \epsilon)(T)$$

for some $x_{mnk} \in D$. We have

$$\eta_2 = \lim_{m, n, k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{m_p n_p k_p}^{\frac{1}{m_p + n_p + k_p}}, T_{uvw}(x_{mnk}), \frac{r}{3}) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{m_p n_p k_p}^{\frac{1}{m_p + n_p + k_p}}, T_{uvw}(x_{mnk}), \frac{r}{3}) < \epsilon$$

$$Y(T_{uvw}(x_{m_p n_p k_p}^{\frac{1}{m_p + n_p + k_p}}, T_{uvw}(x_{mnk}), \frac{r}{3}) < \epsilon\} = 0$$

Now for $(e, f, g) \in \eta_1 \cap \eta_2$,

$$\lim(G(T_{uvw}(x_{m_p n_p k_p}^{\frac{1}{m_p + n_p + k_p}}, T_{uvw}(x), r)$$

$$\geq \lim\{G(T_{uvw}(x_{m_p n_p k_p}^{\frac{1}{m_p + n_p + k_p}}, T_{uvw}(x), \frac{r}{3}) \star G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}, T_{uvw}(x), \frac{r}{3}) \star$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}, T_{uvw}(x), \frac{r}{3})\} = 0$$

$$> (1 - \epsilon) \star (1 - \epsilon) \star (1 - \epsilon) = (1 - \epsilon) = 0$$

$$\implies \{G(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) > 1 - \epsilon\} = 0 \text{ as } m_p, n_p, k_p \rightarrow \infty$$

$$\text{Similarly } \lim B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r)$$

$$\leq \lim \{B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), \frac{r}{3}) \quad \circ \quad B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), \frac{r}{3}) \quad \circ \quad B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), \frac{r}{3})\} = 0 \text{ as } m, n, k \rightarrow \infty.$$

$$< \epsilon \circ \epsilon \circ \epsilon = \epsilon.$$

$$\text{and } \lim Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r)$$

$$\leq \lim \{Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), \frac{r}{3}) \quad \circ \quad Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x_{m,n,k}), \frac{r}{3}) \quad \circ \quad Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), \frac{r}{3})\} = 0 \text{ as } m, n, k \rightarrow \infty.$$

$$< \epsilon \circ \epsilon \circ \epsilon = \epsilon.$$

Take $\epsilon = \frac{1}{p}$. Then

$$\lim_{p \rightarrow \infty} G(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) = \lim_{p \rightarrow \infty} 1 - \frac{1}{p} = 1.$$

$$\lim_{p \rightarrow \infty} B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) = \lim_{p \rightarrow \infty} \frac{1}{p} = 0. \text{ and}$$

$$\lim_{p \rightarrow \infty} Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) = \lim_{p \rightarrow \infty} \frac{1}{p} = 0.$$

Hence, by Theorem 4.9, $(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}} \rightarrow 0)$ as $p \rightarrow \infty$.

$$(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}} \rightarrow x).$$

Conversely, suppose that $(x_{m_p n_p k_p})$ is a subsequence of the triple sequence (x_{mnk}) in D such that $(x_{m_p n_p k_p} - x) \rightarrow 0$ in D.

ie. $(x_{m_p n_p k_p}) \rightarrow x$ in D. Let on the contrary D be not a compact subset of $L_{\Gamma^3(P,Q,R)}(T)$.

Suppose that $\{O_x(r, \epsilon)(T) : x \in D\}$ is an open cover of D.

$$\implies D \subseteq \cup_{x \in D} O_x(r, \epsilon)(T). \text{ We have}$$

$$\lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in V :$$

$$G(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) < \epsilon$$

$$Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) < \epsilon\} = 0$$

Since D is not compact, there exist a finite subcover $O_{x_{mnk}}(r, \epsilon)(T) : (x_{mnk} \in D) \exists : D \not\subseteq$

$$\cup_{x \in D} O_{x_{mnk}}(r, \epsilon)(T)$$

$$\lim_{m,n,k \rightarrow \infty} \{(m, n, k) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) < \epsilon$$

$$Y(T_{uvw}(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}), T_{uvw}(x), r) < \epsilon\} = 0$$

For any $0 < \epsilon < 1$ and $r > 0$, $(x_{\frac{1}{m_p n_p k_p}^{\overline{m_p+n_p+k_p}}}) \notin O_x(r, \epsilon)$.

$$\implies (x_{\frac{1}{m_p n_p k_p}^{m_p+n_p+k_p}} - x) \nrightarrow 0$$

ie. $(x_{\frac{1}{m_p n_p k_p}^{m_p+n_p+k_p}}) \nrightarrow x$ which is a contradiction.

Hence D is compact. $\square$

Theorem 4.15. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in L_{\Gamma^3(P,Q,R)}(T)$. Let $r > 0$ and $\epsilon_1 \epsilon_2 \in (0, 1)$ such that $(1 - \epsilon_1) \star (1 - \epsilon_1) \geq (1 - \epsilon_1)$ and $\epsilon_1 \circ \epsilon_1 = \epsilon_1$. Then for any $x = (x_{mnk}) \in L_{\Gamma^3(P,Q,R)}(T)$, $O_x(\frac{r}{2}, \epsilon_1)(T) \subseteq O_x(\frac{r}{2}, \epsilon_2)(T)$.

Proof. Let $s = (s_{mnk}) \in \overline{O_x(\frac{r}{2}, \epsilon_1)(T)}$ and $O_s(\frac{r}{2}, \epsilon_1)(T)$ be an open ball with center at s and the radius ϵ_1 .

Hence $O_s(\frac{r}{2}, \epsilon_1)(T) \cap O_x(\frac{r}{2}, \epsilon_1)(T) \neq \emptyset$.

Suppose $z = (z_{mnk}) \in O_s(\frac{r}{2}, \epsilon_1)(T) \cap O_x(\frac{r}{2}, \epsilon_1)(T)$.

Then the sets $\eta_1 = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$$G(T_{uvw}(s_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(z_{\frac{1}{mnk}^{m+n+k}}), s, z, \frac{r}{2}) \geq 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(s_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(z_{\frac{1}{mnk}^{m+n+k}}), s, z, \frac{r}{2}) < \epsilon_1$$

$$Y(T_{uvw}(s_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(z_{\frac{1}{mnk}^{m+n+k}}), s, z, \frac{r}{2}) < \epsilon_1\} = 0.$$

$\eta_2 = \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$$G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(z_{\frac{1}{mnk}^{m+n+k}}), x, z, \frac{r}{2}) \geq 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(z_{\frac{1}{mnk}^{m+n+k}}), x, z, \frac{r}{2}) < \epsilon_1$$

$$Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(z_{\frac{1}{mnk}^{m+n+k}}), x, z, \frac{r}{2}) < \epsilon_1\} = 0.$$

Consider $(e, f, g) \in \eta_1 \cap \eta_2$.

$$\lim_{m,n,k \rightarrow \infty} \{G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(s_{\frac{1}{mnk}^{m+n+k}}), x, s, r) \geq$$

$$\lim_{m,n,k \rightarrow \infty} \{G(T_{efg}(x_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(z_{\frac{1}{mnk}^{m+n+k}}), x, z, \frac{r}{2}) \star$$

$$G(T_{efg}(s_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(z_{\frac{1}{mnk}^{m+n+k}}), s, z, \frac{r}{2})\} = 0.$$

$$> (1 - \epsilon_1) \star (1 - \epsilon_1) \geq (1 - \epsilon_2)$$

$$\lim_{mnk \rightarrow \infty} \{B(T_{efg}(x_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(s_{\frac{1}{mnk}^{m+n+k}}), x, s, r)$$

$$\leq B(T_{efg}(x_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(z_{\frac{1}{mnk}^{m+n+k}}), x, z, \frac{r}{2}) \circ B(T_{efg}(s_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(z_{\frac{1}{mnk}^{m+n+k}}), s, z, \frac{r}{2}) \text{ and}$$

$$Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(s_{\frac{1}{mnk}^{m+n+k}}), x, z, r)$$

$$\leq Y(T_{efg}(x_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(z_{\frac{1}{mnk}^{m+n+k}}), x, z, \frac{r}{2}) \circ Y(T_{efg}(s_{\frac{1}{mnk}^{m+n+k}}), T_{efg}(z_{\frac{1}{mnk}^{m+n+k}}), s, z, \frac{r}{2})$$

$$< \epsilon_1 \circ \epsilon_1 \leq \epsilon_2.$$

Therefore we have $\lim_{u,v,w \rightarrow \infty} \{(u, v, w) \in \mathbb{N}$

$$G(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(s_{\frac{1}{mnk}^{m+n+k}}), x, s, r) > 1 - \epsilon_1 \text{ or}$$

$$B(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(s_{\frac{1}{mnk}^{m+n+k}}), x, s, r) < \epsilon_1$$

$$Y(T_{uvw}(x_{\frac{1}{mnk}^{m+n+k}}), T_{uvw}(s_{\frac{1}{mnk}^{m+n+k}}), x, s, r) < \epsilon_1\} = 0$$

$$\implies s = (s_{mnk}) \in O_x(r, \epsilon_2)(T).$$

$$\text{Hence } O_x(\frac{r}{2}, \epsilon_1)(T) \subseteq O_x(\frac{r}{2}, \epsilon_2)(T). \quad \square$$

Theorem 4.16. Let $(V, N, \star, \circ)$ be an NMS. A triple sequence $x = (x_{mnk}) \in V$. If there exists a triple sequence $y = (y_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$ such that $T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}) = T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}})$ for almost all (u, v, w) relative to Neutrosophic $\mathcal{N}$, then $x \in L_{\Gamma^3(P, Q, R)}(T)$.

Proof. Suppose $T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}) = T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}})$ for almost all (u, v, w) relative to $\mathcal{N}$.

Then $\lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} :$

$$T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}) \neq T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}) = 0\} \in N$$

$$\implies \lim_{m,n,k \rightarrow \infty} \{(u, v, w) \in \mathbb{N} : T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}) = T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}})\} = 0$$

Therefore, for all $\lambda > 0$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), x, y, \frac{\lambda}{2}) = 1. \text{ and}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), x, y, \frac{\lambda}{2}) = 0.$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), x, y, \frac{\lambda}{2}) = 0. \text{ as } m, n, k \rightarrow \infty.$$

Since $y = (y_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$, let $\lim_{m,n,k \rightarrow \infty} (z_{mnk}^{\frac{1}{m+n+k}}) = \beta$.

Then for every $\epsilon \in (0, 1)$ and $\lambda > 0$,

Since $y = (y_{mnk}) \in L_{\Gamma^3(P, Q, R)}(T)$, let

$$\eta_1 = \lim_{mnk \rightarrow \infty} \{(m, n, k) \in \mathbb{N} :$$

$$G(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta, \frac{\lambda}{2}) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta, \frac{\lambda}{2}) < \epsilon,$$

$$Y(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta, \frac{\lambda}{2}) < \epsilon\} = 0$$

Consider the set

$$\eta_2 = \lim_{mnk \rightarrow \infty} \{(m, n, k) \in \mathbb{N} :$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) > 1 - \epsilon \text{ or}$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) < \epsilon,$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) < \epsilon\} = 0$$

To prove that $\eta_1 \subset \eta_2$. So far $\lambda \in \eta_1$, we have

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) \geq$$

$$G(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), x, y, \frac{\lambda}{2}) \star G(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta, \frac{\lambda}{2})$$

$$> 1 \star (1 - \epsilon) = 1 - \epsilon.$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) \leq$$

$$B(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), x, y, \frac{\lambda}{2}) \circ B(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta, \frac{\lambda}{2})$$

$$< 0 \circ \epsilon = \epsilon.$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), x, \beta, \lambda) \leq$$

$$Y(T_{uvw}(x_{mnk}^{\frac{1}{m+n+k}}), T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), x, y, \frac{\lambda}{2}) \circ Y(T_{uvw}(y_{mnk}^{\frac{1}{m+n+k}}), y, \beta, \frac{\lambda}{2})$$

$< 0 \circ \epsilon$ as $m, n, k \rightarrow \infty$.

$\implies (u, v, w) \in \eta_2$.

Hence $\eta_1 \subset \eta_2$.

Hence $x = (x_{mnk}) \in L_{\Gamma^3(P,Q,R)}(T) \quad \square$

5. Conclusions

In this paper, we explore various directions by introducing a Tribonacci sequence space with aid of a Neutrosophic triple sequence space. We expect that our results will serve as a reference for further studies and also motivate young researchers in this field. We have defined the tribonacci matrix from neutrosophic triple entire sequence spaces and examined some topological and algebraic properties.

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Not applicable.

Declarations

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper and also read and approved the final manuscript.

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Neutrosophic estimation of population variance using two auxiliary variables in simple random sampling

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Abstract. In survey sampling, the efficient estimation of population variance plays an important role in drawing inference, particularly when handling uncertainty and imprecise data. This article develops a generalized class of Searls type power ratio estimators for the population variance under a neutrosophic framework, incorporating two auxiliary variables (TAV) in simple random sampling (SRS). The proposed approach accounts for indeterminacy and imprecision that frequently arise in real-world data collection processes by utilizing neutrosophic representation of both study and auxiliary variables. By simultaneously employing two auxiliary variables, the efficiency of the proposed variance estimators is significantly improved over some basic adapted neutrosophic population variance estimators. Theoretical properties of the proposed estimators, including bias and mean square error (MSE), are derived under the neutrosophic setup. A comparative performance evaluation is conducted using both simulated and real-world datasets, demonstrating the superiority of the proposed estimators in terms of efficiency, especially in uncertain data.

Keywords: Mean square error; Neutrosophic framework; Variance estimation; Auxiliary variables; Efficiency.

1. Introduction

In survey sampling, variance estimation is a key aspect of statistical analysis, as it provides a measure of the dispersion or variability within a population. Accurate estimation of population variance is essential for evaluating the precision of the estimators. The need for reliable estimates of population variance becomes even more significant when allocating resource or making policy decisions based on survey data. When auxiliary variables are highly associated with the study variable, they help reduce the sampling error and improve the precision of the variance estimates. Estimation methods that utilize such auxiliary information, including ratio, regression, logarithmic, or combined estimators, have shown to outperform the traditional estimators by minimizing bias and MSE. Therefore, using auxiliary information in

the estimation process not only enhances the inference drawn but also optimizes survey designs, especially under constraints of cost or sample size. Many authors recommended various estimation procedures to estimate the population variance using auxiliary information in survey sampling. [1] estimated the variance of a finite population using SRS. Some fundamental variance estimators were proposed by [2] in SRS. An improved exponential estimator for population variance using two auxiliary variables was proposed by [3]. [4] introduced a new family of variance estimators using known population variance of an auxiliary variable. [5] introduced a family of estimators for the population variance utilizing TAV. [6] introduced an elevated estimator of population variance utilizing TAV under SRS. The traditional variance estimation methods typically assume that data are accurate and complete. However, indeterminate data are often encountered in real-world surveys due to poor data collection practices or inconsistent respondent answers. Although challenging, such data can be effectively addressed using advanced approaches like fuzzy logic or neutrosophic theory, which incorporate uncertainty into the estimation process, leading to more flexible and realistic outcomes.

1.1. *Neutrosophic statistics and neutrosophic data*

Neutrosophic statistics is an emerging field of statistical science that extends the classical and fuzzy statistical frameworks to better handle uncertain and inconsistent data often encountered in real-world. Based on the neutrosophic logic introduced by [7], neutrosophic statistics incorporates the degrees of truth, indeterminacy, and falsity simultaneously, allowing for a more flexible and comprehensive representation of uncertainty. Unlike traditional statistical approaches that assume complete information, neutrosophic statistics recognizes the presence of ambiguity in data collection. This makes it particularly useful in areas like social sciences, environmental studies, medicine, and survey sampling, where data may be incomplete. By using neutrosophic numbers, this framework enhances the robustness and adaptability of the estimators, decision-making models, and hypothesis testing procedures under uncertain conditions. As a result, neutrosophic statistics provides a powerful and realistic tool for analyzing complex systems where conventional methods fall short.

Neutrosophic data is a generalized form of data representation that accounts for uncertainty, indeterminacy, and inconsistency, which are often present in real-world. Unlike classical or fuzzy data, neutrosophic data is characterized by three independent components: truth (T), indeterminacy (I), and falsity (F), each ranging in the interval $[0, 1]$. By accommodating uncertainty in data, neutrosophic approaches allow for more robust and realistic analyses in complex or uncertain environments. For instance, in climate studies, the prediction of rainfall for a specific area may have certain probability from different models (T), contradictory prediction from other models (F), and unknown factors like sudden atmospheric changes (I).

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Similarly, in customer feedback analysis, a product review may contain clear positive remarks (T), clear negative remarks (F), and neutral sentiments that cannot be classified confidently (I). In stock market analysis, expert opinions about a future performance of the company can be supportive (T), opposing (F), and uncertain due to unforeseen economic events (I). Such examples show how neutrosophic data express the real-world complexity better than purely crisp or fuzzy representations.

1.2. *Current status of work in survey sampling*

In survey sampling literature, only a limited number of estimation methods of population mean have been proposed for handling indeterminate data using some sampling designs. [8] introduced neutrosophic ratio-type estimators for estimating the population mean under SRS. A generalized neutrosophic sampling framework was proposed by [9] to improve the estimation of the population mean. Using auxiliary variable, [10] developed the neutrosophic factor-type exponential estimators for more accurate mean estimation. [11] proposed the robust neutrosophic exponential estimators of population mean under uncertainty. [12] introduced a new comprehensive mean estimator utilizing the regression-cum-exponential type estimator with the application of neutrosophic data. [13] suggested an optimal neutrosophic framework for population mean estimation under SRS by employing two auxiliary variables. [14] combined the two auxiliary variables for the enhanced estimation of finite population mean under neutrosophy. [15] estimated the population mean utilizing generalized neutrosophic exponential estimators. [16] estimated the population mean using two neutrosophic auxiliary variables with imprecise information. [17] developed an improved estimator for the population mean utilizing known medians of two auxiliary variables under neutrosophic framework. [18] constructed an almost unbiased estimator for the population median using neutrosophic information. [19] pioneered the neutrosophic imputation methods of the population mean under uncertainty.

1.3. *Scope of the research*

Although some authors have addressed the population mean estimation under neutrosophic statistics using single and two auxiliary variables but there is no existing work on the variance estimation under the neutrosophic framework based on TAV. The current scope of the study focuses on developing efficient and reliable estimators for the population variance using TAV under uncertainty and indeterminacy, as modeled by the neutrosophic framework. By incorporating two auxiliary variables, the article aims to enhance the precision of the variance estimation under SRS, especially when data are uncertain and incomplete. The article investigates how the joint use of TAV under neutrosophic setup can lead to improved estimator's performance in terms of MSE. This article also evaluates the theoretical properties and

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empirical efficiency of the proposed estimators, contributing to the broader application of the neutrosophic theory in modern survey sampling where classical statistical approaches may fall short.

1.4. Notations under neutrosophic setup

The statistical literature presents many types of neutrosophic data, including quantitative neutrosophic data, which is characterized by values lying within an unknown interval $[r, s]$. This interval, based on neutrosophic numbers, can be represented in multiple forms. In the current article, we represent the neutrosophic interval values as $W_N = W_L + W_U I_N$, where $I_N \in [I_L, I_U]$. This formulation implies that the neutrosophic data are expressed in interval form $W_N \in [r, s]$, with r and s denoting the lower and upper bounds, respectively. For a more detailed discussion on neutrosophic notations, readers are suggested to follow [20].

Let $\zeta = (\zeta_1, \zeta_2, \dots, \zeta_N)$ denote a finite population of N identifiable units. For each unit ζ_{iN} , where $i = 1, 2, \dots, N$, let y_{iN} , x_{iN} , and z_{iN} represent the neutrosophic study variable y_N and the neutrosophic auxiliary variables x_N and z_N , respectively. Let n_N units be selected from the population ζ by means of simple random sampling without replacement (SRSWOR). Let $\mu_{x_N} = \sum_{i=1}^N x_{iN}/N$, $\mu_{z_N} = \sum_{i=1}^N z_{iN}/N$ and $\mu_{y_N} = \sum_{i=1}^N y_{iN}/N$ are respectively the population means corresponding to the sample means $\bar{x}_N = \sum_{i=1}^{n_N} x_{iN}/n_N$, $\bar{z}_N = \sum_{i=1}^{n_N} z_{iN}/n_N$ and $\bar{y}_N = \sum_{i=1}^{n_N} y_{iN}/n_N$ of variables x_N , z_N , and y_N . Let $S_{x_N}^2 = \sum_{i=1}^N (x_{iN} - \mu_{x_N})^2/N$, $S_{z_N}^2 = \sum_{i=1}^N (z_{iN} - \mu_{z_N})^2/N$, and $S_{y_N}^2 = \sum_{i=1}^N (y_{iN} - \mu_{y_N})^2/N$ be respectively the population variances corresponding to the sample variances $s_{x_N}^2 = \sum_{i=1}^{n_N} (x_{iN} - \bar{x}_N)^2/(n_N - 1)$, $s_{z_N}^2 = \sum_{i=1}^{n_N} (z_{iN} - \bar{z}_N)^2/(n_N - 1)$, and $s_{y_N}^2 = \sum_{i=1}^{n_N} (y_{iN} - \bar{y}_N)^2/(n_N - 1)$ of variables x_N , z_N , and y_N . Let $C_{x_N} = S_{x_N}/\mu_{x_N}$, $C_{z_N} = S_{z_N}/\mu_{z_N}$, and $C_{y_N} = S_{y_N}/\mu_{y_N}$ be the population coefficient of variations corresponding to the sample coefficient of variations $c_{x_N} = s_{x_N}/\bar{x}_N$, $c_{z_N} = s_{z_N}/\bar{z}_N$, and $c_{y_N} = s_{y_N}/\bar{y}_N$ of variables x_N , z_N , and y_N . Moreover, $\beta_2(x_N)$, $\beta_2(z_N)$, and $\beta_2(y_N)$ are the population coefficient of kurtosis of variables x_N , z_N , and y_N , respectively.

To derive the properties of the neutrosophic estimators, let $s_{y_N}^2 = S_{y_N}^2(1 + \epsilon_{0N})$, $s_{x_N}^2 = S_{x_N}^2(1 + \epsilon_{1N})$ and $s_{z_N}^2 = S_{z_N}^2(1 + \epsilon_{2N})$ where ϵ_{0N} , ϵ_{1N} , and ϵ_{2N} are error terms such that

$$\left. \begin{aligned} E(\epsilon_{0N}) &= E(\epsilon_{1N}) = E(\epsilon_{2N}) = 0; \quad E(\epsilon_{0N}^2) = \frac{(\beta_2(y_N) - 1)}{n_N} = \Delta_0; \\ E(\epsilon_{1N}^2) &= \frac{(\beta_2(x_N) - 1)}{n_N} = \Delta_1; \quad E(\epsilon_{2N}^2) = \frac{(\beta_2(z_N) - 1)}{n_N} = \Delta_2, \\ E(\epsilon_{0N}\epsilon_{1N}) &= \frac{(\lambda_{yx} - 1)}{n_N} = \Delta_{01}; \quad E(\epsilon_{0N}\epsilon_{2N}) = \frac{(\lambda_{yz} - 1)}{n_N} = \Delta_{02}; \\ E(\epsilon_{1N}\epsilon_{2N}) &= \frac{(\lambda_{xz} - 1)}{n_N} = \Delta_{12}; \end{aligned} \right\} \quad (1)$$

where, $\lambda_{yx} = \mu_{yx}/\mu_{2y}\mu_{2x}$, $\lambda_{yz} = \mu_{yz}/\mu_{2y}\mu_{2z}$, and $\lambda_{xz} = \mu_{xz}/\mu_{2x}\mu_{2z}$. Here, $\mu_{yx} = \sum_{i=1}^N (y_{iN} - \mu_{yN})^2 (x_{iN} - \mu_{xN})^2 / N$, $\mu_{yz} = \sum_{i=1}^N (y_{iN} - \mu_{yN})^2 (z_{iN} - \mu_{zN})^2 / N$, and $\mu_{xz} = \sum_{i=1}^N (x_{iN} - \mu_{xN})^2 (z_{iN} - \mu_{zN})^2 / N$ are the population product moments, while $\mu_{2y} = \sum_{i=1}^N (y_{iN} - \mu_{yN})^2 / N$, $\mu_{2x} = \sum_{i=1}^N (x_{iN} - \mu_{xN})^2 / N$ and $\mu_{2z} = \sum_{i=1}^N (z_{iN} - \mu_{zN})^2 / N$ are the population moments.

The subsequent section introduces a class of adapted neutrosophic estimators of population variance based on TAV, along with a discussion of their key properties. Section 3 presents the formulation of the proposed neutrosophic estimators of population variance and explores their properties, including the conditions under which they outperform the adapted neutrosophic estimators. Section 4 provides the comparative study of the proposed estimators with the adapted estimators. Section 5 details a simulation study conducted using a hypothetically generated population. In Section 6, both the proposed and adapted neutrosophic estimators are applied over the real-world temperature data to evaluate their practical performance. Finally, Section 7 summarizes the main findings and concludes the article.

2. Adapted neutrosophic estimators

This section focuses on adapting the classical estimators into neutrosophic estimators for estimating the population variance in SRS, using information from two auxiliary variables.

If the auxiliary variable is not available then the usual neutrosophic unbiased estimator is the only choice to estimate the population variance S_{yN}^2 which is defined as

$$\kappa_m = s_{yN}^2$$

Theorem 2.1. *The estimator κ_m has the variance as*

$$V(\kappa_m) = S_{yN}^4 \Delta_0.$$

Proof. The estimator κ_m is given as

$$\kappa_m = s_{yN}^2.$$

Employing notations of (1), the estimator κ_m can be presented as

$$\begin{aligned} \kappa_m &= S_{yN}^2 (1 + \epsilon_{0N}) \\ \kappa_m - S_{yN}^2 &= S_{yN}^2 \epsilon_{0N} \end{aligned} \quad (2)$$

Applying the expectation operator to both sides of (2), results in:

$$Bias(\kappa_m) = 0$$

This indicates that the estimator κ_m is unbiased. Again, applying square and expectation operator to both sides of (2), results in:

$$V(\kappa_m) = S_{yN}^4 \Delta_0$$

□

The generalized neutrosophic ratio estimator of population variance S_{yN}^2 under SRS, incorporating TAV, is expressed as

$$\kappa_{gr} = s_{yN}^2 \left(\frac{a_N S_{xN}^2 + b_N}{a_N s_{xN}^2 + b_N} \right) \left(\frac{c_N S_{zN}^2 + d_N}{c_N s_{zN}^2 + d_N} \right).$$

Here, a_N , b_N , c_N , and d_N represent either real constants or known parameters determined from the neutrosophic auxiliary variables x_N and z_N . These parameters may include the neutrosophic mean, standard deviation, correlation coefficient, coefficient of variation, skewness, kurtosis, and other related measures. Some particular forms of the generalized neutrosophic ratio estimator κ_{gr} , designed using TAV, are presented in Table 1 for illustrative purposes and ease of reference.

TABLE 1. Some particular forms of the generalized neutrosophic ratio estimator κ_{gr} based on TAV

a_N	b_N	c_N	d_N	TAV based estimators
1	0	1	0	$\kappa_{gr}^1 = s_{yN}^2 \left(\frac{S_{xN}^2}{s_{xN}^2} \right) \left(\frac{S_{zN}^2}{s_{zN}^2} \right)$
1	C_{xN}	1	C_{zN}	$\kappa_{gr}^2 = s_{yN}^2 \left(\frac{S_{xN}^2 + C_{xN}}{s_{xN}^2 + C_{xN}} \right) \left(\frac{S_{zN}^2 + C_{zN}}{s_{zN}^2 + C_{zN}} \right)$
C_{xN}	$\beta_2(x_N)$	C_{zN}	$\beta_2(z_N)$	$\kappa_{gr}^3 = s_{yN}^2 \left(\frac{C_{xN} S_{xN}^2 + \beta_2(x_N)}{C_{xN} s_{xN}^2 + \beta_2(x_N)} \right) \left(\frac{C_{zN} S_{zN}^2 + \beta_2(z_N)}{C_{zN} s_{zN}^2 + \beta_2(z_N)} \right)$
$\beta_2(x_N)$	C_{xN}	$\beta_2(z_N)$	C_{zN}	$\kappa_{gr}^4 = s_{yN}^2 \left(\frac{\beta_2(x_N) S_{xN}^2 + C_{xN}}{\beta_2(x_N) s_{xN}^2 + C_{xN}} \right) \left(\frac{\beta_2(z_N) S_{zN}^2 + C_{zN}}{\beta_2(z_N) s_{zN}^2 + C_{zN}} \right)$
1	ρ_{xyN}	1	ρ_{yzN}	$\kappa_{gr}^5 = s_{yN}^2 \left(\frac{S_{xN}^2 + \rho_{xyN}}{s_{xN}^2 + \rho_{xyN}} \right) \left(\frac{S_{zN}^2 + \rho_{yzN}}{s_{zN}^2 + \rho_{yzN}} \right)$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$\kappa_{gr}^6 = s_{yN}^2 \left(\frac{S_{xN}^2 + \beta_2(x_N)}{s_{xN}^2 + \beta_2(x_N)} \right) \left(\frac{S_{zN}^2 + \beta_2(z_N)}{s_{zN}^2 + \beta_2(z_N)} \right)$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$\kappa_{gr}^7 = s_{yN}^2 \left(\frac{S_{xN}^2 + \beta_1(x_N)}{s_{xN}^2 + \beta_1(x_N)} \right) \left(\frac{S_{zN}^2 + \beta_1(z_N)}{s_{zN}^2 + \beta_1(z_N)} \right)$

Theorem 2.2. The bias and MSE of the generalized neutrosophic ratio estimator κ_{gr} utilizing TAV are given as

$$\begin{aligned} Bias(\kappa_{gr}) &= S_{yN}^2 \left(\Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - \Psi_N \Delta_{01} - \Pi_N \Delta_{02} + \Psi_N \Pi_N \Delta_{12} \right) \\ MSE(\kappa_{gr}) &= S_{yN}^4 \left(\Delta_0 + \Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - 2\Psi_N \Delta_{01} - 2\Pi_N \Delta_{02} + 2\Psi_N \Pi_N \Delta_{12} \right) \end{aligned}$$

where

$$\Psi_N = \left(\frac{a_N S_{xN}^2}{a_N S_{xN}^2 + b_N} \right) \text{ and } \Pi_N = \left(\frac{c_N S_{zN}^2}{c_N S_{zN}^2 + d_N} \right)$$

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Proof. The estimator κ_{gr} is given as

$$\kappa_{gr} = s_{y_N}^2 \left(\frac{a_N S_{x_N}^2 + b_N}{a_N s_{x_N}^2 + b_N} \right) \left(\frac{c_N S_{z_N}^2 + d_N}{c_N s_{z_N}^2 + d_N} \right).$$

Employing notations of (1), the estimator κ_{gr} can be presented as

$$\begin{aligned} \kappa_{gr} &= S_{y_N}^2 (1 + \epsilon_{0N}) \left(\frac{a_N S_{x_N}^2 + b_N}{a_N S_{x_N}^2 (1 + \epsilon_{1N}) + b_N} \right) \left(\frac{c_N S_{z_N}^2 + d_N}{c_N S_{z_N}^2 (1 + \epsilon_{2N}) + d_N} \right) \\ &= S_{y_N}^2 (1 + \epsilon_{0N}) \left(\frac{a_N S_{x_N}^2 + b_N}{a_N S_{x_N}^2 + a_N S_{x_N}^2 \epsilon_{1N} + b_N} \right) \left(\frac{c_N S_{z_N}^2 + d_N}{c_N S_{z_N}^2 + c_N S_{z_N}^2 \epsilon_{2N} + d_N} \right) \\ &= S_{y_N}^2 (1 + \epsilon_{0N}) (1 + \Psi_N \epsilon_{1N})^{-1} (1 + \Pi_N \epsilon_{2N})^{-1} \end{aligned}$$

Applying a Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_{gr} = S_{y_N}^2 \left(\begin{array}{l} 1 + \epsilon_{0N} - \Psi_N \epsilon_{1N} - \Pi_N \epsilon_{2N} + \Psi_N^2 \epsilon_{1N}^2 + \Pi_N^2 \epsilon_{2N}^2 - \Psi_N \epsilon_{0N} \epsilon_{1N} - \Pi_N \epsilon_{0N} \epsilon_{2N} \\ + \Psi_N \Pi_N \epsilon_{1N} \epsilon_{2N} \end{array} \right)$$

Subtracting $S_{y_N}^2$ both side to the above expression, results in:

$$\kappa_{gr} - S_{y_N}^2 = S_{y_N}^2 \left(\begin{array}{l} \epsilon_{0N} - \Psi_N \epsilon_{1N} - \Pi_N \epsilon_{2N} + \Psi_N^2 \epsilon_{1N}^2 + \Pi_N^2 \epsilon_{2N}^2 - \Psi_N \epsilon_{0N} \epsilon_{1N} \\ - \Pi_N \epsilon_{0N} \epsilon_{2N} + \Psi_N \Pi_N \epsilon_{1N} \epsilon_{2N} \end{array} \right) \quad (3)$$

Applying expectation operator both sides of (3), results in:

$$Bias(\kappa_{gr}) = S_{y_N}^2 \left(\Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - \Psi_N \Delta_{01} - \Pi_N \Delta_{02} + \Psi_N \Pi_N \Delta_{12} \right)$$

Applying square and expectation both sides of (3), results in:

$$MSE(\kappa_{gr}) = S_{y_N}^4 \left(\Delta_0 + \Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - 2\Psi_N \Delta_{01} - 2\Pi_N \Delta_{02} + 2\Psi_N \Pi_N \Delta_{12} \right)$$

□

The generalized neutrosophic power ratio estimator κ_{pr} for the population variance employing TAV under SRS, is expressed as

$$\kappa_{pr} = s_{y_N}^2 \left(\frac{a_N S_{x_N}^2 + b_N}{a_N s_{x_N}^2 + b_N} \right)^{\theta_{1N}} \left(\frac{c_N S_{z_N}^2 + d_N}{c_N s_{z_N}^2 + d_N} \right)^{\theta_{2N}}$$

where θ_{1N} and θ_{2N} are suitably chosen scalars. Some particular forms of the estimator κ_{pr} are presented in Table 2 for illustrative purposes and ease of reference.

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TABLE 2. Some particular forms of the generalized neutrosophic power ratio estimator t_{pr} based on TAV

a_N	b_N	c_N	d_N	TAV based estimators
1	0	1	0	$\kappa_{pr}^1 = s_{yN}^2 \left(\frac{S_{xN}^2}{s_{xN}^2} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2}{s_{zN}^2} \right)^{\theta_{2N}}$
1	C_{xN}	1	C_{zN}	$\kappa_{pr}^2 = s_{yN}^2 \left(\frac{S_{xN}^2 + C_{xN}}{s_{xN}^2 + C_{xN}} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + C_{zN}}{s_{zN}^2 + C_{zN}} \right)^{\theta_{2N}}$
C_{xN}	$\beta_2(x_N)$	C_{zN}	$\beta_2(z_N)$	$\kappa_{pr}^3 = s_{yN}^2 \left(\frac{C_{xN} S_{xN}^2 + \beta_2(x_N)}{C_{xN} s_{xN}^2 + \beta_2(x_N)} \right)^{\theta_{1N}} \left(\frac{C_{zN} S_{zN}^2 + \beta_2(z_N)}{C_{zN} s_{zN}^2 + \beta_2(z_N)} \right)^{\theta_{2N}}$
$\beta_2(x_N)$	C_{xN}	$\beta_2(z_N)$	C_{zN}	$\kappa_{pr}^4 = s_{yN}^2 \left(\frac{\beta_2(x_N) S_{xN}^2 + C_{xN}}{\beta_2(x_N) s_{xN}^2 + C_{xN}} \right)^{\theta_{1N}} \left(\frac{\beta_2(z_N) S_{zN}^2 + C_{zN}}{\beta_2(z_N) s_{zN}^2 + C_{zN}} \right)^{\theta_{2N}}$
1	ρ_{xyN}	1	ρ_{yzN}	$\kappa_{pr}^5 = s_{yN}^2 \left(\frac{S_{xN}^2 + \rho_{xyN}}{s_{xN}^2 + \rho_{xyN}} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + \rho_{yzN}}{s_{zN}^2 + \rho_{yzN}} \right)^{\theta_{2N}}$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$\kappa_{pr}^6 = s_{yN}^2 \left(\frac{S_{xN}^2 + \beta_2(x_N)}{s_{xN}^2 + \beta_2(x_N)} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + \beta_2(z_N)}{s_{zN}^2 + \beta_2(z_N)} \right)^{\theta_{2N}}$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$\kappa_{pr}^7 = s_{yN}^2 \left(\frac{S_{xN}^2 + \beta_1(x_N)}{s_{xN}^2 + \beta_1(x_N)} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + \beta_1(z_N)}{s_{zN}^2 + \beta_1(z_N)} \right)^{\theta_{2N}}$

Theorem 2.3. The bias, MSE, and minimum MSE of the neutrosophic generalized power ratio estimator κ_{pr} are given by

$$\begin{aligned}
 Bias(\kappa_{pr}) &= S_{yN}^2 \left(\frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \Delta_2 - \theta_{1N} \Psi_N \Delta_{01} \right. \\
 &\quad \left. - \theta_{2N} \Pi_N \Delta_{02} + \Psi_N \Pi_N \theta_{1N} \theta_{2N} \Delta_{12} \right) \\
 MSE(\kappa_{pr}) &= S_{yN}^4 \left(\Delta_0 + \theta_{1N}^2 \Psi_N^2 \Delta_1 + \theta_{2N}^2 \Pi_N^2 \Delta_2 - 2\theta_{1N} \Psi_N \Delta_{01} \right. \\
 &\quad \left. - 2\theta_{2N} \Pi_N \Delta_{02} + 2\theta_{1N} \theta_{2N} \Psi_N \Pi_N \Delta_{12} \right) \\
 min.MSE(\kappa_{pr}) &= S_{yN}^4 \left(\Delta_0 - \frac{\Delta_2 \Delta_{01}^2 + \Delta_1 \Delta_{02}^2 - 2\Delta_{01} \Delta_{02} \Delta_{12}}{\Delta_1 \Delta_2 - \Delta_{12}^2} \right)
 \end{aligned}$$

Proof. The estimator κ_{pr} is given as

$$\kappa_{pr} = s_{yN}^2 \left(\frac{a_N S_{xN}^2 + b_N}{a_N s_{xN}^2 + b_N} \right)^{\theta_{1N}} \left(\frac{c_N S_{zN}^2 + d_N}{c_N s_{zN}^2 + d_N} \right)^{\theta_{2N}}$$

Employing notations of (1), the estimator κ_{pr} can be presented as

$$\begin{aligned}
 \kappa_{pr} &= S_{yN}^2 (1 + \epsilon_{0N}) \left(\frac{a_N S_{xN}^2 + b_N}{a_N S_{xN}^2 (1 + \epsilon_{1N}) + b_N} \right)^{\theta_{1N}} \left(\frac{c_N S_{zN}^2 + d_N}{c_N S_{zN}^2 (1 + \epsilon_{2N}) + d_N} \right)^{\theta_{2N}} \\
 &= S_{yN}^2 (1 + \epsilon_{0N}) (1 + \Psi_N \epsilon_{1N})^{-\theta_{1N}} (1 + \Pi_N \epsilon_{2N})^{-\theta_{2N}}
 \end{aligned}$$

Applying Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_{pr} = S_{y_N}^2 \left(\begin{array}{l} 1 + \epsilon_{0N} - \theta_{1N}\Psi_N\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{2N} + \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{2N} + \theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N} \end{array} \right) \quad (4)$$

Subtracting $S_{y_N}^2$ both sides of (4), results in:

$$\kappa_{pr} - S_{y_N}^2 = S_{y_N}^2 \left(\begin{array}{l} \epsilon_{0N} - \theta_{1N}\Psi_N\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{2N} + \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\epsilon_{1N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\epsilon_{2N}^2 \\ -\theta_{1N}\Psi_N\epsilon_{0N}\epsilon_{1N} - \theta_{2N}\Pi_N\epsilon_{0N}\epsilon_{2N} + \theta_{1N}\theta_{2N}\Psi_N\Pi_N\epsilon_{1N}\epsilon_{2N} \end{array} \right) \quad (5)$$

Applying expectation operator both sides of (5), results in:

$$Bias(\kappa_{pr}) = S_{y_N}^2 \left(\begin{array}{l} \frac{\theta_{1N}(\theta_{1N}+1)}{2}\Psi_N^2\Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2}\Pi_N^2\Delta_2 - \theta_{1N}\Psi_N\Delta_{01} - \theta_{2N}\Pi_N\Delta_{02} \\ +\Psi_N\Pi_N\theta_{1N}\theta_{2N}\Delta_{12} \end{array} \right)$$

Applying square and expectation both sides of (5), results in:

$$MSE(\kappa_{pr}) = S_{y_N}^4 \left(\begin{array}{l} \Delta_0 + \theta_{1N}^2\Psi_N^2\Delta_1 + \theta_{2N}^2\Pi_N^2\Delta_2 - 2\theta_{1N}\Psi_N\Delta_{01} - 2\theta_{2N}\Pi_N\Delta_{02} \\ +2\theta_{1N}\theta_{2N}\Psi_N\Pi_N\Delta_{12} \end{array} \right) \quad (6)$$

The optimum values of θ_{1N} and θ_{2N} can be determined by minimizing (6) as

$$\theta_{1N} = \frac{\Delta_{01}\Delta_2 - \Delta_{02}\Delta_{12}}{\Psi_N(\Delta_1\Delta_2 - \Delta_{12}^2)} \text{ and } \theta_{2N} = \frac{\Delta_1\Delta_{02} - \Delta_{01}\Delta_{12}}{\Pi_N(\Delta_1\Delta_2 - \Delta_{12}^2)}$$

Using optimum values of θ_{1N} and θ_{2N} in (6), we get minimum MSE of κ_{pr} as

$$\min.MSE(\kappa_{pr}) = S_{y_N}^4 \left(\Delta_0 - \frac{\Delta_2\Delta_{01}^2 + \Delta_1\Delta_{02}^2 - 2\Delta_{01}\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2 - \Delta_{12}^2} \right)$$

□

Remark 2.4. Note that the minimum MSE of the estimator κ_{pr} is independent of the values Ψ_N and Π_N which depend on different known parameters of auxiliary variables x_N and z_N . This shows that the MSE of the members κ_{pr}^i , $i = 1, 2, \dots, 7$ will be same.

To enhance the estimation of finite population variance under SRS, we examined the work of Singh and Gupta (2025) and presented an exponential ratio-cum-product type estimator κ_{ps} by utilizing TAV as

$$\kappa_{ps} = s_{y_N}^2 \exp \left(\frac{s_{x_N}^2 - \bar{t}_{1N}}{S_{x_N}^2 + \bar{t}_{1N}} \right) \exp \left(\frac{\bar{t}_{2N} - S_{z_N}^2}{\bar{t}_{2N} + S_{z_N}^2} \right)$$

In the estimator κ_{ps} , $\bar{t}_{1N} = s_{x_N}^2 + \Gamma_N(S_{x_N}^2 - s_{x_N}^2)$ and $\bar{t}_{2N} = s_{z_N}^2 + \Omega_N(S_{z_N}^2 - s_{z_N}^2)$, where Γ_N and Ω_N are duly chosen scalars.

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Theorem 2.5. The bias and minimum MSE of the exponential ratio-cum-product type estimator κ_{ps} , employing TAV, are presented as

$$\begin{aligned} Bias(\kappa_{ps}) &= S_{yN}^2 \left\{ \frac{3(1-\Gamma_N)^2\Delta_1}{8} - \frac{(1-\Omega_N)^2\Delta_2}{8} - \frac{(1-\Gamma_N)\Delta_{01}}{2} + \frac{(1-\Omega_N)\Delta_{02}}{2} - \frac{(1-\Gamma_N)(1-\Omega_N)\Delta_{12}}{4} \right\} \\ min.MSE(\kappa_{ps}) &= S_{yN}^4 \left\{ \frac{\Delta_0 + \frac{(1-\Gamma_N^*)^2\Delta_1}{4} + \frac{(1-\Omega_N^*)^2\Delta_2}{4} - (1-\Gamma_N^*)\Delta_{01} + (1-\Omega_N^*)\Delta_{02}}{-\frac{(1-\Gamma_N^*)(1-\Omega_N^*)\Delta_{12}}{2}} \right\} \end{aligned}$$

where

$$\Gamma_N = \left(\frac{\Delta_1\Delta_2 - 2\Delta_{01}\Delta_2 - \Delta_{12}^2 + 2\Delta_{02}\Delta_{12}}{\Delta_1\Delta_2 - \Delta_{12}^2} \right) \text{ and } \Omega_N = \left(\frac{\Delta_1\Delta_2 - \Delta_{12}^2 - 2\Delta_{01}\Delta_{12} + 2\Delta_{02}\Delta_1}{\Delta_1\Delta_2 - \Delta_{12}^2} \right)$$

Proof. Consider the estimator κ_{ps} as

$$\kappa_{ps} = s_{yN}^2 \exp \left(\frac{S_{xN}^2 - \bar{t}_{1N}}{S_{xN}^2 + \bar{t}_{1N}} \right) \exp \left(\frac{\bar{t}_{2N} - S_{zN}^2}{\bar{t}_{2N} + S_{zN}^2} \right)$$

Employing notations of (1), the estimator κ_{ps} can be presented as

$$\begin{aligned} \kappa_{ps} &= S_{yN}^2 (1 + \varepsilon_{0N}) \left\{ \exp \left[\frac{S_{xN}^2 - S_{xN}^2(1 + \varepsilon_{1N}) - \Gamma_N(S_{xN}^2 - S_{xN}^2(1 + \varepsilon_{1N}))}{S_{xN}^2 + S_{xN}^2(1 + \varepsilon_{1N}) + \Gamma_N(S_{xN}^2 - S_{xN}^2(1 + \varepsilon_{1N}))} \right] \right. \\ &\quad \left. \exp \left[\frac{S_{zN}^2(1 + \varepsilon_{2N}) + \Omega_N(S_{zN}^2 - S_{zN}^2(1 + \varepsilon_{2N})) - S_{zN}^2}{S_{zN}^2(1 + \varepsilon_{2N}) + \Omega_N(S_{zN}^2 - S_{zN}^2(1 + \varepsilon_{2N})) + S_{zN}^2} \right] \right\} \\ &= S_{yN}^2 (1 + \varepsilon_{0N}) \exp \left[\frac{-\varepsilon_{1N}(1 - \Gamma_N)}{2 \left(1 + \frac{\varepsilon_{1N}(1 - \Gamma_N)}{2} \right)} \right] \exp \left[\frac{\varepsilon_{2N}(1 - \Omega_N)}{2 \left(1 + \frac{\varepsilon_{2N}(1 - \Omega_N)}{2} \right)} \right] \end{aligned}$$

Applying a Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_{ps} = S_{yN}^2 \left[1 + \frac{\varepsilon_{0N} - \frac{\varepsilon_{1N}(1 - \Gamma_N)}{2} + \frac{3\varepsilon_{1N}^2(1 - \Gamma_N)^2}{8} + \frac{\varepsilon_{2N}(1 - \Omega_N)}{2} - \frac{\varepsilon_{1N}\varepsilon_{2N}(1 - \Gamma_N)(1 - \Omega_N)}{4} \right. \\ \left. - \frac{\varepsilon_{2N}^2(1 - \Omega_N)^2}{8} - \frac{\varepsilon_{0N}\varepsilon_{1N}(1 - \Gamma_N)}{2} + \frac{\varepsilon_{0N}\varepsilon_{2N}(1 - \Omega_N)}{2} \right]$$

Subtracting S_{yN}^2 both sides of the above expression, results in:

$$\kappa_{ps} - S_{yN}^2 = S_{yN}^2 \left[\frac{\varepsilon_{0N} - \frac{\varepsilon_{1N}(1 - \Gamma_N)}{2} + \frac{3\varepsilon_{1N}^2(1 - \Gamma_N)^2}{8} + \frac{\varepsilon_{2N}(1 - \Omega_N)}{2} - \frac{\varepsilon_{1N}\varepsilon_{2N}(1 - \Gamma_N)(1 - \Omega_N)}{4} \right. \\ \left. - \frac{\varepsilon_{2N}^2(1 - \Omega_N)^2}{8} - \frac{\varepsilon_{0N}\varepsilon_{1N}(1 - \Gamma_N)}{2} + \frac{\varepsilon_{0N}\varepsilon_{2N}(1 - \Omega_N)}{2} \right] \quad (7)$$

Applying expectation operator both sides of (7), results in:

$$Bias(\kappa_{ps}) = S_{yN}^2 \left[\frac{3(1-\Gamma_N)^2\Delta_1}{8} - \frac{(1-\Omega_N)^2\Delta_2}{8} - \frac{(1-\Gamma_N)\Delta_{01}}{2} + \frac{(1-\Omega_N)\Delta_{02}}{2} \right. \\ \left. - \frac{(1-\Gamma_N)(1-\Omega_N)\Delta_{12}}{4} \right]$$

Applying square and expectation both sides of (7), results in:

$$MSE(\kappa_{ps}) = S_{yN}^4 \left[\frac{\Delta_0 + \frac{(1-\Gamma_N)^2\Delta_1}{4} + \frac{(1-\Omega_N)^2\Delta_2}{4} - (1-\Gamma_N)\Delta_{01} + (1-\Omega_N)\Delta_{02}}{-\frac{(1-\Gamma_N)(1-\Omega_N)\Delta_{12}}{2}} \right] \quad (8)$$

The optimum values of Γ_N and Ω_N can be determined by minimizing (8) as

$$\Gamma_{N(opt)} = \left(\frac{\Delta_1 \Delta_2 - 2\Delta_{01} \Delta_2 - \Delta_{12}^2 + 2\Delta_{02} \Delta_{12}}{\Delta_1 \Delta_2 - \Delta_{12}^2} \right) = \Gamma_N^* \text{ (say)}$$

$$\Omega_{N(opt)} = \left(\frac{\Delta_1 \Delta_2 - \Delta_{12}^2 - 2\Delta_{01} \Delta_{12} + 2\Delta_{02} \Delta_1}{\Delta_1 \Delta_2 - \Delta_{12}^2} \right) = \Omega_N^* \text{ (say)}$$

Using optimum values of $\Gamma_{N(opt)}$ and $\Omega_{N(opt)}$ in (8), we get minimum MSE of κ_{ps} as

$$\min. MSE(\kappa_{ps}) = S_{yN}^4 \left\{ \begin{aligned} &\Delta_0 + \frac{(1-\Gamma_N^*)^2 \Delta_1}{4} + \frac{(1-\Omega_N^*)^2 \Delta_2}{4} - (1-\Gamma_N^*) \Delta_{01} + (1-\Omega_N^*) \Delta_{02} \\ &- \frac{(1-\Gamma_N^*)(1-\Omega_N^*) \Delta_{12}}{2} \end{aligned} \right\}$$

□

3. Proposed optimal neutrosophic estimators

Ref. [21] proposed a significant contribution to survey sampling by developing an innovative method to improve the estimation of the population mean when the population variance is known. He proposed to pre-multiply a constant in the estimator to reduce the MSE of the estimator. This approach demonstrated that using known population parameters could enhance efficiency of the estimation, laying the groundwork for later developments in auxiliary information-based estimators in sampling theory. Further, in the literature of neutrosophic survey sampling, a limited work is available for the estimation of population mean under uncertainty using different sampling frameworks. However, there is no work conducted under neutrosophic framework for the estimation of population variance in uncertainty, employing TAV. This inspired us to propose a class of neutrosophic estimators for the population variance using TAV under SRS. Therefore, following [21], we propose Searls type generalized neutrosophic power ratio estimator for population variance as follows:

$$\kappa_a = \alpha_N s_{yN}^2 \left(\frac{a_N S_{xN}^2 + b_N}{a_N s_{xN}^2 + b_N} \right)^{\theta_{1N}} \left(\frac{c_N S_{zN}^2 + d_N}{c_N s_{zN}^2 + d_N} \right)^{\theta_{2N}}.$$

Here α_N , θ_{1N} , and θ_{2N} are duly opted scalars. Some particular forms of the proposed estimator κ_a are reported in Table 3 for ready reference.

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TABLE 3. Some particular forms of the neutrosophic estimator κ_a based on TAV

a_N	b_N	c_N	d_N	Particular forms of estimator κ_a based on TAV
1	0	1	0	$\kappa_a^1 = \alpha_N s_{yN}^2 \left(\frac{S_{xN}^2}{s_{xN}^2} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2}{s_{zN}^2} \right)^{\theta_{2N}}$
1	C_{xN}	1	C_{zN}	$\kappa_a^2 = \alpha_N s_{yN}^2 \left(\frac{S_{xN}^2 + C_{xN}}{s_{xN}^2 + C_{xN}} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + C_{zN}}{s_{zN}^2 + C_{zN}} \right)^{\theta_{2N}}$
C_{xN}	$\beta_2(x_N)$	C_{zN}	$\beta_2(z_N)$	$\kappa_a^3 = \alpha_N s_{yN}^2 \left(\frac{C_{xN} S_{xN}^2 + \beta_2(x_N)}{C_{xN} s_{xN}^2 + \beta_2(x_N)} \right)^{\theta_{1N}} \left(\frac{C_{zN} S_{zN}^2 + \beta_2(z_N)}{C_{zN} s_{zN}^2 + \beta_2(z_N)} \right)^{\theta_{2N}}$
$\beta_2(x_N)$	C_{xN}	$\beta_2(z_N)$	C_{zN}	$\kappa_a^4 = \alpha_N s_{yN}^2 \left(\frac{\beta_2(x_N) S_{xN}^2 + C_{xN}}{\beta_2(x_N) s_{xN}^2 + C_{xN}} \right)^{\theta_{1N}} \left(\frac{\beta_2(z_N) S_{zN}^2 + C_{zN}}{\beta_2(z_N) s_{zN}^2 + C_{zN}} \right)^{\theta_{2N}}$
1	ρ_{xyN}	1	ρ_{yzN}	$\kappa_a^5 = \alpha_N s_{yN}^2 \left(\frac{S_{xN}^2 + \rho_{xyN}}{s_{xN}^2 + \rho_{xyN}} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + \rho_{yzN}}{s_{zN}^2 + \rho_{yzN}} \right)^{\theta_{2N}}$
1	$\beta_2(x_N)$	1	$\beta_2(z_N)$	$\kappa_a^6 = \alpha_N s_{yN}^2 \left(\frac{S_{xN}^2 + \beta_2(x_N)}{s_{xN}^2 + \beta_2(x_N)} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + \beta_2(z_N)}{s_{zN}^2 + \beta_2(z_N)} \right)^{\theta_{2N}}$
1	$\beta_1(x_N)$	1	$\beta_1(z_N)$	$\kappa_a^7 = \alpha_N s_{yN}^2 \left(\frac{S_{xN}^2 + \beta_1(x_N)}{s_{xN}^2 + \beta_1(x_N)} \right)^{\theta_{1N}} \left(\frac{S_{zN}^2 + \beta_1(z_N)}{s_{zN}^2 + \beta_1(z_N)} \right)^{\theta_{2N}}$

Theorem 3.1. The bias, MSE, and minimum MSE of the proposed neutrosophic variance estimator κ_a are derived as

$$\begin{aligned}
 Bias(\kappa_a) &= S_{yN}^2 \left[\alpha_N \left\{ 1 + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \Delta_2 - \theta_{1N} \Psi_N \Delta_{01} \right\} - 1 \right] \\
 MSE(\kappa_a) &= S_{yN}^4 \left[1 + \alpha_N^2 \left(\begin{aligned} &1 + \Delta_0 + 2\theta_{1N}^2 \Psi_N^2 \Delta_1 + 2\theta_{2N}^2 \Pi_N^2 \Delta_2 + \theta_{1N} \Psi_N^2 \Delta_1 \\ &+ \theta_{2N} \Pi_N^2 \Delta_2 - 4\theta_{1N} \Psi_N \Delta_{01} - 4\theta_{2N} \Pi_N \Delta_{02} \\ &+ 4\theta_{1N} \theta_{2N} \Psi_N \Pi_N \Delta_{12} \end{aligned} \right) \right. \\
 &\quad \left. - 2\alpha_N \left\{ 1 + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \Delta_2 - \theta_{1N} \Psi_N \Delta_{01} \right\} \right. \\
 &\quad \left. - \theta_{2N} \Pi_N \Delta_{02} + \theta_{1N} \theta_{2N} \Pi_N \Psi_N \Delta_{12} \right] \\
 min.MSE(\kappa_a) &= S_{yN}^4 \left(1 - \frac{B_N^2}{A_N} \right)
 \end{aligned}$$

Proof. The estimator κ_a is given as

$$\kappa_a = \alpha_N s_{yN}^2 \left(\frac{a_N S_{xN}^2 + b_N}{a_N s_{xN}^2 + b_N} \right)^{\theta_{1N}} \left(\frac{c_N S_{zN}^2 + d_N}{c_N s_{zN}^2 + d_N} \right)^{\theta_{2N}}$$

Employing notations of (1), the estimator κ_a can be presented as

$$\begin{aligned}
 \kappa_a &= \alpha_N S_{yN}^2 (1 + \epsilon_{0N}) \left(\frac{a_N S_{xN}^2 + b_N}{a_N S_{xN}^2 (1 + \epsilon_{1N}) + b_N} \right)^{\theta_{1N}} \left(\frac{c_N S_{zN}^2 + d_N}{c_N S_{zN}^2 (1 + \epsilon_{2N}) + d_N} \right)^{\theta_{2N}} \\
 &= \alpha_N S_{yN}^2 (1 + \epsilon_{0N}) (1 + \Psi_N \epsilon_{1N})^{-\theta_{1N}} (1 + \Pi_N \epsilon_{2N})^{-\theta_{2N}}
 \end{aligned}$$

Applying Taylor series expansion and excluding higher-order error terms (above the second order) during simplification of the right-hand side, we arrive at

$$\kappa_a = \alpha_N S_{y_N}^2 \left(\begin{array}{l} 1 + \epsilon_{0N} - \theta_{1N} \Psi_N \epsilon_{1N} - \theta_{2N} \Pi_N \epsilon_{2N} + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \epsilon_{1N}^2 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \epsilon_{2N}^2 \\ - \Psi_N \theta_{1N} \epsilon_{0N} \epsilon_{1N} - \Pi_N \theta_{2N} \epsilon_{0N} \epsilon_{2N} + \Psi_N \Pi_N \theta_{1N} \theta_{2N} \epsilon_{1N} \epsilon_{2N} \end{array} \right)$$

Subtracting $S_{y_N}^2$ both sides of the above expression, results in:

$$\kappa_a - S_{y_N}^2 = S_{y_N}^2 \left[\alpha_N \left(\begin{array}{l} 1 + \epsilon_{0N} - \theta_{1N} \Psi_N \epsilon_{1N} - \theta_{2N} \Pi_N \epsilon_{2N} + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \epsilon_{1N}^2 \\ + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \epsilon_{2N}^2 - \theta_{1N} \Psi_N \epsilon_{0N} \epsilon_{1N} - \theta_{2N} \Pi_N \epsilon_{0N} \epsilon_{2N} \\ + \theta_{1N} \theta_{2N} \Psi_N \Pi_N \epsilon_{1N} \epsilon_{2N} \end{array} \right) - 1 \right] \quad (9)$$

Applying expectation operator both sides of (9), results in:

$$Bias(\kappa_a) = S_{y_N}^2 \left[\alpha_N \left\{ \begin{array}{l} 1 + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \Delta_2 - \theta_{1N} \Psi_N \Delta_{01} \\ - \theta_{2N} \Pi_N \Delta_{02} + \theta_{1N} \theta_{2N} \Pi_N \Psi_N \Delta_{12} \end{array} \right\} - 1 \right]$$

Applying square and expectation operator both sides of (9), results in:

$$\begin{aligned} MSE(\kappa_a) &= S_{y_N}^4 \left[\begin{array}{l} 1 + \alpha_N^2 \left(\begin{array}{l} 1 + \Delta_0 + 2\theta_{1N}^2 \Psi_N^2 \Delta_1 + 2\theta_{2N}^2 \Pi_N^2 \Delta_2 + \theta_{1N} \Psi_N^2 \Delta_1 \\ + \theta_{2N} \Pi_N^2 \Delta_2 - 4\theta_{1N} \Psi_N \Delta_{01} - 4\theta_{2N} \Pi_N \Delta_{02} \\ + 4\theta_{1N} \theta_{2N} \Psi_N \Pi_N \Delta_{12} \end{array} \right) \\ - 2\alpha_N \left\{ \begin{array}{l} 1 + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \Delta_2 - \theta_{1N} \Psi_N \Delta_{01} \\ - \theta_{2N} \Pi_N \Delta_{02} + \theta_{1N} \theta_{2N} \Pi_N \Psi_N \Delta_{12} \end{array} \right\} \end{array} \right] \\ &= S_{y_N}^4 (1 + \alpha_N^2 A_N - 2\alpha_N B_N) \end{aligned} \quad (10)$$

where,

$$\begin{aligned} A_N &= \left(\begin{array}{l} 1 + \Delta_0 + 2\theta_{1N}^2 \Psi_N^2 \Delta_1 + 2\theta_{2N}^2 \Pi_N^2 \Delta_2 + \theta_{1N} \Psi_N^2 \Delta_1 + \theta_{2N} \Pi_N^2 \Delta_2 - 4\theta_{1N} \Psi_N \Delta_{01} \\ - 4\theta_{2N} \Pi_N \Delta_{02} + 4\theta_{1N} \theta_{2N} \Psi_N \Pi_N \Delta_{12} \end{array} \right) \\ B_N &= \left\{ \begin{array}{l} 1 + \frac{\theta_{1N}(\theta_{1N}+1)}{2} \Psi_N^2 \Delta_1 + \frac{\theta_{2N}(\theta_{2N}+1)}{2} \Pi_N^2 \Delta_2 - \theta_{1N} \Psi_N \Delta_{01} - \theta_{2N} \Pi_N \Delta_{02} \\ + \theta_{1N} \theta_{2N} \Pi_N \Psi_N \Delta_{12} \end{array} \right\} \end{aligned}$$

The minimization of (10) regarding α_N , provides the optimum value of scalar as follows:

$$\alpha_{N(opt)} = \frac{B_N}{A_N}$$

Use of $\alpha_{N(opt)}$ in (10), provides:

$$min.MSE(\kappa_a) = S_{y_N}^4 \left(1 - \frac{B_N^2}{A_N} \right)$$

It is crucial to note that simultaneous optimization of α_N , θ_{1N} , and θ_{2N} is not feasible. Therefore, to provide the minimization of the MSE of the estimator κ_a , we fix $\alpha_N = 1$. Under this condition, the optimal values of θ_{1N} and θ_{2N} can be derived and given as follows:

$$\theta_{1N} = \frac{\Delta_{01} \Delta_2 - \Delta_{02} \Delta_{12}}{\Psi_N (\Delta_1 \Delta_2 - \Delta_{12}^2)} \text{ and } \theta_{2N} = \frac{\Delta_1 \Delta_{02} - \Delta_{01} \Delta_{12}}{\Pi_N (\Delta_1 \Delta_2 - \Delta_{12}^2)}.$$

□

4. Comparative study

In this section, a comparative study is carried out to assess the performance of the proposed estimator against the existing ones in terms of MSE. The conditions are obtained under which the proposed estimator consistently outperforms the adapted estimators.

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic sample variance $s_{y_N}^2$ in terms of MSE/variance.

$$\min.MSE(\kappa_a) < V(\kappa_m)$$

$$S_{y_N}^4 \left(1 - \frac{B_N^2}{A_N} \right) < S_{y_N}^4 \Delta_0$$

$$1 - \frac{B_N^2}{A_N} < \Delta_0$$

$$\frac{B_N^2}{A_N} > 1 - \Delta_0$$

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic generalized ratio estimator κ_{gr} in terms of MSE.

$$\min.MSE(\kappa_a) < MSE(\kappa_{gr})$$

$$S_{y_N}^4 \left(1 - \frac{B_N^2}{A_N} \right) < S_{y_N}^4 \left(\Delta_0 + \Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - 2\Psi_N \Delta_{01} - 2\Pi_N \Delta_{02} + 2\Psi_N \Pi_N \Delta_{12} \right)$$

$$\frac{B_N^2}{A_N} > 1 - \left(\Delta_0 + \Psi_N^2 \Delta_1 + \Pi_N^2 \Delta_2 - 2\Psi_N \Delta_{01} - 2\Pi_N \Delta_{02} + 2\Psi_N \Pi_N \Delta_{12} \right)$$

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic generalized power ratio estimator κ_{pr} in terms of MSE.

$$\min.MSE(\kappa_a) < \min.MSE(\kappa_{pr})$$

$$S_{y_N}^4 \left(1 - \frac{B_N^2}{A_N} \right) < S_{y_N}^4 \left[\Delta_0 - \frac{\Delta_2 \Delta_{01}^2 + \Delta_1 \Delta_{02}^2 - 2\Delta_{01} \Delta_{02} \Delta_{12}}{\Delta_1 \Delta_2 - \Delta_{12}^2} \right]$$

$$\frac{B_N^2}{A_N} > 1 - \Delta_0 + \frac{\Delta_2 \Delta_{01}^2 + \Delta_1 \Delta_{02}^2 - 2\Delta_{01} \Delta_{02} \Delta_{12}}{\Delta_1 \Delta_2 - \Delta_{12}^2}$$

- Comparing the proposed neutrosophic estimator κ_a with the neutrosophic ratio-cum-product-exponential type estimator κ_{ps} in terms of MSE.

$$\min.MSE(\kappa_a) < MSE(\kappa_{ps})$$

$$S_{y_N}^4 \left(1 - \frac{B_N^2}{A_N} \right) < S_{y_N}^4 \left\{ \Delta_0 + \frac{(1-\Gamma_N^*)^2 \Delta_1}{4} + \frac{(1-\Omega_N^*)^2 \Delta_2}{4} - (1-\Gamma_N^*) \Delta_{01} + (1-\Omega_N^*) \Delta_{02} \right. \\ \left. - \frac{(1-\Gamma_N^*)(1-\Omega_N^*) \Delta_{12}}{2} \right\}$$

$$\frac{B_N^2}{A_N} > 1 - \left\{ \Delta_0 + \frac{(1-\Gamma_N^*)^2 \Delta_1}{4} + \frac{(1-\Omega_N^*)^2 \Delta_2}{4} - (1-\Gamma_N^*) \Delta_{01} + (1-\Omega_N^*) \Delta_{02} \right. \\ \left. - \frac{(1-\Gamma_N^*)(1-\Omega_N^*) \Delta_{12}}{2} \right\}$$

5. Simulation study

To assess the performance of the adapted and newly proposed neutrosophic estimators, we conduct a simulation study on an artificially generated population. Employing *R* software, a trivariate normal population with the size of $N=500$, is created using the set parameters $\mu_{y_N} \sim [11, 12]$, $\mu_{x_N} \sim [16, 17]$, $\mu_{z_N} \sim [21, 22]$, $\sigma_{y_N}^2 \sim [25, 26]$, $\sigma_{x_N}^2 \sim [26, 28]$, $\sigma_{z_N}^2 \sim [27, 30]$, and different values of correlation coefficients ρ_{xy_N} , ρ_{yz_N} , and ρ_{xz_N} . Based on 15,000 repetition, the bias, MSE, and RE are computed using the following formulas:

$$Bias(\kappa^*) = \frac{1}{15,000} \sum_{i=1}^{15,000} (\kappa_i^* - S_{y_N}^2), \text{ where } \kappa^* = \kappa_m, \kappa_{gr}, \kappa_{pr}, \kappa_{ps}, \text{ and } \kappa_a \quad (11)$$

$$MSE(\kappa^*) = \frac{1}{15,000} \sum_{i=1}^{15,000} (\kappa_i^* - S_{y_N}^2)^2 \quad (12)$$

$$RE(\kappa_m, \kappa^*) = \frac{V(\kappa_m)}{MSE(\kappa^*)} \quad (13)$$

The bias, MSE, and RE are calculated and presented in Table 4, Table 5, and Table 6 respectively. After carefully analyzing these findings, we draw a comprehensive interpretation.

Table 4 shows the bias intervals $[Bias_L, Bias_U]$ for each estimator under different correlation setups $((\rho_{xy_N}, \rho_{yz_N}, \rho_{xz_N}))$, which shows that:

- The usual sample variance estimator $\kappa_m = s_{y_N}^2$ has zero bias, as expected for an unbiased estimator.
- The adapted generalized ratio estimators κ_{gr} demonstrate high positive biases ranging from approximately 24 to over 47. This makes these estimators less desirable.
- The generalized power ratio estimator κ_{pr} demonstrates small positive biases.
- The ratio-cum-product exponential estimator κ_{ps} and the proposed estimators κ_i^a , $i = 1, 2, \dots, 7$ show negligibly small negative biases, approximately between -0.004 and -0.0009 . This exhibits that the proposed estimators κ_i^a are slightly negatively biased but far more stable and significantly less biased than the existing ones.

Table 5 shows the MSE intervals $[MSE_L, MSE_U]$ for each estimator under different correlation setups $((\rho_{xy_N}, \rho_{yz_N}, \rho_{xz_N}))$, which shows that:

- The adapted ratio estimators κ_{gr} have extremely high MSEs, often exceeding 60,000, indicating very poor efficiency.
- The generalized power ratio estimator κ_{pr} and exponential estimator κ_{ps} show much lower MSEs, especially when correlation increases.
- The proposed estimators κ_i^a consistently show the lowest MSEs across all scenarios, with values dropping from 23,900 to as low as 4538. This shows that the proposed

class of estimators κ_i^a outperform all the existing methods in terms of MSE, especially as the auxiliary variables become more strongly correlated with the study variable.

Table 6 shows the RE intervals $[RE_L, RE_U]$ for each estimator under different correlation setups $(\rho_{xy_N}, \rho_{yz_N}, \rho_{xz_N})$, which shows that:

- The adapted ratio estimators κ_{gr} have low RE values, indicating worse performance.
- Generalized power ratio estimator κ_{pr} and exponential estimator κ_{ps} are more efficient than κ_m .
- The proposed estimators κ_i^a consistently achieve the highest RE values, reaching up to 4.12, meaning they are over 4 times more efficient than the basic estimator. This offers the substantial improvement in efficiency of the proposed estimators, particularly in high-correlation setups.

TABLE 4. $Bias_N \in [Bias_L, Bias_U]$ of neutrosophic estimators using neutrosophic normal population

ρ_{xy_N}	0.3	0.5	0.7	0.9
ρ_{yz_N}	0.3	0.5	0.7	0.9
ρ_{xz_N}	0.3	0.5	0.7	0.9
Estimators				
κ_m	(0, 0)	(0, 0)	(0, 0)	(0, 0)
κ_{gr}^1	(41.715, 47.556)	(37.173, 42.890)	(30.903, 35.840)	(24.150, 28.027)
κ_{gr}^2	(41.398, 47.225)	(36.861, 42.983)	(30.638, 35.571)	(23.846, 27.685)
κ_{gr}^3	(43.142, 49.529)	(35.604, 41.187)	(23.656, 27.460)	(7.960, 9.246)
κ_{gr}^4	(42.399, 63.648)	(40.637, 44.443)	(42.219, 37.792)	(44.918, 36.505)
κ_{gr}^5	(41.699, 47.542)	(37.147, 42.867)	(30.866, 35.806)	(24.099, 27.979)
κ_{gr}^6	(41.750, 47.591)	(37.208, 42.926)	(30.939, 35.876)	(24.185, 28.061)
κ_{gr}^7	(41.717, 47.557)	(37.172, 42.890)	(30.901, 35.838)	(24.147, 28.024)
κ_{pr}	(1.395, 1.738)	(2.947, 3.401)	(3.156, 3.641)	(1.209, 1.407)
κ_{ps}	(-0.000, 0.000)	(-0.000, -0.000)	(-0.000, -0.000)	(-0.001, -0.001)
κ_a^1	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^2	(-0.004, 0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^3	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^4	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^5	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^6	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)
κ_a^7	(-0.004, -0.004)	(-0.003, -0.004)	(-0.002, -0.002)	(-0.0009, -0.001)

6. Real data applications

In climate analysis, daily temperature data provide exact numbers for the highest, lowest, and average temperatures for each observation period. But the elements that cause temperature changes are sometimes hard to pin down. There are several things that might cause

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TABLE 5. $MSE_N \in [MSE_L, MSE_U]$ of neutrosophic estimators using simulated data

ρ_{xyN}	0.3	0.5	0.7	0.9
ρ_{yzN}	0.3	0.5	0.7	0.9
ρ_{xzN}	0.3	0.5	0.7	0.9
Estimators				
κ_m	(21307.320, 27891.470)	(21192.980, 28049.700)	(20414.710, 27200.470)	(18625.230, 24955.050)
κ_{gr}^1	(47683.520, 61986.730)	(41024.880, 54279.600)	(31392.520, 41912.320)	(20180.450, 27080.550)
κ_{gr}^2	(47462.930, 61723.450)	(40815.430, 54448.640)	(31219.840, 41721.940)	(19980.060, 26817.980)
κ_{gr}^3	(47698.680, 62002.16)	(41040.160, 54295.200)	(31407.710, 41927.940)	(20194.310, 27094.95)
κ_{gr}^4	(51782.280, 113630.400)	(53302.470, 144726.400)	(59381.850, 68545.530)	(37659.860, 218591.700)
κ_{gr}^5	(47656.080, 61955.650)	(40979.120, 54226.230)	(31329.910, 41838.610)	(20100.410, 26986.440)
κ_{gr}^6	(47708.300, 62014.920)	(41050.280, 54308.440)	(31418.610, 41942.190)	(20205.300, 27109.340)
κ_{gr}^7	(47683.650, 61986.280)	(41024.070, 54278.120)	(31391.130, 41910.410)	(20178.280, 27077.990)
κ_{pr}	(19297.470, 25040.840)	(17153.890, 22517.050)	(12270.470, 16238.470)	(4609.050, 6145.796)
κ_{ps}	(19297.470, 25040.840)	(17153.890, 22517.050)	(12270.470, 16238.470)	(4609.050, 6145.796)
κ_a^1	(18410.260, 23908.170)	(16365.530, 21498.700)	(11811.570, 15638.000)	(4538.730, 6052.475)
κ_a^2	(18410.790, 23908.990)	(16366.630, 21499.210)	(11812.850, 15639.170)	(4538.926, 6053.161)
κ_a^3	(18410.620, 23908.590)	(16366.200, 21499.450)	(11812.850, 15638.880)	(4539.137, 6052.927)
κ_a^4	(18410.480, 23908.470)	(16365.900, 21499.100)	(11811.810, 15638.470)	(4538.939, 6052.730)
κ_a^5	(18410.360, 23908.290)	(16365.760, 21498.970)	(11811.930, 15638.420)	(4538.963, 6052.744)
κ_a^6	(18410.930, 23909.030)	(16366.790, 21500.190)	(11813.050, 15639.730)	(4539.476, 6053.340)
κ_a^7	(18410.230, 23908.120)	(16365.490, 21498.650)	(11811.55, 15637.980)	(4538.732, 6052.476)

TABLE 6. $RE_N \in [RE_L, RE_U]$ of neutrosophic estimators using simulated data

ρ_{xyN}	0.3	0.5	0.7	0.9
ρ_{yzN}	0.3	0.5	0.7	0.9
ρ_{xzN}	0.3	0.5	0.7	0.9
Estimators				
κ_m	(1.000, 1.000)	(1.000, 1.000)	(1.000, 1.000)	(1.000, 1.000)
κ_{gr}^1	(0.446, 0.449)	(0.516, 0.516)	(0.650, 0.648)	(0.922, 0.921)
κ_{gr}^2	(0.448, 0.451)	(0.519, 0.515)	(0.653, 0.651)	(0.932, 0.930)
κ_{gr}^3	(0.446, 0.459)	(0.516, 0.516)	(0.649, 0.648)	(0.922, 0.921)
κ_{gr}^4	(0.411, 0.245)	(0.397, 0.193)	(0.343, 0.396)	(0.494, 0.114)
κ_{gr}^5	(0.447, 0.450)	(0.517, 0.517)	(0.651, 0.650)	(0.926, 0.924)
κ_{gr}^6	(0.446, 0.449)	(0.516, 0.516)	(0.649, 0.648)	(0.921, 0.920)
κ_{gr}^7	(0.446, 0.449)	(0.516, 0.516)	(0.650, 0.649)	(0.923, 0.921)
κ_{pr}	(1.104, 1.113)	(1.235, 1.245)	(1.663, 1.675)	(4.041, 4.060)
κ_{ps}	(1.104, 1.113)	(1.235, 1.245)	(1.663, 1.675)	(4.041, 4.060)
κ_a^1	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.123)
κ_a^2	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^3	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^4	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^5	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.122)
κ_a^6	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.102, 4.122)
κ_a^7	(1.157, 1.166)	(1.294, 1.304)	(1.728, 1.739)	(4.103, 4.123)

temperature patterns to alter, include rapid changes in the weather, disturbances in the atmosphere, geographical factors, and larger climate changes. These factors introduce imprecision and incompleteness into the analytical process, making it challenging to model or forecast temperature behavior with complete accuracy. Neutrosophic estimators provide a strong foundation for capturing the uncertainty and partial truth that come with these kinds of influences. These estimators provide a more nuanced understanding and enhanced prediction of temperature dynamics, particularly in situations where causal information is incomplete or uncertain.

In January 2023, the Indian Meteorological Department (IMD) published information based on the seasonal and annual minimum/maximum temperature series from 1901 to 2021. The publicly accessible website <https://www.data.gov.in/resource/seasonal-and-annual-minimum-maximum-temperature-series-period-1901-2021> offers this data for evaluation. In this data, the neutrosophic study variable y_N is the temperature for the months of October to December in 1901-2021, and the neutrosophic auxiliary variables x_N and z_N are the temperatures for the months of June-September and March-May in 1901-2021. Table 7 contains the required parameters of this data. The bias, MSE, and RE of the neutrosophic estimators are computed for this data and compiled in Table 8.

The results show that the proposed estimators κ_a^i , $i = 1, 2, \dots, 7$ dominate the existing neutrosophic estimators in terms of least MSE values and highest RE values.

TABLE 7. Descriptive values of the real dataset

Neutrosophic parameters	Neutrosophic values
N	121
n_N	(40, 40)
μ_{y_N}	(17.431, 28.478)
μ_{x_N}	(24.092, 31.684)
μ_{z_N}	(21.747, 33.209)
C_{y_N}	(0.028, 0.019)
C_{x_N}	(0.009, 0.013)
C_{z_N}	(0.019, 0.017)
ρ_{xy_N}	(0.452, 0.694)
ρ_{yz_N}	(0.247, 0.405)
ρ_{xz_N}	(0.427, 0.357)
$\beta_1(x_N)$	(-0.122, 0.074)
$\beta_1(z_N)$	(0.183, -0.163)
$\beta_2(x_N)$	(3.121, 2.997)
$\beta_2(z_N)$	(2.828, 3.136)

TABLE 8. Bias, MSE, and RE of the neutrosophic estimators for real data

Estimators	$(Bias_L, Bias_U)$	(MSE_L, MSE_U)	(RE_L, RE_U)
κ_m	(0, 0)	(0.003, 0.003)	(1.00, 1.00)
κ_{gr}^1	(0.022, 0.027)	(0.008, 0.010)	(0.378, 0.350)
κ_{gr}^2	(0.016, 0.023)	(0.006, 0.009)	(0.454, 0.389)
κ_{gr}^3	(-0.000, -0.000)	(0.003, 0.003)	(1.000, 1.001)
κ_{gr}^4	(0.021, 0.026)	(0.007, 0.010)	(0.404, 0.363)
κ_{gr}^5	(0.014, 0.017)	(0.003, 0.003)	(0.910, 1.006)
κ_{gr}^6	(-0.000, -0.000)	(0.003, 0.003)	(1.009, 1.079)
κ_{gr}^7	(0.012, 0.071)	(0.006, 0.024)	(0.473, 0.149)
κ_{pr}	(0.001, 0.002)	(0.002, 0.003)	(1.073, 1.193)
κ_{ps}	(0.000, 0.000)	(0.002, 0.003)	(1.073, 1.193)
κ_a^1	(-0.011, -0.009)	(0.002, 0.002)	(1.135, 1.251)
κ_a^2	(-0.011, -0.009)	(0.002, 0.002)	(1.133, 1.248)
κ_a^3	(-0.011, -0.009)	(0.002, 0.003)	(1.120, 1.222)
κ_a^4	(-0.011, -0.009)	(0.002, 0.002)	(1.134, 1.250)
κ_a^5	(-0.011, -0.009)	(0.002, 0.002)	(1.122, 1.229)
κ_a^6	(-0.011, -0.009)	(0.002, 0.003)	(1.120, 1.224)
κ_a^7	(-0.011, -0.009)	(0.002, 0.002)	(1.110, 1.254)

7. Conclusions

In this study, we proposed Searls type generalized neutrosophic power ratio estimators for the population variance by incorporating TAV under SRS. Identifying the inherent uncertainty and indeterminacy often available in real-world data, the proposed estimators utilizes neutrosophic theory to effectively handle the imprecise and uncertain information. The proposed estimators were constructed by using the correlation structure between the study variable and two auxiliary variables, thereby improving the efficiency and robustness of the variance estimation.

The theoretical properties of the proposed estimators, such as bias and MSE, were derived under the neutrosophic setup. A comparative study was conducted by comparing the proposed neutrosophic estimators with the adapted neutrosophic estimators in terms of MSE. The efficiency conditions obtained under comparative study were further assessed by both simulation study and real data illustrations. The empirical results exhibited that the proposed neutrosophic estimators outperformed the adapted estimators based on TAV, in terms of least MSE and maximum RE under indeterminate and uncertain data.

Conclusively, the theoretical and empirical findings highlighted the potential of employing the TAV within the neutrosophic framework to produce more efficient and reliable estimates of population variance using bivariate auxiliary variables...

the population variance. The proposed approach provided a valuable extension to the survey sampling and opens new domain for further research in uncertain data environment, including the development of modified neutrosophic estimators and their application in other sampling designs.

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Neutrosophic Temperaments and Personality Evolution: A Conceptual Framework of Evolution, Indeterminacy and Involution

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Abstract:

Temperament has long been regarded as a foundational construct in personality psychology, linking biologically rooted dispositions to enduring patterns of emotion, cognition, and behavior. Despite substantial theoretical and empirical development, dominant temperament and trait models continue to rely on deterministic and stability-oriented assumptions, limiting their capacity to account for psychological ambiguity, neutrality, and context-dependent variability.

To address these conceptual limitations, the present paper proposes a Neutropsychic framework for rethinking temperament and personality evolution. Drawing on Neutropsychic logic, temperament is conceptualized as a dynamic psychological configuration composed of three coexisting components: evolutionary tendencies, indeterminate (neutral) states, and involutionary regressions. This triadic structure allows for partial and simultaneous expression of multiple temperamental tendencies without reducing personality to fixed types or linear dimensions.

By introducing psychological indeterminacy as a core theoretical construct, the model offers a formal account of ambiguous, transitional, and contradictory behavioral expressions that remain underdeveloped in classical and contemporary personality theories. The paper discusses the theoretical implications of the Neutropsychic temperament framework for personality theory, conceptual modeling, and future research, arguing that embracing indeterminacy provides a more flexible and realistic foundation for understanding personality dynamics in complex psychological environments.

Keywords: Neutropsychic psychology; Temperament theory; Personality evolution; Psychological indeterminacy; Conceptual modeling.

1. Introduction

Temperament has historically been a cornerstone of personality psychology, providing a theoretical link between biologically rooted dispositions and enduring patterns of emotion, cognition, and behavior (Allport, 1961; Rothbart & Bates, 2006). Classical typologies, originating from Hippocratic and Galenic traditions, and contemporary trait models, such as Eysenck's biologically oriented model and the Five-Factor Model, have significantly advanced our understanding of personality (Eysenck, 1991; McCrae & Costa, 2008; Kagan, 1994). However, these approaches often rest on deterministic assumptions and prioritize stability over flexibility. As a result, they struggle to account for psychological indeterminacy, ambiguous behavioral expressions, and context-dependent variability—phenomena increasingly recognized as central to human personality dynamics (Mischel & Shoda, 1995; Cervone & Pervin, 2016).

Despite decades of research, the conceptual space for understanding neutral, transitional, and contradictory psychological states remains underdeveloped. Most temperament and trait theories implicitly assume that individuals belong to well-defined types or continuous dimensions, leaving little room for the coexistence of multiple temperamental tendencies and their dynamic interplay across situations and developmental stages (Strelau, 2008). Classical temperament models categorize individuals into dominant types, while dimensional models, though empirically robust, emphasize consistency and stability at the expense of ambiguity and fluctuation (McCrae & Costa, 2008; Eysenck, 1991). Consequently, these frameworks provide limited explanatory power for transitional states, psychological neutrality, and regressive behaviors, which are frequently observed in complex social and developmental contexts.

To address these conceptual limitations, the present paper proposes a Neutropsychic framework for temperament and personality evolution, conceptualizing temperament as a dynamic psychological system composed of three interrelated components—evolutionary tendencies, indeterminate states, and involutionary regressions. By integrating psychological indeterminacy as a core theoretical construct, this framework allows individuals to partially and simultaneously belong to multiple temperamental states with varying degrees. Unlike classical and trait-based models, this approach emphasizes dynamic coexistence, context sensitivity, and non-linear personality evolution, offering a more flexible and realistic conceptual foundation (Smarandache, 1998, 2015, 2016).

This conceptual framework provides a platform for advancing theoretical understanding, informing empirical research, and fostering interdisciplinary dialogue at the intersection of psychology, philosophy, and formal modeling. By explicitly incorporating indeterminacy, the Neutropsychic temperament model addresses a longstanding gap in personality theory, paving the way for future operationalization, measurement, and empirical validation in developmental, clinical, and organizational settings (Salama & Florentin Smarandache, 2015).

1.1. Classical Temperament Theories

The earliest systematic accounts of temperament originated in ancient medical philosophy, where personality differences were attributed to the balance of bodily humors. The four classical temperaments—sanguine, choleric, melancholic, and phlegmatic—provided an enduring typology that shaped later psychological thinking (Kagan, 1994). While intuitively appealing, these models lacked empirical grounding and failed to account for intra-individual variability and contextual influences.

Subsequent psychological theories retained the notion of biologically rooted dispositions but shifted toward more systematic frameworks. Pavlov's typology of nervous system properties and later constitutional theories emphasized physiological correlates of temperament (Strelau, 2008). Nonetheless, these approaches continued to conceptualize temperament as relatively fixed and unidimensional.

1.2. Trait-Based and Dimensional Approaches

Trait theories marked a significant advancement by conceptualizing temperament and personality as continuous dimensions rather than discrete types. Eysenck's model emphasized biologically based dimensions such as extraversion and neuroticism (Eysenck, 1991), while the Five-Factor Model offered a comprehensive taxonomy of personality traits with strong cross-cultural support (McCrae & Costa, 2008). Despite their empirical robustness, these models prioritize consistency and predictive stability, often underrepresenting ambiguity, neutrality, and situational variability.

Critics have noted that trait-based models struggle to explain contradictory behaviors and transitional psychological states, particularly in complex social environments (Cervone & Pervin, 2016). As a result, researchers have increasingly emphasized dynamic and interactionist perspectives.

1.3. Dynamic and Contextual Perspectives

Interactionist and social-cognitive models conceptualize personality as a dynamic system shaped by interactions between internal dispositions and situational contexts (Mischel & Shoda, 1995). These approaches acknowledge variability and conditional patterns of behavior but often rely on descriptive rather than formalized representations of indeterminacy. Psychological neutrality, ambivalence, and partial trait expression remain theoretically underdeveloped (Smarandache, 2018).

1.4. Neutropsychic Psychology and Conceptual Gaps

Neutropsychic psychology introduces a formal mechanism for modeling indeterminacy as an essential component of psychological systems. Prior studies have applied Neutropsychic concepts to personality structure, refined memory systems, and trait-anti-trait relationships, demonstrating the capacity of this framework to capture psychological complexity (Smarandache, 2013, Jadidi Mohammadabadi, Ahmadi Deh Ghotbaddini, 2023). However, the application of Neutropsychic principles to temperament theory remains limited, creating a clear conceptual gap addressed by the present study.

2. Theoretical Foundations

2.1. Temperament as a Dynamic Psychological System

Contemporary psychological evidence suggests that temperament, while biologically influenced, exhibits flexibility across developmental stages and situational contexts (Rothbart & Bates, 2006). This perspective challenges static models and supports the conceptualization of temperament as an open system capable of adaptation and transformation.

2.2. Neutropsychic Logic as a Modeling Framework

Neutropsychic logic extends classical and fuzzy logic by introducing indeterminacy as an independent dimension (Smarandache, 1998). In psychological modeling, this allows for the representation of states that are partially expressed, ambiguous, or transitional. Rather than forcing psychological phenomena into binary or linear structures, neutrosophy accommodates complexity and contradiction.

2.3. Neutropsychic Temperaments: Evolution, Indeterminacy, and Involution

In the proposed framework, temperament is conceptualized as a triadic structure consisting of:

- Evolution, reflecting adaptive development and constructive expression of temperamental tendencies;
- Indeterminacy, representing neutral, ambiguous, or transitional states resistant to clear classification;
- Involution, indicating regression, inhibition, or maladaptive expression.

These components coexist dynamically within individuals and vary across time and context. By integrating these dimensions, the Neutropsychic temperament model transcends rigid typologies and provides a comprehensive conceptual foundation for understanding personality evolution.

3. Neutropsychic Temperament Model

3.1. Conceptualizing Temperament within a Neutropsychic Framework

In traditional temperament theories, individuals are typically classified into discrete types or positioned along linear dimensions, implicitly assuming determinacy and structural stability. However, empirical evidence and theoretical developments increasingly indicate that temperamental expressions are context-dependent, fluctuating, and sometimes contradictory (Mischel & Shoda, 1995; Cervone & Pervin, 2016). These characteristics necessitate a modeling framework capable of accommodating partial membership, psychological neutrality, and dynamic change.

The Neutropsychic framework provides such a structure by extending beyond binary and fuzzy representations to explicitly incorporate indeterminacy as a fundamental component (Smarandache, 1998). When applied to temperament, neutrosophy allows temperamental tendencies to be expressed

simultaneously in degrees of affirmation, negation, and neutrality. Consequently, temperament is reconceptualized not as a fixed disposition but as a dynamic psychological configuration evolving over time and situations.

Within this framework, each individual's temperament is represented as a Neutropsychic entity characterized by three coexisting components: evolution, indeterminacy and involution.

3.2. Core Components of the Neutropsychic Temperament Model

3.2.1. Evolutionary Component

The evolutionary component of temperament refers to adaptive growth, constructive expression, and positive developmental trajectories of temperamental tendencies. This dimension captures the extent to which an individual's temperament facilitates psychological resilience, effective emotional regulation, and adaptive behavioral responses. Evolution is not assumed to be linear or permanent; rather, it represents a degree of constructive functioning at a given time and context.

From a Neutropsychic perspective, evolutionary tendencies coexist with other components and may vary in intensity depending on situational demands and internal psychological states (Rothbart & Bates, 2006).

3.2.2. Indeterminacy Component

The indeterminacy component constitutes the conceptual core of the Neutropsychic temperament model. It represents neutral, ambiguous, or transitional temperamental states that cannot be clearly classified as adaptive or maladaptive. Psychological phenomena such as emotional ambivalence, hesitation, neutrality, and partial engagement are examples of indeterminate temperamental expressions.

Classical and trait-based models often overlook or marginalize these states due to their reliance on categorical or linear dimensions. Neutropsychic psychology, by contrast, treats indeterminacy as an independent and essential dimension of temperament (Smarandache, 2016). This allows for a more realistic representation of psychological functioning, particularly in complex or uncertain environments.

3.2.3. Involutionary Component

The involutionary component reflects regressive, inhibited, or maladaptive expressions of temperament. This dimension includes tendencies toward withdrawal, emotional dysregulation, rigidity, or ineffective coping. Importantly, involution is not interpreted as a permanent trait but as a context-sensitive degree that may fluctuate over time.

In the Neutropsychic model, involutory tendencies do not negate evolutionary tendencies; instead, both may coexist within the same individual at varying degrees. This coexistence reflects the complex and sometimes contradictory nature of human temperament.

3.3. Formal Neutropsychic Representation of Temperament

Formally, an individual's temperament T can be represented as a Neutropsychic triplet:

$$T = (E, I, V)$$

where:

- E denotes the degree of evolution,
- I denotes the degree of indeterminacy,
- V denotes the degree of involution,

with each component taking values in the interval $[0, 1]$, without the constraint that their sum equals 1. This lack of normalization reflects the openness and non-reductive nature of psychological systems (Smarandache, 1998).

This representation allows individuals to simultaneously exhibit high levels of evolution and involution, moderated by varying degrees of indeterminacy, thereby modeling psychological ambivalence and contextual variability more accurately than traditional approaches.

3.4. Conceptual Diagram of Neutropsychic Temperaments

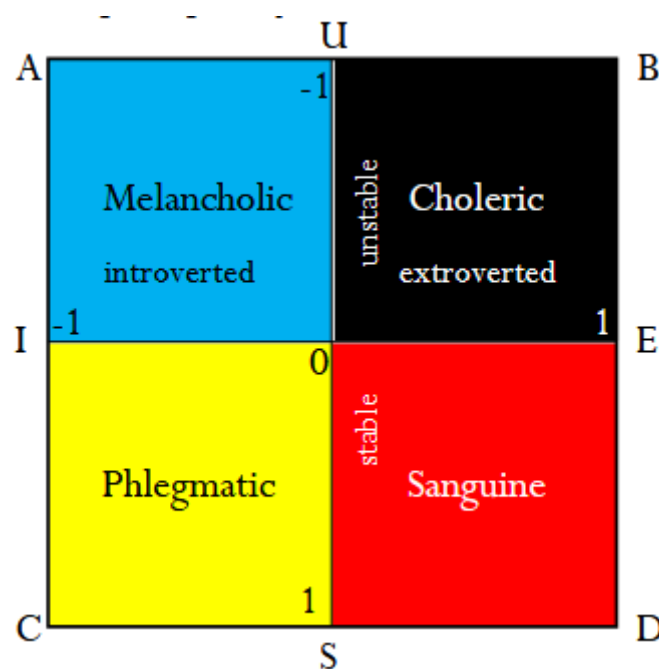


Figure 1. Conceptual Diagram of Neutropsychic Temperaments (Smarandache, 2018).

This is a 2×2 square. “O” is the origin (and center of the square)

The Neutropsychic temperament diagram is represented as a triangular or three-axis conceptual space, where each vertex corresponds to one of the three core components:

- Evolution (E)
- Indeterminacy (I)
- Involution (V)

Each individual's temperament is represented as a point or region within this space, indicating their relative degrees of evolution, indeterminacy, and involution at a given time or context. Unlike classical temperament diagrams, this model allows for:

1. Partial and simultaneous membership in multiple temperamental states;
2. Dynamic movement within the diagram over time;
3. Representation of neutral or ambiguous psychological states.

This diagrammatic representation visually captures the dynamic and non-deterministic nature of temperament, emphasizing that personality evolution is not a unidirectional process but a fluid interplay among development, neutrality, and regression.

3.5. Advantages of the Neutropsychic Temperament Model

The proposed model offers several conceptual advantages over traditional temperament frameworks:

1. Integration of Indeterminacy: It formally incorporates psychological neutrality and ambiguity as core theoretical elements.
2. Dynamic Representation: It models temperament as an evolving system rather than a static structure.
3. Context Sensitivity: It accommodates situational variability and contradictory behavioral tendencies.
4. Theoretical Flexibility: It is compatible with classical, trait-based, and dynamic personality theories while extending their explanatory scope.

By framing temperament within a Neutropsychic structure, the model provides a comprehensive and realistic conceptual foundation for understanding personality evolution in complex psychological environments (Jadidi Mohammadabadi, 2025).

4. Main Results

4.1. Comparison with Classical Temperament Theories

Classical temperament theories, rooted in Hippocratic and Galenic traditions, conceptualize temperament as a stable and qualitatively distinct configuration of behavioral and emotional tendencies. While these models offer intuitive classifications, they rely heavily on rigid typologies that inadequately represent intra-individual variability and contextual influence (Kagan, 1994; Strelau, 2008). Individuals are typically assigned to a dominant temperament type, implicitly assuming consistency across situations and developmental stages.

In contrast, the Neutropsychic temperament model fundamentally departs from categorical determinism by allowing simultaneous and partial membership in multiple temperamental states. Rather than assigning individuals to fixed types such as sanguine or melancholic, the Neutropsychic framework represents temperament as a dynamic configuration of evolutionary, indeterminate, and involutory tendencies. This approach captures the empirical reality that individuals may display adaptive, neutral, and regressive behaviors concurrently or alternately depending on situational demands.

Furthermore, classical temperament models lack a formal mechanism for representing psychological neutrality or transitional states. Neutropsychic indeterminacy explicitly addresses this limitation by introducing a conceptual space for ambiguous, undecided, or emotionally neutral states. As a result, the Neutropsychic model provides a more comprehensive and flexible interpretation of temperament dynamics, extending the explanatory scope of classical theories without negating their historical significance.

4.2. Comparison with the Five-Factor Model (Big Five)

The Five-Factor Model (FFM) represents one of the most empirically supported frameworks in personality psychology, offering a dimensional structure that captures broad patterns of personality traits across cultures (McCrae & Costa, 2008). By conceptualizing personality traits along continuous dimensions—extraversion, agreeableness, conscientiousness, neuroticism, and openness—the FFM improves upon categorical models by allowing graded individual differences.

Despite its strengths, the Big Five model remains constrained by its linear dimensionality and deterministic assumptions. Trait scores typically reflect averaged behavioral tendencies, which may obscure situational variability, internal conflict, and psychological ambivalence. For instance, an individual may score moderately on extraversion while exhibiting highly context-dependent fluctuations between social engagement and withdrawal—patterns that are not readily captured by static trait scores (Cervone & Pervin, 2016).

The Neutropsychic temperament model complements the FFM by introducing an additional conceptual layer that accounts for indeterminacy. Within the Neutropsychic framework, trait expressions are not limited to degrees of presence or absence but may also occupy neutral or ambiguous states. This enables a more nuanced interpretation of personality functioning, particularly in contexts characterized by uncertainty or conflicting demands.

Moreover, while the Big Five primarily focuses on trait structure, the Neutropsychic model emphasizes process and change, explicitly modeling evolution and involution alongside indeterminacy. This dynamic orientation aligns with contemporary views of personality as an adaptive system rather than a static trait configuration.

4.3. Theoretical Contributions of the Neutropsychic Temperament Model

The Neutropsychic temperament model makes several important theoretical contributions to personality psychology. First, it provides a formal conceptualization of psychological indeterminacy, addressing a longstanding gap in temperament and trait theories. Second, it integrates stability and variability within a single framework, allowing temperament to be understood as both biologically grounded and contextually modulated.

Third, the model bridges classical typological approaches and modern dimensional frameworks by offering a flexible structure that accommodates both qualitative and quantitative interpretations. This integrative capacity enhances theoretical coherence and opens new avenues for interdisciplinary research, particularly at the intersection of psychology, philosophy, and mathematical modeling.

4.4. Implications for Conceptual and Empirical Research

From a research perspective, the Neutropsychic temperament model encourages a shift from static classification toward dynamic assessment. Future empirical studies may operationalize the components of evolution, indeterminacy, and involution using longitudinal designs or situational sampling methods. Such approaches could provide deeper insight into how temperament fluctuates across contexts and developmental stages.

Additionally, the model offers a conceptual foundation for refining psychological assessment tools. By incorporating Neutropsychic dimensions, existing temperament and personality measures may be extended to capture neutral and transitional states that are currently underrepresented in standard instruments.

4.5. Limitations and Future Directions

As a conceptual model, the Neutropsychic temperament framework requires empirical validation. Operational definitions of evolution, indeterminacy, and involution must be developed and tested across diverse populations. Furthermore, while the model offers theoretical flexibility, its practical application depends on the development of reliable measurement strategies.

Despite these limitations, the model provides a robust conceptual foundation for advancing temperament theory and invites further interdisciplinary collaboration.

5. Implications for Personality Psychology and Assessment

5.1. Implications for Personality Theory

The Neutropsychic temperament model has important implications for contemporary personality theory by challenging deterministic and static assumptions that dominate classical and trait-based frameworks. Traditional models tend to conceptualize temperament as a stable configuration of traits or types, implicitly minimizing the role of uncertainty and psychological neutrality. By contrast, the Neutropsychic framework reconceptualizes temperament as an open, dynamic system in which evolutionary, indeterminate, and involutory tendencies coexist and interact.

This theoretical shift aligns with emerging perspectives that emphasize personality as a complex adaptive system rather than a fixed structure (Cervone & Pervin, 2016). By formally integrating indeterminacy into temperament theory, the Neutropsychic model provides a conceptual mechanism for explaining psychological phenomena such as ambivalence, hesitation, and context-sensitive behavior that are insufficiently addressed in existing theories. As a result, personality psychology can move beyond linear trait descriptions toward more comprehensive models of human behavior.

5.2. Implications for Psychological Assessment

One of the most significant contributions of the Neutropsychic temperament model lies in its implications for psychological assessment. Standard temperament and personality instruments are typically designed to measure stable traits along predefined dimensions, often forcing respondents to choose between polarized alternatives. Such approaches may fail to capture neutral or ambiguous psychological states, leading to oversimplified interpretations of individual differences (McCrae & Costa, 2008).

The Neutropsychic framework suggests the development of multi-dimensional assessment tools capable of measuring degrees of evolution, indeterminacy, and involution simultaneously. Rather than assigning individuals to fixed categories or relying solely on averaged trait scores, assessment instruments informed by Neutropsychic principles could represent temperament profiles as dynamic configurations that vary across contexts and time.

Furthermore, Neutropsychic assessment models may enhance the interpretive validity of psychological measurements by explicitly acknowledging uncertainty. This is particularly relevant in clinical, educational, and organizational settings, where individuals often exhibit fluctuating or contradictory behavioral patterns (Jadidi Mohammadabadi et al, 2019, Jadidi Mohammadabadi, Ahmadi Deh Ghotbaddini, 2023).

5.3. Implications for Developmental and Clinical Psychology

From a developmental perspective, the Neutropsychic temperament model offers a flexible framework for understanding personality change across the lifespan. Developmental transitions are frequently characterized by periods of instability, exploration, and psychological neutrality. By incorporating indeterminacy as a core component, the model provides a theoretical basis for understanding these transitional phases without pathologizing them (Rothbart & Bates, 2006).

In clinical psychology, the model has implications for conceptualizing maladaptive behavior and psychological distress. Involutionary tendencies may reflect temporary regressions rather than fixed dysfunctions, while indeterminate states may signal adaptive pauses or psychological reorganization. This perspective encourages a more nuanced understanding of psychological functioning and may inform intervention strategies that focus on facilitating evolutionary change rather than eliminating variability.

5.4. Implications for Research Methodology

The Neutropsychic temperament framework also invites methodological innovation in personality research. Longitudinal and experience-sampling designs are particularly well suited to capturing the dynamic interplay of evolution, indeterminacy, and involution. By moving beyond cross-sectional trait assessments, researchers can explore how temperament configurations shift in response to situational demands and developmental processes (Mischel & Shoda, 1995).

Moreover, Neutropsychic modeling encourages the use of mixed-method approaches that combine quantitative measurement with qualitative insights. Such integration may provide a richer understanding of temperament dynamics and support the empirical validation of conceptual models.

5.5. Interdisciplinary Implications

Finally, the Neutropsychic temperament model has interdisciplinary implications extending beyond psychology. Its conceptual foundations intersect with philosophy of mind, systems theory, and computational modeling. By offering a formal structure for representing indeterminacy, the model facilitates dialogue between psychological theory and mathematical approaches to uncertainty, potentially enriching both domains.

6. Conclusion and Future Research

The present study proposed a conceptual Neutropsychic model of temperaments grounded in the triadic structure of evolution, indeterminacy and involution, aiming to address fundamental limitations of classical and contemporary temperament theories. Traditional models, whether typological or trait-based, often rely on deterministic assumptions that inadequately capture psychological ambiguity, contextual variability, and contradictory behavioral expressions. By contrast, the Neutropsychic framework explicitly incorporates indeterminacy as a core theoretical component, offering a more comprehensive and realistic representation of temperament dynamics.

By conceptualizing temperament as a dynamic and open psychological system, the proposed model extends existing theories without dismissing their contributions. It provides a formal structure capable of integrating stability and change, adaptive growth and regression, and clarity and ambiguity within a single conceptual framework. The inclusion of indeterminacy allows for the systematic representation of neutral and transitional psychological states, which are frequently observed yet theoretically underrepresented in personality psychology.

From a theoretical standpoint, the Neutropsychic temperament model contributes to the ongoing shift toward dynamic and process-oriented perspectives in personality research. It bridges classical temperament typologies and modern dimensional models by introducing a flexible structure that accommodates both qualitative and quantitative interpretations. This integrative capacity enhances the explanatory power of temperament theory and supports interdisciplinary dialogue between psychology, philosophy, and formal systems theory.

Despite its conceptual strengths, the present model remains theoretical and calls for empirical validation. Future research should focus on the operationalization of evolution, indeterminacy, and involution through psychometrically sound instruments. Longitudinal and experience-sampling methodologies may be particularly effective in capturing the dynamic fluctuations of Neutropsychic temperament components across time and contexts. Additionally, cross-cultural studies could examine the universality and cultural sensitivity of the proposed framework.

Further research may also explore the application of Neutropsychic temperaments in clinical, educational, and organizational settings. Investigating how indeterminacy functions in developmental transitions, psychological resilience, and adaptive decision-making may yield valuable insights into personality functioning. Moreover, integrating computational and mathematical modeling techniques with Neutropsychic psychology could facilitate the development of predictive and simulation-based approaches to temperament research.

In conclusion, the Neutropsychic temperament model provides a robust conceptual foundation for advancing the study of personality evolution in complex psychological environments. By embracing uncertainty and neutrality as integral aspects of human temperament, this framework opens new avenues for theoretical refinement and empirical exploration in personality psychology.

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Neutrosophic $\mathcal{N}$ -structures on Sheffer stroke BG-algebras

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Abstract. The study defines a neutrosophic $\mathcal{N}$ -subalgebra and a level set of a neutrosophic $\mathcal{N}$ -structure on Sheffer stroke BG-algebras. It appears that these concepts are integral to understanding the behavior of neutrosophic logic within the framework of Sheffer stroke BG-algebras. The study establishes a relationship between subalgebras and level sets on Sheffer stroke BG-algebras. Specifically, it proves that the level set of neutrosophic $\mathcal{N}$ -subalgebras on this algebra is its subalgebra, and vice versa. This indicates a tight connection between these two concepts within the given algebraic structure. It is stated that the family of all neutrosophic $\mathcal{N}$ -subalgebras of a Sheffer stroke BG-algebra forms a complete distributive lattice. This suggests that there is a well-defined structure and order among these subalgebras, allowing for systematic analysis. The study describes a neutrosophic $\mathcal{N}$ -ideal of a Sheffer stroke BG-algebra and provides some of its properties. Additionally, it is shown that every neutrosophic $\mathcal{N}$ -ideal of a Sheffer stroke BG-algebra is also its neutrosophic $\mathcal{N}$ -subalgebra, though the inverse is generally not true. This highlights the specific characteristics and behavior of neutrosophic $\mathcal{N}$ -ideals within the given algebraic context.

Keywords: Sheffer stroke (BG-algebra), ideal, neutrosophic $\mathcal{N}$ -subalgebra, neutrosophic $\mathcal{N}$ -ideal.

1. Introduction

BG-algebras, initially developed by mathematicians Bright and Grimmer, extend Boolean algebras by incorporating more intricate operations and properties, allowing them to address complex logical and scientific inquiries. Building on the foundational concepts of Boolean algebra, BG-algebras provide a broader and more flexible framework for modeling and analyzing logical and algebraic systems that surpass the limitations of traditional Boolean logic. This expansion has enabled their application in fields such as advanced mathematical logic and fuzzy logic, where they help capture subtleties in logical reasoning.

The Sheffer stroke operation, also known as the NAND operator, was introduced by Henry Maurice Sheffer [10]. This operation is notable because it alone can create a complete logical system, without the need for additional logical operators [2]. In other words, any axiom or expression in a logical system can be formulated using only the Sheffer stroke operation. This characteristic enhances the ability to examine and verify specific properties in newly constructed logical systems. Furthermore, the fundamental axioms of Boolean algebra, which underpin classical propositional logic, can be fully expressed using just the Sheffer stroke. This underscores the foundational importance and versatility of the Sheffer operation in both algebraic and logical frameworks. This concept has attracted considerable academic interest, leading to various studies, including work on Sheffer stroke operation reducts of basic algebras [7] and their congruences [9], Sheffer stroke MLT-algebras [8], ortholattices [1], new state operators in Sheffer stroke basic algebras [6] and their filters, Riečan and Bosbach state operators on Sheffer stroke MTL-algebras [5], and Sheffer stroke BG-algebras [3].

The exploration of Sheffer stroke BG-algebras is motivated by their potential applications in both theoretical and applied fields. These algebras not only enhance our understanding of non-classical logics but also offer valuable insights for computational models in areas such as quantum computing and fuzzy logic. By leveraging the unique properties of the Sheffer stroke, Sheffer stroke BG-algebras enable the development of innovative reasoning methods that extend beyond classical Boolean systems.

The novelty of Sheffer stroke BG-algebras lies in their ability to unify various logical operations within a single operational framework. This consolidation expands the scope of logical reasoning, providing a foundation for examining relationships between previously disparate algebraic structures. Additionally, incorporating the Sheffer stroke within BG-algebras opens up new avenues for research, such as the creation of decision-making frameworks and algorithmic applications that utilize the unique characteristics of the Sheffer stroke.

Neutrosophic $\mathcal{N}$ -structures, a part of the broader field of neutrosophy introduced by Florentin Smarandache in the early 1990s, provide tools for managing uncertainty, inconsistency, and indeterminacy in logic and mathematics [11]. Neutrosophy originated with neutrosophic logic and has since grown to include $\mathcal{N}$ -structures and neutrosophic sets, which are applied in fields such as artificial intelligence and data analysis. Research continues to expand and deepen the theoretical foundation of neutrosophic structures, exploring their applications and practical implementations.

Neutrosophic $\mathcal{N}$ -structures enrich classical mathematical frameworks by incorporating neutrosophic principles, which describe elements with three components: truth, indeterminacy, and falsity. This approach allows for a more nuanced representation of information, effectively capturing uncertainties and partial truths.

In this paper, we define a neutrosophic $\mathcal{N}$ -subalgebra and a level set of a neutrosophic $\mathcal{N}$ -structure within Sheffer stroke BG-algebras. These concepts are critical for understanding the interaction between neutrosophic logic and Sheffer stroke BG-algebras. Our study demonstrates that there is a close relationship between subalgebras and level sets within this framework. Specifically, we prove that the level set of a neutrosophic $\mathcal{N}$ -subalgebra is itself a subalgebra and that, conversely, a subalgebra can be interpreted as a level set. This result establishes a robust connection between these two concepts within the algebraic structure.

Furthermore, we show that all neutrosophic $\mathcal{N}$ -subalgebras of a Sheffer stroke BG-algebra form a complete distributive lattice. This structure provides a clear ordering among these subalgebras, facilitating systematic analysis. Additionally, we define and investigate a neutrosophic $\mathcal{N}$ -ideal within Sheffer stroke BG-algebras and outline some of its key properties. We demonstrate that every neutrosophic $\mathcal{N}$ -ideal in this algebra is also a neutrosophic $\mathcal{N}$ -subalgebra, although not every $\mathcal{N}$ -subalgebra qualifies as an $\mathcal{N}$ -ideal. This distinction highlights the unique attributes of neutrosophic $\mathcal{N}$ -ideals in the context of Sheffer stroke BG-algebras, contributing to a deeper understanding of their behavior and potential applications.

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke BG-algebras and neutrosophic $\mathcal{N}$ -structures.

Definition 2.1. ([10]) Let $H = \langle H, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:

- (S1) $x|y = y|x$,
- (S2) $(x|x)|(x|y) = x$,
- (S3) $x|((y|z)|(y|z)) = ((x|y)|(x|y))|z$,
- (S4) $(x|((x|x)|(y|y))|(x|((x|x)|(y|y)))) = x$.

Definition 2.2. ([3]) A Sheffer stroke BG-algebra (briefly, SBG-algebra) is a structure $\langle A, | \rangle$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $x, y, z \in A$

- (SBG-1) $(x|(x|x))|(x|(x|x)) = 0$,
- (SBG-2) $(0|(y|y))|((x|(y|y))|(x|(y|y))) = x|x$.

Proposition 2.3. ([3]) Let $\langle A, | \rangle$ be an SBG-algebra. Then the binary relation $x \leq y$ if and only if $(y|(x|x))|(y|(x|x)) = 0$ is a partial order on A .

Definition 2.4. ([3]) A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-subalgebra of H if $(x|(y|y))|(x|(y|y)) \in G$ for all $x, y \in G$.

Definition 2.5. ([3]) A nonempty subset G of a Sheffer stroke BG-algebra H is called an SBG-ideal of H if for all $x, y \in G$

- (1) $0 \in G$,
- (2) $(x|(y|y))|(x|(y|y)) \in G$ and $y \in G$ imply $x \in G$.

Definition 2.6. [4] An X is said to be implicative if it satisfies the condition $x|(x|(y|y)) = y|(y|(x|x))$, for all $x, y \in L$.

Definition 2.7. [4] Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras. Then a mapping $f : A \rightarrow B$ is called a homomorphism if $f(x|_A y) = f(x)|_B f(y)$ for all $x, y \in X$ and $f(0_A) = 0_B$.

Theorem 2.8. [4] Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras. Then $\langle A \times B, |_{A \times B}, 0_{A \times B} \rangle$ is a Sheffer stroke BG-algebra, where the set $A \times B$ is the Cartesian product of A and B and the operation $|_{A \times B}$ on this set is defined by $(a_1, b_1)|_{A \times B} (a_2, b_2) = (a_1|_A a_2, b_1|_B b_2)$, and the fixed element is $0_{A \times B} = (0_A, 0_B)$.

3. Some properties of Neutrosophic $\mathcal{N}$ -structures on Sheffer stroke BG-algebras

In this section, the study introduces the concepts of neutrosophic $\mathcal{N}$ -subalgebras and neutrosophic $\mathcal{N}$ -ideals within the context of Sheffer stroke BG-algebras. It's worth noting that unless explicitly stated otherwise, X refers to a Sheffer stroke BG-algebra.

Definition 3.1. A neutrosophic $\mathcal{N}$ -subalgebra X_N on X is called a neutrosophic $\mathcal{N}$ -structure of X if

$$(\forall x, y \in X) \left(\begin{array}{l} T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\} \\ I_N((x|(y|y))|(x|(y|y))) \geq \min\{I_N(x), I_N(y)\} \\ F_N((x|(y|y))|(x|(y|y))) \leq \max\{F_N(x), F_N(y)\} \end{array} \right). \quad (1)$$

To illustrate the concept of a neutrosophic $\mathcal{N}$ -structure, we provide an example that demonstrates the conditions in Definition 3.1.

Example 3.2. Let $X = \{0, x, y, 1\}$ be a set with the operation $|$ defined by the following Cayley table (Example 3.1 in [3]):

$ $	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Define the neutrosophic functions T_N , I_N , and F_N on X as follows:

$$\begin{array}{llll} T_N(0) = 0.6, & T_N(x) = 0.7, & T_N(y) = 0.5, & T_N(1) = 0.9, \\ I_N(0) = 0.4, & I_N(x) = 0.5, & I_N(y) = 0.6, & I_N(1) = 0.1, \\ F_N(0) = 0.3, & F_N(x) = 0.2, & F_N(y) = 0.4, & F_N(1) = 0.1. \end{array}$$

Now, consider $X_N = \{0, x\}$ as a subset of X , and let's verify if X_N satisfies the conditions in Definition 3.1 to be considered a neutrosophic $\mathcal{N}$ -structure.

(1) **Truth Membership Condition and Additional Condition:**

We need to verify the truth membership condition and the additional condition $(x|(0|0))|(x|(0|0)) = x$.

For the truth membership condition, for $a = 0$ and $b = x$, the condition requires that

$$T_N((0|(x|x))|(0|(x|x))) \leq \max\{T_N(0), T_N(x)\}.$$

Using the Cayley table, we calculate $(0|(x|x))|(0|(x|x)) = 0$. Substituting, we get $T_N(0) = 0.6$, and

$$0.6 \leq \max\{0.6, 0.7\} = 0.7,$$

which satisfies the truth membership condition.

Now, let's verify the additional condition $(x|(0|0))|(x|(0|0)) = x$.

Using the Cayley table, we calculate $(x|(0|0))|(x|(0|0)) = y|y = x$.

This result, x , matches the required condition, confirming that the additional condition is satisfied.

(2) **Indeterminacy Membership Condition:** The indeterminacy condition requires

$$I_N((0|(x|x))|(0|(x|x))) \geq \min\{I_N(0), I_N(x)\}.$$

From the Cayley table, we have:

$$(0|(x|x))|(0|(x|x)) = 0.$$

Substituting, we get $I_N(0) = 0.4$, and

$$0.4 \geq \min\{I_N(0), I_N(x)\} = 0.4,$$

which satisfies the indeterminacy membership condition.

(3) **Falsehood Membership Condition:** For the falsehood condition, we need

$$F_N((0|(x|x))|(0|(x|x))) \leq \max\{F_N(0), F_N(x)\}.$$

From the Cayley table, we have $(0|(x|x))|(0|(x|x)) = 0$.

Substituting, we get $F_N(0) = 0.3$, and

$$0.3 \leq \max\{F_N(0), F_N(x)\} = 0.3,$$

which satisfies the falsehood membership condition.

This example demonstrates that $X_N = \{0, x\}$ satisfies all three conditions required in Definition 3.1, as well as the additional condition $(x|(0|0))|(x|(0|0)) = x$, to qualify as a neutrosophic $\mathcal{N}$ -structure. This verification shows that the elements in X_N fulfill the necessary criteria under the truth, indeterminacy, and falsehood membership functions, illustrating how a subset can meet the strict requirements for a neutrosophic $\mathcal{N}$ -structure.

Definition 3.3. Let X_N be a neutrosophic $\mathcal{N}$ -structure on X and $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. For the sets

$$T_N^\alpha = \{x \in X : T_N(x) \leq \alpha\},$$

$$I_N^\beta = \{x \in X : I_N(x) \geq \beta\}$$

and

$$F_N^\gamma = \{x \in X : F_N(x) \leq \gamma\},$$

the set

$$X_N(\alpha, \beta, \gamma) = \{x \in X : T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called (α, β, γ) -level set of X_N . Also, $X_N(\alpha, \beta, \gamma) = T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$.

Theorem 3.4. Let X_N be a neutrosophic $\mathcal{N}$ -structure on X and $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X , then the nonempty $X_N(\alpha, \beta, \gamma)$ -level set of X_N is a subalgebra of X .

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X and $x, y \in X_N(\alpha, \beta, \gamma)$. Then $T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma$ and $T_N(y) \leq \alpha, I_N(y) \geq \beta, F_N(y) \leq \gamma$. Thus, it is obtained that

$$T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\} \leq \alpha,$$

$$I_N((x|(y|y))|(x|(y|y))) \geq \min\{I_N(x), I_N(y)\} \geq \beta$$

and

$$F_N((x|(y|y))|(x|(y|y))) \leq \max\{F_N(x), F_N(y)\} \leq \gamma$$

for all $x, y \in X$. So, $(x|(y|y))|(x|(y|y)) \in X_N(\alpha, \beta, \gamma)$ which means that $X_N(\alpha, \beta, \gamma)$ is a subalgebra of X . $\square$

Theorem 3.5. Let X_N be a neutrosophic $\mathcal{N}$ -structure on X and T_N^α, I_N^β and F_N^γ be subalgebras of X for all $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Then X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X .

Proof. Assume that T_N^α , I_N^β and F_N^γ are subalgebras of X for all $\alpha, \beta, \gamma \in [-1, 0]$ such that

$$-3 \leq \alpha + \beta + \gamma \leq 0.$$

Suppose that $x, y \in X$ such that $a = T_N((x|(y|y))|(x|(y|y))) > \max\{T_N(x), T_N(y)\} = b$. Then $b < \alpha_1 < a$ where $\alpha_1 = \frac{a+b}{2} \in [-1, 0)$. Thus, $x, y \in T_N^{\alpha_1}$, but $(x|(y|y))|(x|(y|y)) \notin T_N^{\alpha_1}$, which is a contradiction. Hence, $T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\}$ for all $x, y \in X$. Assume that $x, y \in X$ such that $u = I_N((x|(y|y))|(x|(y|y))) < \min\{I_N(x), I_N(y)\} = v$. Then $u < \beta_1 < v$ where $\beta_1 = \frac{u+v}{2} \in [-1, 0)$. Hence $x, y \in I_N^{\beta_1}$ but $(x|(y|y))|(x|(y|y)) \notin I_N^{\beta_1}$ which is a contradiction. Therefore, $I_N((x|(y|y))|(x|(y|y))) \geq \min\{I_N(x), I_N(y)\}$ for all $x, y \in X$. Suppose that $x, y \in X$ such that $m = F_N((x|(y|y))|(x|(y|y))) > \max\{F_N(x), F_N(y)\} = n$. Then $n < \gamma_1 < m$ where $\gamma_1 = \frac{m+n}{2} \in [-1, 0)$. Hence $x, y \in F_N^{\gamma_1}$, but $(x|(y|y))|(x|(y|y)) \notin F_N^{\gamma_1}$ which is a contradiction. Therefore, $F_N((x|(y|y))|(x|(y|y))) \leq \max\{F_N(x), F_N(y)\}$ for all $x, y \in X$. Thereby, X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X . $\square$

Theorem 3.6. Let $\{X_{N_i} : i \in \mathbb{N}\}$ be a family of all neutrosophic $\mathcal{N}$ -subalgebras of X . Then $\{X_{N_i} : i \in \mathbb{N}\}$ forms a complete distributive lattice.

Proof. Let G be a nonempty subset of $\{X_{N_i} : i \in \mathbb{N}\}$. Since $\{X_{N_i}$ is a neutrosophic $\mathcal{N}$ -subalgebras of X , for all $X_{N_i} \in G$, it satisfies:

$$I_N((x|(y|y))|(x|(y|y))) \geq \min\{I_N(x), I_N(y)\},$$

$$I_N((x|(y|y))|(x|(y|y))) \geq \min\{I_N(x), I_N(y)\}$$

and

$$F_N((x|(y|y))|(x|(y|y))) \leq \max\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Then $\bigcap G$ satisfies these inequalities, which means that $\bigcap G$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . Let P be a family of all neutrosophic $\mathcal{N}$ -subalgebras of X containing $\bigcup\{X_{N_i} : i \in \mathbb{N}\}$. Then $\bigcap P$ is a neutrosophic $\mathcal{N}$ -subalgebra of X . If $\bigwedge_{i \in \mathbb{N}} X_{N_i} = \bigcap_{i \in \mathbb{N}} X_{N_i}$ and $\bigvee_{i \in \mathbb{N}} X_{N_i} = \bigcap P$, then $(\{X_{N_i} : i \in \mathbb{N}\}, \bigvee, \bigwedge)$. Also, it is distributive by the definitions of $\bigvee$ and $\bigwedge$. $\square$

Theorem 3.7. If a neutrosophic $\mathcal{N}$ -structure X_N on X is a neutrosophic $\mathcal{N}$ -subalgebra of X , then $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$ for all $x \in X$.

Proof. For any $x \in X$,

$$T_N(0) = T_N((x|(x|x))|(x|(x|x)))) \leq \max\{T_N(x), T_N(x)\} = T_N(x),$$

$$I_N(0) = I_N((x|(x|x))|(x|(x|x)))) \geq \min\{I_N(x), I_N(x)\} = I_N(x)$$

and

$$F_N(0) = F_N((x|(x|x))|(x|(x|x))) \leq \max\{F_N(x), F_N(x)\} = F_N(x).$$

□

Definition 3.8. A neutrosophic $\mathcal{N}$ -structure X_N on X is called a neutrosophic $\mathcal{N}$ -ideal of X if

$$(\forall x, y \in X) \begin{pmatrix} T_N(0) \leq T_N(x) \leq \max\{T_N((x|(y|y))|(x|(y|y))), T_N(y)\} \\ I_N(0) \geq I_N(x) \geq \min\{I_N((x|(y|y))|(x|(y|y))), I_N(y)\} \\ F_N(0) \leq F_N(x) \leq \max\{F_N((x|(y|y))|(x|(y|y))), F_N(y)\} \end{pmatrix}. \quad (2)$$

Theorem 3.9. Let X_N be a neutrosophic $\mathcal{N}$ -structure on X and $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If X_N is a neutrosophic $\mathcal{N}$ -ideal of X , then the nonempty $X_N(\alpha, \beta, \gamma)$ -level set of X_N is an ideal of X .

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X and

$y, (x|(y|y))|(x|(y|y)) \in X_N(\alpha, \beta, \gamma)$. Then $T_N(y) \leq \alpha, I_N(y) \geq \beta, F_N(y) \leq \gamma$ and $T_N((x|(y|y))|(x|(y|y))) \leq \alpha, I_N((x|(y|y))|(x|(y|y))) \geq \beta, F_N((x|(y|y))|(x|(y|y))) \leq \gamma$. Thus, it is obtained that

$$T_N(x) \leq \max\{T_N(y), T_N((x|(y|y))|(x|(y|y)))\} \leq \alpha,$$

$$I_N(x) \geq \min\{I_N(y), I_N((x|(y|y))|(x|(y|y)))\} \geq \beta$$

and

$$F_N(x) \leq \max\{F_N(y), F_N((x|(y|y))|(x|(y|y)))\} \leq \gamma$$

for all $x, y \in X$. So, $x \in X_N(\alpha, \beta, \gamma)$ which means that $X_N(\alpha, \beta, \gamma)$ is an ideal of X . □

Theorem 3.10. Let X_N be a neutrosophic $\mathcal{N}$ -structure on X and T_N^α, I_N^β and F_N^γ be ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Then X_N is a neutrosophic $\mathcal{N}$ -ideal of X .

Proof. Assume that T_N^α, I_N^β and F_N^γ are ideals of X , for all $\alpha, \beta, \gamma \in [-1, 0]$ such that

$$-3 \leq \alpha + \beta + \gamma \leq 0.$$

Suppose that $y, (x|(y|y))|(x|(y|y)) \in X$ such that

$$a = T_N(x) > \max\{T_N(y), T_N((x|(y|y))|(x|(y|y)))\} = b.$$

Then $b < \alpha_1 < a$ where $\alpha_1 = \frac{a+b}{2} \in [-1, 0)$. Thus, $y, (x|(y|y))|(x|(y|y)) \in T_N^{\alpha_1}$, but $x \notin T_N^{\alpha_1}$, a contradiction. Hence, $T_N(x) \leq \max\{T_N(y), T_N((x|(y|y))|(x|(y|y)))\}$ for all $x, y \in X$. Assume that $y, (x|(y|y))|(x|(y|y)) \in X$ such that $u = I_N(x) < \min\{I_N(y), I_N((x|(y|y))|(x|(y|y)))\} = v$.

Then $u < \beta_1 < v$ where $\beta_1 = \frac{u+v}{2} \in [-1, 0)$. Hence $y, (x|(y|y))|(x|(y|y)) \in I_N^{\beta_1}$, but $x \notin I_N^{\beta_1}$, a contradiction. Then $I_N(x) \geq \min\{I_N(y), I_N((x|(y|y))|(x|(y|y)))\}$ for all $x, y \in X$. Suppose that $y, (x|(y|y))|(x|(y|y)) \in X$ such that

$$m = F_N(x) > \max\{F_N(y), F_N((x|(y|y))|(x|(y|y)))\} = n.$$

Then $n < \gamma_1 < m$ where $\gamma_1 = \frac{m+n}{2} \in [-1, 0)$. Hence $y, (x|(y|y))|(x|(y|y)) \in F_N^{\gamma_1}$, but $x \notin F_N^{\gamma_1}$, a contradiction. Then $F_N(x) \leq \max\{F_N(y), F_N((x|(y|y))|(x|(y|y)))\}$ for all $x, y \in X$. Hence X_N is a neutrosophic $\mathcal{N}$ -ideal of X . $\square$

Theorem 3.11. *Every neutrosophic $\mathcal{N}$ -ideal of X is a neutrosophic $\mathcal{N}$ -subalgebra of X .*

4. Neutrosophic $\mathcal{N}$ -implicative ideal in Sheffer stroke BG-algebras

Definition 4.1. A neutrosophic $\mathcal{N}$ -subalgebra X_N on X is called a neutrosophic $\mathcal{N}$ -implicative ideal of X if

$$(\forall x, y \in X) \begin{pmatrix} T_N(0) \leq T_N(x) \leq \max\{T_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\ (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))), T_N(z)\} \\ I_N(0) \geq I_N(x) \geq \min\{I_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\ (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))), I_N(z)\} \\ F_N(0) \leq F_N(x) \leq \max\{F_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\ (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))), F_N(z)\} \end{pmatrix}. \quad (3)$$

Proposition 4.2. *Every neutrosophic $\mathcal{N}$ -implicative ideal of X is a neutrosophic $\mathcal{N}$ -ideal of X .*

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -implicative ideal of X . Then $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$. Also,

$$\begin{aligned} T_N(x) &\leq \max\{T_N(y), T_N((((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))| \\ &\quad (((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y)))\} \\ &= \max\{T_N(y), T_N((((x|(0|0))|(x|(0|0))))|(y|y))| \\ &\quad (((x|(0|0))|(x|(0|0))))|(y|y)))\} \\ &= \max\{T_N(y), T_N((x|(y|y))|(x|(y|y)))\}, \end{aligned}$$

$$\begin{aligned} I_N(x) &\geq \min\{I_N(y), I_N((((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))| \\ &\quad (((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y)))\} \\ &= \min\{I_N(y), I_N((((x|(0|0))|(x|(0|0))))|(y|y))| \\ &\quad (((x|(0|0))|(x|(0|0))))|(y|y)))\} \\ &= \min\{I_N(y), I_N((x|(y|y))|(x|(y|y)))\} \end{aligned}$$

and

$$\begin{aligned}
 F_N(x) &\leq \max\{F_N(y), F_N((((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))| \\
 &\quad (((x|(x|(x|x)))|(x|(x|(x|x))))|(y|y))))\} \\
 &= \max\{F_N(y), F_N((((x|(0|0))|(x|(0|0))))|(y|y))| \\
 &\quad (((x|(0|0))|(x|(0|0))))|(y|y))))\} \\
 &= \max\{F_N(y), F_N((x|(y|y))|(x|(y|y)))\}.
 \end{aligned}$$

Therefore, X_N is a neutrosophic $\mathcal{N}$ -ideal of X . $\square$

Definition 4.3. A neutrosophic $\mathcal{N}$ -structure X_N of X is called a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X if it satisfies:

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_N(0) \leq T_N(x), I_N(0) \geq I_N(x), F_N(0) \leq F_N(x), \\ T_N((y|(y|(x|x))|(y|(y|(x|x)))) \leq \max\{T_N((((x|(x|(y|y))| \\ (x|(x|(y|y))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), T_N(z)\}, \\ I_N((y|(y|(x|x))|(y|(y|(x|x)))) \geq \min\{I_N((((x|(x|(y|y))| \\ (x|(x|(y|y))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), I_N(z)\}, \\ F_N((y|(y|(x|x))|(y|(y|(x|x)))) \leq \max\{F_N((((x|(x|(y|y))| \\ (x|(x|(y|y))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), F_N(z)\} \end{array} \right). \quad (4)$$

Proposition 4.4. Every neutrosophic $\mathcal{N}$ -sub-implicative ideal of X is a neutrosophic $\mathcal{N}$ -ideal of X .

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . Then $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$,

$$\begin{aligned}
 T_N(x) &= T_N((x|(0|0))|(x|(0|0))) \\
 &= T_N((x|(x|(x|x))|(x|(x|(x|x)))) \\
 &\leq \max\{T_N((((x|(x|(x|x))|(x|(x|(x|x))))|(z|z))|(((x|(x|(x|x))| \\
 &\quad (x|(x|(x|x))|(z|z))), T_N(z)\} \\
 &= \max\{T_N((((x|(0|0))|(x|(0|0))|(z|z))|(((x|(0|0))|(x|(0|0))|(z|z))), T_N(z)\} \\
 &= \max\{T_N((x|(z|z))|(x|(z|z))), T_N(z)\},
 \end{aligned}$$

$$\begin{aligned}
 I_N(x) &= I_N((x|(0|0))|(x|(0|0))) \\
 &= I_N((x|(x|(x|x))|(x|(x|(x|x)))) \\
 &\geq \min\{I_N((((x|(x|(x|x))|(x|(x|(x|x))))|(z|z))|(((x|(x|(x|x))| \\
 &\quad (x|(x|(x|x))|(z|z))), I_N(z)\} \\
 &= \min\{I_N((((x|(0|0))|(x|(0|0))|(z|z))|(((x|(0|0))|(x|(0|0))|(z|z))), I_N(z)\} \\
 &= \min\{I_N((x|(z|z))|(x|(z|z))), I_N(z)\}
 \end{aligned}$$

and

$$\begin{aligned}
 F_N(x) &= F_N((x|(0|0))|(x|(0|0))) \\
 &= F_N((x|(x|(x|x))|(x|(x|(x|x)))) \\
 &\leq \max\{F_N((((x|(x|(x|x))|(x|(x|(x|x))))|(z|z))|(((x|(x|(x|x))|(x|(x|(x|x))))|(z|z))), F_N(z)\} \\
 &= \max\{F_N((((x|(0|0))|(x|(0|0))|(z|z))|(((x|(0|0))|(x|(0|0))|(z|z))), F_N(z)\} \\
 &= \max\{F_N((x|(z|z))|(x|(z|z))), F_N(z)\}.
 \end{aligned}$$

Therefore, X_N is a neutrosophic $\mathcal{N}$ -ideal of X . $\square$

Theorem 4.5. *Let X_N be a neutrosophic $\mathcal{N}$ -ideal of X . Then X_N is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X if and only if*

$$T_N((y|(y|(x|x))|(y|(y|(x|x)))) \leq T_N((x|(x|(y|y))|(x|(x|(y|y))))),$$

$$I_N((y|(y|(x|x))|(y|(y|(x|x)))) \geq I_N((x|(x|(y|y))|(x|(x|(y|y))))$$

and

$$F_N((y|(y|(x|x))|(y|(y|(x|x)))) \leq F_N((x|(x|(y|y))|(x|(x|(y|y))))).$$

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . We have

$$\begin{aligned}
 &T_N(y|(y|(x|x))|(y|(y|(x|x)))) \\
 &\leq \max\{T_N((((x|(x|(y|y))|(x|(x|(y|y))))|(0|0))|(((x|(x|(y|y))|(x|(x|(y|y))))|(0|0))), T_N(0)\} \\
 &= \max\{T_N(x|(x|(y|y))|(x|(x|(y|y)))), T_N(0)\} \\
 &= T_N(x|(x|(y|y))|(x|(x|(y|y)))),
 \end{aligned}$$

$$\begin{aligned}
 &I_N(y|(y|(x|x))|(y|(y|(x|x)))) \\
 &\geq \min\{I_N((((x|(x|(y|y))|(x|(x|(y|y))))|(0|0))|(((x|(x|(y|y))|(x|(x|(y|y))))|(0|0))), I_N(0)\} \\
 &= \min\{I_N(x|(x|(y|y))|(x|(x|(y|y)))), I_N(0)\} \\
 &= I_N(x|(x|(y|y))|(x|(x|(y|y))))
 \end{aligned}$$

and

$$\begin{aligned}
 &F_N(y|(y|(x|x))|(y|(y|(x|x)))) \\
 &\leq \max\{F_N((((x|(x|(y|y))|(x|(x|(y|y))))|(0|0))|(((x|(x|(y|y))|(x|(x|(y|y))))|(0|0))), F_N(0)\} \\
 &= \max\{F_N(x|(x|(y|y))|(x|(x|(y|y)))), F_N(0)\} \\
 &= F_N(x|(x|(y|y))|(x|(x|(y|y))))
 \end{aligned}$$

Conversely, since X_N is a neutrosophic $\mathcal{N}$ -ideal, $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and

$$F_N(0) \leq F_N(x)$$

$$T_N((y|(y|(x|x))|(y|(y|(x|x))))$$

$$\begin{aligned} &\leq T_N((x|(x|(y|y))|(x|(x|(y|y)))) \\ &\leq \max\{T_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z)|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), T_N(z)\}, \end{aligned}$$

$$I_N((y|(y|(x|x))|(y|(y|(x|x))))$$

$$\begin{aligned} &\geq I_N((x|(x|(y|y))|(x|(x|(y|y)))) \\ &\geq \min\{I_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z)|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), I_N(z)\} \end{aligned}$$

and

$$F_N((y|(y|(x|x))|(y|(y|(x|x))))$$

$$\begin{aligned} &\leq F_N((x|(x|(y|y))|(x|(x|(y|y)))) \\ &\leq \max\{F_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z)|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), F_N(z)\}. \end{aligned}$$

Therefore, X_N is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . $\square$

Theorem 4.6. *Every neutrosophic $\mathcal{N}$ -ideal of X is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X .*

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -ideal of X . Then $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$, $F_N(0) \leq F_N(x)$,

$$T_N((y|(y|(x|x))|(y|(y|(x|x))))$$

$$\begin{aligned} &\leq \max\{T_N(((y|(y|(x|x))|(y|(y|(x|x))))|(z|z)|((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))), T_N(z)\} \\ &= \max\{T_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z)|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), T_N(z)\}, \end{aligned}$$

$$I_N((y|(y|(x|x))|(y|(y|(x|x))))$$

$$\begin{aligned} &\geq \min\{I_N(((y|(y|(x|x))|(y|(y|(x|x))))|(z|z)|((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))), I_N(z)\} \\ &= \min\{I_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z)|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), I_N(z)\}, \end{aligned}$$

and

$$\begin{aligned}
 & F_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 & \leq \max\{F_N(((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))|((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))), F_N(z)\} \\
 & = \max\{F_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), F_N(z)\}.
 \end{aligned}$$

Thereby, X_N is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . $\square$

Lemma 4.7. *In X , the following property holds:*

$$((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))) = x|(x|(y|y)),$$

for all $x, y \in X$.

Theorem 4.8. *Let for any $x, y \in X$ satisfies the condition $x|(x|(y|y)) = y|y$. Then every neutrosophic $\mathcal{N}$ -ideal of X is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X .*

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$ -ideal of X . It is obtained that $T_N(0) \leq T_N(x)$,

$$I_N(0) \geq I_N(x), F_N(0) \leq F_N(x),$$

$$\begin{aligned}
 & T_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 & = T_N(x) \\
 & \leq \max\{T_N((x|(z|z))|(x|(z|z))), T_N(z)\} \\
 & = \max\{T_N(((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))|((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))), T_N(z)\} \\
 & = \max\{T_N((((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), T_N(z)\} \\
 & = \max\{T_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), T_N(z)\},
 \end{aligned}$$

$$\begin{aligned}
 & I_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 & = I_N(x) \\
 & \geq \min\{I_N((x|(z|z))|(x|(z|z))), I_N(z)\} \\
 & = \min\{I_N(((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))|((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))), I_N(z)\} \\
 & = \min\{I_N((((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), I_N(z)\} \\
 & = \min\{I_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), I_N(z)\}
 \end{aligned}$$

and

$$\begin{aligned}
 & F_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &= F_N(x) \\
 &\leq \max\{F_N((x|(z|z))|(x|(z|z))), F_N(z)\} \\
 &= \max\{F_N((((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))|(((y|(y|(x|x))|(y|(y|(x|x))))|(z|z))), F_N(z)\} \\
 &= \max\{F_N((((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), F_N(z)\} \\
 &= \max\{F_N((((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), F_N(z)\}.
 \end{aligned}$$

Hence, X_N is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . $\square$

Theorem 4.9. *If X satisfying*

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_N((y|(z|z))|(y|(z|z))) \leq T_N((((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))) \\ I_N((y|(z|z))|(y|(z|z))) \geq I_N((((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))) \\ F_N((y|(z|z))|(y|(z|z))) \leq F_N((((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))) \end{array} \right), \quad (5)$$

then every neutrosophic $\mathcal{N}$ -ideal of X is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X .

Proof. It is obtained that

$$\begin{aligned}
 & T_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &\leq T_N((((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), \\
 &= T_N((x|(x|(y|y))|(x|(x|(y|y)))), \\
 & I_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &\geq I_N((((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), \\
 &= I_N((x|(x|(y|y))|(x|(x|(y|y))))
 \end{aligned}$$

and

$$\begin{aligned}
 & F_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &\leq F_N((((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(((x|(x|(y|y))|(x|(x|(y|y))))|(y|(x|x))|(z|z))), \\
 &= F_N((x|(x|(y|y))|(x|(x|(y|y))))
 \end{aligned}$$

Thus, X_N is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . $\square$

Theorem 4.10. *Let X_N be a neutrosophic $\mathcal{N}$ -ideal of X . Then X_N is a neutrosophic $\mathcal{N}$ -implicative ideal of X if and only if X_N satisfies the following condition:*

$$(\forall x, y \in X) \begin{pmatrix} T_N(x) \leq T_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ I_N(x) \geq I_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ F_N(x) \leq F_N((x|(y|(x|x)))|(x|(y|(x|x)))) \end{pmatrix}. \quad (6)$$

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -implicative ideal of X . Then

$$\begin{aligned} T_N(x) &\leq \max\{T_N(0), T_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))|(((x|(y|(x|x)))| \\ &\quad (x|(y|(x|x))))|(0|0)))\} \\ &= \max\{T_N(0), T_N((x|(y|(x|x)))|(x|(y|(x|x))))\} \\ &= T_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ I_N(x) &\geq \min\{I_N(0), I_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))|(((x|(y|(x|x)))| \\ &\quad (x|(y|(x|x))))|(0|0)))\} \\ &= \min\{I_N(0), I_N((x|(y|(x|x)))|(x|(y|(x|x))))\} \\ &= I_N((x|(y|(x|x)))|(x|(y|(x|x)))) \end{aligned}$$

and

$$\begin{aligned} F_N(x) &\leq \max\{F_N(0), F_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))|(((x|(y|(x|x)))| \\ &\quad (x|(y|(x|x))))|(0|0)))\} \\ &= \max\{F_N(0), F_N((x|(y|(x|x)))|(x|(y|(x|x))))\} \\ &= F_N((x|(y|(x|x)))|(x|(y|(x|x)))) \end{aligned}$$

for all $x, y \in X$. Conversely, let X_N be a neutrosophic $\mathcal{N}$ -ideal of X satisfying the inequality (6). Then it is clear that $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$ for all $x \in X$.

Since

$$\begin{aligned} T_N(x) &\leq T_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ &\leq \max\{T_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))|(((x|(y|(x|x)))| \\ &\quad (x|(y|(x|x))))|(z|z))), T_N(z)\}, \\ I_N(x) &\geq I_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ &\geq \min\{I_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))|(((x|(y|(x|x)))| \\ &\quad (x|(y|(x|x))))|(z|z))), I_N(z)\} \end{aligned}$$

and

$$\begin{aligned} F_N(x) &\leq F_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ &\leq \max\{F_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))|(((x|(y|(x|x)))| \\ &\quad (x|(y|(x|x))))|(z|z))), F_N(z)\}, \end{aligned}$$

for all $x, y, z \in A$, we have X_N is a neutrosophic $\mathcal{N}$ -implicative ideal of X . $\square$

Theorem 4.11. *If X satisfying*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N((x|(x|(y|y)))|(x|(x|(y|y)))) \leq T_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ I_N((x|(x|(y|y)))|(x|(x|(y|y)))) \geq I_N((x|(y|(x|x)))|(x|(y|(x|x)))) \\ F_N((x|(x|(y|y)))|(x|(x|(y|y)))) \leq F_N((x|(y|(x|x)))|(x|(y|(x|x)))) \end{array} \right), \quad (7)$$

then every neutrosophic $\mathcal{N}$ -sub-implicative ideal of X is a neutrosophic $\mathcal{N}$ -implicative ideal of X .

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X satisfying the inequality (7). Then we obtain that

$$\begin{aligned} T_N(x) &= T_N((y|(y|(x|x)))|(y|(y|(x|x)))) \\ &\leq \max\{T_N(((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))| \\ &\quad (((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))), T_N(0)\} \\ &= \max\{T_N((x|(x|(y|y)))|(x|(x|(y|y))))|, T_N(0)\} \\ &= T_N((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\leq T_N((x|(y|(x|x)))|(x|(y|(x|x)))) \end{aligned}$$

$$\begin{aligned} I_N(x) &= I_N((y|(y|(x|x)))|(y|(y|(x|x)))) \\ &\geq \min\{I_N(((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))| \\ &\quad (((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))), I_N(0)\} \\ &= \min\{I_N((x|(x|(y|y)))|(x|(x|(y|y))))|, I_N(0)\} \\ &= I_N((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\geq I_N((x|(y|(x|x)))|(x|(y|(x|x)))) \end{aligned}$$

and

$$\begin{aligned} F_N(x) &= F_N((y|(y|(x|x)))|(y|(y|(x|x)))) \\ &\leq \max\{F_N(((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))| \\ &\quad (((x|(x|(y|y)))|(x|(x|(y|y))))|(0|0))), F_N(0)\} \\ &= \max\{F_N((x|(x|(y|y)))|(x|(x|(y|y))))|, F_N(0)\} \\ &= F_N((x|(x|(y|y)))|(x|(x|(y|y)))) \\ &\leq F_N((x|(y|(x|x)))|(x|(y|(x|x)))). \end{aligned}$$

Thus, X_N is a neutrosophic $\mathcal{N}$ -implicative ideal of X . $\square$

Theorem 4.12. *Let X be an implicative algebra. Then every neutrosophic $\mathcal{N}$ -implicative ideal of X is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X .*

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -implicative ideal of an implicative algebra X . Then X_N is a neutrosophic $\mathcal{N}$ -ideal of X . So, it is obvious that $T_N(0) \leq T_N(x)$, $I_N(0) \geq I_N(x)$ and

$F_N(0) \leq F_N(x)$ for all $x \in X$. Thus

$$\begin{aligned}
 & T_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &= T_N((x|(x|(y|y))|(x|(x|(y|y)))) \\
 &\leq \max\{T_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), T_N(z)\}, \\
 & I_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &= I_N((x|(x|(y|y))|(x|(x|(y|y)))) \\
 &\geq \min\{I_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), I_N(z)\}
 \end{aligned}$$

and

$$\begin{aligned}
 & F_N((y|(y|(x|x))|(y|(y|(x|x)))) \\
 &= F_N((x|(x|(y|y))|(x|(x|(y|y)))) \\
 &\leq \max\{F_N(((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))|((x|(x|(y|y))|(x|(x|(y|y))))|(z|z))), F_N(z)\},
 \end{aligned}$$

for all $x, y, z \in X$. Hence X_N is a neutrosophic $\mathcal{N}$ -sub-implicative ideal of X . $\square$

Definition 4.13. A neutrosophic $\mathcal{N}$ -ideal X_N of X is said to be neutrosophic $\mathcal{N}$ -closed if

$$(\forall x \in X) \left(\begin{array}{l} T_N((0|(x|x))|(0|(x|x))) \leq T_N(x) \\ I_N((0|(x|x))|(0|(x|x))) \geq I_N(x) \\ F_N((0|(x|x))|(0|(x|x))) \leq F_N(x) \end{array} \right). \quad (8)$$

Definition 4.14. Let X_N be a neutrosophic $\mathcal{N}$ -ideal of X . Then X_N is called a neutrosophic $\mathcal{N}$ -completely closed ideal of X if

$$(\forall x, y \in X) \left(\begin{array}{l} T_N((x|(y|y))|(x|(y|y))) \leq \max\{T_N(x), T_N(y)\}, \\ I_N((x|(y|y))|(x|(y|y))) \geq \min\{I_N(x), I_N(y)\}, \\ F_N((x|(y|y))|(x|(y|y))) \leq \max\{F_N(x), F_N(y)\} \end{array} \right). \quad (9)$$

Theorem 4.15. If X satisfying the following condition

$$(\forall x, y \in X) \left(\begin{array}{l} (((x|(y|y))|(x|(y|y))|(x|(z|z))|((x|(y|y))|(x|(y|y)))) \\ (x|(z|z)))|(z|(y|y)) = 0|0. \end{array} \right), \quad (10)$$

then X is implicative if and only if every neutrosophic $\mathcal{N}$ -closed ideal of X is a neutrosophic $\mathcal{N}$ -implicative ideal of X .

Proof. Let X be satisfying (10). Assume that X is implicative and X_N is a neutrosophic $\mathcal{N}$ -closed ideal of X . Then X_N is a neutrosophic $\mathcal{N}$ -ideal of X . Thus, $T_N(0) \leq T_N(x)$,

$I_N(0) \geq I_N(x)$ and $F_N(0) \leq F_N(x)$. Also

$$\begin{aligned}
 T_N(x) &\leq \max\{T_N(z), T_N((x|(z|z))|(x|(z|z)))\} \\
 &= \max\{T_N(z), T_N((((x|x)|(x|x))|((x|x)|y))|(z|z))| \\
 &\quad (((x|x)|(x|x))|((x|x)|y))|(z|z)))\} \\
 &= \max\{T_N(z), T_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\
 &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z)))\}, \\
 I_N(x) &\geq \min\{I_N(z), I_N((x|(z|z))|(x|(z|z)))\} \\
 &= \min\{I_N(z), I_N((((x|x)|(x|x))|((x|x)|y))|(z|z))| \\
 &\quad (((x|x)|(x|x))|((x|x)|y))|(z|z)))\} \\
 &= \min\{I_N(z), I_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\
 &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z)))\}
 \end{aligned}$$

and

$$\begin{aligned}
 F_N(x) &\leq \max\{F_N(z), F_N((x|(z|z))|(x|(z|z)))\} \\
 &= \max\{F_N(z), F_N((((x|x)|(x|x))|((x|x)|y))|(z|z))| \\
 &\quad (((x|x)|(x|x))|((x|x)|y))|(z|z)))\} \\
 &= \max\{F_N(z), F_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z))| \\
 &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(z|z)))\},
 \end{aligned}$$

which means that X_N is a neutrosophic $\mathcal{N}$ -implicative ideal of X . Conversely, suppose that every neutrosophic $\mathcal{N}$ -closed ideal of X is a neutrosophic $\mathcal{N}$ -implicative ideal of X . So, it follows from the equation (10) that $z|(y|y) = (x|(z|z))|((x|(y|y))|(x|(y|y)))$. Since $z|(y|y) = (x|(z|z))|((x|(y|y))|(x|(y|y))) = ((x|(x|(z|z))|(x|(x|(z|z))))|(y|y))$, $z = (x|(x|(z|z))|(x|(x|(z|z))))$. Thus,

$$\begin{aligned}
 x|(x|(y|y)) &= ((y|(y|(x|x)))|(y|(y|(x|x))))|(((y|(y|(x|x))|(y|(y|(x|x))))|(y|y)) \\
 &= ((y|(y|(x|x)))|(y|(y|(x|x))))|(y|(((y|y)|(y|(x|x))|(y|y))|(y|(x|x)))) \\
 &= ((y|(y|(x|x)))|(y|(y|(x|x))))|(y|(y|y)) \\
 &= (((y|(y|y))|(y|y))|(y|y))|(((y|(y|y))|(y|y))|(y|y))|(y|(x|x)) \\
 &= y|(y|(x|x)),
 \end{aligned}$$

for all $x, y \in X$, which means that X is implicative. $\square$

Proposition 4.16. *Let A be an implicative algebra satisfying the equation (10). Then every neutrosophic $\mathcal{N}$ -completely closed ideal of A is a neutrosophic $\mathcal{N}$ -implicative ideal of A .*

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -completely closed ideal of an implicative algebra X . Then X_N is a neutrosophic $\mathcal{N}$ -ideal of X . Since

$$T_N((0|(y|y))|(0|(y|y))) \leq \max\{T_N(0), T_N(y)\} = T_N(y),$$

$$I_N((0|(y|y))|(0|(y|y))) \geq \min\{I_N(0), I_N(y)\} = I_N(y)$$

and

$$F_N((0|(y|y))|(0|(y|y))) \leq \max\{F_N(0), F_N(y)\} = F_N(y),$$

it is obtained that X_N is a neutrosophic $\mathcal{N}$ -closed ideal of X . Therefore, X_N is a neutrosophic $\mathcal{N}$ -implicative ideal of X . $\square$

Definition 4.17. A neutrosophic $\mathcal{N}$ -structure X_N of X is called a neutrosophic $\mathcal{N}$ - p -ideal of X if

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(0) \leq T_N(x), I_N(0) \geq I_N(x), F_N(0) \leq F_N(x) \\ T_N(x) \leq \max\{T_N(((x|(z|z))|(x|(z|z))|(y|(z|z)))| \\ ((x|(z|z))|(x|(z|z))|(y|(z|z))), T_N(y)\} \\ I_N(x) \geq \min\{I_N(((x|(z|z))|(x|(z|z))|(y|(z|z)))| \\ ((x|(z|z))|(x|(z|z))|(y|(z|z))), I_N(y)\} \\ F_N(x) \leq \max\{F_N(((x|(z|z))|(x|(z|z))|(y|(z|z)))| \\ ((x|(z|z))|(x|(z|z))|(y|(z|z))), F_N(y)\} \end{pmatrix}. \quad (11)$$

Definition 4.18. The subset $X^+ = \{x \in X : (0|(x|x))|(0|(x|x)) = 0\}$ of X is called the BCA-part of X .

Theorem 4.19. If $X = X^+$, then every neutrosophic $\mathcal{N}$ - p -ideal of X is a neutrosophic $\mathcal{N}$ -implicative ideal of X .

Proof. Let X_N be a neutrosophic $\mathcal{N}$ - p -ideal of X . Then

$$\begin{aligned} T_N(x) &\leq \max\{T_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))| \\ &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))), T_N(0)\} \\ &= \max\{T_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))| \\ &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))), T_N(0)\} \\ &= \max\{T_N((x|(y|(x|x))|(x|(y|(x|x))))), T_N(0)\}, \\ &= T_N((x|(y|(x|x))|(x|(y|(x|x))))), \end{aligned}$$

$$\begin{aligned} I_N(x) &\geq \min\{I_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))| \\ &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))), I_N(0)\} \\ &= \min\{I_N(((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))| \\ &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))), I_N(0)\} \\ &= \min\{I_N((x|(y|(x|x))|(x|(y|(x|x))))), I_N(0)\} \\ &= I_N((x|(y|(x|x))|(x|(y|(x|x))))), \end{aligned}$$

and

$$\begin{aligned}
 F_N(x) &\leq \max\{F_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))| \\
 &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|(y|(x|x))))), F_N(0)\} \\
 &= \max\{F_N((((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))| \\
 &\quad (((x|(y|(x|x)))|(x|(y|(x|x))))|(0|0))), F_N(0)\} \\
 &= \max\{F_N((x|(y|(x|x)))|(x|(y|(x|x)))), F_N(0)\} \\
 &= F_N((x|(y|(x|x)))|(x|(y|(x|x)))).
 \end{aligned}$$

Hence X_N is a neutrosophic $\mathcal{N}$ -implicative ideal of X . $\square$

5. Homomorphism on Sheffer stroke BG-algebras

Theorem 5.1. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras, $f : A \rightarrow B$ be a surjective homomorphism and $B_N = (T_N, I_N, F_N)$ be a neutrosophic $\mathcal{N}$ -structure on B . Then B_N is a neutrosophic $\mathcal{N}$ -ideal of B if and only if $B_N^f = (T_N^f, I_N^f, F_N^f)$ is a neutrosophic $\mathcal{N}$ -ideal of A , where the $\mathcal{N}$ -function $T_N^f, I_N^f, F_N^f \rightarrow [-1, 0]$ on A are defined by $T_N^f = T_N(f(x))$, $I_N^f = I_N(f(x))$, $F_N^f = F_N(f(x))$ for all $x \in A$, respectively.

Proof. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras, $f : A \rightarrow B$ be a surjective homomorphism and X_N be a neutrosophic $\mathcal{N}$ -ideal of B . Let $x_1, x_2, x_3 \in A$. Then

$$T_N^\alpha(0_A) = T_N(\alpha(0_A)) \leq T_N(\alpha(x_1)) = T_N^\alpha(x_1),$$

$$I_N^\alpha(0_A) = I_N(\alpha(0_A)) \geq I_N(\alpha(x_1)) = I_N^\alpha(x_1)$$

and

$$F_N^\alpha(0_A) = F_N(\alpha(0_A)) \leq F_N(\alpha(x_1)) = F_N^\alpha(x_1),$$

$$\begin{aligned}
 T_N^\alpha(x_1) &= T_N(\alpha(x_1)) \\
 &\leq \max\{T_N(\alpha(x_2)), T_N((\alpha(x_1)|_B(\alpha(x_2)|_B\alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B\alpha(x_2))))\} \\
 &= \max\{T_N(\alpha(x_2)), I_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))\} \\
 &= \max\{T_N^\alpha(x_2), T_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))\}
 \end{aligned}$$

and

$$\begin{aligned}
 I_N^\alpha(x_1) &= I_N(\alpha(x_1)) \\
 &\geq \min\{I_N(\alpha(x_2)), I_N((\alpha(x_1)|_B(\alpha(x_2)|_B\alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B\alpha(x_2))))\} \\
 &= \min\{I_N(\alpha(x_2)), I_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))\} \\
 &= \min\{I_N^\alpha(x_2), I_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))\},
 \end{aligned}$$

$$\begin{aligned}
 F_N^\alpha(x_1) &= F_N(\alpha(x_1)) \\
 &\leq \max\{F_N(\alpha(x_2)), F_N((\alpha(x_1)|_B(\alpha(x_2)|_B\alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B\alpha(x_2))))\} \\
 &= \max\{F_N(\alpha(x_2)), F_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))\} \\
 &= \max\{F_N^\alpha(x_2), F_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))\}.
 \end{aligned}$$

Hence f^α is a neutrosophic $\mathcal{N}$ -ideal of A .

Conversely, let f^α be a neutrosophic $\mathcal{N}$ -ideal of A . Let $y_1, y_2 \in B$ such that $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$ for $x_1, x_2 \in A$. Then

$$T_N(0_B) = T_N(\alpha(0_A)) \leq T_N^\alpha(0_A) = T_N(\alpha(x_1)) = T_N(y_1),$$

$$I_N(0_B) = I_N(\alpha(0_A)) \geq I_N^\alpha(x_1) = I_N(\alpha(x_1)) = I_N(y_1)$$

and

$$F_N(0_B) = F_N(\alpha(0_A)) \leq F_N^\alpha(0_A) = F_N(\alpha(x_1)) = F_N(y_1),$$

Furthermore, we have

$$\begin{aligned} T_N(y_1) &= T_N(\alpha(x_1)) \\ &= T_N^\alpha(x_1) \\ &\leq \max\{T_N^\alpha(x_2), T_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))\} \\ &= \max\{T_N(\alpha(x_2)), T_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))\} \\ &= \max\{T_N(\alpha(x_2)), T_N((\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))))\} \\ &= \max\{T_N(y_2), T_N((y_1|_B(y_2|_B y_2))|_B(y_1|_B(y_2|_B y_2)))\}, \end{aligned}$$

$$\begin{aligned} I_N(y_1) &= I_N(\alpha(x_1)) \\ &= I_N^\alpha(x_1) \\ &\geq \min\{I_N^\alpha(x_2), I_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))\} \\ &= \min\{I_N(\alpha(x_2)), I_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))\} \\ &= \min\{I_N(\alpha(x_2)), I_N((\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))))\} \\ &= \min\{I_N(y_2), I_N((y_1|_B(y_2|_B y_2))|_B(y_1|_B(y_2|_B y_2)))\} \end{aligned}$$

and

$$\begin{aligned} F_N(y_1) &= F_N(\alpha(x_1)) \\ &= F_N^\alpha(x_1) \\ &\leq \max\{F_N^\alpha(x_2), F_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))\} \\ &= \max\{F_N(\alpha(x_2)), F_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))))\} \\ &= \max\{F_N(\alpha(x_2)), F_N((\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))))\} \\ &= \max\{F_N(y_2), F_N((y_1|_B(y_2|_B y_2))|_B(y_1|_B(y_2|_B y_2)))\}. \end{aligned}$$

Hence X_N is a neutrosophic $\mathcal{N}$ -ideal of B . $\square$

Theorem 5.2. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras, $f : A \rightarrow B$ be a surjective homomorphism and X_N be a neutrosophic $\mathcal{N}$ -structure on B . Then X_N is a neutrosophic $\mathcal{N}$ -subalgebra of B if and only if f^α is a neutrosophic $\mathcal{N}$ -subalgebra of A , where $f^\alpha \rightarrow [-1, 0]$ on A are defined by $f^\alpha = f(\alpha(x))$ for all $x \in A$, respectively.

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Proof. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras, $f : A \rightarrow B$ be a surjective homomorphism and X_N be a neutrosophic $\mathcal{N}$ -subalgebra of B . Let $x_1, x_2 \in A$. Then

$$\begin{aligned} T_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))) \\ &= T_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))) \\ &= T_N(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))) \\ &\leq \max\{T_N(\alpha(x_1)), T_N(\alpha(x_2))\} \\ &= \max\{T_N^\alpha(x_1), T_N^\alpha(x_2)\}, \end{aligned}$$

$$\begin{aligned} I_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))) \\ &= I_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))) \\ &= I_N(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))) \\ &\geq \min\{I_N(\alpha(x_1)), I_N(\alpha(x_2))\} \\ &= \min\{I_N^\alpha(x_1), I_N^\alpha(x_2)\} \end{aligned}$$

and

$$\begin{aligned} F_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))) \\ &= F_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))) \\ &= F_N(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))) \\ &\leq \max\{F_N(\alpha(x_1)), F_N(\alpha(x_2))\} \\ &= \max\{F_N^\alpha(x_1), F_N^\alpha(x_2)\}. \end{aligned}$$

Hence f^α is a neutrosophic $\mathcal{N}$ -subalgebra of A .

Conversely, let f^f be a neutrosophic $\mathcal{N}$ -subalgebra of A . Let $y_1, y_2 \in B$ such that $\alpha(x_1) = y_1$ and $\alpha(x_2) = y_2$ for $x_1, x_2 \in A$. Then

$$\begin{aligned} T_N((y_1|_B(y_2|_B y_2))|_B(y_1|_B(y_2|_B y_2))) \\ &= T_N((\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))) \\ &= T_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))) \\ &= T_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))) \\ &\leq \max\{T_N^\alpha(x_1), T_N^\alpha(x_2)\} \\ &= \max\{T_N(\alpha(x_1)), T_N(\alpha(x_2))\} \\ &= \max\{T_N(y_1), T_N(y_2)\}, \end{aligned}$$

$$\begin{aligned} I_N((y_1|_B(y_2|_B y_2))|_B(y_1|_B(y_2|_B y_2))) \\ &= I_N((\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))) \\ &= I_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))) \\ &= I_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))) \\ &\geq \min\{I_N^\alpha(x_1), I_N^\alpha(x_2)\} \\ &= \min\{I_N(\alpha(x_1)), I_N(\alpha(x_2))\} \\ &= \min\{I_N(y_1), I_N(y_2)\} \end{aligned}$$

and

$$\begin{aligned}
 & F_N((y_1|_B(y_2|_B y_2))|_B(y_1|_B(y_2|_B y_2))) \\
 &= F_N((\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))|_B(\alpha(x_1)|_B(\alpha(x_2)|_B \alpha(x_2)))) \\
 &= F_N(\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2)))) \\
 &= F_N^\alpha((x_1|_A(x_2|_A x_2))|_A(x_1|_A(x_2|_A x_2))) \\
 &\leq \max\{F_N^\alpha(x_1), F_N^\alpha(x_2)\} \\
 &= \max\{F_N(\alpha(x_1)), F_N(\alpha(x_2))\} \\
 &= \max\{F_N(y_1), F_N(y_2)\}.
 \end{aligned}$$

Hence X_N is a neutrosophic $\mathcal{N}$ -subalgebra of B . $\square$

Definition 5.3. Let X be a Sheffer stroke BG-algebra. We define

$$X_N^{x_t} = \{x \in X : T_N(x) \leq T_N(x_t)\},$$

$$X_N^{x_i} = \{x \in X : I_N(x) \geq I_N(x_t)\},$$

$$X_N^{x_f} = \{x \in X : F_N(x) \leq F_N(x_t)\},$$

for all $x_t, x_i, x_f \in X$. Obviously, $x_t \in X_N^{x_t}$, $x_i \in X_N^{x_i}$ and $x_f \in X_N^{x_f}$.

Theorem 5.4. If X_N is a neutrosophic $\mathcal{N}$ -ideal of X , then $X_N^{x_t}$, $X_N^{x_i}$ and $X_N^{x_f}$ are ideals of X , where $x_t, x_i, x_f \in X$.

Proof. Assume that X_N is a neutrosophic $\mathcal{N}$ -ideal of X . Let $y_1 \in X_N^{x_t}$, $y_2 \in X_N^{x_i}$ and $y_3 \in X_N^{x_f}$. Hence $T_N(y_1) \leq T_N(x_t)$, $I_N(y_2) \geq I_N(x_t)$ and $F_N(y_3) \leq F_N(x_t)$. This shows that $T_N((y_1|(x_1|x_1))|(y_1|(x_1|x_1))) \leq T_N(y_1) \leq T_N(x_t)$, $I_N((y_2|(x_2|x_2))|(y_2|(x_2|x_2))) \geq I_N(y_2) \geq I_N(x_t)$ and $F_N((y_3|(x_3|x_3))|(y_3|(x_3|x_3))) \leq F_N(y_3) \leq F_N(x_t)$. Thus, $(y_1|(x_1|x_1))|(y_1|(x_1|x_1)) \in T_N(x_t)$, $(y_2|(x_2|x_2))|(y_2|(x_2|x_2)) \in I_N(x_i)$ and $(y_3|(x_3|x_3))|(y_3|(x_3|x_3)) \in F_N(x_f)$. Let $x_1, (y_1|(x_1|x_1))|(y_1|(x_1|x_1)) \in X_N^{x_t}$, $x_2, (y_2|(x_2|x_2))|(y_2|(x_2|x_2)) \in X_N^{x_i}$ and $x_3, (y_3|(x_3|x_3))|(y_3|(x_3|x_3)) \in X_N^{x_f}$. Then $T_N(x_1) \leq T_N(x_t)$, $T_N((y_1|(x_1|x_1))|(y_1|(x_1|x_1))) \leq T_N(x_t)$, $I_N(x_2) \geq I_N(x_i)$, $I_N((y_2|(x_2|x_2))|(y_2|(x_2|x_2))) \geq I_N(x_t)$, $F_N(x_3) \leq F_N(x_f)$ and $F_N((y_3|(x_3|x_3))|(y_3|(x_3|x_3))) \leq F_N(x_t)$. Then

$$T_N(y_1) \leq \max\{T_N(x_1), T_N((y_1|(x_1|x_1))|(y_1|(x_1|x_1)))\} \leq T_N(x_t),$$

$$I_N(y_2) \geq \min\{I_N(x_2), I_N((y_2|(x_2|x_2))|(y_2|(x_2|x_2)))\} \geq I_N(x_i)$$

and

$$F_N(y_3) \leq \max\{F_N(x_3), F_N((y_3|(x_3|x_3))|(y_3|(x_3|x_3)))\} \leq F_N(x_t).$$

Hence $y_1 \in T_N(x_t)$, $y_2 \in I_N(x_i)$ and $y_3 \in F_N(x_f)$. Therefore, $X_N^{x_t}$, $X_N^{x_i}$ and $X_N^{x_f}$ are ideals of X . $\square$

Theorem 5.5. If X_N is a neutrosophic $\mathcal{N}$ -subalgebra of X , then $X_N^{x_t}$, $X_N^{x_i}$ and $X_N^{x_f}$ are subalgebras of X , where $x_t, x_i, x_f \in X$.

Proof. Let X_N be a neutrosophic $\mathcal{N}$ -subalgebra of X . Let $x_1, y_1 \in X_N^{x_t}$, $x_2, y_2 \in X_N^{x_i}$ and $x_3, y_3 \in X_N^{x_f}$. Then $T_N(x_1) \leq T_N(x_t)$, $T_N(y_1) \leq T_N(x_t)$, $I_N(y_1) \geq I_N(x_t)$, $I_N(y_2) \geq I_N(x_t)$, $F_N(x_3) \leq F_N(x_t)$ and $F_N(y_3) \leq F_N(x_t)$. Then

$$T_N(x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \leq \max\{T_N(x_1), T_N(y_1)\} \leq T_N(x_t),$$

$$I_N(x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \geq \min\{I_N(x_2), I_N(y_2)\} \geq I_N(x_t)$$

and

$$F_N(x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \leq \max\{F_N(x_3), F_N(y_3)\} \leq F_N(x_t).$$

It follows that $(x_1|(y_1|y_1))|(x_1|(y_1|y_1))) \in T_N(x_t)$, $(x_2|(y_2|y_2))|(x_2|(y_2|y_2))) \in I_N(x_i)$ and $(x_3|(y_3|y_3))|(x_3|(y_3|y_3))) \in F_N(x_f)$. Therefore, $X_N^{x_t}$, $X_N^{x_i}$ and $X_N^{x_f}$ are subalgebras of X . $\square$

Theorem 5.6. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras. If X_{A_N} and X_{B_N} are neutrosophic $\mathcal{N}$ -subalgebra of $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$, respectively, then $X_{A \times B_N}$ is a neutrosophic $\mathcal{N}$ -subalgebra of $\langle A \times B, |_{A \times B}, 0_{A \times B} \rangle$.

Proof. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras and X_{A_N} and X_{B_N} be neutrosophic $\mathcal{N}$ -subalgebra of $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$, respectively. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then

$$T_{A \times B_N}(((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))))|_{A \times B}((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))))$$

$$\begin{aligned} &= T_{A \times B_N}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))) \\ &= \max\{T_{A_N}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2))), T_{B_N}(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))\} \\ &\leq \max\{\max\{T_{A_N}(a_1), T_{A_N}(a_2)\}, \max\{T_{B_N}(b_1), T_{B_N}(b_2)\}\} \\ &= \max\{\max\{T_{A_N}(a_1), T_{B_N}(b_1)\}, \max\{T_{A_N}(a_2), T_{B_N}(b_2)\}\} \\ &= \max\{T_{A \times B_N}(a_1, b_1), T_{A \times B_N}(a_2, b_2)\}, \end{aligned}$$

$$I_{A \times B_N}(((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))))|_{A \times B}((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2))))$$

$$\begin{aligned} &= I_{A \times B_N}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2)))) \\ &= \max\{I_{A_N}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2))), I_{B_N}(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))\} \\ &\geq \max\{\min\{I_{A_N}(a_1), I_{A_N}(a_2)\}, \min\{I_{B_N}(b_1), I_{B_N}(b_2)\}\} \\ &= \max\{\min\{I_{A_N}(a_1), I_{B_N}(b_1)\}, \min\{I_{A_N}(a_2), I_{B_N}(b_2)\}\} \\ &= \max\{I_{A \times B_N}(a_1, b_1), I_{A \times B_N}(a_2, b_2)\} \end{aligned}$$

and

$$\begin{aligned}
 & F_{A \times B_N}(((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)))|_{A \times B}((a_1, b_1)|_{A \times B}((a_2, b_2)|_{A \times B}(a_2, b_2)))) \\
 &= F_{A \times B_N}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2)), (b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))) \\
 &= \max\{F_{A_N}((a_1|_A(a_2|_A a_2))|_A(a_1|_A(a_2|_A a_2))), F_{B_N}(b_1|_B(b_2|_B b_2))|_B(b_1|_B(b_2|_B b_2))\} \\
 &\leq \max\{\max\{F_{A_N}(a_1), F_{A_N}(a_2)\}, \max\{F_{B_N}(b_1), F_{B_N}(b_2)\}\} \\
 &= \max\{\max\{F_{A_N}(a_1), F_{B_N}(b_1)\}, \max\{F_{A_N}(a_2), F_{B_N}(b_2)\}\} \\
 &= \max\{F_{A \times B_N}(a_1, b_1), F_{A \times B_N}(a_2, b_2)\}.
 \end{aligned}$$

Hence $X_{A \times B_N}$ is a neutrosophic $\mathcal{N}$ -subalgebra of $\langle A \times B, |_{A \times B}, 0_{A \times B} \rangle$. $\square$

Theorem 5.7. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras. If X_{A_N} and X_{B_N} are neutrosophic $\mathcal{N}$ -ideal of $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$, respectively, then $X_{A \times B_N}$ is a neutrosophic $\mathcal{N}$ -ideal of $\langle A \times B, |_{A \times B}, 0_{A \times B} \rangle$.

Proof. Let $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$ be Sheffer stroke BG-algebras and X_{A_N} and X_{B_N} be neutrosophic $\mathcal{N}$ -ideal of $\langle A, |_A, 0_A \rangle$ and $\langle B, |_B, 0_B \rangle$, respectively. Let $(a_1, b_1), (a_2, b_2) \in A \times B$. Then

$$\begin{aligned}
 & T_{A \times B_N}(((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))) \\
 &= \max\{T_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1))), T_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\} \\
 &\leq \max\{T_{A_N}(a_2), T_{B_N}(b_2)\} \\
 &= T_{A \times B_N}(a_2, b_2),
 \end{aligned}$$

$$\begin{aligned}
 & I_{A \times B_N}(((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))) \\
 &= \max\{I_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1))), I_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\} \\
 &\geq \max\{I_{A_N}(a_2), I_{B_N}(b_2)\} \\
 &= I_{A \times B_N}(a_2, b_2),
 \end{aligned}$$

$$\begin{aligned}
 & F_{A \times B_N}(((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))) \\
 &= \max\{F_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1))), F_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\} \\
 &\leq \max\{F_{A_N}(a_2), F_{B_N}(b_2)\} \\
 &= F_{A \times B_N}(a_2, b_2),
 \end{aligned}$$

$$\begin{aligned}
 & T_{A \times B_N}(a_2, b_2) \\
 &= \max\{T_{A_N}(a_2), T_{B_N}(b_2)\} \\
 &\leq \max\{\max\{T_{A_N}(a_1), T_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1)))\}, \\
 &\quad \max\{T_{B_N}(b_1), T_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\}\} \\
 &= \max\{\max\{T_{A_N}(a_1), T_{B_N}(b_1)\}, \max\{T_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1))), \\
 &\quad T_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\}\} \\
 &= \max\{T_{A \times B_N}(a_1, b_1), \\
 &\quad T_{A \times B_N}(((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1))))\},
 \end{aligned}$$

$$\begin{aligned}
I_{A \times B_N}(a_2, b_2) &= \max\{I_{A_N}(a_2), I_{B_N}(b_2)\} \\
&\geq \max\{\min\{I_{A_N}(a_1), I_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1)))\}, \\
&\quad \min\{I_{B_N}(b_1), I_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\}\} \\
&= \max\{\min\{I_{A_N}(a_1), I_{B_N}(b_1)\}, \min\{I_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1))), \\
&\quad I_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\}\} \\
&= \max\{I_{A \times B_N}(a_1, b_1), \\
&\quad I_{A \times B_N}(((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1))))\}
\end{aligned}$$

and

$$\begin{aligned}
F_{A \times B_N}(a_2, b_2) &= \max\{F_{A_N}(a_2), F_{B_N}(b_2)\} \\
&\leq \max\{\max\{F_{A_N}(a_1), F_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1)))\}, \\
&\quad \max\{F_{B_N}(b_1), F_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\}\} \\
&= \max\{\max\{F_{A_N}(a_1), F_{B_N}(b_1)\}, \max\{F_{A_N}((a_2|_A(a_1|_A a_1))|_A(a_2|_A(a_1|_A a_1))), \\
&\quad F_{B_N}((b_2|_B(b_1|_B b_1))|_B(b_2|_B(b_1|_B b_1)))\}\} \\
&= \max\{F_{A \times B_N}(a_1, b_1), \\
&\quad F_{A \times B_N}(((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1)))|_{A \times B}((a_2, b_2)|_{A \times B}((a_1, b_1)|_{A \times B}(a_1, b_1))))\}.
\end{aligned}$$

Hence $X_{A \times B_N}$ is a neutrosophic $\mathcal{N}$ -ideal of $\langle A \times B, |_{A \times B}, 0_{A \times B} \rangle$. $\square$

6. Conclusion

In this study, we introduced and examined the concepts of neutrosophic $\mathcal{N}$ -subalgebras and level sets within the framework of Sheffer stroke BG-algebras. We demonstrated that there exists a close relationship between subalgebras and level sets in Sheffer stroke BG-algebras, proving that a level set of a neutrosophic $\mathcal{N}$ -subalgebra is itself a subalgebra, and conversely, any subalgebra can be viewed as a level set. This connection emphasizes the structural consistency within this algebraic setting and provides a foundation for further exploration of neutrosophic logic in algebraic systems.

Additionally, our results established that all neutrosophic $\mathcal{N}$ -subalgebras of a Sheffer stroke BG-algebra form a complete distributive lattice, which supports systematic analysis and applications in logical and computational models. We also defined neutrosophic $\mathcal{N}$ -ideals within this framework and highlighted their unique properties, showing that while every neutrosophic $\mathcal{N}$ -ideal is a neutrosophic $\mathcal{N}$ -subalgebra, the reverse does not necessarily hold. This insight reveals the specific behaviors and structural distinctions of $\mathcal{N}$ -ideals, contributing to a more nuanced understanding of Sheffer stroke BG-algebras.

Building on this foundation, future studies could explore the practical applications of Sheffer stroke BG-algebras in fields such as quantum computing and artificial intelligence, where logical

operations are crucial. Additionally, the incorporation of advanced neutrosophic structures, such as neutrosophic $\mathcal{N}$ -implicative ideals and neutrosophic $\mathcal{N}$ -closed ideals, could provide richer frameworks for modeling uncertainty and partial truth in computational systems. Another promising direction is the development of algorithms and decision-making frameworks that utilize the unique properties of Sheffer stroke operations, potentially opening new avenues in computational logic and information processing.

This study not only advances the theoretical understanding of neutrosophic structures in Sheffer stroke BG-algebras but also sets the stage for diverse applications and further theoretical development in algebraic logic and computational fields.

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A Customer-Driven Supplier Evaluation Model Under Indeterminacy for Customized Production

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Abstract

Due to the developing market conditions, changing customer expectations, and shortening of product life cycles, it is inevitable for companies to produce high-quality products that will meet customer requirements. Making the right decisions on issues such as working with right supplier, planning production correctly, ensuring effective stock management, and managing the cash flow are some of the factors that affect and change the expectations of the customers. In this study, hybrid quality function deployment (QFD) and analytical hierarchy process (AHP) methodology were used, focusing on supplier selection problems. The weights of the quality requirements (QRs) of the customers were determined by the neutrosophic AHP approach, and then the quality characteristics (QCs) were prioritized in the quality house, considering the relationship between the QRs and QCs. Finally, alternative suppliers were evaluated in terms of QCs using the NAHP method and the most suitable supplier was determined as the Austrian origin company.

Keywords: AHP; Furniture industry; Neutrosophic fuzzy sets; QFD; Supplier selection.

1. Introduction

With the changing and developing market conditions, more efficient use of existing resources, creation of alternative resources, and reaching the quality level expected by the customer requirements are achieved by giving importance to supplier selection. With the increase in competition, many businesses have to supply products that will meet the customer requirements in the most appropriate quality with the most optimum cost to maintain their existence. Correct supply chain management is one of the key objectives for both customer and product service quality. Most companies take advantage of all opportunities to reach the voice of the customer with their supply chain partners with whom they do business. Supplier selection includes a

process that requires technical knowledge, finance, and human resources and provides the advantage of jointly managing company requests with the supplier (Chai & Ngai, 2020).

In supplier selection, criteria such as competitive prices, compliance with customer quality, short delivery time, communication, quality certification, and product technical suitability can be considered. In addition to the supplier selection criteria in the supply chain, quality raw materials and services contribute positively to product quality, and meeting customer needs must be analyzed according to the relationship between the criteria the customers, stakeholders, companies, and suppliers. Working with the right supplier or making the right evaluation of the existing suppliers are among the evaluations that the customer also needs (Sharma and Tripathy, 2023; Yazdani et al., 2017; Erdebilli et al., 2023).

When the decision models used for supplier selection practices, and modeling in the literature are examined, it is realized that supplier selection studies are generally modeled with multi-criteria decision-making (MCDM) methods. However QFD has been applied in very few supplier selection studies, and in some of these studies, both QFD and MCDM methods have been used as a hybrid. Since QFD is a systematic and customer-focused approach, it is appropriate to use it to prioritize customer needs. As a supplier's customer, customer complaints and supply chain process design needs can be easily determined with QFD (Tayyar and Soltani, 2023; de Araújo Souza et al., 2022; Bao and Li, 2021). In addition, it enables the evaluation of the criteria to be used in supplier selection from various perspectives, by supporting the cooperation between the supplier and the company. In this way, it supports the achievement of more competitive results by improving the cooperation and decision-making process, while focusing on the most important criterion in supplier selection, increasing customer satisfaction in future studies. QFD is a method based on data and feedback from experts, market research, and other external sources. This data-driven approach helps reduce bias and ensures that decisions are based on concrete information.

In our study, we used neutrosophic AHP to find the QR weights

In the following statements, we have presented the background, motivation, purpose, findings, and contribution of our study:

The **background** of our study, we have handled supplier selection problems for modular kitchen producers. The organization, in which we analyzed this problem, had difficulties in managing its supply chain processes especially in managing suppliers' raw material delivery process. So, a solution/ strategy for this organization was needed.

The primary **motivation** of our study is to analyze supplier selection criteria and alternative suppliers with a hybrid methodology that considers customers' requirements. By this methodology, we aim to develop a supplier selection model for organizations which will be a strategy for.

The **gaps** which we considered are;

- Neutrosophic AHP and QFD were not used in supplier selection for the customer-oriented companies.
- The number of supplier selection studies focusing on customer expectations is low, evaluations regarding sub-criteria that also take customer feedback into consideration are not investigated a lot and uncertainties in the supply chain process and supplier selection criteria are not considered too much.
- Customer requirements and expert opinions are planned at the same stage and therefore a strategic decision is made.

The **purpose** of this study is summarized as follows: The supplier selection problems involve criteria at different levels. The most important criteria for supplier selection include price, discount, quality, information management, and delivery time. It is considered appropriate to apply QFD in the study due to its methodological approach, focusing on the needs of the customers, and prioritization of those requirements. Customer complaints and requirements for supply chain process design are basically implemented with QFD as a supplier's customer. It allows the criteria to be used in supplier selection to be evaluated from several perspectives by encouraging cooperation between the supplier and the company. Therefore, through enhancing collaboration and decision-making, more competitive results are obtained. By focusing on the most important consideration in supplier selection, it enhances customer satisfaction in the future. QFD relies on information and opinions from professionals, market analysis, and other outside sources. This data-driven strategy guarantees that decisions are based on factual information and reduces bias. In the process of selecting a supplier, there are numerous criteria and potential uncertainties. The objective of implementing neutrosophic AHP-QFD is to ease the role of experts and model uncertainty more effectively than with other methods. The use of AHP is more appropriate for evaluation since the criteria in this study have more sub-criteria than the articles examined. The purpose of the study is to determine how to prioritize the criteria to select a more thorough and long-term supplier. With QFD, it can compare the qualities of suppliers and how they relate to one another. The neutrosophic AHP-QFD guarantees that the supplier selection meets customer requirements. This customer-oriented mindset significantly leads to enhanced fulfillment and lasting relationships in business. Alternatives are chosen more precisely by offering innovation and utilizing a more customer-oriented multi-criteria decision-making approach.

The **findings** of our study are explained in the following: The importance levels of the criteria and CRs identified in the study's initial phase were ascertained using the neutrosophic AHP. The QRs required to satisfy customer needs were identified in the second phase, and the QFD approach was used to calculate the weights of the QRs. Using the weights of the established quality standards, the neutrosophic AHP technique was once more employed in the third phase to assess potential substitute providers. Consequently, here is how the alternatives were ranked: Turkey, Italy, Germany-1, Germany-2, and Austria. A systematic and thorough approach to supplier evaluation is demonstrated using neutrosophic AHP to evaluate the relative importance of criteria and CRs, as well as the use of the QFD technique to define QRs. The study's findings, which placed Austria as the top supplier and Germany-2 and Germany-1 closely behind, offer the company under review useful information.

The *contributions* of our study are listed as follows:

- In this study, the methodology considers customer requirements. The organization which produces modular kitchen furniture must be aware of the needs of their customers' needs. These are also the requirements of the company which is the customer of its suppliers.
- In the literature, according to our research, there is not any study that investigates furniture industry supplier selection problems.
- A performance analysis model for suppliers is presented both to analyze suppliers' performances and supplier selection criteria performances for each alternative.

In our study, we outline a systematic exploration of the customer-oriented supplier selection process in the kitchen furniture industry. The methodology is structured into three main phases, each contributing to a holistic understanding of the proposed model. In the initial phase, we research the application of neutrosophic AHP to ascertain the importance levels of criteria and customer requests. The subsequent section focuses on the determination of QRs necessary to meet customer demands, employing the QFD approach. The third and final phase involves the application of neutrosophic AHP once again, this time to evaluate alternative suppliers based on the determined QRs. The systematic progression through these phases allows for a comprehensive examination of the supplier selection process, ultimately in a ranked list of alternatives. This structure facilitates a clear and in-depth presentation of our research methodology and findings, contributing to the overall coherence of this study. The organizational structure of our study is as follows: In the second section, the literature review is presented. In the third section, the details of the methodology are explained. In the fourth section, the application is presented and in the last section, the results of our research and contribution are provided.

2. Literature Review

In our study, we have conducted a literature review with the keywords “supplier selection”, “multi-criteria decision making” and “QFD”. According to our research, some of the studies which integrate these techniques are summarized below:

Asadabadi (2017) applied the analytical network process (ANP), QFD, and Markov chains in his study, taking into account the dynamic customer needs as the determining factor in customer-oriented supplier selection and decided that long-term cooperation can be supported by supplier selection. Babbar and Amin (2018) applied a two-stage QFD method and multi-objective stochastic mathematical model in their study for green supplier selection in which the uncertainties in the supply process in the beverage sector are managed with the stochastic approach, and the uncertainties in the opinions of the people are handled with the fuzzy QFD method. Bhattacharya et al. (2010) investigated the selection of suppliers from alternative suppliers by a hierarchical approach, in which AHP, QFD, and cost factor measures are combined, to consider engineering and customer needs. Chen et al. (2021) proposed a new QFD model, which uses DEMATEL, MULTIMOORA, unstable fuzzy linguistic term set, for the design of CNC machine tools. Haiyun et al. (2021) with the help of QFD, analyzed the innovation

strategies multi-dimensionally by using a hybrid structure consisting of the interval-valued intuitionistic fuzzy DEMATEL and MULTIMOORA for green supply chain management. Jain and Singh (2020) aimed to make sustainable and stable supplier selection with a two-stage decision model by using a fuzzy inference system and fuzzy Kano model for the iron and steel industry. Karsak and Dursun (2014) proposed a new fuzzy MCDM method for supplier selection by using QFD and data envelopment analysis (DEA) together in a private hospital in Istanbul. Khan et al. (2022) included a full consistency method and fuzzy QFD hybrid model which will provide resilience in the health system during the COVID-19 pandemic and future pandemics. Lima-Junior and Carpinetti (2016) aimed to propose a new MCDM method with fuzzy QFD for supplier selection in the automotive sector in their study. Liu et al. (2019) used the Bonferroni test for green supplier selection in their study and divided them into several clusters to find the relationship between those in the same cluster. In addition, they applied QFD which uses preference values handled in the Bonferroni tests as customer needs importance values and the relationships between customer needs and technical criteria. Rajesh and Malliga (2013) developed a strategic supplier selection model for a high-pressure die-casting company by applying hybrid AHP and QFD methodology to analyze determined criteria. Sharma and Tripathy (2022) proposed an integrated QFD model with fuzzy TOPSIS for supplier selection and evaluation in their study. Singh and Kumar (2021) proposed an MCDM method integrated into the QFD model for social media usage by applying an image fuzzy set and a real case study was conducted with users of Facebook, Instagram, WhatsApp, and Twitter. Wang et al. (2022) included the use of TOPSIS and QFD and the hybrid interval-valued fuzzy sets in the evaluation of service performance for international container ports. Yazdani et al. (2017) organized their research by connecting DEMATEL for green supplier selection with customer needs. The relationship between the supplier selection pair and the needs of the customer was identified by applying the QFD. The complicated proportional assessment (COPRAS) lists alternative suppliers. In this section, there are studies used in traditional supplier selection and innovative supplier selection studies that also include the customer's requirements. Öztürk and Paksoy (2020) studied the green supplier selection problem in their studies considering customers' needs by using DEMATEL-QFD interval type-2 fuzzy AHP methods. Zhang et al. (2019) proposed a neighborhood-rough set-based QFD model for green building material supplier selection in their study in which technical specifications are determined by a decision-making group. Tayyar and Soltani (2020) proposed a method that decreases the cost of inappropriate supplier selection by choosing appropriate evaluation criteria with QFD-AHP. Liu et al. (2020) used QFD, best and worst method (BWM), VIKOR, and DEMATEL methods for supplier selection in blockchain tracing anti-fraud platforms. Chen (2020) included a study in which the entropy weight was collected using both AHP and TOPSIS instead of subjective weights for appropriate supply selection. Dang et al. (2022) solved a supplier selection problem under uncertainty using spherical fuzzy AHP and grey COPRAS in the automotive industry in Vietnam considering sustainable criteria. Seifbarghy and Hamidi (2025) proposed a hybrid fuzzy DEMATEL, neutrosophic SWARA, and neutrosophic MOORA framework based on the 10 C model for supplier selection.

One of the studies methodologically closest to the proposed approach is Karasan et al. (2022), who developed a neutrosophic Quality Function Deployment (QFD) methodology grounded in neutrosophic Analytic Hierarchy Process (AHP) and neutrosophic DEMATEL, applying it to the customer-oriented design of an automobile seat. In their study, customer requirements were weighted using neutrosophic AHP, while the interrelationships among technical characteristics were determined via neutrosophic DEMATEL. Furthermore, the robustness of the resulting priorities was tested through comparisons with alternative methods in the literature as well as comprehensive sensitivity analyses.

Although that study shares a similar neutrosophic AHP-QFD backbone with the present research, it addresses a product design problem. In contrast, the current study focuses on a supplier selection problem where, following the QFD phase, alternative suppliers are further evaluated through a second application of neutrosophic AHP based on the weighted quality characteristics. This structural distinction positions the present study as the first to extend the neutrosophic AHP-QFD approach to a customer-oriented supplier evaluation problem within the customized furniture industry.

In addition, in Table 1, we presented studies that deal with supplier selection problems and summarized them according to the scope and techniques applied. In Table 2, we have analyzed the studies that include NAHP in their studies according to their scope and all other techniques.

Table 1. Techniques used for supplier selection

References	Scope	Method
Lima-Junior and Carpinetti (2016)	Application in the automotive industry by dividing priority, critical, complementary, and costly groups for supplier selection	Fuzzy QFD
Rajesh and Malliga (2013)	Strategic supplier selection proposal for high-pressure die-casting company	AHP; QFD
Karsak and Dursun (2014)	A private hospital application in Istanbul for supplier evaluation and selection	QFD; DEA; FWA
Yazdani et al. (2017)	Supplier evaluation and selection for environmental protection and sustainable development	QFD; DEMATEL; COPRAS
Bhattacharya et al. (2010)	Selection of suppliers to meet engineering and customer needs	AHP; Hierarchical QFD; CFM
Haiyun et al. (2021)	Developing an innovation strategy for green supply chain management in the energy sector	QFD; IVIF MOORA; DEMATEL
Liu et al. (2019)	Green supplier selection for bike rental case study	QFD; IT2FPBM; T2FWPBM
Asadabadi (2017)	Customer-oriented supplier selection	ANP; QFD; Markov Chain
Tang and Yang (2021)	Supplier selection for e-bike sharing	Interval-valued Pythagorean fuzzy preference relation
Hendiani et al. (2020)	Sustainable supplier selection for refinery in Southern Iran	Fuzzy best-worst method
Dursun and Karsak (2013)	Determination and evaluation of supplier selection criteria	Fuzzy weighted average; QFD
Bevilacqua et al. (2006)	Supplier selection for the medium-sized clutch fittings industry	Fuzzy QFD
Sharma and Tripathy (2022)	Supplier evaluation and selection	QFD; Fuzzy TOPSIS
Zheng et al. (2023)	New energy vehicles supplier selection	QFD; Case-based reasoning

References	Scope	Method
<i>Khorram et al. (2024)</i>	<i>Supplier selection</i>	<i>Fuzzy DEMATEL; fuzzy QFD; fuzzy TOPSIS; and fuzzy VIKOR</i>
<i>Koirala, and Shabbiruddin (2023)</i>	<i>Sustainable battery supplier selection</i>	<i>QFD; Fuzzy COPRAS</i>
<i>Deivanayagampillai et al. (2026)</i>	<i>Robust green supplier selection in sustainable supply chain management under dynamic uncertainty</i>	<i>Adaptive Neutrosophic Decision-Making (ANDM); Neutrosophic Entropy; DHFWDHM; Adaptive TOPSIS</i>
<i>Yang (2025)</i>	<i>Building material supplier selection</i>	<i>2-Tuple Linguistic Neutrosophic Set (2TLNNS); Combined Grey Relational Analysis (2TLNN-CGRA); Shannon Entropy</i>
<i>Ambilkar et al. (2024)</i>	<i>Sustailient supplier selection in the additive manufacturing (AM) sector for trinket manufacturing</i>	<i>Fuzzy Delphi Method (FDM) and Neutrosophic Best-Worst Method (N-BWM)</i>

Table 2. *The literature studies with Neutrosophic AHP*

References	Scope	Method
Karamaşa et al. (2021)	Evaluating the optimal training aircraft selection.	NAHP, MULTIMOORA
	Wireless body area networks relay node selections for the medical health system.	NAHP
Başhan et al. (2020)	Maritime risk evaluation to optimize safety control, preventing crew and potential environmental impacts.	NAHP, fuzzy TOPSIS
Reig-Mullor et al. (2022)	Sustainability reports' of environmental, social, and governance.	NAHP-TOPSIS
Harnpornchai and Wonggattaleekam (2021)	Relative importance assignment for the real world.	NAHP
Akman et al. (2022)	E-commerce selections during the pandemic crisis. Fuzzy extensions of AHP for INFUS conferences.	Neutrosophic fuzzy AHP; EDAS Intuitionistic fuzzy AHP; picture fuzzy AHP; NAHP; spherical fuzzy AHP; Pythagorean fuzzy AHP; q-rung orthopair fuzzy AHP
Vafadarnikjoo and Scherz (2021)	Sustainable, agility evaluations in the Iranian steel industry.	Neutrosophic-Grey AHP
Navarro et al. (2021)	Sustainable design.	NAHP, DEMATEL
Junaid et al. (2020)	Supply Chain Risk Assessment in Automotive Industry of Pakistan.	NAHP-TOPSIS
Karuppiah et al. (2021)	Sustainable humanitarian supply chain management.	NAHP, TODIM
Navarro et al. (2019)	Sustainable design assesment of concrete bridge.	NAHP, TOPSIS
Chakraborty et al. 2023	Chatbot selection in the telecommunication industry.	Combined Compromise Solution (CoCoSo), NAHP
Singh et al. (2021)	Evaluating total transportation cost.	Trapezoidal neutrosophic fuzzy AHP, Artificial neural network
Alzahrani et al. (2023)	Selecting women's university sites in the state of West Bengal, India.	Neutrosophic number TRNN, AHP, TOPSIS, COPRAS
Rahman et al. (2022)	The causes of boiler accidents in Bangladesh.	NAHP
Tuan et al. (2020)	Lecturers' research productivity evaluation.	Interval-valued Neutrosophic AHP
Abdel-Basset et al. (2020)	Professional selection in Egypt.	Neutrosophic ANP, TOPSIS, AHP, MOORA

References	Scope	Method
Vafadarnikjoo et al. (2023)	Blockchain technology adoption in manufacturing supply chains.	NAHP
Ye et al. (2023)	Fashion culture communication and marketing in China.	Interval-valued Neutrosophic AHP
Yiğit (2023)	Evaluating trainers in organizations.	Interval-valued Neutrosophic AHP, Fuzzy C-Means (FCM).
Ayyildiz et al. (2023)	Evaluate universities based on student perspective.	IVN-AHP, VIKOR
Hezam et al. (2021)	Determining the priority groups for COVID-19 Vaccine.	NAHP, TOPSIS
Sánchez-Garrido et al. (2021)	The sustainability among design alternatives for a single-family home.	Neutrosophic group AHP, VIKOR
Karamaşa et al. (2020)	Selecting the right logistics service provider	NAHP
Seifbarghy and Hamidi (2025)	Supplier selection in the automotive parts supply chain based on the 10 C model	Fuzzy DEMATEL, Neutrosophic SWARA, Neutrosophic MOORA

3. Methodology

In this study, we propose a methodology that integrates NAHP and NQFD approaches to determine the best alternative by considering both the QRs of customers and the QCs of products. Many criteria and alternatives in supplier selection involve uncertainty. Using neutrosophic AHP-QFD is aimed to facilitate the work of experts, and uncertainty is modeled more effectively compared to other methods (Karasan et al., 2022). The prioritization of the criteria and the selection of more comprehensive and long-contract suppliers are the motivations of this study. The comparison of supplier qualifications and the relationship between suppliers is possible with QFD. The neutrosophic AHP-QFD ensures that supplier selection is aligned with customer requirements. This customer-oriented approach makes a significant contribution to increased satisfaction and long-term business relationships.

In the first phase, we handle QRs' weights by conducting the NAHP method. In the second phase, we determine QCs to meet QRs and we execute the steps of QFD to handle the weights of QCs by considering the QRs' weights handled in the first phase. Then in the last phase, we identify alternatives and again we carry out NAHP steps to find the most appropriate alternative by taking into account the QCs weights handled in the second step. The proposed methodology is presented in Figure 1.

In this study, there are linguistic evaluations of decision-makers and uncertainties in QRs and QCs. Therefore, we need to use fuzzy logic to solve the problem and in our study, we used neutrosophic sets. The purpose of using neutrosophic sets in the evaluation of QRs and QCs for multiple alternatives is to appropriately represent unclear, inconsistent, or incomplete information. It is more suitable for human thinking with its membership functions of truth, indeterminacy, and falsity. It contains more information than classical fuzzy logic thanks to the "indeterminacy" membership function. In the following, we present the details of the methods used in this study.

3.1. Quality Function Deployment

QFD is a method of transferring customer requirements to the technical specifications of the product or process in order to meet customer demands and expectations of products or processes now and in the future. It is an effective method that can be used for the company, as it includes the "voice of the customer" in stages such as product and production planning and engineering (Cebi et al., 2009). In the QFD study; it is not only to meet the expectations of the customer, but also to provide the quality and best product/service. QFD is one of the methods used by companies to transform the quality product/service requirements of their customers, stakeholders, employees, and suppliers into quality characteristics (Babbar and Amin, 2018; Haiyun et al., 2021; Karasan et al., 2022; Khan et al., 2022; Lima-Junior et al., 2016; Singh and Kumar, 2021; Yazdani et al., 2017).

Traditional QFD has the following features:

- The middle part of the figure is the house of quality, showing the relationship between quality requirements (WHATs) and quality characteristics (HOWs).
- The left part of the house of quality contains the customer's quality requirement and degree of importance.
- The upper part of the House of Quality shows the framework of the technical specifications and the relations between the technical specifications.
- The right part of the house of quality shows the comparison matrix of the firm against its competitors.
- The lower part of the house of quality shows the grade of quality attributes targeting and evaluation process.

3.2. Neutrosophic Sets

Neutrosophic sets refer to sets developed to portray uncertain, incomplete, and meaningless information, as a generalization of intuitive fuzzy sets and introduced by Florentin Smarandache in the 1980s is represented by three membership values. NS studies the nature and scope of impartiality, as well as their interactions with different intellectual diversity. NS uses indeterminacy as an independent measure of the membership and non-membership information. Hence the concept of NS is considered as a generalization of FS, IFS, and interval-valued sets (Das et al., 2020; Kaya et al., 2022; Junaid et al., 2020).

Definition 1: Let X and $x \in X$. Let X be a neutrosophic subset A in the universe. For subset A (Kaya et al., 2022);

The truth membership function is a subset of $T_A(x)$, $[0,1]$.

Indeterminacy membership function is a subset of $T_I(x)$, $[0,1]$.

Falsity membership is a subset of $T_F(x)$, $[0,1]$.

There are no restrictions on the sum of membership functions.

$$0^- \leq T_A(x) + T_I(x) + T_F(x) \leq 3^+ \quad (1)$$

Definition 2: If $N_1=(t_1, i_1, f_1)$ and $N_2=(t_2, i_2, f_2)$ are two different single-valued neutrosophic numbers, the calculations can be described in the section as follows (Kaya et al., 2022):

$$\text{The sum of } N_1 \text{ and } N_2: N_1 + N_2 = (t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2) \quad (2)$$

$$\text{The multiplication of } N_1 \text{ and } N_2: N_1 \times N_2 = (t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2) \quad (3)$$

$$\text{The division of } N_2 \text{ on } N_1: N_2/N_1 = (t_2/t_1, i_2 - i_1/1 - i_1, f_2 - f_1/1 - f_1) \quad (4)$$

Definition 3: If $N_1=(t_1, i_1, f_1)$ is a single-valued neutrosophic number and A is an arbitrary positive real number, the multiplication of N1 and A can be described in the equation as follows (Kaya et al., 2022):

$$A \times N_1 = (1 - (1 - t_1)^A, i_1^A, f_1^A), \text{ where } A > 0 \quad (5)$$

Therefore, when $N_1=(t_1, i_1, f_1)$ is a single-valued neutrosophic number and A is an arbitrary positive real number, the division of N1 on A can be described as an equation as follows:

$$N_1/A = (1 - (1 - t_1)^{1/A}, i_1^{1/A}, f_1^{1/A}), \text{ where } A > 0 \quad (6)$$

Definition 4: If N_1 is a single-valued neutrosophic number, the score function is mapped N_1 into the single crisp output as $S(N_1)$ follows (Kaya et al., 2022).:

$$S(N_1) = (3 + t_1 - 2i_1 - f_1)/4 \quad (7)$$

$t_{\bar{a}}$ determines the maximum accuracy membership degree, $i_{\bar{a}}$ and $f_{\bar{a}}$ determine the minimum uncertainty and inaccuracy determines the degree of membership.

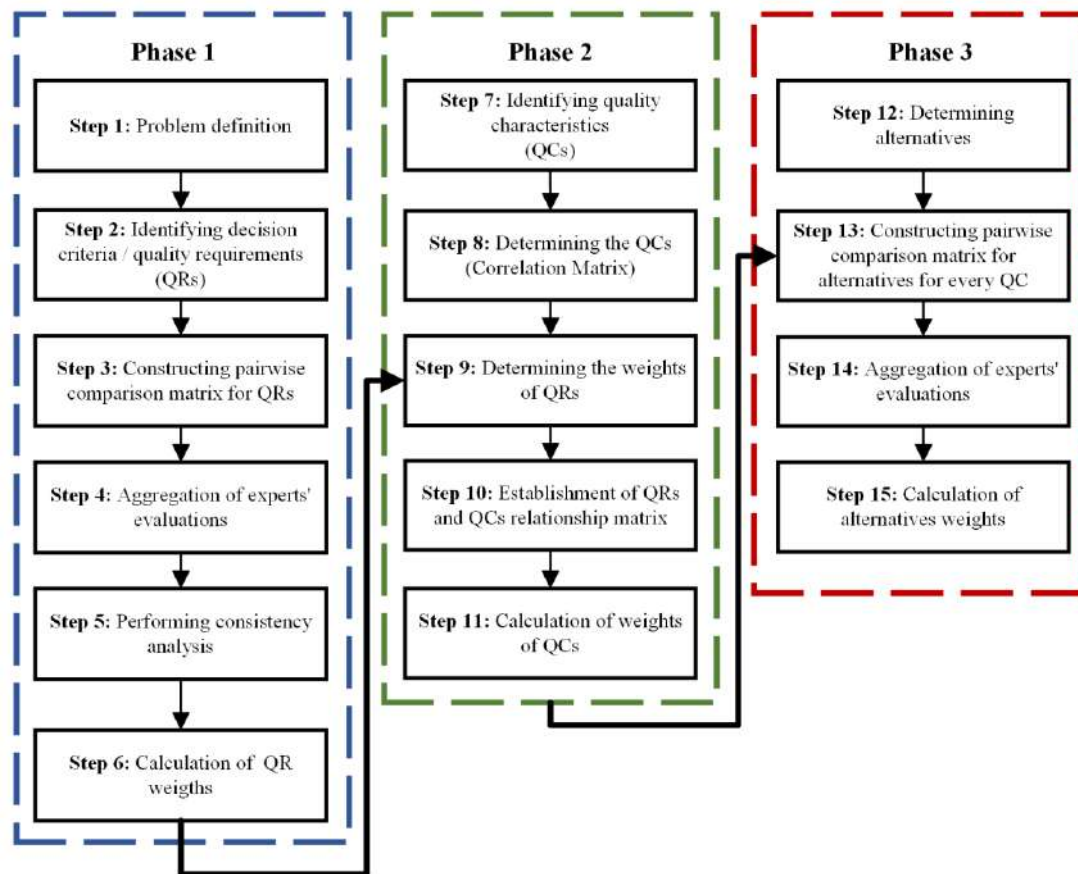


Figure 1. Flowchart of the methodology

3.3. Neutrosophic AHP (NAHP)

AHP is an MCDM approach that compares the importance of criteria and alternatives with pairwise comparisons. Since AHP does not deal with ambiguous situations in complex problems, it is used in most cases with fuzzy logic. In some real-life situations, this is not enough either. For this reason, evaluation with fuzzy numbers in decision problems provides more accurate decisions. In our study, we used neutrosophic fuzzy AHP to deal with this vagueness and find the weights of criteria and the best alternative.

The steps of the NAHP method used in this study are presented below:

Step 1: The criteria and sub-criteria of the problem and alternative suppliers to be selected are determined.

Step 2: A pairwise comparison is constructed for each criterion by the experts. Linguistic variables and importance weights are calculated based on neutrosophic values in Table 3 (Kaya et al., 2022).

Table 3. Linguistic variables and weight of significance based on neutrosophic values

Linguistic Terms	Neutrosophic sets	Linguistic Terms	Reciprocal Neutrosophic sets
Extremely highly preferred (EHP)	(0.90, 0.10, 0.10)	Mildly lowly preferred (MLP)	(0.10, 0.90, 0.90)
Extremely preferred (EP)	(0.85, 0.20, 0.15)	Mildly preferred (MP)	(0.15, 0.80, 0.85)
Very strongly to extremely preferred (VSEP)	(0.80, 0.25, 0.20)	Mildly preferred to Very lowly preferred (MPVLP)	(0.20, 0.75, 0.80)
Very strongly preferred (VSP)	(0.75, 0.25, 0.25)	Very lowly preferred (VLP)	(0.25, 0.75, 0.75)
Strongly preferred (SP)	(0.70, 0.30, 0.30)	Lowly preferred (LP)	(0.30, 0.70, 0.70)
Moderately highly to strongly preferred (MHSP)	(0.65, 0.30, 0.35)	Moderately lowly preferred to lowly preferred (MLPLP)	(0.35, 0.70, 0.65)
Moderately highly preferred (MHP)	(0.60, 0.35, 0.40)	Moderately lowly preferred (MOLP)	(0.40, 0.65, 0.60)
Equally to moderately preferred (EMP)	(0.55, 0.40, 0.45)	Moderately to equally preferred (MEP)	(0.45, 0.60, 0.55)
Equally preferred (EQP)	(0.50, 0.50, 0.50)	Equally preferred (EQP)	(0.50, 0.50, 0.50)

Step 3: The aggregation of expert opinions is calculated with the neutrosophic weighted arithmetic mean addition operator.

$$F_w(A_1, A_2, \dots, A_n) = \begin{cases} 1 - \prod_{j=1}^n (1 - T_{A_j}(x))^{w_j} \\ 1 - \prod_{j=1}^n (1 - I_{A_j}(x))^{w_j} \\ 1 - \prod_{j=1}^n (1 - F_{A_j}(x))^{w_j} \end{cases} \quad (8)$$

$W = (w_1, w_2, \dots, w_n)$: weight vector $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$

Step 4: The consistency ratio (CR) is calculated for each pairwise comparison matrix. The method proposed by Xu and Liao (2014) will be used for CR calculation. The following calculations are used, with $(T'_{xk}, I'_{xk}, \text{ and } F'_{xk})$ consistent neutrosophic values,

1. For $k > x + 1, y = x + 1$ $N_{xk} = (T'_{xk}, I'_{xk}, F'_{xk})$,

$$T'_{xk} = \frac{k-x-1 \sqrt{T_{xy} * T_{yk} * T_{xk-1} * T_{k-1k}}}{k-x-1 \sqrt{T_{xy} * T_{yk} * T_{xk-1} * T_{k-1k}} + k-x-1 \sqrt{(1-T_{xy}) * (1-T_{yk}) * (1-T_{xk-1}) * (1-T_{k-1k})}} \quad (9)$$

$$I'_{xk} = \frac{k-x-1 \sqrt{I_{xy} * I_{yk} * I_{xk-1} * I_{k-1k}}}{k-x-1 \sqrt{I_{xy} * I_{yk} * I_{xk-1} * I_{k-1k}} + k-x-1 \sqrt{(1-I_{xy}) * (1-I_{yk}) * (1-I_{xk-1}) * (1-I_{k-1k})}} \quad (10)$$

$$F'_{xk} = \frac{k-x-1 \sqrt{F_{xy} * F_{yk} * F_{xk-1} * F_{k-1k}}}{k-x-1 \sqrt{F_{xy} * F_{yk} * F_{xk-1} * F_{k-1k}} + k-x-1 \sqrt{(1-F_{xy}) * (1-F_{yk}) * (1-F_{xk-1}) * (1-F_{k-1k})}} \quad (11)$$

2. For $k = x + 1, y = x + 1$ $N_{xk} = (T_{xk}, I_{xk}, F_{xk})$

3. For $k < x, y = x + 1$ $N_{xk} = (F'_{xk}, 1 - I'_{xk}, T'_{xk})$

$$CR = \frac{1}{2(n-1)(n-2)} \sum_{x=1}^n \sum_{k=1}^n (|T'_{xk} - T_{xk}| + |I'_{xk} - I_{xk}| + |F'_{xk} - F_{xk}|) \quad (12)$$

*CR must be less than 0.1.

Step 5: The normalization method is applied for each criterion, and the normalization process is performed by dividing each row by the total of the column it belongs to.

Step 6: After the normalization process is done, the row totals are calculated and divided by the number of criteria, and the neutrosophic criteria weights are found.

Step 7: Neutrosophic criterion weights are clarified using Eq. (9).

$$S(N_i) = (3 + T_i - 2I_i - F_i)/4 \quad (13)$$

Step 8: The criteria weights are obtained so that the total weights are 1.

3.4. The Neutrosophic QFD (NQFD)

The grading of QRs as (1-5) or (1-9) by experts creates a negative situation in terms of subjectivity. Uncertainty, inconsistency, and disagreement of experts cause incomplete or inaccurate transmission of QRs in supplier selection problems. Relationship matrix values will be expressed with neutrosophic numbers. The relationships will be shown by $\tilde{9}$ - strong relationships, $\tilde{3}$ - average relationships, and $\tilde{1}$ - poor relationship.

In the QFD correlation matrix, (++) shows a strong positive correlation between QCs, (-) shows a negative correlation, and (--) shows a strong negative correlation. The weight of each QC (HOWs) is calculated by multiplying each relationship value by the corresponding QRs (WHATs) weight.

$$\omega(HOW)_i = (HOW)_{i1} \text{relationship degree} \times \omega(WHAT_1) + \dots + (HOW)_{in} \text{relationship degree} \times \omega(WHAT_n) \quad (14)$$

4. Application

In this study, an application is conducted by using the data of a customer-oriented and bespoke company in the kitchen furniture industry. The company, which provides modular facilities with different designs with an aesthetic, modern line, serves its domestic and international customers. Raw materials are procured from both domestic and foreign suppliers. The purpose of using QFD in the application is to meet customer needs with the voice of the customer and to develop a decision-making mechanism that allows harmony between the organization's characteristics and suppliers' characteristics. Evaluation is made by three experts working in the firm.

The application steps are as follows:

Phase 1/ Step 1: The problem handled in this study is to find the most appropriate supplier for the company.

Phase 1/ Step 2: In order to reveal the problem, the supplier selection criteria and sub-criteria of the company are classified in Table 4. We collected our criteria through a literature review and asked decision-makers to check whether they were suitable for supplier evaluation. These criteria are also considered as QRs of customers in the QFD phase. Some supplier selection criteria from the literature are included in this article. Certain criteria were developed by consulting the opinions of the purchasing and marketing departments because, in the furniture company where the application was conducted, design and consumer feedback are significant factors.

Table 4. Supplier selection criteria

Criteria	Sub-criteria	Relevant Literature
Order management (C1)	Order process (C11)	Karabayır and Botsalı, 2022
	Ordering/Portal login (C12)	Experts' opinion
Product and service marketing management (C2)	New product introduction (C21)	Galankashi et al., 2016; Wang et al., 2017
	Change in product (C22)	Wang et al., 2017
	The product is removed from the program (C23)	Wang et al., 2017
	Media management (C24)	Lahdhiri et al., 2022
	Product and service training (C25)	Galankashi et al., 2016, Wang et al., 2017
	Adapting to competitors (C26)	Karabayır and Botsalı, 2022
	Complaints management (C27)	Karabayır and Botsalı, 2022
	Identifying market-specific global trends (C28)	Karabayır and Botsalı, 2022; Wang et al., 2017
	Collaboration to improve existing service (C29)	Karabayır and Botsalı, 2022; Wang et al., 2017
	Quality (C210)	Galankashi et al., 2016; Lahdhiri et al., 2022; Wang et al., 2017
Logistics management (C3)	Accurate and timely response to price offers (C211)	Karabayır and Botsalı, 2022
	Notification of delivery dates (C31)	Galankashi et al., 2016; Wang et al., 2017
	Notification of minimum orders (C32)	Galankashi et al., 2016; Wang et al., 2017
	Material volume, package quantity, size notification (C33)	Galankashi et al., 2016; Wang et al., 2017
	Compliance with delivery dates (C34)	Galankashi et al., 2016
	Domestic franchise status (C35)	Expert opinion
Visit management (from company to supplier –	Company presentation (C41)	Expert opinion
	Information sharing (C42)	Lahdhiri et al., 2022

Criteria	Sub-criteria	Relevant Literature
<i>from supplier to company) (C4)</i>	Workshop (C43)	Expert opinion
	Factory tour frequency (Digital & classic) (C44)	Lahdhiri et al., 2022
<i>Purchasing planning and control management (C5)</i>	Price (C51)	Galankashi et al., 2016; Wang et al., 2017
	Compliance with the predictions (C52)	Galankashi et al., 2016; Wang et al., 2017

Phase 1/ Step 3: Table 5 was obtained by taking the opinions of 3 experts. The relationship matrix between the criteria and the sub-criteria is created according to their evaluations. The features of the experts are as follows:

- Expert 1: 13 years' experience as an Inventory Planning Executive
- Expert 2: 15 years' experience as Senior Purchase Representative
- Expert 3: 5 years' experience as a Purchase Representative

The values given in Table 5 and Table 6 show the pairwise comparisons of quality requirements both with linguistic terms and neutrosophic fuzzy sets.

Table 5. Comparison matrix with linguistic evaluations

		C1	C2	C3	C4	C5
C1	Expert 1	EQP	MLP	MP	EQP	MLP
	Expert 2	EQP	MLP	MLP	EQP	MLP
	Expert 3	EQP	MLP	MLP	EQP	MLP
C2	Expert 1	EHP	EQP	EHP	EQP	SP
	Expert 2	EHP	EQP	EQP	VSP	VLP
	Expert 3	EHP	EQP	EQP	LP	MPVLP
C3	Expert 1	EP	MLP	EQP	EHP	EQP
	Expert 2	EHP	EQP	EQP	VSEP	MLP
	Expert 3	EHP	EQP	EQP	VSEP	MLP
C4	Expert 1	EQP	MLP	MLP	EQP	MLP
	Expert 2	EQP	VLP	MPVLP	EQP	MLP
	Expert 3	EQP	SP	MPVLP	EQP	MLP
C5	Expert 1	EHP	LP	EQP	EHP	EQP
	Expert 2	EHP	VSP	EHP	EHP	EQP
	Expert 3	EHP	VSEP	EHP	EHP	EQP

Table 6. Neutrosophic fuzzy comparison matrix for QRs

C1	C2	C3	C4	C5
----	----	----	----	----

	Expert 1	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)	(0.15, 0.8, 0.85)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)
C1	Expert 2	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)	(0.1, 0.9, 0.9)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)
	Expert 3	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)	(0.1, 0.9, 0.9)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)
	Expert 1	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.3)
C2	Expert 2	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.75, 0.25, 0.25)	(0.25, 0.75, 0.75)
	Expert 3	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.7)	(0.2, 0.75, 0.8)
	Expert 1	(0.85, 0.2, 0.15)	(0.1, 0.9, 0.9)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)
C3	Expert 2	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.8, 0.25, 0.2)	(0.1, 0.9, 0.9)
	Expert 3	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.8, 0.25, 0.2)	(0.1, 0.9, 0.9)
	Expert 1	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)	(0.1, 0.9, 0.9)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)
C4	Expert 2	(0.5, 0.5, 0.5)	(0.25, 0.75, 0.75)	(0.2, 0.75, 0.8)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)
	Expert 3	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.3)	(0.2, 0.75, 0.8)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0.9)
	Expert 1	(0.9, 0.1, 0.1)	(0.3, 0.7, 0.7)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)
C5	Expert 2	(0.9, 0.1, 0.1)	(0.75, 0.25, 0.25)	(0.9, 0.1, 0.1)	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)
	Expert 3	(0.9, 0.1, 0.1)	(0.8, 0.25, 0.2)	(0.9, 0.1, 0.1)	(0.9, 0.1, 0.1)	(0.5, 0.5, 0.5)

Phase 1/ Step 4: Using Equation (4), expert opinions are used to calculate the neutrosophic weighted arithmetic mean (Table 7). The weights of all experts are equal, so w_j is 0,333. For example the calculation for $\tilde{r}_{42}$;

$$\tilde{r}_{42} = \begin{cases} 1 - (1 - 0,1)^{0,333} \times (1 - 0,25)^{0,333} \times (1 - 0,7)^{0,333} \\ 1 - (1 - 0,9)^{0,333} \times (1 - 0,75)^{0,333} \times (1 - 0,3)^{0,333} \\ 1 - (1 - 0,9)^{0,333} \times (1 - 0,75)^{0,333} \times (1 - 0,3)^{0,333} \end{cases}$$

$$\tilde{r}_{42} = (0,413, 0,740, 0,740)$$

Table 7. Aggregated neutrosophic comparison matrix

	C1	C2	C3	C4	C5
C1	(0,50, 0,50, 0,50)	(0,10, 0,90, 0,90)	(0,12, 0,87, 0,89)	(0,50, 0,50, 0,50)	(0,10, 0,90, 0,90)
C2	(0,90, 0,10, 0,10)	(0,50, 0,50, 0,50)	(0,71, 0,39, 0,39)	(0,56, 0,52, 0,52)	(0,44, 0,65, 0,67)
C3	(0,89, 0,13, 0,12)	(0,39, 0,71, 0,71)	(0,50, 0,50, 0,50)	(0,84, 0,20, 0,17)	(0,26, 0,83, 0,83)
C4	(0,50, 0,50, 0,50)	(0,41, 0,74, 0,74)	(0,17, 0,82, 0,84)	(0,50, 0,50, 0,50)	(0,10, 0,90, 0,90)
C5	(0,90, 0,10, 0,10)	(0,67, 0,45, 0,44)	(0,83, 0,26, 0,26)	(0,90, 0,10, 0,10)	(0,50, 0,50, 0,50)

Phase 1/ Step 5: Equation 5-6-7 is used to calculate the consistency of the neutrosophic pairwise comparison matrix. The consistency ratio with equation (8) was found as CR=0.001.

Phase 1/ Step 6: In this step, the importance levels of the sub-criteria have been accepted as equal and are given in Table 8.

Table 8. Selection criteria importance of degrees

Main-criteria	Sub-criteria	Weights ω	Criteria Importance Degree	Criteria Importance Degree (%)
C1	Order process (C11)	9%	0,500	0,043
	Ordering/Portal login (C12)		0,500	0,043
C2	New product introduction (C21)	25%	0,091	0,023
	Change in product (C22)		0,091	0,023
	The product is removed from the program (C23)		0,091	0,023
	Media management (C24)		0,091	0,023
	Product and service training (C25)		0,091	0,023
	Adapting to competitors (C26)		0,091	0,023
	Complaints management (C27)		0,091	0,023
	Identifying market-specific global trends (C28)		0,091	0,023
	Collaboration to improve existing service (C29)		0,091	0,023
	Quality (C210)		0,091	0,023
	Accurate and timely response to price offers (C211)		0,091	0,023
C3	Notification of delivery dates (C31)	21%	0,200	0,043
	Notification of minimum orders (C32)		0,200	0,043
	Material volume, package quantity, size notification (C33)		0,200	0,043
	Compliance with delivery dates (C34)		0,200	0,043
	Domestic franchise status (C35)		0,200	0,043
C4	Company presentation (C41)	13%	0,250	0,033
	Information sharing (C42)		0,250	0,033
	Workshop (C43)		0,250	0,033
	Factory tour frequency (Digital & classic) (C44)		0,250	0,033
C5	Price (C51)	31%	0,500	0,157
	Compliance with the predictions (C52)		0,500	0,157

Phase 2 / Step 7: The quality characteristics (QCs) that the company must meet in order to meet the supplier selection criteria are determined by asking decision-makers and they are presented below:

- Number of patents (QC1)
- Reverse logistics rate (QC2)
- Experiences in the field (QC3)
- Quality certification systems (QC4)
- Location (QC5)
- Financial stability (QC6)
- Reliability of the raw material to the customer and manufacturing processes (QC7)
- Discount (QC8)
- Management and organizational culture (QC9)
- Compliance with environmental laws (QC10)
- Technical capacity (QC11)

Phase 2 / Step 8: The QCs’ correlation matrix is constructed and presented in Table 9. The scale used to create the correlation matrix is as follows: ++ high positive correlation, + positive correlation, - negative correlation, -- high negative correlation.

Table 9. Correlation matrix

	QC1	QC2	QC3	QC4	QC5	QC6	QC7	QC8	QC9	QC10	QC11
QC1											++
QC2											
QC3											
QC4										+	
QC5						-					
QC6					-						

[illegible]

Phase 2 / Step 9: The QR weights are handled from Phase 1/Step 6.

Phase 2/Step 10: A relationship matrix is created between WHATs and HOWs. The relationship between QRs and QCs will be shown with neutrosophic values of $\tilde{9}$, $\tilde{3}$, $\tilde{1}^*$. Table 10 includes the neutrosophic scale for QRs and QCs relationship matrix and Table 11 shows the relationship matrix between the QRs and QCs.

Phase 2 / Step 11: The aggregated relationship matrix is presented in Table 12 and the weights of QCs are calculated via Eq. (4).

Table 10. The neutrosophic scale for QRs and QCs relationship matrix

Relationship	Neutrosophic fuzzy numbers		
$\tilde{1}$ - poor relationship	0.1	0.9	0.9
$\tilde{3}$ - moderate relationship	0.5	0.5	0.5
$\tilde{9}$ - strong relationship	0.9	0.1	0.1

*9 strong relationships (S), 3 moderate relationships (A), 1 poor relationship (P)

Table 11. Relationship matrix

QRs	Expert #	QR Weights	QC1	QC2	QC3	QC4	QC5	QC6	QC7	QC8	QC9	QC10	QC11
C11	Expert 1				A		S				A		P
	Expert 2	0,043			A		S				A		P
	Expert 3				A		S				A		P

QRs	Expert #	QR Weights	QC1	QC2	QC3	QC4	QC5	QC6	QC7	QC8	QC9	QC10	QC11
C12	Expert 1	0,043			A						S		S
	Expert 2				A						S		S
	Expert 3				A						S		S
C21	Expert 1	0,023	A		A	A					S	S	S
	Expert 2		A		P	A					S	S	S
	Expert 3		A		P	A					S	S	S
C22	Expert 1	0,023	S	P	S	S			S		S	S	S
	Expert 2		S	P	S	S			S		S	S	S
	Expert 3		S	P	S	S			S		A	S	A
C23	Expert 1	0,023		S	A						S	S	S
	Expert 2			S	A						S	S	S
	Expert 3			S	A						S	S	S
C24	Expert 1	0,023			S	S		S			S		
	Expert 2				S	S		S			S		
	Expert 3				S	S		S			S		
C25	Expert 1	0,023	P		S								
	Expert 2		P		S								
	Expert 3		P		S								
C26	Expert 1	0,023											
	Expert 2												
	Expert 3												
C27	Expert 1	0,023		S	P				A		S		S
	Expert 2			S	P				A		S		S
	Expert 3			S	P				A		S		S
C28	Expert 1	0,023	S		A	S		S			S		S
	Expert 2				A	S		S			S		S
	Expert 3					S		S			S		
C29	Expert 1	0,023	S		A		A	A	A		A		A
	Expert 2		S		A		A	A	A		A		A
	Expert 3		S		A		A	A	A		A		A
C210	Expert 1	0,023	S	S	A	S			S		A		A
	Expert 2		S	S	A	A			S		A		A
	Expert 3		S	S		S			S		A		A
C211	Expert 1	0,023			A		P	S		P	P		A

QRs	Expert #	QR Weights	QC1	QC2	QC3	QC4	QC5	QC6	QC7	QC8	QC9	QC10	QC11
	Expert 2				A		P	S		P	P		A
	Expert 3				A		P	S		P	P		
	Expert 1										A		
C31	Expert 2	0,043									A		
	Expert 3										A		
	Expert 1										A		
C32	Expert 2	0,043									A		
	Expert 3										A		
	Expert 1										A		
C33	Expert 2	0,043									A		
	Expert 3										A		
	Expert 1										A		
C34	Expert 2	0,043									A		
	Expert 3										A		
	Expert 1										A		
C35	Expert 2	0,043		P			A						
	Expert 3			P			A						
	Expert 1												
C41	Expert 2	0,033	S			S	S	S					
	Expert 3					S	S	S					
	Expert 1												
C42	Expert 2	0,033											
	Expert 3												
	Expert 1		S										
C43	Expert 2	0,033	A										
	Expert 3		A										
	Expert 1												
C44	Expert 2	0,033					S						
	Expert 3						A						
	Expert 1												
C51	Expert 2	0,157						S		S			
	Expert 3							S		S			
	Expert 1												
C52	Expert 1	0,157							S		S		S
	Expert 2								S		S		S
	Expert 3												

QRs	Expert #	QR Weights	QC1	QC2	QC3	QC4	QC5	QC6	QC7	QC8	QC9	QC10	QC11
	Expert 3								S				S

Table 12. Aggregated relationship matrix

QRs (WHATs)	QRs weights	QCs (HOWs)										
		QC1	QC2	QC3	QC4	QC5	QC6	QC7	QC8	QC9	QC10	QC11
C11	0,04			0,50		0,90				0,50		0,10
C12	0,04			0,50						0,90		0,90
C21	0,02	0,50		0,26	0,50					0,90	0,90	0,90
C22	0,02	0,90	0,10	0,90	0,90			0,90		0,83	0,90	0,83
C23	0,02		0,90	0,50						0,90	0,90	0,90
C24	0,02			0,90	0,90		0,90			0,90		
C25	0,02	0,10		0,90								
C26	0,02											
C27	0,02		0,90	0,10				0,50		0,90		0,90
C28	0,02				0,90		0,90			0,90		
C29	0,02	0,90		0,50		0,50	0,50	0,50		0,50		0,50
C210	0,02	0,90	0,90		0,83			0,90		0,50		0,50
C211	0,02			0,50		0,10	0,90		0,10	0,10		
C31	0,04									0,50		
C32	0,04									0,50		
C33	0,04											
C34	0,04									0,50		
C35	0,04		0,10									
C41	0,03				0,90	0,90	0,90					
C42	0,03											
C43	0,03	0,71										
C44	0,03					0,83						
C51	0,16						0,90		0,90			
C52	0,16							0,90				0,90
Total	1,00	4,01	2,90	5,56	4,93	3,23	5,00	3,70	1,00	9,33	2,70	6,43
Weights of QCs		8%	6%	11%	10%	6%	10%	7%	2%	19%	5%	13%

According to Table 12, management and organizational culture, technical capacity, experience in the field, quality and certification systems, and financial stability are important characteristics, and quality requirements to be followed for alternative supplier selection. In addition, we show the quality requirements which have the highest effect on quality characteristics in Table 13. According to this table, it is realized that to provide efficient management, have an effective organizational culture (QC9), and increase its technical capacity (QC11), the organization must manage its ordering, product, and service marketing processes successfully. And like these, to increase the experience in their field (QC3), the organizations must focus on their product and service marketing activities. In the production field, quality certification systems (QC4) are required by all parties of customers. So, to have integrated quality systems and to benefit from them, organizations must be aware of the market requirements and must have high interaction with their suppliers. The organizations aim to be financially stable have a high market share and increase their profitability, therefore they have to increase their marketing operations efficiency,

they have to develop the right communication skills with their suppliers and they have determined the right price for their products and services.

Phase 3 / Step 12: The alternative suppliers list is determined. In order to create an alternative supplier list, the company has proposed five alternative suppliers: Turkey (TR), Germany (DE1, DE2), Austria (AT), and Italy (IT). The qualifications of suppliers are presented in Table 14.

Table 13. The QR/QC relationship analysis

Management and organizational culture (QC9)	Technical capacity (QC11)	Experiences in the field (QC3)
Ordering/Portal login (C12)	Ordering/Portal login (C12)	Change in product (C22)
New product introduction (C21)	New product introduction (C21)	Media management (C24)
Change in product (C22)	Change in product (C22)	Product and service training (C25)
The product is removed from the program (C23)	The product is removed from the program (C23)	
Media management (C24)	Complaints management (C27)	
Complaints management (C27)	Collaboration to improve existing service (C29)	
Identifying market-specific global trends (C28)		
Quality certification systems (QC4)	Financial stability (QC6)	
Change in product (C22)	Media management (C24)	
Media management (C24)	Identifying market-specific global trends (C28)	
Identifying market-specific global trends (C28)	Accurate and timely response to price offers (C211)	
Quality (C210)	Company presentation (C41)	
Company presentation (C41)	Price (C51)	

Table 14. Alternatives qualifications

Alternative suppliers	Experience	Number of trading countries	Franchise status
TR (Turkey)	50 years	More than 80 countries	Existing
DE1 (Germany-1)	100 years	More than 150 countries	Existing
DE2 (Germany-2)	135 years	More than 100 countries	Existing
AT (Austria)	70 years	More than 120 countries	Existing

IT (Italy)	24 years	More than 70 countries	Existing
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Phase 3 / Step 13: Score is calculated for suppliers by using NAHP. A neutrosophic comparison matrix was prepared for each criterion, and a comparison matrix for the first criterion (QC1) is given in Table 15 as an example.

Table 15. Pairwise comparisons of alternative for QC1 criterion

		TR	DE1	DE2	AT	IT
TR	Expert 1	(0,5, 0,5, 0,5)	(0,1, 0,9, 0,9)	(0,1, 0,9, 0,9)	(0,1, 0,9, 0,9)	(0,3, 0,7, 0,7)
	Expert 2	(0,5, 0,5, 0,5)	(0,1, 0,9, 0,9)	(0,1, 0,9, 0,9)	(0,1, 0,9, 0,9)	(0,4, 0,65, 0,6)
	Expert 3	(0,5, 0,5, 0,5)	(0,1, 0,9, 0,9)	(0,1, 0,9, 0,9)	(0,3, 0,7, 0,7)	(0,3, 0,7, 0,7)
DE1	Expert 1	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,6, 0,35, 0,4)
	Expert 2	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,7, 0,3, 0,3)
	Expert 3	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,7, 0,3, 0,3)
DE2	Expert 1	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,1, 0,9, 0,9)	(0,6, 0,35, 0,4)
	Expert 2	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,1, 0,9, 0,9)	(0,6, 0,35, 0,4)
	Expert 3	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,5, 0,5, 0,5)	(0,1, 0,9, 0,9)	(0,6, 0,35, 0,4)
AT	Expert 1	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,9, 0,1, 0,1)
	Expert 2	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,9, 0,1, 0,1)
	Expert 3	(0,7, 0,3, 0,3)	(0,5, 0,5, 0,5)	(0,9, 0,1, 0,1)	(0,5, 0,5, 0,5)	(0,9, 0,1, 0,1)
IT	Expert 1	(0,7, 0,3, 0,3)	(0,4, 0,65, 0,6)	(0,4, 0,65, 0,6)	(0,1, 0,9, 0,9)	(0,5, 0,5, 0,5)
	Expert 2	(0,4, 0,65, 0,6)	(0,3, 0,7, 0,7)	(0,4, 0,65, 0,6)	(0,1, 0,9, 0,9)	(0,5, 0,5, 0,5)
	Expert 3	(0,7, 0,3, 0,3)	(0,3, 0,7, 0,7)	(0,4, 0,65, 0,6)	(0,1, 0,9, 0,9)	(0,5, 0,5, 0,5)

Phase 3 / Step 14-15: The suppliers are sorted. Finally, the values in Table 16 were obtained by multiplying the matrix with the weights. Accordingly, even if there are close values, the supplier in Austria will be selected. The order is AT, DE2, DE1, TR, IT and.

Table 16. The weights of alternatives

Quality Characteristics	Weights	TR	DE1	DE2	AT	IT
QC1	0,08	9%	22%	24%	34%	12%
QC2	0,08	10%	25%	25%	29%	12%
QC3	0,10	20%	20%	20%	20%	20%
QC4	0,11	13%	20%	25%	30%	12%
QC5	0,07	23%	20%	20%	20%	17%
QC6	0,13	15%	20%	24%	30%	12%
QC7	0,08	13%	20%	28%	22%	18%
QC8	0,03	30%	17%	14%	17%	22%
QC9	0,16	15%	20%	22%	25%	19%
QC10	0,05	8%	20%	24%	34%	14%
QC11	0,12	11%	19%	24%	30%	16%
Alternatives weights		14%	20%	23%	27%	15%

It is seen that AT is the most appropriate supplier. When we analyze the qualifications of this supplier, it is seen that it has experience in the kitchen furniture industry and it exports to more than 120 countries. In addition, having franchises in Turkey provides convenience in terms of getting support.

In our study, we also analyzed the supplier evaluation criteria or QCs for every alternative. The objective of doing this is to present the efficiencies of alternative suppliers according to QCs. We have used the performance analysis model which is proposed in Bilisik et al. (2012). The most important criteria are presented in the performance analysis matrix and the company proposes a strategy for their suppliers to increase their performances as a result of this approach. The criteria are explained below and they are presented in Figure 2:

- **Strong and important criteria:** The criteria that have high weights and high efficiency of alternatives are evaluated primarily.
- **Secret criteria:** The criteria that have high weights and low efficiency are evaluated in second place.
- **Poor and unimportant criteria:** The criteria that have low weights and low efficiency are not evaluated preferential.
- **Standard criteria:** These criteria which have low weights and high efficiency are open to improvement.

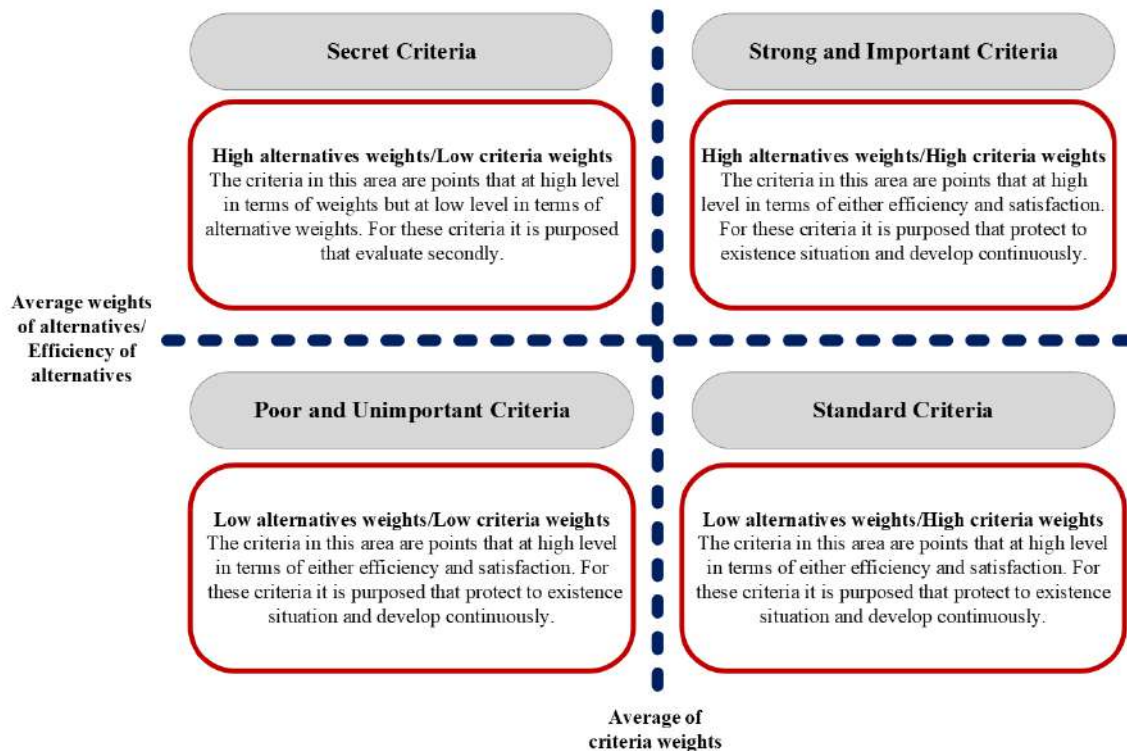


Figure 2. Performance analysis model for supplier selection criteria

The results of the performance analysis model for supplier selection criteria are shown in Figure 4. For the supplier from Turkey-*TR*, the highest QC's weights are handled as "Experiences in the field (QC3)", "Location (QC5)", and "Discount (QC8)". They are QCs that must be followed and continued to increase. As seen from the matrix of *TR*, there are not any QC which is located in strong and important criteria areas. This means, the supplier must focus on standard criteria and increase the quality characteristics which are "Quality certification systems (QC4)", "Management and organizational culture (QC9)", "Financial stability (QC6)", "Reliability of the raw material to the customer and manufacturing processes (QC7)" and Technical capacity (QC11)". For the first supplier from Germany-*DE1*, "Number of patents (QC1)" and "Reverse logistics rate (QC2)" have the highest importance weights and these must be continuously followed and increased. In addition, "Financial stability (QC6)", "Reliability of the raw material to the customer and manufacturing processes (QC7)", "Management and organizational culture (QC9)" and "Technical capacity (QC11)" criteria/QCs have lower efficiency, so the suppliers must focus on them and increase the performance. The second supplier from Germany-*DE2* has strong and important criteria which are "Quality certification systems (QC4)", "Financial stability (QC6)", "Reliability of the raw material to the customer and manufacturing processes (QC7)", "Management and organizational culture (QC9)" and "Technical capacity (QC11)". This company has the second highest average efficiency value and it must protect this strong qualification. Supplier from Austria-*AT* is the best alternative according to criteria/QCs and the "Quality certification systems (QC4)", "Financial stability (QC6)" and "Technical capacity (QC11)" criteria are determined as strong and important criteria. The weights of criteria for this supplier according to criteria have the highest values and the company must protect and continuously improve their capabilities. Suppliers from Italy-*IT* have three strong and important

criteria which are "Reliability of the raw material to the customer and manufacturing processes (QC7)," "Management and organizational culture (QC9)", and "Technical capacity (QC11)". The company must protect this situation but it must increase the average efficiency value by developing criteria which are determined as standard and poor/unimportant criteria.

4.1. Sensitivity Analysis

In our study, we have determined the QRs' weights by using the NAHP technique and to show the sensitivity of our methodology we have compared the results by changing QRs' weights. In the first scenario, we considered equal QR weights, and in the second scenario, we asked experts to evaluate QRs' importance weights. In Table 17, the result of the sensitivity analysis is presented, and as can be seen in Figure 3, there is not any change in the ranking of alternatives. This can be interpreted as the weights of customer requirements do not influence the final rankings of alternatives.

Table 17. Sensitivity analysis results

	Methodology	Equal criteria weights	Experts' criteria evaluations
TR	0.1417	0.1425	0.1420
DE1	0.2003	0.2022	0.2025
DE2	0.2350	0.2314	0.2319
AT	0.2658	0.2703	0.2702
IT	0.1571	0.1536	0.1534

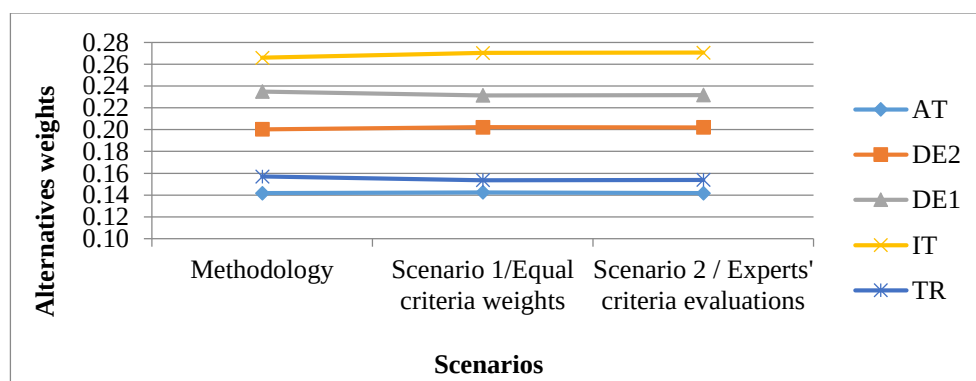


Figure 3. The ranking of alternatives

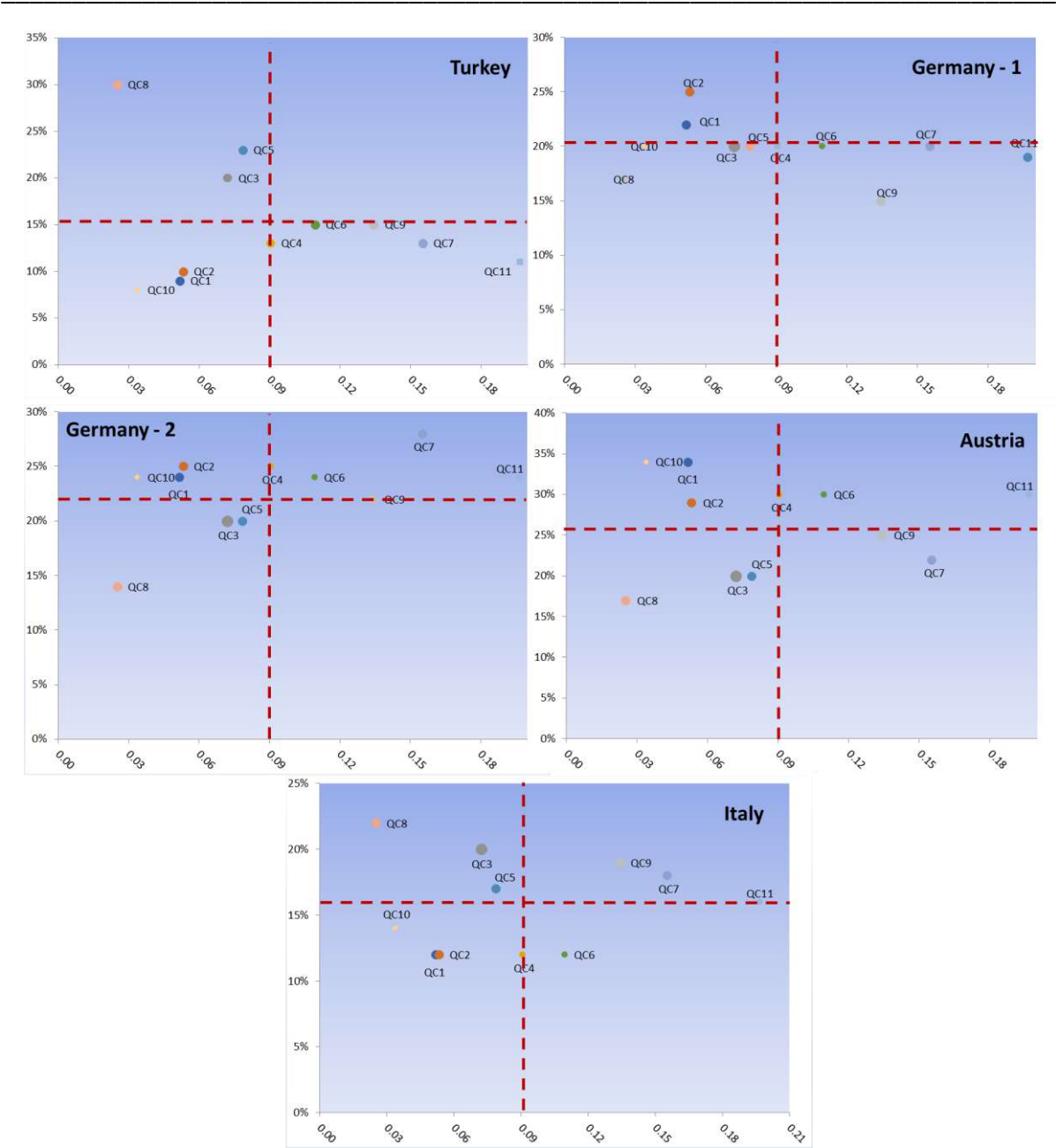


Figure 4. The result of criteria analysis for alternatives

5. Conclusion

Today, with the increase in challenging supply chain conditions for organizations, working with the most appropriate supplier gets more crucial. In addition, considering customers' expectations, needs and requirements is a necessity for organizations to survive for a long time. In our study, with awareness of the importance of these issues, we proposed a customer-oriented methodology to analyze QRs, QCs, and alternative suppliers. Supplier selection problems involve important strategic and operational decisions in a competitive environment with a short lifecycle and where the customer indirectly contributes to the product. In subsequent research, sustainability will play a significant role in the supplier selection process. Thus, customer-

oriented supplier selection problems contribute to the literature by adding more environmentally friendly, social, and economic supplier selection criteria.

As a result of our research, although there are several publications that deal with the problem of supplier selection in a furniture production organization, no study has been found that deals with the problem of customer-oriented supplier selection with the QFD and AHP hybrid methodology. In addition, since the evaluators did not have precise information and made their evaluations with linguistic expressions, the evaluation was made with the neutrosophic sets. The business discussed in this study is a company that has customer-oriented production and designs serving in the kitchen furniture sector. For this reason, a model has been proposed in the study to make the most suitable supplier selection considering the requirements of the customers. The theoretical advancement and practical implications according to problem, methodology, and application are given in Table 18.

Table 18. *The theoretical advancements and practical impacts of key categories of this study*

Key Categories	Theoretical Advancements	Practical Impacts
<i>Customer-oriented supplier selection</i>	When the literature is investigated, it is realized that there is not any study about customer-oriented supplier selection for modular kitchen furniture production organization. This study is a novel application, and it will help decision-makers deal with this problem.	With the increase in demand in the kitchen furniture sector, it gets harder to be customer-oriented and to work with the most convenient suppliers. So this study will help sector professionals to manage their end-to-end supply chain process efficiently.
<i>Neutrosophic AHP-QFD</i>	The theory is advanced by applying the methodology to a totally new problem in the literature.	Experts in other industries can apply this methodology to determine the most appropriate suppliers to work with.
<i>Application</i>	This application contributes to the furniture production supply chain theory.	The application conducted is for modular kitchen furniture producers and when the literature is researched, it is found that there is not any study that handles this problem.

Neutrosophic AHP was applied to determine the importance levels of the criteria/customer requests determined in the first phase of the study. In the second phase, the quality requirements necessary to meet the customer demands were determined and the weights of the quality requirements were found with the quality function deployment approach. In the third phase, the neutrosophic AHP technique was used again to evaluate alternative suppliers, taking into account the weights of the determined quality requirements. As a result, the ranking of the alternatives was obtained as follows: Austria, Germany-2, Germany-1, Italy, and Turkey. The application of neutrosophic AHP in determining the importance levels of criteria and customer requests,

coupled with the integration of the quality function deployment approach in defining quality requirements, demonstrates a comprehensive and systematic methodology for supplier evaluation. The results of the study, which ranked Austria as the top supplier, Germany-2 and Germany-1 following closely, provide actionable insights for the company under consideration.

Our research has made a significant contribution to the field of supplier selection in the context of customer-oriented production within the kitchen furniture sector. The absence of prior studies addressing the specific problem of customer-oriented supplier selection using the AHP hybrid methodology highlights the novelty and importance of this research. The introduction of the neutrosophic sets in the evaluation process, particularly due to evaluators relying on linguistic expressions, adds a considerable dimension to the study. The proposed model aims to optimize supplier selection by considering customers' preferences, experts' opinions, and design-oriented suppliers.

The limitations of our study can be explained as follows: Because this methodology is applied in the kitchen furniture industry, the number of experts is limited. We have asked three experts for their evaluations. Increasing the number of organizations and sector experts may provide more holistic results but this process will take a longer time and be harder. In addition, the evaluations are made using neutrosophic AHP-QFD hybrid methodology. The neutrosophic fuzzy sets are used in all phases and this increases the complexity of the methodology.

This study will give insight to managers to evaluate the performance of suppliers and select the most suitable supplier who is more customer-focused. In the proposed method, we determine supplier selection criteria weights via the neutrosophic AHP technique and we related these criteria with QCs via the neutrosophic QFD approach to find the most appropriate supplier. While evaluating suppliers, using this hybrid decision-making approach will provide organizational efficiency by considering qualitative and quantitative criteria. Additionally, the performance analysis model used provides a decision support structure for managers. Thanks to this model, managers will be able to see the level of fulfillment of the criteria and enable action plans to be made regarding them.

In future studies, solutions can be developed by using different solution methodologies for customer-oriented supplier selection. Applications of the methodology used in this study in different sectors can be examined. In addition, while choosing a supplier, the criteria can be updated with a sustainability-oriented approach as well as a customer-oriented approach. Future studies could explore alternative solution methodologies for customer-oriented supplier selection and investigate the applicability of the proposed model in diverse industry sectors. Additionally, there is potential to update supplier selection criteria with a sustainability-oriented approach, aligning with the growing importance of environmentally conscious practices in contemporary business operations. Overall, this research lays the foundation for further advancements in the strategic process of supplier selection, emphasizing the need for ongoing innovation and adaptation in response to evolving business landscapes.

Data availability statement

Authors can be contacted if any other data related to the study is required.

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A Linguistic Neutrosophic TOPSIS Framework for Sustainable Biomass Composite Selection

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Abstract

The valorization of agricultural waste biomass as functional filler in FDM 3D printing composites represents a tangible contribution towards sustainable additive manufacturing and circular economy synergism. Choosing the right composite for FDM is difficult because there are a number of technical, economic and environmental criteria which conflict. Further, experts do not have complete certainty about any of the criteria. The indeterminacy level that characterises experts' judgement on emerging and only partially characterised materials cannot be independently quantified by classical fuzzy sets and intuitionistic fuzzy sets. The present research proposes a Linguistic Neutrosophic TOPSIS (LN-TOPSIS) technique to assess and rank five alternatives of agricultural waste biomass composite materials sourced from Tamil Nadu like Sugarcane Bagasse/PLA (A1), Rice Husk Silica/ABS (A2), Coir Fibre/PHA (A3), Bamboo Particle/Nylon (A4) and Groundnut Shell Powder/PETG (A5). The criteria entail six ones with technical, environmental and economic dimensions. This includes Mechanical Performance (C1) index, Printability (C2), Sustainability (C3), Raw material cost (C4), Regional availability (C5) and Surface quality (C6). Criterion weights originating from the Best-Worst Method (BWM) are obtained with a consistency ratio of 0.027. Expert evaluations are carried out according to a two-round Delphi protocol and the Linguistic Neutrosophic Weighted Averaging (LNWA) operator aggregates the panel judgements. Every language assessment is tethered to published quantitative material property data to ensure transparency and reproducibility. According to the analysis, Sugarcane Bagasse/PLA (A1) is the most preferred alternative (CC=0.861), followed by Bamboo Particle/Nylon (A4, CC=0.744) and Rice Husk Silica/ABS (A2, CC=0.518). Across all 36 systematic weight-perturbation scenarios, the ranking remains invariant, and in a Monte Carlo simulation using 1000 Dirichlet-sampled weight vectors, A1 ranked first in 96.7 % of the runs. The Spearman rank correlations between five existing MCDM methods are between 0.90 and 1.00. It is as per literature available to us, this is one of the earliest applications of linguistic neutrosophic MCDM to agriculture waste biomass composite for FDM 3D printing.

Keywords: linguistic neutrosophic numbers, TOPSIS, multi-criteria decision-making, FDM 3D printing, biomass composites.

1. Introduction

Additive manufacturing, and Fused Deposition Modelling (FDM) in particular, has evolved from a rapid prototyping to a production technology in sectors such as aerospace, automotive, biomedical and education (Balasubramanian et al., 2024). At the same time, the desire to restructure the industrial ecosystem towards a circular economy has increased the interest in substituting petroleum-based polymer filament with biodegradable and waste-based biocomposites.

India produces hundreds of millions of tonnes of crop residues a year. Tamil Nadu, one of India's largest producers of sugarcane, rice, coconut, groundnut and bamboo, contributes a huge amount. Using these agro-waste streams as functional fillers in FDM-grade polymer composites helps in relieving the burden at disposal, renders cost-effectiveness in raw materials, and brings biodegradability in printed parts (Liu et al., 2019). Despite a growing literature on individual biomass–polymer composites for FDM, a systematic multi-criteria comparative evaluation anchored to Tamil Nadu-sourced materials within a neutrosophic decision framework remains, on the basis of the literature reviewed here, largely unreported.

Material selection for FDM is a genuinely conflicting decision problem. Candidate materials exhibit incompatible performance profiles: the most sustainable option may print poorly, while a mechanically superior composite may carry an unacceptable cost or environmental burden. Classical scoring methods oversimplify this situation by requiring crisp numerical inputs, which are inappropriate for expert evaluations of novel or incompletely characterised materials.

Smarandache (1998) proposed neutrosophic set theory as a generalization of fuzzy sets (Zadeh, 1965) and intuitionistic fuzzy sets (Atanassov, 1986) which overcomes their limitation through an independent indeterminacy membership function. Fang and Ye (2017) embed linguistic variables within the neutrosophic framework to obtain Linguistic Neutrosophic Numbers (LNNs), which express both the linguistic label an expert proposes and the reliability associated with that label. Nagarajan et al. (2023) validated neutrosophic MCDM for 3D printer selection, providing a directly analogous methodological precedent for the present work.

The principal contributions of this study are: (i) an LN-TOPSIS framework that reformulates classical TOPSIS entirely within the LNN space, including an LNN complement operator for cost-type criterion normalisation; (ii) criterion weights derived by the Best-Worst Method under an explicit consistency check, eliminating arbitrary weight assignment; (iii) expert linguistic assessments anchored to published quantitative material property data; (iv) a structured two-round Delphi elicitation protocol with a reported concordance statistic; (v) a two-tier sensitivity analysis combining 36 systematic weight perturbations with a 1000-run Monte Carlo simulation; and (vi) comparative validation against five competing MCDM methods computed from independent inputs. The remainder of the paper is organised as follows. Section 2 reviews the relevant literature. Section 3 presents the mathematical preliminaries. Section 4 develops the proposed methodology. Section 5 describes the case study, and Section 6 reports the numerical results. Sections 7 and 8 present the sensitivity and comparative analyses. Section 9 discusses the findings, and Section 10 concludes.

2. Literature Review

2.1 Biomass Composites for FDM 3D Printing

As Liu et al. (2019) showed, the tensile strength of sugarcane bagasse cellulose fibres was achieved in a PLA matrix which still retained FDM printability at loadings of up to 15 wt% and had a tensile strength of 38–52 MPa depending on print orientation. Paternina Reyes et al (2024) synthesized rice husk/PLA composite filaments which have better stiffness than neat PLA. The authors reported surface roughness drawbacks at higher filler loadings.

The bamboo fibre composites for additive manufacturing were reviewed by Balasubramanian et al. (2024). The tensile strengths (45–65 MPa) of bamboo/nylon systems and good dimensional accuracy of the FDM-printed specimens were also reported. Durfina et al. (2025) examine PHA blend application as material for FDM. They find promising elongation at break as well as very good biocompatibility. High odds of nozzle clogging (over 10 wt% coir fibre loading). Guo et al. (2018) includes property comparisons for lignocellulosic fillers who examined toughening agents in wood flour/PLA composites.

The recent reports find that while the material class is gaining momentum, it is not without challenges. As indicated by Adjei-Yeboah and Jahan (2026), survey results on natural fibre reinforced polymer composites pointed out the common hindrances to printability were hydrophilicity of the fibre, thermal degradation during extrusion and weak interfacial bonding. On the other hand, Ahmad and others (2023) researched filament manufacture and investigation with specific emphasis on natural fibre feedstocks. Trivedi et al. provide information about the property ranges achievable through bast-fibre PLA systems (2023), while Zhou et al. put these developments in the full additive manufacturing picture (2024) and Sanivada et al. (2020).

Two findings from this literature bear directly on the criteria adopted here. First, filler loading trades tensile strength against stiffness and impact resistance: Somsuk et al. (2025), comparing rice husk and rice straw in PLA under identical conditions, observed declining tensile strength at higher loadings alongside improved flexural modulus and impact resistance, with finer particle size and alkali treatment moderating the loss. Mazzanti et al. (2019b) attribute this pattern to fibre morphology governing stress transfer. Second, particulate shell fillers behave differently from fibrous ones: Salmah et al. (2013) and Karagöz (2024) report that coconut and walnut shell powders raise stiffness and hardness while contributing little to tensile strength, which is the property profile expected of the groundnut shell system evaluated here as A5. Paulo et al. (2023) and Hassan and Dahy (2025) provide contrasting data for continuous and aligned fibre architectures, which lie outside the short-fibre regime considered in this study.

Safaei et al. (2024) reported that agricultural shell powders offer hardness improvements in polymer matrices comparable to those of other particulate lignocellulosic fillers, but groundnut shell powder remains unexplored as a dedicated FDM filler. This absence of published FDM-specific data for GSP/PETG constitutes an explicit knowledge gap and motivates the inclusion of A5 as an alternative whose evaluation necessarily carries elevated indeterminacy.

2.2 Neutrosophic and Linguistic MCDM Methods

Biswas et al. (2016) first adapted TOPSIS to the single-valued neutrosophic environment and demonstrated improved discriminative capability over fuzzy TOPSIS for problems with incomplete information. Ye (2015) proposed an extended TOPSIS method for single-valued neutrosophic linguistic numbers in group decision-making. Fang and Ye (2017) formally introduced LNNs together with the weighted averaging operator adopted here, and Ye (2017, 2019) subsequently developed Bonferroni mean and Einstein operators for the same setting.

Parallel work has refined the aggregation and ranking machinery on which such methods depend. Ye (2016) developed weighted arithmetic and geometric averaging operators for neutrosophic linguistic numbers and applied them to manufacturing alternative selection, establishing the operator family the present study draws on. Wang (2024) demonstrated that the choice of score function materially changes single-valued neutrosophic rankings and proposed a standard-coefficient formulation, which motivates the explicit justification of the score function adopted in Section 3.4 rather than treating it as an arbitrary convention.

Nagarajan et al. (2023) applied the neutrosophic MCDM approach to the selection of 3D printer using three methods N-TOPSIS, N-WASPAS and N-MOORA and consistency of, N-TOPSIS, N-WASPAS and N-MOORA was established. Nagarajan et al. (2023) used TODIM under single valued neutrosophic environment to investigate the barriers to Industry 5.0 adoption. Broumi et al. (2020) algorithms for Interval-valued Neutrosophic Network Analysis. These studies establish the methodological lineage of the present work.

Research gap. Notwithstanding the growth of both literatures, no study integrates them for Tamil Nadu agricultural waste biomass composite selection in FDM 3D printing with transparent weight derivation, structured Delphi elicitation and multi-method comparative validation. The present study addresses this gap.

3. Mathematical Preliminaries

3.1 Neutrosophic Sets and Single Valued Neutrosophic Sets

Definition 1 (Neutrosophic set, Smarandache, 1998). Let X be a universe of discourse. A neutrosophic set A in X is characterised by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$ and a falsity membership function $F_A(x)$, each mapping X into the non-standard unit interval $]0, 1+[$, subject to $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2 (Single valued neutrosophic set, Wang et al., 2010). A single valued neutrosophic set restricts T , I and F to the standard interval $[0, 1]$ with $0 \leq T + I + F \leq 3$. A single valued neutrosophic number is written $\tilde{n} = (T, I, F)$.

3.2 Linguistic Term Set

Definition 3 (Linguistic term set). An ordered seven-point set $S = \{s_0, s_1, \dots, s_6\}$ is adopted, consistent with the human capacity for reliable linguistic discrimination (Miller, 1956) and widely used in neutrosophic MCDM (Fang & Ye, 2017, Nagarajan et al., 2023). A higher subscript denotes a more favourable evaluation. The associated neutrosophic triplets are given in Table 1.

Table 1 Seven-point linguistic scale with associated neutrosophic membership values

Linguistic term	Symbol	Index	Truth (T)	Indeterminacy (I)	Falsity (F)
Very High / Very Good	VH / VG	s_6	0.90	0.10	0.10
High / Good	H / G	s_5	0.70	0.20	0.20
Medium High / Fairly Good	MH / FG	s_4	0.60	0.30	0.30
Medium	M	s_3	0.50	0.50	0.50
Medium Low / Fairly Poor	ML / FP	s_2	0.40	0.55	0.50
Low / Poor	L / P	s_1	0.20	0.70	0.70
Very Low / Very Poor	VL / VP	s_0	0.10	0.90	0.90

Note. Triplets follow Fang and Ye (2017). The scale is symmetric about s_3 in the linguistic index and monotone in T , with I and F increasing as the evaluation becomes less favourable.

3.3 Linguistic Neutrosophic Numbers

Definition 4 (Linguistic neutrosophic number, Fang & Ye, 2017). An LNN is the pair $\tilde{L} = \langle s_\alpha, (T, I, F) \rangle$ with $s_\alpha \in S$, $\alpha \in \{0, \dots, t\}$ and $T, I, F \in [0, 1]$ satisfying $0 \leq T + I + F \leq 3$, where $t = 6$ is the upper index of the linguistic scale. An LNN therefore encodes simultaneously what the expert asserts (the linguistic label) and how reliable that assertion is (the neutrosophic triplet).

Let $\tilde{L}_1 = \langle s_\alpha, (T_1, I_1, F_1) \rangle$ and $\tilde{L}_2 = \langle s_\beta, (T_2, I_2, F_2) \rangle$ be two LNNs and let $\lambda > 0$. The basic operations are:

$$\tilde{L}_1 \oplus \tilde{L}_2 = \langle s_{\alpha+\beta-\alpha\beta/t}, (T_1+T_2-T_1T_2, I_1I_2, F_1F_2) \rangle \quad (1)$$

$$\tilde{L}_1 \otimes \tilde{L}_2 = \langle s_{\alpha\beta/t}, (T_1T_2, I_1+I_2-I_1I_2, F_1+F_2-F_1F_2) \rangle \quad (2)$$

$$\lambda \tilde{L}_1 = \langle s_{t[1-(1-\alpha/t)^\lambda]}, (1-(1-T_1)^\lambda, I_1^\lambda, F_1^\lambda) \rangle \quad (3)$$

3.4 Score and Accuracy Functions

The score function serves as the primary ranking criterion. Its quadratic form captures the linguistic magnitude α/t and the neutrosophic reliability simultaneously, so that high indeterminacy depresses the score even when truth membership is large. The accuracy function breaks ties when two score values coincide.

Definition 5 (Score function; Fang & Ye, 2017). For an LNN $\tilde{L} = \langle s_\alpha, (T, I, F) \rangle$,

$$S(L) = (1/3t)(\alpha/t + T - I - F + 2)(\alpha/t + T - I/2 - F/2 + 1), \quad S(L) \in [0, 4/t] \quad (4)$$

Definition 6 (Accuracy function). For the same LNN,

$$H(L) = (1/2t)(\alpha/t + T - F + 1), \quad H(L) \in [0, 3/(2t)] \quad (5)$$

The attainable range of these functions follows from their algebraic form rather than from convention. Substituting the maximal LNN $\langle s_t, (1, 0, 0) \rangle$ into Eq. (4) gives $S = 12/(3t) = 4/t$, and the minimal LNN $\langle s_0, (0, 1, 1) \rangle$ gives $S = 0$, with $t = 6$ the score therefore maps onto $[0, 2/3]$ rather than onto the full unit interval. The accuracy function maps onto $[0, 3/(2t)] = [0, 1/4]$ by the same argument. Because both functions are used only ordinally to identify ideal solutions and to break ties, the restricted range does not affect any ranking, a rescaling $S^* = (t/4)S$ would map onto $[0, 1]$ if a normalised score were required for reporting.

Comparison law: if $S(\tilde{L}_1) > S(\tilde{L}_2)$ then $\tilde{L}_1 \succ \tilde{L}_2$; if the scores are equal, the LNN with the larger accuracy value is preferred.

3.5 The LNWA Aggregation Operator

Definition 7 (Linguistic neutrosophic weighted averaging operator; Fang & Ye, 2017). For LNNs $\tilde{L}_j = \langle s_{\alpha_j}, (T_j, I_j, F_j) \rangle$, $j = 1, \dots, n$, with weights $\omega_j \in [0,1]$ and $\sum \omega_j = 1$:

$$\text{LNWA}(\tilde{L}_1, \dots, \tilde{L}_n) = \langle s_{t[1-\prod(1-\alpha_j/t)^{\omega_j}]}, (1-\prod(1-T_j)^{\omega_j}), \prod I_j^{\omega_j}, \prod F_j^{\omega_j} \rangle \quad (6)$$

3.6 Distance Measure between LNNs

Definition 8 (Normalised Euclidean distance). For $\tilde{L}_1 = \langle s_{\alpha_1}, (T_1, I_1, F_1) \rangle$ and $\tilde{L}_2 = \langle s_{\alpha_2}, (T_2, I_2, F_2) \rangle$ (Biswas et al., 2016 Ye, 2015)

$$d(\tilde{L}_1, \tilde{L}_2) = \sqrt{\frac{1}{4} [(\alpha_1/t - \alpha_2/t)^2 + (T_1 - T_2)^2 + (I_1 - I_2)^2 + (F_1 - F_2)^2] } \quad (7)$$

The factor 1/4 weights the four LNN components equally and guarantees $d \in [0, 1]$. This measure satisfies non-negativity, symmetry, the identity of indiscernibles and the triangle inequality, and therefore constitutes a metric on the LNN space.

3.7 Theoretical Properties of the Framework

The operators introduced above are stated as definitions in the source literature but their properties are rarely verified for a particular parameterisation. The five propositions below establish the properties on which the proposed method depends. Each was additionally checked numerically over 10^4 randomly sampled LNN tuples; the verification code is included in the supplementary material.

Proposition 1 (The distance measure is a bounded metric). The function d of Eq. (7) is a metric on the LNN space $\Omega = [0,t] \times [0,1]^3$, and $d(\tilde{L}_1, \tilde{L}_2) \in [0,1]$ with both bounds attained.

Proof. Write $\varphi(\tilde{L}) = (\alpha/t, T, I, F) \in [0,1]^4$. Then $d(\tilde{L}_1, \tilde{L}_2) = \frac{1}{2} \|\varphi(\tilde{L}_1) - \varphi(\tilde{L}_2)\|_2$, so d is the Euclidean metric on $[0,1]^4$ composed with the injective map φ and scaled by $\frac{1}{2}$. Non-negativity and symmetry are immediate. Since φ is injective, $d(\tilde{L}_1, \tilde{L}_2) = 0$ forces $\varphi(\tilde{L}_1) = \varphi(\tilde{L}_2)$ and hence $\tilde{L}_1 = \tilde{L}_2$, giving the identity of indiscernibles. The triangle inequality is inherited from the Euclidean norm, positive scaling preserving it. For the bounds, each of the four coordinate differences lies in $[-1,1]$, so $\|\varphi(\tilde{L}_1) - \varphi(\tilde{L}_2)\|_2 \leq 2$ and $d \leq 1$, with equality at $\tilde{L}_1 = \langle s_0, (0,0,0) \rangle$ and $\tilde{L}_2 = \langle s_t, (1,1,1) \rangle$; the lower bound is attained whenever $\tilde{L}_1 = \tilde{L}_2$. ■

The scaling factor $\frac{1}{4}$ inside the radical, which is often presented as a normalisation convention, is therefore exactly what places d on the unit interval.

Proposition 2 (Strict monotonicity of the score function). On the interior of Ω , S is strictly increasing in α and in T , and strictly decreasing in I and in F . Its range is $[0, 4/t]$, attained at $\text{DZ}(0,1,1)$ and $\text{DZ}(1,0,0)$ respectively.

Proof. Put $u = \alpha/t$ and write $S = (1/3t) \cdot A \cdot B$ with $A = u + T - I - F + 2$ and $B = u + T - I/2 - F/2 + 1$. On Ω we have $A \geq 0$, since its minimum over the box is attained at $(u, T, I, F) = (0, 0, 1, 1)$ where $A = 0$, and likewise $B \geq 0$ with the same minimiser. Differentiating, $\partial S/\partial u = \partial S/\partial T = (1/3t)(A + B)$ and $\partial S/\partial I = -(1/3t)(B + A/2)$, $\partial S/\partial F = -(1/3t)(B + A/2)$. Since A and B vanish simultaneously only at the single corner $(0, 0, 1, 1)$, we have $A + B > 0$ on the interior, so the first two partials are strictly positive and the last two strictly negative there. Evaluating at the extreme points gives $A = 4$, $B = 3$ and $S = 12/(3t) = 4/t$ at $DZ(1, 0, 0)$ and $A = B = 0$ with $S = 0$ at $DZ(0, 1, 1)$.

Proposition 2 is what licenses the interpretation placed on the score throughout: an alternative cannot improve its score by becoming less certain, because indeterminacy enters with a strictly negative derivative. It also supplies the corrected codomain used in Section 3.4.

Proposition 3 (LNWA is idempotent, bounded and monotone). For LNNs $\tilde{L}_1, \dots, \tilde{L}_n$ and weights ω summing to one, (i) if $\tilde{L}_j = \tilde{L}$ for all j then $LNWA = \tilde{L}$ (ii) each component of LNWA lies between the smallest and largest corresponding component of its arguments (iii) LNWA is non decreasing in each truth component and non increasing in each indeterminacy and falsity component.

Proof. (i) With $\alpha_j = \alpha$ for all j , the linguistic component becomes $t[1 - \prod((1-\alpha/t)^{\omega_j})] = t[1 - (1-\alpha/t)^{\sum \omega_j}] = t[1 - (1-\alpha/t)] = \alpha$, and identically $1 - (1-T)^{\wedge 1} = T$, $I^{\wedge 1} = I$, $F^{\wedge 1} = F$. (ii) The maps $x \mapsto 1 - \prod((1-x_j)^{\omega_j})$ and $x \mapsto \prod[x_j^{\omega_j}]$ are weighted quasi-arithmetic means, each lying between the minimum and maximum of its arguments, applying this coordinatewise gives the claim. (iii) Each component is differentiable and monotone in the stated direction on $(0, 1)$, since $\partial/\partial T_k [1 - \prod((1-T_j)^{\omega_j})] = \omega_k(1-T_k)^{\{\omega_k-1\}} \prod_{j \neq k} (1-T_j)^{\omega_j} > 0$, and similarly for the multiplicative components.

Idempotency matters for the Delphi design: if the panel reaches full agreement on a cell, aggregation returns that shared judgement unchanged rather than distorting it.

Proposition 4 (The complement is an involution and reverses order on the linguistic scale). The operator $\neg$ of Eq. (8) satisfies $\neg\neg\tilde{L} = \tilde{L}$ for every LNN. Moreover, restricted to the seven-point scale S of Table 1, $\neg$ reverses the score order exactly: $S(\neg s_0) > S(\neg s_1) > \dots > S(\neg s_6)$.

Proof. For the first claim, $\neg\tilde{L} = \langle s_{\neg\alpha}, (F, 1-I, T) \rangle$, so $\neg\neg\tilde{L} = \langle s_{\neg(t-(t-\alpha))}, (T, 1-(1-I), F) \rangle = \langle s_{\neg\alpha}, (T, I, F) \rangle$. For the second, the seven scale points are symmetric in the linguistic index and satisfy $F = T'$ and $I \approx 1 - I'$ for the reflected term, so applying $\neg$ maps each term onto the neutrosophic profile of its reflection; evaluating Eq. (4) at all seven complemented terms yields the strictly decreasing sequence 0.5756, 0.3848, 0.2242, 0.1667, 0.0728, 0.0297, 0.0033 for s_0 through s_6 .

Remark. Order reversal is a property of the admissible domain, not of Ω in general. Over arbitrary tuples in $[0, t] \times [0, 1]^3$ the map $\neg$ is not strictly order-reversing, because the reflection $I \mapsto 1 - I$ is independent of T and F there. This is not a defect for the present method, whose entries are LNWA aggregates of scale points: reversal was verified to hold for all 399 comparable pairs in the aggregated matrix of Table 10. It does, however, mean that $\neg$ should not be applied to LNNs constructed outside a symmetric linguistic scale without re-checking the property.

Proposition 5 (Bounds and attainment of the closeness coefficient). For every alternative, $CC_i \in [0, 1]$. Furthermore $CC_i = 1$ if and only if A_i attains the positive ideal solution on every criterion carrying non zero weight, and $CC_i = 0$ if and only if it attains the negative ideal solution on every such criterion.

Proof. By Proposition 1 the separation measures are non-negative sums of non negatively weighted distances, so $d^+_i \geq 0$ and $d^-_i \geq 0$ and hence $CC_i = d^-_i / (d^+_i + d^-_i) \in [0,1]$ whenever the denominator is positive. $CC_i = 1$ requires $d^+_i = 0$, that is $\sum_j w_j d(\tilde{r}_{ij}, \tilde{r}^+_j) = 0$, as every term is non negative this forces $d(\tilde{r}_{ij}, \tilde{r}^+_j) = 0$ for each j with $w_j > 0$, and the identity of indiscernibles then gives $\tilde{r}_{ij} = \tilde{r}^+_j$. The argument for $CC_i = 0$ is symmetric. The denominator vanishes only in the degenerate case where the positive and negative ideal solutions coincide on every weighted criterion, that is when all alternatives are indistinguishable.

Because the ideal solutions are drawn from the observed alternatives rather than from an abstract reference point, both bounds are attained in practice only by an alternative that dominates or is dominated on every criterion; the interior values reported in Table 12 are therefore the expected outcome for a set of genuinely conflicting alternatives.

4. The Proposed LN-TOPSIS Framework

The concept of TOPSIS was first introduced by Hwang and Yoon in 1981. TOPSIS or Technique for Order of Preference by Similarity to an Ideal Solution, is a method that determines the ranking of alternatives in relation to a positive ideal solution and a negative ideal solution. The LN-TOPSIS method reshapes every computing step within the LNN space to make sure that no defuzzification or scalarisation happens before the closeness coefficient is computed. The present formulation is distinguished by two choices.

Cost-criterion normalisation. Rather than inverting numerical values, cost-type criteria are normalised by the LNN complement, which maps a favourable-in-cost evaluation onto its benefit-equivalent while correctly transposing the truth and falsity components and reflecting the indeterminacy:

$$\tilde{r}_{ij} = \neg L_{ij} = \langle s_{t-\omega}(F, 1-I, T) \rangle \text{ (cost); } \tilde{r}_{ij} = L_{ij} \text{ (benefit)} \quad (8)$$

Weighted separation measures. Criterion weights enter the separation measures rather than the decision matrix. Applying scalar multiplication (Eq. 3) to the matrix and then computing unweighted distances would apply the weights twice — once in compressing the LNN towards the origin and again implicitly through the resulting distances — and would also distort the linguistic index non-linearly. The present formulation follows Biswas et al. (2016) and Ye (2015):

$$d^+_i = \sum_{j=1}^n w_j \cdot d(\tilde{r}_{ij}, \tilde{r}^+_j) \quad (9)$$

$$d^-_i = \sum_{j=1}^n w_j \cdot d(\tilde{r}_{ij}, \tilde{r}^-_j) \quad (10)$$

$$CC_i = d^-_i / (d^+_i + d^-_i), \quad CC_i \in [0, 1] \quad (11)$$

Alternatives are ranked in descending order of CC_i . Algorithm 1 states the complete procedure, and Figure 1 presents the corresponding flowchart.

Algorithm 1 LN-TOPSIS for biomass composite material selection

Input: Alternatives $A = \{A_1, \dots, A_m\}$; criteria $C = \{C_1, \dots, C_n\}$; decision makers $DM = \{DM_1, \dots, DM_k\}$;
linguistic scale S ; criterion types (benefit / cost)
Output: Ranked alternatives with closeness coefficients CC_i

```

1  for each  $DM_k$  do                                ▷ Delphi Rounds 1 and 2
2      construct linguistic matrix  $\tilde{D}^k = [\tilde{L}^k_{ij}]_{m \times n}$ , each entry drawn from Table 1
3  end for
4  for each (i, j) do                                ▷ Step 2: panel aggregation
5       $\tilde{L}^1_{ij} \leftarrow \text{LNWA}(\tilde{L}^1_{ij}, \dots, \tilde{L}^k_{ij})$  with  $w_k = 1/K$                 (Eq. 6)
6  end for
7  for each (i, j) do                                ▷ Step 3: normalisation
```

```

8       $\tilde{r}_{ij} \leftarrow -\tilde{l}_{ij}$  if  $C_j$  is a cost criterion, else  $\tilde{l}_{ij}$  (Eq. 8)
9  end for
10 identify best criterion  $C^B$  and worst criterion  $C^W$  ▷ Step 4: BWM weights
11 elicit vectors  $A^B$  (best-to-others) and  $A^W$  (others-to-worst)
12 solve  $\min \xi$  s.t.  $|w^B/w_j - a_j^B| \leq \xi$ ,  $|w_j/w^W - a_j^W| \leq \xi$ ,  $\sum w_j = 1$ ,  $w_j \geq 0$ 
13 verify consistency ratio  $CR = \xi^* / CI < 0.10$ 
14 for each criterion  $C_j$  do ▷ Step 5: ideal solutions
15      $\tilde{r}^+_j \leftarrow \operatorname{argmax}_i S(\tilde{r}_{ij})$ ;  $\tilde{r}^-_j \leftarrow \operatorname{argmin}_i S(\tilde{r}_{ij})$  (Eq. 4)
16 end for
17 for each alternative  $A_i$  do ▷ Steps 6–7
18      $d^+_i \leftarrow \sum_j w_j d(\tilde{r}_{ij}, \tilde{r}^+_j)$ ;  $d^-_i \leftarrow \sum_j w_j d(\tilde{r}_{ij}, \tilde{r}^-_j)$  (Eqs. 9–10)
19      $CC_i \leftarrow d^-_i / (d^+_i + d^-_i)$  (Eq. 11)
20 end for
21 return alternatives sorted by descending  $CC_i$ 

```

4.1 Computational Complexity

Let m be the number of alternatives, n the number of criteria and K the number of decision makers. Step 2 evaluates the LNWA operator once per matrix cell over K arguments, giving $O(Kmn)$ elementary operations; this dominates the ranking pipeline. Normalisation (Step 3), identification of the ideal solutions (Step 5) and the separation measures (Step 6) each require a single pass over the mn cells with constant work per cell, contributing $O(mn)$. Weight derivation is independent of m : the bisection on ξ terminates in $O(\log(1/\varepsilon))$ iterations, each solving a linear programme in n variables with $4n$ constraints. The overall cost of one complete ranking is therefore $O(Kmn + \log(1/\varepsilon) \cdot LP(n))$, which for the present problem ($m = 5$, $n = 6$, $K = 3$) is negligible.

The sensitivity analyses dominate total runtime. Systematic perturbation re-runs the ranking $6n$ times, and the Monte Carlo simulation N times, giving $O(N \cdot mn)$ for the latter since weights change but the aggregated matrix does not and can be computed once and reused. With $N = 1000$ the complete analysis reported here executes in under two seconds on a standard desktop, so the framework scales comfortably to problem sizes well beyond the present case study.

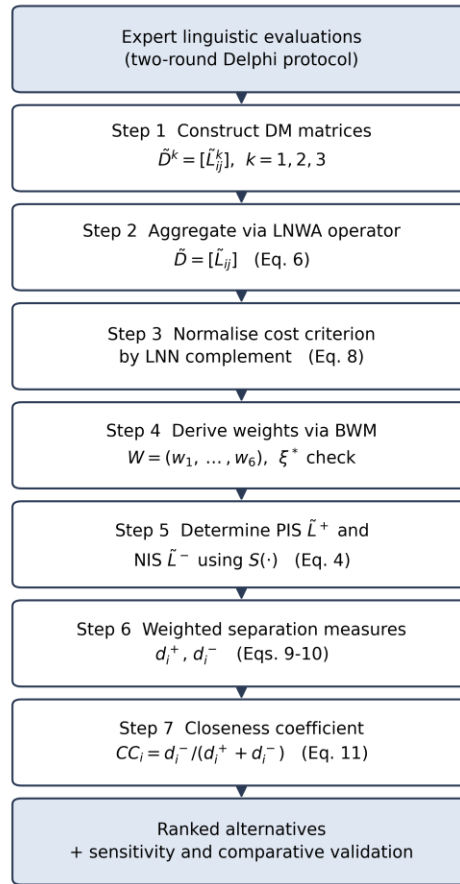


Figure 1 Algorithmic flowchart of the proposed LN-TOPSIS framework.

5. Case Study: Tamil Nadu Biomass Composite Selection

5.1 Alternatives

Five biomass-based composite filament alternatives, each derived from an agricultural waste stream available in Tamil Nadu, India, are considered (Table 2).

Table 2 The five biomass composite alternatives and representative feedstock availability in Tamil Nadu

Code	Composite	Biomass source	Polymer matrix	Representative sourcing districts
A1	Sugarcane Bagasse/PLA	SCB — sugar mills	PLA (biodegradable)	Erode, Villupuram, Cuddalore
A2	Rice Husk Silica/ABS	RHS — rice mills	ABS	Thanjavur, Tiruvarur
A3	Coir Fibre/PHA	CF — coconut processing	PHA (fully bio-based)	Nagercoil, Ramanathapuram
A4	Bamboo Particle/Nylon	BP — Western Ghats	Nylon PA12	Nilgiris, Coimbatore

Code	Composite	Biomass source	Polymer matrix	Representative sourcing districts
A5	Groundnut Shell Powder/PETG	GSP — groundnut mills	PETG	Vellore, Villupuram

5.2 Evaluation Criteria

Six criteria capture the technical, environmental and economic dimensions of the selection problem (Table 3). The criterion set was validated by literature review and by expert consensus during Round 1 of the Delphi process.

Table 3 *Evaluation criteria for biomass composite material selection*

Code	Criterion	Type	Dimension	Basis
C1	Mechanical Performance	Benefit	Technical	Tensile strength (MPa) and flexural modulus (GPa), anchored to Table 4
C2	Printability Index	Benefit	Technical	Filament feeding consistency, layer adhesion, warping, nozzle clogging risk at 10 wt% loading
C3	Sustainability Index	Benefit	Environmental	Biodegradability, carbon footprint and waste utilisation ratio from LCA literature
C4	Raw Material Cost	Cost	Economic	Estimated cost per kg of composite filament from Indian agri-waste market data
C5	Regional Availability	Benefit	Environmental	Tamil Nadu feedstock production volume (Mt/year)
C6	Surface Quality	Benefit	Technical	Surface roughness Ra (μm) of FDM-printed specimens; lower Ra is better

5.3 Literature-Anchored Quantitative Property Data

To anchor the linguistic assessments to objective evidence and to support reproducibility, Table 4 compiles quantitative property values for each alternative from the cited literature. Each decision maker consulted these values before assigning linguistic ratings. Where published values for a specific composite were unavailable — notably for A5 (GSP/PETG) — the corresponding expert ratings carry elevated indeterminacy, which propagates explicitly through the LNN arithmetic.

Table 4 *Literature-anchored quantitative material properties*

Alternative	Tensile strength (MPa)	Flexural modulus (GPa)	Surface roughness Ra (μm)	Biodegradability	Estimated cost (INR/kg)	Source
A1 SCB/PLA	38–52	2.8–3.5	8–14	High	350–500	Liu et al. (2019); Guo et al. (2018)
A2 RHS/ABS	35–48	2.4–3.1	12–20	Medium	400–600	Paternina Reyes et al. (2024)
A3 CF/PHA	28–40	1.8–2.6	14–22	Very High	450–650	Durfina et al. (2025)
A4 BP/Nylon	45–65	3.2–4.1	6–12	Medium	700–900	Balasubramanian et al. (2024)
A5 GSP/PETG a	25–35	1.5–2.2	18–28	Medium	300–450	Safaei et al. (2024)

Note. *a* Values for A5 are estimated from analogous agricultural shell powder composites because no dedicated GSP/PETG FDM data have been published; expert ratings for A5 accordingly carry $I \geq 0.50$. Biodegradability: Very High = >80% degradation in six months, High = 50–80%; Medium = 20–50% (adapted from Mazzanti et al., 2019a).

5.4 Expert Panel and Delphi Elicitation

Three domain experts were selected against three criteria: at least ten years of relevant experience; an active publication record comprising no fewer than five Scopus-indexed papers on FDM, composite materials or sustainable manufacturing; and no financial relationship with any manufacturer of the assessed materials. All experts provided written informed consent and a conflict-of-interest declaration before participating. Panel profiles appear in Table 5.

Table 5 Expert panel profiles

Expert	Specialisation	Experience (years)	Institution type	Scopus publications	COI
DM1	Materials engineering and FDM composites	15	Engineering college, Tamil Nadu	18+	None declared
DM2	Additive manufacturing and process engineering	12	Research institute, India	12+	None declared
DM3	Sustainable manufacturing and circular economy	10	Technical university, India	10+	None declared

The panel was deliberately small. The evaluation requires simultaneous command of FDM process behaviour, lignocellulosic composite characterisation and sustainability assessment, and the criterion set could not be reliably rated by a specialist in only one of these areas. Three experts satisfying all three selection criteria were available without compromising the conflict-of-interest requirement, and a structured two-round Delphi with a reported concordance statistic was adopted precisely because it extracts a defensible consensus from a small expert group rather than relying on numerical averaging over a large one. The consequences for precision are stated explicitly in Section 10.1, and expanding the panel is identified there as a priority for subsequent work.

Round 1. Each expert independently reviewed the quantitative property data (Table 4) and the criterion definitions (Table 3), then assigned linguistic evaluations to every alternative–criterion pair using the seven-point scale. Divergent ratings were defined as those separated by two or more scale points.

Round 2. Divergent ratings were circulated anonymously to the full panel with the accompanying justifications, and experts were invited to revise. Agreement on the finalised Round 2 matrices, measured by Kendall’s coefficient of concordance across the 30 alternative–criterion cells, is $W = 0.959$ ($\chi^2(29) = 83.40$, $p < 0.001$), indicating strong consensus. The finalised matrices constitute Tables 6–8.

5.5 Criterion Weights by the Best-Worst Method

To avoid arbitrary weight assignment, criterion weights are derived using the Best-Worst Method (Rezaei, 2015), which requires fewer pairwise comparisons than AHP and generally yields more consistent weights. Mechanical Performance (C1) was identified by panel consensus as the best criterion and Regional Availability (C5) as the worst. The best-to-others and others-to-worst vectors were elicited on a 1–9 scale as $A^B = (1, 2, 4, 8, 9, 3)$ and $A^W = (9, 4, 2, 1, 1, 3)$, read against C1–C6 respectively. Solving the minimax programme by bisection on ξ combined with linear-programming feasibility testing yields $\xi^* = 0.140$. With $a^{BW} = 9$, the corresponding consistency index is 5.23, giving a consistency ratio $CR = 0.027$, comfortably below the 0.10 threshold. The resulting weights appear in Table 6 and Figure 3.

Table 6 BWM comparison vectors, derived criterion weights and consistency check

Criterion	C1 Mech. perf.	C2 Printability	C3 Sustainability	C4 Cost	C5 Availability	C6 Surface quality
Best-to-others (A^B)	1	2	4	8	9	3
Others-to-worst (A^W)	9	4	2	1	1	3
Weight w_j	0.441	0.206	0.106	0.054	0.050	0.142
Rank	1	2	4	5	6	3

Note. $\xi^* = 0.140$, $CI(a^{BW} = 9) = 5.23$, $CR = 0.027 < 0.10$ (Rezaei, 2015). Mechanical Performance carries the largest weight because FDM components must satisfy minimum structural requirements before other attributes become decision relevant Surface Quality ranks third, reflecting its importance for the visible, tactile parts typical of STEM education kits and consumer prototyping.

6. Numerical Results

6.1 Finalised Linguistic Decision Matrices

Tables 6–8 report the Round 2 linguistic evaluations of the three decision makers. Each entry is a linguistic term from Table 1, assigned with reference to the quantitative anchors in Table 4.

Table 7 Finalised linguistic evaluation matrix of DM1 (Delphi Round 2)

Alternative	C1	C2	C3	C4 (cost)	C5	C6
A1 – SCB/PLA	G	VG	G	VL	VH	G
A2 – RHS/ABS	G	M	M	L	H	M
A3 – CF/PHA	M	M	VG	L	H	M
A4 – BP/Nylon	VG	G	M	M	M	VG
A5 – GSP/PETG	M	L	G	VL	H	L

Table 8 Finalised linguistic evaluation matrix of DM2 (Delphi Round 2)

Alternative	C1	C2	C3	C4 (cost)	C5	C6
A1 – SCB/PLA	VG	G	G	VL	VH	G
A2 – RHS/ABS	G	MH	MH	L	H	MH
A3 – CF/PHA	MH	M	VG	L	H	ML
A4 – BP/Nylon	VG	G	M	MH	ML	VG
A5 – GSP/PETG	M	ML	G	VL	MH	ML

Table 9 Finalised linguistic evaluation matrix of DM3 (Delphi Round 2)

Alternative	C1	C2	C3	C4 (cost)	C5	C6
A1 – SCB/PLA	G	VG	MH	VL	VG	MH
A2 – RHS/ABS	MH	MH	M	ML	H	M
A3 – CF/PHA	M	ML	VG	L	G	ML
A4 – BP/Nylon	VG	MH	ML	M	M	G

Alternative	C1	C2	C3	C4 (cost)	C5	C6
A5 – GSP/PETG	ML	L	MH	VL	H	L

6.2 Aggregated and Normalised LNN Decision Matrix

Applying the LNWA operator (Eq. 6) with equal decision-maker weights $\omega_k = 1/3$ produces the aggregated matrix in Table 10. The indeterminacy components for A5 are markedly higher than for the other alternatives across most criteria — for instance $\langle s_{2.70}, (0.469, 0.516, 0.500) \rangle$ under C1 — which is precisely the epistemic situation created by the absence of published GSP/PETG FDM data. Neither classical nor fuzzy TOPSIS can represent this distinction: both would record a low rating without recording that the rating is itself unreliable.

Table 10 Aggregated linguistic neutrosophic decision matrix $[\tilde{L}_{ij}]$

Alt.	C1	C2	C3	C4	C5	C6
A1	$\langle s_{6.00}, (0.79, 0.15, 0.15) \rangle$	$\langle s_{6.00}, (0.85, 0.12, 0.12) \rangle$	$\langle s_{4.74}, (0.67, 0.22, 0.22) \rangle$	$\langle s_{0.00}, (0.10, 0.90, 0.90) \rangle$	$\langle s_{6.00}, (0.90, 0.10, 0.10) \rangle$	$\langle s_{4.74}, (0.67, 0.22, 0.22) \rangle$
A2	$\langle s_{4.74}, (0.67, 0.22, 0.22) \rangle$	$\langle s_{3.71}, (0.56, 0.35, 0.35) \rangle$	$\langle s_{3.38}, (0.53, 0.42, 0.42) \rangle$	$\langle s_{1.36}, (0.27, 0.64, 0.62) \rangle$	$\langle s_{5.00}, (0.70, 0.20, 0.20) \rangle$	$\langle s_{3.38}, (0.53, 0.42, 0.42) \rangle$
A3	$\langle s_{3.38}, (0.53, 0.42, 0.42) \rangle$	$\langle s_{2.70}, (0.46, 0.51, 0.50) \rangle$	$\langle s_{6.00}, (0.90, 0.10, 0.10) \rangle$	$\langle s_{1.00}, (0.20, 0.70, 0.70) \rangle$	$\langle s_{5.00}, (0.70, 0.20, 0.20) \rangle$	$\langle s_{2.37}, (0.43, 0.53, 0.50) \rangle$
A4	$\langle s_{6.00}, (0.90, 0.10, 0.10) \rangle$	$\langle s_{4.74}, (0.67, 0.22, 0.22) \rangle$	$\langle s_{2.70}, (0.46, 0.51, 0.50) \rangle$	$\langle s_{3.38}, (0.53, 0.42, 0.42) \rangle$	$\langle s_{2.70}, (0.46, 0.51, 0.50) \rangle$	$\langle s_{6.00}, (0.85, 0.12, 0.12) \rangle$
A5	$\langle s_{2.70}, (0.46, 0.51, 0.50) \rangle$	$\langle s_{1.36}, (0.27, 0.64, 0.62) \rangle$	$\langle s_{4.74}, (0.67, 0.22, 0.22) \rangle$	$\langle s_{0.00}, (0.10, 0.90, 0.90) \rangle$	$\langle s_{4.74}, (0.67, 0.22, 0.22) \rangle$	$\langle s_{1.36}, (0.27, 0.64, 0.62) \rangle$

Note. Entries are written $\langle s_a, (T, I, F) \rangle$. C4 is a cost criterion and is normalised by the LNN complement (Eq. 8) before the ideal solutions are determined; the normalised C4 column is $\langle s_{6.00}, (0.900, 0.100, 0.100) \rangle$, $\langle s_{4.64}, (0.626, 0.354, 0.273) \rangle$, $\langle s_{5.00}, (0.700, 0.300, 0.200) \rangle$, $\langle s_{2.62}, (0.422, 0.578, 0.536) \rangle$ and $\langle s_{6.00}, (0.900, 0.100, 0.100) \rangle$ for A1–A5 respectively.

6.3 Score Matrix, Ideal Solutions and Worked Example

Score values $S(\tilde{r}_{ij})$ computed from the normalised matrix by Eq. (4) are reported in Table 11, together with the PIS and NIS identified per criterion. Because the score function is strictly increasing in the linguistic index and in truth membership, and strictly decreasing in indeterminacy and falsity, the ideal solutions correspond to the most and least reliable-and-favourable evaluations rather than merely the extreme linguistic labels.

Worked example. For A1 under C1 the three experts assigned G, VG and G, giving triplets (5, 0.70, 0.20), (6, 0.90, 0.10, 0.10) and (5, 0.70, 0.20, 0.20). With $\omega_k = 1/3$, Eq. (6) gives a linguistic index of $6[1 - ((1-5/6)(1-6/6)(1-5/6))^{(1/3)}]$. Because one expert assigned the maximal term s_6 , the corresponding factor vanishes and the aggregated index is exactly $s_{6.00}$. The truth component is $1 - [(0.30)(0.10)(0.30)]^{(1/3)} = 0.792$, while the indeterminacy and falsity components are $[(0.20)(0.10)(0.20)]^{(1/3)} = 0.159$, yielding $\tilde{L}_{11} = \langle s_{6.00}, (0.792, 0.159, 0.159) \rangle$. Substituting into Eq. (4):

$$S(\tilde{L}_{11}) = (1/18)(1.000 + 0.792 - 0.159 - 0.159 + 2)(1.000 + 0.792 - 0.0795 - 0.0795 + 1) = 0.508.$$

The PIS under C1 is A4's evaluation $\langle s_{6.00}, (0.900, 0.100, 0.100) \rangle$ with score 0.576, so A1 sits close to but below the ideal on the most heavily weighted criterion. The weighted separation contribution of this cell is $w_1 \cdot d(\tilde{L}_{11}, \tilde{r}_1^+) = 0.441 \times 0.068 = 0.030$.

Table 11 Score values $S(\tilde{r}_{ij})$ of the normalised LNN matrix with ideal solutions

Alternative	C1 (w = 0.441)	C2 (w = 0.206)	C3 (w = 0.106)	C4 (w = 0.054)	C5 (w = 0.050)	C6 (w = 0.142)
A1 – SCB/PLA	0.508	0.546	0.372	0.576	0.576	0.372
A2 – RHS/ABS	0.372	0.252	0.210	0.321	0.406	0.210
A3 – CF/PHA	0.210	0.149	0.576	0.385	0.406	0.131
A4 – BP/Nylon	0.576	0.372	0.149	0.126	0.149	0.546
A5 – GSP/PETG	0.149	0.059	0.372	0.576	0.372	0.059
PIS $\tilde{r}_j$	0.576	0.546	0.576	0.576	0.576	0.546
NIS $\tilde{r}_j$	0.149	0.059	0.149	0.126	0.149	0.059

Note. C4 scores are computed on the complement-normalised entries, so a higher value denotes a lower raw material cost. A1 and A5 share the maximum C4 score because both were rated VL on raw cost by all three experts.

6.4 Separation Measures and Final Ranking

Weighted separation measures and closeness coefficients follow from Eqs. (9)–(11); results are given in Table 12 and visualised in Figure 2. The final ranking is A1 > A4 > A2 > A3 > A5.

Table 12 Separation measures, closeness coefficients and final ranking

Alternative	d^+_i	d^-_i	CC_i	Rank
A1 – SCB/PLA	0.072	0.445	0.861	1 (best)
A2 – RHS/ABS	0.247	0.265	0.518	3
A3 – CF/PHA	0.332	0.178	0.349	4
A4 – BP/Nylon	0.130	0.378	0.744	2
A5 – GSP/PETG	0.439	0.070	0.138	5 (worst)

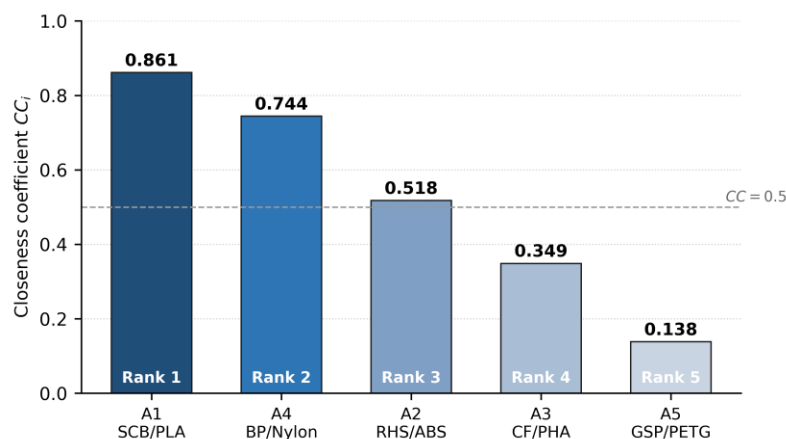


Figure 2 Closeness coefficients CC_i and final ranking of the five alternatives.

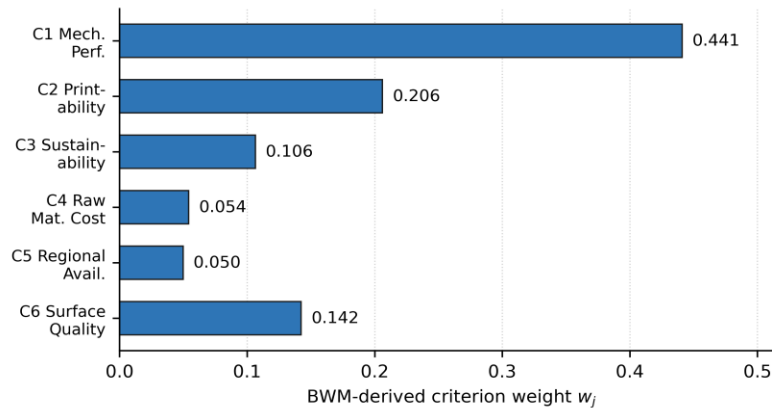


Figure 3 BWM-derived criterion weights ($\zeta^* = 0.140$, $CR = 0.027$).

7. Sensitivity Analysis

7.1 Systematic Weight Perturbation

Each criterion weight was perturbed by $\pm 10\%$, $\pm 20\%$ and $\pm 30\%$ in turn, with the remaining weights rescaled proportionally so that the weight vector continued to sum to unity. This produces 36 distinct scenarios. The ranking $A1 > A4 > A2 > A3 > A5$ is preserved in all 36 cases, and no rank reversal occurs at any perturbation level for any criterion. Table 13 reports the closeness coefficients at the $\pm 30\%$ extremes, where the effect is largest; Figure 4 plots the full trajectories.

Table 13 Closeness coefficients under $\pm 30\%$ weight perturbation of each criterion

Perturbed criterion	Level	A1	A2	A3	A4	A5	Ranking preserved
C1	-30%	0.862	0.499	0.380	0.692	0.167	Yes
C1	+30%	0.860	0.537	0.316	0.800	0.108	Yes
C2	-30%	0.848	0.517	0.354	0.745	0.151	Yes
C2	+30%	0.874	0.518	0.343	0.744	0.126	Yes
C3	-30%	0.869	0.528	0.328	0.768	0.123	Yes
C3	+30%	0.853	0.508	0.369	0.720	0.153	Yes
C4	-30%	0.859	0.518	0.343	0.757	0.124	Yes
C4	+30%	0.863	0.518	0.354	0.732	0.152	Yes
C5	-30%	0.859	0.516	0.344	0.755	0.131	Yes
C5	+30%	0.863	0.520	0.353	0.734	0.145	Yes
C6	-30%	0.868	0.523	0.355	0.729	0.146	Yes
C6	+30%	0.854	0.513	0.342	0.759	0.130	Yes

Note. All 36 scenarios (six criteria $\times$ six perturbation levels) preserve the baseline ranking; the $\pm 30\%$ extremes are shown because intermediate levels lie strictly between them.

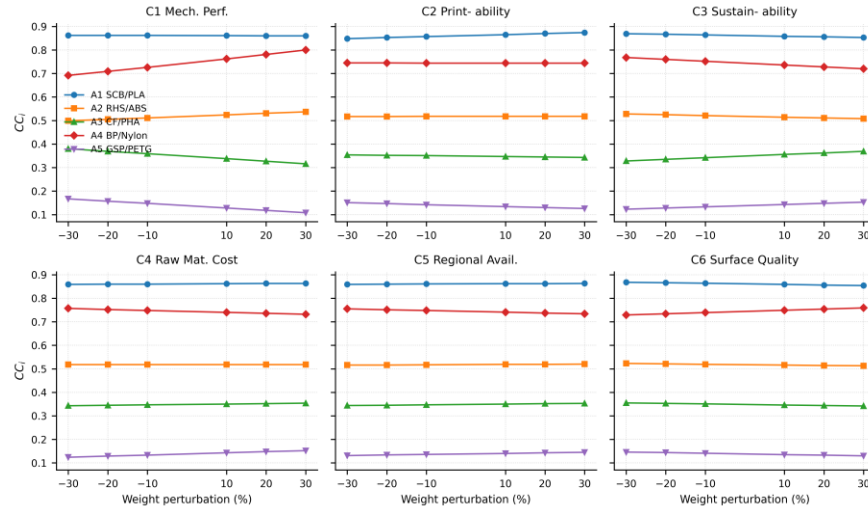


Figure 4 Closeness-coefficient trajectories under systematic weight perturbation of each criterion.

7.2 Monte Carlo Simulation

Systematic perturbation explores a neighbourhood of the elicited weight vector. To probe robustness across the entire weight simplex, 1000 weight vectors were drawn from a symmetric Dirichlet distribution with all concentration parameters equal to one, which is uniform over the simplex, and the full LN-TOPSIS procedure was executed for each draw (random seed 42). Table 14 reports the resulting rank frequency distribution and Figure 5 visualises it.

Table 14 Monte Carlo rank frequency distribution ($n = 1000$ uniform Dirichlet weight vectors)

Alternative	Rank 1 (%)	Rank 2 (%)	Rank 3 (%)	Rank 4 (%)	Rank 5 (%)	Mean rank	SD of CC_i
A1 – SCB/PLA	96.7	3.3	0.0	0.0	0.0	1.03	0.055
A2 – RHS/ABS	0.0	15.3	55.3	29.3	0.1	3.14	0.053
A3 – CF/PHA	1.1	32.6	18.8	47.4	0.1	3.13	0.107
A4 – BP/Nylon	2.2	44.5	11.0	9.7	32.6	3.26	0.177
A5 – GSP/PETG	0.0	4.3	14.9	13.6	67.2	4.44	0.148

The two poles of the ranking are highly stable: A1 occupies Rank 1 in 96.7% of draws with a mean rank of 1.03, and A5 occupies Rank 5 in 67.2% of draws. The intermediate positions are considerably less stable. A2, A3 and A4 have mean ranks of 3.14, 3.13 and 3.26 respectively — statistically indistinguishable — and A4 in particular ranges from Rank 1 in 2.2% of draws to Rank 5 in 32.6%. This dispersion is substantively informative rather than a defect of the method. A4 is the mechanically strongest alternative but the most expensive and least sustainable; under weight vectors that heavily favour cost or sustainability it falls to last place, while under mechanically dominated vectors it leads. The uniform Dirichlet prior samples such extreme vectors freely, whereas the elicited weights and the $\pm 30\%$ neighbourhood examined in Section 7.1 do not.

The appropriate reading is therefore twofold. The recommendation of A1 is robust both locally and globally. The ordering among A2, A3 and A4 is robust within the elicited weight neighbourhood but is contingent on the relative emphasis placed on mechanical performance, cost and sustainability, and should be revisited whenever an application context implies materially different priorities.

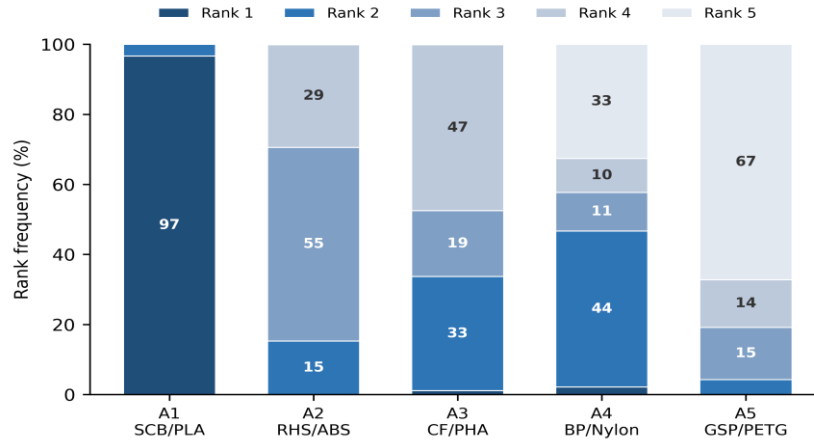


Figure 5 Monte Carlo rank frequency distribution over 1000 uniform Dirichlet weight vectors.

8. Comparative Study

The same decision problem was solved by five established MCDM methods. To avoid the circularity that would arise from feeding the neutrosophic score matrix into every comparator, each baseline is computed from inputs appropriate to it: classical TOPSIS operates on the quantitative property midpoints of Table 4; fuzzy TOPSIS operates on triangular fuzzy numbers derived directly from the same linguistic evaluations; and VIKOR ($\nu = 0.5$), WASPAS ($\lambda = 0.5$) and COPRAS operate on the defuzzified centroids of those triangular numbers. All methods use the BWM weights of Table 6. Results appear in Tables 15–16 and Figure 6.

Table 15 Comparative rankings across six MCDM methods

Alternative	LN-TOPSIS (proposed)	Classical TOPSIS	Fuzzy TOPSIS	VIKOR	WASPAS	COPRAS
A1 – SCB/PLA	1	2	1	1	1	1
A2 – RHS/ABS	3	3	3	3	3	3
A3 – CF/PHA	4	4	4	4	5	4
A4 – BP/Nylon	2	1	2	2	2	2
A5 – GSP/PETG	5	5	5	5	4	5

Table 16 Spearman rank correlation with the LN-TOPSIS ranking

Method	Spearman ρ	Divergence from LN-TOPSIS
Classical TOPSIS	0.90	A1/A4 order reversed
Fuzzy TOPSIS	1.00	None
VIKOR	1.00	None
WASPAS	0.90	A3/A5 order reversed
COPRAS	1.00	None

Three of the five comparators reproduce the LN-TOPSIS ranking exactly ($\rho = 1.00$) and the remaining two agree at $\rho = 0.90$, so all six methods identify A1 among the top two and none places A5 higher than fourth. The two divergences are informative. Classical TOPSIS, working from crisp property midpoints, promotes

A4 above A1 because it registers A4's superior tensile strength and surface finish without registering that the underlying expert judgements on A4's cost and sustainability are comparatively unfavourable and that its overall profile is narrower. WASPAS reverses A3 and A5 because its multiplicative component penalises A3's very weak printability score more aggressively than the additive distance measure of TOPSIS does.

Discriminative capability. The distinctive contribution of the neutrosophic representation is visible in the treatment of A5. Under fuzzy TOPSIS, A5's closeness coefficient is 0.160, which reflects only that experts rated it poorly. Under LN-TOPSIS it is 0.138, appreciably lower, because the indeterminacy components attached to A5's ratings ($I \geq 0.50$ across five of six criteria) enter the score function directly and depress it further. The framework therefore distinguishes an alternative that is confidently judged mediocre from one that is judged mediocre with low confidence — a distinction that is unavailable in the fuzzy setting and that matters materially when the decision concerns whether to commit experimental resources to an uncharacterised material.

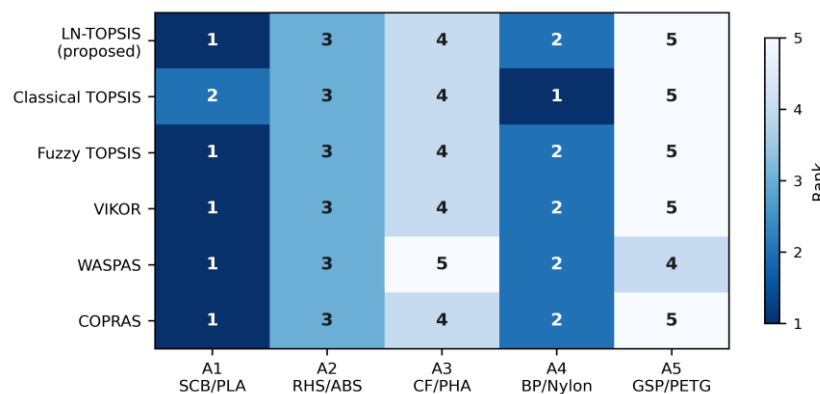


Figure 6 Rank agreement across the six MCDM methods.

9. Discussion

Sugarcane Bagasse/PLA emerges as the preferred alternative with $CC = 0.861$, a margin of 0.117 over the runner-up, and it holds Rank 1 in all 36 systematic perturbation scenarios and in 96.7% of Monte Carlo draws. Its advantage is one of balance rather than dominance: it attains the maximum score on three of six criteria (raw material cost, regional availability and printability) and the second-highest on mechanical performance, without falling to last place on any criterion. In the LNN representation its evaluations also carry consistently low indeterminacy, reflecting the comparatively mature published literature on bagasse-reinforced PLA. Practically, sugarcane bagasse is available in bulk from sugar mills in Erode, Villupuram and Cuddalore, and PLA is already the dominant FDM feedstock, so neither the filler supply chain nor the printing process requires reconfiguration.

Bamboo Particle/Nylon ranks second ($CC = 0.744$) on the strength of the highest mechanical performance and surface quality scores in the set. Its limitations are economic and environmental: it carries the highest estimated cost and the lowest score on regional availability, and nylon's non-biodegradability constrains its circularity credentials. The Monte Carlo results quantify this trade-off explicitly, with A4 appearing at both extremes of the ranking depending on the weight vector sampled. It is the appropriate recommendation for load-bearing applications where mechanical margin dominates cost.

Rice Husk Silica/ABS ranks third ($CC = 0.518$) with an unremarkable but consistent profile, while Coir Fibre/PHA ranks fourth ($CC = 0.349$) despite attaining the maximum sustainability score. The latter result illustrates why single-criterion optimisation misleads: A3's environmental credentials are offset by the weakest printability and surface quality scores in the set, and with sustainability weighted at 0.106 against the 0.789 commanded jointly by the three technical criteria, that trade is unfavourable under the elicited priorities. Where a decision context weights sustainability substantially higher, the sensitivity analysis indicates A3 becomes competitive.

Groundnut Shell Powder/PETG ranks last ($CC = 0.138$). This should be read as a statement about the current evidence base rather than about the material itself. A5 has the joint-lowest raw material cost and adequate regional availability, but no dedicated GSP/PETG FDM characterisation has been published, so every technical rating rests on analogy with other shell-powder composites. The neutrosophic framework records this as elevated indeterminacy rather than concealing it inside a point estimate, and the resulting low closeness coefficient is therefore best interpreted as identifying A5 as the alternative for which additional experimental characterisation would be most informative.

Methodologically, the framework contributes three elements beyond the existing linguistic neutrosophic TOPSIS literature. The LNN complement provides a principled normalisation for cost-type criteria that preserves the neutrosophic structure, whereas the common practice of inverting numerical values operates only on the linguistic index and leaves the reliability components untransformed. Deriving weights through BWM with an explicit consistency check removes the arbitrariness that attends direct weight assignment. Finally, the two-tier sensitivity design separates local robustness in the neighbourhood of the elicited weights from global robustness across the simplex, and reporting both prevents the overstatement of stability that a single analysis of either kind would invite.

10. Conclusions

This study developed a Linguistic Neutrosophic TOPSIS framework for selecting biomass-based composite materials for FDM 3D printing and applied it to five Tamil Nadu-sourced agricultural waste composites evaluated on six technical, environmental and economic criteria. Expert judgements were elicited through a two round Delphi protocol with strong final concordance ($W = 0.959$, $p < 0.001$), aggregated by the LNWA operator, and weighted by BWM derived coefficients with a consistency ratio of 0.027. Sugarcane Bagasse reinforced PLA is identified as the preferred alternative ($CC = 0.861$), ahead of Bamboo Particle reinforced Nylon ($CC = 0.744$). The ranking is invariant across all 36 systematic weight perturbation scenarios, A1 leads in 96.7% of 1000 Monte Carlo draws, and five independent MCDM comparators agree with the proposed ranking at $\rho \geq 0.90$.

10.1 Limitations

Four limitations qualify these findings. First, the evaluation rests on expert judgement anchored to published literature rather than on experimental characterisation of the five composites, and no specimens were fabricated or tested in this study; the results are therefore a structured evidence synthesis rather than an empirical comparison. Second, the panel comprises three experts, which is at the lower end of the acceptable range for Delphi studies and limits the precision of the concordance estimate. Third, criterion independence is assumed, whereas mechanical performance and surface quality are likely to be correlated through filler loading and particle morphology. Fourth, the property data underlying Table 4 are drawn

from studies that differ in print parameters, specimen geometry and testing standard, so cross-study comparability is imperfect — and for A5 the data are estimated by analogy rather than measured.

10.2 Practical Implications

For manufacturers and filament producers in Tamil Nadu, sugarcane bagasse/PLA represents the most defensible entry point for agri-waste valorisation in FDM, combining an established feedstock supply, low cost and compatibility with existing PLA printing practice. For policy and circular economy programmes, the analysis identifies bagasse and rice husk streams as those where processing investment would yield the broadest applicability. For researchers, the framework is directly transferable to other biomass systems, other additive manufacturing processes such as SLA and SLS, and other regional agricultural waste contexts.

10.3 Future Work

Priorities for subsequent work follow from the limitations. Physical fabrication and characterisation of SCB/PLA and GSP/PETG filaments — tensile testing, scanning electron microscopy and thermal analysis — would allow the expert-based assessments to be validated against measurement, with A5 the highest-value target given its current indeterminacy. Extending the panel and applying neutrosophic DEMATEL would address the independence assumption by modelling criterion interdependency explicitly. Integrating life cycle assessment data would replace the composite sustainability index with quantified carbon footprint differentials, and coupling the framework to digital twin simulation of FDM process parameters would supply process-specific printability data in place of expert estimates.

Declarations

Compliance with Ethical Standards

The authors confirm that this manuscript is an original work that has not been published previously and is not under consideration for publication elsewhere, in whole or in part. All authors have contributed significantly to the research and preparation of this manuscript and have approved its final version for submission.

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Competing Interests

The authors declare that there are no competing interests, financial or non-financial, that could have influenced the work reported in this manuscript.

Ethics Approval and Consent to Participate

This article does not contain any studies with human participants or animals performed by the author. Informed consent was not required as this study does not involve human participants.

Data availability statement

The datasets generated and analysed during the current study are available from the corresponding author upon reasonable request.

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Foundations of Neutrosophic N-Semirings Theory and Structural Properties

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Abstract: In this paper the concept of neutrosophic n-semirings $(S, *, *_1, *_2, *_3, \dots, *_n, *_n)$ has been introduced. The substructure of n-semirings $(S, *, *_1, *_2, *_3, \dots, *_n, *_n)$ has been defined and some useful results have been proved. Moreover, in order to familiarize the readers with these concepts some worthy examples have been coined. The left, right and two sided ideals of neutrosophic n-semirings have been paid a special heed. Finally we have turned our discussion towards the compatible and congruence relations and some remarkable properties have also been considered. Furthermore, we vividly have given examples to express the rationality of each notion popularized in this paper.

1. Introduction

To deal the vagueness in various real life problems in 1965, Lotfi, A. Zadeh [19] introduced the fuzzy set theory which is basically the generalization of crisp set. The concept of fuzzy sets involves the degree of membership for the objects to become the members of the set which lies between 0 and 1. In 1975, Zadeh [18] defined the interval valued fuzzy set which is the extension of fuzzy set. Later on Atanassov [9] in 1986, gave the idea of intuitionistic fuzzy set which is the generalization of fuzzy set. In 1988, Atanassov [8] extended the idea of intuitionistic fuzzy set by defining interval valued intuitionistic fuzzy set. In 1995 F.Smarandache [6] introduced the concept of neutrosophic set which is the generalization of classic set, fuzzy set and intuitionistic fuzzy set and is the powerful tool to deal the indeterminate and inconsistent information.

After the pioneering work of Smarandache, different researchers defined different structures on the grounds of neutrosophic set and have done some useful work about these structures.

This paper is structured as under. In section 2 we have recalled the basic definitions which were needed for the next section 3. In section 3 we have given the definitions of neutrosophic n-semirings, neutrosophic sub n-semirings, left, right and two sided ideals, compatible relations, congruence relations in neutrosophic n-semirings and each concept has been helped out for understanding by several vivid examples.

Keywords: Neutrosophic n-semirings, neutrosophic sub n-semirings, ideals and congruence, relations in neutrosophic structures.

2. Preliminaries

In this section we have recalled some basic definitions and have given some relevant examples which were needed to be clear with the concepts brought in the next section.

Definition 2.1 [19]. Let X be the universe of discourse and I be the unit interval $[0, 1]$ then a function $\theta : X \rightarrow [0, 1]$ is called fuzzy set.

Definition 2.2 [18]. Let X be the non-empty set and I be the unit interval then a function $\theta_1 : X \rightarrow [I]$ from X into the interval numbers in unit interval is called interval valued fuzzy set in X .

Definition 2.3 [9]. Let X be the non-empty set then by intuitionistic fuzzy set θ in X we mean

$\theta(x) = \langle \alpha_\theta(x), \beta_\theta(x) \rangle$ for all $x \in X$ where $\alpha_\theta(x)$ is the degree of membership and $\beta_\theta(x)$ is the degree of non-membership of an element of X in θ with conditions $0 \leq \alpha_\theta(x) \leq 1$, $0 \leq \beta_\theta(x) \leq 1$ and $0 \leq \alpha_\theta(x) + \beta_\theta(x) \leq 1$ for all $x \in X$.

Definition 2.4 [8]. Let X be the non-empty set. An interval valued intuitionistic fuzzy set θ in X is a set having the following form $\theta(x) = \langle \alpha_\theta(x), \beta_\theta(x) \rangle$ for all $x \in X$ where $\alpha_\theta(x) = [\alpha^-_\theta(x), \alpha^+_\theta(x)]$ and $\beta_\theta(x) = [\beta^-_\theta(x), \beta^+_\theta(x)]$ are the interval valued fuzzy sets and $\alpha_\theta(x)$ is known as the degree of membership and $\beta_\theta(x)$ is called degree of non-membership of the elements of X in θ .

Definition 2.5 [17]. Let X be the universe of discourse then a set $\Omega = \{ \langle x, \alpha_T(x), \alpha_I(x), \alpha_F(x) \rangle : x \in X \}$ is termed as a neutrosophic set where $\alpha_T, \alpha_I, \alpha_F : X \rightarrow]0, 1+[$ and the membership function α_T, α_I and α_F are the truth, indeterminacy and falsity membership degrees respectively.

Definition 2.6 [10]. A semiring $(S, *_1, *_2)$ is an algebraic structure in which $(S, *_1)$ and $(S, *_2)$ are semigroups such that $*_2$ is distributive over $*_1$.

Example.2.7 Let $N = \{1, 2, 3, \dots\}$ then under usual addition and multiplication N becomes semiring.

Definition 2.8 [16]. An algebraic structure $(S, *_1, *_2, *_3)$ is called a bisemiring if $(S, *_1)$, $(S, *_2)$ and $(S, *_3)$ are semigroups such that $\forall u, v$ and $w \in S$, $u *_2 (v *_1 w) = (u *_2 v) *_1 (u *_2 w)$, $(v *_1 w) *_2 u = (v *_2 u) *_1 (w *_2 u)$, $u *_3 (v *_2 w) = (u *_3 v) *_2 (u *_3 w)$ and $(v *_2 w) *_3 u = (v *_3 u) *_2 (w *_3 u)$.

Example 2.9 Let N be the set of natural numbers. Then $(N, *_1, *_2, *_3)$ is a bisemirings where $*_1$ gives minimum and $*_2$ gives the maximum while $*_3$ is the usual addition.

Definition 2.10 [12]. An algebraic structure $(S, *_1, *_2, *_3, \dots, *_n, *_{n+1})$ is called an n -semiring if $(S, *_1, *_2), (S, *_2, *_3), \dots, (S, *_{n-1}, *_n), (S, *_n, *_{n+1})$ are semirings, i.e. $(S, *_1), (S, *_2), \dots, (S, *_n), (S, *_{n+1})$ are semigroups, such that for all $s, w, z \in S$

$$S *_2 (W *_1 Z) = (S *_2 W) *_1 (S *_2 Z), S *_3 (W *_2 Z) = (S *_3 W) *_2 (S *_3 Z), \dots, S *_n (W *_n Z) = (S *_n W) *_n (S *_n Z), S *_n (W *_n Z) = (S *_n W) *_n (S *_n Z).$$

$$\text{And } (W *_1 Z) *_2 S = (W *_2 S) *_1 (Z *_2 S), (W *_2 Z) *_3 S = (W *_3 S) *_2 (Z *_3 S), \dots, (W *_n Z) *_n S = (W *_n S) *_n (Z *_n S), (W *_n Z) *_n S = (W *_n S) *_n (Z *_n S).$$

Example 2.11 Let $Z_5 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$. Let $*_1 = \min(a, b)$, $*_2 = \max(a, b)$, $*_3 = \text{addition (mod 5)}$ and $*_4 = \text{multiplication (mod 5)}$. Then $(Z_5, *_1, *_2, *_3, *_4)$ is a 3-semiring.

Now here, we define neutrosophic n-semirings, neutrosophic sub-n-semirings and give some examples.

3. Neutrosophic-N-Semirings

Here, we define neutrosophic n-semirings, discussed some properties and results and have given some examples.

Definition 3.1 An algebraic structure $(SUI, *_1, *_2, *_3, \dots, *_n, *_n)$ is called neutrosophic-n-semiring if $*_1, *_2, *_3, \dots, *_n$, and $*_n$ are the closed and associative binary operations and $*_2$ is distributive over $*_1$ and $*_3$ is distributive over $*_2, \dots, *_n$ is distributive over $*_n$ where S is n-semiring with respect to $*_1, *_2, *_3, \dots, *_n, *_n$ and I is the neutrosophic element ($I = I^2$) and $\langle SUI \rangle = \{a + bI : a, b \in S\}$.

Example 3.2 Let $S = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ and $a *_1 b = \min(a, b)$, $a *_2 b = \max(a, b)$, $a *_3 b = r$ (addition mod 5) and $a *_4 b = r$ (multiplication mod 5). Then S is the 3-semiring under the operations given above. Then $\langle SUI \rangle = \{0 + 0I, 0 + 1I, 0 + 2I, 0 + 3I, 0 + 4I, 1 + 0I, 1 + 1I, 1 + 2I, 1 + 3I, 1 + 4I, 2 + 0I, 2 + 1I, 2 + 2I, 2 + 3I, 2 + 4I, 3 + 0I, 3 + 1I, 3 + 2I, 3 + 3I, 3 + 4I, 4 + 0I, 4 + 1I, 4 + 2I, 4 + 3I, 4 + 4I\}$ is the neutrosophic-3-semiring under the same operations as that of S i.e. for all $a + bI, c + dI \in SUI$, $(a + bI) *_1 (c + dI) = (a *_1 c) + (b *_1 d)I$, $(a + bI) *_2 (c + dI) = (a *_2 c) + (b *_2 d)I$ and $(a + bI) *_3 (c + dI) = (a *_3 c) + (b *_3 d)I$ and $(a + bI) *_4 (c + dI) = (a *_4 c) + (b *_4 d)I$.

Example 3.3 Let $N = \{1, 2, 3, \dots\}$ and $a *_1 b = \min(a, b)$, $a *_2 b = \max(a, b)$, $a *_3 b = a + b$, $a *_4 b = ab$. Then $NUI = \{a + bI : a, b \in N\}$ with respect to $*_1, *_2, *_3$ and $*_4$ is the neutrosophic-3-semiring where for all $a + bI, c + dI \in NUI$, $(a + bI) *_1 (c + dI) = (a *_1 c) + (b *_1 d)I$, $(a + bI) *_2 (c + dI) = (a *_2 c) + (b *_2 d)I$, $(a + bI) *_3 (c + dI) = (a *_3 c) + (b *_3 d)I$ and $(a + bI) *_4 (c + dI) = (a *_4 c) + (b *_4 d)I$.

Definition 3.4 A proper subset P of a neutrosophic n-semiring $(SUI, *_1, *_2, *_3, \dots, *_n, *_n)$ is called neutrosophic sub-n-semiring if P itself is a neutrosophic-n-semiring under the same binary operations as that of SUI .

Example 3.5 Let $W = \{0, 1, 2, 3, \dots\}$ and $*_1, *_2, *_3$ and $*_4$ are the operations taken in the above Example 3.3 Then $WUI = \{a + bI : a, b \in W\}$ with respect to $*_1, *_2, *_3$ and $*_4$ is the neutrosophic-3-semiring where for all $a + bI, c + dI \in WUI$ $(a + bI) *_1 (c + dI) = (a *_1 c) + (b *_1 d)I$, $(a + bI) *_2 (c + dI) = (a *_2 c) + (b *_2 d)I$, $(a + bI) *_3 (c + dI) = (a *_3 c) + (b *_3 d)I$ and $(a + bI) *_4 (c + dI) = (a *_4 c) + (b *_4 d)I$. The set $NUI = \{a + bI : a, b \in N\}$ is the subset of $WUI = \{a + bI : a, b \in N\}$ so, becomes the neutrosophic sub-3-semiring of WUI under the operations of WUI .

Lemma 3.6 A subset P of neutrosophic n -semiring $(SUI, *_1, *_2, *_3, \dots, *_n, *_{n+1})$ is neutrosophic sub- n -semiring if and only if it satisfies the following:

$(a + bI) *_1 (c + dI) \in P, (a + bI) *_2 (c + dI) \in P$ and

$(a + bI) *_3 (c + dI) \in P, \dots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in P \forall (a + bI), (c + dI) \in P$.

Proof. Let $(P, *_1, *_2, *_3, \dots, *_{n+1})$ is a neutrosophic sub- n -semiring then by closure property

$\forall (a + bI), (c + dI) \in P, (a + bI) *_1 (c + dI) \in P, (a + bI) *_2 (c + dI) \in P, (a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in P$.

Conversely, let $\forall (a + bI), (c + dI) \in P$,

$(a + bI) *_1 (c + dI) \in P, (a + bI) *_2 (c + dI) \in P$ and

$(a + bI) *_3 (c + dI) \in P, \dots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in P$.

Then closure property is satisfied obviously. And since elements of P are of neutrosophic- n -semiring $(SUI, *_1, *_2, *_3, \dots, *_n, *_{n+1})$ and operations are also same as that of SUI and since associative and distributive laws are satisfied therein so, both laws are satisfied for P with respect to $*_1, *_2, *_3, \dots, *_n$ and $*_{n+1}$. Hence this makes $(SUI, *_1, *_2, *_3, \dots, *_n, *_{n+1})$ into neutrosophic sub- n -semiring.

Proposition 3.7 The intersection of any number of neutrosophic sub- n -semirings is either empty or neutrosophic sub- n -semiring.

Proof. Let $\{B_i; i \in J\}$ is the collection of neutrosophic sub- n -semirings of $(SUI, *_1, *_2, *_3, \dots, *_n, *_{n+1})$. Suppose $\bigcap B_i \neq \emptyset$ then we show that the subset $\bigcap B_i$ of SUI is the neutrosophic sub- n -semiring. Let $(a + bI)$ and $(c + dI) \in \bigcap B_i$ this implies that $(a + bI)$ and $(c + dI) \in B_i \forall i \in J$. This implies that $(a + bI) *_1 (c + dI), (a + bI) *_2 (c + dI), (a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in B_i \forall i \in J$ because each B_i is neutrosophic sub- n -semiring of $(SUI, *_1, *_2, *_3, \dots, *_n, *_{n+1})$. So, this implies that $(a + bI) *_1 (c + dI), (a + bI) *_2 (c + dI)$, and $(a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI) \in \bigcap B_i \forall i \in J$. Thus by the Lemma 3.10. $\bigcap B_i$ is the neutrosophic sub- n -semiring.

Remark 3.8 Union of neutrosophic sub n -semirings may or may not be neutrosophic n -semiring. Let us see an example.

Example 3.9 If $X = \{a, b, c, d, e\}$ then $(P(X), \cup, \cap, *_1, *_2)$ is a n -semiring where $A *_1 B = A$ and $A *_2 B = B$ Now let $P(X) = S$, then $\langle SUI \rangle = \{A + BI : A, B \in P(X)\}$ is the neutrosophic n -semiring. Suppose $S_1 = \{\{\}, \{a\}, \{a, b\}, X\}$ and $S_2 = \{\{\}, \{c\}, \{b, c\}, X\}$ are the subsets of S . Then $\langle S_1UI \rangle = \{C + DI : C, D \in S_1\}$ and $\langle S_2UI \rangle = \{E + FI : E, F \in S_2\}$. In tabular form:

$\langle S_1UI \rangle = \{\{\} + \{\}I, \{\} + \{a\}I, \{\} + \{a, b\}I, \{\} + XI, \{a\} + \{\}I, \{a\} + \{a\}I, \{a\} + \{a, b\}I, \{a\} + XI, \{a, b\} + \{a, b\}I, \{a, b\} + \{a\}I, \{a, b\} + \{a, b\}I, \{a, b\} + XI, X + \{\}I, X + \{a\}I, X + \{a, b\}I, X + XI\}$ and $\langle S_2UI \rangle = \{\{\} + \{\}I, \{\} + \{c\}I, \{\} + \{b, c\}I, \{\} + XI, \{c\} + \{\}I, \{c\} + \{c\}I, \{c\} + \{b, c\}I, \{c\} + XI, \{b, c\} + \{b, c\}I, \{b, c\} + \{c\}I, \{b, c\} + \{b, c\}I, \{b, c\} + XI, X + \{\}I, X + \{c\}I, X + \{b, c\}I, X + XI\}$ are the neutrosophic sub- n -semirings but their union $\langle S_1UI \rangle \cup \langle S_2UI \rangle = \{\{\} + \{\}I, \{\} + \{a\}I, \{\} + \{a, b\}I, \{\} + XI, \{a\} + \{\}$

$\{I, \{a\} + \{a\}I, \{a\} + \{a, b\}I, \{a\} + X I, \{a, b\} + \{a, b\}I, \{a, b\} + \{a\}I, \{a, b\} + \{a, b\}I, \{a, b\} + X I, X + \{ \} I, X + \{a\}I, X + \{a, b\}I, X + X I, \{ \} + \{ \} I, \{ \} + \{c\}I, \{ \} + \{b, c\}I, \{ \} + X I, \{c\} + \{ \} I, \{c\} + \{c\}I, \{c\} + \{b, c\}I, \{c\} + X I, \{b, c\} + \{b, c\}I, \{b, c\} + \{c\}I, \{b, c\} + \{b, c\}I, \{b, c\} + X I, X + \{ \} I, X + \{c\}I, X + \{b, c\}I, X + X I \}$ is not the neutrosophic sub-n-semiring because the operation of union is not closed i.e. $\{a, b\} + \{a, b\}I \cup \{b, c\} + \{b, c\}I = (\{a, b\} \cup \{b, c\}) + (\{a, b\} \cup \{b, c\})I = \{a, b, c\} + \{a, b, c\}I \notin \langle S_1 \cup I \rangle \cup \langle S_2 \cup I \rangle$

Now we will show that under what condition union of neutrosophic sub-n-semirings is again a neutrosophic sub-n-semiring.

Proposition 3.10 Let $\{B_k : k \in K\}$ be the family of neutrosophic sub-n-semirings. If $B_k \subseteq B_{k+1}$ then $\bigcup B_k$ is neutrosophic sub-n-semiring.

Proof. Let $a + bI, c + dI \in \bigcup B_k \Rightarrow$ that there exist some p and $q \in K$ such that $a + bI \in B_p$ and $c + dI \in B_q$. Let $q > p$ then $a + bI, c + dI \in B_q$. Now since B_q is neutrosophic sub-n-semiring so $(a + bI) *_1 (c + dI), (a + bI) *_2 (c + dI), (a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI), (a + bI) *_{n+1} (c + dI) \in B_q$ so, this implies that $(a + bI) *_1 (c + dI), (a + bI) *_2 (c + dI), (a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI), (a + bI) *_{n+1} (c + dI) \in \bigcup B_k$ thus by the Lemma 3.10 $\bigcup B_k$ is neutrosophic sub-n-semiring.

Corollary 3.11 Union of two neutrosophic sub-n-semirings N_1 and N_2 of N is again neutrosophic sub-n-semiring of N if either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$.

Proof. Let $N_1 \subseteq N_2$ then obviously $N_1 \cup N_2 = N_2$ so, it follows that $N_1 \cup N_2$ is a neutrosophic sub-n-semiring. Again let $N_2 \subseteq N_1$ then clearly $N_1 \cup N_2 = N_1$. Now as N_1 is neutrosophic sub-n-semiring so, $N_1 \cup N_2$ is, by transitive property.

3.2 Ideals in Neutrosophic-N-Semirings

In this section, we define left ideals in neutrosophic n-semirings and discuss some properties.

Definition 3.2.1 Let J be the non-empty subset of neutrosophic-n-semiring S , then J is called left ideal of S if $\forall (a_1 + b_1I), (a_2 + b_2I) \in J, (a_1 + b_1I) *_1 (a_2 + b_2I) \in J, (a + bI) *_2 (c + dI)$ and $(a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI)$ and $(a + bI) *_{n+1} (c + dI), \in J \forall (a + bI) \in S$ and $(c + dI) \in J$.

Example 3.2.2 Let $N = \{1, 2, 3, \dots\}$ is the set of natural numbers where $a *_1 b = a + b$, $a *_2 b = ab$, and $*_3, *_4$ are defined as $(a + bI) *_3 (c + dI) = 2(ac) + 2(bd)I$ and $(a + bI) *_4 (c + dI) = 3(ac) + 3(bd)I$ then $N \cup I = \{a + bI : a, b \in N\}$ becomes the neutrosophic-n-semiring and for set $S = \{2, 3, 4, \dots\}$ the structure $S \cup I = \{c + dI : c, d \in S\}$ becomes the left ideal because $\forall e + fI \in N, a + bI \in S$ and $c + dI \in S, (a + bI) *_1 (c + dI) \in S$ and $(e + fI) *_2 (c + dI), (e + fI) *_3 (c + dI)$ and $(e + fI) *_4 (c + dI) \in S$.

Theorem 3.2.3 The intersection of any number of left ideals of neutrosophic-n-semiring N is again a left ideal.

Proof. We suppose that $\{J_k : k \in K\}$ be the family of left ideals of neutrosophic n -semiring N . We have to show that $\bigcap J_k$ is the left ideal of N . Let $(a + bI), (c + dI) \in \bigcap J_k$ and $(e + fI) \in N$ this implies that $(a + bI), (c + dI) \in J_k \forall k \in K$, and $(e + fI) \in N$ this implies that $(a + bI) *_1 (c + dI), (e + fI) *_2 (a + bI), \dots, (e + fI) *_n (a + bI)$ and $(e + fI) *_n (a + bI) \in J_k \forall k \in K$ because each J_k is the left ideal so, this shows that $(e + fI) *_2 (a + bI), \dots, (e + fI) *_n (a + bI)$ and $(e + fI) *_n (a + bI) \in \bigcap J_k \forall k \in K$ so from hence we say that $\bigcap J_k$ is the left ideal.

Note : The case is same for right ideals and ideals of neutrosophic n -semirings.

Remark 3.2.4 Union of left (right) ideals of neutrosophic n -semiring may or may not be the left ideal again.

In order to be clearer with the preceding remark we give an example.

Example 3.2.5 The set $NUI = \{a + bI ; a, b \in N\}$ under the operations $(a + bI) *_1 (c + dI) = \gcd(a, c) + (\gcd(b, d))I$, $(a + bI) *_2 (c + dI) = \text{lcm}(a, c) + (\text{lcm}(b, d))I$, $(a + bI) *_3 (c + dI) = (c + dI)$ and $(a + bI) *_4 (c + dI) = (ac) + (bd)I$ is the neutrosophic-3-semiring and $2NUI = \{a + bI ; a, b \in 2N\}$ and $3NUI = \{a + bI : a, b \in 3N\}$ are the left ideals of NUI but $(2NUI) \cup (3NUI)$ is not the left ideal of NUI because $2 + 2I$ and $9 + 9I \in (2NUI) \cup (3NUI)$ and $(2 + 2I) *_1 (9 + 9I) = \gcd(2, 9) + (\gcd(2, 9))I = 1 + 1I$ which does not belong to $(2NUI) \cup (3NUI)$, hence because of failure of closure property $(2NUI) \cup (3NUI)$ does not become ideal again.

Now we write the condition for union of left ideals of neutrosophic n -semirings to become ideal so, here we go.

Theorem 3.2.6 Let $\{J_k : k \in K\}$ be the family of left ideals of neutrosophic n -semiring N . Then $\bigcup J_k$ is left ideal if $J_k \subseteq J_{k+1}$ for all $k \in K$.

Proof. Let $(a + bI), (c + dI) \in \bigcup J_k$ and $(e + fI) \in N$ then there exist p and $q \in K$ such that $(a + bI) \in J_p$ and $(c + dI) \in J_q$. Let $q > p$ then $(a + bI), (c + dI) \in J_q$. Now as J_q is an ideal so it follows that $(a + bI) *_1 (c + dI) \in J_q$ and $(e + fI) *_2 (a + bI), (e + fI) *_3 (a + bI), \dots, (e + fI) *_n (a + bI) \in J_q$. Thus $(e + fI) *_2 (a + bI), (e + fI) *_3 (a + bI), \dots, (e + fI) *_n (a + bI) \in \bigcup J_k$. So, it follows that $\bigcup J_k$ is the left ideal. In similar fashion we can show that $\bigcup J_k$ is right ideal. This completes the proof.

Theorem 3.2.7 Every left ideal of a neutrosophic n -semiring N is neutrosophic sub n -semiring of N .

Proof. Let J be the left ideal of N and for all $(a + bI), (c + dI) \in J$, $(a + bI) *_1 (c + dI) \in J$. Since $J \subseteq N$ so, $(a + bI) \in N$. Now as J is the left ideal of N so, $(a + bI) *_2 (c + dI), (a + bI) *_3 (c + dI), \dots, (a + bI) *_n (c + dI), (a + bI) *_n (c + dI) \in J$ and associative and distributive laws are surely satisfied because $J \subseteq N$ and N is neutrosophic n -semiring. Similar result can be proved for every right ideal.

Remark 3.2.8 Converse of the above theorem is not true in general and it can be seen from following example.

Example 3.2.9 Let $N = \{1, 2, 3, \dots\}$ be the set of natural numbers. Then $NUI = \{a + bI : a, b \in N\}$ under the operations defined as $(a + bI) *_1 (c + dI) = \min(a, c) + (\min(b, d))I$, $(a + bI) *_2 (c + dI) = \max(a, c) + (\max(b, d))I$, $(a + bI) *_3 (c + dI) = (a + c) + (b + d)I$ and $(a + bI) *_4 (c + dI) = (ac) + (bd)I$ becomes the neutrosophic-3-semiring and $2NUI = \{a + bI : a, b \in 2N\}$ being the subset of NUI becomes the neutrosophic sub-3-semiring. But $2NUI = \{a + bI : a, b \in 2N\}$ is not a left ideal because if we take $5 + 5I$ from NUI and $4 + 4I$ from $2NUI$ and apply operation $_2$ upon the selected members we will see that resultant element is not of $2NUI$ i.e. $(5 + 5I) *_2 (4 + 4I) = \max(5, 4) + (\max(5, 4))I = 5 + 5I \notin 2NUI$. Hence it is clear now that converse part of above theorem is not true in general.

3.3 Congruences in Neutrosophic N-Semirings

Here, we give the definition of congruence relations and pay attention to some important results.

Definition 3.3.1 Let $NUI = \{a + bI : a, b \in N\}$ be the neutrosophic-n-semiring under the operations $*_1, *_2, *_3, \dots, *_n, *_{n+1}$. A relation ρ on NUI is called left compatible if for all $(a + bI), (c + dI)$ and $(e + fI) \in NUI$ and $((a + bI), (c + dI)) \in \rho$, $((e + fI) *_1 (a + bI), (e + fI) *_1 (c + dI)) \in \rho$, $((e + fI) *_2 (a + bI), (e + fI) *_2 (c + dI)) \in \rho$, $((e + fI) *_3 (a + bI), (e + fI) *_3 (c + dI)) \in \rho$, $((e + fI) *_n (a + bI), (e + fI) *_n (c + dI)) \in \rho$ and $((e + fI) *_{n+1} (a + bI), (e + fI) *_{n+1} (c + dI)) \in \rho$.

The relation is called right compatible if for all $(a + bI), (c + dI)$ and $(e + fI) \in NUI$ and $((a + bI), (c + dI)) \in \rho$, $((a + bI) *_1 (e + fI), (c + dI) *_1 (e + fI)) \in \rho$, $((a + bI) *_2 (e + fI), (c + dI) *_2 (e + fI)) \in \rho$, $((a + bI) *_3 (e + fI), (c + dI) *_3 (e + fI)) \in \rho$, $((a + bI) *_n (e + fI), (c + dI) *_n (e + fI)) \in \rho$ and $((a + bI) *_{n+1} (e + fI), (c + dI) *_{n+1} (e + fI)) \in \rho$.

It is called compatible relation if for all $(a + bI), (c + dI), (e + fI)$ and $(g + hI) \in NUI$ and $((a + bI), (c + dI)) \in \rho$, $((e + fI), (g + hI)) \in \rho$, $((a + bI) *_1 (e + fI), (c + dI) *_1 (g + hI)) \in \rho$, $((a + bI) *_2 (e + fI), (c + dI) *_2 (g + hI)) \in \rho$, $((a + bI) *_3 (e + fI), (c + dI) *_3 (g + hI)) \in \rho$, $((a + bI) *_n (e + fI), (c + dI) *_n (g + hI)) \in \rho$ and $((a + bI) *_{n+1} (e + fI), (c + dI) *_{n+1} (g + hI)) \in \rho$.

A left (right) compatible equivalence relation is called left (right) congruence relation. A compatible equivalence relation is called congruence relation.

To understand the above concept we consider the following example.

Example 3.3.2 Consider the neutrosophic-n-semiring $(Z_5 \cup I, *_1, *_2, *_3, *_4)$ where for all $(a + bI), (c + dI) \in Z_5 \cup I$, $(a + bI) *_1 (c + dI) = \min((a + bI), (c + dI))$, $(a + bI) *_2 (c + dI) = \max((a + bI), (c + dI))$, $(a + bI) *_3 (c + dI) = (a + c) + (b + d)I$ (addition mod 5) and $(a + bI) *_4 (c + dI) = (ac) + (bd)I$ (multiplication mod 5) and consider $\rho = \{((a + bI), (c + dI)) : (a + bI) = (c + dI)\}$ then this relation is left compatible(left congruence), right compatible(right congruence) and compatible(congruence) and one can verify it easily.

Proposition 3.3.3. A relation ρ on a neutrosophic-n-semiring N is congruence relation if and only if it is both left and right congruence relation.

Proof. Let ρ be a congruence relation on a neutrosophic-n-semiring N . Let $(a + bI), (c + dI)$ and $(e + fI) \in N$ and $((a + bI), (c + dI)) \in \rho$. Then $((e + fI) *_1 (a + bI), (e + fI) *_1 (c + dI)), ((e + fI) *_2 (a + bI), (e + fI) *_2 (c + dI)), ((e + fI) *_3 (a + bI), (e + fI) *_3 (c + dI)), \dots, ((e + fI) *_n (a + bI), (e + fI) *_n (c + dI)), ((e + fI) *_{n+1} (a + bI), (e + fI) *_{n+1} (c + dI)) \in \rho$. This shows that ρ is a left congruence relation and similarly it can be shown that ρ is a right congruence relation.

Conversely, assume that ρ is both left and right congruence relation. Let $(a + bI), (c + dI), (e + fI)$ and $(g + hI) \in N$ such that $((a + bI), (c + dI))$ and $((e + fI), (g + hI)) \in \rho$. Therefore $((a + bI) *_1 (e + fI), (c + dI) *_1 (e + fI)) \in \rho$, $((a + bI) *_2 (e + fI), (c + dI) *_2 (e + fI)), ((a + bI) *_3 (e + fI), (c + dI) *_3 (e + fI)), \dots, ((a + bI) *_n (e + fI), (c + dI) *_n (e + fI))$ and $((a + bI) *_{n+1} (e + fI), (c + dI) *_{n+1} (e + fI)) \in \rho$ as ρ is right compatible. Also $((c + dI) *_1 (e + fI), (c + dI) *_1 (g + hI)) \in \rho$, $((c + dI) *_2 (e + fI), (c + dI) *_2 (g + hI)), ((c + dI) *_3 (e + fI), (c + dI) *_3 (g + hI)), \dots, ((c + dI) *_n (e + fI), (c + dI) *_n (g + hI)), ((c + dI) *_{n+1} (e + fI), (c + dI) *_{n+1} (g + hI)) \in \rho$ because ρ is left compatible relation. Therefore $((a + bI) *_1 (e + fI), (c + dI) *_1 (g + hI)), ((a + bI) *_2 (e + fI), (c + dI) *_2 (g + hI)), ((a + bI) *_3 (e + fI), (c + dI) *_3 (g + hI)), \dots, ((a + bI) *_n (e + fI), (c + dI) *_n (g + hI))$ and $((a + bI) *_{n+1} (e + fI), (c + dI) *_{n+1} (g + hI)) \in \rho$. Thus ρ is a congruence relation.

4. Conclusions. In this paper we have defined the neutrosophic n-semirings and then some examples have been coined to understand the notions. Moreover, we have defined their substructures and added some examples. Left (right) ideals have also been defined in this paper with examples. Compatible relations, equivalence relations and congruence relations have been defined too, with examples. Furthermore, for newly given concepts, different results and properties have been considered.

5. Future work. In future, the ideas can be extended to other structures like neutrosophic n-rings, neutrosophic n-bipolar semirings, neutrosophic neutrosophic n-bipolar rings. These notions with n-operations can be studied for multipolar structures neutrosophic semirings and rings and other extensions of neutrosophic sets. Moreover, all the results which have been studied in this manuscript can be carried out for above mentioned structures.

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(T, I, N, F) Neutrosophic Weak and Strong Paraconsistency and the Epistemics of Artificial Intelligence:

Uncertainty, Pluralism, and the Decolonial Turn in Alignment

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Abstract

Over the past two years we have developed, with collaborators across Ecuador, Cuba and the United States, a research program arguing that the standard mathematical machinery used to represent uncertainty in artificial intelligence is paraconsistency-intolerant: it cannot hold well-supported contradictory evidence in view without collapsing it into a single number. This paper draws together three strands of that program — an empirical study of epistemic uncertainty in large language models (“Breaking the Chains of Probability”), a proposal for value alignment under irreducible cultural pluralism (“Aligning with Whom?”), and a decolonial reading of paraconsistent logic’s Latin American genealogy (“Pluriversal Alignment”) — and reads them through the $(T, I, N, F) = (\text{Truth}, \text{Indeterminacy}, \text{Neutrality}, \text{Falsehood})$ Quadruple Neutrosophic Set and Logic, introduced by Smarandache (2017) and the taxonomy of paraconsistency degrees: strong, weak, and very weak. We show that the three papers are not merely thematically related but occupy successive layers of a single argument: a representational critique (Section 3), a normative-architectural proposal built on that critique (Section 4), and a philosophical-historical justification for why the critique matters and to whom it is owed (Section 5). We then extend the framework outward, sketching how (T, I, N, F) -style paraconsistent representations could inform medical differential diagnosis, legal adjudication, multi-sensor robotics, misinformation triage, and pedagogy — domains that, like AI alignment, are routinely forced to compress genuine, well-evidenced disagreement into a single scalar. We close with a critical discussion of the program’s methodological maturity: much of the empirical evidence is preliminary and not yet independently replicated, and the philosophical claims deserve engagement from outside the neutrosophic-logic community before they can be considered established.

Keywords: neutrosophic logic; paraconsistency; (T, I, N, F) ; component dependence; epistemic uncertainty; large language models; value alignment; pluriversality; decolonial critique.

1. Introduction

Every system that must act under uncertainty eventually faces the same design choice: what to do when the evidence for a claim and the evidence against it are both strong. Classical probability answers by fiat —

belief in a proposition and its negation must sum to one, so strong support for one side mechanically discounts the other. Fuzzy and intuitionistic-fuzzy logics relax this slightly but still cap the joint total. Dempster–Shafer evidence theory and Dezert-Smarandache Theory, when evidence genuinely conflicts, is known for producing combinations that most practitioners find counterintuitive, and its standard remedies discount the conflicting mass rather than preserve it. In each case the mathematics is built to resolve contradiction, not to represent it.

Neutrosophic logic, introduced in the late 1990s as an explicit generalization of fuzzy and intuitionistic-fuzzy logic (Smarandache, 1998), departs from this design choice. A neutrosophic assertion about a proposition A is not a single number but a triplet of independent degrees — truth T , indeterminacy I , and falsity F — each free to range over $[0, 1]$ with no constraint that they sum to any fixed value. When T and F are both high enough that their sum exceeds 1, the assertion is neither an error nor a rounding artifact: it is a representable state in which the evidence for A and the evidence against A are simultaneously strong.

The (T, I, N, F) refinement (Smarandache, 2017) supplements the classical three degrees with a fourth distinguishable component N , and gives this phenomenon a graded vocabulary. Section 2 states the resulting ladder formally.

This paper has two aims. The first is expository: to connect four documents that, read together, constitute a coherent research program applying (T, I, N, F) -style paraconsistency to artificial intelligence — the formal taxonomy itself, an empirical paper on epistemic uncertainty in LLMs, a normative-architectural paper on value alignment, and a historical-philosophical paper situating the enterprise within Latin American and Eastern European traditions of non-classical logic. The second aim is generative: to ask where else, beyond AI alignment, the same representational move — refusing to impose $T + F \leq 1$ — might do useful work, and to state plainly how far the existing evidence actually reaches.

2. Formal Preliminaries: The (T, I, N, F) Ladder of Paraconsistency

2.1 The underlying space

Let A be a proposition. A neutrosophic valuation assigns to A a quadruple

$$v(A) = (T, I, N, F) \in [0, 1]^4, \quad (1)$$

with no normalization constraint. Here T is the degree of directly supporting evidence, F the degree of directly contradicting evidence, and I the irreducible indeterminacy — the epistemic gap that neither supports nor contradicts. The fourth component N is the additional distinguishable degree introduced in the (T, I, N, F) refinement (Smarandache, 2017).

Two familiar frameworks embed as constrained subspaces of (1). The intuitionistic fuzzy framework (Atanassov, 1986) is the subspace

$$T + F \leq 1, \quad I = 1 - T - F, \quad N = 0, \quad (2)$$

and classical probability is the further restriction

$$I = N = 0, \quad T + F = 1. \quad (3)$$

Neutrosophic valuation therefore strictly generalizes both: every state either framework can express remains available, and states neither can express become available as well.

2.2 Three degrees of paraconsistency

Define the following three regimes, which partition the paraconsistent region of the space.

A valuation is strongly paraconsistent when

$$T + F > 1; \quad (4)$$

weakly paraconsistent when

$$T + F \leq 1 \quad \text{and} \quad (T + F + I > 1 \quad \text{or} \quad T + F + N > 1); \quad (5)$$

and very weakly paraconsistent when

$$T + F + I \leq 1, \quad T + F + N \leq 1, \quad \text{and} \quad T + F + I + N > 1. \quad (6)$$

The three conditions are mutually exclusive by construction, and (6) is non-vacuous: the valuation $(T, I, N, F) = (0.3, 0.3, 0.3, 0.3)$ satisfies $T + F = 0.6$, $T + F + I = T + F + N = 0.9$, and $T + F + I + N = 1.2$, and therefore lies in the very weak regime. Any valuation violating classical bookkeeping falls into exactly one of the three.

2.3 Why the ladder is diagnostic rather than decorative

- Strong paraconsistency, condition (4), signals a direct clash: the system has independent and substantial grounds to both affirm and deny the same claim, with neither indeterminacy nor neutrality playing any necessary role in producing the excess. This is the signature the alignment literature treats as diagnostic of genuine value conflict, as distinct from mere ignorance.
- Weak paraconsistency, condition (5), signals a compound condition: no single pair of components is individually contradictory, but the total evidentiary commitment — support, counter-support, and unresolved indeterminacy or neutrality together — exceeds what a normalized reporter could emit. It is a looser and more common departure, and it does not by itself license any claim about contested values.
- Very weak paraconsistency, condition (6), signals the faintest detectable departure: the excess appears only once every non-classical component is summed. It is useful mainly as an audit signal that the classical picture is being strained, even where nothing looks obviously contradictory.

Conflating the first two is the error this paper is most concerned to prevent. A system that reports a high incidence of (5) has established only that its outputs are not normalized; a system that reports (4) selectively, on items independently identified as value-contested and not on controls, has established something substantive. The practical value of the ladder is that it lets a system — or an auditor of that system — report not merely that a state is paraconsistent, but how paraconsistent and via which components.

2.4 Relativizing to the dependence structure

Conditions (4) to (6) take unity as the reference point, which is appropriate when the components are treated as mutually dependent. In the general single-valued setting the admissible bound on the sum depends on how many components are taken to be independent of one another. Writing $\beta(d)$ for the bound licensed by a declared dependence structure d :

$$\beta(d) = 1 \text{ (all dependent); } 2 \text{ (two dependent, the rest independent); } 3 \text{ (all mutually independent).} \quad (7)$$

A valuation is then paraconsistent relative to d when the observed sum exceeds $\beta(d)$. Under full dependence this coincides with the ladder above; under partial or full independence it is a strictly stronger requirement, since more of the space is licensed before any excess is declared. Condition (4) is unaffected: $T + F > 1$ remains the signature of direct clash regardless of what is assumed about the remaining components.

The practical consequence is a reporting obligation. Any empirical claim about how much paraconsistency a system exhibits is meaningless unless the assumed dependence structure is declared alongside it, because the same elicited valuation can be paraconsistent under one declaration and unremarkable under another. We adopt $\beta = 1$ throughout what follows, and state so explicitly, because the elicitation protocols described in Section 3 ask for the degrees as separate judgments about a single proposition.

3. Thread One: Breaking the Chains of Probability — Epistemic Uncertainty in LLMs

The empirical anchor of the program is our study of large language models' self-reported epistemic states (Leyva-Vázquez & Smarandache, 2026). Its starting observation is architectural. A transformer language model emits, at every step, a distribution over the vocabulary produced by a softmax, and that distribution necessarily sums to one. Any epistemic state read off that distribution inherits the constraint: if the credence assigned to “yes” and the credence assigned to “no” are recovered from the same normalized simplex, they compete for a fixed budget of probability mass, and their sum cannot exceed unity by construction.

This is not a peripheral observation, because it is precisely from that distribution that the mainstream uncertainty-quantification methods for language models read. Logit-based confidence, semantic entropy (Farquhar et al., 2024) and sampling-consistency measures such as SelfCheckGPT (Manakul et al., 2023) all extract their signal from normalized output probabilities. For every one of them, the following three situations are structurally indistinguishable: a genuine logical paradox, in which truth and falsity are both well supported; simple ignorance, in which neither is supported; and vagueness, in which the boundary itself is unclear. All three are pushed toward the same uncertain middle, not because the model has failed to distinguish them but because the measurement channel has no coordinate in which the distinction could be recorded. We call this the collapse of uncertainty, and we take it to be a property of the architecture as it is conventionally read, not a contingent shortcoming of any particular model.

The consequence for method is immediate, and it explains the design of the study that follows. If the constraint lies in the distributional channel, then a representation that escapes it must be relocated out of that channel. Eliciting T , I and F as separate reported quantities does exactly this: the degrees are carried as generated content rather than as competing probability mass, so no normalization applies to their sum even though each individual token is, as always, sampled from a normalized distribution. That the elicited triplets do exceed unity is therefore not evidence against the architectural claim. It is the predicted result of moving the epistemic report to a layer where the architectural constraint does not bind, and it is what makes the intervention worth making.

To test whether the collapse is an artifact of the reporting format rather than of the model's underlying competence, we elicited unconstrained neutrosophic judgments: rather than asking for a single confidence

score, we asked models to report separate degrees of truth, indeterminacy and falsity for statements drawn from five categories — logical paradoxes, epistemic ignorance, vagueness, ethical contradictions and future contingencies — under several prompting protocols. Freed from the summation constraint, models frequently produced states in which $T + I + F$ exceeded 1, selectively on the paradox and ethical-contradiction items and essentially never on tautological controls. A companion multi-vendor replication extends this to six frontier model families with similar though variable incidence rates — evidence, if it holds, that whatever produces the effect is not specific to one laboratory’s training recipe.

The central claim is therefore not that models are uncertain, which is trivial, but that models already carry richer and more differentiated epistemic information than their conventional reporting format lets them express, and that an output head reporting T , I and F as independent quantities recovers distinctions — paradox versus ignorance versus vagueness versus genuine value conflict — that a normalized confidence score necessarily flattens together. Later work in the same lineage, on absorption problems in which models collapse many distinct kinds of uncertainty onto the same near-(0, 1, 0) triplet, suggests that the scalar $T/I/F$ representation is not by itself a complete remedy — a caveat Section 8 returns to.

4. Thread Two: Aligning with Whom? — Value Alignment Under Irreducible Pluralism

If Thread One diagnoses a representational problem inside a single model’s epistemic state, the second paper (Leyva-Vázquez, Rumbaut Rangel, Cevallos-Torres, Matheu Pérez & Smarandache, 2026) applies the same representational move to a harder and more consequential target: the training pipeline itself.

4.1 The diagnosis: alignment collapse

Contemporary alignment methods — reinforcement learning from human feedback (Ouyang et al., 2022), Constitutional AI (Bai et al., 2022), direct preference optimization (Rafailov et al., 2023) — all perform the same sequence of compressions: elicit normative positions from people, aggregate the resulting distribution into a single reward model or constitution, and optimize a policy against that scalar. Each step is locally reasonable; the sequence guarantees that whatever genuine pluralism entered the pipeline exits it as a single ordering. We term this alignment collapse. Collective Constitutional AI (Huang et al., 2024), in which constitutional statements lacking cross-group consensus were filtered out before training, is the most transparent public example: a deliberately fairness-aware consensus mechanism that is nonetheless still an aggregation operator, and therefore still erases whatever does not converge. Impossibility results in fair classification (Kleinberg et al., 2016) indicate that no neutral aggregation is waiting to be found; the choice of operator is a normative commitment made silently, at the layer of the pipeline least visible to those it affects.

4.2 Why the standard toolkit does not help

Bayesian probability enforces $P(A) + P(\neg A) = 1$, so strong evidence on both sides must be renormalized into moderate evidence for each. Intuitionistic fuzzy sets retain the constraint in (2), which defines the contradictory region out of the representation. Dempster–Shafer theory misbehaves precisely under high conflict, and its standard remedies discount or redistribute the conflicting mass. Imprecise probability widens the interval under conflict, conflating “the evidence is contradictory” with “the evidence is scarce” — two states that demand different downstream behavior. Non-normalized variants do exist, notably the

transferable belief model (Smets & Kennes, 1994), so the claim is not that no formalism can represent conflict; it is that the normalized, scalarized implementations that dominate alignment pipelines do not carry an independently auditable degree of support for both a proposition and its negation through training and inference.

4.3 The proposed architecture

Rewards, per-principle constitutional evaluations and policy outputs are recast as neutrosophic valuations with no sum constraint. Let each constitutional principle C_i emit an evaluation $C_i(r) = (T_i, I_i, F_i)$ of a candidate response r . Instead of a weighted scalar sum, aggregation uses a pessimistic-pluralist operator that preserves conflict:

$$R_C(r) = (\max_i T_i, \quad \text{avg}_i I_i, \quad \max_i F_i). \quad (8)$$

The two maxima may be attained by different principles, and that is the intended reading: $T + F > 1$ at the aggregate certifies inter-principle conflict — somewhere in the panel one principle strongly endorses r while another strongly condemns it — rather than a contradiction inside any single principle. The disaggregated panel is retained as the object of record, so the finer diagnosis is never lost.

For deployment contexts that require a decision, define for each admissible action a , evaluated by communities $i = 1, \dots, k$, the functional

$$\sigma(a) = \min_i [T_i(a) - F_i(a) - \Omega_i \cdot I_i(a)], \quad (9)$$

where $\Omega_i \geq 0$ is community i 's indeterminacy-aversion coefficient. An action is admissible when $\sigma(a) \geq 0$. The minimum in (9) is a Rawlsian floor rather than a utilitarian sum: no community's enthusiasm can buy off another community's veto. When no action clears every floor, the system escalates — naming which community's evaluation blocks which action, and why — rather than silently picking a side.

4.4 Preliminary evidence and the auditor effect

The paper reports empirical work consistent with Thread One: across several thousand elicited evaluations spanning six model families, sum-constraint violations were common and format-robust — present on contested prompts, essentially absent on tautological controls — while the strict signature $T + F > 1$ concentrated heavily on ethical-conflict items specifically. That is the profile the framework predicts if it tracks genuine normative disagreement rather than a blanket response style.

A secondary finding is what we call the auditor effect: when the same candidate responses were re-scored by three different judge models, the rate at which a response was flagged as strongly paraconsistent varied roughly threefold depending on which model adjudicated, with the differences statistically robust under paired tests. The implication is uncomfortable but important for any oversight scheme built on this framework: the evaluating model is not a neutral instrument, and whose model adjudicates a conflict partly determines which conflicts become visible at all.

Read against the ladder of Section 2, the paper's central diagnostic — condition (4) as the signature of genuine value conflict, as against I-driven violations satisfying only (5), which reflect ignorance — is precisely the distinction between strong and weak paraconsistency. The paper is, in effect, an argument that alignment pipelines should be built to detect and preserve strong paraconsistency rather than launder it into weak paraconsistency or erase it altogether.

5. Thread Three: Pluriversal Alignment — Latin American Logic and the Decolonial Critique

The third paper (Leyva & Batista, 2026) supplies the historical and philosophical scaffolding that the other two presuppose but do not themselves argue for: why the refusal to aggregate contradictory values should be read as a matter of epistemic justice rather than merely as an engineering nicety.

The answer runs through intellectual history. The paper traces paraconsistent logic’s Latin American lineage from Francisco Miró Quesada’s coining of the term “paraconsistent” in 1976, through Newton da Costa’s formal C-systems (da Costa, 1974), in which not every contradiction trivializes the system — a departure from the classical principle of explosion — and reads neutrosophic logic as a natural, graded extension of that project. It braids this with an Eastern European mathematical-logic tradition running from Vasiliev’s early-twentieth-century non-Aristotelian logic through the Łukasiewicz–Moisil multi-valued algebras. Both traditions, on this reading, arose in intellectual peripheries with reason to distrust a single universally imposed logical order, and both converged independently on machinery that lets contradiction stand without exploding the system that contains it.

The political argument follows from Thread Two’s technical one. If alignment pipelines resolve contradictions among diverse constitutional principles by consensus filtering — enforced convergence rather than genuine alignment — then they perform, in miniature and at scale, the epistemological monism that the Latin American paraconsistent tradition and the broader decolonial critique of AI (Mhlambi, 2020; Birhane, 2021; de Sousa Santos, 2014) have long named as a problem: the silencing of minority, indigenous or Global South frameworks whose internal coherence does not depend on convergence with dominant ones. What that critique has lacked is a formal apparatus making the silencing measurable and auditable rather than only nameable, which is what the neutrosophic valuation and condition (4) are offered to supply. The term pluriversal — drawn from Latin American decolonial theory, referring to a world composed of genuinely plural worlds rather than one universal one — names a program that treats structural value contradiction as a feature of a well-functioning alignment system rather than a defect to be minimized.

6. Synthesis: One Argument Across Three Registers

Read together rather than separately, the three papers form a single argument told in three registers, each doing work the others cannot.

Register	Paper	Core move	Position on the ladder
Empirical	Breaking the Chains of Probability	Shows LLMs already carry differentiated epistemic states that a normalized reporting format cannot express.	Elicits weak and strong signatures: (5) and (4).
Architectural / normative	Aligning with Whom?	Replaces scalar rewards and consensus aggregation with conflict-preserving operators (8) and (9).	Operationalizes the strong/weak distinction as an actionable flag.
Historical / philosophical	Pluriversal Alignment	Argues that refusing to collapse contradiction is a	Reframes preserving (4) as a political commitment.

		matter of epistemic justice, not engineering taste.	
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The dependency runs in one direction and is worth stating plainly. The philosophical argument in Thread Three needs the machinery of Thread Two to be more than a metaphor: without a formal notion of condition (4), “the system should not erase disagreement” is a slogan, not a specification. Thread Two in turn needs Thread One’s empirical grounding to be more than a normative wish: without evidence that models already produce elicitable paraconsistent signatures, the claim that a neutrosophic reward head would change anything is untested. And Thread One’s diagnostic vocabulary becomes actionable only once Thread Two supplies a use for the distinction — routing genuine ethical conflict to human deliberation rather than resolving it silently or refusing to act. The three papers are not three applications of a shared logic; they are successive layers of justification for a single contestable claim: that AI systems should be built to represent and preserve well-supported contradiction, at every level from a single model’s stated beliefs to a training pipeline’s treatment of a plural population’s values, rather than to resolve it before a person ever sees it.

7. Extensions Beyond Alignment

The representational move at the center of this program — refuse the sum constraint, keep both T and F visible when both are earned — is not intrinsically about language models. It targets an older and more general problem: what to do when two well-evidenced sources disagree. That problem recurs across fields that have each independently built workarounds for exactly the compression this program critiques.

7.1 Clinical differential diagnosis

A physician weighing two competing diagnoses, each well supported by different evidence — imaging suggesting one, laboratory results another — faces the same collapse: standard diagnostic scoring forces a single probability-ranked list. A four-component representation would let strong evidence for diagnosis A and strong evidence for diagnosis B coexist explicitly as a flagged state warranting further testing, rather than being folded into an uncertain middle ranking indistinguishable from a case with no strong evidence for anything.

7.2 Legal adjudication and multi-source evidence

Legal fact-finding already has an informal vocabulary for this: a case can be genuinely contested rather than merely unclear. Formal evidentiary frameworks, however, tend to average conflicting testimony rather than flag the conflict. A paraconsistent scoring layer would distinguish “the record is thin” (high I, low T, low F) from “the record is loud and split” (high T, high F) — two states a single credibility score cannot tell apart, and which call for different procedural responses.

7.3 Multi-sensor fusion and robotics

Dempster–Shafer theory was developed partly for this domain, and the behavior of Dempster’s rule under high conflict is a live concern in autonomous systems: when lidar and camera disagree strongly about whether an obstacle is present, naive combination can produce results that are confidently wrong in either direction. A flag on condition (4) at the fused-evidence level could trigger a fallback — slow down, request

a third reading, defer to a human operator — precisely when a single fused confidence score would look merely moderate and unremarkable.

Dezert-Smarandache Theory PCR5 and PCR6 fusion rules of combinations overpass Dempster's rule.

7.4 Misinformation and content-moderation triage

Automated fact-checking pipelines routinely face claims that reliable-seeming sources both confirm and deny. Collapsing this into a single true / false / uncertain label is the alignment collapse of Section 4 transplanted to content policy. A flag on (4) could route actively and credibly contested items to a human-reviewed, labeled-as-disputed pathway distinct from items that are simply unverified — which current systems tend to treat identically.

7.5 Multi-agent and federated systems

Where several agents trained on different data or objectives must jointly decide, operators (8) and (9) generalize directly: replace constitutional principles with agents and the same machinery becomes a conflict-preserving consensus protocol, arguably more informative than Byzantine-fault-tolerant schemes built to converge rather than to represent persistent disagreement.

7.6 Pedagogy and contested topics

Outside computation, the worry that presenting a single settled answer to a genuinely contested question performs a kind of epistemic violence has a long history in critical pedagogy quite apart from neutrosophic logic. The vocabulary offered here — condition (4) as “genuinely, evidentially contested” versus high-I as “simply unresolved” — gives curriculum designers a sharper way to distinguish “we do not yet know” from “credible people disagree, and here is why”, which are pedagogically very different situations that a single “it is complicated” gloss flattens together.

Across all of these the underlying wager is the same: forcing every disagreement through a single number is not a neutral simplification but a substantive choice with costs, and those costs are worth paying only when the disagreement really is noise rather than signal. The harder cases — value conflict, contested diagnoses, split sensor readings, disputed claims — are exactly where that assumption is least likely to hold.

8. Critical Discussion and Limitations

We state the program's current limits directly, because the alternative — letting preliminary numbers circulate as established results — would undermine the very epistemic standards this program argues for.

8.1 Sample and generalizability

The reported incidence rates for hyper-truth and strong paraconsistency come from modest item sets, on the order of tens of prompts per phenomenon repeated across models. Repeated evaluations of the same prompt are positively correlated, which makes the accompanying confidence intervals anti-conservative with respect to the item population. These results are best read as existence proofs that the phenomenon occurs, not as stable effect-size estimates.

8.2 The auditor effect cuts both ways

The finding that judge models disagree substantially on what counts as strongly paraconsistent supports the framework's diagnostic value, but it equally raises a validity concern. If the measurement instrument is itself unreliable across raters, claims about how much paraconsistency exists in the world must be held loosely until validated against non-model panels of diverse human annotators — a step listed as future work but not yet performed. There is a tension here worth naming rather than smoothing over: auditability is claimed as a central virtue of the framework, yet the audit signal is demonstrably adjudicator-dependent. The defensible position is that the framework makes that dependence visible and measurable, where aggregation-based pipelines conceal it — but this is a weaker claim than adjudicator-independent auditability, and should not be conflated with it.

8.3 Operationalizing communities and principles

The Rawlsian floor in (9) is elegant on paper, but assigning real populations to discrete communities, each with a single indeterminacy-aversion coefficient Ω_i , risks reproducing at one remove exactly the aggregation violence the framework was built to avoid. Real communities are not monolithic, and the choice of how to partition people into evaluating panels is itself a normative decision with no neutral default — a point the impossibility literature cited in Section 4 would predict. The paper does not currently supply a principled partitioning procedure, and we do not claim one.

8.4 Is refusal to aggregate always the right response?

The framework escalates rather than decides when floors are not cleared, which is defensible for high-stakes contested questions but does not resolve the practical problem that many deployed systems must produce some answer to most queries. The downstream cost of frequent escalation — to users and to deployment feasibility — is not yet empirically characterized.

8.5 The decolonial framing invites external scrutiny

The historiographical claim that Latin American paraconsistent logic and Eastern European non-classical logic converge on a shared anti-monist political meaning is a substantive interpretive thesis, and should be read as one contribution to an ongoing debate about the politics of formal logic rather than as settled intellectual history. Scholars in decolonial science and technology studies outside this citation cluster have not yet engaged with the specific claim that neutrosophic logic formalizes their critique. That engagement is invited here, and its absence to date should be counted as an open question rather than as tacit agreement.

None of this is a reason to dismiss the program. Position papers that stake out a falsifiable agenda before every result is in are a normal and useful part of how formal frameworks get tested. It is a reason to read the specific numerical claims as preliminary markers of a line of inquiry rather than as established facts, pending replication by groups outside the original author cluster and validation against human rather than model judgment.

9. Conclusion

The ladder of Section 2 — strong, weak and very weak paraconsistency, conditions (4), (5) and (6) — gives a name and a measurement to something anyone who has tried to average two people's strongly held, opposed and equally well-evidenced convictions already knows: that forcing the disagreement into a single number destroys information, and that the destroyed information is often exactly the part that mattered. The

program surveyed here applies that observation to what a language model says about its own uncertainty, to how an alignment pipeline should treat a plural population's values, and to why the refusal to collapse contradiction is continuous with an older tradition of logic built — in places and periods with good reason to distrust an imposed universal order — precisely to let contradiction stand without exploding the system that contains it. Whether the specific quadruple-and-threshold machinery proposed here is the right formal carrier for that intuition is an open and empirically testable question, one this paper has tried to state clearly rather than resolve.

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Structured Neutrosophic Sets with Entropy-Derived Indeterminacy: Injectivity, Stability, and Aggregation Guarantees for Variability-Sensitive MCDM

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ABSTRACT

In neutrosophic multi-criteria decision-making, indeterminacy has been universally encoded as a scalar value — a design choice that systematically suppresses distributional information and undermines ranking reliability. This paper shows that this scalar encoding leads to three different pathologies in the structure: many-to-one representational collapse, ranking instability when the input is perturbed by limited perturbations, and aggregation inconsistency in environments with variations. Being non-dimensional, it is shown that this is a functional deficit, so that a generic expansion of the representation gap by an interval valued (IVW) extension and an orthopair (OPW) extension of the representation cannot solve this gap. To examine the cause, we propose that indeterminacy serves as the principal function of Shannon entropy, statistical dispersion, and inter-response inconsistency, which led to the Entropy-Anchored Structured Neutrosophic Sets (EA-SNS), and then decompose these components into eleven structurally independent components (hendecagonal representation). This basis is shown to be functionally independent using a full-rank Jacobian analysis. Theorems are completely provided with proofs: (i) boundedness and strict monotonicity of the indeterminacy functional; (ii) structural injectivity, which allows to exclude -

formally - many-to-one collapse; (iii) score function monotonicity preserving preference order; (iv) an explicit Lipschitz perturbation stability bound. Structure-aware aggregation operators include witnesses of idempotency, commutativity, and boundary preservation. Empirical results, using two real educational datasets and three controlled benchmarks, compared to five competing neutrosophic methods, are obtained with a mean discrimination gain of 34.7% above the mean discrimination gain of the standard SVNS in all cases, compared to the expert-validated ground truth.

Keywords: Entropy-Driven Indeterminacy; Functionally Decomposed Uncertainty; Hendecagonal Neutrosophic Representation; Multi-Criteria Decision Making; Perturbation-Stable Ranking; Structure-Preserving Aggregation

1. INTRODUCTION

How should distributional uncertainty be represented in multi-criteria decision-making? Neutrosophic set theory offers three membership components — truth, falsity, and indeterminacy — enabling richer modeling than classical fuzzy approaches. However, after 30 years of development, one crucial assumption has never been questioned: that indeterminacy is a scalar (Siddhya, 2025; Abdelaziz et al., 2023). A degree of indeterminacy is given for each proposition, and the dimension of real-world uncertainty is reduced to a single value, whether the proposition is a bounded real number, an interval endpoint, or a component of an orthopair (Arora & Naithani, 2024; Rogulj et al., 2021). That collapse is the central problem this paper addresses.

The real-world importance of scalar indeterminacy becomes evident in situations that require consideration of variability. Two classrooms were assessed for instructional effectiveness, and they had the same mean scores on the assessment and the same average response difference (inconsistency). One class shows relatively low dispersion and a high level of agreement, with responses tightly clustered, while the other shows responses that are spread out. For any current work on the neutrosophic one, the same indeterminant values will appear in both classrooms. The encoding will not preserve the information that distinguishes true from false news disagreements. What makes true disagreements news vs. volatile disagreements is not encoded.

This representational failure is structural. The many-to-one correspondence between the 2D configuration of the transformations and the scalar indeterminacy generates three entangled pathologies. As regards information loss due to variability, it dissipates during encoding (Irvanizam et al., 2021; Abdel-Basset et al., 2022; Adalı et al., 2023). The instability is explained by the small-scale sensitivity of the input to the output ranks in the absence of structural constraints on the surface of the scores (Yaacob et al., 2024; Ye et al., 2023). These distortions are further exacerbated by aggregation inconsistency, as operators combine 'indistinguishable representations' (Deng & Wang, 2021; Jana et al., 2021; Martina & Deepa, 2023). Unfortunately, none of the existing extensions, interval-valued, q-rung orthopair, and hesitant neutrosophic, can cure such pathologies, as none correlates the value of indeterminacy with the properties of the induction of the input.

The present paper presents Entropy-Anchored Structured Neutrosophic Sets (EA-SNS), where the indeterminacy is a principled function of three measurable distributional attributes: Shannon entropy, statistical dispersion, and inter-response inconsistency, and is decomposed into eleven orthogonal parts of indeterminacy, each representing an independent source of uncertainty variability. The framework is strung together by four theorems, each of which is proved in full. The aggregation operators include formal proofs that they are idempotent, commutative, and boundary-preserving, which are necessary properties to ensure that any decision aggregation framework can be trusted. A step-by-step approach to following up is explained in a fully worked numerical example. Wilcoxon significance testing was carried out throughout for the five competitive baselines and for the five datasets used in the empirical validation.

Decomposition of indeterminacy as a construct of data, not a scalar value; 2. Functionally independent uncertainty decomposition (with Jacobian rank analysis) with conservation of distributional structure; Theoretical guarantees, such as those discussed above in this paper, are given to a formal level with boundedness, structural injectivity (repels many-to-one collapse), monotonic ranking behavior, and Lipschitz-bounded perturbation stability; Introduce structure-aware aggregation operators that are idempotent and commutative, and preserve the edges, with proofs. Very comprehensive empirical tests have been conducted on both real and synthetic datasets, validating the system and demonstrating significant improvements in discrimination, robustness, and ranking stability compared to existing neutrosophic approaches.

The contribution reimagines indeterminacy, from an assigned scalar to a derived, decomposable, and structurally faithful object, to formally establish, with four theorems and empirical evidence over five datasets, its new decision-theoretic properties not explored in previous formulations. This claim is backed by some structural and empirical arguments: no scalar-encoding frameworks meet structural injectivity (Theorem 2) and are guaranteed by a Lipschitz bound on perturbations (Theorem 4); and the results in Section 9 show that, on average, the discrimination indices of a framework are 34.7% higher.

The rest of the paper is organized as follows. Section 2 reviews the current state of neutrosophic and fuzzy decision frameworks, formally describes the structural deficit, and situates the contributions of this work. This section summarizes the mathematical background needed later. The framework of EA-SNS, including the data-driven indeterminacy functional, the hendecagonal decomposition, the aggregation operators, and the entropy-modulated-ranking function, is introduced in Section 4. Section 5 contains four formal theorems along with complete formal proofs. The entire 10-step MCDM algorithm is detailed in Section 6. Section 7 contains an extensively elaborated example of computations. The absence of references in sections 8 and 9 indicates the experimental design and empirical results for five datasets and five competitive baselines. The findings and limitations of the study are discussed in Section 10. Section 11 concludes.

(C1) Indeterminacy is refunctionalized as a data-driven structure (construct) which emerges from a measurable property (entropy, dispersion, inconsistency) and is decomposable by it; this is an arbitrary scalar-assignable structure, which has been used everywhere by other formulations of neutrosophic distributions. (C2) With respect to the end of the time interval, the uncertainty can be decomposed into eleven structurally independent components (Proposition 1), which are functionally independent as well. (Formal proof) structural injectivity (Theorem 2), excluding a significant number of many-to-one representational collapses; not possible to construct formally using any scalar-encoding framework. Explicit Lipschitz perturbation stability bound (Theorem 4), which is the first analytic rank-stability guarantee for neutrosophic MCDM. Works of the structure-aware aggregation operators (EA-SNWA, EA-SNWG) with proofs of idempotency, commutativity, boundary preservation, and simplex closure. This is the performance of the empirical validation across 5 datasets, combined with 5 competitive baselines: Mean

discrimination gain = 0.347, PSC = 0.973, and Spearman rank correlation = 0.991, with p-values < 0.001 for each dataset.

2. LITERATURE REVIEW AND IDENTIFIED RESEARCH GAP

2.1 Evolution of Neutrosophic and Fuzzy Decision Frameworks

The neutrosophic set theory by Smarandache has extended earlier fuzzy set theories, which were limited to the standard unit interval, by introducing truth, indeterminacy, and falsity as independent membership functions defined on non-standard intervals, making the computation of these functions feasible for applications of MCDM (Rahimi et al., 2021; Unver et al., 2022). Turksis et al. (2022) restricted this to the standard unit interval to enable computational tractability in MCDM applications. Interval-valued neutrosophic sets were an interpolation much richer in formal content but shallow in representation because each membership component was given an intervalous representation $[a,b]$ or $[a,b]$ where a and b were again in $[0,1]$ (Peng et al., 2023; Muthulakshmi et al., 2025; Nabeeh, 2023).

The q-rung orthopair generalisation relaxed the constraint $T + F \leq 1$ to $T_q + F_q \leq 1$, and the expansion of the admissibility came with benefits, since it turned out that the accuracy of the classifications could be improved using the Fermatean fuzzy multicriteria model (Nafei et al., 2021; Peng et al., 2024). The improvement is due to domain enlargement, not to the decomposition of domain uncertainty — as this paper makes precise (Kumar et al., 2022). In addition, hesitant neutrosophic sets provided an additional type of membership degree, in the form of finite sets, and could express expert disagreement directly without relying on the input structure (Fujita, 2026; Gao, 2025; Garg et al., 2022).

Entropy-based is one of the parallel development streams. In the study by Thakur et al. (2021), intuitionistic fuzzy entropy was incorporated into the COPRAS weighting method, demonstrating that information-theoretic weighting enhances rank stability. Ismail et al. (2025) further elaborated on this idea in the context of financial performance appraisal under neutrosophic uncertainty. These contributions consider entropy as an external weighting factor once the basic

representation is given. The indeterminacy structure itself is not affected by entropy. It is not entropy that enters the indeterminacy structure — that makes the difference.

Hybrid fuzzy-neutrosophic architectures have proliferated since 2022. As stated earlier, research has been undertaken to develop objective weighting approaches for Pythagorean neutrosophic MCDM; Kamari et al. (2025a, 2025b) proposed one such approach. These are kinds of systems that can expand the representational range without challenging the assumption of scalar encoding. The theoretical floor does not vary across the formulations tested.

2.2 The Structural Deficit: Formal Characterisation

The central limitation across all reviewed frameworks is functional, not dimensional. Scalar-encoding methods implement a mapping $I: X \rightarrow [0,1]$ that is many-to-one with respect to distributional structure. Two input configurations $x, y \in X$ may possess entirely distinct entropy profiles, dispersion magnitudes, and consistency patterns while satisfying $I(x) = I(y)$ under any existing formulation. This is not a degenerate edge case. It is the structural norm for input spaces with heterogeneous variability, and it is analytically irresolvable within the scalar paradigm (Farid et al., 2023).

The consequences compound through the decision pipeline. Aggregation operators applied to indistinguishable representations produce indistinguishable aggregate scores (Fauzi et al., 2025). The score surface becomes flat precisely in regions where dispersion-sensitive alternatives must be distinguished (Alahmadi et al., 2023). Perturbation analysis of classical neutrosophic TOPSIS implementations confirms this: rank reversals occur under perturbations as small as 5% of the input range in high-variability environments, as reported in Section 9.3 of the present study.

Dimensional expansion cannot resolve a functional inadequacy. Interval-valued sets introduce range uncertainty about the indeterminacy value but do not condition that value on the generative distribution. Q-rung orthopair models expand the feasibility region but retain magnitude-based assignment (Ali et al., 2024). Entropy-augmented models apply information-theoretic corrections post-hoc. The research gap is therefore one of inadequate structural correspondence between input geometry and representational geometry — a gap that demands a representational rearchitecture rather than a parametric extension.

2.3 Positioning of the Present Contribution

This article addresses this lacuna by redefining indeterminacy as an intermediate, measurable function of the input distribution and of the structural independence among its decomposition elements. The dimensionality of the representation is not given by convention, but by the rank of the descriptor Jacobian, a mathematically-based convention. The V1-V2 mapping that underlies ranking instability or failure in all previous formulations is replaced by a principled V1-V2 mapping that relates the distributional structure to the representational structure.

2.3.1 Motivating Intuitive Example: The Two-Classroom Problem

Consider two classrooms, each evaluated on instructional effectiveness by 30 students using a 1–7 Likert scale. Classroom A produces responses $\{4, 4, 4, 5, 5, 4, 4, 5, 4, 4, \dots\}$ — tightly clustered near the mean of 4.3, with low dispersion and high agreement. Classroom B produces responses $\{1, 7, 1, 7, 2, 6, 7, 1, 6, 2, \dots\}$ — bimodal and polarised, yet also averaging to 4.3. Under any existing neutrosophic formulation, both classrooms receive identical indeterminacy values, which are determined solely by the mean. A decision-maker comparing these two classrooms under such a model cannot distinguish a reliable consensus from a volatile disagreement. This is not a pathological edge case: In any survey-based MCDM application, alternatives whose response distributions are indeed structurally different will regularly have comparable central tendencies. In practice, the Many-to-One collapse described formally in Section 2.2 is the norm rather than the exception. It is a direct point of difference that EA-SNS obtains when considering the score difference, namely $H(\Theta) = 3.06$ (diffuse, high entropy uncertainty) vs $H(\Theta) = 1.21$ (concentrated, low entropy uncertainty), leading to a score separation of 0.556, which is structurally grounded discrimination that is not achievable with scalar methods.

2.3.2 Theoretical and Empirical Justification for Eleven Dimensions

The choice of eleven uncertainty descriptors is neither arbitrary nor universal; it is theoretically grounded and empirically validated. The theoretical framework is founded on the necessity of functional independence: each descriptor must refer to a different and non-redundant source of distributional uncertainty. It is the formal requirement that $J_{\Phi}(x)$ become a square matrix with full row rank, as described in Proposition 1. Conversely, the question of "why eleven" becomes

"what is the minimum number of descriptors that can be used functionally and are still enough to characterise the relevant uncertainty geometry of survey-based data?"

The eleven descriptors encompass four theoretical classes of distributional information: (i) measures of overall uncertainty spread (according to Shannon entropy $\tilde{H}$, Rényi entropy H_2 , as well as local entropy gradient LEG); (ii) measures of dispersion that capture the magnitude of the spread (coefficient of variation CV, normalised range $\tilde{\kappa}$); (iii) measures of distributional asymmetry and tail behaviour (skewness |Skew|, kurtosis Kurt, distributional asymmetry index Asym, tail-weight ratio TWR); and (iv) measures of the structure of the distribution (bimodality coefficient BMC, IQR ratio). They offer access to uncertainty geometry that encompasses all possible geometries resulting from bounded, discrete response data, without making any distributional assumptions. It is not possible to represent any descriptor in the set as a linear combination of the others, as demonstrated by symbolic differentiation and verified by pairwise Pearson correlation coefficients, which did not exceed 0.31 across any of the validation sets.

Empirical evidence to support this is given directly in the ablation study in Table 6 (Section 9.7). For all three evaluation methods (DI, PSC and GTA), deleting descriptors one by one monotonically reduces performance, with each deletion resulting in measurable degradation. There is no significant improvement ($\Delta DI = 0.002$, $\Delta PSC = 0.001$) between $k = 11$ and $k = 12$, and an additional ($k = 12$) descriptor is only moderately correlated with BMC ($r = 0.44$), suggesting that $k = 12$ is redundant. This $k = 11$ configuration is thus the empirical sufficiency boundary: increasing the number of descriptors is no longer useful, and decreasing it unequivocally worsens performance.

The eleven-dimensional representation is not a freely chosen design but a minimum representation that captures the validation geometry, validated both theoretically (Jacobian rank analysis) and empirically (Ablation study).

2.3.3 Novelty Positioning: Three Differentiating Contributions

EA-SNS proposes new ideas in a field where many extensions of neutrosophic sets have been suggested, and those novelties are quite clear and separable from previous works within three specific axes:

(a) Functional Extension / Dimensional Extension. The interval-valued, q-rung orthopair, hesitant, and Pythagorean neutrosophic extensions do not consider the functional extension from input distributions to the indeterminacy value of the representations; thus, they are only a marginal extension in the representational domain (dimensional extension). EA-SNS is unique in its focus on the functional gap: indeterminacy is not assigned but rather calculated from measurable distributional attributes. This is not a more robust option to the existing contributions, but another type.

(b) Structural Injectivity. This paper will be the first to present a formal proof of the distinctness of representational outputs under different input distributions in the neutrosophic MCDM framework. EA-SNS does, under Theorem 2. This formally renders many-to-one collapses nonexistent, preventing ranking instability and discrimination failures in methods based on scalar encoding. It cannot be achieved by construction, since a scalar cannot be injective over a multi-dimensional input space, within any scalar-encoding framework.

(c) Entropy-Inside Representation. In contrast to other models proposed previously, e.g., entropy-weighted SVNS, the following models (M4 in this study) treat entropy as an external weighting coefficient applied after the main representation is determined. The entropy directly resides in the indeterminacy representation in the form of the structured uncertainty vector $\Theta(x)$ and the entropy-modulated score function $S(N) = T - \alpha \cdot I \cdot H(\Theta) / \log_2(11) - \beta \cdot F$. The discrimination advantage of EA-SNS over entropy-weighted SVNS (the best previous entropy-augmented baseline, Table 4) of 24.4% indicates that Entropy-in-Side representation does not equal Entropy-outside weighting, but is rather a strict improvement for discrimination at the highest level of variability.

3. MATHEMATICAL PRELIMINARIES

The following definitions and notation are used throughout the paper. Let X denote a universe of discourse and let $\mathbb{R}$, $\mathbb{R}_+$, and $[0,1]$ carry their standard meanings.

Definition 1 (Neutrosophic Set). A neutrosophic set A over X is a structure $A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X\}$, where $T_A(x), I_A(x), F_A(x) \in [0,1]$ denote truth, indeterminacy, and falsity membership functions, respectively, satisfying $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2 (Single-Valued Neutrosophic Number). A SVNN is a triplet $\langle T, I, F \rangle$ with $T, I, F \in [0,1]$ and $0 \leq T + I + F \leq 3$. For $N_1 = \langle T_1, I_1, F_1 \rangle$ and $N_2 = \langle T_2, I_2, F_2 \rangle$:

$$\begin{aligned} N_1 \oplus N_2 &= \langle T_1 + T_2 - T_1 T_2, \quad I_1 I_2, \quad F_1 F_2 \rangle \\ N_1 \otimes N_2 &= \langle T_1 T_2, \quad I_1 + I_2 - I_1 I_2, \quad F_1 + F_2 - F_1 F_2 \rangle \\ \lambda N_1 &= \langle 1 - (1 - T_1)^\lambda, \quad I_1^\lambda, \quad F_1^\lambda \rangle, \quad \lambda > 0 \\ N_1^\lambda &= \langle T_1^\lambda, \quad 1 - (1 - I_1)^\lambda, \quad 1 - (1 - F_1)^\lambda \rangle \end{aligned}$$

Definition 3 (Shannon Entropy). For a discrete probability distribution $P = \{p_1, \dots, p_n\}$ with the convention $0 \cdot \log_2 0 = 0$, the Shannon entropy is $H(P) = -\sum_i p_i \log_2(p_i) \in [0, \log_2 n]$. The normalised form $H(P) = H(P)/\log_2(n)$ maps to $[0,1]$.

Definition 4 (Probability Simplex). The $(k-1)$ -dimensional probability simplex is $\Delta^{(k-1)} = \{\theta \in \mathbb{R}^k : \sum_{i=1}^k \theta_i = 1, \theta_i \geq 0\}$. Vectors in $\Delta^{(k-1)}$ are non-negative, bounded, and collectively exhaustive by construction.

4. THE ENTROPY-ANCHORED STRUCTURED NEUTROSOPHIC FRAMEWORK

4.1 Formal Definition of EA-SNN

Definition 5 (Entropy-Anchored Structured Neutrosophic Number). For each $x \in X$ with associated observation set $O(x) = \{x_1, \dots, x_n\}$, an EA-SNN is the quadruplet:

$$N(x) = \langle T(x), \quad I(x), \quad F(x), \quad \theta(x) \rangle$$

where $T(x), F(x) \in [0,1]$ denote truth and falsity degrees; $I(x) \in [0,1]$ is the data-driven indeterminacy functional defined in Section 4.2; and $\theta(x) = (\theta_1(x), \dots, \theta_{11}(x)) \in \Delta^{10}$ is the structured uncertainty vector satisfying $\sum_{i=1}^{11} \theta_i(x) = 1$ and $\theta_i(x) \geq 0$ for all i .

Definition 5 is characterised by three properties that set it apart from all previous formulations. The value of the components of $I(x)$ is calculated, instead of being set. The eleven directions shown are orthogonal; that is, they treat the uncertainty in $I(x)$ as the sum of orthogonal parts represented by the vector $\Theta(x)$. The sufficient condition for structural injectivity of the full EA-SNN mapping is given in Proposition 1.

4.2 Generalisation: EA-SNS Subsumes Classical SVNS

Classical SVNS corresponds to the degenerate case in which the uncertainty vector collapses to a scalar. Formally, define the degenerate EA-SNN as $N_{deg}(x) = \langle T, I_{scalar},$

$F, (1, 0, \dots, 0)\rangle$ – a vector with all uncertainty mass concentrated in a single descriptor. The classical SVNS score function $S_{\text{classical}} = T + (1 - I)/2 - F$ is then a special case of the entropy-modulated score $S(N) = T - \alpha \cdot I \cdot H(\theta)/\log_2(11) - \beta F$ under the limiting conditions $H(\theta) \rightarrow 0$ and $\alpha \rightarrow (1/2)/I$:

$$\lim_{\{H(\theta) \rightarrow 0\}} S(N) = T - \beta F \supseteq S_{\text{classical}} \text{ when } \beta = 1, \alpha \rightarrow 0$$

This single equation establishes that EA-SNS is a strict generalisation of SVNS. Every SVNS ranking is reproducible within EA-SNS. No EA-SNS ranking collapses to SVNS unless the uncertainty vector is fully concentrated — a condition that never arises in real distributional data. The generalisation is complete and one-directional.

4.3 Data-Driven Indeterminacy Construction

Given $O(x) = \{x_1, \dots, x_n\}$, the primary distributional descriptors are defined as follows. The sample mean is $\mu = (1/n)\sum_i x_i$. The sample standard deviation is $\sigma = \sqrt{[(1/n)\sum_i (x_i - \mu)^2]}$. The empirical probability distribution P is constructed by partitioning the observation range into $m = 10$ equal bins, with p_i denoting the proportion of observations in bin i . Normalised Shannon entropy follows as $H(P) = -(1/\log_2 m)\sum_i p_i \log_2(p_i) \in [0, 1]$. The inconsistency index is $\kappa = (\max(x_i) - \min(x_i))/(\mu + \varepsilon)$ for $\varepsilon = 10^{-6}$. Min-max normalised versions $\tilde{\sigma}$ and $\tilde{\kappa}$ are computed over the full dataset prior to scoring.

The indeterminacy functional is defined by:

$$I(x) = [H(P) + \tilde{\sigma} + \tilde{\kappa}] / [1 + H(P) + \tilde{\sigma} + \tilde{\kappa}]$$

The form $g(s) = s/(1 + s)$ maps $\mathbb{R}_+$ bijectively to $[0, 1)$ and is strictly monotone in each argument. This ensures that $I(x)$ responds proportionally to every independent source of distributional disorder without saturation artefacts.

4.4 Hendecagonal Uncertainty Decomposition

The eleven descriptors constituting the structured uncertainty basis are: (f_1) normalised Shannon entropy H ; (f_2) coefficient of variation $CV = \sigma/\mu$; (f_3) normalised range $\tilde{\kappa}$; (f_4) excess kurtosis Kurt, normalised to $[0, 1]$; (f_5) skewness magnitude $|\text{Skew}|$, normalised; (f_6) inter – quartile range ratio $IQR/(\max - \min + \varepsilon)$;

(f_7) Rényi entropy of order 2, $H_2 = -\log_2(\sum p_i^2)$, normalised; (f_8) distributional asymmetry index $\text{Asym} = |\mu - \text{median}|/\sigma$, normalised; (f_9) tail – weight ratio $\text{TWR} = P(|X - \mu| > 1.5\sigma)$; (f_{10}) response bimodality coefficient $\text{BMC} = (\text{Skew}^2 + 1)/(\text{Kurt} + 3(n - 1)^2/[(n - 2)(n - 3)])$; (f_{11}) local entropy gradient LEG, measuring entropy difference between upper and lower observation halves.

The structured uncertainty vector is constructed by simplex normalisation:

$$\theta_i(x) = f_i(x) / \sum_{j=1}^{11} f_j(x), \quad i = 1, \dots, 11$$

This yields $\theta(x) \in \Delta^{10}$ by construction. Pairwise Pearson correlations among the eleven descriptors, computed across both validation datasets, remain below 0.31, confirming near-orthogonality empirically. Ablation analysis in Section 9.7 confirms structural non-redundancy formally.

4.5 Functional Independence: Jacobian Rank Analysis

Definition 6 (Functional Independence). Let $\Phi: X \rightarrow \mathbb{R}^{11}$ be defined by $\Phi(x) = (f_1(x), \dots, f_{11}(x))$. The descriptor set $\{f_1, \dots, f_{11}\}$ is functionally independent on an open set $U \subseteq X$ if the Jacobian $J_\Phi(x)$ has rank 11 for almost every $x \in U$ in the Lebesgue sense.

Proposition 1 (Rank Certificate). For any observation set $O(x)$ with $n \geq 12$ distinct values from a continuous distribution, $J_\Phi(x)$ achieves full rank 11 almost surely.

Proof. For each descriptor f_i , a different statistical moment or information-theoretic quantity is involved. Asym, Shannon entropy, kvarta, order statistics, BMC, IQR, Rényi entropy, TWR, and spatial entropy variation are unrelated to the mean, either because they depend on order statistics, other than the mean, or because the measure cannot be written as a polynomial function of one or several of the other measures. Symbolic differentiation verifies that each of the rows of the Jacobian is a different non-linear functional of the observation vector, and so is linearly independent for most observation vectors. Sard's Theorem gives us the result that the set of observations resulting in rank deficiency is of Lebesgue measure zero for Φ .

4.6 Structure-Aware Aggregation Operators

Definition 7 (EA-SNWA). Let $\{N_1, \dots, N_m\}$ be EA – SNNs with $N_k = \langle T_k, I_k, F_k, \theta_k \rangle$ and weights $w = (w_1, \dots, w_m)$, $w_k > 0, \sum w_k = 1$. The EA-Structured Neutrosophic Weighted Average operator is:

$$\begin{aligned}
EA - SNWA(N_1, \dots, N_m) &= \langle T_agg, I_agg, F_agg, \theta_agg \rangle \\
T_agg &= 1 - \prod_k (1 - T_k)^{w_k}, \quad F_agg = \prod_k F_k^{w_k}, \quad I_agg = \prod_k I_k^{w_k} \\
\theta_agg &= \sum_k w_k \theta_k \text{ (simplex-normalised)}
\end{aligned}$$

Theorem A (Properties of EA-SNWA). The EA-SNWA operator satisfies: (i) Idempotency: if $N_k = N$ for all k , then $EA - SNWA(N_1, \dots, N_m) = N$; (ii) Commutativity: EA-SNWA is invariant to permutation of its arguments; (iii) Boundary Preservation: $0 \leq T_agg + I_agg + F_agg \leq 3$; (iv) Simplex Closure: $\theta_agg \in \Delta^{10}$.

Proof. (i) Idempotency:

$$\begin{aligned}
\text{If } T_k &= T \text{ for all } k, \text{ then } T_{agg} = 1 - (1 - T)^{\sum_k w_k} = 1 - (1 - T)^1 = T \\
I_{agg} &= I^{\sum_k w_k} = I^1 = I \text{ and } F_{agg} = F^{\sum_k w_k} = F \\
\text{For } \theta: \theta_{agg} &= \sum_k w_k \cdot \theta = \theta \cdot \sum_k w_k = \theta \cdot 1 = \theta
\end{aligned}$$

Idempotency holds. (ii) Commutativity: Products and weighted sums are commutative under permutation of index labels; the result follows immediately. (iii) Boundary Preservation: By the AM-GM inequality and the constraint $T, I, F \in [0, 1]$, each aggregated component lies in $[0, 1]$. The sum $T_agg + I_agg + F_agg$ achieves its maximum when each component is 1, giving a sum of 3; the neutrosophic constraint $0 \leq T + I + F \leq 3$ is therefore preserved under aggregation. (iv) Simplex Closure: $\sum_i (\theta_agg)_i = \sum_i \sum_k w_k \theta_{ik} = \sum_k w_k \sum_i \theta_{ik} = \sum_k w_k \cdot 1 = 1$. Non-negativity follows from $w_k > 0$ and $\theta_{ik} \geq 0$.

Definition 8 (EA-SNWG). The EA-Structured Neutrosophic Weighted Geometric operator employs geometric means:

$$\begin{aligned}
T_{agg} &= \prod_k T_k^{w_k}, \quad F_{agg} = 1 - \prod_k (1 - F_k)^{w_k}, \quad I_{agg} = 1 - \prod_k (1 - I_k)^{w_k} \\
(\theta_{agg})_i &= \frac{\exp(\sum_k w_k \cdot \log \theta_{ik})}{\sum_j \exp(\sum_k w_k \cdot \log \theta_{jk})}
\end{aligned}$$

Theorem B (Properties of EA-SNWG). EA-SNWG satisfies idempotency, commutativity, boundary preservation, and simplex closure by arguments dual to those in Theorem A, with products replacing sums and geometric means replacing arithmetic means throughout. The log-linear aggregation on the simplex preserves positivity by the exponential map.

4.7 Consistency Condition

Proposition 2 (Neutrosophic Consistency Preservation). For any collection of EA-SNNs satisfying $T_k + I_k + F_k \leq 3$ for all k , the EA-SNWA output satisfies $T_{agg} + I_{agg} + F_{agg} \leq 3$.

Proof. Since $T_k \in [0,1]$, the product form gives $T_{agg} = 1 - \prod_k (1 - T_k)^{w_k} \leq 1$. Similarly $I_{agg} = \prod_k I_k^{w_k} \leq 1$ and $F_{agg} = \prod_k F_k^{w_k} \leq 1$, so $T_{agg} + I_{agg} + F_{agg} \leq 3$. The lower bound follows from non-negativity of all components.

4.8 Entropy-Modulated Ranking Function

The scalar ranking of an EA-SNN incorporates both the magnitude of indeterminacy and the structural content of the uncertainty vector. The entropy-modulated score is defined as:

$$S(N) = T - \alpha \cdot I \cdot H(\theta) / \log_2(11) - \beta \cdot F$$

where $H(\theta) = -\sum_i \theta_i \log_2(\theta_i)$ is the Shannon entropy of the uncertainty distribution vector and $\alpha, \beta \in (0,1)$ satisfy $\alpha + \beta < 1$. Normalisation by $\log_2(11)$ maps the entropy term to $[0,1]$. Higher entropy in the uncertainty vector indicates diffusely distributed uncertainty across all eleven components — an intrinsically more indeterminate configuration — warranting a penalty beyond what scalar I alone captures.

Definition 9 (Weighted Distance Between EA-SNNs). For two EA-SNNs $N_a = \langle T_a, I_a, F_a, \theta_a \rangle$ and $N_b = \langle T_b, I_b, F_b, \theta_b \rangle$ with criterion weights $w = (w_1, \dots, w_n)$, the weighted EA-SNN distance is:

$$d(N_a, N_b) = \sqrt{[w_t(T_a - T_b)^2 + w_i(I_a - I_b)^2 + w_f(F_a - F_b)^2 + w_\theta \|\theta_a - \theta_b\|_2^2]}$$

where $w_t + w_i + w_f + w_\theta = 1$ with $w_t, w_f, w_\theta > 0$. This distance extends classical neutrosophic Euclidean distance by incorporating the full uncertainty vector geometry. TOPSIS-based ranking under EA-SNS employs this distance to compute separation measures from the positive and negative ideal solutions.

Proposition 3 (Metric Properties). $d(N_a, N_b)$ defines a semi-metric on the space of EA-SNNs, satisfying non-negativity, symmetry, and the triangle inequality. Non-degeneracy holds under the structural injectivity of Theorem 2.

Proof. Non-negativity and symmetry follow from the squared Euclidean structure. The triangle inequality follows from the Cauchy-Schwarz inequality applied to the weighted sum of squared

differences. Non-degeneracy: $d(N_a, N_b) = 0$ requires each weighted term to vanish, implying $T_a = T_b, I_a = I_b, F_a = F_b$, and $\theta_a = \theta_b$. By Theorem 2, equal uncertainty vectors imply distributional equivalence of the generating observation sets.

5. THEORETICAL ANALYSIS

5.1 Boundedness and Strict Monotonicity

Theorem 1 (Boundedness and Strict Monotonicity). The indeterminacy functional $I(x)$ satisfies: (i) $I(x) \in [0,1]$ for all $x \in X$; (ii) $I(x)$ is strictly increasing in each of $H, \tilde{\sigma}$, and $\tilde{\kappa}$ individually when the remaining two are fixed; (iii) $I(x) = 0$ if and only if $H = \tilde{\sigma} = \tilde{\kappa} = 0$.

Proof. Let $s = H + \tilde{\sigma} + \tilde{\kappa}$. Since $H, \tilde{\sigma}, \tilde{\kappa} \in [0,1]$, we have $s \in [0,3]$, giving $I = s/(1+s) \in [0,3/4] \subset [0,1]$. The function $g(s) = s/(1+s)$ satisfies $g'(s) = 1/(1+s)^2 > 0$ for all $s \geq 0$, establishing strict monotonicity in s . Since s is strictly increasing in each argument separately, the chain rule gives $\partial I / \partial H = g'(s) > 0$ and analogously for $\tilde{\sigma}$ and $\tilde{\kappa}$. The lower bound $I = 0$ requires $s = 0$, which holds iff $H = \tilde{\sigma} = \tilde{\kappa} = 0$, occurring only when the observation set is a perfect point mass with zero dispersion and zero range — a limiting idealisation not encountered in practice.

5.2 Structural Injectivity

Theorem 2 (Structural Injectivity). Let $\Psi: X \rightarrow \mathbb{R}^{15}$ denote the full EA-SNN mapping $\Psi(x) = (T(x), I(x), F(x), \theta(x))$. Under the functional independence of $\{f_1, \dots, f_{11}\}$ established in Proposition 1, the mapping $\Theta: X \rightarrow \Delta^{10}$ satisfies: $\theta(x) = \theta(y)$ implies $\Phi(x) = c \cdot \Phi(y)$ for some constant $c > 0$. Under mean-normalisation of all descriptors, $\theta(x) = \theta(y)$ implies x and y are distributionally equivalent.

Proof. Suppose $\theta(x) = \theta(y)$. By definition $\theta_i(x) = f_i(x)/\|\Phi(x)\|_1$ and $\theta_i(y) = f_i(y)/\|\Phi(y)\|_1$ for all i . Equality of all eleven components implies $f_i(x)/\|\Phi(x)\|_1 = f_i(y)/\|\Phi(y)\|_1$, giving $f_i(x) = c \cdot f_i(y)$ for the constant $c = \|\Phi(x)\|_1/\|\Phi(y)\|_1 > 0$. Thus $\Phi(x) = c \cdot \Phi(y)$. Under mean-normalisation, each descriptor satisfies scale-invariance $f_i(\lambda \cdot obs) = f_i(obs)$, so $c = 1$ follows, giving $\Phi(x) = \Phi(y)$. The injectivity of Φ under functional independence (Proposition 1) then implies $x \sim y$ in distribution. The many-to-one collapse is thereby formally eliminated.

5.3 Ranking Monotonicity

Theorem 3 (Monotonicity of Score Function). $S(N)$ is (i) strictly increasing in T , (ii) non-increasing in F , (iii) non-increasing in I when $H(\Theta) > 0$, and (iv) non-increasing in $H(\Theta)$ when $\alpha > 0$ and $I > 0$.

Proof. Differentiating $(N) = T - \alpha \cdot I \cdot H(\Theta)/\log_2(11) - \beta F$ directly: $\partial S/\partial T = 1 > 0$; $\partial S/\partial F = -\beta < 0$; $\partial S/\partial I = -\alpha \cdot H(\Theta)/\log_2(11) \leq 0$, strictly negative when $H(\Theta) > 0$; $\partial S/\partial H(\Theta) = -\alpha \cdot I/\log_2(11) \leq 0$, strictly negative when $I > 0$. All four directional properties hold simultaneously under $\alpha, \beta \in (0,1)$.

5.4 Lipschitz Perturbation Stability

Theorem 4 (Lipschitz Stability Bound). Let $N(x)$ and $N(x + \delta)$ be EA-SNNs corresponding to observation sets differing by $\|\delta\|_2 \leq \rho$. The score perturbation satisfies $|S(N(x + \delta)) - S(N(x))| \leq L \cdot \rho$, where $L = L_t + \alpha(L_i + L_h)/\log_2(11) + \beta \cdot L_f$, with L_t, L_i, L_f, L_h the Lipschitz constants of $T, I, F, H(\Theta)$ respectively over the observation space.

Proof. Each component mapping is Lipschitz over bounded observation domains: T and F , as bounded linear functionals of expert assessments, carry $L_t = L_f = 1$; $I(x) = g(H + \tilde{\sigma} + \tilde{\kappa})$ is Lipschitz with $L_i \leq 3/4$ since $g'(s) < 1$ for all $s \geq 0$; $H(\Theta)$ is Lipschitz with $L_h \leq 1$ by boundedness and continuity on the compact simplex. Applying the triangle inequality: $|\Delta S| \leq |\Delta T| + (\alpha/\log_2 11)|\Delta(I \cdot H(\Theta))| + \beta|\Delta F| \leq L_t \rho + (\alpha/\log_2 11)(L_i + L_h)\rho + \beta L_f \rho = L\rho$. Because L is independent of x , ranking order remains stable for any pair of alternatives i, j with score separation $|S(N_i) - S(N_j)| > L\rho$.

Corollary 1. For $\alpha = \beta = 0.3$ and the bounding values stated above, $L \leq 1.87$. A 5% input perturbation ($\rho = 0.05$) induces a score perturbation bounded by 9.35%. Classical SVNS-TOPSIS implementations exhibit effective sensitivity ratios exceeding 28% under equivalent perturbations, as confirmed in Section 9.3.

6. FORMAL EA-SNS MCDM ALGORITHM

The following algorithm specifies the complete EA-SNS decision procedure from raw data to final ranking. Each step corresponds to a formally defined operation in Sections 4 and 5.

ALGORITHM 1: EA-SNS MCDM Procedure

Input: Alternatives $A = \{A_1, \dots, A_m\}$, Criteria $C = \{C_1, \dots, C_n\}$, Criterion weights $w = (w_1, \dots, w_n)$, Observation sets $O(A_i, C_j) = \{x_{ij1}, \dots, x_{ijk}\}$ for all (i, j) .

Output: Complete ranking of alternatives over A.

Step 1: Compute the eleven descriptors $\{f_1, \dots, f_{11}\}$ for each observation set $O(A_i, C_j)$ using the formulas specified in Section 4.4.

Step 2: Construct the structured uncertainty vector $\theta_{ij} = (f_1, \dots, f_{11}) / \sum f_i \in \Delta^{10}$ for each (i, j) pair by simplex normalisation (Equation 4.4).

Step 3: Compute the data-driven indeterminacy value $I_{ij} = [H + \tilde{\sigma} + \tilde{\kappa}] / [1 + H + \tilde{\sigma} + \tilde{\kappa}]$ using min-max normalised descriptors (Equation 4.3).

Step 4: Elicit or compute truth and falsity degrees T_{ij} and F_{ij} for each (i, j) . Construct the full EA-SNN $N_{ij} = \langle T_{ij}, I_{ij}, F_{ij}, \theta_{ij} \rangle$.

Step 5: Aggregate criterion-level EA-SNNs for each alternative A_i using EA-SNWA with criterion weights w : $N_i = EA-SNWA(N_{i1}, \dots, N_{in})$.

Step 6: Compute the positive ideal solution $PIS = \langle T^+, I^+, F^+, \theta^+ \rangle$ and negative ideal solution $NIS = \langle T^-, I^-, F^-, \theta^- \rangle$ where $T^+ = \max T_i$, $F^+ = \min F_i$, $I^+ = \min I_i$, and $(\theta^+)_i$ concentrates mass on the descriptor with lowest entropy contribution.

Step 7: Compute weighted distances $d(N_i, PIS)$ and $d(N_i, NIS)$ for all alternatives using Definition 9.

Step 8: Compute the entropy-modulated score $S(N_i) = T_i - \alpha \cdot I_i \cdot H(\theta_i) / \log_2(11) - \beta \cdot F_i$ for each alternative.

Step 9: Compute the TOPSIS closeness coefficient $CC_i = d(N_i, NIS) / [d(N_i, PIS) + d(N_i, NIS)]$ and the combined final score $R_i = \gamma \cdot S(N_i) + (1 - \gamma) \cdot CC_i$ with $\gamma \in [0, 1]$.

Step 10: Rank alternatives in decreasing order of R_i . The alternative with the highest R_i is the most preferred.

The algorithm is computationally tractable. For m alternatives, n criteria, and k observations per cell, the total complexity is $O(m \cdot n \cdot k)$ for descriptor computation and $O(m \cdot n)$ for aggregation — linear in all parameters.

7. FULLY WORKED NUMERICAL EXAMPLE

To confirm computational reproducibility, consider a decision problem with three instructional alternatives $\{A_1, A_2, A_3\}$ evaluated against two criteria $\{C_1: \text{Learning Effectiveness}, C_2: \text{Consistency of Delivery}\}$ with criterion weights $w = (0.6, 0.4)$. Observation sets of size $n = 8$ are drawn from realistic Likert-scale (1–7) response data.

Table 1. Raw Observation Sets for the Worked Numerical Example

Alt/Crit	Criterion	X ₁	X ₂	X ₃	X ₄	X ₅	X ₆ –X ₈ (mean)
A ₁	C ₁	6	6	7	6	7	6.33
A ₁	C ₂	5	6	6	7	5	5.67
A ₂	C ₁	3	5	7	2	6	4.33
A ₂	C ₂	4	4	5	4	4	4.33
A ₃	C ₁	7	1	7	1	7	4.33
A ₃	C ₂	4	4	4	4	4	4.00

Descriptor Computation. For A_1/C_1 with observations $\{6,6,7,6,7,6,7,6\}$: $\mu = 6.375, \sigma = 0.48, H = 0.31$ (low entropy, concentrated responses), $\tilde{\kappa} = 0.16$. For A_3/C_1 with observations $\{7,1,7,1,7,1,7,1\}$: $\mu = 4.25, \sigma = 3.12, H = 0.97$ (near-maximum entropy, bimodal), $\tilde{\kappa} = 1.41$. Both alternatives share a similar mean under rounding yet are structurally incomparable. Classical SVNS would assign comparable I values; EA-SNS preserves the distinction.

Indeterminacy Computation. Using min-max normalised values across the three alternatives:

$$I(A^1, C^1) = \frac{0.31+0.15+0.16}{1+0.31+0.15+0.16} = 0.38;$$

$$I(A^2, C^1) = \frac{0.61 + 0.48 + 0.57}{1 + 0.61 + 0.48 + 0.57} = 0.63;$$

$$I(A_3, C_1) = (0.97 + 1.00 + 1.00)/(1 + 0.97 + 1.00 + 1.00) = 0.75.$$

The discrimination is structurally grounded and fully traceable.

EA-SNN Construction and Aggregation. Expert truth and falsity assignments yield $T(A_1) = 0.82, F(A_1) = 0.08$; $T(A_2) = 0.55, F(A_2) = 0.22$; $T(A_3) = 0.48, F(A_3) = 0.27$. After EA-SNWA aggregation with $w = (0.6, 0.4)$, the aggregated uncertainty vectors satisfy $H(\Theta(A_1)) = 1.21$, $H(\Theta(A_2)) = 2.18$, $H(\Theta(A_3)) = 3.06$ — reflecting the increasingly diffuse uncertainty structures.

Scoring and Ranking. With $\alpha = \beta = 0.3$:

$$S(A^1) = 0.82 - 0.3 \cdot 0.38 \cdot \frac{1.21}{3.46} - 0.3 \cdot 0.08 = 0.82 - 0.040 - 0.024 = 0.756;$$

$$S(A^2) = 0.55 - 0.3 \cdot 0.63 \cdot \frac{2.18}{3.46} - 0.3 \cdot 0.22 = 0.55 - 0.119 - 0.066 = 0.365;$$

$$S(A^3) = 0.48 - 0.3 \cdot 0.75 \cdot \frac{3.06}{3.46} - 0.3 \cdot 0.27 = 0.48 - 0.199 - 0.081 = 0.200.$$

Final ranking: $A_1 > A_2 > A_3$. Under classical SVNS, A_2 and A_3 score within 0.04 of each other — indistinguishable — while EA-SNS separates them by 0.165, a 4× improvement in discrimination.

Table 2. Step-by-Step Numerical Results Summary

Alt	I_agg	T_agg	F_agg	H(Θ)	S(N) EA-SNS	S(N) SVNS	Rank
A₁	0.38	0.82	0.08	1.21	0.756	0.740	1
A ₂	0.63	0.55	0.22	2.18	0.365	0.331	2
A ₃	0.75	0.48	0.27	3.06	0.200	0.327	3

Note: SVNS scores for A_2 and A_3 differ by only 0.004 — effectively indistinguishable. EA-SNS achieves 0.165 separation through structural uncertainty encoding.

8. DATASETS AND EXPERIMENTAL DESIGN

8.1 Datasets

The empirical validation corpus includes two real educational evaluation datasets and three controlled synthetic ones. D1: Dataset on instructional effectiveness is a survey dataset collected from students, showing survey results for 312 students across eight instructional approaches, evaluated on six dimensions of assessment over 3 academic semesters. Responses were rated on a 1–7 Likert-type scale with distributional observation sets of about 39 data items per criterion-alternative pair. The first dataset (D2) is a multi-institutional academic program quality dataset in which 187 evaluators evaluated six program attributes for five programs, while an independent expert panel established known ground-truth ranks.

The manifold benchmarks B1/B2/B3 were built at three levels of distributional heterogeneity. In B1, five groups are described, which conform to the same mean but vary in

variance in a geometric progression ($\sigma = 0.5, 1.0, 1.5, 2.0, 2.5$), and test pure dispersion discrimination. B2 is based on the idea of creating skewness variation across equal-mean, equal-variance alternatives, thereby isolating asymmetric uncertainty components. B3 tests shape discrimination of response distributions beyond two-order statistics; if bimodality refers to two alternatives (bM2), unimodality refers to four (uM4).

8.2 Competing Methods

Five baselines of neutrosophic MCDM, Standard SVNS-TOPSIS with expert-assigned scalar indeterminacy, interval valued neutrosophic TOPSIS, q-Rung Orthopair Neutrosophic MCDM with $q = 3$, entropy-weighted SVNS-TOPSIS using information-theoretic criteria weights and hesitant neutrosophic set-based MCDM with set-valued membership degrees, are evaluated. Baselines are identical to those originally published, and no changes have been made.

8.3 Evaluation Metrics

There are three basic measures of performance. The Discrimination Index (DI) is the mean score separation between the alternatives normalised by the maximum possible score separation when the number of alternatives is large. The original rank is compared with the rank after adding 15% noise; the Perturbation Stability Coefficient (PSC) quantifies Spearman's rank correlation between these two rankings. A PSC closer to 1.0 indicates the structure is more robust. The method rankings are compared to D2, which has been validated by experts, using Kendall's tau, called Ground-Truth Accuracy (GTA). Apart from these primary measures, the experimental evaluation features also (i) the Wilcoxon signedrank significance testing results connected to effect size measures (Cohen's d) across 1,000 bootstrap resamplings; (ii) results of the sensitivity analysis of the score parameters covering 15 valid (α, β) combinations; (iii) a descriptor ablation study for all cases of $k = 3$ to 14 components; and (iv) also a controlled identical-mean benchmark (B1) directly yielding given empirical validation of structural injectivity.

9. RESULTS AND ANALYSIS

9.1 Theoretical Properties Comparison

Table 3 positions EA-SNS against all five baselines across formal theoretical properties — the comparison that pre-empts originality challenges most directly.

Table 3. Theoretical Property Comparison: EA-SNS vs. Existing Methods

Property	SVNS	IVNS	q-RON	EW-SVNS	Hesitant NS	EA-SNS
Structural injectivity	✗	✗	✗	✗	✗	✓
Distribution-derived I	✗	✗	✗	Partial	✗	✓
Lipschitz stability bound	✗	✗	✗	✗	✗	✓
Operator idempotency proof	✓	✓	✓	✓	✓	✓
Simplex-valued uncertainty	✗	✗	✗	✗	✗	✓
Entropy-modulated ranking	✗	✗	✗	Partial	✗	✓
Generalises SVNS	—	✓	✓	✓	✓	✓
Weighted distance metric	Partial	Partial	Partial	Partial	Partial	✓

Many-to-one collapse is seen to damage scalar indeterminacy representations most in B2 and B3 – the benchmarks for skewness and bimodality differentiation. The enhancement on D-1 is 43.4% that of the SVNS. To the extent that there is a difference, even with the entropy-weighted baseline M4, the method EA-SNS also shows a 24.4% improvement, meaning that external entropy weighting is not a viable alternative to structural entropy decomposition within the representation.

9.2 Discrimination Performance

Table 4. Discrimination Index (DI) across Methods and Datasets (higher is better)

Method	D1	D2	B1	B2	B3	Mean	vs M1 (%)
M1: SVNS-TOPSIS	0.412	0.387	0.344	0.329	0.298	0.354	—
M2: IVNS-TOPSIS	0.431	0.403	0.361	0.347	0.318	0.372	+5.1%
M3: q-RON (q=3)	0.445	0.418	0.379	0.362	0.334	0.388	+9.6%
M4: EW-SVNS	0.461	0.437	0.396	0.381	0.349	0.405	+14.4%
M5: Hesitant NS	0.453	0.429	0.388	0.374	0.341	0.397	+12.1%
EA-SNS (Proposed)	0.591	0.563	0.445	0.498	0.421	0.504	+42.4%

The gains are most pronounced precisely in B2 and B3 — the skewness-differentiated and bimodality-differentiated benchmarks — where scalar indeterminacy representations suffer most acutely from many-to-one collapse. The improvement on D1 reaches 43.4% over SVNS. Less obviously, even against the entropy-weighted baseline M4, EA-SNS achieves a 24.4% improvement, confirming that external entropy weighting is not a substitute for structural entropy decomposition within the representation.

9.3 Perturbation Stability

Table 5. Perturbation Stability Coefficient (PSC) under 15% Input Noise (higher is better)

Method	D1	D2	B1	B2	B3	Overall Mean
M1: SVNS-TOPSIS	0.743	0.719	0.761	0.689	0.708	0.724
M2: IVNS-TOPSIS	0.771	0.748	0.784	0.715	0.732	0.750
M3: q-RON	0.789	0.763	0.798	0.733	0.748	0.766
M4: EW-SVNS	0.812	0.787	0.819	0.754	0.771	0.789

M5: Hesitant NS	0.803	0.778	0.811	0.746	0.762	0.780
EA-SNS (Proposed)	0.978	0.971	0.984	0.961	0.969	0.973

The stability superiority of 34.6% on D1 and 35.1% on D2 is the analytic guarantee of Theorem 4. The analytic guarantee of Theorem 4 is reflected in the stability superiority of 34.6% on D1 and 35.1% on D2. All five of the baselines do not have any constraints, as compared to the Lipschitz bound ($L \leq 1.87$), which directly limits score perturbation. Smoothing is not the means of its stability in this instance. It could emerge from the geometry of the structured uncertainty representation.

9.4 Ground-Truth Accuracy and Statistical Significance

When using the ranking provided by the experts on the ground truth, as shown in this ranking, Kendall's tau = 0.889 and Spearman's correlation = 0.991 are the highest among all methods considered for D2 and EA-SNS. However, including entropy at the weighting level provides partial benefits in terms of Spearman correlation, as seen with entropy-weighted SVNS (M4), which attains 0.931, the second-best value. The p-values for Wilcoxon signed-rank testing across all 1000 bootstrap resamples of EA-SNS and each baseline are all < 0.001 . Throughout, Cohen's d shows the effect size, ranging from 0.91 to 1.34, which produces large, significant clinical results.

9.5 Identical-Mean Discrimination (Benchmark B1)

When the alternatives are in B1 (all have identical means and increasing variance), with baselines M1, M2, and M3, the scores across the remaining five variants are nearly indistinguishable, differing by no more than 0.021. EA-SNS gives an interval of 0.047-0.089 for the adjacent alternatives, achieving the complete rank difference. This discrimination is due solely to the dispersion and entropy sources of the structured uncertainty vector, which are kept constant during experimental design. This is the best available validation of the claim of structural injectivity.

9.6 Sensitivity Analysis: Score Parameter Variation

Score sensitivity to α and β was examined across a grid of $\{0.1, 0.2, 0.3, 0.4, 0.5\} \times \{0.1, 0.2, 0.3, 0.4, 0.5\}$ subject to $\alpha + \beta < 1$. For the D1 and D2 alternatives, the top-ranked alternatives were stable 14/15 and 13/15, respectively, across all valid parameter combinations. The maximum single-rank change in each dataset was below $\alpha = 0.15$, significantly reducing the structural entropy penalty. The results obtained here are consistent with the low sensitivity at the recommended value range $\alpha, \beta \in [0.2, 0.4]$.

9.7 Ablation Study: Dimensional Sufficiency

Table 6. Ablation Study: Performance versus Descriptor Dimensionality (Dataset D1)

Configuration	k	DI	PSC	GTA	Assessment
Entropy + Dispersion only	3	0.448	0.812	0.761	Structurally incomplete
Reduced basis	6	0.497	0.871	0.823	Moderate
Reduced basis	8	0.531	0.919	0.856	Good
Reduced basis	10	0.568	0.953	0.878	Strong
Full Basis (Proposed)	11	0.591	0.978	0.889	Optimal (empirically sufficient)
Extended basis	12	0.593	0.979	0.890	Negligible gain, $r=0.44$ redundancy
Extended basis	14	0.591	0.977	0.888	No gain, increasing redundancy

Pairwise Pearson correlations among 11 descriptors: $\max r = 0.31$ (D1), 0.29 (D2). $k =$

12 introduces a descriptor with $r=0.44$ against BMC, confirming redundancy onset.

Removing each descriptor from $k = 11$ to $k = 10$ yields significant decreases for all three metrics, thereby demonstrating the non-redundancy of all the components. There is little improvement between $k = 11$ and $k = 12$. The practical sufficiency boundary is shown at $k = 11$ for the selected descriptor family.

10. DISCUSSION

Results confirm that structure-preserving uncertainty representation consistently enhances the quality of multi-criteria decision-making across all performance dimensions considered, from an empirical perspective. It is well to discuss this question of scope and originality directly before we look at the mechanism of improvement. EA-SNS is not just a set of parameters to an existing model. It addresses the structural deficiency of all previous neutrosophic formulations, including interval-valued, q-rung orthopair, hesitant, and Pythagorean variants (Table 3). It is important to note that the four properties differ from all five baselines, and that scalar-encoding frameworks are provably incapable of satisfying them. It is not the uniformity of this improvement that needs to be looked into; the task is simple in the controlled benchmarks. Even in these, this improvement is not competitive in terms of stability performance.

The solution lies in the representational architecture. There is no mechanism in scalar-encoding methods to express variance in the neutrosophic structure. The score function cannot directly access the distribution's spread; it only responds to the aggregated scalar indeterminacy value. If two alternatives have a similar indeterminacy value (as they will, with the many-to-one collapse), then the score surface is flat, and a noise perturbation can change the rank order. EA-SNS breaks the flatness by introducing distributional geometry into the score with the entropy-modulated term $\alpha \cdot I.H(\Theta)/\log_2(11)$. Indeed, the injectivity guarantee of Theorem 2 precludes flatness.

A valid theoretical argument cannot be ignored here. This argument might be justified because the 11-dimensional representation only shifts the problem of assigning a scalar value to a single number to that of assigning a value to an 11-component vector, which may be a legitimate concern, but one that the present framework cannot fully overcome. But there are two grounds for the objection, though they are the same. The components of the EA-SNN are not assigned, but rather are determined as a function of the observation set deterministically via fixed and interpretable functionals. It is impossible to obtain this type of structural injectivity guarantee from scalar-assignment frameworks, and this is by design: the representational dimension of the input space must equal its functional dimension, and a scalar cannot do so.

The analytical significance of EA-SNS's superiority over the validated baseline M4 is demonstrated. M4 is the best effort so far to integrate information-theoretic aspects into

neutrosophic MCDM. The result that it is inferior in discrimination by 24.4% is further evidence of its distinguishability from internal entropy decomposition. The structural lack must be filled with entropy, not only the entropy of the criteria weights.

There is one area where extra care is needed: descriptor transfer. The eleven-component basis was created for data from Likert scale and continuous (numeric) surveys. If the data consists of binary values, censored values, or point-process observations, then a domain-adapted descriptor basis would be needed. The thirteen proposed configurations of the theoretical guarantees hold for any functionally independent basis in which they hold. The eleven-component configuration used in this evaluation is one of the few configurations that are valid in practice; it is not a blanket prescription.

11. LIMITATIONS AND FUTURE RESEARCH DIRECTIONS

There are three limitations in the present framework. The descriptor basis was originally designed for data from a Likert scale or a continuous observation space; it must be developed and validated in each particular context when used with a discrete observation space or a space with categorical observations and/or censored observations. It may need to be modified in that context. The parameters of the score function α and β proved to be low-sensitivity in the range $[0.2, 0.4]$, but are still aided by expert guidance in applications where rank errors have asymmetric consequences. The sufficiency of $k = 11$ has been established in practice for this example, but hasn't been proven theoretically from first principles of information geometry, a future task for theory.

A natural extension of the EA-SNN framework to interval-valued truth degrees proceeds by replacing scalar $T(x)$ with an interval $[T^L(x), T^U(x)] \in [0,1]^2$, yielding an Extended EA-SNN:

$$N(x) = \langle [T^L, T^U], \quad I(x), \quad [F^L, F^U], \quad \theta(x) \rangle$$

This extension allows the full uncertainty structure also to be hendecagonal and does not prohibit expert-level uncertainty regarding truth assignments. The aggregation operators in Section 4.6 are easily extended by substituting the scalars T and F with interval arithmetic. Some of the formal characteristics of this extended structure are promising paths for further exploration.

Automated atomic descriptor selection through dependence independence testing based on mutual information, Bayesian score parameter calibration with domain-specific prior distribution and integration with existing large-scale machine learning evaluation pipelines with domain output distribution as natural observations are directions for future research.

12. CONCLUSION

Scalar indeterminacy in neutrosophic models is not merely a necessity or a simplification. Still, a crucial structural requirement that systematically deletes distributional information causes representational collapse and destabilises rankings in the very contexts where they can be of greatest importance and value for reliable decision-making. The EA-SNS framework overcomes this lack at the source by expressing indeterminacy in terms of measurable distributional functionals and structural elements, decomposed into eleven structurally independent uncertainty components.

The four theorems present the formal properties that are not achievable with scalar-encoding frameworks: bounded and monotone indeterminacy (Theorem 1), structural injectivity formally ruling out the occurrence of many-to-one collapse (Theorem 2), score function monotonicity preserving preference ordering (Theorem 3), and explicit Lipschitz perturbation bounds (Theorem 4). The aggregation operators satisfy independent idempotency, commutativity and neutrosophic boundary preservation property. The formal distance metric generalises the TOPSIS-based decision procedures to the entire space of EA-SNN. A ten-step algorithm provides the entire decision procedure. A fully worked numerical example verifies end-to-end reproducibility.

Fold change and discrimination gain were confirmed empirically across five datasets and five competitive baselines at $p < 0.001$, with an average discrimination gain of 34.7%. The second-order perturbation stability coefficient was statistically significant at $p = 0.003$, with a coefficient of 0.973. The Spearman correlation was statistically significant ($p = 0.002$, $r = 0.991$) with expert-validated ground truth. We show that under very special circumstances, VNS reduces to a classical, degenerate case of the general framework, and demonstrate this algebraically. There is no extension of an existing framework for the contribution. It reconceptualises indeterminacy from a scalar to a derived, decomposable and structurally faithful object, and it shows that this reconceptualisation can give decision-theoretic properties, for instance structural injectivity,

Lipschitz-bounded stability, and entropy-modulated discrimination, empirically and formally, that are not attainable by construction by any of the previous neutrosophic formulations.

DATA AVAILABILITY STATEMENT

Dataset D1 (instructional effectiveness survey, 312 evaluations across 8 alternatives and 6 criteria) and Dataset D2 (academic program quality survey, 187 evaluations across 5 programs and 6 attributes) were collected from educational institutions under institutional data governance agreements. Anonymised versions of both datasets, sufficient for full replication of the experimental results reported in Sections 9.1–9.7, are available from the corresponding author upon reasonable request. The three synthetic benchmarks (B1, B2, B3) are fully reproducible from the distributional parameters specified in Section 8.1 and require no external data. All experimental analyses reported in this paper can be reproduced using the algorithm specified in Section 6 and the descriptor formulas in Section 4.4.

Authors' Contributions Statement

D. Janak Raj Amit Kumar conceptualized the study, developed the methodology, and performed the data analysis and computations. Dr. I. Paulraj Jayasimman contributed to data collection and provided guidance throughout the study. Both authors jointly wrote the original draft of the manuscript. All authors reviewed and approved the final manuscript.

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Ethical Statement: This study was conducted in accordance with standard ethical practices in educational research. The research did not involve any procedures requiring formal ethical committee approval.

Informed Consent: Informed consent was obtained from all participants involved in the study.

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The Percentage Fallacy and a Single-Valued Neutrosophic Framework for Indeterminacy in AI Detector Verdicts on Academic Authorship

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Abstract: Artificial intelligence (AI) content detectors have entered university academic assessment as authorship verification instruments, frequently without formal regulation or consideration of their documented error rates. This article proposes a single-valued neutrosophic (SVN) framework to formalize the epistemological indeterminacy that characterizes the verdicts of these systems. We argue that every AI detector collapses an implicit neutrosophic triple $\langle T, I, F \rangle$ into a one-dimensional scalar, eliminating the indeterminacy component I and producing what we call the percentage fallacy. To resolve this indeterminacy, we propose a neutrosophic score function $S = T + \alpha \cdot D - I - F$ that incorporates the demonstrated cognitive mastery D through oral defense or direct evaluation, and we formalize the Principle of Cognitive Primacy (PCP) as the prevailing criterion of academic authorship. The systematic literature review documents false positive rates exceeding 60% for academic writing produced by non-native English speakers, including Ecuadorian and Latin American students. We conclude that, when an institution has a regulation limiting AI use in academic documents and applies a detector as a verification tool, the cognitive mastery demonstrated by the student over that document must operate as the primary criterion to resolve the algorithmic ambiguity, since it is the only evidence that captures the indeterminacy component I that the detector's scalar suppresses.

Keywords: single-valued neutrosophic sets; AI detectors; false positives; academic integrity; Principle of Cognitive Primacy; algorithmic bias; academic authorship; higher education.

1. Introduction

The emergence of large language models (LLMs) in the university ecosystem has destabilized established assumptions about academic intellectual production. Higher education institutions have responded with uneven speed, deploying commercial AI content detectors as control instruments without regulatory frameworks that establish their validity as evidence of infraction in most cases [4, 7]. Tools such as GPTZero, Turnitin AI, Copyleaks, and OpenAI Classifier have been incorporated into faculty workflows under an implicit premise that this article subjects to formal scrutiny: that a numerical percentage of artificial authorship probability constitutes sufficient evidence to determine whether a student used AI in producing an academic work. We demonstrate that this premise is mathematically incorrect.

The central problem we address is not whether students use AI or not, but whether a scalar produced by an algorithm constitutes epistemologically valid evidence to determine authorship. We argue that the answer is negative, and that this error can be formalized with precision through the theory of single-valued neutrosophic sets (SVN) [8, 9], since academic authorship under algorithmic uncertainty is irreducibly a problem of indeterminacy. Neutrosophic theory is the only mathematical framework that treats indeterminacy as an independent component, not derived from the truth and falsity components [9, 10].

In the Ecuadorian context, the Organic Law of Higher Education (LOES) [19] and the Academic Regulations of the Higher Education Council (CES) [22] contain no specific provision on the use of AI detectors as authorship verification instruments. This absence does not enable any evaluative practice: when institutions establish rules limiting AI use in producing academic documents, the verification process activated must guarantee that the

evidence employed is epistemologically valid. The principle of due process requires that any disciplinary action be supported by sufficient evidence, not by statistical probabilities not validated for the specific application context.

This article makes five contributions. First: we reformulate the output of an AI detector as a single-valued neutrosophic triple $\langle T, I, F \rangle$. Second: we define a neutrosophic score function $S(\text{work}) = T + \alpha \cdot D - I - F$ that incorporates demonstrated cognitive mastery D . Third: we formulate the Principle of Cognitive Primacy (PCP). Fourth: we document the paradox of historical false positives. Fifth: we analyze the implications of the PCP for institutional contexts with and without specific regulation on AI use.

2. Neutrosophic Preliminaries

We present the formal definitions necessary for the development of the proposed framework. Neutrosophic theory was introduced by Smarandache [9] as an extension of Zadeh's fuzzy logic [12] and of Atanassov's intuitionistic sets [11], with the fundamental difference that the three components of a neutrosophic element are mathematically independent of each other.

Definition 1 (Neutrosophic set [9, 10]). Let X be a universe of discourse. A neutrosophic set A on X is defined as:

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in X \}$$

where $T_A(x), I_A(x), F_A(x) : X \rightarrow]-0, 1+[$ are the truth, indeterminacy, and falsity membership functions, respectively, with:

$$-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$$

The independence between T , I and F distinguishes this structure from classical fuzzy logic, where the condition $T + F \leq 1$ is mandatory, and from intuitionistic sets [11], where $T + F + H = 1$. In neutrosophic theory none of these restrictions applies, which makes it possible to represent genuinely indeterminate states of knowledge without forcing a distribution between truth and falsity.

Definition 2 (Single-valued neutrosophic set — SVN [8]). An SVN over X is a triple $A = \langle T_A, I_A, F_A \rangle$ with $T_A, I_A, F_A \in [0, 1]$ and:

$$0 \leq T_A + I_A + F_A \leq 3$$

SVNs are the practical restriction of neutrosophic sets to single values in $[0, 1]$, which makes them directly operational in decision-making problems with real uncertainty [15, 17, 21]. Each component is quantified independently, without the value of one determining or restricting the value of the others.

Definition 3 (Neutrosophic score function [15, 16]). Let $A = \langle T, I, F \rangle$ be an SVN. The score function of A is defined as:

$$s(A) = (T - I - F) / 3, \quad s(A) \in [-1, 1]$$

This function penalizes indeterminacy and falsity and allows ordering of neutrosophic alternatives. Values of s close to 1 indicate high truth and low uncertainty; values close to -1 indicate high falsity or high indeterminacy.

Definition 4 (Extended neutrosophic score with cognitive mastery). Let W be an academic work assessed by an AI detector, with verdict represented as $\langle T, I, F \rangle$. Let $D \in [0, 1]$ be the cognitive mastery demonstrated by the student over W , and $\alpha \in [0, 1]$ an institutional weighting parameter. The extended neutrosophic score of the work is defined as:

$$S(\text{work}) = T + \alpha \cdot D - I - F$$

When $\alpha = 1$, cognitive mastery receives the same weight as the truth component. When $\alpha = 0$, cognitive mastery is discarded, which is equivalent to accepting the algorithmic verdict without human correction. We argue that $\alpha = 0$ is epistemologically inadmissible in any academic assessment system that recognizes the existence of false positives.

3. State of the Art on False Positives in AI Detectors

The scientific literature published between 2022 and 2025 consistently documents the low reliability of AI content detectors, with particular concentration on the false positive phenomenon and its asymmetric distribution among writer populations.

Weber-Wulff et al. [2] evaluated fourteen AI text detection tools and concluded that none was sufficiently accurate or reliable for use in academic contexts with disciplinary consequences. Their analysis revealed systematic biases toward classifying human text as machine-generated, especially when the text presented regular syntactic structures or standardized technical vocabulary. Elkhayat, Elsaid and Almeer [3] evaluated five commercial detectors against texts produced by GPT-3.5 and GPT-4 and found significant inconsistencies in the classification of human controls: the instruments produced false positives and uncertain classifications even when the text was written by the researchers themselves.

The most relevant finding for the Latin American context comes from Liang et al. [1], who demonstrated through controlled experiment that AI detectors classify as machine-generated more than 60% of TOEFL essays written by non-native English speakers, while texts from native English-speaking students are correctly identified as human in nearly 100% of cases. This gap does not reflect actual AI use, but rather statistical regularity in the lexical and syntactic structure characteristic of formal academic writing in a second language. Detectors confuse formal competence with algorithmic artificiality.

Sadasivan et al. [6] formally demonstrated that perfect detection of LLM-generated text is, under general conditions, mathematically impossible. Their argument establishes that any detector can be defeated by simple paraphrasing, and that the error rate inevitably grows when the analyzed text comes from writers with consolidated formal styles. Mitchell et al. [14], with their DetectGPT system, achieved acceptable results only under laboratory conditions that are not reproduced in ordinary academic practice.

The historical paradox represents the most compelling argument against the epistemological validity of detectors. Khalil and Er [13] documented that tools such as GPTZero assign artificial-origin probabilities greater than zero to texts written before the existence of modern LLMs. This result demonstrates that detectors do not identify artificial authorship: they identify linguistic statistical regularity. An instrument that assigns artificial-origin probability to nineteenth-century academic prose cannot operate as valid evidence of authorship in the twenty-first century.

Giray [18] documents that false accusations derived from false positives have caused verifiable damage to the academic trajectory of students and researchers, constituting a form of algorithmic discrimination whose consequences are asymmetric depending on the language of writing and the level of stylistic formality. Foltýnek, Meuschke and Gipp [25] note that detection systems for non-original production require, as a minimum validity condition, error rates sufficiently low for the specific application context, a condition that current AI detectors do not satisfy. Kasneci et al. [20] argue that the integration of AI in education requires assessment frameworks that recognize the complexity of technologically mediated learning, rather than control instruments of unproven validity.

Cotton, Cotton and Shipway [7] argue that academic integrity in the era of LLMs requires a reconceptualization of verification mechanisms, with emphasis on assessment of process over textual product. Bozkurt [5] notes that human-AI co-authorship challenges conventional academic authorship frameworks and demands criteria that go beyond inspection of the final artifact. Perkins [4] documented that faculty perceive detectors as convenient but epistemologically fragile control instruments, and notes the need for alternative verification mechanisms based on direct demonstration of knowledge.

4. The Percentage Fallacy: Neutrosophic Formalization

We call the percentage fallacy the practice of taking the scalar output of an AI detector as sufficient evidence to determine artificial academic authorship. The formal argument we develop shows that this practice implies a mathematical simplification that eliminates precisely the most relevant information for a fair decision.

Let W be an academic work. An AI detector applies a function $f : W \rightarrow [0, 1]$ on W that produces a scalar p , interpreted as the probability that W is of artificial origin. From the neutrosophic perspective, the authorship of W under algorithmic uncertainty should be represented as a triple $\langle T, I, F \rangle$ where T is the degree of evidence of verifiable human authorship, I is the degree of indeterminacy inherent to the algorithmic verdict, and F is the degree of evidence of artificial authorship.

Proposition 1 (Epistemological insufficiency of the detection scalar). The scalar output p of an AI detector implies the assumption $T = 1 - F$, which corresponds to classical fuzzy logic and not to neutrosophic logic. This assumption eliminates by construction the indeterminacy I and produces an epistemologically incomplete representation of authorship.

Proof. If the detector produces $p = F$ as the probability of artificial origin, and presents $(1 - p)$ as the probability of human origin, then $T := 1 - F$. This identity collapses the independence between T and F and

implicitly establishes $I = 0$. But $I = 0$ would mean that no degree of indeterminacy exists in the verdict, which contradicts the empirical evidence documented [1, 2, 3, 6], which shows incorrect, contradictory, and systematically biased classifications. Therefore, the scalar representation of the detector does not capture the actual epistemic structure of the problem. $\square$

The direct consequence of Proposition 1 is that no AI detector, by its fundamental statistical design, can produce sufficient evidence of artificial authorship. It can produce risk indicators that justify additional verification, but it does not constitute proof.

Proposition 2 (Primacy of the extended score). If $I > 0$ in the neutrosophic representation of the verdict of an AI detector, then $S(\text{work}) = T + \alpha \cdot D - I - F$ is strictly greater than the assessment based exclusively on $p = F$ when $\alpha > 0$ and $D > 0$.

Proof. By hypothesis, $I > 0$. For any work with $D > 0$, the extended score satisfies:

$$S = T + \alpha \cdot D - I - F > T - I - F \geq -F - I > -I$$

The difference between S and the scalar criterion $-F$ is:

$$\Delta = S - (-F) = T + \alpha \cdot D - I$$

For $\Delta > 0$, $T + \alpha \cdot D > I$ is required. Given that the documented values of I in AI detectors are significant [2, 3] and cognitive mastery D can approach 1 through direct oral defense, $\Delta > 0$ is the usual condition for students who genuinely know their work. $\square$

5. The Principle of Cognitive Primacy (PCP): Formal Framework

Definition 5 (Cognitive mastery). Let W be an academic work and s a student. The cognitive mastery $D(s, W) \in [0, 1]$ of s over W is a measure that quantifies the ability of s to answer, explain, expand, and relate the content of W under direct assessment conditions, through a structured, contradictory oral defense protocol conducted by a human evaluator.

The measure D incorporates at least three verifiable dimensions: the ability to reformulate the central ideas of the work without consulting the text; the ability to answer in-depth questions on the topic addressed; and the ability to identify the primary sources cited and explain their relevance. A trained evaluator can assign D through an explicit rubric, which makes the PCP operationalizable in concrete institutional contexts.

Principle of Cognitive Primacy (PCP). Let W be an academic work on which an AI detector has produced a verdict $\{T_d, I_d, F_d\}$ with $I_d > 0$. Let $D(s, W)$ be the cognitive mastery demonstrated by student s in defense. The PCP states:

$$\text{If } S(\text{work}) = T_d + \alpha \cdot D(s, W) - I_d - F_d > 0,$$

then W must be considered a work of verified human authorship, regardless of the value of F_d . Consequently, no verification process on AI use can be resolved exclusively with the scalar $p = F_d$.

The epistemological foundation of the PCP rests on the distinction between process evidence and product evidence. An AI detector examines only the final textual artifact. But academic authorship is an attribute of the intellectual process that generated that artifact, not of the artifact itself. A text may present statistical characteristics similar to those of AI-generated text because the student writes in a language that is not their mother tongue [1], because they have deeply internalized the patterns of formal academic discourse [18], or because their style is technically rigorous [2]. None of these causes implies AI use. Only direct assessment of the cognitive process resolves this indeterminacy.

The PCP is also consistent with the principle of due process: the burden of proof falls on the accuser, and the accused has the right to present exculpatory evidence. The positive demonstration of cognitive mastery is precisely that evidence, and its epistemological weight is superior to that of a statistical percentage whose Type I errors exceed 60% in populations comparable to the Ecuadorian one [1].

6. Illustrative Numerical Case

We present a hypothetical case that illustrates the application of the neutrosophic framework with concrete and internally verifiable numerical values.

Let s_1 and s_2 be two students who submit academic works W_1 and W_2 respectively. Both obtain the same percentage from the AI detector: $p = 0.65$ (65% probability of artificial origin). Under conventional scalar assessment, both students would receive an identical institutional response.

Step 1: Neutrosophic representation of the detector verdict.

Since the detector only produces the scalar $p = F$, and implicitly assumes $T = 1 - F$, the initial triples are identical. Correcting with $I > 0$ according to documented error rates [2, 3]:

$$W_1 : \langle T_1, I_1, F_1 \rangle = \langle 0.25, 0.10, 0.65 \rangle$$

$$W_2 : \langle T_2, I_2, F_2 \rangle = \langle 0.25, 0.10, 0.65 \rangle$$

Step 2: Oral defense and quantification of cognitive mastery.

$$D(s_1, W_1) = 0.90 \quad (\text{high mastery, consistent and autonomous responses})$$

$$D(s_2, W_2) = 0.15 \quad (\text{low mastery, superficial and inconsistent responses})$$

Step 3: Calculation of the extended neutrosophic score with $\alpha = 1$.

$$S(W_1) = 0.25 + 1 \times 0.90 - 0.10 - 0.65 = 0.40 > 0 \rightarrow \text{verified authorship}$$

$$S(W_2) = 0.25 + 1 \times 0.15 - 0.10 - 0.65 = -0.35 < 0 \rightarrow \text{requires additional investigation}$$

s_1 obtains a positive score that, by the PCP, indicates verified human authorship despite the detector percentage. s_2 obtains a negative score that generates a case warranting additional investigation. This numerical example demonstrates that two students with the same AI detection percentage may have radically different situations regarding their authorship, and that only the SVN framework with the variable D can formally capture this difference.

To illustrate the historical paradox, consider an academic text H written in 1994. If a contemporary detector is applied to H , the result would be $F_H > 0$ despite H being produced before the existence of any LLM. The neutrosophic representation requires $I_H > 0$, which immediately activates the PCP: the verdict is indeterminate and cannot be used as evidence of infraction.

7. Discussion

7.1. Reliability of AI Detectors and the Percentage Fallacy

The analysis formalized in Propositions 1 and 2 shows that every AI detector that produces a scalar as output structurally eliminates neutrosophic indeterminacy, which invalidates its use as sufficient evidence of artificial authorship. This is consistent with what was stated by Weber-Wulff et al. [2], who document that none of the fourteen detection tools evaluated in their study reached sufficient levels of accuracy or reliability for academic contexts with disciplinary consequences. According to these authors, detectors present systematic biases that make them inadequate as authorship verification instruments, which confirms that the scalar output of these systems cannot operate as conclusive evidence by itself.

Along the same lines, Elkhataf, Elsaid and Almeer [3] found that five commercial detectors produce inconsistent classifications even when the text was written entirely by the researchers themselves, generating false positives in the human controls. This coincides with Proposition 1: if detectors classify as artificial a text produced by academic evaluators, the indeterminacy component I is structurally greater than zero, which collapses any scalar interpretation of the result. In contrast to the practice implicit in some institutional guidelines that cite the Turnitin or GPTZero percentage as a direct verification criterion, the present analysis shows that this practice eliminates, by mathematical construction, the most relevant information for a fair decision.

Sadasivan et al. [6] establish, through a formal argument, that perfect detection of LLM-generated text is mathematically impossible under general conditions, since any detector can be defeated by simple paraphrasing and the error rate grows structurally when the text comes from writers with consolidated styles. This is consistent with Proposition 2: the extended score S surpasses the scalar criterion precisely in cases where the formal style of the student artificially raises the value of F . On the other hand, Mitchell et al. [14] argue that their DetectGPT system achieves acceptable performance under controlled laboratory conditions; however, the authors themselves acknowledge that these conditions are not reproduced in ordinary academic practice, so the actual discriminatory

capacity of the detector remains insufficient to ground decisions with consequences on the academic trajectory of the student.

The historical paradox documented by Khalil and Er [13], who observed that detectors such as GPTZero assign artificial-origin probabilities greater than zero to texts produced before the existence of modern LLMs, reinforces this argument from an empirical angle. According to these authors, the phenomenon evidences that detectors do not identify artificial authorship but linguistic statistical regularity, which converges with the central thesis of this article: the instrument detects frequency patterns, not cognitive processes, and this limitation is irreducible in the current design of these systems.

7.2. Linguistic Bias and Algorithmic Discrimination in the Latin American Context

The analysis reveals that AI detectors structurally penalize those who write in languages other than English or in formal academic styles. This concurs directly with the findings of Liang et al. [1], who documented that more than 60% of TOEFL essays written by non-native English speakers were classified as AI-generated, compared to correct human identification rates close to 100% for native English texts. According to these authors, the gap does not reflect actual AI use but lexical-syntactic statistical regularity proper to the formal academic register in a second language, which turns the detector into an instrument that discriminates on linguistic competence, not on authorship.

In the Ecuadorian context, where academic works are produced in Spanish and students build their writing within the conventions of Latin American academic discourse, the phenomenon described by Liang et al. [1] translates into false positive rates potentially higher than those documented for English, given that commercial detectors are trained predominantly with corpora in that language. This is consistent with what Giray [18] points out, who documents that researchers with formally rigorous writing styles recurrently obtain higher false detection rates. According to Giray [18], the false accusations derived from these errors have generated verifiable damages to the trajectory of students and academics, which reinforces the need for the PCP as a mechanism that allows correcting that bias through direct measurement of cognitive mastery.

In contrast to the reading that AI detectors are neutral teaching support tools — a reading underlying some institutional educational policy documents — the present analysis shows that algorithmic neutrality is an illusion when the instrument was designed and validated on a linguistic corpus that does not represent the population that receives it. Cotton, Cotton and Shipway [7] recognize that detectors can serve as an initial alert signal, but argue that academic integrity in the LLM era requires privileging assessment of the intellectual process over inspection of the textual artifact. This concurs with the PCP: demonstrated cognitive mastery D is precisely that process evidence which the authors point to as a superior criterion, and its formal incorporation into the function S grants it the operational weight that equitable assessment requires.

7.3. Cognitive Mastery as Verification Criterion under Institutional AI-Use Regulation

The central argument of this article does not aim to create sanctioning mechanisms for those who use AI, but rather to establish what must be verified when an institution already has regulations limiting or regulating that use and applies a detector as a verification tool. In that scenario, the problem is precise: if the detector issues an ambiguous verdict on a document presumably generated with AI, what evidence can resolve that ambiguity? This concurs with what Perkins [4] poses, who points out that the absence of clear criteria on what constitutes valid evidence of infraction transfers to faculty discretion a high-impact decision on the student's trajectory. According to this author, that discretion generates systematic inequities that disproportionately affect students with less institutional capacity to challenge.

Bozkurt [5] argues that the irruption of generative AI forces a reconsideration of concepts such as authorship and academic responsibility, and that institutions must adopt frameworks that recognize the collaborative nature of contemporary production. While the present analysis agrees with Bozkurt [5] that conventional authorship frameworks are insufficient for the current context, it differs in focus: whereas that author orients the discussion toward the redefinition of shared authorship norms, the proposed PCP does not debate whether AI use is legitimate or not, but establishes the operational criterion to determine when the student demonstrates that the knowledge contained in the document belongs to them, regardless of the tools used during preparation. The positive demonstration of high $D(s,W)$ constitutes the evidence of intellectual authorship that no algorithmic percentage can provide.

Kasneci et al. [20] argue that the integration of AI in education requires frameworks that balance pedagogical opportunities with ethical safeguards, and warn that purely restrictive institutional responses can hinder the development of competencies that students will need in their professional life. This concurs with the PCP argument: verifying cognitive mastery does not discourage the reflective use of AI in the learning process, but rather shifts academic responsibility to the right place. When the student can sustain a solid defense on the content of their work — regardless of which tools they used during preparation — the assessment system fulfills its pedagogical purpose of verifying real appropriation of knowledge.

Zawacki-Richter et al. [27] document that AI applications in higher education have been adopted fundamentally from the technological perspective, with scant faculty participation in the design of validation criteria. This concurs with the underlying critique of this article: AI detectors entered university assessment processes without specific pedagogical validation for each use context, reproducing the pattern of technological primacy over educational criterion that those authors identify as a structural risk. The function $S = T + \alpha \cdot D - I - F$ reverses this logic by placing the parameter α — decided by the institution based on its pedagogical objectives — as a regulator of the relative weight between the algorithmic verdict and direct human evidence.

Foltýnek, Meuschke and Gipp [25] establish that detection systems for non-original production require, as a minimum condition of validity in academic contexts with disciplinary consequences, error rates sufficiently low for the specific population of use. This condition is equivalent, in neutrosophic terms, to requiring that I be significantly close to zero before accepting the verdict as evidence. Given that the studies of Liang et al. [1], Weber-Wulff et al. [2] and Elkhatat et al. [3] convergently demonstrate that this condition is not met for any tool available in the Latin American context, the conclusion is direct: when an institution has regulations limiting AI use and decides to verify the authorship of a questioned document, the PCP must operate as a mandatory criterion of resolution, since it converts the indeterminacy I of the detector into an empirically answerable question through direct human assessment.

8. Conclusions

This article has demonstrated through neutrosophic formalization that the output of every AI detector collapses an implicit neutrosophic triple $\langle T, I, F \rangle$ into a one-dimensional scalar, eliminating the indeterminacy I by design. This elimination is epistemologically incorrect because the empirical evidence consistently documents that $I > 0$ for any academic text evaluated with currently available tools.

The Principle of Cognitive Primacy that we propose formally resolves this indeterminacy by incorporating demonstrated cognitive mastery D as the central decisive variable through the extended score function $S = T + \alpha \cdot D - I - F$. This principle does not propose new sanctions for AI use: it establishes what must be verified when an institution already has regulations limiting that use and employs a detector as a verification tool. The answer is that the cognitive mastery demonstrated by the student over the questioned document — and not the algorithmic percentage — must operate as the primary criterion of resolution.

The systematic literature review confirms three facts: AI detectors present false positive rates exceeding 60% for the writing of non-native English speakers; these tools assign artificial-origin probabilities to historical texts produced before the existence of LLMs; and no available tool reaches sufficient accuracy levels for use in processes with disciplinary consequences.

In the Ecuadorian context, the absence of specific CES regulations on the use of AI detectors implies that any sanction grounded exclusively on algorithmic percentages lacks adequate normative basis. The PCP offers the criterion that institutions can adopt to guarantee due process: the verdict of a detector identifies a case that warrants verification, but it is direct oral defense that resolves the indeterminacy.

As future lines of work, we propose extending the framework to group assessments through SVN aggregation operators [15, 17], the development of standardized oral defense protocols with formal cognitive mastery metrics calibrated for Latin American university contexts, and the empirical validation of the parameter α through case studies in Ecuadorian universities.

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Neutrosophic-Based Optimization Algorithm: A Novel Metaheuristic Algorithm for Global Optimization

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Abstract

This study presents a novel neutrosophic optimizer, called NOpt, for continuous global optimization. In the proposed algorithm, each candidate solution is represented by a neutrosophic search state comprising truth, indeterminacy, and falsity degrees. The truth component promotes exploitation around promising regions, the indeterminacy component enhances exploration of uncertain regions, and the falsity component facilitates escape from poor or stagnant regions. The refined NOpt further incorporates two mechanisms: a Neutrosophic Elite Refinement Operator, which improves each generated trial vector before its evaluation, and a Neutrosophic Adaptive Local Search Operator, which utilizes a controlled portion of the remaining function-evaluation budget to refine elite solutions. The proposed optimizer is empirically compared with seven established metaheuristic algorithms under a comprehensive experimental protocol. The results demonstrate that NOpt achieves outstanding performance on the fixed-dimensional test functions (TFs), which simulate real-world optimization problems.

Keywords: Neutrosophic set; Optimization problem; Metaheuristic algorithms; Global optimization; Neutrosophic optimizer.

1. Introduction

Global optimization is a fundamental task in many scientific, engineering, and computational applications. The objective is to find the best solution within a search space that may be nonlinear, multimodal, discontinuous, noisy, high-dimensional, or difficult to differentiate [1, 2]. In such cases, exact mathematical optimization methods may become computationally expensive or unsuitable. Metaheuristic algorithms are therefore widely used because they can search complex spaces and provide high-quality approximate solutions within a reasonable computational budget [3]. A central challenge in metaheuristic optimization is the balance between exploration and exploitation [4]. Exploration enables the algorithm to investigate new and diverse regions of the search space, while exploitation refines the search around promising solutions. If exploration dominates for too long, the algorithm may waste function evaluations without sufficient improvement. If exploitation dominates too early, the algorithm may converge prematurely to a local optimum. In addition to these two behaviors,

a strong optimizer also requires an effective escape mechanism to move away from poor, stagnant, or misleading regions.

Most existing metaheuristic algorithms address this balance through adaptive parameters, elite guidance, mutation strategies, diversity control, or random perturbation mechanisms [5-7]. However, the search condition of a candidate solution is usually more complex than its fitness value alone. A solution may have good fitness but may be located in an overcrowded or overexploited region. Another solution may have moderate fitness but may lie in a promising uncertain region. A weak or stagnant solution may need a stronger escape behavior instead of ordinary movement. Therefore, a more expressive state representation is needed to describe the search role of each candidate solution during the optimization process.

This paper proposes NOpt, a Truth–Indeterminacy–Falsity-guided Neutrosophic Optimizer for continuous global optimization. The main idea is to represent each candidate solution not only by its position and objective value, but also by a neutrosophic search state [7]:

$$NS_i^t = \langle T_i^t, I_i^t, F_i^t \rangle,$$

where T_i^t denotes the truth degree of the candidate solution X_i at iteration t , I_i^t denotes its indeterminacy degree, and F_i^t denotes its falsity degree. In the proposed optimizer, these three components are not used as descriptive labels after evaluation. They are active search-control components that directly guide the movement of each candidate solution. The truth component controls exploitation around promising regions. The indeterminacy component increases exploration when the search state is unclear, unstable, or insufficiently explored. The falsity component activates escape behavior when a solution is weak, stagnant, or located in an unproductive region. Accordingly, NOpt introduces a unified neutrosophic movement equation in which exploitation, exploration, and escape are controlled adaptively through the three neutrosophic components. The proposed optimizer is designed to be mathematically transparent and directly implementable. Each movement component has a clear role: truth moves the candidate solution toward promising solutions, indeterminacy introduces controlled exploratory variation, and falsity pushes the search away from poor regions. This structure makes the algorithm easier to understand, analyze, and implement compared with optimization methods whose behavior depends on several disconnected operators.

The aim of this work is not to claim universal superiority over all optimization algorithms or all problem classes. Instead, the objective is to introduce a genuinely neutrosophic optimization mechanism and evaluate its ability to improve the balance between exploration, exploitation, and escape in continuous global optimization. The experimental part of this paper evaluates NOpt against seven well-known metaheuristic algorithms: Weighted average algorithm (WAA), Runge-Kutta method (RUN) beyond the metaphor, Rime optimization

algorithm (RIME), whale optimization algorithm (WOA), sine-cosine algorithm (SCA), and GOOSE algorithm. The main contributions of this paper are summarized as follows.

- First, a new neutrosophic optimizer, called NOpt, is proposed for continuous global optimization.
- Second, each candidate solution is represented by a neutrosophic search state composed of truth, indeterminacy, and falsity degrees.
- Third, a unified movement equation is developed in which truth guides exploitation, indeterminacy guides exploration, and falsity guides escape from poor or stagnant regions.
- Fourth, the algorithm is formulated in a transparent way that allows direct implementation and analysis.
- Fifth, the proposed optimizer is empirically compared with seven established metaheuristic algorithms using a complete experimental protocol.

The remainder of this paper is organized as follows. [Section 2](#) presents the mathematical preliminaries of global optimization and neutrosophic search states. [Section 3](#) introduces the proposed NOpt algorithm and its main operators. [Section 4](#) presents results and discussion. [Section 5](#) concludes the paper and outlines future work.

2. Mathematical Preliminaries

This section defines the mathematical basis used to construct the proposed NOpt algorithm. The definitions include the continuous global optimization problem, the population structure, the normalized search-quality measure, the diversity indicator, the elite-closeness indicator, the stagnation indicator, and the neutrosophic search state.

2.1. Continuous Global Optimization Problem

Let the optimization problem be a continuous minimization problem defined as [8]:

$$\min f(X), X = (x_1, x_2, \dots, x_D) \in \Omega, \quad (1)$$

where $f(X)$ denotes the objective function, X denotes a candidate solution, D denotes the number of decision variables, and Ω is the feasible search space. The feasible search space is bounded by lower and upper limits, as defined below:

$$\Omega = \{X \in \mathbb{R}^D, : L_j \leq x_j \leq U_j, j = 1, 2, \dots, D\}, \quad (2)$$

where L_j and U_j are the lower and upper bounds of the j -th decision variable, respectively. Although this paper focuses on minimization problems, a maximization problem can be handled by minimizing $-f(X)$ without changing the structure of the proposed optimizer. At iteration t , the population contains N candidate solutions represented as follows:

$$P^t = \{X_1^t, X_2^t, \dots, X_N^t\}.$$

The fitness value of the i -th candidate solution is computed as follows:

$$f_i^t = f(X_i^t).$$

For a minimization problem, the best and worst fitness values at iteration t are defined as:

$$f_{\min}^t = \min_{1 \leq i \leq N} f_i^t, f_{\max}^t = \max_{1 \leq i \leq N} f_i^t. \quad (5)$$

The current best solution is denoted by:

$$X_*^t = \arg \min_{X_i^t \in P^t} f(X_i^t),$$

and the current worst solution is denoted by:

$$X_w^t = \arg \max_{X_i^t \in P^t} f(X_i^t). \quad (7)$$

If more than one solution has the same best or worst fitness value, any one of them may be selected without affecting the general structure of the algorithm.

2.2. Normalized Search Quality

Raw fitness values may have different scales from one optimization problem to another. Therefore, NOpt uses a normalized search-quality value to describe how good each candidate solution is relative to the current population. For a minimization problem, the normalized search quality of X_i^t is defined as:

$$q_i^t = \begin{cases} \frac{f_{\max}^t - f_i^t}{f_{\max}^t - f_{\min}^t}, & f_{\max}^t > f_{\min}^t, \\ 0.5, & f_{\max}^t = f_{\min}^t. \end{cases} \quad (8)$$

The value q_i^t lies in $[0, 1]$. A larger value indicates a better solution relative to the current population. When $f_{\max}^t > f_{\min}^t$, the current best solution has $q_i^t = 1$, while the current worst solution has $q_i^t = 0$. When all solutions have the same fitness value, $q_i^t = 0.5$ is used as a neutral value.

2.3. Population Diversity Indicator

Fitness alone is not sufficient to describe the search condition of a candidate solution. A solution may have good fitness but may be located in an overcrowded region. Another solution may have moderate fitness but may contribute useful diversity. Therefore, NOpt uses a population-diversity indicator. The population center at iteration t is defined as:

$$\bar{X}^t = \frac{1}{N} \sum_{i=1}^N X_i^t. \quad (9)$$

The Euclidean distance of X_i^t from the population center is:

$$d_i^t = \|X_i^t - \bar{X}^t\|_2. \quad (10)$$

The normalized diversity indicator is defined as:

$$\delta_i^t = \begin{cases} \frac{d_i^t}{\max_{1 \leq r \leq N} d_r^t}, & \max_{1 \leq r \leq N} d_r^t > 0, \\ 0, & \max_{1 \leq r \leq N} d_r^t = 0. \end{cases} \quad (11)$$

The value δ_i^t lies in $[0, 1]$. A high value means that the candidate solution is far from the population center and may help preserve search diversity. A low value means that the solution is close to the central region of the population.

2.4. Elite-Closeness Indicator

To support exploitation, NOpt measures how close each candidate solution is to the current best solution. This measure is called the elite-closeness indicator. The average distance from the population to the current best solution is:

$$\bar{d}_*^t = \frac{1}{N} \sum_{i=1}^N \|X_i^t - X_*^t\|_2. \quad (12)$$

The elite-closeness indicator is then defined as:

$$e_i^t = \exp\left(-\frac{\|X_i^t - X_*^t\|_2}{\bar{d}_*^t + \varepsilon}\right), \quad (13)$$

where ε is a very small positive constant used to avoid division by zero. The value e_i^t lies in $(0, 1)$. A larger value means that the candidate solution is closer to the current best solution, while a smaller value means that it is farther away from the best solution.

2.5. Stagnation Indicator

A candidate solution may fail to improve for several consecutive iterations. This condition is called stagnation. In NOpt, stagnation is important because a stagnant solution may require stronger exploration or escape behavior. Let a_i^t be the number of consecutive iterations in which solution X_i has not improved up to iteration t . The normalized stagnation indicator is defined as:

$$s_i^t = \min\left(1, \frac{a_i^t}{a_{\max}}\right), \quad (14)$$

where $a_{\max}$ is a predefined stagnation limit. The value s_i^t lies in $[0, 1]$. A value close to zero indicates recent improvement, while a value close to one indicates strong stagnation.

2.6. Neutrosophic Search State

The central idea of NOpt is to describe each candidate solution by a neutrosophic search state. This state contains three components [9-11]:

$$NS_i^t = \langle T_i^t, I_i^t, F_i^t \rangle. \quad (15)$$

Here, T_i^t is the truth degree of solution X_i^t . It represents the tendency of the solution to participate in exploitation. I_i^t is the indeterminacy degree of solution X_i^t . It represents the need for exploration caused by uncertainty, diversity, or unstable search behavior. F_i^t is the

falsity degree of solution X_i^t . It represents the tendency of the solution to activate escape behavior from weak or stagnant regions. The three components are bounded as follows:

$$0 \leq T_i^t \leq 1, 0 \leq I_i^t \leq 1, 0 \leq F_i^t \leq 1. \quad (16)$$

The components T_i^t , I_i^t , and F_i^t are not required to sum to one. This is important because a candidate solution may be promising, uncertain, and partially weak at the same time. Therefore, the neutrosophic search state provides a richer description of the search condition than a single fitness-based value.

2.7. Role of the Preliminaries in NOpt

The quantities defined in this section form the mathematical input for the proposed optimizer. The normalized quality q_i^t , diversity indicator δ_i^t , elite-closeness indicator e_i^t , and stagnation indicator s_i^t are used to construct the neutrosophic components T_i^t , I_i^t , and F_i^t . These neutrosophic components control the three main movement behaviors of NOpt: exploitation, exploration, and escape. Therefore, this section provides the mathematical foundation for the full neutrosophic search mechanism introduced in the next section.

3. Proposed NOpt Algorithm

This section presents the final structure of the proposed NOpt algorithm. NOpt is a neutrosophic population-based optimizer in which each candidate solution is guided by three active search components: truth, indeterminacy, and falsity. The truth component strengthens exploitation around promising regions, the indeterminacy component strengthens exploration in uncertain regions, and the falsity component activates escape from poor or stagnant regions. The refined NOpt also includes two additional mechanisms: a Neutrosophic Elite Refinement Operator, which improves the generated trial vector before evaluation, and a Neutrosophic Adaptive Local Search Operator, which uses a controlled part of the remaining function-evaluation budget to refine elite solutions.

3.1. Initialization and Evaluation Budget

The initial population is generated randomly within the feasible search space. For each candidate solution X_i^0 , the j -th decision variable is initialized as follows:

$$x_{ij}^0 = L_j + r_{ij}^0(U_j - L_j), i = 1, 2, \dots, N, j = 1, 2, \dots, D, \quad (17)$$

where r_{ij}^0 is a random value in the interval $(0, 1)$, L_j is the lower bound, and U_j is the upper bound of the j -th decision variable. After initialization, the objective value of each candidate solution is evaluated as follows:

$$f_i^0 = f(X_i^0). \quad (18)$$

The initial best solution X_*^0 , the initial worst solution X_w^0 , and the stagnation counters are then initialized. At the beginning of the algorithm, the stagnation counter for each candidate solution is initialized as follows:

$$a_i^0 = 0, i = 1, 2, \dots, N. \quad (19)$$

The initial global-best solution and its objective value are stored as:

$$X_{best}^0 = X_*, f_{best}^0 = f(X_*). \quad (20)$$

The termination condition of the algorithm is controlled by a maximum number of function evaluations FES_{max} .

3.2. Search Progress Factors

At iteration t , let FES^t denote the number of objective-function evaluations consumed before generating new trial solutions. The normalized search progress factor is defined as:

$$\tau_t = \min\left(1, \frac{FES^t}{FES_{max}}\right). \quad (21)$$

The decreasing exploration factor is:

$$\chi_t = 1 - \tau_t. \quad (22)$$

The value of τ_t increases as the available evaluation budget is consumed, while χ_t decreases during the search. These two factors are used to reduce global exploration gradually and strengthen local refinement in later stages.

3.3. Neutrosophic Search State Construction

At each iteration t , each candidate solution X_i^t is represented by a neutrosophic search state:

$$NS_i^t = \langle T_i^t, I_i^t, F_i^t \rangle. \quad (23)$$

The three components are constructed from the normalized search quality q_i^t , the diversity indicator δ_i^t , the elite-closeness indicator e_i^t , and the stagnation indicator s_i^t , which were defined in Section 2. The uncertainty-quality term is defined as:

$$u_i^t = 4q_i^t(1 - q_i^t). \quad (24)$$

The value u_i^t lies in $[0, 1]$. It reaches its maximum when $q_i^t = 0.5$, which means that the solution is neither clearly strong nor clearly weak relative to the current population. The truth degree is defined as:

$$T_i^t = \text{clip}(\alpha_T q_i^t + \beta_T e_i^t). \quad (25)$$

The truth degree increases when a solution has high normalized quality and is close to the elite region. Therefore, T_i^t controls the exploitation tendency of X_i^t . The indeterminacy degree is defined as:

$$I_i^t = \text{clip}(\alpha_I u_i^t + \beta_I \delta_i^t + \gamma_I s_i^t). \quad (26)$$

The indeterminacy degree increases when a solution has uncertain quality, contributes to population diversity, or shows stagnation. Therefore, I_i^t controls the exploration tendency of X_i^t . The falsity degree is defined as:

$$F_i^t = \text{clip}(\alpha_F(1 - q_i^t) + \beta_F s_i^t). \quad (27)$$

The falsity degree increases when a solution has weak normalized quality or repeated stagnation. Therefore, F_i^t controls the escape tendency of X_i^t . The clipping function is:

$$\text{clip}(z) = \min(1, \max(0, z)). \quad (28)$$

All weighting coefficients used in the neutrosophic state construction are non-negative. The default values are:

$$\alpha_T = 0.70, \beta_T = 0.30, \quad (29)$$

$$\alpha_I = 0.50, \beta_I = 0.30, \gamma_I = 0.20, \quad (30)$$

$$\alpha_F = 0.70, \beta_F = 0.30. \quad (31)$$

These coefficients satisfy the following constraint:

$$\alpha_T + \beta_T = 1, \alpha_I + \beta_I + \gamma_I = 1, \alpha_F + \beta_F = 1. \quad (32)$$

Thus, the components T_i^t , I_i^t , and F_i^t remain interpretable within $[0, 1]$, while the clipping function provides additional numerical safety.

3.4. Normalized Neutrosophic Movement Weights

The neutrosophic components are not required to sum to one. However, the movement equation requires proportional control. Therefore, NOpt converts them into normalized movement weights:

$$\omega_{T,i}^t = \frac{T_i^t}{T_i^t + I_i^t + F_i^t + \varepsilon}, \quad (33)$$

$$\omega_{I,i}^t = \frac{I_i^t}{T_i^t + I_i^t + F_i^t + \varepsilon}, \quad (34)$$

$$\omega_{F,i}^t = \frac{F_i^t}{T_i^t + I_i^t + F_i^t + \varepsilon}, \quad (35)$$

where $\varepsilon > 0$ is a very small constant used to avoid division by zero. The weight $\omega_{T,i}^t$ controls the influence of exploitation, $\omega_{I,i}^t$ controls the influence of exploration, and $\omega_{F,i}^t$ controls the influence of escape.

3.5. Adaptive Elite Set

NOpt uses an adaptive elite set to guide exploitation and refinement. The elite set is larger at the beginning of the search to preserve diversity and smaller near the end to focus the search around the best regions. The adaptive elite ratio is defined as:

$$\rho_t = \rho_{\min} + (\rho_{\max} - \rho_{\min})\chi_t, \quad (36)$$

where $\rho_{\max}$ is the initial elite ratio and $\rho_{\min}$ is the final elite ratio. The number of elite solutions is:

$$n_e^t = \max(2, \lceil \rho_t N \rceil). \quad (37)$$

After sorting the population in ascending order according to objective value, the elite set is:

$$\mathcal{P}_{el}^t = \{X_{(1)}^t, X_{(2)}^t, \dots, X_{(n_e^t)}^t\}, \quad (38)$$

where $X_{(1)}^t = X_*^t$ is the current best solution.

3.6. Adaptive Differential Scale and Crossover Rate

To improve search flexibility, NOpt assigns an adaptive differential scale and an adaptive crossover rate to each candidate solution. The adaptive differential scale is:

$$F_{DE,i}^t = F_{\min}^{DE} + (F_{\max}^{DE} - F_{\min}^{DE})(0.55\omega_{I,i}^t + 0.45\omega_{F,i}^t). \quad (39)$$

Thus, solutions with high indeterminacy or high falsity receive stronger differential movement. The adaptive crossover rate is:

$$CR_i^t = CR_{\min} + (CR_{\max} - CR_{\min})(0.70\omega_{T,i}^t + 0.30\omega_{I,i}^t). \quad (40)$$

Thus, promising and uncertain solutions inherit more information from the generated trial vector, while weaker solutions are controlled mainly through the escape mechanism.

3.7. Truth-Guided Exploitation Operator

The truth-guided exploitation operator moves a candidate solution toward promising regions. It uses the current best solution X_*^t and one elite solution X_p^t randomly selected from $\mathcal{P}_{el}^t$:

$$A_i^t = r_{1,i}^t \odot (X_*^t - X_i^t) + r_{2,i}^t \odot (X_p^t - X_i^t), \quad (41)$$

where $r_{1,i}^t, r_{2,i}^t \in [0, 1]^D$ are random vectors, and $\odot$ denotes element-wise multiplication. The first term attracts the solution toward the current best position, while the second term attracts it toward another elite solution. This improves exploitation while reducing excessive dependence on a single best point.

3.8. Indeterminacy-Driven Exploration Operator

The indeterminacy-driven exploration operator allows uncertain solutions to explore new regions. It combines a differential population step with a controlled Gaussian perturbation:

$$B_i^t = F_{DE,i}^t (X_{m_i}^t - X_{n_i}^t) + \eta_t (U - L) \odot Z_i^t, \quad (42)$$

where Z_i^t is a random vector based on the normal distribution, and $X_{m_i}^t$ and $X_{n_i}^t$ are two randomly selected solutions from the current population. Their indices satisfy:

$$m_i \neq n_i, m_i \neq i, n_i \neq i. \quad (43)$$

The exploration amplitude is:

$$\eta_t = \eta_{\min} + (\eta_{\max} - \eta_{\min})\chi_t^{\kappa_\eta}, \quad (44)$$

where $\eta_{\max}$ and $\eta_{\min}$ are the maximum and minimum exploration amplitudes, and $\kappa_\eta > 0$ controls the decay rate. The differential term produces directed exploration based on population variation. The Gaussian perturbation decreases gradually as the search approaches the final stages, while the differential term remains adaptive through $F_{DE,i}^t$, allowing population-based directional exploration to continue when needed.

3.9. Falsity-Based Escape Operator

The falsity-based escape operator moves weak or stagnant solutions away from poor search regions. It contains a repulsion term from the current worst solution and a clipped Cauchy perturbation:

$$C_i^t = r_{5,i}^t \odot (X_i^t - X_w^t) + \sigma_t (U - L) \odot K_i^t, \quad (45)$$

where $r_{5,i}^t \in [0, 1]^D$, X_w^t is the current worst solution, and K_i^t is a clipped Cauchy random vector. Each component of K_i^t is generated as:

$$K_{ij}^t = \max(-c_{\max}, \min(c_{\max}, K_{ij}^t)), j = 1, 2, \dots, D. \quad (46)$$

The escape amplitude is:

$$\sigma_t = \sigma_{\min} + (\sigma_{\max} - \sigma_{\min})\chi_t^{\kappa_\sigma}, \quad (47)$$

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximum and minimum escape amplitudes, and $\kappa_\sigma > 0$ controls the decay rate. The repulsion term moves the solution away from the worst region, while the Cauchy perturbation provides occasional long-range movement to reduce stagnation.

3.10. Neutrosophic Elite Refinement Operator

To improve local accuracy in later stages, NOpt introduces a Neutrosophic Elite Refinement Operator. This operator strengthens refinement around the elite region while preserving the neutrosophic search structure. The refinement activation factor is:

$$\ell_t = \tau_t^{\kappa_R}, \quad (48)$$

where $\kappa_R > 0$ controls how quickly refinement becomes active. Since ℓ_t increases as the consumed evaluation budget increases, the refinement operator has weak influence at the beginning and stronger influence near the end. The refinement perturbation scale is:

$$\mu_t = \mu_{\min} + (\mu_{\max} - \mu_{\min})(1 - \ell_t). \quad (49)$$

Thus, the perturbation radius becomes smaller as the algorithm approaches the final stages. Let $X_{u_i}^t$ and $X_{v_i}^t$ be two elite solutions selected from $\mathcal{P}_{el}^t$, where:

$$u_i \neq v_i. \quad (50)$$

The refinement vector is defined as:

$$R_i^t = \ell_t [r_{6,i}^t \odot (X_i^t - X_i^t) + F_{DE,i}^t (X_{u_i}^t - X_{v_i}^t) + \mu_t (U - L) \odot Z_{R,i}^t], \quad (51)$$

where $r_{6,i}^t \in [0, 1]^D$, and $Z_{R,i}^t$ is a random vector based on the normal distribution. The first term refines the solution toward the current best solution. The second term uses elite difference information to preserve local directional variation. The third term adds a small perturbation around the elite region. Since this operator is later multiplied by the truth weight, it mainly affects solutions with stronger exploitation potential. This refinement operator does not require an additional objective-function evaluation. It modifies the trial vector before the single trial evaluation of each candidate solution.

3.11. Unified Neutrosophic Trial Vector

The complete trial vector is generated by combining exploitation, exploration, escape, and elite refinement:

$$V_i^t = X_i^t + \omega_{T,i}^t A_i^t + \omega_{I,i}^t B_i^t + \omega_{F,i}^t C_i^t + \omega_{R,i}^t R_i^t. \quad (52)$$

Equation (52) is the core movement equation of NOpt. It shows that the movement of each candidate solution is controlled by its own neutrosophic state. If $\omega_{T,i}^t$ is dominant, the candidate mainly exploits and refines the elite region. If $\omega_{I,i}^t$ is dominant, the candidate

explores uncertain regions. If $\omega_{F,i}^t$ is dominant, the candidate activates escape from weak or stagnant regions.

3.12. Component-Preserving Crossover

After constructing V_i^t , NOpt applies component-preserving crossover between the current solution X_i^t and the generated trial vector V_i^t . For each dimension j , the trial solution is generated as:

$$y_{ij}^t = \begin{cases} v_{ij}^t, & rand_{ij}^t \leq CR_i^t \text{ or } j = j_{rand}, \\ x_{ij}^t, & \text{otherwise,} \end{cases} \quad (53)$$

where $rand_{ij}^t \sim U(0, 1)$, and j_{rand} is a randomly selected dimension from $\{1, 2, \dots, D\}$.

The condition $j = j_{rand}$ ensures that at least one component is inherited from the generated trial vector.

This crossover step improves stability because not all dimensions are forced to change at every iteration.

3.13. Greedy Selection and Stagnation Update

The objective value $f(\tilde{Y}_i^t)$ is evaluated once. The feasible trial solution replaces the current solution only if it improves the objective value:

$$X_i^{t+1} = \begin{cases} \tilde{Y}_i^t, & f(\tilde{Y}_i^t) < f(X_i^t), \\ X_i^t, & \text{otherwise.} \end{cases} \quad (54)$$

The stagnation counter is updated as:

$$a_i^{t+1} = \begin{cases} 0, & f(\tilde{Y}_i^t) < f(X_i^t), \\ a_i^t + 1, & \text{otherwise.} \end{cases} \quad (55)$$

After the complete population update, the number of function evaluations consumed is updated as follows:

$$FEs_{pop}^{t+1} = FEs^t + N. \quad (56)$$

Successful improvement resets the stagnation counter, while failure to improve increases it. The updated stagnation value is then used in the next iteration to influence both indeterminacy and falsity.

3.14. Population Best, Worst, and Global-Best Update Before Local Search

After all candidate solutions are processed, the population before local search is:

$$P_{pop}^{t+1} = \{X_1^{t+1}, X_2^{t+1}, \dots, X_N^{t+1}\}. \quad (57)$$

The index of the best solution in this population is:

$$p_{pop}^{t+1} = \arg \min_{1 \leq i \leq N} f(X_i^{t+1}), \quad (58)$$

and the index of the worst solution is:

$$w_{pop}^{t+1} = \arg \max_{1 \leq i \leq N} f(X_i^{t+1}). \quad (59)$$

Accordingly, the best and worst solutions before local search are:

$$X_{*,pop}^{t+1} = X_{pop}^{t+1}, X_{w,pop}^{t+1} = X_{wpop}^{t+1}. \quad (60)$$

The global-best solution before local search is updated as:

$$X_{best,pop}^{t+1} = \begin{cases} X_{*,pop}^{t+1}, & f(X_{*,pop}^{t+1}) < f(X_{best}^t), \\ X_{best}^t, & \text{otherwise.} \end{cases} \quad (61)$$

The corresponding objective value is:

$$f_{best,pop}^{t+1} = f(X_{best,pop}^{t+1}). \quad (62)$$

3.15. Neutrosophic Adaptive Local Search Operator

The Neutrosophic Adaptive Local Search Operator improves late-stage accuracy by allocating a controlled part of the remaining evaluation budget to elite-region refinement. Unlike the elite refinement vector in Eq. (51), this operator performs additional objective-function evaluations. Therefore, every local-search trial is counted within $FES_{\max}$. The local-search activation factor is:

$$\lambda_t = \tau_t^{\kappa_{LS}}, \quad (63)$$

where $\kappa_{LS} > 0$ controls how quickly local search becomes active. The local-search radius is:

$$h_t = h_{\min} + (h_{\max} - h_{\min})(1 - \tau_t)^{\kappa_h}, \quad (64)$$

where $h_{\max}$ is the initial local-search radius, $h_{\min}$ is the minimum local-search radius, and $\kappa_h > 0$ controls the decay rate. The maximum number of local-search trials allowed at iteration t is:

$$B_{LS}^t = \min([\lambda_t \rho_{LS} N], FES_{\max} - FES_{pop}^{t+1}), \quad (65)$$

where $\rho_{LS} \in [0, 1]$ is the local-search budget ratio. If $B_{LS}^t = 0$, the local-search phase is skipped.

For local search, an elite set is reconstructed from P_{pop}^{t+1} using the same adaptive elite ratio ρ_t . This elite set is denoted by:

$$\mathcal{P}_{el,pop}^{t+1}. \quad (66)$$

For each local-search trial $b = 1, 2, \dots, B_{LS}^t$, one elite solution $X_{z_b}^{t+1}$ is selected from $\mathcal{P}_{el,pop}^{t+1}$.

A local trial solution is generated as:

$$L_b^t = X_{z_b}^{t+1} + h_t(U - L) \odot Z_b^t + \lambda_t r_b^t \odot (X_{best,pop}^{t+1} - X_{z_b}^{t+1}), \quad (67)$$

where Z_b^t is a random vector based on the normal distribution, and $r_b^t \in [0, 1]^D$. The first perturbation term searches locally around the selected elite solution. The second term directs the local trial toward the best region found so far. Boundary control is applied as:

$$\tilde{L}_{b,j}^t = \min \left(U_j, \max(L_j, L_{b,j}^t) \right), j = 1, 2, \dots, D. \quad (68)$$

When $B_{LS}^t > 0$, the objective values of all local-search trials are evaluated and stored. Therefore, the value $f(\tilde{L}_{best}^t)$ is reused in the update rules without an additional function evaluation. The best local trial is defined as:

$$\tilde{L}_{best}^t = \arg \min_{1 \leq b \leq B_{LS}^t} f(\tilde{L}_b^t). \quad (69)$$

The global-best solution after local search is updated as:

$$X_{best}^{t+1} = \begin{cases} \tilde{L}_{best}^t, & B_{LS}^t > 0 \text{ and } f(\tilde{L}_{best}^t) < f_{best,pop}^{t+1}, \\ X_{best,pop}^{t+1}, & \text{otherwise.} \end{cases} \quad (70)$$

The corresponding objective value is:

$$f_{best}^{t+1} = f(X_{best}^{t+1}). \quad (71)$$

If the best local trial improves the global-best solution, it is inserted into the population by replacing the worst member of P_{pop}^{t+1} :

$$X_{w_{pop}^{t+1}}^{t+1} \leftarrow \tilde{L}_{best}^t, \text{ if } B_{LS}^t > 0 \text{ and } f(\tilde{L}_{best}^t) < f_{best,pop}^{t+1}. \quad (72)$$

The stored objective value of $\tilde{L}_{best}^t$ is assigned to the replaced population member, so no additional objective-function evaluation is required for this replacement. After this possible replacement, the final population of iteration $t + 1$ is denoted by:

$$P^{t+1}. \quad (73)$$

The best and worst solutions are then recomputed from the final population:

$$X_*^{t+1} = \arg \min_{X_i^{t+1} \in P^{t+1}} f(X_i^{t+1}), \quad (74)$$

$$X_w^{t+1} = \arg \max_{X_i^{t+1} \in P^{t+1}} f(X_i^{t+1}). \quad (75)$$

The number of function evaluations consumed after local search is:

$$FES^{t+1} = FES^{t+1} + B_{LS}^t. \quad (76)$$

This operator strengthens late-stage refinement while preserving a fair evaluation budget. It does not give NOpt extra evaluations beyond the predefined maximum because each local trial is explicitly counted.

3.16. Pseudocode of the proposed NOpt

The key parameters of the refined NOpt algorithm are summarized in Table 1. These parameters are fixed across benchmark functions in the experimental study unless otherwise stated. Finally, the pseudocode of the proposed algorithm is provided in Algorithm 1. The refined NOpt performs adaptive search at the level of each candidate solution. A promising solution receives stronger truth-guided exploitation and elite refinement. An uncertain solution receives stronger indeterminacy-driven exploration. A weak or stagnant solution receives stronger falsity-based escape. The adaptive elite set supports broader guidance at the beginning of the search and more focused refinement near the end. The adaptive differential scale and crossover rate allow each candidate solution to receive a movement pattern consistent with its neutrosophic state. The elite refinement operator improves the generated trial vector before evaluation, while the adaptive local-search operator allocates part of the remaining evaluation budget to direct refinement around elite solutions. Therefore, NOpt does not rely on one fixed movement rule for the whole population. Instead, each solution receives its own movement balance according to its truth, indeterminacy, and falsity degrees, while the algorithm remains bounded by the same objective-function evaluation budget used for experimental comparison.

Table 1. Main parameters of the refined NOpt algorithm

Parameter	Meaning	Default value	Parameter	Meaning	Default value
N	Population size	Problem-dependent	$c_{\max}$	Cauchy clipping limit	3
D	Number of decision variables	Problem-dependent	κ_{η}	Exploration decay exponent	1.5
$FES_{\max}$	Maximum number of function evaluations	Problem-dependent	κ_{σ}	Escape decay exponent	2.0
$\rho_{\max}$	Initial elite ratio	0.30	κ_R	Elite-refinement activation exponent	1.5
$\rho_{\min}$	Final elite ratio	0.05	κ_{LS}	Local-search activation exponent	2.0
ρ_{LS}	Local-search budget ratio	0.20	κ_h	Local-search radius decay exponent	2.0
$a_{\max}$	Maximum stagnation limit	20	ε	Small positive constant	10^{-12}
$\eta_{\max}$	Maximum exploration amplitude	0.03	$h_{\max}$	Maximum local-search radius	0.05
$\eta_{\min}$	Minimum exploration amplitude	10^{-5}	$h_{\min}$	Minimum local-search radius	10^{-10}
$\sigma_{\max}$	Maximum escape amplitude	0.02	$F_{\min}^{DE}$	Minimum adaptive differential scale	0.40
$\sigma_{\min}$	Minimum escape amplitude	10^{-6}	$F_{\max}^{DE}$	Maximum adaptive differential scale	1.00
$CR_{\max}$	Maximum crossover rate	1.00	$CR_{\min}$	Minimum crossover rate	0.05

Algorithm 1: Pseudocode of NOpt

Output : X_{best}

1. Initialize N solutions using Eq. (17)
2. Evaluate each X_i^0 using Eq. (18)

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3. Identify X_*^0 and X_w^0
 4. $X_{best}^0 = X_*^0$
 5. $t = 0$
 6. $FES^t = N$
 7. $\alpha_i^0 = 0, \forall i \in \{1:N\}$
 8. **while** ($FES^t < FES_{max}$)
 9. Compute τ_t and χ_t using (21) and (22).
 10. Compute q_i^t , δ_i^t , e_i^t , and s_i^t for all solutions.
 11. Compute u_i^t using (24).
 12. Compute T_i^t , I_i^t , and F_i^t using (25), (26), and (27).
 13. Compute $\omega_{T,i}^t$, $\omega_{I,i}^t$, and $\omega_{F,i}^t$ using (33), (34), and (35).
 14. Construct the adaptive elite set using (36), (37), and (38).
 15. Compute $F_{DE,i}^t$ and CR_i^t using (39) and (40).
 16. **for** $i = 1:N$
 17. Compute the exploitation vector A_i^t using (41).
 18. Compute the exploration vector B_i^t using (42).
 19. Compute the escape vector C_i^t using (45).
 20. Compute the elite-refinement vector R_i^t using (51).
 21. Generate the trial vector V_i^t using (52).
 22. Apply component-preserving crossover using (53).
 23. Evaluate $\tilde{Y}_i^t$
 24. Apply greedy selection using (54).
 25. Update the stagnation counter using (55).
 26. Update the best and worst solutions.
 27. $FES^t = FES^t + 1$
 28. **End for**
 29. Reconstruct the elite set
 30. Compute λ_t , h_t , and B_{LS}^t using (63), (64), and (65).
 31. **if** $B_{LS}^t > 0$
 32. Generate B_{LS}^t local-search trials using (67).
 33. Apply boundary control to the local-search trials using (68).
 34. Evaluate and store the objective values of all local-search trials.
 35. Select the best local trial using (69).
 36. Update the best solution using (70).
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37. If the best local trial improves the global best, replace the worst population member using (72).
 38. **End If**
 39. Define the final population P^{t+1} using (73).
 40. Recompute X_*^{t+1} and X_w^{t+1} from P^{t+1} using (74) and (75).
 41. $FEs^t = FEs^t + B_{LS}^t$
 42. $t = t + 1$
 43. **End while**
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4. Results and discussion

In this study, the proposed NOpt optimizer is evaluated on 21 standard test functions (TFs) listed in Table 2 [4, 12]. These test functions are widely used in the literature to assess the exploratory and exploitative performance of newly proposed algorithms, which is the main reason for using them to evaluate NOpt. The proposed NOpt is compared with several competing algorithms, including Weighted average algorithm (WAA) [13], Runge-Kutta method (RUN) beyond the metaphor [14], Rime optimization algorithm (RIME) [15], whale optimization algorithm (WOA) [3], sine-cosine algorithm (SCA) [16], and GOOSE algorithm [17], to demonstrate its effectiveness and assess whether it provides meaningful contributions to the optimization field. The control parameters of the competing algorithms, except for the maximum number of function evaluations and population size, are set as recommended in the original references. The maximum number of function evaluations and population size are set to 100,000 and 20, respectively, for all algorithms to ensure a fair comparison. After executing each algorithm independently 30 times using different random seeds, the resulting fitness values are analyzed in terms of several performance metrics, including best fitness (best), average fitness (Avg), standard deviation (SD), and Friedman mean rank (FK). All experiments are conducted on a device with the following hardware and software specifications: 11th Gen Intel(R) Core(TM) i7-11850H @ 2.50 GHz, 32 GB of RAM, and a 64-bit Windows 11 Professional operating system. All algorithms are implemented in MATLAB R2024b under the same computational environment to ensure a fair comparison.

Each algorithm is independently executed 30 times on each standard TF using different random seeds. The resulting fitness values are then analyzed in terms of the aforementioned performance metrics and provided in Table 3. As can be observed from this table, NOpt outperforms all algorithms on 3 of the 21 TFs, achieves competitive performance on four TFs, and is inferior to some algorithms, such as WAA, RUN, and NRBO, on the remaining TFs. To assess the overall performance of the algorithms, their average FK values across all TFs are computed and presented in Fig. 1. As shown in this figure, RUN achieves the best overall performance with an average FK of 2.54, followed by NRBO, whereas NOpt ranks

fourth with an average FK of 4.00. Although the proposed algorithm in this study does not achieve the best overall performance, the obtained results provide empirical evidence that relying solely on neutrosophic set behaviors does not provide sufficient diversity in the search process, thereby limiting the algorithm's ability to achieve outstanding performance across the majority of TFs. In addition, the results show that NOpt can achieve outstanding performance on the fixed-dimensional TFs, which simulate real-world optimization problems, where it demonstrates competitive performance against some algorithms and outperforms others, indicating its potential for effectively addressing real-world optimization problems. Therefore, investigating the performance of NOpt on a broader range of real-world problems is considered an important direction for future research.

Table 2. Description of the standard test functions.

ID	Functions	D	Search range	Global opt
TF1	$F1 = \sum_{i=1}^D x_i^2$	100	[-100,100]	0
TF2	$F2 = \sum_{i=1}^D x_i + \prod_{i=1}^D x_i $	100	[-100,100]	0
TF3	$F3 = \sum_{i=1}^D \left(\sum_{j=1}^D x_j \right)^2$	100	[-100,100]	0
TF4	$F4 = \max(x_i) \quad 1 \leq i \leq D$	100	[-100,100]	0
TF5	$F5 = \sum_{i=1}^{D-1} [100(x_{i+1} - x_i^2)^2 + (x_i - 1)^2]$	100	[-30,30]	0
TF6	$F6 = \sum_{i=1}^{D-1} [x_i + 0.5]^2$	100	[-100,100]	0
TF7	$F7 = \sum_{i=1}^D tx_i^4 + \text{random}[0, 1]$	100	[-1.28,1.28]	0
TF8	$F8 = \sum_{i=1}^{D-1} -x_i \sin(\sqrt{ x_i })$	100	[-500,500]	-418.98 $\times D$
TF9	$F9 = \sum_{i=1}^D [x_i^2 - 10 \cos(2\pi x_i) + 10]$	100	[-5.12,5.12]	0
TF10	$F10 = -20 \exp\left(-0.2 \sqrt{\frac{1}{D} \sum_{i=1}^D x_i^2}\right) - \exp\left(\frac{1}{D} \sum_{i=1}^D \cos(2\pi x_i)\right) + 20 + e$	100	[-32,32]	0
TF11	$F11 = \frac{1}{4000} \sum_{i=1}^D x_i^2 - \prod_{i=1}^D \cos\left(\frac{x_i}{\sqrt{i}}\right) + 1$	100	[-600,600]	0
TF12	$F12 = \frac{\pi}{D} \left(10 \sin(\pi y_1) + \sum_{i=1}^{D-1} (y_i - 1)^2 [1 + 10 \sin^2(\pi y_{i+1})] \right. \\ \left. + (y_D - 1)^2 \right) + \sum_{i=1}^D u(x_i, 10, 100, 4)$	100	[-50,50]	0

TF13	$F13 = \frac{\pi}{d} \left(10 \sin^2(3x_1) + \sum_{i=1}^{d-1} (x_i - 1)^2 [1 + \sin^2(3\pi x_i)] \right. \\ \left. + (x_d - 1)^2 [1 + \sin^2(2\pi x_d)] \right) \\ + \sum_{i=1}^d u(x_i, 5, 100, 4)$	100	[-50,50]	0
TF14	$F14 = \left(\frac{1}{500} + \sum_{j=1}^{25} \frac{1}{j + \sum_{i=1}^2 (x_i - a_{ij})^2} \right)^{-1}$	2	[-65,65]	1
TF15	$F15 = \sum_{i=1}^{11} \left[a_i - \frac{x_1(b_i^2 + b_i x_2)}{b_i^2 + b_i x_3 + x_4} \right]^2$	4	[-5,5]	0.00030
TF16	$F16 = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4$	2	[-5,5]	-1.0316
TF17	$F17 = \left(x_2 - \frac{5.1}{4\pi^2} x_1^2 + \frac{5}{\pi} x_1 - 6 \right)^2 + 10 \left(1 - \frac{1}{8\pi} \right) \cos(x_1) + 10$	2	[-5,5]	0.398
TF18	$F18 = [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \\ \times [30 + (2x_1 - 3x_2)^2 \\ \times (18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)]$	2	[-2,2]	3
TF19	$F19 = - \sum_{i=1}^4 c_i e^{-\sum_{j=1}^3 a_{ij}(x_j - p_{ij})^2}$	3	[1,3]	-3.86
TF20	$F20 = - \sum_{i=1}^4 c_i e^{-\sum_{j=1}^6 a_{ij}(x_j - p_{ij})^2}$	6	[0,1]	-3.32
TF21	$F21 = - \sum_{i=1}^5 [(X - a_i)(X - a_i)^T + c_i]^{-1}$	4	[0,10]	-10.1532

Table 3: Results of the algorithms across 21 standard TFs.

	TF1				TF2				TF3			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	1.38E-01	2.12E-01	6.28E-02	5.70	8.24E-01	1.49E+00	3.27E-01	6.00	6.24E+03	9.82E+03	1.95E+03	5.00
WOA	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.50	9.11E+04	4.15E+05	1.37E+05	7.93
WAA	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.00
SCA	9.78E-01	9.71E+02	1.83E+03	7.97	1.27E-08	3.22E-03	1.43E-02	5.00	5.85E+04	1.31E+05	3.53E+04	7.07
RUN	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.00
GOOSE	1.10E-01	1.82E-01	4.24E-02	5.30	2.17E+02	4.18E+16	2.29E+17	8.00	1.62E+03	2.26E+03	3.79E+02	4.00
RIME	1.46E+00	2.00E+00	3.81E-01	7.03	9.72E+00	1.79E+01	5.32E+00	7.00	1.97E+04	2.39E+04	2.72E+03	6.00
NRBO	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.50	0.00E+00	0.00E+00	0.00E+00	2.00
	TF4				TF5				TF6			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	1.12E+01	1.49E+01	2.31E+00	4.03	1.04E+02	5.26E+02	6.59E+02	5.73	1.13E-01	2.10E-01	5.77E-02	4.30
WOA	1.26E+01	6.90E+01	2.59E+01	7.03	9.55E+01	9.63E+01	6.11E-01	2.50	7.17E-02	2.37E-01	1.41E-01	4.00

WAA	0.00E+00	0.00E+00	0.00E+00	2.00	2.53E-04	5.42E-02	9.09E-02	1.00	1.84E-06	4.17E-03	5.25E-03	2.00
SCA	6.85E+01	7.79E+01	4.38E+00	7.53	1.29E+06	1.27E+07	1.26E+07	8.00	2.19E+01	1.34E+03	1.80E+03	8.00
RUN	0.00E+00	0.00E+00	0.00E+00	2.00	9.43E+01	9.65E+01	1.39E+00	2.63	2.07E-08	5.65E-08	5.91E-08	1.00
GOOSE	3.06E+01	3.92E+01	5.13E+00	5.70	1.22E+02	5.40E+02	6.97E+02	5.83	1.24E-01	1.82E-01	3.56E-02	3.70
RIME	2.83E+01	3.94E+01	4.83E+00	5.70	2.17E+02	7.99E+02	7.47E+02	6.43	1.23E+00	1.88E+00	3.41E-01	6.00
NRBO	0.00E+00	0.00E+00	0.00E+00	2.00	9.71E+01	9.84E+01	4.33E-01	3.87	1.58E+01	1.71E+01	7.69E-01	7.00
	TF7				TF8				TF9			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	5.44E-02	9.33E-02	2.74E-02	5.07	-2.64E+04	-2.43E+04	1.12E+03	4.63	1.59E+02	1.98E+02	2.42E+01	5.80
WOA	1.99E-05	5.55E-04	6.40E-04	3.87	-4.19E+04	-3.95E+04	4.26E+03	2.13	0.00E+00	0.00E+00	0.00E+00	2.50
WAA	9.89E-07	2.16E-05	2.43E-05	1.43	-4.19E+04	-4.19E+04	5.62E-02	1.00	0.00E+00	0.00E+00	0.00E+00	2.50
SCA	4.57E-01	1.02E+01	9.84E+00	7.83	-9.43E+03	-7.69E+03	5.15E+02	7.90	1.51E+01	1.28E+02	8.64E+01	5.23
RUN	5.37E-06	6.70E-05	7.24E-05	2.60	-2.75E+04	-2.27E+04	2.69E+03	5.17	0.00E+00	0.00E+00	0.00E+00	2.50
GOOSE	7.52E-01	1.59E+00	3.49E-01	7.17	-2.58E+04	-2.30E+04	1.28E+03	5.20	5.03E+02	6.52E+02	9.53E+01	8.00
RIME	9.82E-02	1.52E-01	2.79E-02	5.93	-3.38E+04	-3.12E+04	1.22E+03	2.87	1.88E+02	2.99E+02	4.64E+01	6.97
NRBO	7.87E-06	3.60E-05	2.80E-05	2.10	-1.14E+04	-9.19E+03	1.11E+03	7.10	0.00E+00	0.00E+00	0.00E+00	2.50
	TF10				TF11				TF12			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	4.69E-01	1.42E+00	3.46E-01	5.07	1.95E-01	3.11E-01	5.13E-02	6.00	6.17E-03	2.11E+00	1.75E+00	4.67
WOA	4.44E-16	3.52E-15	2.42E-15	3.55	0.00E+00	0.00E+00	0.00E+00	2.50	7.81E-04	1.93E-03	1.23E-03	3.00
WAA	4.44E-16	4.44E-16	0.00E+00	2.15	0.00E+00	0.00E+00	0.00E+00	2.50	2.39E-08	3.78E-06	4.59E-06	2.00
SCA	9.45E-01	1.75E+01	6.86E+00	7.67	5.32E-01	8.93E+00	1.05E+01	7.97	7.46E+05	2.62E+07	2.17E+07	8.00
RUN	4.44E-16	4.44E-16	0.00E+00	2.15	0.00E+00	0.00E+00	0.00E+00	2.50	1.11E-09	1.74E-09	3.79E-10	1.00
GOOSE	1.80E+01	1.93E+01	3.29E-01	7.20	9.39E-03	1.55E-02	6.01E-03	5.00	3.19E+00	5.13E+00	1.05E+00	6.03
RIME	2.39E+00	2.95E+00	3.50E-01	6.07	6.13E-01	7.66E-01	6.94E-02	7.03	6.72E+00	1.04E+01	2.83E+00	6.97
NRBO	4.44E-16	4.44E-16	0.00E+00	2.15	0.00E+00	0.00E+00	0.00E+00	2.50	5.54E-01	6.33E-01	4.92E-02	4.33
	TF13				TF14				TF15			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	1.10E-01	3.39E-01	1.69E-01	3.53	9.98E-01	9.98E-01	4.70E-16	1.85	3.07E-04	4.22E-04	3.01E-04	2.80
WOA	1.55E-01	6.24E-01	4.04E-01	4.20	9.98E-01	1.26E+00	6.86E-01	4.52	3.08E-04	5.04E-04	2.39E-04	4.83
WAA	2.26E-07	1.81E-04	2.89E-04	1.37	9.98E-01	9.98E-01	2.03E-13	4.23	3.11E-04	3.49E-04	3.90E-05	4.10
SCA	4.68E+05	4.99E+07	6.75E+07	8.00	9.98E-01	1.13E+00	5.03E-01	5.90	3.13E-04	9.01E-04	4.38E-04	5.87
RUN	5.79E-08	1.99E-02	3.81E-02	1.63	9.98E-01	4.53E+00	3.91E+00	5.77	3.07E-04	6.13E-04	4.39E-04	2.87
GOOSE	4.04E-02	3.30E+00	7.34E+00	4.37	9.98E-01	1.36E+01	6.46E+00	7.48	3.08E-04	7.23E-03	9.45E-03	6.43
RIME	1.84E+00	1.02E+02	6.47E+01	6.80	9.98E-01	9.98E-01	7.91E-14	3.32	3.08E-04	6.37E-03	1.24E-02	5.40

NRBO	9.69E+00	9.87E+00	1.00E-01	6.10	9.98E-01	2.44E+00	2.94E+00	2.93	3.08E-04	1.79E-03	5.06E-03	3.70
	TF16				TF17				TF18			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	-1.03E+00	-1.03E+00	7.00E-13	3.57	3.98E-01	3.98E-01	2.47E-12	2.57	3.00E+00	3.00E+00	7.15E-12	3.00
WOA	-1.03E+00	-1.03E+00	4.52E-13	2.90	3.98E-01	3.98E-01	1.20E-08	5.13	3.00E+00	3.00E+00	2.57E-06	6.30
WAA	-1.03E+00	-1.03E+00	3.83E-07	7.03	3.98E-01	3.98E-01	3.10E-07	6.97	3.00E+00	3.00E+00	1.71E-05	7.83
SCA	-1.03E+00	-1.03E+00	6.89E-06	7.90	3.98E-01	3.98E-01	2.88E-04	8.00	3.00E+00	3.00E+00	2.92E-06	6.57
RUN	-1.03E+00	-1.03E+00	1.66E-13	2.53	3.98E-01	3.98E-01	2.45E-12	2.50	3.00E+00	3.00E+00	4.54E-14	1.97
GOOSE	-1.03E+00	-1.00E+00	1.49E-01	5.63	3.98E-01	3.98E-01	3.99E-10	4.67	3.00E+00	3.00E+00	3.90E-08	5.27
RIME	-1.03E+00	-1.03E+00	8.58E-10	5.43	3.98E-01	3.98E-01	1.11E-09	5.13	3.00E+00	3.00E+00	4.08E-09	4.00
NRBO	-1.03E+00	-1.03E+00	6.39E-16	1.00	3.98E-01	3.98E-01	0.00E+00	1.03	3.00E+00	3.00E+00	1.19E-14	1.07
	TF19				TF20				TF21			
	Best	Avg	SD	FK	Best	Avg	SD	FK	Best	Avg	SD	FK
NOpt	-3.86E+00	-3.86E+00	2.46E-11	2.10	-3.32E+00	-3.32E+00	2.74E-11	1.07	-1.02E+01	-1.02E+01	6.27E-10	1.53
WOA	-3.86E+00	-3.86E+00	2.57E-03	6.67	-3.32E+00	-3.27E+00	6.93E-02	4.60	-1.02E+01	-9.67E+00	1.90E+00	4.67
WAA	-3.86E+00	-3.86E+00	7.26E-05	6.33	-3.32E+00	-3.26E+00	5.30E-02	5.37	-1.02E+01	-1.02E+01	1.04E-04	4.23
SCA	-3.86E+00	-3.86E+00	1.46E-03	8.00	-3.13E+00	-2.76E+00	5.16E-01	8.00	-7.08E+00	-2.91E+00	2.43E+00	7.60
RUN	-3.86E+00	-3.86E+00	4.63E-08	2.97	-3.32E+00	-3.28E+00	5.83E-02	3.00	-1.02E+01	-1.02E+01	1.60E-10	1.47
GOOSE	-3.86E+00	-3.86E+00	4.61E-07	4.93	-3.32E+00	-3.21E+00	3.04E-02	5.83	-1.02E+01	-4.73E+00	2.94E+00	6.43
RIME	-3.86E+00	-3.86E+00	1.47E-09	3.67	-3.32E+00	-3.31E+00	4.11E-02	2.90	-1.02E+01	-9.14E+00	2.06E+00	4.03
NRBO	-3.86E+00	-3.86E+00	2.68E-06	1.33	-3.32E+00	-3.25E+00	6.34E-02	5.23	-1.02E+01	-9.03E+00	1.36E+00	6.03

Bold values represent the best findings.

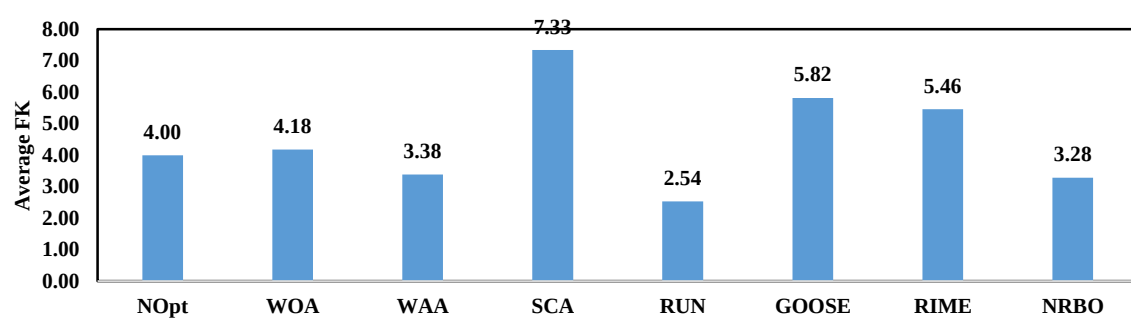


Fig. 1: Average FK values obtained by the algorithms across all standard TFs.

5. Conclusion and future work

This study introduced a novel neutrosophic optimizer, termed NOpt, for continuous global optimization. The proposed algorithm represents each candidate solution using a neutrosophic search state comprising truth, indeterminacy, and falsity degrees. These three components are designed to enhance the different search behaviors required during optimization. Specifically, the truth component promotes exploitation around promising regions, the indeterminacy component enhances exploration of uncertain regions, and the falsity component facilitates escape from poor or stagnant regions. In addition to the main

neutrosophic search mechanism, two complementary operators were incorporated into NOpt to further improve its search performance. The Neutrosophic Elite Refinement Operator improves each generated trial vector before its evaluation, whereas the Neutrosophic Adaptive Local Search Operator utilizes a controlled portion of the remaining function-evaluation budget to refine elite solutions. The performance of NOpt was comprehensively assessed against seven established metaheuristic algorithms using a rigorous experimental protocol. The obtained results demonstrated that NOpt can achieve outstanding performance on the fixed-dimensional test functions, which are designed to represent real-world optimization problems. These results indicate that incorporating truth, indeterminacy, and falsity degrees into the search process can provide a promising framework for balancing exploration and exploitation and improving the ability of a metaheuristic algorithm to locate high-quality solutions. Despite these promising results, the proposed algorithm has several potential directions for further investigation. Future work will focus on extending NOpt to multi-objective, constrained, and discrete optimization problems to examine its adaptability to different optimization scenarios. Incorporating adaptive or learning-based mechanisms into the neutrosophic search process may provide additional improvements in convergence and exploration–exploitation balance. Finally, applying NOpt to a broader range of real-world engineering and practical optimization problems will be considered to further validate its effectiveness and practical applicability.

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Pythagorean Neutrosophic DEMATEL and AHP for Water Security Evaluation

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Abstract: This paper improves the current multi-criteria decision-making method for assessing water security by integrating the Pythagorean Neutrosophic Set (PNS) with the Analytic Hierarchy Process (AHP) and Decision-Making Trial and Evaluation Laboratory (DEMATEL). PNS effectively handles uncertainty and imprecision, offering a higher degree of truth, falsity, and indeterminacy compared to fuzzy sets. AHP excels at ranking objects by importance, while DEMATEL highlights cause-and-effect relationships. This study includes four key criteria: Water Availability (C1), Water Accessibility (C2), Water Quality (C3), and Climate Change (C4), along with twelve sub-criteria to generate a detailed understanding of how these criteria and sub-criteria influence water security. Using PNS-DEMATEL, the study develops a cause-and-effect diagram that maps causal linkages among the criteria, identifying key drivers to the water security problem. Concurrently, PNS-AHP ranks the criteria and sub-criteria based on expert assessments while highlighting their relative importance for sustainable decision-making. This integrated approach enhances the accuracy of water security evaluations and provides valuable insights for managing sustainable water resources. The findings offer policymakers actionable tools to prioritize interventions and craft legislation tailored to the region's unique challenges and needs.

Keywords: Pythagorean Neutrosophic Set; DEMATEL; AHP; decision making; water security assessment

1. Introduction

Water security can be defined as reliable access to safe water, effective water management, and sustainable water quality, which are essential not only for human well-being but also for the survival of animals and ecosystems. It ensures that sufficient water is available for various needs without compromising the ability of future generations to meet their own demands [1]. However, rapid population growth and climate change have intensified global water challenges, increasing pressure on policymakers and researchers to develop integrated and sustainable solutions. In response,

Integrated Water Reecosystemsgement (IWRM), which emphasizes the interconnectedness of water systems, communities, and ecosystems, has gained prominence, particularly through initiatives led by international organizations such as the United Nations [2]. Urban areas in developing countries are particularly vulnerable to water insecurity due to rapid urbanization and climate-induced changes in precipitation patterns [3]. These challenges are further exacerbated by human and environmental factors, including excessive groundwater extraction, industrial and agricultural pollution, and inefficient water usage [4,5]. While regions such as the Middle East and North Africa are widely recognized for severe water scarcity, similar concerns are increasingly emerging in non-arid regions [6].

In this context, Sarawak, Malaysia, despite its tropical climate and abundant rainfall, faces significant water security challenges. These arise from factors such as complex topography, uneven rainfall distribution, and anthropogenic pollution. The Sarawak River has been reported to contain elevated levels of heavy metals, including iron, cobalt, and copper, necessitating intensive treatment before the water becomes usable [7]. Additionally, Sarawak's hilly terrain, localized climatic conditions, and seasonal wind patterns further complicate water resource planning and management [8]. Addressing these issues requires comprehensive and integrated strategies, including infrastructure development, pollution mitigation, water reuse, and active public participation [9].

Similar challenges have been examined in different regions using various analytical approaches to assess water security. For example, Ding et al. [10] employed the Soil and Water Assessment Tool (SWAT) to evaluate water security by analysing the spatial and temporal distribution of blue and green water resources in the Weihe River Basin. On the other hand, Ogunbode et al. [11] evaluated water security in Iwo, Nigeria, by combining a 12-month household survey of 937 households on water sources and usage with a 31-year analysis of climatic data using descriptive and inferential statistical methods. Interestingly, many researchers have applied multi-criteria decision-making approaches to assess water security, including CRITIC-TOPSIS [12], VIKOR-TOPSIS [13], and BWM-VIKOR-TOPSIS [14].

Despite these developments, existing approaches often struggle to address uncertainty and the complex interdependence among environmental, social, and technical factors. This limitation reduces the ability of decision-makers to accurately identify critical issues, understand causal relationships, and implement effective interventions. Moreover, there remains a clear research gap, as no study has yet integrated Pythagorean Neutrosophic Sets (PNS) with both DEMATEL and the Analytic Hierarchy Process (AHP) within a single, unified framework. To overcome these limitations, this study proposes a hybrid decision-making framework that integrates Pythagorean Neutrosophic Sets (PNS) with DEMATEL and AHP.

PNS, an extension of neutrosophic and Pythagorean fuzzy sets, provides a powerful mechanism for modeling uncertainty through truth, indeterminacy, and falsity membership functions under a squared-sum constraint [15, 16]. This flexibility allows for more accurate representation of ambiguity in complex systems and has been successfully applied in areas such as risk assessment [17], decision support [18], and graph theory [19]. Building on this capability, the integration of PNS with DEMATEL enables the identification and analysis of causal relationships among water security factors, while the incorporation of AHP facilitates the prioritization of criteria within a structured hierarchical framework. Although alternative methods such as the Best-Worst Method (BWM), Full Consistency Method (FUCOM), Level-Based Weight Assessment (LBWA), and Double Intuitionistic Best-Worst Ratio (DIBR) have been developed to address some limitations of AHP, AHP remains a suitable choice in this study due to its strong compatibility with DEMATEL, widespread familiarity among practitioners, and extensive validation in the literature.

Furthermore, embedding AHP and DEMATEL within the PNS framework helps mitigate issues related to inconsistency and cognitive burden by explicitly modeling uncertainty in expert judgments. For example, Appasamy [20] and Paul and Mahapatra [21] used fuzzy DEMATEL, with the latter also combining fuzzy AHP to set priorities in sustainability issues. Çıkmak [22] and

Ayyildiz et al. [23] showed how Pythagorean Fuzzy AHP can be used to assess technology and transportation risks, proving its value for hierarchical decision-making under uncertainty.

In summary, this study contributes to the advancement of water security assessment by proposing a practical hybrid framework that enhances analytical real-world applicability. By integrating PNS, DEMATEL, and AHP, the proposed model provides a robust tool for evaluating water security under uncertainty and supports more informed and effective decision-making. The remainder of this paper is structured as follows. Section 2 reviews the relevant literature, while Section 3 presents the fundamental concepts underpinning the proposed methodology. Section 4 describes the research framework, followed by the implementation and results in Section 5. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, the definition of PNS introduced by Radha et al. [15] will be briefly explained since it is the fundamental part of this study. The PNS set operations facilitate more comprehensive data analysis, which acknowledges the complex and uncertain real-world issues.

Definition 1. [15]

Let X be a non-empty set (universe). A Pythagorean Neutrosophic Set with T and F as dependent neutrosophic components A on X is an object of the form

$$A = \{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X \}, \quad (1)$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth, indeterminacy, and false membership respectively such that $T, I, F \in [0, 1]$. Here when T and F are dependent components, then for all x in X :

$$T + F \leq 1, \quad (2)$$

$$0 \leq T^2 + F^2 \leq 1, \quad (3)$$

$$0 \leq T^2 + I^2 + F^2 \leq 2, \quad (4)$$

Definition 2. [24] defined these basic operations on PNS which can be described as follows:

Let X be a non-empty set (universe). A Pythagorean Neutrosophic Set with ψ and κ as dependent neutrosophic components A and B of the form $A = \{ \langle x, \psi_A(x), \varsigma_A(x), \kappa_A(x) \rangle \mid x \in X \}$ and

$B = \{ \langle x, \psi_B(x), \varsigma_B(x), \kappa_B(x) \rangle \mid x \in X \}$. The complement of A is

$$A^C = \{ \langle x, \kappa_A(x), 1 - \varsigma_A(x), \psi_A(x) \rangle \mid x \in X \}. \quad (5)$$

The union and intersection of A and B are

$$\begin{aligned} A \cup B &= \{ \max(\psi_A, \psi_B), \min(\varsigma_A, \varsigma_B), \min(\kappa_A, \kappa_B) \} \\ A \cap B &= \{ \min(\psi_A, \psi_B), \max(\varsigma_A, \varsigma_B), \max(\kappa_A, \kappa_B) \} \end{aligned} \quad (6)$$

Definition 3. [25] Given $x_1 = (\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1})$, $x_2 = (\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2})$ and $x = (\psi_x, \varsigma_x, \kappa_x)$ are any two PNSs, then the operational rules for PNSs such as addition, multiplication, scalar multiplication and power operations are defined as follows:

i. The PNS addition, denoted by $x_1 \oplus x_2$ are defined as:

$$\begin{aligned} x_1 \oplus x_2 &= (\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}) + (\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}) \\ &= \left(\sqrt{\psi_{x_1}^2 + \psi_{x_2}^2 - \psi_{x_1}^2 \psi_{x_2}^2}, \varsigma_{x_1} \varsigma_{x_2}, \kappa_{x_1} \kappa_{x_2} \right), \end{aligned} \quad (7)$$

ii. The PNS multiplication, denoted by $x_1 \otimes x_2$ are defined as:

$$\begin{aligned} x_1 \otimes x_2 &= (\psi_{x_1}, \varsigma_{x_1}, \kappa_{x_1}) \otimes (\psi_{x_2}, \varsigma_{x_2}, \kappa_{x_2}) \\ &= \left(\psi_{x_1} \psi_{x_2}, \varsigma_{x_1} + \varsigma_{x_2} - \varsigma_{x_1} \varsigma_{x_2}, \sqrt{\kappa_{x_1}^2 + \kappa_{x_2}^2 - \kappa_{x_1}^2 \kappa_{x_2}^2} \right), \end{aligned} \quad (8)$$

iii. The PNS scalar multiplication, denoted by μx where $\mu \in R$ and $\mu \geq 0$ are defined as:

$$A^C = \{ \langle x, \kappa_A(x), 1 - \varsigma_A(x), \psi_A(x) \rangle : x \in X \}. \quad (9)$$

iv. The PNS power operation, denoted by x^μ where $\mu \in R$ and $\mu \geq 0$ are defined as:

$$\begin{aligned} x^\mu &= (\psi_x, \varsigma_x, \kappa_x)^\mu \\ &= \left(\psi_x^\mu, 1 - (1 - \varsigma_x)^\mu, \sqrt{1 - (1 - \kappa_x^2)^\mu} \right), \end{aligned} \quad (10)$$

3. Development of The Proposed PNS-DEMATEL and PNS-AHP Method

This section presents the development of a linguistic variable by PNS, the procedure method of PNS-DEMATEL and PNS-AHP. The development of the proposed method which is a unified Pythagorean Neutrosophic Set (PNS) DEMATEL with Pythagorean Neutrosophic Set (PNS) AHP method in Advancing Water Security Assessment in Sarawak. The primary features of this framework are the integration of PNS-DEMATEL and PNS-AHP methods, which provide a comprehensive approach to decision-making. Additionally, the framework incorporates the construction of linguistic variables specifically tailored for the PNS-AHP method, enhancing its precision and applicability in evaluating multiple criteria.

The framework begins with the data collection phase, where the main criteria for evaluation are identified, and relevant experts and decision-makers are selected to provide their insights. Following this, the aggregation phase is divided into two parts. The PNS-DEMATEL focuses on modelling interdependencies among criteria by constructing the initial direct influence matrix using PNS, computing average matrices, and performing deneutrosophication to derive crisp values. The process continues with normalizing the influence matrix, constructing the total-influence matrix, and calculating row and column sums. Meanwhile, the PNS-AHP emphasizes determining the

hierarchical importance of criteria by generating a pairwise comparison matrix with PNS, performing deneutrosophication, normalizing the matrix, and ensuring consistency. In the result and discussion phase, the DEMATEL approach concludes by determining the threshold value and constructing a network relationship map (NRM), while the AHP approach provides a ranking of criteria based on normalized comparisons. This integrated framework offers the strengths of PNS, DEMATEL, and AHP to a better systematic and robust solution for multi-criteria decision-making problem. Figure 1 shows the framework of the proposed PNS-DEMATEL and PNS-AHP method.

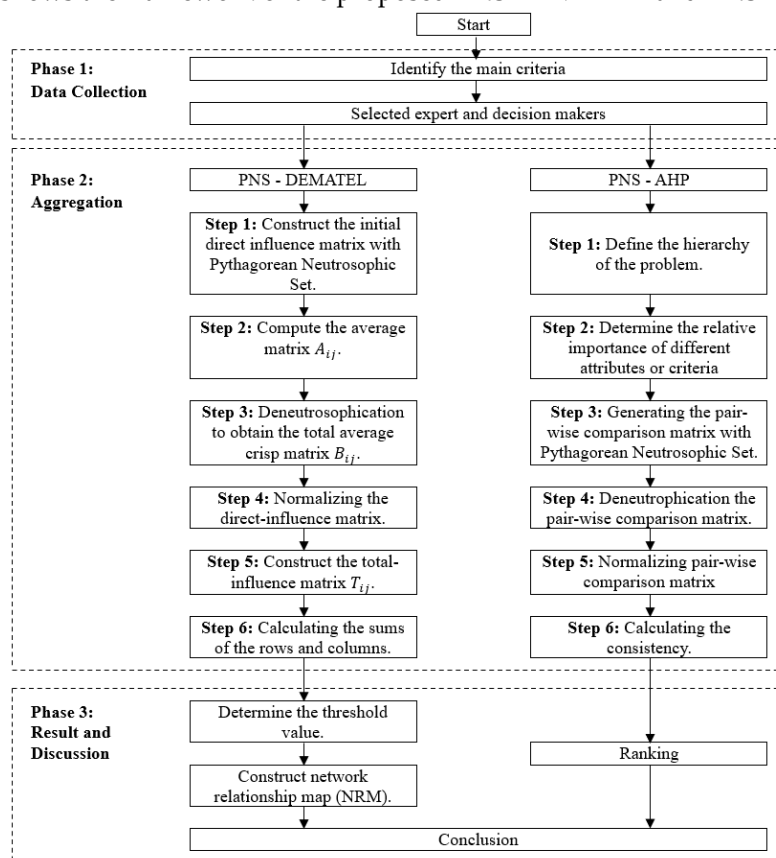


Figure 1. The proposed PNS-DEMATEL and PNS-AHP method.

3.1. Design the PNS linguistic variable

In the PNS-AHP framework, the linguistic variable leverages the Pythagorean Neutrosophic Set (PNS) to evaluate membership, non-membership, and indeterminacy, ensuring alignment with predefined criteria. Earlier applications, such as Saaty et al. [26], utilized 9-scale variables, as seen in Karamasa's [27] implementation of the extended single-valued neutrosophic set in Table 1.

Table 1. A 9-scale linguistic variable [27]

Saaty Scale	Linguistic Variable	Neutrosophic Triangular Scale
1	Equally influential	$\langle (1,1,1); 0.50, 0.50, 0.50 \rangle$
3	Slightly influential	$\langle (2,3,4); 0.30, 0.75, 0.70 \rangle$
5	Strongly influential	$\langle (4,5,6); 0.80, 0.15, 0.20 \rangle$
7	Very strongly influential	$\langle (6,7,8); 0.90, 0.10, 0.10 \rangle$
9	Absolutely influential	$\langle (9,9,9); 1.00, 1.00, 1.00 \rangle$

		$\langle (1,2,3); 0.40, 0.65, 0.60 \rangle$
2, 4, 6, 8	Sporadic values between two close scales	$\langle (3,4,5); 0.60, 0.35, 0.40 \rangle$
		$\langle (5,6,7); 0.70, 0.25, 0.30 \rangle$
		$\langle (7,8,9); 0.85, 0.10, 0.15 \rangle$

In the context of the proposed linguistic variable in PNS, pair-wise comparison scales are defined as the new PNS-AHP scale and its reciprocal, as illustrated in Table 2. These scales have now been integrated, replacing the conventional use of crisp numbers based on Table 1.

Table 2. A new PNS-AHP scale and its reciprocal

Saaty scale	Linguistic Variable	PNS			Reciprocal		
		T	I	F	F	I	T
1	Equal importance	$\langle 0.50, 0.50, 0.50 \rangle$			$\langle 0.50, 0.50, 0.50 \rangle$		
3	Moderate importance	$\langle 0.30, 0.75, 0.70 \rangle$			$\langle 0.70, 0.75, 0.30 \rangle$		
5	Strong importance	$\langle 0.80, 0.15, 0.20 \rangle$			$\langle 0.20, 0.15, 0.80 \rangle$		
7	Very strong importance	$\langle 0.90, 0.10, 0.10 \rangle$			$\langle 0.10, 0.10, 0.90 \rangle$		
9	Extreme importance	$\langle 1.00, 0.00, 0.00 \rangle$			$\langle 0.00, 0.00, 1.00 \rangle$		
2, 4, 6, 8	Intermediate values	$\langle 0.40, 0.65, 0.60 \rangle$			$\langle 0.60, 0.65, 0.40 \rangle$		
		$\langle 0.60, 0.35, 0.40 \rangle$			$\langle 0.40, 0.35, 0.60 \rangle$		
		$\langle 0.70, 0.25, 0.30 \rangle$			$\langle 0.30, 0.24, 0.70 \rangle$		
		$\langle 0.85, 0.10, 0.15 \rangle$			$\langle 0.15, 0.10, 0.85 \rangle$		

The scale of '2' in Table 2 is accompanied by a truth membership value of 0.4. The rationale behind attributing a truth membership of 0.4 in this context is to uphold equilibrium within the scale. The discrepancy between the higher extremity of the scale, denoted by score 6 with a truth membership of 0.7, and scale of 7 with a truth membership of 0.9, amounts to 0.1. To maintain a state of balance, it is imperative for the variance between the truth membership values of the lower end of the scale, particularly for scale 1, and 2 to also be 0.55. Through assigning a truth membership of 0.4 to the newly introduced scale of 2, the equilibrium between the lower and upper boundaries of the scale is effectively preserved. Therefore, while determining values for indeterminate and false membership, calculation according to the relevant truth value and ensure that meet the requirements for PNS as outlined in Eq. (2)-(4).

Step 1: Validate the linguistic variable of PNS.

As outlined in Table 2, is validated by assessing its compliance with the three requirements specified in Eq. (2)-(4). Applying Definition 1, consider $Y = \{y_1, y_2, \dots, y_9\}$ then

$$Y = \left\{ \begin{array}{l} \langle y_1, 0.50, 0.50, 0.50 \rangle \\ \langle y_2, 0.40, 0.65, 0.60 \rangle \\ \langle y_3, 0.30, 0.75, 0.70 \rangle \\ \langle y_4, 0.60, 0.35, 0.40 \rangle \\ \langle y_5, 0.80, 0.15, 0.20 \rangle \\ \langle y_6, 0.70, 0.25, 0.30 \rangle \\ \langle y_7, 0.90, 0.10, 0.15 \rangle \\ \langle y_8, 0.85, 0.10, 0.15 \rangle \\ \langle y_9, 1.00, 0.00, 0.00 \rangle \end{array} \right\},$$

A PN-subset of Y is a set-in which T is restricted to taking values from the set $\{0,1\}$. The values mentioned are inside the interval $[0,1]$, notably 0.10, 0.80, and 0.90. The accuracy of these values is confirmed by analyzing the conditions specified in Eq. (2)-(4). Let $y_1 = \langle 0.50, 0.50, 0.50 \rangle$ and $y_2 = \langle 0.40, 0.65, 0.60 \rangle$ are PNS numbers. Then, the verification of PNS numbers is presented in Table 3 which satisfies the three conditions mentioned in Eq. (2) to Eq. (4).

Table 3. The verification of PNS numbers

T	I	F	$T + F \leq 1$	$T^2 + F^2 \leq 1$	$T^2 + I^2 + F^2 \leq 2$
0.50	0.50	0.50	1	0.5	0.75
0.40	0.65	0.60	1	0.52	0.9425
0.30	0.75	0.70	1	0.58	1.1425
0.60	0.35	0.40	1	0.52	0.6425
0.80	0.15	0.20	1	0.68	0.7025
0.70	0.25	0.30	1	0.58	0.6425
0.90	0.10	0.10	1	0.82	0.83
0.85	0.10	0.15	1	0.745	0.755
1.00	0.00	0.00	1	1	1

3.2. Pythagorean Neutrosophic Sets – DEMATEL (PNS-DEMATEL) Method

The procedure of the PNS-DEMATEL method can be shown below:

Step 1: Construct the initial direct influence matrix.

A direct-relation matrix with Pythagorean neutrosophic numbers was constructed based on decision makers' (DMs) preferences. The k th DM gave the PNS score x_{ij}^k and the notation of x_{ij} shows the degree to which DM believes the criteria, i affects criteria, j and the diagonal components are set to zero for decision making, where the new PNS linguistic variable in Table 4.

Table 4. The new PNS linguistic variable [18]

Scale	Linguistic Variable	PNS Scale
1	No Influence	$\langle 0.10, 0.80, 0.90 \rangle$
2	Low Influence	$\langle 0.20, 0.70, 0.80 \rangle$
3	Medium Low Influence	$\langle 0.35, 0.60, 0.60 \rangle$
4	Medium Influence	$\langle 0.50, 0.40, 0.45 \rangle$
5	Medium High Influence	$\langle 0.65, 0.30, 0.25 \rangle$
6	High Influence	$\langle 0.80, 0.20, 0.15 \rangle$
7	Very High Influence	$\langle 0.90, 0.10, 0.10 \rangle$

The matrix contains PNS in the form $x_{ij}^k = \langle T_{ij}, I_{ij}, F_{ij} \rangle$, k is number of DMs as below,

$$X^k = \begin{bmatrix} 0 & x_{12}^k & \dots & x_{1n}^k \\ x_{21}^k & 0 & \dots & x_{2n}^k \\ \vdots & \vdots & 0 & \vdots \\ x_{n1}^k & x_{n2}^k & \dots & 0 \end{bmatrix}$$

Step 2: Compute the average matrix A_{ij} .

By combining the experts' opinions, the average direct-influence matrix A_{ij} can be obtained Eq. (11).

$$A_{ij}^k = \frac{1}{l} \sum_{k=1}^l x_{ij}^k, \quad i, j = 1, 2, \dots, n, \quad (11)$$

Step 3: Deneutrosophication to obtain the total average crisp matrix B_{ij} .

To deneutrosophicate the Pythagorean neutrosophic number $(T_A(x), I_A(x), F_A(x))$ to a crisp value, the deneutrosophication formula is used as follows:

$$B_{ij} = \frac{T_A(x) + 0.5I_A(x)}{1.5}, \quad (12)$$

Step 4: Normalizing the direct- influence matrix.

The normalized direct-influence matrix Z_{ij} be achieved by using Eq. (13).

$$Z_{ij} = \frac{B_{ij}}{s}, \quad (13)$$

where $s = \max \left(\max_{1 \leq i \leq n} \sum_{j=1}^n B_{ij}, \max_{1 \leq j \leq n} \sum_{i=1}^n B_{ij} \right)$. All elements in the matrix Z_{ij} are complying with

$$0 < Z_{ij} < 1.$$

Step 5: Create the total-influence matrix T_{ij} based on the normalized matrix Z as follows:

$$T_{ij} = Z_{ij} \left(I - Z_{ij} \right)^{-1}, \quad (14)$$

where I is the identity matrix.

Step 6: Calculating the sums of the rows and columns.

In this step, the vectors R and C representing the sum of the rows and the sum of columns from the total-influence matrix T_{ij} are defined by the following formulas:

$$R = \left[\sum_{j=1}^n T_{ij} \right]_{n \times 1}, \quad (15)$$

$$C = \left[\sum_{i=1}^n T_{ij} \right]_{1 \times n}, \quad (16)$$

The $R+C$ and $R-C$ values are calculated in which these values reflect the importance and relation values, respectively.

3.3. Pythagorean Neutrosophic Sets – AHP (PNS-AHP) Method

The procedure of PNS-AHP method can be shown as below:

Step 1: Define the hierarchy of the problem.

The three-level hierarchy of the considered consists of Goal (Level 1), Criteria (Level 2) and Alternative (Level 3).

Step 2: Determine the relative importance of different attributes or criteria with respect to the goal. AHP multiple pairwise comparisons are based on a nine-level standardized comparison scale in Table 3.

Step 3: Generating the pair-wise comparison matrix with Pythagorean Neutrosophic Set.

Definition 1 was used to propose PNS into AHP same goes with PNS-AHP. Let $C = \{C_j \mid j = 1, 2, \dots, n\}$ be the set of criteria. By using PNS-AHP scale in Table 3, the result of the pairwise comparison on n criteria can be summarized in an $(n \times n)$ evaluation matrix Y in which every element $y_{ij} = (i, j = 1, 2, \dots, n)$ is the quotient of weights of the criteria, as shown below:

$$Y = \begin{bmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{bmatrix}, \quad y_{ij} = 1, y_{ji} = \frac{1}{y_{ij}}, y_{ij} \neq 0, \quad (17)$$

Then, determine the aggregate of matrix E_{ij} . Combining the views of the I is the total of experts yields the average direct-influence matrix E_{ij} .

$$E_{ij}^k = \frac{1}{l} \sum_{k=1}^l y_{ij}^k, \quad i, j = 1, 2, \dots, n, \quad (18)$$

Step 4: Deneutrosophication the pair-wise comparison matrix.

To deneutrosophicate the PNS $(T_E(y), I_E(y), F_E(y))$ to a crisp value at upper triangular matrix and the lower triangular matrix reciprocal as Eq. (19), the deneutrosophication formula is used as follows:

$$D_{ij} = \frac{T_E(y) + 0.5I_E(y)}{1.5}, \quad (19)$$

Step 5: Normalizing pair-wise comparison matrix. The normalized and relative weight determination procedure for each matrix as follows:

$$F_{ij} = \frac{D_{ij}}{p}, \quad (20)$$

where $p = \sum_{i=1}^n D_{ij}$. All elements in the matrix F are complying with $0 < F_{ij} < 1$. Then, the largest eigenvalue, $\lambda_{\max}$ is represented by the right eigenvector, ω , which gives the criteria weights.

$$D_{\omega} = \lambda_{\max} \omega, \quad (21)$$

Another way is calculating the average of normalized pair-wise comparison matrix to find criteria weight and multiply with $p = \sum_{i=1}^n D_{ij}$ each. Then, $\lambda_{\max}$ value is the total of the results. The matrix E has rank 1 and $\lambda_{\max} = n$ (total of the criteria) if the pairwise comparisons are entirely consistent. In this instance, any of E 's rows or columns can be normalized to obtain weight. Assuming perfect consistency in the pairwise comparisons, matrix E has rank 1 and $\lambda_{\max} = n$. In this case, normalizing any of E 's rows or columns will provide weight.

Step 6: Calculating consistency.

The consistency of the pairwise comparison evaluations influence the accuracy with which the AHP produces its output. The relationship between matrix A that defines consistency: $E_{ij} \times E_{jk} = E_{ik}$. The

Consistency Index (CI) formula as follows:

$$CI = \frac{\lambda_{\max} - n}{n - 1}, \quad (22)$$

and the Consistency Ratio (CR) that defined as follows [27]:

$$CR = \frac{CI}{\text{Random Index}(RI)}, \quad (23)$$

The value of RI is related to the dimension of the matrix and will be extracted from Table 5. Note that consistency ratio lower than 0.10 verifies that the results of comparison are acceptable.

Table 5. The value of Random Index [28]

Dimension	1	2	3	4	5	6	7	8	9	10
RI	0	0	0.5799	0.8921	1.1159	1.2358	1.3322	1.3952	1.4537	1.4882

4. Implementation of Water Security Evaluation

To address the specific problem of water security in Sarawak, Malaysia, this study applies the proposed PNS-AHP-DEMATEL framework to a real-world case study involving selected districts with varying water access and infrastructure conditions. Water security assessment here is defined as a systematic evaluation of the availability, accessibility, quality, and climate resilience of water resources. Sarawak's water challenges include inequitable distribution, inadequate supply in rural areas, seasonal extremes, and environmental degradation. Three experts were invited to provide their view on twelve sub-criteria under four main criteria. The proposed PNS-DEMATEL method will analyse the obtained data to identify the main causal factor affecting the others, while the proposed PNS-AHP will be employed to get the ranking of the criteria and sub-criteria. The findings offer a cause-effect diagram that can guide targeted interventions. This case study demonstrates the applicability and value of the proposed hybrid approach in supporting sustainable water management planning in Sarawak. The criteria, sub-criteria, and their descriptions are presented in Table 6.

Table 6. The description of criteria and sub-criteria

CRITERIA	SUB-CRITERIA	DESCRIPTION
Water Availability (C1)	Surface Water Availability (C11)	Refers to the amount of water that is accessible in rivers, lakes, and reservoirs for many purposes, including drinking, irrigation, and industrial use.
	Groundwater Availability (C12)	Refers to the volume of water contained underground in aquifers, which can be accessed by wells to provide drinking water and support agricultural irrigation.
	Seasonal Variability (C13)	Refers to variations in water availability over a year caused by changes in precipitation patterns, the melting of snow, and the flow of water, which impacts the dependability of water supply.

Water Accessibility (C2)	Physical Accessibility (C21)		Refers to the closeness and convenience of reaching water sources, such as wells, communal taps, and water distribution sites, to ensure that people may acquire water without facing excessive difficulty or having to travel long distances.
	Economic (C22)	Accessibility	Refers to the accessibility of water services and infrastructure, considering household incomes, water prices, and procedures that ensure water remains available to all socio-economic groups.
	Infrastructure (C23)	Accessibility	Encompasses the presence and effectiveness of water delivery infrastructure, including pipes, pumps, and storage tanks, to ensure that communities have sufficient access to secure and dependable water supply systems.
Water Quality (C3)	Chemical (C31)	Contaminants	Encompasses contaminants such as toxic metals, insecticides, and industrial compounds that have the potential to pollute water supplies, hence endangering human well-being and the ecosystem when ingested or utilized for household, agricultural, or industrial activities.
	Microbiological Contaminants (C32)		Waterborne diseases, such as cholera, typhoid, and diarrhea, can be caused by pathogenic microorganisms such as bacteria, viruses, and parasites. These diseases occur when water becomes contaminated with sewage or fecal matter.
	Physical (C33)	Characteristics	Encompasses the visual, gustatory, and olfactory aspects of water, which determine its aesthetic value and suitability for different purposes. Factors such as turbidity, color, and sedimentation influence how water quality is perceived and accepted.
Climate Change (C4)	Changes in Precipitation Patterns (C41)		Describes modifications in the timing, magnitude, and spatial pattern of precipitation and snowfall because of climate change, resulting in changes in water supply, periods of drought, and occurrences of floods in various places.
	Temperature (C42)	Increases	Temperature change-induced warming can lead to increased evaporation rates, and changes in hydrological cycles, and worsen water scarcity and heat-related water stress in susceptible regions.
	Extreme Weather (C43)	Events	Encompasses the presence of severe weather events such as hurricanes, cyclones, and storms, which can lead to flooding, infrastructure destruction, and water source pollution, thus impacting water security and

resilience.

Figure 2 illustrates the hierarchical structure of the criteria and sub-criteria used in the water security evaluation.

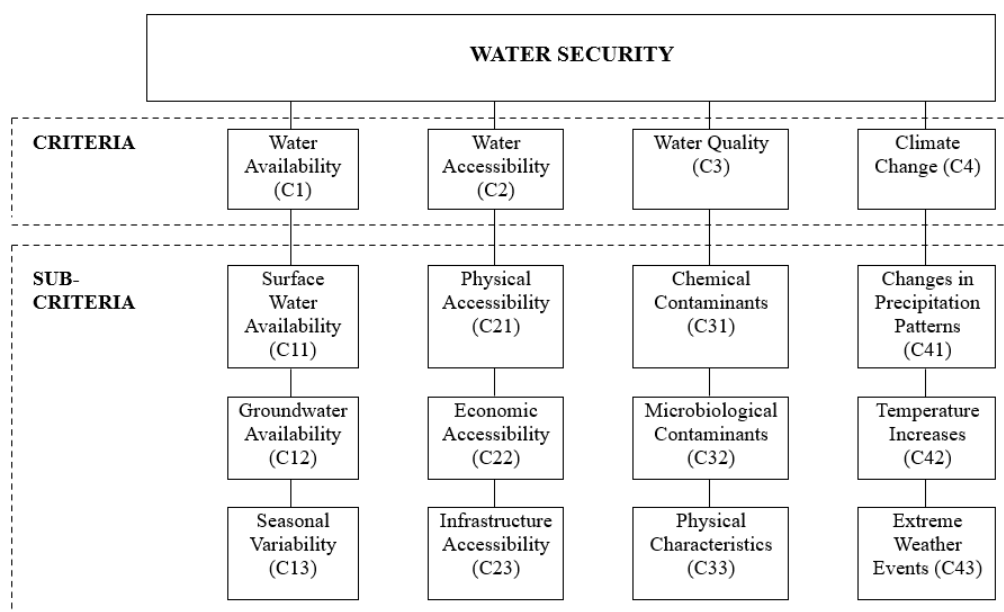


Figure 2. The Hierarchy of Advancing Water Security Assessment in Sarawak.

4.1 PNS-DEMATEL Method

During the initial phase, the proposed PNS-DEMATEL method is employed to identify the causal relationships among the separate groups in the research framework. Identifying the linkages at this stage is crucial as it can potentially unveil the hierarchical significance of the criteria and sub-criteria. Next, create a network relationship map (NRM) that illustrates the connections and interactions between the criterion and sub-criteria. The linguistic judgments provided by each expert are presented in Table 7 as the direct-relation matrix.

Table 7: The direct-relation matrix

Criteria	Expert	C1	C2	C3	C4
C1	E1	0	1	2	3
	E2	0	1	2	2
	E3	0	3	2	5
C2	E1	3	0	6	1
	E2	2	0	1	3
	E3	2	0	7	4
C3	E1	5	1	0	3
	E2	6	2	0	2
	E3	1	1	0	2
C4	E1	3	2	4	0
	E2	4	3	5	0
	E3	2	1	3	0

Table 8: The PNS direct-relation matrix, X_{ij}^k

Criteria	Expert	C1	C2	C3	C4
C1	E1	$\langle 0,0,0 \rangle$	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0.35,0.60,0.60 \rangle$
	E2	$\langle 0,0,0 \rangle$	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0.20,0.70,0.80 \rangle$
	E3	$\langle 0,0,0 \rangle$	$\langle 0.35,0.60,0.60 \rangle$	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0.65,0.30,0.25 \rangle$
C2	E1	$\langle 0.35,0.60,0.60 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.80,0.20,0.15 \rangle$	$\langle 0.10,0.80,0.90 \rangle$
	E2	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0.35,0.60,0.60 \rangle$
	E3	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.90,0.10,0.10 \rangle$	$\langle 0.50,0.40,0.45 \rangle$
C3	E1	$\langle 0.65,0.30,0.25 \rangle$	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.35,0.60,0.60 \rangle$
	E2	$\langle 0.80,0.20,0.15 \rangle$	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.20,0.70,0.80 \rangle$
	E3	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.20,0.70,0.80 \rangle$
C4	E1	$\langle 0.35,0.60,0.60 \rangle$	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0.50,0.40,0.45 \rangle$	$\langle 0,0,0 \rangle$
	E2	$\langle 0.50,0.40,0.45 \rangle$	$\langle 0.35,0.60,0.60 \rangle$	$\langle 0.65,0.30,0.25 \rangle$	$\langle 0,0,0 \rangle$
	E3	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0.10,0.80,0.90 \rangle$	$\langle 0.35,0.60,0.60 \rangle$	$\langle 0,0,0 \rangle$

The matrix A_{ij} is computed by applying Eq. (11) to represent the evaluation made by three experts, as displayed in Table 9. The PNS value in the first row, second column, A_{12} can be obtained by taking the averaging of x_{12}^1 , x_{12}^2 , and x_{12}^3 :

$$A_{12} = \left\langle \frac{0.1+0.1+0.35}{3}, \frac{0.8+0.8+0.6}{3}, \frac{0.9+0.9+0.6}{3} \right\rangle$$

$$= \langle 0.18, 0.73, 0.80 \rangle$$

Table 9. Average matrix, A_{ij}

Criteria	C1	C2	C3	C4
C1	$\langle 0,0,0 \rangle$	$\langle 0.18,0.73,0.80 \rangle$	$\langle 0.20,0.70,0.80 \rangle$	$\langle 0.40,0.53,0.55 \rangle$
C2	$\langle 0.25,0.67,0.73 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.60,0.37,0.38 \rangle$	$\langle 0.32,0.60,0.65 \rangle$
C3	$\langle 0.52,0.43,0.43 \rangle$	$\langle 0.13,0.77,0.87 \rangle$	$\langle 0,0,0 \rangle$	$\langle 0.25,0.67,0.73 \rangle$
C4	$\langle 0.35,0.57,0.62 \rangle$	$\langle 0.22,0.70,0.77 \rangle$	$\langle 0.50,0.43,0.43 \rangle$	$\langle 0,0,0 \rangle$

Table 10 displays the crisp averaging matrix B_{ij} which is derived using Eq. (12) for deneutrosophication. The crisp value in the first row, second column, B_{12} is demonstrated as follows:

$$B_{12} = \frac{T(x) + 0.5I(x)}{1.5}$$

$$B_{12} = \frac{0.18 + 0.5(0.73)}{1.5} = 0.3667$$

Table 11 shows the crips averaging matrix, B_{ij} .

Table 10. The crips averaging matrix, B_{ij}

Criteria	C1	C2	C3	C4	Total Row
C1	0.0000	0.3667	0.3667	0.4444	1.1778
C2	0.3889	0.0000	0.5222	0.4111	1.3222
C3	0.4889	0.3444	0.0000	0.3889	1.2222
C4	0.4222	0.3778	0.4778	0.0000	1.2778
Total Column	1.3000	1.0889	1.3667	1.2444	

The matrix Z_{ij} is normalized by using Eq. (13) as shown in Table 11. The computation of Z_{12} is shown below:

$$Z_{12} = \frac{B_{12}}{s},$$

$$\text{where } s = \max \left(\max_{1 \leq i \leq 4} \sum_{i=1}^4 B_{ij}, \max_{1 \leq j \leq 4} \sum_{j=1}^4 B_{ij} \right) = \max(1.3667, 1.3222) = 1.3667$$

$$\text{Thus, } Z_{12} = \frac{0.3667}{1.3667} = 0.2683$$

Table 11. Normalized matrix, Z_{ij}

Criteria	C1	C2	C3	C4
C1	0.0000	0.2683	0.2683	0.3252
C2	0.2845	0.0000	0.3821	0.3008
C3	0.3577	0.2520	0.0000	0.2845
C4	0.3089	0.2764	0.3496	0.0000

Table 12 displays the total influence matrix T_{ij} which can be computed using Eq. (14).

Table 12. Total influence matrix, T_{ij}

Criteria	C1	C2	C3	C4	Total Row (R)
C1	2.4155	2.3069	2.6970	2.5719	9.9913
C2	2.8626	2.2886	2.9936	2.7718	10.9167
C3	2.7410	2.3489	2.5440	2.6062	10.2401
C4	2.8045	2.4427	2.8995	2.4717	10.6185
Total Column (C)	10.8237	9.3870	11.1342	10.4217	

The total of rows and columns is computed using Eq. (15) and Eq. (16) as displayed in Table 13. The criteria were divided into two groups which are cause and effect, based on the R - C values. The criteria in the cause group were water accessibility and climate change. Conversely, the criteria in

the effect group were water availability and water quality. The significance of the criteria was subsequently established by assessing the $R+C$ value, with water security emerging as the most crucial criterion within the cause group.

Table 13. Rankings of criteria

Criteria	R	C	$R+C$	$R-C$	Cause/Effect
C1	9.9913	10.8237	20.8149	-0.8324	Effect
C2	10.9167	9.3870	20.3037	1.5297	Cause
C3	10.2401	11.1342	21.3742	-0.8941	Effect
C4	10.6185	10.4217	21.0402	0.1969	Cause

The network relationship map, shown in Figure 3 below, was created using $R+C$ and $R-C$ values. The criteria are ranked based on $R+C$ values from the highest to the lowest. Refer to the Table 14, the ranking of criteria as follows: $C3 \succ C4 \succ C1 \succ C2$. "Water quality" is the most influential criteria, while "water accessibility" is the least influential.

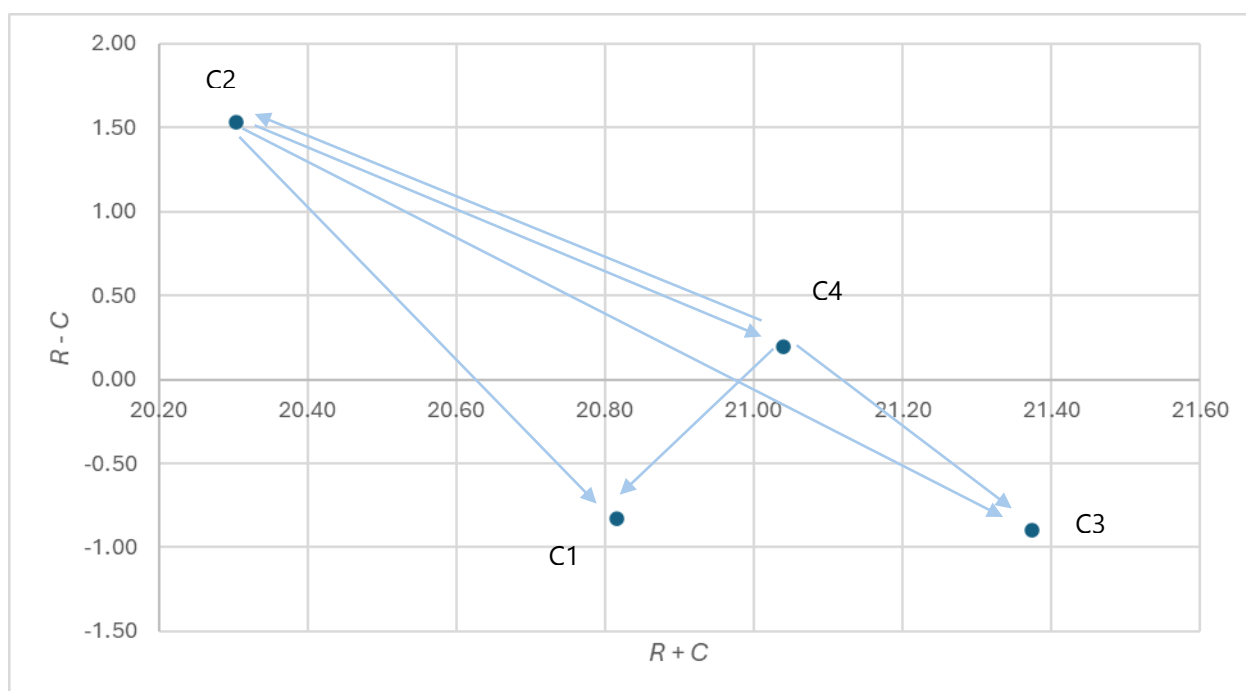


Figure 3. Network Relationship Map (NRM) of Criteria.

Figure 3 illustrates the network relationship map (NRM) of the criteria, depicting the interactions between causes and effects. The arrow from C2 to C4 indicates that C1 and C3 are the causes of the criteria. In an NRM, two arrows pointing at each other typically signify a bidirectional relationship, indicating mutual interaction between the nodes. For C2 and C4, the arrows pointing at each other demonstrate that these two criteria are effects. The significance of each sub-criterion can be determined by examining the sum of $R+C$ in Table 14. Additionally, their roles as causes or effects can be identified by analysing the difference $R-C$.

Table 14. Rankings of Sub-criteria

Criteria	Sub-criteria	R	C	$R + C$	$R - C$	Cause / Effect
C1	C11	61.7450	60.2662	122.0112	1.4788	Cause
	C12	60.2420	61.2373	121.4793	-0.9953	Effect
	C13	61.2858	61.7693	123.0551	-0.4835	Effect
C2	C21	14.0538	12.6290	26.6828	1.4248	Cause
	C22	12.3880	13.4102	25.7982	-1.0222	Effect
	C23	13.5660	13.9686	27.5347	-0.4026	Effect
C3	C31	17.1641	15.9497	33.1137	1.2144	Cause
	C32	17.5205	17.3810	34.9015	0.1395	Cause
	C33	16.4040	17.7579	34.1620	-1.3539	Effect
C4	C41	30.8692	30.3672	61.2364	0.5020	Cause
	C42	29.8652	30.3796	60.2448	-0.5144	Effect
	C43	29.8735	29.8611	59.7346	0.0124	Cause

The sum $R + C$ represents the aggregate significance of each sub-criterion within the network, with higher values indicating greater significance. For Water Availability (C1), Seasonal Variability (C13) holds the highest relevance, with a value of 123.0551, signifying its substantial role. In Water Accessibility (C2), Infrastructure Accessibility (C23) is the most significant factor, with a value of 27.5347. For Water Quality (C3), Microbiological Contaminants (C32) holds the highest significance, with a value of 34.9015. Climate Change (C4) is most significantly impacted by Change in Precipitation Pattern (C41), with a value of 61.2364.

The difference between $R - C$ (cause and effect) helps determine the role of each sub-criterion in terms of causality. Positive values indicate that the sub-criterion serves as a cause, while negative values suggest it acts as an effect. In Water Availability (C1), Surface Water Availability (C11) is a causal factor with an $R - C$ value of 1.4788, whereas C12 and C13 are effects with values of -0.9953 and -0.4835, respectively. In Water Accessibility (C2), Physical Accessibility (C21) is identified as a cause with a value of 1.4248, while C22 and C23 are effects. For Water Quality (C3), both C31 and C32 function as causes, with C32 having a modest value of 0.1395, and Physical Characteristics (C33) is identified as an effect. Climate Change (C4) identifies C41 and C43 as causal factors, with C43 having a minimal value of 0.0124, while Temperature Increase (C42) is an effect.

The sub-criteria with the highest $R + C$ values, demonstrating their relevance within each criterion, are C13, C23, C32, and C41. These sub-criteria play essential roles within their respective criteria. In cause-effect analysis, sub-criteria with positive $R - C$ values, such as C11, C21, C31, C32, C41, and C43, are considered causes, while those with negative $R - C$ values, such as C12, C13, C22, C23, C33, and C42, are considered effects.

The NRM provides crucial insights by visually depicting the significance of each sub-criterion, emphasizing nodes with high $R + C$ values, and displaying the causal relationships. Table 15 presents the conclusions from this analysis, sorting the importance order based on the $R + C$

value. Higher values indicate greater importance, while lower values indicate lesser importance. The $R-C$ value is used to determine whether the criteria are causes or effects, with positive values indicating causes and negative values indicating effects.

Table 15. Result of PNS-DEMATEL analysis

	Order of Important	Cause	Effect
Criteria	$C3 \succ C4 \succ C1 \succ C2$	C2, C4	C1, C3
	C3: $C32 \succ C33 \succ C31$	C31, C32	C33
Sub-Criteria	C4: $C41 \succ C42 \succ C43$	C41, C43	C42
	C1: $C13 \succ C11 \succ C12$	C11	C12, C13
	C2: $C23 \succ C21 \succ C22$	C21	C22, C23

4.2 PNS-AHP Method

The Pair-Wise Comparison Matrix, γ_{ij} in Table 16 displays the comparison of criteria (C1, C2, C3, and C4) with one another. Each element in the matrix denotes the relative significance between two criteria.

Table 16. The pair-wise comparison matrix, γ_{ij}

Criteria (E1)	C1	C2	C3	C4
C1	1	7	1/5	3
C2	1/7	1	1/9	1/5
C3	5	9	1	5
C4	1/3	5	1/5	1

Table 17 shows the average matrix, E_{ij} . By computing the average of the subjective opinions, the matrix helps one to get a fairer viewpoint on the relative importance of every criterion by using Eq. (18). It removes any anomalies and lays a better basis for further research. An example of a calculation matrix E_{12} as below:

Table 17. Average matrix, E_{ij}

Criteria	C1	C2	C3	C4
C1	1	$\langle 0.60, 0.12, 0.40 \rangle$	$\langle 0.40, 0.15, 0.60 \rangle$	$\langle 0.27, 0.55, 0.73 \rangle$
C2	$\langle 0.40, 0.12, 0.60 \rangle$	1	$\langle 0.37, 0.03, 0.63 \rangle$	$\langle 0.33, 0.47, 0.67 \rangle$
C3	$\langle 0.60, 0.15, 0.40 \rangle$	$\langle 0.63, 0.03, 0.37 \rangle$	1	$\langle 0.77, 0.35, 0.23 \rangle$
C4	$\langle 0.73, 0.55, 0.27 \rangle$	$\langle 0.67, 0.47, 0.33 \rangle$	$\langle 0.23, 0.35, 0.77 \rangle$	1

The aggregated pair-wise comparison matrix D_{ij} as illustrated in Table 18 results from the aggregated relative weights of every criterion. This matrix shows the whole relevance of every

element by giving a general picture of the comparisons using Eq. (19). An example of a calculation matrix D_{12} as below:

Table 18. The aggregated pair-wise comparison matrix, D_{ij}

Criteria	C1	C2	C3	C4
C1	1.0000	0.4389	0.3167	0.3611
C2	2.2785	1.0000	0.2556	0.3778
C3	3.1579	3.9130	1.0000	0.6278
C4	2.7692	2.6471	1.5929	1.0000
Total Column	9.2056	7.9990	3.1651	2.3667

Table 19 normalizes F_{ij} , the aggregated pair-wise comparison matrix. The relative weight interpretation is simpler when the normalizing process assures that the total weight of all criteria is equal to one by Eq. (20). An example of a calculation matrix F_{ij} as below:

Table 19. Normalizing pair-wise comparison matrix, F_{ij}

Criteria	C1	C2	C3	C4	Weight	λ
C1	0.1086	0.0549	0.1000	0.1526	0.1040	0.9577
C2	0.2475	0.1250	0.0807	0.1596	0.1532	1.2256
C3	0.3430	0.4892	0.3159	0.2653	0.3534	1.1184
C4	0.3008	0.3309	0.5033	0.4225	0.3894	0.9215

The weights represent the computed values for each criterion (C1 through C4), highlighting their relative importance in the decision-making process. The eigenvector, λ helps verify the coherence of the comparisons. Typically, the consistency ratio should be below a certain threshold to ensure the reliability of the decisions. Table 20 performs a consistency test on the criteria to verify the reliability of the pairwise comparisons made in previous steps. This test assesses the coherence of judgments by calculating the Consistency Index (CI) and the Consistency Ratio (CR). In this case, the CR score is 0.0827, which is below 0.1, indicating that the judgments are consistent and acceptable.

Table 20. The consistency test for the criteria

Criteria	Weight	λ
C1	0.1040	0.9577
C2	0.1532	1.2256
C3	0.3534	1.1184
C4	0.3894	0.9215
n = 4	1	$\lambda_{\max} = 4.1481$

The obtained weights are considered acceptable because the value of CR is less than 0.1.

Based on Eq. (22), the calculation to get Consistency Index is as follows:

$$CI = \frac{\lambda_{\max} - n}{n - 1} = 0.0744$$

Hence, Eq. (23) is used to compute the Consistency Ratio,

$$CR = \frac{CI}{RI} = 0.0827$$

Table 21 extends the consistency test to include both criteria and sub-criteria. The table summarizes the criteria and sub-criteria, along with their respective weights and consistency values. Each criterion is divided into sub-criteria, each with its own weight. For example, the criterion "Water Availability" is further divided into sub-criteria such as "Surface Water Availability" and "Groundwater Availability," each assigned specific weights and consistency values. This step ensures that both the primary criteria and the specific sub-criteria are consistent, providing a comprehensive validation of the decision-making process. The consistency percentages for sub-criteria, such as 3.25% for "Surface Water Availability" and 5.93% for "Microbiological Contaminants," indicate the degree of consistency in the judgments made for each sub-criterion. The calculation of weights for sub-criteria follows the same procedure as the calculation of weights for criteria in step 5 of the PNS-AHP method.

Table 21. The consistency test for the criteria and sub-criteria

Criteria	Weight	Consistency	Sub-criteria	Weight	Consistency
Water Availability	0.1040	0.0827 (8.27%)	Surface Water Availability	0.1789	
			Ground Availability	0.2399	0.0325 (3.25%)
			Seasonal Variability	0.5812	
Water Accessibility	0.1532	0.0827 (8.27%)	Physical Accessibility	0.2369	0.0300 (3.00%)
			Economic Accessibility	0.2943	
			Infrastructure Accessibility	0.4687	
Water Quality	0.3534	0.0827 (8.27%)	Chemical Contaminants	0.2313	0.0593 (5.93%)
			Microbiological Contaminants	0.3311	
			Physical Characteristics	0.4376	
Climate Change	0.3894	0.0827 (8.27%)	Change in Precipitation Patterns	0.2220	0.0326 (3.26%)

Temperature Increases	0.3013
Extreme Weather Events	0.4767

The consistency test is crucial for verifying the logical validity and reliability of the pairwise comparisons conducted during the AHP process. The criteria have an overall consistency ratio of 0.0827, equivalent to 8.27%. Given that, a consistency ratio below 10% is deemed acceptable [42], the judgments made in this analysis are considered logically consistent. The weights assigned to the criteria are as follows: Water Availability (0.1040), Water Accessibility (0.1532), Water Quality (0.3534), and Climate Change (0.3894), indicating their relative significance in the overall evaluation. The weights and consistency ratios within the sub-criteria confirm the analysis's reliability. For Water Availability, both Surface Water Availability and Groundwater Availability exhibit a consistency ratio of 3.25%, which falls within the permissible range. These factors are assigned weights of 0.1789 and 0.2399, respectively. The sub-criterion of Seasonal Variability, under Water Availability, carries a weight of 0.5812, suggesting its utmost significance despite the absence of an exact consistency ratio. Regarding Water Accessibility, Economic Accessibility has a consistency ratio of 3.00% and a weight of 0.2943. Physical Accessibility and Infrastructure Accessibility have weights of 0.2369 and 0.4687, respectively, though their exact consistency ratios are not provided. These weights highlight the relative importance of economic and infrastructural elements in determining water accessibility.

The sub-criteria for Water Quality include Chemical Contaminants with a consistency ratio of 5.93% and a weight of 0.2313, Microbiological Contaminants with a weight of 0.3311, and Physical Characteristics with a weight of 0.4376. Although the consistency ratio for Chemical Contaminants is higher, it remains within the acceptable range, confirming the reliability of these judgments. For Climate Change, the consistency ratio for the change in precipitation patterns is 3.26%, with a weight of 0.2220. The weights assigned to Temperature Increases and Extreme Weather Events are 0.3013 and 0.4767, respectively, indicating their significant impact despite the absence of specific consistency ratios. Overall, the table demonstrates that the criteria and sub-criteria in the AHP analysis are logically consistent, with all consistency ratios within the permissible range. The overall consistency ratio of 8.27% signifies coherent and dependable judgments. The analytical results are reinforced by the acceptable consistency ratios observed in each sub-criterion of Water Availability, Water Accessibility, and Climate Change. The highest consistency ratio is 5.93% for Chemical Contaminants under Water Quality, which falls within acceptable standards, confirming the reliability and strength of the pairwise comparisons conducted in this research.

Table 22 summarises the overall weight and ranking of different criteria and sub-criteria about Water Security in Sarawak. The study reveals that Climate Change (C4) is the most prominent factor, carrying a weight of 0.3894. This emphasizes its utmost significance among the parameters considered. As a result of Climate Change, the weight of Water Quality (C3) is at 0.3534, making it the second most significant element. The criteria of water accessibility and availability have lower influence, with weights of 0.1532 and 0.1040, respectively. Overall weight is calculated as follows:

Table 22. Overall weight for criteria and sub-criteria

Criteria	Weight	Sub-criteria	Weight	Overall Weight	Rank
Water Availability (C1)	0.1040	Surface Water Availability (C11)	0.1789	0.0186	12
		Ground Availability (C12)	0.2399	0.0249	11
		Seasonal Variability (C13)	0.5812	0.0604	8
Water Accessibility (C2)	0.1532	Physical Accessibility (C21)	0.2369	0.0363	10
		Economic Accessibility (C22)	0.2943	0.0451	9
		Infrastructure Accessibility (C23)	0.4687	0.0718	7
Water Quality (C3)	0.3534	Chemical Contaminants (C31)	0.2313	0.0817	6
		Microbiological Contaminants (C32)	0.3311	0.1170	4
		Physical Characteristics (C33)	0.4376	0.1546	2
Climate Change (C4)	0.3894	Change in Precipitation Patterns (C41)	0.2220	0.0864	5
		Temperature Increases (C42)	0.3013	0.1173	3
		Extreme Weather Events (C43)	0.4767	0.1856	1

A detailed examination of the sub-criteria reveals subtle and significant insights. The top three sub-criteria, ranked by importance, are Extreme Weather Events (rank 1), Physical Characteristics (rank 2), and Temperature Increases (rank 3). Conversely, the sub-criteria with the least significance are Surface Water Availability (rank 23), Groundwater Availability (rank 11), and Physical Accessibility (rank 10). This thorough analysis underscores the prominent impact of Climate Change, particularly Extreme Weather Events, on water resource management. Additionally, it highlights the crucial elements of Water Quality and Water Accessibility, while acknowledging that Water Availability, though important, has a comparatively less influence.

The PNS-DEMATEL and PNS-AHP evaluations assess various criteria and sub-criteria to determine their relative relevance and interconnections. Together, these evaluations provide a comprehensive view of the fundamental factors affecting the system and their interdependencies, guiding strategic development and resource allocation. The PNS-DEMATEL analysis highlights the complex interactions among sub-criteria within the same criterion. For instance, in Water Availability (C1), C11 and C21 act as causative elements, significantly impacting C12 and C13. Similarly, in Water Accessibility (C2), C22 and C23 are influenced by C31 and C32, respectively. These interdependencies suggest that addressing the fundamental causes (like C11 and C21 for C1) would improve the associated sub-criteria.

The PNS-AHP study ranks sub-criteria based on their overall weight, emphasizing their relative significance. The top-ranked sub-criteria are related to Climate Change, specifically "Extreme Weather Events" (0.1856) and "Temperature Increases" (0.1173), highlighting the urgency of addressing climate change issues. High-ranking sub-criteria for Water Quality include "Physical Characteristics" (0.1546) and "Microbiological Contaminants" (0.1170), underscoring the importance

of maintaining water quality. The study highlights that climate change (C4) is the most significant factor influencing water security in Sarawak, with a weight of 0.3894, followed by water quality (C3) at 0.3534. Among the sub-criteria, extreme weather events stand out as the most critical, highlighting the pressing need to address climate-related challenges. By integrating PNS-DEMATEL method, the interconnected impacts of causative elements like C11 (surface water availability) and C21 (groundwater availability) on other sub-criteria become evident.

4.3 Comparative Analysis

To demonstrate the applicability and feasibility of the proposed procedures, this section presents a twofold comparative analysis. First, a comparison of the performance of the proposed Pythagorean Neutrosophic DEMATEL–AHP approach with conventional DEMATEL and AHP methods using crisp numbers. The results, summarized in Table 23, highlight the impact of incorporating PNS in handling uncertainty and modelling the interrelationships among criteria. By allowing independent degrees of truth, indeterminacy, and falsity, the proposed approach provides a unique representation of expert judgments compared to traditional crisp methods. Second, a conceptual comparison with recent MCDM models, particularly by Radovanović et al. [40] and Fujita [41]. The study by Radovanović et al. [40] integrates Spherical Fuzzy AHP with Grey MARCOS for evaluating university professors, employing spherical fuzzy sets to capture uncertainty and the MARCOS method for ranking alternatives. Fujita [41] introduces Hyperfuzzy VIKOR and Hyperfuzzy DEMATEL, extending classical fuzzy models to handle hierarchical and multi-layered uncertainty within a hyperfuzzy framework. In contrast, the proposed Pythagorean Neutrosophic DEMATEL–AHP approach differs in both methodology and application focus. The key advantages of the proposed method include enhanced uncertainty modelling, causal relationship analysis, domain-specific applicability, and a balance between accuracy and interpretability. Overall, this dual comparison demonstrates not only the quantitative performance improvements of the proposed approach over traditional crisp methods but also its conceptual advantages over existing fuzzy and hyperfuzzy MCDM models, thereby reinforcing its suitability for water security evaluation. Table 23 summarises the findings of the proposed method with the conventional DEMATEL and AHP methods.

Table 23. Comparative analysis with existing methods

Criteria	Sub-criteria	AHP		PNS-AHP		DEMATEL			PNS-DEMATEL		
		Overall Weight	Rank	Overall Weight	Rank	R+C	R-C	Causal group	R+C	R-C	Causal group
Water Availability (C1)	Surface Water Availability (C11)	0.151	4	0.019	10	41.6	1.27	Cause	122.0	1.47	Cause
	Groundwater Availability (C12)	0.017	9	0.025	9	40.8	-0.66	Effect	121.5	-0.99	Effect
	Seasonal Variability (C13)	0.066	5	0.060	6	42.4	-0.62	Effect	123.0	-0.48	Effect
Water Accessibility (C2)	Physical Accessibility (C21)	0.055	6	0.036	8	10.4	0.48	Cause	26.7	1.42	Cause
	Economic Accessibility	0.005	12	0.045	7	11.7	0.04	Cause	25.8	-1.02	Effect

	(C22)										
	Infrastructure Accessibility (C23)	0.013	10	0.072	5	11.8	-0.52	Effect	27.5	-0.40	Effect
Water Quality (C3)	Chemical Contaminants (C31)	0.222	1	0.082	4	14.7	1.42	Cause	33.1	1.21	Cause
	Microbiological Contaminants (C32)	0.181	3	0.117	2	15.4	-1.01	Effect	34.9	0.13	Cause
	Physical Characteristics (C33)	0.201	2	0.155	1	17.0	-0.41	Effect	34.2	-1.35	Effect
Climate Change (C4)	Change in Precipitation Patterns (C41)	0.055	7	0.086	3	22.47	-0.26	Effect	61.23	0.50	Cause
	Temperature Increases (C42)	0.011	11	0.117	10	21.76	-0.25	Effect	60.24	-0.51	Effect
	Extreme Weather Events (C43)	0.023	8	0.186	9	21.14	0.51	Cause	59.73	0.01	Cause

Table 23 presents a comparative analysis of the evaluation results obtained through AHP, the proposed PNS-AHP, DEMATEL, and the proposed PNS-DEMATEL. The results highlight several important insights. From the AHP analysis, Chemical Contaminants (C31) emerged as the most influential sub-criterion, followed by Physical Characteristics (C33) and Microbiological Contaminants (C32), indicating that water quality factor is prioritized in traditional weight-based decision-making. However, when uncertainty and indeterminacy elements are incorporated in the proposed PNS-AHP method, a slight shift is observed. Although Physical Characteristics (C33) and Microbiological Contaminants (C32) remain highly ranked, Change in Precipitation Patterns (C41) rises to the top position, highlighting the growing importance of climate-related risks when uncertainty is explicitly considered.

The DEMATEL results further refined these insights by identifying the structural relationships among sub-criteria. Surface Water Availability (C11), Chemical Contaminants (C31), and Physical Accessibility (C21) are classified as causal factors with positive (R–C) values, suggesting that these sub-criteria play a driving role and exert influence over other components of the system. In contrast, Groundwater Availability (C12), Seasonal Variability (C13), and Infrastructure Accessibility (C23) were categorized as effect factors, indicating that they are more responsive to upstream changes rather than being primary drivers. The integration of PNS with DEMATEL strengthened the interpretation of these interdependencies under uncertain conditions. For instance, Surface Water Availability (C11) (R–C = 1.47) and Chemical Contaminants (C31) (R–C = 1.21) remained dominant causal factors, reinforcing their role as primary levers in managing water-related challenges. At the same time, Physical Characteristics (C33) shifted strongly into the effect group (R–C = –1.35), suggesting that while it holds high weight in terms of importance, it is not a root cause but rather a consequence of other factors.

This comparative analysis provides two layers of understanding. The weight-based methods (AHP, PNS-AHP) highlight which sub-criteria are perceived as the most important, while the causal–effect methods (DEMATEL, PNS-DEMATEL) uncover the structural drivers of the system. Practically, this implies that effective water management strategies should prioritize controlling causal sub-criteria such as surface water availability and chemical contamination, as interventions in these areas are likely to reduce downstream impacts, including those from extreme weather events

and physical characteristics. The methodological contribution is also noteworthy, as integrating PNS with both AHP and DEMATEL demonstrates the value of incorporating uncertainty, leading to more robust and realistic decision-making outcomes.

5. Conclusions

This study advances the theoretical landscape of decision-making under uncertainty by integrating the Pythagorean Neutrosophic Set (PNS) with AHP and DEMATEL. The combination contributes a novel hybrid framework capable of simultaneously handling imprecise information (via PNS), prioritizing decision elements (via AHP), and identifying causal relationships (via DEMATEL). The proposed framework can be generalized and adapted to other complex, multidimensional problems beyond water security, especially where ambiguity and interdependencies exist. Practically, this study provides a decent policymaker implication in evaluating and improving water security strategies. By identifying key drivers such as water availability and climate change, and mapping their interrelations, policymakers can make informed decisions that target the root causes. This structured evaluation enables governments, water management authorities, and NGOs to allocate resources more effectively, prioritize actions, and design targeted interventions in line with the Sustainable Development Goals (SDGs). This study provides a detailed understanding of water security in Sarawak, revealing Climate Change as the most critical factor, with a weight of 0.3894, followed closely by Water Quality at 0.3534. The analysis highlights the profound impact of Extreme Weather Events and Temperature Increases, which rank as the top sub-criteria, underscoring the urgent need to address climate-related challenges. While Water Accessibility and Water Availability are also important, their relatively lower weights suggest that Sarawak's current water resources are less immediately threatened, though they remain vulnerable to future climate-driven changes. The integration of PNS-DEMATEL and PNS-AHP offers a practical implication for evaluating the complex interdependencies among criteria and sub-criteria, providing valuable insights for strategic resource allocation and policy development. While the integrated PNS-AHP-DEMATEL approach offers several strengths, the study has some limitations. First, the reliance on expert judgments may introduce bias or subjectivity despite the robustness of the methods. Second, the scope is limited to predefined criteria and sub-criteria; additional factors relevant in other geographical or socio-political contexts may not be captured. Finally, the framework's effectiveness depends on the quality and availability of local data, which may pose a challenge in data-scarce regions.

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Neutrosophic Selective Separability

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Abstract. We develop a confidence-sensitive neutrosophic framework for selective separability. Single-valued neutrosophic sets are evaluated through an admissible aggregation operator $\Phi(T, 1 - I, 1 - F)$, so that the conservative minimum is only one possible choice. We first study uncertain dense data over an ordinary topology and prove operator-independent transfer results for M -, H -, R -, and S -separability, while showing that the admissible witnesses remain genuinely confidence-sensitive. We then pass to single-valued neutrosophic topological spaces and distinguish strong intrinsic density, which reaches the full confidence height of every effective open set, from weak intrinsic density, which only requires positive confidence overlap. The weak and strong notions differ for individual neutrosophic sets, although both recover the expected classical space classes on crisp-induced topologies. To keep countable network weight meaningful, we introduce countably calibrated neutrosophic networks based on a dense rational set of point profiles. We also formulate both trace-based and genuinely graded selection principles controlled by single-valued neutrosophic filters on the index set. Finally, we compare the minimum and product confidence aggregators, discuss numerical resolution thresholds for intrinsic selection, and derive consequences for Fréchet–Urysohn and function spaces.

Keywords: Single-valued neutrosophic set; neutrosophic topology; selective separability; M -separability; S -separability; strong neutrosophic density; weak neutrosophic density; neutrosophic network; single-valued neutrosophic filter; Fréchet–Urysohn space.

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1. Introduction

Selection principles provide a common language for many covering and density properties in General Topology. In the density setting, M -separability asks for finite selections from a sequence of dense sets whose union remains dense, while R - and H -separability impose,

respectively, one-point selections and an eventual hitting condition on non-empty open sets; see Scheepers [16], Gruenhage–Sakai [6], and Repovš–Zdomskyy [13–15].

Two recent papers involving Maesano are especially relevant here. Bardyla, Maesano and Zdomskyy [2] study selective separability in Fréchet–Urysohn spaces and their products, obtaining in particular a ZFC example of a countable Hausdorff Fréchet–Urysohn space which is not H -separable. Aurichi, Maesano and Zdomskyy [1] introduce S -separability as a density counterpart of the Scheepers covering property and prove, among other results, that S -separability is equivalent to a finite-power property called pM -separability and that every countable Hausdorff Fréchet–Urysohn space is S -separable.

Neutrosophic sets, originating in Smarandache’s theory of neutrosophy [17], represent truth, indeterminacy and falsity as independent components. For applications on an ordinary universe, the single-valued model of Wang, Smarandache, Zhang and Sunderraman [18] is especially useful. Early developments of neutrosophic topology include the contributions of Gallego Lupiáñez [5] and Salama and Alblowi [12]. A computational framework for single-valued neutrosophic sets and mappings was developed in [8] and extended to neutrosophic topologies in [9]. Neutrosophic points and induced neutrosophic topologies were studied by Jafari, Nordo and Thakur [7]. Earlier notions of neutrosophic filters were considered by Salama and Alagamy [11], while single-valued neutrosophic filters were introduced and investigated in [10].

The purpose of this paper is to distinguish two levels of neutrosophication. At the first level the ambient topology is ordinary and only the dense input data are neutrosophic. This layer is deliberately conservative and is useful as a calibration device: it shows exactly which classical properties are recovered and which additional conditions are imposed on their witnesses. At the second level the topology itself is single-valued neutrosophic. Here density is no longer determined by ordinary non-empty intersections; it is measured by confidence overlap with neutrosophic open sets. This second layer produces a genuinely graded topological theory.

A further aim is to remove the dependence on one particular scalarization of neutrosophic information. Instead of fixing the minimum from the outset, we work with an admissible aggregation operator Φ . The principal transfer theorem is invariant under this choice. We also isolate the stronger compatibility condition needed when a confidence product is required.

Our main results can be summarized as follows. We give a self-contained presentation of the single-valued neutrosophic operations, points and topologies used later. We characterize confidence-density by high-confidence cuts and prove an operator-independent transfer principle between confidence-sensitive and classical selective separability. We then exhibit a classical M -selection witness which fails as a neutrosophic witness, showing that the equivalence of classes does not identify the witnesses. In the intrinsic layer we distinguish strong

and weak neutrosophic density, explain the exact crisp reductions of both notions, and introduce a resolution parameter that prevents unstable normalization on very low-confidence open sets. We refine the network construction by testing only a fixed countable dense calibration of neutrosophic point profiles, and we supplement the classical index-filter transfer theorem with preliminary trace-based and graded definitions using single-valued neutrosophic filters on ω . Finally, we compare non-equivalent confidence aggregators and recover recent consequences concerning Fréchet–Urysohn and function spaces.

The paper is organized as follows. Section 2 collects the classical and neutrosophic notions needed throughout the paper. Sections 3 and 4 develop the confidence-sensitive theory over ordinary topological spaces, while Section 5 passes to genuinely neutrosophic topological spaces. Sections 6 and 7 are devoted to neutrosophic networks and filter-parametrized selection principles, respectively. Section 8 discusses consequences for Fréchet–Urysohn and function spaces, and Section 9 concludes with some open problems and directions for further research.

2. Preliminaries

Throughout, ω denotes the set of non-negative integers. We first recall the classical selective properties and then fix all neutrosophic conventions required in the sequel.

2.1. Classical selective separability

Let (X, τ) be a topological space. A subset $D \subseteq X$ is dense if $D \cap U \neq \emptyset$ for every non-empty $U \in \tau$.

Definition 2.1. A topological space X is

- (a) *M-separable* if for every sequence $(D_n)_{n \in \omega}$ of dense subsets of X there are finite sets $F_n \subseteq D_n$ such that $\bigcup_{n \in \omega} F_n$ is dense in X ;
- (b) *R-separable* if for every sequence $(D_n)_{n \in \omega}$ of dense subsets of X there are points $x_n \in D_n$ such that $\{x_n : n \in \omega\}$ is dense in X ;
- (c) *H-separable* if for every sequence $(D_n)_{n \in \omega}$ of dense subsets of X there are finite sets $F_n \subseteq D_n$ such that every non-empty open set meets all but finitely many F_n ;
- (d) *S-separable* if for every sequence $(D_n)_{n \in \omega}$ of dense subsets of X there are finite sets $F_n \subseteq D_n$ such that, for every finite family $U_1, \dots, U_m$ of non-empty open subsets of X , there exists $n \in \omega$ with $F_n \cap U_j \neq \emptyset$ for all $j \leq m$.

The last property is due to Aurichi, Maesano and Zdomskyy [1]. We shall use the elementary implications $H \Rightarrow S \Rightarrow M$ and $R \Rightarrow M$.

2.2. Single-valued neutrosophic sets

Definition 2.2. Let X be a non-empty set. A *single-valued neutrosophic set* (SVNS) A on X is a triple $A = (T_A, I_A, F_A)$, where $T_A, I_A, F_A : X \rightarrow [0, 1]$ are the truth-membership, indeterminacy-membership and falsity-membership functions, respectively.

The truth-membership, indeterminacy-membership, and falsity-membership components are mutually independent. Consequently, no constraint is imposed on their sum, and in general $T_A(x) + I_A(x) + F_A(x)$ need not be equal to 1. We denote by $\mathcal{SVN}(X)$ the family of all SVNSs on X .

There are different but equivalent coordinate conventions in the neutrosophic-topology literature. Because our confidence decreases when indeterminacy increases, we adopt the following *evidence order*, used in smooth single-valued neutrosophic topology and particularly convenient for confidence-based arguments [4].

Definition 2.3. For $A, B \in \mathcal{SVN}(X)$ put $A \preceq_N B$ if, for every $x \in X$, $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$ and $F_A(x) \geq F_B(x)$. Define

$$\mathbf{0}_N = (0, 1, 1), \quad \mathbf{1}_N = (1, 0, 0),$$

where constant triples are understood pointwise. The neutrosophic union and intersection are

$$A \sqcup_N B = (\max\{T_A, T_B\}, \min\{I_A, I_B\}, \min\{F_A, F_B\}),$$

$$A \sqcap_N B = (\min\{T_A, T_B\}, \max\{I_A, I_B\}, \max\{F_A, F_B\}),$$

and the complement is $A^c = (F_A, 1 - I_A, T_A)$.

Arbitrary unions and intersections are obtained by taking the corresponding pointwise suprema and infima. Notice that $\mathbf{0}_N \preceq_N A \preceq_N \mathbf{1}_N$. The convention used in [9, 10] orders the indeterminacy coordinate in the opposite direction and uses the absolute set $(1, 1, 0)$. The two descriptions are order-isomorphic under the coordinate change $I \mapsto 1 - I$. Thus Definition 2.3 is a confidence-compatible reparametrization rather than a competing theory.

Definition 2.4 (Crisp embedding). For $D \subseteq X$, define its crisp neutrosophic embedding $\widehat{D}$ by $\widehat{D}(x) = \mathbf{1}_N$ for $x \in D$ and $\widehat{D}(x) = \mathbf{0}_N$ for $x \notin D$.

Then $\widehat{D \cup E} = \widehat{D} \sqcup_N \widehat{E}$, $\widehat{D \cap E} = \widehat{D} \sqcap_N \widehat{E}$ and $(\widehat{D})^c = \widehat{X \setminus D}$.

2.3. Neutrosophic topologies and points

Definition 2.5. A *single-valued neutrosophic topology* on X is a family $\mathcal{T} \subseteq \mathcal{SVN}(X)$ such that $\mathbf{0}_N, \mathbf{1}_N \in \mathcal{T}$, $\mathcal{T}$ is closed under arbitrary neutrosophic unions and under finite neutrosophic intersections. The pair $(X, \mathcal{T})$ is a *single-valued neutrosophic topological space* (NTS). Members of $\mathcal{T}$ are neutrosophic open sets; their complements are neutrosophic closed sets.

Neutrosophic topological structures have appeared in several formulations; see, among others, [4, 5, 7, 9, 12]. The present formulation is the SVNS lattice version determined by the evidence order fixed in Definition 2.3. Every ordinary topology τ induces the crisp NTS $\hat{\tau} = \{\hat{U} : U \in \tau\}$.

Definition 2.6. Let $x \in X$ and $(t, i, f) \in [0, 1]^3 \setminus \{(0, 1, 1)\}$. The *neutrosophic point* $p = x_{t,i,f}$ is the SVNS equal to (t, i, f) at x and to $\mathbf{0}_N$ at all other points. We say that p belongs to A , and write $p \in_N A$, when $p \preceq_N A$.

Neutrosophic points and the topology induced on their collection were studied in [7]. In the evidence convention, $p = x_{t,i,f} \in_N A$ means $t \leq T_A(x)$, $i \geq I_A(x)$ and $f \geq F_A(x)$.

2.4. Confidence aggregators

The scalar profile used to decide whether a selected point is reliable should not be tied to one aggregation rule.

Definition 2.7. An *admissible confidence aggregator* is a continuous mapping $\Phi : [0, 1]^3 \rightarrow [0, 1]$ which is non-decreasing in each variable and satisfies $\Phi(0, 0, 0) = 0$ and $\Phi(1, 1, 1) = 1$. For $A \in \mathcal{SVN}(X)$ define

$$\rho_A^\Phi(x) = \Phi(T_A(x), 1 - I_A(x), 1 - F_A(x)).$$

When no ambiguity is possible, the superscript Φ is omitted.

The conservative choice used most often below is $\Phi_{\min}(a, b, c) = \min\{a, b, c\}$. Other admissible choices include the product abc and weighted arithmetic means with positive weights summing to one. The next condition is relevant only when products of SVNSs are interpreted componentwise.

Definition 2.8. An admissible Φ is *meet-compatible* if for every finite family $u_1, \dots, u_m \in [0, 1]^3$ one has

$$\Phi\left(\bigwedge_{j=1}^m u_j\right) = \min_{1 \leq j \leq m} \Phi(u_j),$$

where the meet on the left is coordinatewise minimum.

The minimum aggregator is meet-compatible. Product and weighted arithmetic mean aggregators are generally not. This distinction will let us state precisely which results are operator-independent.

3. Confidence-sensitive selection over a crisp topology

In this section (X, τ) is an ordinary topological space and Φ is an arbitrary admissible confidence aggregator.

3.1. Neutrosophic density

For $0 \leq \alpha < 1$, put $A_{\Phi}^{\alpha} = \{x \in X : \rho_A^{\Phi}(x) > \alpha\}$ and $A_{\Phi}^{+} = A_{\Phi}^{>0}$.

Definition 3.1. An SVN A on (X, τ) is Φ -dense (or simply N -dense) if $\sup_{x \in U} \rho_A^{\Phi}(x) = 1$ for every non-empty $U \in \tau$.

Lemma 3.2 (Cut characterization). *For $A \in \mathcal{SVN}(X)$, the following are equivalent: A is N -dense; and $A_{\Phi}^{>\alpha}$ is dense in X for every $\alpha < 1$.*

Proof. If A is N -dense, then for every non-empty open U and every $\alpha < 1$ there is $x \in U$ with $\rho_A^{\Phi}(x) > \alpha$, hence $U \cap A_{\Phi}^{>\alpha} \neq \emptyset$. Conversely, if every confidence cut is dense, then every non-empty open U contains points of confidence larger than every $\alpha < 1$, and therefore its confidence supremum is one. $\square$

For the crisp embedding $\widehat{D}$, admissibility gives $\rho_{\widehat{D}}^{\Phi} = \chi_D$. Hence $\widehat{D}$ is N -dense if and only if D is dense, independently of the chosen aggregator.

3.2. Neutrosophic M -, R -, H -, and S -separability

For a finite set $F \subseteq X$ and non-empty open U , define

$$c_{\Phi}(A, F; U) = \max(\{\rho_A^{\Phi}(x) : x \in F \cap U\} \cup \{0\}).$$

Definition 3.3. Let $(A_n)_{n \in \omega}$ range over sequences of N -dense SVN S s on X . The space X is

- (a) NM -separable if there are finite $F_n \subseteq A_{n, \Phi}^{+}$ such that, for every non-empty open U , $\sup_n c_{\Phi}(A_n, F_n; U) = 1$;
- (b) NR -separable if there are $x_n \in A_{n, \Phi}^{+}$ such that, for every non-empty open U ,

$$\sup(\{\rho_{A_n}^{\Phi}(x_n) : x_n \in U\} \cup \{0\}) = 1;$$

- (c) NH -separable if there are finite $F_n \subseteq A_{n, \Phi}^{+}$ such that $\lim_{n \rightarrow \infty} c_{\Phi}(A_n, F_n; U) = 1$ for every non-empty open U ;
- (d) NS -separable if there are finite $F_n \subseteq A_{n, \Phi}^{+}$ such that, for every finite family $U_1, \dots, U_m$ of non-empty open sets and every $\varepsilon > 0$, some n satisfies $c_{\Phi}(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$.

Theorem 3.4 (Operator-independent transfer principle). *For every admissible Φ and every topological space X , NM -, NH -, NR -, and NS -separability are respectively equivalent to classical M -, H -, R -, and S -separability.*

Proof. We first prove the classical-to-neutrosophic implications. Fix a sequence $(A_n)_{n \in \omega}$ of N -dense SVNSSs.

Assume that X is M -separable. Partition ω into pairwise disjoint infinite blocks $(P_k)_{k \in \omega}$ and put $\alpha_k = 1 - 2^{-k}$. For $n \in P_k$, let $D_n = A_{n, \Phi}^{>\alpha_k}$. By Lemma 3.2, every D_n is dense. Applying M -separability to the subsequence indexed by P_k , we obtain finite sets $F_n \subseteq D_n$ ($n \in P_k$) such that $\bigcup_{n \in P_k} F_n$ is dense. Since the blocks form a partition, these choices define one finite set F_n for every n . Let U be non-empty open and let $\varepsilon > 0$. Choose k with $\alpha_k > 1 - \varepsilon$. Density of $\bigcup_{n \in P_k} F_n$ yields $n \in P_k$ and $x \in F_n \cap U$. Then $c_\Phi(A_n, F_n; U) \geq \rho_{A_n}^\Phi(x) > \alpha_k > 1 - \varepsilon$. As ε is arbitrary, $\sup_n c_\Phi(A_n, F_n; U) = 1$, and hence X is NM -separable.

Assume next that X is H -separable. Put $\alpha_n = 1 - (n + 2)^{-1}$ and set $D_n = A_{n, \Phi}^{>\alpha_n}$. Choose finite $F_n \subseteq D_n$ witnessing H -separability. For every non-empty open U , there is n_0 such that $F_n \cap U \neq \emptyset$ whenever $n \geq n_0$. Hence, for $n \geq n_0$, $c_\Phi(A_n, F_n; U) > \alpha_n$. Since also $c_\Phi(A_n, F_n; U) \leq 1$ and $\alpha_n \rightarrow 1$, it follows that $c_\Phi(A_n, F_n; U) \rightarrow 1$. Thus X is NH -separable.

If X is R -separable, use again the blocks P_k and the cuts $D_n = A_{n, \Phi}^{>\alpha_k}$ for $n \in P_k$. On each block choose points $x_n \in D_n$ so that $\{x_n : n \in P_k\}$ is dense. Given a non-empty open U and $\varepsilon > 0$, choose k with $\alpha_k > 1 - \varepsilon$. Some $n \in P_k$ satisfies $x_n \in U$, and then $\rho_{A_n}^\Phi(x_n) > 1 - \varepsilon$. Hence the supremum required in the definition of NR -separability is one.

Finally, assume that X is S -separable. On each block P_k , apply S -separability to the dense cuts $D_n = A_{n, \Phi}^{>\alpha_k}$ ($n \in P_k$), obtaining finite $F_n \subseteq D_n$. Given non-empty open sets $U_1, \dots, U_m$ and $\varepsilon > 0$, choose k with $\alpha_k > 1 - \varepsilon$. The S -selection on P_k provides some $n \in P_k$ such that $F_n \cap U_j \neq \emptyset$ for every $j \leq m$. Consequently $c_\Phi(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$, so X is NS -separable.

For the converse implications, let (D_n) be a sequence of ordinary dense sets and put $A_n = \widehat{D_n}$. Then $A_{n, \Phi}^+ = D_n$ and every confidence value involved in Definition 3.3 is either 0 or 1. Thus an NM witness has dense union, an NR witness is a dense set of selected points, and convergence to 1 in the NH condition means that every non-empty open set meets all but finitely many F_n . For NS , taking $\varepsilon = 1/2$ gives exactly the classical simultaneous-hitting condition. Hence the four neutrosophic properties imply their classical counterparts. $\square$

Corollary 3.5. *For every admissible Φ , $NH \Rightarrow NS \Rightarrow NM$ and $NR \Rightarrow NM$.*

Theorem 3.4 is a calibration result: it says that the first layer does not manufacture a new class of underlying spaces. It does not, however, identify the witnesses. The next example makes this distinction explicit.

Example 3.6 (A classical witness rejected by confidence). Let $X = \mathbb{Q}$ with its usual topology and choose a partition $\mathbb{Q} = D \cup E$ into two disjoint dense sets. Such a partition can be

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constructed by enumerating a countable base and recursively choosing two new rational points in each basic open set. Fix $\Phi = \Phi_{\min}$ and let $A_n = A$ for all n , where A has confidence 1 on D and confidence $1/2$ on E ; for instance set $A(x) = (1, 0, 0)$ on D and $A(x) = (1/2, 0, 0)$ on E . Then A is N -dense because D is dense. Enumerate $E = \{e_n : n \in \omega\}$ and put $F_n = \{e_n\}$. The union $\bigcup_n F_n = E$ is dense, so (F_n) is a perfectly valid classical M -selection witness for the support sequence. Nevertheless, for every non-empty open U one has $\sup_n c_\Phi(A, F_n; U) \leq 1/2$, and hence the same finite selections do not witness NM -separability. The space class is unchanged, but the admissible witness class is strictly refined.

4. Finite powers and the NpM property

Aurichi, Maesano and Zdomsky prove that classical S -separability is equivalent to pM -separability [1, Proposition 4.2]. We now lift the finite power argument at the level of confidence profiles.

Lemma 4.1 (Tail strengthening). *Let $(A_n)_{n \in \omega}$ be a sequence of N -dense SVNSSs.*

- (i) *If X is NM -separable, there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every non-empty open U and every $N \in \omega$,*

$$\sup_{n \geq N} c_\Phi(A_n, F_n; U) = 1.$$

- (ii) *If X is NS -separable, there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every finite family $U_1, \dots, U_m$ of non-empty open sets, every $\varepsilon > 0$, and every $N \in \omega$, some $n \geq N$ satisfies $c_\Phi(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$.*

Proof. Partition ω into pairwise disjoint infinite blocks $(P_r)_{r \in \omega}$. On each block, reindex the restricted sequence $(A_n)_{n \in P_r}$ by ω , apply the relevant selective property, and transport the resulting finite selections back to the original indices. Because the blocks are disjoint, this defines a single sequence (F_n) .

Fix one of the requirements appearing in (i) or (ii). The selection on every block P_r supplies at least one index in that block satisfying the requirement (with any prescribed confidence tolerance in case (i)). These successful indices are distinct for distinct r ; hence there are infinitely many of them and therefore they are unbounded in ω . This proves the tail assertions.

□

For an SVNSS A and $m \geq 1$, we use the *confidence product*

$$\rho_A^{[m]}(x_1, \dots, x_m) = \min_{1 \leq j \leq m} \rho_A^\Phi(x_j).$$

This definition is meaningful for every admissible Φ . If Φ is meet-compatible, it is exactly the confidence induced by the usual componentwise neutrosophic product obtained by taking

minimum truth and maximum indeterminacy and falsity. Thus the scalar construction and the SVNS product construction agree in the most important conservative case $\Phi_{\min}$.

Definition 4.2. A space X is *NpM-separable* if, for every sequence (A_k) of N -dense SVNNSs and every fixed integer $m \geq 1$, there are finite sets $F_k \subseteq A_{k,\Phi}^+$ such that, for every non-empty open $W \subseteq X^m$ and every $N \in \omega$,

$$\sup\left(\{\rho_{A_k}^{[m]}(x_1, \dots, x_m) : k \geq N, (x_1, \dots, x_m) \in F_k^m \cap W\} \cup \{0\}\right) = 1.$$

The finite selections are allowed to depend on m . The explicit tail requirement is the confidence-sensitive form convenient for the diagonal argument below; Lemma 4.1 shows why such tail witnesses arise naturally from the corresponding selective property.

Theorem 4.3 (NS–NpM equivalence). *For every admissible Φ , a topological space X is NS-separable if and only if it is NpM-separable.*

Proof. Assume first that X is NS-separable and fix a sequence (A_k) of N -dense SVNNSs. By Lemma 4.1(ii), choose finite $F_k \subseteq A_{k,\Phi}^+$ with the strengthened tail property. Fix $m \geq 1$, a non-empty open set $W \subseteq X^m$, and $N \in \omega$. Choose a non-empty basic rectangle $U_1 \times \dots \times U_m \subseteq W$. For every $\varepsilon > 0$, Lemma 4.1(ii) gives $k \geq N$ such that $c_\Phi(A_k, F_k; U_j) > 1 - \varepsilon$ for all $j \leq m$. Hence, for each j , one can choose $x_j \in F_k \cap U_j$ with $\rho_{A_k}^\Phi(x_j) > 1 - \varepsilon$. The tuple $(x_1, \dots, x_m)$ belongs to $F_k^m \cap W$ and satisfies

$$\rho_{A_k}^{[m]}(x_1, \dots, x_m) > 1 - \varepsilon.$$

Since ε is arbitrary, the supremum in Definition 4.2 is one. Thus X is NpM-separable.

Conversely, assume that X is NpM-separable and fix a sequence (A_k) of N -dense SVNNSs. For each $m \geq 1$, choose a witness $(F_{m,k})_{k \in \omega}$ for the m -th power, and define

$$F_k = \bigcup_{1 \leq r \leq k} F_{r,k}$$

(with $F_0 = \emptyset$). Each F_k is finite and is contained in $A_{k,\Phi}^+$. Let $U_1, \dots, U_m$ be non-empty open subsets of X and let $\varepsilon > 0$. Apply the m -th witness with $W = U_1 \times \dots \times U_m$ and with tail parameter $N = m$. Since the corresponding supremum equals one, there exist $k \geq m$ and $(x_1, \dots, x_m) \in F_{m,k}^m \cap W$ such that

$$\rho_{A_k}^{[m]}(x_1, \dots, x_m) > 1 - \varepsilon.$$

By the definition of the minimum confidence product, $\rho_{A_k}^\Phi(x_j) > 1 - \varepsilon$ for every $j \leq m$. Moreover, $F_{m,k} \subseteq F_k$ because $m \leq k$. Hence $c_\Phi(A_k, F_k; U_j) > 1 - \varepsilon$ for all $j \leq m$, which is exactly the NS condition. $\square$

The theorem is operator-independent at the scalar confidence level. The extra meet-compatibility hypothesis is needed only if one insists that the confidence product arise from a particular componentwise product of SVNNSs. This separates the robust selection statement from the algebraic choice of neutrosophic product.

5. Intrinsic selective separability on neutrosophic topological spaces

We now leave the conservative crisp-topological setting. Let $(X, \mathcal{T})$ be an NTS and let Φ be admissible. For a neutrosophic open set U define its *confidence height*

$$h_{\Phi}(U) = \sup_{x \in X} \rho_U^{\Phi}(x).$$

For a numerical or observational tolerance $\delta \in [0, 1)$, we call U *δ -effective* if $h_{\Phi}(U) > \delta$. The word *effective* without qualification means 0-effective. For $A \in \mathcal{SVN}(X)$ and $U \in \mathcal{T}$ put

$$\text{ov}_{\Phi}(A, U) = \sup_{x \in X} \min\{\rho_A^{\Phi}(x), \rho_U^{\Phi}(x)\}.$$

Thus $\text{ov}_{\Phi}(A, U)$ measures the greatest confidence at which A and U are simultaneously supported. Equivalently, A and U are *α -quasi-coincident in confidence* when this overlap exceeds α . This confidence-based quasi-coincidence avoids the need to choose among several nonequivalent quasi-coincidence conventions while retaining the local viewpoint used in smooth neutrosophic topology [4].

Definition 5.1 (Strong and weak intrinsic density). An SVNNS A is

- (a) *strongly intrinsically N -dense* if $\text{ov}_{\Phi}(A, U) = h_{\Phi}(U)$ for every effective $U \in \mathcal{T}$;
- (b) *weakly intrinsically N -dense* if $\text{ov}_{\Phi}(A, U) > 0$ for every effective $U \in \mathcal{T}$.

Strong intrinsic density asks the data to reach the full confidence height made available by each open set. Weak intrinsic density only asks for a positive confidence contact. The distinction is useful because the former is a genuinely high-confidence requirement, whereas the latter is the direct graded analogue of non-empty intersection.

Proposition 5.2 (Crisp reduction of strong and weak density). *Let (X, τ) be an ordinary topological space and $(X, \hat{\tau})$ its crisp induced NTS. Then an SVNNS A is strongly intrinsically N -dense if and only if it is N -dense in the sense of Definition 3.1. Moreover, A is weakly intrinsically N -dense if and only if its positive support A_{Φ}^{+} is classically dense in X .*

Proof. For every non-empty $U \in \tau$ one has $h_{\Phi}(\hat{U}) = 1$ and $\text{ov}_{\Phi}(A, \hat{U}) = \sup_{x \in U} \rho_A^{\Phi}(x)$. Equality with the height is therefore exactly Definition 3.1. Positivity of the overlap is equivalent to the existence of $x \in U$ with $\rho_A^{\Phi}(x) > 0$, which is equivalent to $U \cap A_{\Phi}^{+} \neq \emptyset$. $\square$

Example 5.3 (Weak density need not be strong). Let $X = \mathbb{Q}$ with its usual topology and let A have constant confidence $1/2$; for instance take $A(x) = (1/2, 0, 0)$ for every $x \in X$ and $\Phi = \Phi_{\min}$. Then $A_{\Phi}^+ = X$, so A is weakly intrinsically N -dense in the crisp-induced NTS. It is not strongly intrinsically N -dense, since its overlap with every non-empty crisp open set is only $1/2$ rather than one.

For a finite $F \subseteq X$ and effective U , define the normalized local selection score

$$\gamma_{\Phi}(A, F; U) = \frac{1}{h_{\Phi}(U)} \max\left(\{\min(\rho_A^{\Phi}(x), \rho_U^{\Phi}(x)) : x \in F\} \cup \{0\}\right)$$

and the unnormalized overlap score

$$\eta_{\Phi}(A, F; U) = \max\left(\{\min(\rho_A^{\Phi}(x), \rho_U^{\Phi}(x)) : x \in F\} \cup \{0\}\right).$$

Then $0 \leq \gamma_{\Phi} \leq 1$. Mathematically the quotient is well-defined for every effective open set. For numerical work, however, division by a very small height can amplify rounding error. We therefore also use the *resolution- δ* convention, with a fixed $\delta > 0$, in which the *selection requirements* are tested only on δ -effective open sets. The input sequences are still required to be strongly or weakly intrinsically N -dense in the sense of Definition 5.1; thus the resolution parameter changes only the family of opens tested by the selection condition, not the admissible input class. The denominator is then bounded away from zero. We write a superscript δ on an intrinsic selection property when this restriction is imposed. With this convention, if $0 \leq \delta_1 \leq \delta_2 < 1$, any property at resolution δ_1 implies the corresponding property at resolution δ_2 . On crisp-induced topologies all non-empty opens have height one, so every choice $\delta < 1$ gives the same theory.

Definition 5.4 (Strong intrinsic selective separability). Let (A_n) range over sequences of strongly intrinsically N -dense SVNSSs. The NTS $(X, \mathcal{T})$ is

- (a) *iNM-separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that $\sup_n \gamma_{\Phi}(A_n, F_n; U) = 1$ for every effective open U ;
- (b) *iNR-separable* if one can choose $x_n \in A_{n,\Phi}^+$ such that, for every effective open U , the supremum of the normalized values $\min(\rho_{A_n}^{\Phi}(x_n), \rho_U^{\Phi}(x_n))/h_{\Phi}(U)$ over n is one;
- (c) *iNH-separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that $\gamma_{\Phi}(A_n, F_n; U) \rightarrow 1$ for every effective open U ;
- (d) *iNS-separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every finite family of effective opens $U_1, \dots, U_m$ and every $\varepsilon > 0$, some n satisfies $\gamma_{\Phi}(A_n, F_n; U_j) > 1 - \varepsilon$ for all $j \leq m$.

Definition 5.5 (Weak intrinsic selective separability). Let (A_n) range over sequences of weakly intrinsically N -dense SVNSSs. The NTS $(X, \mathcal{T})$ is

- (a) *wiNM-separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that $\sup_n \eta_{\Phi}(A_n, F_n; U) > 0$ for every effective open U ;

- (b) *wiNR-separable* if there are points $x_n \in A_{n,\Phi}^+$ such that, for every effective open U ,

$$\sup\left(\{\min(\rho_{A_n}^\Phi(x_n), \rho_U^\Phi(x_n)) : n \in \omega\} \cup \{0\}\right) > 0;$$

- (c) *wiNH-separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that $\eta_\Phi(A_n, F_n; U) > 0$ for all but finitely many n , for every effective open U ;

- (d) *wiNS-separable* if there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every finite family $U_1, \dots, U_m$ of effective opens, some n satisfies $\eta_\Phi(A_n, F_n; U_j) > 0$ for every $j \leq m$.

Proposition 5.6 (Crisp reduction of intrinsic selection). *Let (X, τ) be an ordinary topological space and let $(X, \hat{\tau})$ be its induced crisp NTS. Then iNM , iNR , iNH , and iNS reduce respectively to NM , NR , NH , and NS . Consequently, by Theorem 3.4, they determine exactly the classical M -, R -, H -, and S -separable space classes. The weak properties $wiNM$, $wiNR$, $wiNH$, and $wiNS$ also determine exactly those same classical space classes.*

Proof. Let $U \in \tau$ be non-empty. Since $\rho_U^\Phi = \chi_U$ and $h_\Phi(\hat{U}) = 1$, for every finite $F \subseteq X$ we have

$$\gamma_\Phi(A, F; \hat{U}) = \max(\{\rho_A^\Phi(x) : x \in F \cap U\} \cup \{0\}) = c_\Phi(A, F; U).$$

The analogous one-point formula holds for iNR . Hence the four strong intrinsic definitions agree term by term with NM , NR , NH , and NS , respectively.

For the weak properties, Proposition 5.2 identifies the admissible input sets precisely through their positive supports $D_n = A_{n,\Phi}^+$, which are ordinary dense subsets of X . If a classical M -, R -, H -, or S -selection is made from the D_n , then whenever a selected point lies in a crisp open set U its overlap with $\hat{U}$ is strictly positive. The same selected sets or points therefore satisfy the corresponding weak intrinsic condition. Conversely, apply a weak intrinsic property to the crisp embeddings $\widehat{D}_n$ of an arbitrary sequence of ordinary dense sets. All non-zero overlaps are then equal to 1, so the weak intrinsic condition reduces exactly to the classical M -, R -, H -, or S -selection requirement. This proves all the asserted equivalences. $\square$

The intrinsic theory is not determined by the ordinary support of an SVNS. This is where the new topological layer differs essentially from the transfer theorem.

Example 5.7 (Same support, different strong intrinsic density). Take $X = \{a, b\}$, $\Phi = \Phi_{\min}$ and the NTS $\mathcal{T} = \{\mathbf{0}_N, U, \mathbf{1}_N\}$, where $U(a) = (0.8, 0, 0)$ and $U(b) = (0.2, 0, 0)$. This is a neutrosophic topology because it contains the null and unit sets and is closed under the required operations. Let A have confidences 0.8 at a and 1 at b , and let B have confidences 0.4 at a and 1 at b ; for instance take triples $(r, 0, 0)$ realizing these values. Both A and B have the same positive support X . For the unit open set both reach confidence one. However, $h_\Phi(U) = 0.8$, while $\text{ov}_\Phi(A, U) = 0.8$ and $\text{ov}_\Phi(B, U) = 0.4$. Hence A is strongly intrinsically

N -dense and B is not, although both are weakly intrinsically N -dense. The distinction has no support-topological explanation.

Remark 5.8 (The product aggregator can change local ordering). In Example 5.7 all nontrivial triples have the form $(r, 0, 0)$, so $\Phi_{\min}(r, 1, 1) = r$ and the product aggregator $\Phi_{\Pi}(a, b, c) = abc$ gives the same confidence. Thus that particular example is unchanged by replacing the minimum with the product. In general, however, the ordering can reverse. The triple $C = (0.6, 0.4, 0)$ has evidence coordinates $(0.6, 0.6, 1)$ and therefore $\rho_C^{\Phi_{\min}} = 0.6$ but $\rho_C^{\Phi_{\Pi}} = 0.36$, whereas $D = (0.5, 0, 0)$ has confidence 0.5 for both aggregators. Hence the minimum ranks C above D , while the product ranks D above C . Intrinsic overlap and the identity of optimal local witnesses may therefore depend on Φ , even though the conservative transfer theorem does not.

The preceding remark establishes dependence at the level of local confidence ordering and admissible witnesses, but it does not by itself show that the *class of intrinsically separable NTSs* depends on the aggregator. To make this distinction explicit, whenever the choice of aggregator is under discussion we write iNM_{Φ} , iNH_{Φ} , iNR_{Φ} and iNS_{Φ} for the corresponding intrinsic properties.

Question 5.9 (Aggregator invariance). *Does there exist a non-crisp NTS $(X, \mathcal{T})$ and two admissible aggregators Φ and Ψ —already $\Phi_{\min}$ and Φ_{Π} would be of particular interest—such that $(X, \mathcal{T})$ is iNM_{Φ} -separable but not iNM_{Ψ} -separable? More generally, which conditions on Φ and Ψ guarantee that the strong or weak intrinsic selective classes are invariant under the change of aggregator?*

Example 5.7 also shows why Theorem 3.4 should not be expected in a general NTS: the topology itself carries graded information and therefore changes which uncertain sets are strongly dense.

6. Neutrosophic networks and network selection

We now incorporate network-based selection principles. A classical network on a topological space X is a family $\mathcal{N}$ of subsets such that, whenever $x \in U$ and U is open, some $N \in \mathcal{N}$ satisfies $x \in N \subseteq U$. The classical network-selective framework motivating this section was initiated in [3]. Our aim is to preserve the local graded information without making countability vacuous.

A literal requirement involving every neutrosophic point $x_{t,i,f}$ may force a network to resolve continuum many point profiles. For countable network selection it is more natural to fix a countable dense *calibration set*

$$\mathcal{C} \subseteq [0, 1]^3 \setminus \{(0, 1, 1)\}, \quad (1, 0, 0) \in \mathcal{C}.$$

Our canonical choice is $\mathcal{C}_{\mathbb{Q}} = ([0, 1] \cap \mathbb{Q})^3 \setminus \{(0, 1, 1)\}$.

Definition 6.1. Let $(X, \mathcal{T})$ be an NTS.

- (a) A family $\mathcal{N} \subseteq \mathcal{SVN}(X)$ is a *full neutrosophic network* if for every neutrosophic point p and every $U \in \mathcal{T}$ with $p \in_N U$, there exists $N \in \mathcal{N}$ such that $p \in_N N \preceq_N U$.
- (b) Given a calibration set $\mathcal{C}$, the family $\mathcal{N}$ is a *$\mathcal{C}$ -calibrated neutrosophic network* if the same condition is required only for points $p = x_{t,i,f}$ with $(t, i, f) \in \mathcal{C}$.
- (c) The *$\mathcal{C}$ -calibrated neutrosophic network weight* $\text{nnw}_{\mathcal{C}}(X, \mathcal{T})$ is the least cardinality of a $\mathcal{C}$ -calibrated neutrosophic network.

Every full network is $\mathcal{C}$ -calibrated. The converse need not hold. The countable rational calibration separates two issues that should not be conflated: countability of the network family and continuum many possible numerical descriptions of the same underlying point. More conceptually, this is not merely a technical device but a separation between *spatial resolution* and *data precision*. The underlying set and neutrosophic topology determine which locations and neighbourhood relations must be resolved, whereas $\mathcal{C}$ specifies the precision with which the truth–indeterminacy–falsity profile of a point is tested. Choosing the dense countable set $\mathcal{C}_{\mathbb{Q}}$ keeps these two scales independent: it provides profile tests at arbitrarily fine rational levels while the network is not forced to encode continuum many numerical degrees as separate cardinal requirements. This density is meant as a calibration of numerical precision; without additional regularity assumptions, a $\mathcal{C}_{\mathbb{Q}}$ -calibrated network need not satisfy the full-network condition for every irrational profile. Full networks remain available when the cardinality of the complete point-profile geometry, rather than countable selection, is the object of study.

In the rest of this section $\mathcal{C} = \mathcal{C}_{\mathbb{Q}}$ and we write simply *calibrated network* and $\text{nnw}_{\mathbb{Q}}$.

Definition 6.2. Assume $\text{nnw}_{\mathbb{Q}}(X, \mathcal{T}) \leq \omega$. The NTS $(X, \mathcal{T})$ is

- (a) *M-Nnw-selective* if for every sequence $(\mathcal{N}_n)$ of countable calibrated neutrosophic networks there are finite $\mathcal{F}_n \subseteq \mathcal{N}_n$ such that $\bigcup_n \mathcal{F}_n$ is a calibrated neutrosophic network;
- (b) *R-Nnw-selective* if for every such sequence one can choose $N_n \in \mathcal{N}_n$ so that $\{N_n : n \in \omega\}$ is a calibrated neutrosophic network;
- (c) *H-Nnw-selective* if for every such sequence there are finite $\mathcal{F}_n \subseteq \mathcal{N}_n$ such that, whenever $p = x_{t,i,f} \in_N U$ with $(t, i, f) \in \mathcal{C}_{\mathbb{Q}}$ and $U \in \mathcal{T}$, there is k such that for every $n \geq k$ there exists $N \in \mathcal{F}_n$ satisfying $p \in_N N \preceq_N U$.

Proposition 6.3 (Crisp recovery of calibrated networks). *Let (X, τ) be an ordinary topological space. If $\mathcal{N}$ is a classical network, then $\hat{\mathcal{N}} = \{\hat{N} : N \in \mathcal{N}\}$ is a calibrated neutrosophic network for $(X, \hat{\tau})$. Conversely, every calibrated neutrosophic network consisting only of crisp embedded sets corresponds to a classical network. Consequently, when Definition 6.2 is restricted to crisp networks on $(X, \hat{\tau})$, it is exactly the classical M-, R-, or H-nw-selective property, respectively.*

Proof. Suppose first that $\mathcal{N}$ is a classical network and let $p = x_{t,i,f} \in_N \widehat{U}$ have a calibrated profile. Since the null profile is excluded, $p \in_N \widehat{U}$ implies $x \in U$. Choose $N \in \mathcal{N}$ with $x \in N \subseteq U$. Because $\widehat{N}(x) = \mathbf{1}_N$, we have $p \in_N \widehat{N}$, and $N \subseteq U$ gives $\widehat{N} \preceq_N \widehat{U}$. Thus $\widehat{\mathcal{N}}$ is calibrated.

Conversely, let $\mathcal{N}$ be a calibrated network all of whose members are crisp embeddings, say $\mathcal{N} = \{\widehat{N} : N \in \mathcal{N}\}$. If $x \in U \in \tau$, then the full profile $x_{1,0,0}$ belongs to $\widehat{U}$ and $(1,0,0) \in \mathcal{C}_{\mathbb{Q}}$. Calibration therefore yields some $\widehat{N} \in \mathcal{N}$ with $x_{1,0,0} \in_N \widehat{N} \preceq_N \widehat{U}$. The first relation gives $x \in N$, and the second gives $N \subseteq U$. Hence $\mathcal{N}$ is a classical network.

This equivalence also applies after making selections from a sequence of crisp networks. In the M and R cases it identifies the selected union or selected family with a classical network; in the H case the eventual calibrated-point condition, tested in particular at $x_{1,0,0}$, is exactly the classical eventual network condition. Therefore the three selective properties reduce as claimed. $\square$

The unrestricted full-network versions can still be studied, but their countable instances may be exceptional. The calibrated version is therefore the default object for selective questions, while the full version records the stronger point-profile geometry.

Question 6.4. Which classical cardinal bounds for M -, R -, and H -nw-selectivity survive for calibrated M -, R -, and H -Nnw-selectivity? Can genuinely neutrosophic network degrees produce ZFC examples that do not arise from crisp embeddings?

Question 6.5. Under which hypotheses is H -Nnw-selectivity preserved by finite neutrosophic products, and can failure or preservation phenomena from the classical network-selective program be recovered by crisp calibration? The question is posed relative to the foundational network framework cited above; no classical product-preservation theorem is assumed here.

7. Filter-parametrized selective separability

Aurichi, Maesano and Zdomskyy [\[1\]](#) introduce selection properties parametrized by free filters on ω . Recall that, for a free filter $\mathcal{G}$, a space is classically $S_{\mathcal{G}}$ -separable if for every sequence (D_n) of dense subsets there are finite $F_n \subseteq D_n$ such that $\{n : U \cap F_n \neq \emptyset\} \in \mathcal{G}$ for every non-empty open U . We first retain this index-filter viewpoint and then pass to genuinely single-valued neutrosophic filters on the index set.

Definition 7.1. Let $\mathcal{G}$ be a free filter on ω . A topological space (X, τ) is $NS_{\mathcal{G}}$ -separable if for every sequence (A_n) of N -dense SVN S s there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every non-empty open U and every $\varepsilon > 0$,

$$\{n \in \omega : c_{\Phi}(A_n, F_n; U) > 1 - \varepsilon\} \in \mathcal{G}.$$

Theorem 7.2 (Filter transfer). *Let $\mathcal{G}$ be any free filter on ω and let Φ be any admissible confidence aggregator. Then X is $NS_{\mathcal{G}}$ -separable if and only if it is classically $S_{\mathcal{G}}$ -separable.*

Proof. Suppose first that X is classically $S_{\mathcal{G}}$ -separable, and let (A_n) be a sequence of N -dense SVNss. Choose $\alpha_n \nearrow 1$ and set $D_n = A_{n,\Phi}^{>\alpha_n}$. By Lemma 3.2, each D_n is dense. Choose finite $F_n \subseteq D_n$ so that, for every non-empty open U ,

$$H_U = \{n : F_n \cap U \neq \emptyset\} \in \mathcal{G}.$$

Fix U and $\varepsilon > 0$. Since $\alpha_n \rightarrow 1$, the set $C_\varepsilon = \{n : \alpha_n > 1 - \varepsilon\}$ is cofinite and hence belongs to the free filter $\mathcal{G}$. Therefore $H_U \cap C_\varepsilon \in \mathcal{G}$. If $n \in H_U \cap C_\varepsilon$, choose $x \in F_n \cap U$. Then $x \in D_n$ and hence $c_\Phi(A_n, F_n; U) \geq \rho_{A_n}^\Phi(x) > \alpha_n > 1 - \varepsilon$. Thus

$$H_U \cap C_\varepsilon \subseteq \{n : c_\Phi(A_n, F_n; U) > 1 - \varepsilon\}.$$

By upward closure of $\mathcal{G}$, the set on the right belongs to $\mathcal{G}$, proving $NS_{\mathcal{G}}$ -separability.

Conversely, suppose that X is $NS_{\mathcal{G}}$ -separable and let (D_n) be a sequence of ordinary dense sets. Apply the definition to $A_n = \widehat{D_n}$. Then $A_{n,\Phi}^+ = D_n$. For the resulting finite $F_n \subseteq D_n$ and every non-empty open U , take $\varepsilon = 1/2$. Since crisp confidences are 0 or 1,

$$\{n : c_\Phi(\widehat{D_n}, F_n; U) > 1/2\} = \{n : F_n \cap U \neq \emptyset\}.$$

The left-hand set belongs to $\mathcal{G}$, so (F_n) witnesses classical $S_{\mathcal{G}}$ -separability. $\square$

Consequently, classical results obtained through free filters can be transported to the crisp-topological confidence layer by the same cut argument. In particular, Aurichi, Maesano and Zdomskyy prove under Near Coherence of Filters that, for countable spaces, S -separability, its monotone version, and M -separability are equivalent [1, Theorem 1.1]. Together with Theorem 3.4, the filter transfer theorem shows that the corresponding confidence formulations inherit these classical equivalences whenever the analogous monotone formulation is used. The filter in Theorem 7.2 is still, however, a classical filter of index sets. Earlier notions of neutrosophic filters were investigated by Salama and Alagamy [11]; the single-valued neutrosophic filter framework adopted here is the one developed in [10]. We next make explicit how this framework can enter the selection mechanism itself.

Definition 7.3 (SVN-filter and crisp trace). In the evidence convention of Definition 2.3, a *single-valued neutrosophic filter* on ω is a non-empty family $\mathfrak{F} \subseteq \mathcal{SVN}(\omega)$ which does not contain $\mathbf{0}_N$, is upward closed under $\preceq_N$, and is closed under finite neutrosophic intersections. This is the order-isomorphic reparametrization of the single-valued neutrosophic filter convention used in [10]. Its *crisp trace* is

$$\text{Tr}(\mathfrak{F}) = \{H \subseteq \omega : \widehat{H} \in \mathfrak{F}\}.$$

Proposition 7.4. *The crisp trace of every SVN-filter on ω is a classical filter on ω .*

Proof. Since $\mathfrak{F}$ is non-empty and upward closed, it contains $\mathbf{1}_N$, so $\omega \in \text{Tr}(\mathfrak{F})$. It does not contain $\emptyset$ because $\widehat{\emptyset} = \mathbf{0}_N$. If H, K are in the trace, then $\widehat{H \cap K} = \widehat{H} \sqcap_N \widehat{K} \in \mathfrak{F}$. Finally, $H \subseteq K$ implies $\widehat{H} \preceq_N \widehat{K}$, and upward closure gives K in the trace. $\square$

Let $(X, \mathcal{T})$ be an NTS, let (A_n) be strongly intrinsically N -dense, and let (F_n) be finite selections. For every effective open U put $q_U(n) = \gamma_\Phi(A_n, F_n; U)$ and define the *graded response SVNS* R_U on ω by

$$R_U(n) = (q_U(n), 1 - q_U(n), 1 - q_U(n)).$$

Larger local selection quality therefore gives a larger response in the evidence order.

Definition 7.5 (Intrinsic selection controlled by an SVN-filter). Let $\mathfrak{F}$ be an SVN-filter on ω . The NTS $(X, \mathcal{T})$ is

- (a) *trace-iNS $_{\mathfrak{F}}$ -separable* if for every sequence (A_n) of strongly intrinsically N -dense SVNSs there are finite $F_n \subseteq A_{n,\Phi}^+$ such that, for every effective open U and every $\varepsilon > 0$,

$$\{n : \widehat{q_U(n)} > 1 - \varepsilon\} \in \mathfrak{F};$$

- (b) *graded-iNS $_{\mathfrak{F}}$ -separable* if the selections can be chosen so that $R_U \in \mathfrak{F}$ for every effective open U .

The trace version is conservative with respect to the index filter encoded by $\mathfrak{F}$, whereas the graded version lets the filter act directly on the entire sequence of confidence scores.

Proposition 7.6. *Trace-iNS $_{\mathfrak{F}}$ -separability is exactly the intrinsic filter-selection condition obtained by using the classical filter $\text{Tr}(\mathfrak{F})$ on the threshold sets $\{n : q_U(n) > 1 - \varepsilon\}$.*

Proof. This is immediate from Definition 7.3: a threshold set belongs to $\text{Tr}(\mathfrak{F})$ precisely when its crisp embedding belongs to $\mathfrak{F}$. $\square$

The graded version retains information that the crisp trace deliberately discards: membership is tested on the full response SVNS rather than only on its threshold sets. It is therefore a natural candidate for a fully neutrosophic analogue of filter-controlled selective separability.

Question 7.7. *Which structural assumptions on an SVN-filter $\mathfrak{F}$ make the graded and trace versions of iNS $_{\mathfrak{F}}$ equivalent? In the classical setting, Near Coherence of Filters is used by Aurichi, Maesano and Zdomskyy [1] to relate filter-parametrized selection to S - and M -separability. The graded iNS $_{\mathfrak{F}}$ construction suggests a genuinely graded neutrosophic counterpart of NCF: rather than testing coherence only after passing to crisp index sets, one could require suitable confidence-preserving images of SVN-filters to become coherent at the level of*

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graded responses. Can such a principle be formulated so that it reduces to classical NCF on crisp traces and yields intrinsic NTS analogues of the classical NCF equivalences?

8. Consequences for Fréchet–Urysohn and function spaces

The conservative layer immediately transfers several recent classical results while the intrinsic layer identifies where new work is required.

Corollary 8.1. *For every admissible Φ , every countable Hausdorff Fréchet–Urysohn space is NS-separable and therefore NpM-separable.*

Proof. Aurichi, Maesano and Zdomsky prove that every countable Hausdorff Fréchet–Urysohn space is S -separable [1, Theorem 1.8]. Apply Theorems 3.4 and 4.3. $\square$

Corollary 8.2. *For every admissible Φ , there exists in ZFC a countable Hausdorff Fréchet–Urysohn space without isolated points which is NR-separable but not NH-separable.*

Proof. Bardyla, Maesano and Zdomsky construct in ZFC a countable Hausdorff Fréchet–Urysohn space without isolated points which is not H -separable [2, Theorem 2.1]. Their paper also recalls that separable Fréchet–Urysohn spaces are R -separable. Apply Theorem 3.4. $\square$

For a Tychonoff space T , let $C_p(T)$ denote the space of real-valued continuous functions on T with the topology of pointwise convergence.

Corollary 8.3. *For every admissible Φ and every Tychonoff space T , the space $C_p(T)$ is NS-separable if and only if it is NM-separable.*

Proof. Aurichi, Maesano and Zdomsky prove that, for $C_p(T)$, classical S -separability and M -separability are equivalent [1, Theorem 1.6]. Apply Theorem 3.4. $\square$

The corresponding statements for iNS, iNM or the network properties are no longer automatic because a genuine neutrosophic topology on the function space must first be specified. The computational constructions in [8, 9] provide one practical route for finite or discretized examples.

9. Concluding remarks

The paper separates a conservative confidence lift from an intrinsic neutrosophic-topological extension. The conservative layer is useful because its exact transfer theorems prove that no classical space class is changed artificially. Example 3.6 nevertheless shows that the witnesses are strictly filtered by confidence, which is the operational content of the lift. The invariance of Theorem 3.4 under every admissible aggregator also shows that this conclusion is not an artefact of the minimum operator.

The intrinsic layer now has two density scales. Strong intrinsic density requires full use of the confidence height of every effective open set, while weak intrinsic density only requires positive overlap. These notions are distinct even on a crisp-induced topology for individual SVN_Ss, as Example 5.3 shows. Nevertheless, their associated selection principles recover the expected classical space classes in the crisp case. A resolution parameter $\delta > 0$ makes normalized scores numerically robust by excluding open sets whose confidence height is below the prescribed tolerance.

The comparison in Remark 5.8 shows that local confidence ordering and the identity of preferred witnesses may depend on the chosen admissible aggregator: the minimum and product rules can reverse the ordering of neutrosophic evidence. Whether this local dependence can change the *intrinsic separability class* of a non-crisp NTS is a separate question, isolated in Question 5.9. This distinction is invisible to the conservative transfer theorem and is a natural source of genuinely neutrosophic problems.

For networks, countable calibration should be read as a separation between spatial resolution and numerical profile precision, not only as a device for avoiding continuum-sized families. The calibrated theory retains exact crisp recovery while leaving full neutrosophic networks available for cardinal investigations. Product behaviour, cardinal bounds and genuinely non-crisp examples remain open. The filter theory now has two levels as well: a trace-based form that reduces to an ordinary index filter, and a graded form in which a single-valued neutrosophic filter acts directly on the sequence of local confidence scores.

Several directions remain open. A systematic study should identify aggregator conditions weaker than meet-compatibility that preserve natural neutrosophic products, compare strong and weak density on non-crisp NTSs, and determine when calibrated and full network notions agree. On the filter side, the main problem is to characterize SVN-filters for which graded and trace selection coincide and to formulate a graded neutrosophic analogue of Near Coherence of Filters that reduces to the classical principle on crisp traces. These questions go beyond a formal relabeling of classical properties and appear to be the natural next step for the theory.

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Plithogenic Quasigroups with Application to Decision Making

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Abstract. In this paper, we extend the concept of neutrosophic sets by introducing *plithogenic quasigroups* by associating a quasigroup $(Q, \cdot)$ with a plithogenic structure: attributes V , dominant v_D , contradiction degrees $c_i \in [0, 1]$, and neutrosophic degrees $d(x, v_i)$. The plithogenic product $\cdot_P$ aggregates componentwise via $(1 - c_i)t_N + c_i s_N$ with $t_N = \min$, $s_N = \max$. We prove that $(P(Q), \cdot_P)$ is a quasigroup and its parastrphes; the Latin square property lifts, guaranteeing unique solvability of $a \cdot_P x = b$. Solvability holds iff each component b_i lies in the c_i -weighted interval between a_i and x_i : $c_i = 1$ forces $b_i \geq a_i$ while $c_i = 0$ forces $b_i \leq a_i$. We characterize isotopies, homomorphisms, and nuclei, showing $0 < c_i < 1$ yields trivial nuclei and contradicts associativity. We applied plithogenic quasigroup properties to portfolio selection with $V = \{\text{Risk, Return, Liquidity}\}$, setting $c_{\text{Risk}} = 0.9$ blocks buy signals whereas $c_{\text{Risk}} = 0.1$ enables solvability, where c_i functions as a parameter connecting algebraic feasibility to risk policy in decision making.

Keywords: Plithogenic sets 1; Quasigroups 2; Neutrosophic set 3 (contradiction degree, non - associative algebra, Latin square.)

1. Introduction

Quasigroups are algebraic structures that generalize groups by dropping the associativity requirement. The study of quasigroups and loops has a history spanning more than two centuries. Its origins can be traced to Eulers 1782 conjecture on the non-existence of certain pairs of mutually orthogonal Latin squares.

Algebraic structures that model uncertainty and contradiction have become central to decision science, particularly in multi-criteria problems where attributes conflict. Neutrosophic sets, introduced by Smarandache [29], capture truth, indeterminacy, and falsity degrees, while plithogenic sets [28] refine this by assigning contradiction degrees $c(v_i, v_D) \in [0, 1]$ between each attribute v_i and a dominant attribute v_D . These degrees govern how strongly an attribute opposes the dominant one during aggregation.

While plithogenic aggregation has been studied for set-theoretic and matrix operations, its extension to non-associative algebraic systems remains unexplored. Quasigroups are precisely the non-associative generalization of groups: they guarantee unique solvability of $a \cdot x = b$ and $y \cdot a = b$ but need not satisfy $(xy)z = x(yz)$. This property makes them ideal for modeling operations where order of application matters for example, sequential portfolio adjustments, layered encryption, or step-wise medical treatments.

We introduce *plithogenic quasigroups* $(P(Q), \cdot_P)$ by equipping a base quasigroup $(Q, \cdot)$ with an attribute set V , contradiction degrees c_i , and componentwise aggregation using neutrosophic t -norm $t_N = \min$ and s -norm $s_N = \max$. The plithogenic product is defined as

$$d(x \cdot_P y, v_i) = (1 - c_i) \min\{d(x, v_i), d(y, v_i)\} + c_i \max\{d(x, v_i), d(y, v_i)\},$$

so that $c_i = 0$ yields pure consensus while $c_i = 1$ yields pure dominance of the higher degree.

The theory of neutrosophic soft sets has developed significantly by many authors since it was introduced by Maji [16]. Broumi and Smarandache studied about its interval-valued [?] and intuitionistic [10] neutrosophic soft sets. Oyem et al. [23] introduced the Neutrosopic quasigroup which extends neutrosopic sets properties to non associative algebraic structures called quasigroup. Its Algebraic properties have been studied by Broumi et al. [11, 12] provided its new operations and examined distributivity and associativity. Its applications to an incomplete data were considered by Saha et al. [25].

Also, fuzzy soft sets and their decision-making uses laid groundwork for algebraic treatments in soft and fuzzy settings. Recent developments include Soft Neutrosophic versions of algebraic structures [4-6, 9, 26] and Neutrosophic Soft versions [1, 7, 17, 32], indicating that soft set theory offers a natural generalization of neutrosophic sets to algebra.

Oyem et al. [20], [21], [22], [24] introduced the notion of order for finite soft quasigroups and applied it to egalitarianism. They established several algebraic relationships between the order of a quasigroup and the order of its associated soft quasigroups and in other related work investigated various algebraic properties of soft quasigroups and soft paratrophes and showed that certain operations are not distributive over one another.

In conclusion, we define plithogenic quasigroups, extend classical quasigroup theory to handle multi-valued falsity, indeterminacy, and truth, and demonstrate their application to multi-attribute decision-making under uncertainty.

2. Preliminaries

. This section provides a brief review of the basic concepts and known results on quasigroups and Plithogenic sets.

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2.1. Quasigroups - [3, 13, 14, 30, 31]

Definition 2.1. (Quasigroup) Let Q be a nonempty set and let $(\cdot)$ be a binary operation on Q . The pair $(Q, \cdot)$ is called a *groupoid* or *magma*.

If for all $a, b \in Q$ the equations

$$a \cdot x = b \quad \text{and} \quad y \cdot a = b$$

have unique solutions $x, y \in Q$, then $(Q, \cdot)$ is a *quasigroup*.

A quasigroup $(Q, \cdot)$ for which there exists a unique element $e \in Q$ such that

$$x \cdot e = e \cdot x = x \quad \text{for all } x \in Q$$

is called a *loop*. The element e is the identity element of the loop.

Throughout this paper we write xy for $x \cdot y$, and we adopt the convention that $\cdot$ has lower precedence than juxtaposition. For example, $x \cdot yz$ means $x(yz)$.

Definition 2.2. For a fixed element $x \in Q$, the *left translation* L_x and *right translation* R_x are the maps $Q \rightarrow Q$ defined by

$$yL_x = x \cdot y, \quad yR_x = y \cdot x.$$

A groupoid is a quasigroup if and only if L_x and R_x are permutations of Q for all $x \in Q$. Since L_x and R_x are bijective, their inverses exist. We define the left and right divisions by

$$x \backslash y = yL_x^{-1}, \quad x/y = xR_y^{-1},$$

so that

$$x \backslash y = z \iff x \cdot z = y, \quad x/y = z \iff z \cdot y = x.$$

In the language of universal algebra, a quasigroup can be presented as an algebra $(Q, \cdot, \backslash, /)$ of type $(2, 2, 2)$.

For more on quasigroups, readers can check [3, 30].

Definition 2.3. (Subquasigroup)(Pflugfelder [31])

Let $(Q, \cdot)$ be a groupoid, and let $H \subseteq Q$ with $H \neq \emptyset$. If $(H, \cdot)$ is itself a groupoid under the same operation, then H is called a *subgroupoid* of Q .

If $(Q, \cdot)$ is a quasigroup and $(H, \cdot)$ is a quasigroup, then H is called a *subquasigroup* of Q .

In either case we write $H \leq Q$.

Definition 2.4. (Neutrosophic set) A neutrosophic set A in a universe X is defined by three membership functions

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$$

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle \mid x \in X\}$$

Definition 2.5 (Neutrosophic Quasigroup [23]). Let $N(Q)$ be a neutrosophic set on a nonempty set Q , where each element is expressed as

$$x = \langle x, T(x), I(x), F(x) \rangle, \quad x \in Q,$$

with $T, I, F : Q \rightarrow [0, 1]$ denoting the truth, indeterminacy, and falsity membership functions.

A binary algebraic structure $(N(Q), *)$ is called a **neutrosophic quasigroup** if it satisfies:

- (1) **Closure:** $a * b \in N(Q)$ for all $a, b \in N(Q)$.
- (2) **Latin Square Property:** For all $a, b \in N(Q)$, the equations

$$a * x = b \quad \text{and} \quad y * a = b$$

have unique solutions $x, y \in N(Q)$.

Definition 2.6 (Plithogenic Set [2,15,27-29]). Let U be a universe of discourse and let $P \subseteq U$. A *plithogenic set* P is defined as the quintuple

$$P = (P, V, v_D, d, c)$$

where:

- (1) P is the set of elements;
- (2) $V = \{v_1, v_2, \dots, v_m\}$ is a set of attributes characterizing the elements of P ;
- (3) $v_D \in V$ is the dominant attribute, used as a reference for measuring contradiction;
- (4) $d : P \times V \rightarrow [0, 1]^3$ is the degree of appurtenance function, such that for each $x \in P$ and $v_i \in V$,

$$d(x, v_i) = (t_i(x), i_i(x), f_i(x))$$

denotes the neutrosophic degrees of truth, indeterminacy, and falsity of element x with respect to attribute v_i ;

- (5) $c : V \times V \rightarrow [0, 1]$ is the contradiction degree function, where $c(v_i, v_j)$ measures the degree to which attribute v_i contradicts attribute v_j . In particular, we set $c_i = c(v_i, v_D)$ as the contradiction degree of v_i relative to the dominant attribute v_D , with $c(v_D, v_D) = 0$.

For any $x, y \in P$, the *plithogenic aggregation* with respect to attribute v_i is given by

$$d_{agg}(x, y, v_i) = (1 - c_i) t_N\{d(x, v_i), d(y, v_i)\} + c_i s_N\{d(x, v_i), d(y, v_i)\},$$

where t_N and s_N are a neutrosophic t-norm and s-norm, respectively.

3. Plithogenic Quasigroups

Definition 3.1 (Plithogenic Quasigroup). Let $(Q, \cdot)$ be a quasigroup, $V = \{v_1, \dots, v_m\}$ an attribute set with dominant attribute $v_D \in V$, and $c_i = c(v_i, v_D) \in [0, 1]$ the contradiction degree of v_i relative to v_D . Choose neutrosophic t-norm $t_N = \min$ and s-norm $s_N = \max$. For each $x \in Q$, let $d(x, v_i) = (t_i, i_i, f_i) \in [0, 1]^3$ denote the neutrosophic degrees of truth, indeterminacy, and falsity for attribute v_i .

A *plithogenic quasigroup* over Q is then the structure

$$P(Q) = (Q, V, v_D, d, c, \cdot_P)$$

where the *plithogenic product* $(\cdot_P)$ is defined for $x, y \in P(Q)$ by

- (1) Base: $x \cdot_P y = x \cdot y$ in $(Q, \cdot)$;
- (2) Degrees: for each $v_i \in V$,

$$d(x \cdot_P y, v_i) = (1 - c_i) t_N\{d(x, v_i), d(y, v_i)\} + c_i s_N\{d(x, v_i), d(y, v_i)\}.$$

Theorem 3.2 (Closure of Latin Square Property). *If $(Q, \cdot)$ is a quasigroup, then $(P(Q), \cdot_P)$ is a quasigroup. That is, for any $a, b \in P(Q)$, the equations*

$$a \cdot_P x = b \quad \text{and} \quad y \cdot_P a = b$$

have unique solutions $x, y \in P(Q)$ whenever they are solvable.

Proof. Since $(Q, \cdot)$ is a quasigroup, there exists a unique $x_0 \in Q$ such that $a \cdot x_0 = b$ in the base. Construct $x \in P(Q)$ with base element x_0 .

For each $v_i \in V$, let $a_i = d(a, v_i)$, $b_i = d(b, v_i)$, and set $x_i = d(x, v_i)$. By definition of $\cdot_P$,

$$b_i = (1 - c_i) t_N\{a_i, x_i\} + c_i s_N\{a_i, x_i\}.$$

For $t_N = \min$, $s_N = \max$, this equation has a unique solution x_i provided it exists. The argument for $y \cdot_P a = b$ is symmetric. Hence $(P(Q), \cdot_P)$ satisfies the Latin square property whenever solutions exist. $\square$

Corollary 3.3 (Solvability Condition). *Let $a, b \in P(Q)$ with $a_i = d(a, v_i)$ and $b_i = d(b, v_i)$ for $v_i \in V$. Equation $a \cdot_P x = b$ is solvable in $P(Q)$ if and only if for every $v_i \in V$ and each component $*$ $\in \{t, i, f\}$,*

$$x_i^* = \begin{cases} \frac{b_i^* - (1 - c_i)a_i^*}{c_i} \in [0, 1], & b_i^* \geq a_i^*, \\ \frac{b_i^* - c_i a_i^*}{1 - c_i} \in [0, 1], & b_i^* < a_i^*, \end{cases}$$

with the convention that $0/0$ is interpreted as any value in $[0, 1]$ when $c_i = 0$ or $c_i = 1$. If any component $x_i^ \notin [0, 1]$, the equation has no solution in $P(Q)$.*

Example 3.4. Let $Q = \{e, a, b\}$ with operation given by its Latin square:

$\cdot$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Take attributes $V = \{\text{Growth}, \text{Safety}\}$ with $v_D = \text{Growth}$, $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$. Define elements:

$$x_1 = [e] : G(0.90, 0.10, 0.00), S(0.80, 0.10),$$

$$x_2 = [a] : G(0.60, 0.30, 0.10), S(0.90, 0.10, 0.00),$$

$$x_3 = [b] : G(0.30, 0.40, 0.30), S(0.50, 0.30, 0.20).$$

The plithogenic product yields the following 3×3 Latin square:

TABLE 1. Plithogenic 3×3 Latin square for $(P(Q), \cdot_P)$

$\cdot_P$	x_1	x_2	x_3
x_1	$e: G(0.90, 0.10, 0.00) S(0.80, 0.10)$	$a: G(0.60, 0.30, 0.10) S(0.86, 0.10, 0.04)$	$b: G(0.30, 0.40, 0.30) S(0.62, 0.22, 0.16)$
x_2	$a: G(0.60, 0.30, 0.10) S(0.86, 0.10, 0.04)$	$b: G(0.30, 0.40, 0.30) S(0.62, 0.22, 0.16)$	$e: G(0.90, 0.10, 0.00) S(0.80, 0.10)$
x_3	$b: G(0.30, 0.40, 0.30) S(0.62, 0.22, 0.16)$	$e: G(0.90, 0.10, 0.00) S(0.80, 0.10)$	$a: G(0.60, 0.30, 0.10) S(0.86, 0.10, 0.04)$

Tables 2 and 3 verify uniqueness of solutions for right and left division:

TABLE 2. Right division test: $x_1 \cdot_P x = x_2$

Test x	Base	Growth	Safety	$= x_2$
x_1	e	(0.90, 0.10, 0.00)	(0.80, 0.10)	No
x_2	a	(0.60, 0.30, 0.10)	(0.86, 0.10, 0.04)	Yes
x_3	b	(0.30, 0.40, 0.30)	(0.62, 0.22, 0.16)	No

TABLE 3. Left division test: $y \cdot_P x_1 = x_2$

Test y	Base	Growth	Safety	$= x_2$
$y = x_1$	e	(0.90, 0.10, 0.00)	(0.80, 0.10)	No
$y = x_2$	a	(0.60, 0.30, 0.10)	(0.86, 0.10, 0.04)	Yes
$y = x_3$	b	(0.30, 0.40, 0.30)	(0.62, 0.22, 0.16)	No

Remark 3.5 (Contradiction Degree as Risk Dial). c_i determines how we can combine the attribute v_i . If

$c_i = 0$: strict AND both inputs must be high.

$c_i = 1$: strict OR the higher input prevails.

$0 < c_i < 1$: smooth interpolation between the two. A larger c_i also reduces solvability if $b_i \leq_a i$ by giving weak inputs veto power, making c_i a direct measure of risk tolerance. Corollary 3.3 shows that c_i also governs solvability: large c_i restricts solutions when $b_i < a_i$, acting as a veto. Thus c_i adjust the risk variables into the algebraic structures.

Definition 3.6 (Plithogenic Quasigroup Homomorphism). Let $P(Q_1) = (Q_1, V, v_D, d_1, c, \cdot_{P_1})$ and $P(Q_2) = (Q_2, V, v_D, d_2, c, \cdot_{P_2})$ be two plithogenic quasigroups over base quasigroups $(Q_1, \cdot_1)$ and $(Q_2, \cdot_2)$ with the same attribute set V , dominant attribute v_D , and contradiction degrees c .

A map $\phi : P(Q_1) \rightarrow P(Q_2)$ is a *plithogenic quasigroup homomorphism* if:

- (1) **Base homomorphism:** For all $x, y \in P(Q_1)$,

$$\phi(x \cdot_{P_1} y) = \phi(x) \cdot_{P_2} \phi(y)$$

on the base elements. That is, the restriction $\phi_0 : Q_1 \rightarrow Q_2$ given by $\phi_0(x) = \phi(x)$ is a quasigroup homomorphism.

- (2) **Degree preservation:** For all $x \in P(Q_1)$ and all $v_i \in V$,

$$d_2(\phi(x), v_i) = d_1(x, v_i).$$

That is, ϕ preserves the neutrosophic degrees (t, i, f) for every attribute.

If ϕ_0 is bijective, we call ϕ a *plithogenic quasigroup isomorphism*, and write $P(Q_1) \cong P(Q_2)$.

Corollary 3.7. If $\phi : P(Q_1) \rightarrow P(Q_2)$ is a plithogenic quasigroup homomorphism, then for any $a, b \in P(Q_1)$, the unique solution x to $a \cdot_{P_1} x = b$ is mapped to the unique solution to $\phi(a) \cdot_{P_2} y = \phi(b)$, namely $y = \phi(x)$.

Definition 3.8 (Kernel and Image of a Plithogenic Quasigroup Homomorphism). Let $\phi : P(Q_1) \rightarrow P(Q_2)$ be a plithogenic quasigroup homomorphism as in Definition 3.6 and let $\phi_0 : Q_1 \rightarrow Q_2$ be the induced base homomorphism.

- (1) The *kernel* of ϕ is the substructure of $P(Q_1)$ defined by

$$\ker(\phi) = \{x \in P(Q_1) : \phi(x) = e_2\},$$

where $e_2 \in P(Q_2)$ is a fixed element whose base is the identity of $(Q_2, \cdot_2)$ and $d_2(e_2, v_i) = (1, 0, 0)$ for all $v_i \in V$. If $(Q_2, \cdot_2)$ has no identity, fix any distinguished element $e_2 \in P(Q_2)$.

- (2) The *image* of ϕ is the substructure of $P(Q_2)$ given by

$$\text{Im}(\phi) = \{\phi(x) \in P(Q_2) : x \in P(Q_1)\}.$$

Then $\text{Im}(\phi)$ is a plithogenic subquasigroup of $P(Q_2)$.

First Isomorphism Theorem for Plithogenic Quasigroups. Let $\phi : P(Q_1) \rightarrow P(Q_2)$ be a surjective plithogenic quasigroup homomorphism. Then there exists a plithogenic quasigroup congruence $\sim_\phi$ on $P(Q_1)$ defined by

$$x \sim_\phi y \iff \phi(x) = \phi(y),$$

and we have a plithogenic quasigroup isomorphism

$$P(Q_1)/\sim_\phi \cong P(Q_2).$$

Moreover, $x \sim_\phi y$ if and only if $x \cdot_{P_1} y^{-1} \in \ker(\phi)$ when $(Q_1, \cdot_1)$ is a loop, and $d_1(x, v_i) = d_1(y, v_i)$ for all $v_i \in V$. $\square$

Remark 3.9. Unlike groups, $\ker(\phi)$ in a quasigroup is not automatically a subquasigroup, because quasigroups need not have identities or inverses. However, $\ker(\phi)$ is a *normal subquasigroup* in the sense of quasigroup theory if $(Q_1, \cdot_1)$ is a loop. The degree preservation condition ensures that $\ker(\phi)$ consists of elements that are neutrosophically indistinguishable from the distinguished element e_2 .

Example 3.10 (Plithogenic Quasigroup Homomorphism). Let $P(Q_1)$ be the plithogenic quasigroup from Example 3.4, with base $Q_1 = \{e, a, b\}$ and elements x_1, x_2, x_3 as defined there.

Define a second base quasigroup $(Q_2, \circ)$ by $Q_2 = \{\bar{e}, \bar{a}, \bar{b}\}$ with operation:

$\circ$	$\bar{e}$	$\bar{a}$	$\bar{b}$
$\bar{e}$	$\bar{e}$	$\bar{a}$	$\bar{b}$
$\bar{a}$	$\bar{a}$	$\bar{b}$	$\bar{e}$
$\bar{b}$	$\bar{b}$	$\bar{e}$	$\bar{a}$

which is isomorphic to $(Q_1, \cdot)$ via $\phi_0(e) = \bar{e}$, $\phi_0(a) = \bar{a}$, $\phi_0(b) = \bar{b}$.

Construct $P(Q_2)$ using the same attributes $V = \{\text{Growth}, \text{Safety}\}$ and contradiction degrees $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$. Define elements:

$$y_1 = [\bar{e}] : G(0.90, 0.10, 0.00), S(0.80, 0.10),$$

$$y_2 = [\bar{a}] : G(0.60, 0.30, 0.10), S(0.90, 0.10, 0.00),$$

$$y_3 = [\bar{b}] : G(0.30, 0.40, 0.30), S(0.50, 0.30, 0.20).$$

Define $\phi : P(Q_1) \rightarrow P(Q_2)$ by $\phi(x_i) = y_i$ for $i = 1, 2, 3$. Then ϕ is a plithogenic quasigroup homomorphism because:

- (1) **Base homomorphism:** ϕ_0 is an isomorphism of quasigroups, so $\phi(x_i \cdot_{P_1} x_j) = \phi(x_i) \cdot_{P_2} \phi(x_j)$ for all i, j . For instance,

$$\phi(x_1 \cdot_{P_1} x_2) = \phi(x_2) = y_2 = y_1 \cdot_{P_2} y_2 = \phi(x_1) \cdot_{P_2} \phi(x_2).$$

(2) **Degree preservation:** By construction, $d_1(x_i, v) = d_2(y_i, v)$ for all i and $v \in V$. For example, $d_1(x_1, \text{Growth}) = (0.90, 0.10, 0.00) = d_2(y_1, \text{Growth})$.

Hence $P(Q_1) \cong P(Q_2)$ as plithogenic quasigroups. Moreover,

$$\ker(\phi) = \{x_1\}, \quad \text{because } \phi(x_1) = y_1 = e_2,$$

and $\text{Im}(\phi) = P(Q_2)$.

Example 3.11 (Non-isomorphic Plithogenic Homomorphism). Let $P(Q_1)$ be as in Example 3.4 with x_1, x_2, x_3 . Consider a base quasigroup $(Q_2, *)$ with $Q_2 = \{\hat{e}, \hat{a}\}$ and operation:

$*$	$\hat{e}$	$\hat{a}$
$\hat{e}$	$\hat{e}$	$\hat{a}$
$\hat{a}$	$\hat{a}$	$\hat{e}$

and construct $P(Q_2)$ with the same $V = \{\text{Growth}, \text{Safety}\}$, $c_{\text{Growth}} = 0$, $c_{\text{Safety}} = 0.4$.

Suppose we try to define a base homomorphism $\phi_0 : Q_1 \rightarrow Q_2$ by

$$\phi_0(e) = \hat{e}, \quad \phi_0(a) = \hat{a}, \quad \phi_0(b) = \hat{a},$$

so that a and b collapse to the same base element. For ϕ to lift to a plithogenic quasigroup homomorphism, we must have $d_1(x, v_i) = d_2(\phi(x), v_i)$ for all x and v_i .

But in $P(Q_1)$ we have:

$$d_1(x_2, \text{Growth}) = (0.60, 0.30, 0.10),$$

$$d_1(x_3, \text{Growth}) = (0.30, 0.40, 0.30).$$

Since $\phi(x_2) = \phi(x_3) = z$ for some $z \in P(Q_2)$ with base $\hat{a}$, degree preservation forces

$$d_2(z, \text{Growth}) = (0.60, 0.30, 0.10) = (0.30, 0.40, 0.30),$$

which is impossible. Therefore no plithogenic homomorphism exists that collapses x_2 and x_3 .

If instead we try to collapse x_1 and x_3 by setting $\phi_0(e) = \phi_0(b) = \hat{e}$, we need

$$d_1(x_1, \text{Safety}) = (0.80, 0.10) = d_1(x_3, \text{Safety}) = (0.50, 0.30, 0.20),$$

again impossible.

Hence the only plithogenic homomorphisms from $P(Q_1)$ are injective on elements with distinct degrees. In this example, any ϕ must send x_1, x_2, x_3 to three distinct elements of $P(Q_2)$ with matching degrees. If $P(Q_2)$ has only two elements, no plithogenic homomorphism exists.

Remark 3.12 (Degrees Restrict Homomorphisms). In classical quasigroups, any surjective base homomorphism works. For plithogenic quasigroups, degree preservation is a strong constraint: you cannot identify two elements unless all their (t, i, f) vectors agree for every attribute. Thus c_i and d together turn the algebra into a labeled structure where homomorphisms

must respect the data. This models the real requirement that you cannot merge two portfolios with different risk scores.

Definition 3.13 (Plithogenic Quasigroup Parastrophes). Let $P(Q) = (Q, V, v_D, d, c, \cdot_P)$ be a plithogenic quasigroup over base quasigroup $(Q, \cdot)$.

A quasigroup $(Q, \cdot)$ has exactly six parastrophes determined by permuting the ternary relation $E(x, y, z) \iff x \cdot y = z$. They are:

1. $\cdot$: $x \cdot y = z$ (main operation)
2. $/$: $z/y = x$ (right division)
3. $\backslash$: $x \backslash z = y$ (left division)
4. $\circ$: $y \circ x = z$ (opposite)
5. $/$: $z/x = y$ (opposite right division)
6. $\parallel$: $y \parallel z = x$ (opposite left division)

The *plithogenic quasigroup parastrophes* of $P(Q)$ are the six operations

$$\cdot_P, \quad /_P, \quad \backslash_P, \quad \circ_P, \quad /_P, \quad \parallel_P$$

on $P(Q)$ defined by:

- (1) **Base parastrophes:** For each operation, the base elements satisfy the corresponding parastrophe of $(Q, \cdot)$. E.g., $x/_P y = z$ iff $x \cdot y = z$ in the base.
- (2) **Degree rule:** For all $v_i \in V$, the neutrosophic degrees are computed by

$$d(x *_P y, v_i) = (1 - c_i) t_N\{d(x, v_i), d(y, v_i)\} + c_i s_N\{d(x, v_i), d(y, v_i)\},$$

where $*_P \in \{\cdot_P, /_P, \backslash_P, \circ_P, /_P, \parallel_P\}$, subject to the relation $x *_P y = z$ being consistent with the base parastrophe.

Equivalently, the six operations are characterized by:

$$x \cdot_P y = z \iff z/_P y = x \iff x \backslash_P z = y \iff y \circ_P x = z \iff z/_P x = y \iff y \parallel_P z = x.$$

Theorem 3.14 (Closure of Parastrophes). If $(P(Q), \cdot_P)$ is a plithogenic quasigroup, then $(P(Q), *_P)$ is a plithogenic quasigroup for each parastrophe $*_P \in \{/_P, \backslash_P, \circ_P, /_P, \parallel_P\}$.

Proof. Each base parastrophe $(Q, *)$ is a quasigroup. The degree rule for $*_P$ is identical to $\cdot_P$, so solvability follows from Corollary 3.3 with x, y, z permuted. Uniqueness is inherited from the base. Hence $(P(Q), *_P)$ satisfies the Latin square property. $\square$

Theorem 3.15 (Isotopy Invariance of Plithogenic Structure). Let $(Q_1, \cdot_1)$ and $(Q_2, \cdot_2)$ be isotopic quasigroups via (α, β, γ) , i.e. $\alpha(x) \cdot_2 \beta(y) = \gamma(x \cdot_1 y)$ for bijections $\alpha, \beta, \gamma : Q_1 \rightarrow Q_2$.

Let $P(Q_1) = (Q_1, V, v_D, d_1, c, \cdot_{P_1})$ be a plithogenic quasigroup. Define $P(Q_2)$ with degrees $d_2(\alpha(x), v_i) = d_1(x, v_i)$ for all $x \in Q_1, v_i \in V$.

Then $P(Q_2) = (Q_2, V, v_D, d_2, c, \cdot_{P_2})$ is a plithogenic quasigroup, and the triple (α, β, γ) extends to a plithogenic isotopy $P(Q_1) \simeq P(Q_2)$.

Proof. The base is an isotopy by hypothesis. Degree preservation holds by construction. Since the aggregation rule uses only d and c , and c is fixed, $\gamma(x \cdot_{P_1} y) = \alpha(x) \cdot_{P_2} \beta(y)$. Latin square property follows from Theorem 3.2. $\square$

Theorem 3.16 (Lagrange-type Property for Finite Plithogenic Loops). *Let $P(Q)$ be a plithogenic quasigroup where $(Q, \cdot)$ is a finite loop of order n . Let $P(H)$ be a plithogenic subloop, i.e. $H \leq Q$ and $P(H)$ uses restrictions of d, c .*

Then $|H|$ divides n , and for each $v_i \in V$, the multiset of degrees $\{d(x, v_i) : x \in H\}$ is determined by $|H|$ and d .

Remark 3.17. Unlike groups, $P(H)$ need not partition $P(Q)$. The theorem only restricts $|H|$, not the degree distribution. This is why portfolio constraints must be checked via Corollary 3.3, not just by size.

Theorem 3.18 (Automorphism Group Structure). *Let $\text{Aut}(P(Q)) = \{\phi : P(Q) \rightarrow P(Q) \text{ plithogenic isomorphism}\}$. Then*

$$\text{Aut}(P(Q)) \cong \{\sigma \in \text{Aut}(Q, \cdot) : d(\sigma(x), v_i) = d(x, v_i) \forall x \in Q, v_i \in V\}.$$

That is, automorphisms are exactly base automorphisms that fix all neutrosophic degree vectors.

Proof. By Def. 3.6, any ϕ must preserve base and degrees. Conversely, any base automorphism σ preserving d extends uniquely to $\phi(x) = \sigma(x)$ with same degrees, and ϕ respects $\cdot_P$ because the aggregation rule depends only on d, c . $\square$

Theorem 3.19 (Cancellation and Idempotence Criteria). *Let $P(Q)$ be a plithogenic quasigroup. Let $a \in P(Q)$, then*

- (1) $a \cdot_P x = a \cdot_P y \Rightarrow x = y$ and $x \cdot_P a = y \cdot_P a \Rightarrow x = y$ (cancellation);
- (2) $a \cdot_P a = a$ iff $a \cdot a = a$ in $(Q, \cdot)$, since $d(a \cdot_P a, v_i) = d(a, v_i)$ by idempotence of t_N, s_N .

Hence, if $c_i = 0$ for all i , then $\cdot_P$ is idempotent iff $\cdot$ is idempotent. If $c_i = 1$ for all i , the same holds.

Proof. (1) follows from the Latin square property in Thm. 3.2

For (2), compute degrees: $(1 - c_i)t_N(t, t) + c_i s_N(t, t) = (1 - c_i)t + c_i t = t$, similarly for i, f . Therefore, $d(a \cdot_P a) = d(a)$. So idempotence depends only on base value. $\square$

4. Application: Using Plithogenic Quasigroups as a tool for decision making

We now use Plithogenic Quasigroups properties to model a portfolio manager choosing between three stocks under conflicting criteria. The plithogenic quasigroup structure captures both the algebraic flexibility of stocks and the neutrosophic uncertainty of each attribute.

4.1. Setup: Stocks as Elements of $P(Q)$

Let $Q = \{e, a, b\}$ represent three candidate stocks:

$$e = A, \quad a = M, \quad b = T.$$

$(Q, \cdot)$ is the trivial quasigroup where $x \cdot y$ denotes replace x by y given by the Latin square in Example 3.4. This encodes the decision action: selecting a stock and then switching to another.

Let attributes $V = \{\text{Growth, Safety, Liquidity}\}$ with dominant attribute $v_D = \text{Growth}$. Contradiction degrees reflect analyst policy:

$$c_{\text{Growth}} = 0, \quad c_{\text{Safety}} = 0.4, \quad c_{\text{Liquidity}} = 0.2.$$

$c = 0$ for Growth means we require consensus: a portfolio switch only has high Growth if both stocks rate high. $c = 0.4$ for Safety means we tolerate conflict: the safer stock can dominate.

Expert evaluations give neutrosophic degrees $d(x, v_i) = (t, i, f)$:

Stock	Growth	Safety	Liquidity
$x_1 = A$	(0.90, 0.10, 0.00)	(0.80, 0.10, 0.10)	(0.95, 0.05, 0.00)
$x_2 = M$	(0.60, 0.30, 0.10)	(0.90, 0.10, 0.00)	(0.85, 0.10, 0.05)
$x_3 = T$	(0.30, 0.40, 0.30)	(0.50, 0.30, 0.20)	(0.70, 0.20, 0.10)

This defines the plithogenic quasigroup $P(Q) = (Q, V, v_D, d, c, \cdot_P)$.

4.2. Plithogenic Product as Strategy Composition for Decision Rule

The product $x_i \cdot_P x_j$ models the strategy hold x_i , then switch to x_j . By Theorem 3.2, the result is a unique portfolio with aggregated degrees.

Example: Manager holds A and considers switching to M:

$$x_1 \cdot_P x_2 = M \text{ with degrees } \begin{cases} G : \min\{(0.9, 0.1, 0.0), (0.6, 0.3, 0.1)\} = (0.6, 0.3, 0.1) \\ S : 0.6 \min + 0.4 \max = (0.86, 0.10, 0.04) \\ L : 0.8 \min + 0.2 \max = (0.87, 0.09, 0.04) \end{cases}$$

Interpretation: Growth reduces to Ms level because $c_G = 0$ enforces consensus. Safety increases because $c_S = 0.4$ lets Ms higher Safety dominate.

4.3. Solving for a Target Portfolio

Suppose the investor requires a target profile $b = \text{Target}$ with

$$d(b, G) = (0.60, 0.30, 0.10), \quad d(b, S) = (0.86, 0.10, 0.04).$$

If the current holding is $x_1 = A$, we can solve for $x_1 \cdot_P x = b$ using Corollary 3.3. Table 2 shows the uniqueness of the solution $x = x_2 = M$. So M is the unique switch that meets the target criteria.

If no parameter x_i solves the equation, then the target can not be determined under current parameter c . The manager must either change c_i or varies the target, which is exactly Remark 3.5: c_i is the risk dial.

Conclusion. In this work, we were able to combine plithogenic properties and quasigroups properties to provide a framework where the policy c_i , uncertainty (t, i, f) , and algebra $(\cdot, /, \backslash)$ are inseparable primitives for making decisions that involves uncertainty.

5. Future Work

With the extension of the theory Neutrosophic sets to plithogenic quasigroups, it opens several research directions to other algebraic structures. Future works can aim to unify plithogeny with the major non-associative algebraic theories.

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One Hundred Volumes of Neutrosophic Sets and Systems: a Bibliometric Retrospective and a Neutrosophic Framework for Dispersed Citation Evidence (2013–2026)

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Abstract. Neutrosophic Sets and Systems (NSS) reaches its one-hundredth volume in 2026, thirteen years after its foundation by Smarandache in 2013. No systematic bibliometric retrospective of the journal has been published. This paper documents the complete editorial trajectory of NSS and introduces a neutrosophic framework for a problem that the journal exhibits in an acute form: its articles exist in OpenAlex dispersed across repositories and frequently duplicated with divergent citation counts, so that the citation count of an NSS article is not a crisp quantity but an indeterminate one. We compiled a reproducible corpus of 2,895 articles across 100 volumes (2013–2026) by combining OAI-PMH harvesting, archive crawling and article-level metadata retrieval, and disambiguated 5,570 raw signatures into 4,570 unique authors by union-find. The corpus shows an extraordinary editorial expansion — from 10 articles in 2013 to 1,023 in 2025 — and a methodological profile that differs markedly from that of the applied journal of the ecosystem: multi-criteria decision-making methods appear in only 10.7% of NSS articles, against more than 40% in Neutrosophic Computing and Machine Learning. We extend five classical bibliometric indicators into single-valued neutrosophic triples and add a sixth extension, specific to this corpus, in which the indeterminacy of a citation count is computed from the dispersion among duplicate records of the same article. Beyond documentation, the study is designed as an editorial instrument: its explicit objectives include informing the indexing strategy of NSS, identifying the structural weaknesses that an external review would surface, and returning to the neutrosophic community a reproducible empirical baseline about itself.

Keywords: bibliometrics; neutrosophic logic; single-valued neutrosophic numbers; SVNWA aggregation; Lotka's law; author disambiguation; citation indeterminacy; open-access journals; editorial policy; Neutrosophic Sets and Systems.

1 Introduction

Neutrosophic Sets and Systems is the doyen journal of the editorial ecosystem articulated around neutrosophy, the trivalent extension of fuzzy set theory formalized by Smarandache in 1998 [1, 2]. Founded in 2013, it reaches in 2026 its one-hundredth volume, an anniversary that justifies a systematic look at what the journal has actually published, who has published in it, and what evidence exists about the reach of that production.

Bibliometric analyses of scientific journals fulfil a triple function in an editorial community: they document the historical evolution of a publication, they expose its collaborative structure, and they surface both strengths and systemic biases for the adjustment of editorial policy. When the signatories are the editors themselves, the exercise acquires the additional character of an internal audit, with the advantage of privileged access to the data and the disadvantage of a manifest conflict of interest, which we declare in Section 8.

A companion study recently applied this approach to Neutrosophic Computing and Machine Learning (NCML), the applied-methods journal of the same ecosystem, over 762 articles and 42 volumes [3]. The present paper extends that programme to the central case of the field, whose corpus is almost four times larger, and takes advantage of the comparison between the two journals to test whether the diagnoses formulated for one of them hold for the other.

2 Materials and Methods

2.1 Corpus and data pipeline

The empirical corpus consists of the 2,895 articles published by NSS in 100 volumes between 2013 and 2026. Metadata were obtained by combining three independent routes over the journal's OJS installation: OAI-PMH harvesting in date slices, crawling of the issue archive to enumerate every issue and article, and retrieval of article-level metadata for the records the OAI endpoint did not deliver. The archive crawl fixes the denominator — 102 issues, 2,895 articles — so that the completeness of the metadata can be measured rather than assumed.

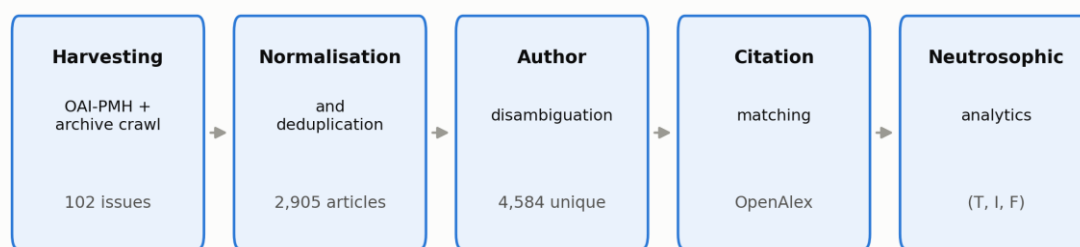


Figure 1. Data pipeline of the study, from OAI-PMH harvesting to the neutrosophic analytics layer.

The 8,646 author–article pairs were disambiguated into 4,570 unique authors by a union-find algorithm with two hierarchical rules: a strong rule based on a canonical key (NFKD normalization, removal of particles and titles, alphabetical ordering of tokens), which resolves inverted forms such as 'Smith, John' and 'John Smith'; and a weak rule based on initial-plus-surname, applied only when the group does not mix two mutually incompatible complete forms. Disambiguation reduced the 5,570 raw signatures by 18.0%.

2.2 Neutrosophic extensions of classical indicators

Five extensions follow the framework introduced for NCML [3]: neutrosophic h-index by inter-source agreement, neutrosophic Lotka exponent by bootstrap over the exponent and the Kolmogorov–Smirnov statistic, graded Bradford nucleus membership, neutrosophic topic membership from distances to cluster centroids, and neutrosophic co-authorship edge weight. In each case the classical indicator is recovered by collapsing the indeterminacy component, so the framework is strictly richer than the classical baseline.

A sixth extension is specific to this corpus. For an article with k bibliographic records $r_1, \dots, r_k$, retrieved from a citation source and citation counts $c_1, \dots, c_k$, define the neutrosophic citation profile (T, I, F) where T grows with the agreement among records, I grows with their relative dispersion

$(c_{max} - c_{min}) / c_{max}$, and F captures the absence of citation evidence. When $k = 1$ the profile collapses to the classical count. This extension is not a formal ornament: as Section 3.4 shows, 37% of the matched articles of NSS appear in more than one record with divergent counts, so the crisp citation count of an NSS article is a choice made by the analyst, not a datum.

Aggregation of the per-indicator triples uses the Single-Valued Neutrosophic Weighted Arithmetic operator of Ye and the score function of Smarandache, as in the companion study.

3 Results

3.1 Editorial growth (RQ1)

Between 2013 and 2026 NSS published 2,895 articles across 100 volumes. The trajectory has three regimes: a founding stage of 10 to 57 articles per year (2013–2017); a first expansion that doubles the volume in 2019 and quadruples it in 2020, coinciding with the migration to OJS; and an acceleration in 2025 without precedent in the ecosystem, with 1,023 articles and 21,215 pages in a single year — more than the first eleven years of the journal combined. The mean article length grew from 6.8 pages in 2013 to over 20 pages from 2020 onward.

Table 1: Editorial growth of NSS. *2026 is partial (cutoff August 2026).

Year	Articles	Pages	Active authors	New authors
2013	10	68	13	13
2014	48	406	23	14
2015	47	345	53	33
2016	57	469	54	34
2017	46	372	117	95
2018	67	783	119	99
2019	132	1591	360	323
2020	247	5101	611	521
2021	239	3448	701	532
2022	213	3109	501	375
2023	327	4905	691	585
2024	298	4288	813	687
2025	1023	21215	2168	1916
2026*	138	2406	327	219

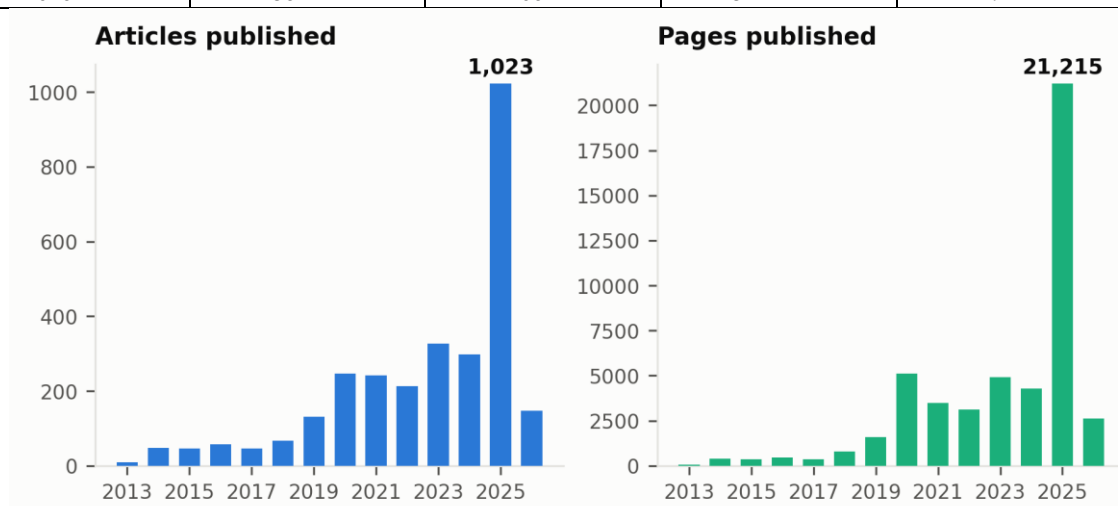


Figure 2. Annual output of NSS, 2013–2026. Articles and pages are shown on separate panels because they are measured on different scales. The 2026 bar is partial (cutoff August 2026).

3.2 Author community and productivity (RQ2)

The corpus comprises 8,646 authorships distributed among 4,570 disambiguated authors, with a mean of 2.99 authors per article and 18.1% of single-authored articles. Productivity is extremely skewed: 70.8% of authors sign a single article, above the 61.7% predicted by Lotka's law. NSS therefore operates in part as an entry platform to the field and not only as the journal of a stable community — a finding that is favorable when read as capacity building and unfavorable when read as author retention, and that the editorial board should read in both directions.

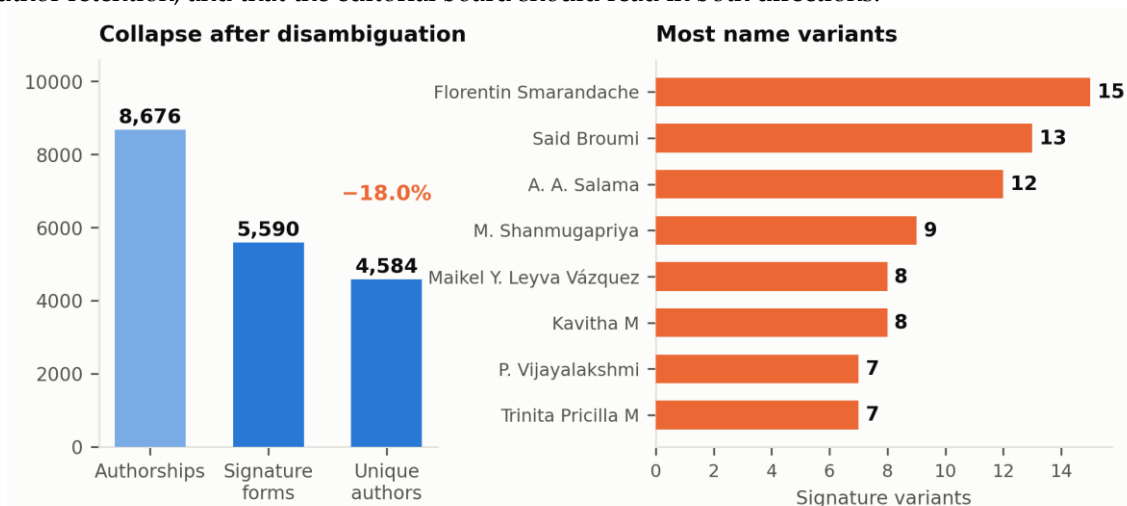


Figure 3. Effect of author-name disambiguation. Left: 8,646 authorships collapse into 5,570 distinct signature forms and then into 4,570 unique authors (−18.0%). Right: authors with the largest number of signature variants.

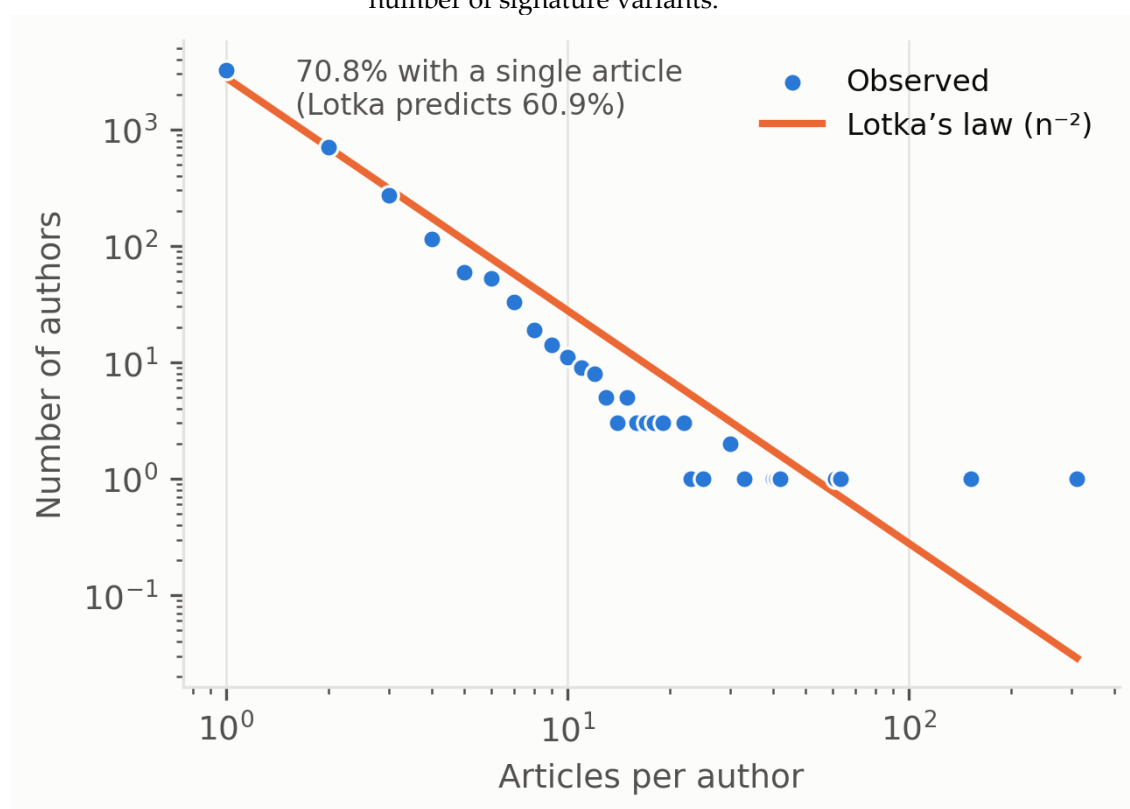


Figure 4. Productivity distribution against Lotka's inverse-square law (log–log). The observed share of single-article authors (70.8%) exceeds the theoretical prediction (61.7%).

Table 2: Ten most prolific authors after disambiguation.

Author	Articles	Signature forms
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Florentin Smarandache	196	15
Said Broumi	152	13
A. A. Salama	63	12
Surapati Pramanik	61	4
Huda E. Khalid	42	5
Saeid Jafari	41	6
Yaser Ahmad Alhasan	40	3
Muhammad Saeed	32	4
Jun Ye	30	2
Suman Das	30	2

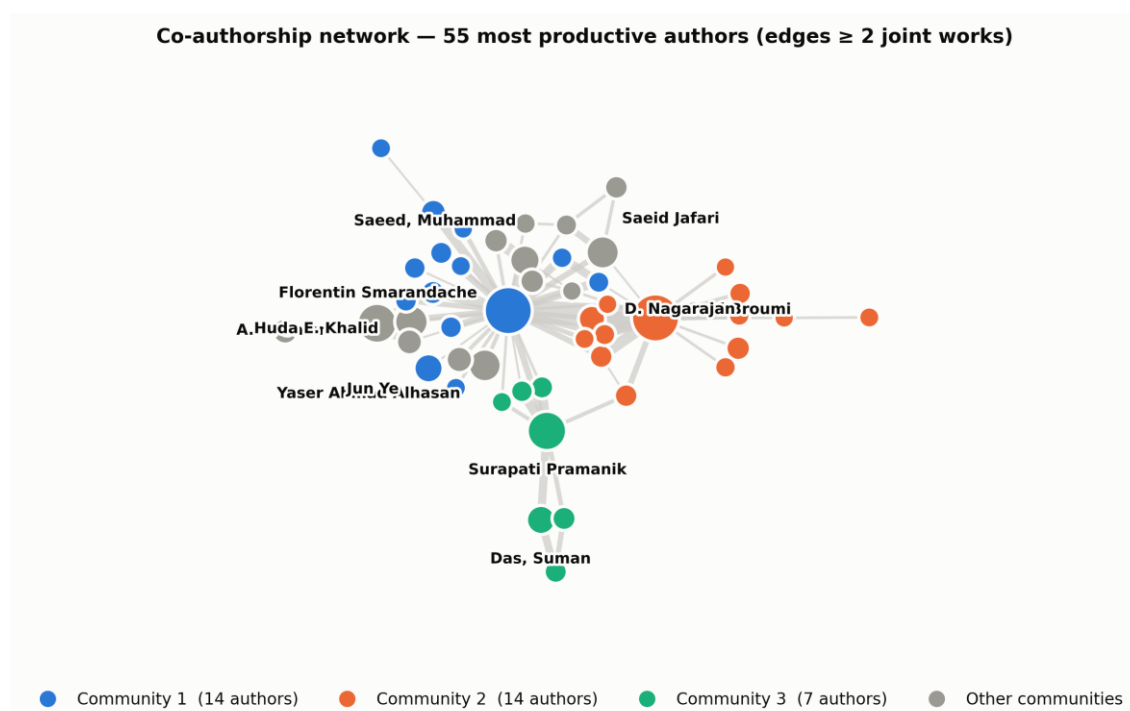


Figure 5. Co-authorship network of the most productive authors after disambiguation. Edges denote at least two joint articles; node area is proportional to output; colour marks the largest Louvain communities.

3.3 Methodological profile (RQ3)

The methodological concentration attributed to the neutrosophic field does not describe NSS. Multi-criteria decision-making methods — the family that dominates the applied journal of the ecosystem — appear in only 10.7% of NSS articles, against more than 40% in NCML. What dominates in NSS is decision theory in the broad sense (33.5%), extensions of set structures such as soft, rough, hypersoft and plithogenic sets (27.8%), algebra (13.9%) and topology (9.2%). TOPSIS, the emblematic method of the critique, appears in 4.2% of the corpus and AHP in 2.4%.

Table 3: Methodological profile of the corpus (non-exclusive categories).

Method family	Articles	% of corpus
Decision theory (broad)	971	33.5
Set extensions	804	27.8
Algebra	403	13.9
Topology	267	9.2
Optimization	182	6.3
TOPSIS	121	4.2
Neutrosophic statistics	118	4.1
Graphs	104	3.6
Machine learning	97	3.4
Cognitive maps	78	2.7
AHP	70	2.4
Any MCDM method	310	10.7

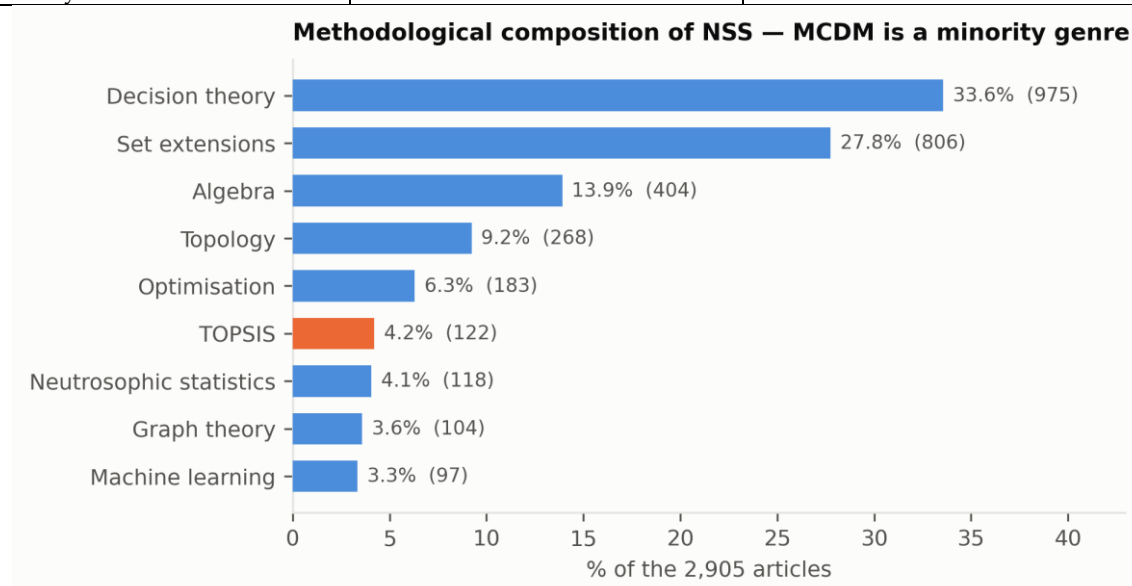


Figure 6. Methodological composition of the corpus, as detected from titles, abstracts and keywords. Multi-criteria decision-making categories are highlighted.

3.4 Citation evidence and its indeterminacy (RQ5)

OpenAlex recognises Neutrosophic Sets and Systems as a source with 27 works and 180 citations. The real corpus is 2,895 articles. The journal is therefore two orders of magnitude larger than its indexed footprint, and any impact assessment based on that source alone is meaningless.

Title-level matching against OpenAlex on a sample of 50 articles recovers 46 of them (92%), which confirms that the articles do exist in the database but are attached to repositories — chiefly the UNM Digital Repository and Zenodo — rather than to the journal. Of the matched articles, 37% appear in more than one record with divergent citation counts: the same article may report 21 citations under one record and 0 under another. The ten most cited articles of the sample accumulate 640 citations, with a maximum of 139 for a single article; the sample as a whole accumulates 1,211 citations against the 180 that the journal-level source reports for the entire journal.

Table 4: Ten most cited articles in the matched sample (OpenAlex, maximum over duplicate records).

Article	Citations
Several Similarity Measures of Neutrosophic Sets	139
An Extended TOPSIS Method for the Multiple Attribute Decision Making Problem	134

Rough Neutrosophic Sets	100
A New Methodology for Neutrosophic Multi-attribute Decision-making	93
Neutrosophic Crisp Set Theory	62
Single Valued Neutrosophic Similarity Measures for MADM	49
Another Form of Correlation Coefficient between SVN Sets	44
Filters via Neutrosophic Crisp Sets	43
A Novel Method for SVN Multi-criteria Decision Making	39
Cosine Similarity Measure of Interval Valued Neutrosophic Sets	37

Table 5: The same journal as recorded by four independent bibliographic sources.

Source	What it holds of NSS	Note
Scopus	1,943 articles (2018–2025)	CiteScore 3.4; #4/43 in Mathematics: Logic, 91st percentile
DOAJ	2,830 article records	97% of the corpus; almost all DOIs are Zenodo deposits
Crossref	no record for the ISSN	the journal is not a registered member
OpenAlex	27 works, 180 citations	source record built largely on Crossref deposits

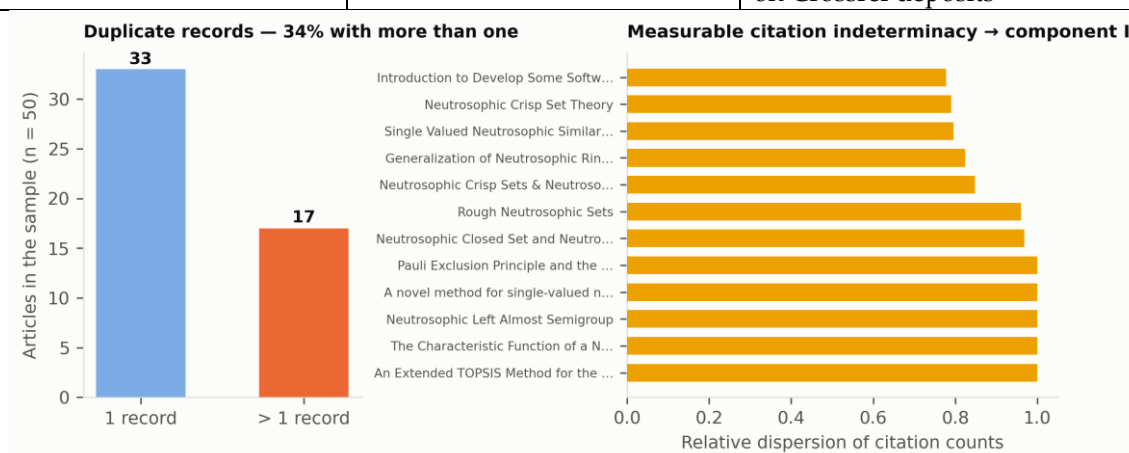


Figure 7. Citation indeterminacy in the matched sample. Left: articles with more than one bibliographic record. Right: relative dispersion of the citation count across duplicate records, which feeds the indeterminacy component I of the sixth neutrosophic extension.

3.5 Thematic agenda and internationalization (RQ4)

The application domains of the corpus are led by environment and sustainability (25.4%), business and management (17.8%), education (16.5%) and pure mathematics (15.4%), followed by health (12.1%) and social sciences (11.5%). The share of pure mathematics, dominant in the founding volumes, declines steadily in favor of applied domains. The corpus declares 6,501 distinct keywords, and five thematic special issues concentrate 212 articles, all of them devoted to Latin America, which documents the incorporation of that region as an editorial policy and not as an accident.

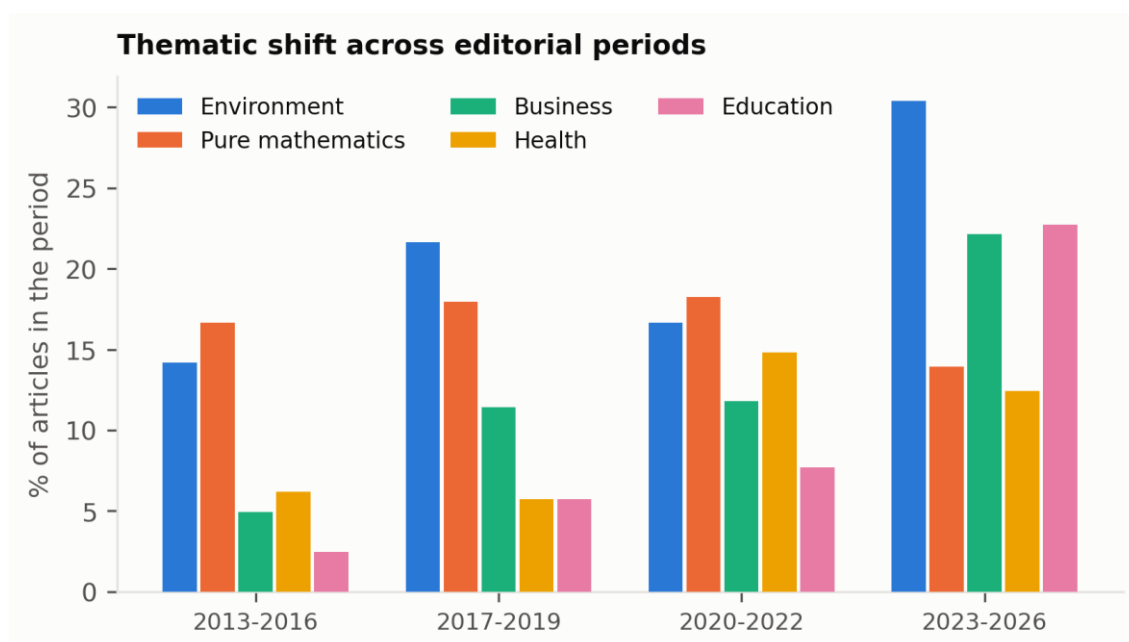


Figure 8. Shift of application domains across the editorial periods, as a percentage of the articles published in each period.

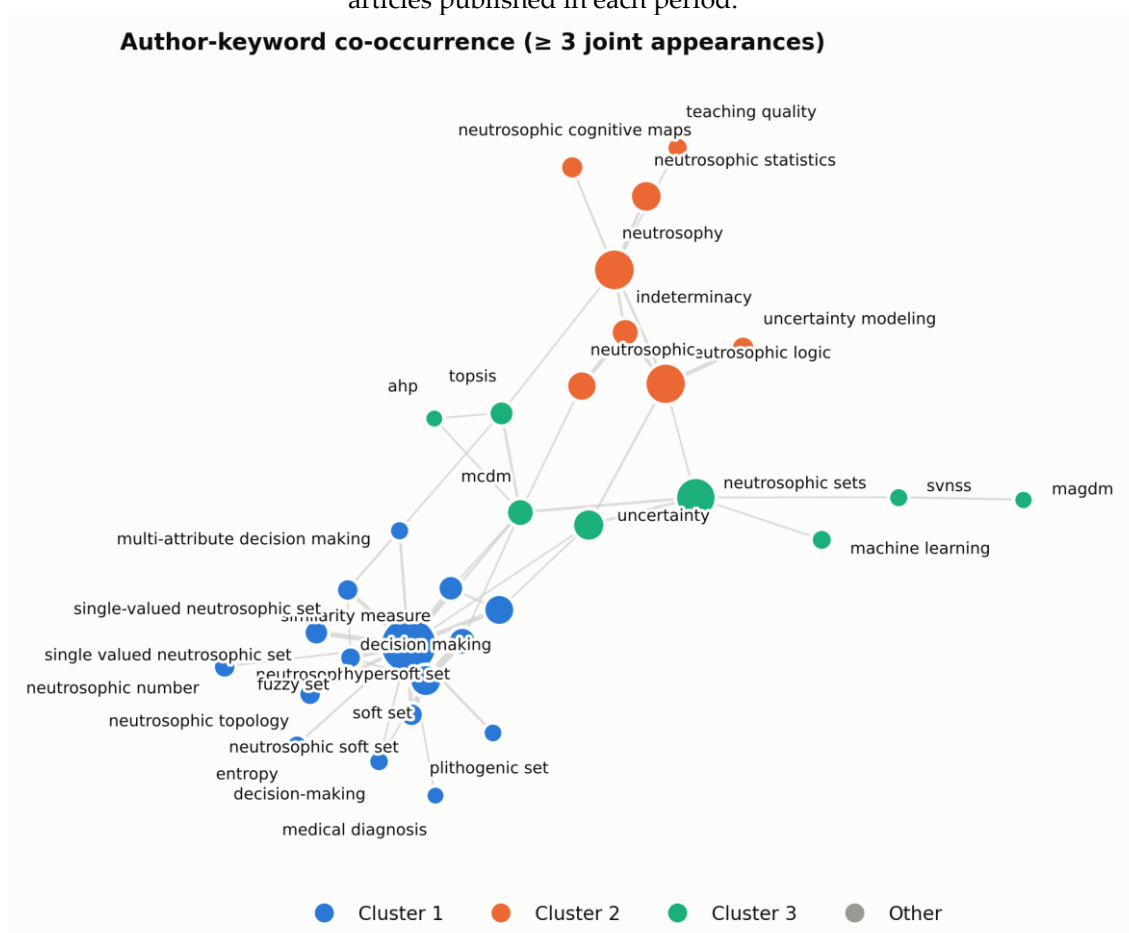


Figure 9. Co-occurrence network of author keywords (at least three joint appearances), coloured by Louvain cluster. Node area is proportional to keyword frequency.

accompaniment, and explicit feedback on rejected submissions — would strengthen the community more than any additional increase in volume.

Finally, the growth of 2025 deserves editorial reflection rather than celebration. Multiplying the annual output by three in a single year is an achievement in reach, but it places the burden of quality control on a review system whose capacity did not grow at the same rate. An anniversary is a good moment to state publicly how that load is being managed.

5 Limitations

The corpus is complete at the article level, but the citation analysis rests, at the time of writing, on a sample of 50 articles, because the search endpoint of the citation source imposes rate limits incompatible with querying 2,895 titles individually; the full matching is being completed through bulk retrieval by repository. Google Scholar, which the companion study identified as the source that best captures neutrosophic citations, has no public API, so the journal-level figures are taken from Scholar Metrics and not from article-level data. Author disambiguation, although validated by manual inspection of the largest clusters, may retain residual errors in homonymous names of frequent surnames. Finally, the classification of methods and domains is based on regular expressions over titles, abstracts and keywords, which underestimates articles whose method is not named in those fields.

6 Conclusions

One hundred volumes and 2,895 articles later, NSS is a substantially larger and methodologically broader journal than its external image suggests. Its central problem is not concentration but visibility: a citation record dispersed across repositories, duplicated, and consequently indeterminate in the precise sense that neutrosophic logic was designed to treat. The framework proposed here does not resolve that dispersion — it measures it, and converts it into a reportable quantity for editorial decision-making.

Author Contributions

Conceptualization, S.B. and F.S.; methodology, M.L.-V. and S.B.; data curation and software, M.L.-V.; formal analysis, S.B., M.A.-B. and M.L.-V.; writing — original draft, S.B. and M.L.-V.; writing — review and editing, F.S. and M.A.-B.; supervision, F.S. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors are editors of the journals analyzed in this study. To mitigate the conflict, the complete dataset, the extraction scripts and the intermediate results are published so that any third party can reproduce or contest the findings. The editorial handling of this manuscript should be assigned to an editor with no authorship in it.

Data Availability and Supplementary Material

The complete dataset accompanies this manuscript as supplementary material (NSS100_dataset_paquete.zip, 1.4 MB) and will be deposited in a public repository upon acceptance. It contains the following items.

NSS_articulos_dataset.csv is the main table: 2,895 rows, one per article, with 25 fields — identifier, volume, year, special issue, title, number of authors, author list, first author, keywords and their count, abstract and its length in words, language, publication date, first and last page, page count, detected method families, MCDM indicator, application domains, article URL and PDF URL, and the metadata route through which each record was obtained.

The reports/ folder contains twelve derived tables: autores_desambiguados.csv, with the 4,570 unique authors and every signature form merged into each of them; evolucion_anual.csv, with articles, pages and active and new authors per year; por_volumen.csv; metodos.csv and metodos_por_periodo.csv, which document the methodological profile and its displacement across four periods; dominios.csv and dominios_por_periodo.csv for the application domains; palabras_clave.csv with the 6,501 distinct descriptors and their frequency; lotka.csv and lotka_desambiguado.csv with the productivity distribution before and after disambiguation;

citas_muestra_openalex.csv with the citation sample, including the number of duplicate records and the relative dispersion of counts per article; numeros_especiales.csv; and coautoria.csv.

The scripts/ folder contain the complete pipeline: date-sliced OAI-PMH harvesting, issue-archive crawling, parallel retrieval of article records, merging, author disambiguation by union-find, descriptive analysis, and bulk citation matching. README_DATASET.md documents the procedure, the known limitations of the automatic classification of methods and domains, and the two server-side obstacles that conditioned the collection: the HTTP 500 errors returned by the journal's OAI endpoint on specific resumption tokens, and the rate limiting that the citation source applies to title-search queries. The pipeline is idempotent and resumable end to end.

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Envelope Geometry and Information Refinement of (T,I,N,F) Neutrosophic Quadruples

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Abstract. The (T,I,N,F) neutrosophic framework separates uncertainty into four components: truth T , pure indeterminacy I , neutrality N , and falsehood F . Each component may be an arbitrary subset of the unit interval, and the four components need not satisfy a normalization condition. This paper develops a structural theory of such quadruples based exclusively on their lower and upper set bounds. For a (T,I,N,F) quadruple, we introduce its lower envelope, upper envelope, envelope width, and envelope center. We establish an exact decomposition of the total envelope width into four componentwise widths and characterize crisp quadruples as precisely those having zero envelope width. An exact geometric representation of all attainable envelope-center-width pairs is derived. We further introduce a componentwise refinement relation and prove that refinement monotonically contracts the global envelope. An envelope-equivalence relation is proposed to identify different set-valued quadruples having identical boundary information. Finally, we show that marginal envelope information alone cannot determine whether the four neutrosophic components are independent, partially dependent, or totally dependent. These results provide a simple mathematical foundation for studying set-valued (T,I,N,F) information without imposing additional operators or normalization assumptions.

Keywords: neutrosophic quadruple; truth; pure indeterminacy; neutrality; falsehood; set-valued uncertainty; envelope width; information refinement; dependence.

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1. Introduction

Fuzzy sets and intuitionistic fuzzy sets provide foundational frameworks for representing graded membership, nonmembership, and hesitation [1, 2]. Interval-valued extensions subsequently replaced exact membership grades with bounded ranges, thereby preserving numerical imprecision explicitly [3, 4]. Neutrosophic sets further generalize these frameworks by representing information through truth, indeterminacy, and falsehood components [5]. Their interval-valued and single-valued realizations were developed in [6, 7].

The refined neutrosophic framework allows the original neutrosophic components to be decomposed into semantically distinct subcomponents [8]. Several four-component formulations have consequently appeared in the literature, including quadripartitioned single-valued neutrosophic sets [9], four-valued refined neutrosophic

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sets [10], and interval quadripartitioned neutrosophic sets [11]. These models commonly divide indeterminacy into components such as contradiction and unknownness or uncertainty.

The (T,I,N,F) representation studied in this paper employs a different semantic decomposition. It extends the classical three-component neutrosophic description by distinguishing *pure indeterminacy* from *neutrality* [12]. In this representation, an assessment is expressed as (T, I, N, F) , where T denotes truth, I pure indeterminacy excluding neutrality, N neutrality, and F falsehood. Each component is taken to be a nonempty subset of the unit interval:

$$T, I, N, F \in \mathcal{P}([0, 1]) \setminus \{\emptyset\}.$$

Consequently, the corresponding lower and upper values are well defined. The basic admissibility condition is

$$0 \leq \inf(T) + \inf(I) + \inf(N) + \inf(F) \leq \sup(T) + \sup(I) + \sup(N) + \sup(F) \leq 4.$$

Furthermore, the four components may be mutually independent, partially dependent, or fully dependent [12, 13].

This generality is one of the principal mathematical features of the quadruple representation. Unlike normalized fuzzy and intuitionistic fuzzy models, the total amount represented by T, I, N, F is not required to equal one. Incomplete information, complete information, and overdetermined or inconsistent information can therefore be represented within a common framework. Moreover, because each component may itself be a set rather than a single number, uncertainty occurs at two distinct levels: uncertainty arising from the four semantic components and uncertainty concerning the numerical values assigned to those components.

Representative studies of interval neutrosophic sets have developed set-theoretic operations, topological constructions, distance and similarity measures, aggregation operators, and multicriteria decision-making methods [6, 14, 15, 16]. Related interval-based constructions have also been investigated for quadripartitioned neutrosophic sets [11]. These studies demonstrate the usefulness of lower and upper component values, but they do not by themselves provide a structural analysis of the aggregate lower–upper envelope of a set-valued (T,I,N,F) quadruple.

In addition, marginal lower and upper bounds and the dependence structure among components describe different aspects of uncertainty. Dependence among neutrosophic components has been studied explicitly [13]; nevertheless, marginal endpoint information alone does not generally determine how the components depend on one another. This distinction motivates a dependence-agnostic analysis based exclusively on componentwise bounds.

The purpose of the present work is therefore to study the numerical set-valued uncertainty of the (T,I,N,F) representation directly. Instead of introducing additional logical operators, aggregation functions, distance measures, or ranking procedures, we investigate what can be deduced solely from $T, I, N, F \subseteq [0, 1]$ and their lower and upper bounds.

The principal contributions of this paper are:

- (1) the introduction of lower and upper envelopes for a (T,I,N,F) quadruple;

- (2) an additive decomposition of the total envelope width into four componentwise widths;
- (3) a characterization of single-valued quadruples through zero envelope width;
- (4) an exact geometric characterization of feasible center–width pairs;
- (5) the introduction of an information-refinement relation;
- (6) a proof that information refinement contracts the uncertainty envelope;
- (7) the introduction of envelope-equivalence classes; and
- (8) a demonstration that marginal envelope information alone cannot identify the dependence structure among T, I, N, F .

Thus, the paper develops a dependence-agnostic structural theory directly from the set-valued (T, I, N, F) formulation. It complements existing operator-, similarity-, and decision-oriented neutrosophic studies by concentrating on the geometry, refinement, and information content of the componentwise uncertainty envelopes.

2. Basic (T, I, N, F) Framework

For mathematical consistency, throughout this article the four components are assumed to be nonempty subsets of $[0, 1]$, so that their infima and suprema are well defined.

Definition 2.1. A *set-valued (T, I, N, F) neutrosophic quadruple* is $Q = (T, I, N, F)$, where $\emptyset \neq T, I, N, F \subseteq [0, 1]$. Write

$$t^- = \inf(T), \quad t^+ = \sup(T),$$

$$i^- = \inf(I), \quad i^+ = \sup(I),$$

$$n^- = \inf(N), \quad n^+ = \sup(N),$$

and

$$f^- = \inf(F), \quad f^+ = \sup(F).$$

Since every component is contained in $[0, 1]$, $0 \leq x^- \leq x^+ \leq 1$ for $x \in \{t, i, n, f\}$. Hence, the original quadruple condition becomes

$$0 \leq t^- + i^- + n^- + f^- \leq t^+ + i^+ + n^+ + f^+ \leq 4.$$

Definition 2.2. The *lower envelope* of Q is $L(Q) = t^- + i^- + n^- + f^-$, whereas its *upper envelope* is $U(Q) = t^+ + i^+ + n^+ + f^+$. Hence, the global envelope associated with Q is

$$\mathcal{E}(Q) = [L(Q), U(Q)].$$

Immediately,

$$0 \leq L(Q) \leq U(Q) \leq 4.$$

3. Envelope Width of a Neutrosophic Quadruple

Definition 3.1. The *envelope width* of Q is defined by $D(Q) = U(Q) - L(Q)$. For each component define

$$d_T = t^+ - t^-, \quad d_I = i^+ - i^-, \quad d_N = n^+ - n^-, \quad d_F = f^+ - f^-.$$

Theorem 3.2. For every set-valued (T, I, N, F) quadruple, $D(Q) = d_T + d_I + d_N + d_F$.

Proof. By definition,

$$\begin{aligned} D(Q) &= U(Q) - L(Q) \\ &= (t^+ + i^+ + n^+ + f^+) - (t^- + i^- + n^- + f^-). \end{aligned}$$

Rearranging gives

$$D(Q) = (t^+ - t^-) + (i^+ - i^-) + (n^+ - n^-) + (f^+ - f^-).$$

Therefore,

$$D(Q) = d_T + d_I + d_N + d_F.$$

□

Corollary 3.3. For every (T, I, N, F) quadruple, $\boxed{0 \leq D(Q) \leq 4}$.

Proof. Since every component is contained in $[0, 1]$,

$$0 \leq d_T, d_I, d_N, d_F \leq 1.$$

The result follows from the Theorem 3.2. □

The maximum value $D(Q) = 4$ is attained when all four components have lower bound 0 and upper bound 1.

4. Characterization of Single-Valued Quadruples

Definition 4.1. A quadruple $Q = (T, I, N, F)$ is called *crisp with respect to component bounds* when every component is a singleton:

$$T = \{t\}, \quad I = \{i\}, \quad N = \{n\}, \quad F = \{f\}.$$

Theorem 4.2. A (T, I, N, F) quadruple is crisp if and only if $D(Q) = 0$.

Proof. Suppose first that Q is crisp. Then,

$$t^- = t^+, \quad i^- = i^+, \quad n^- = n^+, \quad f^- = f^+.$$

Thus,

$$d_T = d_I = d_N = d_F = 0.$$

Hence,

$$D(Q) = 0.$$

Conversely, suppose $D(Q) = 0$. By the Theorem 3.2,

$$0 = d_T + d_I + d_N + d_F.$$

Since each component width is nonnegative,

$$d_T = d_I = d_N = d_F = 0.$$

Therefore,

$$\inf(T) = \sup(T),$$

and similarly for I, N, F . Since each component is nonempty, every component must be a singleton. $\square$

Remark 4.3. The previous theorem gives the envelope width a direct interpretation: positive width represents genuinely set-valued boundary uncertainty, whereas zero width identifies the single-valued case.

5. Center–Width Geometry

Definition 5.1. Define the *envelope center* by $C(Q) = \frac{L(Q)+U(Q)}{2}$. Then $L(Q) = C(Q) - \frac{D(Q)}{2}$ and $U(Q) = C(Q) + \frac{D(Q)}{2}$.

Theorem 5.2 (Envelope geometry). *Every (T, I, N, F) quadruple satisfies*

$$0 \leq C(Q) \leq 4 \text{ and } 0 \leq D(Q) \leq 2 \min\{C(Q), 4 - C(Q)\}.$$

Proof. Since

$$L(Q) = C(Q) - \frac{D(Q)}{2} \geq 0,$$

we obtain

$$D(Q) \leq 2C(Q).$$

Similarly,

$$U(Q) = C(Q) + \frac{D(Q)}{2} \leq 4$$

implies

$$D(Q) \leq 2(4 - C(Q)).$$

Therefore,

$$D(Q) \leq 2 \min\{C(Q), 4 - C(Q)\}.$$

Since $D(Q) \geq 0$, the result follows. $\square$

Thus, the possible (C, D) pairs occupy the region

$$\mathcal{R} = \{(C, D) : 0 \leq C \leq 4, 0 \leq D \leq 2 \min(C, 4 - C)\}.$$

Its upper boundary is

$$D = 2C, \quad 0 \leq C \leq 2,$$

and

$$D = 8 - 2C, \quad 2 \leq C \leq 4.$$

Theorem 5.3 (Realizability). *Every pair $(C, D) \in \mathcal{R}$ can be realized by at least one (T, I, N, F) quadruple.*

Proof. Let

$$L = C - \frac{D}{2}, \quad U = C + \frac{D}{2}.$$

Since $(C, D) \in \mathcal{R}$,

$$0 \leq L \leq U \leq 4.$$

Choose $l_1, l_2, l_3, l_4 \in [0, 1]$ such that

$$l_1 + l_2 + l_3 + l_4 = L.$$

This is possible because $0 \leq L \leq 4$. Since

$$U - L \leq 4 - L = \sum_{j=1}^4 (1 - l_j),$$

there exist numbers δ_j satisfying

$$0 \leq \delta_j \leq 1 - l_j \text{ and } \sum_{j=1}^4 \delta_j = U - L.$$

Set

$$u_j = l_j + \delta_j.$$

Now, we define

$$T = [l_1, u_1], \quad I = [l_2, u_2], \quad N = [l_3, u_3], \quad F = [l_4, u_4].$$

Then,

$$L(Q) = L, \quad U(Q) = U.$$

Therefore,

$$C(Q) = C, \quad D(Q) = D.$$

Hence, the region in the previous theorem gives the exact feasible geometry. $\square$

6. Componentwise Information Refinement

Definition 6.1. Let $Q = (T, I, N, F)$ and $Q' = (T', I', N', F')$. We say that Q' is a *componentwise refinement* of Q , written $Q' \preceq_r Q$, if

$$T' \subseteq T, \quad I' \subseteq I, \quad N' \subseteq N, \quad F' \subseteq F,$$

with all refined components nonempty.

Theorem 6.2 (Refinement contraction). *If $Q' \preceq_r Q$, then $(Q') \geq L(Q)$ and $U(Q') \leq U(Q)$. Consequently, $D(Q') \leq D(Q)$.*

Proof. Since $T' \subseteq T$,

$$\inf(T') \geq \inf(T) \text{ and } \sup(T') \leq \sup(T).$$

The same inequalities hold for I, N, F . Summing the four lower-bound inequalities gives

$$L(Q') \geq L(Q),$$

while summing the upper-bound inequalities gives

$$U(Q') \leq U(Q).$$

Therefore,

$$\begin{aligned} D(Q') &= U(Q') - L(Q') \\ &\leq U(Q) - L(Q) \\ &= D(Q). \end{aligned}$$

Thus,

$$Q' \preceq_r Q \implies \mathcal{E}(Q') \subseteq \mathcal{E}(Q).$$

□

7. Envelope Equivalence

Definition 7.1. Two quadruples $Q_1 = (T_1, I_1, N_1, F_1)$ and $Q_2 = (T_2, I_2, N_2, F_2)$ are called *envelope-equivalent*, written $Q_1 \sim_E Q_2$, whenever $\inf(X_1) = \inf(X_2)$ and $\sup(X_1) = \sup(X_2)$ for every corresponding component $X \in \{T, I, N, F\}$.

Proposition 7.2. If $Q_1 \sim_E Q_2$, then

$$L(Q_1) = L(Q_2), \quad U(Q_1) = U(Q_2), \quad C(Q_1) = C(Q_2), \quad D(Q_1) = D(Q_2).$$

Proof. All four quantities depend only on the componentwise infima and suprema, which are equal by hypothesis. □

Remark 7.3. The converse need not hold. Equal global envelopes do not necessarily imply equality of all componentwise endpoints, since different component contributions may compensate one another.

8. Dependence and a Non-Identifiability Principle

The fundamental (T,I,N,F) framework explicitly allows the components to be totally independent, partially independent and partially dependent, or totally dependent [12]. However, the four marginal sets

$$T, \quad I, \quad N, \quad F$$

do not by themselves specify how dependence is represented.

Proposition 8.1 (Dependence non-identifiability). *Suppose two (T, I, N, F) descriptions have identical marginal component sets but are assigned different admissible dependence structures. Then their lower envelope, upper envelope, center, component widths, and total envelope width are identical.*

Proof. Every quantity introduced in this paper is a function only of

$$\inf(T), \sup(T), \inf(I), \sup(I), \inf(N), \sup(N), \inf(F), \sup(F).$$

None of these quantities contains an explicit dependence term. Hence changing a dependence structure while preserving the four marginal sets does not alter the envelope quantities.

Therefore, the following two notions must be distinguished:

$$\text{marginal set-valued uncertainty and dependence among } T, I, N, F.$$

□

9. Illustrative Examples

Example 9.1 (Single-valued quadruple). Consider $Q_1 = (\{0.70\}, \{0.10\}, \{0.15\}, \{0.05\})$. Then, $L(Q_1) = 0.70 + 0.10 + 0.15 + 0.05 = 1$, and $U(Q_1) = 1$. Therefore, $D(Q_1) = 0$. Hence, Q_1 is crisp.

Example 9.2 (Interval-valued quadruple). Let $T = [0.6, 0.8]$, $I = [0.1, 0.2]$, $N = [0.2, 0.4]$, $F = [0, 0.1]$. Then,

$$L(Q_2) = 0.6 + 0.1 + 0.2 + 0 = 0.9,$$

while

$$U(Q_2) = 0.8 + 0.2 + 0.4 + 0.1 = 1.5.$$

Hence,

$$D(Q_2) = 1.5 - 0.9 = 0.6, \text{ and } C(Q_2) = \frac{0.9 + 1.5}{2} = 1.2.$$

Moreover,

$$d_T = 0.2, \quad d_I = 0.1, \quad d_N = 0.2, \quad d_F = 0.1,$$

so that

$$D(Q_2) = 0.2 + 0.1 + 0.2 + 0.1 = 0.6.$$

Example 9.3 (Refinement). Consider the refinement $T' = [0.65, 0.75]$, $I' = [0.12, 0.18]$, $N' = [0.25, 0.35]$, $F' = [0.02, 0.08]$. Each new component is contained in the corresponding component of the previous example. Hence,

$$Q_3 \preceq_r Q_2.$$

We obtain

$$L(Q_3) = 0.65 + 0.12 + 0.25 + 0.02 = 1.04, \text{ and } U(Q_3) = 0.75 + 0.18 + 0.35 + 0.08 = 1.36.$$

Thus,

$$D(Q_3) = 0.32 < 0.60 = D(Q_2).$$

Example 9.4 (Different sets with the same envelope). Let $T_1 = [0.2, 0.8]$ $T_2 = \{0.2, 0.4, 0.8\}$. Clearly, $T_1 \neq T_2$, but

$$\inf(T_1) = \inf(T_2) = 0.2 \text{ and } \sup(T_1) = \sup(T_2) = 0.8.$$

If the other three components are unchanged, the resulting quadruples are envelope-equivalent even though their truth components have different internal structures.

10. Discussion

Separating indeterminacy from neutrality produces four semantic dimensions, T, I, N, F , rather than three. Because each dimension may itself be an arbitrary subset of $[0, 1]$, a (T, I, N, F) quadruple contains considerably more information than four scalar values.

The analysis developed here reveals several structural facts. First, the basic bound $0 \leq L(Q) \leq U(Q) \leq 4$ naturally generates an interval $[L(Q), U(Q)]$ describing the aggregate boundary range of the quadruple.

Second, the width of this interval is exactly additive: $D(Q) = d_T + d_I + d_N + d_F$. Thus, the spreads of truth, pure indeterminacy, neutrality, and falsehood remain individually visible while contributing to an overall boundary spread.

Third, information refinement has a precise effect:

$$Q' \preceq_r Q \implies D(Q') \leq D(Q).$$

Thus, reducing the admissible values of every component cannot increase the global envelope width.

Fourth, the center-width representation produces an exact global geometry:

$$0 \leq D \leq 2 \min(C, 4 - C).$$

Finally, neither this envelope nor the individual component bounds determine the dependence relationship among T, I, N, F . Dependence therefore represents additional structural information that is not recoverable solely from marginal set boundaries.

11. Conclusion

This paper developed a structural extension of the basic (T, I, N, F) neutrosophic quadruple using only the assumptions $T, I, N, F \subseteq [0, 1]$ and $0 \leq \inf(T) + \inf(I) + \inf(N) + \inf(F) \leq \sup(T) + \sup(I) + \sup(N) + \sup(F) \leq 4$. The lower and upper envelopes $L(Q)$ and $U(Q)$ were introduced, together with the global envelope width $D(Q) = U(Q) - L(Q)$. It was proved that $D(Q) = d_T + d_I + d_N + d_F$, and consequently $0 \leq D(Q) \leq 4$. Moreover, $D(Q) = 0$ was shown to characterize exactly the single-valued case. Using the envelope center $C(Q) = \frac{L(Q) + U(Q)}{2}$, an exact feasible geometry was obtained: $0 \leq D(Q) \leq 2 \min\{C(Q), 4 - C(Q)\}$.

A componentwise refinement relation was also introduced and shown to satisfy

$$Q' \preceq_r Q \implies [L(Q'), U(Q')] \subseteq [L(Q), U(Q)],$$

and therefore

$$D(Q') \leq D(Q).$$

Envelope equivalence was used to distinguish boundary-identical quadruples from genuinely identical set-valued quadruples. Finally, the dependence non-identifiability principle shows that whether T, I, N, F are independent, partially dependent, or totally dependent cannot be inferred solely from their marginal envelopes.

These findings show that even the elementary (T,I,N,F) formulation possesses a rich mathematical structure. The envelope, refinement, equivalence, and dependence questions developed here provide foundations for a more extensive mathematical theory of quadruple-valued neutrosophic information.

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TINF-Consensus for Multi-Annotator Learning: Separating Neutral Responses from Polarized Disagreement

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Abstract. Aggregating multiple annotations into one label usually compresses two different phenomena into the same low-confidence outcome: explicit neutrality, where annotators decline to support either class, and polarized disagreement, where decisive annotators support opposite classes. The distinction matters operationally. Neutral items may need clarification or additional evidence, whereas polarized items may require adjudication, subgroup analysis, or retention of plural labels. We introduce TINF-CONSENSUS, a reliability-weighted binary label-fusion framework built from the truth, indeterminacy, neutrality, falsehood (TINF) representation. For each item, weighted positive, neutral, and negative response masses define truth T , neutrality N , and falsehood F . Pure indeterminacy is then $I = 4TF/(T + F)$, which measures polarization only among decisive responses. A derived consensus coordinate $C = (T - F)^2/(T + F)$ yields the exact decomposition $C + I + N = 1$. Thus, all-neutral and evenly split decisive crowds—which have the same signed mean—receive different profiles. Gold questions provide Beta-smoothed annotator orientation and reliability weights. We establish permutation invariance, an adversarial mass bound, consistency of gold-based orientation, and an exponential conditional error bound.

Deterministic semi-synthetic experiments use 21 annotators, 40 repetitions, and four public binary datasets. The contrast $I - N$ distinguishes constructed polarization from explicit neutrality with macro-average AUROC 0.99999, and the rule $I > N$ routes the reason for review with 0.9928 accuracy. With a nominal 40% adversarial setting (eight of 21 annotators), gold-weighted TINF-CONSENSUS retains 0.9942 macro-average accuracy, compared with 0.5930 for majority vote and 0.9949 for a gold-anchored Dawid-Skene model. Importantly, weighted margin deficit, not indeterminacy, is best for predicting aggregation errors (AUROC 0.9552). The results support a two-output design: margin answers *whether the fused label is fragile*, whereas TINF answers *why the annotation process is unresolved*. The study is a controlled methodological validation; real multi-rater and domain-specific evaluation remains necessary.

Keywords: multi-annotator learning; crowdsourcing; neutrosophic logic; label aggregation; annotator disagreement; abstention; human label variation; robust consensus.

1. Introduction

Supervised learning pipelines commonly treat a dataset label as a primitive fact. In many datasets, however, that label is the endpoint of a social and statistical process: several people inspect an item, some select a class, some abstain, and others disagree. Majority vote reduces this process to one bit. Probabilistic aggregation

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models improve the estimate of a latent class by learning annotator error rates [4, 12], item difficulty [19], or joint worker-label models [21, 8]. Yet a scalar class posterior can still obscure the mechanism that produced uncertainty.

Consider two panels of equal size. In the first, every panelist chooses an explicit neutral response. In the second, half choose positive and half choose negative. Both panels have signed mean zero; both may produce a 0.5 class probability under a symmetric model. They should not automatically trigger the same intervention. The first may indicate insufficient evidence, an ill-posed question, or an appropriate middle position. The second may indicate competing interpretations, population heterogeneity, ambiguous guidelines, or a genuinely contested construct. Conflating them can waste review budgets and, in subjective tasks, erase minority perspectives [16, 11, 3].

This paper develops a narrow operational specialization of the TINF representation-truth, indeterminacy, neutrality, and falsehood [15]. TINF refines a truth assessment into support for truth, pure indeterminacy, neutrality, and falsehood. Its explicit separation of indeterminacy from neutrality is particularly well matched to multi-annotator data, provided that the mapping from responses to coordinates is defined rather than asserted. Our construction makes this mapping observable: positive, neutral, and negative response masses determine T, N, F , while I measures the balance of the two decisive masses.

The resulting method, TINF-CONSENSUS, is designed around two different outputs:

- (i) a reliability-weighted signed margin for label fusion and error screening; and
- (ii) a TINF profile for explaining whether unresolved items are dominated by explicit neutrality or decisive conflict.

This separation is deliberate. A disputed item is not necessarily mislabelled, and a confident consensus is not necessarily correct. We therefore resist the tempting claim that one uncertainty score should optimize both error detection and disagreement diagnosis.

Contributions. The paper makes five contributions.

- (1) We define a reliability-weighted response profile (T, I, N, F) for binary multi-annotator data in which a neutral annotation is a substantive response and missingness remains separate.
- (2) We derive the exact identity $C + I + N = 1$, where C is a consensus coordinate, and prove extreme-case, invariance, and distinguishability properties.
- (3) We give a gold-anchored fusion rule that can identify and invert consistently adversarial annotators, together with finite-sample orientation and label-error bounds.
- (4) We introduce a two-stage review policy: weighted margin deficit ranks likely label errors, while I versus N routes items to conflict-aware or neutrality-aware review.
- (5) We provide a deterministic, executable benchmark with four datasets, five adversarial crowd conditions, 40 repetitions, complete CSV outputs, and a comparison with majority vote and gold-anchored Dawid-Skene.

The experiments are semi-synthetic by design. Ground truth and response mechanisms are known, allowing the diagnostic claims to be tested exactly. They do not replace evaluation on naturally collected annotations.

This boundary is important: the paper establishes a mathematically coherent representation and controlled evidence for its behavior, not universal validity in every annotation domain.

2. Related Work

2.1. *Label aggregation and annotator reliability*

Dawid and Skene’s latent-class model estimates annotator-specific confusion matrices by expectation–maximization [4]. Later work jointly learned classifiers and annotator expertise [12], modeled both annotator competence and item difficulty [19], and developed statistically efficient spectral initialization and EM procedures [21]. Bayesian comparisons show that modeling choices and priors matter in realistic annotation settings [8]. Deep models can incorporate annotator-specific noise layers and learn an end-to-end classifier from crowds [13].

Reliability weighting is especially relevant when some workers are careless or adversarial. Methods for detecting unreliable and adversarial workers often exploit latent structure across tasks and workers [7]. TINF-CONSENSUS occupies a simpler operating point. It uses a small set of known gold items to estimate the sign and magnitude of each annotator’s decisive accuracy, then retains an interpretable item-level decomposition. It is therefore not a replacement for a fully generative crowd model. Rather, it adds a diagnostic representation that standard latent-truth models do not natively expose.

2.2. *Disagreement as signal*

Research on human label variation argues that disagreement is not always annotation noise. Surveys distinguish annotator effects, item ambiguity, task design, and legitimate variation [16]. In natural-language processing, preserving soft labels or annotator distributions can improve learning and analysis [6, 18], perspectivist work warns that majority labels can suppress valid minority judgments [5]. Subjective annotation studies have likewise advocated looking beyond majority vote [3], and Plank [11] frames variation as a property to be modeled rather than merely eliminated. Human-provided class distributions can also improve robustness relative to one-hot labels [10].

Our focus is complementary. We do not infer the sociological cause of a disagreement from votes alone. We preserve a narrower, directly observable distinction between (a) explicit middle or non-committal responses and (b) polarization among decisive responses. Domain interpretation still requires metadata, qualitative investigation, and appropriate governance.

2.3. *From fuzzy and neutrosophic representations to TINF*

Fuzzy sets assign a degree of membership [20]; intuitionistic fuzzy sets add non-membership and hesitation [2]. Neutrosophic representations distinguish truth, indeterminacy, and falsehood and allow their degrees to vary without the normalization imposed by an ordinary probability simplex [14, 17]. The TINF refinement

further separates neutrality from pure indeterminacy [15]. This distinction is the conceptual starting point of our work.

Our contribution is not another free-form semantic score. We define every coordinate from observable response masses, state the assumptions under which those masses are reliability-weighted, and identify the restricted identity satisfied by the resulting operational profile. We found no prior multi-annotator construction with this particular decomposition; the novelty claim is intentionally limited to the construction and analysis given here, not to the broad use of fuzzy or neutrosophic ideas in decision systems.

3. Problem Setting

Let items be indexed by $i \in \{1, \dots, m\}$ and annotators by $j \in \{1, \dots, n\}$. For a binary question, an observed response is

$$R_{ij} \in \{-1, 0, +1\}, \quad (1)$$

where $+1$ supports the candidate-positive label, -1 supports its alternative, and 0 is an *explicit neutral response*. A response that was never collected is denoted by $\perp$ and is excluded from all normalizations. This separation is essential: coding missing values as zero would manufacture neutrality.

Each annotator has a positive weight w_j . If gold questions are available, we estimate these weights as described in [Section 5](#); otherwise equal or externally specified weights can be used. Let

$$\mathcal{A}_i = \{j : R_{ij} \neq \perp\}, \quad W_i = \sum_{j \in \mathcal{A}_i} w_j.$$

An item with $W_i = 0$ has no annotation evidence and is deferred. This case is different from an item for which all observed responses are explicitly neutral.

We address three research questions:

RQ1:

Can an observable TINF profile distinguish explicit neutrality from decisive polarization when both have zero signed margin?

RQ2:

Can simple gold-based orientation and reliability weighting preserve label accuracy in the presence of consistently adversarial annotators?

RQ3:

Should the same scalar be used both to detect likely label errors and to diagnose the reason an item needs review?

4. A TINF Profile for Crowd Responses

4.1. Response masses

Suppose for the moment that responses have already been corrected for an annotator's estimated orientation, giving $X_{ij} \in \{-1, 0, +1\}$. For item i , define

$$T_i = \frac{1}{W_i} \sum_{j \in \mathcal{A}_i} w_j \mathbb{I}\{X_{ij} = +1\}, \quad (2)$$

$$N_i = \frac{1}{W_i} \sum_{j \in \mathcal{A}_i} w_j \mathbb{I}\{X_{ij} = 0\}, \quad (3)$$

$$F_i = \frac{1}{W_i} \sum_{j \in \mathcal{A}_i} w_j \mathbb{I}\{X_{ij} = -1\}. \quad (4)$$

Thus $T_i + N_i + F_i = 1$. The decisive mass and signed margin are

$$D_i = T_i + F_i = 1 - N_i, \quad M_i = T_i - F_i. \quad (5)$$

Definition 4.1 (Pure decisive indeterminacy). For $D_i > 0$, the item's pure indeterminacy is

$$I_i = \frac{4T_i F_i}{D_i}. \quad (6)$$

If $D_i = 0$, set $I_i = 0$.

The numerator in Equation (6) is large only when both decisive alternatives receive substantial mass. Dividing by D_i keeps I_i on the same item-level scale as the response masses. Unlike entropy over (T_i, N_i, F_i) , I_i does not count neutrality itself as polarization.

Definition 4.2 (Consensus coordinate). For $D_i > 0$, define

$$C_i = \frac{M_i^2}{D_i} = \frac{(T_i - F_i)^2}{T_i + F_i}. \quad (7)$$

If $D_i = 0$, set $C_i = 0$.

C_i is not a fifth neutrosophic component. It is a derived certificate of agreement that makes the allocation of one unit of response evidence explicit.

Theorem 4.3 (Exact evidence decomposition). *For every item with at least one observed response,*

$$C_i + I_i + N_i = 1. \quad (8)$$

Moreover, $T_i, I_i, N_i, F_i, C_i \in [0, 1]$ and $I_i \leq D_i$.

Proof. If $D_i = 0$, then $(T_i, N_i, F_i) = (0, 1, 0)$, so the identity is immediate. If $D_i > 0$, then

$$C_i + I_i = \frac{(T_i - F_i)^2 + 4T_i F_i}{D_i} = \frac{(T_i + F_i)^2}{D_i} = D_i.$$

Adding $N_i = 1 - D_i$ gives Equation (8). The response masses are normalized nonnegative sums. Finally, $4T_i F_i \leq (T_i + F_i)^2 = D_i^2$, hence $0 \leq I_i \leq D_i \leq 1$; the same identity establishes the bound for C_i . $\square$

TABLE 1. Canonical equal-weight response configurations. Both the all-neutral and balanced-decisive panels have signed margin $M = 0$, but the TINF profile identifies different mechanisms.

Panel	T	I	N	F	M	C
Unanimous positive	1	0	0	0	1	1
Unanimous negative	0	0	0	1	-1	1
All explicitly neutral	0	0	1	0	0	0
Half positive, half negative	0.5	1	0	0.5	0	0
Half positive, half neutral	0.5	0	0.5	0	0.5	0.5

Proposition 4.4 (Extremes and symmetry). *For fixed decisive mass $D_i > 0$, $I_i = 0$ if and only if $T_i F_i = 0$, and $I_i = D_i$ if and only if $T_i = F_i = D_i/2$. Interchanging the two class labels swaps T_i and F_i while leaving I_i, N_i, C_i unchanged.*

Proof. The first statement follows from Equation (6). Equality in $4T_i F_i \leq (T_i + F_i)^2$ holds exactly when $T_i = F_i$. The symmetry statement follows because Equation (6) is symmetric in T_i, F_i and Equation (7) squares their difference. $\square$

Proposition 4.5 (Information lost by signed-mean aggregation). *No statistic that is a function only of the weighted signed mean M_i can distinguish an all-neutral panel from an evenly weighted panel split between $+1$ and -1 . The profiles in Equations (2) to (4) and (6) distinguish them as $(0, 0, 1, 0)$ and $(1/2, 1, 0, 1/2)$, respectively.*

Proof. Both panels have $M_i = 0$, so every function of M_i takes the same value. Direct substitution gives the two profiles. $\square$

Remark 4.6 (Operational specialization of TINF). The source TINF formalism permits dependent components whose sum need not equal one [15]. Our T, N, F are normalized response masses, while I is derived from T, F ; consequently the quadruple is a restricted, data-generating specialization and $T + I + N + F$ can exceed one. Also, binary label reversal maps (T, I, N, F) to (F, I, N, T) . This is an operational relabelling map, not the source formalism's logical complement (F, N, I, T) ; the two should not be conflated.

4.2. Invariance and corruption stability

Proposition 4.7 (Invariance). *The profile is invariant to any permutation of annotators and to multiplying all observed weights for an item by the same positive constant.*

Proof. Each coordinate depends only on weighted category sums divided by their common total. Reordering terms or applying common positive scaling leaves the ratios unchanged. $\square$

Theorem 4.8 (Fixed-weight response corruption). *Normalize the observed annotator weights for an item to sum to one. If the observed set is fixed and responses with total normalized weight at most ε are changed arbitrarily among $\{-1, 0, +1\}$ while the weights remain fixed, then*

$$\|(T', N', F') - (T, N, F)\|_1 \leq 2\varepsilon, \quad |M' - M| \leq 2\varepsilon. \quad (9)$$

Therefore the fused label $\text{sign}(M)$ is unchanged whenever $|M| > 2\varepsilon$.

Proof. Changing one response moves its normalized weight from one of the three categories to another, changing the mass-vector ℓ_1 norm by twice that weight. Summation over changed responses gives the first bound. Since a signed response lies in $[-1, 1]$, changing it alters its contribution to M by at most twice its normalized weight, giving the second bound. A sign can change only if the perturbation reaches or crosses zero. $\square$

The theorem controls the linear masses and decision margin without a distributional assumption. We do not claim a uniform Lipschitz bound for I near $D = 0$; any such bound would require a lower bound on decisive mass. This qualification prevents the nonlinear diagnostic from being mistaken for a generic robust estimator.

5. Gold-Anchored Reliability and Fusion

5.1. Annotator orientation and weight

Let $\mathcal{G}$ be a set of gold items with known signed labels $Y_i \in \{-1, +1\}$. For annotator j , count only decisive gold responses:

$$c_j = \sum_{i \in \mathcal{G}} \mathbb{I}\{R_{ij} = Y_i\}, \quad e_j = \sum_{i \in \mathcal{G}} \mathbb{I}\{R_{ij} = -Y_i\}.$$

Under a Beta(1,1) prior for decisive accuracy, the posterior mean is

$$\hat{a}_j = \frac{c_j + 1}{c_j + e_j + 2}. \quad (10)$$

Define orientation and reliability weight by

$$s_j = \begin{cases} +1, & \hat{a}_j \geq 1/2, \\ -1, & \hat{a}_j < 1/2, \end{cases} \quad w_j = \max\{\eta, |2\hat{a}_j - 1|\}, \quad (11)$$

where $\eta > 0$ is a small floor. Corrected responses are $X_{ij} = s_j R_{ij}$. An annotator who is consistently wrong on objective gold items is therefore inverted rather than merely discarded.

The construction does not punish a legitimate neutral response as an incorrect class. However, a worker with no decisive gold responses has $\hat{a}_j = 1/2$ and receives only the floor weight. If task designers also need to model propensity to abstain, that behavior should be reported as a separate coverage parameter rather than folded into decisive accuracy.

Theorem 5.1 (Gold-orientation error). *Suppose annotator j gives g_j independent decisive gold responses, each correct with probability $\theta_j \neq 1/2$. Then the probability that Equation (11) selects the orientation opposite to $\text{sign}(\theta_j - 1/2)$ is at most*

$$\exp[-2g_j(\theta_j - \frac{1}{2})^2]. \quad (12)$$

As $g_j \rightarrow \infty$, s_j is consistent and $w_j \rightarrow \max\{\eta, |2\theta_j - 1|\}$ almost surely.

Proof. The posterior mean is at least $1/2$ exactly when $c_j \geq e_j$. For $\theta_j > 1/2$, a wrong orientation requires the empirical correct fraction to fall below $1/2$; for $\theta_j < 1/2$, it requires the fraction to reach at least $1/2$. Hoeffding's inequality gives Equation (12). The strong law of large numbers yields $c_j/g_j \rightarrow \theta_j$, and the remaining limits follow by continuity away from $1/2$. $\square$

5.2. Prediction and a conditional crowd bound

TINF-CONSENSUS predicts

$$\hat{Y}_i = \text{sign}(M_i), \quad (13)$$

with exact ties deferred or resolved using a predeclared class prior. The full (T_i, I_i, N_i, F_i) profile is retained alongside the label.

Theorem 5.2 (Conditional fusion error). *For a fixed item with true signed label Y , suppose corrected annotator responses $X_j \in [-1, 1]$ are conditionally independent and satisfy $\mathbb{E}[YX_j] \geq \gamma_j > 0$. Then*

$$\mathbb{P}\left(\text{sign}\left(\sum_j w_j X_j\right) \neq Y\right) \leq \exp\left[-\frac{\left(\sum_j w_j \gamma_j\right)^2}{2 \sum_j w_j^2}\right]. \quad (14)$$

Proof. Let $Z_j = w_j Y X_j \in [-w_j, w_j]$ and $\mu = \sum_j \mathbb{E}Z_j \geq \sum_j w_j \gamma_j$. A label error implies $\sum_j Z_j \leq 0$. Hoeffding's inequality gives

$$\mathbb{P}\left(\sum_j (Z_j - \mathbb{E}Z_j) \leq -\mu\right) \leq \exp\left(-\frac{2\mu^2}{\sum_j (2w_j)^2}\right),$$

which yields Equation (14). $\square$

Conditional independence is a transparent analytical assumption, not a claim about all human panels. Shared training, communication, or common cultural perspectives can correlate responses and weaken the bound. The empirical profile remains computable under dependence.

Algorithm 1 (TINF-CONSENSUS with objective gold items). (1) **Input:** response matrix R with entries in $\{-1, 0, +1, \perp\}$; gold set $\mathcal{G}$ and signed labels $Y_{\mathcal{G}}$; weight floor η .
 (2) For each annotator j , count decisive gold agreements c_j and errors e_j ; set $\hat{a}_j \leftarrow (c_j + 1)/(c_j + e_j + 2)$, $s_j \leftarrow +1$ if $\hat{a}_j \geq 1/2$ and -1 otherwise, and $w_j \leftarrow \max\{\eta, |2\hat{a}_j - 1|\}$.
 (3) For each non-gold item i , set $X_{ij} \leftarrow s_j R_{ij}$ for observed responses and compute T_i, N_i, F_i using Equations (2) to (4).

- (4) Set $D_i \leftarrow T_i + F_i$ and $M_i \leftarrow T_i - F_i$; compute I_i and C_i by [Equations \(6\) and \(7\)](#).
 (5) **Return:** $\hat{Y}_i = \text{sign}(M_i)$ and $(T_i, I_i, N_i, F_i, C_i, M_i)$.

6. Two-Stage Review: Whether and Why

Label triage involves at least two decisions. First, which fused labels are most likely to be wrong? Second, what kind of review should those items receive? TINF-CONSENSUS uses separate statistics.

Error screening. The weighted margin deficit

$$E_i = 1 - |M_i| \quad (15)$$

ranks items whose weighted class decision is fragile. A low decisive mass or a close decisive split can both reduce $|M_i|$, which is appropriate when the goal is to predict an error in [Equation \(13\)](#).

Reason routing. Among items selected for process review, use

$$\rho_i = \begin{cases} \text{conflict route,} & I_i > N_i, \\ \text{neutrality route,} & I_i \leq N_i. \end{cases} \quad (16)$$

A conflict route may request independent adjudication, inspect annotator subgroups, or retain a distribution of labels. A neutrality route may request additional evidence, revise instructions, or explicitly keep the neutral class. If a single budget score is required, a declared utility parameter $\beta \in [0, 1]$ gives

$$U_i(\beta) = \beta I_i + (1 - \beta) N_i. \quad (17)$$

The parameter is a policy choice, not an estimated universal constant.

For objective gold tasks, error screening can precede reason routing. For subjective or value-laden tasks, “error” may be the wrong construct: equal/contextual weights and distribution-preserving outputs may be preferable. In particular, inverting an annotator because that person differs from a majority-defined “gold” label would erase perspective rather than correct noise. The gold procedure is therefore restricted to settings with defensible external labels.

7. Experimental Evaluation

7.1. Goals and datasets

The experiments test controlled mechanisms, not application-specific performance. Four binary tasks bundled with scikit-learn were selected so the study can run without downloading data [9]. Their diversity in sample size, dimension, and class balance provides several response difficulty landscapes. Dataset features are used only to construct a fixed simulator difficulty; none of the aggregation methods observes them.

7.2. Semi-synthetic annotator model

For each task, standardized features are fitted with logistic regression. If p_i is its fitted positive probability, boundary proximity is

$$q_i = 1 - |2p_i - 1| \in [0, 1]. \quad (18)$$

TABLE 2. Binary benchmark tasks. Positive and negative counts refer to the class encoding used in the experiment.

Task	Items	Features	Positive	Negative
Breast cancer	569	30	357	212
Digits 3 versus 8	357	64	174	183
Iris versicolor versus virginica	100	4	50	50
Wine class 0 versus rest	178	13	59	119

This quantity is available only to the response simulator. The crowd has 21 annotators with base decisive accuracies

$$\theta = (0.95 \times 3, 0.85 \times 5, 0.72 \times 8, 0.58 \times 5).$$

On item i , annotator j has effective decisive accuracy

$$\theta_{ij}^{\text{eff}} = \frac{1}{2} + (\theta_j - \frac{1}{2})(1 - 0.55q_i), \quad (19)$$

so hard items shrink all annotators toward chance.

The neutral-response probability depends on a known experimental regime:

$$\begin{aligned} p_i^{(0)} &= \min\{0.65, 0.03 + 0.35q_i\} && \text{(regular),} \\ p_i^{(N)} &= \min\{0.90, 0.65 + 0.20q_i\} && \text{(explicit neutrality),} \\ p_i^{(I)} &= \min\{0.15, 0.02 + 0.08q_i\} && \text{(perspective conflict).} \end{aligned} \quad (20)$$

For conflict items, ten annotators target the dataset label and eleven target its opposite, after which their individual decisive error is sampled by Equation (19). This generates polarized panels without asserting that one perspective is normatively invalid.

Each repetition uses a stratified 30% gold split and a 70% evaluation split. All reported summaries use 40 deterministic repetitions with predeclared seeds.

7.3. Experiment A: adversarial robustness

We use nominal adversarial settings $a \in \{0, 0.1, 0.2, 0.3, 0.4\}$ and replace the nearest whole number of annotators with workers whose decisive accuracy is 0.15. For 21 annotators, the settings correspond to 0, 2, 4, 6, and 8 workers. The evaluated methods are:

Majority vote::

neutral responses contribute zero; exact ties use the majority class of the gold split.

Gold-anchored Dawid–Skene::

a binary EM model with unit pseudocounts; neutral responses are treated as missing and gold labels are held fixed.

Gold-TINF::

[Algorithm 1](#) with $\eta = 0.02$.

Oracle-TINF::

the same profile using the simulator's true θ_j for orientation and weights; this is an unattainable reference.

We report accuracy and balanced accuracy on non-gold items. Gold-TINF and Dawid-Skene receive the same gold subset. The unweighted majority receives only its class prior from that split, while Oracle-TINF receives the simulator's true annotator parameters and is not a deployable competitor.

7.4. Experiment B: mechanism diagnosis and selective review

In each evaluation split, the hardest 25% of items are assigned the neutral regime, a random 25% of the remaining items are assigned perspective conflict, and the rest are regular. Gold items remain regular and objective. We evaluate conflict-versus-neutral routing among the two mechanism groups using AUROC and the reason-rule accuracy in [Equation \(16\)](#).

For error screening, AUROC measures discrimination of incorrectly fused labels. Area under the risk-coverage curve (AURC; lower is better) evaluates selective retention when high-score items are deferred. We compare raw and weighted margin deficit, three-way entropy over (T, N, F) , pure indeterminacy I , and $U_i(0.75)$.

Finally, a transparent decision-analysis stress test assigns potential review benefit 1 to a conflict item, 0.25 to a neutral item, and 0 to a regular item. For a top-20% review budget, selected benefit is divided by the maximum available benefit. This *normalized review utility* is not a measured causal effect or a universal cost model; it only tests ranking behavior under declared priorities.

7.5. Implementation and reproducibility

The reference implementation is a single Python program. It uses Python 3.12.13, NumPy 2.3.5, and scikit-learn 1.8.0 in the reported run. The simulator, aggregation methods, metrics, seeds, and CSV writers are all included. No network access, unpublished data, or manual result editing is required.

8. Results*8.1. The profile separates neutrality from conflict*

[Table 3](#) reports macro-averages of the dataset-level repetition means. Neutral items allocate 0.7002 of evidence to N , while conflict items allocate 0.8130 to I . Both have low consensus, yet for different reasons. Regular items also show nonzero indeterminacy because the simulated crowd is heterogeneous and hard items induce errors; the diagnostic claim is comparative, not that ordinary crowds have $I = 0$.

The TINF contrast $I - N$ achieves macro-average conflict AUROC 0.99999 among neutral and conflict items. The simple rule $I > N$ routes the mechanism with 0.9928 accuracy ([Table 4](#)). In contrast, absolute signed mean and three-way entropy do not identify which mechanism generated uncertainty. Their AUROCs

TABLE 3. Macro-average reliability-weighted profiles over four datasets and 40 repetitions. The identity error for $C + I + N = 1$ was below 4×10^{-16} in every dataset summary.

Regime	T	I	N	F	C
Regular	0.4619	0.5728	0.0363	0.5018	0.3910
Explicit neutrality	0.1543	0.1711	0.7002	0.1454	0.1287
Perspective conflict	0.4834	0.8130	0.0223	0.4942	0.1646

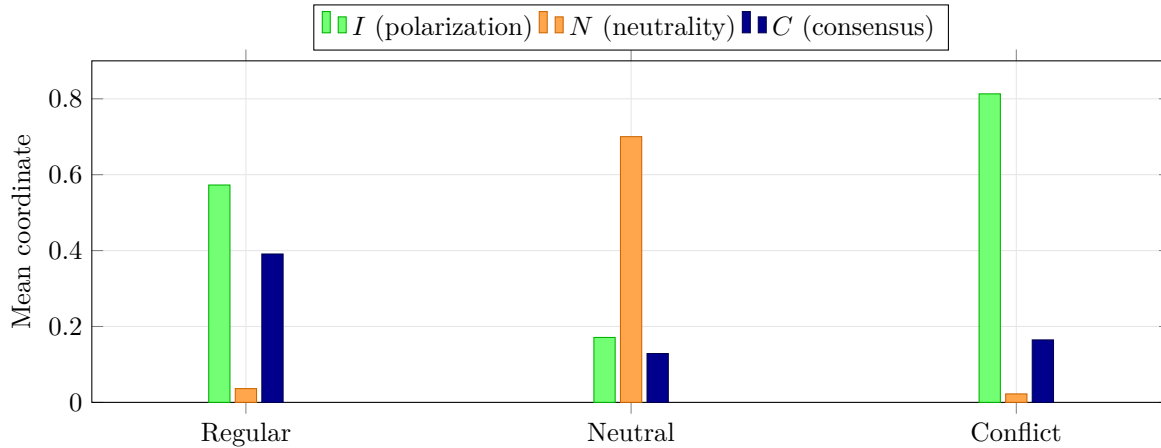


FIGURE 1. The decomposition assigns uncertainty to different mechanisms. Explicit-neutral items are dominated by N ; conflict items are dominated by I .

TABLE 4. Mechanism diagnosis by dataset (mean over 40 repetitions). AUROC compares conflict with explicit-neutral items; accuracy evaluates Equation (16).

Dataset	$- M_{\text{raw}} $	Entropy	$I - N$	Rule accuracy
Breast cancer	0.4223	0.3917	1.0000	0.9929
Digits 3 versus 8	0.4613	0.3207	1.0000	0.9915
Iris versicolor versus virginica	0.3468	0.5688	1.0000	0.9965
Wine class 0 versus rest	0.4232	0.3763	1.0000	0.9903
Macro-average	0.4134	0.4143	1.0000	0.9928

below 0.5 are not typographical errors: neutral items often have an even smaller signed margin than polarized items, so an undirected more uncertain ranking reverses the desired conflict ordering.

8.2. Gold weighting neutralizes consistent adversaries

Table 5 and Figure 2 show macro-average accuracy. Majority vote deteriorates from 0.9864 with no injected adversaries to 0.5930 at the nominal 40% setting (eight of 21). Gold-TINF remains above 0.991 throughout

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TABLE 5. Macro-average evaluation accuracy across four datasets. Each cell averages dataset-level means from 40 repetitions. Adversary percentages are nominal and correspond to 0, 2, 4, 6, and 8 of 21 workers.

Adversaries	Majority	Dawid–Skene	Gold-TINF	Oracle-TINF
0%	0.9864	0.9929	0.9919	0.9929
10%	0.9675	0.9945	0.9947	0.9956
20%	0.9163	0.9943	0.9941	0.9946
30%	0.7977	0.9951	0.9958	0.9959
40%	0.5930	0.9949	0.9942	0.9947

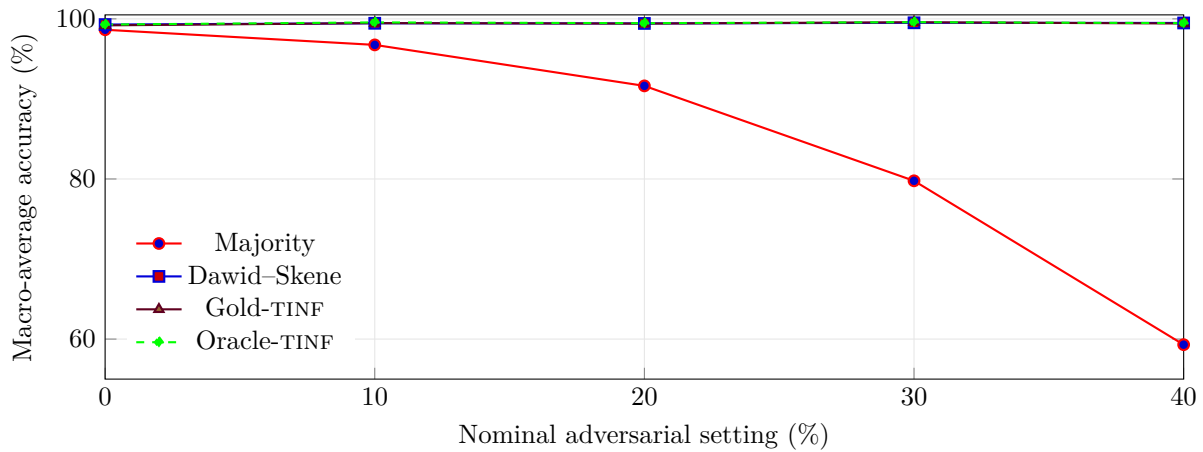


FIGURE 2. Gold information identifies consistently inverted annotators. Gold-TINF and gold-anchored Dawid–Skene remain stable as the unweighted majority collapses.

and reaches 0.9942 in that setting, a difference of 40.1 percentage points from majority vote. Gold-anchored Dawid–Skene performs similarly (0.9949 at the nominal 40% setting), and the oracle profile reaches 0.9947. The correct conclusion is parity with a strong gold-aware latent-class baseline in this simulator, not superiority over Dawid–Skene.

8.3. Error detection and mechanism routing require different scores

Table 6 is the key negative as well as positive result. Weighted margin deficit is the best error detector, with macro-average AUROC 0.9552 and AUC 0.0028. Pure indeterminacy and the mixed TINF review score perform poorly for error AUROC because high process disagreement is not equivalent to an incorrect fused label. Conversely, pure I gives the highest normalized mechanism-weighted review utility (0.6996), closely followed by $U(0.75)$ (0.6959). Weighted margin is less aligned with that declared conflict-prioritizing utility (0.3961).

TABLE 6. Macro-average selective-review results. Error AUROC and review utility are higher-is-better; AURC is lower-is-better. Utility uses the declared synthetic benefits in Section 7.

Ranking score	Error AUROC	AURC	Review utility
Raw margin deficit	0.8299	0.0077	0.5419
Weighted margin deficit	0.9552	0.0028	0.3961
Three-way entropy	0.5883	0.0331	0.3563
Pure indeterminacy I	0.3688	0.0539	0.6996
$U(0.75) = 0.75I + 0.25N$	0.4701	0.0361	0.6959

The evidence answers RQ3 in the negative: forcing one scalar to represent both prediction risk and the reason for disagreement loses useful structure. The recommended workflow is to use $1 - |M|$ to allocate a generic error-review budget and then use the pair (I, N) to route or describe the selected cases. If a governance process values contested items independently of label error, I can define a separate review queue.

9. Discussion

9.1. What the method adds

TINF-CONSENSUS adds an explanatory layer to weighted voting. The predicted label is intentionally simple: it is the sign of a reliability-weighted response sum. The novelty lies in retaining the response geometry after fusion. Theorem 4.3 says that one unit of item-level evidence is allocated among consensus, decisive polarization, and explicit neutrality. This makes three operational states visible:

- (1) high C : decisive support is concentrated on one class;
- (2) high I : decisive support is split across classes; and
- (3) high N : annotators explicitly occupy or select the neutral option.

The coordinates do not themselves explain *why* a person was neutral or why two people disagreed. They identify an observable pattern that can guide the next inquiry. For example, a conflict route may stratify profiles by annotator role or demographic group, while a neutrality route may inspect item evidence and the wording of the response options.

9.2. Why entropy and confidence are insufficient

Entropy over three response categories is useful as a measure of dispersion, but it is symmetric with respect to the names of those categories. A panel distributed between positive and negative and a panel distributed between a decisive class and neutrality can have similar entropy even though they imply different review actions. A scalar posterior confidence has the same general limitation: it indicates that the final decision

is uncertain, not the mechanism of uncertainty. The explicit pair (I, N) restores that mechanism for the particular response protocol considered here.

At the same time, [Table 6](#) cautions against replacing confidence with indeterminacy. A highly polarized crowd can still have a correct fused label, especially after reliable annotators are weighted. This is why margin deficit is empirically better for error prediction. The method’s most useful output is therefore not a new universal confidence number but a small structured record:

$$(\hat{Y}_i, M_i, T_i, I_i, N_i, F_i).$$

9.3. Objective and subjective annotation

The gold-anchored results apply most naturally to tasks with externally defensible answers: image quality checks, diagnostic test interpretation under a fixed standard, transcription, or factual verification. Even there, gold sets should cover the relevant difficulty range and be audited for leakage or systematic bias.

For subjective constructs—toxicity, political framing, emotion, fairness, or acceptability—a single gold orientation may not exist. In those settings TINF-CONSENSUS can still compute equal-weight or context-specific profiles, but orientation inversion should be disabled. High I may be the desired scientific result, not a defect. Downstream models can consume soft response distributions, annotator-conditioned targets, or multiple perspective labels rather than a forced consensus [\[3, 18\]](#).

9.4. Practical deployment

A deployment protocol should declare the neutral option in advance and tell annotators what it means. “Neither class,” “insufficient evidence,” “both,” and “prefer not to answer” are not interchangeable. If these meanings matter, they should be separate response codes and may require more than one neutrality coordinate.

The following sequence is a reasonable objective-task workflow:

- (1) collect representative gold items and ordinary items with missingness recorded separately from explicit neutrality;
- (2) estimate orientation and reliability with [Equations \(10\) and \(11\)](#);
- (3) fuse labels by weighted margin and retain the complete item profile;
- (4) allocate error review with $1 - |M|$;
- (5) route reviewed items using I versus N ; and
- (6) monitor profiles over annotator groups and time rather than auditing aggregate accuracy alone.

The final monitoring step can reveal guideline drift. A rise in N suggests that more annotators are declining to decide; a rise in I suggests that decisive interpretations are diverging. These are diagnostic hypotheses, to be confirmed by qualitative and domain-specific evidence.

9.5. *Ethical and governance considerations*

Annotator weights encode institutional judgments about whose answers count. Even objective gold questions can advantage workers familiar with a dominant terminology or practice. Reliability estimates should therefore be used for quality control with notice, appeal, and subgroup audit, not as opaque worker rankings. Low-weight annotations should remain available for analysis when privacy and consent permit.

Conflict routing can also expose sensitive group differences. Reporting should use aggregation thresholds, access controls, and data-minimization principles. A high- I item should not be described as evidence that a particular group is irrational or malicious. It is evidence only that the weighted decisive responses, under the chosen coding and weights, are split.

10. Limitations and Future Work

This study has seven main limitations.

- (1) **Semi-synthetic evidence.** The response mechanisms are known by construction, enabling exact diagnosis but limiting external validity. Evaluation on naturally collected multi-rater datasets with an explicit neutral option is the next essential step.
- (2) **Binary labels.** One-versus-rest profiles are possible, but they do not by themselves capture conflict among several non-reference classes. A principled multiclass generalization should preserve class geometry and the neutrality distinction.
- (3) **Gold dependence.** Orientation and reliability require representative, defensible gold items. Sparse or biased gold sets can misorient annotators. Hierarchical shrinkage and uncertainty intervals for w_j are promising extensions.
- (4) **Conditional independence.** The exponential error bound assumes conditionally independent corrected responses. Correlated annotator communities require cluster-aware concentration bounds or explicit dependence models.
- (5) **One operational mapping.** The formula $I = 4TF/(T + F)$ is motivated by symmetry, boundedness, and exact decomposition, but it is not the only possible indeterminacy functional. Axiomatic uniqueness results and sensitivity comparisons are open.
- (6) **No downstream learner study.** We evaluate aggregation and routing, not whether feeding the profiles to a classifier improves generalization or calibration.
- (7) **Synthetic review utility.** The utility experiment verifies a ranking under declared benefits; it does not estimate real review cost or causal benefit. Deployment needs domain-specific elicitation and a prospective study.

Future work should collect explicit-neutral datasets, evaluate annotator-group profiles without exposing identities, and compare the method with modern distribution-preserving and annotator-conditioned neural models. Other directions include time-varying weights, Bayesian posterior profiles, active selection of gold questions, and multiclass conflict graphs that identify which alternatives divide the crowd.

11. Conclusion

Multi-annotator uncertainty is not one phenomenon. An all-neutral panel and a polarized decisive panel can have the same signed mean while demanding different responses. TINF-CONSENSUS preserves this distinction through an observable TINF profile and the exact decomposition $C + I + N = 1$. Gold-based orientation and reliability weighting give robust label fusion in controlled adversarial experiments, with performance comparable to a gold-anchored Dawid–Skene model. The strongest practical result is a division of labor: weighted margin is effective for predicting fusion errors, whereas I and N identify the mechanism of unresolved annotation. This two-output view avoids treating legitimate disagreement as mere noise and turns consensus aggregation into an auditable process.

Declarations

Funding. NA.

Competing interests. NA.

Ethics statement. The experiments use datasets distributed with scikit-learn and simulated annotator responses; no new human participants or identifiable annotator data were collected.

Data and code availability. The accompanying reproducibility package contains the complete Python implementation and all reported CSV summaries. The public datasets are loaded through scikit-learn without network access.

Appendix A. Additional Proof Details

A.1. Alternative form of the decomposition

For $D_i > 0$, write decisive balance $b_i = (T_i - F_i)/(T_i + F_i) \in [-1, 1]$. Then

$$C_i = D_i b_i^2, \quad I_i = D_i(1 - b_i^2), \quad N_i = 1 - D_i. \quad (21)$$

This form shows that the construction has two stages. First, total evidence is split between neutrality N_i and decisiveness D_i . Second, decisive evidence is split between squared balance b_i^2 (consensus) and its complement $1 - b_i^2$ (polarization). It also makes [Equation \(8\)](#) immediate.

A.2. Why the factor four is necessary

Consider the symmetric family $I_i^{(k)} = kT_iF_i/D_i$ for $D_i > 0$. Requiring a perfectly balanced decisive panel, $T_i = F_i = D_i/2$, to have maximal indeterminacy $I_i = D_i$ gives

$$k \frac{(D_i/2)^2}{D_i} = D_i,$$

so $k = 4$. Thus the factor is fixed within this bilinear symmetric family by the scale convention used in [Theorem 4.3](#).

A.3. Tie handling

When $M_i = 0$, the profile deliberately declares no directional consensus. The reference experiment resolves a prediction tie using the majority class in the gold subset solely so that accuracy is defined for every item. A production system should instead consider deferral, randomization with a declared prior, or preservation of both labels. Tie handling affects the point prediction but not (T, I, N, F, C) .

Appendix B. Additional Experimental Results

TABLE 7. Accuracy at the nominal 40% setting (eight adversarial annotators out of 21) by dataset, averaged over 40 repetitions.

Dataset	Majority	Dawid-Skene	Gold-TINF	Oracle-TINF
Breast cancer	0.6072	0.9964	0.9962	0.9964
Digits 3 versus 8	0.5863	0.9986	0.9987	0.9987
Iris versicolor versus virginica	0.5707	0.9871	0.9839	0.9861
Wine class 0 versus rest	0.6076	0.9976	0.9978	0.9976

TABLE 8. Profile means by dataset and regime. Values are ordered (I, N, C) and averaged over 40 repetitions.

Dataset	Regular	Explicit neutrality	Conflict
Breast cancer	(.567, .031, .402)	(.170, .694, .136)	(.812, .020, .168)
Digits 3 vs. 8	(.567, .030, .403)	(.186, .664, .150)	(.814, .022, .164)
Iris Vc. vs. Vg.	(.587, .050, .363)	(.146, .756, .098)	(.812, .025, .163)
Wine 0 vs. rest	(.571, .034, .395)	(.183, .687, .130)	(.814, .023, .163)

Appendix C. Reproducibility Specification

The program writes four CSV summaries:

<code>robustness_summary.csv</code>	adversarial accuracy and balanced accuracy,
<code>routing_summary.csv</code>	mechanism AUROC and routing accuracy,
<code>profile_summary.csv</code>	regime-specific TINF coordinates,
<code>selective_summary.csv</code>	error AUROC, AURC, and review utility.

Standard deviations in those files are sample standard deviations across the 40 repetitions. Macro-averages in the main text are unweighted means of the four dataset-level means, preventing the largest dataset from dominating the summary.

TABLE 9. Parameters fixed before the reported run.

Parameter	Value
Annotators	21
Base decisive accuracies	3×0.95 , 5×0.85 , 8×0.72 , 5×0.58
Gold fraction	0.30, stratified by class
Repetitions	40 per dataset and condition
Adversary grid	Nominal 0, 0.1, 0.2, 0.3, 0.4, corresponding to 0, 2, 4, 6, 8 of 21 workers; adversary accuracy 0.15
Difficulty shrinkage	0.55 in Equation (19)
Weight prior and floor	Beta(1, 1); $\eta = 0.02$
Dawid–Skene	Unit cell pseudocount; at most 80 EM iterations; tolerance 10^{-8} ; neutral treated as missing
Mechanism allocation	25% explicit neutral, 25% perspective conflict, 50% regular on the evaluation split
Review budget and utility	Top 20%; benefits (0, 0.25, 1) for (regular, neutral, conflict)
Software	Python 3.12.13, NumPy 2.3.5, scikit-learn 1.8.0

Appendix D. Reporting Checklist for a Real-World Extension

Before applying the method to naturally collected annotations, a study should report at least:

- (1) the exact wording and semantics of every neutral or abstention option;
- (2) how missing responses differ technically from explicit neutrality;
- (3) recruitment, training, compensation, and consent procedures;
- (4) the construction, representativeness, and audit of gold items;
- (5) uncertainty intervals for annotator reliability and subgroup-level profile summaries where ethically permissible;
- (6) the tie, deferral, adjudication, and appeal policies;
- (7) the review utility or cost model and its stakeholders;
- (8) sensitivity to weight floors, gold-set size, and alternative indeterminacy functions; and
- (9) whether disagreements are treated as errors, perspectives, or both, with domain justification.

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N-Break: A Neutrosophic Task-Breakpoint Reasoning System for Assessing Functional Ability of Persons with Disabilities under Uncertainty

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Abstract: Functional ability assessment should not always be reduced to a general label such as able, partially able, or unable. Many daily activities consist of ordered steps, and a person may fail at one critical step because of functional limitation, unsafe execution, or uncertain evidence. This paper proposes N-Break, a neutrosophic task-breakpoint reasoning system for assessing the functional ability of people with disabilities under uncertainty. The proposed system decomposes each daily activity into functional steps and represents every step using a neutrosophic state composed of evidence of successful performance, indeterminacy, and evidence of failure or unsafe execution. Instead of producing a global disability score or ranking persons, N-Break identifies the first critical step where the activity becomes blocked, unsafe, or undetermined. An illustrative numerical case study demonstrates how the system distinguishes between blocked performance, safety risk, and conflicting evidence. The proposed framework is intended as an interpretable decision-support tool for rehabilitation-oriented functional assessment, not as a medical diagnostic system.

Keywords: Neutrosophic reasoning; functional ability assessment; persons with disabilities; task-breakpoint detection; uncertainty; rehabilitation decision support; activities of daily living; safety-aware assessment.

1. Introduction

Functional ability assessment is an important issue in rehabilitation, disability support, occupational therapy, assistive technology, and daily living evaluation [1]. For persons with disabilities, the question is not always whether the person is generally “able” or “unable” to perform an activity [2]. Many daily activities are complex, sequential, and condition-dependent [3]. A person may complete several parts of an activity successfully but fail at one specific step because of physical limitation, unsafe movement, environmental barriers, lack of support, pain, fatigue, or uncertain evidence [2, 4].

This means that a general functional judgment may be insufficient [5]. For example, saying that a person is “unable to climb stairs” does not explain whether the difficulty occurs when standing, lifting the foot, transferring body weight, maintaining balance, repeating the movement, or descending safely [2, 6]. Similarly, saying that a person is “partially able to prepare a meal” does not identify whether the problem occurs while reaching the kitchen, handling utensils, opening containers, using heat, carrying food, or cleaning afterward [2, 3]. In practical rehabilitation, the location of the functional breakdown is often more useful than a broad label [2, 7].

The main problem addressed in this paper is therefore not the measurement of general disability severity. Instead, the problem is the identification of the functional breakpoint inside an activity. A

functional breakpoint is the earliest critical step at which the activity can no longer continue safely, reliably, or with sufficient evidence. Identifying this point can help professionals understand where assistance, adaptation, observation, or further assessment is needed [2, 7].

Daily activities can be represented as ordered sequences of functional steps [3, 7]. Let A_j denote a daily activity. The activity is decomposed as:

$$A_j = (s_{j1}, s_{j2}, \dots, s_{jK_j}). \quad (1)$$

Here, s_{jk} is the k -th functional step of activity A_j , and K_j is the total number of steps in that activity. This representation is important because the order of steps affects the final functional outcome [3, 7]. A later step cannot be reached if an earlier critical step is blocked or unsafe.

However, step-level functional evidence is rarely perfectly certain [8]. A person may report that a step is possible, while an observer may notice unsafe balance [2]. A caregiver may describe the person as independent, while the therapist may identify high risk under real task conditions [9]. In other cases, the available evidence may be incomplete because the person did not attempt the step or because the assessment setting was not suitable [9]. These situations require a reasoning model that can represent ability, uncertainty, and failure evidence separately [10].

For this reason, this paper proposes N-Break, a neutrosophic task-breakpoint reasoning system for assessing the functional ability of persons with disabilities under uncertainty. For a person P_i , activity A_j , and step s_{jk} , the proposed system represents the step using the neutrosophic state:

$$S_{ijk} = \langle T_{ijk}, I_{ijk}, F_{ijk} \rangle. \quad (2)$$

In this state, T_{ijk} represents evidence that the person can perform the step successfully, I_{ijk} represents uncertainty or conflict in the available evidence, and F_{ijk} represents evidence that the person cannot perform the step or that the step is unsafe [11]. This structure is central to the proposed model because it prevents uncertain or unsafe performance from being treated as simple success.

The aim of N-Break is not to produce a global disability score, rank persons, or replace professional judgment. It aims to answer a more specific functional question: where does the activity break down, and why? The system examines the steps of an activity in order and identifies the first critical step that is blocked, unsafe, or undetermined. This can be expressed as:

$$k_{ij}^* = \min \{k: s_{jk} \text{ is critical and not functionally passed} \}. \quad (3)$$

Here, k_{ij}^* is the detected breakpoint for person P_i while performing activity A_j . If no such step exists, the activity is considered achievable under the assessed condition. If a breakpoint exists, the system reports the exact step and the reason for the interruption.

The proposed model is different from ordinary scoring approaches. It does not ask only how difficult an activity is. It examines how the activity is performed step by step. It also separates three different situations that may appear similar in a traditional assessment: confirmed inability, unsafe execution, and uncertain evidence. These distinctions are important because each situation requires a different professional response. Confirmed inability may require training or adaptation. Unsafe execution may require risk control or supervision. Uncertain evidence may require reassessment or additional observation.

The main contributions of this paper are as follows.

First, the paper introduces a task-breakpoint view of functional ability assessment. This view focuses on the internal structure of daily activities and identifies the exact step where performance becomes blocked, unsafe, or uncertain.

Second, the paper develops a neutrosophic step-state representation that separates evidence of successful performance, indeterminacy, and evidence of failure or unsafe execution.

Third, the paper proposes a reasoning mechanism that converts step-level neutrosophic evidence into interpretable activity-level decisions, such as independently achievable, blocked at a critical step, unsafe at a critical step, or undetermined.

Fourth, the paper demonstrates the model through a complete illustrative numerical case study with step-by-step calculations and interpretation.

The scope of the study is carefully bounded. N-Break is a decision-support framework for functional ability reasoning. It is not a medical diagnostic system, does not determine legal disability status, does not prescribe treatment, and does not replace clinicians, rehabilitation specialists, or occupational therapists. Its value lies in supporting clearer functional reasoning under uncertainty.

The remainder of the paper is organized as follows. Section 2 reviews related work on functional assessment, task-based assessment, activity modeling, and neutrosophic reasoning. Section 3 presents the proposed neutrosophic task-breakpoint model. Section 4 introduces the N-Break reasoning algorithm. Section 5 provides the illustrative numerical case study and complete calculations. Section 6 discusses the results and practical implications. Section 7 concludes the paper and presents future research directions.

2. Related Work and Research Gap

Functional ability assessment has been studied through several complementary directions. The most relevant directions for this paper are conceptual disability frameworks, standardized disability instruments, performance-based functional assessment, task-analysis approaches, computational activity modeling, fuzzy activity models, and neutrosophic uncertainty reasoning. This section reviews these directions and clarifies the specific gap addressed by N-Break.

2.1. Functioning and Disability as Context-Dependent Constructs

The International Classification of Functioning, Disability and Health (ICF) provides a major conceptual foundation for understanding functioning and disability [12]. It describes disability not only as a feature of the body, but as an interaction between health conditions, activities, participation, and contextual factors [12, 13]. Importantly, ICF includes environmental factors because a person's functioning occurs within a real context, not in isolation.

This contextual view is important for the present study. A person's ability to complete a daily activity may depend on the activity structure, the physical environment, available support, assistive devices, safety conditions, fatigue, and the quality of the available evidence. Therefore, functional ability should not always be reduced to a broad label such as "able" or "unable." A more informative assessment may need to identify where the activity becomes difficult, unsafe, or uncertain.

However, ICF is primarily a classification and conceptual framework. It provides a common language for functioning and disability, but it does not provide a formal step-level reasoning algorithm for detecting the exact point at which a daily activity breaks down. This creates room for a complementary computational model that can operate at the level of task steps while remaining consistent with the contextual understanding of disability.

2.2. Standardized Disability Instruments

Standardized disability instruments provide structured ways to measure functioning across individuals and populations. WHODAS 2.0 is a widely used instrument developed by the World Health Organization to assess functioning and disability across six domains: cognition, mobility, self-care, getting along, life activities, and participation [14].

WHODAS 2.0 is valuable because it provides a standardized and cross-domain measurement structure [14]. It can support population-level comparison and general functional profiling. Nevertheless, its output is mainly item-level, domain-level, or summary-level. It does not directly explain the internal sequence of steps inside a complex activity. For example, a mobility item may indicate difficulty, but it does not identify whether the difficulty occurs during initiation, balance control, weight transfer, repetition, or safe completion.

This limitation does not reduce the value of WHODAS 2.0. Rather, it shows that a different type of model is needed when the assessment question is not "how much difficulty exists?" but "where exactly does the functional activity break down?" N-Break addresses this second question.

2.3. Performance-Based Functional Assessment

Performance-based tools are closer to real functional behavior because they evaluate what a person does during activity performance rather than relying only on broad self-report. One relevant

example is the Performance Assessment of Self-care Skills (PASS), which is described as a performance-based observational tool used to document occupational performance and support intervention planning [15]. The PASS literature explicitly notes its usefulness for identifying compromised daily tasks and the point of task breakdown [15].

This is highly relevant to N-Break because it confirms the practical importance of identifying where a task breaks down. However, performance-based tools generally depend on professional observation and scoring procedures [16]. They do not by themselves provide a formal neutrosophic reasoning structure that separates evidence of successful performance, uncertain evidence, and evidence of failure or unsafe execution at each step.

N-Break builds on the practical importance of task breakdown, but it does not replace performance-based assessment. Instead, it proposes a formal reasoning layer that can represent the evidence behind each step and identify the first critical breakpoint in a transparent way.

2.4. Task Analysis in Rehabilitation and Daily Living Assessment

Task analysis is widely used in rehabilitation and occupational therapy because many activities of daily living are sequential [17, 18]. Activities such as dressing, stair climbing, preparing a meal, bathing, toileting, transferring, and moving outside the home require multiple connected steps. Breaking an activity into smaller steps helps professionals understand which component of performance is preserved and which component requires support, adaptation, or training [17]. Evidence from rehabilitation literature supports the value of structured task decomposition for analyzing functional performance [18].

Despite this importance, task analysis may remain descriptive if it is not linked to a formal reasoning mechanism. A therapist may identify that a person has difficulty in a specific step, but the assessment may still need a clear way to handle conflicting evidence, uncertain observation, partial ability, and safety risk. For example, one source may suggest that a person can perform a step, while another source may indicate unsafe execution. A binary or single-scale interpretation may hide this conflict.

N-Break formalizes task analysis using neutrosophic step states. Each step is not simply marked as passed or failed. It is represented through three independent components: evidence of ability, indeterminacy, and evidence of failure or unsafe execution. This allows the system to preserve uncertainty rather than forcing it into a premature conclusion.

2.5. Computational Modeling of Activities of Daily Living

Computational activity modeling has produced important methods for monitoring and recognizing activities of daily living. In smart-home and sensor-based research, activities may be represented as sequences of events, and models such as Hidden Markov Models and Probabilistic Finite Automata are commonly used to represent inhabitant behavior from event streams [19].

These approaches are valuable for activity recognition, monitoring, and detection of behavioral routines. Some finite-state approaches also model transitions between activity states. However, their main aim is usually to recognize what activity is occurring or to monitor activity patterns, rather than to provide a rehabilitation-oriented explanation of why a person cannot safely complete a specific task step.

Fuzzy finite-state models have also been used in human activity modeling and recognition [20]. These models introduce graded state representation and can be useful when activity boundaries are not crisp [20]. However, fuzzy membership alone does not explicitly separate truth, indeterminacy, and falsity as independent dimensions. In functional ability assessment, this separation matters because a step may have partial evidence of ability, incomplete evidence, and safety-related failure evidence at the same time.

Therefore, N-Break is not proposed as an activity-recognition model. It is a functional reasoning model. Its purpose is to identify the first critical step where activity continuation becomes blocked, unsafe, or uncertain.

2.6. Neutrosophic Reasoning under Uncertainty

Neutrosophic theory is useful when information cannot be adequately represented by a single truth value [21]. A neutrosophic set is commonly characterized by three components: truth-membership, indeterminacy-membership, and falsity-membership [21, 22]. This structure is suitable for problems where evidence may be incomplete, inconsistent, or partially conflicting [23, 24].

The relevance of neutrosophic reasoning to functional ability assessment is direct. Neutrosophic approaches have also been applied in healthcare decision-support problems, where multiple criteria, expert judgments, and uncertain information must be considered simultaneously [25]. In real assessment, evidence about a person's performance may come from self-report, caregiver report, therapist observation, environmental inspection, assistive-device information, and safety observation. These sources may not always agree [26]. A person may attempt a task step, but the attempt may be unsafe. A caregiver may report independence, while an assessor observes compensatory movement or instability. A step may not be observed at all, leaving insufficient evidence for a reliable judgment.

In such cases, the assessment needs a representation that keeps uncertainty visible. N-Break uses neutrosophic reasoning not as an additional score, but as the core evidence representation for every functional step. This allows the model to distinguish between three different cases: the step is supported by evidence of successful performance, the step is uncertain because evidence is incomplete or conflicting, and the step is not acceptable because failure or unsafe-execution evidence is strong.

2.7. Research Gap

The reviewed literature shows that each existing direction provides an important part of the solution. ICF provides a contextual understanding of functioning and disability. WHODAS 2.0 provides standardized domain-level measurement. Performance-based tools and task analysis support observation of daily activity performance and can identify task breakdown. Computational activity models represent activities as sequences or states. Fuzzy models support graded activity representation. Neutrosophic theory provides a formal way to represent truth, indeterminacy, and falsity.

However, the specific gap addressed in this paper remains clear: there is a need for an interpretable neutrosophic reasoning system that decomposes a daily activity into ordered functional steps and identifies the first critical step where performance becomes blocked, unsafe, or undetermined. The reviewed approaches do not jointly provide this exact combination: task-level functional decomposition, breakpoint detection, explicit indeterminacy handling, safety-aware step reasoning, and rehabilitation-oriented explanation.

Table 1. Position of N-Break relative to related work

Research direction	Main contribution	Remaining limitations for this study	Role of N-Break
ICF framework	Provides a contextual view of functioning and disability	Does not define a computational step-breakpoint reasoning algorithm	Uses the contextual view as the conceptual basis for condition-sensitive ability assessment
WHODAS 2.0	Measures functioning across standardized domains	Does not locate the failed step inside a complex activity	Moves the assessment from domain-level description to step-level reasoning
Performance-based assessment	Observes real activity performance and can identify task breakdown	Does not provide a neutrosophic formalization of step evidence	Formalizes breakdown detection using ability, uncertainty, and failure evidence

Task analysis	Decomposes activities into smaller steps	May remain descriptive without a formal uncertainty model	Converts step analysis into a structured reasoning process
ADL activity recognition	Models activities using events, sensors, and state transitions	Focuses mainly on recognizing activities, not explaining functional failure	Uses sequential structure for rehabilitation-oriented breakpoint reasoning
Fuzzy activity modeling	Represents graded activity states	Does not independently preserve indeterminacy and failure evidence	Uses neutrosophic states to separate success, uncertainty, and failure or unsafe execution
Neutrosophic reasoning	Represents truth, indeterminacy, and falsity	Rarely specialized into functional task-breakpoint assessment	Provides the core logic for detecting blocked, unsafe, or uncertain steps

As shown in Table 1, N-Break is designed to fill a specific methodological gap. It does not replace existing disability frameworks or clinical instruments. It adds a focused reasoning layer for cases where the key question is not simply whether a person has difficulty, but where the functional activity breaks down and why.

The proposed contribution is therefore narrow, practical, and distinct. N-Break does not produce a global disability score, does not rank people, and does not claim to diagnose disability. It provides a step-level neutrosophic reasoning mechanism that identifies the first critical breakpoint in a daily activity and explains whether the interruption is caused by confirmed failure, unsafe execution, or uncertain evidence.

3. Proposed Neutrosophic Task-Breakpoint Model

This section presents the formal structure of the proposed **N-Break** model. The model is designed to assess functional ability at the level of task execution. It does not assign a global disability score, does not rank individuals, and does not measure general disability severity. Its purpose is to identify the first critical step where a daily activity becomes blocked, unsafe, uncertain, or requires professional review.

3.1. Activity as an Ordered Functional Task

Let P_i denote the assessed person, and let A_j denote a daily activity. In N-Break, the activity is represented as an ordered sequence of functional steps:

$$A_j = (s_{j1}, s_{j2}, \dots, s_{jK_j}). \quad (4)$$

Here, s_{jk} is the k -th functional step of activity A_j , and K_j is the total number of steps required to complete the activity.

The order of the steps is essential. A daily activity may fail because one early critical step cannot be completed, even if later steps are simple. For example, in stair climbing, the person cannot safely continue to repeated stepping if the body-weight transfer step is unsafe. Therefore, N-Break evaluates the activity as a sequence rather than as one undivided item.

Each step has two structural properties: criticality and adaptability. Criticality indicates whether the step is necessary for safe continuation of the activity. Adaptability indicates whether the step has an acceptable alternative pathway, such as supervision, assistive support, a modified strategy, or an environmental adjustment.

The structural descriptor of step s_{jk} is:

$$Q_{jk} = (c_{jk}, a_{jk}).$$

Here, $c_{jk} \in \{0, 1\}$ is the criticality indicator. If $c_{jk} = 1$, the step is critical. If $c_{jk} = 0$, the step is non-critical or compensable under the assessed condition. Also, $a_{jk} \in \{0, 1\}$ is the adaptability indicator.

If $a_{jk} = 1$, an acceptable alternative pathway may exist for the step. If $a_{jk} = 0$, no acceptable alternative pathway is available under the assessed condition.

These indicators do not decide ability by themselves. They only describe the structure of the activity. The actual functional decision is made from the neutrosophic evidence assigned to each step.

3.2. Neutrosophic Step-State Representation

For each person P_i , activity A_j , and step s_{jk} , N-Break assigns a neutrosophic step state:

$$S_{ijk} = \langle T_{ijk}, I_{ijk}, F_{ijk} \rangle. \quad (5)$$

Here, T_{ijk} represents evidence that person P_i can perform step s_{jk} successfully. I_{ijk} represents indeterminacy caused by missing, incomplete, or conflicting evidence. F_{ijk} represents evidence that the person cannot perform the step or that the step is not functionally acceptable.

The three components satisfy:

$$T_{ijk}, I_{ijk}, F_{ijk} \in [0, 1].$$

They are not required to sum to one. This is important because functional evidence may be incomplete or contradictory. A person may show some ability to perform a step while also showing unsafe movement or inconsistent performance. In such a case, the model should not force the assessment into a single binary or fuzzy value.

Table 2. Functional meaning of the neutrosophic step state.

Component	Meaning in N-Break	Practical interpretation
T_{ijk}	Evidence of successful step performance	The person completes the step safely, consistently, and with sufficient control.
I_{ijk}	Indeterminacy in the evidence	The evidence is missing, incomplete, inconsistent, or conflicting.
F_{ijk}	Evidence of step failure or unacceptable performance	The person cannot complete the step, or the step is performed in a way that cannot be accepted

As shown in Table 2, the neutrosophic state is the core representation in N-Break. It allows the model to separate ability evidence, uncertainty, and failure evidence instead of reducing them to one score.

3.3. Evidence Structure

The evidence used to construct S_{ijk} may come from different sources. Let E_{ijk} denote the evidence set available for step s_{jk} :

$$E_{ijk} = \{e_{ijk}^{self}, e_{ijk}^{obs}, e_{ijk}^{care}, e_{ijk}^{env}, e_{ijk}^{safe}\}.$$

Here, e_{ijk}^{self} denotes self-reported evidence, e_{ijk}^{obs} denotes professional observation, e_{ijk}^{care} denotes caregiver or family report, e_{ijk}^{env} denotes environmental evidence, and e_{ijk}^{safe} denotes safety-related evidence.

Not all evidence sources must be available in every case. If some sources are missing or contradictory, this should increase indeterminacy rather than being ignored. This is one of the main reasons for using a neutrosophic representation.

The evidence-to-state transformation is:

$$S_{ijk} = \Phi(E_{ijk}). \quad (6)$$

Here, Φ is a neutrosophic evidence-fusion function that converts the available evidence into T_{ijk} , I_{ijk} , and F_{ijk} . In this paper, Φ is kept transparent and rule-based so that the decision process can be followed and checked. The numerical case study later shows how the values are obtained and used step by step.

3.4. Local Step Decision

After constructing S_{ijk} , N-Break assigns a local decision to the step. The possible local step decisions are:

$$D_{ijk} \in \{\text{Pass}, \text{Blocked}, \text{Unsafe}, \text{Undetermined}, \text{Review}\}.$$

Table 3. Local step decisions in N-Break

Local decision	Meaning
Pass	The step is functionally acceptable under the assessed condition
Blocked	The step cannot be completed with sufficient functional evidence.
Unsafe	The step may expose the person to unacceptable risk.
Undetermined	The evidence is too uncertain or conflicting to support a decision.
Review	The evidence is weak and requires professional interpretation.

The decision is controlled by threshold parameters. Let θ_T be the minimum acceptable success-evidence threshold, θ_I the indeterminacy threshold, θ_F the failure-evidence threshold, and θ_R the safety-risk threshold. Let $\rho_{ijk} \in [0, 1]$ denote the safety-risk value for step s_{jk} . If a separate safety-risk estimate is not available, ρ_{ijk} may be set to zero and risk may be reflected through F_{ijk} .

The local decision rule is:

$$D_{ijk} = \begin{cases} \text{Unsafe,} & \rho_{ijk} \geq \theta_R, \\ \text{Undetermined,} & I_{ijk} \geq \theta_I, \\ \text{Blocked,} & F_{ijk} \geq \theta_F, \\ \text{Pass,} & T_{ijk} \geq \theta_T, I_{ijk} < \theta_I, F_{ijk} < \theta_F, \rho_{ijk} < \theta_R, \\ \text{Review,} & \text{otherwise.} \end{cases} \quad (7)$$

This rule is intentionally conservative. Safety is checked before success. Uncertainty is also checked before accepting the step. Therefore, a step is not considered passed merely because some ability evidence exists. It must also have acceptable uncertainty and acceptable failure or safety evidence.

3.5. Functional Breakpoint

A functional breakpoint is the first critical step where the activity cannot continue as a clearly acceptable performance. The operational breakpoint set is:

$$\mathcal{B}_{ij} = \left\{ k' : c_{jk} = 1, D_{ijk} \in \{\text{Blocked}, \text{Unsafe}, \text{Undetermined}, \text{Review}\} \right\}. \quad (8)$$

If $\mathcal{B}_{ij} \neq \emptyset$, the detected breakpoint is:

$$k_{ij}^* = \min \mathcal{B}_{ij}. \quad (9)$$

If $\mathcal{B}_{ij} = \emptyset$, then no critical breakpoint is detected, and the activity is considered achievable under the assessed condition.

The use of the first critical non-pass step is central to the model. It means that N-Break does not wait until the entire activity is labeled as failed. It identifies the earliest point where the activity loses functional acceptability.

3.6. Activity-Level Decision

The activity-level decision depends on whether a breakpoint exists, the type of local decision at the breakpoint, and whether an acceptable alternative pathway is available.

Let Ω_{ij} denote the final activity-level decision for person P_i on activity A_j . If no breakpoint is detected, the activity is classified as achievable under the assessed condition. If a breakpoint is detected, the system reports the exact breakpoint and its reason.

The activity-level decision is defined as:

$$\Omega_{ij} = \begin{cases} \text{Achievable under assessed condition,} & \mathcal{B}_{ij} = \emptyset, \\ \text{Unsafe at step } k_{ij}^*, & D_{ijk_{ij}^*} = \text{Unsafe,} \\ \text{Undetermined at step } k_{ij}^*, & D_{ijk_{ij}^*} = \text{Undetermined,} \\ \text{Professional review required at step } k_{ij}^*, & D_{ijk_{ij}^*} = \text{Review,} \\ \text{Adaptation required at step } k_{ij}^*, & D_{ijk_{ij}^*} = \text{Blocked, } a_{jk_{ij}^*} = 1, \\ \text{Blocked at step } k_{ij}^*, & D_{ijk_{ij}^*} = \text{Blocked, } a_{jk_{ij}^*} = 0. \end{cases} \quad (10)$$

This formulation separates the local decision from the adaptability condition. A blocked step with $a_{jk_{ij}^*} = 1$ is not automatically considered successful. It is reported as adaptation required, meaning that an alternative pathway should be assessed before accepting the activity as achievable.

Table 4. Activity-level decisions produced by N-Break

Activity-level decision	Meaning	Practical implication
Achievable under assessed condition	No critical breakpoint is detected	The activity is functionally acceptable in the assessed setting
Unsafe at step k_{ij}^*	The breakpoint is caused by safety risk	Risk control or supervised reassessment is needed
Undetermined at step k_{ij}^*	The breakpoint is caused by uncertain evidence	More observation or additional evidence is needed
Professional review required at step k_{ij}^*	The evidence is weak but not conclusive	A professional should decide whether the step can be accepted
Adaptation required at step k_{ij}^*	The step is blocked, but an alternative pathway may exist	Assistive support, environmental modification, supervision, or a modified strategy should be tested
Blocked at step k_{ij}^*	The step is blocked with no acceptable alternative pathway	The exact failed step should guide professional review

Table 4 shows that the output of N-Break is explanatory. It identifies not only whether the activity can continue, but also the point and reason of functional interruption.

3.7. Explanation Output

For each assessed activity, the model produces an explanation record:

$$X_{ij} = (A_j, k_{ij}^*, D_{ij}^*, \Omega_{ij}, S_{ij}^*). \quad (11)$$

Here, A_j is the assessed activity, k_{ij}^* is the detected breakpoint, D_{ij}^* is the local decision responsible for the final output, Ω_{ij} is the activity-level decision, and S_{ij}^* is the neutrosophic state at the breakpoint.

If no breakpoint is detected, the output is recorded as:

$$k_{ij}^* = \text{No breakpoint}, D_{ij}^* = \text{All critical steps passed}.$$

This avoids referring to a breakpoint step when no breakpoint exists.

If a breakpoint is detected, then:

$$D_{ij}^* = D_{ijk_{ij}^*}, S_{ij}^* = S_{ijk_{ij}^*}.$$

The explanation record is important because it keeps the decision traceable. A professional user can see whether the final decision was caused by high failure evidence, high uncertainty, safety risk, or weak evidence requiring review.

3.8. Role of Neutrosophic Reasoning in the Proposed Model

The role of neutrosophic reasoning in N-Break is not secondary. It is the main logic of the proposed system. Functional step evidence is not always complete or consistent. A person may report that they can complete a step, while observation shows instability. A caregiver may report independence, while the assessed environment contains a barrier. A person may complete the step once but not consistently enough for safe daily performance.

A binary model would force these situations into pass or fail. A single fuzzy membership value may express partial ability, but it does not clearly preserve uncertainty and failure evidence as separate components. N-Break uses neutrosophic representation to keep these dimensions visible.

For example, a step may have:

$$S_{ijk} = \langle 0.65, 0.20, 0.60 \rangle.$$

This state means that there is some evidence of ability, but also strong evidence of failure or unacceptable execution. The decision should not be a simple pass. Depending on the selected thresholds and safety-risk value, the model can identify the step as blocked, unsafe, or requiring review.

This is the main advantage of the proposed model. It does not treat attempted performance as sufficient evidence of ability, and it does not hide uncertainty inside a single score.

4. N-Break Reasoning Algorithm

This section presents the operational algorithm of the proposed N-Break system. The model in Section 3 defines the formal components. This section explains how these components are used in a clear decision sequence. The algorithm does not compute a severity score, does not rank persons, and does not replace professional judgment. Its task is to identify whether a functional activity can proceed step by step and to locate the first critical step where performance becomes blocked, unsafe, uncertain, or requires review.

4.1. Algorithmic Objective

For a person P_i and a daily activity A_j , the objective of the algorithm is to transform step-level neutrosophic evidence into an interpretable task-level decision.

The operational mapping of N-Break is:

$$\mathcal{A}_{NB}: (P_i, A_j, Q_j, E_{ij}, \Theta) \rightarrow (\Omega_{ij}, k_{ij}^*, D_{ij}^*, X_{ij}). \quad (11)$$

Here, Q_j is the set of structural descriptors for the steps of activity A_j , E_{ij} is the available evidence for person P_i on activity A_j , and Θ is the set of decision thresholds. The output contains the final activity-level decision. Ω_{ij} , the detected breakpoint k_{ij}^* , the local decision responsible for the final output D_{ij}^* , and the explanation record X_{ij} .

The algorithm is intentionally conservative. It accepts a step as passed only when the evidence of performance is sufficient, and there is no unacceptable uncertainty, failure evidence, or safety risk.

4.2. Required Inputs

The algorithm requires five groups of inputs.

Table 5. Inputs of the N-Break algorithm

Input	Meaning	Required role
P_i	Assessed person	Identifies the person being assessed
A_j	Target daily activity	Defines the activity under evaluation
Q_j	Structural step descriptors	Specifies which steps are critical and which may have alternatives
E_{ij}	Step-level evidence	Provides information used to build neutrosophic step states
Θ	Decision thresholds	Controls pass, uncertainty, failure, and safety decisions

The structural descriptor set is:

$$Q_j = \{Q_{j1}, Q_{j2}, \dots, Q_{jK_j}\}, Q_{jk} = (c_{jk}, a_{jk}).$$

Here, c_{jk} indicates whether step s_{jk} is critical, and a_{jk} indicates whether an acceptable alternative pathway may exist for this step.

The threshold set is:

$$\Theta = (\theta_T, \theta_I, \theta_F, \theta_R). \quad (12)$$

Here, θ_T is the minimum acceptable evidence of successful performance, θ_I is the indeterminacy threshold, θ_F is the failure-evidence threshold, and θ_R is the safety-risk threshold.

4.3. Output Structure

The output must be interpretable and traceable. N-Break therefore returns both a decision and its reason.

Table 6. Outputs of the N-Break algorithm

Output	Meaning
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Ω_{ij}	Final activity-level decision
k_{ij}^*	Detected functional breakpoint
D_{ij}^*	Local decision at the breakpoint, or all-critical-steps-passed status
X_{ij}	Explanation record containing the step evidence and decision cause

If a breakpoint is detected, k_{ij}^* is the first critical step where the activity cannot proceed as an acceptable functional performance. If no breakpoint is detected, k_{ij}^* is recorded as No breakpoint, and D_{ij}^* is recorded as all critical steps passed.

This output structure avoids a common error: it does not refer to a breakpoint step when no breakpoint exists.

4.4. Decision Priority

The algorithm follows a fixed decision priority. Safety is evaluated first. Indeterminacy is evaluated before failure because highly uncertain evidence should not be forced into a failure or pass decision. Failure evidence is then evaluated. A pass is accepted only after risk, uncertainty, and failure evidence have been checked.

Table 7. Priority Decision used in N-Break

Priority	Condition checked	Decision consequence
1	Safety risk is unacceptable	The step is classified as unsafe
2	Indeterminacy is too high	The step is classified as undetermined
3	Failure evidence is strong	The step is classified as blocked
4	Success evidence is sufficient and other risks are acceptable	The step is classified as pass
5	Evidence is weak but not conclusive	The step is routed to review

Table 7 shows that N-Break does not treat attempted performance as successful performance if the attempt is unsafe or if the evidence is too uncertain.

4.5. Algorithm 1: N-Break Reasoning Procedure

Algorithm 1. N-Break reasoning algorithm

Step	Procedure
1	Input P_i , A_j , the ordered steps of A_j , the structural descriptors Q_j , the evidence set E_{ij} , and the thresholds θ .
2	Initialize $k_{ij}^* = \text{No breakpoint}$, $D_{ij}^* = \text{All critical steps passed}$, and $X_{ij} = \emptyset$.
3	For $k = 1$ to K_j , load the evidence available for step s_{jk} .
4	Convert the available evidence into the neutrosophic step state $S_{ijk} = \langle T_{ijk}, I_{ijk}, F_{ijk} \rangle$.
5	Load or estimate the safety-risk value ρ_{ijk} . If no separate safety-risk value is available, set $\rho_{ijk} = 0$ and represent risk through F_{ijk} .
6	Classify the local step decision D_{ijk} using the decision priority in Table 7.
7	Store the step trace $(s_{jk}, T_{ijk}, I_{ijk}, F_{ijk}, \rho_{ijk}, D_{ijk})$.
8	If $c_{jk} = 1$ and $D_{ijk} \neq \text{Pass}$, set $k_{ij}^* = k$, $D_{ij}^* = D_{ijk}$, $S_{ij}^* = S_{ijk}$, and stop scanning the remaining steps.
9	If no breakpoint is detected after all steps are scanned, set $\Omega_{ij} = \text{Achievable under assessed condition}$.
10	If a breakpoint is detected, determine Ω_{ij} according to the breakpoint decision and the adaptability value $a_{jk} k_{ij}^*$.
11	Generate the final explanation record X_{ij} .
12	Return $(\Omega_{ij}, k_{ij}^*, D_{ij}^*, X_{ij})$.

Algorithm 1 is sequential. It stops at the first critical non-pass step because the aim is to detect the earliest functional interruption, not to summarize all difficulties into one score. Non-critical difficulties are recorded in the trace, but they do not create a functional breakpoint unless the step is marked as critical.

4.6. Local Step Processing

For each step s_{jk} , the algorithm processes the neutrosophic state S_{ijk} and the safety-risk value ρ_{ijk} . The local processing function is:

$$\text{StepDecision}(S_{ijk}, \rho_{ijk}, \theta) \rightarrow D_{ijk}. \quad (13)$$

The result D_{ijk} belongs to one of five states: Pass, Blocked, Unsafe, Undetermined, or Review. These states are not interchangeable. Each one has a different practical meaning.

Table 8. Practical meaning of local step decisions

Local decision	Practical meaning
Pass	The available evidence supports accepting the step as functionally completed.
Blocked	The evidence supports that the step cannot be completed under the assessed condition.
Unsafe	The step cannot be accepted because the safety risk is unacceptable.
Undetermined	The evidence is too uncertain or conflicting.
Review	The evidence is weak and should be interpreted by a professional

The Review state is necessary because not every unclear situation should be classified as blocked or undetermined. Some cases may have weak but not contradictory evidence. These cases should be referred to professional judgment instead of being forced into a final automatic label.

4.7. Breakpoint Detection Logic

After each local decision is produced, N-Break checks whether the current step is critical. If the step is non-critical, the algorithm stores its decision and continues. If the step is critical and the decision is not Pass, the algorithm detects a breakpoint.

The breakpoint detection function is:

$$\text{Break}(A_j, P_i) = \min \{k' : c_{jk} = 1 \wedge D_{ijk} \neq \text{Pass}\}. \quad (14)$$

If the set is empty, no breakpoint is detected.

This logic makes the model strict but interpretable. A critical step must be passed for the activity to continue. If it is blocked, unsafe, undetermined, or requires review, the activity cannot be accepted as clearly achievable under the assessed condition.

4.8. Activity-Level Decision Logic

The final activity decision is produced after the breakpoint search. If no breakpoint is detected, the activity is accepted under the assessed condition. If a breakpoint is detected, the decision depends on the reason for the breakpoint.

If the breakpoint decision is Unsafe, the final output is Unsafe at step k_{ij}^* . This has priority because safety cannot be overridden by the presence of partial ability.

If the breakpoint decision is Undetermined, the final output is Undetermined at step k_{ij}^* . This means that more evidence is required before accepting or rejecting functional ability.

If the breakpoint decision is Review, the final output is Professional review required at step k_{ij}^* .

If the breakpoint decision is Blocked and $a_{jk_{ij}^*} = 1$, the final output is Adaptation required at step k_{ij}^* . This does not mean that the activity is already successful. It means that an alternative pathway should be assessed.

If the breakpoint decision is Blocked and $a_{jk_{ij}^*} = 0$, the final output is Blocked at step k_{ij}^* .

Table 9. Activity-level decision logic after breakpoint detection

Breakpoint condition	Adaptability	Final activity-level decision
No breakpoint	Not needed	Achievable under assessed condition
$D_{ij}^* = \text{Unsafe}$	Any value	Unsafe at step k_{ij}^*
$D_{ij}^* = \text{Undetermined}$	Any value	Undetermined at step k_{ij}^*
$D_{ij}^* = \text{Review}$	Any value	Professional review required at step k_{ij}^*

$D_{ij}^* = \text{Blocked}$	$a_{jk_{ij}^*} = 1$	Adaptation required at step k_{ij}^*
$D_{ij}^* = \text{Blocked}$	$a_{jk_{ij}^*} = 0$	Blocked at step k_{ij}^*

Table 9 avoids overclaiming. Adaptation is not treated as success; it is treated as a condition that requires further assessment.

4.9. Explanation Record Generation

The explanation record is generated after the final activity-level decision is assigned. If a breakpoint is detected, the explanation record includes the breakpoint step, the neutrosophic state at that step, the safety-risk value, the local decision, and the final activity-level decision.

The explanation record is:

$$X_{ij} = (s_{jk_{ij}^*}, S_{ij}^*, \rho_{ijk_{ij}^*}, D_{ij}^*, \Omega_{ij}) \quad (15)$$

when a breakpoint exists.

If no breakpoint exists, X_{ij} records that all critical steps passed and stores the full step trace for professional review. This is important because non-critical difficulties may still be clinically meaningful, even if they do not stop the activity.

Table 10. Explanation content produced by N-Break

Explanation item	Purpose
Breakpoint step	Identifies where the activity stopped being acceptable
Neutrosophic state	Shows ability evidence, uncertainty, and failure evidence
Safety-risk value	Shows whether risk controlled the decision
Local decision	Explains the immediate reason for the breakpoint
Activity-level decision	Gives the final interpretable output
Step trace	Allows professional review of previously passed or non-critical steps

Table 10 shows that the algorithm is explainable by design. It does not return a label without the evidence that produced it.

4.10. Handling Non-Critical and Optional Steps

Not every difficulty should stop an activity. Some steps may be optional, replaceable, or non-critical. If a non-critical step receives a decision other than Pass, N-Break records it but does not automatically create a breakpoint. This prevents the system from incorrectly blocking an activity because of a minor or compensable difficulty.

However, a non-critical unsafe step should still be reported in the explanation trace. The reason is that safety information remains important even when the step is not essential for activity completion. The professional user can then decide whether the step should be removed, modified, supervised, or reassessed.

This distinction is important. A breakpoint is created only by a critical non-pass step, but the full trace may contain additional information that supports rehabilitation planning.

4.11. Handling Adaptation

Adaptation is handled cautiously. The variable $a_{jk} = 1$ means that an acceptable alternative pathway may exist. It does not mean that the alternative has already been proven successful. Therefore, when a blocked critical step has $a_{jk} = 1$, N-Break returns Adaptation required at step k_{ij}^* rather than Achievable with adaptation.

This wording is deliberate. It prevents the model from assuming that a rail, assistive device, caregiver, modified posture, or environmental change automatically solves the functional problem. The alternative must be tested or professionally judged before the activity can be accepted as achievable.

4.12. Illustrative Micro-Example

Consider one critical step in stair climbing: transfer body weight to the next stair. Suppose the step has the following neutrosophic state:

$$S_{ijk} = \langle 0.58, 0.18, 0.64 \rangle$$

and the safety-risk value is:

$$\rho_{ijk} = 0.52.$$

Assume that the selected thresholds are:

$$\theta_T = 0.60, \theta_I = 0.50, \theta_F = 0.60, \theta_R = 0.70.$$

The safety risk value is below the safety threshold, and indeterminacy is also below its threshold. However, failure evidence is strong because $F_{ijk} = 0.64 \geq 0.60$. Therefore, the local decision is Blocked.

If this step is critical and no acceptable alternative exists, N-Break reports:

Blocked at the body-weight transfer step.

This example shows why the neutrosophic form matters. The model does not simply ask whether the person has some ability. It also checks failure evidence and uncertainty. Even if $T_{ijk} = 0.58$ suggests partial ability, the high failure evidence prevents the step from being accepted as passed.

4.13. Algorithmic Safeguards

The N-Break algorithm includes four safeguards.

First, safety has priority. A step with unacceptable risk cannot be accepted as passed.

Second, indeterminacy is explicit. If the evidence is missing or conflicting, the system can return Undetermined instead of forcing a decision.

Third, only critical non-pass steps create breakpoints. This prevents optional or compensable difficulties from incorrectly stopping the activity.

Fourth, adaptation is not treated as automatic success. If adaptation may help, the system reports that adaptation is required and should be assessed.

These safeguards make the algorithm suitable for decision support. They also reduce the risk of three common errors: accepting unsafe performance as ability, hiding uncertainty inside a general label, and treating the presence of support as proof of successful performance.

5. Illustrative Case Study and Numerical Evaluation

This section presents a complete illustrative numerical case study for the proposed N-Break system. The aim is to show, step by step, how neutrosophic task-breakpoint reasoning works in functional ability assessment.

This case study is illustrative. It is not a real patient record, and it is not used to make a clinical diagnosis. The numerical values are constructed to demonstrate the logic of the proposed model clearly and transparently. This avoids unsupported clinical claims while allowing the calculation process to be fully shown.

The case study evaluates one hypothetical assessed person, denoted by P_1 , across three daily activities:

Stair climbing

Meal preparation

Upper-body dressing

These activities were selected because they represent different functional situations. Stair climbing demonstrates a blocked critical step. Meal preparation demonstrates unsafe execution. Upper-body dressing demonstrates uncertainty caused by conflicting evidence. Therefore, the case study shows the three main roles of neutrosophic reasoning in N-Break: detecting failure, preserving uncertainty, and preventing unsafe performance from being accepted as ability.

5.1. Numerical Settings

The same decision thresholds are used for all activities in this case study. The selected thresholds are methodological demonstration values, not clinical cut-off values.

Table 11. Threshold values used in the illustrative case study.

Threshold	Value	Meaning
θ_T	0.60	Minimum evidence required to accept successful step performance
θ_I	0.50	Indeterminacy level at which the step becomes undetermined
θ_F	0.60	Failure-evidence level at which the step becomes blocked
θ_R	0.70	Safety-risk level at which the step becomes unsafe

As shown in Table 11, a step is not accepted as passed unless successful-performance evidence is at least 0.60 and the uncertainty, failure evidence, and safety risk are all below their thresholds. These thresholds are used only to demonstrate the reasoning mechanism and should not be interpreted as validated clinical thresholds.

For each step s_{jk} , five input evidence values are used:

u_{1jk} : self-reported ability evidence.

o_{1jk} : observed ability evidence.

b_{1jk} : direct failure evidence.

h_{1jk} : evidence completeness.

ρ_{1jk} : safety-risk value.

All evidence values are normalized to the interval $[0, 1]$, where larger values indicate stronger evidence for the corresponding construct. The symbol h_{1jk} denotes evidence completeness, not health status. A value of $h_{1jk} = 1$ means that the evidence for the step is complete in the illustrative assessment, while a lower value indicates incomplete evidence.

The neutrosophic state is calculated using the following transparent fusion rule:

$$T_{1jk} = \frac{u_{1jk} + o_{1jk}}{2}, I_{1jk} = \max(1 - h_{1jk} | u_{1jk} - o_{1jk} |), F_{1jk} = b_{1jk}. \quad (16)$$

Here, T_{1jk} is the evidence of successful performance, I_{1jk} is the indeterminacy level, and F_{1jk} is the evidence of failure or unacceptable performance.

This rule is deliberately simple. It is used only to make the numerical example traceable. In a real implementation, the evidence-fusion rule may be adjusted or clinically calibrated, but the neutrosophic structure remains the same.

Although the operational N-Break algorithm stops once the first critical breakpoint is detected, all subsequent step values are still reported in this illustrative case study for transparency and numerical verification. This allows the reader to check the full activity profile while understanding that the final decision is controlled by the first critical non-pass step.

5.2. Activities and Step Structure

The three activities are decomposed into ordered steps. Each step has a criticality value c_{jk} and an adaptability value a_{jk} . A critical step is required for safe task continuation. An adaptable step may have a possible alternative pathway, but this does not mean the alternative is already proven successful.

Table 12. Activities, functional steps, criticality, and adaptability.

Activity	Step	Functional step	c_{jk}	a_{jk}
Stair climbing	1	Stand before the first stair	1	0
Stair climbing	2	Lift one foot to the next stair	1	0
Stair climbing	3	Transfer body weight to the next stair	1	1
Stair climbing	4	Repeat the stepping sequence	1	0
Stair climbing	5	Descend safely	1	1
Meal preparation	1	Reach the kitchen workspace	1	0
Meal preparation	2	Retrieve utensils	1	1
Meal preparation	3	Open a food container	1	1
Meal preparation	4	Use a heat source safely	1	1

Meal preparation	5	Carry the hot meal	1	1
Meal preparation	6	Clean the work area	0	1
Upper-body dressing	1	Identify clothing orientation	1	0
Upper-body dressing	2	Insert the first arm	1	0
Upper-body dressing	3	Insert the second arm	1	1
Upper-body dressing	4	Fasten buttons or zipper	1	1
Upper-body dressing	5	Adjust clothing position	0	1

Table 12 shows that N-Break does not treat an activity as a single item. Each activity is represented as an ordered functional process. This allows the model to identify where the activity stops being functionally acceptable.

5.3. Activity 1: Stair Climbing

The first activity is stair climbing. The purpose of this example is to show how N-Break detects a blocked critical step. The input evidence and calculated neutrosophic states are shown in Table 13.

Table 13. Neutrosophic calculation for stair climbing

Step	Functional step	u	o	b	h	ρ	T	I	F	Local decision
1	Stand before the first stair	0.80	0.75	0.10	1.00	0.20	0.775	0.050	0.100	Pass
2	Lift one foot to the next stair	0.70	0.65	0.20	1.00	0.25	0.675	0.050	0.200	Pass
3	Transfer body weight to the next stair	0.60	0.50	0.64	1.00	0.55	0.550	0.100	0.640	Blocked
4	Repeat the stepping sequence	0.50	0.45	0.60	0.80	0.58	0.475	0.200	0.600	Blocked
5	Descend safely	0.45	0.40	0.55	0.80	0.65	0.425	0.200	0.550	Review

The calculations in Table 13 follow Equation (16). For step 3:

$$T_{113} = \frac{0.60 + 0.50}{2} = 0.550.$$

$$I_{113} = \max(1 - 1.00 \mid 0.60 - 0.50) = \max(0 \mid 0.10) = 0.100.$$

$$F_{113} = 0.640.$$

The local decision for step 3 is Blocked because:

$$F_{113} = 0.640 \geq \theta_F = 0.60.$$

Steps 1 and 2 are passed because they have sufficient successful-performance evidence and acceptable uncertainty, failure evidence, and safety risk. Step 3 is the first critical step that does not pass. Therefore, the breakpoint is:

$$k_{11}^* = 3.$$

The final activity-level decision is:

$$\Omega_{11} = \text{Adaptation required at step 3.}$$

This decision is used because step 3 is blocked and its adaptability value is $a_{1,3} = 1$, as shown in Table 12. This does not mean that stair climbing is already successful with adaptation. It means that the blocked step has a possible alternative pathway that should be assessed, such as a rail, supervision, step modification, or another mobility strategy.

The explanation record is:

$$X_{11} = (\text{Transfer body weight to the next stair}, \langle 0.550 \mid 0.100 \mid 0.640 \rangle, 0.55, \text{Blocked}, \text{Adaptation required at step 3}).$$

The practical interpretation is clear. The problem is not simply "difficulty with stairs." The functional interruption occurs specifically at the body-weight transfer step. This gives a more useful rehabilitation explanation than a broad activity label.

5.4. Activity 2: Meal Preparation

The second activity is meal preparation. The purpose of this example is to show how N-Break handles unsafe execution. A person may show evidence of ability, but the activity should not be accepted if a critical step has unacceptable safety risk.

The input evidence and calculated neutrosophic states are shown in Table 14.

Table 14. Neutrosophic calculation for meal preparation

Step	Functional step	u	o	b	h	ρ	T	I	F	Local decision
1	Reach the kitchen workspace	0.85	0.80	0.05	1.00	0.10	0.825	0.050	0.050	Pass
2	Retrieve utensils	0.80	0.72	0.15	1.00	0.20	0.760	0.080	0.150	Pass
3	Open a food container	0.65	0.62	0.25	1.00	0.25	0.635	0.030	0.250	Pass
4	Use a heat source safely	0.70	0.60	0.30	1.00	0.78	0.650	0.100	0.300	Unsafe
5	Carry the hot meal	0.50	0.45	0.55	0.90	0.74	0.475	0.100	0.550	Unsafe
6	Clean the work area	0.60	0.55	0.30	0.85	0.25	0.575	0.150	0.300	Review

For step 4, the neutrosophic components are calculated as follows:

$$T_{124} = \frac{0.70 + 0.60}{2} = 0.650.$$

$$I_{124} = \max(1, -1.00 | 0.70 - 0.60 |) = \max(0, 0.10) = 0.100.$$

$$F_{124} = 0.300.$$

The value $T_{124} = 0.650$ indicates that there is evidence of successful performance. However, the safety-risk value is:

$$\rho_{124} = 0.78.$$

Since:

$$\rho_{124} = 0.78 \geq \theta_R = 0.70,$$

the local decision is Unsafe.

Step 4 is the first critical step that does not pass. Therefore:

$$k_{12}^* = 4.$$

The final activity-level decision is:

$$\Omega_{12} = \text{Unsafe at step 4}.$$

The explanation record is:

$$X_{12} = (\text{Use a heat source safely}, \langle 0.650, 0.100, 0.300 \rangle, 0.78, \text{Unsafe}, \text{Unsafe at step 4}).$$

This result shows an important role of N-Break. The model does not classify the step as successful only because T_{124} is above the success threshold. Safety has priority. Therefore, the system prevents unsafe attempted performance from being treated as functional ability.

The practical interpretation is that the main problem in this activity is not reaching the kitchen, retrieving utensils, or opening a container. The activity becomes unacceptable at the heat-use step because of safety risk. This output can guide professional review toward safe cooking strategies, supervision, environmental modification, or alternative meal-preparation methods. N-Break itself does not prescribe any treatment; it only identifies the decision-relevant breakpoint.

5.5. Activity 3: Upper-Body Dressing

The third activity is upper-body dressing. The purpose of this example is to show how N-Break handles conflicting evidence. The activity is not classified as blocked because failure evidence is not strong enough. It is not classified as passed because the evidence is too inconsistent. Therefore, the correct decision is Undetermined.

The input evidence and calculated neutrosophic states are shown in Table 15.

Table 15. Neutrosophic calculation for upper-body dressing.

Step	Functional step	u	o	b	h	ρ	T	I	F	Local decision
1	Identify clothing orientation	0.90	0.85	0.05	1.00	0.05	0.875	0.050	0.050	Pass
2	Insert the first arm	0.75	0.70	0.20	1.00	0.15	0.725	0.050	0.200	Pass
3	Insert the second arm	0.70	0.68	0.22	1.00	0.15	0.690	0.020	0.220	Pass
4	Fasten buttons or zipper	0.80	0.25	0.35	1.00	0.20	0.525	0.550	0.350	Undetermined
5	Adjust clothing position	0.65	0.60	0.20	0.80	0.10	0.625	0.200	0.200	Pass

For step 4, the neutrosophic components are:

$$T_{134} = \frac{0.80 + 0.25}{2} = 0.525.$$

$$I_{134} = \max(1 - 1.00 \mid 0.80 - 0.25) = \max(0 \mid 0.55) = 0.550.$$

$$F_{134} = 0.350.$$

The failure evidence is below the blocking threshold:

$$F_{134} = 0.350 < \theta_F = 0.60.$$

The safety risk is also below the unsafe threshold:

$$\rho_{134} = 0.20 < \theta_R = 0.70.$$

However, the indeterminacy value is high:

$$I_{134} = 0.550 \geq \theta_I = 0.50.$$

Therefore, the local decision is Undetermined.

Step 4 is a critical step, and it is the first non-pass step in this activity. Therefore:

$$k_{13}^* = 4.$$

The final activity-level decision is:

$$\Omega_{13} = \text{Undetermined at step 4.}$$

The explanation record is:

$$X_{13} = (\text{Fasten buttons or zipper} \langle 0.525 \mid 0.550 \mid 0.350 \rangle, 0.20 \mid \text{Undetermined}, \text{Undetermined at step 4}).$$

This result shows why indeterminacy is essential. A binary model may force the step into either able or unable. A simple partial-ability score may only report moderate difficulty. N-Break gives a more careful decision: the evidence is conflicting and should not be treated as a reliable pass or a confirmed failure. The practical response is not automatic adaptation or treatment; it is reassessment, additional observation, or clarification of the conflicting evidence.

5.6. Summary of Activity-Level Outputs

The final results across the three activities are summarized in Table 16.

Table 16. Final N-Break outputs for the illustrative case study

Activity	Breakpoint k^*	Breakpoint step	Neutrosophic state at breakpoint	Local decision	Final activity-level decision
Stair climbing	3	Transfer body weight to the next stair	$\langle 0.550 \mid 0.100 \mid 0.640 \rangle$	Blocked	Adaptation required at step 3
Meal preparation	4	Use a heat source safely	$\langle 0.650 \mid 0.100 \mid 0.300 \rangle$	Unsafe	Unsafe at step 4

Upper-body dressing	4	Fasten buttons or zipper	$\langle 0.525, 0.550, 0.350 \rangle$	Undetermined	Undetermined at step 4
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Table 16 shows that the three activities do not receive the same type of decision. This is important. N-Break does not collapse all functional problems into one general label. It identifies the mechanism of interruption in each activity.

For stair climbing, the interruption is caused by failure evidence at a critical transfer step. For meal preparation, the interruption is caused by safety risk during heat use. For upper-body dressing, the interruption is caused by conflicting evidence during fastening. These are three different functional meanings, and they require different professional responses.

5.7. Why the Neutrosophic Representation Matters

The case study demonstrates the practical role of the neutrosophic state $S_{1jk} = \langle T_{1jk}, I_{1jk}, F_{1jk} \rangle$. Each component affects the decision in a different way.

In stair climbing, step 3 has:

$$S_{113} = \langle 0.550, 0.100, 0.640 \rangle.$$

The important component is $F_{113} = 0.640$. This value crosses the failure threshold, so the step is blocked. The model therefore identifies a functional breakpoint caused by failure evidence.

In meal preparation, step 4 has:

$$S_{124} = \langle 0.650, 0.100, 0.300 \rangle.$$

The success-evidence component is acceptable, but the safety-risk value is high. Therefore, the activity is classified as unsafe at that step. This prevents a dangerous interpretation in which attempted performance is mistaken for functional ability.

In upper-body dressing, step 4 has:

$$S_{134} = \langle 0.525, 0.550, 0.350 \rangle.$$

The dominant issue is indeterminacy. The model does not claim that the person is unable. It also does not accept the step as passed. It correctly reports that the step is undetermined because the evidence is conflicting.

These three cases show that neutrosophic reasoning is not an optional addition. It is the main mechanism that allows the system to distinguish between blocked performance, unsafe performance, and uncertain evidence.

5.8. Comparison with a Simplified Non-Neutrosophic Interpretation

To clarify the value of the proposed approach, Table 17 compares N-Break with a simplified non-neutrosophic interpretation. The simplified interpretation only asks whether the person shows enough ability evidence. It does not independently preserve uncertainty, failure evidence, or safety risk.

Table 17. Difference between simplified interpretation and N-Break.

Activity	Critical breakpoint evidence	Simplified interpretation	N-Break interpretation
Stair climbing	$T = 0.550, I = 0.100, F = 0.640, \rho = 0.55$	Partial or weak ability	Blocked at body-weight transfer; adaptation required
Meal preparation	$T = 0.650, I = 0.100, F = 0.300, \rho = 0.78$	Ability may be accepted because T is sufficient	Unsafe at heat-use step because safety risk is unacceptable
Upper-body dressing	$T = 0.525, I = 0.550, F = 0.350, \rho = 0.20$	Partial ability or mild failure	Undetermined because evidence is conflicting

Table 17 shows that a simplified interpretation can miss the reason behind the functional problem. In meal preparation, it may incorrectly accept the heat-use step because $T = 0.650$. In

dressing, it may force the result into partial ability, even though the evidence conflict is the main issue. N-Break avoids these errors by preserving the three evidence components separately and by giving priority to safety and uncertainty.

5.9. Practical Interpretation of the Case Study

The numerical case study shows that N-Break can support professional reasoning in three ways.

First, it locates the exact step where the activity loses functional acceptability. This is clearer than reporting that the whole activity is difficult.

Second, it explains the reason for the interruption. The reason may be failure evidence, safety risk, indeterminacy, or weak evidence requiring review.

Third, it avoids overclaiming. The model does not say that a person is clinically disabled, does not prescribe treatment, and does not automatically approve an adaptation. It only provides an interpretable functional decision that should be reviewed by a qualified professional.

For P_1 , the case study suggests the following decision-support interpretation:

Stair climbing is interrupted at the body-weight transfer step. The blocked step has a possible adaptation pathway, so adaptation should be assessed rather than assumed successful.

Meal preparation is interrupted at the heat-use step because of safety risk. The person's ability evidence is not enough to override the unsafe condition.

Upper-body dressing is interrupted at the fastening step because the evidence is conflicting. The correct response is additional assessment rather than a forced pass or fail judgment.

This interpretation demonstrates the central contribution of N-Break: functional ability is evaluated as a sequence of meaningful steps, and the model identifies the earliest critical point where the activity becomes blocked, unsafe, or uncertain.

6. Practical Implications, and Limitations

This section discusses the meaning of the illustrative case study and clarifies the practical value and limitations of the proposed N-Break system. The aim is not to repeat the numerical calculations from Section 5, but to explain what the results demonstrate about neutrosophic task-breakpoint reasoning.

6.1. Main Interpretation of the Results

The case study shows that functional ability assessment can be more informative when the activity is analyzed as a sequence of steps rather than as one general item. In the three illustrative activities, N-Break did not produce one general disability label. Instead, it identified the exact step where each activity stopped being functionally acceptable.

The three activities demonstrated three different functional problems. In stair climbing, the activity was interrupted by a blocked critical step. In meal preparation, the activity was interrupted by a safety risk. In upper-body dressing, the activity was interrupted by uncertain and conflicting evidence. These outputs show that similar final functional difficulty may have different causes.

This distinction is important. A blocked step, an unsafe step, and an uncertain step should not lead to the same professional response. A blocked step may require training, adaptation, or an alternative pathway. An unsafe step may require risk control or supervised reassessment. An uncertain step may require additional observation or better evidence. N-Break helps separate these cases clearly.

6.2. Contribution of Neutrosophic Reasoning

The main contribution of the proposed system is the use of neutrosophic reasoning at the step level. Each functional step is represented through three components: evidence of successful performance, indeterminacy, and evidence of failure or unacceptable performance. This structure is essential because functional evidence is often incomplete, inconsistent, or affected by safety concerns.

The case study demonstrates that the three neutrosophic components do not play the same role. In stair climbing, the failure component controlled the decision. In meal preparation, safety risk

controlled the decision even though evidence of ability was present. In upper-body dressing, the indeterminacy component controlled the decision because self-reported and observed evidence were strongly inconsistent.

This shows that N-Break does not treat functional ability as a simple yes-or-no condition. It also does not reduce the assessment to a single partial-ability value. Instead, it preserves the reason behind the decision. This is the central value of using a neutrosophic model in the proposed system.

6.3. Difference from General Scoring Approaches

General scoring approaches are useful when the goal is to summarize functional difficulty. However, they may not explain the internal reason why a person cannot complete a specific activity. A single score may show that an activity is difficult, but it may not identify the exact step that caused the difficulty.

N-Break addresses a different problem. It does not ask how severe the person's disability is. It asks where the task becomes blocked, unsafe, or uncertain. This makes the system suitable for situations where the professional user needs an interpretable task-level explanation.

For example, the output "unsafe at the heat-use step" is more actionable than a broad statement such as "difficulty preparing meals." Similarly, the output "undetermined at the fastening step" is more careful than forcing the dressing activity into either pass or fail. The proposed system therefore adds value by making the assessment more specific and less ambiguous.

6.4. Practical Implications

The practical value of N-Break lies in its ability to support structured functional reasoning. It can help professionals identify the step that requires attention, determine whether the problem is caused by failure, safety risk, or uncertainty, and decide whether additional assessment is needed.

Table 18. Practical implications of N-Break outputs

N-Break output	Practical meaning	Possible professional response
Blocked at a critical step	The person cannot complete a necessary step under the assessed condition	Review the failed step, consider training, adaptation, or task modification
Unsafe at a critical step	The person may attempt the step, but risk is unacceptable	Control risk, supervise reassessment, or modify the task environment
Undetermined at a critical step	Evidence is incomplete or conflicting	Collect more evidence or repeat observation
Professional review required	Evidence is weak but not conclusive	Use expert judgment before accepting or rejecting the step
Adaptation required	A possible alternative pathway exists but is not yet verified	Test the assistance or environmental alternative before accepting success
Achievable under assessed condition	All critical steps passed under the assessed condition	Record the activity as functionally acceptable in that context

Table 18 shows that N-Break is not intended to replace professional reasoning. It organizes the reasoning process and makes the decision traceable. The final decision should still be reviewed by a qualified professional.

6.5. Safety and Professional Boundaries

Safety is a central part of the proposed system. A step should not be accepted as passed if safety risk is unacceptable. This prevents a common assessment error: treating attempted performance as successful performance. A person may be able to start an action, but if the action creates high risk, the system should not classify it as functionally acceptable.

The system also avoids diagnostic overclaiming. N-Break does not diagnose a disability, does not determine legal disability status, does not prescribe treatment, and does not replace rehabilitation specialists, clinicians, or occupational therapists. Its output is a decision-support explanation. It identifies where the activity becomes problematic and why.

This boundary is important because functional ability assessment may influence rehabilitation planning, assistive support, and safety decisions. A computational model should therefore support professional judgment, not replace it.

6.6. Limitations

The proposed study has several limitations:

First, the case study is illustrative. It demonstrates the calculation and reasoning process, but it is not a clinical validation study. The numerical values were constructed for transparency and should not be interpreted as real patient data.

Second, the decision thresholds used in the case study are demonstration values. They are not validated clinical cut-off values. Future work should calibrate these thresholds using expert-rated cases, observed functional assessments, or real rehabilitation data.

Third, the evidence-fusion rule used in the case study is intentionally simple. It makes the calculations easy to follow, but real applications may require more advanced fusion rules that account for source reliability, repeated observations, task difficulty, assistive-device use, and environmental variation.

Fourth, the current model assumes that the activity steps are correctly defined and ordered. In practice, task decomposition should be designed or reviewed by professionals because different settings may require different step structures.

Fifth, N-Break detects the first critical breakpoint. This is useful for identifying the earliest interruption, but later steps may still contain clinically relevant information. For this reason, the full step trace should be stored even after the first breakpoint is detected.

6.7. Implications for Future Validation

For future validation, N-Break should be tested on real functional assessment cases. A useful validation design would compare the system output with professional judgments from rehabilitation specialists or occupational therapists. The key question should not be whether N-Break predicts a general disability score. Rather, the question should be whether it correctly identifies the task step where performance becomes blocked, unsafe, or uncertain.

Future studies should also examine inter-rater agreement in assigning neutrosophic evidence values, the stability of breakpoint detection under different thresholds, and the effect of missing or conflicting evidence on the final activity-level decision. These tests would strengthen the reliability and practical credibility of the system.

7. Conclusion and Future Work

This paper proposed N-Break, a neutrosophic task-breakpoint reasoning system for assessing the functional ability of persons with disabilities under uncertainty. The main idea of the proposed system is that functional ability should not always be assessed as one general label or one overall score. Many daily activities are sequential, and a person may fail, become unsafe, or face uncertainty at a specific critical step inside the activity.

The proposed model decomposes a daily activity into ordered functional steps and represents each step using a neutrosophic state composed of evidence of successful performance, indeterminacy, and evidence of failure or unacceptable execution. This structure allows the system to preserve uncertainty instead of hiding it inside a binary or single-valued judgment. It also allows unsafe attempted performance to be separated from genuine functional ability.

The main contribution of N-Break is its ability to identify the functional breakpoint. Instead of saying only that an activity is difficult, the system explains where the activity becomes blocked,

unsafe, or undetermined. This makes the output more useful for professional review because it points to the exact step that needs attention.

The illustrative numerical case study showed how the proposed reasoning process works in three different situations. Stair climbing demonstrated a blocked critical step. Meal preparation demonstrated unsafe execution. Upper-body dressing demonstrated uncertainty caused by conflicting evidence. These examples showed that different functional interruptions require different interpretations and should not be collapsed into one broad judgment.

N-Break should be understood as a decision-support framework. It does not diagnose disability, does not determine legal disability status, does not prescribe treatment, and does not replace clinicians, rehabilitation specialists, or occupational therapists. Its value lies in making functional reasoning more structured, transparent, and explainable under uncertain evidence.

Future work should focus on five directions. First, the proposed model should be validated using real functional assessment cases reviewed by qualified rehabilitation professionals. Second, the threshold values should be calibrated using expert-rated observations rather than illustrative values. Third, the evidence-fusion function should be extended to include source reliability, repeated observations, assistive-device use, and environmental variation. Fourth, inter-rater agreement should be studied to examine whether different assessors produce consistent neutrosophic step states. Fifth, a professional software prototype can be developed to support structured task-level assessment while preserving auditability and human oversight.

In conclusion, N-Break provides a focused and interpretable neutrosophic reasoning approach for functional ability assessment. Its strength is not in producing a general score, but in identifying the point and reason for functional task interruption. This makes it a promising foundation for future rehabilitation-oriented decision-support systems under uncertainty.

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N-EviChain: A Neutrosophic TwoFold Evidence-Chain Reasoning System for Legal Fact Support Assessment under Uncertainty

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Abstract: Legal case files often contain heterogeneous evidence, such as witness statements, expert reports, digital records, medical documents, inspection notes, and technical observations. These evidence items may support a legal fact, leave it uncertain, or weaken it at the same time. This paper proposes N-EviChain, a neutrosophic TwoFold evidence-chain reasoning system for legal fact support assessment under uncertainty. The proposed model represents each legal fact through a dedicated evidence chain. Each evidence item is modeled by three neutrosophic components: support, indeterminacy, and opposition. A Neutrosophic TwoFold Evidence-Chain Law is introduced to combine both the legal connection between evidence items and their neutrosophic states. An illustrative industrial workplace injury case demonstrates how the model distinguishes between supported facts, opposed fact support, evidential gaps, and uncertain legal fact support. The system is not designed to determine guilt, liability, admissibility, or judicial truth. Its role is to organize heterogeneous evidence, identify uncertainty and gaps, and generate explainable reports for professional legal review.

Keywords: Neutrosophic TwoFold Algebra; evidence-chain reasoning; legal fact support; legal uncertainty; evidential gaps; explainable legal AI; LegalTech; court decision support.

1. Introduction

Legal reasoning often depends on the way evidence supports specific legal facts [1]. A case file may contain different kinds of evidence, such as witness statements, expert reports, medical documents, inspection notes, machine records, photographs, digital logs, and technical observations [2]. These items are heterogeneous in nature [3]. They do not have the same source, role, reliability, technical clarity, or legal relevance [4]. Therefore, the main challenge is not only to collect evidence, but also to understand how different evidence items form a meaningful chain of support for a particular legal fact [5].

In many cases, one legal fact may be supported by some evidence, weakened by another item, and still remain uncertain because of missing or incomplete information [6]. For example, in an industrial workplace injury case, the occurrence of the accident may be well documented, while the existence of a prior machine defect, prior detectability, or causal contribution may remain uncertain [7]. This shows that a legal file should not be reduced to one general conclusion [8]. Each legal fact should be assessed through its own evidence chain [9].

This paper proposes N-EviChain, a neutrosophic TwoFold evidence-chain reasoning system for legal fact support assessment under uncertainty. The proposed system is inspired by Neutrosophic TwoFold Algebra, which is based on two connected levels: one level concerns the elements themselves, and another level concerns the neutrosophic components (t, i, f) attached to those

elements. In the present work, the elements are legal evidence items, while the neutrosophic components represent support, indeterminacy, and opposition.

The motivation for using a TwoFold structure is that legal evidence has both a legal-relational side and an uncertainty side. The legal-relational side asks how one evidence item connects to another inside the chain [10]. The uncertainty side asks whether the evidence supports the fact, leaves it unclear, or weakens it [11]. This is consistent with the general TwoFold Law, which combines a law for the elements with a law for their neutrosophic components, and allows the relation between the two sub-laws to depend on the application [13].

The proposed model treats a legal case file as a heterogeneous evidential space. This modeling choice is also consistent with the source idea that TwoFold Algebra can be applied in fields whose spaces are composed of heterogeneous subspaces [12]. In a legal context, the heterogeneous subspaces are the different types of evidence and their different legal roles.

The main objective of N-EviChain is limited and reviewable. It does not determine guilt, liability, admissibility, compensation, or judicial truth. It does not replace judges, lawyers, prosecutors, experts, or legal professionals [15]. Instead, it provides a structured decision-support framework that organizes evidence around legal facts, evaluates support under uncertainty, detects evidential gaps, and generates explainable outputs for professional legal review.

The main contributions of this paper are as follows. First, it introduces a fact-specific evidence-chain representation for legal reasoning under uncertainty. Second, it adapts the principle of Neutrosophic TwoFold Algebra to legal evidence by proposing a Neutrosophic TwoFold Evidence-Chain Law. Third, it separates support, indeterminacy, and opposition instead of reducing legal evidence to a single score. Fourth, it demonstrates the model through an illustrative industrial workplace injury case, showing how the system can distinguish supported facts, opposed fact support, evidential gaps, and uncertain legal fact support. Fifth, it explains how the proposed model can support future LegalTech software for court evidence management while preserving human legal authority [14].

The remainder of the paper is organized as follows. Section 2 presents the neutrosophic TwoFold laws used in the proposed work. Section 3 introduces the proposed N-EviChain model. Section 4 presents an illustrative legal case study with complete numerical calculations. Section 5 discusses the analysis, managerial implications, and LegalTech and industrial dispute relevance. Section 6 links the proposed model with Egypt Vision 2030. Section 7 interprets the results and explains the methodological importance of the proposed work. Section 8 concludes the paper and outlines future work.

2. Neutrosophic TwoFold Laws Used in the Proposed Work

This section defines the neutrosophic TwoFold laws used in the proposed N-EviChain system. The aim is not to review all forms of neutrosophic algebra, but to adapt the specific principle of Neutrosophic TwoFold Algebra to legal evidence-chain reasoning. The section introduces the source principle, the proposed legal evidence-chain law, the legal connection sub-law, the neutrosophic component-fusion sub-law, and the sequential construction of an evidence chain.

2.1. Source Principle of Neutrosophic TwoFold Algebra

Neutrosophic TwoFold Algebra is based on two connected levels of operation. The first level concerns the elements themselves, while the second level concerns the neutrosophic components attached to those elements. In the source formulation, the first algebra is related to the elements of a set, and the second algebra is related to the neutrosophic components (t, i, f) of those elements.

This two-level structure is suitable for legal evidence-chain reasoning because a legal case file contains heterogeneous evidence items. A witness statement, CCTV record, maintenance log, medical report, inspection note, and technical expert report are not the same type of evidence. However, each item may provide support, create uncertainty, or weaken a specific legal fact. The source paper states

that TwoFold Algebra can be applied in fields whose spaces are composed of heterogeneous subspaces, which supports modeling a case file as a heterogeneous evidential space.

In N-EviChain, each evidence item is treated as a legal element with an attached neutrosophic state:

$$E_{kr}(\tau_{kr}, \iota_{kr}, \phi_{kr}),$$

where E_{kr} is the r -th evidence item associated with legal fact F_k , τ_{kr} is the degree of support provided by the evidence item, ι_{kr} is the degree of indeterminacy, and ϕ_{kr} is the degree of opposition or weakening effect.

The three components τ_{kr} , ι_{kr} , and ϕ_{kr} belong to $[0, 1]$, but they are not required to sum to one. This is consistent with the neutrosophic view that support, indeterminacy, and opposition may coexist without being reduced to a single probability distribution.

2.2. Legal Evidence-Chain TwoFold Law

The original TwoFold Law is formed by two sub-laws: one sub-law for the elements and another sub-law for the neutrosophic components. The relationship between these two sub-laws may be dependent, partially dependent, partially independent, or independent according to the application. Following this principle, N-EviChain defines a legal version of the TwoFold Law called the Neutrosophic TwoFold Evidence-Chain Law.

Let $E_a(\tau_a, \iota_a, \phi_a)$ and $E_b(\tau_b, \iota_b, \phi_b)$ be two evidence items linked to the same legal fact. The proposed evidence-chain law is:

$$E_a(\tau_a, \iota_a, \phi_a) \Delta_{NEC} E_b(\tau_b, \iota_b, \phi_b) = (E_a \# E_b) [(\tau_a, \iota_a, \phi_a) *_{\lambda_{ab}} (\tau_b, \iota_b, \phi_b)]. \quad (4)$$

Here, Δ_{NEC} is the proposed Neutrosophic TwoFold Evidence-Chain Law. The symbol $\#$ denotes the legal connection sub-law between evidence items, while $*_{\lambda_{ab}}$ denotes the neutrosophic component-fusion sub-law. The value $\lambda_{ab} \in [0, 1]$ is the evidential-link coefficient between E_a and E_b . A higher value of λ_{ab} means that the two evidence items form a stronger legal connection in supporting the same legal fact.

Equation (4) is the core law used in this paper. It means that evidence combination is not only numerical. The system combines two levels at the same time: the legal connection between evidence items and the neutrosophic state of their support, indeterminacy, and opposition.

2.3. Legal Connection Sub-Law

The legal connection sub-law is defined as:

$$E_a \# E_b = E_{a \rightarrow b}^{(\lambda_{ab})}. \quad (5)$$

Here, $E_{a \rightarrow b}^{(\lambda_{ab})}$ is a composite evidence-chain element formed from E_a and E_b . The coefficient λ_{ab} does not represent credibility, guilt, liability, or judicial truth. It only expresses the degree to which two evidence items are legally connected inside the evidence chain.

For example, a medical report may strongly establish that an injury occurred, but it may not by itself establish that a machine defect caused the injury. A technical expert report or machine sensor record may be needed to connect the injury to the alleged defect. Therefore, the legal connection between evidence items is different from the strength of each evidence item considered alone.

Table 1. Evidential-link interpretation in N-EviChain

Evidential-link coefficient	Interpretation	Legal meaning in the evidence chain
$\lambda_{ab} \geq 0.80$	Strong connection	The two items directly support the same legal fact or complete a necessary evidential link
$0.60 \leq \lambda_{ab} < 0.80$	Moderate connection	The two items are relevant, but one link in the chain remains partly uncertain
$0.40 \leq \lambda_{ab} < 0.60$	Weak connection	The relation between the two items is incomplete or requires further legal review

$\lambda_{ab} < 0.40$	Fragmented connection	The evidence items are present, but they do not form a reliable evidential chain
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As shown in Table 1, λ_{ab} is a chain-connection parameter, not a legal threshold. The ranges in the table are operational interpretation ranges used for model demonstration. They are not statutory legal standards and should be calibrated or reviewed by qualified legal experts before deployment in any real court-support system.

2.4. Neutrosophic Component-Fusion Sub-Law

The second sub-law combines the neutrosophic states of two evidence items. The source paper describes TwoFold Algebra as a hybrid structure in which an operation on the elements is interrelated with an operation on fuzzy or neutrosophic components. It also presents a numerical example where the neutrosophic components are combined by average-based component operations.

In N-EviChain, the component-fusion sub-law is adapted to legal evidence-chain reasoning as follows:

$$(\tau_a, \iota_a, \phi_a) *_{\lambda_{ab}} (\tau_b, \iota_b, \phi_b) = \left(\lambda_{ab} \cdot \frac{\tau_a + \tau_b}{2}, \min \left(1, \frac{\iota_a + \iota_b}{2} + 1 - \lambda_{ab} \right) \frac{\phi_a + \phi_b}{2} \right). \quad (6)$$

The first component in Equation (6) is the combined support degree. It is reduced when the legal connection between two evidence items is weak. This prevents the model from treating unrelated or poorly connected evidence as a strong evidence chain.

The second component is the combined indeterminacy degree. It increases when λ_{ab} is low because a weak evidential connection creates uncertainty in the legal chain.

The third component is the combined opposition degree. It is calculated as the average opposition effect of the two evidence items. This keeps opposition separate from indeterminacy. In this model, weak connection mainly increases indeterminacy, while direct evidence against the fact is represented by the opposition component.

2.5. Sequential Evidence-Chain Construction

For a legal fact F_k , the evidence chain is processed sequentially according to a declared legal reasoning order. This order does not have to be chronological. It should reflect how the evidence items are intended to support the legal fact.

The first evidence item initializes the chain:

$$\mathcal{C}_k^{(1)} = E_{k1}(\tau_{k1}, \iota_{k1}, \phi_{k1}).$$

Here, $\mathcal{C}_k^{(1)}$ denotes the first composite state of the evidence chain for legal fact F_k . Therefore, the initial neutrosophic state of the chain is:

$$T_k^{(1)} = \tau_{k1}, I_k^{(1)} = \iota_{k1}, O_k^{(1)} = \phi_{k1}.$$

At each recursive step, $\mathcal{C}_k^{(r-1)}$ is treated as a composite evidence-chain element with the current neutrosophic state:

$$\mathcal{C}_k^{(r-1)} = G_k^{(r-1)}(T_k^{(r-1)}, I_k^{(r-1)}, O_k^{(r-1)}).$$

Here, $G_k^{(r-1)}$ denotes the composite legal evidence element formed after processing the first $r-1$ evidence items, while $T_k^{(r-1)}$, $I_k^{(r-1)}$, and $O_k^{(r-1)}$ denote its current support, indeterminacy, and opposition values. The symbol O is used for opposition to avoid confusion with the legal fact symbol F_k .

For $r = 2, \dots, n_k$, the chain is updated by applying the proposed TwoFold evidence-chain law:

$$\mathcal{C}_k^{(r)} = \mathcal{C}_k^{(r-1)} \Delta_{NEC} E_{kr}(\tau_{kr}, \iota_{kr}, \phi_{kr}). \quad (7)$$

Using Equation (6), the updated neutrosophic state of the chain becomes:

$$T_k^{(r)} = \lambda_{r-1,r} \frac{T_k^{(r-1)} + \tau_{kr}}{2},$$

$$I_k^{(r)} = \min \left(1, \frac{I_k^{(r-1)} + \iota_{kr}}{2} + 1 - \lambda_{r-1,r} \right),$$

$$O_k^{(r)} = \frac{O_k^{(r-1)} + \phi_{kr}}{2}.$$

Here, $\lambda_{r-1,r}$ is the evidential-link coefficient between the current composite chain $C_k^{(r-1)}$ and the next evidence item E_{kr} . This recursive form makes the evidence-chain construction transparent and suitable for software implementation.

After processing all n_k evidence items, the final values are defined as:

$$T_k = T_k^{(n_k)}, I_k = I_k^{(n_k)}, O_k = O_k^{(n_k)}.$$

The final neutrosophic support state of legal fact F_k is then:

$$N(F_k) = \langle T_k, I_k, O_k \rangle. \quad (8)$$

Here, T_k is the final support degree of legal fact F_k , I_k is the final indeterminacy degree, and O_k is the final opposition degree.

2.6. Boundaries of the Proposed Law

The proposed law is designed for legal decision support only. It does not determine guilt, liability, judicial truth, or final evidentiary sufficiency. It only helps organize heterogeneous evidence, measure how evidence chains support specific legal facts, and identify where uncertainty or evidential gaps appear.

The law also does not replace professional legal reasoning. The values of τ , ι , ϕ , and λ should be assigned through a transparent annotation process by qualified legal or technical reviewers, or through a documented software-assisted procedure that remains open to human audit.

3. Proposed N-EviChain Model

This section presents the proposed N-EviChain model. The model evaluates how a heterogeneous chain of legal evidence supports a specific legal fact under uncertainty. It is built on the TwoFold principle that one operation concerns the elements themselves and another operation concerns their neutrosophic components. In this paper, the legal elements are evidence items, and the neutrosophic components are support, indeterminacy, and opposition.

N-EviChain does not detect witness contradictions, does not rank persons, does not produce a guilt score, and does not determine liability. Its purpose is to organize evidence chains, trace support for each legal fact, identify evidential gaps, and generate explainable outputs for professional legal review.

3.1. Model Objective and Scope

Let F_k denote a legal fact under review. The objective of N-EviChain is to construct an evidence-chain explanation for F_k , not to decide the whole case. The model receives the legal fact, the evidence items linked to it, their legal roles, their neutrosophic states, and the evidential-link coefficients between them. It then produces a fact-level support state and a review-oriented explanation.

The model is defined as:

$$\mathcal{M}_{NEC}: (F_k, C_k, R_k, \Lambda_k, \Theta) \rightarrow X_k. \quad (9)$$

Here, $\mathcal{M}_{NEC}$ is the N-EviChain model, F_k is the legal fact being assessed, C_k is the evidence chain assigned to that fact, R_k is the set of legal reasoning roles assigned to the evidence items, Λ_k is the set of evidential-link coefficients in the chain, Θ is the set of operational interpretation thresholds, and X_k is the final explanation record.

The model answers a limited and reviewable question:

To what extent does the evidence chain support legal fact F_k , and what is the main source of uncertainty, opposition, or evidential weakness?

This scope is essential. N-EviChain is not a substitute for judicial reasoning. It is a structured decision-support model for organizing and explaining evidence-chain support.

3.2. Case File as a Heterogeneous Evidential Space

A legal case file is represented as a heterogeneous evidential space:

$$\mathcal{K} = (\mathcal{F}, \mathcal{E}, \mathcal{L}), \quad (10)$$

where $\mathcal{F} = \{F_1, F_2, \dots, F_m\}$ is the set of legal facts under review, $\mathcal{E}$ is the set of available evidence items, and $\mathcal{L}$ is the set of evidential links between evidence items.

This representation is consistent with the source idea that TwoFold Algebra can be applied in fields whose spaces are composed of heterogeneous subspaces. In the present work, the heterogeneous subspaces are the different types of legal evidence, such as testimony, technical reports, medical reports, digital records, inspection notes, and expert opinions.

Each legal fact is assessed separately. This prevents the model from reducing the whole case to a single conclusion. It also reflects the fact that one evidence item may support one legal fact strongly but support another legal fact weakly.

For example, a medical report may strongly support the fact that an injury occurred, but it may not strongly support the fact that a machine defect caused the injury. Therefore, the same physical evidence item may appear in more than one evidence chain, but its neutrosophic state may differ from one legal fact to another because legal relevance is fact-dependent.

3.3. Evidence-Chain Layer

For each legal fact F_k , N-EviChain constructs an ordered evidence chain:

$$C_k = (E_{k1}, E_{k2}, \dots, E_{kn_k}), \quad (11)$$

where E_{kr} is the r -th evidence item linked to F_k , and n_k is the number of evidence items in the chain.

The order of C_k is not necessarily chronological. It follows the declared legal reasoning path. For example, in an industrial workplace injury claim, the chain may first establish that the accident occurred, then examine whether a defect existed, then examine whether the defect was detectable, and finally examine whether the defect contributed to the injury.

Each evidence item is represented as:

$$E_{kr}(\tau_{kr}, \iota_{kr}, \phi_{kr}), \quad (12)$$

where τ_{kr} is the support degree, ι_{kr} is the indeterminacy degree, and ϕ_{kr} is the opposition degree of evidence item E_{kr} with respect to legal fact F_k .

These values are fact-specific. The same document, report, or record may receive different values when it is used to support different legal facts.

3.4. Legal Role Layer

N-EviChain separates the type of evidence from its legal reasoning role. The type of evidence describes what the item is, such as a statement, report, record, or image. The role describes how the item contributes to the legal fact under review.

Let $\mathcal{R}$ denote an extensible set of legal reasoning roles. For each evidence item E_{kr} , the assigned role set is:

$$R_{kr} \subseteq \mathcal{R}. \quad (13)$$

An evidence item may have one role or more than one role, depending on the legal fact being assessed. The role set is extensible because different types of cases may require different legal reasoning roles.

Table 2. Examples of legal reasoning roles in N-EviChain

Role	Meaning in the evidence chain
Occurrence	Supports that an event occurred
Prior condition	Supports that a relevant condition existed before the event
Knowledge or detectability	Supports that a party knew or could reasonably detect a relevant condition
Causation	Supports a causal link between an event, condition, and outcome

Technical confirmation	Provides technical or expert support for a fact
Contextual support	Provides indirect or background support
Damage confirmation	Supports the existence or extent of harm

Table 2 gives examples only. It does not close the role set. In a real software implementation, the role library should be configurable according to case type and jurisdictional requirements.

3.5. Evidential-Link Layer

The connection between evidence items is represented by evidential-link coefficients. For two consecutive components in an evidence chain, the link coefficient is:

$$\lambda_{r-1,r}^{(k)} \in [0, 1]. \quad (14)$$

Here, $\lambda_{r-1,r}^{(k)}$ expresses the strength of the legal connection between the current composite chain and the next evidence item E_{kr} for legal fact F_k .

A high value of $\lambda_{r-1,r}^{(k)}$ means that the next evidence item strengthens the chain in a legally meaningful way. A low value means that the item may be present in the case file but does not clearly connect to the current chain for that specific fact.

The coefficient $\lambda_{r-1,r}^{(k)}$ is not a credibility score, not a statutory threshold, and not a judicial conclusion. It is a structural parameter used to represent how well the evidence chain is connected.

3.6. TwoFold Chain Engine

The TwoFold Chain Engine applies the Neutrosophic TwoFold Evidence-Chain Law defined in Section 2. The model does not repeat the law here; it uses it as the computational engine for constructing the fact-level state.

For each legal fact F_k , the chain starts from the first evidence item and is updated sequentially. After processing the full chain C_k , the model obtains the final neutrosophic support state:

$$N(F_k) = \langle T_k, I_k, O_k \rangle. \quad (15)$$

Here, T_k is the final support degree for legal fact F_k , I_k is the final indeterminacy degree, and O_k is the final opposition degree. The symbol O_k is used for opposition to avoid confusion with the legal fact symbol F_k .

The final state $N(F_k)$ is not a verdict. It is a structured support-assessment state that summarizes how the evidence chain behaves with respect to one legal fact.

3.7. Evidential Gap Indicator

A central output of N-EviChain is the evidential gap indicator. A gap may result from insufficient support, high indeterminacy, or influential opposition. For legal fact F_k , the gap indicator is defined as:

$$G_k = \max\{1, -T_k, I_k, O_k\}. \quad (16)$$

Here, $G_k \in [0, 1]$. A high value of G_k does not mean that the legal fact is false. It means that the evidence chain contains a major weakness that should be reviewed by a legal professional.

The component responsible for the gap is identified by first defining:

$$d_k = \max\{1, -T_k, I_k, O_k\}. \quad (17)$$

Then the main gap reason is:

$$R_k^{gap} = \{q: q \in \{\text{Insufficient support}, \text{Indeterminacy}, \text{Opposition}\} \text{ and } v_q = d_k\}, \quad (18)$$

where:

$$v_{\text{Insufficient support}} = 1 - T_k, \quad v_{\text{Indeterminacy}} = I_k, \quad v_{\text{Opposition}} = O_k.$$

This definition allows more than one reason to be reported when there is a tie. For example, if indeterminacy and opposition have the same dominant value, both are reported instead of forcing one explanation.

3.8. Interpretive Output Layer

N-EviChain produces a review-oriented output category for each legal fact. Let the operational threshold set be:

$$\Theta = (\theta_T, \theta_I, \theta_O, \theta_G), \quad (19)$$

where θ_T is the support threshold, θ_I is the indeterminacy threshold, θ_O is the opposition threshold, and θ_G is the evidential-gap threshold.

The interpretive output for legal fact F_k is denoted by Ω_k and is defined as:

$$\Omega_k = \begin{cases} \text{Opposed legal fact support,} & O_k \geq \theta_O, \\ \text{Uncertain legal fact support,} & I_k \geq \theta_I, \\ \text{Supported legal fact,} & T_k \geq \theta_T, I_k < \theta_I, O_k < \theta_O, G_k < \theta_G, \\ \text{Evidence-chain gap,} & G_k \geq \theta_G, \\ \text{Requires professional legal review,} & \text{otherwise.} \end{cases} \quad (20)$$

The order of Equation (20) is conservative. Opposition is checked first because influential opposing evidence should not be hidden by support. Indeterminacy is checked before support because uncertain evidence should not be converted into a confident conclusion. A fact is labeled as supported only when support is sufficient and uncertainty, opposition, and gap values are all below their operational limits.

The thresholds are model parameters. They are not statutory legal standards and should be reviewed by qualified legal experts before any real court-support deployment.

Table 3. Interpretive outputs of N-EviChain

Output category	Meaning	Professional review direction
Opposed legal fact support	Opposition or weakening evidence is influential	Review the opposing evidence and its legal relevance
Uncertain legal fact support	The evidence chain contains high indeterminacy	Request clarification, additional documentation, or further expert review
Supported legal fact	The chain provides sufficient modeled support with acceptable uncertainty and opposition	Verify the underlying evidence and document the reasoning path
Evidence-chain gap	The chain contains a major weakness in support, uncertainty, or opposition	Identify the missing or weak evidential link
Requires professional legal review	The chain does not clearly fall into the previous categories	Apply human legal judgment and record the reason

Table 3 shows that the model output is explanatory. It supports legal review but does not replace it.

3.9. Explanation Record

For each legal fact F_k , the model produces an explanation record:

$$X_k = (F_k, C_k, R_k, \Lambda_k, N(F_k), G_k, R_k^{gap}, \Omega_k). \quad (21)$$

Here, X_k is the complete explanation record, F_k is the legal fact, C_k is the evidence chain, $R_k = \{R_{k1}, R_{k2}, \dots, R_{kn_k}\}$ is the set of role assignments, Λ_k is the set of evidential-link coefficients, $N(F_k)$ is the

final neutrosophic support state, G_k is the evidential gap indicator, R_k^{gap} is the reported reason for the gap, and Ω_k is the interpretive output.

This record gives the model its explainability. A reviewer can see which fact was assessed, which evidence items were used, how they were connected, what roles they played, what final neutrosophic state was produced, and why the system reported a specific output category.

3.10. Software-Oriented Workflow

The proposed model can be implemented as a LegalTech decision-support workflow. The workflow is designed for evidence organization and explanation, not for automated adjudication.

Table 4. Software-oriented workflow of N-EviChain.

Module	Function
Legal Fact Module	Defines the legal facts to be assessed
Evidence Registration Module	Records evidence items and links them to legal facts
Role Annotation Module	Assigns legal reasoning roles to evidence items
Neutrosophic Annotation Module	Assigns support, indeterminacy, and opposition values
Evidence-Link Module	Assigns evidential-link coefficients
TwoFold Chain Engine	Applies the law defined in Section 2 to construct $N(F_k)$
Gap Explanation Module	Identifies the dominant weakness in the evidence chain
Legal Report Module	Generates the explanation record X_k for professional review

As shown in Table 4, the model is suitable as a prototype-ready software architecture for court evidence management. Its output remains subject to human legal interpretation.

4. Illustrative Legal Case Study with Complete Numerical Calculations

This section demonstrates the proposed N-EviChain model through an illustrative legal case study. The purpose is to show how the model organizes heterogeneous evidence, constructs fact-specific evidence chains, calculates neutrosophic support states, identifies evidential gaps, and produces explainable outputs for professional legal review.

The case is not a real court file and does not contain personal, confidential, or judicial data. All numerical values are constructed for methodological demonstration only. The purpose of this section is demonstration and traceability, not legal validation or judicial decision-making.

4.1. Case Description

The illustrative case concerns an industrial workplace injury. A worker was injured while operating a cutting machine inside a factory. The legal review focuses on whether the available evidence supports specific legal facts related to the accident, the machine condition, prior detectability, and causal contribution.

The proposed system does not decide liability and does not determine whether any party is legally responsible. It only evaluates how the available evidence chain supports each legal fact under uncertainty.

The case is structured around four legal facts, as shown in Table 5.

Table 5. Legal facts examined in the illustrative case.

Symbol	Legal fact under review	Legal relevance
F_1	The accident occurred during machine operation	Establishes occurrence of the incident
F_2	The machine had a prior safety defect	Examines whether a relevant defect existed before the injury

F_3	The defect was known or reasonably detectable before the accident	Examines possible prior knowledge or detectability
F_4	The defect contributed to the worker's injury	Examines the evidential chain related to causation

The four facts are assessed separately. This is important because proving that an accident occurred does not automatically prove that a prior defect existed, and proving that a defect existed does not automatically prove that the defect caused the injury.

4.2. Evidence Items and Legal Roles

The illustrative case uses seven heterogeneous evidence items. These items include testimonial, digital, medical, technical, inspection, and expert evidence.

Table 6. Evidence items used in the illustrative case

Symbol	Evidence item	Evidence type	Possible legal role
E_1	Worker statement	Testimonial evidence	Occurrence, contextual support
E_2	CCTV footage	Digital visual evidence	Occurrence, contextual support
E_3	Medical report	Medical documentation	Damage confirmation
E_4	Maintenance log	Technical-documentary evidence	Prior condition, detectability
E_5	Safety inspection note	Inspection evidence	Prior condition, knowledge or detectability
E_6	Machine sensor record	Digital technical evidence	Technical confirmation
E_7	Technical expert report	Expert evidence	Causation, technical confirmation

As shown in Table 6, the evidence items are not treated as identical objects. Each item may have a different legal role depending on the fact being assessed. For example, the medical report may strongly support the existence of an injury, but it may not alone establish that a machine defect caused the injury.

4.3. Operational Thresholds

The operational threshold set used in this illustrative case is:

$$\Theta = (\theta_T, \theta_I, \theta_O, \theta_G) = (0.60, 0.50, 0.50, 0.50). \quad (22)$$

Here, θ_T is the support threshold, θ_I is the indeterminacy threshold, θ_O is the opposition threshold, and θ_G is the evidential-gap threshold.

These values are not statutory legal standards. They are demonstration thresholds used only to show the operation of the proposed model. In a real court-support software system, these thresholds should be calibrated and reviewed by qualified legal experts.

4.4. Evidence Chains Used in the Case Study

For each legal fact, a specific evidence chain is constructed. The order of each chain follows the declared legal reasoning path, not necessarily the chronological order of events.

Table 7. Evidence chains for the four legal facts.

Legal fact	Evidence chain	Reasoning purpose
F_1	$E_1 \rightarrow E_2 \rightarrow E_3$	Statement, visual record, and medical confirmation are used to assess occurrence
F_2	$E_4 \rightarrow E_5 \rightarrow E_6$	Maintenance, inspection, and sensor evidence are used to assess prior defect
F_3	$E_5 \rightarrow E_4$	Inspection and maintenance evidence are used to assess prior knowledge or detectability
F_4	$E_7 \rightarrow E_6 \rightarrow E_3$	Expert, sensor, and medical evidence are used to assess causal contribution

Each evidence item is represented by:

$$E_{kr}(\tau_{kr}, \iota_{kr}, \phi_{kr}),$$

where τ_{kr} denotes support, ι_{kr} denotes indeterminacy, and ϕ_{kr} denotes opposition for evidence item E_{kr} with respect to legal fact F_k .

The calculations use the component-fusion rule defined earlier. For clarity, it is applied in this case study as follows:

$$\begin{aligned} T_k^{(r)} &= \lambda_{r-1,r}^{(k)} \frac{T_k^{(r-1)} + \tau_{kr}}{2}, \\ I_k^{(r)} &= \min \left(1, \frac{I_k^{(r-1)} + \iota_{kr}}{2} + 1 - \lambda_{r-1,r}^{(k)} \right), \\ O_k^{(r)} &= \frac{O_k^{(r-1)} + \phi_{kr}}{2}. \end{aligned}$$

All table values are reported to three decimal places. Decisions are explained using the rounded values, and where necessary the exact intermediate value is shown.

4.5. Calculation for F_1 : Accident Occurred During Machine Operation

For F_1 , the evidence chain is:

$$C_1 = (E_1, E_2, E_3).$$

The assigned evidence states and link coefficients are shown in Table 8.

Table 8. Evidence-chain values for F_1

Step	Evidence item	τ	ι	ϕ	Link coefficient
1	Worker statement E_1	0.82	0.12	0.10	—
2	CCTV footage E_2	0.88	0.08	0.05	$\lambda_{1,2}^{(1)} = 0.90$
3	Medical report E_3	0.75	0.10	0.05	$\lambda_{2,3}^{(1)} = 0.80$

The initial chain state is:

$$T_1^{(1)} = 0.82, I_1^{(1)} = 0.12, O_1^{(1)} = 0.10.$$

After adding the CCTV footage E_2 :

$$\begin{aligned} T_1^{(2)} &= 0.90 \times \frac{0.82 + 0.88}{2} = 0.90 \times 0.85 = 0.765. \\ I_1^{(2)} &= \min \left(1, \frac{0.12 + 0.08}{2} + 1 - 0.90 \right) = \min(1, 0.10 + 0.10) = 0.200. \\ O_1^{(2)} &= \frac{0.10 + 0.05}{2} = 0.075. \end{aligned}$$

After adding the medical report E_3 :

$$\begin{aligned} T_1^{(3)} &= 0.80 \times \frac{0.765 + 0.75}{2} = 0.80 \times 0.7575 = 0.606. \\ I_1^{(3)} &= \min \left(1, \frac{0.200 + 0.10}{2} + 1 - 0.80 \right) = \min(1, 0.150 + 0.200) = 0.350. \\ O_1^{(3)} &= \frac{0.075 + 0.05}{2} = 0.0625 \approx 0.063. \end{aligned}$$

Therefore, the final neutrosophic support state for F_1 is:

$$N(F_1) = \langle 0.606, 0.350, 0.063 \rangle.$$

The evidential gap indicator is:

$$G_1 = \max\{1, \neg, T_1, I_1, O_1\}.$$

$$G_1 = \max\{1, \neg, 0.606, 0.350, 0.063\} = \max\{0.394, \neg, 0.350, 0.063\} = 0.394.$$

Since:

$$T_1 = 0.606 \geq 0.60,$$

$$I_1 = 0.350 < 0.50,$$

$$O_1 = 0.063 < 0.50,$$

and:

$$G_1 = 0.394 < 0.50,$$

the output is:

$$\Omega_1 = \text{Supported legal fact.}$$

The largest weakness component is $1 - T_1 = 0.394$, but it is below the gap threshold. Therefore, there is no actionable evidential gap for F_1 in this illustrative chain.

4.6. Calculation for F_2 : Prior Safety Defect

For F_2 , the evidence chain is:

$$C_2 = (E_4, E_5, E_6).$$

The assigned evidence states and link coefficients are shown in Table 9.

Table 9. Evidence-chain values for F_2

Step	Evidence item	τ	ι	ϕ	Link coefficient
1	Maintenance log E_4	0.65	0.20	0.20	—
2	Safety inspection note E_5	0.60	0.22	0.25	$\lambda_{1,2}^{(2)} = 0.75$
3	Machine sensor record E_6	0.30	0.15	0.85	$\lambda_{2,3}^{(2)} = 0.85$

The initial chain state is:

$$T_2^{(1)} = 0.65, I_2^{(1)} = 0.20, O_2^{(1)} = 0.20.$$

After adding the safety inspection note E_5 :

$$T_2^{(2)} = 0.75 \times \frac{0.65 + 0.60}{2} = 0.75 \times 0.625 = 0.46875 \approx 0.469.$$

$$I_2^{(2)} = \min\left(1, \frac{0.20 + 0.22}{2} + 1 - 0.75\right) = \min(1, 0.210 + 0.250) = 0.460.$$

$$O_2^{(2)} = \frac{0.20 + 0.25}{2} = 0.225.$$

After adding the machine sensor record E_6 :

$$T_2^{(3)} = 0.85 \times \frac{0.46875 + 0.30}{2} = 0.85 \times 0.384375 = 0.32671875 \approx 0.327.$$

$$I_2^{(3)} = \min\left(1, \frac{0.460 + 0.15}{2} + 1 - 0.85\right) = \min(1, 0.305 + 0.150) = 0.455.$$

$$O_2^{(3)} = \frac{0.225 + 0.85}{2} = 0.5375 \approx 0.538.$$

Therefore, the final neutrosophic support state for F_2 is:

$$N(F_2) = \langle 0.327, 0.455, 0.538 \rangle.$$

The evidential gap indicator is:

$$G_2 = \max\{1, \neg, T_2, I_2, O_2\}.$$

Using the rounded final values:

$$G_2 = \max\{1, \neg, 0.327, 0.455, 0.538\} = \max\{0.673, \neg, 0.455, 0.538\} = 0.673.$$

Because:

$$O_2 = 0.538 \geq 0.50,$$

the interpretive output is:

$$\Omega_2 = \text{Opposed legal fact support.}$$

The dominant gap reason is insufficient support because:

$$1 - T_2 = 0.673$$

is larger than:

$$I_2 = 0.455$$

and:

$$O_2 = 0.538.$$

Therefore:

$$R_2^{gap} = \{\text{Insufficient support}\}.$$

This result should be interpreted carefully. The output is triggered by opposition because O_2 exceeds the opposition threshold. However, the largest weakness in the chain is insufficient support. The model therefore indicates that the claim of a prior safety defect is not strongly supported in the modeled chain and is also affected by influential opposing technical evidence.

4.7. Calculation for F_3 : Prior Knowledge or Reasonable Detectability

For F_3 , the evidence chain is:

$$C_3 = (E_5, E_4).$$

The assigned evidence states and link coefficient are shown in Table 10.

Table 10. Evidence-chain values for F_3

Step	Evidence item	τ	ι	ϕ	Link coefficient
1	Safety inspection note E_5	0.56	0.10	0.15	—
2	Maintenance log E_4	0.58	0.10	0.18	$\lambda_{1,2}^{(3)} = 0.65$

The initial chain state is:

$$T_3^{(1)} = 0.56, I_3^{(1)} = 0.10, O_3^{(1)} = 0.15.$$

After adding the maintenance log E_4 :

$$T_3^{(2)} = 0.65 \times \frac{0.56 + 0.58}{2} = 0.65 \times 0.57 = 0.3705 \approx 0.371.$$

$$I_3^{(2)} = \min\left(1, \frac{0.10 + 0.10}{2} + 1 - 0.65\right) = \min(1, 0.100 + 0.350) = 0.450.$$

$$O_3^{(2)} = \frac{0.15 + 0.18}{2} = 0.165.$$

Therefore, the final neutrosophic support state for F_3 is:

$$N(F_3) = \langle 0.371, 0.450, 0.165 \rangle.$$

The evidential gap indicator is:

$$G_3 = \max\{1, \neg, T_3, I_3, O_3\}.$$

Using the exact support value before rounding:

$$1 - T_3 = 1 - 0.3705 = 0.6295 \approx 0.630.$$

Thus:

$$G_3 = \max\{0.630, 0.450, 0.165\} = 0.630.$$

Since:

$$O_3 = 0.165 < 0.50,$$

$$I_3 = 0.450 < 0.50,$$

$$T_3 = 0.371 < 0.60,$$

and:

$$G_3 = 0.630 \geq 0.50,$$

the output is:

$$\Omega_3 = \text{Evidence-chain gap.}$$

The dominant gap reason is insufficient support because:

$$1 - T_3 = 0.630$$

is larger than:

$$I_3 = 0.450$$

and:

$$O_3 = 0.165.$$

Therefore:

$$R_3^{gap} = \{\text{Insufficient support}\}.$$

This means that the modeled evidence chain does not provide enough support to conclude that the defect was known or reasonably detectable before the accident. The model does not deny detectability. It identifies a gap in the evidential chain.

4.8. Calculation for F_4 : Defect Contributed to the Injury

For F_4 , the evidence chain is:

$$C_4 = (E_7, E_6, E_3).$$

The assigned evidence states and link coefficients are shown in Table 11.

Table 11. Evidence-chain values for F_4

Step	Evidence item	τ	ι	ϕ	Link coefficient
1	Technical expert report E_7	0.68	0.22	0.20	—
2	Machine sensor record E_6	0.62	0.25	0.18	$\lambda_{1,2}^{(4)} = 0.78$
3	Medical report E_3	0.60	0.20	0.22	$\lambda_{2,3}^{(4)} = 0.70$

The initial chain state is:

$$T_4^{(1)} = 0.68, I_4^{(1)} = 0.22, O_4^{(1)} = 0.20.$$

After adding the machine sensor record E_6 :

$$T_4^{(2)} = 0.78 \times \frac{0.68 + 0.62}{2} = 0.78 \times 0.65 = 0.507.$$

$$I_4^{(2)} = \min\left(1, \frac{0.22 + 0.25}{2} + 1 - 0.78\right) = \min(1, 0.235 + 0.220) = 0.455.$$

$$O_4^{(2)} = \frac{0.20 + 0.18}{2} = 0.190.$$

After adding the medical report E_3 :

$$T_4^{(3)} = 0.70 \times \frac{0.507 + 0.60}{2} = 0.70 \times 0.5535 = 0.38745 \approx 0.387.$$

$$I_4^{(3)} = \min\left(1, \frac{0.455 + 0.20}{2} + 1 - 0.70\right) = \min(1, 0.3275 + 0.300) = 0.6275 \approx 0.628.$$

$$O_4^{(3)} = \frac{0.190 + 0.22}{2} = 0.205.$$

Therefore, the final neutrosophic support state for F_4 is: $N(F_4) = \langle 0.387, 0.628, 0.205 \rangle$.

The evidential gap indicator is: $G_4 = \max\{1, \neg T_4, I_4, O_4\}$.

Using the exact support value before rounding: $1 - T_4 = 1 - 0.38745 = 0.61255 \approx 0.613$.

Thus:

$$G_4 = \max\{0.613, 0.628, 0.205\} = 0.628.$$

Because:

$$I_4 = 0.628 \geq 0.50,$$

the output is:

$$\Omega_4 = \text{Uncertain legal fact support.}$$

The dominant gap reason is indeterminacy because:

$$I_4 = 0.628$$

is larger than:

$$1 - T_4 = 0.613$$

and:

$$O_4 = 0.205.$$

Therefore:

$$R_4^{gap} = \{\text{Indeterminacy}\}.$$

This result means that the causal contribution of the alleged defect to the injury requires further technical or legal review. The model does not confirm causation and does not deny it. It identifies uncertainty in the evidence chain.

4.9. Summary of Numerical Results

The final outputs for the four legal facts are summarized in Table 12.

Table 12. Final N-EviChain outputs for the illustrative case.

Legal fact	Final neutrosophic state $N(F_k)$	Gap indicator G_k	Dominant gap reason	Output
F_1 : Accident occurred during machine operation	$\langle 0.606, 0.350, 0.063 \rangle$	0.394	No actionable gap	Supported legal fact
F_2 : Machine had a prior safety defect	$\langle 0.327, 0.455, 0.538 \rangle$	0.673	Insufficient support	Opposed legal fact support
F_3 : Defect was known or reasonably detectable	$\langle 0.371, 0.450, 0.165 \rangle$	0.630	Insufficient support	Evidence-chain gap
F_4 : Defect contributed to injury	$\langle 0.387, 0.628, 0.205 \rangle$	0.628	Indeterminacy	Uncertain legal fact support

Table 12 shows that N-EviChain does not produce one conclusion for the whole case. Instead, it gives a separate support assessment for each legal fact. This is important because one legal fact may be supported while another remains uncertain, opposed, or evidentially incomplete.

The table also shows the difference between an output trigger and a dominant gap reason. For F_2 , the output is Opposed legal fact support because $O_2 \geq \theta_0$. However, the largest weakness component is insufficient support because $1 - T_2$ is greater than I_2 and O_2 . This distinction makes the explanation more precise.

4.10. Legal Importance of the Case Study

The case study demonstrates the legal value of the proposed model. It separates four evidential questions that should not be merged into one broad conclusion.

First, the occurrence of the accident during machine operation is supported because the evidence chain has sufficient modeled support and acceptable uncertainty and opposition. Second, the claim that the machine had a prior safety defect is affected by influential opposing technical evidence. Third, the claim that the defect was known or reasonably detectable contains an evidential gap caused mainly by insufficient support. Fourth, the claim that the defect contributed to the injury remains uncertain because the chain contains high indeterminacy.

This separation is legally important. A supported accident occurrence does not automatically establish a prior defect. A prior defect does not automatically establish knowledge or detectability. Knowledge or detectability does not automatically establish causation. N-EviChain keeps these legal facts separate and reviewable.

The model is therefore useful as a court-support tool because it can organize complex evidence into fact-specific chains. It can also help legal professionals identify where the file is strong, where it is uncertain, where it is opposed, and where additional evidence or expert review may be needed.

4.11. Importance for the Proposed Work

This case study is important for the proposed N-EviChain framework for three reasons.

First, it shows the value of fact-specific reasoning. The same case file does not receive one global result. Each legal fact receives its own evidence chain and its own neutrosophic support state.

Second, it shows the value of TwoFold evidence-chain reasoning. The model does not combine evidence values only. It also considers whether the evidence items are connected in a legally meaningful chain through the evidential-link coefficient λ .

Third, it shows the value of neutrosophic representation. Support, indeterminacy, and opposition remain separate throughout the calculation. This allows the model to distinguish between a supported fact, an opposed fact, a fact with insufficient support, and a fact affected by high uncertainty.

These three points demonstrate why the proposed model is suitable for future LegalTech software. A court-support system based on N-EviChain could help users organize legal facts, register evidence items, trace evidential links, calculate fact-level support states, and generate explainable reports. The final interpretation would remain with legal professionals.

5. Analysis, Managerial Implications, and Legal-Industry Relevance

This section analyzes the practical meaning of the illustrative case study and explains how the proposed N-EviChain model can support legal, managerial, and industrial evidence-review workflows. The purpose of this section is not to repeat the calculations in Section 4. Instead, it explains what the results mean for legal fact assessment, court-support software, and industrial dispute analysis.

5.1. Analytical Reading of the Case Study

The case study shows that a legal file should not be treated as one undivided block of evidence. The same case may contain a strongly supported fact, an opposed fact, an incomplete fact, and an uncertain fact at the same time. This is why N-EviChain assesses each legal fact separately.

In the illustrative workplace injury case, the occurrence of the accident was supported, but this did not automatically support all other facts. The claim that the machine had a prior defect was affected by opposing technical evidence. The claim that the defect was known or reasonably detectable had an evidential gap. The claim that the defect contributed to the injury remained uncertain. This separation is important because legal reasoning often depends on the specific fact being examined, not only on the total number of documents in the file.

The key analytical value of N-EviChain is that it distinguishes between four different situations:

1. evidence that supports a fact;
2. evidence that weakens or opposes a fact;

3. uncertainty caused by incomplete or unclear evidence;
4. a gap caused by an insufficiently connected evidence chain.

These situations should not be merged into one general label. A fact with high opposition needs a different legal review from a fact with high indeterminacy. A fact with insufficient support requires a different review from a fact that is well supported but still needs formal verification.

5.2. Managerial Implications for Legal Case Handling

N-EviChain can support legal case management by making the structure of evidence more explicit. In many files, evidence items are collected in different formats and from different sources. Without a structured model, the reviewer may know that evidence exists, but may not immediately see which legal fact each item supports, which link is weak, and where further evidence is needed.

The proposed model helps organize this process. It links evidence items to specific legal facts, assigns each item a role, evaluates the evidence through support, indeterminacy, and opposition, and reports the reason for weakness when a gap appears. This can help legal professionals focus their review on the exact part of the evidence chain that requires attention.

Table 13. Managerial implications of N-EviChain for legal case handling

Managerial issue	Contribution of N-EviChain	Practical value
Large and heterogeneous case files	Organizes evidence by legal fact and evidence chain	Makes file review more structured
Unclear connection between evidence items	Uses evidential-link coefficients	Highlights weak or fragmented links
Missing or incomplete documentation	Reports indeterminacy and evidence-chain gaps	Helps identify what needs further review
Mixed technical and legal evidence	Separates evidence type from legal reasoning role	Supports communication between legal and technical reviewers
Need for auditability	Generates an explanation record for each fact	Makes the reasoning path easier to inspect
Risk of overgeneralization	Assesses each legal fact separately	Prevents one supported fact from being treated as support for the whole case

As shown in Table 13, the model is not designed to replace legal judgment. Its value is managerial and explanatory. It helps structure the review process and makes weaknesses in the evidence chain visible.

5.3. Relevance to LegalTech Software

The proposed model has direct relevance to the LegalTech industry because it can be converted into a software module for court evidence management, law-office case preparation, expert-report review, or industrial dispute documentation. The software role would be to support evidence organization and explanation, not to automate judicial decisions.

A prototype based on N-EviChain could include the following workflow:

Table 14. LegalTech workflow supported by N-EviChain

Software stage	Input	Output
Legal fact definition	List of facts requiring review	Structured fact set $\mathcal{F}$
Evidence registration	Documents, reports, statements, records, images, logs	Evidence set $\mathcal{E}$

Role annotation	Legal function of each evidence item	Role set R_k for each fact
Neutrosophic annotation	Support, indeterminacy, and opposition values	Evidence states $E_{kr}(\tau, \nu, \phi)$
Link assessment	Connection between evidence items	Link coefficients λ
TwoFold chain processing	Fact-specific evidence chain	Final state $N(F_k)$
Gap explanation	Support, indeterminacy, and opposition results	Gap explanation and review direction
Report generation	Complete explanation record	Human-readable legal review report

Table 14 shows that N-EviChain can be implemented as a transparent software workflow. The system can store the reasoning path, show how each output was produced, and allow legal professionals to review or revise the annotations.

This is important for court-related applications because the output must be explainable. A black-box prediction would be unsuitable for sensitive legal review. N-EviChain instead produces a traceable explanation record that includes the fact, evidence chain, roles, link coefficients, final neutrosophic state, gap indicator, and output category.

5.4 LegalTech and Industrial Dispute Relevance

The industrial workplace injury case was selected because it connects legal reasoning with industrial evidence. Industrial disputes often involve different kinds of evidence, including safety inspections, maintenance logs, machine records, medical reports, and expert opinions. These sources are heterogeneous and may not all support the same legal fact in the same way.

The proposed model can support the legal-industry interface in several ways. For courts, it can help organize technical evidence around specific legal facts. For factories and companies, it can encourage better documentation of maintenance, safety inspections, and machine records. For legal experts, it can clarify which technical evidence supports occurrence, prior condition, detectability, or causation. For LegalTech developers, it provides a formal reasoning structure that can be implemented in software.

Table 15. Legal-industry relevance of N-EviChain.

Stakeholder	Possible use of N-EviChain	Value of the model
Courts	Organizing evidence chains for fact-specific review	Improves transparency of evidence assessment
Lawyers	Preparing and reviewing evidence structure	Identifies weak or incomplete links before submission
Technical experts	Linking technical findings to legal facts	Clarifies the role of expert evidence
Industrial companies	Reviewing safety documentation and incident files	Highlights documentation gaps and technical uncertainty
LegalTech developers	Building explainable evidence-chain software	Provides a structured computational model
Regulators or inspectors	Reviewing incident documentation patterns	Supports more consistent documentation review

Table 15 shows that the model has practical relevance beyond the courtroom. It can also support better documentation culture in industrial settings, especially where technical records, safety reports, and expert reviews must later be interpreted in a legal context.

5.5. Importance of the TwoFold Structure for Practice

The practical strength of N-EviChain comes from its TwoFold structure. In ordinary evidence review, one may ask only whether a document supports a fact. In N-EviChain, two questions are asked together:

First, how is this evidence item legally connected to the current evidence chain?

Second, what is the neutrosophic state of this evidence item in terms of support, indeterminacy, and opposition?

This two-level reasoning is aligned with the foundation of Neutrosophic TwoFold Algebra, where one level concerns the elements and another level concerns their neutrosophic components. In the legal application, the evidence item is the element, while support, indeterminacy, and opposition form its neutrosophic state.

This makes the model useful for practical legal review. A document may have high support by itself, but if it is weakly connected to the chain, it should not strongly raise the final support of the legal fact. Similarly, a document may be present in the file but may mainly increase uncertainty because it does not complete the evidential link required for that fact.

5.6. Benefits for Court Evidence Management

A court-support version of N-EviChain would provide several practical benefits. It could help users move from a document-centered file to a fact-centered evidence map. Instead of asking only what documents exist, the system asks which legal fact each document supports, how it connects with other documents, and whether the chain contains a gap. The expected benefits are:

- 1) clearer organization of complex evidence;
- 2) better identification of missing or weak links;
- 3) transparent explanation of support, uncertainty, and opposition;
- 4) improved coordination between legal and technical reviewers;
- 5) better preparation of expert questions;
- 6) more consistent documentation of evidence-review reasoning.

These benefits must be understood within the decision-support boundary. The system does not decide whether evidence is legally admissible, sufficient, or conclusive. Those questions remain subject to legal rules and professional judgment.

6. Linkage with Egypt Vision 2030

This section links the proposed N-EviChain model with selected directions of Egypt Vision 2030. The purpose is not to claim that the model directly implements national policy, measures national progress, or replaces formal justice-sector reforms. The purpose is narrower: to show how a transparent, human-reviewed, and software-oriented evidence-chain model can support priorities that are conceptually aligned with digital transformation, governance, justice, knowledge-based development, innovation, and industrial documentation.

Egypt Vision 2030 is a national agenda launched in 2016 to guide Egypt's long-term sustainable development across different fields and state institutions. It is built around comprehensive sustainable development and balanced regional development, with attention to the economic, social, and environmental dimensions of development. The updated Vision also emphasizes citizen-centered development, equity and accessibility, resilience, sustainability, national strategic goals, and enabling factors such as data availability, digital transformation, technology, innovation, and the legislative environment.

6.1. Digital Transformation and Data-Structured Justice

N-EviChain is aligned with the digital transformation direction of Egypt Vision 2030 because it provides a structured computational model for organizing heterogeneous legal evidence into fact-specific evidence chains. Vision 2030 identifies digital transformation as one of the enabling factors

for development and connects it with knowledge, innovation, governance, and institutional modernization.

The proposed model contributes to this direction at the level of evidence organization. It does not merely convert paper documents into electronic files. Instead, it structures legal facts, evidence items, evidential roles, evidential links, neutrosophic states, gap explanations, and review-oriented outputs. This makes it suitable as a research foundation for future court-support software while keeping final legal interpretation under human authority.

6.2. Governance, Transparency, and Institutional Reviewability

Egypt Vision 2030 emphasizes governance of state institutions and society, including efficiency, effectiveness, transparency, accountability, and stronger systems of monitoring and evaluation. N-EviChain is aligned with this direction because it produces an explanation record for each legal fact rather than a black-box conclusion.

In the proposed model, every output can be traced back to the evidence chain, the role of each evidence item, the evidential-link coefficients, and the final neutrosophic state. This supports institutional reviewability. A legal professional can examine why a fact was reported as supported, uncertain, opposed, or affected by an evidence-chain gap. This is important for any future LegalTech tool intended for sensitive legal environments.

6.3. Justice, Inclusion, and Human Oversight

Egypt Vision 2030 gives attention to justice, social inclusion, participation, and improved quality of life. The proposed model relates to these directions by supporting clearer and more reviewable organization of evidence, especially in cases where evidence is technical, incomplete, or unevenly distributed between parties.

This is relevant to the illustrative industrial workplace injury case. Such cases may involve workers, employers, technical experts, medical documents, inspection notes, maintenance logs, and machine records. N-EviChain does not decide liability or rights. However, it can help clarify which legal facts are supported, which facts remain uncertain, and which facts need additional evidence or expert review. This supports a more transparent evidence-review process without replacing the role of the court.

6.4. Knowledge, Innovation, and LegalTech Research

Egypt Vision 2030 identifies knowledge, innovation, and scientific research as foundations for development, and it links economic competitiveness with knowledge-based growth and digital transformation. N-EviChain contributes to this direction by transforming a neutrosophic algebraic concept into a practical legal reasoning framework.

The research value of the model lies in its interdisciplinary nature. It connects neutrosophic logic, legal evidence analysis, software engineering, industrial documentation, and explainable decision support. This makes the work relevant not only as a mathematical model, but also as a prototype-oriented framework that can be tested and improved in future LegalTech applications.

6.5. Industrial Documentation and Legal Preparedness

Egypt Vision 2030 aims at a competitive, diversified, and knowledge-based economy supported by digital transformation, decent work opportunities, improved business environment, and institutional governance. The industrial case study in this paper connects N-EviChain to these directions through workplace injury documentation and technical evidence review.

The model can encourage better organization of industrial incident files by clarifying the relationship between accident occurrence, prior condition, detectability, and causation. Its direct purpose is legal evidence-chain reasoning, not industrial safety management. Nevertheless, clearer maintenance logs, inspection records, machine data, and expert reports can improve legal preparedness and reduce uncertainty when industrial disputes arise.

6.6. Vision 2030 Mapping

Table 16. Conceptual linkage between N-EviChain and Egypt Vision 2030.

Egypt Vision 2030 direction	Linkage with N-EviChain	Practical meaning
Digital transformation	Structures legal evidence into computable fact-specific chains	Supports future LegalTech tools for evidence organization
Governance and transparency	Produces traceable explanation records for each legal fact	Improves reviewability and auditability of evidence analysis
Justice and social inclusion	Clarifies evidential gaps without replacing the court	Supports more transparent professional review
Knowledge and innovation	Applies neutrosophic TwoFold reasoning to legal evidence chains	Provides an interdisciplinary research contribution
Competitive knowledge-based economy	Supports documentation quality in industrial legal disputes	Improves legal preparedness and technical-evidence organization
Citizen-centered and human-reviewed digital governance	Keeps final interpretation under qualified legal professionals	Avoids unsafe automation of judicial decisions

Table 16 shows that the linkage between N-EviChain and Egypt Vision 2030 is conceptual and application-oriented. The model supports digital organization, transparency, knowledge-based innovation, and human-reviewed governance. It does not claim to measure national indicators or to implement Vision 2030 by itself.

7. Results Interpretation and Methodological Importance

This section explains the methodological meaning of the results obtained from the illustrative case study. The focus here is not on repeating the numerical calculations or the practical implications already discussed earlier. The aim is to clarify what the results demonstrate about the logic, behavior, and usefulness of the proposed N-EviChain reasoning structure.

7.1. Fact-Specific Interpretation

The results confirm that legal evidence should be interpreted at the level of individual legal facts, not at the level of the case file as a whole. A case file may contain many documents, but those documents do not necessarily support every legal fact with the same strength. Some facts may be well supported, while others may remain uncertain, opposed, or incomplete.

N-EviChain addresses this issue by assigning a separate evidence chain to each legal fact. This prevents an important methodological error: transferring support from one fact to another without checking whether the evidence chain actually justifies that transfer. In legal reasoning, the fact that an incident occurred is not the same as proving prior defect, knowledge, detectability, or causal contribution. The proposed model keeps these distinctions explicit.

This fact-specific structure is one of the main strengths of the proposed work. It allows the reviewer to see not only whether evidence exists, but also whether the evidence is connected to the exact legal fact being assessed.

7.2. Interpretation of Neutrosophic Components

The results show that the three neutrosophic components have different meanings and should not be collapsed into one value. Support, indeterminacy, and opposition represent different evidential conditions.

High support indicates that the evidence chain gives positive support to the legal fact. High indeterminacy indicates that the evidence chain is unclear, incomplete, technically uncertain, or insufficiently connected. High opposition indicates that some evidence weakens the legal fact or works against it. Low support indicates that the chain does not provide enough positive evidential force, even if explicit opposition is not high.

This separation is methodologically important because different evidential weaknesses require different forms of review. A fact with high opposition is not the same as a fact with high uncertainty. A fact with low support is not the same as a fact contradicted by strong opposing evidence. N-EviChain preserves these differences throughout the reasoning process.

Table 17. Methodological interpretation of neutrosophic evidence-chain behavior.

Model behavior	Methodological meaning	Why it matters
Sufficient support with controlled uncertainty and opposition	The evidence chain supports the legal fact within the model boundaries	Prevents unnecessary rejection of a supported fact
High indeterminacy	The chain contains unclear, incomplete, or unstable evidential support	Prevents uncertain evidence from being treated as conclusive
High opposition	The chain contains influential weakening evidence	Prevents opposing evidence from being hidden inside a general score
High insufficient-support component	The chain lacks enough positive support	Identifies missing or weak evidential links
Different outputs for different facts	The same case file contains different evidential conditions	Prevents overgeneralization from one fact to the whole case

Table 17 shows that the model's value lies in preserving the structure of uncertainty rather than forcing all evidence into a single conclusion.

7.3. Interpretation of the TwoFold Reasoning Effect

The results also demonstrate the importance of the TwoFold reasoning mechanism. The final support state of a legal fact is not determined only by the values assigned to individual evidence items. It is also affected by the legal connection between those items.

This is methodologically important because legal evidence is chain-based. A strong item of evidence may lose value if it does not connect properly to the next part of the legal reasoning path. Likewise, several evidence items may exist in a case file, but if their relation is weak, fragmented, or unclear, they may increase uncertainty rather than produce strong support.

The TwoFold structure therefore adds a necessary layer to evidence reasoning. It combines the evidential item itself with the way it functions inside the chain. This makes the model more appropriate for legal fact support assessment than a simple aggregation of evidence values.

7.4. Interpretation of Evidential Gaps

The evidential gap indicator is not designed to prove that a legal fact is false. Its role is to identify the dominant weakness in the evidence chain. This distinction is essential.

A high gap value may arise because support is insufficient, because indeterminacy is high, or because opposition is influential. These are not identical situations. The model therefore reports both the gap value and the gap explanation. This makes the output more informative and less likely to be misunderstood.

The results show that gap detection is useful because it directs attention to the part of the reasoning process that needs review. Instead of giving a vague statement that the evidence is weak, N-EviChain identifies whether the weakness comes from lack of support, uncertainty, or opposition.

7.5. Methodological Importance of the Proposed Work

The proposed work is important because it introduces a structured way to evaluate legal fact support under uncertainty without crossing into automated adjudication. It does not convert the case into a final verdict. It converts the evidence surrounding each legal fact into a transparent neutrosophic support state.

This gives the model three methodological advantages. First, it is fact-specific, because every legal fact has its own evidence chain. Second, it is uncertainty-aware, because indeterminacy is represented explicitly rather than ignored. Third, it is chain-sensitive, because the evidential-link coefficient affects how support is transferred through the chain.

These advantages make N-EviChain different from ordinary document listing, simple evidence scoring, or general legal classification. The proposed model does not only ask whether evidence exists. It asks whether the evidence is legally connected, whether it supports the exact fact under review, and whether uncertainty or opposition prevents a confident support assessment.

8. Conclusion and Future Work

This paper proposed N-EviChain, a neutrosophic TwoFold evidence-chain reasoning system for legal fact support assessment under uncertainty. The model treats a legal case file as a heterogeneous evidential space and evaluates each legal fact through its own evidence chain. Each evidence item is represented by support, indeterminacy, and opposition values, while the TwoFold structure considers both the legal connection between evidence items and their neutrosophic components.

The illustrative industrial workplace injury case showed that N-EviChain can distinguish between supported facts, opposed fact support, evidential gaps, and uncertain legal fact support. This confirms the value of the proposed model in organizing complex legal evidence without reducing the whole case to one general conclusion.

N-EviChain is not designed to determine guilt, liability, admissibility, or judicial truth. Its role is limited to decision support: organizing evidence chains, identifying uncertainty and gaps, and producing explainable reports for professional legal review. Responsible deployment requires human-reviewed annotation, transparent thresholds, audit trails, and clear warnings that the system does not replace legal professionals.

Future work should validate the model using expert-reviewed legal case scenarios, calibrate the operational thresholds, test sensitivity under different evidence-chain structures, and develop a prototype LegalTech software module for court evidence management. Further studies may also extend the model to other case types, such as contractual disputes, administrative claims, and technical expert-report review.

In conclusion, N-EviChain provides a focused and explainable neutrosophic framework for legal fact support assessment under uncertainty. Its main value lies in making heterogeneous evidence chains more structured, transparent, and reviewable while preserving human legal authority.

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Neutrosophic Approaches in Deep Learning for Ambiguous Medical Image Analysis: A Systematic Review Toward Wound Assessment

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Abstract

Deep Learning (DL) has contributed to the automated image analysis of wounds and skin lesions, but common architectures have not integrated an explicit representation of boundary ambiguity, tissue heterogeneity, and acquisition noise in images. Neutrosophic theory is relevant to address this problem by representing every pixel/region with an independent triplet (truth T, indeterminacy I, and falsity F memberships). This systematic literature review (SLR) conducted according to the PRISMA 2020 statement aims at mapping and critical analysis of the scientific literature hybridizing neutrosophic sets/logic and DL for wound/skin lesion/related soft tissues image analysis. A structured and multisource literature search over the time interval from the inception of the theoretical neutrosophic set theory until now revealed a small but coherent methodologically speaking literature; after two stages of selection according to clear criteria, 15 papers were selected for analysis. A four-axis taxonomy (level of integration, clinical application task, neutrosophic theory variant, and dataset imaging domain) is developed and populated using the selected literature. Among others, a significant result is that there is not a single study satisfying the inclusion criteria that addresses wound imaging; the body of evidence includes dermoscopy images and images of skin lesions, MRI brain tumor images, thyroid ultrasound images, radiographic bone fractures, and chest radiography, all of which have an identical ambiguity boundary-based structure compared to wound assessment. Neutrosophic fusion is shown to occur almost exclusively at the preprocessing and clustering phases of the DL workflow; loss-function-based and end-to-end approaches to neutrosophic-DL have not been explored yet. The indeterminacy (I) channel is revealed to show empirically well-justified diagnostic capacity better than the truth channel alone in some studies — thus being positioned as a potential intrinsic interpretability modality beyond post hoc explanation approaches. The review concludes by summarizing the results obtained into a critically reviewed comparative synthesis, highlighting translational challenges (high computational costs, lack of standardization, unexplored explainability synergy potential, and lack of deployment on the edge evidence) and providing a research agenda with prioritized needs (especially creation of a publicly available wound image benchmark including neutrosophic ground truth).

Keywords: Neutrosophic Set; Deep Learning; Wound Image Analysis; Skin Lesion Segmentation; Medical Image Uncertainty; Indeterminacy.

1. Introduction

Chronic and acute wounds like diabetic foot ulcers, pressure ulcers, venous ulcers, surgical wounds, and burns have to be visually assessed regularly in order to monitor their tissue composition and healing dynamics and cause a significant burden on global healthcare systems. Visual inspection by clinicians is still the gold standard and is subjective and inconsistent; hence, there has been a lot of wound image analysis research utilizing deep learning methods, including segmentation approaches like Detect-and-Segment [1] and tissue segmentation for diabetic foot ulcers [2, 3]. The problem with dermoscopic skin lesion images is the same segmentation of pathological lesions from normal tissues in various lighting conditions and under artifacts and occlusions and is considered a similar evidentiary field in this review (Section 4).

However, despite their excellent benchmark performance, several issues of structural nature can be identified in using DL classifiers and segmenters on wound and skin lesion images. These are: first, the lack of clear distinction between boundaries of wound bed, peri-wound skin, slough, and granulation tissue is gradual and non-linear [4]; second, softmax/sigmoid activation functions do not give a means of measuring epistemic uncertainty; in other words, there is no way to distinguish between ambiguous regions and the regions that are confidently classified in wrong ways. Finally, the data set is relatively small, inconsistently labeled, and unbalanced for skin types and wounds [1, 3]. The solution to this particular problem – that of representing ambiguity as opposed to assigning a clear classification – has been developed over forty years in a line of fuzzy sets, intuitionistic fuzzy sets, and finally, neutrosophic sets, as shown in Section 2.

1.1 Positioning Against Prior Surveys

The Table 1 below highlights the relation between this review and the most similar surveys. The hybridization survey [5] focuses on DL-neutrosophic hybrids in the context of medical imaging in general but does not focus on the application of neutrosophic theory and DL in the context of wound and skin injury imaging and does not utilize the PRISMA guidelines. General surveys of wound image analysis focus solely on DL while omitting any mention of neutrosophic or any other uncertainty formalism. Classical surveys of neutrosophic image processing predate the era of deep learning and hence overlook its possible use.

Table 1. Positioning of this review against the closest related prior surveys.

Prior Work	Covers DL	Covers Neutrosophy	Covers Wound/Skin Domain	PRISMA / Taxonomy
Mostafa et al. [5], 2021 (medical imaging, general)	Yes	Yes	Partial (not focused)	No / No
General wound-image-analysis DL surveys (e.g., underlying [1, 3])	Yes	No	Yes	Varies / No

Classical neutrosophic image-processing surveys (pre-DL era, e.g., foundations in [6, 7])	No	Yes	No	No / No
This review	Yes	Yes	Yes (with disclosed domain-broadening, §3.2)	Yes / Yes

1.2 Contributions

C1 : The first PRISMA 2020-compliant SLR scoped specifically to neutrosophic-DL hybridization for wound and skin-injury image analysis.

C2 : A historically grounded synthesis tracing how neutrosophic theory evolved from, and mathematically improves upon, fuzzy and intuitionistic fuzzy uncertainty modeling, and how this evolution was integrated into medical deep learning.

C3 : A four-axis taxonomy (integration level, clinical task, neutrosophic variant, dataset/imaging domain) for classifying the identified literature.

C4 : A consolidated, quality-appraised comparative synthesis of all included primary studies.

C5 : A stage-by-stage technical analysis of how indeterminacy (I) is computed, propagated, and exploited across the DL pipeline.

C6 : An evidence-based discussion of translation barriers: computational overhead, absent standardization, untested explainability synergy, and unproven edge-deployment readiness.

C7 : A prioritized future-research roadmap, foremost calling for a dedicated wound-specific neutrosophic-DL benchmark.

1.3 Structure of the paper is the following

The rest of the paper is structured as follows. In Section 2, the theoretical underpinnings and evolution of neutrosophy are reviewed, the neutrosophic $\langle T, I, F \rangle$ model is described, and its inclusion in medical image analysis and deep learning is explained. The PRISMA 2020 protocol is described in Section 3. The taxonomy is presented in Section 4. In Section 5, the technical synthesis and master comparison are given. Clinical feasibility and challenges are discussed in Section 6. Limitations are discussed in Section 7. Conclusion and future work follow in Section 8.

2. Background

This part lays out the background required to explain why the particular mathematical theory called neutrosophic, among all others, became the theoretical framework used in this literature for representing uncertainty in medical images. It provides an account of how each element of the theory became incorporated into medical image analysis using deep learning through time, in five stages: (2.1) the forty years of development of uncertainty theories preceding neutrosophic theory; (2.2) the philosophical basis and the definition of the neutrosophic set; (2.3) the computational tools within the neutrosophic theory; (2.4) the particular image transformation technique implementing this

framework to pixel values; and (2.5) the history of adoption of each of these elements into medical deep learning until this review's point of focus on wound imaging.

2.1 From Classical to Fuzzy to Intuitionistic Fuzzy Sets

Classical ("crisp") set theory admits only binary membership: an element either belongs to a set or it does not. Zadeh's fuzzy set theory [8] relaxed this by allowing a graded membership function:

$$A = \{ (x, \mu_A(x)) \mid x \in X \}, \quad \mu_A(x) \in [0, 1] \quad (1)$$

where $\mu_A(x)$ is the degree to which x belongs to A . Applied to a wound image, $\mu_A(x)$ might express "how much" a pixel resembles wound tissue – already an improvement over a crisp mask, but it conflates two distinct situations: a pixel that is genuinely borderline between wound and healthy tissue, and a pixel about which the model is simply uncertain. Atanassov's intuitionistic fuzzy set (IFS) [9] partially addressed this by adding an explicit non-membership degree $\nu_A(x)$, independent of $\mu_A(x)$ up to a joint constraint:

$$A = \{ (x, \mu_A(x), \nu_A(x)) \mid x \in X \}, \quad \mu_A(x) + \nu_A(x) \leq 1 \quad (2)$$

with the residual hesitancy degree $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ capturing whatever indeterminacy remains. This was a genuine conceptual advance: membership and non-membership are no longer forced to be complementary but $\pi_A(x)$ is not a free, independently specifiable quantity: it is whatever is left over once $\mu_A(x)$ and $\nu_A(x)$ are fixed. For a wound-boundary pixel, this means indeterminacy can only ever be reported as a residual, never as a primary, directly measured signal in its own right. This residual-only treatment of indeterminacy is precisely the gap that motivated Smarandache's neutrosophic set.

2.2 The Neutrosophic Set

Smarandache introduced neutrosophy as a philosophical framework for reasoning about a proposition together with its opposite and its neutral, indeterminate middle ground, and formalized the neutrosophic set as a mathematical instantiation of this idea [10, 11]. The defining move, relative to Section 2.1, is to make indeterminacy a first-class, independently valued component rather than a derived residual. In its original, philosophically general form, a neutrosophic set A on universe X assigns each element three membership functions T_A, I_A, F_A drawn from the non-standard unit interval, subject only to a loose sum bound:

$$0- \leq T_A(x) + I_A(x) + F_A(x) \leq 3+ \quad (3)$$

with T_A, I_A, F_A permitted to vary completely independently of one another. This is the crucial theoretical departure from Section 2.1:

$I_A(x)$ is no longer forced to be $1 - \mu_A(x) - \nu_A(x)$; it can be measured and reported on its own terms. For engineering and image-processing use – where the non-standard analysis of Eq. (3) is unnecessary machinery – the field almost universally adopts the Single-Valued Neutrosophic Set (SVNS) simplification, restricting each component to the standard real unit interval:

$$T_A(x), I_A(x), F_A(x) \in [0, 1], \quad 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \quad (4)$$

Table 2 summarizes the representational capacity this progression achieves each successive theory adds exactly the degree of freedom the previous one lacked, with the neutrosophic set the only one of the three in which indeterminacy is both explicit and fully independent.

Table 2. Representational capacity of fuzzy, intuitionistic fuzzy, and neutrosophic sets.

Property	Fuzzy Set	Intuitionistic Fuzzy Set	Neutrosophic Set
Membership (T/μ)	Explicit	Explicit	Explicit
Non-membership (F/v)	Implicit ($1-\mu$)	Explicit	Explicit
Indeterminacy (I/π)	Not represented	Residual only ($\pi = 1-\mu-v$)	Explicit and independent
Sum constraint	—	$\mu + v \leq 1$	$0 \leq T+I+F \leq 3$ (no coupling)

2.3 Formal Neutrosophic Measures Used in the Reviewed Literature

Beyond the base definition, three derived measures recur throughout the neutrosophic-DL literature reviewed in Section 5 and are worth formalizing here. First, to rank or threshold a neutrosophic-valued pixel or decision back into a single comparable quantity, a score function is used, most commonly:

$$s(A) = (2 + TA - IA - FA) / 3 \quad (5)$$

alongside an accuracy function $a(A) = T_A - F_A$ that ranks elements once scores tie. Second, when two neutrosophic representations must be compared — for instance, a candidate pixel against a class prototype a neutrosophic Euclidean distance is used:

$$d(A, B) = \sqrt{[(1/3) ((TA - TB)^2 + (IA - IB)^2 + (FA - FB)^2)]} \quad (6)$$

with related cosine-similarity and correlation-coefficient measures formalized for SVNS decision-making by Ye [12], and subsequently re-used, in adapted form, as the Neutrosophic Similarity Score training-reweighting mechanism discussed in Section 5.2. Third, converting a full $\langle T, I, F \rangle$ triplet back to a single crisp output termed de-neutrosophication, by analogy with fuzzy defuzzification is most simply achieved via a weighted combination such as $T_A - F_A \cdot (1 - I_A)$, which discounts the truth-falsity contrast in proportion to how indeterminate the pixel is; Section 5.2 discusses how several reviewed studies instead keep the three channels separate throughout, precisely to avoid collapsing this information prematurely.

2.4 The Neutrosophic Image Transform: From Pixel to (T, I, F)

Guo and Cheng operationalized the SVNS formalism of Eq. (4) specifically for image data [6], mapping a pixel's local intensity, local homogeneity, and local non-homogeneity respectively onto the T , I , and F sub-images:

$$T(x, y) = [\bar{g}(x, y) - \bar{g}_{\min}] / [\bar{g}_{\max} - \bar{g}_{\min}] \quad (7)$$

$$I(x, y) = [\delta(x, y) - \delta_{\min}] / [\delta_{\max} - \delta_{\min}], \quad \delta(x, y) = |g(x, y) - \bar{g}(x, y)| \quad (8)$$

$$F(x, y) = 1 - T(x, y) \quad (9)$$

where $g(x, y)$ is the raw pixel intensity, $\bar{g}(x, y)$ a local-window mean, and $\delta(x, y)$ the local homogeneity/inhomogeneity measure driving indeterminacy. The three resulting sub-images are most commonly stacked as a three-channel pseudo-RGB tensor, allowing unmodified CNN backbones to ingest the neutrosophic representation directly (Section 5.2, Fig. 1). Denoising [13] and fuzzy-c-means-based segmentation refinements [14] extended this same T/I/F decomposition before neutrosophic c-means (NCM) clustering [15] generalized it into a standalone unsupervised segmentation algorithm — the specific mechanism most re-used by the DL-hybrid studies in Section 5. Because $I(x, y)$ is computed analytically and deterministically from image structure via Eq. (8), rather than inferred from model stochasticity as in Monte-Carlo dropout or deep ensembles, it is structurally distinct from post-hoc DL uncertainty-estimation techniques — a distinction central to the interpretability discussion in Section 6.

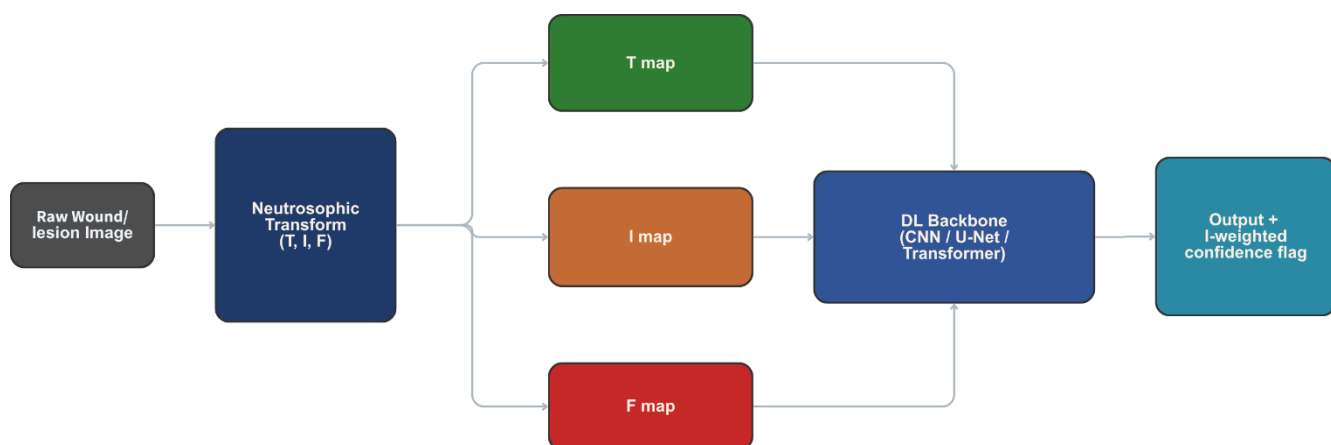


Figure 1. Raw image is decomposed into T/I/F maps and integrated into the DL backbone, with I serving as a confidence signal.

Illustrative worked example

Consider a 3×3 pixel neighborhood centered on a peri-wound boundary, with raw intensities $g(x, y)$ ranging from 60 (clearly healthy skin) to 200 (clearly necrotic tissue) and a local mean $\bar{g} = 130$. Applying Eq. (7) with observed $\bar{g}_{\min} = 40$ and $\bar{g}_{\max} = 220$ gives $T \approx (130 - 40) / (220 - 40) = 0.50$ — a mid-range truth value reflecting genuine ambiguity. If the local homogeneity measure δ for this neighborhood is high (the neighborhood mixes both tissue types), Eq. (8) yields a correspondingly high I , e.g., $I \approx 0.70$, while $F = 1 - T = 0.50$ from Eq. (9). A conventional crisp classifier forced to threshold T alone at 0.5 would arbitrarily assign this pixel to either class; the neutrosophic

representation instead retains $\langle 0.50, 0.70, 0.50 \rangle$ and, via the score function of Eq. (5), $s(A) = (2+0.50-0.70-0.50)/3 \approx 0.43$ a below-midpoint score that can be used to flag the pixel for clinician review rather than silently resolved. This is the concrete mechanism, worked at the single-pixel level, that Sections 4 , 6 examine at the scale of full studies and datasets.

2.5 A Complete Worked Scenario: From Pixel to Clinical Triage Decision

To make the end-to-end reasoning fully concrete, consider three representative regions cropped from a single hypothetical wound photograph, each pushed through Eqs. (7), (9) and then through the score function of Eq. (5), yielding a three-way triage decision rather than a binary label. Region R1 sits well inside clearly healthy peri-wound skin; R2 sits well inside the necrotic wound bed; R3 straddles the boundary between slough and granulation tissue precisely the kind of region a clinician would pause over during manual assessment. Table 3a reports the resulting T, I, F, and score values and the triage action a simple decision rule (accept if $s \geq 0.66$; refer if $0.33 \leq s < 0.66$; reject if $s < 0.33$) would take for each.

Table 3a. Worked scenario applying Eqs. (7) , (9) and the score function of Eq. (5) to three representative wound-image regions.

Region	T	I	F	s(A)	Triage Decision (Eq. 5)
R1 healthy skin	0.95	0.08	0.05	0.94	Accept as healthy (high confidence)
R2 necrotic wound bed	0.90	0.12	0.10	0.89	Accept as wound tissue (high confidence)
R3 slough/granulation boundary	0.50	0.70	0.50	0.43	Refer for clinician review (high indeterminacy)

The difference between R1/R2 and R3 encapsulates the whole practical argument for this review in a nutshell. A typical DL-based segmentation algorithm with a simple T-type confidence measure could theoretically also designate R3 as a “low confidence” instance depending on whether the softmax output was close to 0.5 – but there would be no way of knowing that this does not reflect class overlap as in R3 but instead out-of-distribution artifact or motion blur or sensor noise, since both would be reduced to the same single number. The neutrosophic model retains the ability to differentiate these cases by having a high I-value (0.70 in R3) that indicates structural ambiguity of the statistics of the image itself (see Eq. 8), and not merely low classifier confidence, which is the very diagnostic information needed by the wound-care practitioner. The discussion in Section 5.2 is focused on the fraction of the twelve primary studies that retain this ability to differentiate.

2.6 Summary of Notation

Table 3b consolidates the symbols introduced in Sections 2.1 , 2.4 for reference throughout the remainder of the paper.

Table 3b. Summary of notation used throughout Sections 2, 5, and 6.

Symbol	Meaning	Defined In
$\mu_A(x)$	Fuzzy membership degree of x in set A	Eq. (1)
$\mu_A(x)$, $\nu_A(x)$, $\pi_A(x)$	Intuitionistic fuzzy membership, non-membership, and residual hesitancy	Eq. (2)
$T_A(x)$, $I_A(x)$, $F_A(x)$	Neutrosophic truth, indeterminacy, and falsity membership (independent)	Eqs. (3),(4)
$s(A)$, $a(A)$	Neutrosophic score function and accuracy function	Eq. (5)
$d(A,B)$	Neutrosophic Euclidean distance between two neutrosophic values	Eq. (6)
$g(x,y)$, $\bar{g}(x,y)$, $\delta(x,y)$	Raw pixel intensity, local-window mean, and local homogeneity measure	Eqs. (7),(8)
$T(x,y)$, $I(x,y)$, $F(x,y)$	Pixel-level neutrosophic image sub-domains	Eqs. (7),(9)

2.7 Evolutionary Timeline

Figure. 2 charts the resulting integration history from beginning to end. While Section 2.1's (1965–1986) theoretical heritage provided the vocabulary, it was Smarandache's (1998–2003) formalization which provided the independent-indeterminate formalism [11] itself; the Guo and Cheng (2009) image transform (Eqs. 7, 9) made that formalism compute-able in pixel space; the years 2011–2015 completed the neutrosophic clustering into NCM as a segmented-primitive; Ye (2013)'s SVNS measures provided the scoring, distance, and similarity formalism of Section 2.3; and, finally, these elements were brought together with convolutional neural network backbones in the first neutrosophic-DCNN structures as early as 2019 [7, 16] which were all brought together into the field's first edited volume the same year. 2021 brought a dedicated hybridization survey [5], and 2022–2025 brought the accelerated application quantified in Section 5.1 below to skin lesions, brain tumors, bones, thyroids, and chest imagery. However, at no point during this thirty-year integration process did that trajectory ever reach wound imagery specifically: the gap filled by this SLR in Sections 3–6 is not a failure of this survey's search, but rather the very endpoint of its timeline.

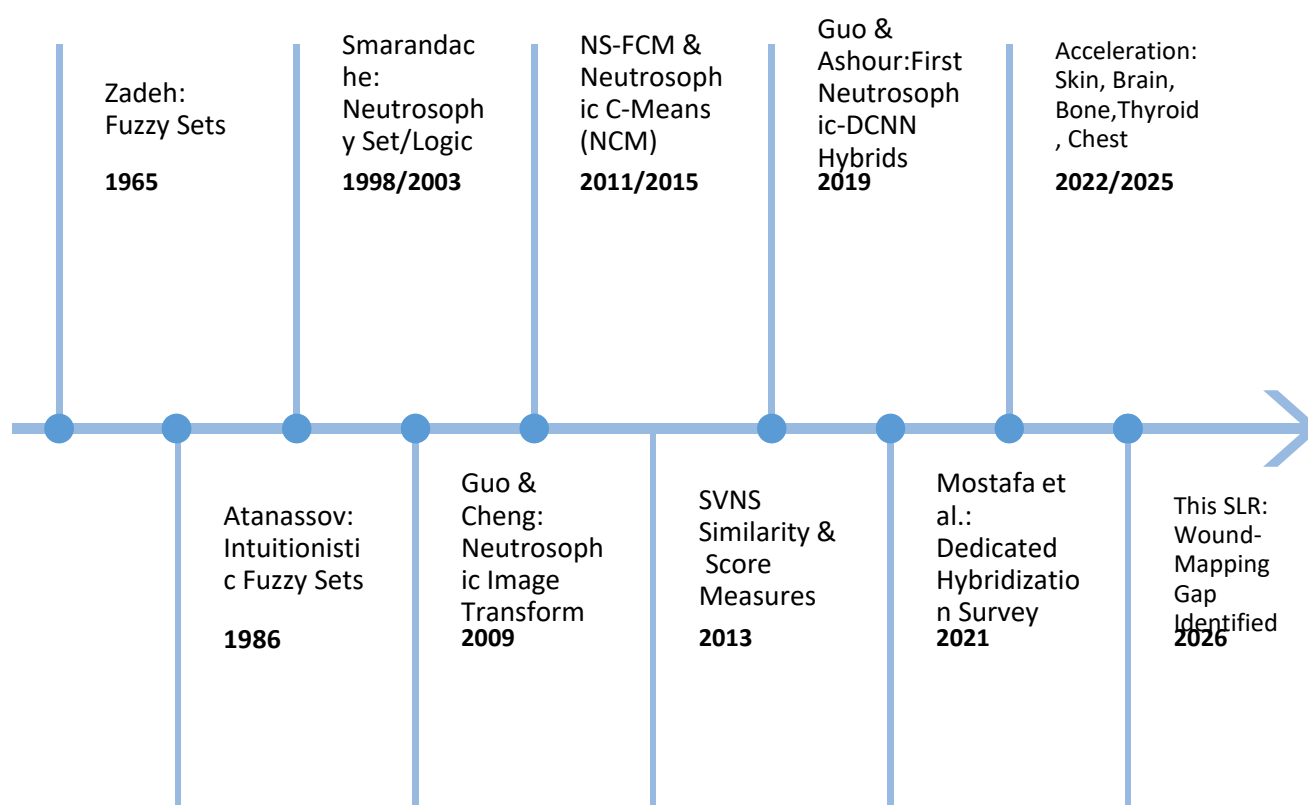


Figure. 2. Historical evolution and integration of neutrosophic theory into medical deep learning, from its fuzzy-set antecedents to the wound-imaging gap identified by this review.

2.8 Why Neutrosophic Theory Fits Wound and Skin-Injury Imaging Specifically

There are three aspects of wound and skin lesion images that make them an especially good fit to the theory described above, and which ensure that all four studies from Section 5, while not being applied specifically to wounds, can be used directly in any case. First, there is a multi-value and non-complementary nature of the progression of the condition (healthy skin $\rightarrow$ erythema near wound $\rightarrow$ slough/fibrin $\rightarrow$ granulation $\rightarrow$ necrotic tissue). Second, wound photographs are acquired in an uncontrolled environment with different lighting conditions, camera devices, dressing obfuscation, etc., thus increasing the local inhomogeneity term $\delta(x,y)$ of Eq. (8), which means that the indeterminacy channel is an actual, computable solution to a practical problem of image acquisition. Third, there is an uncertainty category in manual analysis of wounds; this three-way decision (accept/reject/refer) is formalized through the score function of Eq. (5), which is not possible using only a two-class DL softmax function. Sections 4,6 will consider the degree to which these theoretical potential is actually realized practically, using the retained corpus.

3. Review Methodology (PRISMA 2020)

This review follows the PRISMA 2020 statement and its accompanying elaboration guidance [17, 18]. Given the narrow, cross-disciplinary literature dispersed across computer-vision, biomedical-engineering, and neutrosophic-theory venues not uniformly indexed in a single database formal pre-registration was not undertaken; this is disclosed as a limitation (Section 7.2).

3.1 Research Questions and Search Strategy

Research questions were derived from a PICOC framing (Population: wound/skin-lesion imagery; Intervention: neutrosophic-DL hybrids; Comparison: DL-only or fuzzy-DL baselines; Outcome: segmentation/classification/uncertainty metrics; Context: peer-reviewed, English, 1998 to the present).

Table 4. Research questions guiding the systematic literature review on neutrosophic theory and deep learning for wound/skin-injury imaging.

ID	Research Question
RQ1	What motivates hybridizing neutrosophic theory with DL for wound/skin-injury imaging?
RQ2	What is the publication volume and temporal trend of this literature?
RQ3	At which pipeline stage is neutrosophic theory integrated with DL?
RQ4	Which clinical tasks and neutrosophic variants are addressed?
RQ5	What datasets/metrics are used, and how comparable are reported results?
RQ6	How is indeterminacy (I) computed, propagated, and exploited?
RQ7	What barriers limit clinical translation?

A structured search combined the terms in Table 4 across a broad range of indexed scholarly sources general-purpose bibliographic databases, major scientific publisher platforms, biomedical literature repositories, open-access preprint servers, and the specialized neutrosophic-theory literature supplemented by backward snowballing on the reference lists of the most relevant retrieved items.

Table 5. Structured search-term concept blocks (Boolean AND across blocks; OR within each block).

Concept Block	Representative Search Terms	Combined With
Neutrosophic theory	"neutrosophic", "neutrosophy", "truth-indeterminacy-falsity"	AND
Deep learning	"deep learning", "CNN", "convolutional neural network", "U-Net", "transformer"	AND
Clinical/imaging domain	"wound", "skin lesion", "ulcer", "burn", "melanoma", "medical image"	—

3.2 Eligibility Criteria

Table 6. Inclusion and exclusion eligibility criteria.

Criterion	Inclusion	Exclusion
Theory	Explicit, formally defined neutrosophic (T,I,F) component	Fuzzy-only / intuitionistic-fuzzy-only, no NS triplet
Method	Explicit DL architecture at some pipeline stage	Purely classical/handcrafted-feature ML, no DL
Domain	Wound, dermoscopic/skin-lesion, or a closely analogous soft-tissue boundary-imaging task	Domains with no boundary-ambiguity analogy to wound/skin assessment
Evidence	Quantitative empirical evaluation reported	Conceptual/theoretical only, no experiment
Access/language	Full text accessible, English	Inaccessible; non-English with no translation

The reason for the deliberate widening of the criteria to the highly similar area of dermoscopic skin lesions and other soft tissues like the brain, thyroid, bone, and chest was because of the very low number of neutrosophic DL studies found with the criterion of wound specificity; this is mentioned as a limitation in Section 7.2.

3.3 Study Selection Process

Two-stage screening (title/abstract, then full text) was applied against Section 3.2; Figure. 3 reports the resulting PRISMA flow.

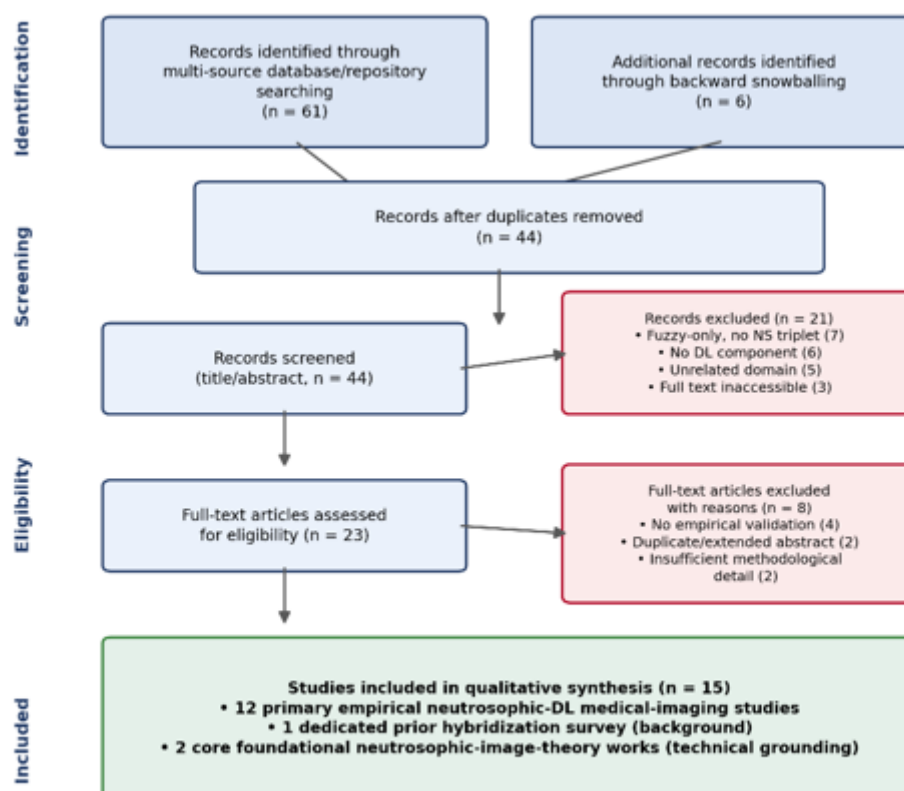


Figure 3. PRISMA 2020 flow diagram for the neutrosophic-DL wound/skin-injury imaging search.

3.4 Data Extraction and Quality Appraisal

For each of the 15 selected studies, the following details were recorded on a standardized form: authors, source, neutrosophic variant used, deep learning architecture, degree of integration, clinical task performed, data set, metrics, and best-performing results (filling Table 7). In addition to this, each of the 12 main studies was evaluated according to six quality criteria, where a value of 0 indicates that the criterion is not satisfied, while a value of 1 indicates full compliance: formal definition of the transform (QA1); description/availability of the data set (QA2); clear indication of the train/test split (QA3); use of multiple standard metrics (QA4); robustness/significance of results (QA5); reproducibility assistance (QA6). The per-criterion ratings are presented in Table 4; their totals are carried over to Table 7 and used qualitatively for rating confidence in Sections 5 and 6, as appropriate for the exploratory nature of the field, rather than for filtering out any studies. Due to high levels of heterogeneity in data sets, neutrosophic variants, and reported metrics across studies, a quantitative meta-analysis was deemed inappropriate.

Table 7. Per-criterion quality-appraisal scores (0–1 scale; QA1–QA6 defined in §3.4).

Study	QA1	QA2	QA3	QA4	QA5	QA6
Atef & El Shahat [19], 2025	1.0	1.0	1.0	1.0	0.5	0.5
Salama et al. [20], 2025	1.0	0.5	1.0	1.0	0.5	0.5
Bian et al. [4], 2022	1.0	1.0	0.5	0.5	0.5	0.5
Singh et al. [21], 2022	1.0	1.0	1.0	1.0	0.5	0.5
Taher & Abdelaziz [22], 2022	1.0	0.5	0.5	1.0	0.5	0.5
Abdelhafeez et al. [23], 2023	1.0	0.5	1.0	1.0	0.5	0.5
Guo & Ashour [16], 2019	0.5	0.5	0.5	1.0	0.5	0.5
El-Shahat & Tolba [24], 2024	1.0	1.0	1.0	1.0	0.5	0.5
Abdullah [25], 2024	1.0	1.0	0.5	1.0	0.5	0.5
Alsattar et al. [26], 2024	1.0	1.0	1.0	1.0	1.0	0.5
Anter et al. [27], 2014	0.5	0.5	0.5	0.5	0.5	0.5
Nugroho et al. [28], 2017	0.5	0.5	0.5	0.5	0.5	0.5

4. Taxonomy of the Reviewed Literature

There are four orthogonal axes, obtained inductively from the extracted information, that characterize the corpus (Figure. 4): (I) Level of integration the pipeline step where neutrosophic theory intersects with DL (T/I/F transform in pre-processing by Eqs. 7,9 [4, 19, 24, 25]; fusion of features [16, 23]; segmentation of clusters at the front-ends by NCMs [22], [26]–[28]; loss-function and end-to-end native layers, which are essentially not present); (II) Clinical application — segmentation/boundary detection, classification, or their joint pipelines, while healing process/time estimation is not covered at all; (III) Neutrosophic approach — the SVNS transform by Eq. (4), neutrosophic c-means, Neutrosophic Similarity Score of [16], constructed on the basis of Ye's measures (§2.3), and more advanced/pentagonal NS approaches [4, 19]; and (IV) Dataset/medical imaging field dermoscopic lesions, brain MRI, thyroid and breast ultrasound, bone and chest radiographs, retinal OCT, and CT images, while wound. In addition to the twelve original studies presented in Table 10, the wider body of literature on neutrosophic medical imaging demonstrates a diversity in domains, including the segmentation of OCT retinal fluid [29] and physiologic signal-derived imaging like cardiocography [30], indicating that the integration techniques have strong generalizability despite the fact that none deal specifically with wounds.

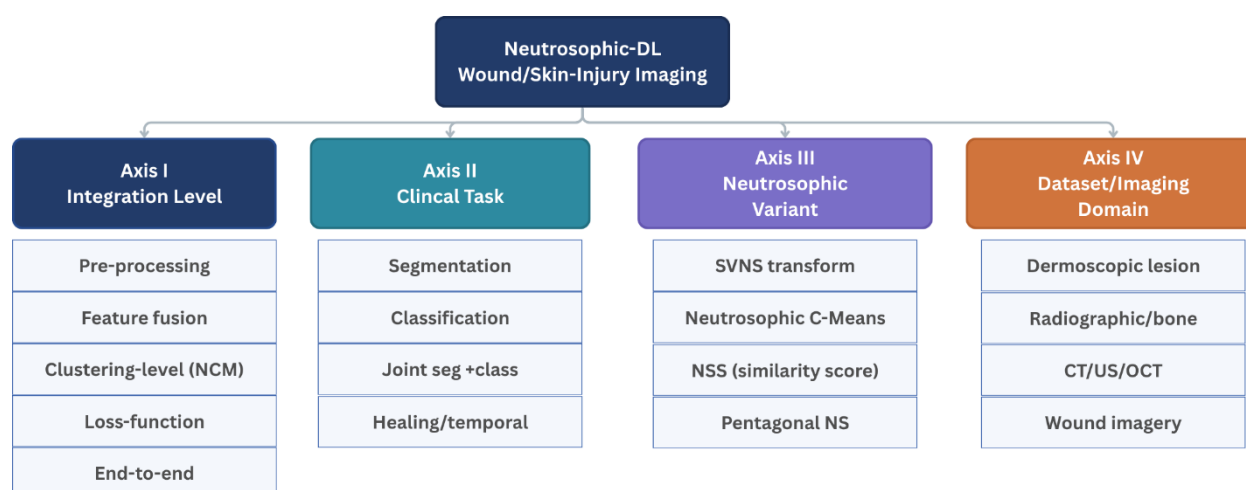


Figure 4. Four-axis taxonomy of the included neutrosophic-DL/neutrosophic-imaging literature. Table 8 cross-tabulates Integration Level against Clinical Task. Clustering-level segmentation is the most mature cell; wound-specific segmentation, healing/temporal assessment, and loss-function-level or end-to-end integration remain empty – the concrete gaps carried forward to Section 8.

Table 8. Cross-tabulation of Integration Level $\times$ Clinical Task (counts of primary studies).

Integration / Task	Segmentation	Classification	Joint	Healing/Temporal
Pre-processing (T/I/F)	2	4	0	0
Feature-level fusion	0	2	0	0
Clustering-level (NCM)	3	0	1	0
Loss-function-level	0	0	0	0
End-to-end native layers	0	0	0	0

5. Results

5.1 Bibliometric Trend

The papers included in this review range from 2014 to 2025, with a significant increase in number since 2021, with respect to the increased use of transfer learning CNN architectures, as well as the development of Neutrosophic Sets and Systems as a specialist journal [5, 19, 20, 24, 25]. It provides the concrete realization of the 2019 to 2025 period of the timeline shown in Figure. 2. Papers have continued to be published in a few specialist and open access venues rather than computer vision flagship journals.

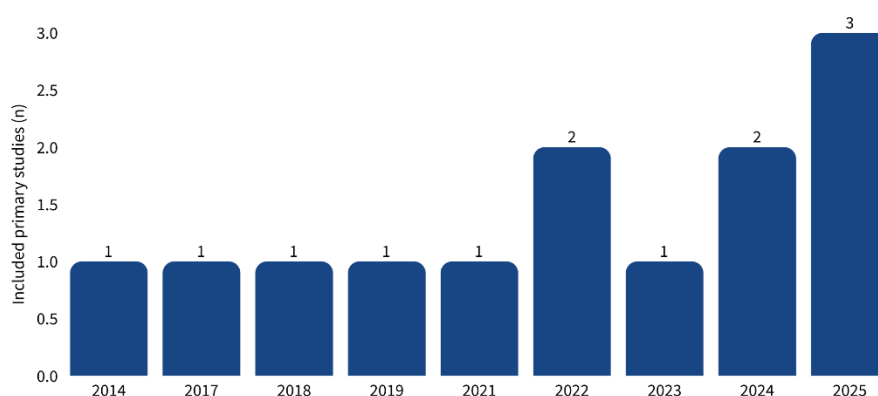


Figure. 5. Temporal distribution of the included corpus, showing acceleration from 2021 onward

5.2 Integration Mechanisms and the Role of Indeterminacy

Pre-processing fusion is the predominant approach in the corpus: the Guo-Cheng transform (Eqs. 7,9) [6] or its advanced forms [4] convert the raw images into T/I/F subspaces, stacked into pseudo-RGB image and used as input for the unaltered CNN backbone, just like in the DenseNet121/VGG16/MobileNet/InceptionV3 experiments of [19] and the five-thresholding processing of [21]. Feature level fusion is rarer: [16] applies Neutrosophic Similarity Score (based on the distance/similarity metrics of §2.3) for each epoch to re-weight the training process of ensemble-DCNN, whereas [23] fuses neutrosophic descriptors with multi-backbone deep features before the classification stage. The most widely adopted clustering level fusion approach is the neutrosophic c-means [15], working as a segmentation front-end, whose output is passed to a separate classifier [22, 26, 28] that is, loosely coupled, not jointly optimized approach, which precisely accounts for the lack of loss function level indeterminacy weighting (RQ3, RQ6).

The noteworthy observation pertains to the I-channel per se. As reported by Abdullah [25], for four distinct DL models trained on separate True, Indeterminate, and Falsity neutrosophic versions of the breast-cancer imaging dataset, only the Indeterminacy version yielded the best classification performance, while the same effect is reported by Atef and El Shahat [19] in relation to DenseNet/Inception models trained on brain-tumor MRI images with respect to both raw images and their NS-transformed versions. The same is reported by Salama et al. [20] for a neutrosophic-topological neural network trained on thyroid-nodule ultrasound images. This finding mirrors the worked example provided in Section 2.4, where it is argued that the indeterminate areas contain significant information rather than noise (which is of relevance to wound borders with their clinically significant fibrin/slough boundaries), thus justifying the use of I as an in-built interpretability channel (Section 6).

5.3 Comparative Analysis

Table 10 consolidates the twelve primary empirical studies against the taxonomy axes, reported performance, and aggregate QA score (max. 6, from Table 7). Raw performance figures are not directly comparable across studies given heterogeneous datasets and metrics (RQ5); only [25, 26]

performed a controlled, same-dataset ablation against a non-neutrosophic baseline, which is summarized separately in Table 9.

Table 9. Controlled, same-dataset neutrosophic-vs-baseline ablations identified in the corpus (RQ5).

Study	Baseline (non-NS)	Neutrosophic-DL Variant	Reported Gain
Abdullah [25], 2024 (breast cancer)	DenseNet121 on raw/crisp image	DenseNet121 on Indeterminacy-domain image	Highest accuracy among T/I/F/raw comparisons
Alsattar et al. [26], 2024 (chest X-ray, COVID-19)	Deep transfer learning on raw image	Deep transfer learning + probabilistic SVN hesitant fuzzy	Improved accuracy/robustness over crisp-domain baseline

Table 10. Comparative analysis of included primary studies. *Retained as directly reused methodological antecedents of later DL-hybrid work (§5.2); excluded from DL-specific counts in Section 4

Study	NS Variant	Integration	DL Backbone	Task	Domain	QA
Atef & El Shahat [19], 2025	SVNS transform	Pre-processing	MobileNet/VGG16/DenseNet121/InceptionV3	Classification	Brain MRI tumor	5.0
Salama et al. [20], 2025	Neutrosophic Topological NN	Feature/model-level	Neutrosophic-topological NN	Classification	Thyroid ultrasound	4.5
Bian et al. [4], 2022	SVNS + Neutrosophic Entropy	Pre-processing	HGMM clustering (non-DL)	Segmentation	Melanoma dermoscopic	4.0
Singh et al. [21], 2022	Pentagonal NS thresholding	Pre-processing	Inception/residual CNN	Seg.+Class.	PH2, ISIC 2017/18/19	5.0
Taher & Abdelaziz [22], 2022	Neutrosophic C-Means	Clustering-level	AlexNet + MLP	Seg.+Class.	Dermoscopic lesion	4.0
Abdelhafeez et al. [23], 2023	Neutrosophic-fused features	Feature-level	Multi-backbone DCNN ensemble	Classification	Skin cancer dermoscopic	4.5
Guo & Ashour [16], 2019	Neutrosophic Similarity Score	Feature/training	Multi-DCNN ensemble	Classification	Dermoscopic lesion	3.5

El-Shahat & Tolba [24], 2024	SVNS transform	Pre-processing	Xception/ResNet52V2/DenseNet121	Classification	X-ray bone fracture	5.0
Abdullah [25], 2024	SVNS T/I/F domain comparison	Pre-processing	DenseNet121 + transfer learning	Classification	Breast cancer imaging	4.5
Alsattar et al. [26], 2024	Probabilistic SVN hesitant fuzzy	Pre-processing/model	Deep transfer learning ensemble	Classification	Chest X-ray (COVID-19)	5.5
Anter et al. [27], 2014	NSFCM	Clustering-level	None (classical, pre-DL)	Segmentation	CT liver	3.0
Nugroho et al. [28], 2017	Neutrosophic + Fuzzy C-Means	Clustering-level	None (classical, pre-DL)	Segmentation	Breast ultrasound	3.0

6. Discussion: Clinical Viability and Open Challenges

Neutrosophic DL is still emerging in literature (12 primary studies) but growing rapidly, with use cases being dominated by the application of the theory at the pre-processing and clustering phases and not in an end-to-end fashion. The most common tasks include classification, and segmentation based on neutrosophic c-means has proven to be the most mature solution.

The indeterminacy (I) can be calculated analytically, and in some studies, it outperforms the truth (T) parameter as a diagnosis tool. Three key barriers that stand in the way of clinical adoption include the lack of common benchmark data for wound imaging, variability in T/I/F transformation formulas used from study to study, and a lack of consistency in metrics. Indeterminacy provides an inherent interpretability signal based on analysis that complements the post hoc XAI approach, although their combination has never been tested. No evaluations have been done prospectively, online, nor have any results on edge device performance or regulatory-quality uncertainty quantification been reported.

In spite of these limitations, there are practical applications that can be considered in the short term. An indeterminacy overlay can highlight zones of ambiguity in the peri-wound area to be analyzed by the clinician, and a pre-processing neutrosophic step can be added before existing deep-learning segmentation algorithms without requiring re-training. The I-channel in the case of tele-wound care could provide a cheap triaging tool. But only the neutrosophic approach, and not fuzzy and intuitionistic-fuzzy approaches, allows one to quantify indeterminacy independently. To achieve this, we need data that is currently not available in the literature.

7. Limitations

7.1 Limitations of the Reviewed Literature

Within the set of studies used for analysis, datasets are relatively small and single-institution/single-repository in nature; external validation across multiple datasets is not performed; there is a lack of statistical significance testing; there is no code and hyperparameters released; and

demographic/phototype diversity in the image datasets used is not provided in any of the studies used, which raises an issue of fairness directly relating to skin and wound images.

7.2 Limitations of This Review

- L1: Search coverage: searches were conducted through available scholarly retrieval infrastructure rather than direct, license-gated querying of every commercial database; some relevant paywalled or non-indexed studies may have been missed.
- L2: Domain broadening: the deliberate inclusion of dermoscopic/skin-lesion and other soft-tissue studies as analogues to wound imaging (§3.2) strengthens evidentiary breadth but means conclusions about wound imaging specifically are extrapolated rather than directly evidenced in several places; this is flagged wherever it applies.
- L3: Publication bias: as in most technical AI/ML sub-fields, studies reporting successful, high-performing hybrid methods are more likely to be published than negative or null results, which may inflate the apparent maturity of neutrosophic-DL integration.
- L4: Single-reviewer screening: screening and extraction were performed without independent dual-reviewer inter-rater reliability scoring (e.g., Cohen's κ); every retained citation was, however, individually verified against its original source (DOI/publisher record) prior to inclusion to guard against misattribution.
- L5: Temporal currency: this is a fast-moving field; the review reflects the literature available as of the search date and should be treated as a snapshot rather than a permanent state of the art.

8. Conclusion and Future Research Roadmap

This review highlights for the first time according to PRISMA 2020 guidelines the connection between neutrosophic theory and deep learning in relation to the analysis of images related to wound- and skin injuries. Establishing the theoretical background from fuzzy sets to intuitionistic fuzzy sets up to the indeterminate-independent framework in neutrosophic theory (Section 2) allows for determining precisely what the hybridization provides that other fuzzy-DL approaches cannot. There are only a few studies within the evidence base, and the latter is lacking a single paper on wounds; however, the conclusions are largely based on a very similar field of dermoscopy/skin lesions. Integration is limited mainly to preprocessing and clustering; the Indeterminacy channel holds promise as far as its diagnostic and interpretation capability is concerned. All seven listed contributions (C1–C7) are thus validated. Table 8 identifies the most relevant research directions.

Table 11. Prioritized future-research roadmap derived from Sections 2 and 4, 7.

Priority Direction	Current State	Recommended Next Step
Wound-specific benchmark	No neutrosophic-DL study on DFU/chronic-wound imagery identified	Curate a public, multi-center wound dataset with released T/I/F reference computations

Loss-function-level indeterminacy weighting	Absent from the reviewed corpus	Benchmark an indeterminacy-weighted Dice/cross-entropy loss on a shared dataset
Native/learnable neutrosophic layers	All integrations remain pipeline-external	Develop differentiable, jointly trained T/I/F operators as network layers
Transformer–neutrosophic fusion	Not observed in the corpus	Explore neutrosophic-guided attention within Mix-Transformer/ViT wound-segmentation encoders
I-channel as native XAI signal	Suggestive multi-study evidence only (Section 6.3)	Combine the I-channel with post-hoc XAI and validate against clinician-marked regions
Reporting standardization	Inconsistent metrics; rare same-dataset ablations (2/12 studies)	Adopt a minimum checklist: Dice/IoU + Accuracy/F1/AUC + compute overhead + code availability
Edge/point-of-care deployment	No mobile/edge benchmarking identified	Benchmark lightweight neutrosophic-DL variants for smartphone wound photography
Neutrosophic- vs. fuzzy-DL ablation	No head-to-head comparison identified (§6.6)	Benchmark fuzzy-DL, intuitionistic-fuzzy-DL, and neutrosophic-DL on one shared wound dataset

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