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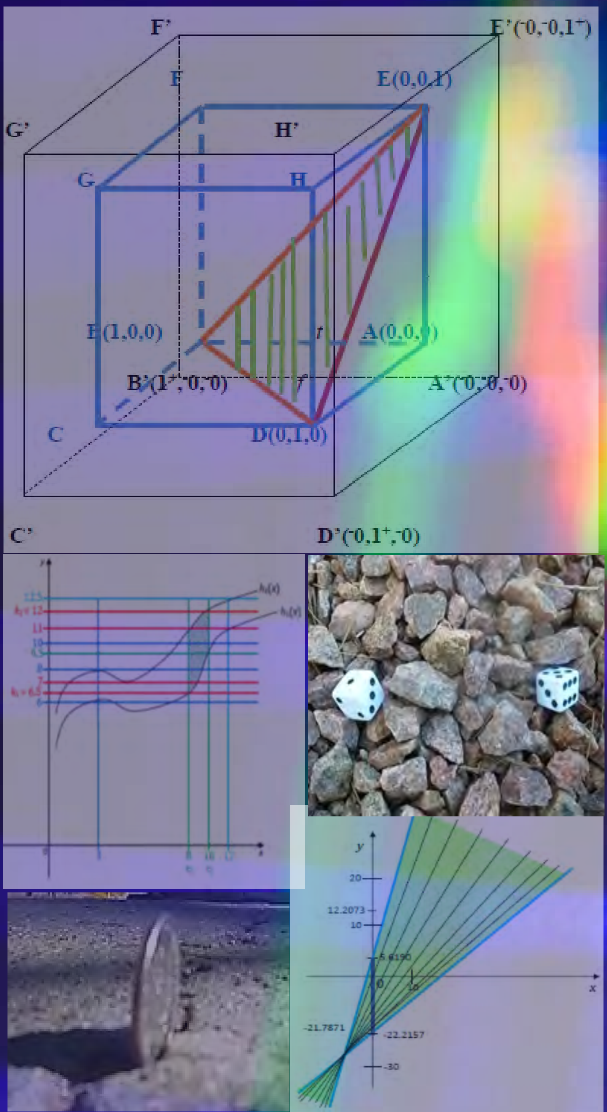

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset
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Information for Authors and Subscribers

"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

Recently, NSS was also approved by Clarivate Analytics for Emerging Sources Citation Index (ESCI) available on the Web of Science platform, starting with Vol. 15, 2017.

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March 20, 2019

Prof. Florentin Smarandache
Univ New Mexico, Gallup Campus

Dear Prof. Florentin Smarandache,

I am pleased to inform you that ***Neutrosophic Sets and Systems*** has been selected for coverage in Clarivate Analytics products and services. Beginning with V. 15 2017, this publication will be indexed and abstracted in:

- Emerging Sources Citation Index

If possible, please mention in the first few pages of the journal that it is covered in these Clarivate Analytics services.

Would you be interested in electronic delivery of your content? If so, we have attached our Journal Information Sheet for your review and completion.

In the future ***Neutrosophic Sets and Systems*** may be evaluated and included in additional Clarivate Analytics products to meet the needs of the scientific and scholarly research community.

Thank you very much.

Sincerely,

Marian Hollingsworth

Marian Hollingsworth
Director, Publisher Relations

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Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures (revisited)

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Abstract: In all classical algebraic structures, the *Laws of Compositions* on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call *NeutroDefined*, or totally undefined (totally false) that we call *AntiDefined*.
Again, in all classical algebraic structures, the *Axioms* (Associativity, Commutativity, etc.) defined on a set are totally true, but it is again a restrictive case, because similarly there are numerous situations in science and in any domain of knowledge when an Axiom defined on a set may be only partially-true (and partially-false), that we call *NeutroAxiom*, or totally false that we call *AntiAxiom*. Therefore, we open for the first time in 2019 new fields of research called *NeutroStructures* and *AntiStructures* respectively.

Keywords: Neutrosophic Triplets, (Axiom, NeutroAxiom, AntiAxiom), (Law, NeutroLaw, AntiLaw), (Associativity, NeutroAssociaticity, AntiAssociativity), (Commutativity, NeutroCommutativity, AntiCommutativity), (WellDefined, NeutroDefined, AntiDefined), (Semigroup, NeutroSemigroup, AntiSemigroup), (Group, NeutroGroup, AntiGroup), (Ring, NeutroRing, AntiRing), (Algebraic Structures, NeutroAlgebraic Structures, AntiAlgebraic Structures), (Structure, NeutroStructure, AntiStructure), (Theory, NeutroTheory, AntiTheory), S-denying an Axiom, S-geometries, Multispace with Multistructure.

1. Introduction

For the necessity to more accurately reflect our reality, Smarandache [1] introduced for the first time in 2019 the NeutroDefined and AntiDefined Laws, as well as the NeutroAxiom and AntiAxiom, inspired from Neutrosophy ([2], 1995), giving birth to new fields of research called NeutroStructures and AntiStructures.

Let's consider a given classical algebraic Axiom. We defined for the first time the neutrosophic triplet corresponding to this Axiom, which is the following: *(Axiom, NeutroAxiom, AntiAxiom)*; while the classical Axiom is 100% or totally true, the NeutroAxiom is partially true and partially false (the degrees of truth and falsehood are both > 0), while the AntiAxiom is 100% or totally false [1].

For the classical algebraic structures, on a non-empty set endowed with well-defined binary laws, we have properties (axioms) such as: associativity & non-associativity, commutativity & non-commutativity, distributivity & non-distributivity; the set may contain a neutral element with respect to a given law, or may not; and so on; each set element may have an inverse, or some set elements may not have an inverse; and so on.

Consequently, we constructed for the first time the neutrosophic triplet corresponding to the Algebraic Structures [1], which is this: *(Algebraic Structure, NeutroAlgebraic Structure, AntiAlbegraic Structure).*

Therefore, we had introduced for the first time [1] the *NeutroAlgebraic Structures* & the *AntiAlgebraic Structures*. A (classical) Algebraic Structure is an algebraic structure dealing only with (classical) Axioms (which are totally true). Then a NeutroAlgebraic Structure is an algebraic structure that has at least one NeutroAxiom, and no AntiAxioms.

While an AntiAlgebraic Structure is an algebraic structure that has at least one AntiAxiom.
These definitions can straightforwardly be extended from Axiom/NeutroAxiom/AntiAxiom to any Property/NeutroProperty/AntiProperty, Proposition/NeutroProposition/AntiProposition, Theorem/NeutroTheorem/AntiTheorem, Theory/NeutroTheory/AntiTheory, etc. and from Algebraic Structures to other Structures in any field of knowledge.

2. Neutrosophy

We recall that in neutrosophy we have for an item <A>, its opposite <antiA>, and in between them their neutral <neutA>.

We denoted by <nonA> = <neutA> ∪ <antiA>, where ∪ means union, and <nonA> means what is not <A>.

Or <nonA> is refined/split into two parts: <neutA> and <antiA>.

The neutrosophic triplet of <A> is: $(\langle A\rangle, \langle neutA\rangle, \langle antiA\rangle)$, *with* $\langle neutA\rangle \cup \langle antiA\rangle = \langle nonA\rangle$.

3. Definition of Neutrosophic Triplet Axioms

Let $\mathcal{U}$ be a universe of discourse, endowed with some well-defined laws, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and an Axiom α, defined on S, using these laws. Then:

1) If all elements of $\mathcal{S}$ verify the axiom α, we have a *Classical Axiom*, or simply we say *Axiom*.
2) If some elements of $\mathcal{S}$ verify the axiom α and others do not, we have a *NeutroAxiom* (which is also called *NeutAxiom*).
3) If no elements of $\mathcal{S}$ verify the axiom α, then we have an *AntiAxiom*.

The Neutrosophic Triplet Axioms are:
(Axiom, NeutroAxiom, AntiAxiom) with

NeutroAxiom ∪ *AntiAxiom* = *NonAxiom*,
and *NeutroAxiom* ∩ *AntiAxiom* = φ (empty set),
where ∩ means intersection.

Theorem 1: The Axiom is 100% true, the NeutroAxiom is partially true (its truth degree > 0) and partially false (its falsehood degree > 0), and the AntiAxiom is 100% false.

Proof is obvious.

Theorem 2: Let *d: {Axiom, NeutroAxiom, AntiAxiom}* → *[0 ,1]* represent the degree of negation function.

The NeutroAxiom represents a degree of partial negation {*d* ∊ *(0, 1)*} of the Axiom, while the AntiAxiom represents a degree of total negation {*d* = *1*} of the Axiom.

Proof is also evident.

4. Neutrosophic Representation

We have: $\langle A \rangle$ = *Axiom;*

$\langle neutA \rangle$ = *NeutroAxiom (or NeutAxiom);*

$\langle antiA \rangle$ = *AntiAxiom;* and $\langle nonA \rangle$ = *NonAxiom.*

Similarly, as in *Neutrosophy*, *NonAxiom* is refined/split into two parts: *NeutroAxiom* and *AntiAxiom*.

5. Application of NeutroLaws in Soft Science

In *soft sciences* the laws are interpreted and re-interpreted; in social and political legislation the laws are flexible; the same law may be true from a point of view, and false from another point of view. Thus, the law is partially true and partially false (it is a *Neutrosophic Law*).

For example, "gun control". There are people supporting it because of too many crimes and violence (and they are right), and people that oppose it because they want to be able to defend themselves and their houses (and they are right too).

We see two opposite propositions, both of them true, but from different points of view (from different criteria/parameters; plithogenic logic may better be used herein). How to solve this? Going to the middle, in between opposites (as in neutrosophy): allow military, police, security, registered hunters to bear arms; prohibit mentally ill, sociopaths, criminals, violent people from bearing arms; and background check on everybody that buys arms, etc.

6. Definition of Classical Associativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is associative on the set $\mathcal{S}$, iff $\forall\, a, b, c \in \mathcal{S}$, $a * (b * c) = (a * b) * c$.

7. Definition of Classical NonAssociativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is non-associative on the set $\mathcal{S}$, iff $\exists a, b, c \in \mathcal{S}$, such that $a * (b * c) \neq (a * b) * c$.

So, it is sufficient to get a single triplet a, b, c (where a, b, c may even be all three equal, or only two of them equal) that doesn't satisfy the associativity axiom.

Yet, there may also exist some triplet $d, e, f \in \mathcal{S}$ that satisfies the associativity axiom: $d * (e * f) = (d * e) * f$.

The classical definition of NonAssociativity does not make a distinction between a set $(\mathcal{S}_1,*)$ whose all triplets $a, b, c \in \mathcal{S}_1$ verify the non-associativity inequality, and a set $(\mathcal{S}_2,*)$ whose some triplets verify the non-associativity inequality, while others don't.

8. NeutroAssociativity & AntiAssociativity

If $\langle \mathrm{A} \rangle$ = (classical) Associativity, then $\langle \mathrm{nonA} \rangle$ = (classical) NonAssociativity.
But we refine/split $\langle \mathrm{nonA} \rangle$ into two parts, as above:

$\langle neutA \rangle$ = *NeutroAssociativity;*
$\langle antiA \rangle$ = *AntiAssociativity.*

Therefore, *NonAssociativity* = *NeutroAssociativity* $\cup$ *AntiAssociativity.*

The Associativity's neutrosophic triplet is: <*Associativity, NeutroAssociativity, AntiAssociativity*>.

9. Definition of NeutroAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.
The set $(\mathcal{S},*)$ is NeutroAssociative if and only if:
there exists at least one triplet $a_1, b_1, c_1 \in \mathcal{S}$ such that: $a_1 * (b_1 * c_1) = (a_1 * b_1) * c_1$; and there exists at least one triplet $a_2, b_2, c_2 \in \mathcal{S}$ such that: $a_2 * (b_2 * c_2) \neq (a_2 * b_2) * c_2$. Therefore, some triplets verify the associativity axiom, and others do not.

10. Definition of AntiAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.
The set $(\mathcal{S},*)$ is AntiAssociative if and only if: for any triplet $a, b, c \in \mathcal{S}$ one has $a * (b * c) \neq (a * b) * c$. Therefore, none of the triplets verify the associativity axiom.

11. Example of Associativity

Let *N = {0, 1, 2, …, ∞}*, the set of natural numbers, be the universe of discourse, and the set

$\mathcal{S} = \{0, 1, 2, \dots, 9\} \subset$ N, also the binary law $*$ be the classical addition modulo 10 defined on N.

Clearly the law * is well-defined on S, and associative since:

$a + (b + c) = (a + b) + c$ (mod 10), for all $a, b, c \in \mathcal{S}$.

The degree of negation is 0%.

12. Example of NeutroAssociativity

$\mathcal{S} = \{0, 1, 2, \dots, 9\}$, and the well-defined binary law $*$ constructed as below:
$a * b = 2a + b$ (mod 10).
Let's check the associativity: $a * (b * c) = 2a + (b * c) = 2a + 2b + c$
$(a * b) * c = 2(a * b) + c = 2(2a + b) + c = 4a + 2b + c$
The triplets that verify the associativity result from the below equality: $2a + 2b + c = 4a + 2b + c$ or $2a = 4a$ (mod 10) or $0 = 2a$ (mod 10), whence $a \in \{0, 5\}$.

Hence, two general triplets of the form: $\{(0, b, c), (5, b, c), \text{where } b, c \in \mathcal{S}\}$ verify the associativity.

The degree of associativity is $\frac{2}{10} = 20\%$, corresponding to the two numbers $\{0, 5\}$ out of ten.

While the other general triplet: $\{(a, b, c), \text{where } a \in \mathcal{S} \setminus \{0, 5\}, \text{while } b, c \in \mathcal{S}\}$

do not verify the associativity.

The degree of negation of associativity is $\frac{8}{10} = 80\%$.

13. Example of AntiAssociativity

$\mathcal{S} = \{a, b\}$, and the binary law $*$ well-defined as in the below Cayley Table:

*	a	b
a	b	b
b	a	a

Theorem 3. For any $x, y, z \in \mathcal{S}$, $x * (y * z) \neq (x * y) * z$.

Proof. We have $2^3 = 8$ possible triplets on $\mathcal{S}$:

1) (a, a, a)

$a * (a * a) = a * b = b$

while $(a * a) * a = b * a = a \neq b$.

2) (a, a, b)

$a * (a * b) = a * b = b$

$(a * a) * b = b * b = a \neq b$.

3) (a, b, a)

$a * (b * a) = a * a = b$

$(a * b) * a = b * a = a \neq b$.

4) (b, a, a)

$b * (a * a) = b * b = a$

$(b * a) * a = a * a = b \neq a$.

5) (a, b, b)

$a * (b * b) = a * a = b$

$(a * b) * b = b * b = a \neq b$.

6) (b, a, b)

$b * (a * b) = b * b = a$

$(b * a) * b = a * b = b \neq a$.

7) (b, b, a)

$b * (b * a) = b * a = a$

$(b * b) * a = a * a = b \neq a$.

8) (b, b, b)

$b * (b * b) = b * a = a$

$(b * b) * b = a * b = b \neq a$.

Therefore, there is no possible triplet on $\mathcal{S}$ to satisfy the associativity. Whence the law is AntiAssociative. The degree of negation of associativity is $\frac{8}{8} = 100\%$.

14. Definition of Classical Commutativity

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is Commutative on the set $\mathcal{S}$, iff $\forall\, a, b \in \mathcal{S},\ a * b = b * a$.

15. Definition of Classical NonCommutativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is NonCommutative on the set $\mathcal{S}$, iff $\exists a, b \in \mathcal{S}$, such that $a * b \neq b * a$.
So, it is sufficient to get a single duplet $a, b \in \mathcal{S}$ that doesn't satisfy the commutativity axiom.
However, there may exist some duplet $c, d \in \mathcal{S}$ that satisfies the commutativity axiom: $c * d = d * c$.

The classical definition of NonCommutativity does not make a distinction between a set $(\mathcal{S}_1,*)$ whose all duplets $a, b \in \mathcal{S}_1$ verify the NonCommutativity inequality, and a set $(\mathcal{S}_2,*)$ whose some duplets verify the NonCommutativity inequality, while others don't.
That's why we refine/split the NonCommutativity into NeutroCommutativity and AntiCommutativity.

16. NeutroCommutativity & AntiCommutativity

Similarly to Associativity we do for the Commutativity:
If $\langle \mathrm{A} \rangle$ = (classical) *Commutativity*, then $\langle \mathrm{nonA} \rangle$ = (classical) *NonCommutativity*.
But we refine/split $\langle \mathrm{nonA} \rangle$ into two parts, as above:

$\langle neutA \rangle$ = *NeutroCommutativity;*

$\langle antiA \rangle$ = *AntiCommutativity*.

Therefore, *NonCommutativity* = *NeutroCommutativity* $\cup$ *AntiCommutativity*.
The Commutativity's neutrosophic triplet is:

<*Commutativity, NeutroCommutativity, AntiCommutativity*>.

In the same way, Commutativity means all elements of the set commute with respect to a given binary law, NeutroCommutativity means that some elements commute while others do not, while AntiCommutativity means that no elements commute.

17. Example of NeutroCommutativity

$\mathcal{S} = \{a, b, c\}$, and the well-defined binary law $*$.

$*$	a	b	c
a	b	c	c
b	c	b	a
c	b	b	c

$a * b = b * a = c$ (commutative);

$\begin{cases} a * c = c \\ c * a = b \neq c \end{cases}$ (not commutative);

$\begin{cases} b * c = a \\ c * b = b \neq a \end{cases}$ (not commutative).

We conclude that $(\mathcal{S},*)$ is $\frac{1 \text{ pair}}{3 \text{ pairs}} \approx 33\%$ commutative, and $\frac{2 \text{ pair}}{3 \text{ pairs}} \approx 67\%$ not commutative.

Therefore, the degree of negation of the commutativity of $(\mathcal{S},*)$ is 67%.

18. Example of AntiCommutativity

$\mathcal{S} = \{a, b\}$, and the below binary well-defined law $*$.

*	a	b
a	b	b
b	a	a

where $a * b = b$, $b * a = a \neq b$ (not commutative)

Other pair of different element does not exist, since we cannot take $a * a$ nor $b * b$. The degree of negation of commutativity of this $(\mathcal{S},*)$ is 100%.

19. Definition of Classical Unit-Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $\mathcal{S}$ has a classical unit element $e \in \mathcal{S}$, iff e is unique, and for any $x \in \mathcal{S}$ one has $x * e = e * x = x$.

20. Partially Negating the Definition of Classical Unit-Element

It occurs when at least one of the below statements occurs:

1) There exists at least one element $a \in \mathcal{S}$ that has no unit-element.

2) There exists at least one element $b \in \mathcal{S}$that has at least two distinct unit-elements e_1 $e_2 \in \mathcal{S}$, $e_1 \neq e_2$, such that:

$b * e_1 = e_1 * b = b$,

$b * e_2 = e_2 * b = b$.

3) There exists at least two different elements $c, d \in \mathcal{S}$, $c \neq d$, such that they have different unit-elements $e_c, e_d \in \mathcal{S}$, $e_c \neq e_d$, with $c * e_c = e_c * c = c$, and $d * e_d = e_d * d = d$.

21. Totally Negating the Definition of Classical Unit-Element

The set $(\mathcal{S},*)$ has *AntiUnitElements*, if:

Each element $x \in \mathcal{S}$ has either no unit-element, or two or more unit-elements (unicity of unit-element is negated).

22. Definition of NeutroUnitElements

The set $(\mathcal{S},*)$ has *NeutroUnit Elements*, if:

1) [Degree of Truth] There exist at least one element $a \in S$ that has a single unit-element.

2) [Degree of Falsehood] There exist at least one element $b \in S$ that has either no unit-element, or at least two distinct unit-elements.

23. Definition of AntiUnit Elements

The set $(\mathcal{S},*)$ has *AntiUnit Elements*, if:

Each element $x \in \mathcal{S}$ has either no unit-element, or two or more distinct unit-elements.

24. Example of NeutroUnit Elements

$\mathcal{S} = \{a, b, c\}$, and the well-defined binary law $*$:

*	a	b	c
a	b	b	a
b	b	b	a
c	a	b	c

Since,

$a * c = c * a = a$

$c * c = c$

the common unit element of *a* and *c* is *c* (two distinct elements $a \neq c$ have the same unit element *c*).

From $b * a = a * b = b$

$b * b = b$

we see that the element b has two distinct unit elements a and b.

Since only one element *b* does not verify the classical unit axiom (i.e. to have a unique unit), out of 3 elements, the degree of negation of unit element axiom is $\frac{1}{3} \approx 33\%$, while $\frac{2}{3} \approx 67\%$ is the degree of truth (validation) of the unit element axiom.

25. Example of AntiUnit Elements

$\mathcal{S} = \{a, b, c\}$, endowed with the well-defined binary law $*$ as follows:

$*$	a	b	c
a	a	a	a
b	a	c	b
c	a	c	b

Element a has 3 unit-elements: a, b, c, because:

$a * a = a$

$a * b = b * a = a$

and $a * c = c * a = a$.

Element b has no u-it element, since:

$b * a = a \neq b$

$b * b = c \neq b$

and $b * c = b$, but $c * b \neq b$.

Element c has no unit-element, since:

$c * a = a \neq c$

$c * b = c$, but $b * c = b \neq c$,

and $c * c = b \neq c$.

The degree of negation of the unit-element axiom is $\frac{3}{3} = 100\%$.

26. Definition of Classical Inverse Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non − empty set $\mathcal{S} \subseteq \mathcal{U}$.

Let $e \in \mathcal{S}$ be the classical unit element, which is unique.

For any element $x \in \mathcal{S}$, there exists a unique element, named the inverse of x, denoted by x^{-1}, such that:

$x * x^{-1} = x^{-1} * x = e$.

27. Partially Negating the Definition of Classical Inverse Element

It occurs when at least one statement from below occurs:

1) There exists at least one element $a \in \mathcal{S}$ that has no inverse with respect to no ad-hoc unit-element;

or

2) There exists at least one element $b \in \mathcal{S}$ that has two or more inverses with respect to some ad-hoc unit-elements.

28. Totally Negating the Definition of Classical Inverse Element

Each element has either no inverse, or two or more inverses with respect to some ad-hoc unit-elements respectively.

29. Definition of NeutroInverse Elements

The set $(\mathcal{S},*)$ has NeutroInverse Elements if:

1) [Degree of Truth] There exist at least one element that has a unique inverse with respect to some ad-hoc unit-element.

2) [Degree of Falsehood] There exists at least one element $c \in \mathcal{S}$ that does not have any inverse with respect to no ad-hoc unit element, or has at least two distinct inverses with respect to some ad-hoc unit-elements.

30. Definition of AntiInverse Elements

The set $(\mathcal{S},*)$ has AntiInverse Elements, if: each element has either no inverse with respect to no ad-hoc unit-element, or two or more distinct inverses with respect to some ad-hoc unit-elements.

31. Example of NeutroInverse Elements

$S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

*	a	b	c
a	a	b	c
b	b	a	a
c	b	b	b

Because $a * a = a$, hence its ad-hoc unit/neutral element $neut(a) = a$ and correspondingly its inverse element is $inv(a) = a$.

Because $b * a = a * b = b$, hence its ad-hoc inverse/neutral element $neut(b) = a$;

from $b * b = a$, we get $inv(b) = b$.

No $neut(c)$, hence no $inv(c)$.

Hence *a* and *b* have ad-hoc inverses, but *c* doesn't.

32. Example of AntiInverse Elements

Similarly, $S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

*	a	b	c
a	b	b	c
b	a	a	a
c	c	a	a

There is no *neut(a)* and no *neut(b)*, hence: no *inv(a)* and *no inv(b)*.

$c * a = a * c = c$, hence: $neut(c) = a$.

$c * b = b * c = a$, hence: $inv(c) = b$;

$c * c = c * c = a,$ hence: $inv(c) = c$; whence we get two inverses of c.

33. Cases When Partial Negation (NeutroAxiom) Does Not Exist

Let's consider the classical geometric *Axiom*:

On a plane, through a point exterior to a given line it's possible to draw a single parallel to that line.

The *total negation* is the following *AntiAxiom*:

On a plane, through a point exterior to a given line it's possible to draw either no parallel, or two or more parallels to that line.

The *NeutroAxiom* does not exist since it is not possible to partially deny and partially approve this axiom.

34. Connections between the neutrosophic triplet *(Axiom, NeutroAxiom, AntiAxiom)* and the *S-denying an Axiom*

The *S-denying of an Axiom* was first defined by Smarandache [3, 4] in 1969 when he constructed *hybrid geometries* (or *S-geometries*) [5 – 18].

35. Definition of S-denying an Axiom

An *Axiom* is said *S-denied* [3, 4] if in the same space the axiom behaves differently (i.e., validated and invalided; or only invalidated but in at least two distinct ways). Therefore, we say that an axiom is partially or totally negated { or there is a degree of negation in (0, 1] of this axiom }: http://fs.unm.edu/Geometries.htm.

36. Definition of S-geometries

A geometry is called *S-geometry* [5] if it has at least one *S*-denied axiom.

Therefore, the Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries were united altogether for the first time, into the same space, by some *S*-geometries. These *S*-geometries could be partially Euclidean and partially Non-Euclidean, or only Non-Euclidean but in multiple ways.

The most important contribution of the *S*-geometries was the introduction of the *degree of negation of an axiom* (and more general the degree of negation of any theorem, lemma, scientific or humanistic proposition, theory, etc.).

Many geometries, such as pseudo-manifold geometries, Finsler geometry, combinatorial Finsler geometries, Riemann geometry, combinatorial Riemannian geometries, Weyl geometry, Kahler geometry are particular cases of *S*-geometries. (Linfan Mao).

37. Connection between *S-denying an Axiom* and *NeutroAxiom / AntiAxiom*

"Validated and invalidated" Axiom is equivalent to NeutroAxiom. While "only invalidated but in at least two distinct ways" Axiom is part of the AntiAxiom (depending on the application).

"Partially negated" (or $0 < d < 1$, where d is the degree of negation) is referred to NeutroAxiom. While "there is a degree of negation of an axiom" is referred to both NeutroAxiom (when $0 < d < 1$) and AntiAxiom (when $d = 1$).

38. Connection between NeutroAxiom and MultiSpace

In any domain of knowledge, a *S-multispace with its multistructure* is a finite or infinite (countable or uncountable) union of many spaces that have various structures (Smarandache, 1969, [19]). The multi-spaces with their multi-structures [20, 21] may be non-disjoint. The multispace with multistructure form together a *Theory of Everything*. It can be used, for example, in the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak, and strong interactions in physics.

Therefore, a NeutroAxiom splits a set *M*, which it is defined upon, into two subspaces: one where the Axiom is true and another where the Axiom is false. Whence *M* becomes a BiSpace with BiStructure (which is a particular case of MultiSpace with MultiStructure).

39. (Classical) WellDefined Binary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a binary law $*$ defined on U. For any $x, y \in \mathcal{S}$, one has $x * y \in \mathcal{S}$.

40. NeutroDefined Binary Law

There exist at least two elements (that could be equal) $a, b \in \mathcal{S}$ such that $a * b \in \mathcal{S}$. And there exist at least other two elements (that could be equal too) $c, d \in \mathcal{S}$ such that $c*d \notin S$.

41. Example of NeutroDefined Binary Law

Let *U* = *{a, b, c}* be a universe of discourse, and a subset $\mathcal{S} = \{a, b\}$, endowed with the below NeutroDefined Binary Law $*$:

*	a	b
a	b	b
b	a	c

We see that: $a * b = b \in S$, $b * a = a \in S$, but $b * b = c \notin S$.

42. AntiDefined Binary Law

For any $x, y \in \mathcal{S}$ one has $x * y \notin \mathcal{S}$.

43. Example of AntiDefined Binary Law

Let *U* = *{a, b, c, d}* a universe of discourse, and a subset $\mathcal{S} = \{a, b\}$, and the below binary well-defined law $*$.

*	a	b
a	c	d
b	d	c

where all combinations between *a* and *b* using the law * give as output *c* or *d* who do not belong to S.

44. Theorem 4 (The Degenerate Case)

If a set is endowed with AntiDefined Laws, all its algebraic structures based on them will be AntiStructures.

45. WellDefined n-ary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a n-ary law, for n integer, $n \geq 1$, defined on $\mathcal{U}$.

$L: \mathcal{U}^n \to \mathcal{U}$.

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \ldots, x_n) \in \mathcal{S}$.

46. NeutroDefined n-ary Law

There exists at least a n-plet $a_1, a_2, \ldots, a_n \in \mathcal{S}$ such that $L(a_1, a_2, \ldots, a_n) \in S$. The elements $a_1, a_2, \ldots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \ldots, b_n \in \mathcal{S}$ such that $L(a_1, a_2, \ldots, a_n) \notin S$. The elements $b_1, b_2, \ldots, b_n$ may be equal or not among themselves.

47. AntiDefined n-ary Law

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \ldots, x_n) \notin \mathcal{S}$.

48. WellDefined n-ary HyperLaw

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subset_{\neq} \mathcal{U}$, and a n-ary hyperlaw, for n integer, $n \geq 1$:

$H: \mathcal{U}^n \to \mathcal{P}(\mathcal{U})$, where $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$.

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \ldots, x_n) \in \mathcal{P}(\mathcal{S})$.

49. NeutroDefined n-ary HyperLaw

There exists at least a n-plet $a_1, a_2, \ldots, a_n \in \mathcal{S}$ such that $H(a_1, a_2, \ldots, a_n) \in \mathcal{P}(\mathcal{S})$. The elements $a_1, a_2, \ldots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \ldots, b_n \in \mathcal{S}$ such that $H(b_1, b_2, \ldots, b_n) \notin \mathcal{P}(\mathcal{S})$. The elements $b_1, b_2, \ldots, b_n$ may be equal or not among themselves.

50. AntiDefined n-ary HyperLaw

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \ldots, x_n) \notin \mathcal{P}(\mathcal{S})$.

*

The most interesting are the cases when the composition law(s) are well-defined (classical way) and neutro-defined (neutrosophic way).

51. WellDefined NeutroStructures

Are structures whose laws of compositions are well-defined, and at least one axiom is NeutroAxiom, while not having any AntiAxiom.

52. NeutroDefined NeutroStructures

Are structures whose at least one law of composition is NeutroDefined, and all other axioms are NeutroAxioms or Axioms.

53. Example of NeutroDefined NeutroGroup

Let U = {a, b, c, d} be a universe of discourse, and the subset

$S = \{a, b, c\}$, endowed with the binary law $*$:

$*$	a	b	c
a	a	c	c
b	a	a	a
c	c	a	d

NeutroDefined Law of Composition:

Because, for example: $a*b = c \in S$, but $c*c = d \notin S$.

NeutroAssociativity:

Because, for example: $a*(a*c) = a*c = c$ and $(a*a)*c = a*c = c$;

while, for example: $a*(b*c) = a*a = a$ and $(a*b)*c = c*c = d \neq a$.

NeutroCommutativity:

Because, for example: $a*c = c*a = c$, but $a*b = c$ while $b*a = a \neq c$.

NeutroUnit Element:

There exists the same unit-element a for a and c, or $neut(a) = neut(c) = a$, since $a*a = a$ and $c*a = a*c = c$.

But there is no unit element for b, because $b*x = a$, not b, for any $x \in S$ (see the above Cayley Table).

NeutroInverse Element:

With respect to the same unit element a, there exists an inverse element for a, which is a, or $inv(a) = a$, because $a*a = a$, and an inverse element for c, which is b, or $inv(c) = b$, because $c*b = b*c = a$.

But there is no inverse element for b, since b has no unit element.

Therefore $(S, *)$ is a NeutroDefined NeutroCommutative NeutroGroup.

54. WellDefined AntiStructures

Are structures whose laws of compositions are well-defined, and have at least one AntiAxiom.

55. NeutroDefined AntiStructures

Are structures whose at least one law of composition is NeutroDefined and no law of composition is AntiDefined, and has at least one AntiAxiom.

56. AntiDefined AntiStructures

Are structures whose at least one law of composition is AntiDefined, and has at least one AntiAxiom.

57. Conclusion

The neutrosophic triplet (<A>, <neutA>, <antiA>), where <A> may be an "Axiom", a "Structure", a "Theory" and so on, <antiA> the opposite of <A>, while <neutA> (or <neutroA>) their neutral in between, are studied in this paper.

The NeutroAlgebraic Structures and AntiAlgebraic Structures are introduced now for the first time, because they have been ignored by the classical algebraic structures. Since, in science and technology and mostly in applications of our everyday life, the laws that characterize them are not necessarily well-defined or well-known, and the axioms / properties / theories etc. that govern their spaces may be only partially true and partially false (as *<neutA>* in neutrosophy, which may be a blending of truth and falsehood).

Mostly in idealistic or imaginary or abstract or perfect spaces we have rigid laws and rigid axioms that totally apply (that are 100% true). But the laws and the axioms should be more flexible in order to comply with our imperfect world.

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References

1. Smarandache, Florentin, *Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures*, in his book "Advances of Standard and Nonstandard Neutrosophic Theories", Pons Ed., Brussels, European Union, 2019.
2. Smarandache, F., *Neutrosophy. / Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998.
3. Bhattacharya, S., *A Model to The Smarandache Geometries (S-denied, or smarandachely-denied)*, Journal of Recreational Mathematics, Vol. 33, No. 2, p. 66, 2004-2005; updated version in Octogon Mathematical Magazine, Vol. 14, No. 2, 690-692, October 2006.
4. Smarandache, F., *S-Denying a Theory*, International J.Math. Combin. Vol. 2(2013), 01-07, https://zenodo.org/record/821509#.XjR5QTJKgs4.
5. Kuciuk, L., Antholy M., *An Introduction to Smarandache Geometries* (*S-geometries*), Mathematics Magazine, Aurora, Canada, Vol. 12, 2003, and online: http://www.mathematicsmagazine.com/1-2004/Sm_Geom_1_2004.htm; also presented at New Zealand Mathematics Colloquium, Massey University, Palmerston North, New Zealand, December 3-6, 2001; also presented at the International Congress of Mathematicians (ICM2002), Beijing, China, 20-28 August 2002, http://www.icm2002.org.cn/B/Schedule_Section04.htm and in 'Abstracts of Short Communications to the International Congress of Mathematicians', International Congress of Mathematicians, 20-28 August 2002, Beijing, China; and in JP Journal of Geometry and Topology, Allahabad, India, Vol. 5, No. 1, 77-82, 2005.
6. Mao Linfan, *An introduction to Smarandache Geometries on Maps*, 2005 International Conference on Graph Theory and Combinatorics, Zhejiang Normal University, Jinhua, Zhejiang, P. R. China, June 25-30, 2005; also appeared in "Smarandache geometries & map theory with applications" (I), Chinese Branch Xiquan House, 2007.
7. Ashbacher, C., *Smarandache Geometries*, Smarandache Notions Journal, Vol. 8, 212-215, No. 1-2-3, 1997.
8. Chimienti, S. and Bencze, M., *Smarandache Paradoxist Geometry*, Bulletin of Pure and Applied Sciences, Delhi, India, Vol. 17E, No. 1, 123-1124, 1998; http://fs.unm.edu/prd-geo1.txt.
9. Mao, Linfan, *An introduction to Smarandache geometries on maps*, 2005 International Conference on Graph Theory and Combinatorics, Zhejiang Normal University, Jinhua, Zhejiang, P. R. China, June 25-30, 2005.
10. Mao, Linfan, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, partially post-doctoral research, Chinese Academy of Science, Am. Res. Press, Rehoboth, 2005.
11. Mao, Linfan, *Selected Papers on Mathematical Combinatorics (I),* World Academic Press, Liverpool, U.K., 2006.
12. Iseri, H., *Partially Paradoxist Smarandache Geometries*, http://fs.unm.edu/Howard-Iseri-paper.pdf.
13. Iseri, H., *Smarandache Manifolds*, Am. Res. Press, Rehoboth, 2002, http://fs.unm.edu/Iseri-book1.pdf
14. Perez, M., *Scientific Sites*, in 'Journal of Recreational Mathematics', Amityville, NY, USA, Vol. 31, No. 1, 86, 2002-20003.

15. Smarandache, F., *Paradoxist Mathematics*, in Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.
16. Mao, Linfan, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Partially postdoctoral research for the Chinese Academy of Sciences), Beijing, 2005, http://fs.unm.edu/Geometries.htm.
17. Smarandache, F., *Paradoxist Mathematics (1969)*, in Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.
18. Smarandache, F., *Paradoxist Geometry*, State Archives from Valcea, Rm. Valcea, Romania, 1969.
19. Smarandache, Florentin, *Neutrosophic Transdisciplinarity (MultiSpace & MultiStructure),* Arhivele Statului, Filiala Valcea, Romania, 1969 http://fs.unm.edu/NeutrosophicTransdisciplinarity.htm**.**
20. Smarandache, F., *Multi-space and Multi-structure*, in "Neutrosophy. Neutrosophic Logic, Set, Probability and Statistics", ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998, http://fs.unm.edu/Multispace.htm.
21. Rabounski, D., *Smarandache Spaces as a new extension of the basic space-time of general relativity*, Progress in Physics, Vol. 2, L1-L2, 2010.

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A Hybrid Neutrosophic Approach of DEMATEL with AR-DEA in Technology Selection

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Abstract: Technology selection is a leading step for decision makers throughout the technology selection process. The extraction of convenient technology is pretended to be a real challenge that faces decision makers. The technology selection considers the qualitative and quantitative criteria which needs to a special representation due to the conditions of non-compensation and uncertainty on real life. The objectives of this study is to make a hybrid approach using decision making trial and evaluation laboratory (DEMATEL) for detecting the positive and negative regions, and assurance region data envelopment analysis (AR-DEA) for evaluating the efficiency of Decision Making Units (DMUs). The hybrid model is protracted with neutrosophic philosophy in representing the perspectives of specialists and experts to achieve the most optimized outputs. An illustrative case study, about technology revolution and digital transformation in EGYPT, is presented to demonstrate the proposed model.

Keywords: Neutrosophic sets; Technology Selection; DEMATEL; Assurance Region; Data Envelopment Analysis.

1. Introduction

Technology has been an innovative manner that facilitates human life activities in real life. The selection of the appropriate technology is pretended to be a hard targets for experts. The selected technology will directly influence on the competitive advantages for organizations. Indeed, technology not only has valuable benefits, but also has susceptible weakness. Due to the technology complexity of operational and strategic distinctive, the technology selection can aids decision makers to build a vision to be able to choose the appropriate candidates of technologies [1]. The technology can be prescribed in many dimensionality terms such as cost, flexibility, quick delivery, and time [2].

The process of technology selection addressed by multiple methodologies over time, the classical approaches used was the mathematical programming [3]. The mathematical programming objective is to select the most convenient technology with lowest production cost by the use of non-linear 0-1 programming model [4]. Considering the complexity of technology selection, a fuzzy GP approach is presented to select the most appropriate machine tool and to allocate to a flexible manufacturing systems technology [5]. Data envelopment analysis (DEA) is a nonparametric efficiency method, such that data is not necessary to fit normal distribution [6]. The DEA can be used efficiently in technology selection. The DEA can assign weights for inputs and outputs to achieve to the maximum level of efficiency. In [7] presents a methodology consists of two phases for solving the technology problem process. The first phase, the data envelopment analysis (DEA) is focused on extracting the best vendor's solutions with respect to various technology parameters. The second stage, multi-attribute decision making model is used to prioritize and metric the outputted technology selection from first phase. The objective of decision-making units (DMUs) is to be efficient by producing the maximized outcomes and minimized incomes. The efficiency of DMUs can be evaluated with DEA as a powerful tool. In DEA, the input and outputs must be determined. In [8] proposes an innovative model, IDEA (Imprecise Data Envelopment Analysis) model to rank the technology suppliers. In [9] illustrated a weight multi-criteria decision-making (MCDM) methodology to evaluate the relative efficiency of DMUs according to various outputs and one determined input. The efficiency of DUMs is a model derived from of DEA methodology to extract exact and ordinal outcomes. When importance of preferences information between inputs and outputs are combined in multiple models, the resulted model is called Assurance region (AR) models. The efficiency problem includes technological and commercial aspects. A study about Superconducting Super Collider (SSC) in United States is conducted to reduce the number of site location [10]. By applying DEA on case study's data, the output included five out of six solutions were efficient. However, by including more analytical bounds, AR decreased the output to be one out of six. The AR is applied in another case study, about an efficient analysis for the possible linear production sets to make a real reduction on candidates [11].

The process of technology selection includes many technical and operational comparisons such as: cost, capacity, load, velocity, and etc. Many studies focus on the efficiency to enhance the decisions for the technology selection [12, 13]. The DEMTAL is a kind of structural modeling suggested to solve complex and interrelated problems [12]. The DEMTAL can formulate and analyze the problem into relationships between the correlated and complex criterions in order to attain the best solutions. Many decision-making methods are provided to organizations to choose the best technology [1, 3, 4, 7, 8]. However, the statement of any decision is a surrounded with environment of vague, impression, inconsistency, and uncertainty. According to the complex considerations of the environmental conditions in technology selection, researchers integrate fuzzy to DEMATEL method to attain more accurate analysis [14-17]. Actually, the fuzzy set considered the degree of membership function and neglected the degree of non- membership, and indeterminate [18]. Hence, the fuzzy DEMTAL con not addressed the decisions which are associated with uncertainty and inconsistency. To overcome fuzzy set limitations, neutrosophic sets proposed to address the conditions of uncertainty and inconsistency [19, 33-39].

Neutrosophic sets are a novel aspect in philosophy that investigates the scope and origin of neutralities [20, 21]. The neutrosophic sets are used in many complex applications and achieved awesome results such as in IoT influential factors [22] , IoT Transitions difficulties on enterprises [19] personnel selection [23], cloud services [24], supplier selection [18, 25-27], supply chain management (SCM) [25]. In real life situations, the preferences and correlations between criterions cannot be easily determined by decision makers. Hence neutrosophic can deal with uncertainty and inconsistency conditions. Neutrosophic aids decision makers to find compensations methodology to the indeterminate decision cases. Therefore, the research aims to propose a novel methodology that integrates the assurance region- data envelopment analysis (AR-DEA) with neutrosophic DEMTAL to enhance the technology selection process. Some basic and important definitions about neutrosophic sets are provided in [22].

For clarity, the reset of research is organized as follows: Section 2 mentions neutrosophic DEMTAL methodology. Section 3 represents basic steps of (AR-DEA). Section 4 illustrates the integrated methodology for technology selection. Section 5 presents a numerical example. Finally, section 6 ends with the conclusions and future work.

2. The Neutrosophic DEMATEL Methodology

The neutrosophic sets developed to cover the current conditional environmental of uncertainty and inconsistency that cannot be covered with other methods such as fuzzy and intuitionistic fuzzy [28]. The neutrosophic sets can apply compensatory methods for the indeterminate situations for decision judgments. DEMATEL is a methodology used to analyze the preferences between complex criterions by building well-structural model [2]. It is very hard task to take decision of preferences of various criterions. Hence, the research proposes to extend the traditional DEMTEL with neutrosophic set theory in order add valuable advantages:

1. Neutrosophic can present various expert judgments for a specific problem.
2. Neutrosophic can support perspectives of experts with compensatory values for the degree of true, false decisions. In addition to indeterminate decisions.
3. Neutrosophic can definitely represent different expert's perspectives to demonstrate if any anomalies found in the general judgments, such as: less experience, or biasness.
4. Neutrosophic can represent expert judgments in real situations of uncertainty and inconsistency of information

Therefore, the current study integrates neutrosophic with DEMATEL methodology in order to attain more accurate analysis. The steps of neutrosophic DEMATEL are mentioned as follows:

Step 1. Determine the aim of your study and detect the following issues:

- The decision maker experts in the proposed study.
- Identify the basic criterions related to study

Step 2. Construct decision judgments of the current study in a pairwise comparison matrix

- Construct the pairwise comparison matrix from decision judgments for the preferences scale mentioned in Table 1 [23]. Experts should determine their perspectives and expectation of the problem to detect maximum truth, minimum indeterminacy, and minimum false membership function.

Table 1. The Linguistics phrase and corresponding NTS

Score	*Linguistic Phrase*	*NTS*
1	Equally significant	$1 = \langle\langle 1, 1, 1\rangle; 0.50, 0.50, 0.50\rangle$
3	Slightly significant	$3 = \langle\langle 2, 3, 4\rangle; 0.30, 0.75, 0.70\rangle$
5	Strongly significant	$5 = \langle\langle 4, 5, 6\rangle; \langle 0.80, 0.15, 0.20\rangle$
7	very strongly significant	$7 = \langle\langle 6, 7, 8\rangle, 0.90, 0.10, 0.10\rangle$
9	Absolutely significant	$9 = \langle\langle 9, 9, 0\rangle; 1.00, 0.00, 0.00\rangle$
2	sporadic values between two close scales	$2 = \langle\langle 1, 2, 3\rangle; 0.40, 0.60, 0.65\rangle$
4		$4 = \langle\langle 3, 4, 5\rangle; 0.35, 0.60, 0.40\rangle$
6		$6 = \langle\langle 5, 6, 7\rangle; 0.70, 0.25, 0.30\rangle$
8		$8 = \langle\langle 7, 8, 9\rangle; 0.85, 0.10, 0.15\rangle$

Step 3. Construct initial direct relation

- Construct a general vision for your study from aggregating decision makers' perspectives. The averaged aggregated pairwise comparison matrix is formulated by the use of the following equation r_{ij} .

$$r_{ij} = \frac{\sum_{z=1}^{z} (z_{ij}^{\ z})}{z} \tag{1}$$

- The general vision are constructed by the estimated preferences and resulted in an aggregated pairwise comparison matrix as follows in (2):

$$A = \begin{pmatrix} r_{11} & \cdots & r_{1n} \\ \vdots & \ddots & \vdots \\ r_{n1} & \cdots & r_{mn} \end{pmatrix} \tag{2}$$

- Change the aggregates pairwise comparison matrix from the form of triangular neutrosophic scale to the form of crisp value by the use of the following score function [19]:

$$s(r_{ij}) = \left| l_{ij} \times m_{j} \times u_{ij}) \frac{T_{ij} + I_{ij} + F_{ij}}{9} \right| , \tag{3}$$

where l, m, u denotes lower, median, upper of the scale neutrosophic numbers, T, I, F are the truth-membership, indeterminacy, and falsity membership functions respectively of triangular neutrosophic number.

Step 4. **Construct the normalized direct relation matrix**

The initial direct relation is represented in the form of (2). According to previous step (3), the normalized direct relation matrix can be computed as follows:

$$B= 1 \Big/ \max_{1 \le i \le m} \sum_{j=1}^{n} r_{ij} ; i = 1,2,3,\ldots m; \ j = 1,2,3,\ldots, n \tag{4}$$

$$Y = B \times R \tag{5}$$

Step 5. **Obtain the total relation matrix.**

Apply the following equation to produce the total relation matrix from the generalized direct relation matrix Y. The total matrix relation is computed as follows [12]:

$$\sum_{n=1}^{\infty} Y_i = Y + Y^2 + Y^3 \ldots Y^m$$

$$= Y(1 + Y + Y^2 + \ldots + Y^{n-1})$$

$$= Y(I - Y)^{-1}(I - Y)(I + Y + Y^2 + \ldots + Y^{n-1})$$

$$= Y(1 - Y)^{-1}(I - Y^n) = Y(I - Y)^{-1}$$

$$T = Y \times (I - Y)^{-1} \quad , \tag{6}$$

such that I denotes to identity matrix, and T is the matrix of total relation

Step 6. Identify the cause effect relationship using the function of summation of rows and columns

The cause effect relationship is detected by using the summation of rows (R_i), of columns (C_j) form total matrix relation T as follows in next equations [14]:

$$T = \lfloor t_{ij} \rfloor_{m \times m}; i, j = 1,2,\ldots n \tag{7}$$

$$R_i = \sum_{1 \le j \le m}^{m} t_{ij}, \forall i \tag{8}$$

$$C_j = \sum_{1 \le i \le n} t_{ij}, \forall j \tag{9}$$

Step 7. Build the casual effect relationship diagram

The analysis of cause effect diagram two axes denotes the followings:

- **Horizontal axes:** represents the summation of rows and columns ($R_i + C_j$), and refers to the importance of the proposed criteria.
- **Vertical axes:** represents the subtraction of rows and columns ($R_i - C_j$), and refers to the degree of influence of the selected criteria

3. The AR-DEA methodology

Considering the whole decision maker units (DMU) in the decision maker process for AR-DEA methodology, the decision maker is influenced with other complementary players such as [28] and modeled in Fig.1:

- **Buyers:** anybody requests for a service according to considered contract. .
- **Users:** anybody actually receives and use the service.
- **Influencers**: anybody affects sales by supplying information or advice
- **Gatekeepers:** anybody controls the follow of information for the suppliers.

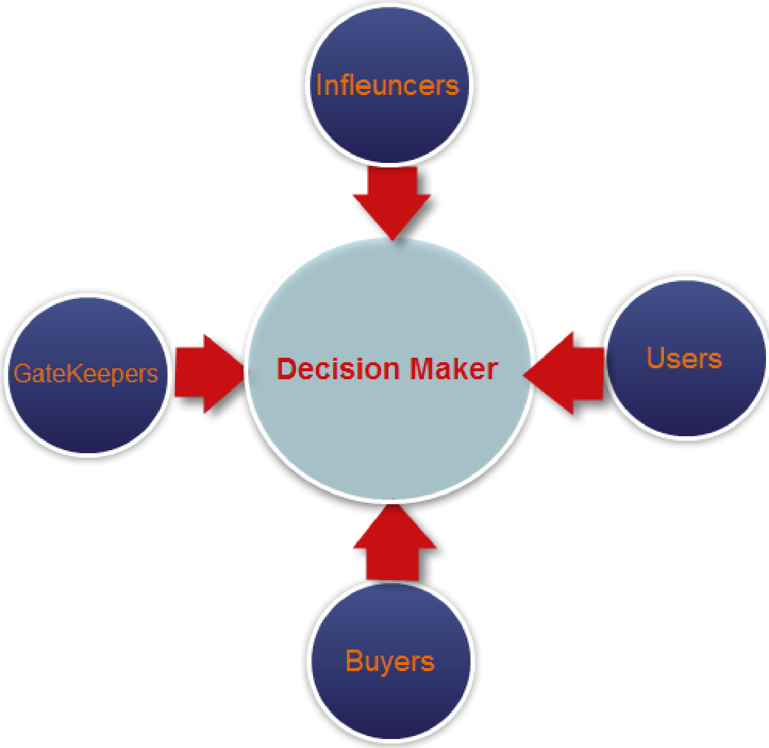

Figure 1. Decision makers unit

The DEA is an approach used to evaluate the efficiencies for DMUs [6]. The challenge in DMUs of technology selection is the absence for decision maker's judgments and preferences. The weight restriction inclusion in DEA model allows the integration of relative important between inputs and outputs for technology selection problem. The extension of DEA method with further calculations led to the development of the AR model [10]. The AR introduces a domain of possible candidates for multiple virtual suppliers. The next steps are discussed the scale of input and output levels, NB. The DMUs are strict to be in positive manner.

Step 8: Transform problem scale from ordinal to interval

The proposed study uses a novel weight technique which is so-called ordinal weight restriction assurance region [2]. The decision problem affected with various incomes and outcome. By the use of neutrosophic DEMATEL, the input and output weights can be obtained by the following equations:

$$X_1 \geq X_2 \geq \cdots \geq X_i \tag{10}$$

$$Y_1 \geq Y_2 \geq \cdots \geq Y_j \tag{11}$$

The preceding Eq. (10), and Eq. (11) represent ordinal scale. For using DEA, novel methods proposed to transform ordinal scale into cardinal scale [29]. The proposed study uses the following equations to transform ordinal scale into interval scale:

$$\boldsymbol{X_i} \in [\boldsymbol{\delta u^{m-i}}, \boldsymbol{u^{1-i}}];\ \boldsymbol{i = 1, \cdots, m\,;\, \delta \leq u^{1-m}} \quad , \tag{12}$$

$$\boldsymbol{Y_j} \in [\boldsymbol{\delta u^{n-j}}, \boldsymbol{u^{1-j}}];\ \boldsymbol{j = 1, \cdots, n\,;\, \delta \leq u^{1-n}} \quad , \tag{13}$$

where $\mathbf{X_i}$, $\mathbf{Y_j}$ represents the interval scale lower and upper bounds for inputs/outputs, $\boldsymbol{u}$ is a parameter indicates the preference intensity given by decision makers and must be greater than 1. $\boldsymbol{\delta}$ is a ratio parameter indicates by decision makers, and $\boldsymbol{i}, \boldsymbol{j}$ represents the ordinal scale of DEMATEL final ranking.

Step 9: The weight restrictions to solve AR-DEA methodology

The final output from the proposed Eq. (12), Eq. (13) presents the absolute number for interval scale of lower and upper bounds for the input/output weight priorities. In addition, the use of interval scale for weights substitutes the linear programming methods [29]. Unlike [2] AR without weight restrictions, and linear programming method [29], the proposed final type of AR is introduced in form. (14). Such that the weight restriction AR is added and modeled as follows:

$$
\begin{aligned}
&E_{0=max \sum_{j=1}^{s} wy_j\, y_{j0}} \,, \\
&s.t \ \sum_{i=1}^{m} wx_i\, x_{i0} \,, \\
&\sum_{j=1}^{s} wy_j\, y_{jz} - \sum_{i=1}^{m} wx_i\, x_{iz} \leq 1 \,, \forall_z \,, \\
&\partial_i \leq wx_i \leq \gamma_i, \qquad \forall_i, \\
&\beta_j \leq wy_j \leq \omega_j, \qquad \forall_i,
\end{aligned}
\tag{14}
$$

where $\mathrm{wx_i}$ is the weight for input, $\mathrm{wy_j}$ is the weight of output, $\partial_\mathrm{i}, \gamma_\mathrm{i}, \beta, \omega_\mathrm{j}$ are user specified constants. The weight restrictions a raise some challenges such as problem may not be solves, relative efficiency may not be computed. So [30] proposes to multiply constants of restricts A and B as follows in form (15):

$$
\begin{aligned}
&E_{0=max \sum_{j=1}^{s} wy_j\, y_{j0}} \,, \\
&s.t \ \sum_{i=1}^{m} wx_i\, x_{i0} \,, \\
&\sum_{j=1}^{s} wy_j\, y_{jz} - \sum_{i=1}^{m} wx_i\, x_{iz} \leq 1 \,, \forall_z \,, \\
&\partial_i A \leq wx_i \leq \gamma_i A, \qquad \forall_i, \\
&\beta_j B \leq wy_j \leq \omega_j B, \qquad \forall_i,
\end{aligned}
\tag{15}
$$

4. The Proposed hybrid methodology

The environment of decision making is surrounded with vague, impression, uncertainty, incomplete information, and non-compensatory. The integrated methodology of decision maker's judgments of DEMATEL and AR-DEA is modeled and summarized in the Fig.2. The steps of the proposed study have been mentioned in details in the previous two sections and will be summarized in Fig.3

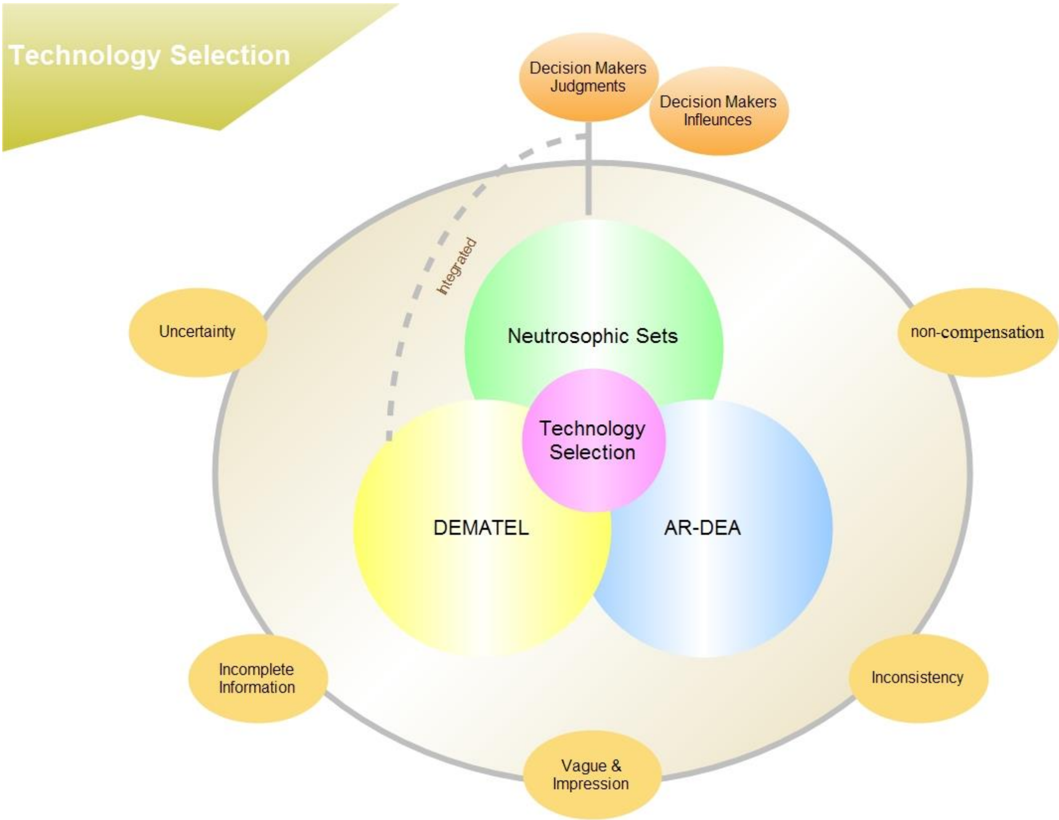

Figure 2. The hybrid methodology of neutrosophic DEMATEL with AR-DEA

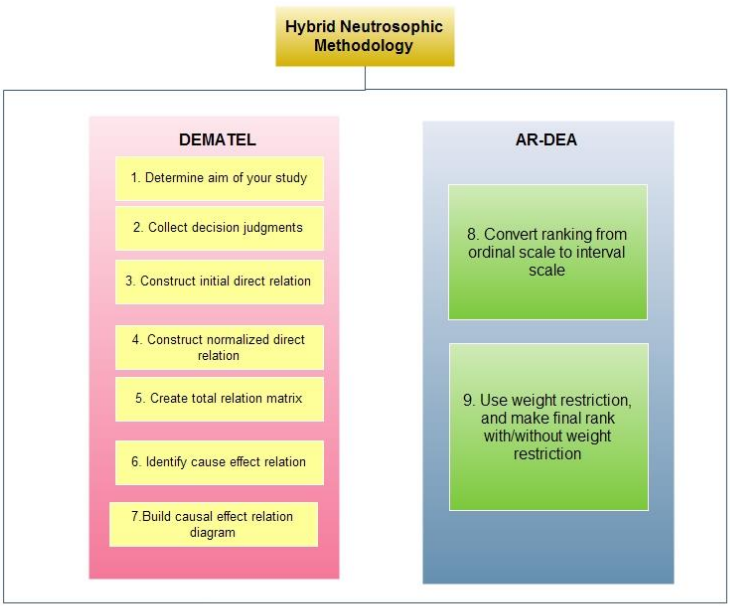

Figure 3. Steps for the proposed hybrid methodology

5. A case study for the proposed hybrid methodology

The proposed hybrid methodology is applied in a wide range of technology selection in Egypt. Egypt is going towards a huge information technology revolution and digital transformation on the practices for many sector of the Egyptian state. The technology revolution contains several axes, including recent developments in information and communications technology. The digital transformation revolution is including the fifth generation of communications, artificial intelligence, and cloud computing. Hence, the current decision makers faces a huge challenges for selecting the most appropriate and efficient technology that will cause a direct influence on the Egyptian state. Hence, we used to apply the proposed hybrid methodology of neutrosophic DEMTAL and AR-DEA. A standard input and output parameters are used in [1, 2]. We consider cost as input, while consider repeatability, load, capacity, velocity, and amount of know-how transfer as outputs for technology selection as mentioned in table 2.

Table 2. The description for the main criterions for technology selection

Criteria	Type	Symbol	Description
Cost	Input	X_1	The disbursement correlated with technology life cycle of introduction, growth, maturity, and decline [31].
Repeatability	Output	Y_1	The degree of closeness of the convention between outcomes under same measurements and conditions [1].
Load Capacity	Output	Y_2	The maximum load for intended property to achieve to the intended expectations with a given distinct amount of weight [32].
Know- how amount transfer	Output	Y_3	The use of distinct technology in a way to operate in such an efficient and effective manner [2].

Step 1: Determine decision makers experts whom are the actual input paramter for the hybird propsed methodology.

Step 2: The decision maker judgements are collected and scaled by the neutrosophic scale mentioned in table 1.

Step 3: Obtain the intial direct relation matrix. The aggregatd paire-wise comparison matrix is obtained by applying Eq.(1) and formed in (2) as depicated in table 3. Apply the score function on the aggregated pair-wise comparison matrix mentioned in Eq.(3) to change the neutrosophic scale to crisp values as mentioned in table 4.

Step 4: Construct th normaized direct matrix by apply Eq.(4) and Eq.(5). The results are mentioned table 5.

Step 5: The total relation matrix is computed by the useof Eq.(6) and mentioned in table 6

Step 6: The cause effect relation is presented by the detection of total matrix relation T by the use of Eq.(7), Eq. (8), Eq(9). The resuls of cause effect relation in table 7. According to table 7 the priotorize in importance are Y_1, Y_2, and Y_3, and the less important are Y_3, Y_2, and Y_1.

Step 7: The cause effect diagram is denoted as ($R_i + C_j$) horizontally, and ($R_i - C_j$) vertically ,and illustrated in Fig 4.

***Step 8*:** The ranking from the previous step is Transformed by the use of Eq. (12), Eq. (13) from ordinal scale to interval scale as mentioned in table 8.

Step 9: Considering the DMUs possible scenarios, the use of weight restriction for efficiency is to solve the hybrid neutrosophic AR-DEA methodology. To focus on the importance of the proposed study, ranking computed with/without weight restrictions and results mentioned in table 9. The without weight restriction is computed from [6], and with weight restriction computed according to Eq. (15). Indeed, a difference between $rank_1$, and $rank_2$ notified which lead to the great important for the proposed method as mentioned in Fig.5. By the way, the increase of the amount of parameters in the proposed demonstrates the influence of decision makers than other traditional methods.

Table 3. The initial aggregated pairwise comparison matrix for decision maker's experts

Criteria	Y_1	Y_2	Y_3
Y_1	$\langle\langle 1,1,1\rangle;0.50,0.50,0.50\rangle$	$\langle\langle 2,3,4\rangle;0.30,0.75,0.70\rangle$	$\langle\langle 5,6,7\rangle;0.70,0.25,0.30\rangle$
Y_2	$1/\langle\langle 2,3,4\rangle;0.30,0.75,0.70\rangle$	$\langle\langle 1,1,1\rangle;0.50,0.50,0.50\rangle$	$\langle\langle 1,2,3\rangle;0.40,0.65,0.60\rangle$
Y_3	$1/\langle\langle 5,6,7\rangle;0.70,0.25,0.430\rangle$	$1/\langle\langle 1,2,3\rangle;0.40,0.65,0.60\rangle$	$\langle\langle 1,1,1\rangle;0.50,0.50,0.50\rangle$

Table 4. The crisp values for initial aggregated pairwise comparison matrix

Criteria	Y_1	Y_2	Y_3
Y_1	1	1.855	2.101
Y_2	0.539	1	1.388
Y_3	0.475	0.720	1

Table 5. The normalized direct matrix

Criteria	Y_1	Y_2	Y_3
Y_1	0.20175	0.374272	0.423978
Y_2	0.108752	0.20175	0.280204
Y_3	0.096003	0.145262	0.20175

Table 6. The total relation matrix

Criteria	Y_1	Y_2	Y_3
Y_1	0.512384	0.913638	1.123984
Y_2	0.288305	0.512387	0.684009
Y_3	0.234351	0.385095	0.512388

Table 7. The cause effect relation of total relation

Rows / Columns	Ri	Cj	$R_i + C_j$	$R_i - C_j$	Rank
1	2.550	1.035	3.585046	1.514966	1
2	1.484	1.811	3.29582	-0.32642	3
3	1.131	2.320	3.452215	-1.18855	2

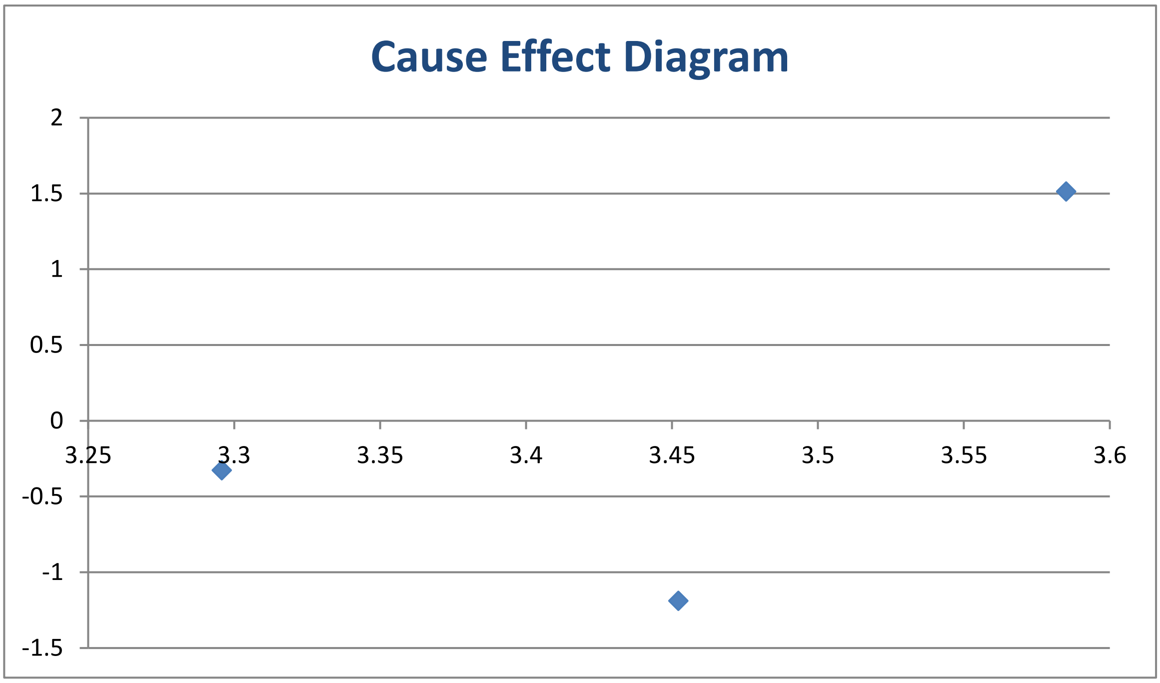

Figure 4. The cause effect diagram

Table 8. The transformation of ordinal scale to interval scale for U_r

Outputs	Ordinal Scale	Lower bound of output weight	Upper bound of output weight
U_1	1	0.22	1
U_2	3	0.1	0.44
U_3	2	0.15	0.66

Table 9. Efficiency score with consideration of with/without weight restrictions

DMU	Without weight restriction	$Rank_1$	With weight restriction	$Rank_2$
1	1.00	1	1.00	1
2	0.731	3	0.664	3
3	0.881	2	0.748	2
4	0.730	4	0.544	5
5	0.650	5	0.530	4

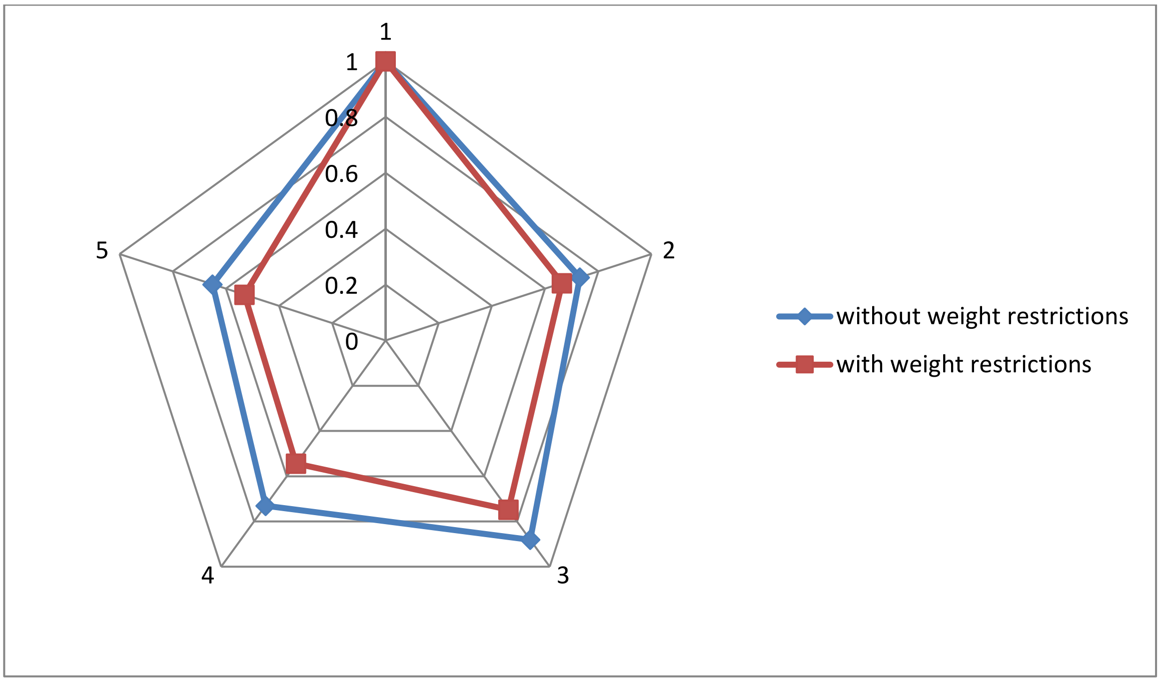

Figure 5. The ranking with/without weight restrictions

6. Conclusion

In this study, a hybrid neutrosophic DEMATEL with AR-DEA for technology selection is proposed. First, the DEMATEL aggregate the decision judgments in conditions of non-compensation, uncertainty, and incomplete information by the use of neutrosophic scale. The DEMATEL detect positive and negative regions in the form of cause effect relation, and introduce ranking for relations of inputs and outputs effects for technology selection process. Second the use of AR-DEA evaluate the efficiency for DMUs according to weight restrictions of AR to involve many influences of decision makers, rather than the traditional method of non-considering weight restrictions. A case study is applied on technology revolution and digital transformation in EGYPT that demonstrates the importance for the proposed study. For future trends, we can extend study by use of TOPSIS and MUTLIMOORA methods and make comparisons among ranking results.

References

1. Farzipoor, S.R. Technology selection in the presence of imprecise data, weight restrictions and nondiscretionary factors. Int. J. Adv. Manuf. Technol., 2009, DOI 10.1007/s00170-008-1514-5, 41:827-838.
2. Mohaghar, A., Jafarzadeh,A., Fathi,M.,Faghih. An Integrated Approach with AR-DEA and Fuzzy DEMATEL for Technology Selection. World Applied Sciences Journal, 2012, 16(11), 1649-1656.
3. Hassan, H., ORKCU, M. Data Envelopment Analysis Cross Efficiency Evaluation Approach to the Technology Selection. *University Journal of SciencePart A: Engineering And Innovation GU J Sci Part:A, 2015, 3(1):1-14.*
4. Singh, N., Sushil,P. Technology selection models for multi-stage production systems: Joint application of physical system theory and mathematical programming. European Journal of Operational Research, 1990, 47(2).
5. Chan, F.T.S, Swarnkar, R., Tiwari, M.K. Fuzzy goal-programming model with an artificial immune system (AIS) Approach for a machine tool selection and operation allocation problem in a flexible manufacturing system, *International Journal of Production Research*,2005, 43(19): 4147–4163.
6. Charnes, A., Cooper, W.W., Rhodes. E. Measuring the efficiency of decision making units. European Journal of Operational Research, 1978, 2(6), 429-444.
7. Khouja, M. The use of data envelopment analysis for technology selection. Comput. Ind., 1995 , 28 (1): 123-132.
8. Farzipoorsaen, F. An algorithm for ranking technology suppliers in the presence of nondiscretionary factors, Elsevier, 2006, 181(2),1616-1623

9. Karsak, E.E., S.S. Ahiska. Practical common weight multicriteria decision-making approach with an improved discriminating power for technology selection. Int. J. Prod. Res., 2005, 43 (8):
10. Thompson, R.G., Langemeier, L.N., Lee, C.T., Lee, E., and Thrall, R.M. The role of multiplier bounds in efficiency analysis with application to kansas farming. J. Econ., , 1990, 46 (1/2): 93-108
11. Wang, Y., Chin,K., Poon, G. A data envelopment analysis method with assurance region for weight generation in the analytic hierarchy process,Decision Support Systems, 2008,45(4), 913-921.
12. Falatoonitoosi,Leman,E., Sorooshian,Z., Salimi,S., Meysam. Decision-Making Trial and Evaluation Laboratory. Research Journal of Applied Sciences, Engineering and Technology. 2013, 5, 3476-3480.
13. Dinmohammadi, A., Shafiee, M. Determination of the Most Suitable Technology Transfer Strategy for Wind Turbines Using an Integrated AHP-TOPSIS Decision Model. Energies, 2017, 10(642).
14. Seker, S., Zavadskas , K. Application of Fuzzy DEMATEL Method for Analyzing Occupational Risks on Construction Sites, Sustainability, 2017, 9(2083).
15. Chang, E., Chang, C.,Wu, C. Fuzzy DEMATEL method for developing supplier selection criteria, Expert Systems with Applications, 2011, 38(3).
16. Ehsanifar, Arash, M. Applying Fuzzy DEMATEL Method to Analyze Supplier Selection Criteria (Case Study: WagonPars Company). International Research Journal of Finance and Economics, 2013.
17. Jaganathan S., Erinjeri, J.J., Ker J. Fuzzy analytic hierarchy process based group decision support system to select and evaluate new manufacturing technologies. International *Journal Advanced Manufacturing Technology*, 2007, 32(11–12): 1253–1262.
18. Abdel-Basset, M., Manogaran, G., Gamal, A., Smarandache, F. A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. Design Automation for Embedded Systems, 2018, 22(3): 257-278.
19. Abdel-Basset, M., Nabeeh, N. A., El-Ghareeb, H. A , Aboelfetouh. A. Utilizing Neutrosophic Theory to Solve Transition Difficulties of IoT-Based Enterprises. Enterprise Information Systems, 2019.
20. Smarandache, F. *Neutrosophy*, *Neutrosophic Logic*, *Neutrosophic Set*, *Neutrosophic Probability and Statistics*. Ann Arbor, MI, USA: ProQuest, 1998.
21. El-Ghareeb, H. A. Novel Open Source Python Neutrosophic Package. Neutrosophic Sets and Systems, Vol.25, pp. 136.2019.
22. Nabeeh, N. A., Abdel-Basset, M., El-Ghareeb, H. A., Aboelfetouh. A. Neutrosophic Multi-Criteria Decision Making Approach for IoT-Based Enterprises. IEEE Access, 2019, doi: 10.1109/ACCESS.2019.2908919.
23. Nabeeh, N. A., Abdel-Basset, M., El-Ghareeb, H. A., Aboelfetouh. A. An Integrated Neutrosophic-TOPSIS Approach and its Application to Personnel Selection: A New Trend in Brain Processing and Analysis. IEEE Access, 2019, doi: 10.1109/ACCESS.2019.2899841
24. Abdel-Basset, M., Mohamed, M., Chang,V. NMCDA: A framework for evaluating cloud computing services. *Future Gener. Comput. Syst.*, 2018m 86, 12◎29, Sep. 2018.
25. Abdel-Basset, M., Mohamed, M., Smarandache, F. A hybrid neutrosophic group ANP-TOPSIS framework for supplier selection problems. *Symmetry*, 2018, 10(6), 226, 2018.
26. Abdel-Basset, M., Manogaran, G., Mohamed, M., Chilamkurti, N. Three-way decisions based on neutrosophic sets and AHP-QFD framwork for supplier selection problem," *Future Gener. Comput. Syst.*, 2018,. 89, 19◎30.
27. Muralidharan, C., Anantharaman, N.,. Deshmukh, S. G. A multi-criteria group decision making model for supplier rating," *J. Supply Chain Manage.*, 2002, 38, (3), 22◎33, 2002.
28. Smarandache F (2010) Neutrosophic set-a generalization of the intuitionistic fuzzy set. J Defense ResourManag 1(1):107
29. Wang, Y.M., R. Greatbanks ,J. B. Yang. Interval efficiency assessment using data envelopment analysis. Fuzzy Sets and Systems, 2005, 153: 347-370.
30. Podinovski, V.V. Side effects of absolute weight bounds in DEA models. Eur. J. Oper. Res.,1999, 115 (3): 583-595.
31. Folgado, R., Pe◎as, P., Henriques, E. Life cycle cost for technology selection: A Case study in the manufacturing of injection moulds. International Journal of Production Economics, 2010, 128(1), 368-378.ISSN 0925-5273.
32. El-Khattam,W., Salama, M.M.A. Distributed generation technologies, definitions and benefits, Electric Power Systems Research, 2004, 71(2), 119-128
33. Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F. (2019). A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine, 100, 101710.
34. Abdel-Baset, M., Chang, V., & Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry, 108, 210-220.

35. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 77, 438-452.
36. Abdel-Basset, M., Atef, A., & Smarandache, F. (2019). A hybrid Neutrosophic multiple criteria group decision making approach for project selection. Cognitive Systems Research, 57, 216-227.
37. Abdel-Basset, Mohamed, Mumtaz Ali, and Asma Atef. "Resource levelling problem in construction projects under neutrosophic environment." The Journal of Supercomputing (2019): 1-25.
38. Abdel-Basset, Mohamed, and Rehab Mohamed. "A novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management." Journal of Cleaner Production 247 (2020): 119586.
39. Abdel-Basset, Mohamed, Mumtaz Ali, and Asmaa Atef. "Uncertainty Assessments of Linear Time-Cost Tradeoffs using Neutrosophic Set." Computers & Industrial Engineering (2020): 106286.

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BMBJ-neutrosophic subalgebra in BCI/BCK-algebras

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Abstract: For the first time Smarandache introduced neutrosophic sets which can be used as a mathematical tool for dealing with indeterminate and inconsistent information. the notion of BMBJ-neutrosophic set and subalgebra, as a generalization of a neutrosophic set, is introduced, and it's application to BCI/BCK-algebras is investigated. The concept of BMBJ-neutrosophic subalgebras in BCI/BCK-algebras is introduced, and related properties are investigated. New BMBJ-neutrosophic subalgebra is established by using an BMBJ-neutrosophic subalgebra of a BCI/BCK-algebra. Alos, homomorphic (inverse) image of BMBJ-neutrosophic subalgebra and translation of BMBJ-neutrosophic subalgebra is investigated. At the end, we provided conditions for an BMBJ-neutrosophic set to be an BMBJ-neutrosophic subalgebra.

Keywords: BMBJ-neutrosophic set; BMBJ-neutrosophic subalgebra; BMBJ-neutrosophic S-extension.

1 Introduction

Different types of uncertainties are encountered in some complex system and many fields like biological, behavioural and chemical etc. L.A. Zadeh [33] in 1965 introduced the fuzzy set for the first time to handle uncertainties in many applications. Also K. Atanassov introduced the intuitionistic fuzzy set on the universe X as a generalisation of fuzzy set [6] in 1986. The concept of neutrosophic set is developed by Smarandache ([27], [28] and [29]), and this is a more general platform that extends the notions of classic set like (intuitionistic) fuzzy set and interval valued (intuitionistic) fuzzy set. Neutrosophic set theory is applied to various fields which is referred to the [1], [2], [3], [4], [5] [8], [9], [22] and [24]. Neutrosophic algebraic structures in BCI/BCK-algebras are discussed in the papers [7], [13], [14], [15], [19], [16], [17], [18], [20], [25], [26], [30], [31] and [32].

In this paper, we introduce the notion of BMBJ-neutrosophic sets and subalgebra, as a generalisation of neutrosophic set, and we investigate it's application and related properties it to BCI/BCK-algebras. We provide some characterizations of BMBJ-neutrosophic subalgebra, and by using an BMBJ-neutrosophic subalgebra of a BCI/BCK-algebra, a new BMBJ-neutrosophic subalgebra will be propose. We consider the homomorphic inverse image of BMBJ-neutrosophic subalgebra, and consider translation of BMBJ-neutrosophic subalgebra. At the last step, we provide some conditions for an BMBJ-neutrosophic set to be an BMBJ-neutrosophic subalgebra.

2 Preliminaries

A BCI/BCK-algebra is an important class of logical algebras introduced by K. Iséki (see [11] and [12]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)\ (x * x = 0)$,

(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra*. Any BCI/BCK-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{2.1}$$
$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2.2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{2.3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{2.4}$$

where $x \leq y$ if and only if $x * y = 0$. Any BCI-algebra X satisfies the following conditions (see [10]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{2.5}$$
$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)). \tag{2.6}$$

A nonempty subset S of a BCI/BCK-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

By an *interval number* we mean a closed subinterval $\tilde{a} = [a^-, a^+]$ of I, where $0 \leq a^- \leq a^+ \leq 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as *refined minimum* (briefly, rmin) and *refined maximum* (briefly, rmax) of two elements in $[I]$. We also define the symbols "$\succeq$", "$\preceq$", "$=$" in case of two elements in $[I]$. Consider two interval numbers $\tilde{a}_1 := \left[a_1^-, a_1^+\right]$ and $\tilde{a}_2 := \left[a_2^-, a_2^+\right]$. Then

$$\begin{aligned}
&\mathrm{rmin}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\min\left\{a_1^-, a_2^-\right\}, \min\left\{a_1^+, a_2^+\right\}\right], \\
&\mathrm{rmax}\left\{\tilde{a}_1, \tilde{a}_2\right\} = \left[\max\left\{a_1^-, a_2^-\right\}, \max\left\{a_1^+, a_2^+\right\}\right], \\
&\tilde{a}_1 \succeq \tilde{a}_2\ \Leftrightarrow\ a_1^- \geq a_2^-,\ a_1^+ \geq a_2^+,
\end{aligned}$$

and similarly we may have $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 = \tilde{a}_2$. To say $\tilde{a}_1 \succ \tilde{a}_2$ (resp. $\tilde{a}_1 \prec \tilde{a}_2$) we mean $\tilde{a}_1 \succeq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$ (resp. $\tilde{a}_1 \preceq \tilde{a}_2$ and $\tilde{a}_1 \neq \tilde{a}_2$). Let $\tilde{a}_i \in [I]$ where $i \in \Lambda$. We define

$$\operatorname*{rinf}_{i\in\Lambda} \tilde{a}_i = \left[\inf_{i\in\Lambda} a_i^-, \inf_{i\in\Lambda} a_i^+\right] \quad \text{and} \quad \operatorname*{rsup}_{i\in\Lambda} \tilde{a}_i = \left[\sup_{i\in\Lambda} a_i^-, \sup_{i\in\Lambda} a_i^+\right].$$

Let X be a nonempty set. A function $A : X \to [I]$ is called an *interval-valued fuzzy set* (briefly, an *IVF set*) in X. Let $[I]^X$ stand for the set of all IVF sets in X. For every $A \in [I]^X$ and $x \in X$, $A(x) = [A^-(x), A^+(x)]$ is called the *degree* of membership of an element x to A, where $A^- : X \to I$ and $A^+ : X \to I$ are fuzzy sets in X which are called a *lower fuzzy set* and an *upper fuzzy set* in X, respectively. For simplicity, we denote $A = [A^-, A^+]$.

Let X be a non-empty set. A *neutrosophic set* (NS) in X (see [28]) is a structure of the form:

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

where $A_T : X \to [0,1]$ is a truth membership function, $A_I : X \to [0,1]$ is an indeterminate membership function, and $A_F : X \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A = (A_T, A_I, A_F)$ for the neutrosophic set

$$A := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}.$$

We refer the reader to the books [10, 21] for further information regarding BCi/BCK-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3 BMBJ-neutrosophic structures with applications in BCI/BCK-algebras

Definition 3.1. Let X be a non-empty set. By an *MBJ-neutrosophic set* in X, we mean a structure of the form:

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}$$

where M_A and J_A are fuzzy sets in X, which are called a truth membership function and a false membership function, respectively, and $\tilde{B}_A$ is an IVF set in X which is called an indeterminate interval-valued membership function.

For the sake of simplicity, we shall use the symbol $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ for the MBJ-neutrosophic set

$$\mathcal{A} := \{\langle x; M_A(x), \tilde{B}_A(x), J_A(x)\rangle \mid x \in X\}.$$

Definition 3.2. Let X be a BCI/BCK-algebra. An MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is called an *BMBJ-neutrosophic subalgebra* of X if it satisfies:

$$(\forall x, y \in X) \left(\begin{array}{l} M_A(x * y) \geq \min\{M_A(x), M_A(y)\}, \\ \tilde{B}_A^-(x * y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ \tilde{B}_A^+(x * y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ J_A(x * y) \leq \max\{J_A(x), J_A(y)\}, \\ M_A(x) + \tilde{B}_A^-(x) \leq 1, \tilde{B}_A^+(x) + J_A(x) \geq 1\}. \end{array} \right) \tag{3.1}$$

Example 3.3. Consider a set $X = \{0, a, b, c\}$ with the binary operation $*$ which is given in Table 1. Then

Table 1: Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

$(X; *, 0)$ is a BCK-algebra (see [21]). Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an MBJ-neutrosophic set in X defined by Table 2. It is routine to verify that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X.

Table 2: MBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$

X	$M_A(x)$	$\tilde{B}_A(x)$	$J_A(x)$
0	0.7	$[0.3, 0.8]$	0.2
a	0.3	$[0.1, 0.5]$	0.6
b	0.1	$[0.3, 0.8]$	0.4
c	0.5	$[0.1, 0.5]$	0.7

In what follows, let X be a BCI/BCK-algebra unless otherwise specified.

Proposition 3.4. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then $M_A(0) \geq M_A(x)$, $\tilde{B}_A^-(0) \leq \tilde{B}_A^-(x)$, $\tilde{B}_A^+(0) \geq \tilde{B}_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$.*

Proof. For any $x \in X$, we have

$$M_A(0) = M_A(x * x) \geq \min\{M_A(x), M_A(x)\} = M_A(x),$$

$$\tilde{B}_A^-(0) = \tilde{B}_A^-(x * x) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(x)\} = \tilde{B}_A^-(x),$$

$$\tilde{B}_A^+(0) = \tilde{B}_A^-(x * x) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^-(x)\} = \tilde{B}_A^+(x)$$

and

$$J_A(0) = J_A(x * x) \leq \max\{J_A(x), J_A(x)\} = J_A(x).$$

This completes the proof. □

Proposition 3.5. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic subalgebra of X. If there exists a sequence $\{x_n\}$ in X such that*

$$\lim_{n\to\infty} M_A(x_n) = 1,\ \lim_{n\to\infty} \tilde{B}_A^-(x_n) = 0,\ \lim_{n\to\infty} \tilde{B}_A^+(x_n) = 1 \text{ and } \lim_{n\to\infty} J_A(x_n) = 0, \tag{3.2}$$

then $M_A(0) = 1$, $\tilde{B}_A^-(0) = 0$, $\tilde{B}_A^+(0) = 1$ and $J_A(0) = 0$.

Proof. Using Proposition 3.4, we know that $M_A(0) \geq M_A(x)$, $\tilde{B}_A^-(0) \leq \tilde{B}_A^-(x)$, $\tilde{B}_A^+(0) \geq \tilde{B}_A^+(x)$ and $J_A(0) \leq J_A(x)$ for all $x \in X$. for every positive integer n. Note that

$$\begin{aligned}
1 \geq M_A(0) &\geq \lim_{n\to\infty} M_A(x_n) = 1,\\
0 \leq \tilde{B}_A^-(0) &\leq \lim_{n\to\infty} \tilde{B}_A^-(x_n) = 0,\\
1 \geq \tilde{B}_A^+(0) &\geq \lim_{n\to\infty} \tilde{B}_A^+(x_n) = 1,\\
0 \leq J_A(0) &\leq \lim_{n\to\infty} J_A(x_n) = 0.
\end{aligned}$$

Therefore $M_A(0) = 1$, $\tilde{B}_A^-(0) = 0$, $\tilde{B}_A^+(0) = 1$ and $J_A(0) = 0$. □

Theorem 3.6. *Given an BMBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, if (M_A, J_A) is an intuitionistic fuzzy subalgebra of X, and B_A^- and B_A^+ are fuzzy subalgebras of X, then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X.*

Proof. It is sufficient to show that $\tilde{B}_A$ satisfies the condition

$$(\forall x, y \in X)(\tilde{B}_A^-(x * y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}), \tag{3.3}$$

$$(\forall x, y \in X)(\tilde{B}_A^+(x * y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}). \tag{3.4}$$

For any $x, y \in X$, we get

$$\begin{aligned}
\tilde{B}_A(x * y) &= [\tilde{B}_A^-(x * y), \tilde{B}_A^+((x * y)]\\
&\geq [\max \tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}].
\end{aligned}$$

Therefore $\tilde{B}_A$ satisfies the condition (3.3), and so $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. □

If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then

$$\begin{aligned}
[B_A^-(x * y), B_A^+(x * y)] &= \tilde{B}_A(x * y) \succeq \operatorname{rmin}\{\tilde{B}_A(x), \tilde{B}_A(y)\}\\
&= \operatorname{rmin}\{[B_A^-(x), B_A^+(x), [B_A^-(y), B_A^+(y)]\}\\
&= [\min\{B_A^-(x), B_A^-(y)\}, \min\{B_A^+(x), B_A^+(y)\}]
\end{aligned}$$

for all $x, y \in X$. It follows that $B_A^-(x * y) \geq \min\{B_A^-(x), B_A^-(y)\}$ and $B_A^+(x * y) \geq \min\{B_A^+(x), B_A^+(y)\}$. Thus B_A^- and B_A^+ are fuzzy subalgebras of X. But (M_A, J_A) is not an intuitionistic fuzzy subalgebra of X as seen in Example 3.3. This shows that the converse of Theorem 3.6 is not true.

Given an BMBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X, we consider the following sets.

$$\begin{aligned}
&U(M_A; t) := \{x \in X \mid M_A(x) \geq t\},\\
&L(\tilde{B}_A^-; \delta_1) := \{x \in X \mid \tilde{B}_A^-(x) \leq \delta_1\},\\
&U(\tilde{B}_A^+; \delta_2) := \{x \in X \mid \tilde{B}_A^+(x) \geq \delta_2\},\\
&L(J_A; s) := \{x \in X \mid J_A(x) \leq s\}
\end{aligned}$$

where $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$.

Theorem 3.7. *An BMBJ-neutrosophic set $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ in X is an BMBJ-neutrosophic subalgebra of X if and only if the non-empty sets $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are subalgebras of X for all $t, \delta_1, \delta_2, \in [0, 1]$.*

Proof. Suppose that $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. Let $t, s \in [0, 1]$ and $[\delta_1, \delta_2] \in [I]$ be such that $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are non-empty. For any $x, y, a, b, u, v \in X$, if $x, y \in U(M_A; t)$, $a, b \in L(\tilde{B}_A^-; \delta_1)$, $c, d \in U(\tilde{B}_A^+; \delta_2)$ and $u, v \in L(J_A; s)$, then

$$\begin{aligned}
&M_A(x * y) \geq \min\{M_A(x), M_A(y)\} \geq \min\{t, t\} = t,\\
&\tilde{B}_A^-(a * b) \leq \max\{\tilde{B}_A^-(a), \tilde{B}_A^-(b)\} \leq \max\{\delta_1, \delta_1\} = \delta_1,\\
&\tilde{B}_A^+(c * d) \geq \min\{\tilde{B}_A^+(c), \tilde{B}_A^+(d)\} \geq \min\{\delta_2, \delta_2\} = \delta_2,\\
&J_A(u * v) \leq \max\{J_A(u), J_A(v)\} \leq \min\{s, s\} = s,
\end{aligned}$$

and so $x * y \in U(M_A; t)$, $a * b \in L(\tilde{B}_A^-; \delta_1)$, $c * d \in U(\tilde{B}_A^+; \delta_2)$ and $u * v \in L(J_A; s)$. Therefore $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are subalgebras of X.

Conversely, assume that the non-empty sets $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are subalgebras of X for all $t, s, \delta_1, \delta_2 \in [0, 1]$. If $M_A(a_0 * b_0) < \min\{M_A(a_0), M_A(b_0)\}$ for some $a_0, b_0 \in X$, then $a_0, b_0 \in U(M_A; t_0)$ but $a_0 * b_0 \notin U(M_A; t_0)$ for $t_0 := \min\{M_A(a_0), M_A(b_0)\}$. This is a contradiction, and thus $M_A(a * b) \geq \min\{M_A(a), M_A(b)\}$ for all $a, b \in X$. Similarly, we can show that $\tilde{B}_A^-(a * b) \leq \max\{\tilde{B}_A^-(a), \tilde{B}_A^-(b)\}$, $\tilde{B}_A^+(c * d) \geq \min\{\tilde{B}_A^+(c), \tilde{B}_A^+(d)\}$ and $J_A(a * b) \leq \max\{J_A(a), J_A(b)\}$ for all $a, b \in X$. $\square$

Using Proposition 3.4 and Theorem 3.7, we have the following corollary.

Corollary 3.8. *If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then the sets $X_{M_A} := \{x \in X \mid M_A(x) = M_A(0)\}$, $X_{\tilde{B}_A^-} := \{x \in X \mid \tilde{B}_A^-(x) = \tilde{B}_A^-(0)\}$, $X_{\tilde{B}_A^+} := \{x \in X \mid \tilde{B}_A^+(x) = \tilde{B}_A^+(0)\}$, and $X_{J_A} := \{x \in X \mid J_A(x) = J_A(0)\}$ are subalgebras of X.*

We say that the subalgebras $U(M_A; t)$, $L(\tilde{B}_A^-; \delta_1)$, $U(\tilde{B}_A^+; \delta_2)$ and $L(J_A; s)$ are *BMBJ-subalgebras* of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.

Theorem 3.9. *Every subalgebra of X can be realized as BMBJ-subalgebras of an BMBJ-neutrosophic subalgebra of X.*

Proof. Let K be a subalgebra of X and let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic set in X defined by

$$M_A(x) = \begin{cases} t & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases} \quad \tilde{B}_A^-(x) = \begin{cases} \gamma_1 & \text{if } x \in K, \\ 1 & \text{otherwise,} \end{cases} \quad \tilde{B}_A^+(x) = \begin{cases} \gamma_2 & \text{if } x \in K, \\ 0 & \text{otherwise,} \end{cases} \quad J_A(x) = \begin{cases} s & \text{if } x \in K, \\ 1 & \text{otherwise,} \end{cases} \tag{3.5}$$

where $t \in (0, 1]$, $s \in [0, 1)$ and $\gamma_1, \gamma_2 \in (0, 1]$ with $\gamma_1 < \gamma_2$. It is clear that $U(M_A; t) = K$, $L(\tilde{B}_A^-; \gamma_1) = K$, $U(\tilde{B}_A^+; \gamma_2) = K$ and $L(J_A; s) = K$. Let $x, y \in X$. If $x, y \in K$, then $x * y \in K$ and so

$$\begin{aligned} &M_A(x * y) = t = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A^-(x * y) = \gamma_1 = \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ &\tilde{B}_A^+(x * y) = \gamma_2 = \max\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ &J_A(x * y) = s = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

If any one of x and y is contained in K, say $x \in K$, then $M_A(x) = t$, $\tilde{B}_A^-(x) = \gamma_1$, $\tilde{B}_A^+(x) = \gamma_2$, $J_A(x) = s$, $M_A(y) = 0$, $\tilde{B}_A^-(y) = 0$, $\tilde{B}_A^+(y) = 0$ and $J_A(y) = 1$. Hence

$$\begin{aligned} &M_A(x * y) \geq 0 = \min\{t, 0\} = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A^-(x * y) \leq 1 = \max\{\gamma_1, 1\} = \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ &\tilde{B}_A^+(x * y) \geq 0 = \min\{\gamma_2, 0\} = \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ &J_A(x * y) \leq 1 = \max\{s, 1\} = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

If $x, y \notin K$, then $M_A(x) = 0 = M_A(y)$, $\tilde{B}_A^-(x) = 1 = \tilde{B}_A^-(y)$, $\tilde{B}_A^+(x) = 0 = \tilde{B}_A^+(y)$ and $J_A(x) = 1 = J_A(y)$. It follows that

$$\begin{aligned} &M_A(x * y) \geq 0 = \min\{0, 0\} = \min\{M_A(x), M_A(y)\} \\ &\tilde{B}_A^-(x * y) \leq 1 = \max\{1, 1\} = \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}, \\ &\tilde{B}_A^+(x * y) \geq 0 = \min\{0, 0\} = \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}, \\ &J_A(x * y) \leq 1 = \max\{1, 1\} = \max\{J_A(x), J_A(y)\}. \end{aligned}$$

Therefore $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. □

Theorem 3.10. *For any non-empty subset K of X, let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic set in X which is given in (3.5). If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then K is a subalgebra of X.*

Proof. Let $x, y \in K$. Then $M_A(x) = t = M_A(y)$, $\tilde{B}_A^-(x) = \gamma_1 = \tilde{B}_A^-(y)$, $\tilde{B}_A^+(x) = \gamma_2 = \tilde{B}_A^+(y)$ and $J_A(x) = s = J_A(y)$. Thus

$$\begin{aligned} &M_A(x * y) \geq \min\{M_A(x), M_A(y)\} = t, \\ &\tilde{B}_A^-(x * y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\} = \gamma_1, \\ &\tilde{B}_A^+(x * y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\} = \gamma_2, \\ &J_A(x * y) \leq \max\{J_A(x), J_A(y)\} = s, \end{aligned}$$

and therefore $x * y \in K$. Hence K is a subalgebra of X. □

Using an BMBJ-neutrosophic subalgebra of a BCI-algera, we establish a new BMBJ-neutrosophic subalgebra.

Theorem 3.11. *Given an BMBJ-neutrosophic subalgebra $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ of a BCI-algebra X, let $\mathcal{A}^* = (M_A^*,\, \tilde{B}_A^*,\, J_A^*)$ be an BMBJ-neutrosophic set in X defined by $M_A^*(x) = M_A(0 * x)$, $\tilde{B}_A^*(x) = \tilde{B}_A(0 * x)$ and $J_A^*(x) = J_A(0 * x)$ for all $x \in X$. Then $\mathcal{A}^* = (M_A^*,\, \tilde{B}_A^*,\, J_A^*)$ is an BMBJ-neutrosophic subalgebra of X.*

Proof. Note that $0 * (x * y) = (0 * x) * (0 * y)$ for all $x, y \in X$. We have

$$\begin{aligned} M_A^*(x * y) &= M_A(0 * (x * y)) = M_A((0 * x) * (0 * y)) \\ &\geq \min\{M_A(0 * x), M_A(0 * y)\} \\ &= \min\{M_A^*(x), M_A^*(y)\}, \end{aligned}$$

$$\begin{aligned} (\tilde{B}_A^-)^*(x * y) &= \tilde{B}_A^-(0 * (x * y)) = \tilde{B}_A^-((0 * x) * (0 * y)) \\ &\leq \max\{\tilde{B}_A^-(0 * x), \tilde{B}_A^-(0 * y)\} \\ &= \max\{(\tilde{B}_A^-)^*(x), (\tilde{B}_A^-)^*(y)\} \end{aligned}$$

$$\begin{aligned} (\tilde{B}_A^+)^*(x * y) &= \tilde{B}_A^+(0 * (x * y)) = \tilde{B}_A^+((0 * x) * (0 * y)) \\ &\geq \min\{\tilde{B}_A^+(0 * x), \tilde{B}_A^+(0 * y)\} \\ &= \min(\{\tilde{B}_A^+)^*(x), (\tilde{B}_A^+)^*(y)\}, \end{aligned}$$

and

$$\begin{aligned} J_A^*(x * y) &= J_A(0 * (x * y)) = J_A((0 * x) * (0 * y)) \\ &\leq \max\{J_A(0 * x), J_A(0 * y)\} \\ &= \max\{J_A^*(x), J_A^*(y)\} \end{aligned}$$

for all $x, y \in X$. Therefore $\mathcal{A}^* = (M_A^*,\, \tilde{B}_A^*,\, J_A^*)$ is an BMBJ-neutrosophic subalgebra of X. □

Theorem 3.12. *Let $f : X \to Y$ be a homomorphism of BCK/BCI-algebras. If $\mathcal{B} = (M_B,\, \tilde{B}_B,\, J_B)$ is an MBJ-neutrosophic subalgebra of Y, then $f^{-1}(\mathcal{B}) = (f^{-1}(M_B),\, f^{-1}(\tilde{B}_B),\, f^{-1}(J_B))$ is an BMBJ-neutrosophic subalgebra of X, where $f^{-1}(M_B)(x) = M_B(f(x))$, $f^{-1}(\tilde{B}_B)(x) = \tilde{B}_B(f(x))$ and $f^{-1}(J_B)(x) = J_B(f(x))$ for all $x \in X$.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned} f^{-1}(M_B)(x * y) &= M_B(f(x * y)) = M_B(f(x) * f(y)) \\ &\geq \min\{M_B(f(x)), M_B(f(y))\} \\ &= \min\{f^{-1}(M_B)(x), f^{-1}(M_B)(y)\}, \end{aligned}$$

$$\begin{aligned}
f^{-1}(\tilde{B}_B^-)(x*y) &= \tilde{B}_B^-(f(x*y)) = \tilde{B}_B^-(f(x)*f(y)) \\
&\leq \max\{\tilde{B}_B^-(f(x)), \tilde{B}_B^-(f(y))\} \\
&= \max\{f^{-1}(\tilde{B}_B^-)(x), f^{-1}(\tilde{B}_B^-)(y)\},
\end{aligned}$$

$$\begin{aligned}
f^{-1}(\tilde{B}_B^+)(x*y) &= \tilde{B}_B^+(f(x*y)) = \tilde{B}_B^+(f(x)*f(y)) \\
&\geq \min\{\tilde{B}_B^+(f(x)), \tilde{B}_B^+(f(y))\} \\
&= \min\{f^{-1}(\tilde{B}_B^+)(x), f^{-1}(\tilde{B}_B^+)(y)\},
\end{aligned}$$

and

$$\begin{aligned}
f^{-1}(J_B)(x*y) &= J_B(f(x*y)) = J_B(f(x)*f(y)) \\
&\leq \max\{J_B(f(x)), J_B(f(y))\} \\
&= \max\{f^{-1}(J_B)(x), f^{-1}(J_B)(y)\}.
\end{aligned}$$

Hence $f^{-1}(\mathcal{B}) = (f^{-1}(M_B),\, f^{-1}(\tilde{B}_B),\, f^{-1}(J_B))$ is an BMBJ-neutrosophic subalgebra of X. □

Let $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ be an BMBJ-neutrosophic set in a set X. We denote

$$\begin{aligned}
&\top := 1 - \sup\{M_A(x) \mid x \in X\}, \\
&\Pi := \inf\{\tilde{B}_B^-(x) \mid x \in X\}. \\
&\pi := 1 - \sup\{\tilde{B}_B^+(x) \mid x \in X\}. \\
&\perp := \inf\{J_A(x) \mid x \in X\}.
\end{aligned}$$

For any $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \perp]$, we define $\mathcal{A}^T = (M_A^p,\, \tilde{B}_A^a,\, \tilde{B}_A^b,\, J_A^q)$ by $M_A^p(x) = M_A(x) + p$, $\tilde{B}_A^a(x) = \tilde{B}_A^-(x) + a$, $\tilde{B}_A^b(x) = \tilde{B}_A^+(x) + b$ and $J_A^q(x) = J_A(x) - q$. Then $\mathcal{A}^T = (M_A^p,\, \tilde{B}_A^a,\, \tilde{B}_A^b,\, J_A^q)$ is an BMBJ-neutrosophic set in X, which is called a (p, a, b, q)-*translative BMBJ-neutrosophic set* of $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$.

Theorem 3.13. *If $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then the (p, a, b, q)-translative BMBJ-neutrosophic set of $\mathcal{A} = (M_A,\, \tilde{B}_A,\, J_A)$ is also an BMBJ-neutrosophic subalgebra of X.*

Proof. For any $x, y \in X$, we get

$$\begin{aligned}
M_A^p(x*y) &= M_A(x*y) + p \geq \min\{M_A(x), M_A(y)\} + p \\
&= \min\{M_A(x) + p, M_A(y) + p\} = \min\{M_A^p(x), M_A^p(y)\},
\end{aligned}$$

$$\begin{aligned}
\tilde{B}_A^a(x*y) &= \tilde{B}_A^-(x*y) + a \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\} + a \\
&= \max\{\tilde{B}_A^-(x) + a, \tilde{B}_A^-(y) + a\} = \max\{\tilde{B}_A^a(x), \tilde{B}_A^a(y)\},
\end{aligned}$$

$$\begin{aligned}\tilde{B}_A^b(x*y) &= \tilde{B}_A^+(x*y)+b \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}+b \\ &= \min\{\tilde{B}_A^+(x)+b, \tilde{B}_A^+(y)+b\} = \max\{\tilde{B}_A^b(x), \tilde{B}_A^b(y)\},\end{aligned}$$

and

$$\begin{aligned}J_A^q(x*y) &= J_A(x*y)-q \leq \max\{J_A(x), J_A(y)\}-q \\ &= \max\{J_A(x)-q, J_A(y)-q\} = \max\{J_A^q(x), J_A^q(y)\}.\end{aligned}$$

Therefore $\mathcal{A}^T = (M_A^p, \tilde{B}_A^a, \tilde{B}_A^b, J_A^q)$ is an BMBJ-neutrosophic subalgebra of X. □

Theorem 3.14. *Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ be an BMBJ-neutrosophic set in X such that its (p, a, b, q)-translative BMBJ-neutrosophic set is an BMBJ-neutrosophic subalgebra of X for $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \perp]$. Then $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X.*

Proof. Assume that $\mathcal{A}^T = (M_A^p, \tilde{B}_A^a, \tilde{B}_A^b, J_A^q)$ is an BMBJ-neutrosophic subalgebra of X for $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \perp]$. Let $x, y \in X$. Then

$$\begin{aligned}M_A(x*y)+p &= M_A^p(x*y) \geq \min\{M_A^p(x), M_A^p(y)\} \\ &= \min\{M_A(x)+p, M_A(y)+p\} \\ &= \min\{M_A(x), M_A(y)\}+p,\end{aligned}$$

$$\begin{aligned}\tilde{B}_A^a(x*y)-a &= \tilde{B}_A^-(x*y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\} \\ &= \max\{\tilde{B}_A^a(x)-a, \tilde{B}_A^a(y)-a\} \\ &= \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}-a.\end{aligned}$$

$$\begin{aligned}\tilde{B}_A^b(x*y)-b &= \tilde{B}_A^+(x*y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\} \\ &= \min\{\tilde{B}_A^b(x)-b, \tilde{B}_A^b(y)-b\} \\ &= \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}-b.\end{aligned}$$

and

$$\begin{aligned}J_A(x*y)-q &= J_A^q(x*y) \leq \max\{J_A^q(x), J_A^q(y)\} \\ &= \max\{J_A(x)-q, J_A(y)-q\} \\ &= \max\{J_A(x), J_A(y)\}-q.\end{aligned}$$

It follows that $M_A(x*y) \geq \min\{M_A(x), M_A(y)\}$, $\tilde{B}_A^-(x*y) \leq \max\{\tilde{B}_A^-(x), \tilde{B}_A^-(y)\}$, $\tilde{B}_A^+(x*y) \geq \min\{\tilde{B}_A^+(x), \tilde{B}_A^+(y)\}$ and $J_A(x*y) \leq \max\{J_A(x), J_A(y)\}$ for all $x, y \in X$. Hence $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X. □

Definition 3.15. Let $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ and $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ be BMBJ-neutrosophic sets in X. Then $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is called an *BMBJ-neutrosophic S-extension* of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ if the following assertions are valid.

(1) $M_B(x) \geq M_A(x)$, $\tilde{B}_A^-(x) \leq \tilde{B}_A^-(x)$, $\tilde{B}_A^+(x) \geq \tilde{B}_A^+(x)$ and $J_B(x) \leq J_A(x)$ for all $x \in X$,

(2) If $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic subalgebra of X, then $\mathcal{B} = (M_B, \tilde{B}_B, J_B)$ is an BMBJ-neutrosophic subalgebra of X.

Theorem 3.16. *Given $p \in [0, \top]$, $a \in [0, \Pi]$, $b \in [0, \pi]$ and $q \in [0, \perp]$, the (p, a, b, q)-translative BMBJ-neutrosophic set $\mathcal{A}^T = (M_A^p, \tilde{B}_A^a, \tilde{B}_A^b, J_A^q)$ of an BMBJ-neutrosophic subalgebra $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$ is an BMBJ-neutrosophic S-extension of $\mathcal{A} = (M_A, \tilde{B}_A, J_A)$.*

Proof. Straightforward. □

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References

[1] M. Abdel-Basset, M. Saleh, A. Gamal, A. and F. Smarandache. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 77 (2019), 438-452.

[2] M. Abdel-Baset, V. Chang, A. Gamal and F. Smarandach. An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. Computers in Industry, 106 (2019), 94-110.

[3] M. Abdel-Basset, G. Manogaran, A. Gamal, and F. Smarandache. A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of medical systems, 43(2), 38. (2019).

[4] M. Abdel-Baset, V. Chang and A. Gamal. Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry, 108 (2019), 210-220.

[5] M. Abdel-Basset, G. Manogaran, A. Gamal, A and F. Smarandache. A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. Design Automation for Embedded Systems (2019), 1-22.

[6] K. Atanassov, "Intuitionistic fuzzy sets," Fuzzy Sets and Systems, vol. 20, p. 87-96 (1986).

[7] R.A. Borzooei, X.H. Zhang, F. Smarandache and Y.B. Jun, Commutative generalized neutrosophic ideals in BCK-algebras, Symmetry 2018, 10, 350; doi:10.3390/sym10080350.

[8] S. Broumi, A. Dey, M. Talea, A. Bakali, F. Smarandache, D. Nagarajan, M. Lathamaheswari and Ranjan Kumar(2019), "Shortest Path Problem using Bellman Algorithm under Neutrosophic Environment," Complex and amp; Intelligent Systems ,pp-1-8, https://doi.org/10.1007/s40747-019-0101-8 ,

[9] S. Broumi, M.Talea, A. Bakali, F. Smarandache, D.Nagarajan, M. Lathamaheswari and M.Parimala, "Shortest path problem in fuzzy, intuitionistic fuzzy and neutrosophic environment" .an overview, Complex and amp; Intelligent Systems ,2019,pp 1-8, https://doi.org/10.1007/s40747-019-0098-z

[10] Y.S. Huang, BCI-algebra, Beijing: Science Press (2006).

[11] K. Iséki, On BCI-algebras, Math. Seminar Notes 8 (1980), 125–130.

[12] K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon. 23 (1978), 1–26.

[13] Y.B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform. 14(1) (2017), 75–86.

[14] Y.B. Jun, S.J. Kim and F. Smarandache, Interval neutrosophic sets with applications in BCK/BCI-algebra, Axioms 2018, 7, 23.

[15] Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras, Information 2017, 8, 128.

[16] Y. B. Jun, S. Z. Song, F. Smarandache and H. Bordbar Neutrosophic Quadruple BCK/BCI-Algebras, Axioms 2018, 7, 2.

[17] Y.B. Jun, F. Smarandache, S.Z. Song and H. Bordbar, Neutrosophic Permeable Values and Energetic Subsets with Applications in BCK/BCI-Algebras, Mathematics 6 (5), 74

[18] Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic falling shadows applied to subalgebras and ideals in BCK/BCI-algebras, Annals of Fuzzy Mathematics and Informatics

[19] Y.B. Jun, F. Smarandache, S.Z. Song and M. Khan, Neutrosophic positive implicative $\mathcal{N}$-ideals in BCK/BCI-algebras, Axioms 2018, 7, 3.

[20] M. Khan, S. Anis, F. Smarandache and Y.B. Jun, Neutrosophic $\mathcal{N}$-structures and their applications in semigroups, Ann. Fuzzy Math. Inform. 14(6) (2017), 583–598.

[21] J. Meng and Y.B. Jun, BCK-algebras, Kyung Moon Sa Co., Seoul (1994).

[22] M. Mohseni Takallo, H. Bordbar, R.A. Borzooei, Y. B. Jun BMBJ-neutrosophic ideals in BCK/BCI-algebras Neutrosophic Sets and Systems, Vol. 7, 2019

[23] G. Muhiuddin, H. Bordbar, F. Smarandache and Y. B. Jun, Further results on $(\in, \in)$-neutrosophic subalgebras and ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, Vol. 20, 2018.

[24] N. A. Nabeeh, F. Smarandache, M. Abdel-Basset, H. A. El-Ghareeb and Aboelfetouh, A. An Integrated Neutrosophic-TOPSIS Approach and Its Application to Personnel Selection: A New Trend in Brain Processing and Analysis. IEEE Access, 7 (2019), 29734-29744.

[25] M.A. Öztürk and Y.B. Jun, Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points, J. Inter. Math. Virtual Inst. 8 (2018), 1–17.

[26] A.B. Saeid and Y.B. Jun, Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points, Ann. Fuzzy Math. Inform. 14(1) (2017), 87–97.

[27] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).

[28] F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, Rehoboth: American Research Press (1999).

[29] F. Smarandache, Neutrosophic set, a generalization of intuitionistic fuzzy sets, International Journal of Pure and Applied Mathematics, 24(5) (2005), 287–297.

[30] S.Z. Song, H. Bordbar and Y.B. Jun, Quotient Structures of BCK/BCI-Algebras Induced by Quasi-Valuation Maps Axioms 2018, 7(2), 26; https://doi.org/10.3390/axioms7020026

[31] S.Z. Song, M. Khan, F. Smarandache and Y.B. Jun, A novel extension of neutrosophic sets and Fs application in BCK/BI-algebras, New Trends in Neutrosophic Theory and Applications (Volume II), Pons Editions, Brussels, Belium, EU 2018, 308–326.

[32] S.Z. Song, F. Smarandache and Y.B. Jun, , Neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras, Information 2017, 8, 130.

[33] L.A. Zadeh, Fuzzy sets, Information and Control, 8(3) (1965), 338–353.

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New Open Sets in N-Neutrosophic Supra Topological Spaces

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Abstract: The neutrosophic set is an imprecise set to deal the concepts of uncertainty, vagueness and irregularity, which consists of three independent functions called truth-membership, indeterminacy-membership and falsity-membership. This set is a generalization of Atanassov's intuitionistic fuzzy sets. The neutrosophic supra topological space is a set together with neutrosophic supra topology. The intension of this paper is to develop the concept of N-neutrosophic supra topological spaces. We further investigate the closure and interior operators in N-neutrosophic supra topological spaces. Moreover, some weak form of N-neutrosophic supra topological open sets are defined and establish their relations with suitable examples.

Keywords: N-neutrosophic supra topology; N-neutrosophic supra α-open set; N-neutrosophic supra semi- open set; N-neutrosophic supra pre-open set; N-neutrosophic supra β-open set.

1. Introduction

A. Lottif Zadeh[1] developed a new set to analyze imprecise, vagueness and ambiguity information, namely fuzzy set, it discuss each element along with the membership value. Fuzzy set theory [2, 3, 4, 5] was applied in various fields such control systems, artificial intelligence, biology, medical diagnosis, economics and probability. C. L. Chang [6] introduced the concept of fuzzy topological space. R. Lowen [7] further studied about the fuzzy topological compactness. AbdMonsef and Ramadan [9] introduced fuzzy supra topological spaces and its continuous mappings. In 1986, K. Atanassov [10] introduced intuitionistic fuzzy set as a generalization of the fuzzy set, by taking into account both the degrees of membership and of non-membership of an element subject to the condition that their sum does not exceed 1. Some researchers [11, 12, 13, 14, 15, 16, 17] used the intuitionistic fuzzy sets in pattern recognition, medical diagnosis, data mining process. Dogan Coker [18] generalized the fuzzy topological spaces into intuitionistic fuzzy topological spaces and further Reza Saadati and Jin Han Park [19] studied the properties of intuitionistic fuzzy topological spaces. The concept of intuitionistic fuzzy supra topological space was initiated by N. Turnal [20]. Neutrosophic set is the generalization of Atanassov's intuitionistic fuzzy set, developed by Florentin Samarandache [21, 22, 23] which is a set considering the degree of membership, the degree of indeterminacy-membership and the degree of falsity-membership whose values are real standard or non-standard subset of unit interval] 0^- ; 1^+[. Recently many researchers [24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37] introduced neutrosophic numbers, several similarity measures and single-valued neutrosophic sets, which are applied in attribute decision making, information system quality, medical diagnosis, control systems, artificial intelligence, etc. Salama et al. [38, 39] defined the neutrosophic crisp set and neutrosophic topological space. In 1963, Norman Levine [40] initiated the concept of semi open sets and discussed the continuous functions in classical spaces. O.Njastad [41] showed that the family of all α-open sets forms a topology. Mashhour et al. [42] investigated the properties of pre open sets. Andrijevic [43] discussed the behavior of β-open sets in classical topology. By relaxing one of the topological axioms, Mashhour et al. [44] further developed the concept of supra topological space with the properties. Devi et al. [45] introduced the properties of α-open sets and α-continuous functions in supra topological spaces. Supra topological pre-open sets and its continuous functions are defined by O.R.Sayed [46]. Saeid Jafari et al. [47] investigated the properties of supra β-open sets and its continuity. In 2016, Lellis Thivagar et al. [48] developed a new theory called N-topological spaces and its own open sets. Apart from this, M. Lellis Thivagar and M.Arockia Dasan [49] derived some new N-topologies by the help of weak open sets and mappings in N-topological spaces. Recently, G.Jayaparthasarathy et al. [50] defined the concept of neutrosophic supra topological spaces and proposed a new method to solve medical diagnosis problems by using single valued neutrosophic score function.

The present paper is organized as follows: The second section gives some basic properties of fuzzy, intuitionistic, neutrosophic sets and neutrosophic supra topological spaces. The third section extends the concept of neutrosophic supra topological spaces into N-neutrosophic supra topological spaces with the properties of closure and interior operators. In the next section, we introduce some weak open sets in N-neutrosophic supra topological spaces, namely N-neutrosophic supra α-open sets, N-neutrosophic supra semi-open sets, N-neutrosophic supra pre-open sets and N-neutrosophic supra β-open sets. The fifth section discusses the relationship between N-neutrosophic supra topological closed sets. In the next section, we compare the neutrosophic supra topological spaces and N-neutrosophic supra topological spaces with their limitations. The seventh section states the conclusion and future work of this paper. Finally all the necessary references of this paper are given.

2. Preliminaries

In this section, we discuss some basic definitions and properties of fuzzy, intuitionistic, neutrosophic sets and neutrosophic supra topological spaces which are useful in sequel.

Definition 2.1 [1] Let X be a non empty set and a fuzzy set A on X is of the form $A = \{(x, \mu_A(x)) : x \in X\}$, where $0 \leq \mu_A(x) \leq 1$ represents the degree of membership function of each $x \in X$ to the set A. For X, I^X denotes the collection of all fuzzy sets of X.

Definition 2.2 [10] Let X be a non empty set. An intuitionistic set A is of the form $A = \{(x, \mu_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x)$ and $\gamma_A(x)$ represent the degree of membership and non membership function respectively of each $x \in X$ to the set A and

$0 \leq \mu_A(x) + \gamma_A(x) \leq 1$ for all $x \in X$. The set of all intuitionistic sets of X is denoted by I(X).

Definition 2.3 [21] Let X be a non empty set. A neutrosophic set A having the form $A = \{(x, \mu_A(x), \sigma_A(x), \gamma_A(x)) : x \in X\}$, where $\mu_A(x), \sigma_A(x)$ and $\gamma_A(x) \in]^-0, 1^+[$ represent the degree of membership (namely $\mu_A(x)$), the degree of indeterminacy (namely $\sigma_A(x)$) and the degree of non membership (namely $\gamma_A(x)$) respectively of each $x \in X$ to the set A such that $-_0 \leq \mu_A(x) + \sigma_A(x) + \gamma_A(x) \leq 3^+$ for all $x \in X$. For X, $N(X)$ denotes the collection of all neutrosophic sets of *X*.

Definition 2.4. [22] The following statements are true for neutrosophic sets *A* and *B* on *X*:

$\mu_A(x) \leq \mu_B(x)$, $\sigma_A(x) \leq \sigma_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$ for all $x \in X$ if and only if $A \subseteq B$.

$A \subseteq B$ and $B \subseteq A$ if and only if $A = B$.

$A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{\sigma_A(x), \sigma_B(x)\}, \max\{\gamma_A(x), \gamma_B(x)\}) : x \in X\}$.

$A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \max\{\sigma_A(x), \sigma_B(x)\}, \min\{\gamma_A(x), \gamma_B(x)\}) : x \in X\}$.

More generally, the intersection and the union of a collection of neutrosophic sets $\{A_i\}_{i \in \Lambda}$, are defined by $\bigcap_{i \in \Lambda} A_i = \{(x, inf_{i \in \Lambda}\{\mu_{A_i}(x)\}, inf_{i \in \Lambda}\{\sigma_{A_i}(x)\}, sup_{i \in \Lambda}\{\gamma_{A_i}(x)\}) : x \in X\}$ and $\bigcup_{i \in \Lambda} A_i = \{(x, sup_{i \in \Lambda}\{\mu_{A_i}(x)\}, sup_{i \in \Lambda}\sigma_{A_i}(x)\}, inf_{i \in \Lambda}\{\gamma_{A_i}(x)\}) : x \in X\}$.

Corollary 2.5. [23] The following statements are true for the neutrosophic sets A, B, C and D on X :

$A \cap C \subseteq B \cap D$ and $A \cup C \subseteq B \cup D$, if $A \subseteq B$ and $C \subseteq D$.

$A \subseteq B \cap C$, if $A \subseteq B$ and $A \subseteq C$. $A \cup B \subseteq C$, if $A \subseteq C$ and $B \subseteq C$.

$A \subseteq C$, if $A \subseteq B$ and $B \subseteq C$.

Definition 2.6. [50] Let A, B be two neutrosophic sets of X, then the difference of A and B is a neutrosophic set on X, defined as $A \setminus B = \{(x, |\mu_A(x) - \mu_B(x)|, |\sigma_A(x) - \sigma_B(x)|, 1 - |\gamma_A(x) - \gamma_B(x)|) : x \in X\}$. Clearly $X^c = X \setminus X = (x, 0,0,1) = \emptyset$ and $\emptyset^c = X \setminus \emptyset = (x, 1,1,0) = X$.

Notation 2.7. Let *X* be a non empty set. We consider the neutrosophic empty set as $\emptyset = \{(x, 0, 0, 1) : x \in X\}$ and the neutrosophic whole set as $X = \{(x, 1, 1, 0) : x \in X\}$.

Corollary 2.8. [50] The following statements are true for the neutrosophic sets $\{A\}_{i=1}^{\infty}$, A, B on X:

$(i)\ \cap_{i\in\Lambda} (A_i)^c = \cup_{i\in\Lambda} Ai^c, (\cup_{i\in\Lambda}\ Ai)^c = \cap_{i\in\Lambda} Ai^c$.

(ii) $(A^c)^c = A$.

iii) $B^c \subseteq A^c$, if $A \subseteq B$.

Definition 2.9. [39] Let X be a non empty set. A subfamily τ_n of $N(X)$ is said to be a neutrosophic topology on X if the neutrosophic sets $X$and $\emptyset$ belong to τ_n, $\tau_n$is closed under arbitrary union and $\tau_n$is closed under finite intersection. Then (X, τ_n) is called neutrosophic topological space (shortly nts), members of $\tau_n$are known as neutrosophic open sets and their complements are neutrosophic closed sets. For a neutrosophic set $A$of X, the interior and closure of A are respectively defined as: $int_n(A) = \cup\{G : G \subseteq A, G \in \tau_n\}$ and $cl_n(A) = \cap\{F : A \subseteq F, F^c \in \tau_n\}$.

Definition 2.10. [50] Let Xbe a non empty set. A sub collection$\tau_n{}^* \subseteq N(X)$ is said to be a neutrosophic supra topology on X if the sets $\emptyset$, X$\in \tau_n{}^*$and $\tau_n{}^*$is closed under arbitrary union. Then the ordered pair $(X, \tau_n{}^*)$ is called neutrosophic supra topological space on X(for short nsts). The elements of $\tau_n{}^*$are known as neutrosophic supra open sets and its complement is called neutrosophic supra closed. Let (X, τ_n) be a neutrosophic topological space, then a neutrosophic supra topology $\tau_n{}^*$on $X$is said to be an associated neutrosophic supra topology with τ_n if $\tau_n \subseteq \tau_n{}^*$. Every neutrosophic topology on X is neutrosophic supra topology on X.

Definition 2.11. [50] Let A be a neutrosophic set on nsts$(X, \tau_n{}^*)$, then the $int_{\tau_n{}^*}(A)$ and $cl\tau_n{}^*(A)$ are respectively defined as: $int_{\tau_n{}^*}(A) = \cup\{G : G \subseteq A$ and $G \in \tau_n{}^*\}$ and $cl_{\tau_n{}^*}(A) = \cap\{F : A \subseteq F$ and$F^c \in \tau_n{}^*\}$.

3. *N*-Neutrosophic Supra Topological Spaces

In this section, we introduce N-neutrosophic supra topological spaces and investigate the properties of closure, interior operators in *N*-neutrosophic supra topological spaces.

Definition 3.1. Let X be a non empty set, $\tau_{n_1}{}^*, \tau_{n_2}{}^*, \ldots, \tau_{n_N}{}^*$ be *N*-arbitrary neutrosophic supra topologies defined on X. Then the collection $N\tau_n{}^* = \{S \subseteq X : S = \cup_{i=1}^{N} Ai, Ai \in \tau_{n_i}{}^*\}$ is said to be a *N*-neutrosophic supra topology if it satisfies the following axioms:

$X, \emptyset \in N\tau_n{}^*$.

$\cup_{i=1}^{\infty} Si \in N\tau_n{}^*\ for\ all\ S_i \in N\tau_n{}^*$.

Then the *N*-neutrosophic supra topological space is the non empty set X together with the collection $N\tau_n{}^*$,denoted by $(X, N\tau_n{}^*)$ and its elements are known as $N\tau_n{}^*$-open sets on X. A neutrosophic subset A of X is said to be $N\tau_n{}^*$-closed on X if $X \setminus A$ is $N\tau_n{}^*$-open on X. The set of all $N\tau_n{}^*$-open sets on X and the set of all $N\tau_n{}^*$-closed sets on X are respectively denoted by $N\tau_n{}^*O(X)$ and $N\tau_n{}^*C(X)$.

Remark 3.2. For instance, if $N = 1$, then $(X, 1\tau_n{}^* = \tau_n{}^*)$ is called the classical neutrosophic supra topological space [50]. If $N = 2$, then $(X, 2\tau_n{}^*)$ is called the bi neutrosophic supra topological space. If $N = 3$, then $(X, 3\tau_n{}^*)$ is called the tri neutrosophic supra topological space defined on X and so on.

Example 3.3. Let $X = \{a, b, c\}, N = 4$, assume the neutrosophic supra topologies $\tau_{n_1}{}^* = \{\emptyset, X, ((0.5, 0.5, 0.5), (1,1,0), (0,0,1))\}, \tau_{n_2}{}^* = \{\emptyset, X, ((0.25, 0.25, 0.75),$

$(0,0,1), (1,1,0))\}$, $\tau_{n_3}{}^* = \{\emptyset, X, ((0.5, 0.5, 0.5), (1,1,0), (1,1,0))\}$ and

$\tau_{n_4}{}^* = \{\emptyset, X, ((0.5, 0.5, 0.5), (1,1,0), (0,0,1)), ((0.5, 0.5, 1), (1,1,0), (1,1,0))\}$.

$4\tau_n{}^* = \{\emptyset, X, ((0.5, 0.5, 0.5), (1,1,0), (0,0,1)), ((0.25, 0.25, 0.75), (0,0,1), (1,1,0)),$

$((0.5, 0.5, 0.5), (1,1,0), (1,1,0))\}$ and $(4\tau_n{}^*)^c = \{X, \emptyset, ((0.5, 0.5, 0.5), (0,0,1), (1,1,0)),$

$((0.75, 0.75, 0.25), (1,1,0), (0,0,1)), ((0.5, 0.5, 0.5), (0,0,1), (0,0,1))\}$. Therefore $(X, 4\tau_n{}^*)$ is a quad neutrosophic supra topological space on X.

Remark 3.4. (i) If $N = 1$, then $N\tau_n{}^* = \tau_n{}^*$.

(ii) Union of two N-neutrosophic supra topologies is again an N-neutrosophic supra topology.

(iii) Intersection of two N-neutrosophic supra topologies is again an N-neutrosophic supra topology.

Proof. (i): The proof is trivial.

(ii): Let $(N\tau_n{}^*)_1$ and $(N\tau_n{}^*)_2$ be two N-neutrosophic supra topologies on X. Clearly, X and $\emptyset$ are the elements of $(N\tau_n{}^*)_1 \cup (N\tau_n{}^*)_2$. Let $\{A_i\}_{i\in\Lambda} \in (N\tau_n{}^*)_1 \cup (N\tau_n{}^*)_2$, then by definition of N-neutrosophic supra topology, $\cup_{i\in\Lambda} A_i \in (N\tau_n{}^*)_1 \cup (N\tau_n{}^*)_2$. Thus the union of two N-neutrosophic supra topologies is a N-neutrosophic supra topology.

(iii): Let $(N\tau_n{}^*)_1$ and $(N\tau_n{}^*)_2$ be two N-neutrosophic supra topologies on X. Clearly, X and $\emptyset$ are the elements of $(N\tau_n{}^*)_1 \cap (N\tau_n{}^*)_2$. Let $\{A_i\}_{i\in\Lambda} \in (N\tau_n{}^*)_1 \cap (N\tau_n{}^*)_2$, then, $\cup_{i\in\Lambda} A_i \in (N\tau_n{}^*)_1$, $\cup_{i\in\Lambda} A_i \in (N\tau_n{}^*)_2$ and so $\cup_{i\in\Lambda} A \in (N\tau_n{}^*)_1 \cap (N\tau_n{}^*)_2$. Thus the intersection of two N-neutrosophic supra topologies is a N-neutrosophic supra topology.

Remark 3.5. In classical N-topological spaces, the union of two N-topologies need not be a N-topology. But this statement is not true in N-neutrosophic supra topological spaces as proved above. Thus the union of two N-neutrosophic supra topologies is a N-neutrosophic supra topology.

Definition 3.6. Let $(X, N\tau_n^{\ *})$ be a N-neutrosophic supra topological space and A be a neutrosophic set of X. Then

$N\tau_n^{\ *}$ -interior of A is defined by $int_{N\tau_n^*}(A) = \cup\{G : G \subseteq S$ and G is $N\tau_n^{\ *}$-open$\}$.

$N\tau_n^{\ *}$-closure of A is defined by $cl_{N\tau_n^*}(A) = \cap\{F : A \subseteq F$ and F is $N\tau_n^{\ *}$-closed$\}$.

Theorem 3.7. The following are true for neutrosophic sets A and B of N-neutrosophic supra topological space $(X, N\tau_n^{\ *})$:

$A = cl_{N\tau_n^*}(A)$ if and only if A is N-neutrosophic supra closed.

$A = int_{N\tau_n^*}(A)$ if and only if A is N-neutrosophic supra open.

$cl_{N\tau_n^*}(A) \subseteq cl_{N\tau_n^*}(B)$, if $A \subseteq B$.

$int_{N\tau_n^*}(A) \subseteq int_{N\tau_n^*}(A)$, if $A \subseteq B$.

$cl_{N\tau_n^*}(A) \cup cl_{N\tau_n^*}(B) \subseteq cl_{N\tau_n^*}(A \cup B)$.

$int_{N\tau_n^*}(A) \cup int_{N\tau_n^*}(B) \subseteq int_{N\tau_n^*}(A \cup B)$.

$cl_{N\tau_n^*}(A) \cap cl_{N\tau_n^*}(B) \supseteq cl_{N\tau_n^*}(A \cap B)$.

$int_{N\tau_n^*}(A) \cap int_{N\tau_n^*}(B) \supseteq int_{N\tau_n^*}(A \cap B)$.

$int_{N\tau_n^*}(A^c) = (cl_{N\tau_n^*}(A))^c$.

$(int_{N\tau_n^*}(A))^c = cl_{N\tau_n^*}(A^c)$.

Proof. (i): Since $A = cl_{N\tau_n^*}(A)$ and by definition $cl_{N\tau_n^*}(A)$ is N-neutrosophic supra closed, then A is N-neutrosophic supra closed. Conversely, if B is any N-neutrosophic supra closed containing A, and since $cl_{N\tau_n^*}(A)$ is the intersection of all N-neutrosophic supra closed sets containing A, then $cl_{N\tau_n^*}(A) \subseteq B$ and $cl_{N\tau_n^*}(A)$ is the smallest N-neutrosophic supra closed set containing A. Since A is N-neutrosophic supra closed, then the smallest N-neutrosophic supra closed set containing A is A itself. Therefore, $A = cl_{N\tau_n^*}(A)$.

(ii): Since $A = int_{N\tau_n^*}(A)$ and by definition $int_{N\tau_n^*}(A)$ is N-neutrosophic supra open, then A is N-neutrosophic supra open. Conversely, if B is any N-neutrosophic supra open contained in A, and since $int_{N\tau_n^*}(A)$ is the union of all N-neutrosophic supra open sets contained in A, then $int_{N\tau_n^*}(A) \supseteq B$ and $int_{N\tau_n^*}(A)$ is the largest N-neutrosophic supra open set contained in A. Since A is N-neutrosophic supra open, then the largest N-neutrosophic supra open set contained in A is A itself. Therefore, $A = int_{N\tau_n^*}(A)$.

(iii):
$cl_{N\tau_n^*}(B) = \cap\{G : G^c \in N\tau_n^*, B \subseteq G\} \supseteq \cap\{G : G^c \in N\tau_n^*, A \subseteq G\} = cl_{N\tau_n^*}(A)$.

Thus, $cl_{N\tau_n^*}(A) \subseteq cl_{N\tau_n^*}(B)$.

(iv):
$int_{N\tau_n^*}(B) = \cup\{G : G \in N\tau_n^*, B \supseteq G\} \supseteq \cup\{G : G \in N\tau_n^*, A \supseteq G\} = int_{N\tau_n^*}(A)$.
Thus, $int_{N\tau_n^*}(A) \subseteq int_{N\tau_n^*}(B)$.

(v): Since $A \cup B \supseteq A, B$, then by part (iii) $cl_{N\tau_n^*}(A) \cup cl_{N\tau_n^*}(B) \subseteq cl_{N\tau_n^*}(A \cup B)$.

(vi): Since $A \cup B \supseteq A, B$, then by part (iv) $int_{N\tau_n^*}(A) \cup int_{N\tau_n^*}(B) \subseteq int_{N\tau_n^*}(A \cup B)$.

(vii): Since $A \cap B \subseteq A, B$, then by part (iii) $int_{N\tau_n^*}(A) \cap int_{N\tau_n^*}(B) \supseteq int_{N\tau_n^*}(A \cap B)$.

(viii): Since $A \cap B \subseteq A, B$, then by part (iv) $int_{N\tau_n^*}(A) \cap int_{N\tau_n^*}(B) \subseteq int_{N\tau_n^*}(A \cap B)$.

(ix): $cl_{N\tau_n^*}(A) = \cap\{G : G^c \in N\tau_n^*, G \supseteq A\}$, $(cl_{N\tau_n^*}(A))^c = \cup\{G^c : G^c$ is a N-neutrosophic supra open in X and $G^c \subseteq A^c\} = int_{N\tau_n^*}(A^c)$. Thus, $(cl_{N\tau_n^*}(A))^c = int_{N\tau_n^*}(A^c)$

(x): $int_{N\tau_n^*}(A) = \cup\{G : G \in N\tau_n^*, G \subseteq A\}, (int_{N\tau_n^*}(A))^c = \cap\{G^c : G^c$ is a N-neutrosophic supra closed in X and $G^c \supseteq A^c\} = cl_{N\tau_n^*}(A^c)$. Thus, $(int(A))^c = int_{N\tau_n^*}(A^c)$.

Remark 3.8. If we take complement of either side of (ix) and (x) of previous theorem, we get

(i) $cl_{N\tau_n^*}(A) = (int_{N\tau_n^*}(A^c))^c$.

(ii) $int_{N\tau_n^*}(A) = (cl_{N\tau_n^*}(A^c))^c$.

Theorem 3.9. Let $(X, N\tau_n^*)$ be a *N*-neutrosophic supra topological space and *A* be a neutrosophic set of *X*. Then

(i) $int_{N\tau_n^*}(A) \supseteq int_{\tau_{n_1}^*}(A) \cup int_{\tau_{n_2}^*}(A) \cup \ldots \cup int_{\tau_{n_N}^*}(A)$.

(ii) $cl_{N\tau_n^*}(A) \subseteq cl_{\tau_{n_1}^*}(A) \cap cl_{\tau_{n_2}^*}(A) \cap \ldots \cap cl_{\tau_{n_N}^*}(A)$.

Proof. (i): By definition of N-neutrosophic supra topological space, we have $N\tau_n^* = \{S \subseteq X: S = \cup_{i=1}^{N} A_i,\ A_i \in \tau_{n_i}^*\} \supseteq \tau_{n_1}^* \cup \tau_{n_2}^* \cup \ldots \cup \tau_{n_N}^*$.

Therefore, $int_{N\tau_n^*}(A) \supseteq int_{\tau_{n_1}^*}(A) \cup int_{\tau_{n_2}^*}(A) \cup \ldots \cup int_{\tau_{n_N}^*}(A)$.

(ii): Since $int_{N\tau_n^*}(A^c) \supseteq int_{\tau_{n_1}^*}(A^c) \cup int_{\tau_{n_2}^*}(A^c) \cup \ldots \cup int_{\tau_{n_N}^*}(A^c)$, then $(cl_{N\tau_n^*}(A))^c \supseteq (cl_{\tau_{n_1}^*}(A))^c \cup (cl_{\tau_{n_2}^*}(A))^c \cup \ldots \cup (cl_{\tau_{n_N}^*}(A))^c$ which implies $cl_{N\tau_n^*}(A) \subseteq cl_{\tau_{n_1}^*}(A) \cap cl_{\tau_{n_2}^*}(A) \cap \ldots \cap cl_{\tau_{n_N}^*}(A)$.

4. N-Neutrosophic Supra Topological Weak Open Sets

In this section, we introduce some new classes of N-neutrosophic supra topological open sets and discuss the relationship between them.

Definition 4.1. A neutrosophic set A of a N-neutrosophic supra topological space $(X, N\tau_n^*)$ is called

N-neutrosophic supra α-open set if $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A)))$.

N-neutrosophic supra semi-open set if $A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A))$.

N-neutrosophic supra pre-open set if $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(A))$.

N-neutrosophic supra β-open set if $A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(cl_{N\tau_n^*}(A)))$.

The set of all N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open and N-neutrosophic supra β-open) sets of $(X, N\tau_n^*)$ is denoted by $N\tau_n^*O(X)$ (resp. $N\tau_n^*SO(X), N\tau_n^*PO(X)$ and $N\tau_n^*\beta O(X)$.

Theorem 4.2. Let A be a subset of N-neutrosophic supra topological space $(X, N\tau_n^*)$. Then

every N-neutrosophic supra open set is N-neutrosophic supra α-open.

every N-neutrosophic supra α-open set is N-neutrosophic supra semi-open.

every N-neutrosophic supra α-open set is N-neutrosophic supra pre-open every N-neutrosophic supra semi-open set is N-neutrosophic supra β-open.

every N-neutrosophic supra pre-open set is N-neutrosophic supra β-open.

Proof.(i): Assume A is N-neutrosophic supra open, $int_{N\tau_n^*}(A) = A$.

Since $A \subseteq cl_{N\tau_n^*}(A)$, $int_{N\tau_n^*}(A) \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A))$.

Then $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A)))$. Therefore, A is N-neutrosophic supra semi-open.

(ii): Assume A is N-neutrosophic supra α-open and since $int_{N\tau_n^*}(A) \subseteq A$, then $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A))) \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A))$. Therefore, A is N-neutrosophic supra semi-open.

(iii): Assume A is N-neutrosophic supra α-open and since $int_{N\tau_n^*}(A) \subseteq A$, then

$cl_{N\tau_n^*}(int_{N\tau_n^*}(A)) \subseteq cl_{N\tau_n^*}(A)$. Then $A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A))) \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(A))$. Therefore, A is N-neutrosophic supra pre-open.

(iv): Assume A is N-neutrosophic supra semi-open and since $A \subseteq cl_{N\tau_n^*}(A)$, then $int_{N\tau_n^*}(A) \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(A)$). Then $A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A)) \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}\left(cl_{N\tau_n^*}(A)\right))$. Therefore, A is N-neutrosophic supra β-open.

(v): Assume A is N-neutrosophic supra pre-open and since $A \subseteq cl_{N\tau_n^*}(A)$, then $A \subseteq cl_{N\tau_n^*}(A) \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(cl_{N\tau_n^*}(A)))$. Therefore, A is N-neutrosophic supra β-open.

The converse of the above theorem need not be true as shown in the following examples.

Example4.3. Let $X = \{a, b\}$ and $N = 2$, assume $\tau_{n_1}{}^* = \{\emptyset, X, ((0.3, 0.4), (0.3, 0.4), (0.4,0.5))\}$

$\tau_{n_2}{}^* = \{\emptyset, X, \left((0.4, 0.2), (0.4, 0.2), (0.5, 04)\right)\}$. Then

$2\tau_n{}^* = \{\emptyset, X, ((0.3, 0.4), (0.3,0.4), (0.4, 0.5)), ((0.4, 0.2), (0.4, 0.2), (0.5, 0.4)),$

$((0.4, 0.4), (0.4, 0.4), (0.4, 0.4))\}$ is a bi neutrosophic supra topology on X. Then the neutrosophic set $A = ((0.4, 0.6), (0.4, 0.6), (0.3, 0.4))$ is 2-neutrosophic supra α-open but not 2-neutrosophic supra open.

Example4.4. Let $X = \{a, b\}$ and $N = 2$, assume $\tau_{n_1}{}^{*} = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5))\}$, $\tau_{n_2}{}^{*} = \{\emptyset, X, ((0.4, 0.3), (0.4, 0.3), (0.5, 0.2))\}$. Then

$2\tau_n{}^{*} = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5)), ((0.4, 0.3), (0.4, 0.3), (0.5, 0.2)),$

$((0.4, 0.5), (0.4, 0.5), (0.4, 0.2))\}$ is a bi neutrosophic supra topology on X. Then the neutrosophic set $A = ((0.4, 0.5), (0.4, 0.5), (0.5, 0.4))$ is 2-neutrosophic supra pre-open, 2-neutrosophic supra β-open, but not 2-neutrosophic supra α-open and not 2-neutrosophic supra semi-open.

Example4.5. Let $X = \{a, b\}$ and $N = 3$, assume

$\tau_{n_1}{}^{*} = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5))\}$, $\tau_{n_2}{}^{*} =$ $\{\emptyset, X, ((0.4, 0.3), (0.4, 0.3), (0.5, 0.6))\}$

and $\tau_{n_3}{}^{*} = \{\emptyset, X, ((0.4, 0.5), (0.4, 0.5), (0.4, 0.5))\}$. Then $3\tau_n{}^{*} = \{\emptyset, X, ((0.3, 0.5), (0.3, 0.5), (0.4, 0.5)),$

$((0.4, 0.3), (0.4, 0.3), (0.5, 0.6)), ((0.4, 0.5), (0.4, 0.5), (0.4, 0.5))\}$ is a tri neutrosophic supra topology on X. Then $A = ((0.4, 0.5), (0.4, 0.5), (0.4, 0.5))$ is 3-neutrosophic supra semi-open and 3-neutrosophic supra β-open, but not 3-neutrosophic supra α-open and not 3-neutrosophic supra pre-open.

Theorem 4.6. A neutrosophic set A in a N-neutrosophic supra topological space $(X, N\tau_n{}^{*})$ is N-neutrosophic supra α-open set if and only if A is both N-neutrosophic supra semi-open and N-neutrosophic supra pre-open.

Proof. Assume that A is N-neutrosophic supra α-open set, then $A \subseteq int_{N\tau_n{}^{*}}(cl_{N\tau_n{}^{*}}(int_{N\tau_n{}^{*}}(A))) \subseteq cl_{N\tau_n{}^{*}}(int_{N\tau_n{}^{*}}(A))$. Since $int_{N\tau_n{}^{*}}(A) \subseteq A$, then

$A \subseteq int_{N\tau_n{}^{*}}(cl_{N\tau_n{}^{*}}(int_{N\tau_n{}^{*}}(A))) \subseteq int_{N\tau_n{}^{*}}(cl_{N\tau_n{}^{*}}(A))$. Therefore, A is both N-neutrosophic supra semi-open and N-neutrosophic supra pre-open. On the other hand, assume that A is both N-neutrosophic supra semi-open and N-neutrosophic supra pre-open. Then $A \subseteq int_{N\tau_n{}^{*}}(cl_{N\tau_n{}^{*}}(A)) \subseteq int_{N\tau_n{}^{*}}(cl_{N\tau_n{}^{*}}(int_{N\tau_n{}^{*}}(A)))$. Therefore, A is N-neutrosophic supra α-open.

Lemma 4.7. The arbitrary union of N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open, N-neutrosophic supra β-open) sets is

N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open, N-neutrosophic supra β-open).

Proof. Here we only prove for N-neutrosophic supra α-open sets and similarly we can prove for N-neutrosophic supra semi-open, N-neutrosophic supra pre-open, N-neutrosophic supra β-open sets. Assume that $\{A_i\}_{i\in\Lambda} \in N\,\tau_n{}^*\alpha O(X)$, then $A_i \subseteq int_{N\tau_n{}^*}(cl_{N\tau_n{}^*}(int_{N\tau_n{}^*}(A_i)))$. Since $\cup_{i\in\Lambda} int_{N\tau_n{}^*}(A_i) \subseteq int_{N\tau_n{}^*}(\cup_{i\in\Lambda} A_i)$,
$\cup_{i\in\Lambda} cl_{N\tau_n{}^*}(int_{N\tau_n{}^*}(A_i)) \subseteq cl_{N\tau_n{}^*}(int_{N\tau_n{}^*}(\cup_{i\in\Lambda} A_i))$.
Then $\cup_{i\in\Lambda} A_i \subseteq \cup_{i\in\Lambda} int_{N\tau_n{}^*}(cl_{N\tau_n{}^*}(int_{N\tau_n{}^*}(A_i))) \subseteq int_{N\tau_n{}^*}(cl_{N\tau_n{}^*}(int_{N\tau_n{}^*}(\cup_{i\in\Lambda} A_i)))$.
Therefore, $\cup_{i\in\Lambda} A_i$ is a N-neutrosophic supra α-open set.

Remark 4.8. Intersection of any two N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open, N-neutrosophic supra β-open) sets need not be a N-neutrosophic supra α -open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open, N-neutrosophic supra β-open) set.

Example 4.9. Let $X = \{a, b\}$ and $N = 3$, assume $\tau_{n_1}{}^* = \{\emptyset, X, ((0.3,0.5),(0.3,0.5),(0.4,0.5))\}$, $\tau_{n_2}{}^* = \{\emptyset, X, ((0.4,0.3),(0.4,0.3),(0.5,0.4))\}$ and

$\tau_{n_3}{}^* = \{\emptyset, X, ((0.4,0.5),(0.4,0.5),(0.4,0.4))\}$. Then $3\,\tau_n{}^* = \{\emptyset, X,$

$((0.3,0.5),(0.3,0.5),(0.4,0.5)),((0.4,0.3),(0.4,0.3),(0.5,0.4)),$
$((0.4,0.5),(0.4,0.5),(0.4,\ 0.4))\}$
is a tri neutrosophic supra topology on X and $(X, 3\tau_n{}^*)$ is a tri neutrosophic supra topological space on X. Here $A = ((0.3,\ 0.5),(0.3,0.5),(0.4,$ 0.5)) and $B = ((0.4,0.3),(0.4,\ 0.3),(0.5,0.4))$ are both 3-neutrosophic supra α -open and 3-neutrosophic supra semi open, but $A \cap B$ is not 3-neutrosophic supra α-open and not 3-neutrosophic supra semi-open.

Example4.10. Let $X = \{a, b, c\}, N = 3$, assume the neutrosophic supra topologies $\tau_{n_1}{}^* = \{\emptyset, X\}, \tau_{n_2}{}^* = \{\emptyset, X, ((0.6,0,0),(0.4,0.1,0),(0,0,1))\}$, $\tau_{n_3}{}^* = \{\emptyset, X, ((0.3,0.7,1),$

$(0.7,0.6,1),(1,1,0))\}$. Then $3\,\tau_n{}^* = \{\emptyset, X, ((0.6,0,0),(0.4,0.1,0),(0,0,1)),$

$((0.3,0.7,1),(0.7,0.6,1),(1,1,0)),((0.6,0.7,1),(0.7,0.6,1),(0,0,0))\}$ is a tri neutrosophic supra topology on X and $(X, 3\,\tau_n{}^*)$ is a tri neutrosophic supra topological space on X. Here the neutrosophic sets $A = ((0.6,0,0),(0.4,0.1,0),(0,0,1))$ and $B = ((0.3,0.7,1),(0.7,0.6,1),(1,1,0))$ are 3-neutrosophic supra pre-open and

3-neutrosophic supra β-open, but $A \cap B$ is not 3-neutrosophic supra pre-open and 3-neutrosophic supra β-open.

Remark 4.11. In classical topological spaces, O. Njastad [41] proved that the collection of all α-open sets form a topology which is finer than the collection of all open sets. This statement need not be true in neutrosophic topological spaces as shown in the following example, that is, the collection of all neutrosophic α-open sets need not be a neutrosophic topology, but this collection forms a neutrosophic supra topology.

Example4.12. Let $X = \{a, b\}$, assume the neutrosophic topology $\tau_n = \{\emptyset, X, ((0.3,0.6),(0.5,0.2),(0.4,0.5)), ((0.2,0.5), (0.6, 0.3),(0.7,0.1)), ((0.3, 0.6),(0.6,0.3),(0.4,0.1)),((0.2,0.5),(0.5,0.2),(0.7,0.5))\}$ and (X, τ_n) is a neutrosophic topological space on X. Here $A = ((0.4,0.8),(0.6, 0.3),(0.4,0.4))$ and $B = ((1,0.5),(0.9,0.7),(0.2,0))$ are neutrosophic α-open, but $A \cap B$ is not neutrosophic α-open.

Lemma 4.13. Let $A, B \in N(X)$ and A be a N-neutrosophic supra open set such that $int_{N\tau_n^*}(A) \subseteq B \subseteq A$, then B is N-neutrosophic supra open.

Proof. Assume that A is a N-neutrosophic supra open set such that $int_{N\tau_n^*}(A) \subseteq B \subseteq A$. Then $B \subseteq A = int_{N\tau_n^*}(A) = int_{N\tau_n^*}(B) \subseteq B$. Therefore, B is N-neutrosophic supra open.

Lemma 4.14. Let $A, B \in N(X)$ and A be a N-neutrosophic supra α-open set such that $int_{N\tau_n^*}(A) \subseteq B \subseteq A$, then B is N-neutrosophic supra α-open.

Proof. Assume that A is a N-neutrosophic supra α-open set such that $int_{N\tau_n^*}(A) \subseteq B \subseteq A$. Then $B \subseteq A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A))) = int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(B)))$. Therefore, B is N-neutrosophic supra α-open.

Lemma 4.15. Let $A, B \in N(X)$ and A be a N-neutrosophic supra semi-open set such that $int_{N\tau_n^*}(A) \subseteq B \subseteq A$, then B is N-neutrosophic supra semi-open.

Proof. Assume that A is a N-neutrosophic supra semi-open set such that $int_{N\tau_n^*}(A) \subseteq B \subseteq A$. Then $B \subseteq A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A)) = cl_{N\tau_n^*}(int_{N\tau_n^*}(B))$. Therefore, B is N-neutrosophic supra semi-open.

Lemma 4.16. Let $A, B \in N(X)$ and A be a N-neutrosophic supra pre-open set such that $cl_{N\tau_n^*}(A) \subseteq B \subseteq A$, then B is N-neutrosophic supra pre-open.

Proof. Assume that A is a N-neutrosophic supra pre-open set such that $cl_{N\tau_n^*}(A) \subseteq B \subseteq A$. Then $B \subseteq A \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(A)) \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(B))$. Therefore, B is N-neutrosophic supra pre-open.

Lemma 4.17. Let $A, B \in N(X)$ and A be a N-neutrosophic supra β-open set such that $cl_{N\tau_n^*}(A) \subseteq B \subseteq A$, then B is N-neutrosophic supra β-open.

Proof. Assume that A is a N-neutrosophic supra β-open set such that $cl_{N\tau_n^*}(A) \subseteq B \subseteq A$. Then $B \subseteq A \subseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(cl_{N\tau_n^*}(A))) \subseteq cl_{N\tau_n^*}(A)(int_{N\tau_n^*}(int_{N\tau_n^*}(B)))$. Therefore, B is N-neutrosophic supra β-open.

5. N-Neutrosophic Supra Topological Weak Open Sets

In this section, we introduce some weak closed sets in N-neutrosophic supra topological spaces and investigate the relationship between them.

Definition 5.1. A neutrosophic set A of a N-neutrosophic supra topological space $(X, N\,\tau_n{}^*)$ is called N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, Nneutrosophic supra pre-closed and N-neutrosophic supra β-closed) if the complement of A is N-neutrosophic supra α-open (resp. N-neutrosophic supra semi-open, N-neutrosophic supra pre-open and N-neutrosophic supra β-open). The set of all N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, N-neutrosophic supra pre-closed and N-neutrosophic supra β-closed) sets of $(X, N\,\tau_n{}^*)$ is denoted by $N\,\tau_n{}^*\alpha C(X)$ (resp. $N\,\tau_n{}^*SC(X), N\,\tau_n{}^*PC(X)$ and $N\,\tau_n{}^*\beta C(X)$.

Theorem 5.2. A neutrosophic set A of a N-neutrosophic supra topological space$(X, N\,\tau_n{}^*)$ is

N-neutrosophic supra α-closed if $cl_{N\tau_n^*}(int_{N\tau_n^*}(cl_{N\tau_n^*}(A))) \subseteq A$.

N-neutrosophic supra semi-closed if $int_{N\tau_n^*}(cl_{N\tau_n^*}(A)) \subseteq A$.

N-neutrosophic supra pre-closed if $cl_{N\tau_n^*}(int_{N\tau_n^*}(A)) \subseteq A$.

N-neutrosophic supra β-closed if $int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A))) \subseteq A$.

Proof. : Here we shall prove parts (i) only and the remaining parts similarly follows. Assume A is N-neutrosophic supra α-closed, then A^c is N-neutrosophic supra α-open and $A^c \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A^c)))$. Then $A \supseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(cl_{N\tau_n^*}(A)))$.

Theorem 5.3. Let A be a subset of N-neutrosophic supra topological space $(X, N\,\tau_n{}^*)$. Then

every N-neutrosophic supra closed set is N-neutrosophic supra α-closed.

every N-neutrosophic supra α-closed set is N-neutrosophic supra semi-closed.

every N-neutrosophic supra α-closed set is N-neutrosophic supra pre-closed.

every N-neutrosophic supra semi-closed set is N-neutrosophic supra β-closed.

every N-neutrosophic supra pre-closed set is N-neutrosophic supra β-closed.

Proof. The proof follows from theorem 4.2 and definition 5.1.

The converse of the above theorem need not be true as shown in the following examples.

Example 5.4. Consider example 4.3, the neutrosophic set $A = ((0.6, 0.4), (0.6, 0.4), (0.7, 0.6))$ is 2-neutrosophic supra α-closed but not 2-neutrosophic supra closed. Consider example 4.4, the neutrosophic set $B = ((0.6, 0.5), (0.6, 0.5), (0.5, 0.6))$ is 2-neutrosophic supra pre-closed, 2-neutrosophic supra β-closed, but not 2-neutrosophic supra α-closed and not 2-neutrosophic supra semi-closed. Consider example 4.5, the neutrosophic set $C = ((0.6, 0.6), (0.6, 0.6), (0.5, 0.6))$ is 3-neutrosophic supra semi-closed and 3-neutrosophic supra β-closed, but not 3-neutrosophic supra α-closed and not 3-neutrosophic supra pre-closed.

Theorem 5.5. A neutrosophic set A in a N-neutrosophic supra topological space$(X, N\,\tau_n{}^*)$ is N-neutrosophic supra α-closed set if and only if A is both N-neutrosophic supra semi-closed and N-neutrosophic supra pre-closed.

Proof. The proof follows directly from theorem 4.6 and definition 5.1.

Lemma 5.6. The arbitrary intersection of N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, N-neutrosophic supra pre-closed, N-neutrosophic supra β-closed) sets is N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, N-neutrosophic supra pre-closed, N-neutrosophic supra β-closed).

Proof. The proof follows directly from lemma 4.7 and definition 5.1.

Remark 5.7. Union of any two N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, N-neutrosophic supra pre-closed, N-neutrosophic supra β-closed) sets need not be a N-neutrosophic supra α-closed (resp. N-neutrosophic supra semi-closed, N-neutrosophic supra pre-closed, N-neutrosophic supra β-closed) set.

Example5.8. Consider example 4.9, the neutrosophic sets $A = ((0.7,\ 0.5), (0.7, 0.5), (0.6, 0.5))$ and $B = ((0.6, 0.7), (0.6, 0.7), (0.5, 0.6))$ are both 3-neutrosophic supra α-closed and 3-neutrosophic supra semi-closed, but $A \cup B$ is not 3-neutrosophic supra α-closed and not 3-neutrosophic supra semi-closed. Consider example 4.10, the neutrosophic sets $A = ((0.4, 1, 1), (0.6, 0.9, 1), (1, 1, 0))$ and $B = ((0.7, 0.3, 0), (0.3, 0.4, 0), (0, 0, 1))$ are 3-neutrosophic supra pre-closed and 3-neutrosophic supra β-closed, but $A \cup B$ is not 3-neutrosophic supra pre-closed and 3-neutrosophic supra β-closed.

Lemma5.9. Let $A, B \in N(X)$ and A be a N-neutrosophic supra α-closed set such that $A \subseteq B \subseteq cl_{N\tau_n^*}(A)$, then B is N-neutrosophic supra α-closed.

Proof. Assume that A is a N-neutrosophic supra α-closed set such that$A \subseteq B \subseteq cl_{N\tau_n^*}(A)$. Then $cl_{N\tau_n^*}\left(int_{N\tau_n^*}\left(cl_{N\tau_n^*}(B)\right)\right) \subseteq cl_{N\tau_n^*}\left(int_{N\tau_n^*}\left(cl_{N\tau_n^*}(A)\right)\right) \subseteq A \subseteq B$. Therefore, B is N-neutrosophic supra α-closed.

Lemma 5.10. Let $A, B \in N(X)$ and A be a N-neutrosophic supra semi-closed set such that $A \subseteq B \subseteq cl_{N\tau_n^*}(A)$, then B is N-neutrosophic supra semi-closed.

Proof. Assume that A is a N-neutrosophic supra semi-closed set such that $A \subseteq B \subseteq cl_{N\tau_n^*}(A)$. Then $cl_{N\tau_n^*}(B) \subseteq cl_{N\tau_n^*}(A)$ and $int_{N\tau_n^*}(cl_{N\tau_n^*}(B)) \subseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(A)) \subseteq A \subseteq B$. Therefore, B is N-neutrosophic supra semi-closed.

Lemma 5.11. Let $A, B \in N(X)$ and A be a N-neutrosophic supra pre-closed set such that $int_{N\tau_n^*}(A) \supseteq B \supseteq A$, then B is N-neutrosophic supra pre-closed.

Proof. Assume that A is a N-neutrosophic supra pre-closed set such that $int_{N\tau_n^*}(A) \supseteq B \supseteq A$. Then $B \supseteq A \supseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(A)) \supseteq cl_{N\tau_n^*}(int_{N\tau_n^*}(B))$. Therefore, B is N-neutrosophic supra pre-closed.

Lemma 5.12. Let $A, B \in N(X)$ and A be a N-neutrosophic supra β-closed set such that $int_{N\tau_n^*}(A) \supseteq B \supseteq A$, then B is N-neutrosophic supra β-closed.

Proof. Assume that A is a N-neutrosophic supra β-closed set such that $int_{N\tau_n^*}(A) \supseteq B \supseteq A$. Then $B \supseteq A \supseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(A))) \supseteq int_{N\tau_n^*}(cl_{N\tau_n^*}(int_{N\tau_n^*}(B)))$. Therefore, B is N-neutrosophic supra β-closed.

6.Comparison and Limitations

S.No	Neutrosophic supra topological spaces	N-Neutrosophic supra topological spaces
1	A sub collection τ_n^* of neutrosophic sets on a non empty set X is said to be a neutrosophic supra topology on X if the sets $\emptyset, X \in \tau_n^*$ and $\bigcup_{i=1}^{\infty} A_i \in \tau_n^*$, for $\{A_i\}_{i=1}^{\infty} \in \tau_n^*$. A non empty set X together with the collection τ_n^* is called neutrosophic supra topological	Let X be a non empty set, $\tau_{n_1}{}^*$, $\tau_{n_2}{}^*, \ldots, \tau_{n_N}{}^*$ be N-arbitrary neutrosophic supra topologies defined on X. Then the collection

	space on X (for short nsts) denoted by the ordered pair(X, τ_n^*). The members of τ_n^* are known as neutrosophic supra open sets.	$N\tau_n{}^* = \{S \subseteq X : S = \cup_{i=1}^{N} A_i, A_i \in \tau_{n_i}{}^*\}$ is said to be a N-neutrosophic supra topology if it satisfies the following axioms: (i) $X, \emptyset \in N\tau_n{}^*$. (ii) $\cup_{i=1}^{\infty} S_i \in N\tau_n{}^* for\ all\ S_i \in N\tau_n{}^*$ The *N*-neutrosophic supra topological space is the non empty set X together with the collection $N\tau_n{}^*$, denoted by $(X, N\tau_n{}^*)$. The elements of $N\tau_n{}^*$are known as $N\tau_n{}^*$-open sets on X.
2	It is a generalization of intuitionistic supra topological spaces.	It is an extension of neutrosophic supra topological spaces.
3	Every neutrosophic topology is neutrosophic supra topology.	Every *N*-neutrosophic topology is *N*-neutrosophic supra topology.
4	It is a particular case of *N*-neutrosophic supra topology, that is if *N*=1, then we have neutrosophic supra topology.	It is a general form of neutrosophic supra topology.
5	Union of two neutrosophic supra topologies is again a neutrosophic supra topology. Intersection of two neutrosophic supra topologies is again a neutrosophic supra topology. These two properties may not true in neutrosophic topology.	Union of two *N*-neutrosophic supra topologies is again an *N*-neutrosophic supra topology. Intersection of two *N*-neutrosophic supra topologies is again an *N*-neutrosophic supra topology. These two properties may not true in *N*-neutrosophic topology.
6	The collection of neutrosophic supra α -open sets need not form a neutrosophic topology, but it is a neutrosophic supra topology.	The collection of *N*-neutrosophic supra α-open sets need not form an *N*-neutrosophic topology, but this collection is an *N*-neutrosophic supra topology.

7. Conclusions and Future Work

Neutrosophic topological space is a generalization intuitionistic fuzzy topological space to deal the concept of vagueness. This paper has developed N -neutrosophic supra topological spaces and its closure operator. Moreover, we have defined some weak form of open sets in N-neutrosophic supra topological spaces and established their relations. Apart from this, we have observed that the collection of weak open sets in N-neutrosophic supra topological spaces need not form an N-neutrosophic topology, but this forms an N-neutrosophic supra topology. We can be developed and implement these N-neutrosophic supra topological open sets to other research areas of topology such as Nano topology, Rough topology, Digital topology and so on.

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References

1. Zadeh, L.A. Probability measures of fuzzy events, Journal of Mathematical Analysis and Applications, **1968**, Volume 23, pp. 421 – 427.
2. Adlassnig, K.P. Fuzzy set theory in medical diagnosis, IEEE Transactions on Systems, Man, and Cybernetics, **1986**, Volume 16 (2), pp. 260 – 265.
3. Sugeno, M. An Introductory survey of fuzzy control, Information sciences,**1985**, Volume 36, pp. 59 – 83.
4. Innocent, P.R.; John, R.I. Computer aided fuzzy medical diagnosis, Information Sciences, **2004**, Volume 162, pp. 81 – 104.
5. Roos, T.J. Fuzzy Logic with Engineering Applications, McGraw Hill P.C., New York, **1994**.
6. Chang, C.L. Fuzzy topological spaces, J. Math. Anal. and Appl., **1968**, Volume 24, pp. 182 – 190.
7. Lowen, R. Fuzzy topological spaces and fuzzy compactness, J. Math. Anal. Appl., **1976**, Volume 56, pp. 621 – 633.
8. Mashhour, A.S.; Allam, A.A.; Mohmoud, F.S.; Khedr, F.H. On supra topological spaces, Indian J.Pure and Appl.Math.,**1983**, Volume 14(4), pp. 502 – 510.
9. Abd El-monsef, M.E.; Ramadan, A.E. On fuzzy supra topological spaces, Indian J. Pure and Appl.Math., **1987**, Volume 18(4), pp. 322 – 329.
10. Atanassov, K. Intuitionistic fuzzy sets, Fuzzy Sets and Systems, **1986**, Volume 20, pp. 87 – 96.
11. De, S.K.; Biswas, A.; Roy, R. An application of intuitionistic fuzzy sets in medical diagnosis, Fuzzy Sets and System, **2001**, Volume 117(2), pp. 209–213.
12. Biswas, P.; Pramanik, S.; Giri, B.C. A study on information technology professionals' health problem based on intuitionistic fuzzy cosine similarity measure, Swiss Journal of Statistical and Applied Mathematics, **2014**, Volume 2(1), pp. 44–50.
13. Khatibi, V.; Montazer, G.A. Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition, Artificial Intelligence in Medicine, **2009**, Volume 47(1), pp. 43–52.
14. Hung, K.C.; Tuan, H.W. Medical diagnosis based on intuitionistic fuzzy sets revisited, Journal of Interdisciplinary Mathematics, **2013**, Volume 16(6), pp. 385 – 395.
15. Szmidt, E.; Kacprzyk, J. Intuitionistic fuzzy sets in some medical applications, In International Conference on Computa-tional Intelligence, Springer, Berlin, Heidelberg, **2001**, pp. 148 – 151.
16. De, S.K.; Biswas, A.; Roy, R. An application of intuitionistic fuzzy sets in medical diagnosis, Fuzzy Sets and System, **2001**, Volume 117(2), pp. 209 – 213.
17. Khatibi, V.; Montazer, G.A. Intuitionistic fuzzy set vs. fuzzy set application in medical pattern recognition, Artificial Intelligence in Medicine, **2009**, Volume 47(1), pp. 43–52.
18. Dogan Coker. An introduction to intuitionistic fuzzy topological spaces, Fuzzy sets and system, **1997**, Volume 88(1), pp. 81 – 89.
19. Reza Saadati.; Jin Han Park. On the intuitionistic fuzzy topological space, Chaos, Solitons and Fractals, **2006**, Volume 27(2), pp. 331 – 344.

20. Turnal, N. An over view of Intuitionistic fuzzy Supra topological Spaces, Hacettepe Journal of Mathematics and statistics, **2003**, Volume 32, pp. 17-26.
21. Smarandache, F. A unifying field of logics. Neutrosophy: neutrosophic probability, set and logic, American Research Press, Rehoboth, **1998**.
22. Smarandache, F.; Pramanik, S. New trends in neutrosophic theory and applications, Brussels, Belgium, EU: Pons Editions, **2016**.
23. Smarandache, F. Neutrosophic set, a generalization of the intuitionistic fuzzy sets, Int. J. Pure. Appl. Math., **2005**, Volume 24, pp. 287 – 297.
24. Wang, H.; Smarandache, F.; Zhang, Y.; Sunderraman, R. Single valued neutrosophic sets, Multi-space and Multi-structure, **2010**, Volume 4, pp. 410–413.
25. Ye, J. Neutrosophic tangent similarity measure and its application to multiple attribute decision making, Neutrosophic Sets and Systems, **2015**, Volume 9, pp. 85–92.
26. Ye, J.; Ye, S. Medical diagnosis using distance-based similarity measures of single valued neutrosophic multisets, Neutrosophic Sets and Systems, **2015**, Volume 7, pp. 47–54.
27. Broumi, S.; Smarandache, F. Several similarity measures of neutrosophic sets, Neutrosophic Sets and Systems, **2013**, Volume 1, pp. 54–62.
28. Pramanik, S.; Mondal, K. Cotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis, Journal of New Theory, **2015**, Volume 4, pp. 90–102.
29. Abdel-Basset, M.; Mohamed, R.; Zaied, A. E. N. H.; Smarandache, F. A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics, Symmetry, **2019**, Volume 11(7), 903.
30. Abdel-Baset, M.; Chang, V.; Gamal, A. Evaluation of the green supply chain management practices: A novel neutrosophic approach, Computers in Industry, **2019**, Volume 108, 210-220.
31. Abdel-Basset, M.; Saleh, M.; Gamal, A.; Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number, Applied Soft Computing, **2019**, Volume 77, 438-452.
32. Abdel-Basset, M.; Manogaran, G.; Gamal, A.; Smarandache, F. A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection, Journal of medical systems, **2019**, Volume 43(2), 1-13.
33. Abdel-Basset, M.; Atef, A.; Smarandache, F. A hybrid Neutrosophic multiple criteria group decision making approach for project selection, Cognitive Systems Research, **2019**, Volume 57, 216-227.
34. Abdel-Basset, M.; Mumtaz, A.; Atef, A. Resource levelling problem in construction projects under neutrosophic environment, The Journal of Supercomputing, **2019**: 1-25.
35. Karaaslan, F. Gaussian single-valued neutrosophic numbers and its application in multi-attribute decision making, Neutrosophic Sets and Systems, **2018**, Volume 22, pp.101–117.
36. Giri, B. C.; Molla, M. U.; Biswas, P. TOPSIS Method for MADM based on Interval Trapezoidal Neutrosophic Number, Neutrosophic Sets and Systems, **2018**, Volume 22, pp. 151-167.
37. Aal, S. I. A.; Ellatif, A.M.A.A.; Hassan, M.M. Two Ranking Methods of Single Valued Triangular Neutrosophic Numbers to Rank and Evaluate Information Systems Quality, Neutrosophic Sets and Systems, **2018**, Volume 19, pp. 132-141.
38. Salama, A. A.; Alblowi, S.A. Neutrosophic Set and Neutrosophic Topological Spaces, IOSR Journal of Mathematics, 2012, Volume 3(4), pp. 31–35.
39. Salama, A.A.; Smarandach, F.; Valeri Kroumov. Neutrosophic Crisp Sets and Neutrosophic Crisp Topological Spaces, Neutrosophic Sets and Systems, **2014**, Volume 2, pp. 25–30.
40. Levine, N. Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, **1963**, Volume 70, pp. 36 – 41.
41. Njastad, O. On some classes of nearly open sets, Pacific J. Math., **1965**, Volume 15, pp. 961 – 970.
42. Mashhour, A.S.; Abd El-Monsef, M.E.; El-Deeb, S.N. On pre continuous and weak pre continuous mappings, Proc. Math. Phys. Soc., Egypt, **1982**, Volume 53, pp. 47 – 53.
43. Andrijevic, D. Semi-preo pen sets, Mat. Vesnik, **1986**, Volume 38(1), pp. 24 – 32.
44. Mashhour, A.S.; Allam, A.A.; Mohmoud, F.S.; Khedr, F.H. On supra topological spaces, Indian J. Pure and Appl.Math., 1983, Volume 14(4), pp. 502–510.
45. Devi, R.; Sampathkumar, S.; Caldas, M. On supra α-open sets and supr α-continuous functions. General Mathematics,16 (2), 77-84.

46. Sayed, O. R. Supra pre open sets and supra pre-continuity on topological spaces, VasileAlecsandri University of Bacau Faculty of Sciences, Scientific Studies and Research Series Mathematics and Informatics, **2010**, Volume 20(2), pp. 79-88.
47. Saeid Jafari.; Sanjay Tahiliani. Supra β-open sets and supra β-continuity on topological spaces, Annales Univ. SCI. Budapest., **2013**, Volume 56, pp. 1–9.
48. LellisThivagar, M.; Ramesh, V.; Arockia Dasan, M. On new structure of N-topology, Cogent Mathematics, **2016**, Volume 3, pages- 10.
49. LellisThivagar, M.; Arockia Dasan, M. New Topologies via Weak N-Topological Open Sets and Mappings, Journal of New Theory, **2019**, 29, pp. 49-57.
50. Jayaparthasarathy, G.; Little Flower, V.F.; Arockia Dasan, M. Neutrosophic Supra Topological Applications in Data Mining Process, Neutrosophic Sets and System, **2019**, Volume 27, pp. 80 – 97.

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A Novel Methodology for Assessment of Hospital Service according to BWM, MABAC, PROMETHEE II

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Abstract: In this study, a proposed methodology of Best Worst Method (BWM), Multi-Attributive Border Approximation Area Comparison (MABAC), and Preference Ranking Organization Method for Enrichment Evaluations (PROMETHEE II) are suggested to achieve a methodical and systematic procedure to assess the hospital serving under the canopy of neutrosophic theory. The assessing of hospital serving challenges of ambiguity, vagueness, inconsistent information, qualitative information, imprecision, subjectivity and uncertainty are handled with linguistic variables parameterized by bipolar neutrosophic scale. Hence, the hybrid methodology of Bipolar Neutrosophic Linguistic Numbers (BNLNs) of BWM is suggested to calculate the significance weights of assessment criteria, and MABAC as an accurate method is presented to assess hospital serving. In addition to consider the qualitative criteria compensation in hospital service quality in MABAC in order to overcome drawbacks PROMETHEE II of non-compensation to reinforce the serving effectiveness arrangements of the possible alternatives. An experiential case including 9 assessment criteria, 2 public and 3 private hospitals in Sharqiyah EGYPT assessed by 3 evaluators from several scopes of medical industry to prove validity of the suggested methodology. The case study shows that the service effectiveness of private hospitals is superior to public hospitals, since the public infirmaries are scarcely reinforced by governmental institutions.

Keywords: Hospital service; Neutrosophic Sets; Bipolar; BWM; MABAC; PROMETHEE II

1. Introduction

Nowadays, the achievements of best service are regarded as the key success for organizations. The major aim to estimate service fitness is to measure service execution, detect service trouble, spun service allocation, and deliver the best service for users[1]. Several studies have been performed to gauge service efficiency of different scopes. e.g. web [2], airport [3, 4], transportation [5], bank [6] and healthcare [7]. In healthcare, control and service efficiency rating are very important for hospitals and medical centers fields. There are more than 50 generic and private hospitals in Sharqiyah EGYPT with tackled unceasing competitive pressure. The medical providers claim that the ability to deliver an efficient healthcare service to patients grantee the future achievement in healthcare[8].

For patients, who looking for healthcare services there are two main anxieties superiority and efficiency of the hospitals and medical centers. Hospitals and medical centers have to augment their healthcare value and effectiveness to help patients to achieve the most desirable service [9]. The managements of hospital try to fulfill the requirements of patients [10]. Such that, the hospital and medical centers are the service that directly connect, interact, and supply people with medical facilities [11]. The main goal for hospitals includes hold and engage more patients as possible by achieving their latent requirements and desires [11]. The main challenge for healthcare in hospitals is the service value given for patients [11]The growing of service value includes assessment the value of connecting with the doctors, employers, mangers, physicians, surgeons and nurses with patients in an efficient manner [12].

The hospital service value can be described according to various criterions either qualitative or quantitative. Hence, the hospital services are a problem of multi-criteria decision making (MCDM) with multiple criterions, alternatives, and decision makers. Researches illustrated various methodologies evaluate the service value [13-15] . However, the environment of hospital services is surrounded with complexity conditions of ambiguity, vagueness, inconsistent information, qualitative information, imprecision, subjectivity and uncertainty. Hence, the study proposed a hybrid methodology of BWM, MABAC, and PROMETHEE II as an effective tool in multi-criteria decision making based on BNLNs to make assessment of hospital services. The traditional BWM is extended with BNLNs terms to facilitate the description of qualitative criterions and alternatives [16]. The MABAC is suggested as an influential methodology to handle the complex and uncertain decision making problems [17]. The PROMETTEE is a methodology depends on non-compensation of criteria. The MABAC is combined with PROMETTEE to overcome the limitations of non-compensation and challenges of hospital service problems and recommend the final rankings to assess service value in Sharqiyah EGYPT.

The article is planned as follows: Section 2 presents the literature review. Section 3 presents the hybrid methodology of decision making for assessing of hospital serving by the use of neutrosophic theory by the integration of the BWM, MABAC and PROMETHEE II. Section 4 presents a case study to validate the proposed model and achieve to a final efficient rank. Section 5 summarizes the aim of the proposed study and the future work.

2. Related Studies

In this section, a review of literature will be displayed about the environment assessment of hospital service quality. The SERVQUAL is a well-defined methodology used to evaluate service effectiveness. The SERVQUAL has been applied in several aspects which comprise education [18], retail [19-21] and healthcare [22]. The MABAC been extended under various fuzzy environments [23]. E.g. combined interval fuzzy rough sets with the MABAC method to rank the firefighting chopper [24]. [25] presented rough numbers with the MABAC for sustainable system evaluation. Hence, to beat limitations of MABAC method the concept of PROMETHE II has been presented. Many of studies have been enhanced the PROMETHEE II method to solve decision making issues under ambiguous contexts [26]. In [27], presented the PROMETHEE II method under hesitant fuzzy linguistic circumstances to choose green logistic suppliers. Due to conditions of uncertainty and incomplete information, a novel PROMETHEE II method is proposed to solve decision making issues under probability multi-valued neutrosophic situation [28]. Usually, it is hard for DMs to straight allocate the weight values of assessment criteria in advance. [16] presented a novel weights calculation method, the BWM approach. In [29], combined the BWM method with grey system to calculate the weights of criteria. In [30], the BWM method enhanced with applying hesitant fuzzy numbers to explain criteria relative significance grades. In real life situations decisions, alternatives, criterions are taken in conditions of ambiguity, vagueness, inconsistent information, qualitative information, imprecision, subjectivity and uncertainty. In [31-43], proposes LNNs based on descriptive expressions to describe the judgments of decision makers, criterions, and alternatives. We propose to build a hybrid methodology of BNLNs based on BWM, MABAC, and PROMETHEE II.

3. Methodology

In this study, a hybrid methodology for assessment of hospital service quality is based on BNLNs.

The traditional BWM method is extended with descriptive BNLNs to prioritize the problem's criterions. The MABAC is proposed to deal with the complexity and uncertainty hospital service quality. The PROMETHEE II is used to solve the non-compensation problem of criteria. Hence, a hybrid methodology is built by using BWM, MABAC and PROMETHEE II as mentioned in **Figure 1**.

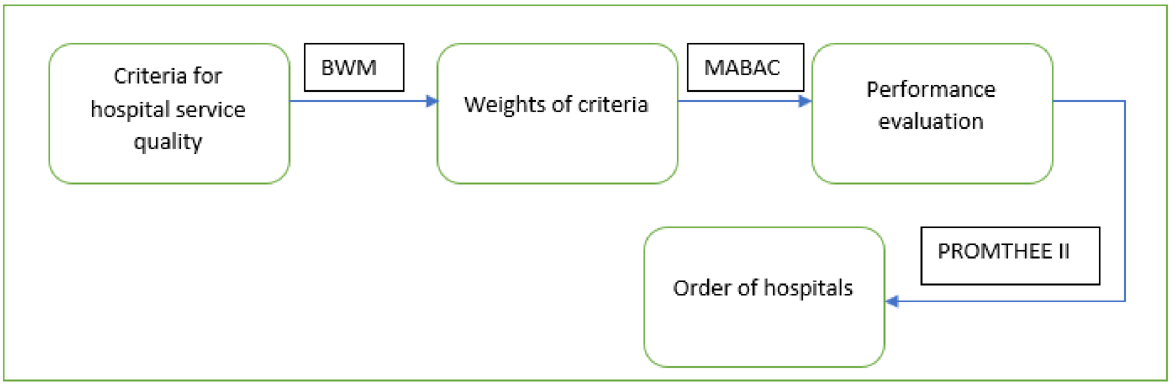

Figure.1. The overall conceptualization of the proposed approach

In this section, a hybrid decision making framework is designed built on the integration of extended BWM, MABAC and PROMETHEE II methodologies to determine the desirable substitute hospital that achieves the requirements and the expectation of patients by evaluating a group of candidate hospitals. The steps of the proposed bipolar neutrosophic with BWM, MABAC and PROMETHEE II are modeled in **Figure 2** and mentioned in details as following

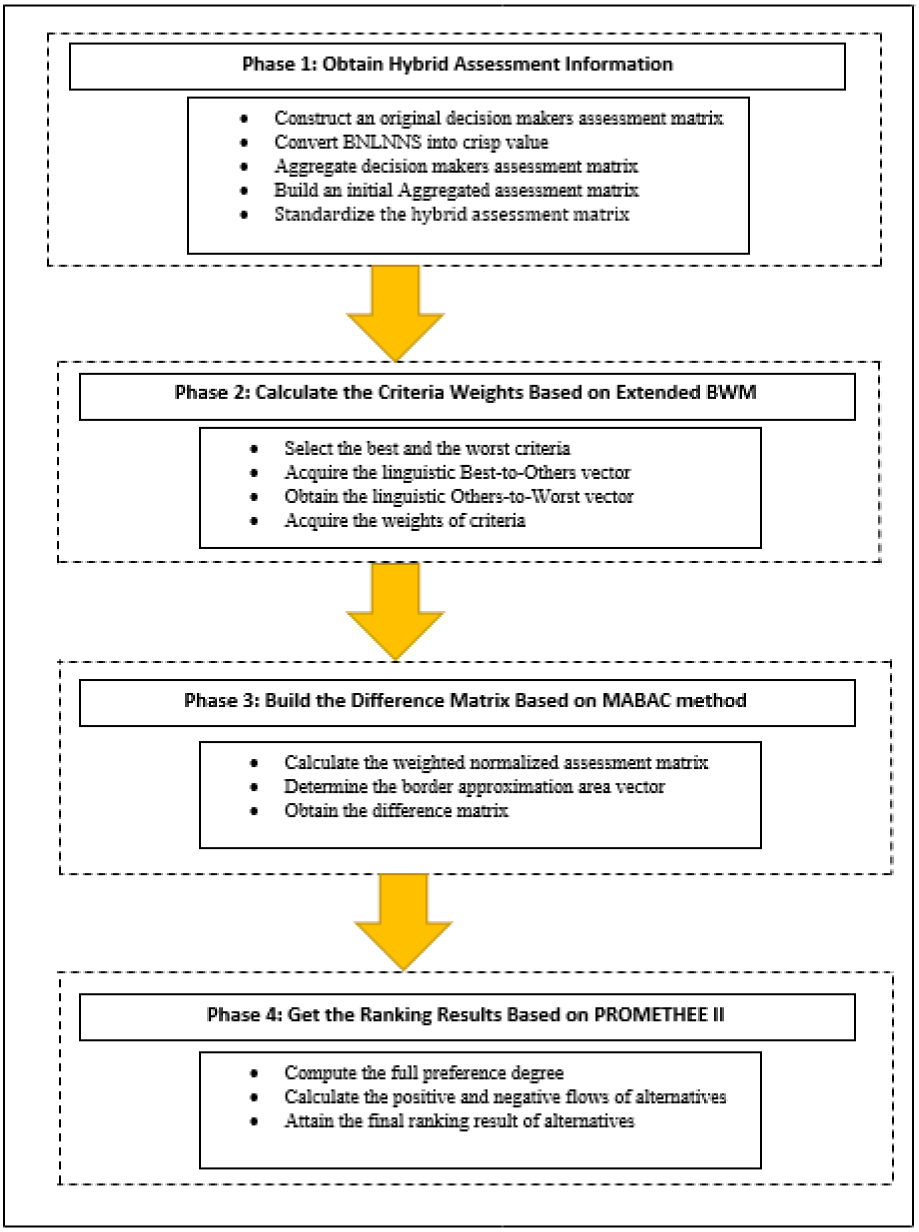

Figure 2. Framework of hybrid decision making

Phase 1: Obtain Hybrid Assessment Information

The goal from this phase is to obtain the hybrid assessment information:

Step 1: Construct an original decision makers assessment matrix

The linguistic term (LTS) provided by DMs using descriptive expressions such as: (Extremely important, Very important, Midst important, Perfect, Approximately similar, Poor, Midst poor, Very poor, Extremely poor. The BNLNS is an extension of fuzzy set, bipolar fuzzy set, intuitionistic fuzzy set, LTS, and neutrosophic sets is introduced by [35]. Bipolar Neutrosophic is $[T^{+}, I^{+}, F^{+}, T^{-}, I^{-}, F^{-}]$ where T^{+}, I^{+}, F^{+} range in [1,0] and T^{-}, I^{-}, F^{-} range in [-1,0]. T^{+}, I^{+}, F^{+} is the positive membership degree indicating the truth membership, indeterminacy membership and falsity membership and T^{-}, I^{-}, F^{-} is the negative membership degree indicates the truth membership, indeterminacy membership and falsity membership. E.g. [0.3, 0.2, 0.6, -0.2, -0.1, -0.5] is a bipolar neutrosophic number.

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For BNLNS qualitive criteria values can be calculated by decision makers under a predefined the LTS. Then, an initial hybrid decision making matrix is built as [32]

$$G^D = \begin{array}{c} \\ H_1 \\ \vdots \\ H_o \end{array} \begin{array}{c} \begin{array}{ccc} C1 & \dots & C_p \end{array} \\ \begin{bmatrix} b_{11}^D & \cdots & b_{1p}^D \\ \vdots & \ddots & \vdots \\ b_{o1}^D & \cdots & b_{op}^D \end{bmatrix} \end{array} \tag{1}$$

Where b_{sr}^D is a BNLNS, representing the assessment result of alternative $H_s(s = 1,2, \dots . o)$ with respect to criterion $C_r(r = 1,2, \dots . p)$ and $D = 1,2,3$ represent number of decision maker.

Step 2: Convert BNLNs into crisp value using score function mentioned as [36]

$$s(b_{op}) = \left(\frac{1}{6}\right) * (T^+ + 1 - I^+ + 1 - F^+ + 1 + T^- - I^- - F^-) \tag{2}$$

Step 3: Aggregate decision makers assessment matrix [34]

$$b_{sr} = \frac{\sum_{D=1}^{D}(b_{op}^D)}{D} \tag{3}$$

Where T_{sr}^{+D} is a truth degree in positive membership for decision makers (D), I_{sr}^{+D} is a indeterminacy degree and F_{sr}^{+D} the falsity degree. T_{sr}^{-D} the truth degree in negative membership for decision maker (D), I_{sr}^{-D} the indeterminacy degree and F_{sr}^{-D} the falsity degree.

Step 4: Build an initial aggregated assessment matrix

$$G = \begin{array}{c} \\ H_1 \\ \vdots \\ H_o \end{array} \begin{array}{c} \begin{array}{ccc} C1 & \dots & C_p \end{array} \\ \begin{bmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \ddots & \vdots \\ b_{s1} & \cdots & b_{sr} \end{bmatrix} \end{array} \tag{4}$$

Step 5: Standardize the hybrid assessment matrix.

Normalize the positive and negative criteria of the decision matrix as follows:

For crisp value, the assessment data $b_{sr}(s = 1,2, \dots \dots . o,\ r = 1,2, \dots \dots . p)$ can be normalized with [17]:

$$N_{sr} = \begin{cases} \dfrac{b_{sr} - \min\limits_{r}(b_{sr})}{\max\limits_{r}(b_{sr}) - \min\limits_{r}(b_{sr})}, & For\ benefit\ criteria \\ \dfrac{\max\limits_{r}(b_{sr}) - b_{sr}}{\max\limits_{r}(b_{sr}) - \min\limits_{r}(b_{sr})}, & For\ cost\ criteria \end{cases} \tag{5}$$

Then, a normalized hybrid assessment matrix is formed as

$$N = \begin{array}{c} \\ H_1 \\ \vdots \\ H_o \end{array} \begin{array}{c} \begin{array}{ccc} C1 & \dots & C_p \end{array} \\ \begin{bmatrix} N_{11} & \cdots & N_{1p} \\ \vdots & \ddots & \vdots \\ N_{o1} & \cdots & N_{op} \end{bmatrix} \end{array} \tag{6}$$

Where N_{sr} shows the normalized value of the decision matrix of S^{th} alternative in R^{th} criteria

Phase 2: Calculate the Criteria Weights Based on Extended BWM

In this study, the BWM is extended with LTS to obtain the weights of criteria given linguistic expressions.

Step 6: Select the best and the worst criteria

When calculated the assessment criteria $\{C1 \;\; \dots \;\; C_p\}$, decision makers need to choose the best (namely, the most significant) criterion, denoted as C_B. Meanwhile the worst (namely, the least significant) criterion should be selected and represented as C_W.

Step 7: Acquire the linguistic Best-to-Others vector

Make pairwise comparison between the most important criterion C_B and the other criteria, then a linguistic Best to-Others vector is obtained with [16]:

$$LC_B = \left(C_{B1}, C_{B2} \dots\dots\dots\dots \; C_{Bp}\right) \tag{7}$$

Where C_{Br} is a linguistic term within a certain LTS, representing the preference degree of the best criterion C_B over criterion $c_r(r = 1,2, \dots\dots p)$ In specific, $C_{BB} = 1$.

Step 8: Obtain the linguistic Others-to-Worst vector.

Similarly, make pairwise comparison between the other criteria and the worst criterion C_W, then a linguistic Others-to-Worst vector is obtained with [16]:

$$LC_W = \left(C_{1W}, C_{2W} \dots\dots\dots\dots \; C_{pW}\right) \tag{8}$$

Where C_{rW} is a linguistic term within a certain LTS, representing the preference degree of criterion $c_r(r = 1,2, \dots\dots p)$ over the worst criterion C_W in precise, $C_{WW} = 1$.

Step 9: Acquire the weights of criteria

The goal from this step to obtain optimal weights of criteria so that the BWM is extended with crisp number for nonlinear programming model as mentioned [16]:

min ε

S.t.

$$\begin{cases} \left|\frac{w_B}{w_r} - C_{Br}\right| \le \varepsilon \text{ For all r} \\ \left|\frac{w_r}{w_W} - C_{rW}\right| \le \varepsilon \text{ For all r} \end{cases} \tag{9}$$

Where w_r is the weight of criterion C_r , w_B is the weight of the best criteria C_B and, w_W is the weight of the worst criteria C_W. when solving model (9) the weight of w_r and optimal consistency index ε can be computed.

Phase 3: Build the Difference Matrix Based on MABAC method

Build difference matrix built on the idea of MABAC method

Step 10: Calculate the weighted normalized assessment matrix

Given the normalized values of assessment and the weights of criteria. The weighted normalized values of each criterion are got as follow [17]:

$$\widehat{N}_{sr} = (w_r + N_{sr} * w_r, \quad s = 1,2, \dots.o, r = 1,2, \dots.p \tag{10}$$

Where w_r is a weight of criteria r and N_{sr} is a normalized value of s and r.

Step 11: Determine the border approximation area vector

The border approximation area vector X is computed as [17]:

$$X_r = \frac{1}{p}\sum_{s=1}^{p} \widehat{N}_{sr} \quad s = 1,2, \ldots . o, r = 1,2, \ldots . p \tag{11}$$

By calculating the values of the border approximation area matrix, a $o \times 1$ matrix is obtained.

Step 12: Obtain the difference matrix

The difference degree between the border approximation area X_r and each element $\widehat{N}_{sr}$ in the weighted normalized assessment matrix can be calculated with [17]:

$$S_{sr} = \widehat{N}_{sr} - X_r \tag{12}$$

Hence, the difference matrix $S = (S_{sr})_{oxp}$ is accomplished.

Phase 4: Get the Ranking Results Based on PROMETHEE II

Attain the rank of hospitals based on PROMETHEE II method

Step 13: Compute the full preference degree

Compute the alternative difference of s^{th} alternative with respect to other alternative. the preference function is used in this study as follows [37]:

$$P_r(H_s, H_t) = \begin{cases} 0 & \text{if } S_{sr} - S_{tr} \le 0 \\ S_{sr} - S_{tr} & \text{if } S_{sr} - S_{tr} > 0 \end{cases} \quad s, t = 1,2, \ldots . . o \tag{13}$$

Then the aggregated preference function can be computed as:

$$P(H_s, H_t) = \sum_p^o w_r * P_r(H_s, H_t) / \sum_p^o w_r \tag{14}$$

Step 14: Calculate the positive and negative flows of alternatives

The positive fl0w (namely, the outgoing flow) $\psi^+(H_i)$ [37]:

$$\psi^+(H_i) = \frac{1}{n-1}\sum_{t=1, t \ne s}^{o} P(H_s, H_t) \quad s = 1,2, \ldots \ldots . . o \tag{15}$$

The negative fl0w (namely, the entering flow) $\psi^-(H_i)$ [37]:

$$\psi^-(H_i) = \frac{1}{n-1}\sum_{t=1, t \ne s}^{o} P(H_t, H_s) \quad s = 1,2, \ldots \ldots . . o \tag{16}$$

Step 15: Attain the final ranking result of alternatives

The net outranking $\psi(H_i)$ of alternative H_i [37]:

$$\psi(H_i) = \psi^+(H_i) - \psi^-(H_i) \quad s = 1,2, \ldots . o \tag{17}$$

Hence, the final ranking order can be achieved according to the net outranking flow value of each alternative. The larger the value of $\psi(H_i)$, the better the alternative H_i.

4. Case Study

In this section, a case of hospital service quality for 2 public and 3 private hospitals in Sharqiyah EGYPT is presented to verify the applicability for the method. The hybrid methodology aims to provide best medical and health-care serving performance for patients. Two governmental hospitals: Zagazig University Hospital (ZUH, H_1) and MABARRA Hospital (MH, H_2), and three private hospitals - El-Salam Hospital (ESH, H_3), GAWISH hospital (GH, H_4) and EL-HARAMAIN hospital (EHH, H_5). The proposed hospitals are selected to be assessed by 3 evaluators with regard to 9 assessing criteria. The 3 evaluators notice that the actual state of affairs, meeting patients people, doctors, and nurses of these 5 hospitals with regard to 15 criteria to measure the service performance. The suggested approach integrates the BWM, MABAC and PROMETHEE II with BNLNs in order to make assessing for hospital service

The main and sub-criteria of hospital service quality is decided by the aid of consultation involving healthcare managers, experts and academicians. Therefore, the study considers the four main criteria and 9 sub-criteria as shown in **Figure 3**, and described in **Table 1.**

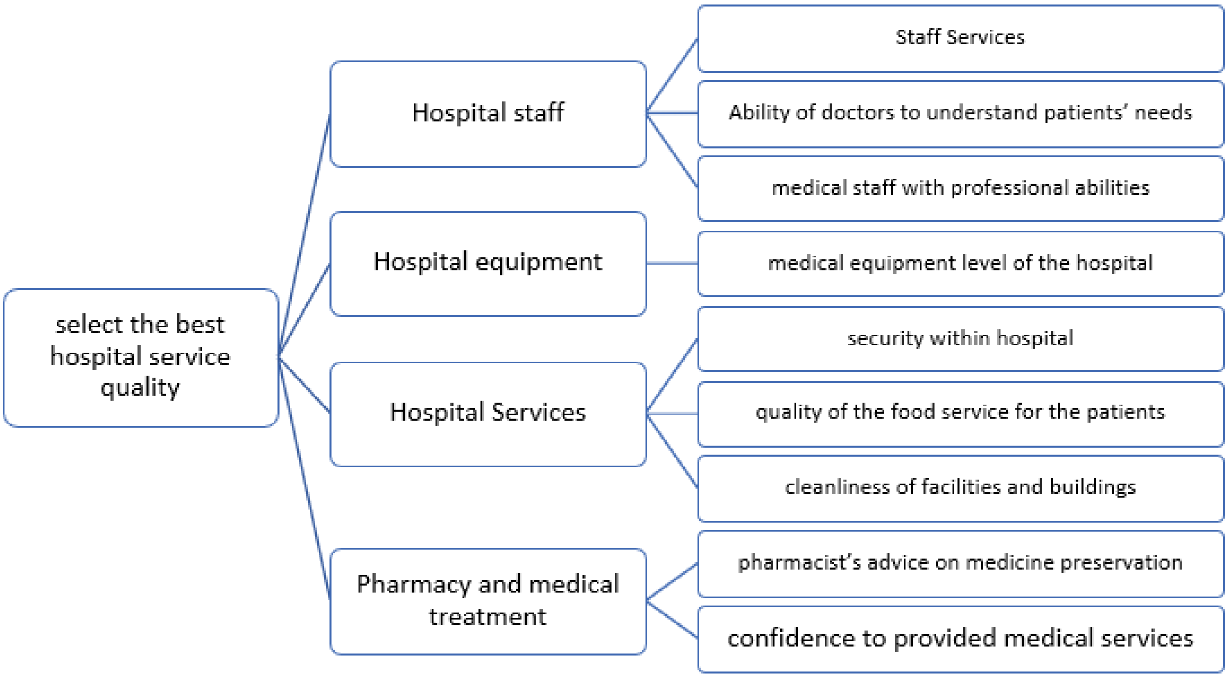

Figure. 3. The structure for assessing the hospitals service quality.

Table 1. hospital of service quality criteria

Factor	**Criteria**	**Description**
Hospital staff	C_1	Staff Services
	C_2	Ability of doctors to understand patients' needs
	C_3	Medical staff with professional abilities
Hospital equipment	C_4	Medical equipment level of the hospital
Hospital services	C_5	Security within hospital
	C_6	Quality of the food service for the patients
	C_7	Cleanliness of facilities and buildings
pharmacy and medical treatment	C_8	Pharmacist's advice on medicine preservation
	C_9	Confidence to provided medical services

In phase 1. Experts make assessment with respect to the evaluation criteria in table 1. As criteria C_1 to C_9 are qualitative factors, evaluation information of these subjective criteria is by means of BNLNs. Even though all the 9 criteria belong to benefit type, their scopes are different.

Step 1: Construct an original decision makers assessment matrix

calculate the suitable LTS for weights of criteria and alternatives with respect to every criterion. Each LTS is extended by bipolar neutrosophic sets to be BNLNs as mentioned in table 2. The BNLNs is described as follows [36]: Extremely important = [0.9,0.1,0.0,0.0, -0.8, -0.9] Where the first three numbers present the positive membership degree. $(T^+(x), I^+(x), F^+(x))$ 0.9, 0.1 and 0.1, such that $T^+(x)$ the truth degree in positive membership. $I^+(x)$ the indeterminacy degree and $F^+(x)$ the falsity degree. The last three numbers present the negative membership degree. $(T^-(x), I^-(x), F^-(x))$ 0.0, -0.8, and -0.9, $T^-(x)$ the truth degree in negative membership, such that $I^-(x)$ the indeterminacy degree and $F^-(x)$ the falsity degree. Table 1, table 2, and table 3 represent the assessments for the three evaluators by the use of Eq. (1).

Step 2: Convert BNLNs into crisp value using score function

Convert BNLNs to crisp value in table 2 by using score function in Eq. (2).

Step 3: Aggregate decision makers assessment matrix using Eq. (3).

Step 4: Build an initial Aggregated assessment matrix using Eq. (4), and shown in table 6.

Step 5: Standardize the hybrid assessment matrix

Normalized the aggregated decision matrix, given the positive or negative type of the criteria using Eq. (5), the normalized values of the aggregated decision matrix using Eq. (6) are shown as in Table 11.

Table 2. Bipolar neutrosophic numbers scale

LTS	Bipolar neutrosophic numbers scale $[T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)]$	Crisp value
Extremely important	[0.9,0.1,0.0,0.0, -0.8, -0.9]	0.92
Very important	[1.0,0.0,0.1, -0.3, -0.8, -0.9]	0.73
Midst important	[0.8,0.5,0.6, -0.1, -0.8, -0.9]	0.72
Perfect	[0.7,0.6,0.5, -0.2, -0.5, -0.6]	0.58
Approximately similar	[0.5,0.2,0.3, -0.3, -0.1, -0.3]	0.52
Poor	[0.2,0.3,0.4, -0.8, -0.6, -0.4]	0.45
Midst poor	[0.4,0.4,0.3, -0.5, -0.2, -0.1]	0.42
Very poor	[0.3,0.1,0.9, -0.4, -0.2, -0.1]	0.36
Extremely poor	[0.1,0.9,0.8, -0.9, -0.2, -0.1]	0.13

In Phase 2. The goal from this phase determine the weights of criteria based on evaluation of decision maker. Use BWM to calculate weights of criteria.

Step 6: Select the best and the worst criteria

At the beginning C_3 is the best criteria and the C_1 is the worst criteria.

Step 7: Acquire the linguistic Best-to-Others vector

Construct pairwise comparison vector for the best criteria using Eq. (7) in table 7.

Step 8: Obtain the linguistic Others-to-Worst vector

Construct pairwise comparison vector for the worst criteria using Eq. (8) in table 8.

Step 9: Acquire the weights of criteria

By applying best to others and worst to others using Eq. (9) the weights are computed in table 10. Figure 4 shows priority of criteria. Compute consistency ratio: $\varepsilon = 0.05$. For the consistency ratio, as $C_{BW} = 0.7$ the consistency index for this problem is 3.73 from table 9 and the consistency ratio 0.05/3.73 = 0.013, which indicates a desirable consistency.

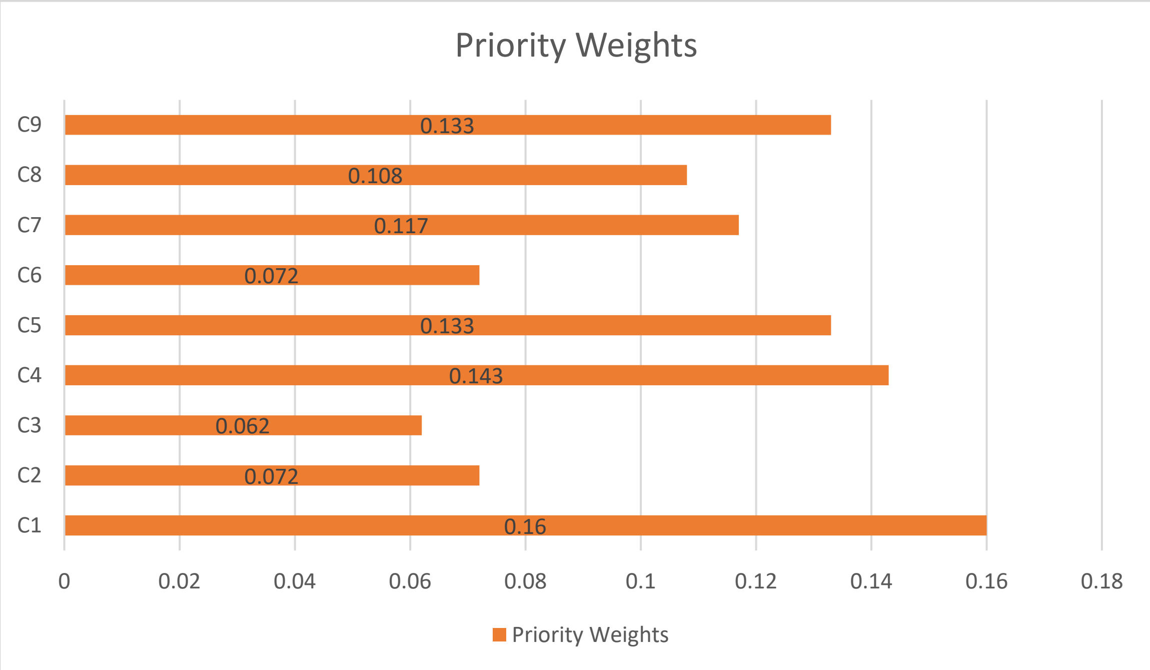

Figure 4. Priority weights of criteria

Table 3. Assessment of hospitals services by the first evaluator

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	0.13	0.36	0.92	0.73	0.52	0.36	0.52	0.92	0.73
H_2	0.36	0.42	0.52	0.36	0.42	0.52	0.73	0.42	0.36
H_3	0.72	0.73	0.92	0.73	0.73	0.73	0.52	0.72	0.73
H_4	0.36	0.42	0.52	0.36	0.42	0.52	0.73	0.42	0.36
H_5	0.92	0.73	0.52	0.92	0.73	0.52	0.73	0.72	0.92

Table 4. Assessment of hospitals service by the second evaluator

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	0.42	0.13	0.92	0.72	0.36	0.36	0.13	0.92	0.73
H_2	0.36	0.42	0.52	0.36	0.42	0.52	0.73	0.42	0.36
H_3	0.72	0.73	0.73	0.92	0.73	0.73	0.72	0.72	0.73
H_4	0.36	0.42	0.52	0.36	0.42	0.52	0.73	0.42	0.36
H_5	0.92	0.73	0.52	0.92	0.73	0.52	0.73	0.72	0.92

Table 5. Assessment of hospitals service by the third evaluator.

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	0.36	0.42	0.92	0.73	0.42	0.36	0.52	0.73	0.73
H_2	0.36	0.52	0.52	0.42	0.73	0.52	0.52	0.42	0.73
H_3	0.72	0.73	0.73	0.72	0.73	0.52	0.52	0.72	0.73
H_4	0.36	0.42	0.52	0.36	0.42	0.52	0.73	0.42	0.36
H_5	0.92	0.73	0.52	0.92	0.73	0.52	0.73	0.72	0.92

Table 6. Aggregation values of ranking alternatives by all decision makers

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	0.30	0.30	0.92	0.73	0.43	0.36	0.39	0.86	0.73
H_2	0.36	0.45	0.52	0.38	0.52	0.52	0.66	0.42	0.48
H_3	0.72	0.73	0.79	0.79	0.73	0.66	0.56	0.72	0.73
H_4	0.36	0.42	0.52	0.36	0.42	0.52	0.73	0.42	0.36
H_5	0.92	0.73	0.52	0.92	0.73	0.52	0.73	0.72	0.92

Table 7. pairwise comparison vector for the best criterion

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
C_5	0.72	0.13	1	0.13	0.58	0.45	0.52	0.42	0.36

Table 8. pairwise comparison vector for the worst criterion

Criteria	C_3
C_1	1
C_2	0.13
C_3	0.72
C_4	0.58
C_5	0.52
C_6	0.13
C_7	0.42
C_8	0.36
C_9	0.52

Table 9. The consistency Index

Criteria	1	2	3	4	5	6	7	8	9
Weights	0.00	0.44	1.00	1.63	2.30	3.00	3.73	4.47	5.23

Table 10. Weights of criteria based on BWM

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
Weights	0.16	0.072	0.062	0.143	0.133	0.072	0.117	0.108	0.133

In Phase 3: Build the Difference Matrix Based on MABAC method:

Step 10: Calculate the weighted normalized assessment matrix

Construct the weighted normalized decision matrix using Eq. (10). E.g. the weighted normalized values of the first criteria are as follows:

$\widehat{N}_{11} = w_1 + N_{11} * w_1 = 0.16 * (1 + 0) = 0.16$

$\widehat{N}_{21} = w_1 + N_{21} * w_1 = 0.16 * (1 + 0) = 0.175$

$\widehat{N}_{31} = w_1 + N_{31} * w_1 = 0.16 * (1 + 0) = 0.268$

$\widehat{N}_{41} = w_1 + N_{41} * w_1 = 0.16 * (1 + 0) = 0.175$

$\widehat{N}_{51} = w_1 + N_{51} * w_1 = 0.16 * (1 + 0) = 0.32$

The other weighted normalized values of the criteria are determined in table 12.

Step 11: Determine the border approximation area vector

Compute the border approximate area matrix using Eq. (11). The amounts of the border approximate area matrix are as follows:

Criteria	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
Approximation area	0.2196	0.1098	0.0826	0.2132	0.1954	0.1092	0.1939	0.1588	0.2

Figure 5 show amount of the border approximate area.

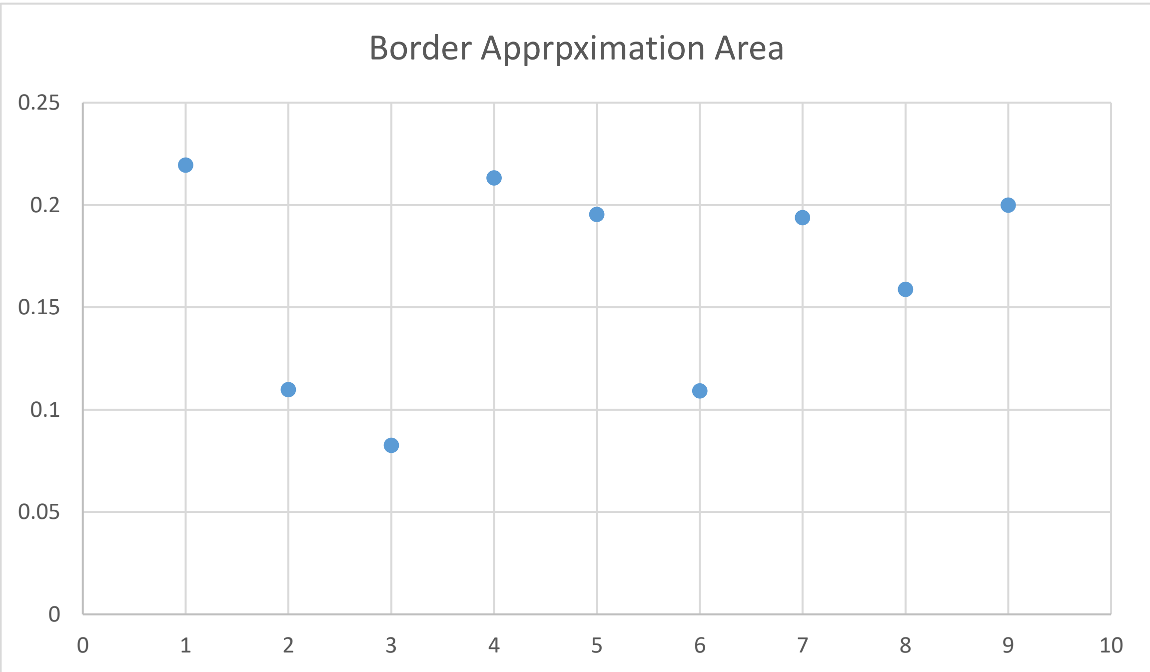

Figure 5. Border approximation area

Step 12: Obtain the difference matrix

Compute The distance from the border approximate area using Eq. (12). The distance of each alternative from the border approximate area, is shown in table 13.

Table 11. Normalized values of the Aggregated decision matrix

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	0	0	1	0.660	0.032	0	0	1	0.660
H_2	0.096	0.348	0	0.035	0.322	0.533	0.794	0	0.214
H_3	0.677	1	0.675	0.767	1	1	0.5	0.681	0.660
H_4	0.096	0.279	0	0	0	0.533	1	0	0
H_5	1	1	0	1	1	0.533	1	0.681	1

Table 12. Values of the weighted normalized decision matrix

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	0.16	0.072	0.124	0.237	0.137	0.072	0.117	0.216	0.220
H_2	0.175	0.097	0.062	0.148	0.175	0.110	0.209	0.108	0.161
H_3	0.268	0.144	0.103	0.252	0.266	0.144	0.1755	0.181	0.220
H_4	0.175	0.092	0.062	0.143	0.133	0.110	0.234	0.108	0.133
H_5	0.32	0.144	0.062	0.286	0.266	0.110	0.234	0.181	0.266

Table 13. Distance from the border approximate area

Criteria/Alternatives	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
H_1	-0.05	-0.03	0.04	0.02	-0.05	-0.03	-0.07	0.05	0.02
H_2	-0.04	-0.01	-0.02	-0.06	-0.02	0.0008	0.01	-0.05	-0.03
H_3	0.04	0.03	0.02	0.03	0.07	0.03	-0.01	0.02	0.02
H_4	-0.04	-0.01	-0.02	-0.07	-0.06	0.0008	0.04	-0.05	-0.06
H_5	0.10	0.03	-0.02	0.07	0.07	0.0008	0.04	0.02	0.06

In phase 4: Get the Ranking Results Based on PROMETHEE II

Step 13: Compute the full preference degree

Calculate the evaluative differences of s^{th} alternative with respect to other alternatives. Compute the preference function using Eq. (13). Calculate the aggregated preference function using Eq. (14) in table 14.

Step 14: Calculate the positive and negative flows of alternatives

Calculate the positive and negative flows of alternatives using Eq. (15) Eq. (16) in table 14. Calculate the net outranking flow of each alternative in the fourth column using Eq. (17) in table 14. Indicates that $\psi(H_5) > \psi(H_3) > \psi(H_1) > \psi(H_2) > \psi(H_4)$.

Step 15: Attain the final ranking result of alternatives

Determine the ranking of all the considered alternatives in table 15 depending on the values of net flow in last column in table 14. The ranking order is $H_5 \succ H_3 \succ H_1 \succ H_2 \succ H_4$. Hence, the best hospital alternative isH_5. Figure 6 shows the order of hospitals.

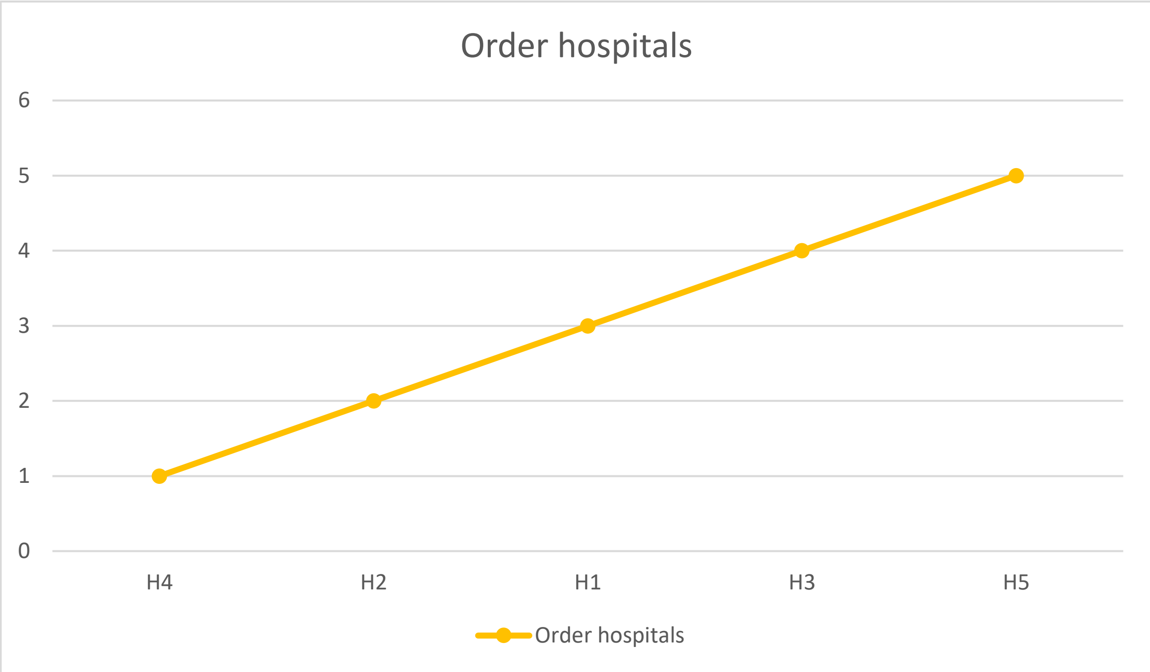

Figure 6. Order of hospitals

Table 14. The aggregated preference function

Alternatives	H_1	H_2	H_3	H_4	H_5	Leaving flow $\psi^+(H_i)$	Entering flow $\psi^-(H_i)$	Net flow
H_1	0	0.03261	0.00448	0.03936	0.00696	0.020853	0.039006	-0.01815
H_2	0.018608	0	0.00234	0.01074	0	0.007922	0.039006	-0.03108
H_3	0.04745	0.059312	0	0.070052	0.004582	0.045349	0.039006	0.006343
H_4	0.018128	0.00351	0.00585	0	0	0.006872	0.039006	-0.03213
H_5	0.071838	0.07888	0.02649	0.08611	0	0.06583	0.039006	0.026824

Table 15. Final Rank Of alternatives

Alternatives	Rank
H_1	3
H_2	4
H_3	2
H_4	5
H_5	1

5. Conclusion

The study proposes a hybrid methodology of neutrosophic set with BWM, MABAC and PROMETHEE II to assess a set of possible hospitals in an effort to reach to the superior qualified substitute that pleases the desires and the anticipations for patients. Consequently, raw data surveyed from 3 evaluators and assessed by the neutrosophic BWM, MABAC and PROMETHEE model to measure the proportional healthcare service effectiveness performance of 5 hospitals. The outcomes display that the 5 most significant criteria for assessing the hospital service effectiveness are: Staff Services, medical equipment level of the hospital, security within hospital, confidence to provided medical services and cleanliness of facilities and buildings. Particularly, because the private infirmaries are hardly supported by government intuitions, they are prompted to provide superior services than public infirmaries in order to enhance patients' gratification and consequently keep allegiance to the hospital. The future work includes using other applicable methodologies such as TOPSIS and making comparative studies that reflect on the assessing of hospital services.

References

1. Li, L.X., Relationships between determinants of hospital quality management and service quality performance—a path analytic model. Omega, 1997. 25(5): p. 535-545.
2. Cardoso, J., et al., Quality of service for workflows and web service processes. Web Semantics: Science, Services and Agents on the World Wide Web, 2004. 1(3): p. 281-308.
3. Chou, C.-C., et al., An evaluation of airline service quality using the fuzzy weighted SERVQUAL method. Applied Soft Computing, 2011. 11(2): p. 2117-2128.
4. Kuo, M.-S. and G.-S. Liang, Combining VIKOR with GRA techniques to evaluate service quality of airports under fuzzy environment. Expert Systems with Applications, 2011. 38(3): p. 1304-1312.
5. Awasthi, A., et al., A hybrid approach based on SERVQUAL and fuzzy TOPSIS for evaluating transportation service quality. Computers & Industrial Engineering, 2011. 61(3): p. 637-646.
6. Karimi, M., et al., A hybrid approach based on SERVQUAL and fuzzy TOPSIS for evaluating banking service quality. Decision Science Letters, 2015. 4(3): p. 349-362.
7. Büyüközkan, G., G. Çifçi, and S. Güleryüz, Strategic analysis of healthcare service quality using fuzzy AHP methodology. Expert systems with applications, 2011. 38(8): p. 9407-9424.
8. Min, H., A. Mitra, and S. Oswald, Competitive benchmarking of health care quality using the analytic hierarchy process: An example from Korean cancer clinics. Socio-economic planning sciences, 1997. 31(2): p. 147-159.
9. Chang, S.-J., et al., Taiwan quality indicator project and hospital productivity growth. Omega, 2011. 39(1): p. 14-22.
10. Teng, C.-I., et al., Development of service quality scale for surgical hospitalization. Journal of the Formosan Medical Association, 2007. 106(6): p. 475-484.
11. Shieh, J.-I., H.-H. Wu, and K.-K. Huang, A DEMATEL method in identifying key success factors of hospital service quality. Knowledge-Based Systems, 2010. 23(3): p. 277-282.
12. Lee, M.A. and Y.-H. Yom, A comparative study of patients' and nurses' perceptions of the quality of nursing services, satisfaction and intent to revisit the hospital: A questionnaire survey. International journal of nursing studies, 2007. 44(4): p. 545-555.
13. Akdag, H., et al., The evaluation of hospital service quality by fuzzy MCDM. Applied Soft Computing, 2014. 23: p. 239-248.
14. Chang, T.-H., Fuzzy VIKOR method: a case study of the hospital service evaluation in Taiwan. Information Sciences, 2014. 271: p. 196-212.

15. Zhang, W., T.-b. YANG, and Z.-j. WU, Comprehensive evaluation on quality of hospital medical services by using TOPSIS method. Practical Preventive Medicine, 2007. 5: p. 1-25.
16. Rezaei, J., Best-worst multi-criteria decision-making method. Omega, 2015. 53: p. 49-57.
17. Alinezhad, A. and J. Khalili, MABAC Method, in New Methods and Applications in Multiple Attribute Decision Making (MADM). 2019, Springer. p. 193-198.
18. Lupo, T., A fuzzy ServQual based method for reliable measurements of education quality in Italian higher education area. Expert systems with applications, 2013. 40(17): p. 7096-7110.
19. de Ruyter, K. and M. Wetzels, On the perceived dynamics of retail service quality. Journal of Retailing and Consumer Services, 1997. 4(2): p. 83-88.
20. Sweeney, J.C., G.N. Soutar, and L.W. Johnson, Retail service quality and perceived value: A comparison of two models. Journal of Retailing and Consumer Services, 1997. 4(1): p. 39-48.
21. Vazquez, R., et al., Service quality in supermarket retailing: identifying critical service experiences. Journal of retailing and consumer services, 2001. 8(1): p. 1-14.
22. Altuntas, S., T. Dereli, and M.K. Yilmaz, Multi-criteria decision making methods based weighted SERVQUAL scales to measure perceived service quality in hospitals: A case study from Turkey. Total Quality Management & Business Excellence, 2012. 23(11-12): p. 1379-1395.
23. Wang, L., J.-j. Peng, and J.-q. Wang, A multi-criteria decision-making framework for risk ranking of energy performance contracting project under picture fuzzy environment. Journal of cleaner production, 2018. 191: p. 105-118.
24. Pamučar, D., I. Petrović, and G. Ćirović, Modification of the Best–Worst and MABAC methods: A novel approach based on interval-valued fuzzy-rough numbers. Expert systems with applications, 2018. 91: p. 89-106.
25. Yazdani, M., et al., Development of a decision support framework for sustainable freight transport system evaluation using rough numbers. International Journal of Production Research, 2019: p. 1-27.
26. Wu, Y., et al., An intuitionistic fuzzy multi-criteria framework for large-scale rooftop PV project portfolio selection: Case study in Zhejiang, China. Energy, 2018. 143: p. 295-309.
27. Liao, H., et al., Green logistic provider selection with a hesitant fuzzy linguistic thermodynamic method integrating cumulative prospect theory and PROMETHEE. Sustainability, 2018. 10(4): p. 1291.
28. Liu, P., S. Cheng, and Y. Zhang, An Extended Multi-criteria Group Decision-Making PROMETHEE Method Based on Probability Multi-valued Neutrosophic Sets. International Journal of Fuzzy Systems, 2019. 21(2): p. 388-406.
29. Amoozad Mahdiraji, H., et al., A hybrid fuzzy BWM-COPRAS method for analyzing key factors of sustainable architecture. Sustainability, 2018. 10(5): p. 1626.
30. Mi, X. and H. Liao, An integrated approach to multiple criteria decision making based on the average solution and normalized weights of criteria deduced by the hesitant fuzzy best worst method. Computers & Industrial Engineering, 2019. 133: p. 83-94.
31. Nabeeh, N.A., A. Abdel-Monem, and A. Abdelmouty, A Hybrid Approach of Neutrosophic with MULTIMOORA in Application of Personnel Selection. Neutrosophic Sets and Systems, 2019: p. 1.
32. Nabeeh, N.A., et al., Neutrosophic multi-criteria decision making approach for iot-based enterprises. IEEE Access, 2019. 7: p. 59559-59574.
33. Abdel-Basset, M., et al., Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises. Enterprise Information Systems, 2019: p. 1-21.

34. Nabeeh, N.A., et al., An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis. IEEE Access, 2019. 7: p. 29734-29744.

35. Deli, I., M. Ali, and F. Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. in 2015 International Conference on Advanced Mechatronic Systems (ICAMechS). 2015. IEEE.

36. Abdel-Basset, M., et al., A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of medical systems, 2019. 43(2): p. 38.

37. Athawale, V.M. and S. Chakraborty. Facility location selection using PROMETHEE II method. in Proceedings of the 2010 international conference on industrial engineering and operations management. 2010. Bangladesh Dhaka.

38. Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F. (2019). A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine, 100, 101710.

39. Abdel-Baset, M., Chang, V., & Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry, 108, 210-220.

40. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 77, 438-452.

41. Abdel-Basset, M., Atef, A., & Smarandache, F. (2019). A hybrid Neutrosophic multiple criteria group decision making approach for project selection. Cognitive Systems Research, 57, 216-227.

42. Abdel-Basset, Mohamed, Mumtaz Ali, and Asma Atef. "Resource levelling problem in construction projects under neutrosophic environment." The Journal of Supercomputing (2019): 1-25.

43. Abdel-Basset, Mohamed, and Rehab Mohamed. "A novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management." Journal of Cleaner Production 247 (2020): 119586.

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Some Results on Single Valued Neutrosophic Hypergroup

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Abstract: We introduced the theory of Single valued neutrosophic hypergroup as the initial theory of single valued neutrosophic hyper algebra and also developed some results on single valued neutrosophic hypergroup.

Keywords: Hypergroup; Level sets; Single valued neutrosophic sets; Single valued neutrosophic hypergroup.

1. Introduction

Florentin Smarandache introduced Neutrosophic sets in 1998 [16], which is the generalization of the intuitionistic fuzzy sets. In some real time situations, decision makers faced some difficulties with uncertainty and inconsistency values. Neutrosophic sets helped the decision makers to deal with uncertainty values. Abdel-Basset et.al. used neutrosophic concept in real life decision-making problems [1-7]. The concept of single valued neutrosophic set was introduced by Wang. et. al [17].

As a generalization of classical algebraic structure, Algebraic hyper structure was introduced by F. Marty [11]. Corsini and Leoreanu-Fotea developed the applications of hyper structure [9]. Algebraic hyperstructures has many applications in fuzzy sets, lattices, artificial intelligence, automation, combinatorics. Corsini introduced hypergroup theory [8]. After while the hyperstructure theory has seen broader applications in many fields. Some of the recent works on hyperstructures related to vague soft groups, vague soft rings and vague soft ideals can be found in [12, 13].

In this paper we develop the theory of single valued neutrosophic hypergroup and also established some results on single valued neutrosophic hypergroup.

2. Preliminaries

Definition 2.1 [17] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function T_A, an indeterminancy-membership function I_A and a falsity-membership function F_A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$.

$T_A: X \to]0^-, 1^+[$

$I_A: X \to]0^-, 1^+[$

$F_A: X \to]0^-, 1^+[$

There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.

Definition 2.2 [17] Let X be a space of points (objects),with a generic element of X denoted by x. A single valued neutrosophic set (SVNS) A in X is characterized by T_A, I_A and F_A. For each point x in X, $T_A, I_A, F_A \in [0,1]$.

Definition 2.3 [17] The complement of a SVNS A is denoted by c(A) and is defined by

$T_{c(A)}(x) = F_A(x)$

$I_{c(A)}(x) = 1 - I_A(x)$

$F_{c(A)}(x) = T_A(x)$, for all x in X.

Definition 2.4 [17] A SVNS A is contained in the other SVNS B, $A \subseteq B$, if and only if,

$T_A(x) \leq T_B(x)$

$I_A(x) \geq I_B(x)$

$F_A(x) \geq F_B(x)$, for all x in X.

Definition 2.5 [17] The union of two SVNS s A and B is a SVNS C, written as $C = A \cup B$, whose truth, indeterminancy and falsity-membership functions are defined by,

$T_C(x) = \max(T_A(x), T_B(x))$

$I_C(x) = \min(I_A(x), I_B(x))$

$F_C(x) = \min(F_A(x), F_B(x))$, for all x in X.

Definition 2.6 [17] The intersection of two SVNS s A and B is a SVNS C, written as $C = A \cap B$, whose truth, indeterminancy and falsity-membership functions are defined by,

$T_C(x) = \min(T_A(x), T_B(x))$

$I_C(x) = \max(I_A(x), I_B(x))$

$F_C(x) = \max(F_A(x), F_B(x))$, for all x in X.

Definition 2.7 [17] The falsity-favorite of a SVNS B, written as $B\nabla A$, whose truth and falsity-membership functions are defined by

$T_B(x) = T_A(x)$

$I_B(x) = 0$

$F_B(x) = \min\{F_A(x) + I_A(x), 1\}$, for all x in X.

Definition 2.8 [13] A hypergroup $\langle H, \circ \rangle$ is a set H equipped with an associative hyperoperation $(\circ): H \times H \to P(H)$ which satisfies $x \circ H = H \circ x = H$ for all $x \in H$ (Reproduction axiom)

Definition 2.9 [13] A hyperstructure $\langle H, \circ \rangle$ is called an H_v-group if the following axioms hold:

(i) $x \circ (y \circ z) \cap (x \circ y) \circ z \neq \emptyset$ for all $x, y, z \in H$,

(ii) $x \circ H = H \circ x = H$ for all $x \in H$.

If $\langle H, \circ \rangle$ only satisfies (i), then $\langle H, \circ \rangle$ is called a H_v- semigroup.

Definition 2.10 [13] A subset K of H is called a subhypergroup if $\langle K, \circ \rangle$ is a hypergroup of $\langle H, \circ \rangle$.

3. Single Valued Neutrosophic Hypergroup.

Throughout this section H denotes the hypergroup $< H, \circ >$

Definition 3.1 *Let $\mathcal{A}$ be a single valued neutrosophic set over H. Then $\mathcal{A}$ is called a single valued neutrosophic hypergroup over H, if the following conditions are satisfied* (i) $\forall\ p, q \in H$,

$min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \leq inf\{T_{\mathcal{A}}(r):\ r \in p \circ q\}$,

$max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\} \geq sup\{I_{\mathcal{A}}(r):\ r \in p \circ q\}$ *and*

$$max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} \geq sup\{F_{\mathcal{A}}(r):\ r \in p \circ q\}$$

$(ii)\ \forall\ l, p \in H, there\ exists\ q \in H\ such\ that\ p \in l \circ q\ and$

$$min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \leq T_{\mathcal{A}}(q),$$
$$max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(p)\} \geq I_{\mathcal{A}}(q)\ and$$
$$max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(p)\} \geq F_{\mathcal{A}}(q)$$

$(iii)\ \forall\ l, p \in H, there\ exists\ r \in H\ such\ that\ p \in r \circ l\ and$

$$min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \leq T_{\mathcal{A}}(r),$$
$$max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(p)\} \geq I_{\mathcal{A}}(r)\ and$$
$$max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(p)\} \geq F_{\mathcal{A}}(r)$$

If $\mathcal{A}$ satisfies condition (i) then $\mathcal{A}$ is a single valued neutrosophic semihypergroup over H. Condition (ii) and (iii) represent the left and right reproduction axioms respectively. Then $\mathcal{A}$ is a single valued neutrosophic subhypergroup of H.

Example 3.2 If the family of t-level sets of SVNS $\mathcal{A}$ over H

$\mathcal{A}_t = \{p \in H\ |\ T_{\mathcal{A}}(p) \geq t, I_{\mathcal{A}}(p) \leq t\ \text{and}\ F_{\mathcal{A}}(p) \leq t\}$ is a subhypergroup of H then,

$\mathcal{A}$ is a single valued neutrosophic hypergroup over H.

Theorem 3.3 Let $\mathcal{A}$ be a SVNS over H. Then $\mathcal{A}$ is a single valued neutrosophic hypergroup over H iff $\mathcal{A}$ is a single valued neutrosophic semihypergroup over H and also $\mathcal{A}$ satisfies the left and right reproduction axioms.

Proof. The proof is obvious from Definition: 3.1

Theorem 3.4 Let $\mathcal{A}$ be a SVNS over H. If $\mathcal{A}$ is a single valued neutrosophic hypergroup over H ,then $\forall\ t \in [0,1]\ \mathcal{A}_t \neq \emptyset$ is a subhypergroup of H.

Proof. Let $\mathcal{A}$ be a single valued neutrosophic hypergroup over H and let $p, q \in \mathcal{A}_t$, then

$T_{\mathcal{A}}(p), T_{\mathcal{A}}(q) \geq t, I_{\mathcal{A}}(p), I_{\mathcal{A}}(q) \leq t$ and $F_{\mathcal{A}}(p), F_{\mathcal{A}}(q) \leq t$.

Then we have,

$\inf\{T_{\mathcal{A}}(r): r \in p \circ q\} \geq \min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \geq \min\{t, t\} = t$

$\sup\{I_{\mathcal{A}}(r): r \in p \circ q\} \leq t$ and

$\sup\{F_{\mathcal{A}}(r): r \in p \circ q\} \leq t$

This implies $r \in \mathcal{A}_t$. Then $\forall\ r \in p \circ q$, $p \circ q \subseteq \mathcal{A}_t$.

Thus $\forall r \in \mathcal{A}_t$, we obtain $r \circ \mathcal{A}_t \subseteq \mathcal{A}_t$

Now, Let $l, p \in \mathcal{A}_t$, then there exist $q \in H$ such that $p \in l \circ q$ and

$\{T_{\mathcal{A}}(q)\} \geq \min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \geq \min\{t, t\} = t$

$\{I_{\mathcal{A}}(q)\} \leq t$ and

$\{F_{\mathcal{A}}(q)\} \leq t$. This implies $q \in \mathcal{A}_t$

This proves that $\mathcal{A}_t \subseteq r \circ \mathcal{A}_t$. As such $\mathcal{A}_t = r \circ \mathcal{A}_t$

Which proves that $\mathcal{A}_t$ is a subhypergroup of H.

Theorem 3.5 Let $\mathcal{A}$ be a SVNS over H. Then the following are equivalent,

(i) $\mathcal{A}$ is a single valued neutrosophic hypergroup over H

(ii) $\forall\ t \in [0,1]\ \mathcal{A}_t \neq \emptyset$ is a subhypergroup of H.

Proof. (i) $\Rightarrow$ (ii) The proof is obvious from Theorem : 3.4.

(ii) ⇒ (i) Now assume that $\mathcal{A}_t$ is a subhypergroup of H.

Let $p, q \in \mathcal{A}_{t_0}$ and let $\min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} = \max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\} = \max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} = t_0$

Since $p \circ q \subseteq \mathcal{A}_{t_0}$, then for every $r \in p \circ q$, $T_{\mathcal{A}}(r) \geq t_0$, $I_{\mathcal{A}}(r) \leq t_0$, $F_{\mathcal{A}}(r) \leq t_0$

$$\min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \leq \inf\{T_{\mathcal{A}}(r):\ r \in p \circ q\},$$
$$\max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\} \geq \sup\{I_{\mathcal{A}}(r):\ r \in p \circ q\} \text{ and}$$
$$\max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} \geq \sup\{F_{\mathcal{A}}(r):\ r \in p \circ q\}$$

Condition (i) is verified.

Next, let $l, p \in \mathcal{A}_{t_1}$, for every $t_1 \in [0,1]$ and

let $\min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(q)\} = \max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(p)\} = \max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(q)\} = t_1$

Then there exist $q \in \mathcal{A}_{t_1}$ such that $p \in l \circ q \subseteq \mathcal{A}_{t_1}$. Since $q \in \mathcal{A}_{t_1}$,

$$T_{\mathcal{A}}(q) \geq t_1 = \min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(q)\}$$
$$I_{\mathcal{A}}(q) \leq t_1 = \max\{I_{\mathcal{A}}(l), I_{\mathcal{A}}(q)\}$$
$$F_{\mathcal{A}}(q) \leq t_1 = \max\{F_{\mathcal{A}}(l), F_{\mathcal{A}}(q)\}$$

Condition (ii) is verified. Similarly, (iii) .

Theorem 3.6 Let $\mathcal{A}$ be a SVNS over H. Then $\mathcal{A}$ be a single valued neutrosophic hypergroup over H iff $\forall\ \alpha, \beta, \gamma \in [0,1]$, $\mathcal{A}_{(\alpha,\beta,\gamma)}$ is a subhypergroup of H.

Proof. The proof is straight forward.

Theorem 3.7 Let $\mathcal{A}$ be a single valued neutrosophic hypergroup over H and $\forall\ t_1, t_2 \in [0,1]$ $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ be the t-level sets of $\mathcal{A}$ with $t_1 \geq t_2$, then $\mathcal{A}_{t_1}$ is a subhypergroup of $\mathcal{A}_{t_2}$.

Proof. $\forall t_1, t_2 \in [0,1]$, $\mathcal{A}_{t_1}$ and $\mathcal{A}_{t_2}$ be the t-level sets of $\mathcal{A}$ with $t_1 \geq t_2$

This implies that $\mathcal{A}_{t_1} \subseteq \mathcal{A}_{t_2}$

By Theorem 3.4. $\mathcal{A}_{t_1}$ is a subhypergroup of $\mathcal{A}_{t_2}$.

Theorem 3.8 Let $\mathcal{A}$ and $\mathcal{B}$ be single valued neutrosophic hypergroups over H. Then $\mathcal{A} \cap \mathcal{B}$ is a single valued neutrosophic hypergroup over H if it is non-null.

Proof. Suppose $\mathcal{A}$ and $\mathcal{B}$ be single valued neutrosophic hypergroups over H.

By Definition: 2.6. $\mathcal{A} \cap \mathcal{B} = \{< p, T_{\mathcal{A}\cap\mathcal{B}}(p), I_{\mathcal{A}\cap\mathcal{B}}(p), F_{\mathcal{A}\cap\mathcal{B}}(p) >:\ p \in H\}$

where $T_{\mathcal{A}\cap\mathcal{B}}(p) = T_{\mathcal{A}}(p) \wedge T_{\mathcal{B}}(p)$, $I_{\mathcal{A}\cap\mathcal{B}}(p) = I_{\mathcal{A}}(p) \vee I_{\mathcal{B}}(p)$ and $F_{\mathcal{A}\cap\mathcal{B}}(p) = F_{\mathcal{A}}(p) \vee F_{\mathcal{B}}(p)$

For all $p, q \in H$

(i) $\min\{T_{\mathcal{A}\cap\mathcal{B}}(p), T_{\mathcal{A}\cap\mathcal{B}}(q)\} = \min\{T_{\mathcal{A}}(p) \wedge T_{\mathcal{B}}(p), T_{\mathcal{A}}(q) \wedge T_{\mathcal{B}}(q)\}$

$$\leq \min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \wedge \min\{T_{\mathcal{B}}(p), T_{\mathcal{B}}(q)\}$$
$$\leq \inf\{T_{\mathcal{A}}(r):\ r \in p \circ q\} \wedge \inf\{T_{\mathcal{B}}(r):\ r \in p \circ q\}$$
$$\leq \inf\{T_{\mathcal{A}}(r) \wedge T_{\mathcal{B}}(r):\ r \in p \circ q\}$$
$$= \inf\{T_{\mathcal{A}\cap\mathcal{B}}(r):\ r \in p \circ q\}$$

Similarly, we can prove that $\max\{I_{\mathcal{A}\cap\mathcal{B}}(p), I_{\mathcal{A}\cap\mathcal{B}}(q)\} \geq \sup\{I_{\mathcal{A}\cap\mathcal{B}}(r):\ r \in p \circ q\}$

$$\max\{F_{\mathcal{A}\cap\mathcal{B}}(p), F_{A\cap B}(q)\} \geq \sup\{F_{\mathcal{A}\cap\mathcal{B}}(r):\ r \in p \circ q\}$$

(ii) $\forall\ l, p \in H$, there exists $q \in H$ such that $p \in l \circ q$,

$\min\{T_{\mathcal{A}\cap\mathcal{B}}(l), T_{\mathcal{A}\cap\mathcal{B}}(p)\} = \min\{T_{\mathcal{A}}(l) \wedge T_{\mathcal{B}}(l)\}, \{T_{\mathcal{A}}(p) \wedge T_{\mathcal{B}}(p)\}$

$$= \min\{T_{\mathcal{A}}(l), T_{\mathcal{A}}(p)\} \wedge \min\{T_{\mathcal{B}}(l), T_{B}(p)\}$$
$$\leq T_{\mathcal{A}}(q) \wedge T_{\mathcal{B}}(q) = T_{\mathcal{A}\cap\mathcal{B}}(q)$$

Therefore, $\mathcal{A} \cap \mathcal{B}$ is a single valued neutrosophic hypergroup over H.

Theorem 3.9 Let $\mathcal{A}$ and $\mathcal{B}$ be single valued neutrosophic hypergroups over H. Then $\mathcal{A} \cup \mathcal{B}$ is a single valued neutrosophic hypergroup over H.

Proof. By Definition: 2.5.

$$\mathcal{A} \cup \mathcal{B} = \{< p, T_{\mathcal{A}\cup\mathcal{B}}(p), I_{\mathcal{A}\cup\mathcal{B}}(p), F_{\mathcal{A}\cup\mathcal{B}}(p) > : p \in H\}$$

where $T_{\mathcal{A}\cup\mathcal{B}}(p) = T_{\mathcal{A}}(p) \vee T_{\mathcal{B}}(p)$, $I_{\mathcal{A}\cup\mathcal{B}}(p) = I_{\mathcal{A}}(p) \wedge I_{\mathcal{B}}(p)$ and $F_{\mathcal{A}\cup\mathcal{B}}(p) = F_{\mathcal{A}}(p) \wedge F_{\mathcal{B}}(p)$

For all $p, q \in H$,

$$\begin{aligned}
\min\{T_{\mathcal{A}\cup\mathcal{B}}(p), T_{\mathcal{A}\cup\mathcal{B}}(q)\} &= \min\{T_{\mathcal{A}}(p) \vee T_{\mathcal{B}}(p), T_{\mathcal{A}}(q) \vee T_{\mathcal{B}}(q)\} \\
&\leq \min\{T_{\mathcal{A}}(p), T_{\mathcal{A}}(q)\} \vee \min\{T_{\mathcal{B}}(p), T_{\mathcal{B}}(q)\} \\
&\leq \inf\{T_{\mathcal{A}}(r) : r \in p \circ q\} \vee \inf\{T_{\mathcal{B}}(r) : r \in p \circ q\} \\
&\leq \inf\{T_{\mathcal{A}}(r) \vee T_{\mathcal{B}}(r) : r \in p \circ q\} \\
&= \inf\{T_{\mathcal{A}\cup\mathcal{B}}(r) : r \in p \circ q\}
\end{aligned}$$

Similarly, the other holds.

Theorem 3.10 Let $\mathcal{A}$ be a single valued neutrosophic hypergroup over H. Then the falsity- favorite of $\mathcal{A}$ (ie., $\nabla\mathcal{A}$) is also a single valued neutrosophic hypergroup over H.

Proof. By Definition: 2.7. $\mathcal{B} = \nabla\mathcal{A}$, where the membership values are $T_{\mathcal{B}}(x) = T_{\mathcal{A}}(x)$, $I_{\mathcal{B}}(x) = 0$ and $F_{\mathcal{B}}(x) = \min\{F_{\mathcal{A}}(x) + I_{\mathcal{A}}(x), 1\}$

Then we have to prove for $F_{\mathcal{B}}$, $\forall p, q \in H$

$$\begin{aligned}
\max\{F_{\mathcal{B}}(p), F_{\mathcal{B}}(q)\} &= \max\{F_{\mathcal{A}}(p) + I_{\mathcal{A}}(p) \wedge 1, F_{\mathcal{A}}(q) + I_{\mathcal{A}}(q) \wedge 1\} \\
&= \max\{F_{\mathcal{A}}(p) + I_{\mathcal{A}}(p), F_{\mathcal{A}}(q) + I_{\mathcal{A}}(q)\} \wedge 1 \\
&\geq (\max\{F_{\mathcal{A}}(p), F_{\mathcal{A}}(q)\} + \max\{I_{\mathcal{A}}(p), I_{\mathcal{A}}(q)\}) \wedge 1 \\
&\geq (\sup\{F_{\mathcal{A}}(r) : r \in p \circ q\} + \sup\{I_{\mathcal{A}}(r) : r \in p \circ q\}) \wedge 1 \\
&= \sup\{F_{\mathcal{A}}(r) + I_{\mathcal{A}}(r) \wedge 1 : r \in p \circ q\} \\
&= \sup\{F_{\mathcal{B}}(r) : r \in p \circ q\})
\end{aligned}$$

In similar manner the other conditions holds.

4. Conclusions

In this paper, we have developed the theory of hypergroup for the single-valued neutrosophic set by introducing several hyperalgebraic structures and some results were verified. The future research related to this work involve the development of other hyperalgebraic theory for the single-valued neutrosophic sets and interval-valued neutrosophic sets.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., amp; Smarandache, F. (2019). A Hybrid Plithogenic Decision- Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics, Symmetry, 11(7), 903.
2. Abdel-Basset, M., Ali, M., & Atef, A. (2020). Uncertainty Assessments of Linear Time-Cost Tradeoffs using Neutrosophic Set. Computers & Industrial Engineering, 106286.
3. Abdel-Basset, Mohamed, and Rehab Mohamed. "A novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management." Journal of Cleaner Production 247 (2020): 119586.
4. Abdel-Baset, M., Chang, V., amp; Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach, Computers in Industry, 108, 210-220.
5. Abdel-Basset, M., Saleh, M., Gamal, A., amp; Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under *type-2 neutrosophic number*. Applied Soft Computing, 77, 438-452.
6. Abdel-Baset, M., Chang, V., Gamal, A., amp; Smarandache, F. (2019). *An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field.,* Computers in Industry, 106, 94- 110.
7. Abdel-Basset, M., Manogaran, G., Gamal, A., amp; Smarandache, F. (2019). *A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection*, Journal of medical systems, 43(2), 38
8. P. Corsini, *Prolegomena of Hypergroup Theory,* Aviani Editor, Tricesimo, Italy, 2nd edition, 1993.
9. P.Corsini and V. Leoreanu-Fotea*, Application of Hyperstructures Theory: Advances in Mathematics*, Kluwer Academic, Dodrecht, The Netherlands, 2003.
10. Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics* ,Pons Publishing House, Brussels, Belgium, 141 p., 2017; arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx:
11. F. Marty, Sur *une Generalisation de la Notion de Groupe* , in Proceedings of the 8th Congress Mathematiciens Scandinaves, pp. 45-49, Stockholm, Sweden, 1934.
12. G. Selvachandran and A. R. Salleh. *Vague soft hypergroups and vague soft hypergroup homomorphism*, Advances in Fuzzy Systems, vol. 2014, Article ID 758637, 10 pages, 2014.
13. G.Selvachandran and A. R. Salleh, *Algebraic Hyperstructures of Vague Soft Sets Associated with Hyperrings and Hyperideals,* Hindawi Publishing Corporation The Scientific World Journal Volume 2015, Article ID 780121, 12 pages.
14. G. Selvachandran, *Introduction to the theory of soft hyperrings and soft hyperring homomorphism* , JPJournal ofAlgebra, Number Theory and Applications. In press.
15. G. Selvachandran and A.R. Salleh, *Hypergroup theory applied to fuzzy soft sets*. Global Journal of Pure and Applied Mathematics 11(2) (2015) 825-835.
16. Smarandache. F. *Neutrosophy: Neutrosophic Probability, Set and logic*, Ann Arbor, Michigan, USA, 2002; 105.
17. Wang, H., Smarandache ,F., Zhang, Y., Sunderraman, R. *Single Valued Neutrosophic Sets*, Technical Sciences and Applied Mathematics.
18. Zadeh, L. (1965). *Fuzzy sets* ,Inform and Control 8 338-353.

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Neutrosophic Bipolar Fuzzy Set and its Application in Medicines Preparations

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Abstract: To tackle the real life problems we come across, in various fields like computer sciences, medical sciences, social sciences and engineering works where we are facing many ambiguities and imprecisions. Here we bring an idea of neutrosophic bipolar fuzzy decision making where hybridized multi-attributes are involved, which is a very helpful tool to tackle the ambiguities and imprecisions. We present the neutrosophic bipolar fuzzy transformation techniques. The different types of attributes are transformed into unified neutrosophic bipolar fuzzy values. It includes the group decision making mode based on hybrid decision making problems with exact values, interval values and linguistic variables. Calculations of weights by decision makers, composition of aggregated weighted neutrosophic bipolar fuzzy decision matrices, determination of entropy weights, finding positive ideal solution(PIS),and negative ideal solution(NIS), calculation of grey relational coefficient ,calculation of degree of weighted grey relational coefficient of each alternative, determination of relative relational degree of each alternative from the positive ideal solution (PIS) and negative ideal solution (NIS) and ranking of the alternatives are the concepts which are introduced in the case of neutrosophic bipolar fuzzy hybrid multi-attribute group decision making. Eventually, we apply these concepts and techniques upon hybrid multi-attributes decision making problem of selecting the best medicine to cure some particular diseases and develop an algorithm for neutrosophic bipolar fuzzy hybrid multi-attribute group decision making.

Keywords: Neutrosophic bipolar fuzzy sets; multi-attribute group decision making; neutrosophic bipolar fuzzy transformation techniques; interval values and linguistic variables.

1. Introduction

The concept of fuzzy set theory was basically given by Zadeh [1]. The idea of fuzzy set theory has been extended to vague fuzzy set [2-5], interval-valued fuzzy set, intuitionistic fuzzy set [6], L-fuzzy set, Q-fuzzy set [7-11], probabilistic fuzzy set and so on, [12-19]. All these versions had limitations in different situations. Smarandache [20], gave the idea of neutrosophic set which is the generalization of all previous versions of fuzzy sets. Unfortunately, these, models were handling the problems involving only positive preferences and opinions, whereas human mind tends to work in both directions, positive and negative, in order to come up with a decision. Therefore, to bridge up this deficiency Zhang [21], introduced the notion of bipolar fuzzy sets. The features of bipolar fuzzy sets were considered and discussed in detail by Naveed at al. [22-24], Dubois et al. [25] and Silva et al. [26]. The applications of neutrosophic set theory are found in various fields of life, like computer sciences, physical sciences, medical sciences, social sciences, engineering and multi-criteria group decision making problems. The uses of neutrosophic theory for sets in decision making problems (DMP) have been considered by Basset et al. [27-31]. Qun et al. [32] and many others in many [33-36], they gave the idea of linguistic multiple attribute group decision making (LMAGDM). Chen [37] and Hung [38], introduced the idea of manipulation of multiple attribute decision making problems depends upon fuzzy sets. Later on Zhan et al. [39] applied the neutrosophic cubic sets in multi-criteria decision-making issues. Gulistan et al. [40] discussed the notion of neutrosophic cubic graphs and gave the real-life applications in industrial areas. Applications of neutrosophic sets in different directions can be seen in [41-44] and [45-52].

Neutrosophic sets are more general versions to handle the uncertain data problems when compared to the different versions of fuzzy sets. When handling uncertain issues where both positive and negative characteristics are involved, the bipolar fuzzy sets are found to be helpful. In propensity to take decisions considering both positive and negative preferences, we [45], recently defined the concept of neutrosophic bipolar fuzzy sets. We also defined neutrosophic bipolar fuzzy weighted averaging and neutrosophic bipolar fuzzy ordered weighted averaging operators.

In this paper, we will extend the neutrosophic bipolar fuzzy set by introducing the idea of neutrosophic bipolar fuzzy hybrid multi-attribute group decision making where we use the different neutrosophic bipolar fuzzy transformation techniques. We give the new conversion techniques between the exact values and neutrosophic bipolar fuzzy numbers. The conversion techniques between interval values and neutrosophic bipolar fuzzy numbers have also been considered and likewise we also discuss the transformations techniques between linguistic variables and neutrosophic bipolar fuzzy numbers. Graphical representations of the notions in this paper have been considered as well. Finally, numerical example related to a medicine company which intends to prepare three different types of medicines for a certain type of disease.

2. Preliminaries

In this section we provide some of the precursors in developing our new concept.

Definition 2.1. [1] A fuzzy set maps the elements of a universe X to the unit interval $[0,1]$.

Definition 2.2. [13] Let X be a universe of discourse. An intuitionistic fuzzy set, A in X is an object having the following form $A = \{\langle x, \mu(x), \nu(x)\rangle : x \in X\}$

where $\mu_A(x)$ is known as a degree of membership and $\nu_A(x)$ is known as a degree of non-membership of the element X to the IFS A with the condition, $0 \leq \mu(x) \leq 1$,

$0 \leq \nu(x) \leq 1$, $0 \leq \mu(x) + \nu(x) \leq 1$. For each IFS A in X. The hesitancy indeterminacy degree measure as follows, $\pi_A(x) = 1 - \mu(x) - \nu(x)$. Then $\pi_A(x)$ is known as degree of indeterminacy membership of x to the set A and $\forall\ x \in X$.

Definition 2.3. [21] Let X be a non-empty set. Then a bipolar fuzzy set, is an object of the form $B = \langle x, \langle \mu^+(x),\ \mu^-(x)\rangle : x \in X\rangle$, where $\mu^+(x): X \to [0,1]$ and $\mu^-(x): X \to [-1,0]$, $\mu^+(x)$ is a positive material and $\mu^-(x)$ is a negative material of $x \in X$. For simplicity, we write the bipolar fuzzy set as $B = \langle \mu^+,\ \mu^-\rangle$ instead of $B = \langle x, \langle \mu^+(x),\ \mu^-(x)\rangle : x \in X\rangle$.

Definition 2.4. [32, 34, 41] A single valued neutrosophic set, is defined as;

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where X be the universe of discourse and A is characterized by a t-membership function $T_A: X \to [0,1]$, an i-membership function $I_A: X \to [0,1]$ and a f-membership function $F_A: X \to [0,1]$, where $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.5. [6] A neutrosophic set, is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$$

and X is a universe of discourse and A is characterized by a t-membership function $T_A : X \to]0^-, 1^+[$, an i-membership function $I_A : X \to]0^-, 1^+[$ and a f-membership function $F_A : X \to]0^-, 1^+[$. There is no condition on the sum of $T_A(x), I_A(x), F_A(x)$, so $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 2.6. [45] Let X be a non-vacuous set. Then a neutrosophic bipolar fuzzy set, is an object of the form $NB = (NB^+, NB^-)$where $NB^+ = \langle y, \langle T_{NB^+}, I_{NB^+}, F_{NB^+}\rangle : x \in X\rangle$, $, NB^- = \langle y, \langle T_{NB^-}, I_{NB^-}, F_{NB^-}\rangle : x \in X\rangle$ such that $T_{NB^+}, I_{NB^+}, F_{NB^+} : X \to [0,1]$ and $T_{NB^-}, I_{NB^-}, F_{NB^-} : X \to [-1,0]$.

Definition 2.7. [45] Let $NB_j = (NB_j^+, NB_j^-)$ be the collection of neutrosophic bipolar fuzzy values. Then a mapping $NBFWA_\omega : \Omega^n \to \Omega$ defined by

$$NBFWA_\omega(NB_1, NB_2, \dots, NB_n) = \omega_1 NB_1 \oplus \omega_2 NB_2 \oplus, \dots, \oplus\ \omega_n NB_n$$

is called a neutrosophic bipolar fuzzy weighted averaging $(NBFWA)$ operator of dimension n, where $w = (w_1, w_2, \dots, w_n)^T$ is the weight vector of $NB_j (j = 1,2,\dots,n)$, with $\omega_j \in [0,1]$ and $\Sigma_{j=1}^n w_j = 1$.

Especially, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then the $NBFWA$ operator is reduced to a neutrosophic bipolar fuzzy averaging $(NBFA)$ operator of dimension n, which is defined as follows:

$$NBFA(NB_1, NB_2, \dots, NB_n) = \frac{1}{n}(NB_1 \oplus NB_2 \oplus, \dots, \oplus\ NB_n).$$

Definition 2.8. [45] Let $NB_j = (NB_j^+, NB_j^-)$ be a collection of neutrosophic bipolar fuzzy values. A neutrosophic bipolar fuzzy ordered weighted averaging $(NBFOWA)$operator of n dimension is a mapping $NBFOWA : \Omega^n \to \Omega$, that has an associated vector:
$\omega = (\omega_1, \omega_2, \dots, \omega_n)^T$ such that $\omega_j \in [0,1]$ and $\Sigma_{j=1}^n \omega_j = 1$. Furthermore

$$NBFOWA_\omega(NB_1^+, NB_2^+, \dots, NB_n^+) = \omega_1 NB_{\sigma(1)}^+ \oplus \omega_2 NB_{\sigma(2)}^+ \oplus, \dots, \oplus\ \omega_n NB_{\sigma(n)}^+$$
$$NBFOWA_\omega(NB_1^-, NB_2^-, \dots, NB_n^-) = \omega_1 NB_{\sigma(1)}^- \oplus \omega_2 NB_{\sigma(2)}^- \oplus, \dots, \oplus\ \omega_n NB_{\sigma(n)}^-$$

where $(\sigma(1), \sigma(2), \dots, \sigma(n))$ is a permutation of $(1,2,\dots,n)$ such that $NB_{\sigma(j-1)} \ge NB_{\sigma(j)}$ for all j .

Especially, if $\omega = \left(\frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n}\right)^T$, then the $NBFOWA$ operator is reduced to a bipolar fuzzy averaging

(NBFA) operator of dimension n .

Definition 2.9. [17] A linguistic variable, is a variable whose values are words or sentences in natural or artificial language.

3. Neutrosophic Bipolar Fuzzy Transformations Techniques

In this section we develop the neutrosophic bipolar fuzzy hybrid (MADM) with different types of data values. The neutrosophic bipolar fuzzy hybrid (MADM) problem based on four different data types, exact values, intervals, NBFNs and linguistic terms. Let $NB = \{NB_{1,}NB_{2,}, \ldots, NB_{n,}\}$ be a finite set of alternatives, and let $C = \{c_1, c_2, \ldots c_n\}$ be a set of attributes with weight vector $w = (w_1, w_2, \ldots, w_m)$, where $w \geq 0\ (j = 1,2,\ldots,m)$ and

$$\sum_{j=1}^{m} w_j = 1.$$

Let $R^k = (a_{ij}^{(k)})_{n\times m}$ be a neutrosophic bipolar fuzzy hybrid decision matrix, where $(a_{ij}^{(k)})$ will be the exact values, intervals, NBFNs, and linguistic terms. We need to transform three other types of attributed values in R^k into unified NBFNs. In the following discussion, we will explore the transformation techniques for each of the data types.

3.1. Conversion between exact values and NBFNs

The values of different attributes have different dimensions. Thus, the real numbers in the hybrid decision making need to be standardized in order to eliminate interference in the results. Generally, there are two kinds of attributes, the benefit type and the cost. The higher the benefit type value is, the better it is. While in the cost type, it is the opposite. For the benefit type, formula is

$$b_{ij}^{(k)} = \frac{a_{ij}^{(k)}}{\sqrt{\sum_{i=1}^{m} (a_{ij}^{(k)})^2}}. \tag{1}$$

The cost type formula is;

$$b_{ij}^{(k)} = \frac{\left(\frac{1}{a_{ij}^{(k)}}\right)}{\sqrt{\sum_{i=1}^{m} \left(\frac{1}{(a_{ij}^{(k)})}\right)^2}}. \tag{2}$$

Standardized precise number can be transformed into neutrosophic bipolar fuzzy numbers as

$$a_{ij}^{(k)} = ((\mu_{ij}^{+(k)}, I_{ij}^{+(k)}, F_{ij}^{+(k)}), (\mu_{ij}^{-(k)}, I_{ij}^{-(k)}, F_{ij}^{-(k)}))$$

$$\mu_{ij}^{+(k)} = b_{ij}^{(k)}, F_{ij}^{(k)} = \frac{\mu_{ij}^{(k)}}{2}, I_{ij}^{(k)} = \frac{\mu_{ij}^{(k)}}{3}, \mu_{ij}^{-(k)} = -1 + b_{ij}^{(k)},$$

$$F_{ij}^{-(k)} = \frac{\mu_{ij}^{-(k)}}{2}, I_{ij}^{-(k)} = \frac{\mu_{ij}^{-(k)}}{3} \tag{3}$$

For intervals and NBFNs, for the benefit type formula is,

$$b_{ij}^{L(k)} = \frac{a_{ij}^{L(k)}}{\sqrt{\sum_{i=1}^{m} (a_{ij}^{U(k)})^2}}, \quad b_{ij}^{U(k)} = \frac{a_{ij}^{U(k)}}{\sqrt{\sum_{i=1}^{m} (a_{ij}^{L(k)})^2}}. \tag{4}$$

For the cost type formula is;

$$b_{ij}^{L(k)} = \frac{\left(\frac{1}{a_{ij}^{U(k)}}\right)}{\sqrt{\sum_{i=1}^{m} \left(\frac{1}{(a_{ij}^{L(k)})}\right)^2}}, \quad b_{ij}^{U(k)} = \frac{a_{ij}^{L(k)}}{\sqrt{\sum_{i=1}^{m} \left(\frac{1}{(a_{ij}^{U(k)})}\right)^2}}. \tag{5}$$

Standardized interval numbers can be transformed into neutrosophic bipolar fuzzy numbers as follows;

$$a_{ij}^{(k)} = \left(\left(\mu_{ij}^{+(k)}, I_{ij}^{+(k)}, F_{ij}^{+(k)}\right), \left(\mu_{ij}^{-(k)}, I_{ij}^{-(k)}, F_{ij}^{-(k)}\right)\right), \quad \mu_{ij}^{(k)} = b_{ij}^{L(k)}, F_{ij}^{(k)} = \frac{\mu_{ij}^{(k)}}{3}, I_{ij}^{(k)} = \frac{\mu_{ij}^{(k)}}{2}$$

$$\mu_{ij}^{-(k)} = -1 + b_{ij}^{U(k)}, F_{ij}^{-(k)} = \frac{\mu_{ij}^{-(k)}}{3}, I_{ij}^{-(k)} = \frac{\mu_{ij}^{-(k)}}{2} \tag{6}$$

Note: The indeterminacy $I \neq 1 - \mu - F$. We have defined functions F and I as in [3,6] to be used in this paper.

3.2. Conversion between linguistic variables and NBFNs

Linguistic variables are used usually when situations are complex or not well defined. The words or sentences given by the decision makers for rating or ranking like very good, good, fine, poor, very poor etc., can be converted into, and expressed as a quantities (NBFNs). The linguistic variables for the position of the decision makers can be expressed in NBFNs in Table 1 and shown as in Figure 1.

Table 1. Linguistic variable for the important of decision makers

Linguistic variable	NBFNs
Very important	((0.85, 0.42, 0.28), (-0.10, -0.05, -0.03))
Important	((0.70, 0.35, 0.23), (-0.2, -0.10, -0.06))
Medium	((0.55, 0.27,0.18), (-0.30, -0.15, -0.10))
Unimportant	((0.30, 0.15, 0.10), (-0.60, -0.30, -0.20))
Very unimportant	((0.10, 0.05, 0.03), (-0.90, -0.45, -0.30))

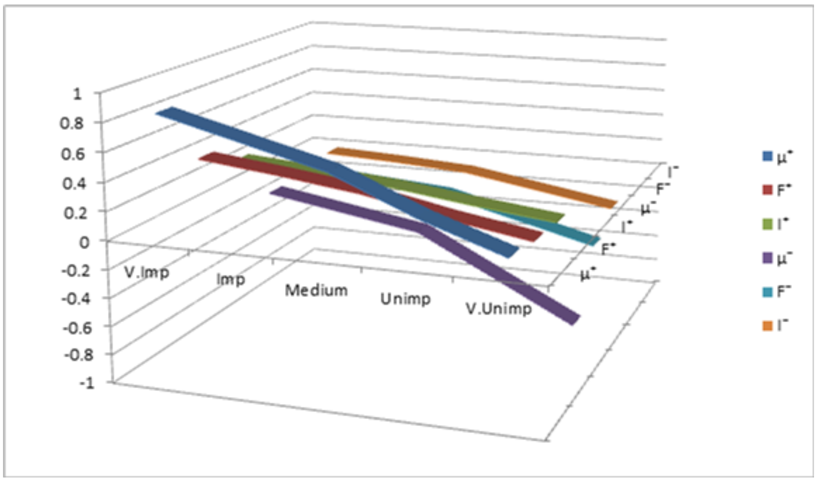

Figure 1. Graphical representation of importance of linguistic variables

Table 2. Conversion of linguistic variable into NBFNs

Linguistic variable	NBFNs
Extremely high (EH)	((0.95,0.47,0.31), (-0.03,-0.015,-0.01))
Very very high (VVH)	((0.83,0.41,0.27), (-0.10,-0.05,-0.03))
Very high (VH)	((0.77,0.38,0.25), (-0.12,-0.06,-0.04))
High (H)	((0.65,0.32,0.21), (-0.21,-0.10,-0.07))
Medium high (MH)	((0.55,0.27,0.18), (-0.32,-0.16,-0.10))
Medium (M)	((0.50,0.25,0.16), (-0.38,-0.19,-0.12))
Medium low (ML)	((0.35,0.17,0.11), (-0.45,-0.22,-0.15))
Low (L)	((0.22,0.11,0.07), (-0.3,-0.15,-0.1))
Very low (VL)	((0.12,0.06,0.04), (-0.87,-0.43,-0.29))
Very very low (VVL)	((0.06,0.03,0.02), (-0.93,-0.46,-0.31))

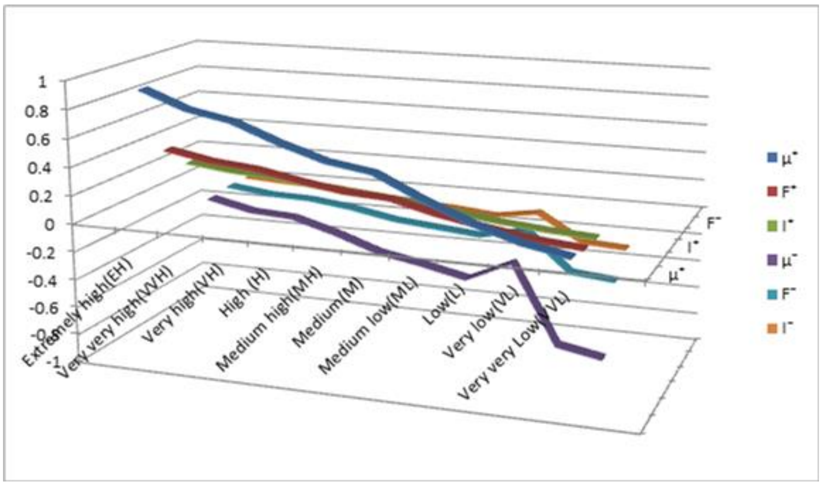

Figure 2. The rating of alternatives

The ratings of alternatives with respect to qualitative criteria can be converted into NBFNs as shown in Table 2 and shown as in Figure 2.

4. Neutrosophic Bipolar Fuzzy Hybrid Multi-Attribute Decision-Making

Neutrosophic bipolar fuzzy hybrid multi-attribute decision making problems are defined on a set of alternatives, from which the decision makers must select the best alternative according to some criteria. Suppose that there exists an alternative set $NB = \{NB_{1,}NB_{2,},\dots,NB_{n,}\}$ which consists of n alternatives, the decision makers will choose the best one from NB according to an attribute set $C = \{c_1, c_2, \dots, c_m\}$ in which m attributes are there. For convenience, we denote the weight vector of attribute by $w = \{w_1, w_2, \dots, w_m\}^T$, where $w_j \geq 0\ (j = 1,2,\dots,m)$ and

$$\sum_{j=1}^{m} w_j = 1.$$

We develop an algorithm for neutrosophic bipolar fuzzy hybrid MADM as follows:

Step 1. Consider the neutrosophic bipolar fuzzy hybrid decision matrix of each decision maker. The neutrosophic bipolar fuzzy hybrid decision matrix involves four different data types: exact values, intervals, NBFNs, and linguistic terms.

Step 2. In this step we use the transformation techniques to transform exact values, interval values, and linguistic variables, into neutrosophic bipolar fuzzy information. Assume that the rating of alternative $A_i(j = 1,2,\dots,n)$ with respect to attribute c_j given by the kth experts e_k can be expressed in $a_{ij}^{(k)} = ((\mu_{ij}^{+(k)}, I_{ij}^{+(k)}, F_{ij}^{+(k)}), (\mu_{ij}^{-(k)}, I_{ij}^{-(k)}, F_{ij}^{-(k)}))$. Hence a hybrid multiattribute group decision-making problem can be concisely expressed in a matrix format as:

$$R^{(k)} = (\alpha_{ij}^{(k)})_{n\times m} = \begin{bmatrix} \alpha_{11}^{(k)} & \alpha_{11}^{(k)} & \cdots & \alpha_{1m}^{(k)} \\ \alpha_{21}^{(k)} & \alpha_{22}^{(k)} & \cdots & \alpha_{2m}^{(k)} \\ . & . & & . \\ . & . & \cdots & . \\ . & . & & . \\ \alpha_{n1}^{(k)} & \alpha_{n2}^{(k)} & \cdots & \alpha_{nm}^{(k)} \end{bmatrix} \tag{7}$$

where $a_{ij}^{(k)} = ((\mu_{ij}^{+(k)}, I_{ij}^{+(k)}, F_{ij}^{+(k)}), (\mu_{ij}^{-(k)}, I_{ij}^{-(k)}, F_{ij}^{-(k)}))$.

Step 3. In this step we calculate the weight of each decision maker. Calculate the weight with respect to the Kth decision maker e_k . Determine the weights of decision makers, let $D_k = ((\mu_{ij}^{+(k)}, I_{ij}^{+(k)}, F_{ij}^{+(k)}), (\mu_{ij}^{-(k)}, I_{ij}^{-(k)}, F_{ij}^{-(k)}))$ be a neutrosophic bipolar fuzzy number for rating of the Kth decision maker. Then the weight of the Kth decision maker can be obtained as follows:

$$\lambda_k = \frac{(\mu_k^+ + I_k^+(\mu_k^+/(\mu_k^+ + F_k^+))) + |(\mu_k^- + I_k^-(\mu_k^-/(\mu_k^- + F_k^-)))|}{\sum_{k=1}^{t} (\mu_k^+ + I_k^+(\mu_k^+/(\mu_k^+ + F_k^+))) + |(\mu_k^- + I_k^-(\mu_k^-/(\mu_k^- + F_k^-)))|} \quad \text{where} \quad \sum_{k=1}^{t} \lambda_k = 1 \tag{8}$$

Step 4. Compose the aggregated weighted neutrosophic bipolar fuzzy decision matrix. In this step, aggregated weighted neutrosophic bipolar fuzzy decision matrix R is formed by considering the aggregated neutrosophic bipolar fuzzy decision matrix and weights vector of decision maker. The aggregated neutrosophic bipolar fuzzy decision matrix (ANBFDM) was formed by applying the neutrosophic bipolar fuzzy weighted averaging operator (NBFWAO). By considering weights $\lambda_k(k = 1,2,\ldots,t)$ of decision makers, elements β_{ij} of (ANBFDM) can be calculated by using (NBFWA) as follows:

$$\beta_{ij} = [(\mu_{ij}^{+'} = 1 - \prod_{k=1}^{t}(1-\mu_{ij}^{+(k)})^{\lambda_k}, I_{ij}^{+'} = \frac{1-\prod_{k=1}^{t}(1-\mu_{ij}^{+(k)})^{\lambda_k}}{2},$$

$$F_{ij}^{+'} = \frac{1-\prod_{k=1}^{t}(1-\mu_{ij}^{+(k)})^{\lambda_k}}{3})^{\lambda k}), (\mu_{ij}^{-'} = -\prod_{k=1}^{t}(1-\mu_{ij}^{-(k)})^{\lambda_k},$$

$$I_{ij}^{-'} = \frac{-\prod_{k=1}^{t}(1-\mu_{ij}^{-(k)})^{\lambda_k}}{2}, F_{ij}^{-'} = \frac{-\prod_{k=1}^{t}(1-\mu_{ij}^{-(k)})^{\lambda_k}}{3})]. \tag{9}$$

where

$$R = (\beta_{ij})_{n\times m} \quad = ((\mu_{ij}^{+'}, I_{ij}^{+'}, F_{ij}^{+'}), (\mu_{ij}^{-'}, I_{ij}^{-'}, F_{ij}^{-'}))_{n\times m}$$

Step 5. Determine the entropy weights of the selection criteria. In this step, all criteria may not be assumed to be of equal importance. w represents a set of grades of importance. Let w_j be the weights of the criteria, the neutrosophic bipolar fuzzy entropy H_j is calculated by equations;

$$H_j = \frac{1}{n}\sum_{i=1}^{n}\frac{\min((\mu_{ij}^{+'}, I_{ij}^{+'}, F_{ij}^{+'}),(|\mu_{ij}^{-'}|,|I_{ij}^{-'}|,|F_{ij}^{-'}|))}{\max((\mu_{ij}^{+'}, I_{ij}^{+'}, F_{ij}^{+'}),(|\mu_{ij}^{-'}|,|I_{ij}^{-'}|,|F_{ij}^{-'}|))}. \tag{10}$$

The entropy weights of the jth criteria can be calculated as follows:

$$w_j = \frac{1-H_j}{m-\sum_{j=1}^{m} H_j} \tag{11}$$

Step 6. Determine the positive ideal solution (PIS) and the negative ideal solution (NIS) based on neutrosophic bipolar fuzzy numbers. Both solutions are vectors of NBFN elements, and they are resulting AWNBFDM matrix as follows:

$$\begin{aligned} r^{+} &= ((\mu_1^{+'}, I_1^{+'}, F_1^{+'}), (\mu_1^{-'}, I_1^{-'}, F_1^{-'}))^{+}, ((\mu_2^{+'}, I_2^{+'}, F_2^{+'}), (\mu_2^{-'}, I_2^{-'}, F_2^{-'}))^{+}, \ldots \\ &\quad ,\ldots, ((\mu_m^{+'}, I_m^{+'}, F_m^{+'}), (\mu_m^{-'}, I_m^{-'}, F_m^{-'}))^{+}. \\ r^{-} &= ((\mu_1^{+'}, I_1^{+'}, F_1^{+'}), (\mu_1^{-'}, I_1^{-'}, F_1^{-'}))^{-}, ((\mu_2^{+'}, I_2^{+'}, F_2^{+'}), (\mu_2^{-'}, I_2^{-'}, F_2^{-'}))^{-}, \ldots \\ &\quad ,\ldots, ((\mu_m^{+'}, I_m^{+'}, F_m^{+'}), (\mu_m^{-'}, I_m^{-'}, F_m^{-'}))^{-}. \end{aligned} \tag{12}$$

where

$$\begin{aligned} &((\mu_j^{+'}, I_j^{+'}, F_j^{+'}), (\mu_j^{-'}, I_j^{-'}, F_j^{-'}))^{+} \\ &= (\overset{i}{\max}(\mu_{ij}^{+'}), \overset{i}{\min}(I_{ij}^{+'}), \overset{i}{\min}(F_{ij}^{+'}))\overset{i}{\min}(\mu_{ij}^{-'}), \overset{i}{\max}(I_{ij}^{-'}), \overset{i}{\max}(F_{ij}^{-'}))),\ j = 1,2,\ldots,m, \\ &((\mu_j^{+'}, I_j^{+'}, F_j^{+'}), (\mu_j^{-'}, I_j^{-'}, F_j^{-'}))^{-} \\ &= (\overset{i}{\min}(\mu_{ij}^{+'}), \overset{i}{\max}(I_{ij}^{+'}), \overset{i}{\max}(F_{ij}^{+'})), \overset{i}{\max}(\mu_{ij}^{-'}), \overset{i}{\min}(I_{ij}^{-'}), \overset{i}{\min}(F_{ij}^{-'}))),\ j = 1,2,\ldots,m. \end{aligned} \tag{13}$$

Step 7. Find the grey relational coefficient of each evaluation value from positive ideal solution

(PIS) and negative ideal solution (NIS) by using the following equations, respectively. The grey relational coefficients of each evaluation value from PIS and NIS are defined as:

$$\xi_{ij}^{+}=\frac{\min\limits_{1\le i\le n}\min\limits_{1\le j\le m} d\left(\gamma_{ij},r_j^{+}\right)+\tau\max\limits_{1\le i\le n}\max\limits_{1\le j\le m} d\left(\gamma_{ij},r_j^{+}\right)}{d\left(\gamma_{ij},r_j^{+}\right)+\tau\max\limits_{1\le i\le n}\max\limits_{1\le j\le m} d\left(\gamma_{ij},r_j^{+}\right)},$$
$$i=1,2,\ldots,n,\ j=1,2,\ldots,m,$$
$$\xi_{ij}^{-}=\frac{\min\limits_{1\le i\le n}\min\limits_{1\le j\le m} d(\gamma_{ij},r_j^{-})+\tau\max\limits_{1\le i\le n}\max\limits_{1\le j\le m} d(\gamma_{ij},r_j^{-})}{d(\gamma_{ij},r_j^{-})+\tau\max\limits_{1\le i\le n}\max\limits_{1\le j\le m} d(\gamma_{ij},r_j^{-})}, \tag{14}$$
$$i=1,2,\ldots,n,\ j=1,2,\ldots,m,$$

where $\tau\in[0,1]$. Generally, $\tau=0.5$ is used.

Step 8. Find out the degree of weighted grey relational coefficient of each alternative as follows:

$$\xi_i^{+}=\sum_{j=1}^{m} w_j\xi_{ij}^{+},\quad \xi_i^{-}=\sum_{j=1}^{m} w\xi_{ij}^{-},\ \text{where } i=1,2,\ldots,n. \tag{15}$$

Step 9. Find out the relative relational degree of each alternative from the positive ideal solution (PIS) and negative ideal solution (NIS) by using the formula as follows:

$$\xi_i=\frac{\xi_i^{+}}{\xi_i^{+}+\xi_i^{-}},\ i=1,2,\ldots,n. \tag{16}$$

Step 10. Rank of alternatives. We rank the alternatives according to the $\xi_i, i=1,2,\ldots,n,$ in descending order and choose the alternative with the maximum ξ_i.

5. Numerical Applications

A medicine company intends to prepare three different types of medicines A_1, A_2 and A_3 (Alternatives) depending upon different compositions, to cure some ailment. Three attributes are involved to select the best medicine for the treatment,

(i). Effectiveness (c_1), (ii). Economy (c_2), (iii). Timings (c_3) .

The positive effects of the medicines on the person who needs medical care, are taken as a positive truth membership functions while negative effects of adverse reactions, are the negative truth membership functions, less time consumption to cure the ailment is taken as a positive indeterminacy function whereas more time consumption is taken as negative indeterminacy functions. likewise, positive and negative economic factors are placed as a positive and negative falsity functions.

This is a hybrid MADM problem involving three different data types: exact values, intervals and linguistic terms. To resolve this matter, we apply the developed method for the ranking and selection of the more effective, fast acting and more economic medicine (alternative). Three experts (e_1, e_2, e_3) are involved in the selection process. Each expert expresses his/her preferences depending upon the worth of the alternatives and upon his/her own knowledge over them. The hybrid decision matrices R^1, R^2 and R^3 given by the experts e_1, e_2 and e_3 are shown in Tables 3, 4 and 5.

Step 1. Consider the neutrosophic bipolar fuzzy hybrid decision matrix of each decision maker. The neutrosophic bipolar fuzzy hybrid decision matrix involves four different data types: exact values, intervals, NBFNs, and linguistic terms.

Step 2. Transform the hybrid decision matrix of each decision maker into neutrosophic bipolar fuzzy decision matrix. The exact values and intervals in the hybrid decision matrices given by the decision makers shown in Tables $3-6$ are standardized and then transformed into a neutrosophic bipolar fuzzy number. The linguistic evaluations shown in Tables $3-6$ are converted into NBFNs by using Table 1. Then, the neutrosophic bipolar fuzzy decision matrix $R^{(k)}(k=1,2,3,4)$ of each decision maker shown in Tables $6,7,8$ and 9 .

Step 3. Determine the weights of decision makers. The importance of the decision makers in the group decision making process is shown in Table 9. These linguistic variables used can be converted into NBFNs by utilizing Table 2. In order to obtain the weights $\lambda_k(k=1,2,3,4)$ of the decision makers, and formula (11) is used:

Table 3. HDM R^1 by e_1

	C_1	C_2	C_3
A_1	VH	2	[20,30]
A_2	H	3	[15,25]
A_3	M	4	[18,24]

Table 4. HDM R^2 by e_2

	C_1	C_2	C_3
A_1	VH	5	[12,24]
A_2	H	3	[18,26]
A_3	M	4	[16,22]

Table 5. HDM R^3 by e_3

	C_1	C_2	C_3
A_1	VH	6	[20,22]
A_2	H	4	[15,18]
A_3	M	3	[12,20]

Table 6. Neutrosophic bipolar fuzzy decision matrix R^1 given by the expert e_1

	C_1	C_2	C_3
A_1	$(0.85,0.42,0.28,-0.1,-0.05,-0.03)$	$(0.78,0.39,0.26,-0.22,-0.11,-0.07)$	$(0.44,0.22,0.15,-0.03,-0.02,-0.01)$
A_2	$(0.70,0.35,0.23,-0.20,-0.10,-0.06)$	$(0.51,0.26,0.17,-0.49,-0.24,-0.16)$	$(0.33,0.16,0.11,-0.19,-0.10,-0.06)$
A_3	$(0.45,0.22,0.15,-0.30,-0.15,-0.10)$	$(0.39,0.2,0.13,-0.61,-0.30,-0.20)$	$(0.40,0.2,0.13,-0.12,-0.06,-0.04)$

Table 7. Neutrosophic bipolar fuzzy decision matrix R^2 given by the expert e_2

	C_1	C_2	C_3
A_1	$(0.85,0.42,0.28,-0.1,-0.05,-0.03)$	$(0.43,0.22,0.14,-0.57,-0.28,-0.19)$	$(0.29,0.14,0.10,-0.11,-0.06,-0.04)$
A_2	$(0.70,0.35,0.23,-0.20,-0.10,-0.06)$	$(0.71,0.36,0.24,-0.29,-0.14,-0.10)$	$(0.43,0.22,0.14,-0.03,-0.02,-0.01)$
A_3	$(0.55,0.27,0.18,-0.30,-0.15,-0.10)$	$(0.54,0.27,0.18,-0.46,-0.23,-0.15)$	$(0.38,0.19,0.13,-0.18,-0.09,-0.06)$

Table 8. Neutrosophic bipolar fuzzy decision matrix R^3 given by the expert e_3

	C_1	C_2	C_3
A_1	$(0.85,0.42,0.28,-0.1,-0.05,-0.03)$	$(0.37,0.18,0.12,-0.63,-0,32,-0.21)$	$(0.57,0.28,0.19,-0.21,-0.10,-0.07)$
A_2	$(0.70,0.35,0.23,-0.02,-0.10,-0.06)$	$(0.55,0.28,0.18,-0.45,-0.22,-0.15)$	$(0.43,0.22,0.14,-0.35,-0.18,-0.12)$
A_3	$(0.55,0.27,0.18,-0.30,-0.15,-0.10)$	$(0.73,0.36,0.24,-0.27,-0.14,-0.09)$	$(0.34,0.17,0.11,-0.28,-0.14,-0.10)$

Table 9. The importance of decision makers

	Linguistic variable	
d_1	Very important	$k=1$
d_2	Important	$k=2$
d_3	Medium	$k=3$

Using (8) we calculate the λ_k which are $\lambda_1 = 0.353, \lambda_2 = 0.334, \lambda_3 = 0.312$ as shown in Figure 3.:

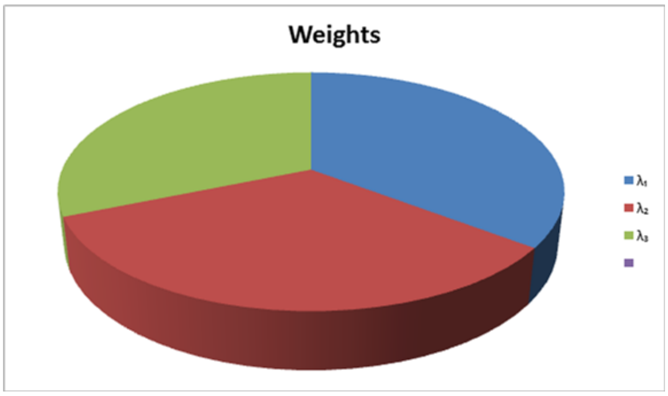

Figure 3. The weight vector

Step 4. Construct the aggregated neutrosophic bipolar fuzzy decision matrix based on the ideas of decision makers. By formula (9), we get the bipolar fuzzy decision matrix R by aggregating all the neutrosophic bipolar fuzzy decision matrices $R^{(K)}(K = 1,2,3)$. The neutrosophic bipolar fuzzy decision matrix R is shown in Table 10.

Table 10. Neutrosophic bipolar fuzzy decision matrix R,

	C_1	C_2	C_3
A_1	$(0.85, 0.42, 0.28, -0.1, -0.05, -0.03)$	$(0.58, 0.29, 0.19, -0.41, -0.20, -0.14)$	$(0.44, 0.22, 0.15, -0.08, -0.04, -0.03)$
A_2	$(0.70, 0.35, 0.23, -0.20, -0.10, -0.06)$	$(0.60, 0.30, 0.20, -0.40, -0.20, -0.13)$	$(0.39, 0.02, 0.13, -0.12, -0.06, -0.04)$
A_3	$(0.45, 0.22, 0.15, -0.30, -0.15, -0.10)$	$(0.57, 0.28, 0.19, -0.43, -0.22, -0.14)$	$(0.37, 0.18, 0.12, -0.18, -0.09, -0.06)$

Step 5. Calculate the entropy weights of the criteria. Use formula (10) to calculate the neutrosophic bipolar fuzzy entropy H_j $(j = 1,2,3)$,

$$H_1 = 0.72, H_2 = 0.78, H_3 = 0.86.$$

Then, use formula (11) to obtain the entropy weights below which are shown in Figure 4.

$$w_1 = 0.44, w_2 = 0.34, w_3 = 0.22.$$

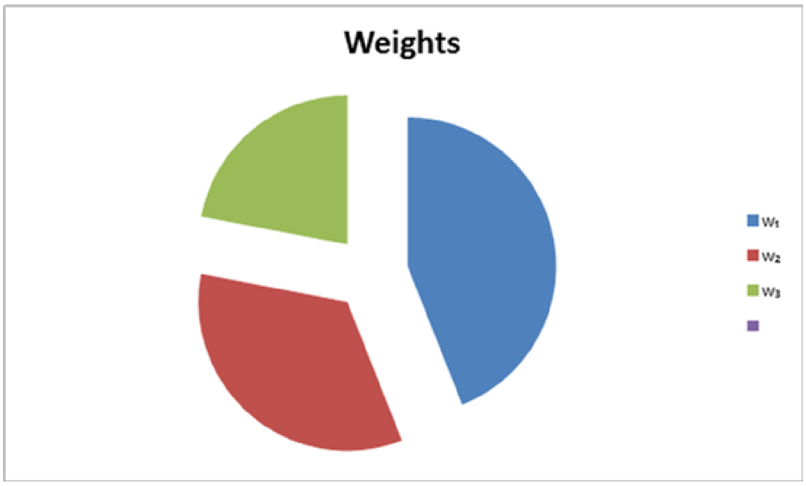

Figure 4. The entropy weight vector

Step 6. The neutrosophic bipolar fuzzy positive ideal solution (PIS) and neutrosophic bipolar fuzzy negative ideal solution (NIS) were obtained as;

$$r^{+} = ((0.85,0.22.0.15,-0.30,-0.05,-0.03)),$$
$$((0.60,0.28,0.19,-0.43,-0.20,-0.13)),(0.44,0.18,0.12,-0.18,-0.04,-0.03).$$

$$r^{-} = ((0.45,0.42.0.28,-0.10,-0.15,-0.10)),$$
$$((0.57,0.30,0.20,-0.40,-0.22,-0.14)),(0.37,0.22,0.15,-0.08,-0.09,-0.05).$$

Step 7. Find out the grey relational coefficient of each alternative from PIS and NIS respectively as in the positive ideal solution ξ^{+} and the negative ideal solution ξ^{-}.

$$\text{Positive ideal solution} \quad \xi^{+} = (\xi_{ij}^{+})_{3\times 3} = \begin{bmatrix} 0.47 & 0.85 & 0.77 \\ 0.40 & 1.00 & 0.71 \\ 0.40 & 1.00 & 0.77 \end{bmatrix}$$

$$\text{Negative ideal solution} \quad \xi^{-} = (\xi_{ij}^{-})_{3\times 3} = \begin{bmatrix} 0.40 & 0.89 & 0.77 \\ 0.40 & 1.00 & 0.77 \\ 0.42 & 1.00 & 0.77 \end{bmatrix}$$

Step 8. According to the above step, the attributes weight vector is:

$$w = (0.44,0.34,0.22)$$

then the degree of grey relational coefficient of each alternative from positive ideal solution (PIS) and negative ideal solution (NIS) can be calculated and are;

$$\xi_1^{+} = 0.67, \xi_2^{+} = 0.68, \xi_3^{+} = 0.69.$$
$$\xi_1^{-} = 0.65, \xi_2^{-} = 0.69, \xi_3^{-} = 0.70.$$

Step 9. Calculate the relative relational degree of each alternative below and shown in Figure 5.

$$\xi_1 = 0.507, \xi_2 = 0.496, \xi_3 = 0.500$$

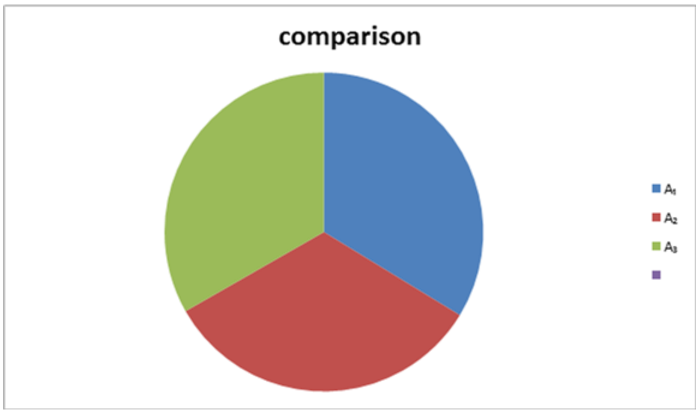

Figure 5. The relative relational degree of alternatives

Step 10. Rank the alternatives. The relative relational degree of alternatives is determined, and then six alternatives are ranked as; $A_1 > A_3 > A_2$. So the alternative A_1 is selected as an appropriate alternative.

6. Comparison Analysis

There is no doubt about that fuzzy sets and all models of fuzzy sets, are helping us out in variety of fields. Amidst of other applications, the decision-making problems are rendered to all versions of fuzzy sets for resolution and can be seen in [27, 29, 30, 34, 45, 47]. Similarity measures have been studied in [16, 45, 49]. Bipolarity in human reasoning and affective decision making studied in [26]. Hybrid multi-attribute group decision making based on intuitionistic fuzzy information and GRA method, discussed in [33]. Recently, [45] defined neutrosophic bipolar fuzzy set and neutrosophic bipolar fuzzy weighted averaging (NBFWA) and neutrosophic bipolar fuzzy ordered weighted averaging (NBFOWA) operators, similarity measures and gave an algorithm and application of neutrosophic bipolar fuzzy sets in decision making in case of multi-attributes.

7. Conclusions

Continuing the work on neutrosophic bipolar fuzzy sets we discussed hybrid multi-attributes group decision making based on neutrosophic bipolar fuzzy sets with different neutrosophic bipolar fuzzy transformation techniques. We apply these concepts and techniques upon hybrid multi-attributes decision making problem of selecting the best medicine to cure some diseases and develop an algorithm for neutrosophic bipolar fuzzy hybrid multi-attribute group decision making. In future the developed technique and procedure can be used in different decision-making problems, like numerical analysis for root convergence [53-58], signature theory, signal processing and operations management [59].

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REFERENCES

1. Zadeh, L. A. Fuzzy sets. Inf. Control, 1965; **8**, pp.338-353.
2. Alhazaymeh, K.; Hassan, N. Vague soft set relations and functions. J. Intell. Fuzzy Systems, 2015; **28**(3), pp. 1205-1212.
3. Alhazaymeh, K.; Hassan, N. Mapping on generalized vague soft expert set, Int. J. Pure Appl. Math., 2014; **93**(3), pp. 369-376.
4. Alhazaymeh, K.; Hassan, N. Generalized vague soft expert set, Int. J. Pure Appl. Math., 2014; **93**(3), 351-360.
5. Alhazaymeh, K.; Hassan, N. Application of generalized vague soft expert set in decision making, Int. J. Pure Appl. Math., 2014; **93**(3), 361-367.
6. Alhazaymeh, K.; Halim, S.A.; Salleh, A.R.; Hassan, N. Soft intuitionistic fuzzy sets, Appl. Math. Sci., 2012; **6**(54), pp. 2669-2680.
7. Adam, F.; Hassan, N. Q-fuzzy soft set, Appl. Math. Sci., 2014; **8**(174), pp. 8689-8695.
8. Adam, F.; Hassan, N. Operations on Q-fuzzy soft set, Appl. Math. Sci., 2014; **8**(175), pp. 8697-8701.
9. Adam, F.; Hassan, N. Q-fuzzy soft matrix and its application, AIP Conf. Proc., 2014; **1602**, pp. 772-778.
10. Adam, F.; Hassan, N. Properties on the multi Q-fuzzy soft matrix, AIP Conf. Proc., 2014; **1614**, pp. 834-839.
11. Adam, F.; Hassan, N. Multi Q-fuzzy soft set and its application, Far East J. Math. Sci., 2015; **97**(7), pp. 871-881.
12. Lee, K.M. Comparison of interval-valued fuzzy sets, intuitionistic fuzzy sets, and bipolar-valued fuzzy sets. J. Fuzzy Logic Intelligent Systems, 2004; **14**, pp.125-129
13. Atanassov, K. Intuitionistic fuzzy sets, Fuzzy Sets Syst.,1986; **20**, pp.87-96.
14. Kandel, A.; Byatt,W. Fuzzy sets, fuzzy algebra and fuzzy statistics. Proc. of the IEEE, 1978; **66**, pp. 1619-1639.

15. Meghdadi, A. H.; Akbarzadeh, M. Probabilistic fuzzy logic and probabilistic fuzzy systems. The 10th IEEE International Conference on Fuzzy Systems, 2001; **3**, pp.1127-1130.
16. Hung,W. L.; Yang, M. S. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern Recognit. Lett., 2004; **25**, pp.1603-1611.
17. Delgado. M.; Verdegay,J. L.; Vila, M. A. Linguistic decision making models. Int. J. Intell. Syst., 1992; **7**, pp.479-492.
18. Truck, I. Comparison and links between two 2-tuple linguistic models for decision making. Knowledge-Based Systems, 2015; **87**, pp.61-68.
19. Varnamkhasti, M.J.; Hassan, N. Neurogenetic algorithm for solving combinatorial engineering problems, J. Appl. Math., 2012, 253714.
20. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, 1999.
21. Zhang, W. R. "Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis," in Proceedings of the Industrial Fuzzy Control and Intelligent Systems Conference, and the NASA Joint Technology Workshop on Neural Networks and Fuzzy Logic and Fuzzy Information Processing Society Biannual Conference, pp. 305–309, San Antonio, Tex, USA, December 1994.
22. Yaqoob, N.; Ansari, M. A. Bipolar, -fuzzy ideals in ternary semigroups. Int. J. Math. Anal., 2013; **7**, pp.1775-1782.
23. Yaqoob, N.; Aslam, M.; Davvaz, B.; Ghareeb, A. Structures of bipolar fuzzy -hyperideals in -semihypergroups. J. Intell. Fuzzy Systems, 2014; **27**, pp.3015-3032.
24. Yaqoob, N.; Aslam, M.; Rehman, I.; Khalaf, M.M. New types of bipolar fuzzy sets in -semihypergroups, Songklanakarin J. Sci. Technol., 2016; **38**, pp.119-127.
25. Dubois, D.; Kaci, S.; Prade, H. Bipolarity in reasoning and decision, an Introduction. International Conference on Inf. Pro. Man. Unc. IPMU, 2004; 04, pp.959-966.
26. Silva Da, R.; Livet, P. Bipolarity in human reasoning and affective decision making. Int. J. Intell. Syst., 2008; **23**, pp.898-922.
27. Abdel-Basset, M.; Mohamed, M. A novel and powerful framework based on neutrosophic sets to aid patients with cancer. Future Generation Computer Systems, 2019; **98**, pp. 144-153.
28. Abdel-Basset, M.; Manogaran, G.; Gamal, A.; Smarandache, F. A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. Design Automation for Embedded Systems, 2018; **22**, pp. 257-278.
29. Abdel-Basset, M.; Manogaran, G.; Gamal, A.; Smarandache, F. A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of Medical Systems, 2019; **43**(38), pp. 1-13. doi:10.1007/s10916019-1156-1
30. Abdel-Basset, M.; Saleh, M.; Gamal, A.; Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Appl. Soft Comput., 2019; **77**, pp. 438-452.
31. Abdel-Basset, Mohamed, and Rehab Mohamed. "A novel plithogenic TOPSIS-CRITIC model for sustainable supply chain risk management." Journal of Cleaner Production **247** (2020): 119586.
32. Qun, W.; Peng, Wu.; Ligang, Z.; Huayou, C.; Xianjun, G. Some new Hamacher aggregation operators under single-valued neutrosophic 2-tuple linguistic environment and their applications to multi-attribute group decision making, Computers & Industrial Engineering, 2018; **116**, pp. 144-162.
33. Guo, J. Hybrid multiattribute group decision making based on intuitionistic fuzzy information and GRA Method. ISRN Applied Mathematics, 2013; **2013**, 146026.
34. Liu, P.; Wang, Y. Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean. Neural Comput. Appl., 2014; **25**, pp.2001-2010.
35. Liu, P.; Shi, L. The generalized hybrid weighted average operator based on interval neutrosophic hesitant set and its application to multiple attribute decision making. Neural Comput. Appl. 2015; **26**, pp.457-471.
36. Peng, J. J. J.; Wang, Q.; Wang, J.; Zhang, H.Y.; Chen, X.H. Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. Int. J. System Sciences, 2015; **8**, pp. 2342-2358.
37. Chen, S. M. A new approach to handling fuzzy decision-making problems. IEEE Trans. Syst. Man Cybern.,1988; **18**, pp.1012-1016.
38. Hung, W. L; Yang, M. S. Similarity measures of intuitionistic fuzzy sets based on Hausdorff distance. Pattern Recognit. Lett., 2004; **25**, pp.1603-1611.
39. Zhan, J.; Khan, M.; Gulistan, M.; Ali, A. Applications of neutrosophic cubic sets in multi-criteria decision making. Int. J. for Uncertainty Quantification, 2017; **7**, pp.377-394.

40. Gulistan, M.; Yaqoob, N.; Rashid, Z.; Smarandache, F.; Wahab, H. A. A Study on Neutrosophic Cubic Graphs with Real Life Applications in Industries. Symmetry 2018; **10**, 203.
41. Sahin,R. ; Kucuk, A. Subsethood measure for single valued neutrosophic sets. J. Intell. Fuzzy Systems, 2015; **29**, pp.525-530.
42. Zavadskas, E. K; Bausys, R.; Kaklauskas, A.; Ubarte, I.; Kuzminske, A.; Gudiene, N. Sustainable market valuation of buildings by the single-valued neutrosophic MAMVA method. Appl. Soft Comput., 2017; **57**, pp.74-87.
43. Zhan, J.; Khan, M.; Gulistan, M.; Ali, A. Applications of neutrosophic cubic sets in multi-criteria decision making. Int. J. for Uncertainty Quantification, 2017; **7**, pp.377-394.
44. Zhang, H. Y.; Wang, J.Q.; Chen, X.H. Interval neutrosophic sets and their application in multicriteria decision making problems. The Scientific World Journal, 2014; **2014**, 645953.
45. Hashim, R. M.; Gulistan, M.; Smarandache, F. Applications of Neutrosophic Bipolar fuzzy Sets in HOPE Foundation for Planning to Build a children Hospital with Different Types of similarity measures. Symmetry 2018; **10**, 331.
46. Zhang, W. R. Bipolar fuzzy sets, Proc. of FUZZ-IEEE, 1998, pp. 835-840.
47. Biswas, P.; Pramanik, S.; Giri, B. C. Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2014; **2**, pp. 102–110.
48. Majumdar, P.; Samanta, S.K. On similarity and entropy of neutrosophic sets. J. Intell. Fuzzy Systems, 2014; **26**, pp.1245-1252.
49. Abu Qamar, M.; Hassan, N. Entropy, measures of distance and similarity of Q-neutrosophic soft sets and some applications. Entropy 2018; **20**, 672.
50. Abu Qamar, M.; Hassan, N. Characterizations of group theory under Q-neutrosophic soft environment. Neutrosophic Sets and Systems, 2019; **27**, pp. 114–130.
51. Biswas, P.; Pramanik, S.; Giri, B. C. Aggregation of triangular fuzzy neutrosophic set information and its application to multi-attribute decision making. Neutrosophic Sets and Systems, 2016; **12**, pp. 20-40.
52. Biswas, P.; Pramanik, S.; Giri, B. C. Entropy based grey relational analysis method for multi-attribute decision making under single valued neutrosophic assessments. Neutrosophic Sets and Systems, 2014; **2**, pp. 102–110.
53. Jamaludin, N.; Monsi, M.; Hassan, N.; Suleiman, M. Modification on interval symmetric single-step procedure ISS-5δ for bounding polynomial zeros simultaneously, AIP Conf. Proc., 2013; **1522**, pp. 750-756.
54. Jamaludin, N.; Monsi, M.; Hassan, N.; Kartini, S. On modified interval symmetric single-step procedure ISS2-5D for the simultaneous inclusion of polynomial zeros, Int. J. Math. Anal., 2013; **7**(20), pp. 983-988.
55. Monsi, M.; Hassan, N.; Rusli, S.F. The point zoro symmetric single-step procedure for simultaneous estimation of polynomial zeros, J. Appl. Math., 2012; **2012**, 709832.
56. Sham, A.W.M.; Monsi, M.; Hassan, N.; Suleiman, M. A modified interval symmetric single step procedure ISS-5D for simultaneous inclusion of polynomial zeros, AIP Conf. Proc., 2013; **1522**, pp. 61-67.
57. Sham, A.W.M.; Monsi, M.; Hassan, N. An efficient interval symmetric single step procedure ISS1-5D for simultaneous bounding of real polynomial zeros, Int. J. Math. Anal., 2013; **7**(20), pp. 977-981.
58. Abu Bakar, N.; Monsi, M.; Hassan, N. An improved parameter regula falsi method for enclosing a zero of a function, Appl. Math. Sci., 2012; **6**(28), pp. 1347-1361.
59. Biswas, P.; Pramanik, S.; Giri, B. C. TOPSIS method for multi-attribute group decision-making under single valued neutrosophic environment. Neural Comput. Appl., 2016; **27** (3), pp. 727-737.

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ELECTRE Approach for Multi-attribute Decision-making in Refined Neutrosophic Environment

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Abstract: Uncertainty, imprecise, incomplete, and inconsistent information can be found in many real-life systems and may enter some problems in a much more complex way. Neutrosophic set is the effective and useful tool to describe problems with Uncertainty, imprecise, incomplete, and inconsistent information. In this regard, the present study is trying to present a neutrosophic electrode model through an example to demonstrate the efficiency of the proposed model. In this example, 3 alternatives were evaluated on 5 criteria by 4 experts based on the neutrosophic linguisting variables. After converting the neutrosophic linguisting variables to neutrosophic numbers, it is paid to calculate the integrated matrix and after that, weights of criteria and experts. In the next steps, the concordance and disconcordance matrices are calculated and after that the calculations are done based on the description of section 3. Finally, are ranked the alternatives in this numerical example. The results show that A_3, A_2 and A_1 were ranked first to third respectively.

Keywords: ELECTRE; Multi-attribute Decision Making; Refined Neutrosophic Environment

1. Introduction

In fact, we have partial, approximate or inaccurate information about the phenomena around ourselves. Uncertainty may occur due to addressing to this inaccurate or partial information. Moreover, Xu and Yager (2006) pointed out that lack of awareness about exact result of a particular choice due to lack of time, lack of accessible information, and insufficient attention of decision makers to the information caused uncertainty. It seems a framework is required to overcome this uncertainty [1]. Liu and lin (2006) classified different uncertainty frameworks into following categories: probability, gray system theory, and fuzzy set theory. Fuzzy set theory is one of the widely accepted frameworks for uncertainty [2]. The general form of this theory is considered as the degree of membership for each set of elements from the reference set, so that there is a large distinction between membership and non-membership of the elements. In fact, determining membership degree for elements is difficult and is accompanied with a degree of hesitation. Considering hesitation, Atanassov (1986) introduced the concept of the intuitive fuzzy set as generalization of fuzzy set [3]. The inventive fuzzy set (IFS) will be defined with three continuous members: the degree of membership, the degree of non-membership, and the degree of hesitation [4], which is the most ideal measure of fuzzy set to describe the information of an uncertain and inaccurate decision [3].

Comparing to fuzzy sets, IFS is more efficient in terms of ambiguity and uncertainty. IFS is confusing and unreliable as the intuitive fuzzy set takes into account membership and non-membership degree as well as hesitation degree which seems to be one of the elements of real-world data. On the other hand, it is difficult to identify “exact values” for membership and non-membership degrees of an element due to the complexity and diversity of real-life management conditions. Therefore, presentation of membership and non-membership degrees as distance may provide appropriate measure for uncertainty, inaccuracy or ambiguity. Atanassov and Gargov (1989) introduced the concept of Interval Valued Intuitionistic Fuzzy Sets (IVIFS) with the degree of membership and the degree of non-membership, whose values are relative to real numbers as interval [5]. IVIFS is the development of a normal distance fuzzy set using the concept of the inventive fuzzy set. Intuitional fuzzy set is a new and effective tool for dealing with a variety of obscure and inaccurate variables for solving decision problems that deals with more vague and uncertain data relative to the intuitive fuzzy set [6].

Although fuzzy sets developed and prevailed, in reality, they could not handle problems with a variety of uncertainty conditions; particularly problems with indeterminate and inconsistent information are not solvable by fuzzy sets. In decision-making problems, fuzzy sets could not handle all types of uncertainty, including indeterminate and inconsistent information, in the real world [7]. In many situations, decision makers have incomplete, indeterminate, and inconsistent options relative to criteria. It has been determined that intuitive fuzzy and fuzzy decision-making analyses are inadequate to handle incomplete, indeterminate, and inconsistent information [8]. Recently Smarandache (1999) has proposed the concepts of non-rooted logic and the neutrosophic set to control these conditions [9]. The set is most appropriate tool for dealing with decision-making problems with incomplete, indeterminate, and inconsistent information while the intuitionistic fuzzy set cannot represent and handle indeterminacy and inconsistent information [10]. The neutrosophic set is a powerful framework that incorporates all the concepts of a definitive set, Fuzzy sets and Fuzzy Intuitionistic sets. The neutrosophic set is identified by three independent degrees, called the degree of accuracy, lack of reliability, and the degree of inaccuracy. These three elements are completely independent. One of the important features of this set is that each of the elements of this set not only has a certain degree of membership, but also have a definite degree of inaccuracy and lack of reliability [11]. It is important to note that, unlike IFS and IVIFS, the uncertainty gap in a neutrosophic set is clearly defined. The neutrosophic set has applications in various fields, including image processing ([12-13]), medical artificial ([14-15]), cluster analyses [16] and supplier selection [17]. Other collections have arisen since the neutrosophic collection is not easy to use in the empirical and practical problems. Wang et al. (2010) introduced a single-value neutrosophic set (SVNS) which is a specific example of a non-stereoscopic set used to handle real-life science and engineering problems [7]. The increasing growth of the neutrosophic collection as well as the pervasiveness of decision-making has led neutrosophic set to be used extensively in decision-making problems. Some uses of this collection in the decision-making process are mentioned in the following.

Ye (2013) examined multi-criteria decision-making problems by using the correlation coefficient in neutrosophic sets [18]. Ye (2014) also introduced a non-stereospecific cross-entropy cross-decision in multi-criteria decision-making problems [19]. Biswas et al. (2014) proposed a gray-based entropy method for solving multiple-decision decision problems in neutrosophic single-value sets. Biswas et al (2014) also proposed a new method for solving multi-criteria decision-making problems based on single-valued neutrosophic sets with specific weights [11].

Also In recent years, several studies have been carried out on multi-criteria decision-making techniques in the neutroscopic environment, including:

Sodenkamp et al., (2018) in a research developed a novel method that uses single-valued neutrosophic sets (NSs) to handle independent multi-source uncertainty measures affecting the reliability of experts’ assessments in group multi-criteria decision-making (GMCDM) problems. In the proposed approach, the neutrosophic indicators are defined to explicitly reflect DMs’ credibility (voting power), inconsistencies/errors inherent to the assessing process, and DMs' confidence in their own evaluation abilities [20]. Liu et al., (2019) in their extended the SS TN and TCN to single-valued numbers (SVNN) and proposed the SS operational laws for SVNNs. Then, they merged the prioritized aggregation (PRA) operator with SS operations, and developed the single valued neutrosophic Schweizer Sklar prioritized weighted averaging (SVNSSPRWA) operator, single valued neutrosophic Schweizer- Sklar prioritized ordered weighted averaging (SVNSSPROWA) operator, single-valued neutrosophic Schweizer-Sklar prioritized weighted geometric (SVNSSPRWG) operator, and single-valued neutrosophic Schweizer-Sklar prioritized ordered weighted geometric (SVNSSPROWG) operator. Moreover, they study some useful characteristics of these proposed aggregation operators (AOs) and proposed two decision making models to deal with multiple-attribute decision making (MADM) problems under SVN information based on the SVNSSPRWA and SVNSSPRWG operators [21]. Liu & you (2019) in their study defined a new distance measure between two linguistic neutrosophic sets (LNSs), and build a model based on the maximum deviation to obtain fuzzy measure, further, they developed the bidirectional projection-based MCGDM method with LNNs in which a weight model based on fuzzy measure is proposed where the weights of evaluation criteria is partial unknown and the interactions among criteria are considered[22]. Thong et al., (2019) in their study proposed a new concept called the Dynamic Interval-valued Neutrosophic Set (DIVNS) for such the dynamic decision-making applications [23]. In the same vein, Abdul Basset et al., have done many studies in the neutrosophic environment such as: supplier selection with group TOPSIS technique under type-2 neutrosophic number[24], project selection with a hybrid neutrosophic multiple criteria group decision making[25], evaluation Hospital medical care systems based on plithogenic sets[26], selecting supply chain with a hybrid plithogenic decision-making approach[27], solve transition difficulties with Utilizing neutrosophic theory[28], Evaluation of the green supply chain management practices[29].

ELECTRE method was introduced by Benayoun, Roy and Sussmann in 1966[30], and has been successfully and widely used in many decision-making problems including agricultural [31], medical science [32], financial [33], economics [34], project selection [35], communication and transportation ([35-36]). The origin of ELECTRE method dates back to 1965, when an European consulting firm employed a team of researchers to make a decision on real multi-criteria problems on innovation in new activities of institutions [37]. ELECTRE method uses the concept of outranking comparisons. This idea relates to the concepts of coordination, inconsistency, and non-rank, deriving from real world applications [38]. The method uses the consistency and inconsistency indices for analyzing non-ranked comparisons between the options [39]. ELECTRE method was developed and different types of this method which are proposed to overcome in decision making conditions are among these methods ELECTRE I, ELECTRE II, ELECTRE III, ELECTRE IV, ELECTRE TRI-C and ELECTRE IS ([37],[39],[40-41]) .

Given the extension of this method, it is worth noting that the ELECTRE method as an efficient and useful method in management research has not yet been developed in the context of the neutrosophic ambiguity. For this purpose, the present paper seeks to develop a neutrosophic ELECTRE method based on intuitive fuzzy ELECTRE method.

2. Refined Neutrosophic Environment

Neutrosophy has been proposed by Smarandache [42-43] as a new branch of philosophy, with ancient roots, dealing with "the origin, nature and scope of neutralities, as well as their interactions

with different ideational spectra". The fundamental thesis of neutrosophy is that every idea has not only a certain degree of truth, as is generally assumed in many-valued logic contexts, but also a falsity degree and an indeterminacy degree that have to be considered independently from each other. Smarandache seems to understand such "indeterminacy" both in a subjective and an objective sense, i.e. as uncertainty as well as imprecision, vagueness, error, doubtfulness, etc [44].
In this section, some basic concepts and definitions of NSs and SNSs are briefly reviewed.

2.1. NS and SNSs

In this subsection, the definitions and operations of NSs and SNSs are introduced.

Definition 1. Let X is a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy- membership function $I_A(X)$ and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(X)$ and $F_A(x)$ are real standard or nonstandard subsets of $]0-,1+[$ [9, 45]. In other words, $T_A(x): X \to]0-,1+[$, $I_A(x): X \to]0-,1+[$, and $F_A(x): X \to]0-,1+[$. We have no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$; thus, $0- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3+$ [46].

In other form, the neutrosophic set A is an object having the following form $A = \{\langle T_A(X), I_A(X), F_A(X), x \in X \rangle\}$.

The set $I_A(X)$ may represent not only indeterminacy, but also vagueness, uncertainty, imprecision, error, contradiction, undefined, unknown, incompleteness, redundancy, etc.[44],[47]. In order to catch up vague information, an indeterminacy-membership degree can be split into subcomponents, such as ''contradiction,'' ''uncertainty'', and ''unknown''[48].

Definition 2. A neutrosophic set A is contained in the other neutrosophic set B, denoted by $A \subseteq B$ if and only if $\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$, $\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$, $\inf F_A(x) \ge \inf F_B(x)$, and $\sup F_A(x) \ge \sup F_B(x)$ for every x in X [9].

Definition 3. The complement of a neutrosophic set A is denoted by A^c and is defined as $T_A^c(x) = \{1^+\} - T_A(x)$, $I_A^c(x) = \{1^+\} - I_A(x)$, and $F_A^c(x) = \{1^+\} - F_A(x)$ for every x in X [9].

Since it is hard to use NSs to solve practical problems, so Wang et al introduced Single-valued neutrosophic sets that can be used in real scientific and engineering applications.

2.2. Single-valued neutrosophic sets

Single-valued neutrosophic set is a special case of neutrosophic set. In this section, some basic definitions, operations, and properties regarding single valued neutrosophic sets are introduced.

Definition 4. Let X be a space of points (objects) with generic elements in X denoted by x. An $SVNS$ A in X is characterized by the truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. For each point x in X, $T_A(x), I_A(x), F_A(x) \in [0,1]$ [7].

Therefore, an $SVNS$ A can be written as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in X\}$$

The following expressions are defined in[7] for $SVNSs$ A, B:

1- $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$ for any x in X, 2- $A = B$ if and only if $A \subseteq B$, $B \subseteq A$,

3- $A^c = \{\langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle | x \in X\}$.

For convenience, an $SVNS$ A is denoted by the simplified symbol $A = \{T_A(x), I_A(x), F_A(x)\}$ for any x in X. For two $SVNSs$ A and B, the operational relations are defined by [7].

1- $A \cup B = \langle \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_B(x)) \rangle$ for any x in X,

2- $A \cap B = \langle \min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_B(x)) \rangle$ for any x in X,

3- $A \oplus B = \langle T_A(x) + T_B(x) - T_A(x).T_B(x), I_A(x).I_B(x), F_A(x).F_B(x) \rangle$ for any x in X,

4- $A \otimes B = \langle T_A(x).T_B(x), I_A(x) + I_B(x) - I_A(x).I_B(x), F_A(x) + F_B(x) - F_A(x).F_B(x) \rangle$ for any x in X,

5. $\lambda A = \left\langle 1 - (1 - T_A(x))^{\lambda}, (I_A(x))^{\lambda}, (F_A(x))^{\lambda} \right\rangle, \lambda > 0$ for any x in X [35],

6. $A^{\lambda} = \left\langle (T_A(x))^{\lambda}, 1 - (1 - I_A(x))^{\lambda}, 1 - (1 - F_A(x))^{\lambda} \right\rangle, \lambda > 0$ for any x in X [35],

7- $\Delta A = \langle \min(T_A(x) + I_A(x), 1), 0, F_A(x) \rangle$ for any x in X,

8- $\nabla A = \langle T_A(x), 0, \min(F_A(x) + I_A(x), 1) \rangle$ for any x in X.

2.3. Neutrosophic refined set

Let A be a neutrosophic refined set.

$A = \left\{ \left\langle x, \left(T_A^1(x_i), T_A^2(x_i), \ldots, T_A^m(x_i)\right), \left(I_A^1(x_i), I_A^2(x_i), \ldots, I_A^m(x_i)\right), \left(F_A^1(x_i), F_A^2(x_i), \ldots, F_A^m(x_i)\right) \right\rangle : x \in X \right\}$

where $T_A^j(x_i)$: $X \in [0,1]$, $I_A^j(x_i)$: $X \in [0,1]$, $F_A^j(x_i)$: $X \in [0,1]$, $j = 1, 2, \ldots, m$ such that $0 \le \sup T_A^j(x_i) + \sup I_A^j(x_i) + \sup F_A^j(x_i) \le 3$, $j = 1, 2, \ldots, m$ for any $x \in X$. Now, $\left(T_A^j(x_i), I_A^j(x_i), F_A^j(x_i)\right)$ are the truth-membership sequence, indeterminacy-membership sequence, and falsity-membership sequence of the element x, respectively. Also, m is called the dimension of neutrosophic refined sets A [50].

2.4. Distance between two SVNSs

Majumdar and Samanta [51] studied similarity and entropy measure by incorporating Euclidean distances of neutrosophic sets.

2.4.1. Euclidean distance between two SVNSs

Let $A = \{\langle x_i : T_A(x_i), I_A(x_i), F_A(x_i) \rangle, i = 1, 2, \ldots, n\}$ and

$B = \{\langle x_i : T_B(x_i), I_B(x_i), F_B(x_i) \rangle, i = 1, 2, \ldots, n\}$ be $SVNSs$. Then the Euclidean distance between two $SVNSs$ A and B can be defined as follows[48]:

$$E(A, B) = \sqrt{\sum_{i=1}^{n} \left((T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2 \right)} \tag{1}$$

The normalized Euclidean distance between two $SVNSs$ A and B can be defined as follows:

$$E_N(A, B) = \sqrt{\frac{1}{3n} \sum_{i=1}^{n} \left((T_A(x_i) - T_B(x_i))^2 + (I_A(x_i) - I_B(x_i))^2 + (F_A(x_i) - F_B(x_i))^2 \right)} \tag{2}$$

2.4.2. The Hamming distance between two SVNSs

the Hamming distance between two $SVNSs$ A and B can be defined as follows[51]:

$$L_{Ham}(A,B)=\sum_{i=1}^{n}\{|T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)|+|F_A(x_i)-F_B(x_i)|\} \tag{3}$$

The normalized Hamming distance between two SVNSs A and B can be defined as follows:

$$L_{Ham(N)}(A,B)=\frac{1}{3n}\sum_{i=1}^{n}\{|T_A(x_i)-T_B(x_i)|+|I_A(x_i)-I_B(x_i)|+|F_A(x_i)-F_B(x_i)|\} \tag{4}$$

2.5. Crispfication of a neutrosophic set

Let $A=\{\langle x_i : T_{A_j}(x_i), I_{A_j}(x_i), F_{A_j}(x_i)\rangle, j=1,2,\ldots,n\}$ be n $SVNSs$. The equivalent crisp number of each W_j can be defined as [11]:

$$W_j^c=\frac{1-\sqrt{\frac{\left((1-T_{A_j}(x_i))^2+(I_{A_j}(x_i))^2+(F_{A_j}(x_i))^2\right)}{3}}}{\sum_{i=1}^{n}\left\{1-\sqrt{\frac{\left((1-T_{A_j}(x_i))^2+(I_{A_j}(x_i))^2+(F_{A_j}(x_i))^2\right)}{3}}\right\}} \tag{5}$$

$W_j^c \geq 0$, $\sum_{k=1}^{p} W_j^c = 1$

3. ELECTRE approach

The ELECTRE approach is employed to identify the best alternative. The ELECTRE approach can be presented as follows (including 9 steps):

Step 1. Determining the decision matrix: Assume that $A=\{A_1, A_2,\ldots, A_m\}$ is the set of alternatives with the set C of n criteria, $C=\{C_1, C_2,\ldots, C_n\}$, $D=(d_{ij})_{m\times n}$ is the decision matrix, and $W=\{W_1, W_2,\ldots, W_n\}$ is the weight vector of criteria that the sum of weight of all criteria is equal to 1.

Table 1. Single-valued neutrosophic set decision matrix

$D=(d_{ij})_{m\times n}=$

Criteria / alternatives	C_1	C_2	$\cdots$	C_n
A_1	$\langle d_{11}\rangle$	$\langle d_{12}\rangle$	$\cdots$	$\langle d_{1n}\rangle$
A_2	$\langle d_{21}\rangle$	$\langle d_{22}\rangle$	$\cdots$	$\langle d_{2n}\rangle$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
A_m	$\langle d_{m1}\rangle$	$\langle d_{m2}\rangle$	$\cdots$	$\langle d_{mn}\rangle$
W_j	w_1	w_2	$\cdots$	w_n

Here, $d_{ij}\left(i=1,2,\ldots,m \ \ and \ \ j=1,2,\ldots,n\right)$ are all single-valued neutrosophic numbers.

Here, λ is the vector of experts' weight, based on which the opinion of experts is aggregated.

Step 2. Aggregate the decision makers (DMs') opinion to construct an neutrosophic decision matrix

Let $r_{ij}^{k}=\left(T_{ij}^{k},I_{ij}^{k},F_{ij}^{k}\right)$ be the neutrosophic number provided by DM_k on the assessment of A_i with respect to C_j. The aggregated neutrosophic rating of alternatives with respect to each criterion is calculated based on neutrosophic weighted averaging (NWA) operator as:

$$r_{ij}^{k}=NWA\left(r_{ij}^{(1)},r_{ij}^{(2)},\ldots,r_{ij}^{(l)}\right)=\left\langle 1-\prod_{k=1}^{l}\left(1-T_{ij}^{(k)}\right)^{\lambda_k},\prod_{k=1}^{l}\left(I_{ij}^{(k)}\right)^{\lambda_k},\prod_{k=1}^{l}\left(F_{ij}^{(k)}\right)^{\lambda_k}\right\rangle \tag{6}$$

Step 3. Determining the weights of criteria: There are various ways to determine the weights of the criteria.

Let $w_j^k=\left(T_j^k,I_j^k,F_j^k\right)$ be the weight of criterion C_j given by K^{th} decision-maker DM. The aggregated neutrosophic weights $\left(w_j\right)$ of criteria are calculated by

$$w_j=\lambda_1 w_j^{(1)}\cup\lambda_2 w_j^{(2)}\cup\ldots\cup\lambda_k w_j^{(k)}=\left\langle 1-\prod_{k=1}^{l}\left(1-T_{ij}^{(k)}\right)^{\lambda_k},\prod_{k=1}^{l}\left(I_{ij}^{(k)}\right)^{\lambda_k},\prod_{k=1}^{l}\left(F_{ij}^{(k)}\right)^{\lambda_k}\right\rangle \text{ where } w_j=\left(T_j,I_j,F_j\right),\ j=1,2,\ldots,n$$

Step 4. Determining the concordance and discordance sets: In this step the concordance and discordance sets are determined. The concordance set can be classified in different types of the concordance sets as strong concordance set, moderate concordance set and weak concordance set. It is the same for the discordance sets.

the strong concordance set is determined as follows:

$$C_{kl}=\left\{j\middle|T_{kj}\geq T_{lj},F_{kj}<F_{lj},I_{kj}<I_{lj}\right\} \tag{7}$$

moderate concordance set is as follows:

$$C'_{kl}=\left\{j\middle|T_{kj}\geq T_{lj},F_{kj}<F_{lj},I_{kj}\geq I_{lj}\right\} \tag{8}$$

weak concordance set is as follows:

$$C''_{kl}=\left\{j\middle|T_{kj}\geq T_{lj},F_{kj}\geq F_{lj}\right\} \tag{9}$$

The strong discordance set can be determined in ELECTRE method as follows:

$$D_{kl}=\left\{j\middle|T_{kj}<T_{lj},F_{kj}\geq F_{lj},I_{kj}\geq I_{lj}\right\} \tag{10}$$

moderate discordance set is as follows:

$$D'_{kl}=\left\{j\middle|T_{kj}<T_{lj},F_{kj}\geq F_{lj},I_{kj}<I_{lj}\right\} \tag{11}$$

weak discordance set is as follows:

$$D''_{kl} = \{ j | T_{kj} < T_{lj}, F_{kj} < F_{lj} \} \tag{12}$$

Decision makers give weights in different sets. W_C , $W_{C'}$, $W_{C''}$, W_D , $W_{D'}$ and $W_{D''}$ are the weights of the strong concordance, moderate concordance, weak concordance, strong discordance, moderate discordance and weak discordance sets, respectively.

The concepts of concordance sets and discordance sets are used for calculating concordance sets and discordance matrixes and then determining the aggregate dominance matrix.

Step 5. Constructing the concordance and discordance matrixes: The relative value of the concordance set is measured through the concordance index. the concordance index shows that the relative dominance of certain alternative over a competing alternative. The concordance index g_{kl} between A_k and A_l is defined as:

$$C_{kl} = w_C \times \sum_{j \in C_{kl}} w_j + w_{C'} \times \sum_{j \in C'_{kl}} w_j + w_{C''} \times \sum_{j \in C''_{kl}} w_j \tag{13}$$

The concordance matrix C is defined as follows:

$$C = \begin{bmatrix} - & c_{12} & \cdots & \cdots & c_{1m} \\ c_{21} & - & c_{23} & \cdots & c_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ c_{(m-1)1} & \cdots & \cdots & - & c_{(m-1)m} \\ c_{m1} & c_{m2} & \cdots & c_{m(m-1)} & - \end{bmatrix}$$

It is obvious that a higher value of c_{kl} indicates that A_k is preferred to A_l. The discordance index d_{kl} between A_k and A_l is defined as:

$$d_{kl} = \frac{\max_{j \in D_{kl}} w_D{}^* \times dis(X_{kj}, X_{lj})}{\max_{j \in J} dis(X_{kj}, X_{lj})} \tag{14}$$

$$dis(X_{kj}, X_{lj}) = \sqrt{\frac{1}{2}\left((T_{kj} - T_{lj})^2 + (I_{kj} - I_{lj})^2 + (F_{kj} - F_{lj})^2\right)}$$

$w^*{}_D$ is equal to W_D, $W_{D'}$ and $W_{D''}$ depending on the different types of discordance sets. The discordance matrix D is defined as follows:

$$D = \begin{bmatrix} - & d_{12} & \cdots & \cdots & d_{1m} \\ d_{21} & - & d_{23} & \cdots & d_{2m} \\ \cdots & \cdots & - & \cdots & \cdots \\ d_{(m-1)1} & \cdots & \cdots & - & d_{(m-1)m} \\ d_{m1} & d_{m2} & \cdots & d_{m(m-1)} & - \end{bmatrix}$$

Step 6. Constructing the concordance and discordance dominance matrixes: The concordance dominance matrix F can be calculated with aid of a threshold value for the concordance index.

When concordance index of c_{kl} does not exceed the minimum specified boundary value, or $c_{kl} \geq \bar{c}$, only A_k has the chance of mastery over A_l.

$$\bar{c} = \frac{\sum_{k=1,k\neq l}^{m} \sum_{l=1,l\neq k}^{m} c_{kl}}{m \times (m-1)} \tag{15}$$

Based on the boundary value of Boolean F matrix, each element of this matrix is as follows:

If $c_{kl} \geq \bar{c}$ $f_{kl} = 1$

If $c_{kl} < \bar{c}$ $f_{kl} = 0$

In this matrix, element 1 indicates mastery of an option with respect to other elements.

The discordance dominance matrix G can be calculated with aid of a threshold value for the discordance index.

This matrix is built for discordance index of d_{kl} like F matrix with a boundary value of $\bar{d}$. g_{kl} element of discordance dominance matrix G is measured as follows:

$$\bar{d} = \frac{\sum_{k=1,k\neq l}^{m} \sum_{l=1,l\neq k}^{m} d_{kl}}{m \times (m-1)} \tag{16}$$

The following equations are established:

If $d_{kl} \geq \bar{d}$ $g_{kl} = 1$

If $d_{kl} < \bar{d}$ $g_{kl} = 0$

Each element of matrix G indicates mastery relations between two options.

Step 7. Determining the aggregate dominance matrix: Thus, step is to calculated the intersection of the concordance dominance matrix F and the discordance dominance matrix G . Each of elements of this matrix e_{kl} is defined as follows:

$$e_{kl} = f_{kl} \times g_{kl} \tag{17}$$

Step 8. Eliminate the less favorable alternatives: The aggregate dominance matrix E provides orders of relative preferences of options. If $e_{kl} = 1$, it means that A_k is preferable to A_l for both concordance and disharmony criteria, but A_k still has a chance of mastery over other options. Conditions where A_k cannot be mastered in ELECTERE method are as follows:

When at least a l is equal to one. $e_{kl} = 1,\ l = 1,2,\ldots,m,\ k \neq l$

For all of i $e_{kl} = 0,\ i = 1,2,\ldots,m,\ i \neq k, i \neq l$

Application of these conditions seems difficult, but mastery options can be easily identified in E matrix. If each column of matrix E has at least an element with value 1, this column is mastered by its other studied rows. Therefore, columns with element 1 will be easily removed.

Step 9. Using the ranking process proposed by Wu and Chen**:** Since ELECTERE method cannot rank all options, we use proposed method by Wu and Chen[52] for ranking options. Steps of this method are as follows.

Step 9.1. Determining concordance matrix c' :This step uses ideal TOPSIS solution method. If c^* is the largest value of concordance matrix, matrix c' will be obtained by calculation of the following equation.

$$c'_{kl} = c^* - c_{kl} \tag{18}$$

Step 9.2. Determining discordance matrix d': If d^* is the largest value of discordance matrix, matrix d' will be obtained by calculation of the following equation.

$$d'_{kl} = d^* - d_{kl} \tag{19}$$

Step 9.3. Determining the aggregate dominance matrix P:

$$P = \begin{bmatrix} - & p_{12} & \cdots & p_{1m} \\ p_{21} & - & p_{23} & p_{2m} \\ \vdots & & & \\ p_{m1} & p_{m2} & p_{m(m-1)} & - \end{bmatrix}$$

Each element of matrix P is defined according to the following equation.

$$p_{kl} = \frac{d'_{kl}}{c'_{kl} + d'_{kl}} \tag{20}$$

Here, c'_{kl} is the element of concordance dominance matrix, and d'_{kl} is the element of discordance dominance matrix.

Step 9.4. Determining the best alternative: According to results of Step 9-3, we can obtain the combinatorial evaluation of options through Equation 21.

$$\bar{p}_k = \frac{1}{m-1} \sum_{l=1, l \neq k}^{m} p_{kl}, \; k = 1, 2, \ldots, m \tag{21}$$

Then, the best option is specified according to Equation 22, and finally options are ranked incrementally.

$$A^* = \max\{\bar{p}_k\} \tag{22}$$

A^* is the best alternative.

The process summary of the proposed method is shown in Figure 1.

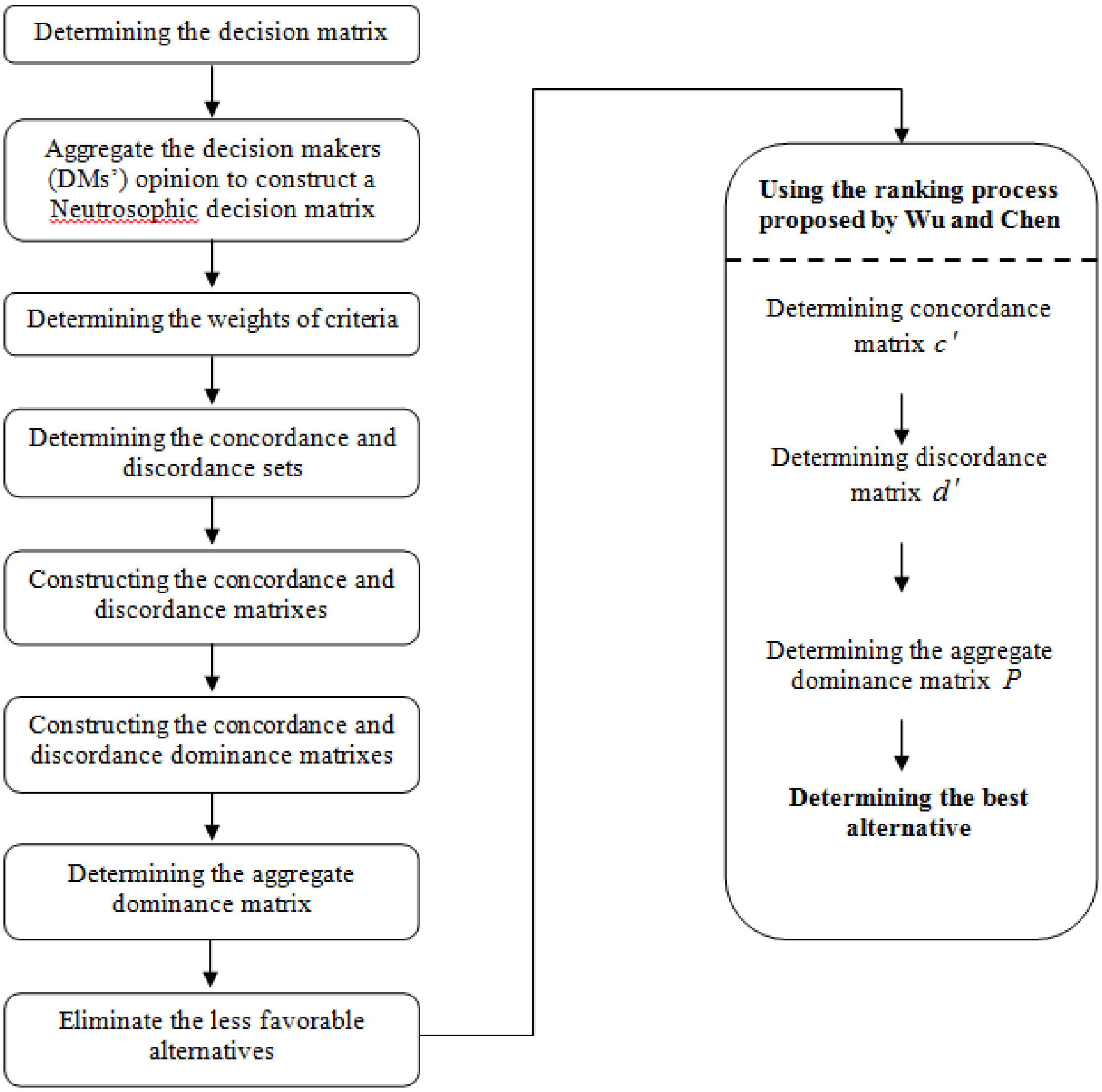

Figure 1: The proposed model of Neutrosophic ELECTRE

4. Numerical example

In this section, we solve a problem to show the effectiveness of the proposed approach. There are three alternatives A_1, A_2, A_3 and five criteria C_1, C_2, C_3, C_4, C_5. Then, the proposed procedure for solving the problem is provided using the following steps.

Step 1. Constructing the decision matrix: The results of the evaluation of alternatives by four experts, based on the criteria, are shown in the table below:

Table 2. Evaluation of alternatives by neutrosophic numbers

D_1	C_1	C_2	C_3	C_4	C_5
A_1	(0.7,0.2,0.1)	(0.8,0.3,0.3)	(0.4,0.1,0.2)	(0.5,0.1,0.1)	(0.6,0.4,0.1)
A_2	(0.6,0.2,0.1)	(0.7,0.4,0.2)	(0.3,0.2,0.1)	(0.3,0.1,0.2)	(0.8,0.2,0.2)
A_3	(0.7,0.1,0.2)	(0.6,0.2,0.2)	(0.4,0.4,0.4)	(0.6,0.1,0.1)	(0.7,0.1,0.1)
D_2	C_1	C_2	C_3	C_4	C_5
A_1	(0.8,0.2,0.1)	(0.7,0.1,0.2)	(0.5,0.1,0.1)	(0.6,0.2,0.3)	(0.5,0.6,0.1)
A_2	(0.7,0.3,0.2)	(0.6,0.1,0.1)	(0.6,0.2,0.3)	(0.5,0.1,0.2)	(0.4,0.5,0.2)
A_3	(0.6,0.2,0.2)	(0.8,0.2,0.1)	(0.6,0.1,0.2)	(0.7,0.1,0.1)	(0.5,0.5,0.1)
D_3	C_1	C_2	C_3	C_4	C_5
A_1	(0.9,0.1,0.1)	(0.5,0.3,0.2)	(0.6,0.4,0.1)	(0.2,0.5,0.3)	(0.4,0.4,0.4)
A_2	(0.8,0.2,0.1)	(0.6,0.3,0.1)	(0.5,0.4,0.1)	(0.4,0.2,0.1)	(0.5,0.3,0.2)
A_3	(0.8,0.1,0.2)	(0.7,0.1,0.1)	(0.6,0.3,0.2)	(0.4,0.1,0.1)	(0.6,0.1,0.2)
D_4	C_1	C_2	C_3	C_4	C_5
A_1	(0.6,0.1,0.1)	(0.8,0.2,0.1)	(0.9,0.2,0.3)	(0.7,0.4,0.3)	(0.7,0.3,0.4)
A_2	(0.7,0.2,0.01)	(0.7,0.1,0.3)	(0.7,0.3,0.1)	(0.6,0.5,0.1)	(0.6,0.2,0.4)
A_3	(0.7,0.1,0.2)	(0.6,0.1,0.2)	(0.6,0.2,0.1)	(0.7,0.1,0.3)	(0.7,0.3,0.2)

Step 2. Aggregate the decision makers (DMs') opinion to construct a neutrosophic decision matrix: The aggregated decision matrix can be determined by applying the aggregated operator (6) and is calculated as shown below:

Table 2. The aggregated neutrosophic decision matrix

	C_1	C_2	C_3	C_4	C_5
A_1	(0.738,0.144,0.1)	(0.695,0.203,0.187)	(0.57,0.162,0.158)	(0.465,0.244,0.225)	(0.543,0.414,0.193)
A_2	(0.693,0.222,0.067)	(0.65,0.184,0.158)	(0.499,0.259,0.133)	(0.436,0.175,0.144)	(0.559,0.278,0.238)
A_3	(0.693,0.12,0.2)	(0.67,0.144,0.143)	(0.54,0.219,0.201)	(0.593,0.1,0.132)	(0.619,0.201,0.139)

Step 3. Determining the weights of the criteria: The weight matrix (see Table 3) of the criteria described in this problem can be displayed as follows:

Table 3. Weight matrix of criteria

	C_1	C_2	C_3	C_4	C_5
D_1	(0.9,0.1,0.2)	(0.8,0. 2,0.3)	(0.5,0.4,0.3)	(0.5,0.2,0.15)	(0.5,0.4,0.4)
D_2	(0.8,0.2,0.1)	(0.7,0.1,0.3)	(0.6,0.3,0.3)	(0.8,0.25,0.1)	(0.6,0.3,0.4)
D_3	(0.6,0.3,0.2)	(0.5,0.3,0.2)	(0.8,0.2,0.1)	(0.7,0.2,0.1)	(0.4,0.4,0.4)
D_4	(0.6,0.1,0.2)	(0.6,0.1,0.2)	(0.6,0.2,0.3)	(0.5,0.1,0.2)	(0.3,0.2,0.1)

The aggregated weights for all criteria are presented below:

Table 4. The aggregated weights of criteria

	C_1	C_2	C_3	C_4	C_5
	(0.725,0.15,0.166)	(0.653,0.15,0.25)	(0.604,0.27,0.241)	(0.608,0.178,0.133)	(0.444,0.31,0.281)

According to Table.4 and equation 5, the crisp of weights of criteria are presented as following:

Table 6. The crisp of weights of criteria

CRITERA	C_1	C_2	C_3	C_4	C_5
Crisp weight	0.204	0.202	0.200	0.202	0.192

Step 4. Determining the concordance and discordance sets**:** In this step, assume that the subjective importance of attributes, W, is given by the decision maker, the decision maker also gives the relative weight (W')

$$W' = \{w_C, w_{C'}, w_{C''}, w_D, w_{D'}, w_{D''}\} = \left\{1, \frac{2}{3}, \frac{1}{3}, 1, \frac{2}{3}, \frac{1}{3}\right\}$$

The strong concordance set described in this problem can be displayed as follows:

$$C = \begin{bmatrix} - & - & C_3 \\ C_4 & - & - \\ C_4, C_5 & C_2, C_4, C_5 & - \end{bmatrix}$$

The moderate concordance set described in this problem can be displayed as follows:

$$C' = \begin{bmatrix} - & - & C_1 \\ - & - & C_1 \\ - & - & - \end{bmatrix}$$

The weak concordance set described in this problem can be displayed as follows:

$$C'' = \begin{bmatrix} - & C_1, C_2, C_3 & C_2 \\ C_5 & - & - \\ - & C_1, C_3 & - \end{bmatrix}$$

The strong discordance set described in this problem can be displayed as follows:

$$D = \begin{bmatrix} - & C_4 & C_4, C_5 \\ - & - & C_2, C_4, C_5 \\ C_3 & - & - \end{bmatrix}$$

The moderate discordance set described in this problem can be displayed as follows:

$$D' = \begin{bmatrix} - & - & - \\ - & - & - \\ C_1 & - & - \end{bmatrix}$$

The weak discordance set described in this problem can be displayed as follows:

$$D'' = \begin{bmatrix} - & C_5 & - \\ C_1, C_2, C_3 & - & C_3 \\ C_2 & - & - \end{bmatrix}$$

Step 5. Calculating the concordance and discordance matrixes: The concordance matrix described in this problem can be calculated as follows:

$$C = \begin{bmatrix} - & 0.202 & 0.403 \\ 0.266 & - & 0.136 \\ 0.394 & 0.733 & - \end{bmatrix}$$

The discordance matrix described in this problem can be calculated as follows:

$$D = \begin{bmatrix} - & 0.578 & 0.999 \\ 0.289 & - & 0.650 \\ 0.111 & 0 & - \end{bmatrix}$$

Step 6. Determining the concordance and discordance dominance matrixes: The concordance dominance matrix can be determined. The average concordance index is:

$$\bar{c} = \frac{\sum_{k=1,k\neq l}^{3} \sum_{l=1,l\neq k}^{3} c_{kl}}{3\times 2} = 0.356 \qquad F = \begin{bmatrix} - & 0 & 1 \\ 0 & - & 0 \\ 1 & 1 & - \end{bmatrix}$$

The discordance dominance matrix can be determined. The average discordance index is:

$$\bar{d} = \frac{\sum_{k=1,k\neq l}^{3} \sum_{l=1,l\neq k}^{3} d_{kl}}{3\times 2} = 0.438 \qquad G = \begin{bmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix}$$

Step 7. Determining the aggregate dominance matrix: The aggregate dominance matrix can be determined.

$$E = \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 1 & 1 & - \end{bmatrix}$$

Step 8. Eliminating the less favourable alternatives: Using the seventh step, we remove the undesirable alternative. Matrix E provides the following ranking Figure. 2.

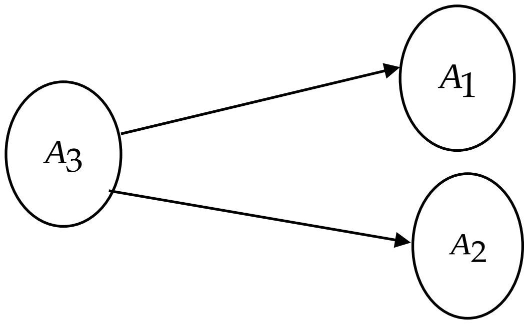

Figure 2. Ranking of Matrix E

It is obvious that A_3 is preferred to A_1 and A_2. But two alternatives of A_1 and A_2 cannot be ranked. This condition appears difficult to apply, but the dominated alternatives can be easily identified in the E matrix. In this section it used ranking process proposed by wu and chen. This process is as following:

Step 9. Using the ranking process:

9.1. Determining concordance matrix c': The concordance dominance matrix can be calculated as follows:($c^* = 0.733$)

$$C' = \begin{bmatrix} - & 0.531 & 0.330 \\ 0.467 & - & 0.597 \\ 0.339 & 0 & - \end{bmatrix}$$

9.2. Determining discordance matrix d': The discordance dominance matrix can be calculated as follows:($d^* = 0.999$)

$$D' = \begin{bmatrix} - & 0.421 & 0 \\ 0.710 & - & 0.349 \\ 0.888 & 0.999 & - \end{bmatrix}$$

9.3. Determining the aggregate dominance matrix P: The aggregate dominance matrix can be calculated as follows:

$$P = \begin{bmatrix} - & 0.442 & 0 \\ 0.603 & - & 0.369 \\ 0.724 & 1 & - \end{bmatrix}$$

9.4. Determining the best alternative: According to the values of $\overline{P}$ the best alternative is determined.

$\overline{P}_1 = 0.221, \overline{P}_2 = 0.486, \overline{P}_3 = 0.862$

The optimal ranking order of the alternatives is given by $A_3 > A_2 > A_1$. The best alternative is A_3.

5. Conclusion

This paper has proposed an approach for solving MCDM problems using neutrosophic and ELECTRE method. In many cases, it is difficult for decision-makers to precisely express a preference when solving Multi-attribute decision making (MADM) problems with uncertain information. SVNSES is an effective and useful decision-making tool to describe indeterminate and inconsistent information and it is also possible for a user to view the opinions of all experts in a single model. Since SVNNs reflect not only the degrees of truth (membership) and falsity (non-membership), but also indeterminacy, the evaluation information was described more comprehensively in the proposed approach. This paper is devoted to present a new ELECTERE-based approach for MADM under neutrosophic environment. In the evaluation process, the ratings of each alternative with respect to each attribute are given as linguistic variables characterized by single-valued neutrosophic numbers. After the formation and integration of the decision matrix, the weights of the criteria were calculated. After that, were determined concordance and discordance sets and matrixes, respectively. Then were formed the concordance and discordance dominance matrixes. In the next step, was created the aggregate dominance matrix and then was paid to eliminating the less favourable alternatives. Finally, by using concordance and discordance matrixes and the aggregate dominance matrix, was donned the ranking of alternatives and it was found the best alternative. The results showed that the A_3 was the best. The advantage of the proposed method is more suitable for solving multiple attribute decision-making problems with neutrosophic information because neutrosophic sets can handle indeterminate and inconsistent information and are the extension of intuitionistic fuzzy sets. The future work is to develop other aggregated algorithms for some other practical decision-making problems, such as supply chain management, personal selection in academia, project evaluation, manufacturing systems, and many other areas of management systems. Also, in the future, the proposed method can be used for dealing with interval-valued neutrosophic soft expert based MCDM problems. Also, this approach can be applied to other multi-criteria decision-making methods, including VIKOR, DEMTEL, PROMOTHEE and etc, also weight determination techniques; It can also be comparing the results of solving these methods with the results of these techniques in fuzzy and intuitionistic fuzzy environments.

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References

1. Xu, Z. and R.R. Yager, *Dynamic intuitionistic fuzzy multi-attribute decision making.* International Journal of Approximate Reasoning, 2008. **48**(1): p. 246-262.
2. Liu, S. and Y. Lin, *Grey information: theory and practical applications*. 2006: Springer Science & Business Media.
3. Atanassov, K.T., *Intuitionistic fuzzy sets.* Fuzzy sets and Systems, 1986. **20**(1): p. 87-96.
4. Atanassov, K., *Intuitionistic fuzzy sets, VII ITKR's Session, Sofia deposed in Central Sci.* Technical Library of Bulg. Acad. of Sci, 1983. **1697**: p. 84.
5. Atanassov, K. and G. Gargov, *Interval valued intuitionistic fuzzy sets.* Fuzzy sets and systems, 1989. **31**(3): p. 343-349.
6. Ma, X., N. Sulaiman, and M. Rani, *Applications of interval-valued intuitionistic fuzzy soft sets in a decision making problem.* Software Engineering and Computer Systems, 2011: p. 642-651.
7. Wang, H., et al., *Single valued neutrosophic sets.* Review of the Air Force Academy, 2010(1): p. 10.

8. Broumi, S., J. Ye, and F. Smarandache, *An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables*. Neutrosophic Sets & Systems, 2015. **8**.
9. Smarandache, F., *Neutrosophy: A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability*. 1999, American Research Press Santa Fe.
10. Ye, J., *A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets.* Journal of Intelligent & Fuzzy Systems, 2014. **26**(5): p. 2459-2466.
11. Biswas, P., S. Pramanik, and B.C. Giri, *A new methodology for neutrosophic multi-attribute decision making with unknown weight information.* Neutrosophic Sets and Systems, 2014. **3**: p. 42-52.
12. Zhang, M., L. Zhang, and H. Cheng, *A neutrosophic approach to image segmentation based on watershed method.* Signal Processing, 2010. **90**(5): p. 1510-1517.
13. Guo, Y. and A. Şengür, *A novel image segmentation algorithm based on neutrosophic similarity clustering.* Applied Soft Computing, 2014. **25**: p. 391-398.
14. Ye, J., *Improved cosine similarity measures of simplified neutrosophic sets for medical diagnoses.* Artificial intelligence in medicine, 2015. **63**(3): p. 171-179.
15. Ansari, A., R. Biswas, and S. Aggarwal, *Proposal for applicability of neutrosophic set theory in medical AI.* International Journal of Computer Applications, 2011. **27**(5): p. 5-11.
16. Guo, Y. and A. Sengur, *NCM: Neutrosophic c-means clustering algorithm.* Pattern Recognition, 2015. **48**(8): p. 2710-2724.
17. Şahin, R. and M. Yiğider, *A Multi-criteria neutrosophic group decision making metod based TOPSIS for supplier selection.* arXiv preprint arXiv:1412.5077, 2014.
18. Ye, J., *Multicriteria decision-making method using the correlation coefficient under single-valued neutrosophic environment.* International Journal of General Systems, 2013. **42**(4): p. 386-394.
19. Ye, J., *Single valued neutrosophic cross-entropy for multicriteria decision making problems.* Applied Mathematical Modelling, 2014. **38**(3): p. 1170-1175.
20. Sodenkamp, M. A., Tavana, M., & Di Caprio, D, *An aggregation method for solving group multi-criteria decision-making problems with single-valued neutrosophic sets*. *Applied Soft Computing*, 2018. *71*: p. 715-727.
21. Liu, P., Khan, Q., & Mahmood, T, *Multiple-attribute decision making based on single-valued neutrosophic Schweizer-Sklar prioritized aggregation operator*. *Cognitive Systems Research*, 2019. *57*: p. 175-196.
22. Liu, P., & You, X, *Bidirectional projection measure of linguistic neutrosophic numbers and their application to multi-criteria group decision making. Computers & Industrial Engineering*, 2019. *128*: p. 447-457.
23. Thong, N. T., Dat, L. Q., Hoa, N. D., Ali, M., & Smarandache, F, *Dynamic interval valued neutrosophic set: Modeling decision making in dynamic environments. Computers in Industry*, 2019. *108*: p. 45-52.
24. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F, *An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number.* Applied Soft Computing, 2019. *77*: p. 438-452.
25. Abdel-Basset, M., Atef, A., & Smarandache, F, *A hybrid Neutrosophic multiple criteria group decision making approach for project selection.* Cognitive Systems Research, 2019. *57*: p.216-227.
26. Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F, *A novel model for evaluation Hospital medical care systems based on plithogenic sets.* Artificial intelligence in medicine, 2019. *100*: p. 101710.

27. Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., & Smarandache, F, *A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics.* Symmetry, 2019. ***11***(7): p. 903.
28. Abdel-Basset, M., Nabeeh, N. A., El-Ghareeb, H. A., & Aboelfetouh, A, *Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises.* Enterprise Information Systems, 2019. p. 1-21.
29. Abdel-Baset, M., Chang, V., & Gamal, A, *Evaluation of the green supply chain management practices: A novel neutrosophic approach.* Computers in Industry, 2019. *108*: p. 210-220.
30. Benayoun, R., B. Roy, and B. Sussman, *ELECTRE: Une méthode pour guider le choix en présence de points de vue multiples.* Note de travail, 1966. **49**.
31. Haurant, P., P. Oberti, and M. Muselli, *Multicriteria selection aiding related to photovoltaic plants on farming fields on Corsica island: A real case study using the ELECTRE outranking framework.* Energy policy, 2011. **39**(2): p. 676-688.
32. Figueira, J.R., et al., *Electre Tri-C, a multiple criteria decision aiding sorting model applied to assisted reproduction.* international journal of medical informatics, 2011. **80**(4): p. 262-273.
33. Pasiouras, F., S. Tanna, and C. Zopounidis, *The identification of acquisition targets in the EU banking industry: An application of multicriteria approaches.* International Review of Financial Analysis, 2007. **16**(3): p. 262-281.
34. Cavallaro, F., *A comparative assessment of thin-film photovoltaic production processes using the ELECTRE III method.* Energy Policy, 2010. **38**(1): p. 463-474.
35. Montazer, G.A., H.Q. Saremi, and M. Ramezani, *Design a new mixed expert decision aiding system using fuzzy ELECTRE III method for vendor selection.* Expert Systems with Applications, 2009. **36**(8): p. 10837-10847.
36. Abedi, M., et al., *ELECTRE III: A knowledge-driven method for integration of geophysical data with geological and geochemical data in mineral prospectivity mapping.* Journal of applied geophysics, 2012. **87**: p. 9-18.
37. Figueira, J., V. Mousseau, and B. Roy, *ELECTRE methods*, in *Multiple criteria decision analysis: State of the art surveys*. 2005, Springer. p. 133-153.
38. Roy, B. and D. Vanderpooten, *An overview on "The European school of MCDA: Emergence, basic features and current works".* European Journal of Operational Research, 1997. **99**(1): p. 26-27.
39. de Almeida, A.T., *Multicriteria decision model for outsourcing contracts selection based on utility function and ELECTRE method.* Computers & Operations Research, 2007. **34**(12): p. 3569-3574.
40. Roy, B., *Partial preference analysis and decision-aid: The fuzzy outranking relation concept.* Conflicting objectives in Decisions, 1977: p. 40-75.
41. Corner, J.L. and C.W. Kirkwood, *Decision analysis applications in the operations research literature, 1970–1989.* Operations Research, 1991. **39**(2): p. 206-219.
42. F. Smarandache, *AUnifying Field in Logics: Neutrosophic Logic.* Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Research Press, Rehoboth, NM, 1999.
43. Pramanik, S., & Mondal, K. *Interval neutrosophic multi-attribute decision-making based on grey relational analysis.* Neutrosophic Sets and Systems, 2015. 9, p. 13-22.
44. Rivieccio, U., *Neutrosophic logics: Prospects and problems.* Fuzzy sets and systems, 2008. **159**(14): p. 1860-1868.

45. Biswas, P., Pramanik, S., & Giri, B. C. *Cosine similarity measure based multi-attribute decision-making with trapezoidal fuzzy neutrosophic numbers.* Neutrosophic sets and systems, 2015. 8, p. 46-56.
46. Uluçay, V., Kiliç, A., Yildiz, I., & Sahin, M. *A new approach for multi-attribute decision-making problems in bipolar neutrosophic sets.* Neutrosophic Sets Systems, 2018. **23**(1): p. 142-159.
47. Ghaderi, S.F., et al., *Behavioral simulation and optimization of generation companies in electricity markets by fuzzy cognitive map.* Expert Systems with Applications, 2012. **39**(5): p. 4635-4646.
48. Smarandache, F., *Neutrosophic set-a generalization of the intuitionistic fuzzy set.* International Journal of Pure and Applied Mathematics, 2005. **24**(3): p. 287.
49. Liu, P. and Y. Wang, *Multiple attribute decision-making method based on single-valued neutrosophic normalized weighted Bonferroni mean.* Neural Computing and Applications, 2014. **25**(7-8): p. 2001-2010.
50. Smarandache, F., *n-Valued Refined Neutrosophic Logic and Its Applications to Physics.* Unmatter Plasma, Relativistic Oblique-Length Contraction Factor, Neutrosophic Diagram and Neutrosophic Degree of Paradoxicity: Articles and Notes, 2013: p. 40.
51. Majumdar, P. and S.K. Samanta, *On similarity and entropy of neutrosophic sets.* Journal of Intelligent & Fuzzy Systems, 2014. **26**(3): p. 1245-1252.
52. Wu, M.-C. and T.-Y. Chen, *The ELECTRE multicriteria analysis approach based on Atanassov's intuitionistic fuzzy sets.* Expert Systems with Applications, 2011. **38**(10): p. 12318-12327.

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A Note on the Concept of α – Level Sets of Neutrosophic Set

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Abstract: Neutrosophic set is a unique concept endowed with unconnected degree of indeterminacy excluded in the non-classical sets it generalizes. This paper communicates shortly on the notions of α - lower level and α - upper level sets of a neutrosophic set and investigates some basic properties.

Keywords: Neutrosophic set; α - lower level and α - upper level sets of a neutrosophic set

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1. Introduction

Uncertainty is unavoidable in real life situations as classical structure cannot handle them. Dealing with vague, uncertain or imperfect information was a huge task for many years. Many models were proposed in order to suitably integrate uncertainty into the system description. Zadeh [12] noticed typically that the collections of objects encountered in real world do not have exactly sharp boundaries of membership as described by a German mathematician, George Cantor (1845-1918). Consequently, he introduced fuzzy set concept and delineated it as a collection of objects with graded membership. However, Atanassov [6] initiated an extension of fuzzy set called intuitionistic fuzzy set. Intuitionistic fuzzy set accommodates additional degrees of freedom (non-membership and hesitation margin) into set description and is broadly used as a tool of intensive research by scholars and scientists.

One of the motivating generalizations of fuzzy set theory and intuitionistic fuzzy set theory is neutrosophic set theory introduced by Smarandache [11]. A neutrosophic set theory is independently characterized by a truth membership function, an indeterminate membership function and a falsity membership function. Therefore, the neutrosophic set theory has become a popular subject of research in problems associated with uncertainty.

Very recently, the scholarly world has witnessed growing research interests in the theory of neutrosophic sets such as medical diagnosis [1, 4, 5], database [7], topology [10], image processing [8], and decision-making problem [2, 3, 9].

The paper attempts to develop the concepts of α - lower level and α - upper level sets of a neutrosophic set and investigates some basic properties based on the related research of fuzzy sets and intuitionistic fuzzy sets with the aim to create a paradigm shift in the aspects of algebra.

2. Preliminaries

In this section, we will give some preliminary information that will be useful in the sequel of the paper

Definition 2.1 [11] A neutrosophic set (NS) A in a non-empty set X is a structure of the form

$A = \{ \langle x, T_A(x),\ I_A(x),\ F_A(x) \rangle \mid x \in X \}$, where $T_A,\ I_A,\ F_A : X \longrightarrow]^-0, 1^+[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $x \in X$ to the set $x \in A$ with the condition $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.
Here, $1^+ = 1 + c$, where 1 is its standard part and c its non-standard part. Analogously, $^-0 = 0 - c$ is expressed in turn.
The above definition has been used by several authors in literature with sizable number of publications. On the contrary, the results presented in this paper are devoid of non-standard and restricted to the interval $[0, 1]$ for practical techniques.

As an illustration, let us consider the following example.
Example 2.1 Assume that $X = \{a, b, c\}$, where a characterizes the competence, b characterizes the reliability and c indicates the costs of the objects. It may be further assumed that the values of a, b and c are in $[0, 1]$ and they are obtained from some surveys of some connoisseurs. The connoisseurs may impose their view in three components viz. the degree of goodness, the degree of indeterminacy and that of poorness to describe the characteristics of the objects. Suppose A is a neutrosophic set in X, such that,

$A = \{(a, \langle 0.3, 0.4, 0.5 \rangle), (b, \langle 0.5, 0.2, 0.3 \rangle), (c, \langle 0.7, 0.2, 0.2 \rangle)\}$, where the degree of goodness of capability is 0.3, degree of indeterminacy of capability is 0.4 and degree of falsity of capability is 0.5 implying $T_A(a) = 0.3,\ I_A(b) = 0.4,\ F_A(c) = 0.5$ etc.

For simplicity, $A = \{ \langle x,\ T_A(x),\ I_A(x),\ F_A(x) \rangle \mid x \in X \}$, can be expressed as $A(x) = \left(T_A(x),\ I_A(x),\ F_A(x) \right)$ since the membership functions $T_A,\ I_A,\ F_A$ are defined from X into the unit interval $[0, 1]$.

Definition 2.2 [11] Let A and B be two neutrosophic sets in a non-empty set X. Then

$(\boldsymbol{i})$ $\boldsymbol{A \subseteq B \Leftrightarrow T_A(x) \leq T_B(x),\ I_A(x) \leq I_B(x),\ F_A(x) \geq F_B(x).}$

$(\boldsymbol{ii})$ $\boldsymbol{A = B \Leftrightarrow T_A(x) = T_B(x),\ I_A(x) = I_B(x),\ F_A(x) = F_B(x).}$

$(\boldsymbol{iii})$ $\boldsymbol{A \cap B} = \{ \langle x,\ \wedge(T_A(x),\ T_B(x)),\ \wedge(I_A(x),\ I_B(x)),\ \vee(F_A(x),\ F_B(x)) \rangle \mid x \in X \}$.

(iv) $\boldsymbol{A \cup B} = \{ \langle x,\ \vee(T_A(x),\ T_B(x)),\ \vee(I_A(x),\ I_B(x)),\ \wedge(F_A(x),\ F_B(x)) \rangle \mid x \in X \}$, where $\wedge$ and $\vee$ are minimum and maximum operations.

(v) $A^c = \{ \langle x,\ F_A(x),\ 1 - I_A(x),\ T_A(x)) \rangle \mid x \in X \}$.

(vi) $A \setminus B = \{ \langle x,\ T_A \wedge F_B(x),\ I_A(x) \wedge 1 - I_B(x),\ F_A(x) \vee T_B(x)) \rangle \mid x \in X \}$.

With reference to Definition 2.2 (v), $(A^c)^c = A$.

Remark 2.1 If $\{ A_i \mid i \in J \}$ is a family of neutrosophic sets, then $\left(\bigcup_{i \in J} A_i \right)^c = \bigcap_{i \in J} A_i^c$ and $\left(\bigcap_{i \in J} A_i \right)^c = \bigcup_{i \in J} A_i^c$.

Proposition 2.1 Let A, B, C, D be any neutrosophic sets in a non-empty set X, we have

(i) if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

(ii) if $A \subseteq B$, then $A^c \subseteq B^c$.

(iii) if $A \subseteq B$ and $A \subseteq C$, then $A \subseteq B \cap C$.

(iv) if $A \subseteq B$ and $C \subseteq B$, then $A \cup C \subseteq B$.

(v) if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$ and $A \cap C \subseteq B \cap D$.

Proof. Immediate from definitions.

Definition 2.3 [11] A neutrosophic set A in a non-empty set X is said to be universe neutrosophic set if $T_A(x) = I_A(x) = 1,\ F_A(x) = 0,\ \forall\, x \in X$. It is denoted by 1_N.

A neutrosophic set A in a non-empty set X is said to be null neutrosophic set if $T_A(x) = I_A(x) = 0$, $F_A(x) = 1$, $\forall\, x \in X$. It is denoted by 0_N.

3. Main Results

Definition 3.1 Let A be any neutrosophic set in a non-empty set X. Then for any $\alpha \in [0,1]$, the α – lower level and the α – upper level sets of A denoted by $L(A,\alpha)$ and $U(A,\alpha)$ are respectively defined as follows:

$$L(A,\alpha) = \{\, x \in X \mid T_A(x) \geq \alpha,\ I_A(x) \geq \alpha,\ F_A(x) \leq \alpha \,\} \text{ and}$$

$$U(A,\alpha) = \{\, x \in X \mid T_A(x) \leq \alpha,\ I_A(x) \leq \alpha,\ F_A(x) \geq \alpha \,\}.$$

Example 3.1 Let $A = \{(a, \langle 0.4, 0.3, 0.5\rangle), (b, \langle 0.5, 0.3, 0.1\rangle), (c, \langle 0.2, 0.5, 0.9\rangle)\}$ and $\alpha \in [0,1]$. Then $L(A, 0.1) = L(A, 0.2) = L(A, 0.3) = \{b\}$, $L(A, 0.4) = \{\emptyset\}$, $\alpha \geq 0.4$. However, $U(A,\alpha) = \{\emptyset\}$, $0.1 \leq \alpha \leq 0.3$, $U(A, 0.4) = \{a\}$, $U(A, 0.5) = \{a, c\}$, $U(A, 0.6) = \{c\}$, $\alpha \geq 0.6$.

If A, B, C are neutrosophic sets in a non-empty X and $\alpha, \beta \in [0,1]$, then the results in the following proposition are not difficult to verify from definitions.

Proposition 3.1

(i) $A \subseteq B \Longrightarrow L(A,\alpha) \subseteq L(B,\alpha)$.

(ii) $\alpha \geq \beta \Longrightarrow L(A,\alpha) \supseteq L(A,\beta)$.

(iii) $L\big(\bigcap_{i\in J} A_i, \alpha\big) = \bigcap_{i\in J} L(A_i, \alpha)$.

(iv) $U(A,\alpha) \subseteq L(A,\alpha)$.

Proposition 3.2

$(\boldsymbol{i})$ $L(A \cup B, \alpha) = L(A,\alpha) \cup L(B,\alpha)$.

(ii) $L(A \cap B, \alpha) = L(A,\alpha) \cap L(B,\alpha)$.

(iii) $A = B \Longleftrightarrow L(A,\alpha) = L(B,\alpha)$, $\forall\, \alpha \in [0,1]$.

Proof.

(i) $L(A \cup B, \alpha) = \{\, x \in X \mid T_{A\cup B}(x) \geq \alpha,\ I_{A\cup B}(x) \geq \alpha,\ F_{A\cup B}(x) \leq \alpha \,\}$

$= \{\, x \in X \mid T_A(x) \vee T_B(x) \geq \alpha,\ I_A(x) \vee I_B(x) \geq \alpha,\ F_A(x) \wedge F_B(x) \leq \alpha \,\}$

$= \{\, x \in X \mid T_A(x) \geq \alpha \cup T_B(x) \geq \alpha,\ I_A(x) \geq \alpha \cup I_B(x) \geq \alpha,\ F_A(x) \leq \alpha \cup F_B \leq \alpha \,\}$

$=$

$\{\, x \in X \mid T_A(x) \geq \alpha,\ I_A(x) \geq \alpha,\ F_A(x) \leq \alpha \,\} \cup \{\, x \in X \mid T_B(x) \geq \alpha,\ I_B(x) \geq \alpha,\ F_B(x) \leq \alpha \,\}$

$= L(A,\alpha) \cup L(B,\alpha)$

Hence, $L(A \cup B, \alpha) = L(A,\alpha) \cup L(B,\alpha)$.

(ii) Similar to the proof of (i).

(iii) Clearly, $A = B \Longrightarrow T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$ $\forall\, x \in X$.

Undoubtedly, $L(A,\alpha) = \{\, x \in X \mid T_A(x) \geq \alpha,\ I_A(x) \geq \alpha,\ F_A(x) \leq \alpha \,\}$ and $L(B,\alpha) = \{\, x \in X \mid T_B(x) \geq \alpha,\ I_B(x) \geq \alpha,\ F_B(x) \leq \alpha \,\}$.

But $A = B$ $\forall\, x \in X$. Hence, $L(A,\alpha) = L(B,\alpha)$, $\forall\, \alpha \in [0,1]$.

Conversely, suppose that $\forall\, \alpha \in [0,1]$, $L(A,\alpha) = L(B,\alpha)$ but $A \neq B$. Moreover, $A \neq B$ if and only if

there exists some $y \in X$ such that $T_A(y) \neq T_B(y)$, $I_A(y) \neq I_B(y)$, $F_A(y) \neq F_B(y)$. Without loss of generality, assume that $T_A(y) \leq T_B(y)$, $I_A(y) \leq I_B(y)$, $F_A(y) \leq F_B(y)$ and let $\gamma = T_B(y) = I_B(y) =$

$F_B(y)$. It must be that $y \notin L(A,\gamma)$ but $y \in L(B,\gamma)$. Then $L(A,\alpha)$ and $L(B,\alpha)$ are identical, and this
is a contradiction.

The distributive laws are satisfied for α – lower level sets of a neutrosophic set.

Proposition 3.3
(i) $L(A\cup(B\cap C),\alpha) = L(A\cup B,\alpha) \cap L(A\cup C,\alpha)$.
(ii) $L(A\cap(B\cup C),\alpha) = L(A\cap B,\alpha) \cup L(A\cap C,\alpha)$.

Proof. Similar to the proof of Proposition 3.2.

Theorem 3.1 Let A be a neutrosophic set in a non-empty set X and $\alpha,\beta \in [0,1]$. If α comprises all finite values in $[0,1]$ and $\alpha \leq \beta$, then $\cap L(A,\alpha) = L(A,\beta)$.

Proof.
Let $x \in \cap L(A,\alpha)$. Then $x \in L(A,\alpha)\ \forall\, \alpha \in [0,1]$.
$\Rightarrow T_A(x) \geq \alpha,\ I_A(x) \geq \alpha,\ F_A(x) \leq \alpha\ \ \forall\, \alpha \in [0,1],\ x \in X$.
Since $\alpha \leq \beta$, then $T_A(x) \geq \alpha \leq \beta,\ I_A(x) \geq \alpha \leq \beta,\ F_A(x) \leq \alpha \leq \beta\ \ \forall\, \alpha \in [0,1]$.
$\Rightarrow \cap L(A,\alpha) \subseteq L(A,\beta)$.
Conversely, let $x \in L(A,\beta)$, then $T_A(x) \geq \beta,\ I_A(x) \geq \beta,\ F_A(x) \leq \beta,\ \forall\ x \in X$.
$\Rightarrow T_A(x) \geq \beta \geq \alpha,\ I_A(x) \geq \beta \geq \alpha,\ F_A(x) \leq \beta \leq \alpha,\ \forall\, \alpha \in [0,1]$.
$\Rightarrow T_A(x) \geq \alpha,\ I_A(x) \geq \alpha,\ F_A(x) \leq \alpha,\ \forall\, \alpha \in [0,1]$.
$\Rightarrow L(A,\beta) \subseteq \cap L(A,\alpha)$.
Hence, $\cap L(A,\alpha) = L(A,\beta)$.

Proposition 3.4 Let A be a universal neutrosophic set in a non-empty set X and $\alpha \in [0,1]$. Then $L(A,0) = X$.

Proof. Straightforward.

Remark 3.1 If A is a universal neutrosophic set in a non-empty set X and $\alpha \in [0,1]$, then $L(A,0) = L(A,1)$.

Theorem 3.2 If $L(A,\alpha),\ \alpha \in [0,1]$ be the α – lower level sets of a neutrosophic set in a non-empty set X such that $\cap\, \alpha U(F_A,\alpha)$ is restricted to non-zero values, then $A = \cup_{\alpha\in[0,1]} \alpha L(A,\alpha)$.

Proof.
$A(x) = (T_A(x),\ I_A(x),\ F_A(x)) = (a,b,c)$ and for each $\alpha \in (a,1],\ \alpha \in (b,1],\ \alpha \in (0,c)$, we have $T_A(x) = a < \alpha,\ I_A(x) = b < \alpha$ and $F_A(x) = c > \alpha$. Thus, $L(A,\alpha) = (0,0\,,0)$.
However, for each $\alpha \in (0,a],\ \alpha \in (0,b],\ \alpha \in [c,1)$, we have $T_A(x) = a \geq \alpha,\ I_A(x) = b \geq \alpha$ and $F_A(x) = c \leq \alpha$. Thus, $L(A,\alpha) = (1,1,1)$.
Hence, $\cup_{\alpha\in[0,1]} \alpha L(A,\alpha) = (\vee_{\alpha\in(0,a]} \alpha = a = T_A(x),\ \vee_{\alpha\in(0,b]} \alpha = b = I_A(x),\ \wedge_{\alpha\in[c,1)} \alpha = c = F_A(x))$ with the restriction on $\cap\, \alpha U(F_A,\alpha)$ to be considered non-zero values. This completes the proof.

Example 3.2 Let A be any neutrosophic set in a non-empty set X, given by
$A = \{(a,\langle 0.4, 0.3, 0.5\rangle), (b,\langle 0.5, 0.3, 0.1\rangle), (c,\langle 0.2, 0.5, 0.9\rangle)\}$.

For expediency, let us denote A as

$A = \{(0.4, 0.3, 0.5)/a, (0.5, 0.2, 0.3)/b, (0.7, 0.2, 0.2)/c\}$.

Then

$$L(A,0.1) = \{(1,1,0)/a, (1,1,1)/b, (1,1,0)/c\}$$
$$L(A,0.2) = \{(1,1,0)/a, (1,1,1)/b, (1,1,0)/c\}$$
$$L(A,0.3) = \{(1,1,0)/a, (1,1,1)/b, (0,1,0)/c\}$$
$$L(A,0.4) = \{(1,0,0)/a, (1,0,1)/b, (0,1,0)/c\}$$
$$L(A,0.5) = \{(0,0,1)/a, (1,0,1)/b, (0,1,0)/c\}$$
$$L(A,0.9) = \{(0,0,1)/a, (0,0,1)/b, (0,0,1)/c\}$$

It is not difficult to see that

$$A = 0.1L(A,0.1) \cup 0.2L(A,0.2) \cup 0.3L(A,0.3) \cup 0.4L(A,0.4) \cup 0.5\,L(A,0.5) \cup 0.9L(A,0.9).$$

The following results presented below are for α – upper level sets of a neutrosophic set.

Proposition 3.5

(i) $A \subseteq B \Longrightarrow U(B,\alpha) \subseteq U(A,\alpha)$.
(ii) $\alpha \leq \beta \Longrightarrow U(A,\alpha) \subseteq U(A,\beta)$.
(iii) $\bigcap_{i\in J} U(A_i,\alpha) \subseteq U\left(\bigcap_{i\in J} A_i,\alpha\right)$.

Proof. Straightforward.

Proposition 3.6 If A and B are two neutrosophic sets in a non-empty set X and $\alpha \in [0,1]$, then

$(\boldsymbol{i})$ $U(A\cap B,\alpha) \supseteq U(A,\alpha) \cap U(B,\alpha)$.
(ii) $U(A\cup B,\alpha) = U(A,\alpha) \cup U(B,\alpha)$.
(iii) $A = B \Leftrightarrow U(A,\alpha) = U(B,\alpha),\ \forall\, \alpha \in [0,1]$.

Proof.

(i) $U(A\cap B,\alpha) = \{\, x \in X \mid T_{A\cap B}(x) \leq \alpha,\ I_{A\cap B}(x) \leq \alpha,\ F_{A\cap B}(x) \geq \alpha \,\}$
$= \{\, x \in X \mid T_A(x) \wedge T_B(x) \leq \alpha,\ I_A(x) \wedge I_B(x) \leq \alpha,\ F_A(x) \vee F_B(x) \geq \alpha \,\}$
$\geq \{\, x \in X \mid T_A(x) \leq \alpha \cap T_B(x) \leq \alpha,\ I_A(x) \leq \alpha \cap I_B(x) \leq \alpha,\ F_A(x) \geq \alpha \cup F_B \geq \alpha \,\}$
$=$
$\{\, x \in X \mid T_A(x) \leq \alpha,\ I_A(x) \leq \alpha,\ F_A(x) \geq \alpha \,\} \cap \{\, x \in X \mid T_B(x) \leq \alpha,\ I_B(x) \leq \alpha,\ F_B(x) \geq \alpha \,\}$
$= U(A,\alpha) \cap U(B,\alpha)$

Hence, $U(A\cap B,\alpha) \supseteq U(A,\alpha) \cap U(B,\alpha)$.

(ii) It is obtained in a similar way.

(iii) The proof is similar to the proof of Proposition 3.2(iii).

Proposition 3.7

(i) $U(A\cup(B\cap C),\alpha) \subseteq U(A\cup B,\alpha) \cap U(A\cup C,\alpha)$.
(ii) $U(A\cap(B\cup C),\alpha) \subseteq U(A\cap B,\alpha) \cup U(A\cap C,\alpha)$.

Proof. Similar to the proof of Proposition 3.6(i).

Proposition 3.8 Let A be a null neutrosophic set in a non-empty set X and $\alpha \in [0,1]$. Then $U(A,0) = X$.

Proof. Straightforward.

Remark 3.2 If A is a null neutrosophic set in a non-empty set X and $\alpha \in [0,1]$, then $U(A,0) = U(A,1)$.

Theorem 3.3 If $U(A,\alpha)$, $\alpha \in [0,1]$ be the α – upper level sets of a neutrosophic set in a non-empty set X such that $\cap\, \alpha U(T_A,\alpha)$ and $\cap\, \alpha U(I_A,\alpha)$ are restricted to non-zero values, then $A = \cap_{\alpha\in[0,1]} \alpha U(A,\alpha)$.

Proof.
The proof is analogous to the proof of Theorem 3.2.
Let $A(x) = (T_A(x),\ I_A(x),\ F_A(x)) = (a,b,c)$. Then $T_A(x) = a > \alpha$, $I_A(x) = b > \alpha$ and $F_A(x) = c < \alpha$, $\forall\ \alpha \in [0,a)$, $\alpha \in [0,b)$, $\alpha \in (c,1]$. Thus, $U(A,\alpha) = (0,0,0)$.
On the other hand, $T_A(x) = a \le \alpha$, $I_A(x) = b \le \alpha$ and $F_A(x) = c \ge \alpha$, $\forall\ \alpha \in [a,1)\ \alpha \in [b,1)\ \alpha \in (0,c]$. Thus, $U(A,\alpha) = (1,1,1)$.
Hence, $\cap_{\alpha\in[0,1]} \alpha U(A,\alpha) = (\wedge_{\alpha\in[a,1)} \alpha = a = T_A(x),\ \wedge_{\alpha\in[b,1)} \alpha = b = I_A(x),\ \vee_{\alpha\in(0,c]} \alpha = c = F_A(x))$ with the restriction on $\cap\, \alpha U(T_A,\alpha)$ and $\cap\, \alpha U(I_A,\alpha)$ to be considered non-zero values. Hence the proof.

5. Conclusions (authors also should add some future directions points related to her/his research)

The concepts of α – lower level and α – upper level sets and their properties in neutrosophic sets are described. This study is worthy of level sets extension in the hybrid set structures such as neutrosophic multisets, neutrosophic soft sets and rough neutrosophic sets.

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References

1. Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F. (2019). A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine, 100, 101710.
2. Abdel-Basset, M., Mohammed, R., Zaied, A. E. N. H., & Smarandache, F. (2019). A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry , 11(7), 903.
3. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 77, 438-452.
4. Abdel-Basset, M., Manogaran, G., Gamal, A., & Smarandache, F. (2019). A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of medical sysems, 43(2), 38.
5. Ansari, Biwas R., & Aggarwal. (2011). Proposal for applicability of neutrosophic set theory in medical AI. International Journal of Computer Applications, 27(5), 5-11.
6. Atanassov, K. T. (1986). Intuitionistic fuzzy sets. Fuzzy Sets and Systems, 20(1), 87-96.
7. Arora, M., Biwas, R., & Pandy, U. S. (2011). Neutrosophic relational database decomposition. International Journal of Advanced Computer Science and Applications, 2(8), 121-125.
8. Cheng, H. D. & Guo, Y. (2008). A new neutrosophic approach to image thresholding. New Mathematics and Natural Computation, 4(3), 291-308.
9. Kharal, A. (2013). A neutrosophic multicriteria decision making method. New Mathematics and Natural Computation.
10. Lupiez, F. G. (2008). On neutrosophic topology. Kybernetes, 37(6) (2008), 797-800.

11. Smarandache, F. (1999). A unifying field in logics, Neutrosophic Probability, Set and Logic, Rehoboth: American Research Press.
12. Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338-353.

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T-Neutrosophic Cubic Set on BF-Algebra

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Abstract: In this paper, the concept of t-neutrosophic cubic set is introduced and investigated the t-neutrosophic cubic set through subalgebra, ideal and closed ideal of BF-algebra. Homomorphic properties of t-neutrosophic cubic subalgebra and ideal are also investigated with some related properties.

Keywords: BF-algebra, t-neutrosophic cubic set, t-neutrosophic cubic subalgebra, t-neutrosophic cubic closed ideal.

1 Introduction

Zadeh [33, 34] introduced the concept of fuzzy set. Jun et al. [7] defined interval-valued fuzzy set and discussed its properties. Jun et al. [8] presented the notion of cubic subgroups. Senapati et al. [26] generalized the idea of cubic set to subalgebras, ideals and closed ideals of B-algebra. Imai and Iseki [5, 6] introduced the two classes of algebra which were BCK algebra and BCI-algebra. Huang [4] investigated the BCI-algebra. Jun et al. [10, 11] applied the idea of cubic set to subalgebras, ideals and q-ideals in BCK/BCI-algebra. Neggers et al. [13] defined and studied the B-algebra. Cho et al. [3] studied the relations of B-algebra with different topics. Park et al. [15] studied quadratic B-algebra on field X with a BCI-algebra. Saeid [16] was given the idea of interval valued fuzzy subalgebra in B-algebra. Walendziak [32] proved the conditions of B-algebra. Senapati et al. [21, 22, 23, 24, 31] was introduced the fuzzy dot subalgebra of BG-algebra, fuzzy dot subalgebra, fuzzy dot ideals, interval-valued fuzzy closed ideals and fuzzy subalgebra with respect to t-norm in B-algebra. Senapati et. al. [17, 25] was introduced L-fuzzy G-subalgebra of G-algebra and bipolar fuzzy set which was related to B-algebra. Khalid et. al. [20] studied the intuitionistic fuzzy translation. Many researchers [12, 27, 28, 29, 30] have done a lot of work on BG-algebra which was a generalization of B-algebra. Smarandache [18, 19] introduced the concept of neutrosophic set. Jun et al. [9] introduced neutrosophic cubic set. Barbhuiya [2] studied the t-intuitionistic fuzzy BG-subalgebra. Takallo et al. [37] introduced the MBJ-neutrosophic set, BMBJ-neutrosophic subalgebra, BMBJ-neutrosophic ideal and BMBJ-neutrosophic $\circ$-subalgebra. G. Muhiuddin et al. [38] studied the neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, neutrosophic quadruple subalgebra and (positive implicative) neutrosophic quadruple ideal. Park [39] introduced the notion of neutrosophic ideal in subtraction algebra and discussed conditions for a neutrosophic set to be a neutrosophic ideal. Borzooei et al. [40] introduced the concept of MBJ-neutrosophic set, BMBJ-neutrosophic ideal and positive implicative BMBJ-neutrosophic ideal. Jun et al. [41] studied the commutative falling neutrosophic ideals in BCK-algebra. Song et al. [42] investigated the interval neutrosophic set and applied to ideals in BCK/BCI-algebra. Khalid et al. [43] interestingly investigated the neutrosophic soft cubic subalgebra through significant results. Muhiuddin et al. [44] was studied neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, (regular) neutrosophic quadruple ideal and neutrosophic quadruple q-ideal. Muhiuddin et al. [45] investigated the (ϵ, ϵ)-neutrosophic subalgebra, (ϵ, ϵ)-neutrosophic ideal. Akinleye et al. [46] defined the neutrosophic quadruple algebraic structures, also studied neutrosophic quadruple rings and presented their elementary properties. Basset et al. [47] studied integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection. Basset et al. [48] studied the type 2 neutrosophic number, score and accuracy function, multi attribute decision making TOPSIS and T2NN-TOPSIS.

The purpose of this paper is to introduce the idea of t-neutrosophic cubic set [t-NCS] and to investigate this set through the concepts of subalgebra, ideal and closed ideal of BF-algebra. Homomorphic image and inverse homomorphic image of t-neutrosophic cubic subalgebra [t-NCSU] and t-neutrosophic cubic ideal [t-NCID] are also studied.

2 Preliminaries

In this section, basic definitions are cited that are necessary for this paper.

Definition 2.1 [32] A nonempty set X with a constant 0 and a binary operation $*$ is called BF-algebra when it fulfills these axioms.

1. $t_1 * t_1 = 0$
2. $t_1 * 0 = 0$
3. $0 * (t_1 * t_2) = t_2 * t_1$ for all $t_1, t_2 \in X$.

A BF-algebra is denoted by $(X, *, 0)$.

Definition 2.2 [1] A nonempty subset S of G-algebra X is called a subalgebra of X if $t_1 * t_2 \in S$ $\forall$ $t_1, t_2 \in S$.

Definition 2.3 [14] Mapping $f|X \to Y$ of B-algebra is called homomorphism if $f(t_1 * t_2) = f(t_1) * f(t_2)$ $\forall$ $t_1, t_2 \in X$.

Definition 2.4 [23] A nonempty subset I of B-algebra X is called an ideal if for any $t_1, t_2 \in X$, (i) $0 \in I$, (ii) $t_1 * t_2 \in I$ and $t_2 \in I \Rightarrow t_1 \in I$.

An ideal I of B-algebra X is called closed if $0 * t_2 \in I$, $\forall$ $t_2 \in I$.

Definition 2.5 [33] Let X be the set of elements which are denoted generally by t_1. Then a fuzzy set C in X is defined as $C = \{< t_1, \mu_C(t_1) > |t_1 \in X\}$, where $\mu_C(t_1)$ is called the existenceship value of t_1 in C and $\mu_C(t_1) \in [0,1]$.

For a family $C_i = \{< t_1, \mu_{C_i}(t_1) > \ |t_1 \in X\}$ of fuzzy sets in X, where $i \in k$ and k is index set, we define the join (∨) meet (∧) operations as follows:

$$\bigvee_{i\in k} C_i = (\bigvee_{i\in k} \mu_{C_i})(t_1) = \sup\{\mu_{C_i}|i \in k\}$$

and

$$\bigwedge_{i\in k} C_i = (\bigwedge_{i\in k} \mu_{C_i})(t_1) = \inf\{\mu_{C_i}|i \in k\}$$

respectively, $\forall\ t_1 \in X$.

Definition 2.6 [2] Let two elements $D_1, D_2 \in D[0,1]$. If $D_1 = [(t_1)_1^-, (t_1)_1^+]$ and $D_2 = [(t_1)_2^-, (t_1)_2^+]$, then $\text{rmax}(D_1, D_2) = [\max((t_1)_1^-, (t_1)_2^-), \max((t_1)_1^+, (t_1)_2^+)]$ which is denoted by $D_1 \vee^r D_2$ and $\text{rmin}(D_1, D_2) = [\min((t_1)_1^-, (t_1)_2^-), \min((t_1)_1^+, (t_1)_2^+)]$ which is denoted by $D_1 \wedge^r D_2$. Thus, if $D_i = [((t_1)_1)_i^-, ((t_1)_2)^+] \in D[0,1]$ for $i = 1,2,3, \ldots,$ then we define $\text{rsup}_i(D_i) = [\sup_i(((t_1)_1)_i^-), \sup_i(((t_1)_1)_i^+)]$, i.e., $\vee_i^r D_i = [\vee_i ((t_1)_1)_i^-, \vee_i ((t_1)_1)_i^+]$. In the same way we define $\text{rinf}_i(D_i) = [\inf_i(((t_1)_1)_i^-), \inf_i(((t_1)_1)_i^+)]$, i.e., $\wedge_i^r D_i = [\wedge_i ((t_1)_1)_i^-, \wedge_i ((t_1)_1)_i^+]$. Now we call $D_1 \geq D_2 \Leftarrow (t_1)_1^- \geq (t_1)_2^-$ and $(t_1)_1^+ \geq (t_1)_2^+$. Similarly the relations $D_1 \leq D_2$ and $D_1 = D_2$ are defined.

Definition 2.7 [1,22] A fuzzy set $C = \{< t_1, \mu_C(t_1) > |t_1 \in X\}$ is called a fuzzy subalgebra of X if $\mu_C(t_1 * t_2) \geq \min\{\mu_C(t_1), \mu_C(t_2)\}\ \forall\ t_1, t_2 \in X$. A fuzzy set $C = \{< t_1, \mu_C(t_1) > |t_1 \in X\}$ in X is called a fuzzy ideal of X if it satisfies (i) $\mu_C(0) \geq \mu_C(t_1)$ and (ii) $\mu_C(t_1) \geq \min\{\mu_C(t_1 * t_2), \mu_A(t_2)\}\ \forall\ t_1, t_2 \in X$.

Definition 2.8 [33] An IVFS B over X is an object of the form $B = \{< t_1, \mu_B(t_1) > |t_1 \in X\}$ Where $\mu_B(t_1)$: $X \to D[0:1]$, Where D[0,1] is the collection of all subintervals of [0,1]. The interval $\mu_B(t_1)$ shows the interval of the degree of membership of the element t_1 to the set B, Where $\mu_B(t_1) = \{\mu_{LB}(t_1), \mu_{UB}(t_1)\}$, $\forall\ t_1 \in X$.

Definition 2.9 [16] A interval valued fuzzy set $C = \{< t_1, \mu_C(t_1) > |t_1 \in X\}$ is called a interval valued fuzzy subalgebra of X if it satisfies $\mu_C(t_1 * t_2) \geq \text{rmin}\{\mu_C(t_1), \mu_C(t_2)\}\ \forall\ t_1, t_2 \in X$.

Definition 2.10 [15] A pair $\tilde{\mathcal{P}}_k = (A, \Lambda)$ is called NCS where $A = \{\langle t_1; A_T(t_1), A_I(t_1)\ , A_F(t_1)\rangle\ |t_1 \in Y\}$ is an INS in Y and $\Lambda = \{\langle t_1; \lambda_T(t_1), \lambda_I(t_1), \lambda_F(t_1)\rangle |\ t_1 \in Y\}$ is a neutrosophic set in Y.

Definition 2.11 [26] Let $C = \{\langle t_1, \kappa(t_1), \sigma(t_1)\rangle\}$ be a cubic set, where $\kappa(t_1)$ is an interval-valued fuzzy set in X, $\sigma(t_1)$ is a fuzzy set in X. Then C is cubic subalgebra under binary operation $*$ if following axioms are satisfied:

C1: $\kappa(t_1 * t_2) \geq \text{rmin}\{\kappa(t_1), \kappa(t_2)\}$,

C2: $\sigma(t_1 * t_2) \leq \max\{\sigma(t_1), \sigma(t_2)\}\ \forall\ t_1, t_2 \in X$.

Definition 2.12 [9] Suppose X be a nonempty set. A neutrosophic cubic set in X is pair $\mathcal{C} = (\kappa, \sigma)$ where $\kappa = \{\langle t_1; \kappa_E(t_1), \kappa_I(t_1), \kappa_N(t_1)\rangle\ |t_1 \in X\}$ is an interval neutrosophic set in X and $\sigma = \{\langle t_1; \sigma_E(t_1), \sigma_I(t_1), \sigma_N(t_1)\rangle\ |t_1 \in X\}$ is a neutrosophic set in X.

Definition 2.13 [9] For any $\mathcal{C}_i = (\kappa_i, \sigma_i)$ where

$\kappa_i = \{\langle t_1; \kappa_{iE}(t_1), \kappa_{iI}(t_1), \kappa_{iN}(t_1)\rangle\ |t_1 \in X\}$,

$\sigma_i = \{\langle t_1; \sigma_{iE}(t_1), \sigma_{iI}(t_1), \sigma_{iN}(t_1)\rangle\ |t_1 \in X\}$ for $i \in k$, P-union, P-inersection, R-un -ion and R-intersection are defined respectively by

P-union $\bigcup_{P\, i\in k} \mathcal{C}_i = (\bigcup_{i\in k} \kappa_i, \bigvee_{i\in k} \sigma_i)$, **P-intersection** $\bigcap_{P\, i\in k} \mathcal{C}_i = (\bigcap_{i\in k} \kappa_i, \bigwedge_{i\in k} \sigma_i)$,

R-union $\bigcup_{R\, i\in k} \mathcal{C}_i = (\bigcup_{i\in k} \kappa_i, \bigwedge_{i\in k} \sigma_i)$, **R-intersection:** $\bigcap_{R\, i\in k} \mathcal{C}_i = (\bigcap_{i\in k} \kappa_i, \bigvee_{i\in k} \sigma_i)$,

where

$$\bigcup_{i\in k} \kappa_i = \{\langle t_1; (\bigcup_{i\in k} \kappa_{iE})(t_1), (\bigcup_{i\in k} \kappa_{iI})(t_1), (\bigcup_{i\in k} \kappa_{iN})(t_1)\rangle | t_1 \in X\},$$

$$\bigvee_{i\in k} \sigma_i = \{\langle t_1; (\bigvee_{i\in k} \sigma_{iE})(t_1), (\bigvee_{i\in k} \sigma_{iI})(t_1), (\bigvee_{i\in k} \sigma_{iN})(t_1)\rangle | t_1 \in X\},$$

$$\bigcap_{i\in k} \kappa_i = \{\langle t_1; (\bigcap_{i\in k} \kappa_{iE})(t_1), (\bigcap_{i\in k} \kappa_{iI})(t_1), (\bigcap_{i\in k} \kappa_{iN})(t_1)\rangle | t_1 \in X\},$$

$$\bigwedge_{i\in k} \sigma_i = \{\langle t_1; (\bigwedge_{i\in k} \sigma_{iE})(t_1), (\bigwedge_{i\in k} \sigma_{iI})(t_1), (\bigwedge_{i\in k} \sigma_{iN})(t_1)\rangle | t_1 \in X\},$$

Definition 2.14 [36] Let $C = (\mu_C, \nu_C)$ be an IFS in BF-algebra X and $t \in [0,1]$, then the IFS C^t is called the t-intuitionistic fuzzy subset of X w.r.t C and is defined as $C^t = \{< t_1, \mu_{C^t}(t_1), \nu_{C^t}(t_1) > | t_1 \in Y\} = < \mu_{C^t}, \nu_{C^t} >$ where $\mu_{C^t}(t_1) = \min\{\mu_C(t_1), t\}$ and $\mu_{C^t}(t_1) = \max\{\nu_C(t_1), 1 - t\}$ $\forall$ $t_1 \in X$.

Definition 2.15 [36] Let $B^t = (\mu_{B^t}, \nu_{B^t})$ be a t-intuitionistic fuzzy subset of BF-algebra X and $t \in [0,1]$ then B^t is called t-intuitionistic fuzzy subalgebra of X if it fulfills these axioms.

(i) $\mu_{B^t}(t_1 * t_2) \geq \min\{\mu_{B^t}(t_1), \mu_{B^t}(t_2)\}$,

(ii) $\nu_{B^t}(t_1 * t_2) \leq \max\{\nu_{B^t}(t_1), \nu_{B^t}(t_2)\}$, $\forall$ $t_1, t_2 \in X$.

3 t-Neutrosophic Cubic Subalgebra of BF-algebra

Let $\mathcal{C} = (\kappa_{\mathcal{C}}, \sigma_{\mathcal{C}})$ be a neutrosophic cubic set [**NCS**] of BF-algebra X, then the NCS $\mathcal{C}$ is called the t-neutrosophic cubic set (**t-NCS**) of X w.r.t $\mathcal{C}$ and is defined as $\mathcal{C}^t = \{< t_1, \hat{\kappa}^t(t_1), \sigma^t(t_1) > | t_1 \in X\} = < \hat{\kappa}^t, \sigma^t >$ such that $\hat{\kappa}^t(t_1) = \{< \hat{\kappa}^t_E(t_1), \hat{\kappa}^t_I(t_1), \hat{\kappa}^t_N(t_1) > | t_1 \in X\}$ and $\sigma(t_1) = \{< \sigma^t_E(t_1), \sigma^t_I(t_1), \sigma^t_N(t_1) > | t_1 \in X\}$ with two independent components where $\hat{\kappa}^t(t_1) = \{\mathrm{rmin}(\hat{\kappa}_E(t_1), t), \mathrm{rmin}(\hat{\kappa}_I(t_1), t'), \mathrm{rmin}(\hat{\kappa}_N(t_1), 2 - t - t')\}, \sigma^t(t_1) = \{\max(\sigma_E(t_1), t), \max(\sigma_I(t_1), t'), \max(\sigma_N(t_1), 2 - t - t')\}$ and $\forall$ $t, t', 2 - t - t' \in [0,1]$ and now concept of cubic subalgebra can be extended to t-NCSU.

Definition 3.1 Let $\mathcal{C} = (\hat{\kappa}, \sigma)$ be a cubic set, where X is subalgebra. Then $\mathcal{C}$ is t-NCSU under binary operation $*$ if it satisfies the following conditions:

N1:

$\hat{\kappa}^t{}_E(t_1 * t_2) \geq \mathrm{rmin}\{\hat{\kappa}^t_E(t_1), \hat{\kappa}^t_E(t_2)\}$,

$\hat{\kappa}^t{}_I(t_1 * t_2) \geq \mathrm{rmin}\{\hat{\kappa}^t_I(t_1), \hat{\kappa}^t_I(t_2)\}$,

$\hat{\kappa}^t{}_N(t_1 * t_2) \geq \mathrm{rmin}\{\hat{\kappa}^t_N(t_1), \hat{\kappa}^t_N(t_2)\}$,

N2:

$\sigma^t{}_E(t_1 * t_2) \leq \max\{\sigma^t_E(t_1), \sigma^t_E(t_2)\}$

$\sigma^t{}_I(t_1 * t_2) \leq \max\{\sigma^t_I(t_1), \sigma^t_I(t_2)\}$

$\sigma^t{}_N(t_1 * t_2) \leq \max\{\sigma^t_N(t_1), \sigma^t_N(t_2)\}$.

Where **E** means existenceship/membership value, **I** means indeterminacy existenceship/membership value and N means non existenceship/membership value. For our convenience we introduce new notation for t-neutrosophic cubic set as

$$\boldsymbol{\mathcal{C}} = (\boldsymbol{\hat{\kappa}^{t}_{E,I,N}}, \boldsymbol{\sigma^{t}_{E,I,N}}) = \{\langle \mathbf{t_1}, \boldsymbol{\hat{\kappa}^{t}_{E,I,N}}(\mathbf{t_1}), \boldsymbol{\sigma^{t}_{E,I,N}}(\mathbf{t_1})\rangle\} = \{\langle \mathbf{t_1}, \boldsymbol{\hat{\kappa}^{t}_{\Xi}}(\mathbf{t_1}), \boldsymbol{\sigma^{t}_{\Xi}}(\mathbf{t_1})\rangle\}$$

and for conditions N1, N2 as

N1: $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$,

N2: $\sigma^{t}_{\Xi}(t_1 * t_2) \leq \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$.

Example 3.2 Let $X = \{0, t_1, t_2, t_3, t_4, t_5\}$ be a BF-algebra with the following Cayley table.

$*$	0	t_1	t_2	t_3	t_4	t_5
0	0	t_5	t_4	t_3	t_2	t_1
t_1	t_1	0	t_5	t_4	t_3	t_2
t_2	t_2	t_1	0	t_5	t_4	t_3
t_3	t_3	t_2	t_1	0	t_5	t_4
t_4	t_4	t_3	t_2	t_1	0	t_5
t_5	t_5	t_4	t_3	t_2	t_1	0

A t-neutrosophic cubic set $\mathcal{C} = (\hat{\kappa}^{t}{}_{\Xi}, \sigma^{t}_{\Xi})$ of X is defined by

	0	t_1	t_2	t_3	t_4	t_5
$\hat{\kappa}^{t}{}_{E}$	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]
$\hat{\kappa}^{t}{}_{I}$	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]
$\hat{\kappa}^{t}{}_{N}$	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]

	0	t_1	t_2	t_3	t_4	t_5
$\sigma^{t}{}_{E}$	0.1	0.3	0.1	0.3	0.1	0.3
$\sigma^{t}{}_{I}$	0.3	0.5	0.3	0.5	0.3	0.5
$\sigma^{t}{}_{N}$	0.5	0.6	0.5	0.6	0.5	0.6

Both the conditions of definition are satisfied by the set $\mathcal{C}$. Thus $\mathcal{C} = (\hat{\kappa}^{t}{}_{\Xi}, \sigma^{t}_{\Xi})$ is a t-NCSU of X.

Proposition 3.3 Let $\mathcal{C} = \{\langle t_1, \hat{\kappa}^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_1)\rangle\}$ is a t-NCSU of X, then $\forall\ t_1 \in X$, $\hat{\kappa}^{t}_{\Xi}(t_1) \geq \hat{\kappa}^{t}_{\Xi}(0)$ and $\sigma^{t}_{\Xi}(0) \leq \sigma^{t}_{\Xi}(t_1)$. Thus, $\hat{\kappa}^{t}_{\Xi}(0)$ and $\sigma^{t}_{\Xi}(0)$ are the upper bound and lower bound of $\hat{\kappa}^{t}_{\Xi}(t_1)$ and $\sigma^{t}_{\Xi}(t_1)$ respectively.

Proof. $\forall\ t_1 \in X$, we have $\hat{\kappa}^{t}_{\Xi}(0) = \hat{\kappa}^{t}{}_{\Xi}(t_1 * t_1) \geq \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}{}_{\Xi}(t_1)\} = \hat{\kappa}^{t}_{\Xi}(t_1) \Rightarrow \hat{\kappa}^{t}_{\Xi}(0) \geq \hat{\kappa}^{t}_{\Xi}(t_1)$ and $\sigma^{t}_{\Xi}(0) = \sigma^{t}_{\Xi}(t_1 * t_1) \leq \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_1)\} = \sigma^{t}_{\Xi}(t_1) \Rightarrow \sigma^{t}_{\Xi}(0) \leq \sigma^{t}_{\Xi}(t_1)$.

Theorem 3.4 Let $\mathcal{C}$=$\{\langle t_1, \hat{\kappa}^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_1)\rangle\}$ be a t-NCSU of X. If there exists a sequence $\{(t_1)_n\}$ of X such that $\lim_{n\to\infty}\hat{\kappa}^{t}{}_{\Xi}((t_1)_n) = [1,1]$ and $\lim_{n\to\infty}\sigma^{t}_{\Xi}((t_1)_n) = 0$.Then $\hat{\kappa}^{t}{}_{\Xi}(0) = [1,1]$ and $\sigma^{t}_{\Xi}(0) = 0$.

Proof. Using above proposition, $\widehat{\kappa}_{\Xi}^{t}(0) \geq \widehat{\kappa}_{\Xi}^{t}(t_1)\ \forall\ t_1 \in X$, $\therefore\ \widehat{\kappa}_{\Xi}^{t}(0) \geq \widehat{\kappa}_{\Xi}^{t}((t_1)_n)$ for $n \in Z^{+}$. Consider, $[1,1] \geq \widehat{\kappa}_{\Xi}^{t}(0) \geq \lim_{n\to\infty}\widehat{\kappa}_{\Xi}^{t}((t_1)_n) = [1,1]$. Hence $\widehat{\kappa}_{\Xi}^{t}(0) = [1,1]$.

Again, using proposition, $\sigma_{\Xi}^{t}(0) \leq \sigma_{\Xi}^{t}(t_1)\ \forall\ t_1 \in X$, $\therefore\ \sigma_{\Xi}^{t}(0) \leq \sigma_{\Xi}^{t}((t_1)_n)$ for $n \in Z^{+}$. Consider, $0 \leq \sigma_{\Xi}^{t}(0) \leq \lim_{n\to\infty}\sigma_{\Xi}^{t}((t_1)_n) = 0$. Hence $\sigma_{\Xi}^{t}(0) = 0$.

Theorem 3.5 The R-intersection of any set of t-NCSU of X is t-NCSU of X.

Proof. Let $\mathcal{C}_i^t = \{\langle t_1, (\widehat{\kappa}^t{}_i)_\Xi, (\sigma_i^t)_\Xi\rangle | t_1 \in X\}$ where $i \in k$, is family of sets of t-NCSU of X and $t_1, t_2 \in X$ and $t \in [0,1]$ Then

$$(\cap\ (\widehat{\kappa}^t{}_i)_\Xi)(t_1 * t_2) = \mathrm{rinf}(\widehat{\kappa}^t{}_i)_\Xi(t_1 * t_2)$$
$$\geq \mathrm{rinf}\{\mathrm{rmin}\{(\widehat{\kappa}^t{}_i)_\Xi(t_1), (\widehat{\kappa}^t{}_i)_\Xi(t_2)\}\}$$
$$= \mathrm{rmin}\{\mathrm{rinf}(\widehat{\kappa}^t{}_i)_\Xi(t_1), \mathrm{rinf}(\widehat{\kappa}^t{}_i)_\Xi(t_2)\}$$
$$= \mathrm{rmin}\{(\cap\ (\widehat{\kappa}^t{}_i)_\Xi)(t_1), (\cap\ (\widehat{\kappa}^t{}_i)_\Xi)(t_2)\}$$
$$\Rightarrow (\cap\ (\widehat{\kappa}^t{}_i)_\Xi)(t_1 * t_2) \geq \mathrm{rmin}\{(\cap\ (\widehat{\kappa}^t{}_i)_\Xi)(t_1), (\cap\ (\widehat{\kappa}^t{}_i)_\Xi)(t_2)\}$$

and

$$(\vee\ (\sigma_i^t)_\Xi)(t_1 * t_2) = \sup(\sigma_i^t)_\Xi(t_1 * t_2)$$
$$\leq \sup\{\max\{(\sigma_i^t)_\Xi(t_1), (\sigma_i^t)_\Xi(t_2)\}\}$$
$$= \max\{\sup(\sigma_i^t)_\Xi(t_1), \sup(\sigma_i^t)_\Xi(t_2)\}$$
$$= \max\{(\vee\ (\sigma_i^t)_\Xi)(t_1), (\vee\ (\sigma_i^t)_\Xi)(t_2)\}$$
$$\Rightarrow (\vee\ (\sigma_i^t)_\Xi)(t_1 * t_2) \leq \max\{(\vee\ (\sigma_i^t)_\Xi)(t_1), (\vee\ (\sigma_i^t)_\Xi)(t_2)\},$$

which show that R-intersection of $\mathcal{C}_i^t$ is t-NCSU of X.

Remark 3.6 The R-union, P-intersection and P-union of t-NCSU need not to be a t-NCSU which is explained through example.

let $X = \{0, t_1, t_2, t_3, t_4, t_5\}$ be a BF-algebra with the following Caley table.

$*$	0	t_1	t_2	t_3	t_4	t_5
0	0	t_2	t_1	t_3	t_4	t_5
t_1	t_1	0	t_2	t_5	t_3	t_4
t_2	t_2	t_1	0	t_4	t_5	t_3
t_3	t_3	t_4	t_5	0	t_1	t_2
t_4	t_4	t_5	t_3	t_2	0	t_1
t_5	t_5	t_3	t_4	t_1	t_2	0

Let $\mathcal{C}_1^t = ((\widehat{\kappa}^t)_\Xi^1, (\sigma^t)_\Xi^1)$ and $\mathcal{C}_2^t = ((\widehat{\kappa}^t)_\Xi^2, (\sigma^t)_\Xi^2)$ are t-neutrosophic cubic sets of X which are defined by

	0	t_1	t_2	t_3	t_4	t_5
$\hat{\kappa}_1^t E$	[0.4,0.5]	[0.2,0.3]	[0.2,0.3]	[0.4,0.5]	[0.2,0.3]	[0.2,0.3]
$\hat{\kappa}_1^t I$	[0.6,0.7]	[0.3,0.4]	[0.3,0.4]	[0.6,0.7]	[0.3,0.4]	[0.3,0.4]
$\hat{\kappa}_1^t N$	[0.7,0.8]	[0.4,0.5]	[0.4,0.5]	[0.7,0.8]	[0.4,0.5]	[0.4,0.5]
$\hat{\kappa}_2^t E$	[0.7,0.8]	[0.3,0.4]	[0.3,0.4]	[0.3,0.4]	[0.7,0.8]	[0.3,0.4]
$\hat{\kappa}_2^t I$	[0.8,0.7]	[0.2,0.3]	[0.2,0.3]	[0.2,0.3]	[0.8,0.7]	[0.2,0.3]
$\hat{\kappa}_2^t N$	[0.7,0.6]	[0.2,0.4]	[0.2,0.4]	[0.2,0.4]	[0.7,0.6]	[0.2,0.4]

	0	t_1	t_2	t_3	t_4	t_5
$\sigma_1^t E$	0.2	0.9	0.9	0.2	0.9	0.9
$\sigma_1^t I$	0.3	0.8	0.8	0.3	0.8	0.8
$\sigma_1^t N$	0.5	0.7	0.7	0.5	0.7	0.7
$\sigma_2^t E$	0.3	0.6	0.6	0.6	0.3	0.6
$\sigma_2^t I$	0.4	0.8	0.8	0.8	0.4	0.8
$\sigma_2^t N$	0.5	0.8	0.8	0.8	0.3	0.8

$(\cup\ (\hat{\kappa}^t)^i_\Xi)(a_3 * a_4) = ([0.3,0.4],[0.3,0.4],[0.4,0.5])_\Xi \ngeq ([0.7,0.8],[0.6,0.7],[0.5,0.6])_\Xi =$ $\mathrm{rmin}\{(\cup\ (\hat{\kappa}^t)^i_\Xi)(a_3),(\cup\ (\hat{\kappa}^t)^i_\Xi)(a_4)\}$ and $(\wedge\,({\sigma^t}_i)_\Xi)(a_3 * a_4) = (0.5,0.6,0.7)_\Xi \nleq (0.3,0.4,0.5)_\Xi = \max\{(\wedge$ $({\sigma^t}_i)_\Xi)(a_3),(\wedge\,(\sigma^t_i)_\Xi)(a_4)\}$.

Theorem 3.7. Let $\mathcal{C}^t_i = \{\langle t_1,({\hat{\kappa}^t}_i)_\Xi,(\sigma^t_i)_\Xi\rangle | t_1 \in X\}$ be a collection of sets of t-NCSU of X, where $i \in k$ and $t \in [0,1]$. If $\inf\{\max\{(\sigma^t_i)_\Xi(t_1),(\sigma^t_i)_\Xi(t_1)\}\} = \max\{\inf(\sigma^t_i)_\Xi(t_1)$ $,\inf(\sigma^t_i)_\Xi(t_1)\}\ \forall\ t_1 \in X$, then the P-intersection of $\mathcal{C}^t_i$ is also a t-NCSU of X.

Proof. Suppose that $\mathcal{C}^t_i = \{\langle t_1,({\hat{\kappa}^t}_i)_\Xi,(\sigma^t_i)_\Xi\rangle | t_1 \in X\}$ where $i \in k$, be a collection of sets of t-NCSU of X such that $\inf\{\max\{(\sigma^t_i)_\Xi(t_1),(\sigma^t_i)_\Xi(t_1)\}\} = \max\{\inf(\sigma^t_i)_\Xi(t_1),\inf(\sigma^t_i)_\Xi(t_1)\}\ \forall\ a \in X$. Then for $t_1, t_2 \in X$ and $t \in [0,1]$. Then

$$(\cap\ ({\hat{\kappa}^t}_i)_\Xi)(t_1 * t_2) = \mathrm{rinf}\{({\hat{\kappa}^t}_i)_\Xi(t_1 * t_2)\}$$

$$\geq \mathrm{rinf}\{\mathrm{rmin}\{({\hat{\kappa}^t}_i)_\Xi(t_1),({\hat{\kappa}^t}_i)_\Xi(t_2)\}\}$$

$$= \mathrm{rmin}\{\mathrm{rinf}({\hat{\kappa}^t}_i)_\Xi(t_1),\mathrm{rinf}({\hat{\kappa}^t}_i)_\Xi(t_2)\}$$

$$= \mathrm{rmin}\{(\cap\ ({\hat{\kappa}^t}_i)_\Xi)(t_1),(\cap\ ({\hat{\kappa}^t}_i)_\Xi)(t_2)\}$$

$$\Rightarrow (\cap\ ({\hat{\kappa}^t}_i)_\Xi)(t_1 * t_2) \geq \mathrm{rmin}\{(\cap\ ({\hat{\kappa}^t}_i)_\Xi)(t_1),(\cap\ ({\hat{\kappa}^t}_i)_\Xi)(t_2)\}$$

and

$$(\wedge\,(\sigma^t_i))_\Xi)(t_1 * t_2) = \inf(\sigma^t_i)_\Xi(t_1 * t_2)$$

$$\leq \inf\{\max\{(\sigma^t_i)_\Xi(t_1),(\sigma^t_i))_\Xi(t_2)\}\}$$

$$= \max\{\inf(\sigma^t_i)_\Xi(t_1),\inf(\sigma^t_i)_\Xi(t_2)\}$$

$$= \max\{(\wedge\,(\sigma^t_i)_\Xi)(t_1),(\wedge\,(\sigma^t_i))_\Xi)(t_2)\}$$

$$\Rightarrow (\wedge (\sigma_i^t)_\Xi)(t_1 * t_2) \le \max\{(\wedge (\sigma_i^t)_\Xi)(t_1), (\wedge (\sigma_i^t))_\Xi)(t_2)\},$$

which show that P-intersection of $\mathcal{C}_i^t$ is t-NCSU of X.

Theorem 3.8. Let $\mathcal{C}_i^t = \{\langle t_1, (\hat{\kappa}^t{}_i)_\Xi, (\sigma_i^t)_\Xi\rangle | t_1 \in X\}$ where $i \in k$, be a collection of sets of t-NCSU of X. If $\sup\{r\min\{(\hat{\kappa}^t{}_i)_\Xi(t_1), (\hat{\kappa}^t{}_i)_\Xi(t_2)\}\} = r\min\{\sup(\hat{\kappa}^t{}_i)_\Xi(t_1), \sup(\hat{\kappa}^t{}_i)_\Xi(t_2)\}$ and $\inf\{\max\{(\sigma_i^t)_\Xi(t_1), (\sigma_i^t)_\Xi(t_2)\}\} = \max\{\inf(\sigma_i^t)_\Xi(t_1), \inf(\sigma_i^t)_\Xi(t_2)\}$, $\forall\, t_1 \in X$. Then P-union of $\mathcal{C}_i^t$ is t-NCSU of X.

Proof. Let $\mathcal{C}_i^t = \{\langle t_1, (\hat{\kappa}^t{}_i)_\Xi, (\sigma_i^t)_\Xi\rangle | t_1 \in X\}$ where $i \in k$, be a collection of sets of t-NCSU of X such that $\sup\{r\min\{(\hat{\kappa}^t{}_i)_\Xi(t_1), (\hat{\kappa}^t{}_i)_\Xi(t_2)\}\} = r\min\{\sup(\hat{\kappa}^t{}_i)_\Xi(t_1), \sup(\hat{\kappa}^t{}_i)_\Xi(t_2)\}$

$\forall\ t_1 \in X$. Then for $t_1, t_2 \in X$, and $t \in [0,1]$.

$$(\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1 * t_2) = r\sup(\hat{\kappa}^t{}_i)_\Xi(t_1 * t_2)$$
$$\ge r\sup\{r\min\{(\hat{\kappa}^t{}_i)_\Xi(t_1), (\hat{\kappa}^t{}_i)_\Xi(t_2)\}\}$$
$$= r\min\{r\sup(\hat{\kappa}^t{}_i)_\Xi(t_1), r\sup(\hat{\kappa}^t{}_i)_\Xi(t_2)\}$$
$$= r\min\{(\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1), (\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_2)\}$$
$$\Rightarrow (\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1 * t_2) \ge r\min\{(\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1), (\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_2)\}$$

and

$$(\vee (\sigma_i^t)_\Xi)(t_1 * t_2) = \sup(\sigma_i^t)_\Xi(t_1 * t_2)$$
$$\le \sup\{\max\{(\sigma_i^t)_\Xi(t_1), (\sigma_i^t)_\Xi(t_2)\}\}$$
$$= \max\{\sup(\sigma_i^t)_\Xi(t_1), \sup(\sigma_i^t)_\Xi(t_2)\}$$
$$= \max\{(\vee (\sigma_i^t)_\Xi)(t_1), (\vee (\sigma_i^t)_\Xi)(t_2)\}$$
$$\Rightarrow (\vee (\sigma_i^t)_\Xi)(t_1 * t_2) \le \max\{(\vee (\sigma_i^t)_\Xi)(t_1), (\vee (\sigma_i^t)_\Xi)(t_2)\},$$

which show that P-union of $\mathcal{C}_i^t$ is t-NCSU of X.

Theorem 3.9 Let $\mathcal{C}_i^t = \{\langle t_1, (\hat{\kappa}^t{}_i)_\Xi, (\sigma_i^t)_\Xi\rangle | t_1 \in X\}$ where $i \in k$, be a collection of sets of t-NCSU of X. If $\inf\{\max\{(\sigma_i^t)_\Xi(t_1), (\sigma_i^t)_\Xi(t_2)\}\} = \max\{\inf(\sigma_i^t)_\Xi(t_1), \inf(\sigma_i^t)_\Xi(t_2)\}$ and $\sup\{r\min\{(\hat{\kappa}^t{}_i)_\Xi(t_1), (\hat{\kappa}^t{}_i)_\Xi(t_2)\}\} = r\min\{\sup(\hat{\kappa}^t{}_i)_\Xi(t_1), \sup(\hat{\kappa}^t{}_i)_\Xi(t_2)\}$ $\forall\, t_1 \in X$ and $t \in [0,1]$. Then R-union of $\mathcal{C}_i^t$ is a t-NCSU of X.

Proof. Let $\mathcal{C}_i^t = \{\langle t_1, (\hat{\kappa}^t{}_i)_\Xi, (\sigma_i^t)_\Xi\rangle | t_1 \in X\}$ where $i \in k$, and $t \in [0,1]$ be collection of sets of t-NCSU of X such that $\inf\{\max\{(\sigma_i^t)_\Xi(t_1), (\sigma_i^t)_\Xi(t_2)\}\} = \max\{\inf(\sigma_i^t)_\Xi(t_1), \inf(\sigma_i^t)_\Xi(t_2)\}\}$ and $\sup\{r\min\{(\hat{\kappa}^t{}_i)_\Xi(t_1), (\hat{\kappa}^t{}_i)_\Xi(t_2)\}\} = r\min$

$\{\sup(\hat{\kappa}^t{}_i)_\Xi(t_1), \sup(\hat{\kappa}^t{}_i)_\Xi(t_2)\}$ $\forall\ t_1 \in X$. Then for $t_1, t_2 \in X$ and $t \in [0,1]$

$$(\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1 * t_2) = r\sup(\hat{\kappa}^t{}_i)_\Xi(t_1 * t_2)$$
$$\ge r\sup\{r\min\{(\hat{\kappa}^t{}_i)_\Xi(t_1), (\hat{\kappa}^t{}_i)_\Xi(t_2)\}\}$$
$$= r\min\{r\sup(\hat{\kappa}^t{}_i)_\Xi(t_1), r\sup(\hat{\kappa}^t{}_i)_\Xi(t_2)\}$$
$$= r\min\{(\cup\ \hat{\kappa}^t{}_i)_\Xi)(t_1), (\cup\ \hat{\kappa}^t{}_i)_\Xi)(t_2)\}$$
$$\Rightarrow (\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1 * t_2) \ge r\min\{(\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_1), (\cup\ (\hat{\kappa}^t{}_i)_\Xi)(t_2)\}$$

and

$$(\wedge (\sigma_i^t)_\Xi)(t_1 * t_2) = \inf(\sigma_i^t)_\Xi(t_1 * t_2)$$
$$\le \inf\{\max\{(\sigma_i^t)_\Xi(t_1), (\sigma_i^t)_\Xi(t_2)\}\}$$

$= \max\{\inf(\sigma_i^t)_\Xi(t_1), \inf(\sigma_i^t)_\Xi(t_2)\}$

$= \max\{(\wedge(\sigma_i^t)_\Xi)(t_1), (\wedge(\sigma_i^t)_\Xi)(t_2)\}$

$\Rightarrow (\wedge(\sigma_i^t)_\Xi)(t_1 * t_2) \le \max\{(\wedge(\sigma_i^t)_\Xi)(t_1), (\wedge(\sigma_i^t)_\Xi)(t_2)\}$,

which show that R-union of $\mathcal{C}_i^t$ is t-NCSU of X.

Theorem 3.10 If t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}^t{}_\Xi, \sigma_\Xi^t)$ of X is subalgebra, then $\forall\ t_1 \in X$, $\hat{\kappa}^t{}_\Xi(0 * t_1) \ge \hat{\kappa}^t{}_\Xi(t_1)$ and $\sigma_\Xi^t(0 * t_1) \le \sigma_\Xi^t(t_1)$.

Proof. For all $t_1 \in X$, $\hat{\kappa}^t{}_\Xi(0 * t_1) \ge \mathrm{rmin}\{\hat{\kappa}^t{}_\Xi(0), \hat{\kappa}^t{}_\Xi(t_1)\} = \mathrm{rmin}\{\hat{\kappa}^t{}_\Xi(t_1 * t_1), \hat{\kappa}^t{}_\Xi(t_1)\} \ge \mathrm{rmin}\{\mathrm{rmin}\{\hat{\kappa}^t{}_\Xi(t_1), \hat{\kappa}^t{}_\Xi(t_1)\}, \hat{\kappa}^t{}_\Xi(t_1)\} = \hat{\kappa}^t{}_\Xi(t_1)$and similarly $\sigma_\Xi^t(0 * t_1) \le \max\{\sigma_\Xi^t(0), \sigma_\Xi^t(t_1)\} = \sigma_\Xi^t(t_1)$.

Theorem 3.11 If t-netrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}^t{}_\Xi, \sigma_\Xi^t)$ of X is subalgebra then $\mathcal{C}^t(t_1 * t_2) = \mathcal{C}^t(t_1 * (0 * (0 * t_2)))\ \forall\ t_1, t_2 \in X$.

Proof. Let X be a BF-algebra and $t_1, t_2 \in X$. Then we know by above lemma that $t_2 = 0 * (0 * t_2)$. Hence $\hat{\kappa}^t{}_\Xi(t_1 * t_2) = \hat{\kappa}^t{}_\Xi(t_1 * (0 * (0 * t_2)))$ and $\sigma_\Xi^t(t_1 * t_2) = \sigma_\Xi^t(t_1 * (0 * (0 * t_2)))$. Therefore, $\mathcal{C}_\Xi^t(t_1 * t_2) = \mathcal{C}_\Xi^t(t_1 * (0 * (0 * t_2)))$.

Theorem 3.12 If t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}_\Xi^t, \sigma_\Xi^t)$ of X is t-NCSU, then $\forall\ t_1, t_2 \in$, $\hat{\kappa}_\Xi^t\big(t_1 * (0 * t_2)\big) \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}^t{}_\Xi(t_2)\}$ and $\sigma_\Xi^t\big(t_1 * (0 * t_2)\big) \le \max\{\sigma_\Xi^t(t_1),\ \sigma_\Xi^t(t_2)\}$.

Proof. Let $t_1, t_2 \in X$. Then we have $\hat{\kappa}_\Xi^t(t_1 * (0 * t_2)) \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(0 * t_2)\} \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(t_2)\}$ and $\sigma_\Xi^t(t_1 * (0 * t_2)) \le \max\{\sigma_\Xi^t(t_1), \sigma_\Xi^t(0 * t_2)\} \le \max\ \{\sigma_\Xi^t(t_1), \sigma_\Xi^t(t_2)\}$ by definition and proposition.

Theorem 3.13 If a t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}^t{}_\Xi, \sigma_\Xi^t)$ of X satisfies the following conditions, then $\mathcal{C}^t$ refers to a t-NCSU of X:

1. $\hat{\kappa}_\Xi^t(0 * t_1) \ge \hat{\kappa}_\Xi^t(t_1)$ and $\sigma_\Xi^t(0 * t_1) \le \sigma_\Xi^t(t_1)\ \forall\ t_1 \in X$
2. $\hat{\kappa}_\Xi^t(t_1 * (0 * t_2)) \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(t_2)\}$ and $\sigma_\Xi^t(t_1 * (0 * t_2)) \le \max\{\sigma_\Xi^t(t_1), \sigma_\Xi^t(t_2)\}$, $\forall\ t_1, t_2 \in X$ and $t \in [0,1]$.

Proof. Assume that the t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}_\Xi^t, \sigma_\Xi^t)$ of X satisfies the above conditions (1 and 2). Then by lemma, we have $\hat{\kappa}_\Xi^t(t_1 * t_2) = \hat{\kappa}_\Xi^t(t_1 * (0 * (0 * t_2))) \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(0 * t_2)\} \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(t_2)\}$ and $\sigma_\Xi^t(t_1 * t_2) = \sigma_\Xi^t(t_1 * (0 * (0 * t_2))) \le \max\{\sigma_\Xi^t(t_1), \sigma_\Xi^t(0 * t_2)\} \le \max\{\sigma_\Xi^t(t_1), \sigma_\Xi^t(t_2)\}\ \forall\ t_1, t_2 \in X$. Hence $\mathcal{C}^t$ is t-NCSU of X.

Theorem 3.14 A t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}_\Xi^t, \sigma_\Xi^t)$ of X is t-NCSU of X $\Leftarrow$ $\hat{\kappa}_\Xi^{t-}, \hat{\kappa}_\Xi^{t+}$ and σ_Ξ^t are fuzzy subalgebra of X.

Proof. Let $\hat{\kappa}_\Xi^{t-}, \hat{\kappa}_\Xi^{t+}$ and σ_Ξ^t are fuzzy subalgebra of X and $t_1, t_2 \in X$ and $t \in [0,1]$. Then $\hat{\kappa}_\Xi^{t-}(t_1 * t_2) \ge \min\{\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}_\Xi^{t-}(t_2)\}, \hat{\kappa}_\Xi^{t+}(t_1 * t_2) \ge \min\{\hat{\kappa}_\Xi^{t+}(t_1), \hat{\kappa}_\Xi^{t+}(t_2)\}$ and $\sigma_\Xi^t(t_1 * t_2) \le \max\{\sigma_\Xi^t(t_1), \sigma_\Xi^t(t_2)\}$. Now, $\hat{\kappa}_\Xi^t(t_1 * t_2) = [\hat{\kappa}_\Xi^{t-}(t_1 * t_2), \hat{\kappa}_\Xi^{t+}(t_1 * t_2)] \ge [\min\{\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}_\Xi^{t-}(t_2)\}, \min\{\hat{\kappa}_\Xi^{t+}(t_1), \hat{\kappa}_\Xi^{t+}(t_2)\}] \ge \mathrm{rmin}\{[\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}^{t+}{}_\Xi(t_2)], [\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}_\Xi^{t+}$

$(t_2)]\} = \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(t_2)\}$. Therefore, $\mathcal{C}^t$ is t-NCSU of X. Conversely, assume that $\mathcal{C}^t$ is a t-NCSU of X . For any $t_1, t_2 \in X$, $[\hat{\kappa}_\Xi^{t-}(t_1 * t_2), \hat{\kappa}_\Xi^{t+}(t_1 * t_2)] = \hat{\kappa}_\Xi^t(t_1 * t_2) \ge \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1), \hat{\kappa}_\Xi^t(t_2)\} = \mathrm{rmin}\{[\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}^{t+}{}_\Xi(t_1)], [\hat{\kappa}_\Xi^{t-}(t_2),\ \hat{\kappa}_\Xi^{t+}(t_2)]\}$ =$[\min\{\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}_\Xi^{t-}$

$(t_2)\}, \min\{\hat{\kappa}_\Xi^{t+}(t_1), \hat{\kappa}_\Xi^{t+}(t_2)\}]$. Thus, $\hat{\kappa}_\Xi^{t-}(t_1 * t_2) \ge \min\{\hat{\kappa}_\Xi^{t-}(t_1), \hat{\kappa}_\Xi^{t-}(t_2)\}$, $\hat{\kappa}_\Xi^{t+}(t_1 * t_2) \ge \min\{\hat{\kappa}_\Xi^{t+}(t_1), \hat{\kappa}_\Xi^{t+}(t_2)\}$ and $\sigma_\Xi^t(t_1 * t_2) \le \max\{\sigma_\Xi^t(t_1), \sigma_\Xi^t(t_2)\}$. Hence $\hat{\kappa}_\Xi^{t+}, \hat{\kappa}_\Xi^{t-}$ and σ_Ξ^t are fuzzy subalgebra of X.

Theorem 3.15 Let $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ be a t-NCSU of X and $n \in \mathbb{Z}^+$(the set of positive integer). Then

1. $\hat{\kappa}^t_\Xi(Л_n t_1 * t_1) \geq \hat{\kappa}^t_\Xi(t_1)$ for $n \in \mathbb{O}$,
2. $\sigma^t_\Xi(Л_n t_1 * t_1) \leq \sigma^t_\Xi(t_1)$ for $n \in \mathbb{O}$,
3. $\hat{\kappa}^t_\Xi(Л_n t_1 * t_1) = \hat{\kappa}^t_\Xi(t_1)$ for $n \in \mathbb{E}$,
4. $\sigma^t_\Xi(Л_n t_1 * t_1) = \sigma^t_\Xi(t_1)$ for $n \in \mathbb{E}$.

Proof. Let $t_1 \in X$ and n is odd. Then $n = 2q - 1$ for some positive integer q. We prove the theorem by induction. Now $\hat{\kappa}^t_\Xi(t_1 * t_1) = \hat{\kappa}^t_\Xi(0) \geq \hat{\kappa}^t_\Xi(t_1)$ and $\sigma^t_\Xi(t_1 * t_1) = \sigma^t_\Xi(0) \leq \sigma^t_\Xi(t_1)$. Suppose that $\hat{\kappa}^t_\Xi(Л_{2q-1} t_1 * t_1) \geq \hat{\kappa}^t_\Xi(t_1)$ and $\sigma^t_\Xi(Л_{2q-1} t_1 * t_1) \leq \sigma^t_\Xi(t_1)$. Then by assumption, $\hat{\kappa}^t_\Xi(Л_{2(q+1)-1} t_1 * t_1) = \hat{\kappa}^t_\Xi(Л_{2q+1} t_1 * t_1) = \hat{\kappa}^t_\Xi(Л_{2q-1} t_1 * (t_1 * (t_1 * t_1))) = \hat{\kappa}^t_\Xi(Л_{2q-1} t_1 * t_1) \geq \hat{\kappa}^t{}_\Xi(t_1)$ and $\sigma^t_\Xi(Л_{2(q+1)-1} t_1 * t_1) = \sigma^t_\Xi(Л_{2q+1} t_1 * t_1) = \sigma^t_\Xi(Л_{2q-1} t_1 * (t_1 * (t_1 * t_1))) = \sigma^t_\Xi(Л_{2q-1} t_1 * t_1) \leq \sigma^t_\Xi(t_1)$, which prove (1) and (2), similarly we can prove the remaining cases (3) and (4).

Theorem 3.16 The sets denoted by $I_{\hat{\kappa}^t_\Xi}$ and $I_{\sigma^t_\Xi}$ are also subalgebras of X, which are defined as: $I_{\hat{\kappa}^t_\Xi} = \{t_1 \in X | \hat{\kappa}^t_\Xi(t_1) = \hat{\kappa}^t_\Xi(0)\}$, $I_{\sigma^t_\Xi} = \{t_1 \in X | \sigma^t_\Xi(t_1) = \sigma^t_\Xi(0)\}$. Let $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ be a t-NCSU of X. Then the sets $I_{\hat{\kappa}^t_\Xi}$ and $I_{\sigma^t_\Xi}$ are subalgebras of X.

Proof. Let $t_1, t_2 \in I_{\hat{\kappa}^t_\Xi}$. Then $\hat{\kappa}^t_\Xi(t_1) = \hat{\kappa}^t_\Xi(0) = \hat{\kappa}^t_\Xi(t_2)$ and $\hat{\kappa}^t_\Xi(t_1 * t_2) \geq \text{rmin}\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(t_2)\} = \hat{\kappa}^t{}_\Xi(0)$. By using Proposition 3.3, we know that $\hat{\kappa}^t_\Xi(t_1 * t_2) = \hat{\kappa}^t{}_\Xi(0)$ or equivalently $t_1 * t_2 \in I_{\hat{\kappa}^t{}_\Xi}$.

Again let $t_1, t_2 \in I_{\hat{\kappa}^t_\Xi}$. Then $\sigma^t_\Xi(t_1) = \sigma^t_\Xi(0) = \sigma^t_\Xi(t_2)$ and $\sigma^t_\Xi(t_1 * t_2) \leq \max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(t_2)\} = \sigma^t_\Xi(0)$. Again by using Proposition 3.3, we know that $\sigma^t_\Xi(t_1 * t_2) = \sigma^t_\Xi(0)$ or equivalently $t_1 * t_2 \in I_{\hat{\kappa}^t_\Xi}$. Hence the sets $I_{\hat{\kappa}^t_\Xi}$ and $I_{\sigma^t_\Xi}$ are subalgebras of X.

Theorem 3.17 Let A be a nonempty subset of X and $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ be a t-neutrosophic cubic set of X defined by

$$\hat{\kappa}^t_\Xi(t_1) = \begin{cases} [\mu_{\Xi_1}, \mu_{\Xi_2}], & \text{if } t_1 \in A \\ [\nu_{\Xi_1}, \nu_{\Xi_2}], & \text{otherwise,} \end{cases} \quad \sigma^t_\Xi(t_1) = \begin{cases} \phi_\Xi, & \text{if } t_1 \in A \\ \delta_\Xi, & \text{otherwise} \end{cases}$$

, $\forall$ $[\mu_{\Xi_1}, \mu_{\Xi_2}], [\nu_{\Xi_1}, \nu_{\Xi_2}] \in D[0,1]$ and ϕ_Ξ, $\delta_\Xi \in [0,1]$ with $[\mu_{\Xi_1}, \mu_{\Xi_2}] \geq [\nu_{\Xi_1}, \nu_{\Xi_2}]$ and $\phi_\Xi \leq \delta_\Xi$. Then $\mathcal{C}^t$ is a t-NCSU of X $\Leftrightarrow$ A is a subalgebra of X. Moreover, $I_{\hat{\kappa}^t_\Xi} = A = I_{\sigma^t_\Xi}$

Proof. Let $\mathcal{C}^t$ be a t-NCSU of X and $t_1, t_2 \in X$ such that $t_1, t_2 \in A$. Then $\hat{\kappa}^t_\Xi(t_1 * t_2) \geq \text{rmin}\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(t_2)\} = \text{rmin}\{[\mu_{\Xi_1}, \mu_{\Xi_2}], [\mu_{\Xi_1}, \mu_{\Xi_2}]\} = [\mu_{\Xi_1}, \mu_{\Xi_2}]$ and $\sigma^t_\Xi(t_1 * t_2) \leq \max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(t_2)\} = \max\{\phi_\Xi, \phi_\Xi\} = \phi_\Xi$. Therefore $t_1 * t_2 \in A$. Hence A is a subalgebra of X.

Conversely, suppose that A is a subalgebra of X and $t_1, t_2 \in X$. Consider two cases.
Case 1: If $t_1, t_2 \in A$ then $t_1 * t_2 \in A$, thus $\hat{\kappa}^t_\Xi(t_1 * t_2) = [\mu_{\Xi_1}, \mu_{\Xi_2}] = \text{rmin}\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(t_2)\}$ and $\sigma^t_\Xi(t_1 * t_2) = \phi_\Xi = \max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(t_2)\}$.
Case 2: If $t_1 \notin A$ or $t_2 \notin A$, then $\hat{\kappa}^t_\Xi(t_1 * t_2) \geq [\nu_{\Xi_1}, \nu_{\Xi_2}] = \text{rmin}\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(t_2)\}$ and $\sigma^t_\Xi(t_1 * t_2) \leq \delta_\Xi = \max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(t_2)\}$. Hence $\mathcal{C}^t$ is a t-NCSU of X.

Now, $I_{\hat{\kappa}^t_\Xi} = \{t_1 \in X, \hat{\kappa}^t_\Xi(t_1) = \hat{\kappa}^t_\Xi(0)\} = \{t_1 \in X, \hat{\kappa}^t_\Xi(t_1) = [\alpha_{\Xi_1}, \alpha_{\Xi_2}]\} = A$ and $I_{\sigma^t_\Xi} = \{t_1 \in X, \sigma^t_\Xi(t_1) = \sigma^t_\Xi(0)\} = \{t_1 \in X, \sigma^t_\Xi(t_1) = \gamma_\Xi\} = A$.

Definition 3.18 Let $\mathcal{C}^t = (\hat{\kappa}^t{}_\Xi, \sigma^t_\Xi)$ be a t-neutrosophic cubic set of X. For $[s_{E_1}, s_{E_2}], [s_{I_1}, s_{I_2}], [s_{N_1}, s_{N_2}] \in D[0,1]$ and $t_{E_1}, t_{I_1}, t_{N_1} \in [0,1]$, the set $U(\hat{\kappa}^t{}_\Xi | ([s_{E_1}, s_{E_2}], [s_{I_1}, s_{I_2}], [s_{N_1}, s_{N_2}])) = \{t_1 \in X | \hat{\kappa}^t_E(t_1) \geq [s_{E_1}, s_{E_2}], \hat{\kappa}^t_I(t_1) \geq [s_{I_1}, s_{I_2}], \hat{\kappa}^t_N(t_1) \geq [s_{N_1}, s_{N_2}]\}$ is called upper $([s_{E_1}, s_{E_2}], [s_{I_1}, s_{I_2}], [s_{N_1}, s_{N_2}])$-level of $\mathcal{C}^t$ and $L(\sigma^t_\Xi | (t_{E_1}, t_{I_1}, t_{N_1})) = \{t_1 \in X | \sigma^t{}_E(t_1) \leq t_{E_1}, \sigma^t{}_I(t_1) \leq t_{I_1}, \sigma^t{}_N(t_1) \leq t_{N_1}\}$ is called lower $(t_{E_1}, t_{I_1}, t_{N_1})$-level of $\mathcal{C}^t$.

For comfort, we introduce the new notions for upper level and lower level of $\mathcal{C}^{t}$ as, $U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}]=\{t_1 \in X|\hat{\kappa}^{t}_{\Xi}(t_1) \geq [s_{\Xi_1}, s_{\Xi_2}]\}$ is called upper ($[s_{\Xi_1}, s_{\Xi_2}]$)-level of $\mathcal{C}^{t}$ and $L(\sigma^{t}_{\Xi}|t_{\Xi_1})=\{t_1 \in X|\sigma^{t}_{\Xi}(t_1) \leq t_{\Xi_1}\}$ is called lower t_{Ξ_1}-level of $\mathcal{C}^{t}$.

Theorem 3.19 If $\mathcal{C}^{t} = (\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi})$ is t-NCSU of X, then the upper $[s_{\Xi_1}, s_{\Xi_2}]$-level and lower t_{Ξ_1}-level of $\mathcal{C}^{t}$ are subalgebras of X.

Proof. Let $t_1, t_2 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$. Then $\hat{\kappa}^{t}_{\Xi}(t_1) \geq [s_{\Xi_1}, s_{\Xi_2}]$ and $\hat{\kappa}^{t}_{\Xi}(t_2) \geq [s_{\Xi_1}, s_{\Xi_2}]$. It follows that $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\} \geq [s_{\Xi_1}, s_{\Xi_2}] \Rightarrow t_1 * t_2 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$. Hence, $U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}]$ is a subalgebra of X. Let $t_1, t_2 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$. Then $\sigma^{t}_{\Xi}(t_1) \leq t_{\Xi_1}$ and $\sigma^{t}_{\Xi}(t_2) \leq t_{\Xi_1}$. It follows that $\sigma^{t}_{\Xi}(t_1 * t_2) \leq \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\} \leq t_{\Xi_1} \Rightarrow t_1 * t_2 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$. Hence $L(\sigma^{t}_{\Xi}|t_{\Xi_1})$ is a subalgebra of X.

Corollary 3.20 Let $\mathcal{C}^{t} = (\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi})$ is t-NCSU of X. Then $\hat{\kappa}^{t}_{\Xi}([s_{\Xi_1}, s_{\Xi_2}]; t_{\Xi_1})= U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}]) \cap L(\sigma^{t}_{\Xi}|t_{\Xi_1})$ $= \{t_1 \in X|\hat{\kappa}^{t}_{\Xi}(t_1) \geq [s_{\Xi_1}, s_{\Xi_2}], \sigma^{t}_{\Xi}(t_1) \leq t_{\Xi_1}\}$ is a subalgebra of X.

Proof. We can prove it by using above proved Theorem. The converse of above corollary is not valid.

Theorem 3.21 Every subalgebra of X can be realized as both the upper $[s_{\Xi_1}, s_{\Xi_2}]$-level and lower t_{Ξ_1}-level of some t-NCSU of X.

Proof. Let $\mathcal{A}^{t}$ be a t-NCSU of X, and t-neutrosophic cubic set $\mathcal{C}^{t}$ on X is defined by

$$\hat{\kappa}^{t}_{\Xi} = \begin{cases} [\mu_{\Xi_1}, \mu_{\Xi_2}] & \text{if } t_1 \in \mathcal{A}^{t} \\ [0,0] & \text{otherwise}. \end{cases}, \sigma^{t}_{\Xi} = \begin{cases} \nu_{\Xi_1} & \text{if } t_1 \in \mathcal{A}^{t} \\ 0 & \text{otherwise}. \end{cases}$$

$\forall$ $[\mu_{\Xi_1}, \mu_{\Xi_2}] \in D[0,1]$ and $\nu_{\Xi_1} \in [0,1]$. We investigate the following cases.

Case 1 If $\forall$ $t_1, t_2 \in \mathcal{A}^{t}$ then $\hat{\kappa}^{t}_{\Xi}(t_1) = [\mu_{\Xi_1}, \mu_{\Xi_2}]$, $\sigma^{t}_{\Xi}(t_1) = \nu_{\Xi_1}$ and $\hat{\kappa}^{t}_{\Xi}(t_2) = [\mu_{\Xi_1}, \mu_{\Xi_2}]$, $\sigma^{t}_{\Xi}(t_2) = \nu_{\Xi_1}$.Thus $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) = [\mu_{\Xi_1}, \mu_{\Xi_2}] = \text{rmin}\{[\mu_{\Xi_1}, \mu_{\Xi_2}], [\mu_{\Xi_1}, \mu_{\Xi_2}]\} = \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$ and $\sigma^{t}_{\Xi}(t_1 * t_2) = \nu_{\Xi_1} = \max\{\nu_{\Xi_1}, \nu_{\Xi_1}\} = \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$.

Case 2 If $t_1 \in \mathcal{A}^{t}$ and $t_2 \notin \mathcal{A}^{t}$, then $\hat{\kappa}^{t}_{\Xi}(t_1) = [\mu_{\Xi_1}, \mu_{\Xi_2}]$, $\sigma^{t}_{\Xi}(t_1) = \nu_{\Xi_1}$ and $\hat{\kappa}^{t}_{\Xi}(t_2) = [0,0]$, $\sigma^{t}_{\Xi}(t_2) = 1$. Thus $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq [0,0] = \text{rmin}\{[\mu_{\Xi_1}, \mu_{\Xi_2}], [0,0]\} = \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$ and $\sigma^{t}_{\Xi}(t_1 * t_2) \leq 1 = \max\{\nu_{\Xi_1}, 1\} = \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$.

Case 3 If $t_1 \notin \mathcal{A}^{t}$ and $t_2 \in \mathcal{A}^{t}$, then $\hat{\kappa}^{t}_{\Xi}(t_1) = [0,0]$, $\sigma^{t}_{\Xi}(t_1) = 1$ and $\hat{\kappa}^{t}_{\Xi}(t_2) = [\mu_{\Xi_1}, \mu_{\Xi_2}]$, $\sigma^{t}_{\Xi}(t_2) = \nu_{\Xi_1}$. Thus $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq [0,0] = \text{rmin}\{[0,0], [\mu_{\Xi_1}, \nu_{\Xi_2}]\} = \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$ and $\sigma^{t}_{\Xi}(t_1 * t_2) \leq 1 = \max\{1, \nu_{\Xi_1}\} = \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$.

Case 4 If $t_1 \notin \mathcal{A}^{t}$ and $t_2 \notin \mathcal{A}^{t}$, then $\hat{\kappa}^{t}_{\Xi}(t_1) = [0,0]$, $\sigma^{t}_{\Xi}(t_1) = 1$ and $\hat{\kappa}^{t}_{\Xi}(t_2) = [0,0]$, $\sigma^{t}_{\Xi}(t_2) = 1$. Thus $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq [0,0] = \text{rmin}\{[0,0], [0,0]\} = \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$ and $\sigma^{t}_{\Xi}(t_1 * t_2) \leq 1 = \max\{1,1\} = \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$. Therefore, $\mathcal{C}^{t}$ is a t-NCSU of X.

Theorem 3.22 Let $\mathcal{A}^{t}$ be a subset of X and $\mathcal{C}^{t}$ be a t-neutrosophic cubic set on X which is given in the proof of above theorem. If $\mathcal{C}^{t}$ is realized as lower level subalgebra and upper level subalgebra of some t-NCSU of X, then $\mathcal{B}^{t}$ is a t-neutrosophic cubic one of X.

Proof. Let $\mathcal{C}^{t}$ be a t-NCSU of X, and $t_1, t_2 \in \mathcal{C}^{t}$. Then $\hat{\kappa}^{t}_{\Xi}(t_1) = \hat{\kappa}^{t}_{\Xi}(t_2) = [\alpha_{\Xi_1}, \alpha_{\Xi_2}]$ and $\sigma^{t}_{\Xi}(t_1) = \sigma^{t}_{\Xi}(t_2) = \beta_{\Xi_1}$. Thus $\hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq \text{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\} = \text{rmin}\{[\alpha_{\Xi_1}, \alpha_{\Xi_2}],$ $[\alpha_{\Xi_1}, \alpha_{\Xi_2}]\} = [\alpha_{\Xi_1}, \alpha_{\Xi_2}]$ and $\sigma^{t}_{\Xi}(t_1 * t_2) \leq \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\} = \max\{\beta_{\Xi_1}, \beta_{\Xi_1}\} = \beta_{\Xi_1} \Rightarrow t_1 * t_2 \in \mathcal{A}^{t}$. Hence proof is completed.

4 Image and Pre-image of t-Neutrosophic Cubic Subalgebra

In this section, homomorphism of t-neutrosophic cubic subalgebra is defined and some results are studied.

Suppose Γ be a mapping from X into Y and $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ be a t-neutrosophic cubic set in X. Then the inverse-image of $\mathcal{C}^t$ is defined as $\Gamma^{-1}(\mathcal{C}^t) = \{\langle t_1, \Gamma^{-1}(\hat{\kappa}^t_\Xi), \Gamma^{-1}(\sigma^t_\Xi)\rangle | t_1 \in X\}$ and $\Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1) = \hat{\kappa}^t_\Xi(\Gamma(t_1))$ and $\Gamma^{-1}(\sigma^t_\Xi)(t_1) = \sigma^t_\Xi(\Gamma(t_1))$. It can be shown that $\Gamma^{-1}(\mathcal{C}^t)$ is a t-neutrosophic cubic set.

Theorem 4.1 Suppose that $\Gamma | X \to Y$ be a homomorphism of BF-algebra. If $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ is a t-NCSU of Y, then the pre-image $\Gamma^{-1}(\mathcal{C}^t)$=$\{\langle t_1, \Gamma^{-1}(\hat{\kappa}^t_\Xi), \Gamma^{-1}(\sigma^t_\Xi)\rangle | t_1 \in X\}$ of $\mathcal{C}^t$ under Γ is a t-NCSU of X.

Proof. Assume that $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ is a t-NCSU of Y and $t_1, t_2 \in X$. Then $\Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1 * t_2) = \hat{\kappa}^t_\Xi(\Gamma(t_1 * t_2)) = \hat{\kappa}^t_\Xi(\Gamma(t_1) * \Gamma(t_2)) \geq \text{rmin}\{\hat{\kappa}^t_\Xi(\Gamma(t_1)), \hat{\kappa}^t_\Xi(\Gamma(t_2))\} = \text{rmin}\{\Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1), \Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_2)\}$ and $\Gamma^{-1}(\sigma^t_\Xi)(t_1 * t_2) = \sigma^t_\Xi(\Gamma(t_1 * t_2)) = \sigma^t_\Xi(\Gamma(t_1) * \Gamma(t_2)) \leq \max\{\sigma^t_\Xi(\Gamma(t_1)), \sigma^t_\Xi(\Gamma(t_2))\} = \max\{\Gamma^{-1}(\sigma^t_\Xi)(t_1), \Gamma^{-1}(\sigma^t_\Xi)(t_2)\}$. $\therefore \Gamma^{-1}(\mathcal{C}^t) = \{\langle t_1, \Gamma^{-1}(\hat{\kappa}^t{}_\Xi), \Gamma^{-1}(\sigma^t_\Xi)\rangle | t_1 \in X\}$ is t-NCSU of X.

Theorem 4.2 Consider $\Gamma | X \to Y$ be a homomorphism of BF-algebra and $\mathcal{C}^t_j = ((\hat{\kappa}^t_j)_\Xi, (\sigma^t_j)_\Xi)$ be a t-NCSU of Y, where $j \in k$. If $\inf\{\max\{(\sigma^t_j)_\Xi(t_2), (\sigma^t_j)_\Xi(t_2)\}\} = \max\{\inf(\sigma^t_j)_\Xi(t_2), \inf(\sigma^t_j)_\Xi(t_2)\}$, $\forall$ $t_2 \in Y$. Then $\Gamma^{-1}(\bigcap_{R\, j\in k} \mathcal{C}^t_j)$ is t-NCSU of X.

Proof. Let $\mathcal{C}^t_j = ((\kappa^t_j)_\Xi, (\sigma^t_j)_\Xi)$ be a t-NCSU of Y where $j \in k$satisfying $\inf\{\max\{(\sigma^t_j)_\Xi(t_2), (\sigma^t_j)_\Xi(t_2)\}\}$ = $\max\{\inf(\sigma^t_j)_\Xi(t_2), \inf(\sigma^t_j)_\Xi(t_2)\}$, $\forall$ $t_2 \in Y$. Then by Theorem 3.7 we know, $\bigcap_{R\, j\in k} \mathcal{C}^t_j$ is a t-NCSU of Y. Hence $\Gamma^{-1}(\bigcap_{R\, j\in k} \mathcal{C}^t_j)$ is t-NCSU of X.

Theorem 4.3 Let $\Gamma | X \to Y$ be a homomorphism of BF-algebra. Assume that $\mathcal{C}^t_j = ((\hat{\kappa}^t_j)_\Xi, (\sigma^t_j)_\Xi)$ be a collection of sets of t-NCSU of Y where $j \in k$. If $\text{rsup}\{\text{rmin}\{(\hat{\kappa}^t_j)_\Xi(t_2), (\hat{\kappa}^t_j)_\Xi(t_2)\}\} = \text{rmin}\{\text{rsup}(\hat{\kappa}^t_j)_\Xi(t_2), \text{rsup}(\hat{\kappa}^t_j)_\Xi(t_2)\}$, $\forall$ $(t_2), (t_2)' \in Y$. Then $\Gamma^{-1}(\bigcup_{R\, j\in k} \mathcal{C}^t_j)$ is t-NCSU of X.

Proof. Let $\mathcal{C}^t_j = ((\hat{\kappa}^t_j)_\Xi, (\sigma^t_j)_\Xi)$ be a t-NCSU of Y where $j \in k$ satisfying $\text{rsup}\{\text{rmin}\{(\hat{\kappa}^t_j)_\Xi(t_2), (\hat{\kappa}^t_j)_\Xi(t_2')\} = \text{rmin}\{\text{rsup}(\hat{\kappa}^t_j)_\Xi(t_2), \text{rsup}(\hat{\kappa}^t_j)_\Xi(t_2')\}$ $\forall$ $t_2, t_2' \in Y$. Then by Theorem 3.8 we know, $\bigcup_{R\, j\in k} \mathcal{C}^t_j$ is a t-NCSU of Y. Hence $\Gamma^{-1}(\bigcup_{R\, j\in k} \mathcal{C}^t_j)$ is t-NCSU of X.

Definition 4.4 A t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ in BF-algebra X is said to have rsup-property and inf-property for any subset P of X, $\exists$ $p_0 \in T$ such that $\hat{\kappa}^t_\Xi(p_0) = \underset{p_0 \in S}{\text{rsup}} \hat{\kappa}^t_\Xi(p_0)$ and

$\sigma^t_\Xi(s_0) = \inf_{t_0 \in T} \sigma^t_\Xi(t_0)$ respectively.

Definition 4.5 Let Γ be mapping from X to Y. If $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ is neutrosphic cubic set of X, then the image of $\mathcal{C}^t$ under Γ is denoted by $\Gamma(\mathcal{C}^t)$ and is defined as $\Gamma(\mathcal{C}^t)$=$\{\langle t_1, \Gamma_{\text{rsup}}(\hat{\kappa}^t_\Xi), \Gamma_{\inf}(\hat{\kappa}^t_\Xi)\rangle | t_1 \in X\}$, where

$$\Gamma_{\text{rsup}}(\hat{\kappa}^t_\Xi)(t_2) = \begin{cases} \underset{t_1 \in \Gamma^{-1}(t_2)}{\text{rsup}} (\hat{\kappa}^t_\Xi)(t_1), & \text{if } \Gamma^{-1}(t_2) \neq \phi \\ [0,0], & \text{otherwise}, \end{cases}$$

and

$$\Gamma_{\inf}(\sigma^t_\Xi)(t_2) = \begin{cases} \inf_{t_1 \in \Gamma^{-1}(t_2)} \sigma^t_\Xi(t_1), & \text{if } \Gamma^{-1}(t_2) \neq \phi \\ 1, & \text{otherwise}. \end{cases}$$

Theorem 4.6 Suppose $\Gamma | X \to Y$ be a homomorphism from a BF-algebra X onto a BF-algebra Y. If $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ is a t-NCSU of X, then the image $\Gamma(\mathcal{C}^t)$ $=\{\langle t_1, \Gamma_{\text{rsup}}(\hat{\kappa}^t_\Xi), \Gamma_{\inf}(\sigma^t_\Xi)\rangle | t_1 \in X\}$ of $\mathcal{A}$ under Γ is t-NCSU of Y.

Proof. Let $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ be a t-NCSU of X and $t_2, t_2' \in Y$. We know that $\{t_1 * t_1' | t_1 \in \Gamma^{-1}(t_2)$ and $t_1' \in \Gamma^{-1}t_2'\} \subseteq \{t_1 \in X | t_1 \in \Gamma^{-1}(t_2 * t_2')\}$. Now $\Gamma_{rsup}(\hat{\kappa}^t_\Xi)(t_2 * t_2')$=rsup$\{\hat{\kappa}^t_\Xi(t_1) | t_1 \in \Gamma^{-1}(t_2 * t_2')\} \geq$ rsup$\{\hat{\kappa}^t_\Xi(t_1 * t_1') | t_1 \in \Gamma^{-1}(t_2)$ and $t_1' \in \Gamma^{-1}(t_2')\} \geq$ rsup$\{$rmin$\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(t_1')\} | t_1 \in \Gamma^{-1}(t_2)$ and $t_1' \in \Gamma^{-1}(t_2')\} =$ rmin$\{$rsup$\{\hat{\kappa}^t_\Xi(t_1) | t_1 \in \Gamma^{-1}(t_2)\}$, rsup$\{\hat{\kappa}^t_\Xi(t_1') | t_1' \in \Gamma^{-1}(t_2')\}\} =$ rmin$\{\Gamma_{rsup}(\hat{\kappa}^t_\Xi)(t_2)$,

$\Gamma_{rsup}(\hat{\kappa}^t_\Xi)(t_2')\}$ and $\Gamma_{inf}(\sigma^t_\Xi)(t_2 * t_2') = \inf\{\sigma^t_\Xi(t_1) | t_1 \in \Gamma^{-1}(t_2 * t_2')\} \leq \inf\{\sigma^t_\Xi(t_1 * t_1') | t_1 \in \Gamma^{-1}(t_2)$ and $t_1' \in \Gamma^{-1}(t_2')\} \leq \inf\{\max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(t_1')\} | t_1 \in \Gamma^{-1}(t_2)$ and $t_1' \in \Gamma^{-1}(t_2')\} = \max\{\inf\{\sigma^t_\Xi(t_1) | t_1 \in \Gamma^{-1}(t_2)\}, \inf\{\sigma^t_\Xi(t_1') | t_1' \in \Gamma^{-1}(t_2')\}\} = \max\{\Gamma_{inf}(\sigma^t_\Xi)(t_2), \Gamma_{inf}(\sigma^t_\Xi)(t_2')\}$. Hence $\Gamma(\mathcal{C}^t)$=$\{\langle t_1, \Gamma_{rsup}(\hat{\kappa}^t{}_\Xi), \Gamma_{inf}(\sigma^t{}_\Xi) \rangle | t_1 \in X\}$

is a t-NCSU of Y.

Theorem 4.7 Assume that $\Gamma | X \to Y$ is a homomorphism of BF-algebra and $\mathcal{C}^t_i = \{(\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi\}$ is a t-NCSU of X, where $i \in k$. If $\inf\{\max\{(\sigma^t_i)_\Xi(t_1), (\sigma^t_i)_\Xi(t_1)\}\} = \max\{\inf(\sigma^t_i)_\Xi(t_1), \inf(\sigma^t_i)_\Xi(t_1)\}$, $\forall\, t_1 \in X$. Then $\Gamma(\bigcap_{P\, i\in k} \mathcal{C}^t_i)$ is a t-NCSU of Y.

Proof. Let $\mathcal{C}^t_i = \{(\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi\}$ be a collection of sets of t-NCSU of X, where $i \in k$ satisfies $\inf\{\max\{(\sigma^t_i)_\Xi(t_1), (\sigma^t_i)_\Xi(t_1)\}\} = \max\{\inf(\sigma^t_i)_\Xi(t_1), \inf(\sigma^t_i)_\Xi(t_1)\}$ $\forall$ $t_1 \in X$. Then by above stated theorem, $\bigcap_{P\, i\in k} \mathcal{C}^t_i$ is a t-NCSU of X. Hence $\Gamma(\bigcap_{P\, i\in k} \mathcal{C}^t_j)$ is t-NCSU of Y.

Theorem 4.8 Suppose $\Gamma | X \to Y$ be a homomorphism of BF-algebra and $\mathcal{C}^t_i = \{(\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi\}$ be a t-NCSU of X where $i \in k$.If rsup$\{$rmin$\{(\kappa^t_i)_\Xi(t_1), (\hat{\kappa}^t_i)_\Xi(t_1)\}\}$ = rmin$\{$rsup $(\hat{\kappa}^t_i)_\Xi(t_1)$, rsup$(\hat{\kappa}^t_i)_\Xi(t_1')\}$, $\forall$ $t_1, t_1' \in Y$. Then $\Gamma(\bigcup_{P\, i\in k} \mathcal{C}^t_i)$ is also a t-NCSU of Y.

Proof. Let $\mathcal{C}^t_i = \{(\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi\}$ be a collection of sets of t-NCSU of X where $i \in k$ satisfies rsup$\{$rmin$\{(\hat{\kappa}^t_i)_\Xi(t_1), (\hat{\kappa}^t_i)_\Xi(t_1')\}\}$ = rmin$\{$rsup$(\hat{\kappa}^t_i)_\Xi(t_1)$, rsup$(\hat{\kappa}^t_i)_\Xi(t_1')\}$, $\forall\, t_1, t_1' \in X$. Then by above stated theorem we know that $\bigcup_{P\, i\in k} \mathcal{C}^t_i$ is a t-NCSU of X. Hence $\Gamma(\bigcup_{P\, i\in k} \mathcal{C}^t_i)$ is t-NCSU of Y.

Theorem 4.9 For a homomorphism $\Gamma | X \to Y$ of BF-algebra, the following results hold:

1. If $\forall$ $i \in k$, $\mathcal{C}^t_i$ is t-NCSU of X, then $\Gamma(\bigcap_{R\, i\in k} \mathcal{C}^t_i)$ is t-NCSU of Y,
2. If $\forall\, i \in k$, $\mathcal{D}^t_i$ is t-NCSU of Y, then $\Gamma^{-1}(\bigcap_{R\, i\in k} \mathcal{D}^t_i)$ is t-NCSU of X.

Proof. Straightforward.

Theorem 4.10 Let Γ be an isomorphism from a BF-algebra X onto a BF-algebra Y. If $\mathcal{C}^t$ is a t-NCSU of X. Then $\Gamma^{-1}(\Gamma(\mathcal{C}^t)) = \mathcal{C}^t$.

Proof. For any $t_1 \in X$, let $\Gamma(t_1) = t_2$. Since Γ is an isomorphism, $\Gamma^{-1}(t_2) = \{t_1\}$. Thus $\Gamma(\mathcal{C}^t)(\Gamma(t_1)) = \Gamma(\mathcal{C}^t)(t_2) = \bigcup_{t_1 \in \Gamma^{-1}(t_2)} \mathcal{C}^t(t_1) = \mathcal{C}^t(t_1)$.For any $t_2 \in Y$, Γ is an isomorphism, $\Gamma^{-1}(t_2) = \{t_1\}$ so that $\Gamma(t_1) = t_2$. Thus $\Gamma^{-1}(\mathcal{C}^t)(t_1) = \mathcal{C}^t(\Gamma(t_1)) = \mathcal{C}^t(t_2)$. Hence, $\Gamma^{-1}(\Gamma(\mathcal{C}^t)) = \mathcal{C}^t$.

Corollary 4.11 Consider Γ is an Isomorphism from a BF-algebra X onto a BF-algebra Y. If $\mathcal{C}^t$ is a t-NCSU of Y. Then $\Gamma(\Gamma^{-1}(\mathcal{C}^t)) = \mathcal{C}^t$.

Proof. Straightforward.

Corollary 4.12 Let $\Gamma | X \to X$ be an automorphism. If $\mathcal{C}^t$ is a t-NCSU of X. Then $\Gamma(\mathcal{C}^t) = \mathcal{C}^t \Leftarrow \Gamma^{-1}(\mathcal{C}^t) = \mathcal{C}^t$.

5 t-Neutrosophic Cubic Closed Ideal of BF-algebra

In this section, t-neutrosophic cubic ideal and t-neutrosophic cubic closed ideal of BF-algebra are defined and investigated through related results.

Definition 5.1 A t-neutrosophic cubic set $\mathcal{C}^{\mathrm{t}} = (\hat{\kappa}^{\mathrm{t}}_{\Xi}, \sigma^{\mathrm{t}}_{\Xi})$ of X is called a t-NCID of X if it satisfies following axoims:

N3. $\hat{\kappa}^{\mathrm{t}}{}_{\Xi}(0) \geq \hat{\kappa}^{\mathrm{t}}_{\Xi}(\mathrm{t}_1)$ and $\sigma^{\mathrm{t}}_{\Xi}(0) \leq \sigma^{\mathrm{t}}_{\Xi}(\mathrm{t}_1)$,

N4. $\hat{\kappa}^{\mathrm{t}}_{\Xi}(\mathrm{t}_1) \geq \mathrm{rmin}\{\hat{\kappa}^{\mathrm{t}}_{\Xi}(\mathrm{t}_1 * \mathrm{t}_2), \hat{\kappa}^{\mathrm{t}}_{\Xi}(\mathrm{t}_2)\}$,

N5. $\sigma^{\mathrm{t}}_{\Xi}(\mathrm{t}_1) \leq \max\{\sigma^{\mathrm{t}}_{\Xi}(\mathrm{t}_1 * \mathrm{t}_2), \sigma^{\mathrm{t}}_{\Xi}(\mathrm{t}_2)\}$, $\forall\, \mathrm{t}_1, \mathrm{t}_2 \in \mathrm{X}$.

Example 5.2 Consider a BF-algebra $\mathrm{X} = \{0, \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3\}$ and binary operation * is defined on X as

⁕	0	t_1	t_2	t_3
	0	t_1	t_2	t_3
t_1	t_1	0	t_3	t_2
t_2	t_2	t_3	0	t_1
t_3	t_3	t_2	t_1	0

Let $\mathcal{C}^{\mathrm{t}} = \{\hat{\kappa}^{\mathrm{t}}{}_{\Xi}, \sigma^{\mathrm{t}}_{\Xi}\}$ be a t-neutrosophic cubic set in X is defined as,

	0	t_1	t_2	t_3
$\hat{\kappa}^t{}_E$	[1,1]	[0.8,0.7]	[1,1]	[0.4,0.6]
$\hat{\kappa}^t{}_I$	[0.8,0.8]	[0.5,0.7]	[0.8,0.8]	[0.6,0.4]
$\hat{\kappa}^t{}_N$	[0.7,0.8]	[0.4,0.5]	[0.7,0.8]	[0.8,0.4]

and

	0	t_1	t_2	t_3
$\sigma^t{}_E$	0	0.7	0	0.6
$\sigma^t{}_I$	0.1	0.5	0.1	0.6
$\sigma^t{}_N$	0.2	0.3	0.2	0.4

Then it can be easy verify that $\mathcal{C}^{\mathrm{t}}$ satisfies the conditions N3, N4 and N5. Hence $\mathcal{C}^{\mathrm{t}}$ is t-NCID of X.

Definition 5.3 Let $\mathcal{C}^{\mathrm{t}} = \{\hat{\kappa}^{\mathrm{t}}_{\Xi}, \sigma^{\mathrm{t}}_{\Xi}\}$ be a t-neutrosophic cubic set X then it is called t-neutrosophic cubic closed ideal of X if it satisfies N4, N5 and N6. $\hat{\kappa}^{\mathrm{t}}_{\Xi}(0 * \mathrm{t}_1) \geq \hat{\kappa}^{\mathrm{t}}_{\Xi}(\mathrm{t}_1)$ and $\sigma^{\mathrm{t}}_{\Xi}(0 * \mathrm{t}_1) \leq \sigma^{\mathrm{t}}_{\Xi}(\mathrm{t}_1)$, $\forall\, \mathrm{t}_1 \in \mathrm{X}$.

Example 5.4 Let $\mathrm{X} = \{0, \mathrm{t}_1, \mathrm{t}_2, \mathrm{t}_3, \mathrm{t}_4, \mathrm{t}_5\}$ be a BF-algebra as in Example 3.2 and $\mathcal{C}^{\mathrm{t}} = \{\hat{\kappa}^{\mathrm{t}}_{\Xi}, \sigma^{\mathrm{t}}_{\Xi}\}$ be a t-neutrosophic cubic set in X is defined as

	0	t_1	t_2	t_3	t_4	t_5
$\hat{\kappa}^{\mathrm{t}}{}_{\mathrm{E}}$	[0.4,0.7]	[0.3,0.6]	[0.3,0.6]	[0.2,0.4]	[0.2,0.4]	[0.2,0.4]
$\hat{\kappa}^{\mathrm{t}}{}_{\mathrm{I}}$	[0.5,0.8]	[0.4,0.7]	[0.4,0.7]	[0.3,0.6]	[0.3,0.6]	[0.3,0.6]
$\hat{\kappa}^{\mathrm{t}}{}_{\mathrm{N}}$	[0.6,0.9]	[0.5,0.8]	[0.5,0.8]	[0.3,0.4]	[0.3,0.4]	[0.3,0.4]

	0	t_1	t_2	t_3	t_4	t_5
$\sigma^t{}_E$	0.3	0.6	0.6	0.8	0.8	0.8
$\sigma^t{}_I$	0.4	0.5	0.5	0.7	0.7	0.7
$\sigma^t{}_N$	0.5	0.6	0.6	0.9	0.9	0.9

By calculations it is clear that $\mathcal{C}^t$ is a t-neutrosophic cubic closed ideal of X.

Proposition 5.5 Every t-neutrosophic cubic closed ideal is a t-NCID.

Proof The converse of proposition 5.5 is not true in general as shown in the given example.

Example 5.6 Let $X = \{0, t_1, t_2, t_3, t_4, t_5\}$ be a BF-algebra as in Example 3.2 and $\mathcal{C}^t = \{\hat{\kappa}^t_\Xi, \sigma^t_\Xi\}$ be a t-neutrosophic cubic set in X is defined as

	0	t_1	t_2	t_3	t_4	t_5
$\hat{\kappa}^t{}_E$	[0.5,0.7]	[0.4,0.6]	[0.4,0.6]	[0.3,0.4]	[0.3,0.4]	[0.3,0.4]
$\hat{\kappa}^t{}_I$	[0.6,0.8]	[0.5,0.7]	[0.5,0.7]	[0.4,0.6]	[0.4,0.6]	[0.4,0.6]
$\hat{\kappa}^t{}_N$	[0.7,0.9]	[0.6,0.8]	[0.6,0.8]	[0.5,0.4]	[0.5,0.4]	[0.5,0.4]

	0	t_1	t_2	t_3	t_4	t_5
$\sigma^t{}_E$	0.2	0.5	0.5	0.6	0.6	0.6
$\sigma^t{}_I$	0.3	0.4	0.4	0.7	0.7	0.7
$\sigma^t{}_N$	0.3	0.5	0.5	0.8	0.8	0.8

By calculations verify that $\mathcal{C}^t$ is a t-NCID of X. But it is not a t-neutrosophic cubic closed ideal of X since $\hat{\kappa}^t{}_\Xi(0 * t_1) \ngeq \hat{\kappa}^t_\Xi(t_1)$ and $\sigma^t_\Xi(0 * t_1) \nleq \sigma^t_\Xi(t_1)$, $\forall\ t_1 \in X$.

Corollary 5.7 Every t-NCSU which satisfies N4 and N5 becomes a t-neutrosophic cubic closed ideal.

Theorem 5.8 Every t-neutrosophic cubic closed ideal of a BF-algebra X is also a t-NCSU of X.

Proof. Suppose $\mathcal{C}^t = \{\hat{\kappa}^t_\Xi, \sigma^t_\Xi\}$ be a t-neutrosophic cubic closed ideal of X, then for any $t_1 \in X$ we have $\hat{\kappa}^t_\Xi(0 * t_1) \geq \hat{\kappa}^t_\Xi(t_1)$ and $\sigma^t_\Xi(0 * t_1) \leq \sigma^t_\Xi(t_1)$. Now by N4, N6, Proposition 3.3, we know that $\hat{\kappa}^t_\Xi(t_1 * t_2) \geq \text{rmin}\{\hat{\kappa}^t_\Xi((t_1 * t_2) * (0 * t_2)), \hat{\kappa}^t_\Xi(0 * t_2)\} = \text{rmin}\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(0 * t_2)\} \geq \text{rmin}\{\hat{\kappa}^t_\Xi(t_1), \hat{\kappa}^t_\Xi(t_2)\}$ and $\sigma^t_\Xi(t_1 * t_2) \leq \max\{\sigma^t_\Xi((t_1 * t_2) * (0 * t_2)), \sigma^t_\Xi(0 * t_2)\} = \max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(0 * t_2)\} \leq \max\{\sigma^t_\Xi(t_1), \sigma^t_\Xi(t_2)\}$. Hence $\mathcal{C}^t$ is a t-neutrosophic cubic subalgeba of X.

Theorem 5.9 The R-intersection of any set of t-NCIDs of X is a t-NCID of X.

Proof. Let $\mathcal{C}^t_i = \{(\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi\}$ where $i \in k$, be a collection of sets of t-NCID of X and $t_1, t_2 \in X$. Then

$$(\cap\ (\hat{\kappa}^t_i)_\Xi)(0) = \text{rinf}(\hat{\kappa}^t_i)_\Xi(0)$$

$$\geq \text{rinf}(\hat{\kappa}^t_i)_\Xi(t_1)$$

$$= (\cap\ (\hat{\kappa}^t_i)_\Xi)(t_1)$$

$$\Rightarrow (\cap\ (\hat{\kappa}^t_i)_\Xi)(0) \geq (\cap\ (\hat{\kappa}^t_i)_\Xi)(t_1),$$

$(\vee (\sigma_i^t)_\Xi)(0) = \sup(\sigma_i^t)_\Xi(0)$

$\leq (\sigma_i^t)_\Xi(t_1)$

$= (\vee (\sigma_i^t)_\Xi)(t_1)$

$\Rightarrow (\vee (\sigma_i^t)_\Xi)(0) \leq (\vee (\sigma_i^t)_\Xi)(t_1),$

$(\cap (\hat{\kappa}_i^t)_\Xi)(t_1) = \mathrm{rinf}(\hat{\kappa}_i^t)_\Xi(t_1)$

$\geq \mathrm{rinf}\{\mathrm{rmin}\{(\hat{\kappa}_i^t)_\Xi(t_1 * t_2), (\hat{\kappa}_i^t)_\Xi(t_2)\}\}$

$= \mathrm{rmin}\{\mathrm{rinf}(\hat{\kappa}_i^t)_\Xi(t_1 * t_2), \mathrm{rinf}(\hat{\kappa}_i^t)_\Xi(t_2)\}$

$= \mathrm{rmin}\{(\cap (\hat{\kappa}_i^t)_\Xi)(t_1 * t_2), (\cap (\hat{\kappa}_i^t)_\Xi)(t_2)\}$

$\Rightarrow (\cap (\hat{\kappa}_i^t)_\Xi)(t_1) \geq \mathrm{rmin}\{(\cap (\hat{\kappa}_i^t)_\Xi)(t_1 * t_2), (\cap (\hat{\kappa}_i^t)_\Xi)(t_2)\}$

and

$(\vee (\sigma_i^t)_\Xi)(t_1) = \sup(\sigma_i^t)_\Xi(t_1)$

$\leq \sup\{\max\{(\sigma_i^t)_\Xi(t_1 * t_2), (\sigma_i^t)_\Xi(t_2)\}\}$

$= \max\{\sup(\sigma_i^t)_\Xi(t_1 * t_2), \sup(\sigma_i^t)_\Xi(t_2)\}$

$= \max\{(\vee (\sigma_i^t)_\Xi)(t_1 * t_2), (\vee (\sigma_i^t)_\Xi)(t_2)\}$

$\Rightarrow (\vee (\sigma_i^t)_\Xi)(t_1) \leq \max\{(\vee (\sigma_i^t)_\Xi)(t_1 * t_2), (\vee (\sigma_i^t)_\Xi)(t_2)\},$

which show that R-intersection is a t-NCID of X.

Theorem 5.10 The R-intersection of any set of t-neutrosophic cubic closed ideals of X is also a t-neutrosophic cubic closed ideal of X.

Proof. It is similar to the proof of Theorem 5.9.

Theorem 5.11 For a t-neutrosophic cubic ideal $\mathcal{C}^t = \{\hat{\kappa}_\Xi^t, \sigma_\Xi^t\}$ of X, the following assertions are valid:

1. if $t_1 * t_2 \leq z$, then $\hat{\kappa}_\Xi^t(t_1) \geq \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_2), \hat{\kappa}_\Xi^t(t_3)\}$ and $\sigma_\Xi^t(t_1) \leq \max\{\sigma_\Xi^t(t_2), \sigma_\Xi^t(t_3)\}$,

2. if $t_1 \leq t_2$, then $\hat{\kappa}_\Xi^t(t_1) \geq \hat{\kappa}_\Xi^t(t_2)$ and $\sigma_\Xi^t(t_1) \leq \sigma_\Xi^t(t_2)$, $\forall\ t_1, t_2, t_3 \in X$.

Proof. 1. Assume that $t_1, t_2, t_3 \in X$ such that $t_1 * t_2 \leq t_3$. Then $(t_1 * t_2) * t_3 = 0$ and thus $\hat{\kappa}_\Xi^t(t_1) \geq \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1 * t_2), \hat{\kappa}_\Xi^t(t_2)\} \geq \mathrm{rmin}\{\mathrm{rmin}\{\hat{\kappa}_\Xi^t((t_1 * t_2) * t_3), \hat{\kappa}_\Xi^t(t_3)\}, \hat{\kappa}_\Xi^t(t_2)\} = \mathrm{rmin}\{\mathrm{rmin}\{\hat{\kappa}_\Xi^t(0), \hat{\kappa}_\Xi^t(t_3)\}, \hat{\kappa}_\Xi^t(t_2)\} = \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_2), \hat{\kappa}_\Xi^t(t_3)\}$ and $\sigma_\Xi^t(t_1) \leq \max\{\sigma_\Xi^t(t_1 * t_2), \sigma_\Xi^t(t_2)\} \leq \max\{\max\{\sigma_\Xi^t((t_1 * t_2) * t_3), \sigma_\Xi^t(t_3)\}, \sigma_\Xi^t(t_2)\} = \max\{\max\{\sigma_\Xi^t(0), \sigma_\Xi^t(t_3)\},$

$\sigma_\Xi^t(t_2)\} = \max\{\sigma_\Xi^t(b), \sigma_\Xi^t(t_3)\}$.

2. Again, take $t_1, t_2 \in X$ such that $t_1 \leq t_2$. Then $t_1 * t_2 = 0$ and thus $\hat{\kappa}_\Xi^t(t_1) \geq \mathrm{rmin}\{\hat{\kappa}_\Xi^t(t_1 * t_2), \hat{\kappa}_\Xi^t(t_2)\} = \mathrm{rmin}\{\hat{\kappa}_\Xi^t(0), \hat{\kappa}_\Xi^t(t_2)\} = \hat{\kappa}_\Xi^t(t_2)$ and $\sigma_\Xi^t(t_1) \leq \mathrm{rmin}\{\sigma_\Xi^t(t_1 * t_2), \sigma_\Xi^t(t_2)\} = \mathrm{rmin}\{\sigma_\Xi^t(0), \sigma_\Xi^t(t_2)\} = \sigma_\Xi^t(t_2)$.

Theorem 5.12 Let $\mathcal{C}^t = \{\hat{\kappa}_\Xi^t, \sigma_\Xi^t\}$ is a neutrosophic cubic ideal of X. If $t_1 * t_2 \leq t_1, \forall\ t_1, t_2 \in X$. Then $\mathcal{C}^t$ is a t-NCSU of X.

Proof. Assume that $\mathcal{C}^t = \{\hat{\kappa}_\Xi^t, \sigma_\Xi^t\}$ is a t-neutrosophic cubic ideal of X. Suppose that $t_1 * t_2 \leq t_1\ \forall\ t_1, t_2 \in X$. Then

$\hat{\kappa}_\Xi^t(t_1 * t_2) \geq \hat{\kappa}_\Xi^t(t_1)$ ($\because$ By Theorem 5.11)

$\geq \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1 * t_2), \hat{\kappa}^{t}_{\Xi}(t_2)\}$ ($\because$ By N4)

$\geq \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$ ($\because$ By Theorem 5.11)

$\Rightarrow \hat{\kappa}^{t}_{\Xi}(t_1 * t_2) \geq \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1), \hat{\kappa}^{t}_{\Xi}(t_2)\}$

and

$\sigma^{t}_{\Xi}(t_1 * t_2) \leq \sigma^{t}_{\Xi}(t_1)$ ($\because$ By Theorem 5.11)

$\leq \max\{\sigma^{t}_{\Xi}(t_1 * t_2), \sigma^{t}_{\Xi}(t_2)\}$ ($\because$ By N5)

$\leq \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$ ($\because$ By Theorem 5.11)

$\Rightarrow \sigma^{t}_{\Xi}(t_1 * t_2) \leq \max\{\sigma^{t}_{\Xi}(t_1), \sigma^{t}_{\Xi}(t_2)\}$.

Hence $\mathcal{C}^t = \{\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi}\}$ is a t-NCSU of X.

Theorem 5.13 If $\mathcal{C}^t = \{\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi}\}$ is a t-neutrosophic cubic ideal of X, then $(\dots((t_1 * x_1) * x_2) * \dots) * x_n = 0$ for any $t_1, x_1, x_2, \dots, x_n \in X \Rightarrow \hat{\kappa}^{t}{}_{\Xi}(t_1) \geq \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(x_1), \hat{\kappa}^{t}_{\Xi}(x_2), \dots,$ $\hat{\kappa}^{t}_{\Xi}(x_n)\}$ and $\sigma^{t}_{\Xi}(t_1) \leq \max\{\sigma^{t}_{\Xi}(x_1), \sigma^{t}_{\Xi}(x_2), \dots, \sigma^{t}_{\Xi}(x_n)\}$.

Proof. We can prove this theorem by using induction on n and Theorem 5.11.

Theorem 5.14 A t-neutrosophic cubic set $\mathcal{C}^t = (\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi})$ is a t-neutrosophic cubic closed ideal of X $\Leftarrow$ $U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$ and $L(\sigma^{t}_{\Xi}|t_{\Xi_1})$ are closed ideals of X for every $[s_{\Xi_1}, s_{\Xi_2}] \in D[0,1]$ and $t_{\Xi_1} \in [0,1]$.

Proof. Assume that $\mathcal{C}^t = (\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi})$ is a t-neutrosophic cubic closed ideal of X. For $[s_{\Xi_1}, s_{\Xi_2}] \in D[0,1]$, clearly, $0 * t_1 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$, where $t_1 \in X$. Let $t_1, t_2 \in X$ be such that $t_1 * t_2 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$ and $t_2 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$. Then $\hat{\kappa}^{t}_{\Xi}(t_1) \geq \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(t_1 * t_2), \hat{\kappa}^{t}_{\Xi}(t_2)\} \geq [s_{\Xi_1}, s_{\Xi_2}] \Rightarrow t_1 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}]$. Hence $U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$ is a closed ideal of X.

For $t_{\Xi_1} \in [0,1]$. Clearly, $0 * t_1 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$. Let $t_1, t_2 \in X$ be such that $t_1 * t_2 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$ and $t_2 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$. Then $\sigma^{t}_{\Xi}(t_1) \leq \max\{\sigma^{t}_{\Xi}(t_1 * t_2), \sigma^{t}_{\Xi}(t_2)\} \leq t_{\Xi_1} \Rightarrow t_1 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$. Hence $L(\sigma^{t}_{\Xi}|t_{\Xi_1})$ is a t-neutrosophic cubic closed ideal of X.

Conversely, suppose that each nonempty level subset $U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$ and $L(\sigma^{t}_{\Xi}|t_{\Xi_1})$ are closed ideals of X. For any $t_1 \in X$, let $\hat{\kappa}^{t}_{\Xi}(t_1) = [s_{\Xi_1}, s_{\Xi_2}]$ and $\sigma^{t}_{\Xi}(t_1) = t_{\Xi_1}$. Then $t_1 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}])$ and $t_1 \in L(\sigma^{t}_{\Xi}|t_{\Xi_1})$. Since $0 * t_1 \in U(\hat{\kappa}^{t}_{\Xi}|[s_{\Xi_1}, s_{\Xi_2}]) \cap L(\sigma^{t}_{\Xi}|t_{\Xi_1})$, it follows that $\hat{\kappa}^{t}_{\Xi}(0 * t_1) \geq [s_{\Xi_1}, s_{\Xi_2}] = \hat{\kappa}^{t}_{\Xi}(t_1)$ and $\sigma^{t}_{\Xi}(0 * t_1) \leq t_{\Xi_1} = \sigma^{t}_{\Xi}(t_1)$ $\forall$ $t_1 \in X$. If there exists $\alpha_{\Xi_1}, \beta_{\Xi_1} \in X$ such that $\hat{\kappa}^{t}_{\Xi}(\alpha_{\Xi_1}) \leq \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(\alpha_{\Xi_1} * \beta_{\Xi_1}), \beta_{\Xi_1}\}$, then by taking $[s'_{\Xi_1}, s'_{\Xi_2}] = \frac{1}{2}[\hat{\kappa}^{t}_{\Xi}(\alpha_{\Xi_1} * \beta_{\Xi_1}) + \mathrm{rmin}\{\hat{\kappa}^{t}_{\Xi}(\alpha_{\Xi_1}), \hat{\kappa}^{t}_{\Xi}(\beta_{\Xi_1})\}]$.

It follows that $\alpha_{\Xi_1} * \beta_{\Xi_1} \in U(\hat{\kappa}^{t}_{\Xi}|[s'_{\Xi_1}, s'_{\Xi_2}])$ and $\beta_{\Xi_1} \in U(\hat{\kappa}^{t}_{\Xi}|[s'_{\Xi_1}, s'_{\Xi_2}])$, but $\alpha_{\Xi_1} \notin U(\hat{\kappa}^{t}_{\Xi}|[s'_{\Xi_1}, s'_{\Xi_2}])$, which is contradiction. Hence, $U(\hat{\kappa}^{t}_{\Xi}|[s'_{\Xi_1}, s'_{\Xi_2}])$ is not closed ideal of X.

Again, if there exists $\alpha_{\Xi_1}, \beta_{\Xi_1} \in X$ such that $\sigma^{t}_{\Xi}(\alpha_{\Xi_1}) \geq \max\{\sigma^{t}_{\Xi}(\alpha_{\Xi_1} * \beta_{\Xi_1}), \sigma^{t}_{\Xi}(\beta_{\Xi_1})\}$, then by taking $t'_{\Xi_1} = \frac{1}{2}[\sigma^{t}_{\Xi}(\alpha_{\Xi_1} * \beta_{\Xi_1}) + \max\{\sigma^{t}_{\Xi}(\alpha_{\Xi_1}), \sigma^{t}_{\Xi}(\beta_{\Xi_1})\}]$.

It follows that $\alpha_{\Xi_1} * \beta_{\Xi_1} \in L(\sigma^{t}_{\Xi}|t'_{\Xi_1})$ and $\beta_{\Xi_1} \in L(\sigma^{t}_{\Xi}|t'_{\Xi_1})$, but $\alpha_{\Xi_1} \notin L(\sigma^{t}_{\Xi}|t'_{\Xi_1})$, which is contradiction. So $L(\sigma^{t}_{\Xi}|t'_{\Xi_1})$ is not closed ideal of X. Hence $\mathcal{C}^t = (\hat{\kappa}^{t}_{\Xi}, \sigma^{t}_{\Xi})$ is a t-neutrosophic cubic ideal of X because it satisfies N3 and N4.

6 Neutrosophic Cubic Ideals under Homomorphism

In this section, t-neutrosophic cubic ideals are investigated under homomorphism through some results.

Theorem 6.1 Suppose that $\Gamma|X \to Y$ is a homomorphism of BF-algebra. If $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ is a t-NCID of Y. Then pre-image $\Gamma^{-1}(\mathcal{C}^t) = (\Gamma^{-1}(\hat{\kappa}^t_\Xi), \Gamma^{-1}(\sigma^t_\Xi))$ of $\mathcal{C}^t$ under Γ of X is a t-NCID of X.

Proof. For all $t_1 \in X$, $\Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1) = \hat{\kappa}^t_\Xi(\Gamma(t_1)) \le \hat{\kappa}^t_\Xi(0) = \hat{\kappa}^t_\Xi(\Gamma(0)) = \Gamma^{-1}(\hat{\kappa}^t_\Xi)(0)$ and $\Gamma^{-1}(\sigma^t_\Xi)(t_1) = \sigma^t_\Xi(\Gamma(t_1)) \ge \sigma^t_\Xi(0) = \sigma^t_\Xi(\Gamma(0)) = \Gamma^{-1}(\sigma^t_\Xi)(0)$. Let $t_1, t_2 \in X, \Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1) = \hat{\kappa}^t_\Xi(\Gamma(t_1)) \ge \mathrm{rmin}\{\hat{\kappa}^t_\Xi(\Gamma(t_1) * \Gamma(t_2)), \hat{\kappa}^t_\Xi(\Gamma(t_2))\} = \mathrm{rmin}\{\hat{\kappa}^t_\Xi(\Gamma(t_1 * t_2)), \hat{\kappa}^t_\Xi(\Gamma(t_2))\} = \mathrm{rmin}\{\Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1 * t_2), \Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_2)\}$ and $\Gamma^{-1}(\sigma^t_\Xi)(a) = \sigma^t_\Xi(\Gamma(t_1)) \le \max\{\sigma^t_\Xi(\Gamma(t_1) * \Gamma(t_2)), \sigma^t_\Xi(\Gamma(t_2))\} = \max\{\sigma^t_\Xi(\Gamma(t_1 * t_2)), \sigma^t_\Xi(\Gamma(t_2))\} = \max\{\Gamma^{-1}(\sigma^t_\Xi)(t_1 * t_2), \Gamma^{-1}(\sigma^t_\Xi)(t_2)\}$. Hence $\Gamma^{-1}(\mathcal{C}^t) = (\Gamma^{-1}(\hat{\kappa}^t_\Xi), \Gamma^{-1}(\sigma^t_\Xi))$ is a t-NCID of X.

Corollary 6.2 A homomorphic pre-image of a t-neutrosophic cubic closed ideal is a t-NCID.

Proof. Using Proposition 5.5 and Theorem 6.1, we can prove this corollary .

Corollary 6.3 A homomorphic preimage of a t-neutrosophic cubic closed ideal is also a t-NCSU.

Proof. Using Theorem 5.8 and Theorem 6.1, we can prove this corollary.

Corollary 6.4 Let $\Gamma|X \to Y$ be a homomorphism of BF-algebra. If $\mathcal{C}^t_i = ((\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi)$ is a t-NCID of Y where $i \in k$ then the pre image $\Gamma^{-1}(\bigcap_{i\in k_R} (\mathcal{C}^t_i)_\Xi) = (\Gamma^{-1}(\bigcap_{i\in k_R} (\hat{\kappa}^t_i)_\Xi)$, $\Gamma^{-1}(\bigcap_{i\in k_R} (\sigma^t_i)_\Xi))$ is a t-NCID of X.

Proof. Using Theorem 5.9 and Theorem 6.1, we can prove this corollary.

Corollary 6.5 Let $\Gamma|X \to Y$ be a homomorphism of BF-algebra. If $\mathcal{C}^t_i = ((\hat{\kappa}^t_i)_\Xi, (\sigma^t_i)_\Xi)$ is a t-neutrosophic cubic closed ideals of Y where $i \in k$ then the pre-image $\Gamma^{-1}(\bigcap_{i\in k_R} (\mathcal{C}^t_i)_\Xi) = (\Gamma^{-1}(\bigcap_{i\in k_R} (\hat{\kappa}^t_i)_\Xi), \Gamma^{-1}(\bigcap_{i\in k_R} (\sigma^t_i)_\Xi))$ is a t-neutrosophic cubic closed ideal of X.

Proof. Straightforward, using Theorem 5.10 and Theorem 6.1.

Theorem 6.6 Suppose that $\Gamma|X \to Y$ is an epimorphism of BF-algebra. Then $\mathcal{C}^t = (\hat{\kappa}^t_\Xi, \sigma^t_\Xi)$ is a t-NCID of Y, if $\Gamma^{-1}(\mathcal{C}^t) = (\Gamma^{-1}(\hat{\kappa}^t_\Xi), \Gamma^{-1}(\sigma^t_\Xi))$ of $\mathcal{C}^t$ under Γ of X is a t-NCID of X.

Proof. For any $t_2 \in Y, \exists\ t_1 \in X$ such that $t_2 = \Gamma(t_1)$. Then $\hat{\kappa}^t_\Xi(t_2) = \hat{\kappa}^t_\Xi(\Gamma(t_1)) = \Gamma^{-1}(\hat{\kappa}^t_\Xi)(t_1) \le \Gamma^{-1}(\hat{\kappa}^t_\Xi)(0) = \hat{\kappa}^t_\Xi(\Gamma(0)) = \hat{\kappa}^t{}_\Xi(0)$ and $\sigma^t_\Xi(t_2) = \sigma^t_\Xi(\Gamma(t_1)) = \Gamma^{-1}(\sigma^t_\Xi)$

$(t_1) \ge \Gamma^{-1}(\sigma^t_\Xi)(0) = \sigma^t_\Xi(\Gamma(0)) = \sigma^t_\Xi(0)$.

Suppose $(t_2)_1, (t_2)_2 \in Y$. Then $\Gamma((t_1)_1) = (t_2)_1$ and $\Gamma((t_1)_2) = (t_2)_2$ for some $(t_1)_1, (t_1)_2 \in X$. Thus $\hat{\kappa}^t_\Xi((t_2)_1) = \hat{\kappa}^t_\Xi(\Gamma((t_1)_1)) = \Gamma^{-1}(\hat{\kappa}^t_\Xi)((t_1)_1) \ge \mathrm{rmin}\{\Gamma^{-1}(\hat{\kappa}^t_\Xi)$

$((t_1)_1 * (t_1)_2),\ \Gamma^{-1}(\hat{\kappa}^t_\Xi)((t_1)_2)\} = \mathrm{rmin}\{\hat{\kappa}^t_\Xi(\Gamma((t_1)_1 * (t_1)_2)), \hat{\kappa}^t_\Xi(\Gamma((t_1)_2))\} = \mathrm{rmin}\{\hat{\kappa}^t_\Xi$

$(\Gamma((t_1)_1) * \Gamma((t_1)_2)), \hat{\kappa}^t_\Xi(\Gamma((t_1)_2))\} = \mathrm{rmin}\{\hat{\kappa}^t_\Xi((t_2)_1 * (t_2)_2), \hat{\kappa}^t_\Xi((t_2)_2)\}$ and

$$\begin{aligned}\sigma^t_\Xi((t_2)_1) = \sigma^t_\Xi(\Gamma((t_1)_1)) &= \Gamma^{-1}(\sigma^t_\Xi)((t_1)_1) \le \max\{\Gamma^{-1}(\sigma^t_\Xi)((t_1)_1 * (t_1)_2), \Gamma^{-1}(\sigma^t_\Xi)((t_1)_2)\}\\ &= \max\{\sigma^t_\Xi(\Gamma((t_1)_1 * (t_1)_2)), \sigma^t_\Xi(\Gamma((t_1)_2))\} = \max\{\sigma^t_\Xi(\Gamma((t_1)_1) * \Gamma((t_1)_2)), \sigma^t_\Xi(\Gamma((t_1)_2))\}\\ &= \max\{\sigma^t_\Xi((t_2)_1 * (t_2)_2), \sigma^t_\Xi((t_2)_2)\}.\end{aligned}$$

Hence $\mathcal{C}^t = (\hat{\kappa}^t{}_\Xi, \sigma^t_\Xi)$ is a t-NCID of Y.

7 Conclusion

In this paper, the concept of t-neutrosophic cubic set was defined and investigated it on BF-algebra through several useful results. For future work this study will provide base for t-neutrosophic soft cubic set, t-neutrosophic soft cubic (M-subalgebra, normal ideals) and different algebras like G-algebra and B-algebra.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Ahn, S. S. Bang, K. On fuzzy subalgebras in B-algebra, Communications of the Korean Mathematical Society 18 (2003) 429-437.
2. Biswas, R. Rosenfeld's fuzzy subgroup with interval valued membership function, Fuzzy Sets and Systems, 63 (1994) 87-90.
3. Cho, J. R. Kim, H.S. On B-algebras and quasigroups, Quasigroups and Related System 8 (2001) 1-6.
4. Huang, Y. BCI-algebra, Science Press Beijing, 2006.
5. Imai, Y. Iseki, K. On Axiom systems of Propositional calculi XIV, Proc, Japan Academy, 42 (1966) 19-22.
6. Iseki, K. An algebra related with a propositional calculus, Proc. Japan Academy, 42 (1966) 26-29.
7. Jun, Y. B. Kim, C. S. Yang, K. O. Cubic sets, Annuals of Fuzzy Mathematics and Informatics, 4 (2012) 83-98.
8. Jun, Y. B. Jung, S. T. Kim, M. S. Cubic subgroup, Annals of Fuzzy Mathematics and Infirmatics, 2 (2011) 9-15.
9. Jun, Y. B. Smarandache, F. Kim, C. S. Neutrosophic Cubic Sets, New Math. and Natural Computation, (2015) 8-41.
10. Jun, Y. B. Kim, C. S. Kang, M. S. Cubic Subalgebras and ideals of BCK/BCI-algebra, Far East Journal of Mathematical Sciences 44 (2010) 239-250.
11. Jun, Y. B. Kim, C. S. Kang, J. G. Cubic q -Ideal of BCI-algebras, Annals of Fuzzy Mathematics and Informatics 1 (2011) 25-31.
12. Kim, C. B. Kim, H.S. On BG-algebra, Demonstration Mathematica 41 (2008) 497-505.
13. Neggers, J. Kim, H. S. On B-algebras, Mathematichki Vensnik, 54 (2002) 21-29.
14. Neggers, J. Kim, H. S. A fundamental theorem of B-homomorphism for B-algebras, International Mathematical Journal 2 (2002) 215-219.
15. Park, H. K. Kim, H. S On quadratic B-algebras, Qausigroups and Related System 7 (2001) 67-72.
16. Saeid, A. B. Interval-valued fuzzy B-algebras, Iranian Journal of Fuzzy System 3 (2006) 63-73.
17. Senapati, T. Bipolar fuzzy structure of BG-algebras, The Journal of Fuzzy Mathematics 23 (2015) 209-220.
18. Smarandache, F. Neutrosophic set a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math. 24 (3) (2005) 287-297.
19. Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, (American Reserch Press, Rehoboth, NM, 1999).
20. Khalid, M. Iqbal, R. Zafar , S. Khalid, H. Intuitionistic Fuzzy Translation and Multipl-ication of G-algebra, The Journal of Fuzzy Mathematics 27 (3) 17 (2019).
21. Senapati, T. Bhowmik, M. Pal, M. Fuzzy dot subalgebras and fuzzy dot ideals of B-algebra, Journal of Uncertain System 8 (2014) 22-30.
22. Senapati, T. Bhowmik, M. Pal, M. Fuzzy closed ideals of B-algebras, International Journal of Computer Science, Engineering and Technology 1 (2011) 669-673

23. Senapati, T. Bhowmik, M. Pal, M. Fuzzy closed ideals of B-algebras with interval-valued membership function, International Journal of Fuzzy Mathematical Archive 1 (2013) 79-91.
24. Senapati, T. Bhowmik, M. Pal, M. Fuzzy B-subalgebras of B-algebra with resepect to t-norm, Journal of Fuzzy Set Valued Analysis 2012 (2012) 11 pages, doi: 10.5899/2012/jfsva-00111.
25. Senapati, T. Jana, C. Bhowmik, M. Pal, M. L-fuzzy G-subalgebra of G-algebras, Journal of the Egyptian Mathematical Society (2014) http://dx.doi.org/10. 1016 /j.joems .2014.05.010.
26. Senapati, T. Kim, C. H. Bhowmik, M. Pal, M. Cubic subalgebras and cubic closed ideals of B-algebras, Fuzzy. Inform. Eng. 7 (2015) 129-149.
27. Senapati, T. Bhowmik, M. Pal, M. Intuitionistic L-fuzzy ideals of BG-algebras, Afrika Matematika 25 (2014) 577-590.
28. Senapati, T. Bhowmik, M. Pal, M. Interval-valued intuitionistic fuzzy BG-subalgebras, The Journal of Fuzzy Mathematics 20 (2012) 707-720.
29. Senapati, T. Bhowmik, M. Pal, M. Interval-valued intuitionistic fuzzy closed ideals BG-algebras and their products, International Journal of Fuzzy Logic Systems 2 (2012) 27-44.
30. T. Bhowmik, M. Pal, M. Intuitionistic fuzzifications of ideals in BG-algebra, Mathematica Aeterna 2 (2012) 761-778.
31. Senapati, T. Bhowmik, M. Pal, M. Fuzzy dot structure of BG-algrbras, Fuzzy Informa-tion and Engineering 6 (2014) 315-329.
32. Walendziak, A. Some axiomation of B-algebras, Mathematics Slovaca 56 (2006) 301 -306.
33. Zadeh, L. A. Fuzzy sets, Information and control 8 (1965) 338-353.
34. Zadeh, L. A. The concept of a linguistic variable and its application to approximate reasoning, Information science 8 (1975) 199-249.
35. Barbhuiya, S. R. t-intuitionistic Fuzzy Subalgebra of BG-Algebras, Advanced Trends in Mathematics 06-01, Vol. 3 (2015) pp16-24.
36. Sharma, P. K. t-intuitionistic Fuzzy Quotient Group, Advances in Fuzzy Mathematics, 7 (1) (2012) 1-9.
37. Takallo, M. M. Bordbar, H. Borzooei, R. A. Jun, Y. B. BMBJ-neutrosophic ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, vol. 27 (2019) pp. 1-16, DOI: 10.5281/zenodo.3275167.
38. Muhiuddin, G. Smarandache, F. Jun, Y. B. Neutrosophic Quadruple Ideals in Neutrosophic Quadruple BCI-algebras, Neutrosophic Sets and Systems, vol. 25 (2019) pp. 161-173, DOI: 10.5281/zenodo.2631518.
39. Park, C. H. Neutrosophic ideal of Subtraction Algebras, Neutrosophic Sets and Systems, vol. 24 (2019) pp. 36-45, DOI:10.5281/zenodo.2593913.
40. Borzooei, R. A. Takallo, M. M. Smarandache, F. Jun, Y. B. Positive implicative BMBJ -neutrosophic ideals in BCK-algebras, Neutrosophic Sets and Systems, vol. 23 (2018) pp. 126-141, DOI: 10.5281/zenodo.2158370.
41. Jun, Y. B. Smarandache, F. Ozturk, M. A. Commutative falling neutrosophic ideals in BCK-algebras, Neutrosophic Sets and Systems, vol. 20 (2018) pp. 44-53, http://doi.org/ 10.5281/zenodo.1235351.
42. Song, S. Z. Khan, M. Smarandache, F. Jun, Y. B. Interval neutrosophic sets applied to ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, vol. 18 (2017) pp. 16-26, http://doi.org/10.5281/zenodo.1175164.
43. Khalid, M. Iqbal, R. Broumi, S. Neutrosophic soft cubic Subalgebras of G-algebras. 28, (2019), 259-272. 10.5281/zenodo.3382552.
44. Muhiuddin, G. Jun, Y. B. Smarandache, F. Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras, Neutrosophic Sets and Systems, Vol. 25, (2019).

45. G. Muhiuddin, H. Bordbar, F. Smarandache, Y.B. Jun, Further results on (ϵ, ϵ)-neutrosophic subalgebras and ideals in BCK/BCI-algebras, Neutrosophic Sets and Systems, Vol. 20, (2018).
46. Akinleye, S.A. Smarandache, F. Agboola, A.A.A. On neutrosophic quadruple algebraic structures, Neutrosophic Sets and Systems 12 (2016) 122–126.
47. Basset, M. A. Chang, V. Gamal, A., Smarandache, F. An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field, Computers in Industry 106, 94-110, 2019.
48. Basset, M. A. Saleh, M. Gamal, A. Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 77 (2019) 438-452.

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Neutrosophic Inventory Backorder Problem Using Triangular Neutrosophic Numbers

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Abstract: A company may have backorders if they run out of the stock in their stores, in which case, it can just place a new order to restock its shelves. A customer who is willing to wait for some time until the company has restocked the products would have to place a backorder. A backorder only exists if customers are willing to wait for the order. In this paper, a neutrosophic inventory backorder problem using a triangular neutrosophic numbers is introduced. First, we fuzzify the carrying cost and shortage cost as triangular neutrosophic numbers and the signed distance method is used to defuzzify them. From these, we can obtain the neutrosophic optimal shortage quantity and the neutrosophic total cost. A numerical example is provided to illustrate the proposed model in neutrosophic environment.

Keywords: Neutrosophic EOQ; Neutrosophic set; Signed distance method; Triangular neutrosophic numbers.

1. Introduction

Backorders represents any quantity of inventory an enterprise customer have ordered but have not yet received as it presently isn't to be had in stock. An enterprise's backorders are an essential factor in its inventory control evaluation. The quantity of items on backorder and how long it takes to fulfill these customer orders can offer perception into how properly the company manages its stock.
Sen and Malakar [13] considered an EOQ model with shortage, considering the various parameters as triangular, trapezoidal fuzzy number and parabolic fuzzy number. Intuitionistic fuzzy set - a generalization of fuzzy set was introduced by Atanassov [1]. Yao and Lee [15] developed a fuzzy inventory with or without backorder for fuzzy order quantity with trapezoidal fuzzy number. Bulancak and Kirkavak [3] applied trapezoidal fuzzy number for EOQ with backorder.
Fuzzy inventory model without shortages was proposed by Dutta and Kumar[4]. Carrying cost and set up cost are expressed as fuzzy trapezoidal numbers and for defuzzification signed distance method is used by them. Mahuya Deb and Prabjot Kaur[6] developed an intuitionistic fuzzy inventory backorder problem using triangular intuitionistic fuzzy numbers. D. Banerjee and S. Pramanik[2] developed a single-objective linear goal programming problem with neutrosophic numbers. F.

Smarandache[14] introduced neutrosophic set and neutrosophic logic by considering the non-standard analysis. F. Smarandache[16] introduced the plithogenic set -as generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets

Neutrosophic set is the take a look at of neutralities origin, nature and scope and additionally their interactions with exceptional ideational spectra. To deal with unsure information processing, the brand new emerging tool known as neutrosophic set is used. Neutrosophic set is a powerful and popular formal framework that has the potential to address uncertainty analysis in information sets. However, the neutrosophic set desires to be specified detail. So that, we define an example of neutrosophic set called as single-valued neutrosophic set (SVNS). Single valued neutrosophic set is an instance of neutrosophic set. The SVNS is a set of generalization of a classic set, fuzzy set, interval value fuzzy set, intuitionistic fuzzy set and para consistent set. The single-valued neutrosophic set is used in lots of locations like professional machine, information fusion gadget, query answering device, bioinformatics and scientific informatics and many others.

Pranab Biswas, Surapati Pramanik, Bibhas C. Giri [12] introduced multi-attribute group decision making based on expected value of neutrosophic trapezoidal numbers. An exact formula of expected value for neutrosophic trapezoidal number is established. Irfan Deli and Yusuf subas[5] discussed two special forms of single valued neutrosophic numbers such as single valued trapezoidal neutrosophic numbers and single valued triangular neutrosophic numbers. M.Mullai and S.Broumi[7] proposed neutrosophic inventory model without shortages. Also neutrosophic inventory model with price break for finding the optimal solution of the model for the optimal order quantity was established by M.Mullai and R. Surya[8].

In this paper, neutrosophic inventory backorder model is established by taking the parameters as triangular neutrosophic numbers. The neutrosophic optimal shortage quantity and the neutrosophic optimal total cost are derived in this model and signed distance method is used for defuzzification. A neutrosophic set may help in solving membership function when it is not defined accurately. Without difficulty, the work can also manage the inventory system of any company in neutrosophic backorder model. The novelty of this model is to give more accurate results than existing methods whenever uncertain and unexpected situations arise in back order inventory system. To illustrate the results of this model, sensitivity analysis is presented for crisp, fuzzy, intuitionistic fuzzy and neutrosophic sets and the results are discussed briefly.

2. Preliminaries

The basic definitions involving neutrosophic set, single valued neutrosophic sets and triangular neutrosophic numbers which are very useful for the proposed model are outlined here.

Definition 2.1 (Irfan Deli and Yusuf Subas., 2014) (Neutrosophic set)

Let E be a universe. A neutrosophic set A in E is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard elements of [0,1]. It can be written as

$$A=\{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in E, T_A(x), I_A(x), F_A(x) \in]0^-, 1^+[\}.$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq T_A(x)+I_A(x) + F_A(x) \leq 3+$.

Definition 2.2 (Irfan Deli and Yusuf Subas., 2014) (Single-valued neutrosophic set)

Let E be a universe. A single valued neutrosophic set A, which can be used in real scientific and engineering applications, in E is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard elements of [0,1]. It can be written as

$$A=\{\langle x, T_A(x), I_A(x), F_A(x) \rangle : x \in E, T_A(x), I_A(x), F_A(x) \in [0,1]\}.$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0 \le T_A(x)+I_A(x) + F_A(x) \le 3$.

Definition 2.3 (Irfan Deli and Yusuf Subas., 2014) (Triangular neutrosophic numbers)

Let the triangular neutrosophic number $\tilde{a} = \langle (a_1, b_1, c_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership, and falsity-membership functions are defined as follows:

$$\mu\tilde{a}(x) = \begin{cases} (x-a_1)w_{\tilde{a}}/(b_1-a_1) & \text{if } a_1 \le x \le b_1 \\ w_{\tilde{a}} & \text{if } x = b_1 \\ (c_1-x)w_{\tilde{a}}/(c_1-b_1) & \text{if } b_1 \le x \le c_1 \\ 0 & \text{if } otherwise \end{cases}$$

$$\nu\tilde{a}(x) = \begin{cases} (b_1-x+(x-a_1)u_{\tilde{a}})/(b_1-a_1) & \text{if } a_1 \le x \le b_1 \\ u_{\tilde{a}} & \text{if } x = b_1 \\ (x-b_1+(c_1-x)u_{\tilde{a}})/(c_1-b_1) & \text{if } b_1 \le x \le c_1 \\ 1 & \text{if } otherwise \end{cases}$$

$$\lambda\tilde{a}(x) = \begin{cases} (b_1-x+(x-a_1)y_{\tilde{a}})/(b_1-a_1) & \text{if } a_1 \le x \le b_1 \\ y_{\tilde{a}} & \text{if } x = b_1 \\ (x-b_1+(c_1-x)y_{\tilde{a}})/(c_1-b_1) & \text{if } b_1 \le x \le c_1 \\ 1 & \text{if } otherwise \end{cases}$$ respectively.

If $a_1 \ge 0$ and at least $c_1 > 0$ then $\tilde{a} = \langle (a_1, b_1, c_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is called a positive triangular neutrosophic number, denoted by $\tilde{a} > 0$. Likewise, if $c_1 \le 0$ and at least $a_1 < 0$, then $\tilde{a} = \langle (a_1, b_1, c_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ is called a negative triangular neutrosophic number, denoted by $\tilde{a} < 0$. A triangular neutrosophic number $\tilde{a} = \langle (a_1, b_1, c_1); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \rangle$ may express an ill-known quantity about a, which is approximately equal to a.

Definition 2.4 (Sushil Kumar. U and Rajput .S., 2006)(Signed distance method)

Let $\tilde{D} \epsilon F$. We define the signed distance of $\tilde{D}$ measured from $\tilde{0}$ as

$$d(\tilde{D}, \tilde{0}) = \frac{1}{2}\int_0^1 [D_L(\alpha) + D_R(\alpha)]d\alpha$$

Definition 2.5 (Mahuya Deb and Prabjot Kaur., 2016) (Defuzzification)

(i) Defuzzification for Triangular Fuzzy Number

The defuzzification value for a triangular fuzzy number (a_1, a_2, a_3) is given by

$$A = \frac{a_1+2a_2+a_3}{4}$$

(ii) Defuzzification for Triangular Intuitionistic Fuzzy Number

Let $\hat{A} = (a_1, a_2, a_3)(a'_1, a_2, a'_3)$ be a triangular intuitionistic fuzzy number. Then the signed distance of $\hat{A}$ can be calculated as follows

$$D^s(\hat{A},\hat{0}) = \frac{1}{4}\left[\int_0^1 L_\mu(\alpha) + \int_0^1 L_\mu(\alpha) + \int_0^1 L_\mu(\alpha) + \int_0^1 L_\mu(\alpha)\right]$$

$$= \frac{1}{4}\left[\int_0^1 \{a_1 - \alpha(a_2 - a_1)\}\delta\alpha + \int_0^1 \{a_3 - \alpha(a_3 - a_2)\}\delta\alpha + \int_0^1 \{a_2 - (1-\alpha)(a_2 - a'_1)\}\delta\alpha + \int_0^1 \{a_2 + (1-\alpha)(a'_3 - a_2)\}\delta\alpha\right]$$

$$= \frac{a_1 + 2a_2 + a_3 + a'_1 + 2a_2 + a'_3}{8}$$

3. Notations

C_h^N - Neutrosophic carrying cost per unit quantity per unit time

C_s^N - Neutrosophic shortage cost per unit quantity per unit time

D^N - Neutrosophic total demand

$(TC)^N$ - Neutrosophic total cost

Q^N - Neutrosophic order quantity

Q^{*N} - Neutrosophic optimal order quantity

$F(q)^N$ - Defuzzified total neutrosophic cost

4. Assumptions

- At the opening of every cycle, only a single order is produced and the entire lot is delivered in one batch.
- Q^N is the neutrosophic lot-size per cycle whereas S_1^N is the neutrosophic initial inventory level after fulfilling the back-logged quantity of previous cycle and $Q^N - S_1^N$ is the maximum shortage level.
- T^N is the cycle length where t_1^N is the period with no shortage.

5. Neutrosophic model with shortages

This section describes the inventory model with backorder in neutrosophic environment. Since the inventory carrying cost and shortage cost are in neutrosophic numbers, we represent them by triangular neutrosophic numbers as follows:

Let $C_h^N = (C_{h_1}^N, C_{h_2}^N, C_{h_3}^N)(C_{h_1}{'}^N, C_{h_2}^N, C_{h_3}{'}^N)(C_{h_1}{''}^N, C_{h_2}^N, C_{h_3}{''}^N)$

$C_s^N = (C_{s_1}^N, C_{s_2}^N, C_{s_3}^N)(C_{s_1}{'}^N, C_{s_2}^N, C_{s_3}{'}^N)(C_{s_1}{''}^N, C_{s_2}^N, C_{s_3}{''}^N)$

To defuzzify the triangular neutrosophic numbers, the signed distance method is defined as follows:

Let $A^N = (a_1, a_2, a_3)(a'_1, a_2, a'_3)(a''_1, a_2, a''_3)$ be a triangular neutrosophic number. Then the signed distance of A^N is written as

$$D^s(A^N, 0) = \frac{a_1 + 2a_2 + a_3 + a''_1 + 2a_2 + a''_3}{8}$$

The neutrosophic total cost is given by

$$(TC)^N = \frac{1}{T}\left[\frac{C_h^N {s_1^2}^N}{2D^N} + \frac{1}{2D^N} C_s^N (Q^N - s_1^N)^2\right]$$

$$= (C_{h_1}^N, C_{h_2}^N, C_{h_3}^N)(C_{h_1}{''}^N, C_{h_2}^N, C_{h_3}{''}^N)\frac{{s_1^2}^N}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N}(C_{s_1}^N, C_{s_2}^N, C_{s_3}^N)(C_{s_1}{''}^N, C_{s_2}^N, C_{s_3}{''}^N)$$

$$= (C_{h_1}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_1}^N, C_{h_2}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_2}^N, C_{h_3}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_3}^N)(C_{h_1}{''}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_1}{''}^N, C_{h_2}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_2}^N, C_{h_3}{''}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_3}{''}^N)$$

The defuzzified neutrosophic total cost using above signed distance method is given by

$$F(q)^N = \frac{1}{8}[(C_{h_1}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_1}^N) + 2(C_{h_2}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_2}^N) + (C_{h_3}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_3}^N) + (C_{h_1}{''}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_1}{''}^N) + 2(C_{h_2}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_2}^N) + (C_{h_3}{''}^N \frac{s_1^{2N}}{2D^N} + \frac{(Q^N - s_1^N)^2}{2D^N} C_{s_3}{''}^N)]$$

To find the minimum of $D(F(q)^N)$ by taking the derivative $D(F(q)^N)$ and equating it to zero,

(i.e) $\frac{1}{8}\{\frac{s_1^N}{D^N}[(C_{h_1}^N + C_{s_1}^N) + 4(C_{h_2}^N + C_{s_2}^N) + (C_{h_3}^N + C_{s_3}^N) + (C_{h_1}{''}^N + C_{s_1}{''}^N) + (C_{h_3}{''}^N + C_{s_3}{''}^N)] - \frac{Q^N}{D^N}[C_{s_1}^N + 4C_{s_2}^N + C_{s_3}^N + C_{s_1}{''}^N + C_{s_3}{''}^N]\} = 0$, we get

$$s_1^N = \frac{C_{s_1}^N + 4C_{s_2}^N + C_{s_3}^N + C_{s_1}{''}^N + C_{s_3}{''}^N}{(C_{h_1}^N + C_{s_1}^N) + 4(C_{h_2}^N + C_{s_2}^N) + (C_{h_3}^N + C_{s_3}^N) + (C_{h_1}{''}^N + C_{s_1}{''}^N) + (C_{h_3}{''}^N + C_{s_3}{''}^N)} Q^N$$

$$s_1^N = \frac{C_{s_1}^N + 4C_{s_2}^N + C_{s_3}^N + C_{s_1}{''}^N + C_{s_3}{''}^N}{(C_{h_1}^N + C_{s_1}^N) + 4(C_{h_2}^N + C_{s_2}^N) + (C_{h_3}^N + C_{s_3}^N) + (C_{h_1}{''}^N + C_{s_1}{''}^N) + (C_{h_3}{''}^N + C_{s_3}{''}^N)} D^N T^N \quad \cdots\cdots\cdots\cdots\cdots \quad (1)$$

Also at $s_1^N = s_1^{N^*}$, we get $D^2(F(s_1^N)) > 0$

Hence, the minimum neutrosophic total cost is given by

$$F(q^N)^* = \frac{1}{8}\left[\left(C_{h_1}^N \frac{s_1^{2N^*}}{2D^N} + \frac{(Q^N - s_1^{N^*})^2}{2D^N} C_{s_1}^N\right) + 2\left(C_{h_2}^N \frac{s_1^{2N^*}}{2D^N} + \frac{(Q^N - s_1^{N^*})^2}{2D^N} C_{s_2}^N\right) + \left(C_{h_3}^N \frac{s_1^{2N^*}}{2D^N} + \frac{(Q^N - s_1^{N^*})^2}{2D^N} C_{s_3}^N\right) + \left(C_{h_1}{''}^N \frac{s_1^{2N^*}}{2D^N} + \frac{(Q^N - s_1^{N^*})^2}{2D^N} C_{s_1}{''}^N\right) + 2\left(C_{h_2}^N \frac{s_1^{2N^*}}{2D^N} + \frac{(Q^N - s_1^{N^*})^2}{2D^N} C_{s_2}^N\right) + \left(C_{h_3}{''}^N \frac{s_1^{2N^*}}{2D^N} + \frac{(Q^N - s_1^{N^*})^2}{2D^N} C_{s_3}{''}^N\right)\right] \cdots\cdots\cdots \quad (2)$$

6. Numerical Example

A commodity is to be furnished at a constant rate of 20 units per day. A penalty cost will be charged at a rate of Rs 8 per day, if it is past due for missing the scheduled shipping date. The cost of carrying the commodity in inventory is Rs 14 per unit per month. The production process is such that each month (30 days) a batch of items is started and is available for delivery any time after the end of the month. Find the optimal level of inventory at the beginning of each month. Find the optimal level of inventory at the beginning of each month.

Solution:

Given D = 20, T = 30 , C_h = 14/30 = 0.47 and C_s = 8

Using [4], the shortage quantity and minimum total cost for crisp set, fuzzy set and intuitionistic fuzzy sets are calculated. Also, they are compared with neutrosophic optimal shortage quantity and minimum neutrosophic total cost [by equation (1) and (2)] and tabulated as follows:

	Crisp Set	**Fuzzy Set**	**Intuitionistic Fuzzy Set**	**Neutrosophic Set**
D	20	20	20	20
T	30	30	30	30
C_h	14/30 = 0.47	(0.46,0.49,0.51)	(0.44,0.47,0.49) (0.42, 0.47, 0.51)	(0.44, 0.47, 0.49) (0.42, 0.47, 0.51) (0.4, 0.47, 0.53)
C_s	8	(6, 7, 9)	(6, 7, 9) (4, 7, 10)	(6, 7, 9) (4, 7, 10) (5, 7, 9)
Shortage quantity	567.376	563.654	563	563.06
Minimum total cost	260.993	264.917	266.612	266.513

7. Analytical Observations

In this section, the analysis of shortage quantity and minimum total cost for crisp set, fuzzy set, intuitionistic fuzzy set and neutrosophic set for table:1 is shown graphically.

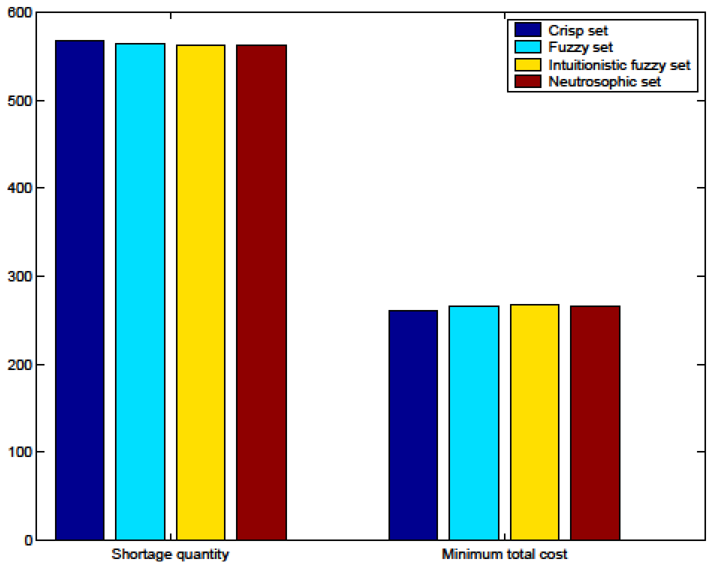

Figure 1: Neutrosophic backorder problem

Also, from the above analytical observations, we conclude that,

- The analysis of the problem under the optimal shortage quantity in neutrosophic environment is closer to crisp, fuzzy and intuitionistic fuzzy environments.

• The optimal shortage quantity in neutrosophic set increases when the optimal shortage quantity in intuitionistic fuzzy set decreases.

• The minimum total cost in neutrosophic set decreases when the minimum total cost in intuitionistic fuzzy set increases.

8. Conclusions

In this proposed model, the neutrosophic total cost and neutrosophic optimal shortage quantity in triangular neutrosophic numbers are obtained. In neutrosophic environment, the shortage quantity is as close to the inuitionistic fuzzy set. The benefit of the neutrosophic inventory model gives better result than fuzzy and intuitionistic fuzzy inventory models. A comprehensive sensitivity analysis has been performed to illustrate the impact of demand on the ordering policy comparing with existing methods. The present proposed work is helpful for business organizations where customer's demands are not fulfilled instantly. In future, the various neutrosophic inventory models will be developed with various limitations such as lead time, backlogging and deteriorating items, etc.

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Conflicts of Interest

The authors declare no conflict of interest.

References

1. Atanassov, T.K. Intuitionistic fuzzy sets. Fuzzy Sets and Systems, **1986,** 20, 87-96.
2. Banerjee, D. Pramanik, S. Single-objective linear goal programming problem with neutrosophic numbers. International Journal of Engineering Science and Research Technology, **2018,** 7(5), 454-469.
3. Bulancak, E. and Krkavak, E. Economic order quantity model with backorders using trapezoidal fuzzy numbers. 5th International Conference on Soft Computing, Computing with Words and Perceptions in System Analysis, Decision and Control, **2009,** 1-4.
4. Dutta, D. and Pavan Kumar. Fuzzy inventory model without shortage using trapezoidal fuzzy number with sensitive analysis. IOSR- Journal of Mathematics, **2012,** 4(3), 32-37.
5. Irfan Deli and Yusuf Subas. Single valued neutrosophic numbers and their applications to multicriteria decision making problem. Journal of Intelligent and Fuzzy Systems, **2014**.
6. Mahuya Deb and Prabjot Kaur. An intuitionistic fuzzy inventory backorder problem using triangular intuitionistic fuzzy numbers. International Journal of Emerging Technology and Advanced Engineering, **2016**.
7. Mullai, M. and Broumi, S. Neutrosophic inventory model without shortages. Asian Journal of Mathematics and Computer Research, **2018,** 23(4), 214-219.
8. Mullai, M. and Surya, R. Neutrosophic EOQ model with price breaks. Neutrosophic sets and systems, volume 19, **2018**.
9. Mullai, M. and Surya, R. Neutrosophic project evaluation and review techniques. Neutrosophic sets and systems, volume 24, **2019**.
10. Abdel-Baset, M.; Chang, V.; Gamal, A.;Smarandache, F. An integrated neutrosophic ANP and VIKOR method for achieving sustainable supplier selection: A case study in importing field. *Computers in Industry* **2019**, 106, 94-110.

11. Abdel-Basset, M.; Saleh, M.;Gamal, A.;Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. *Applied Soft Computing* **2019**, 77, 438-452.
12. Pranab Biswas, Surapati Pramanik, and Bibhas C. Giri. Multi-attribute group decision making based on expected value of neutrosophic trapezoidal numbers. New Trends in Neutrosophic Theory and Applications. 2, 105-124.
13. Sen, N. and Malakar. A Fuzzy inventory model with shortages using different fuzzy numbers. American Journal of Mathematics and Statistics, **2015,** 5(5), 238-248.
14. Smarandache, F. Neutrosophic set - a generalization of the intuitionistic fuzzy set. IEEE International Conference, **2006,** 38 -42.
15. Smarandache, F. and Pramanik, S. (Eds). New trends in neutrosophic theory and applications. Volume 2, **2018.** Brussels: Pons Editions
16. Smarandache, F. Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets-Revisited. Neutrosophic Sets and Systems, Volume 21, **2018**.
17. Sushil Kumar .U and Rajput .S. Fuzzy inventory model for deteriorating items with time dependent demand rate and partial backlogging. Applied Mathematics, **2006**, 6, 496-509. http://www.scirp.org/journal/am
18. Yao S.J and Lee M.H. Fuzzy inventory with or without backorder for fuzzy order quantity with trapezoidal fuzzy number. Fuzzy Sets and Systems, **2000,** 105, 311-337,

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Generalized Neutrosophic Competition Graphs

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Abstract: The generalized neutrosophic graph is a generalization of the neutrosophic graph that represents a system perfectly. In this study, the concept of a neutrosophic digraph, generalized neutrosophic digraph and out-neighbourhood of a vertex of a generalized neutrosophic digraph is studied. The generalized neutrosophic competition graph and matrix representation are analyzed. Also, the minimal graph and competition number corresponding to generalized neutrosophic competition graph are defined with some properties. At last, an application in real life is discussed.

Keywords: Competition graph, neutrosophic graph, generalized neutrosophic competition graph, competition number.

1. Introduction

Graph theory is a significant part of applied mathematics, and it is applied as a tool for solving many problems in geometry, algebra, computer science, social networks [1] and optimization etc. Cohen (1968) introduced the concept of competition graph [2] with application in an ecosystem which was related to the competition among species in a food web. If two species have at least one common prey, then there is a competition between them. Let $\vec{G} = (V, \vec{E})$ be a digraph, which corresponds to a food web. A vertex $x \in V$ represents a species in the food web and an arc $(\overrightarrow{x,s}) \in \vec{E}$ means x preys on the species s. The competition graph $C(\vec{G})$ of a digraph $\vec{G}$ is an undirected graph $G = (V, E)$ which has same vertex set and has an edge between two distinct vertices $x, y \in V$ if there exists a vertex $s \in V$ and arcs $(\overrightarrow{x,s}), (\overrightarrow{y,s}) \in \vec{E}$.
Roberts et al. (1976,1978) studied that for any graph with isolated vertices is the competition graph [3, 4] and the minimum number of such vertices is called competition number. Opsut (1982) discussed the computation of competition number [5] of a graph. Kim et al. (1993,1995) introduced the p-competition graph [6] and also p-competition number [7]. Brigham et al. (1995) introduced $\emptyset - tolerance$ graph as a generalization of p-competition [8]. Cho and Kim (2005) studied competition number [9] of a graph having one hole. Li and Chang (2009) proposed about competition graph [10]

with h holes. Factor and Merz introduced $(1,2)$ step competition graph [11] of a tournament and extended to $(1,2)$ −step competition graph.

In real life, it is full of imprecise data which motivated to define fuzzy graph [12] by Kaufman (1973) where all the vertices and edges of the graph have some degree of memberships. There are lots of research works on fuzzy graphs [13]. In 2006, Parvathi and Karunambigal introduced intuitionistic fuzzy graph [14] where all the vertices and edges of the graph have some degree of memberships and degree of non-memberships. The concepts of interval-valued fuzzy graphs [15] were introduced by Akram and Dubek (2011) where the membership values of vertices and edges are interval numbers. Even the representation of competition by competition does not show the characteristic properly. Considering in food web, species and prey are all fuzzy in nature, Samanta and Pal (2013) represent competition [16] in a more realistic way in fuzzy environment. After that, as a generalization of the fuzzy graph, Samanta and Sarkar (2016, 2018) proposed the generalized fuzzy graph [17] and generalized fuzzy competition graph [18] where the membership values of edges are functions of membership values of vertices. Pramanik et al. introduced fuzzy $\emptyset - tolerance$ competition graphs with the idea of fuzzy tolerance graphs [19].

Smarandache (1998) proposed the concept of a neutrosophic set [20] which has three components: the degree of truth membership, degree of falsity membership and degree of indeterminacy membership. The neutrosophic set is the generalization of fuzzy set [21] and intuitionistic fuzzy set [22].

The neutrosophic environment has several applications in real life including evaluation of the green supply chain management practices [23], evaluation Hospital medical care systems based on plithogenic sets [24], decision-making approach with quality function deployment for selecting supply chain sustainability metrics [25], intelligent medical decision support model based on soft computing and IoT [26], utilizing neutrosophic theory to solve transition difficulties of IoT-based enterprises [27], etc.

As a generalization of the fuzzy graph and intuitionistic fuzzy graph, Broumi et al. (2015) defined the single-valued neutrosophic graph [28]. The definition of a neutrosophic graph by Broumi et al. is different in the definition of neutrosophic graph [29] by Akram. Also, the presentation of competition [30] by neutrosophic graph was introduced by Akram and Siddique (2017). In that paper, the authors did not follow the same definition of Broumi. In these papers, there were restrictions on T, I, F values. To remove the restrictions on T, I, F values, Broumi et al. (2018) introduced the generalized neutrosophic graph [31] using the concept of generalized fuzzy graph. The concepts of generalized neutrosophic graph motivate us to introduce the generalized neutrosophic competition graph. There are few papers available for readers on neutrosophic graph theory [32-34].

The rest of the study is organized as follows. In the second section, the main problem definition is described. In section 3, the basic concepts related to the neutrosophic graph, neutrosophic directed graph, generalized neutrosophic graph, a generalized neutrosophic directed graph is discussed with example. In this section, the generalized neutrosophic competition graph is proposed and corresponding minimal graphs, competition number is studied. In section 4, a matrix representation of the generalized neutrosophic competition graph is proposed with a suitable example. In section 5, an application in economic growth is studied. In the last section, the conclusion of the proposed study and future directions is depicted.

A gist of contribution (Table 1) of authors is presented below.

Table 1. Contribution of authors to competition graphs

Authors	Year	Contributions
Cohen	1968	Introduced competition graph.
Kauffman	1973	Introduced fuzzy graphs
Smarandache	1998	Introduced the concepts of neutrosophic set
Parvathi and Karunambigal	2006	Introduced intuitionistic fuzzy graph
Samanta and Pal	2013	Introduced fuzzy competition graph
Broumi et al.	2015	Introduced neutrosophic graph
Samanta and Sarkar	2016	Introduced the generalized fuzzy graph
Akram and Siddique	2017	Introduced neutrosophic competition graph
Samanta and Sarkar	2018	Introduced representation of competition by a generalized fuzzy graph
Broumi et al.	2018	Introduced Generalized neutrosophic graph
Das et al.	This paper	Introduced generalized neutrosophic competition graph

2. Generalized neutrosophic competition graph

Definition 1.[28] A graph $G = (V, E)$ where $E \subseteq V \times V$ is said to be neutrosophic graph if

i) there exist functions $\rho_T: V \to [0,1], \rho_F: V \to [0,1] and \rho_I: V \to [0,1]$ such that

$0 \le \rho_T(v_i) + \rho_F(v_i) + \rho_I(v_i) \le 3$ for all $v_i \in V\ (i = 1,2,3, \dots, n)$

where $\rho_T(v_i),\ \rho_F(v_i),\ \rho_I(v_i)$ denote the degree of true membership, degree of falsity membership and degree of indeterminacy membership of the vertex $v_i \in V$ respectively.

ii) there exist functions $\mu_T: E \to [0,1], \mu_F: E \to [0,1]\ and\ \mu_I: E \to [0,1]$such that

$$\mu_T(v_i, v_j) \le \min\ [\rho_T(v_i),\ \rho_T(v_j)]$$

$\mu_F(v_i, v_j) \ge max[\rho_F(v_i),\ \rho_F(v_j)]$

$\mu_I(v_i, v_j) \ge max[\rho_I(v_i),\ \rho_I(v_j)]$

and $0 \le \mu_T(v_i, v_j) + \mu_F(v_i, v_j) + \mu_I(v_i, v_j) \le 3$ for all $(v_i, v_j) \in E$

where $\mu_T(v_i, v_j),\ \mu_F(v_i, v_j),\ \mu_I(v_i, v_j)$ denote the degree of true membership, degree of falsity membership and degree of indeterminacy membership of the edge $(v_i, v_j) \in E$ respectively.

Definition 2.[31] A graph $G = (V, E)$ where $E \subseteq V \times V$ is said to be generalized neutrosophic graph if there exist functions

$$\rho_T: V \to [0,1], \rho_F: V \to [0,1] and \rho_I: V \to [0,1],$$

$$\mu_T: E \to [0,1], \mu_F: E \to [0,1]\ and\ \mu_I: E \to [0,1]$$

$$\phi_T: E_T \to [0,1], \phi_F: E_F \to [0,1]\ and\ \phi_I: E_I \to [0,1]$$

such that

$0 \le \rho_T(v_i) + \rho_F(v_i) + \rho_I(v_i) \le 3$ for all $v_i \in V\ (i = 1,2,3, \dots, n)$

and

$$\mu_T(v_i, v_j) = \phi_T(\rho_T(v_i),\ \rho_T(v_j))$$
$$\mu_F(v_i, v_j) = \phi_F(\rho_F(v_i),\ \rho_F(v_j))$$

$\mu_I(v_i, v_j) = \phi_I(\rho_I(v_i),\ \rho_I(v_j))$

where $E_T = \{(\rho_T(v_i),\ \rho_T(v_j)): \mu_T(v_i, v_j) \geq 0\}$, $E_F = \{(\rho_F(v_i),\ \rho_F(v_j)): \mu_F(v_i, v_j) \geq 0\}$, $E_I = \{(\rho_I(v_i),\ \rho_I(v_j)): \mu_I(v_i, v_j) \geq 0\}$ and $\rho_T(v_i),\ \rho_F(v_i),\ \rho_I(v_i)$ denote the degree of true membership, the degree of falsity membership, the indeterminacy membership of vertex $v_i \in V$ respectively and $\mu_T(v_i, v_j),\ \mu_F(v_i, v_j),\ \mu_I(v_i, v_j)$ denote the degree of true membership, the degree of falsity membership and the degree of indeterminacy membership of edge$(v_i, v_j) \in E$ respectively.

Definition 3. A graph $\vec{G} = (\mathrm{V}, \vec{E})$ where $\vec{E} \subseteq V \times V$ is said to be neutrosophic digraph if

i) there exist functions $\rho_T: V \to [0,1], \rho_F: V \to [0,1]$ and $\rho_I: V \to [0,1]$ such that $0 \leq \rho_T(v_i) + \rho_F(v_i) + \rho_I(v_i) \leq 3$ for all $v_i \in V\ (i = 1,2,3, \dots, n)$

where $\rho_T(v_i),\ \rho_F(v_i),\ \rho_I(v_i)$ denote the degree of true membership, degree of falsity membership and degree of indeterminacy membership of the vertex v_i respectively.

ii) there exist functions $\mu_T: \vec{E} \to [0,1], \mu_F: \vec{E} \to [0,1]\ and\ \mu_I: \vec{E} \to [0,1]$such that

$$\mu_T(\overrightarrow{v_i, v_j}) \leq \min\ [\rho_T(v_i),\ \rho_T(v_j)]$$

$\mu_F(\overrightarrow{v_i, v_j}) \geq max[\rho_F(v_i),\ \rho_F(v_j)]$

$\mu_I(\overrightarrow{v_i, v_j}) \geq max[\ \rho_I(v_i),\ \rho_I(v_j)]$

and $0 \leq \mu_T(\overrightarrow{v_i, v_j}) + \mu_F(\overrightarrow{v_i, v_j}) + \ \mu_I(\overrightarrow{v_i, v_j}) \leq 3$ for all $(v_i, v_j) \in E$

where $\mu_T(\overrightarrow{v_i, v_j}),\ \mu_F(\overrightarrow{v_i, v_j}),\ \mu_I(\overrightarrow{v_i, v_j})$ denote the degree of true membership, degree of falsity membership and degree of indeterminacy membership of the edge $(\overrightarrow{v_i, v_j}) \in \vec{E}$ respectively.

Example 1. Consider a graph (Fig.1) $\vec{G} = (V, \vec{E})$ where $V = \{v_1, v_2, v_3, v_4\}$ and $\vec{E} = \{(\overrightarrow{v_1, v_2}), (\overrightarrow{v_1, v_3}), (\overrightarrow{v_2, v_3}), (\overrightarrow{v_3, v_4})\}$. The membership values of vertices (Table 2) and edges (Table 3) and the corresponding graph are given following.

Table 2. Membership values of vertices of a graph (Fig.1)

	v_1	v_2	v_3	v_4
ρ_T	0.4	0.3	0.5	0.3
ρ_F	0.3	0.1	0.6	0.4
ρ_I	0.2	0.4	0.4	0.6

Table 3. membership values of edges of a graph (Fig.1)

	$(\overrightarrow{v_1, v_2})$	$(\overrightarrow{v_1, v_3})$	$(\overrightarrow{v_2, v_3})$	$(\overrightarrow{v_3, v_4})$
μ_T	0.3	0.3	0.2	0.3
μ_F	0.4	0.6	0.6	0.6
μ_I	0.4	0.5	0.5	0.6

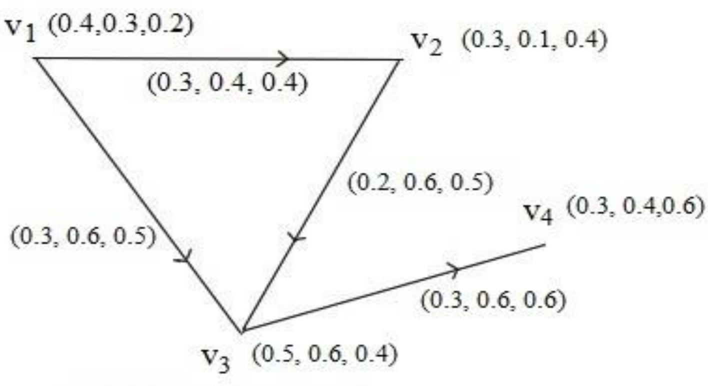

Figure.1. A neutrosophic digraph

Definition 4. A graph $\overrightarrow{G'} = (V,\vec{E})$ where $\vec{E} \subseteq V \times V$ is said to be generalized neutrosophic digraph if there exist functions

$$\rho_T: V \to [0,1], \rho_F: V \to [0,1] and \rho_I: V \to [0,1],$$

$$\mu_T: \vec{E} \to [0,1], \mu_F: \vec{E} \to [0,1] \text{ and } \mu_I: \vec{E} \to [0,1]$$
$$\phi_T: E_T \to [0,1], \phi_F: E_F \to [0,1] \text{ and } \phi_I: E_I \to [0,1]$$

such that

$$0 \leq \rho_T(v_i) + \rho_F(v_i) + \rho_I(v_i) \leq 3 \text{ for all } v_i \in V \ (i = 1,2,3, \ldots., n)$$

and

$$\mu_T(\overrightarrow{v_i, v_j}) = \phi_T(\rho_T(v_i), \rho_T(v_j))$$
$$\mu_F(\overrightarrow{v_i, v_j}) = \phi_F(\rho_F(v_i), \rho_F(v_j))$$
$$\mu_I(\overrightarrow{v_i, v_j}) = \phi_I(\rho_I(v_i), \rho_I(v_j))$$

where $E_T = \{(\rho_T(v_i), \rho_T(v_j)): \mu_T(v_i, v_j) \geq 0\}$, $E_F = \{(\rho_F(v_i), \rho_F(v_j)): \mu_F(v_i, v_j) \geq 0\}$, $E_I = \{(\rho_I(v_i), \rho_I(v_j)): \mu_I(v_i, v_j) \geq 0\}$ and $\rho_T(v_i)$, $\rho_F(v_i)$, $\rho_I(v_i)$ denote the degree of true membership, the degree of falsity membership, the indeterminacy membership of vertex $v_i \in V$ respectively and $\mu_T(\overrightarrow{v_i, v_j})$, $\mu_F(\overrightarrow{v_i, v_j})$, $\mu_I(\overrightarrow{v_i, v_j})$ denote the degree of true membership, the degree of falsity membership and the degree of indeterminacy membership of edge$\overrightarrow{(v_i, v_j)} \in \vec{E}$ respectively.

Example 2. Consider a graph (Fig.2)$\vec{G} = (V, \vec{E})$ where $V = \{v_1, v_2, v_3, v_4\}$ and $\vec{E} = \{(\overrightarrow{v_1, v_2}), (\overrightarrow{v_1, v_3}), (\overrightarrow{v_4, v_1}), (\overrightarrow{v_3, v_2})\}$.

Consider the membership values of vertices (Table 4) are given below:

Table 4. Membership values of vertices of a graph (Fig.2)

	v_1	v_2	v_3	v_4
ρ_T	0.5	0.6	0.2	0.7
ρ_F	0.4	0.5	0.4	0.3
ρ_I	0.3	0.6	0.7	0.4

Consider the membership values of edges (Table 5) as

$$\mu_T(m,n) = \max\{m,n\} = \mu_F(m,n) = \mu_I(m,n)$$

Table 5. Membership values of edges of a graph (Fig.2)

	$(\overrightarrow{v_1,v_2})$	$(\overrightarrow{v_1,v_3})$	$(\overrightarrow{v_4,v_1})$	$(\overrightarrow{v_3,v_2})$
μ_T	0.3	0.3	0.2	0.3
μ_F	0.4	0.6	0.6	0.6
μ_I	0.4	0.5	0.5	0.6

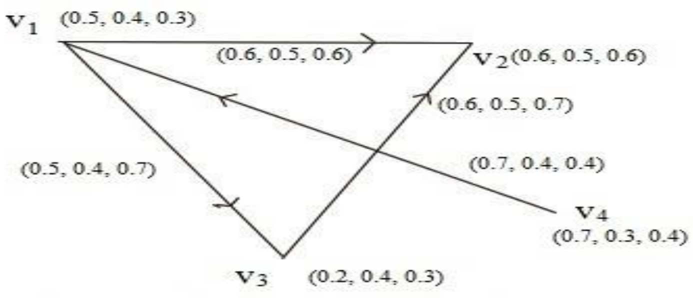

Figure 2. A generalized neutrosophic digraph

Definition 5. Let $\overrightarrow{G'} = (V,\vec{E})$ be a generalized neutrosophic digraph. Then out-neighbourhood $N^+(v_i)$ of a vertex $v_i \in V$ is given by

$$N^+(v_i) = \{v_j, (\mu_T(\overrightarrow{v_i,v_j}), \mu_F(\overrightarrow{v_i,v_j}), \mu_I(\overrightarrow{v_i,v_j})) : (\overrightarrow{v_i,v_j}) \in \vec{E}\}$$

where $\mu_T(\overrightarrow{v_i,v_j}), \mu_F(\overrightarrow{v_i,v_j}), \mu_I(\overrightarrow{v_i,v_j})$ denote the degree of true membership, the degree of falsity membership and indeterminacy membership of edge $(\overrightarrow{v_i,v_j}) \in \vec{E}$.

Example 3. Consider a GN digraph (Fig.3) $\vec{G} = (V,\vec{E})$ where $V = \{v_1, v_2, v_3, v_4\}$ and $\vec{E} = \{(\overrightarrow{v_1,v_2}), (\overrightarrow{v_1,v_3}), (\overrightarrow{v_1,v_4}), (\overrightarrow{v_2,v_3}), (\overrightarrow{v_3,v_4})\}$.

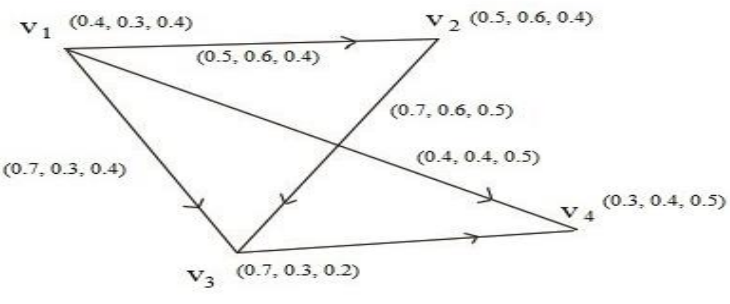

Fig.3. A generalized neutrosophic digraph

$$N^+(v_1) = \{(v_2, (0.5, 0.6, 0.4)), (v_3, (0.7, 0.3, 0.4)), (v_4, (0.4, 0.4, 0.5))\}$$

$$N^+(v_2) = \{(v_3, (0.7,0.6,0.5))\}\ ,\ N^+(v_3) = \{(v_4, (0.7,0.4,0.5))\},\quad N^+(v_4) = \emptyset.$$

Definition 6. Let $\overrightarrow{G'} = (V, \vec{E})$ be a generalized neutrosophic digraph. Then the generalized neutrosophic competition graph $C(\vec{G}')$ of $\vec{G} = (V, \vec{E})$ is a generalized neutrosophic graph which has the same vertex set V and has a neutrosophic edge between u, v if and only if $N^+(u) \cap N^+(v) \neq \emptyset$ and there exist sets $S_1 = \{\gamma_u^T, u \in V\}$, $S_2 = \{\gamma_u^F, u \in V\}$, $S_3 = \{\gamma_u^I, u \in V\}$ and functions $\phi_1: S_1 \times S_1 \to [0,1]$, $\phi_2: S_2 \times S_2 \to [0,1]$, $\phi_3: S_3 \times S_3 \to [0,1]$ such that edge-membership value of an edge $(u, v) \in E'$ is $(\mu_T(u,v),\ \mu_F(u,v),\ \mu_I(u,v))$ where

$$\mu_T(u,v) = \phi_1(\gamma_u^T, \gamma_v^T)$$
$$\mu_F(u,v) = \phi_2(\gamma_u^F, \gamma_v^F)$$
$$\mu_I(u,v) = \phi_3(\gamma_u^I, \gamma_v^I)$$

$$\gamma_u^T = \min\{\mu_T(\overrightarrow{u,w}), \forall w \in N^+(u) \cap N^+(v)\}, \gamma_v^T = \min\{\mu_T(\overrightarrow{u,w}), \forall w \in N^+(u) \cap N^+(v)\},$$
$$\gamma_u^F = \max\{\mu_F(\overrightarrow{u,w}), \forall w \in N^+(u) \cap N^+(v)\},\ \gamma_v^F = \max\{\mu_F(\overrightarrow{u,w}), \forall w \in N^+(u) \cap N^+(v)\},$$
$$\gamma_u^I = \max\{\mu_I(u,w), \forall w \in N^+(u) \cap N^+(v)\},\ \gamma_u^I = \min\{\mu_I(v,w), \forall w \in N^+(u) \cap N^+(v)\}.$$

Example 4. Consider a GN digraph(Fig.3) $G = (V, \vec{E})$ where $V = \{v_1, v_2, v_3, v_4\}$ and
$\vec{E} = \{(\overrightarrow{v_1, v_2}), (\overrightarrow{v_1, v_3}), (\overrightarrow{v_1, v_4}), (\overrightarrow{v_2, v_3}), (\overrightarrow{v_3, v_4})\}$.

Then the corresponding competition graph (Fig.4) with membership values of edges (Table 6) is

Table 6. Membership values of edges a graph (Fig.4)

	(v_1, v_2)	(v_1, v_3)
μ_T	0.7	0.4
μ_F	0.3	0.3
μ_I	0.4	0.2

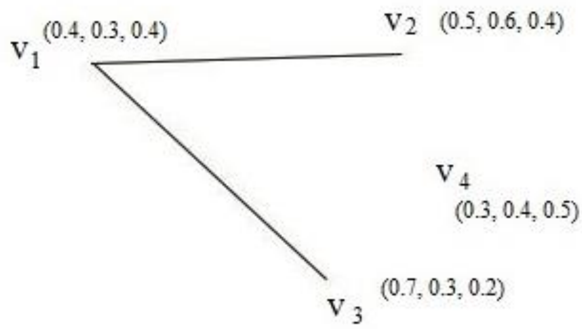

Figure 4. A generalized neutrosophic competition graph of a graph (Fig.3)

Theorem 1. Let G be a generalized neutrosophic graph. Then there exists a generalized neutrosophic digraph $\overrightarrow{G'}$ such that $C(\overrightarrow{G'}) = G$.

Proof. Let $G = (V, E)$ be a generalized neutrosophic graph and (x,y) be an edge in G. Now, a generalized neutrosophic digraph $\overrightarrow{G'}$ is to be constructed such that $C(\overrightarrow{G'}) = G$.

Let $x^{'}, y' \in \overrightarrow{G'}$ be the corresponding vertices of $x, y \in G$. Then we can draw two directed edges from vertices $x^{'}, y$ to a vertex $z^{'} \in \overrightarrow{G'}$ such that $z^{'} \in N^{+}(x') \cap N^{+}(y')$. Similarly, we can do for all vertices and edges of G and hence $C\left(\overrightarrow{G'}\right) = G$.

Definition 7. Let G be a generalized neutrosophic graph. Minimal graph, $\overrightarrow{G'}$ of G is a generalized neutrosophic digraph such that $C\left(\overrightarrow{G'}\right) = G$ and $\overrightarrow{G'}$ has the minimum number of edges i.e. if there exists another graph G'' with $C\left(\overrightarrow{G''}\right) = G$, then number of edges of $\overrightarrow{G''}$ is greater than or equal to the number of edges of $\overrightarrow{G'}$.

Consider a generalized neutrosophic graph. If it is assumed as a generalized neutrosophic competition graph, then our task is to find the corresponding generalized neutrosophic digraph. Then there are a lot of graphs for a single generalized neutrosophic competition graph. We will consider the graph with a minimum number of edges.

Theorem 2. Let G be a generalised neutrosophic connected graph whose underlying graph is a complete graph with n vertices. Then the number of edges in a minimal graph of G is equal to 2n, n ≥ 3.

Proof. Let $G = (V, E)$ be a connected generalized neutrosophic graph whose underlying graph is a complete graph of n vertices so that each vertex of G is connected with each other. Let u, v be two adjacent vertices in G and u_1, v_1 be the corresponding vertices in the minimal graph $\overrightarrow{G}^{'}$. Consider a generalised neutrosophic directed graph $\vec{G}_1^{'}$in such a way that every vertex of $\vec{G}$ other than u_1 has only out-neighbourhood as u_1. Thus $\vec{G}_1^{'}$ has $(n-1)$ edges. Similarly, a generalised neutrosophic directed graph $\vec{G}_2^{'}$ is considered for v_1 and hence $\vec{G}_2^{'}$ has $(n-1)$ edges. Now, consider a generalised neutrosophic directed graph $\vec{G}_3^{'}$ with only edges $(\overrightarrow{u_1, w_1}), (\overrightarrow{v_1, w_1})$. Thus $\overrightarrow{G}^{'} = \vec{G}_1^{'} \cup \vec{G}_2^{'} \cup \vec{G}_3^{'}$. The number of edges in $\overrightarrow{G}^{'}$ is $(n-1) + (n-1) + 2 = 2n$.

Definition 8. Score$s$of an edge (u, v) between two vertices in a generalized neutrosophic graph is given by $s(u, v) = [2\mu_T(1 - \mu_F) + \mu_I]/3$ where μ_T, μ_F and μ_I are the degree of truth membership, degree of falsity membership and degree of indeterminacy membership of the edge (u, v) respectively.

Example 5. Consider a GN graph (Fig.5) $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{(v_1, v_2), (v_1, v_4), (v_2, v_3), (v_3, v_4), (v_2, v_4)\}$.

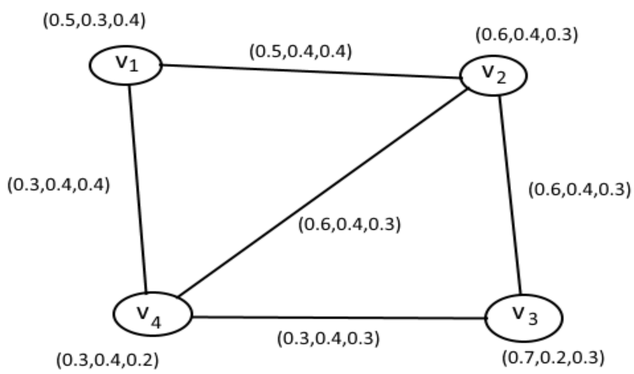

Figure 5. An example of a generalized neutrosophic graph

The score of the edge (v_3, v_4) is 0.42. Similarly, the scores of all edges should be found.

Definition 9. In a generalized neutrosophic graph, a vertex u with adjacent vertices $v_1, v_2, \ldots., v_k$ is said to be isolated if $s(u, v_i) = 0$ for $i = 1,2,3 \ldots.., k$.

Note1. If $\mu_F = 1, \mu_I = 0$, then score = 0 and if $\mu_T = 0 = \mu_I$ then score = 0.

Example 6. Consider a GN graph (Fig.6) $G = (V, E)$ where $V = \{v_1, v_2, v_3, v_4\}$ and

$$E = \{(v_1, v_2), (v_1, v_3), (v_2, v_3), (v_2, v_4)\}$$

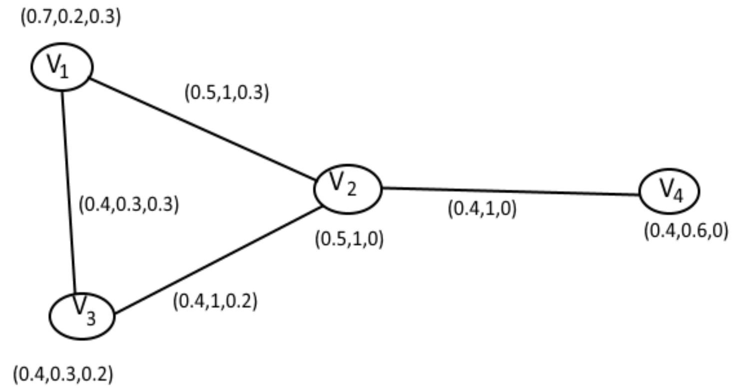

Figure 6. An example of a generalized neutrosophic graph with isolated vertex

The adjacent vertex of v_4 is v_2 and the score of the edge(v_2, v_4) is 0, so v_4 is an isolated vertex.

Definition 10. A cycle of length ≥ 4 in a generalized neutrosophic graph is called a hole if all the edges of this cycle have a non-zero score.

Example 7. Consider the graph in example 5, $v_1 - v_2 - v_3 - v_4 - v_1$is a cycle of length 4 and all the of the cycle have non-zero score and hence it is a hole.

Definition 11. The smallest number of the isolated vertex in a generalized neighbourhood graph is called competition number. It is denoted by $k_N(G)$.

Lemma 1. If a crisp graph has one hole, then its completion number is at most 2. But the Competition number of a generalized neutrosophic graph with exactly one hole may be greater than two. Let us consider a graph (Fig.7) with exactly one hole with competition number 2.

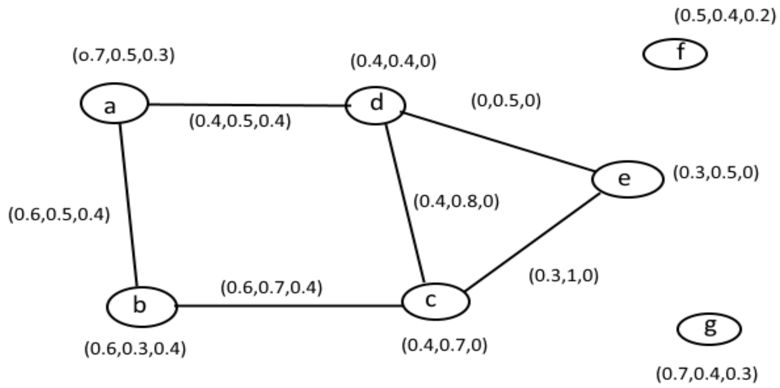

Figure 7. Generalized neutrosophic graph with competition number 2.

It may be noted that scores of edges $(\overrightarrow{a, b}),(\overrightarrow{b, c}), (\overrightarrow{c, d})$and $(\overrightarrow{d, a})$ are non-zero as per definition of the hole. But the score of $(\overrightarrow{d, e})$ and $(\overrightarrow{c, e})$ may be zero. Hence e is an isolated vertex. Thus competition number is 3.

Definition 12. A neutrosophic graph is said to be a neutrosophic chordal graph if all the holes have a chord with score > 0.

Example 10. Consider the graph in example 5, $v_1 - v_2 - v_3 - v_4 - v_1$are only a hole and the edge (v_2, v_4) is a chord with a non-zero score, then the graph is a neutrosophic chordal graph.

Lemma 2. The competition number of a neutrosophic chordal graph with pendant vertex be greater than 1. In the neutrosophic chordal graph (Fig.8) given below, since the vertex e is isolated, then the competition number is greater than 2.

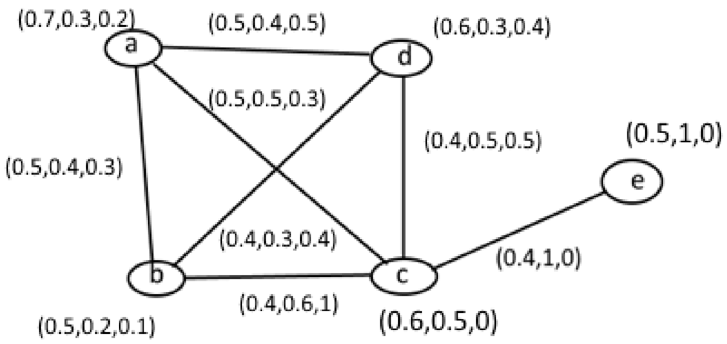

Figure 8. Neutrosophic chordal graph

3. Matrix representation of GNCG

It is one kind of adjacency matrix of the GNCG. The entries of the matrix are calculated as follows:

Step-1: Let us consider a generalized neutrosophic digraph (GNDG).

Step-2: Find the pair of vertices $u_i, v_i (i = 1,2, \ldots, m)$ such that there exist edges $(\overrightarrow{u_\iota, x_k})$, $(\overrightarrow{v_\iota, x_l})$ for $(k, l = 1,2, \ldots, p)$ with $N^+(u_i)$ and $N^+(v_i)$.

Step-3: Find the set $N^+(u_i) \cap N^+(v_i) = \{x_n, n = 1,2, \ldots, q\}, say.$

Step-4: let $\gamma_u^T = \min \{\mu_T(\overrightarrow{u_\iota, x_1}), \mu_T(\overrightarrow{u_\iota, x_2}), \ldots, \mu_T(\overrightarrow{u_\iota, x_q})\}$

$$\gamma_v^T = \min \{\mu_T(\overrightarrow{v_\iota, x_1}), \quad \mu_T(\overrightarrow{v_\iota, x_2}), \ldots, \mu_T(\overrightarrow{v_\iota, x_q})\}$$

$$\gamma_u^F = \max \{\mu_F(\overrightarrow{u_\iota, x_1}), \mu_F(\overrightarrow{u_\iota, x_2}), \ldots, \mu_F(\overrightarrow{u_\iota, x_q})\}$$

$$\gamma_v^F = \max \{\mu_F(\overrightarrow{v_\iota, x_1}), \mu_F(\overrightarrow{v_\iota, x_2}), \ldots, \mu_F(\overrightarrow{v_\iota, x_q})\}$$

$$\gamma_u^I = \min \{\mu_I(\overrightarrow{u_\iota, x_1}), \mu_I(\overrightarrow{u_\iota, x_2}), \ldots, \mu_I(\overrightarrow{u_\iota, x_q})\}$$

$$\gamma_v^I = \max \{\mu_I(\overrightarrow{v_\iota, x_1}), \mu_I(\overrightarrow{v_\iota, x_2}), \ldots, \mu_I(\overrightarrow{v_\iota, x_q})\}.$$

Step-5: Find the degree of true membership, degree of falsity membership and degree of indeterminacy membership by the following formula

$$\mu_T(u, v) = \varphi_1(\gamma_u^T, \gamma_v^T),$$

$$\mu_F(u, v) = \varphi_2(\gamma_u^F, \gamma_v^F),$$

$$\mu_I(u, v) = \varphi_3(\gamma_u^I, \gamma_v^I)$$

For simplification, one function φ may be used in place of $\varphi_1, \varphi_2, \varphi_3$.

Step-6: the competition matrix is a square matrix. Its order equal to the number of vertices. Its entries are given below.

$$a_{ij} = \begin{cases} \left(\varphi_1\left(\gamma_i^T, \gamma_j^T\right), \varphi_2\left(\gamma_i^F, \gamma_j^F\right), \varphi_3\left(\gamma_i^I, \gamma_j^I\right)\right) & if\ there\ is\ an\ edge\ between\ vertex\ i\ and\ j \\ (0,0,0), & if\ there\ is\ no\ edge\ between\ vertex\ i\ and\ j. \end{cases}$$

Example 11. An example of matrix representation is presented with all steps.

Step -1: Consider a GNDG (Fig.9)$\overrightarrow{G'} = (V, \vec{E})$. The membership values of vertices and edges are given in the graph (Fig.)

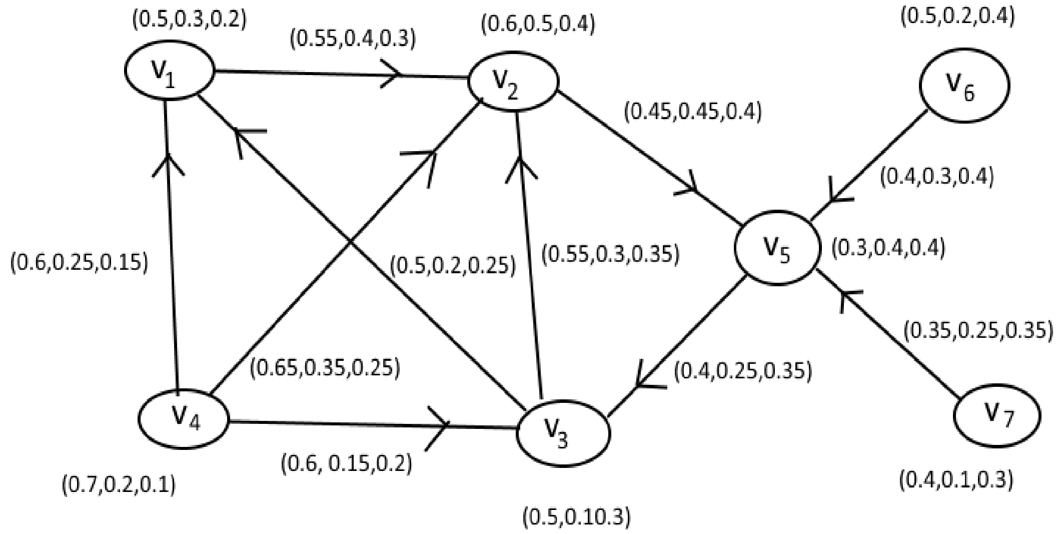

Figure 9. A generalized neutrosophic graph with seven vertices

Step-2:$N^+(v_1) = \{v_2\}N^+(v_2) = \{v_5\}\ N^+(v_3) = \{v_2, v_1\}$

$N^+(v_4) = \{v_1, v_3\}N^+(v_5) = \{v_3\}N^+(v_6) = \{v_5\}N^+(v_7) = \{v_5\}$.

Step-3: $N^+(v_1) \cap N^+(v_2) = \emptyset,\quad N^+(v_1) \cap N^+(v_3) = \{v_2\},\ N^+(v_1) \cap N^+(v_4) = \{v_2\},$

$N^+(v_1) \cap N^+(v_5) = \emptyset,\quad N^+(v_1) \cap N^+(v_6) = \emptyset,\ N^+(v_1) \cap N^+(v_7) = \emptyset,$

$N^+(v_2) \cap N^+(v_3) = \emptyset,\quad N^+(v_2) \cap N^+(v_4) = \emptyset, N^+(v_2) \cap N^+(v_5) = \emptyset,$

$N^+(v_2) \cap N^+(v_6) = \{v_5\}, N^+(v_2) \cap N^+(v_7) = \{v_5\},\ ,\ N^+(v_3) \cap N^+(v_4) = \{v_1\},$

$N^+(v_3) \cap N^+(v_5) = \emptyset, N^+(v_3) \cap N^+(v_6) = \emptyset,\quad N^+(v_3) \cap N^+(v_7) = \emptyset,$

$N^+(v_4) \cap N^+(v_5) = \{v_3\}, N^+(v_4) \cap N^+(v_6) = \emptyset,\quad N^+(v_4) \cap N^+(v_7) = \emptyset,$

$N^+(v_5) \cap N^+(v_6) = \emptyset,\quad N^+(v_5) \cap N^+(v_7) = \emptyset,\quad N^+(v_6) \cap N^+(v_7) = \{v_5\},$

Step-4:

$$\gamma_{12}^T = 0.55,\quad \gamma_{12}^F = 0.4,\quad \gamma_{12}^I = 0.3$$
$$\gamma_{32}^T = 0.55,\quad \gamma_{32}^F = 0.3,\quad \gamma_{32}^I = 0.35$$
$$\gamma_{42}^T = 0.65,\quad \gamma_{42}^F = 0.35,\quad \gamma_{42}^I = 0.25$$
$$\gamma_{25}^T = 0.45,\quad \gamma_{25}^F = 0.45,\quad \gamma_{25}^I = 0.4$$
$$\gamma_{65}^T = 0.4,\quad \gamma_{65}^F = 0.3,\quad \gamma_{65}^T = 0.4$$
$$\gamma_{75}^T = 0.35,\quad \gamma_{75}^F = 0.25,\quad \gamma_{75}^I = 0.35$$
$$\gamma_{31}^T = 0.5,\quad \gamma_{31}^F = 0.2,\quad \gamma_{31}^I = 0.25$$

$$\gamma_{41}^T = 0.6, \quad \gamma_{41}^F = 0.25, \quad \gamma_{41}^I = 0.15$$
$$\gamma_{43}^T = 0.6, \quad \gamma_{43}^F = 0.15, \quad \gamma_{43}^I = 0.2$$
$$\gamma_{53}^T = 0.4, \quad \gamma_{53}^F = 0.25, \quad \gamma_{53}^I = 0.35$$

Step-5:

$$\mu_{13}^T = 0, \quad \mu_{13}^F = 0.1, \quad \mu_{13}^I = 0.05$$
$$\mu_{14}^T = 0.1, \quad \mu_{14}^F = 0.05, \quad \mu_{13}^I = 0.05$$
$$\mu_{34}^T = 0.1, \quad \mu_{34}^F = 0.05, \quad \mu_{34}^I = 0.1$$
$$\mu_{45}^T = 0.2, \quad \mu_{45}^F = 0.1, \quad \mu_{45}^I = 0.15$$
$$\mu_{26}^T = 0.05, \quad \mu_{26}^F = 0.15, \quad \mu_{26}^I = 0$$
$$\mu_{27}^T = 0.1, \quad \mu_{27}^F = 0.2, \quad \mu_{27}^I = 0.05$$
$$\mu_{67}^T = 0.05, \quad \mu_{67}^F = 0.05, \quad \mu_{67}^I = 0.05$$

Step-6: the corresponding matrix is

$$\begin{pmatrix} - & (0,0,0) & (0,0.1,0.05) & (0.1,0.05,0.05) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & - & (0,0,0) & (0,0,0) & (0,0,0) & (0.05,0.15,0) & (0.1,0.2,0.05) \\ (0,0.1,0.05) & (0,0,0) & - & (0.1,0.05,0.1) & (0,0,0) & (0,0,0) & (0,0,0) \\ (0.1,0.05,0.05) & (0,0,0) & (0.1,0.05,0.1) & - & (0.2,0.1,0.15) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) & (0.2,0.1,0.15) & - & (0,0,0) & (0,0,0) \\ (0,0,0) & (0.05,0.15,0) & (0,0,0) & (0,0,0) & (0,0,0) & - & (0.05,0.05,0.05) \\ (0,0,0) & (0.1,0.2,0.05) & (0,0,0) & (0,0,0) & (0,0,0) & (0.05,0.05,0.05) & - \end{pmatrix}$$

4. An application in economic competition

Like competitions in the ecosystem, there are many competitions running in real life. In this study, the competition in economic growth among the countries (Fig.10) are presented in the neutrosophic environment. We consider two factors: GDP and GPI. Gross Domestic Product (GDP) of a country is the total market value of all goods and services produced in a specific time period in the country. The Global Peaceful Index (GPI) of a country is the value of peacefulness in the country relative to global.

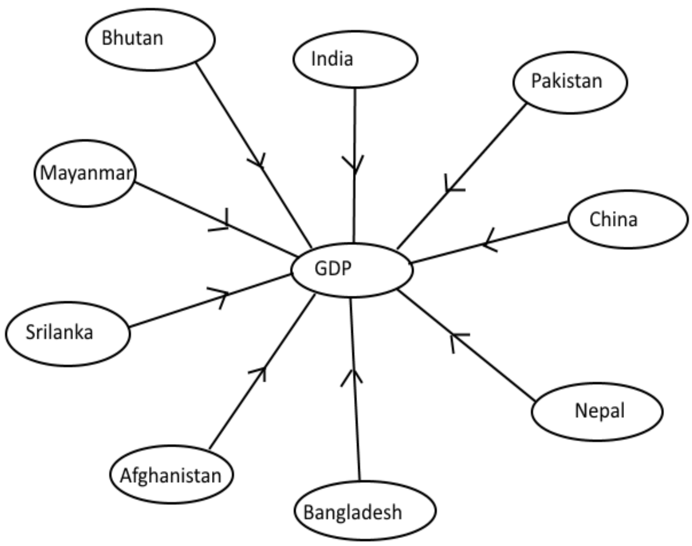

Figure 10. Competition among countries

The GDP growth is taken as the degree of truth membership, GPI is taken as the degree of falsity memberships. The uncertainty causes like flood, elections etc. may be taken as the degree of indeterminacy membership. The data of GDP growth and GPI are collected from internet. The country of India with neighbours countries are competing with each other to become more strong. Since all countries are competing, so the corresponding competition graph is a complete graph.

The membership values of countries (nodes) are given in the tabular form (Table 7, Table 8) and the membership values of edges are calculated by the following formula and are represented by a matrix.

$$\mu_T(u,v) = 1 - |\sigma_T^u - \sigma_T^v|,$$
$$\mu_F(u,v) = 1 - |\sigma_F^u - \sigma_F^v|,$$
$$\mu_I(u,v) = 0$$

Table 7. Countries with GDP and GPI values

SL. No.	Country	GDP	GPI
1	India	7.257	2.605
2	Pakistan	2.905	3.072
3	China	6.267	2.217
4	Nepal	6.536	2.003
5	Bangladesh	7.289	2.128
6	Bhutan	4.816	1.506
7	Myanmar	6.448	2.393
8	Afganistan	3	3.574
9	Srilanka	3.5	1.986

Table 8. Countries with their normalized values of GDP and GPI.

Sl. No.	Country	N GDP	1/GPI	N GPI	N GDP~ N GPI
1	India	0.996	0.38	0.576	0.42
2	Pakistan	0.399	0.33	0.5	0.101
3	China	0.86	0.45	0.682	0.178
4	Nepal	0.897	0.5	0.758	0.139
5	Bangaladesh	1	0.47	0.712	0.288
6	Bhutan	0.661	0.66	1	0.339
7	Mayanmar	0.885	0.42	0.636	0.249
8	Afganistan	0.412	0.28	0.424	0.012
9	Srilanka	0.48	0.5	0.758	0.278

The competition among countries is given above by the matrix form.

$$\begin{pmatrix}
(1,1,0) & (0.924,0.681,0) & (0.864,0.758,0) & (0.901,0.719,0) & (0.996,0.868,0) & (0.665,0.919,0) & (0.889,0.829,0) & (0.416,0.592,0) & (0.484,0.858,0) \\
(0.403,0.681,0) & (1,1,0) & (0.539,0.923,0) & (0.502,0.962,0) & (0.399,0.813,0) & (0.738,0.762,0) & (0.514,0.852,0) & (0.987,0.911,0) & (0.919,0.823,0) \\
(0.864,0.758,0) & (0.894,0.923,0) & (1,1,0) & (0.963,0.961,0) & (0.86,0.89,0) & (0.801,0.839,0) & (0.975,0.929,0) & (0.552,0.834,0) & (0.62,0.9,0) \\
(0.901,0.719,0) & (0.818,0.962,0) & (0.963,0.961,0) & (1,1,0) & (0.897,0.851,0) & (0.764,0.8,0) & (0.988,0.89,0) & (0.515,0.873,0) & (0.583,0.861,0) \\
(0.996,0.868,0) & (0.864,0.813,0) & (0.86,0.89,0) & (0.897,0.851,0) & (1,1,0) & (0.661,0.949,0) & (0.885,0.961,0) & (0.412,0.724,0) & (0.48,0.99,0) \\
(0.665,0.919,0) & (0.576,0.762,0) & (0.801,0.839,0) & (0.764,0.8,0) & (0.661,0.949,0) & (1,1,0) & (0.776,0.91,0) & (0.751,0.673,0) & (0.819,0.939,0) \\
(0.889,0.829,0) & (0.94,0.852,0) & (0.975,0.929,0) & (0.988,0.89,0) & (0.885,0.961,0) & (0.776,0.91,0) & (1,1,0) & (0.527,0.763,0) & (0.595,0.971,0) \\
(0.416,0.592,0) & (0.848,0.911,0) & (0.552,0.834,0) & (0.515,0.873,0) & (0.412,0.724,0) & (0.751,0.673,0) & (0.527,0.763,0) & (1,1,0) & (0.932,0.734,0) \\
(0.004,0.858,0) & (0.424,0.823,0) & (0.62,0.9,0) & (0.583,0.861,0) & (0.48,0.99,0) & (0.819,0.939,0) & (0.595,0.971,0) & (0.932,0.734,0) & (1,1,0)
\end{pmatrix}$$

Conclusion

This study presents the generalization of neutrosophic competition graph where edge restrictions are withdrawn. A representation of GNCG is presented by a square matrix. Also, the minimal graph and competition number are introduced. A real-life application is presented and discussed by the GNCG. In this application, true membership value is taken as GDP, the gross domestic product of countries, and falsity is taken as complement of of GPI, Global Peace Index of such countries. These parameters may be taken differently to capture the competitions among countries. This representation will be helpful to perceive real-life competitions. This study assumed only one step competition. In future, n-step neutrosophic competition graph and several other related notions will be studied. This study will be the backbone of that.

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References

1. Das K., Samanta S. and Pal M., Study on centrality measures in social networks: a survey, Social network analysis and mining, 8, 13, 2018.
2. Cohen J.E., Interval graphs and food webs: a finding and a problem, Document 17696-PR, RAND Corporation, Santa Monica, CA, 1968.
3. Roberts F. S., Discrete Mathematical Models, with Applications to Social, Biological, and Environmental Problems, Prentice-Hall, Englewood Cliffs, NJ, 1976.
4. Roberts F. S., Food webs, competition graphs, and the boxicity of ecological phase space, in Theory and Applications of Graphs, (Y. Alavi and D. Lick, eds.), Springer-Verlag, New York, 477–490,1978.
5. Opsut R. J., On the computation of the competition number of a graph, SIAM Journal on Algebraic Discrete Mathematics, 3, 420-428, 1982.
6. Kim S. R., McKee T. A., McMorris R. R. and Roberts F. S., p-competition graph, Linear Algebra and its Applications, 217, 167-178, 1995.
7. Kim S. R., McKee T. A., McMorris R. R. and Roberts F. S., p-competition number, Discrete Applied Mathematics, 46, 89-92, 1993.
8. Brigham R. C., McMorris F. R. and Vitray R.P., Tolerance competition graphs, Linear Algebra and its Applications, 217, 41- 52, 1995.
9. Cho H. H. and Kim S. R., The competition number of a graph having exactly one hole, Discrete Mathematics, 303, 32-41, 2005.
10. Li B. J. and Chang G. J., The competition number of a graph with exactly one hole, all of a which are independent, Discrete Applied Mathematics, 157, 1337-1341, 2009.
11. Factor K. A. S. and Merz S. K., The (1,2) –step competition graph of a tournament, Discrete Applied mathematics, 159, 100-103, 2011.
12. Kauffman A., Introduction a la Theorie des Sous-emsemblesFlous, Paris: Masson et CieEditeurs, 1973.
13. Mordeson J. N. and Nair P. S., Fuzzy Graphs and Hypergraphs, Physica Verlag, 2000.
14. Parvathi R. and Karunambigai M.G., Intuitionistic fuzzy graphs, Computational Intelligence, Theory and Applications , 38, 139-150, 2006.
15. Akram M. and Dubek W. A., Interval-valued fuzzy graphs, Computer and Mathematics with Applications, 61, 289-299, 2011.
16. Samanta S. and Pal M., Fuzzy k-competition graphs and p-competition fuzzy graphs, Fuzzy Information and Engineering, 5, 191-204, 2013.
17. Samanta S., Sarkar B., Shin D. and Pal M., Completeness and regularity of generalized fuzzy graphs, Springer Plus, 5, 1-14, 2016.
18. Samanta S. and Sarkar B., Representation of competitions by generalized fuzzy graphs, International Journal of Computational Intelligence System, 11, 1005-1015, 2018.
19. Pramanik T., Samanta S., Sarkar B. and Pal M., Fuzzy φ-tolerance competition graphs, Soft Computing, 21, 3723-3734, 2016.
20. Smarandache F., Neutrosophyneutrosophic probability, Set and Logic, Amer Res Press, Rehoboth, USA, 1998.

21. Zadeh L.A., Fuzzy sets, Information and Control, **8**, 338–353, 1965.
22. Atanassov K.T., Intuitionistic fuzzy sets, Fuzzy Set and Systems, **20** ,87–96, 1986.
23. Abdel-Baset M., Chang V. and Gamal A., Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry, 108, 210-220, 2019.
24. Abdel-Basset M., El-hoseny M., Gamal A. and Smarandache F., A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine, 100, 101710, 2019.
25. Abdel-Basset M., Mohamed R., Zaied A. E. N. H. and Smarandache F., A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry, 11(7), 903, 2019.
26. Abdel-Basset M., Manogaran G., Gamal A. and Chang V., A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT. IEEE Internet of Things Journal, 2019.
27. Abdel-Basset M., Nabeeh N. A., El-Ghareeb H. A. and Aboelfetouh, A., Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises. Enterprise Information Systems, 1-21, 2019.
28. Broumi S., Talea M., Bakali A. and Smarandache F., Single valued neutrosophic graphs, Journal of New Theory, 10, 86-101, 2016.
29. Akram M. and Shahzadi G., Operations on single-valued neutrosophic graphs, Journal of Uncertain Systems,11(1), 1-26, 2017.
30. Akram M. and Siddique S., Neutrosophic competition graphs with applications, Journal Intelligence and Applications, 33, 921-935,2017.
31. Broumi S., Bakali A., Talea M., Smarandache F. and Hassan A., Generalized single-valued neutrosophic graphs of first type, Acta Electrotechica, 59, 23-31, 2018.
32. Broumi S., Talea M., Bakali A., Singh P. K., Smarandache F., Energy and Spectrum Analysis of Interval Valued Neutrosophic Graph using MATLAB, Neutrosophic Sets and Systems, 24, 46-60, 2019.
33. Nagarajan D., Lathamaheswari M., Broumi S., Kavikumar J., Dombi Interval Valued Neutrosophic Graph and its Role in Traffic Control Management, Neutrosophic Sets and Systems, 24, 114-133, 2019.
34. Sinha K., Majumdar P., Entropy based Single Valued Neutrosophic Digraph and its applications, Neutrosophic Sets and Systems, 19, 119-126, 2018.

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Operations of Single Valued Neutrosophic Coloring

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Abstract: Smarandache introduced the concept of Neutrosophic which deals with membership, non-membership and indeterminacy values. Wang discussed the Single Valued Neutrosophic sets in 2010. Single Valued Neutrosophic graph was introduced by Broumi and in 2019 Single Valued Neutrosophic coloring was introduced. In this paper, some properties of the Single Valued Neutrosophic Coloring of Strong Single Valued Neutrosophic graph, Complete Single Valued Neutrosophic graph and Complement of Single Valued Neutrosophic graphs are discussed.

Keywords: single-valued neutrosophic graphs; single-valued neutrosophic vertex coloring; strong single-valued neutrosophic graph; complete single-valued neutrosophic graph.

1. Introduction

Francis Guthrie's four-color conjecture was reasoned for the development of the new branch of graph coloring in graph theory. Graph coloring is assigning labels to the vertices or edges or both vertices and edges. Distinct vertices received different colors are called proper coloring. Graph coloring technique used in many areas like telecommunication, scheduling, computer networks etc.

Most of the problems are not only deals the accurate values, sometimes handle vague values. Fuzzy sets were introduced by Zadeh [29] in 1965, dealt imprecise values in his work. Fuzzy graph theory concept was developed by Rosenfeld [25] in 1975. Munoz et al. [27] in 2004 and Eslahchi, Onagh [19] in 2006 discussed the fuzzy chromatic number and its properties.

Kassimir T. Atanassov [11] introduced the concept of intuitionistic fuzzy sets in 1986 and intuitionistic fuzzy graph in 1999. The intuitionistic graphs are handled membership and non-membership values. Vague set concept introduced by Gau and Buehrer [21] in 1993. In 2014, Akram et al. [9] discussed vague graphs and further work extended by Borzooei et al. [12, 13]. Vertex and Edge coloring of Vague graphs were introduced by Arindam Dey et al. [10] in 2018.

Neutrosophic set was introduced by F. Smarandache [25] in 1998, it's a generalization of the intuitionistic fuzzy set. It consists of membership value, indeterminacy value and non-membership value. Neutrosophic logic play a vital role in several of the real valued problems like law, medicine, industry, finance, engineering, IT, etc. Wang et al. [28] worked on Single valued neutrosophic sets in 2010. Strong Neutrosophic graph and its properties were introduced and discussed by Dhavaseelan et al. [20] in 2015 and Single valued neutrosophic concept introduced in 2016 by Akram and Shahzadi [6, 7, 8]. Broumi et al. [14, 15, 16, 17, 18] extended their works in single valued neutrosophic graphs, interval valued neutrosophic graphs (IVNG) and bipolar neutrosophic graphs. Abdel-Basset et al. used Neutrosophic concept in their papers [1, 2, 3, 4, 5] to find the decisions for some real-life operation research and IoT-based enterprises in 2019. In 2019, Jan et al. [23] have reviewed the following definitions: Interval-Valued Fuzzy Graphs (IVFG), Interval-Valued Intuitionistic Fuzzy Graphs (IVIFG), Complement of IVFG, SVNG, IVNG and the complement of SVNG and IVNG. They have modified those definitions, supported with some examples. Neutrosophic graphs happen to play a vital role in the building of neutrosophic models. Also, these graphs can be used in networking, Computer technology, Communication, Genetics, Economics, Sociology, Linguistics, etc., when the concept of indeterminacy is present.

In this research paper, the bounds of single valued neutrosophic vertex coloring for SVNG, Complement of SVNG are determined and discussed some more operations on SVNG.

Definition 1.1. [26] Let X be a space of points(objects). A neutrosophic set A in X is characterized by truth-membership function $t_A(x)$, an indeterminacy-membership function $i_A(x)$ and a falsity-membership function $f_A(x)$. The functions $t_A(x)$, $i_A(x)$, and $f_A(x)$, are real standard or non-standard subsets of $]0^-, 1^+[$. That is, $t_A(x): X \to]0^-, 1^+[$, $i_A(x): X \to]0^-, 1^+[$ and $f_A(x): X \to]0^-, 1^+[$ and $0^- \leq t_A(x) + i_A(x) + f_A(x) \leq 3^+$.

Definition 1.2. [7] A single-valued neutrosophic graphs (SVNG) G = (X, Y) is a pair where X: N → [0,1] is a single-valued neutrosophic set on N and Y: N × N → [0,1] is a single-valued neutrosophic relation on N such that

$$t_Y(xy) \leq \min\{t_X(x), t_X(y)\},$$

$$i_Y(xy) \leq \min\{i_X(x), i_X(y)\},$$

$$f_Y(xy) \leq \max\{f_X(x), f_X(y)\},$$

for all x, y ∈ N. X and Y are called the single-valued neutrosophic vertex set of G and the single-valued neutrosophic edge set of G, respectively. A single-valued neutrosophic relation Y is said to be symmetric if $t_Y(xy) = t_Y(yx)$, $i_Y(xy) = i_Y(yx)$ and $f_Y(xy) = f_Y(yx)$, for all x,y ∈ N. Single-valued neutrosophic be abbreviated here as SVN.

2. Single-Valued Neutrosophic Vertex Coloring (SVNVC)

In this section, discussed the bounds of SVNVC for the resultant SVNG by some operations on SVNG, CSVNG and complement of SVNG. Also discussed some theorems.

Definition 2.1. [24] A family $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_k\}$ of SVN fuzzy set is called a k-SVNVC of a SVNG G = (X, Y) if

1. $\vee \gamma_i(x) = X, \ \forall x \in X$
2. $\gamma_i \wedge \gamma_j = 0$

3. For every incident vertices of edge xy of G, $\min\{\gamma_i(m_1(x)), \gamma_i(m_1(y))\} = 0$, $\min\{\gamma_i(i_1(x)), \gamma_i(i_1(y))\} = 0$ *and* $\max\{\gamma_i(n_1(x)), \gamma_i(n_1(y))\} = 1, (1 \leq i \leq k)$.

This k-SVNVC of G is denoted by $\chi_v(G)$, is called the SVN chromatic number of the SVNG G.

Definition 2.2 A SVNG G = (X, Y) is called complete single-valued neutrosophic graph (CSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$
$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$
$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all x, y ∈ X.

For any single value neutrosophic subgraph H of SVNG G, $\chi_v(H) \leq \chi_v(G)$

Theorem 2.3.

For any SVNG with n vertices $\chi_v(G) \leq n$.

Proof:

By the observation that the CSVNG with n vertices has the SVNVC is n. All the other graphs with n vertices are subgraphs of the CSVNG, it is clear by the above observation. Hence $\chi_v(G) \leq n$.

Definition 2.4 Let $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$ be single-valued neutrosophic graphs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. The union G1 ∪ G2 is defined as a pair (X, Y) such that

$$t_X(x) = \begin{cases} t_{X_1}(x), & if\ x \in V_1 and\ x \notin V_2, \\ t_{X_2}(x), & if\ x \in V_2 and\ x \notin V_1, \\ \max\left(t_{X_1}(x), t_{X_2}(x)\right), & if\ x \in V_1 \cap V_2. \end{cases}$$

$$i_X(x) = \begin{cases} i_{X_1}(x), & if\ x \in V_1 and\ x \notin V_2, \\ i_{X_2}(x), & if\ x \in V_2 and\ x \notin V_1, \\ \max\left(i_{X_1}(x), i_{X_2}(x)\right), & if\ x \in V_1 \cap V_2. \end{cases}$$

$$f_X(x) = \begin{cases} f_{X_1}(x), & if\ x \in V_1 and\ x \notin V_2, \\ f_{X_2}(x), & if\ x \in V_2 and\ x \notin V_1, \\ \min\left(f_{X_1}(x), f_{X_2}(x)\right), & if\ x \in V_1 \cap V_2. \end{cases}$$

$$t_Y(xy) = \begin{cases} t_{Y_1}(xy), & if\ xy \in E_1 and\ x \notin E_2, \\ t_{Y_2}(xy), & if\ xy \in E_2 and\ x \notin E_1, \\ \max\left(t_{Y_1}(x), t_{Y_2}(x)\right), & if\ x \in E_1 \cap E_2. \end{cases}$$

$$i_Y(xy) = \begin{cases} i_{Y_1}(xy), & if\ xy \in E_1 and\ x \notin E_2, \\ i_{Y_2}(xy), & if\ xy \in E_2 and\ x \notin E_1, \\ \max\left(i_{Y_1}(x), i_{Y_2}(x)\right), & if\ x \in E_1 \cap E_2. \end{cases}$$

$$f_Y(xy) = \begin{cases} f_{Y_1}(xy), & if\ xy \in E_1 and\ x \notin E_2, \\ f_{Y_2}(xy), & if\ xy \in E_2 and\ x \notin E_1, \\ \min\left(f_{Y_1}(x), f_{Y_2}(x)\right), & if\ x \in E_1 \cap E_2. \end{cases}$$

For any SVNGs $G_1 = (X_1, Y_1)$ and $G_2 = (X_2, Y_2)$, $\chi_v(G_1 \cup G_2) = max\{\chi_v(G_1), \chi_v(G_2)\}$.

Definition 2.5 [8] The complement of a SVNG G = (X, Y) is a SVNG $\bar{G} = (\bar{X}, \bar{Y})$, where

1. $\bar{X} = X$
2. $\bar{t_X}(x) = t_X(x), \bar{i_X}(x) = i_X(x), \bar{f_X}(x) = f_X(x)\ for\ all\ x \in X$

3. $\bar{t_X}(xy) = \begin{cases} \min\{t_X(x), t_X(y)\} & if\ \ t_Y(xy) = 0 \\ \min\{t_X(x), t_X(y)\} - t_Y(xy) & if\ \ t_Y(xy) > 0 \end{cases}$

$\bar{i_X}(xy) = \begin{cases} \min\{i_X(x), i_X(y)\} & if\ \ i_Y(xy) = 0 \\ \min\{i_X(x), i_X(y)\} - i_Y(xy) & if\ \ i_Y(xy) > 0 \end{cases}$

$\bar{f_X}(xy) = \begin{cases} \max\{f_X(x), f_X(y)\} & if\ \ f_Y(xy) = 0 \\ \max\{f_X(x), f_X(y)\} - f_Y(xy) & if\ \ f_Y(xy) > 0 \end{cases}$

for all $x, y \in X$.

Theorem 2.6. For any SVNG G with n vertices, $2\sqrt{n} \leq \chi_v(G) + \chi_v(\bar{G}) \leq 2n$ and $n \leq \chi_v(G)\chi_v(\bar{G}) \leq n^2$.

Let every vertex of G has n − 1 adjacent vertices, then by the definition of complement of SVNG each vertex of $\bar{G}$ has the lesser than or equal to n − 1 adjacent vertices. Hence, the inequalities true for all SVNG. Thus, $2\sqrt{n} \leq \chi_v(G) + \chi_v(\bar{G}) \leq 2n$ and $n \leq \chi_v(G)\chi_v(\bar{G}) \leq n^2$.

Definition 2.7.

A SVNG G = (X, Y) is called strong single-valued neutrosophic graph (SSVNG) if the following conditions are satisfied:

$$t_Y(xy) = \min\{t_X(x), t_X(y)\},$$
$$i_Y(xy) = \min\{i_X(x), i_X(y)\},$$
$$f_Y(xy) = \max\{f_X(x), f_X(y)\},$$

for all (x,y) ∈ Y .

Observation 2.8

For any SSVNG G with n vertices, $2\sqrt{n} \leq \chi_v(G) + \chi_v(\bar{G}) \leq \mathrm{n} + 1$ and $n \leq \chi_v(G)\chi_v(\bar{G}) \leq (\frac{n+1}{2})^2$.

Given that G is SSVNG and the complement of G is defined by $\bar{G} = (\bar{X}, \bar{Y})$, where

1. $\bar{X} = X$
2. $\bar{t_X}(x) = t_X(x), \bar{i_X}(x) = i_X(x), \bar{f_X}(x) = f_X(x)\ for\ all\ x \in X$
3. $\bar{t_X}(xy) = \begin{cases} \min\{t_X(x), t_X(y)\} & if\ \ t_Y(xy) = 0 \\ 0 & if\ \ t_Y(xy) > 0 \end{cases}$

$\bar{i_X}(xy) = \begin{cases} \min\{i_X(x), i_X(y)\} & if\ \ i_Y(xy) = 0 \\ 0 & if\ \ i_Y(xy) > 0 \end{cases}$

$\bar{f_X}(xy) = \begin{cases} \max\{f_X(x), f_X(y)\} & if\ \ f_Y(xy) = 0 \\ 0 & if\ \ f_Y(xy) > 0 \end{cases}$

for all $x, y \in X$. Hence, the above inequalities hold.

Theorem 2.9. For a path graph P_n, $\chi_v(P_n) = 2$ where $n \geq 2$.

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be a family of SVN fuzzy sets defined on V as follows:

$$\gamma_1(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = odd \\ (0,0,1) & for\ i = even \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for\ i = even \\ (0,0,1) & for\ i = odd \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2\}$ fulfilled the conditions of SVNVC of the graph G. Hence the SVN chromatic number of P_n is $\chi_v(P_n) = 2$.

Theorem 2.10. For a cycle graph C_n, $\chi_v(C_n) = \begin{cases} 2 \ if \ n = even \\ 3 \ if \ n = odd \end{cases}$ where $n \geq 3$.

For n is odd:

Let $\Gamma = \{\gamma_1, \gamma_2, , \gamma_3\}$ be a family of SVN fuzzy sets defined on V as follows:

$$\gamma_1(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \ i = 1,3,5, \dots, n-2 \\ (0,0,1) & for \ others \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \ i = 2,4,6, \dots, n-1 \\ (0,0,1) & for \ others \end{cases}$$

$$\gamma_3(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \ i = n \\ (0,0,1) & for \ others \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2, , \gamma_3\}$ fulfilled the conditions of SVNVC of the graph G. Hence the SVN chromatic number $\chi_v(C_n) = 3$.

For n is even:

Let $\Gamma = \{\gamma_1, \gamma_2\}$ be a family of SVN fuzzy sets defined on V as follows:

$$\gamma_1(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \ i = odd \\ (0,0,1) & for \ i = even \end{cases}$$

$$\gamma_2(x_i) = \begin{cases} \big(t(x_i), i(x_i), f(x_i)\big) & for \ i = even \\ (0,0,1) & for \ i = odd \end{cases}$$

Hence the family $\Gamma = \{\gamma_1, \gamma_2\}$ fulfilled the conditions of SVNVC of the graph G. Hence the SVN chromatic number $\chi_v(C_n) = 2$.

Theorem 2.11. For any graph SVNG, $\chi_v(G) \leq \Delta(G) + 1$.

Here $\Delta(G)$ denotes the number of edges incident with a vertex of SVNG G, hence the result is true for all SVNG.

3. Conclusions

Graph Coloring is an useful technique to solve many real life problems which are easily converted as graph models. SVNG is dealt with vague and imprecise values. Single Valued Neutrosophic Coloring concept was introduced by the authors in [24]. In this paper, we discussed few more results of SVNVC using CSVNG and Complement of SVNG. We have an idea to extend the concept of SVNVC with irregular coloring and dominating coloring technique in future.

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Conflicts of Interest

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References

1. Abdel-Basset. M; Atef. A; Smarandache. F. A hybrid Neutrosophic multiple criteria group decision making approach for project selection, Cognitive System Research, 2019; 57, 216-227.
2. Abdel-Basset. M; El-hoseny. M; Gamal. A; Smarandache. F. A novel model for evaluation Hospital medical care systems based on plithogenic sets, Artificial intelligence in medicine, 2019; 100, 101710.

3. Abdel-Basset. M; Manogaran. G; Gamal. A; Smarandache. F. A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of medical systems, 2019; 43(2), 38.
4. Abdel-Basset. M; Mohamed. R; Zaied. A. E. N. H; Smarandache. F. A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics, Symmetry, 2019; 11(7), 903.
5. Abdel-Basset. M Saleh, M; Gamal. A; Smarandache. F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 2019; 77, 438-452.
6. Akram. M; Siddique. S; Davvaz. B. New concepts in neutrosophic graphs with application, J. Appl. Math. Comput., 2018, 57, 279-302.
7. Akram. M; Shahzadi. S. Neutrosophic soft graphs with application, J. Intell. Fuzzy Syst., 2017, 32(1), 841-858.
8. Akram. M; Shahzadi. S. Operations on single-valued neutrosophic graphs, J. Uncertain Syst., 2016, 11, 1-26.
9. Akram. M; Feng. F; Sarwar. S; Jun. Y. B. Certain types of vague graphs, UPB Sci. Bull. Ser.A, 2014, 76, 141-154.
10. Arindam Dey; Le Hoang Son; Kishore Kumar. P. K; Ganeshsree Selvachandran; Shio Gai Quek. New concepts on vertex and edge coloring of simple vague graphs, MDPI in Symmetry, 2018, 10(9), 373(1-18).
11. Atanassov. K. T. Intuitionistic fuzzy sets, Fuzzy sets and systems, 1986, 20, 87-96.
12. Borzooei. R. A; Rashmanlou. H. More results on vague graphs, UPB Sci. Bull. Ser.A, 2016, 78, 109-122.
13. Borzooei. R. A; Rashmanlou. H. New concepts on vague graphs, Int. J. Mach. Learn. Cybern., 2017, 8, 1081-1092.
14. Broumi. S; Talea. M; Bakali.A; Smarandache. F. Single-valued neutrosophic graphs, J. New Theory, 2016, 10, 86-101.
15. Broumi. S; Talea. M; Bakali.A; Smarandache. F. On bipolar Single valued neutrosophic graphs, J. New Theory, 2016, 11, 84-102.
16. Broumi. S; Talea. M; Bakali.A; Smarandache. F. Interval valued neutrosophic graphs, Crit. Rev., 2016, 12, 5-33.
17. Broumi. S; Talea. M; Smarandache. F; Bakali.A . Single valued neutrosophic graphs: Degree, Order and Size, 2016 IEEE Int. Conf. Fuzzy Systems(FUZZ), 2016, 2444-2451.
18. Broumi. S; Smarandache. F; Bakali. A; Mehra. S; Talea. M; Singh. M. Strong Degrees in Single valued neutrosophic graphs, K. Arai.et.al(EDS): FICC 2018, AISC, 2019, 886, 221-238.
19. Changiz Eslahchi; Onagh. B. N. Vertex strength of fuzzy graphs, Int. J. Math. Mathematical Sci. 2006, 1-9.
20. Dhavaseelan. R; Vikramaprasad. R; Krishnaraj. V Certain types of neutrosophic graphs, Int. J. Math. Sci. Appl., 2015, 5(2), 333-339.
21. Gau. W. L; Buehrer. D. J. Vague sets, IEEE Trans. Syst. Man Cybern. 1993, 23, 610-614.
22. Harary. F. Graph Theory, Narosa Publishing home, New Delhi 1969.
23. Jan. N; Ullah. K; Mahmood. T; Garg. H; Davvaz. B; Saeid. A. B; Broumi. S. Some root level modifications in interval valued fuzzy graphs and their generalizations including neutrosophic graphs, Mathematics, 2019, 7(1), 7010072(1-22).
24. Rohini. A; Venkatachalam. M; Broumi. S; Smarandache. F. Single valued neutrosophic coloring, (Accepted).

25. Rosenfeld. A. Fuzzy graphs, Fuzzy sets and their applications, Academic Press, New York, 1975.
26. Smarandache. F. Neutrosophy: Neutrosophic Probability, Set and logic, Ann Arbor, Michigan, USA, 2002, 105.
27. Susana Munoz; Teresa Ortuno. M; Javier Ramirez; Javier Yanez. Coloring fuzzy graphs, Omega, 2005, 33(3), 211-221.
28. Wang. H; Smarandache. F; Zhang. Y; Sunderrama. R. Single-valued neutrosophic sets, Multisspace Multiscriot, 2010, 4, 410-413.
29. Zadeh. L. A. Fuzzy sets, Information Control, 1965, 8, 338-353.

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Neutrosophic Fuzzy Hierarchical Clustering for Dengue Analysis in Sri Lanka

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Abstract: In the structure of nature, we believe that there is an underlying knowledge in all the phenomena we wish to understand. Mainly in the area of epidemiology we often tend to seek the structure of the data obtained, pattern of the disease, nature or cause of its emergence among living organisms. Sometimes, we could see the outbreak of disease is ambiguous and the exact cause of the disease is unknown. A significant number of algorithms and methods are available for clustering disease data. We could see that literature has no traces of including indeterminacy or vagueness in data which has to be much concentrated in epidemiological field. This study analyzes the attack of dengue in 26 districts of Sri Lanka for the period of seven years from 2012 to 2018. Clusters with low risk, medium risk and high risk areas affected by dengue are identified. In this paper, we propose a new algorithm called Neutrosophic-Fuzzy Hierarchical Clustering algorithm (NFHC) that includes indeterminacy. Proposed algorithm is compared with fuzzy hierarchical clustering algorithm and hierarchical clustering algorithm. Finally the results are evaluated with the benchmarking indexes and the performance of the clustering algorithm is studied. NFHC has performed a way better than the other two algorithms.

Keywords: Dengue; Hierarchical clustering; Fuzzy hierarchical clustering; Neutrosophic Logic

1. Introduction

Emerging and re-emerging infectious diseases which are transmitted to the environment is a great threat to human living. The infections can take many forms and it can seriously affect human health. Dengue is one among the disease which causes severe outbreaks in many regions of the world. Its prevalence, incidence and geographic distribution are demanding a divisive applicable plan for control measures against dengue fever. In this case the complete structure of data and regions affected by dengue has to be known. Many situations exist that the ambiguity arises in finding a solution to the problem. Clustering and Classification are the most commonly encountered knowledge-discovery technique. Clustering is used in numerous applications such as disease detection, market analysis, medical diagnosis etc. The study concentrates on Sri Lankan dengue data analysis. Dengue fever occurs in the background of heavy rain and flooding and has affected almost26 districts in Sri Lanka. The country has reported 51659 cases in the year 2018 and approximately 41.2 % cases identified in western province alone[1]. In Pakistan, dengue has progressed towards becoming a risk for general wellbeing because of inaccessibility of vaccination, unclean water, highly populated territories and low quality of sanitation and sewage [2]. There have been a number of researches done on dengue fever diagnosis and numerous methods have been proposed using classification and clustering techniques for dengue analysis. G.P.Silveria proposed evolution technique of dengue risk analysis or prediction using the model Takagi-Sugeno. Takagi-Sugeno model included parameters such as human population density, density of potential mosquito breeding and rainfall. The fuzzy rules were developed using partial differential equations for Low, Medium and High dengue affected areas. The uncertainty factor considered in this study is the breeding period and the maturation of mosquito eggs and Silveria considered rainfall as a factor for the increase or decrease in the population of mosquitoes [3]. The selection of Neutrosophic approach has increased in group decision making in vague decision environment. Neutrosophic approach with Technique for Order Preference by Similarity to Ideal Solution (TOPSIS)[4] is considered for decision making process to deal with the vagueness and uncertainty by considering the data for the decision criteria. Neutrosophic environment provides a new technique in Multi Criteria Decision Making problem. Author Abdel-Basset M [5], has developed and integrated Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) into Decision-Making Trial and Evaluation Laboratory (DEMATEL) on a neutrosophic set that handles to overcome the ambiguity or the lack of information. He has applied on project selection criteria where the best alternatives are provided by the neutrosophic approach.

This paper mainly focuses on the finding of Dengue affected areas using the clustering technique found. The clusters are formed as low risk, medium risk and high risk areas. It helps the public sectors to concentrate particularly on that area for the remedial measures that are to be considering for the wellbeing of the society. Based on the neutrosophic approach, the clustering for the low risk, medium risk and high risk areas are identified and clustered.

2. Related Work

The ambiguity or uncertainty representation or handling of incomplete knowledge becomes a vital problem in the field of computer science. Researchers from various fields have dealt with vague, indeterminate, imprecise and sometimes insufficient information of uncertain data. The concept of uncertainty is usually handled by probabilistic approach. Soft computing techniques also deals with these problems such as called fuzzy sets [6] and intuitionistic fuzzy sets [7] and rough sets. Fuzzy logic is a collection of mathematical values for representing and understanding is based on membership degrees rather than the crisp membership of traditional binary logic. It leads to more human intelligent machines as fuzzy logic tries to model the human feeling of words, decision-making and common sense[8].

Unlike Boolean's two-valued logic, Fuzzy logic is multi-valued logic. Matrices play an important role in representation of the real world problems of science and engineering. Therefore, a few authors have proposed a matrix representation of fuzzy sets and intuitionistic fuzzy sets [9,10,11,12,13,14,15,16,17]. Fuzzy set and Intuitionistic Fuzzy Set deals with the membership and non-membership values. Membership value shows the truthiness of the algorithm which is classified or clustered. Non-membership values show the falsity of the data that it doesn't belong to that class.

For some reasons, the calculation of non-membership value is not always possible as in the case of membership values. So, there exists some indeterministic that part depicts the ambiguity in fuzzy logic. Subsequently, Smarandache [18, 19] introduced the term Neutrosophic Set (NS), which is formed as a generalization of classical set, fuzzy set, intuitionistic fuzzy set. The literature [20-24] shows the growth of decision-making algorithms over neutrosophical set theory.

Neutrosophic logic that shows the clear separation between the" relative truth" and" absolute truth" while the fuzzy logic does not show any separation. Smarandache Florentine proposed the concept of neutrosophic logic based on nonstandard analysis by Abraham Robinson in 1960s. Generally, we can say that the available disease information in inherently unclear and unpredictable. In real life issues, an element of indeterminacy exists and in this respect, neutrosophic logic can be used. Neutrosophic logic generalizes fuzzy, intuitive, boolean, para-consistent logic etc.

In many medical diagnosis and study of diseases, the indeterminacy or falsity in the input is not captured so far. It is seen from the literature that the concept of neutrosophic logic is not applied much on medical diagnosis. Neutrosophic clustering technique is neither employed nor applied to any medical applications. Some of the applications of neutrosophic logic are Social Network Analysis, Financial Market Information, Neutrosophic Security, Neutrosophic cognitive maps, Application to Robotics etc.

2.1 Machine Learning on Dengue

Many authors have concentrated on Machine Learning algorithms for classification and prediction of various diseases. In over 100 nations, dengue is endemic and causes an estimated 50 million infections per year. Nearly 3.97 billion individuals are at danger of infection from 128 nations [25]. Machine Learning algorithms such as Regression Models, Decision Tree, Artificial Neural Network, Rough Set Theory, Support Vector Machine etc. are successfully applied [26]. Daranee Thitiprayoonwongse et al proposed a hybrid technique combining a decision-making tree with a fuzzy logic approach to constructing a model for dengue infection. Author obtained a set of rules from decision tree and transformed to fuzzy rules. The results were better by combining fuzzy and decision tree approaches [27]. Torra [28], has proposed a fuzzy hierarchical clustering for representing the documents. Fuzzy hierarchical clusters are used in order to assure that the clusters are small enough by giving low information loss.

This research mainly focuses on clustering of Dengue disease in various parts of Sri Lanka. Increased risk to infectious diseases was recognized as one of five main emerging threats to public health resulting from the changes in the natural environment [29]. Diseases caused by mosquitoes are a specific danger to humans. The danger of transmission relies on climate variables that regulate mosquito habitat development [30-32]. This paper discusses the possibilities to exploit neutrosophic logic in epidemiology domain. In many cases, the representational parameters which include temperature and humidity as mentioned by [30-32] the climatic variables could also be a part in spread of disease. Most of the cases are rare that all the external parameters are considered, which leads to a chaos about conclusion to be drawn.

So the developed system should adapt to the conditions that are uncontrollable or unanticipated. In this case indeterminacy plays an important role. The concept of indeterminacy is handled or explained in a improvised way by neutrosophic logic. A better approach for all the above is Neutrosophic logic.

3. Proposed Work

Clustering can be seen as an practical problem in pattern recognition in unsupervised learning. Problems can be size of dataset, number of clusters to be formed, there is no ground truth solution unlike classification problems. The goal is to partition the data set into a certain number of natural and homogeneous sets where each set's elements is as similar as possible and different from the other sets. In real world applications, cluster separation is a fuzzy concept and therefore the idea of fuzzy subsets provides particular benefits over standard clustering [33]. This research proposes a hybridized technique for hierarchical clustering by amalgamation of fuzzy and neutrosophic approach. There by, the proposed algorithm gains the benefits of addressing imprecise, indeterministic, vague and uncertain data.

3.1. Hierarchical Clustering (HC)

In the process of hierarchical clustering, a distance matrix (D) is constructed where; d_{ij} is the distance between the cities. During clustering, i^{th} and j^{th} locations are merged into a cluster and distance matrix is updated. Eventually, the cities are merged based on the similarity measure and the dimension of D gets reduced on every step of merging. Hierarchical clustering is categorized based on the method of merging. It includes Single, Complete, Average, Centroid, Median and Ward. Merging clusters based on minimum distance between each element is called single linkage clustering. Clustering based on maximum distance between each element is complete linkage clustering, clustering the mean distance between each element is average linkage clustering, clustering is done by mean values of one group with the mean values on other group elements is centroid clustering. To overcome the disadvantage of centroid method the median of two groups are clustered is called median linkage clustering. Median linkage clustering is suitable for both similarity and distance measures. Wards method calculates the sum of the squares of the distance between the elements P_i and P_j, where P_i and P_j are the location of the elements in i^{th} and j^{th} positions.

The distance matrix is formed by using the Euclidean equation. Single, complete and average link are defined by the way of merging the cities based on nearest, farthest and average distance respectively.

$$d_{ij} = \sqrt{\sum_{k=1}^{n} (x_{ik} - x_{jk})^2} \tag{3.1}$$

Where i,j are the location of cities and n, k are the number of cities.

Distance matrix here with dimension of 26×26 is formed. It is constructed on the basis of equation 3.1.Once the distance matrix is formed and based upon the method of hierarchical clustering, clusters are generated.

3.2. Fuzzy Hierarchical Clustering(FHC)

Given a set of objects, a fuzzy hierarchical framework has been implemented to construct clusters. The methodbegins to establish a fuzzy partition that uses the membership formula[34]. The membership matrix is calculatedusing the equation 3.2 which gives distance between each of the object, here it represents the cities.

$$\mu_{ik} = \left[\sum_{j=1}^{n} \left(\frac{d_{ik}}{d_{jk}} \right)^{2/m^{`}-1} \right]^{-1} \tag{3.2}$$

where n is the number of locations, m is the weighting parameter or fuzzifier, r is the number of iterations used for convergence. There is no theoretical optimumchoice of m in literature. The range is usually between 1.25 - 2 [35] and here we have choosen value 2. Theinitial membership matrix(μ) is formed using equation (3.2). We have formed a fuzzy measure for objects.Here one object can belong to various clusters with the varying membership values ranging from 0 to 1. Valuesfalling between these endpoints (from low toextremely favorable clustering) were mapped as membershipdegrees. The non-membership value also called as falsity value, represented as $\vee$ [36]. It is calculated using thefollowing equation,

$$\vee_i = \frac{1-\mu_i}{1+\lambda\mu_i} \tag{3.3}$$

where, λ is the weighted parameter value ranging from 0 to 1. Here the value of λ is taken as 0.8.

3.3. Neutrosophic Fuzzy Hierarchical Clustering(NFHC)

The notion of a neutrosophical set was initially proposed by Smarandache [37]. A neutrosophical set A isdefined by a universal set X with truth-membership function T_A, a falsity-membership function F_A and anindeterminacy-membership function I_A. Here,$T_A(x)$,$F_A(x)$ and

$I_A(x)$ are the real standard sets of values]0; 1⁺[, i.e., $T_A(x)$: X $\rightarrow$]0; 1⁺[, $I_A(x)$: X $\rightarrow$]0; 1+[, and $F_A(x)$:X $\rightarrow$]0; 1+[. The indeterminancy-value whichis also denoted by π is given by,

$$\pi_i = 1 - \mu_i - \vee_i = \frac{1-\mu_i}{1+\lambda\mu_i} (or) \pi_i = 1 - \mu_i - \vee_i \tag{3.4}$$

From equation (3.2),(3.3) and (3.4), a neutrosophic triplet matrix is obtained. Table 2A shows a sample tripletmatrix. Before performing clustering, triplet matrix (μ, π, ∨) [38] is converted into scalar value matrix using normalized hamming distance. The normalized hamming distance [39] between two locations P and Q is defined

$$N_d(P,Q) = \frac{1}{3n}\sum_{i=1}^{n}(\left|T_P(w_i - T_Q(w_i)\right| + \left|F_P(w_i - F_Q(w_i)\right| + \left|I_P(w_i - I_Q(w_i)\right|) \tag{3.5}$$

To perform the clustering part. the triplet matrix is converted into a scalar value using equation (3.5)[40]. The neutrosophic weights of a triplet matrix is converted into scalar weights. The resultant matrix is aneutrosophic matrix and HC is applied for clustering, there by we get a neutrosophic fuzzy clusters.

The dataset consists of dengue reported cases in 26 cities of Sri Lanka. Data is collected for six consecutiveyears from 2012 to 2018. First step is finding out the diatnce matrix (D) using the equation (3.1). The matrixformed here is 26×26 as distance matrix. Using equation (3.2), (3.3) and (3.4) triplet matrix of (μ, π, ∨) iscalculated. By using equation (3.5) the neutrosophic triplet matrix is converted to function matrix with scalarvalue upon which hierarchical clustering is formed. Example of the membership matrix obtained for different years. The representation for the year 2012 is given in table 1A.

We then perform the process of hierarchical clustering using algorithm 1, for the results diaplayed in table1A. HC is applied on each year and clusters are formed for each consecutive year from 2012 to 2018. HC hasdifferent methods such as single, complete, wighted, centroid, median and ward.

Algorithm 1: Hierarchical Clustering($N_d(P,Q)$, Method=single linkage)

```
begin
mat[][] ← initialized to N_d(P,Q) values from equation 3.5
disjoint set=[][]
for each city_i in mat[][] do
    for each city_j in mat[][] do
        n=min(max(city_i, city_j))
        merge(city_i.city_j)
    end
    repeat until single cluster
end
```

Algorithm 2: Hierarchical Clustering($N_d(P, Q)$, Method=complete linkage)

```
begin
mat[][] ← initialized to N_d(P, Q) values from equation 3.5
disjoint set=[][]
for each city_i in mat[][] do
    for each city_j in mat[][] do
        n=max(max(city_i, city_j))
        merge(city_i.city_j)
    end
    repeat until single cluster
end
```

In the second step, the value of falsity or the non-membership is determined using the formula (3.3). The set of values in each column of the matrix represents (μ, π, $\vee$) for each location.

Finally, the neutrosophic matrix is constructed using equation (3.4). The obtained result is a triplet of the form (0.9425, 0.0752 and 0.0603). The triplet matrix expresses the truthness, falsity and indeterminacy value of each location paired with all other locations in the dataset. Similar matrix of 26×26 is obtained for all consecutive years starting from 2012 to 2018. Now find the similarity between each pair of objects in and neutrosophic triplet matrix.

The Euclidean distance matrix, membership matrix and triplet matrix is calculated using algorithm 2. The data is taken from the year 2012 to 2017 as training data. Once the algorithm is implemented, it has to be tested for its accuracy and how well the proposed algorithm works. The process is applied on data set for the year 2018 and the clusters are formed. The predicted clusters are compared with the actual data for all the 26cities. Several performance indices techniques are elaborated in section 5.

4. Dataset Descriptions

The data is collected from Epidemiology Unit Ministry of Sri Lanka. The dengue cases are collected for six consecutive years from 2012 to 2017. The data can be downloaded from thesite [41]. Data consist of 26 locations in Sri Lanka such as Colombo, Gampaha, Kalutara, Kandy, Matale, N Eliya, Galle, Hambantota, Matara, Jaffna, Kilinochchi, Mannar, Vavuniya, Mulativu, Batticaloa, Ampara, Trincomalee, Kurunegala, Puttalam, Apura, Polonnaruwa, Badulla, Moneragala, Ratnapura, Kegalle and Kalmunai.

Table 1 List of Cities in Sri Lanka

Cities	Names
1	Colombo
2	Gampaha
3	Kalutara
4	Kandy
5	Matale
6	N Eliya
7	Galle
8	Hambantota
9	Matara
10	Jaffna
11	Kilinochchi
12	Mannar
13	Vavuniya
14	Mulativu
15	Batticola

16	Ampara
17	Trincomalee
18	Kurunegala
19	Puttalam
20	Apura
21	Polonnaruwa
22	Badulla
23	Moneragala
24	Ratnapura
25	Kegalle
26	Kalmunai

Algorithm 3: Neutrosophic Fuzzy Score Calculation

matrix[loc][loc] $\leftarrow$ initialized to distance matrix for all cities in DB
city=[list of all cities]
for *i in city* **do**
    for *j in city* **do**
        $D_{[city_i][city_j]} \leftarrow$ euclidean distance$(city_i, city_j)$
    end
end
for *i in city* **do**
    for *k in i* **do**
        for *j in city* **do**
            $x = \sum_{j=1}^{n}(D[i][k]/D[j][k])^2$
            `// n number of locations`
            $\mu_{ik} = (1/x)$
        end
    end
end
for *i in city* **do**
    for *j in city* **do**
        calculate $\vee(city_{i,j})$ using equation (3.3)
        // $\vee$ is non membership value
        calculate $\pi(city_{i,j})$ using equation (3.4)
        // π is indeterminacy value
    end
end
for *i in city* **do**
    for *j in city* **do**
        $N_d(P,Q) = \frac{1}{3n}\sum_{i=1}^{n}(\mid T_P(w_i) - T_Q(w_i)\mid + \mid T_P(w_i) - T_Q(w_i)\mid + \mid T_P(w_i) - T_Q(w_i)\mid)$
            `// hamming distance formula`
        // $N_d(P,Q)$ resultant scalar matrix
    end
end
Perform Hierarchical Clustering on $(N_d(P,Q)$, method)
`// method = (single, complete, centroid, median, ward)`
`// Perform algorithm 2 for HC`

5. Experimental Results

5.1. Inconsistency Coefficient

The relative consistency of each link in a formed hierarchical cluster is quantified as inconsistency coefficient. When the links are more consistent, the neighboring links have approximately same length. Inconsistency coefficient of each link compares its height with the average height of other links from the same level of hierarchy. When the links have larger the coefficient there exists greater the difference between the objects connected by the link. When the difference between the link values is very small, it is difficult to make conclusions. Hence higher the inconsistency gives better clustering. Inconsistency value for different links is tabulated in Table 2.

Considering the results from table 2, the maximum difference between the links in neutrosophic fuzzy hierarchical clustering is identified. When the tree is cut at maximum linkage, the resulting clusters are found to be three clusters. The number of clusters is identified using inconsistency coefficient. With the inconsistency value and the number of cluster, data is divided into three parts such as low risk, medium risk and highly affected dengue areas in Sri Lanka. Neutrosophic Fuzzy Hierarchical Clustering has shown highest inconsistent values such as **0.9168**, **0.8714**, **0.7721, 0.7428** and **0.7216** for single linkage clustering, complete linkage clustering, centroid, median and ward method respectively. The results are better in a way as NFHC has given the maximum distance between the links compared with other two techniques.

Table 2. Inconsistency Coefficient of a tree cut in Hierarchical Clustering.

	Cluster Link	**Single**	**Complete**	**Centroid**	**Median**	**Ward**
HC	I-2 links	0.7071	0.7083	0.6931	0.6682	0.6581
HC	I-3 links	**0.8913**	**0.9078**	**0.8691**	**0.7671**	**0.7891**
HC	I-4 links	0.6247	0.6901	0.5926	0.6347	0.6874
FHC	I-2 links	0.7629	0.7145	0.7526	0.6921	0.7021
FHC	I-3 links	**0.8970**	**0.8825**	**0.8191**	**0.7421**	**0.7334**
FHC	I-4 links	0.5236	0.6971	0.5626	0.6477	0.6792
NHFC	I-2 links	0.7461	0.7971	0.7526	0.7126	0.6986
NHFC	I-3 links	**0.9168**	**0.8714**	**0.7721**	**0.7428**	**0.7126**
NHFC	I-4 links	0.6326	0.5910	0.6812	0.6809	0.6574

Figure 1 depicts NFHC clustering applied on dataset for the year 2018. The value in the x-axis represents the cities and y-axis represents the tree cut. Figure 1 is visualized in shape map of Sri Lanka. Based on the inconsistency-coefficient the tree is cut into three clusters. Clustering for the year 2012-2018 is given in figure 3. It has shown effective clustering based on the performance indices explained in section 5.2.

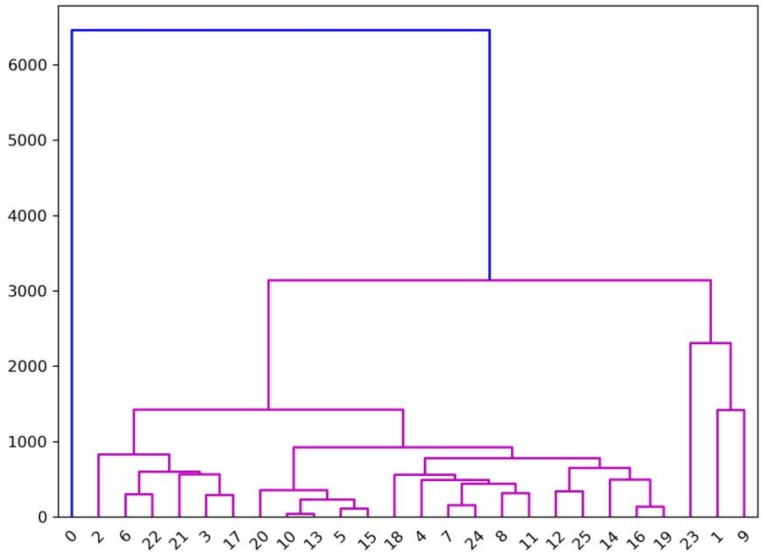


Figure 1: Dendrogram representation of NFHC on dengue data for year 2018

5.2. Performance Indices

Performance indices are used to assess clustering algorithms performance. The literature contains several performance indices. The Silhouette Coefficient [42], Davis-Bouldin (DB) index [43] and Dunn (D) index [44] are some of the most popular indicators of effectiveness assessment.

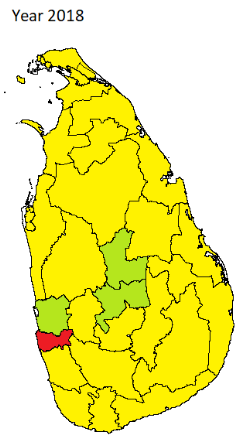


Figure 2: NFHC Cluster Visualization for Year 2018, Green-low risk, Yellow-medium risk, Red-high risk.

5.2.1. Silhouette Coefficient

Silhouettee index is an index of cluster validity used to evaluate the performance of any cluster. An element'ssilhouette index describes its proximity to its own cluster with its proximity to other clusters. A clusters silhouette width s(x) is described as,

$$s(x) = b(x) - \frac{a(x)}{\max[b(x), a(x)]} \tag{5.1}$$

where, a(x) and b(x) are the similarities of the clusters. The average silhouette width of all clusters is the silhouette index of the entire clustering. Silhouette index is used to indicate the compactness and segregation of clusters. The silhouette index value ranges from -1 to 1 and a better clustering outcome is indicated by its greater values. The silhouette coefficient of neutrosophic fuzzy hierarchical clustering is high with the value of **0.7163**, stating that the performance of Neutrosophic fuzzy hierarchical clustering is better than hierarchical clustering and fuzzy hierarchical clustering with the score of 0.6782 and 0.5137 respectively.

5.2.2. Davis-Bouldin (DB) index

The DB index is described as the cluster-to-cluster distance proportion of the amount of data. It is formulated in the following way,

$$DB = \frac{1}{c}\sum_{i-1}^{c} \max_{k \neq i}\left\{\frac{s(v_i)+s(v_k)}{d(v_i, v_k)}\right\} for 1 < i, k < c \tag{5.2}$$

The DB index seeks at minimizing cluster separation and maximizing cluster distance. The lower the DB index shows effective clustering. Our proposed algorithm Neutrosophic fuzzy hierarchical clustering has shown the lowest DB-index value of **2.5725** for the method of Single linkage clustering. Proposed algorithm has shown better results when compared to traditional algorithms. Experiment also reveals that fuzzy hierarchical clustering also performs better than traditional hierarchical clustering. However NFHC outperforms all.

5.2.3. Dunn (D) index

The D index is used to define clusters that are compact and separate. The calculation is as follows,

$$Dunn = \min_i \left\{ \min_{k \neq i} \left\{ \frac{d(v_i, v_{k)}}{\max_l s(v_l)} \right\} \right\} for 1 < k, i, l < c \tag{5.3}$$

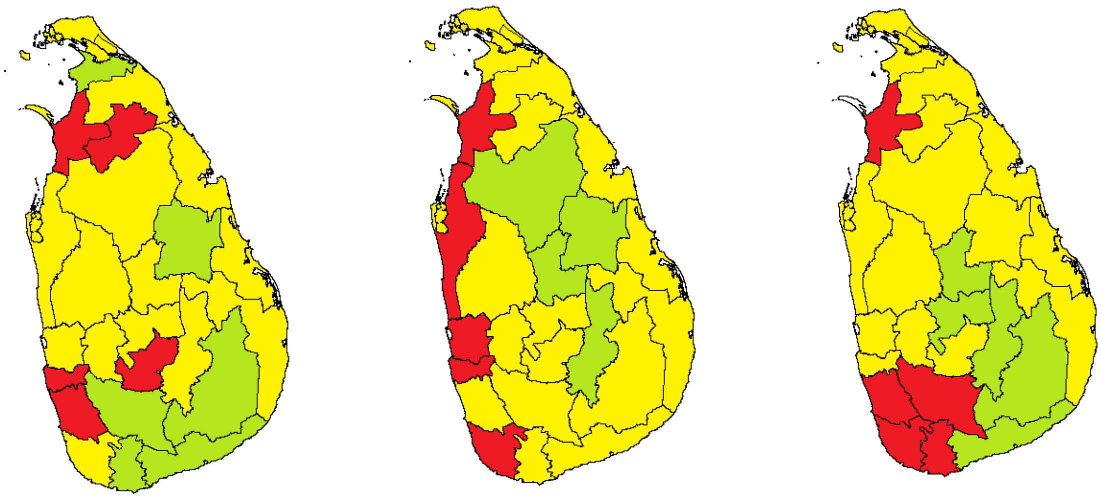
Dunn index's objective is to maximize the distance between the clusters and minimize the distance within the cluster. An elevated D index therefore means better clustering. In our implementation, highest Dunn index is achieved for NFHC algorithm with the number **1.159** of highest among all other methods. It has shown better clustering compared to other algorithms.

Table 3. Performance Metrics of HC, FHC, NFHC

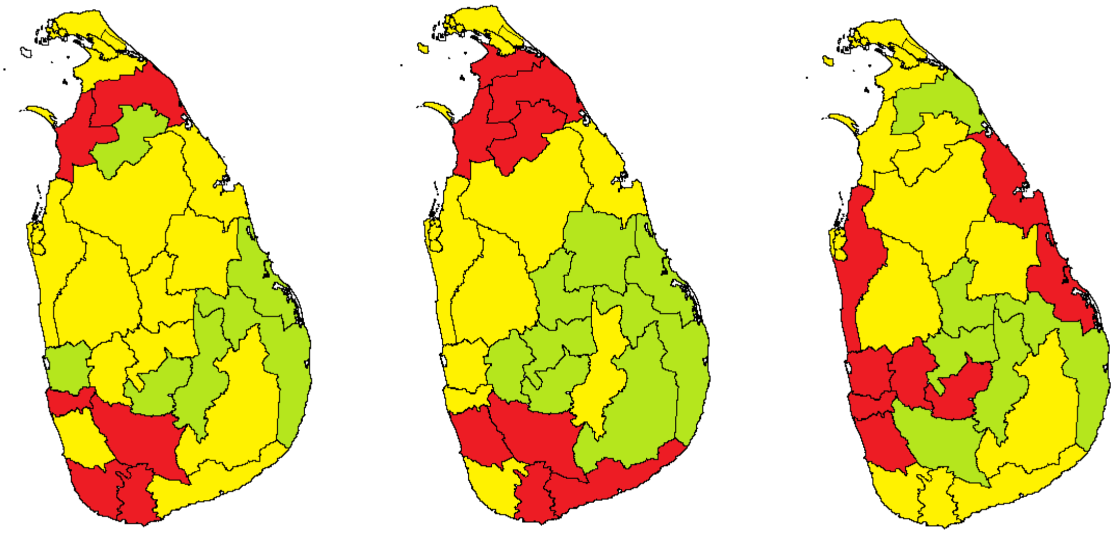
	Method	Clustering		
		HC	FHC	NHFC
Silhouette Coefficient	Single	0.1263	0.6782	**0.7163**
	Complete	0.2455	0.5763	**0.6911**
	Centroid	0.4726	0.5922	**0.6729**
	Median	0.5137	0.5501	**0.6905**
	Ward	0.4968	0.4328	**0.7077**
DB - Index	Single	5.2637	3.4266	**2.5725**
	Complete	4.1258	**2.4611**	2.4627
	Centroid	4.2162	3.1249	**2.6674**
	Median	4.5018	3.6791	**2.0169**
	Ward	4.8679	3.0628	**2.4209**
Dunn Index	Single	0.5671	0.8241	**1.134**
	Complete	0.7744	0.7689	**1.021**
	Centroid	0.8671	0.7749	**1.159**
	Median	0.9632	0.9621	**1.067**
	Ward	0.8940	0.8017	**1.116**

From table 3, we can infer that, the cluster validation of neutrosophic fuzzy hierarchical clustering has shown better results compared with hierarchical clustering and fuzzy hierarchical clustering. The metrics such as silhouette coefficient, DB index and Dunn index states the excellence of thee proposed model. The best values of silhouette cluster analysis is found in NFHC with 0.7163 for single link, 0.6911 for complete link, 0.6729 for centroid method, 0.6905 for median method and 0.7077 in ward method. Silhouette coefficient has shown highest results in NFHC for all 5 methods. DB index has also produced effective results in cluster analysis of NFHC. The lowest value of DB index is centroid method of NFHC with the value 2.6674 whereHC and FHC values for centroid method are 4.2162 and 3.1249 respectively. Other methods such as single,complete, median and ward has also given lowest values on NFHC comparing with FHC and traditional HC.Though DB index of complete method is good in FHC. FHC is also comparatively good when compared with traditional HC, as it has produced effective clustering that HC. Highest recorded Dunn index value is 1.159, for the method of centroid in NFHC. Final inference from NFHC is, it is giving better results on all the methods of clustering such as single, complete, centroid, median and ward when compared with same method on fuzzy hierarchical clustering and hierarchical clustering.

It is evident from the table 3, that the proposed NFHC shows its superiority in its performance compared to other methods. Though the fuzzy hierarchical clustering has considered membership value for clustering and produced better clusters compared with HC clusters, NFHC outperforms the fuzzy results. Thus, proposed NFHC is better in a way as it handles or capable of handling any data even with indeterminacy or inconsistency.



(a) Year 2012 (b) Year 2013 (c) Year 2014



(d) Year 2015 (e) Year 2016 (f) Year 2017

Figure 3: Cluster Plot for NFHC, color depicts Green-low risk, Yellow-medium risk, Red-high risk.

The visualization part in figure 3 clearly says that, the city of Colombo was in high risk area over the past seven years. The trend in Colombo city reveals that it is always in high risk area of dengue. In the year 2018, Colombo is the only highly affected area compared to all other cities in Sri Lanka. If the trend continues, the life of people at Colombo is in great threat. Looking into the cities in the middle of Sri Lanka such as Polonnaruwa, Matale, Polonnaruwa, Trincomalee and Kandy they have crossed the threshold of being in low risk area to medium risk area. This depicts that the states are gradually increasing in its dengue admissions. It is an important issue to be noted by the government, as in future these cities are in high risk of getting into a danger zone of dengue. Considering the southern cities of Sri Lanka, in the year 2012 the number of dengue cases was low. Over the five consecutive years it has shown the mixed results of being in medium and highly affected area. In the area of south, the control measures have to be taken strongly for cutting down the growth of dengue fever. The major pattern that is observed from the year 2012 to 2018 is that, none of the cities had reduced from reporting the dengue cases. It has always increased from one level to next level showing the spread of dengue in a drastic manner.

6. Conclusions

The study mainly identifies the areas that are affected dengue fever. Though many studies have touched the concept of clustering, the area of indeterminacy in clustering for the field of epidemiology is still under research. We used neutrosophic fuzzy hierarchical clustering and fuzzy hierarchical clustering in this article to cluster dengue fever in Sri Lanka. The purpose of neutrosophic fuzzy is, it can handle the indeterminate and inconsistent information where the fuzzy fails to handles that information. Cluster validation metrics has given better results in neutrosophic fuzzy hierarchical clustering than the other two algorithms of fuzzy hierarchical clustering and hierarchical clustering. Some of the findings from this study is that, Colombo is identified as highest dengue affected area, many of the cities are in the peak of threshold that it can move to the danger zone at any point of time. Re-emerging areas such as Galle, Matara, Hambantota, Ratnapura and Badulla are to be concentrated more so that the pattern of occurrence can be controlled in future. This method can be used in other fields so that the break out of any disease can be avoided earlier. In future, the algorithm can be extended for monitoring other diseases that are affected by environmental and climatic variables. This model can also be extended as multi-criteria model for identifying the outbreak of hotspots and early warning systems.

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Conflicts of Interest: The authors declare no conflict of interest.

Appendix A

The following matrices contain the supplementary data for the experimental work carried out. The data is given for the year 2012.

Table A1 (a) represents Membership matrix (μ) for the cities C_1 to C_{14} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
C_1	0	0.5261	0.5423	0.6631	0.6217	0.8431	0.7456	0.4675	0.7634	0.7124	0.6419	0.6787	0.7123	0.6912
C_2	0.5261	0	0.4571	0.5863	0.2413	0.7512	0.6674	0.5931	0.7213	0.8012	0.7632	0.2745	0.5481	0.8456
C_3	0.5423	0.4571	0	0.7512	0.6942	0.4623	0.7561	0.5001	0.6417	0.7812	0.4123	0.8436	0.9845	0.1664
C_4	0.6631	0.5863	0.7512	0	0.8412	0.5679	0.4987	0.6782	0.6034	0.5846	0.3699	0.7415	0.5769	0.8462
C_5	0.6217	0.2413	0.6942	0.8412	0	0.7135	0.5671	0.6746	0.5237	0.5713	0.5712	0.6716	0.9412	0.6565
C_6	0.8431	0.7512	0.4623	0.5679	0.7135	0	0.5172	0.4872	0.5716	0.4872	0.6742	0.4369	0.2145	0.7956
C_7	0.7456	0.6674	0.7561	0.4987	0.5671	0.5172	0	0.6813	0.4213	0.5716	0.7416	0.5716	0.6715	0.6135
C_8	0.4675	0.5931	0.5001	0.6782	0.6746	0.4872	0.6813	0	0.6148	0.5127	0.4137	0.8413	0.8422	0.8436
C_9	0.7634	0.7213	0.6417	0.6034	0.5237	0.5716	0.4213	0.6148	0	0.4219	0.5166	0.7168	0.6479	0.4696
C_{10}	0.7124	0.8012	0.7812	0.5846	0.5713	0.4872	0.5716	0.5127	0.4219	0	0.5712	0.6741	0.9145	0.6713
C_{11}	0.6419	0.7632	0.4123	0.3699	0.5712	0.6742	0.7416	0.4137	0.5166	0.5712	0	0.4193	0.4785	0.6971
C_{12}	0.6787	0.2745	0.8436	0.7415	0.6716	0.4369	0.5716	0.8413	0.7168	0.6741	0.4193	0	0.5136	0.8435
C_{13}	0.7123	0.5481	0.9845	0.5769	0.9412	0.2145	0.6715	0.8422	0.6479	0.9145	0.4785	0.5136	0	0.3469
C_{14}	0.6912	0.8456	0.1664	0.8462	0.6565	0.7956	0.6135	0.8436	0.4696	0.6713	0.6971	0.8435	0.3469	0

Table A1 (b) represents Membership matrix (μ) for the cities C_{15} to C_{26} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
C_{15}	0.5197	0.5966	0.5523	0.8425	0.6656	0.8626	0.5946	0.6816	0.3266	0.3247	0.7486	0.9462	0.5653	0.6556
C_{16}	0.4128	0.4956	0.6595	0.5656	0.9463	0.2176	0.8956	0.6867	0.9562	0.7416	0.9512	0.6821	0.5185	0.5251
C_{17}	0.7946	0.6596	0.2648	0.8746	0.6941	0.1623	0.5952	0.7856	0.7953	0.9451	0.5623	0.1265	0.5659	0.7566
C_{18}	0.6843	0.3266	0.1654	0.6957	0.8946	0.7162	0.3266	0.2185	0.3256	0.1966	0.7152	0.3956	0.6748	0.7465
C_{19}	0.7069	0.8951	0.3261	0.2154	0.1595	0.5451	0.5482	0.1782	0.6816	0.4845	0.7185	0.3497	0.6494	0.4896
C_{20}	0.8431	0.2546	0.3665	0.5955	0.8685	0.1656	0.6595	0.8466	0.4863	0.7566	0.8465	0.6645	0.5867	0.7451
C_{21}	0.7629	0.1655	0.1796	0.6456	0.8562	0.7161	0.6845	0.7136	0.6416	0.4986	0.7856	0.7565	0.3516	0.7413
C_{22}	0.5527	0.4652	0.7656	0.5966	0.7163	0.6145	0.5164	0.5651	0.4516	0.7166	0.6146	0.3556	0.3888	0.7463
C_{23}	0.6237	0.8455	0.5965	0.7465	0.9461	0.6858	0.7465	0.8592	0.4566	0.2156	0.3562	0.4532	0.5666	0.4857
C_{24}	0.5179	0.8665	0.5165	0.6266	0.5169	0.5996	0.3566	0.7415	0.4566	0.6856	0.7164	0.5645	0.5959	0.5165
C_{25}	0.5873	0.4865	0.8698	0.7495	0.9561	0.6515	0.5795	0.5167	0.7866	0.3595	0.2186	0.8465	0.6585	0.4812
C_{26}	0.5766	0.8455	0.5356	0.5486	0.6715	0.6123	0.7155	0.4189	0.6589	0.3658	0.7529	0.6485	0.5568	0.6745

Table A1 (c) represents Membership matrix (μ) for the cities C_1 to C_{14} from Table 1 in section 4.

	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_1	0.5197	0.4128	0.7946	0.6843	0.7069	0.8431	0.7629	0.5527	0.6237	0.5179	0.5873	0.5766
C_2	0.5966	0.4956	0.6596	0.3266	0.8951	0.2546	0.1655	0.4652	0.8455	0.8665	0.4865	0.8455
C_3	0.5523	0.6595	0.2648	0.1654	0.3261	0.3665	0.1796	0.7656	0.5965	0.5165	0.8698	0.5356
C_4	0.8425	0.5656	0.8746	0.6957	0.2154	0.5955	0.6456	0.5966	0.7465	0.6266	0.7495	0.5486
C_5	0.6656	0.9463	0.6941	0.8946	0.1595	0.8685	0.8562	0.7163	0.9461	0.5169	0.9561	0.6715
C_6	0.8626	0.2176	0.1623	0.7162	0.5451	0.1656	0.7161	0.6145	0.6858	0.5996	0.6515	0.6123
C_7	0.5946	0.8956	0.5952	0.3266	0.5482	0.6595	0.6845	0.5164	0.7465	0.3566	0.5795	0.7155
C_8	0.6816	0.6867	0.7856	0.2185	0.1782	0.8466	0.7136	0.5651	0.8592	0.7415	0.5167	0.4189
C_9	0.3266	0.9562	0.7953	0.3256	0.6816	0.4863	0.6416	0.4561	0.4566	0.4566	0.7866	0.6589
C_{10}	0.3247	0.7416	0.9451	0.1966	0.4845	0.7566	0.4986	0.7166	0.2156	0.6856	0.3595	0.3658
C_{11}	0.7486	0.9512	0.5623	0.7152	0.7185	0.8465	0.7856	0.6146	0.3562	0.7164	0.2186	0.7529
C_{12}	0.9462	0.6821	0.1265	0.3956	0.3497	0.6645	0.7565	0.3556	0.4532	0.5645	0.8465	0.6485
C_{13}	0.5653	0.5185	0.5659	0.6748	0.6494	0.5867	0.3516	0.3888	0.5666	0.5959	0.6585	0.5568
C_{14}	0.6556	0.5251	0.7566	0.7465	0.4896	0.7451	0.7413	0.7463	0.4857	0.5165	0.4812	0.6745

Table A1 (d) represents Membership matrix (μ) for the cities C_{15} to C_{26} from Table 1 in section 4.

	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_{15}	0	0.4657	0.6289	0.6465	0.6594	0.8556	0.5162	0.3589	0.9415	0.4565	0.8465	0.7456
C_{16}	0.4657	0	0.8956	0.7441	0.8949	0.3598	0.5716	0.5635	0.4945	0.9452	0.9515	0.9512
C_{17}	0.6289	0.8956	0	0.2156	0.4163	0.6147	0.1897	0.8656	0.3859	0.1763	0.4569	0.3518
C_{18}	0.6465	0.7441	0.2156	0	0.2155	0.5716	0.7166	0.8462	0.6889	0.6455	0.5743	0.4686
C_{19}	0.6594	0.8949	0.4163	0.2155	0	0.6816	0.2965	0.4562	0.3462	0.4655	0.7152	0.8597
C_{20}	0.8556	0.3598	0.6147	0.5716	0.6816	0	0.4859	0.4856	0.5678	0.5615	0.4969	0.7456
C_{21}	0.5162	0.5716	0.1897	0.7166	0.2965	0.4859	0	0.7855	0.4887	0.7416	0.8917	0.2654
C_{22}	0.3589	0.5635	0.8656	0.8462	0.4562	0.4856	0.7855	0	0.8946	0.4852	0.1985	0.6464
C_{23}	0.9415	0.4945	0.3859	0.6889	0.3462	0.5678	0.4887	0.8946	0	0.8561	0.5785	0.4156
C_{24}	0.4565	0.9452	0.1763	0.6455	0.4655	0.5615	0.7416	0.4852	0.8561	0	0.4668	0.5486
C_{25}	0.8465	0.9515	0.4569	0.5743	0.7152	0.4969	0.8917	0.1985	0.5785	0.4668	0	0.5972
C_{26}	0.7456	0.9512	0.3518	0.4686	0.8597	0.7456	0.2654	0.6464	0.4156	0.5486	0.5972	0

Table A2 (a) represents Neutrosophic matrix ($\mu, \pi, \vee$) for the cities C_1 to C_5 from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5
C_1	0,0,0	0.5261,0.1403,0.3335	0.5423,0.1384,0.3192	0.6631,0.1068,0.2300	0.6217,0.1256,0.2526
C_2	0.5261,0.1403,0.3335	0,0,0	0.4571,0.1316,0.4112	0.5863,0.1203,0.2933	0.2413,0.1096,0.6491
C_3	0.5423,0.1384,0.3192	0.4571,0.1316,0.4112	0,0,0	0.7512,0.0857,0.1630	0.6942,0.1000,0.2057
C_4	0.6631,0.1068,0.2300	0.5863,0.1203,0.2933	0.7512,0.0857,0.1630	0,0,0	0.8412,0.0588,0.0999
C_5	0.6217,0.1256,0.2526	0.2413,0.1091,0.6491	0.6942,0.1000,0.2057	0.8412,0.0588,0.0999	0,0,0

Table A2 (b) represents Neutrosophic matrix ($\mu, \pi, \vee$) for the cities C_6 to C_{10} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5
C_6	0.8431,0.0631,0.0937	0.7512,0.0857,0.1630	0.4623,0.1314,0.4062	0.5679,0.1229,0.3091	0,0,0
C_7	0.7456,0.0950,0.1593	0.6674,0.1059,0.2266	0.7561,0.0844,0.1594	0.4987,0.1297,0.3715	0.7135,0.0954,0.1910
C_8	0.4675,0.1449,0.3875	0.5931,0.1193,0.2875	0.5001,0.1296,0.3702	0.6782,0.1035,0.2182	0.5671,0.1230,0.3098
C_9	0.7634,0.0897,0.1468	0.7213,0.0935,0.1851	0.6417,0.1110,0.2472	0.6034,0.1177,0.2788	0.6746,0.10439,0.2210
C_{10}	0.7124,0.1044,0.1831	0.8012,0.0714,0.1273	0.7812,0.0773,0.1414	0.5846,0.1206,0.2947	0.5237,0.1277,0.3485

Table A2 (c) represents Neutrosophic matrix (μ, π, ∨) for the cities C_{11} to C_{20} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5
C_{11}	0.6419,0.1214,0.2366	0.7632,0.0824,0.1543	0.4123,0.1316,0.4560	0.3699,0.1295,0.5005	0.5712,0.1224,0.3063
C_{12}	0.6787,0.1130,0.2082	0.2745,0.1169,0.6085	0.8436,0.0580,0.0983	0.7415,0.0883,0.1701	0.6716,0.1050,0.2233
C_{13}	0.7123,0.1047,0.1832	0.5481,0.1253,0.3265	0.9845,0.0063,0.0091	0.5769,0.1217,0.3013	0.9412,0.0233,0.0354
C_{14}	0.6912,0.1099,0.1988	0.8456,0.0574,0.0969	0.1664,0.0869,0.7466	0.8462,0.0572,0.0965	0.6565,0.1081,0.2353
C_{15}	0.5197,0.1410,0.3392	0.5966,0.1188,0.2845	0.5523,0.1248,0.3228	0.8425,0.0584,0.0990	0.6656,0.1062,0.2281
C_{16}	0.4128,0.1457,0.4414	0.4956,0.1299,0.3744	0.6595,0.1075,0.2329	0.5656,0.1232,0.3111	0.9463,0.0213,0.0323
C_{17}	0.7946,0.0798,0.1255	0.6596,0.1075,0.2328	0.2648,0.1149,0.6202	0.8746,0.0476,0.0777	0.6941,0.1000,0.2058
C_{18}	0.6843,0.1116,0.2040	0.3266,0.1253,0.5480	0.1654,0.0866,0.7479	0.6957,0.0996,0.2046	0.8946,0.0405,0.0648
C_{19}	0.7069,0.1058,0.1872	0.8951,0.0404,0.0644	0.3261,0.1252,0.5486	0.2154,0.1028,0.6817	0.1595,0.0844,0.7560
C_{20}	0.8431,0.0631,0.0937	0.2546,0.1127,0.6326	0.3665,0.1293,0.5041	0.5955,0.1190,0.2854	0.8685,0.0497,0.0817

Table A2 (d) represents Neutrosophic matrix (μ, π, ∨) for the cities C_{21} to C_{26} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5
C_{21}	0.7629,0.0898,0.1472	0.1655,0.0866,0.7478	0.1796,0.0916,0.7287	0.6456,0.1103,0.2440	0.8562,0.0538,0.0899
C_{22}	0.5527,0.1371,0.3101	0.4652,0.1313,0.4034	0.7656,0.0817,0.1526	0.5966,0.1188,0.2845	0.7163,0.0947,0.1889
C_{23}	0.6237,0.1252,0.2510	0.8455,0.0574,0.0970	0.5965,0.1188,0.2846	0.7465,0.0870,0.1664	0.9461,0.0214,0.0324
C_{24}	0.5179,0.1412,0.3408	0.8665,0.0504,0.0830	0.5165,0.1283,0.3551	0.6266,0.1138,0.2595	0.5169,0.1283,0.3547
C_{25}	0.5873,0.1319,0.2807	0.4865,0.1304,0.3830	0.8698,0.0492,0.0809	0.7495,0.0861,0.1643	0.9561,0.0176,0.0262
C_{26}	0.5766,0.1336,0.2897	0.8455,0.0574,0.0970	0.5356,0.1266,0.3377	0.5486,0.1252,0.3261	0.6715,0.1050,0.2234

Table A2 (e) represents Neutrosophic matrix (μ, π, ∨) for the cities C_1 to C_5 from Table 1 in section 4.

	C_6	C_7	C_8	C_9	C_{10}
C_1	0.8431,0.0582,0.0986	0.7456,0.0872,0.1671	0.4675,0.1312,0.4012	0.7634,0.0824,0.1541	0.7124,0.0956,0.1919
C_2	0.7512,0.0857,0.1630	0.6674,0.1059,0.2266	0.5931,0.1193,0.2875	0.7213,0.0935,0.1851	0.8012,0.0714,0.1273
C_3	0.4623,0.1314,0.4062	0.7561,0.0844,0.1594	0.5001,0.1296,0.3702	0.6417,0.1110,0.2472	0.7812,0.0773,0.1414
C_4	0.5679,0.1229,0.3091	0.4987,0.1297,0.3715	0.6782,0.1035,0.2182	0.6034,0.1177,0.2788	0.5846,0.1206,0.2947
C_5	0.7135,0.0954,0.1910	0.5671,0.1230,0.3098	0.6746,0.1043,0.2210	0.5237,0.1277,0.3485	0.5713,0.1224,0.3062

Table A2 (f) represents Neutrosophic matrix (μ, π, ∨) for the cities C_6 to C_{10} from Table 1 in section 4.

	C_6	C_7	C_8	C_9	C_{10}
C_6	0,0,0	0.5172,0.1283,0.3544	0.4872,0.1304,0.3823	0.5716,0.1224,0.3059	0.4872,0.1304,0.3823
C_7	0.5172,0.1283,0.3544	0,0,0	0.6813,0.1029,0.2157	0.4213,0.1320,0.4469	0.5716,0.1224,0.3059
C_8	0.4872,0.1304,0.3823	0.6813,0.1029,0.2157	0,0,0	0.6148,0.1158,0.2693	0.5127,0.1286,0.3586
C_9	0.5716,0.1224,0.3059	0.4213,0.1320,0.4469	0.6148,0.1158,0.2693	0,0,0	0.4219,0.1327,0.4462
C_{10}	0.4872,0.1304,0.3823	0.5716,0.1224,0.3059	0.5127,0.1286,0.3586	0.4219,0.1327,0.4462	0,0,0

Table A2 (g) represents Neutrosophic matrix (μ, π, ∨) for the cities C_{11} to C_{20} from Table 1 in section 4.

	C_6	C_7	C_8	C_9	C_{10}
C_{11}	0.6742,0.1044,0.2213	0.7416,0.0883,0.1700	0.4137,0.1316,0.4546	0.5166,0.1283,0.3550	0.3368,0.1265,0.5366
C_{12}	0.4369,0.1318,0.4312	0.5716,0.1224,0.3059	0.8413,0.0588,0.0998	0.7168,0.0946,0.1885	0.6741,0.1044,0.2215
C_{13}	0.2145,0.1025,0.6829	0.6715,0.1050,0.2234	0.8422,0.0585,0.0992	0.6479,0.1098,0.2422	0.9145,0.0333,0.0521
C_{14}	0.7956,0.0731,0.1312	0.6135,0.1838,0.2703	0.8436,0.0580,0.0983	0.4696,0.1312,0.3991	0.6713,0.1050,0.2236
C_{15}	0.8626,0.0517,0.0856	0.5946,0.1191,0.2862	0.6816,0.1028,0.2155	0.3266,0.1253,0.5480	0.3247,0.1250,0.5502
C_{16}	0.2176,0.1034,0.6789	0.8956,0.0402,0.0641	0.6867,0.1017,0.2115	0.9562,0.0175,0.0262	0.7416,0.0883,0.1700
C_{17}	0.1623,0.0854,0.7522	0.5952,0.1190,0.2857	0.7856,0.0760,0.1383	0.7953,0.0732,0.1314	0.9451,0.0218,0.0330
C_{18}	0.7162,0.0947,0.1890	0.3266,0.1253,0.5480	0.2185,0.1036,0.6778	0.3256,0.1251,0.5492	0.1966,0.0971,0.7062
C_{19}	0.5451,0.1256,0.3292	0.5482,0.1252,0.3265	0.1782,0.0911,0.7306	0.6816,0.1028,0.2155	0.4845,0.1305,0.3849
C_{20}	0.1656,0.0866,0.7477	0.6595,0.1075,0.2329	0.8466,0.0570,0.0963	0.4863,0.1304,0.3832	0.7566,0.0842,0.1591

Table A2 (h) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_{21} to C_{26} from Table 1 in section 4.

	C_6	C_7	C_8	C_9	C_{10}
C_{21}	0.7161,0.0947,0.1891	0.6845,0.1022,0.2132	0.7136,0.0954,0.1909	0.6416,0.1110,0.2473	0.4986,0.1297,0.3716
C_{22}	0.6145,0.1159,0.2695	0.5164,0.1283,0.3552	0.5651,0.1232,0.3116	0.4516,0.1317,0.4166	0.7166,0.0946,0.1887
C_{23}	0.6858,0.1019,0.2122	0.7465,0.0870,0.1664	0.8592,0.0528,0.0879	0.4566,0.1316,0.4117	0.2156,0.1028,0.6815
C_{24}	0.5996,0.1183,0.2820	0.3566,0.1285,0.5148	0.7415,0.0883,0.1701	0.4566,0.1316,0.4117	0.6856,0.1019,0.2124
C_{25}	0.6515,0.1091,0.2393	0.5795,0.1213,0.2991	0.5167,0.1283,0.3549	0.7866,0.0757,0.1376	0.3595,0.1287,0.5117
C_{26}	0.6123,0.1163,0.2713	0.7155,0.0949,0.1895	0.4189,0.1317,0.4493	0.6589,0.1076,0.2334	0.3658,0.1292,0.5049

Table A2 (i) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_1 to C_5 from Table 1 in section 4.

	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_1	0.6419,0.1110,0.2470	0.6787,0.1034,0.2178	0.7123,0.0960,0.1919	0.6912,0.1006,0.2081	0.5197,0.12811,0.3521
C_2	0.7632,0.0824,0.1543	0.2745,0.1169,0.6085	0.5481,0.1253,0.3265	0.8456,0.0574,0.0969	0.5966,0.1188,0.2845
C_3	0.4123,0.1316,0.4560	0.8436,0.0580,0.0983	0.9845,0.0063,0.0091	0.1664,0.0869,0.7466	0.5523,0.1248,0.3228
C_4	0.3699,0.1295,0.5005	0.7415,0.0883,0.1701	0.5769,0.1217,0.3013	0.8462,0.0572,0.0965	0.8425,0.0584,0.0990
C_5	0.5712,0.1224,0.3063	0.6716,0.1050,0.2233	0.9412,0.0233,0.0354	0.6565,0.1081,0.2353	0.6656,0.1062,0.2281

Table A2 (j) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_6 to C_{10} from Table 1 in section 4.

	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_6	0.6742,0.1044,0.2213	0.4369,0.1318,0.4312	0.2145,0.1025,0.6829	0.7956,0.0731,0.1312	0.8626,0.0517,0.0856
C_7	0.7416,0.0883,0.1700	0.5716,0.1224,0.3059	0.6715,0.1050,0.2234	0.6135,0.1838,0.2703	0.5946,0.1191,0.2862
C_8	0.4137,0.1316,0.4546	0.8413,0.0588,0.0998	0.8422,0.0585,0.0992	0.8436,0.0580,0.0983	0.6816,0.1028,0.2155
C_9	0.5166,0.1283,0.3550	0.7168,0.0946,0.1885	0.6479,0.1098,0.2422	0.4696,0.1312,0.3991	0.3266,0.1253,0.5480
C_{10}	0.5712,0.1224,0.3063	0.6741,0.1044,0.2214	0.9145,0.0333,0.0521	0.6713,0.1050,0.2236	0.3247,0.1250,0.5502

Table A2 (k) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_{11} to C_{20} from Table 1 in section 4.

	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_{11}	0,0,0	0.4193,0.1317,0.4489	0.4785,0.1308,0.3906	0.6971,0.0993,0.2035	0.7486,0.0864,0.1649
C_{12}	0.4193,0.1317,0.4489	0,0,0	0.5136,0.1286,0.3577	0.8435,0.0581,0.0983	0.9462,0.0214,0.0323
C_{13}	0.4785,0.1308,0.3906	0.5136,0.1286,0.3577	0,0,0	0.3469,0.1276,0.5254	0.5653,0.1232,0.3114
C_{14}	0.6971,0.0993,0.2035	0.8435,0.0581,0.0983	0.3469,0.1276,0.5254	0,0,0	0.6556,0.1083,0.2360
C_{15}	0.7486,0.0864,0.1649	0.9462,0.0214,0.0323	0.5653,0.1232,0.3114	0.6556,0.1083,0.2360	0,0,0
C_{16}	0.9512,0.0195,0.0292	0.6821,0.1027,0.2151	0.5185,0.1282,0.3532	0.5251,0.1276,0.3472	0.4657,0.1313,0.4029
C_{17}	0.5623,0.1236,0.3140	0.1265,0.0710,0.8024	0.5659,0.1231,0.3109	0.7566,0.0842,0.1591	0.6289,0.1134,0.2576
C_{18}	0.7152,0.0950,0.1897	0.3956,0.1310,0.4733	0.6748,0.1043,0.2208	0.7465,0.0870,0.1664	0.6465,0.1101,0.2433
C_{19}	0.7185,0.0942,0.1872	0.3497,0.1278,0.5224	0.6494,0.1095,0.2410	0.4896,0.1302,0.3801	0.6594,0.1075,0.2330
C_{20}	0.8465,0.0571,0.0963	0.6645,0.1065,0.2289	0.5867,0.1203,0.2929	0.7451,0.0873,0.1675	0.8556,0.0540,0.0903

Table A2 (l) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_{21} to C_{26} from Table 1 in section 4.

	C_{11}	C_{12}	C_{13}	C_{14}	C_{15}
C_{21}	0.7856,0.0760,0.1383	0.7565,0.0843,0.1591	0.3516,0.1280,0.5203	0.7413,0.0883,0.1703	0.5162,0.1284,0.3553
C_{22}	0.6146,0.1159,0.2694	0.3556,0.1284,0.5159	0.3888,0.1307,0.4804	0.7463,0.0870,0.1666	0.3589,0.1287,0.5123
C_{23}	0.3562,0.1284,0.5153	0.4532,0.1316,0.4151	0.5666,0.1230,0.3103	0.4857,0.1304,0.3838	0.9415,0.0232,0.0352
C_{24}	0.7164,0.0947,0.1888	0.5645,0.1233,0.3121	0.5959,0.1189,0.2851	0.5165,0.1283,0.3551	0.4565,0.1316,0.4118
C_{25}	0.2186,0.1037,0.6776	0.8465,0.0571,0.0963	0.6585,0.1077,0.2337	0.4812,0.1307,0.3880	0.8465,0.0571,0.0963
C_{26}	0.7529,0.0852,0.1618	0.6485,0.1097,0.2417	0.5568,0.1242,0.3189	0.6745,0.1043,0.2211	0.7456,0.0872,0.1671

Table A2 (m) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_1 to C_5 from Table 1 in section 4.

	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}
C_1	0.4128,0.1316,0.4555	0.7946,0.07341,0.1319	0.6843,0.1022,0.2134	0.7069,0.0970,0.1960	0.8431,0.0582,0.0986
C_2	0.4956,0.1299,0.3744	0.6596,0.1075,0.2328	0.3266,0.1253,0.5480	0.8951,0.0404,0.0644	0.2546,0.1127,0.6326
C_3	0.6595,0.1075,0.2329	0.2648,0.1149,0.6202	0.1654,0.0866,0.7479	0.3261,0.1252,0.5486	0.3665,0.1293,0.5041
C_4	0.5656,0.1232,0.3111	0.8746,0.0476,0.0777	0.6957,0.0996,0.2046	0.2154,0.1028,0.6817	0.5955,0.1190,0.2854
C_5	0.9463,0.0213,0.0323	0.6941,0.1000,0.2058	0.8946,0.0405,0.0648	0.1595,0.0844,0.7560	0.8685,0.0497,0.0817

Table A2 (n) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_6 to C_{10} from Table 1 in section 4.

	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}
C_6	0.2176,0.1034,0.6789	0.1623,0.0854,0.7522	0.7162,0.0947,0.1890	0.5451,0.1256,0.3292	0.1656,0.0866,0.7477
C_7	0.8956,0.0402,0.0641	0.5952,0.1190,0.2857	0.3266,0.1253,0.5480	0.5482,0.1252,0.3265	0.6595,0.1075,0.2329
C_8	0.6867,0.1017,0.2115	0.7856,0.0760,0.1383	0.2185,0.1036,0.6778	0.1782,0.0911,0.7306	0.8466,0.0570,0.0963
C_9	0.9562,0.0175,0.0262	0.7953,0.0732,0.1314	0.3256,0.1251,0.5492	0.6816,0.1028,0.2155	0.4863,0.1304,0.3832
C_{10}	0.7416,0.0883,0.1700	0.9451,0.0218,0.0330	0.1966,0.0971,0.7062	0.4845,0.1305,0.3849	0.7566,0.0842,0.1591

Table A2 (o) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_{11} to C_{15} from Table 1 in section 4.

	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}
C_{11}	0.9512,0.0195,0.0292	0.5623,0.1236,0.3140	0.7152,0.0950,0.1897	0.7185,0.0942,0.1872	0.8465,0.0571,0.0963
C_{12}	0.6821,0.1027,0.2151	0.1265,0.0710,0.8024	0.3956,0.1310,0.4733	0.3497,0.1278,0.5224	0.6645,0.1065,0.2289
C_{13}	0.5185,0.1282,0.3532	0.5659,0.1231,0.3109	0.6748,0.1043,0.2208	0.6494,0.1095,0.2410	0.5867,0.1203,0.2929
C_{14}	0.5251,0.1276,0.3472	0.7566,0.0842,0.1591	0.7465,0.0870,0.1664	0.4896,0.1302,0.3801	0.7451,0.0873,0.1675
C_{15}	0.4657,0.1313,0.4029	0.6289,0.1134,0.2576	0.6465,0.1101,0.2433	0.6594,0.1075,0.2330	0.8556,0.0540,0.0903
C_{16}	0,0,0	0.8956,0.0402,0.0641	0.7441,0.0876,0.1682	0.8949,0.0404,0.0646	0.3598,0.1288,0.5113
C_{17}	0.8956,0.0402,0.0641	0,0,0	0.2156,0.1028,0.6815	0.4163,0.1317,0.4519	0.6147,0.1159,0.2693
C_{18}	0.7441,0.0876,0.1682	0.2156,0.1028,0.6815	0,0,0	0.2155,0.1028,0.6816	0.5716,0.1224,0.3059
C_{19}	0.8949,0.0404,0.0646	0.4163,0.1317,0.4519	0.2155,0.1028,0.6816	0,0,0	0.6816,0.1028,0.2155
C_{20}	0.3598,0.1288,0.5113	0.6147,0.1159,0.2693	0.5716,0.1224,0.3059	0.6816,0.1028,0.2155	0,0,0

Table A2 (p) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_{21} to C_{26} from Table 1 in section 4.

	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}
C_{21}	0.5716,0.1224,0.3059	0.1897,0.0949,0.7153	0.7166,0.0946,0.1887	0.2965,0.1209,0.5825	0.4859,0.1304,0.3836
C_{22}	0.5635,0.1234,0.3130	0.8656,0.0507,0.0836	0.8462,0.0572,0.0965	0.4562,0.1316,0.4121	0.4856,0.1304,0.3839
C_{23}	0.4945,0.1299,0.3755	0.3859,0.1306,0.4834	0.6889,0.1012,0.2098	0.3462,0.1275,0.5262	0.5678,0.1229,0.3092
C_{24}	0.9452,0.0218,0.0329	0.1763,0.0904,0.7332	0.6455,0.1103,0.2441	0.4655,0.1313,0.4031	0.5615,0.1237,0.3147
C_{25}	0.9515,0.0193,0.0291	0.4569,0.1316,0.4114	0.5743,0.1220,0.3036	0.7152,0.0950,0.1897	0.4969,0.1298,0.3732
C_{26}	0.9512,0.0195,0.0292	0.3518,0.1280,0.5201	0.4686,0.1312,0.4001	0.8597,0.0527,0.0875	0.7456,0.0872,0.1671

Table A2 (q) represents Neutrosophic matrix (μ, π, $\vee$) for the cities C_1 to C_5 from Table 1 in section 4.

	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_1	0.7629,0.0825,0.1545	0.5527,0.1247,0.3225	0.6237,0.1143,0.2619	0.5179,0.1282,0.3538	0.5873,0.1202,0.2924	0.5766,0.1217,0.3016
C_2	0.1655,0.0866,0.7478	0.4652,0.1313,0.4034	0.8455,0.0574,0.0970	0.8665,0.0504,0.0830	0.4865,0.1304,0.3830	0.8455,0.0574,0.0970
C_3	0.1796,0.0916,0.7287	0.7656,0.0817,0.1526	0.5965,0.1188,0.2846	0.5165,0.1283,0.3551	0.8698,0.0492,0.0809	0.5356,0.1266,0.3377
C_4	0.6456,0.1103,0.2440	0.5966,0.1188,0.2845	0.7465,0.0870,0.1664	0.6266,0.1138,0.2595	0.7495,0.0861,0.1643	0.5486,0.1252,0.3261
C_5	0.8562,0.0538,0.0899	0.7163,0.0947,0.1889	0.9461,0.0214,0.0324	0.5169,0.1283,0.3547	0.9561,0.0176,0.0262	0.6715,0.1050,0.2234

Table A2 (r) represents Neutrosophic matrix (μ, π, ∨) for the cities C_6 to C_{10} from Table 1 in section 4.

	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_6	0.7161,0.0947,0.1891	0.6145,0.1159,0.2695	0.6858,0.1019,0.2122	0.5996,0.1183,0.2820	0.6515,0.1091,0.2393	0.6123,0.1163,0.2713
C_7	0.6845,0.1022,0.2132	0.5164,0.1283,0.3552	0.7465,0.0870,0.1664	0.3566,0.1285,0.5148	0.5795,0.1213,0.2991	0.7155,0.0949,0.1895
C_8	0.7136,0.0954,0.1909	0.5651,0.1232,0.3116	0.8592,0.0528,0.0879	0.7415,0.0883,0.1701	0.5167,0.1283,0.3549	0.4189,0.1317,0.4493
C_9	0.6416,0.1110,0.2473	0.4561,0.1316,0.4122	0.4566,0.1316,0.4117	0.4566,0.1316,0.4117	0.7866,0.0757,0.1376	0.6589,0.1076,0.2334
C_{10}	0.4986,0.1297,0.3716	0.7166,0.0946,0.1887	0.2156,0.1028,0.6815	0.6856,0.1019,0.2124	0.3595,0.1287,0.5117	0.3658,0.1292,0.5049

Table A2 (s) represents Neutrosophic matrix (μ, π, ∨) for the cities C_{11} to C_{20} from Table 1 in section 4.

	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_{11}	0.7856,0.0760,0.1383	0.6146,0.1159,0.2694	0.3562,0.1284,0.5153	0.7164,0.0947,0.1888	0.2186,0.1037,0.6776	0.7529,0.0852,0.1618
C_{12}	0.7565,0.0843,0.1591	0.3556,0.1284,0.5159	0.4532,0.1316,0.4151	0.5645,0.1233,0.3121	0.8465,0.0571,0.0963	0.6485,0.1097,0.2417
C_{13}	0.3516,0.1280,0.5203	0.3888,0.1307,0.4804	0.5666,0.1230,0.3103	0.5959,0.1189,0.2851	0.6585,0.1077,0.2337	0.5568,0.1242,0.3189
C_{14}	0.7413,0.0883,0.1703	0.7463,0.0870,0.1666	0.4857,0.1304,0.3838	0.5165,0.1283,0.3551	0.4812,0.1307,0.3880	0.6745,0.1043,0.2211
C_{15}	0.5162,0.1284,0.3553	0.3589,0.1287,0.5123	0.9415,0.0232,0.0352	0.4565,0.1316,0.4118	0.8465,0.0571,0.0963	0.7456,0.0872,0.1671
C_{16}	0.5716,0.1224,0.3059	0.5635,0.1234,0.3130	0.4945,0.1299,0.3755	0.9452,0.0218,0.0329	0.9515,0.0193,0.0291	0.9512,0.0195,0.0292
C_{17}	0.1897,0.0949,0.7153	0.8656,0.0507,0.0836	0.3859,0.1306,0.4834	0.1763,0.0904,0.7332	0.4569,0.1316,0.4114	0.3518,0.1280,0.5201
C_{18}	0.7166,0.0946,0.1887	0.8462,0.0572,0.0965	0.6889,0.1012,0.2098	0.6455,0.1103,0.2441	0.5743,0.1220,0.3036	0.4686,0.1312,0.4001
C_{19}	0.2965,0.1209,0.5825	0.4562,0.1316,0.4121	0.3462,0.1275,0.5262	0.4655,0.1313,0.4031	0.7152,0.0950,0.1897	0.8597,0.0527,0.0875
C_{20}	0.4859,0.1304,0.3836	0.4856,0.1304,0.3839	0.5678,0.1229,0.3092	0.5615,0.1237,0.3147	0.4969,0.1298,0.3732	0.7456,0.0872,0.1671

Table A2 (t) represents Neutrosophic matrix (μ, π, ∨) for the cities C_{21} to C_{26} from Table 1 in section 4.

	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_{21}	0,0,0	0.7855,0.0760,0.1384	0.4887,0.1303,0.3809	0.7416,0.0883,0.1700	0.8917,0.0416,0.0666	0.2654,0.1150,0.6195
C_{22}	0.7855,0.0760,0.1384	0,0,0	0.8946,0.0405,0.0648	0.4852,0.1305,0.3842	0.1985,0.0977,0.7037	0.6464,0.1101,0.2434
C_{23}	0.4887,0.1303,0.3809	0.8946,0.0405,0.0648	0,0,0	0.8561,0.0539,0.0899	0.5785,0.1214,0.3000	0.4156,0.1316,0.4527
C_{24}	0.7416,0.08830.1700	0.4852,0.1305,0.3842	0.8561,0.0539,0.0899	0,0,0	0.4668,0.1313,0.4018	0.5486,0.1252,0.3261
C_{25}	0.8917,0.0416,0.0666	0.1985,0.0977,0.7037	0.5785,0.1214,0.3000	0.4668,0.1313,0.4018	0,0,0	0.5972,0.1187,0.2840
C_{26}	0.2654,0.1150,0.6195	0.6464,0.1101,0.2434	0.4156,0.1316,0.4527	0.5486,0.1252,0.3261	0.5972,0.1187,0.2840	0,0,0

Table A3 (a) represents Neutrosophic matrix after applying hamming distance for the cities C_1 to C_{14} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}	C_{13}	C_{14}
C_1	0	0.4433	0.447	0.5418	0.3582	0.7898	0.5508	0.3305	0.3261	0.6694	0.5571	0.271	0.3458	0.1396
C_2	0.4433	0	0.3353	0.1916	0.1823	0.5309	0.1945	0.1967	0.2858	0.7518	0.4687	0.5588	0.539	0.4353
C_3	0.447	0.3353	0	0.3313	0.4457	0.2289	0.6479	0.2153	0.2834	0.4965	0.6412	0.4136	0.6892	0.6917
C_4	0.5418	0.1916	0.3313	0	0.7432	0.5929	0.5827	0.1459	0.3221	0.6101	0.1763	0.14	0.3278	0.5817
C_5	0.3582	0.1823	0.4457	0.7432	0	0.6959	0.2846	0.5782	0.2531	0.4157	0.3157	0.341	0.5521	0.5674
C_6	0.7898	0.5309	0.2289	0.5929	0.6959	0	0.34	0.2859	0.3219	0.6197	0.7082	0.6185	0.61	0.7526
C_7	0.5508	0.1945	0.6479	0.5827	0.2846	0.34	0	0.3838	0.5086	0.1929	0.2141	0.6446	0.1398	0.26548
C_8	0.3305	0.1967	0.2153	0.1459	0.5782	0.2859	0.3838	0	0.3662	0.1384	0.2855	0.1762	0.4473	0.663
C_9	0.3261	0.2858	0.2834	0.3221	0.2531	0.3219	0.5086	0.3662	0	0.7778	0.498	0.4885	0.3933	0.3152
C_{10}	0.6694	0.7518	0.4965	0.6101	0.4157	0.6197	0.1929	0.1384	0.7778	0	0.604	0.5562	0.7577	0.5133
C_{11}	0.5571	0.4687	0.6412	0.1763	0.3157	0.7082	0.2141	0.2855	0.498	0.604	0	0.4959	0.469	0.1715
C_{12}	0.271	0.5588	0.4136	0.14	0.341	0.6185	0.6446	0.1762	0.4885	0.5562	0.4959	0	0.7279	0.6924
C_{13}	0.3458	0.539	0.6892	0.3278	0.5521	0.61	0.1398	0.4473	0.3933	0.7577	0.469	0.7279	0	0.7494
C_{14}	0.1396	0.4353	0.6917	0.5817	0.5674	0.7526	0.2654	0.663	0.3152	0.5133	0.1715	0.6924	0.7494	0

Table A3 (b) represents Neutrosophic matrix after applying hamming distance for the cities C_{15} to C_{26} from Table 1 in section 4.

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	C_{11}	C_{12}
C_{15}	0.6944	0.7008	0.4806	0.4121	0.1509	0.79	0.4049	0.6515	0.3032	0.5141	0.3544	0.5558
C_{16}	0.4426	0.6747	0.44	0.7544	0.3789	0.1483	0.5688	0.3143	0.2569	0.1429	0.4367	0.7575
C_{17}	0.3075	0.548	0.3728	0.1853	0.1045	0.3206	0.7626	0.7373	0.1398	0.3686	0.7656	0.7036
C_{18}	0.5409	0.2148	0.2086	0.1	0.6134	0.4225	0.1798	0.717	0.1925	0.665	0.7401	0.7935
C_{19}	0.591	0.7089	0.4751	0.3052	0.1797	0.3776	0.1955	0.2037	0.4955	0.212	0.7846	0.39
C_{20}	0.2562	0.3649	0.2452	0.3967	0.3969	0.4713	0.5398	0.103	0.1251	0.1014	0.36	0.4607
C_{21}	0.3619	0.2825	0.7845	0.1805	0.5186	0.3665	0.1048	0.6383	0.1369	0.1912	0.1633	0.1431
C_{22}	0.481	0.3809	0.6533	0.7111	0.3088	0.5711	0.113	0.1428	0.4441	0.5196	0.5764	0.6453
C_{23}	0.4514	0.7166	0.6064	0.3478	0.7938	0.4129	0.5549	0.6055	0.1707	0.5102	0.3787	0.5538
C_{24}	0.5419	0.1217	0.6572	0.5024	0.4357	0.5537	0.5498	0.318	0.3079	0.7958	0.6032	0.7631
C_{25}	0.4644	0.7398	0.1996	0.4928	0.5413	0.6671	0.2003	0.2952	0.4742	0.5949	0.4601	0.2854
C_{26}	0.7894	0.746	0.2488	0.3585	0.6353	0.6826	0.6129	0.539	0.6214	0.4399	0.345	0.6074

Table A3 (c) represents Neutrosophic matrix after applying hamming distance for the cities C_1 to C_{14} from Table 1 in section 4.

	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
C_1	0.6944	0.4426	0.3075	0.5409	0.591	0.2562	0.3619	0.481	0.4514	0.5419	0.4644	0.7894
C_2	0.7008	0.6747	0.548	0.2148	0.7089	0.3649	0.2825	0.3809	0.7166	0.1217	0.7398	0.746
C_3	0.4806	0.44	0.3728	0.2086	0.4751	0.2452	0.7845	0.6533	0.6064	0.6572	0.1996	0.2488
C_4	0.4121	0.7544	0.1853	0.1	0.3052	0.3967	0.1805	0.7111	0.3478	0.5024	0.4928	0.3585
C_5	0.1509	0.3789	0.1045	0.6134	0.1797	0.3969	0.5186	0.3088	0.7938	0.4357	0.5413	0.6353
C_6	0.79	0.1483	0.3206	0.4225	0.3776	0.4713	0.3665	0.5711	0.4129	0.5537	0.6671	0.6826
C_7	0.4049	0.5688	0.7626	0.1798	0.1955	0.5398	0.1048	0.113	0.5549	0.5498	0.2003	0.6129
C_8	0.6515	0.3143	0.7373	0.717	0.2037	0.103	0.6383	0.1428	0.6055	0.318	0.2952	0.539
C_9	0.3032	0.2569	0.1398	0.1925	0.4955	0.1251	0.1369	0.4441	0.1707	0.3079	0.4742	0.6214
C_{10}	0.5141	0.1429	0.3686	0.665	0.212	0.1014	0.1912	0.5196	0.5102	0.7958	0.5949	0.4399
C_{11}	0.3544	0.4367	0.7656	0.7401	0.7846	0.36	0.1633	0.5764	0.3787	0.6032	0.4601	0.345
C_{12}	0.5558	0.7575	0.7036	0.7935	0.39	0.4607	0.1431	0.6453	0.5538	0.7631	0.2854	0.6074
C_{13}	0.3269	0.7296	0.7973	0.4498	0.3054	0.5586	0.5784	0.1679	0.3204	0.3118	0.4626	0.2408
C_{14}	0.2313	0.3992	0.7247	0.3409	0.6391	0.55	0.5596	0.4211	0.3099	0.1127	0.7782	0.4564

Table A3 (d) represents Neutrosophic matrix after applying hamming distance for the cities C_{15} to C_{26} from Table 1 in section 4.

	C_{13}	C_{14}	C_{15}	C_{16}	C_{17}	C_{18}	C_{19}	C_{20}	C_{21}	C_{22}	C_{23}	C_{24}	C_{25}	C_{26}
15	0.3269	0.2313	0.0	0.587	50.4798	0.4102	0.277	0.3913	0.7747	0.2681	0.4932	0.1883	0.1427	0.48
16	0.7296	0.3992	0.5875	0.0	0.3031	0.7394	0.2291	0.1018	0.55	0.478	0.5216	0.6383	0.7097	0.5613
17	0.7973	0.7247	0.4798	0.3031	0.0	0.6038	0.5184	0.6117	0.1639	0.5222	0.48	0.3112	0.4569	0.5877
18	0.4498	0.3409	0.4102	0.7394	0.6038	0.0	0.1905	0.3585	0.7423	0.767	0.6496	0.6868	0.7177	0.5705
19	0.3054	0.6391	0.277	0.2291	0.5184	0.1905	0.0	0.1448	0.336	0.3811	0.2179	0.1647	0.7418	0.7442
20	0.5586	0.55	0.3913	0.1018	0.6117	0.3585	0.1448	0.0	0.2731	0.4678	0.2975	0.7043	0.1269	0.3461
21	0.5784	0.5596	0.7747	0.55	0.1639	0.7423	0.336	0.2731	0.0	0.1003	0.5828	0.1995	0.1118	0.2824
22	0.1679	0.4211	0.2681	0.478	0.5222	0.767	0.3811	0.4678	0.1003	0.0	0.4584	0.6436	0.5071	0.2357
23	0.3204	0.3099	0.4932	0.5216	0.48	0.6496	0.2179	0.2975	0.5828	0.4584	0.0	0.4397	0.5777	0.4602
24	0.3118	0.1127	0.1883	0.6383	0.3112	0.6868	0.1647	0.7043	0.1995	0.6436	0.4397	0.0	0.3931	0.6435
25	0.4626	0.7782	0.1427	0.7097	0.4569	0.7177	0.7418	0.1269	0.1118	0.5071	0.5777	0.3931	0.0	0.7415
26	0.2408	0.4564	0.48	0.5613	0.5877	0.5705	0.7442	0.3461	0.2824	0.2357	0.4602	0.6435	0.7415	0.0

References

1. Government of Sri Lanka, Epidemiology Unit, Ministry of Health: Dengue https://reliefweb.int/report/sri-lanka/epidemiology-unit-ministry-health-dengue-update-10-june-2019
2. Dengue fever infects over 12,000 in Pakistan(2011), Tribune the Express https://tribune.com.pk/story/263068/dengue-fever-infects-over-12000-in-pakistan/
3. Silveira, Graciele P and de Barros, Laecio C Analysis of the dengue risk by means of a Takagi–Sugeno-style model Fuzzy Sets and Systems, 2015, 122-137.
4. Abdel-Basset, M., Manogaran, G., Gamal, A. and Smarandache, F., 2019. A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection. Journal of medical systems, 43(2), 38.
5. Abdel-Basset M, Atef A, Smarandache F. A hybrid Neutrosophic multiple criteria group decision making approach for project selection. Cognitive Systems Research. 2019 Oct 1; 57:216-27.
6. L.A. Zadeh, Fuzzy Sets, Information and Control, (1965), 8, 338-353.
7. K. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy sets and systems, (1986), 20, 87-96.
8. D.W. Patterson Introduction to Artificial Intelligence and Expert Systems, 1990, Prentice-Hall Inc., Englewood Cliffs, N.J USA
9. N. Cagman and S. Enginoglu, Soft matrix theory and its decision making, Comput. Math. Appl., (2010), 59, 3308- 3314.
10. Y. Yong and J. Chenli, Fuzzy soft matrices and their applications, part 1, LNAI, 7002, (2011), 618-627.
11. M. J. Borah, T. J. Neog and D. K. Sut, Fuzzy soft matrix theory and its decision making, IJMER, (2012), 2, 121-127.
12. T. J. Neog and D. K. SutAn application of fuzzy soft sets in decision making problems using fuzzy soft matrices, IJMA, (2011), 2258-2263.
13. S. Broumi, F. Smarandache and M. Dhar, n fuzzy soft matrix based on reference function, Information engineering and electronic business, (2013), 2, 52-59.
14. J. I. Mondal and T. K. Roy, Intuitionistic fuzzy soft matrix theory, Mathematics and statistics, (2013), 1 (2), 43-49, DOI: 10.13189/ms.2013.010205.
15. B. Chetia and P. K. Das, Some results of intuitionistic fuzzy soft matrix theory, Advanced in applied science research, (2012), 3(1), 412-423.
16. T. M. Basu, N. K. Mahapatra and S. K. Mondal, Intuitionistic fuzzy soft matrix and its application in decision making problems, Annals of fuzzy mathematics and informatics, (2014), 7(1), 109-131.
17. P. Rajarajeswari and P. Dhanalakshmi, Intuitionistic fuzzy soft matrix theory and its application in decision making, IJERT, (2013), 2(4), 1100-1111.
18. F. Smarandache, Neutrosophic set, A generalisation of the intuitionistic fuzzy sets, Inter.J.PureAppl.Math., (2005), 24, 287-297.
19. F. Smarandache, Neutrosophy, Neutrosophic Probability, Set and Logic, Amer. Res. Press, Rehoboth, USA., (1998), 105, http://fs.gallup.unm.edu/eBook-neutrosophics4.pdf(fourthversion).
20. S. Das, S. Kumar, S. Kar and T. Pal, Group decision making using neutrosophic soft matrix : An algorithmic approach, Journal of King Saud University - Computer and Information Sciences, (2017), https://doi.org/10.1016/j.jksuci. 2017.05.001.
21. S. Pramanik, P. P. Dey and B. C. Giri, TOPSIS for single valued neutrosophic soft expert set based multi-attribute decision making problems, Neutrosophic Sets and Systems, (2015), 10, 88-95.
22. P. P. Dey, S. Pramanik and B. C. Giri, Generalized neutrosophic soft multi-attribute group decision making based on TOPSIS, Critical Review (2015), 11, 41-55.
23. S. Pramanik and S. Dalapati, GRA based multi criteria decision making in generalized neutrosophic soft set environment, Global Journal of Engineering Science and Research Management, (2016), 3(5), 153-169.
24. P. P. Dey, S. Pramanik and B. C. Giri, Neutrosophic soft multi-attribute group decision making based on grey relational analysis method, Journal of New Results in Science, (2016), 10, 25-37.
25. Bhatt, Samir and Gething, Peter W and Brady, Oliver J and Messina, Jane P and Farlow, Andrew W and Moyes, Catherine L and Drake, John M and Brownstein, John S and Hoen, Anne G and Sankoh, Osman and others, The global distribution and burden of dengue, Nature, 2013, 496, 504.

26. Fatima, Meherwar and Pasha, Maruf, Survey of machine learning algorithms for disease diagnostic, Journal of Intelligent Learning Systems and Applications, 2017, 9, 1-16.
27. Thitiprayoonwongse, Daranee and Suriyaphol, Prapat and Soonthornphisaj, Nuanwan A Data mining framework for building dengue infection disease model, The 26th Annual Conference of the Japanese Society for Artificial Intelligence, 2012, pages 1K2IOS1b7–1K2IOS1b7.
28. Torra, V., 2005, May. Fuzzy c-means for fuzzy hierarchical clustering. In The 14th IEEE International Conference on Fuzzy Systems, 2005. FUZZ'05, 646-651.
29. Myers, Samuel S and Patz, Jonathan A, Emerging threats to human health from global environmental change, Annual Review of Environment and Resources, 2009, 34, 223-252.
30. Hassan, Hafiz and Shohaimi, Shamarina and Hashim, Nor R, Risk mapping of dengue in Selangor and Kuala Lumpur, Malaysia, Geospatial health, 2012, 7, 21-25.
31. Morin, Cory W and Comrie, Andrew C and Ernst, Kacey, Climate and dengue transmission: evidence and implications, Environmental health perspectives, 2013, 121, 1264–1272
32. Hassan, Hafiz and Shohaimi, Shamarina and Hashim, Nor R, Projecting the impact of climate change on dengue transmission in Dhaka, Bangladesh, Environment international, 2014, 63, 137-142.
33. Pal, Sankar K and Pal, Amita, Pattern recognition: from classical to modern approaches, World Scientific, 2001.
34. Ross, T.J., 2005. Fuzzy logic with engineering applications. John Wiley & Sons.
35. Bezdek, James C, Pattern recognition with fuzzy objective function algorithms, Springer Science and Business Media, 2013.
36. B.K. Tripathy and J anuradha, Soft Computing, Advances and Applications, Cengage Learning, 2015, 305-318.
37. Smarandache, Florentin, A unifying field in Logics: Neutrosophic Logic, American Research Press, 1999, 1-141.
38. Patrascu, V., Shannon Entropy for Neutrosophic Information. Infinite Study, 2018, R.C.E.I.T-1.9.18.
39. Hamming RW. Error detecting and error correcting codes. The Bell system technical journal. 1950 Apr; 29(2):147-60.
40. Banerjee, Durga and Giri, Bibhas C and Pramanik, Surapati and SmarandacheFlorentin, GRA for multi attribute decision making in neutrosophic cubic set environment, Neutrosophic Sets and Systems, 2012, 12, 59.
41. Government of Sri Lanka, Epidemiological Unit. Available Online: URL (http://www.epid.gov.lk/web/index.php?option=com_casesanddeaths&Itemid=448&lang=en#)
42. Rousseeuw PJ. Silhouettes: a graphical aid to the interpretation and validation of cluster analysis. Journal of computational and applied mathematics. 1987 Nov 1; 20:53-65.
43. Davies DL, Bouldin DW. A cluster separation measure. IEEE transactions on pattern analysis and machine intelligence. 1979 Apr(2):224-7.
44. Dunn JC. A fuzzy relative of the ISODATA process and its use in detecting compact well-separated clusters, 1973, 32-57.

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On the Isotopy of some Varieties of Fenyves Quasi Neutrosophic Triplet Loop (Fenyves BCI-algebras)

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Abstract: Neutrosophy theory has found application in health sciences in recent years. There is the need to develop neutrosophic algebraic systems which are good and appropriate for studying and understanding the effects of diseases and their possible treatments. In order to achieve this, special types of quasi neutrosophic loops and their isotopy needed to be introduced for this purpose. Fenyves BCI-algebras are BCI-algebras (special types of quasi neutrosophic loops) that satisfy the 60 Bol-Moufang identities. In this paper, the isotopy of BCI-algebras are studied. Neccessary and sufficient conditions for a groupoid isotope of a BCI-algebra to be a BCI-algebra are established. It is shown that p-semisimplicity, quasi-associativity and BCK-algebra are invariant under isotopies which are determined by some regular permutation groups. Furthermore, the isotopy of both the 46 associative and 14 non-associative Fenyves BCI-algebras are also studied. It is shown that for BCI-alegbras, associativity is isotopic invariant. Hence, the following set of Fenyves BCI algebras (F_i-algebras) are invariant under any isotopy: $i \in \{1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,38,40,41,43,44,45,47,48,49,50,51,53,57,58,60\}$. It is shown that the following sets of non-associative Fenyves BCI algebras (F_i-algebras) are invariant under isotopies which are determined by some regular permutation groups: $i \in \{3,5,8,19,21,29,39,42,46,52,55,56,59\}, \{56\}, \{8,19,29,39,46,59\}$. In conclusion, this is the isotopic study of 120 particular types of the 540 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) which were recently discovered, wherein the 14 non-associative Fenyves BCI-algebras do not necessarily have the Iseki's conditions (S). Importantly, applying these results, the initial (old, sick or healthy) state of a person can be represented by a type of Fenyves BCI-algebra, while the Fenyves BCI-algebra isotope will represent the final (new, healthy or sick) state of the person as a result of the prescribed medical treatment, which the isotopism represents. The isotopism is a measure of the change from the old state of body condition to the new state.

Keywords: BCI-algebra; quasi neutrosophic loops; Fenyves identities; Bol-Moufang Type

1. Introduction

The prevalence and spread of diseases among inhabitants of the world, especially tropical regions has raised serious concerns among scientists. In this work, we embarked on an algebraic way of representing the effects of diseases on the health of the people. This is based on the philosophy of representing disease-victim(s) by algebraic structures. These structures represent the state of health before the "invasion" by organisms which cause disease(s). The transformation of the body by these diseases is represented by the isotopisms which form the crux of the study. The isotopisms transform a hitherto healthy person to somebody with health challenges. Other researchers who have worked on neutrosophy theory and its applications to medicine and other fields include Abdel-Basset et al. [1], [2], [3], [4].

1.1. BCI-algebra and BCK-algebra

BCK-algebras and BCI-algebras are abbreviated as two B-algebras. The former was raised in 1966 by Imai and Iseki [16], Japanese mathematicians, and the latter was put forward in the same year by Iseki [17]. The two algebras originated from two different sources: set theory and propositional calculi.

There are some systems which contain the only implicational functor among logical functors, such as the system of weak positive implicational calculus, BCK-system and BCI-system. Undoubtedly, there are common properties among those systems. We know that there are close relationships between the notions of the set difference in set theory and the implication functor in logical systems. For example, we have the following simple inclusion relations in set theory:

$$(A-B)-(A-C)\subseteq C-B, \qquad A-(A-B)\subseteq B.$$

These are similar to the propositional formulas in propositional calculi:

$$(p\to q)\to((q\to r)\to(p\to r)), \qquad p\to((p\to q)\to q),$$

which raise the following questions: What are the most essential and fundamental properties of these relationships? Can we formulate a general algebra from the above consideration? How do we find an axiomatic system to establish a good theory of general algebras? Answering these questions, K.Iseki formulated two kinds of B-algebras, in which BCI-algebras are of wider class than BCK-algebras. Their names are taken from BCK and BCI systems in combinatory logic.

BCI-Algebras are very interesting algebraic structures that have generated wide interest among pure mathematicians. In fact, since late 1970s, much attention has been paid to the study of BCI and BCK algebras. In particular, the participation in the research of polish mathematicians Tadeusz Traczyk and Andrzej Wronski as well as Australian mathematician William H. Cornish and so on, is really making this branch of algebra to develop rapidly. Many interesting and important results are discovered continuously. Now, the theory of BCI-algebras has been widely spread to many areas such as general theory which includes congruences, quotient algebras, BCI-Homomorphisms, direct sums and direct products, commutative BCK-algebras, positive implicative and implicative BCK-algebras, derivations of BCI-algebras, and ideal theory of BCI-algebras ([16], [18], [14], [41], [50]).

1.2. BCI-algebra and the Fenyves Identities

We shall now discuss BCI-algebras in relation to Fenyves identities.

Definition 1 *A triple* $(X,*,0)$ *is called a BCI-algebra if the following conditions are satisfied for any* $x,y,z\in X$:

1. $((x*y)*(x*z))*(z*y)=0$;
2. $x*0=x$;
3. $x*y=0$ and $y*x=0 \Rightarrow x=y$.

We call the binary operation $*$ on X multiplication, and the constant 0 in X the zero element of X. We often write X instead of $(X,*,0)$ for a BCI-algebra in brevity. Juxtaposition xy shall be at times used for $x*y$ and will have preference over $*$ i.e. $xy*z=(x*y)*z$.

Example 1 *Let S be a set. Let 2^S be the power set of S, $-$ the set difference and $\emptyset$ for the empty set. Then $(2^S,-,\emptyset)$ is a BCI-algebra.*

Example 2 *Suppose $(G,\cdot,e)$ is an abelian group with e as the identity element. Define a binary operation $*$ on G by putting $x*y=xy^{-1}$. Then $(G,*,e)$ is a BCI-algebra.*

Example 3 $(\mathbb{Z},-,0)$ *and* $(\mathbb{R}-\{0\},\div,1)$ *are BCI-algebras.*

Example 4 *Let S be a set. Let 2^S be the power set of S, Δ the symmetric difference and $\emptyset$ the empty set. Then $(2^S,\Delta,\emptyset)$ is a BCI-algebra.*

The following theorems give necessary and sufficient conditions for the existence of a BCI-algebra.

Theorem 1 *(Yisheng [51])*

Let X be a non-empty set, $*$ a binary operation on X and 0 a constant element of X. Then $(X,*,0)$ is a BCI- algebra if and only if the following conditions hold:

1. $((x*y)*(x*z))*(z*y)=0;$
2. $(x*(x*y))*y=0;$
3. $x*x=0;$
4. $x*y=0$ and $y*x=0$ imply $x=y$.

Definition 2 *A BCI- algebra $(X,*,0)$ is called a BCK-algebra if $0*x=0$ for all $x\in X$.*

Definition 3 *(Jaiyéolá et al. [36])*

A BCI- algebra $(X,*,0)$ is called a Fenyves BCI-algebra if it satisfies an identity of Bol-Moufang type. The identities of Bol-Moufang type are given below:

$F_1: xy*zx=(xy*z)x$ $F_2: xy*zx=(x*yz)x$ (Moufang identity) $F_3: xy*zx=x(y*zx)$

$F_4: xy*zx=x(yz*x)$ (Moufang identity) $F_5: (xy*z)x=(x*yz)x$ $F_6: (xy*z)x=x(y*zx)$ (extra identity)

$F_7: (xy*z)x=x(yz*x)$ $F_8: (x*yz)x=x(y*zx)$ $F_9: (x*yz)x=x(yz*x)$ $F_{10}: x(y*zx)=x(yz*x)$

$F_{11}: xy\cdot xz=(xy*x)z$ $F_{12}: xy*xz=(x*yx)z$ $F_{13}: xy*xz=x(yx*z)$ (extra identity)

$F_{14}: xy*xz=x(y*xz)$ $F_{15}: (xy*x)z=(x*yx)z$ $F_{16}: (xy*x)z=x(yx*z)$

$F_{17}: (xy*x)z=x(y*xz)$ (Moufang identity) $F_{18}: (x*yx)z=x(yx*z)$

$F_{19}: (x*yx)z=x(y*xz)$ (left Bol identity) $F_{20}: x(yx*z)=x(y*xz)$ $F_{21}: yx*zx=(yx*z)x$

$F_{22}: yx*zx=(y*xz)x$ (extra identity) $F_{23}: yx*zx=y(xz*x)$ $F_{24}: yx*zx=y(x*zx)$

$F_{25}: (yx*z)x=(y*xz)x$ $F_{26}: (yx*z)x=y(xz*x)$ (right Bol identity)

$F_{27}: (yx*z)x=y(x*zx)$ (Moufang identity) $F_{28}: (y*xz)x=y(xz*x)$ $F_{29}: (y*xz)x=y(x*zx)$

$F_{30}: y(xz*x)=y(x*zx)$ $F_{31}: yx*xz=(yx*x)z$ $F_{32}: yx*xz=(y*xx)z$ $F_{33}: yx*xz=y(xx*z)$

$F_{34}: yx*xz=y(x*xz)$ $F_{35}: (yx*x)z=(y*xx)z$ $F_{36}: (yx*x)z=y(xx*z)$ (RC identity)

$F_{37}: (yx*x)z=y(x*xz)$ (C-identity) $F_{38}: (y*xx)z=y(xx*z)$ $F_{39}: (y*xx)z=y(x*xz)$ (LC identity)

$F_{40}: y(xx*z)=y(x*xz)$ $F_{41}: xx*yz=(x*xy)z$ (LC identity) $F_{42}: xx*yz=(xx*y)z$

$F_{43}: xx*yz=x(x*yz)$ $F_{44}: xx*yz=x(xy*z)$ $F_{45}: (x*xy)z=(xx*y)z$

$F_{46}: (x*xy)z=x(x*yz)$ (LC identity) $F_{47}: (x*xy)z=x(xy*z)$ $F_{48}: (xx*y)z=x(x*yz)$ (LC identity)

$F_{49}: (xx*y)z=x(xy*z)$ $F_{50}: x(x*yz)=x(xy*z)$ $F_{51}: yz*xx=(yz*x)x$ $F_{52}: yz*xx=(y*zx)x$

$F_{53}: yz*xx=y(zx*x)$ (RC identity) $F_{54}: yz*xx=y(z*xx)$ $F_{55}: (yz*x)x=(y*zx)x$

$F_{56}: (yz*x)x=y(zx*x)$ (RC identity) $F_{57}: (yz*x)x=y(z*xx)$ (RC identity)

$F_{58}: (y * zx)x = y(zx * x)$ $F_{59}: (y * zx)x = y(z * xx)$ $F_{60}: y(zx * x) = y(z * xx)$

The identities of Bol-Moufang type are sixty in number based on Fenyves [12], [13]. The identities of Bol-Moufang type were investigated in BCI-algebras by Jaiyéolá et al. [36], thereby leading to the study of the sixty varieties of Fenyves BCI -algebras, as well as their holomorphic study in Ilojide et al. [15]. Here are some examples.

Example 5 *Let us assume the BCI-algebra $(G,*,e)$ in Example 2. Then $(G,*,e)$ is an F_8-algebra, F_{19}-algebra, F_{29}-algebra, F_{39}-algebra, F_{46}-algebra, F_{52}-algebra, F_{54}-algebra, F_{59}-algebra.*

Example 6 *Let us assume the BCI-algebra $(2^S,-,\emptyset)$ in Example 1. Then $(2^S,-,\emptyset)$ is an F_3-algebra, F_5-algebra, F_{21}-algebra, F_{29}-algebra, F_{42}-algebra, F_{46}-algebra, F_{54}-algebra and F_{55}-algebra.*

Example 7 *The BCI-algebra $(2^S,\Delta,\emptyset)$ in Example 4 is associative.*

Example 8 *By considering the direct product of the BCI-algebras $(G,*,e)$ and $(2^S,-,\emptyset)$ of Example 2 and Example 1 respectively, we have a BCI-algebra $(G \times 2^S,(*,-),(e,\emptyset))$ which is a F_{29}-algebra and a F_{46}-algebra.*

Remark 1 *The direct product of two or more BCI-algebras which are F_i-algebras will give a BCI-algebra which is an F_i-algebra for distinct i's.*

Definition 4 *A BCI-algebra $(X,*,0)$ is called associative if $(x * y) * z = x * (y * z)$ for all $x,y,z \in X$.*

Definition 5 *A BCI-algebra $(X,*,0)$ is called p-semisimple if $0 * (0 * x) = x$ for all $x \in X$.*

Theorem 2 *(Yisheng [51]) Suppose that $(X,*,0)$ is a BCI-algebra. Define a binary relation $\leq$ on X by which $x \leq y$ if and only if $x * y = 0$ for any $x,y \in X$. Then $(X,\leq)$ is a partially ordered set with 0 as a minimal element(meaning that $x \leq 0$ implies $x = 0$ for any $x \in X$).*

Definition 6 *A BCI-algebra $(X,*,0)$ is called quasi-associative if $(x * y) * z \leq x * (y * z)$ for all $x,y,z \in X$.*

The following theorems give equivalent conditions for associativity, quasi-associativity and p-semisimplicity in a BCI-algebra:

Theorem 3 *(Yisheng [51])*

Given a BCI-algebra X, the following are equivalent $x,y,z \in X$:

1. X is associative.
2. $0 * x = x$.
3. $x * y = y * x \; \forall \, x,y \in X$.

Theorem 4 *(Yisheng [51])*

Let X be a BCI-algebra. Then the following conditions are equivalent for any $x,y,z,u \in X$:

1. X is p-semisimple
2. $(x * y) * (z * u) = (x * z) * (y * u)$.
3. $0 * (y * x) = x * y$.
4. $(x * y) * (x * z) = z * y$.
5. $z * x = z * y$ implies $x = y$. (the left cancellation law)
6. $x * y = 0$ implies $x = y$.

Theorem 5 *(Yisheng [51])*

Given a BCI-algebra X, the following are equivalent for all $x,y \in X$:

1. X is quasi-associative.

2. $x * (0 * y) = 0$ implies $x * y = 0$.
3. $0 * x = 0 * (0 * x)$.
4. $(0 * x) * x = 0$.

Theorem 6 *(Yisheng [51])*
A triple $(X, * ,0)$ is a BCI-algebra if and only if there is a partial ordering $\leq$ on X such that the following conditions hold for any $x, y, z \in X$:

1. $(x * y) * (x * z) \leq z * y$;
2. $x * (x * y) \leq y$;
3. $x * y = 0$ if and only if $x \leq y$.

Theorem 7 *(Yisheng [51])*
Let X be a BCI-algebra. X is p-semisimple if and only if one of the following conditions holds for any $x, y, z \in X$:

1. $x * z = y * z$ implies $x = y$. (the right cancellation law)
2. $(y * x) * (z * x) = y * z$.
3. $(x * y) * (x * z) = 0 * (y * z)$.

Theorem 8 *(Yisheng [51]) Suppose that $(X, * ,0)$ is a BCI-algebra. X is associative if and only if X is p-semisimple and X is quasi-associative.*

Theorem 9 *(Yisheng [51]) Suppose that $(X, * ,0)$ is a BCI-algebra. Then for all $x, y, z \in X$:*

1. $(x * y) * z = (x * z) * y$.
2. $x \geq y$ implies $0 * x = 0 * y$.

Remark 2 *In Theorem 8, quasi-associativity in BCI-algebra plays a similar role which weak associativity (i.e. the F_i identities) plays in quasigroup and loop theory.*

1.3. Isotopy and Autotopy in Quasigroups and Loops

We now move on to quasigroups and loops, their isotopy and autotopy.

Definition 7 *Let L be a non-empty set. Define a binary operation $(\cdot)$ on L. If $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If in a groupoid $(L, \cdot)$, the equations:*

$$a \cdot x = b \qquad and \qquad y \cdot a = b$$

have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. If in a quasigroup $(L, \cdot)$, there exists a unique element e called the identity element such that for all $x \in L$, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop.

Remark 3 *For a groupoid $(G, \cdot)$, $R_x: G \to G$, the right translation is defined by $yR_x = y \cdot x$ and $L_x: G \to G$, the left translation is defined by $yL_x = x \cdot y$ for all $x, y \in G$. This mappings are not necessarily bijections. But for a quasigroup, they are.*

Consider $(G, \cdot)$ and $(H, \circ)$ being two groupoids (quasigroups, loops). Let A, B and C be three bijective mappings, that map G onto H. The triple $\alpha = (A, B, C)$ is called an *isotopism* of $(G, \cdot)$ onto $(H, \circ)$, written as

$$(G, \cdot) \xrightarrow{(A,B,C)} (H, \circ) \text{ if } xA \circ yB = (x \cdot y)C \forall x, y \in G.$$

So, $(H, \circ)$ is called a groupoid (quasigroup, loop) *isotope* of $(G, \cdot)$.

If $C = I$ is the identity map on G so that $H = G$, then the triple $\alpha = (A, B, I)$ is called a *principal isotopism* of $(G,\cdot)$ onto $(G,\circ)$ and $(G,\circ)$ is called a *principal isotope* of $(G,\cdot)$. Eventually, the equation of relationship now becomes

$$x \cdot y = xA \circ yB \forall x, y \in G$$

which is easier to work with. But if $A = R_g$ and $B = L_f$ where $f, g \in G$, the relationship now becomes

$$x \cdot y = xR_g \circ yL_f \forall x, y \in G.$$

With this new form, the triple $\alpha = (R_g, L_f, I)$ is called an f, g-*principal isotopism* of $(G,\cdot)$ onto $(G,\circ)$, f and g are called *translation elements* of G or at times written in the pair form (g, f), while $(G,\circ)$ is called an f, g-*principal isotope* of $(G,\cdot)$.

The following theorem shows that the principal isotopes of a groupoid account for all its isotopes.

Theorem 10 *(Pflugfelder [43])*

If $(G,\cdot)$ and $(H,\circ)$ are isotopic groupoids, then $(H,\circ)$ is isomorphic to some principal isotope $(G, \text{å})$ of $(G,\cdot)$.

Let $(X,*,0)$ be a BCI-algebra and let $x + y = x * (0 * x)$. A groupoid $(X, +)$ is called an associated groupoid of $(X,*,0)$. Based on Theorem 2, Corollaries 3, 4 and 5 of Dudek [9], $x * y = x - y = x + (-y) \Leftrightarrow (x * y)I = xI + yJ$ where $J: x \mapsto -x$. so, we have

Lemma 1 *A BCI-algebra* $(X,*,0)$ *is a quasigroup if and only if there exists an abelian group* $(X, +, 0)$ *such that* $(X, +, 0) \xrightarrow{(I,I,J)} (X,*,0)$.

According to Dudek [9], the variety of all BCI-algebras that are quasigroups (BCI-quasigroups) is selected from the quasivariety of all BCI-algebra by any of the following equivalent laws:

(i) p-semi simplicity law: $0 * (0 * x) = x$

(ii) Semi left inverse property: $x * (x * y) = y$ (SLIP)

(iii) Medial law: $(x * y) * (z * u) = (x * z) * (y * u)$

(iv) $(x * y) * (x * z) = (z * y)$

(v) $0 * (x * z) = z * x$

(vi) $(x * y) * (z * x) = (x * z) * (y * x)$

(vii) $[(x * y) * z] * [(x * u) * y] = (u * z)$

Thus, following Lemma 1, it can further be said that the variety of all BCI-algebras that are quasigroups is determined by abelian group under the isotopy (I, I, J) where J is the inverse mapping on the abelian group.

Dudek [11] showed that a BCI-algebra with the medial law obeys the SLIP and further showed in Dudek [10] that every BCI-algebra that obeys the SLIP has the Iseki's condition (S)-[19] and form a variety characterized with an associated abelian group.

In Theorem 10, if $(G,\cdot) = (H,\circ)$, then the triple $\alpha = (A, B, C)$ of bijections on $(G,\cdot)$ is called an autotopism of the groupoid (quasigroup, loop) $(G,\cdot)$. Such triples form a group $AUT(G,\cdot)$ called the autotopism group of $(G,\cdot)$. Furthermore, if $A = B = C$, then A is called an automorphism of the groupoid (quasigroup, loop) $(G,\cdot)$. Such bijections form a group $AUM(G,\cdot)$ called the automorphism group of $(G,\cdot)$.

The group of all permutation on G is called the permutation group of G and denoted by $SYM(G)$.

1. $U \in SYM(G)$ is called autotopic if there exists $(U, V, W) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\Sigma(G,\cdot)$.
2. $U \in SYM(G)$ is called λ-regular if there exists $(U, I, U) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\Lambda(G,\cdot) \leq \Sigma(G,\cdot)$.
3. $U \in SYM(G)$ is called ρ-regular if there exists $(I, U, U) \in AUT(G,\cdot)$; the set of all such mappings forms a group $\mathcal{P}(G,\cdot) \leq SYM(G)$.
4. $U \in SYM(G)$ is called μ-regular if there exists $U' \in SYM(G)$ such that $(U, U'^{-1}, I) \in AUT(G,\cdot)$. U' is called the adjoint of U. The set of all μ-regular mappings forms a group $\Phi(G,\cdot) \leq \Sigma(G,\cdot)$. The set of all adjoint mapping forms a group $\Psi(G,\cdot) \leq SYM(G)$. Whenever $U' = U$, then U is said to be μ-regular and self adjoint.

1.4. *Quasigroup, Loop and their Universality*

In recent past, and up to the present time, identities of Bol-Moufang type have been studied on the platform of groupoids, quasigroups and loops by Fenyves [12], Phillips and Vojtěchovský, P. [44] , [45], [46], Jaiyeola [20], Robinson [47], Burn [6], [7], [8], Kinyon and Kunen [40] and by several other authors to mention a few. Fenyves [13], Kinyon and Kunen [40], and Phillips and Vojtěchovský [46] found some of these identities to be equivalent to associativity in quasigroups and loops (i.e. groups), and others to describe weak associative laws such as extra, Bol, Moufang, central, flexible laws in quasigroups and loops. These results are tabularly summarised in Jaiyéolá et al. [36].

Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance (universality) has been considered. Some others are flexible loops, F-quasigroups, totally symmetric quasigroups(TSQ), distributive quasigroups, weak inverse property loops(WIPLs), cross inverse property loops(CIPLs), semi-automorphic inverse property loops(SAIPLs) and inverse property loops(IPLs). As shown in Pflugfelder [43], a left(right) inverse property loop is universal if and only if it is a left(right) Bol loop, so an IPL is universal if and only if it is a Moufang loop. Kepka et. al. [37], [38], [39] solved the Belousov problem concerning the universality of F-quasigroup which has been open since 1967. The universality of WIPLs and CIPLs has been addressed by Osborn [42] and Artzy [5] respectively while the universality of elasticity(flexibility) was studied by Syrbu [49]. Jaiyéolá [20], [22], Jaiyéolá and Adéníran [26], [27], [28] studied the universality of central loops while Jaiyéolá [23], [21], [24] , [25], Jaiyéolá and Adéníran [29], [31], [30], [32], and Jaiyéolá et al. [33] studied the universality Osborn loops.

1.5. *Some Existing Results on Fenyves BCI-algebras*

Jaiyéolá et al. [36] investigated Fenyves identities on the platform of BCI-algebras. They classified the Fenyves BCI-algebras into 46 associative and 14 non-associative types and showed that some Fenyves identities played the role of quasi-associativity, vis-a-vis Theorem 8 in

BCI-algebras. Their work clarified the relationship between a BCI-algebra, a quasigroup and a loop. Some of their results are stated below.

Theorem 11 *(Jaiyéolá et al. [36])*

1. A BCI algebra X is a quasigroup if and only if it is p-semisimple.
2. A BCI algebra X is a loop if and only if it is associative.
3. An associative BCI algebra X is a Boolean group.

Theorem 12 *(Jaiyéolá et al. [36])*

Let $(X,*,0)$ be a BCI-algebra. If X is any of the following Fenyves BCI-algebras, then X is associative.

1. F_1-algebra 2. F_2-algebra 3. F_4-algebra 4. F_6-algebra 5. F_7-algebra 6. F_9-algebra 7. F_{10}-algebra 8. F_{11}-algebra 9. F_{12}-algebra 10. F_{13}-algebra 11. F_{14}-algebra 12. F_{15}-algebra 13. F_{16}-algebra 14. F_{17}-algebra 15. F_{18}-algebra 16. F_{20}-algebra 17. F_{22}-algebra 18. F_{23}-algebra 19. F_{24}-algebra 20. F_{25}-algebra 21. F_{26}-algebra 22. F_{27}-algebra 23. F_{28}-algebra 24. F_{30}-algebra 25. F_{31}-algebra 26. F_{32}-algebra 27. F_{33}-algebra 28. F_{34}-algebra 29. F_{35}-algebra 30. F_{36}-algebra 31. F_{37}-algebra 32. F_{38}-algebra 33. F_{40}-algebra 34. F_{41}-algebra 35. F_{43}-algebra 36. F_{44}-algebra 37. F_{45}-algebra 38. F_{47}-algebra 39. F_{48}-algebra 40. F_{49}-algebra 41. F_{50}-algebra 42. F_{51}-algebra 43. F_{53}-algebra 44. F_{57}-algebra 45. F_{58}-algebra 46. F_{60}-algebra.

Remark 4 *All other F_i's which are not mentioned in Theorem 12 were found to be non-associative. Every BCI-algebra is naturally an F_{54} BCI-algebra. A BCI-algebra that obeys any of the F_i's in Theorem 12 is a Boolean group by Theorem 11(3), hence isomorphic to its associated groupoid (the abelian group in Lemma 1).*

Zhang et al. [52] introduced quasi-neutrosophic triplet loops (QNTLs) which is made up of nine main types (cf. Definition 9 of Jaiyéolá et al. [36]). BCI-algebra belong to the class of three of these nine main types of QNTLs: (r-r)-QNT, (r-l)-QNTL and (r-lr)-QNTL. Therefore, any F_i BCI-algebra, $1 \leq i \leq 60$ belongs to at least one of the following varieties of Fenyves quasi neutrosophic triplet loops: (r-r)-FQNTL, (r-l)-FQNTL and (r-lr)-FQNTL. Any associative QNTL is called a quasi neutrosophic triplet group (QNTG).

The variety of quasi neutrosophic triplet loop is a generalization of neutrosophic triplet group (NTG) which was originally introduced by Smarandache and Ali [48]. New results and developments on neutrosophic triplet groups and neutrosophic triplet loop have been reported by Zhang et al. [52], [54], [55], [53], and Smarandache and Jaiyéolá [34], [35].

1.6. Motivation, Problem Statement, Aims and Objectives, Methodology

In this current paper, the isotopy of BCI-algebras is the main focus of this study (an extension of the work in Jaiyéolá et al. [36]). Necessary and sufficient conditions for a groupoid isotope of a BCI-algebra to be a BCI-algebra will be established. It will be shown that p-semisimplicity, quasi-associativity and BCK-algebra are invariant under isotopies which are determined by some regular permutation groups. Furthermore, the isotopy of both the 46 associative and 14 non-associative Fenyves BCI-algebras will also be studied. This is with the view of showing that there exist some other laws aside (i) to (vii) in subsection 1.3 which can be used to select some other varieties of BCI-algebra (e.g. F_i BCI-algebras, which are not necessarily quasigroups) from the quasivariety of all BCI-algebras. Furthermore, this will mean that such varieties of BCI-algebra (which are not necessarily quasigroups) can be determined by another structure under an isotopy which differs from (I, I, J). Consequently, the 14 non-associative Fenyves BCI-algebras do not necessarily have the Iseki's conditions (S) based on the results in Theorem 14 of Jaiyéolá et al. [36].

2. Main Results

2.1. Regular Bijections of BCI-Algebras

We need the following results on regular bijections of BCI-algebras.

Lemma 2 *Let $(G,\cdot,0)$ be a BCI-algebra with $\delta, U \in SYM(G)$. Then the following hold:*

1. δ is λ-regular $\Leftrightarrow \delta R_x = R_x\delta \Leftrightarrow L_{x\delta} = L_x\delta$ for all $x \in G$.
2. δ is ρ-regular $\Leftrightarrow \delta L_x = L_x\delta \Leftrightarrow R_{x\delta} = R_x\delta$ for all $x \in G$.
3. δ is μ-regular and self-adjoint $\Leftrightarrow \delta R_x = R_{x\delta} \Leftrightarrow L_{x\delta} = \delta L_x$ for all $x \in G$.
4. If U is λ-regular, then $L_{0U} = L_0U$, $xU \cdot x = 0U$ for all $x \in G$.
5. If U is ρ-regular, then $U = R_{0U}$, $0 \cdot 0U = 0U$, $UL_0 = L_0U$.
6. If U is μ-regular and self-adjoint, then $0U \cdot 0U^{-1} = 0$, $UR_{0U^{-1}} = I$, $L_{0U} = UL_0$.
7. If U is autotopic, then there exist $V, W \in SYM(G)$ such that $U^{-1}W = R_{0V}$, $VL_{0U} = L_0W$, $xU \cdot xV = 0W$ for all $x \in G$.

Proof.

1. δ is λ-regular $\Leftrightarrow (\delta, I, \delta) \in$ AUT $(G,\cdot) \Leftrightarrow y\delta \cdot xI = (y \cdot x)\delta \Leftrightarrow y\delta R_x = yR_x\delta \Leftrightarrow \delta R_x = R_x\delta \Leftrightarrow y\delta R_x = yR_x\delta \Leftrightarrow y\delta \cdot x = (y \cdot x)\delta \Leftrightarrow xL_{y\delta} = xL_y\delta \Leftrightarrow L_{y\delta} = L_y\delta$.
2. δ is ρ-regular $\Leftrightarrow (I, \delta, \delta) \in$ AUT $(G,\cdot) \Leftrightarrow xI \cdot y\delta = (x \cdot y)\delta \Leftrightarrow y\delta L_x = yL_x\delta \Leftrightarrow \delta L_x = L_x\delta \Leftrightarrow y\delta L_x = yL_x\delta \Leftrightarrow x \cdot y\delta = (x \cdot y)\delta \Leftrightarrow xR_{y\delta} = xR_y\delta \Leftrightarrow R_{y\delta} = R_y\delta$.
3. δ is μ-regular with adjoint $\delta' = \delta \Leftrightarrow (\delta, \delta'^{-1}, I) \in$ AUT $(G,\cdot) \Leftrightarrow x\delta \cdot y\delta'^{-1} = (x \cdot y)I \Leftrightarrow x\delta \cdot y\delta\delta^{-1} = x \cdot y\delta$ (by replacing y by $y\delta$) $\Leftrightarrow x\delta \cdot y = x \cdot y\delta \Leftrightarrow x\delta R_y = xR_{y\delta} \Leftrightarrow \delta R_y = R_{y\delta} \Leftrightarrow x\delta R_y = xR_{y\delta} \Leftrightarrow x\delta \cdot y = x \cdot y\delta \Leftrightarrow yL_{x\delta} = y\delta L_x \Leftrightarrow L_{x\delta} = \delta L_x$.
4. If U is λ-regular, then $xU \cdot y = (xy)U$. Put $x = 0$ in this, then you have $L_{0U} = L_0U$. Putting $y = x$, we have $xU \cdot x = 0U$.
5. If U is ρ-regular, then $x \cdot yU = (xy)U$. Put $y = 0$, then you get $U = R_{0U}$. Putting $x = y = 0$, we have $0 \cdot 0U = 0U$. Substituting $x = 0$, we get $UL_0 = L_0U$.
6. If U is μ-regular with adjoint $U' = U$, then $x \cdot yU^{-1} = x \cdot y$. Put $x = y = 0$ to get $0U \cdot 0U^{-1} = 0$. Put $y = 0$ to get $UR_{0U^{-1}} = I$. Put $x = 0$ to get $L_{0U} = UL_0$.
7. If U is autotopic, then there exist $V, W \in SYM(G)$ such that $xU \cdot yV = x \cdot y$. Putting $y = 0$, we get $U^{-1}W = R_{0V}$. Substituting $x = 0$, we have $VL_{0U} = L_0W$. Substituting $y = x$, we get $xU \cdot xV = 0W$.

2.2. Quasi Neutrosophic Triplet Loop Isotopes of BCI-Algebras

We now present results on isotopy of BCI-algebras.

Theorem 13 *Let $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*)$ where $(G,\cdot,0)$ is a BCI-algebra and $(G,*)$ is a groupoid.*

1. Let $\varepsilon^{-1}\delta = \delta^{-1}\varepsilon$. Then, $(G,*,0)$ is a (r-r)-quasi NTL or (r-l)-quasi NTL or (r-rl)-quasi NTL if

and only if $\delta = \varepsilon$ and $\delta = R_{0\varepsilon^{-1}}$ (i.e. $\exists g \in G \ni \delta = R_g; g = 0\varepsilon^{-1}$).

2. $(G,*,0)$ is a BCI-algebra if and only if the following hold:
   a. $\delta = R_{0\varepsilon^{-1}}$ $(\exists g \in G \ni \delta = R_g; g = 0\varepsilon^{-1})$;
   b. $\delta = \varepsilon$;
   c. $[(x \cdot y) * (x \cdot z)] * (z \cdot y) = 0$.

Proof.

1. $(G,*,0)$ is a (r-r)-quasi NTL or (r-l)-quasi NTL or (r-rl)-quasi NTL if and only if $x * 0 = x$ and $x * x = 0$.
   a. $x * 0 = x \Leftrightarrow (x\delta^{-1} \cdot 0\varepsilon^{-1})I = x \Leftrightarrow x\delta^{-1}R_{0\varepsilon^{-1}} = x \Leftrightarrow \delta^{-1}R_{0\varepsilon^{-1}} = I \Leftrightarrow \delta = R_{0\varepsilon^{-1}}$.
   b. $x * x = 0 \Leftrightarrow x\delta^{-1} \cdot x\varepsilon^{-1} = 0 = x^2$. Replace x by $x\varepsilon^{-1}\delta$ to get $x * x = 0 \Leftrightarrow x\varepsilon^{-1}\delta\delta^{-1} \cdot x\varepsilon^{-1}\delta\varepsilon^{-1} = (x\varepsilon^{-1}\delta)^2 \Leftrightarrow x\varepsilon^{-1} \cdot x\varepsilon^{-1}\delta\varepsilon^{-1} = 0 \Leftrightarrow x\varepsilon^{-1} \cdot x\delta^{-1} = 0$. So, $x\delta^{-1} \cdot x\varepsilon^{-1} = 0$ and $x\varepsilon^{-1} \cdot x\delta^{-1} = 0$ implies that $x\delta^{-1} = x\varepsilon^{-1} \Leftrightarrow \delta = \varepsilon$.

2. For the forward, we shall assume that $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*)$ and $(G,*,0)$ is a BCI-algebra.
   a. As above in 1, $x * 0 = x \Leftrightarrow \delta = R_{0\varepsilon^{-1}}$.
   b. Let $x * y = 0$ and $y * x = 0$, and so $x\delta^{-1} \cdot y\varepsilon^{-1} = 0$ and $y\delta^{-1} \cdot x\varepsilon^{-1} = 0$ respectively. The equation $y\delta^{-1} \cdot x\varepsilon^{-1} = 0$ can be re-written as $y\delta^{-1} \cdot x\varepsilon^{-1} = y^2$. Now, replacing y by $y\varepsilon^{-1}\delta$ to get $y\varepsilon^{-1}\delta\delta^{-1} \cdot x\varepsilon^{-1} = (y\varepsilon^{-1}\delta)^2 \Rightarrow y\varepsilon^{-1} \cdot x\varepsilon^{-1} = 0 \Rightarrow y\varepsilon^{-1} \cdot x\varepsilon^{-1} = x^2$. Furthermore, x by $x\delta^{-1}\varepsilon$ to get $y\varepsilon^{-1} \cdot x\delta^{-1}\varepsilon\varepsilon^{-1} = (x\delta^{-1}\varepsilon)^2 \Rightarrow y\varepsilon^{-1} \cdot x\delta^{-1} = 0$.

   Thus, we have shown that $x\delta^{-1} \cdot y\varepsilon^{-1} = 0$ and $y\varepsilon^{-1} \cdot x\delta^{-1} = 0$. Recall that $x \cdot y = 0$ and $y \cdot x = 0$ imply that $x = y$. So, $x\delta^{-1} = y\varepsilon^{-1} \Rightarrow \delta = \varepsilon$.
   c. $[(x * y) * (x * z)] * (z * y) = 0 \Leftrightarrow [(x\delta^{-1} \cdot y\varepsilon^{-1})\delta^{-1} \cdot (x\delta^{-1} \cdot z\varepsilon^{-1})\varepsilon^{-1}]\delta^{-1} \cdot [(z\delta^{-1} \cdot y\varepsilon^{-1})]\varepsilon^{-1} = 0$. Replace $x\delta^{-1}$ by x, $y\varepsilon^{-1}$ by y, and $z\varepsilon^{-1}$ by z to get $[(x \cdot y)\delta^{-1} \cdot (x \cdot z)\varepsilon^{-1}]\delta^{-1} \cdot [z\varepsilon\delta^{-1} \cdot y]\varepsilon^{-1} = 0 \Rightarrow [(x \cdot y) * (x \cdot z)]\delta^{-1} \cdot [z\varepsilon\delta^{-1} \cdot y]\varepsilon^{-1} = 0 \Rightarrow [(x \cdot y) * (x \cdot z)] * [z\varepsilon\delta^{-1} \cdot y] = 0 \Rightarrow [(x \cdot y) * (x \cdot z)] * [z \cdot y] = 0$.

   For the converse: we shall assume (a), (b) and (c). Following directly the reverse of 2(a), $x * 0 = x$. Since $\delta = \varepsilon$, then $x * y = 0 \Rightarrow x\delta^{-1} \cdot y\varepsilon^{-1} = 0$ and $y * x = 0 \Rightarrow y\delta^{-1} \cdot x\varepsilon^{-1} = 0$ which means that $x\delta^{-1} \cdot y\delta^{-1} = 0$ and $y\delta^{-1} \cdot x\delta^{-1} = 0$ imply $x = y$. Since $\delta = \varepsilon$, then (c) can be reversed to get $[(x * y) * (x * z)] * (z * y) = 0$. $\therefore$ $(G,*,0)$ is a BCI-algebra.

Corollary 1 *Let* $(G,\cdot,0) \xrightarrow{(R_g,R_g,I)} (G,*)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*)$ *is a groupoid.*

1. $(G,*,0)$ is a (r-r)-quasi NTL, (r-l)-quasi NTL and (r-rl)-quasi NTL.
2. $(G,*,0)$ is a BCI-algebra if and only if $[(x \cdot y) * (x \cdot z)] * (z \cdot y) = 0$ holds.

Proof. We shall use Theorem 13. 1 and 2 are true because $R_g = R_{0R_g^{-1}}$ since $g = 0R_g^{-1} \Leftrightarrow g^2 = 0$, which is true in the BCI-algebra $(G,\cdot,0)$.

Theorem 14 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond)$ *such that* $0C = 0'$, *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond)$ *is a groupoid.*

1. Let $A^{-1}B = B^{-1}A$, then $(H,\diamond,0')$ is a (r-r)-quasi NTL or (r-l)-quasi NTL or (r-rl)-quasi NTL if and only if $A = B$ and $A = R_{0'B^{-1}}C$ (i.e. $\exists g \in G \ni A = R_gC, g =$

$0'B^{-1}$).

2. $(H,\diamond,0')$ is a BCI-algebra if and only if the following hold:
   a. $A = R_{0'B^{-1}}C$ $(\exists g \in G \ni A = R_gC, g = 0'B^{-1})$;
   b. $A = B$;
   c. $[(x \diamond y) \diamond (x \diamond z)] \diamond (z \diamond y) = 0'$.

Proof. We make use of Theorem 13. Theorem 10 shall be applied in here as follows: $(G,*)$ is a principal isotope of $(G,\cdot)$ such that $(G,*) \overset{C}{\cong} (H,\diamond)$.

a. is true $\Leftrightarrow AC^{-1} = R_{0(BC^{-1})^{-1}} \Leftrightarrow AC^{-1} = R_{0CB^{-1}} \Leftrightarrow A = R_{0'B^{-1}}C$.

b. is true $\Leftrightarrow AC^{-1} = BC^{-1} \Leftrightarrow A = B$.

c. $[(x\cdot y)*(x\cdot z)]*(z\cdot y) = 0 \Leftrightarrow \{[(x\cdot y)*(x\cdot z)]*(z\cdot y)\}C = 0C \Leftrightarrow [(x\cdot y)*(x\cdot z)]C \diamond (z\cdot y)C = 0' \Leftrightarrow [(x\cdot y)C \diamond (x\cdot z)C] \diamond (z\cdot y)C = 0' \Leftrightarrow [(xA \diamond yB) \diamond (xA \diamond zB)] \diamond (zA \diamond yB) = 0'$.

Replace xA by x, yB by y, and zB by z to get $[(x \diamond y) \diamond (x \diamond z)] \diamond (zB^{-1}A \diamond y) = 0' \Leftrightarrow [(x \diamond y) \diamond (x \diamond z)] \diamond (z \diamond y) = 0'$.

Corollary 2 *Let* $(G,\cdot,0) \xrightarrow{(R_gC,R_gC,C)} (H,\diamond)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond)$ *is a groupoid. Let* $0C = 0'$*, then*

1. $(H,\diamond,0')$ is a (r-r)-quasi NTL, (r-l)-quasi NTL and (r-rl)-quasi NTL.
2. $(H,\diamond,0')$ is a BCI-algebra if and only if $[(x \diamond y) \diamond (x \diamond z)] \diamond (z \diamond y) = 0'$ holds.

Proof. We shall use Theorem 14. 1 and 2 are true because $R_gC = R_{0'(R_gC)^{-1}C}$ since $g = 0'(R_gC)^{-1} \Leftrightarrow g = 0'C^{-1}R_g^{-1} \Leftrightarrow g = 0R_g^{-1} \Leftrightarrow g^2 = 0$, which is true in the BCI-algebra $(G,\cdot,0)$.

2.3. Isotopy of [p-semisimple, quasi-associative] BCI-Algebras and BCK-Algebras

Isotopy of p-semisimple, quasi-associative BCI-algebras and BCK-Algebras is presented.

Theorem 15 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Under any of the following conditions:*

1. $0\delta = 0, \delta \in \mathcal{P}(G,*)$ and $|\delta| = 2$ (i.e. $\delta^2 = I$);
2. $\delta \in \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$ and $|\delta| = 2$;

$(G,\cdot,0)$ is p-semisimple if and only if $(G,*,0)$ is p-semisimple.

Proof. By Theorem 13, $\delta = \varepsilon$.

1. $(G,\cdot,0)$ is p-semisimple if and only if $0\cdot(0\cdot x) = x \Leftrightarrow L_0^2 = I$. $(G,\cdot,0)$ is p-semisimple if and only if $0\delta * (0\delta * x\delta)\delta = x \Leftrightarrow 0 * (0 * x\delta)\delta = x \Leftrightarrow 0 * (0 * x)\delta = x\delta \Leftrightarrow \mathbb{L}_0\delta\mathbb{L}_0 = \delta$. Following 2. of Lemma 2, $(G,\cdot,0)$ is p-semisimple if and only if $\mathbb{L}_0^2 = I \Leftrightarrow (G,*,0)$ is p-semisimple.
2. $(G,\cdot,0)$ is p-semisimple if and only if $(x\cdot y)\cdot(x\cdot z) = z\cdot y \Leftrightarrow L_xL_{x\cdot y} = R_y$. $(G,\cdot,0)$ is p-semisimple if and only if $(x\delta * y\varepsilon)\delta * (x\delta * z\varepsilon)\varepsilon = z\delta * y\varepsilon \Leftrightarrow (x*y)\delta * (x*z)\delta = z*y \Leftrightarrow \mathbb{L}_x\delta\mathbb{L}_{(x*y)\delta} = \mathbb{R}_y$.

Following 3. of Lemma 2, $(G,\cdot,0)$ is p-semisimple if and only if $\mathbb{L}_x\delta^2\mathbb{L}_{(x*y)} = \mathbb{R}_y \Leftrightarrow$ $\mathbb{L}_x\mathbb{L}_{(x*y)} =$
$\mathbb{R}_y \Leftrightarrow (G,*,0)$ is p-semisimple.

Corollary 3 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra, and* $(G,*)$ *is a principal isotope of* $(G,\cdot)$. *Under any of the following conditions:*

1. $0C = 0A, AC^{-1} \in \mathcal{P}(G,*)$ and $CA^{-1}C = A$;
2. $AC^{-1} \in \Phi(G,*)$ with $(AC^{-1})' = AC^{-1} \in \Psi(G,*)$ and $CA^{-1}C = A$;

$(G,\cdot,0)$ is p-semisimple if and only if $(H,\diamond,0')$ is p-semisimple.
Proof. Use the Theorem 15.

Theorem 16 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that* $0\delta = 0$. $(G,\cdot,0)$ *is a BCK-algebra if and only if* $(G,*,0)$ *is a BCK-algebra.*

Proof. $(G,\cdot,0)$ is a BCK-algebra if and only if $0\cdot x = 0 \Leftrightarrow 0\delta * x\varepsilon = 0 \Leftrightarrow 0 * x\delta = 0 \Leftrightarrow 0 * x = 0$ if and only if $(G,*,0)$ is a BCK-algebra.

Corollary 4 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a zero-cancellative BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that* $0C = 0A = 0'$. $(G,\cdot,0)$ *is a BCK-algebra if and only if* $(H,\diamond,0')$ *is a BCK-algebra.*

Proof. Use the Theorem 16.

Theorem 17 *Let* $(G,\cdot,0,\leq) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0,\lessapprox)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Under any of the following conditions:*

1. $\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*)$;
2. $\delta \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$;
3. $\delta \in \Lambda(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$;

$(G,\cdot,0)$ is quasi-associative if and only if $(G,*,0)$ is quasi-associative.

Proof. In the light of Theorem 2, we shall adopt the following representation for any two self maps A and B on G: $A \leq B \Leftrightarrow xA \leq xB$ and $A \lessapprox B \Leftrightarrow xA \lessapprox xB$ for all $x \in G$. Recall that by Theorem 2, $x\cdot y = 0 \Leftrightarrow x \leq y$ and $x * y = 0 \Leftrightarrow x \lessapprox y$. So, $x \leq y \Leftrightarrow x\cdot y = 0 \Leftrightarrow x\delta * y\varepsilon = 0 \Leftrightarrow x\delta \lessapprox y\varepsilon$. Hence, $x \leq y \Leftrightarrow x\delta \lessapprox y\varepsilon$. Note that by Theorem 13, $\delta = \varepsilon$.

1. $(G,\cdot,0)$ is quasi-associative if and only if $(x\cdot y)\cdot z \leq x\cdot(y\cdot z) \Leftrightarrow (x\delta * y\varepsilon)\delta * z\varepsilon \leq x\delta * (y\delta * z\varepsilon)\varepsilon \Leftrightarrow (x*y)\delta * z \leq x * (y*z)\varepsilon \Leftrightarrow \mathbb{R}_y\delta\mathbb{R}_z \leq \mathbb{R}_{(y*z)\delta}$.

Following 1. and 2. of Lemma 2, $(G,\cdot,0)$ is quasi-associative if and only if $\delta\mathbb{R}_y\mathbb{R}_z \leq \delta\mathbb{R}_{y*z} \Leftrightarrow (x\delta * y) * z \leq x\delta * (y*z) \Leftrightarrow (x*y)*z \leq x*(y*z) \Leftrightarrow [(x*y)*z]\cdot[x*(y*z)] = 0 \Leftrightarrow [(x*y)*z]\delta * [x*(y*z)]\varepsilon = 0 \Leftrightarrow [(x*y)*z\delta] * [x*(y*z\varepsilon)] = 0 \Leftrightarrow [(x*y)*z] * [x*(y*z)] = 0 \Leftrightarrow [(x*y)*z] \lessapprox [x*(y*z)]$ if and only if $(G,*,0)$ is quasi-associative.

2. By Lemma 2, $\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*) \Leftrightarrow \delta \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$. Hence, the

conclusion follows by 1.

3. By Lemma 2, $\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*) \Leftrightarrow \delta \in \Lambda(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$. Hence, the conclusion follows by 1.

Corollary 5 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra,* $(H,\diamond,0')$ *is a BCI-algebra and* $(G,*)$ *is a principal isotope of* $(G,\cdot)$ *with* $0C = 0'$. *Under any of the following conditions:*

1. $AC^{-1} \in \mathcal{P}(G,*) \cap \Lambda(G,*)$;
2. $AC^{-1} \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $(AC^{-1})' = AC^{-1} \in \Psi(G,*)$;
3. $AC^{-1} \in \Lambda(G,*) \cap \Phi(G,*)$ with $(AC^{-1})' = AC^{-1} \in \Psi(G,*)$;

$(G,\cdot,0)$ is quasi-associative if and only if $(H,\diamond,0')$ is quasi-associative.

Proof. Use the Theorem 5.

2.4. Isotopy of Associative Fenyves BCI-Algebras

Isotopy of associative Fenyves BCI-algebras is presented. The set $Centrum(G,\cdot)$ of a groupoid $(G,\cdot)$ is defined as $Centrum(G,\cdot) = \{x \in G: xy = yx \forall y \in G\}$.

Theorem 18 *Let* $(G,\cdot,0) \xrightarrow{(\alpha,\alpha,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *and* $(G,*,0)$ *are BCI-algebras.* $(G,*,0)$ *is associative if and only if* $0\alpha^{-1} \in Centrum(G,\cdot)$.

Proof. $0 * x = x \Leftrightarrow 0\alpha^{-1} \cdot x\alpha^{-1} = x \Leftrightarrow \alpha = L_{0\alpha^{-1}} \Leftrightarrow R_{0\alpha^{-1}} = L_{0\alpha^{-1}} \Leftrightarrow 0\alpha^{-1} \in Centrum(G,\cdot)$.

Corollary 6 *Let* $(G,\cdot,0) \xrightarrow{(A,A,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *and* $(H,\diamond,0')$ *are BCI-algebras.* $(H,\diamond,0')$ *is associative if and only if* $0CA^{-1} \in Centrum(G,\cdot)$.

Proof. Use Theorem 18.

Corollary 7 *Let* $(G,\cdot,0) \xrightarrow{(\alpha,\alpha,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *and* $(G,*,0)$ *are BCI-algebras.* $(G,*,0)$ *is an* F_i*-algebra if and only if* $0\alpha^{-1} \in Centrum(G,\cdot)$ *for* $i = 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,$ $23,24,25,26,27,28,30,31,32,33,34,35,36,\ 37,38,40,41,43,44,45,47,48,49,50,51,53,57,58,60$.

Proof. This follows by Theorem 18 and Theorem 12.

Corollary 8 *Let* $(G,\cdot,0) \xrightarrow{(A,A,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *and* $(H,\diamond,0')$ *are BCI-algebras.* $(H,\diamond,0')$ *is an* F_i*-algebra if and only if* $0CA^{-1} \in Centrum(G,\cdot)$ *for* $i = 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,$ $23,24,25,26,27,28,30,31,32,33,34,35,36,\ 37,38,40,41,43,44,45,47,48,49,50,51,53,57,58,60$.

Proof. This follows by Corollary 6 and Theorem 12.

Theorem 19 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is associative if and only if* $(G,*,0)$ *is associative.*

Proof. $(G,\cdot,0)$ is associative if and only if $x \cdot y = y \cdot x \Leftrightarrow x\delta * y\varepsilon = y\delta * x\varepsilon \Leftrightarrow x * y = y * x \Leftrightarrow (G,*,0)$ is associative.

Corollary 9 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is associative if and only if* $(H,\diamond,0')$ *is associative.*

Proof. This follows from Theorem 19.

Corollary 10 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is an* F_i *-algebra if and only if* $(G,*,0)$ *is an* F_i *-algebra,* $i =$ 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,38,40,41, 43,44,45,47,48,49,50,51,53,57,58,60.

Proof. This follows from Theorem 12 and Theorem 19.

Corollary 11 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra. Then* $(G,\cdot,0)$ *is an* F_i *-algebra if and only if* $(H,\diamond,0')$ *is an* F_i *-algebra,* $i =$ 1,2,4,6,7,9,10,11,12,13,14,15,16,17,18,20,22,23,24,25,26,27,28,30,31,32,33,34,35,36,37,38,40,41, 43,44,45,47,48,49,50,51,53,57,58,60.

Proof. This follows from Theorem 12 and Corollary 9.

Remark 5 *Note that those* F_i *identities which are not in Corollary 11, do not necessarily imply associativity in BCI-algebra, hence, they need some isotopic conditions for isotopic invariance. The next subsection addresses this.*

2.5. Isotopy of Non-Associative Fenyves BCI-Algebras

Isotopy of non-associative Fenyves BCI-algebras is presented.

Theorem 20 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that any of the following is true:*

1. .$\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*)$;
2. $\delta \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$;
3. $\delta \in \Lambda(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$.

Then, $(G,\cdot,0)$ is an F_i -algebra if and only if $(G,*,0)$ is an F_i -algebra; where $i = 3,5,8,19,21,29,39,42,46,52,55,56,59$.

Proof. By Lemma 2, $\delta \in \mathcal{P}(G,*) \cap \Lambda(G,*) \Leftrightarrow \delta \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*) \Leftrightarrow \delta \in \Lambda(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$. By Theorem 13, $\delta = \varepsilon$. The arguments of the proof is based on condition 1.

$(G,\cdot,0)$ is an F_3-algebra if and only if $(x \cdot y) \cdot (z \cdot x) = x \cdot [y \cdot (z \cdot x)] \Leftrightarrow (x\delta * y\varepsilon)\delta * (z\delta * x\varepsilon)\varepsilon = x\delta * [y\delta * (z\delta * x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow (x * y)\delta * (z * x)\varepsilon = x * [y * (z * x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{L}_x\delta\mathbb{R}_{(z*x)\varepsilon} = y\mathbb{R}_{(z*x)\varepsilon}\varepsilon\mathbb{L}_x \Leftrightarrow \mathbb{L}_x\delta\mathbb{R}_{(z*x)}\varepsilon = \mathbb{R}_{(z*x)}\varepsilon^2\mathbb{L}_x \Leftrightarrow y\mathbb{L}_x\mathbb{R}_{(z*x)} = y\mathbb{R}_{(z*x)}\mathbb{L}_x \Leftrightarrow [(x * y) * (z * x) = x * [y * (z * x)] \Leftrightarrow (G,*,0)$ is an F_3-algebra.

$(G,\cdot,0)$ is an F_5-algebra if and only if $[(x \cdot y) \cdot z)]x = [x \cdot (y \cdot z)]x \Leftrightarrow [(x * y)\delta * z]\delta * x = [x * (y * z)\varepsilon]\delta * x \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x = y\mathbb{L}_x\delta\mathbb{R}_z\delta\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x = \mathbb{L}_x\delta\mathbb{R}_z\delta\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\varepsilon\delta\mathbb{R}_x = \mathbb{L}_x\mathbb{R}_z\delta^2\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\mathbb{R}_x = \mathbb{L}_x\mathbb{R}_z\mathbb{R}_x \Leftrightarrow y\mathbb{R}_z\mathbb{L}_x\mathbb{R}_x = y\mathbb{L}_x\mathbb{R}_z\mathbb{R}_x \Leftrightarrow [x * (y * z)] * x = [(x * y) * z] * x \Leftrightarrow (G,*,0)$ is an F_5-algebra.

$(G,\cdot,0)$ is an F_8-algebra if and only if $[x \cdot (y \cdot z)] \cdot x = x \cdot [y \cdot (z \cdot x)] \Leftrightarrow [x\delta * (y\delta * z\varepsilon)\varepsilon]\delta * x\varepsilon = x\delta * [y\delta * (z\delta * x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x * (y * z)\varepsilon]\delta * x = x * [y * (z * x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x = y\mathbb{R}_{(z*x)\varepsilon}\varepsilon\mathbb{L}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\varepsilon\delta\mathbb{R}_x = \mathbb{R}_{(z*x)}\varepsilon^2\mathbb{L}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\mathbb{R}_x = \mathbb{R}_{(z*x)}\mathbb{L}_x \Leftrightarrow [x * (y * z)] * x = x * [y * (z * x)] \Leftrightarrow (G,*,0)$ is an F_8-algebra

$(G,\cdot,0)$ is an F_{19}-algebra if and only if $[x \cdot (y \cdot x)] \cdot z = x \cdot [y \cdot (x \cdot z)] \Leftrightarrow [x\delta * (y\delta * x\varepsilon)\varepsilon]\delta * z\varepsilon = x\delta * [y\delta * (x\delta * z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x * (y * x)\varepsilon]\delta * z\varepsilon = x * [y * (x * z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z = y\mathbb{R}_{(x*z)\varepsilon}\varepsilon\mathbb{R}_x \Leftrightarrow \mathbb{R}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z = \mathbb{R}_{(x*z)}\varepsilon^2\mathbb{R}_x \Leftrightarrow \mathbb{R}_x\mathbb{L}_x\mathbb{R}_z = \mathbb{R}_{(x*z)}\mathbb{R}_x \Leftrightarrow [x * (y * x)] * z = x * [y * (x * z)] \Leftrightarrow (G,*,0)$ is an F_{19}-algebra.

$(G,\cdot,0)$ is an F_{21}-algebra if and only if $[(y \cdot x) \cdot (z \cdot x)] = [(y \cdot x) \cdot z] \cdot x \Leftrightarrow (y\delta * x\varepsilon)\delta * (z\delta * x\varepsilon)\varepsilon = [(y\delta * x\varepsilon)\delta * z\varepsilon]\delta * x\varepsilon \Leftrightarrow (y * x)\delta * (z * x)\varepsilon = [(y * x)\delta * z]\delta * x \Leftrightarrow z\mathbb{L}_{y\mathbb{R}_x\delta}\delta\mathbb{R}_x = z\mathbb{R}_x\delta\mathbb{L}_{y\mathbb{R}_x\delta} \Leftrightarrow \mathbb{L}_{y\mathbb{R}_x\delta}\mathbb{R}_x = \mathbb{R}_x\mathbb{L}_{y\mathbb{R}_x\delta} \Leftrightarrow \mathbb{L}_{y\delta\mathbb{R}_x}\mathbb{R}_x = \mathbb{R}_x\mathbb{L}_{y\delta\mathbb{R}_x} \Leftrightarrow z\mathbb{L}_{y\mathbb{R}_x}\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_{y\mathbb{R}_x} \Leftrightarrow [(y * x) * (z * x)] = [(y * x) * z] * x \Leftrightarrow (G,*,0)$ is an F_{21}-algebra.

$(G,\cdot,0)$ is an F_{29}-algebra if and only if $[y \cdot (x \cdot z)] \cdot x = y \cdot [x \cdot (z \cdot x)] \Leftrightarrow [y\delta * (x\delta * z\varepsilon)\varepsilon]\delta * x\varepsilon = y\delta * [x\delta * (z\delta * x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y * (x * z)\varepsilon]\delta * x = y * [x * (z * x)\varepsilon]\varepsilon \Leftrightarrow z\mathbb{L}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_y\varepsilon\delta\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_x\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_y\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_x\mathbb{L}_y \Leftrightarrow [y * (x * z)] * x = y * [x * (z * x)] \Leftrightarrow (G,*,0)$ is an F_{29}-algebra.

$(G,\cdot,0)$ is an F_{39}-algebra if and only if $[y \cdot (x \cdot x)] \cdot z = y \cdot [x \cdot (x \cdot z)] \Leftrightarrow [y\delta * (x\delta * x\varepsilon)\varepsilon]\delta * z\varepsilon = y\delta * [x\delta * (x\delta * z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y * (x * x)\varepsilon]\delta * z = y * [x * (x * z)\varepsilon]\varepsilon \Leftrightarrow z\mathbb{L}_{[y*(x*x)\varepsilon]\delta} = z\mathbb{L}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y \Leftrightarrow$

$\mathbb{L}_{[y*(x*x)\varepsilon\delta]} = \mathbb{L}_x^2\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_{[y*(x*x)]} = \mathbb{L}_x^2\mathbb{L}_y \Leftrightarrow [y*(x*x)]*z = y*[x*(x*z)] \Leftrightarrow (G,*,0)$ is an F_{39}-algebra.

$(G,\cdot,0)$ is an F_{42}-algebra if and only if $(x\cdot x)\cdot(y\cdot z) = [(x\cdot x)\cdot y]\cdot z \Leftrightarrow 0\delta*(y*z)\varepsilon = (0\delta*y)\delta*z \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_{0\delta} = y\mathbb{L}_{0\delta}\delta\mathbb{R}_z \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_0\delta = y\mathbb{L}_0\delta\delta\mathbb{R}_z \Leftrightarrow y\mathbb{R}_z\mathbb{L}_0\varepsilon\delta = y\mathbb{L}_0\mathbb{R}_z \Leftrightarrow y\mathbb{R}_z\mathbb{L}_0 = y\mathbb{L}_0\mathbb{R}_z \Leftrightarrow 0*(y*z) = (0*y)*z \Leftrightarrow (G,*,0)$ is an F_{42}-algebra.

$(G,\cdot,0)$ is an F_{46}-algebra if and only if $[x\cdot(x\cdot y)]\cdot z = x\cdot[x\cdot(y\cdot z)] \Leftrightarrow [x\delta*(x\delta*y\varepsilon)\varepsilon]\delta*z\varepsilon = x\delta*[x\delta*(y\delta*z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x*(x*y)\varepsilon]\delta*z = x*[x*(y*z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{L}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z = y\mathbb{R}_z\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_z \Leftrightarrow \mathbb{L}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z = \mathbb{R}_z\mathbb{L}_x\varepsilon^2\mathbb{L}_z \Leftrightarrow [x*(x*y)]*z = x*[x*(y*z)] \Leftrightarrow (G,*,0)$ is an F_{46}-algebra.

$(G,\cdot,0)$ is an F_{52}-algebra if and only if $(y\cdot z)\cdot(x\cdot x) = [(y\cdot z)\cdot x]\cdot x \Leftrightarrow (y\delta*z\varepsilon)\delta*(x\delta*x\varepsilon)\varepsilon = [(y\delta*z\varepsilon)\delta*x\varepsilon]\delta*x\varepsilon \Leftrightarrow (y*z)\delta*(x*x)\varepsilon = [(y*z)\delta*x]\delta*x \Leftrightarrow y\mathbb{R}_z\delta\mathbb{R}_{(x*x)\varepsilon} = y\mathbb{R}_z\delta\mathbb{R}_x\delta\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\mathbb{R}_{(x*x)}\varepsilon\delta = \mathbb{R}_z\mathbb{R}_x\delta^2\mathbb{R}_x \Leftrightarrow \mathbb{R}_z\mathbb{R}_{(x*x)} = \mathbb{R}_z\mathbb{R}_x^2 \Leftrightarrow (y*z)*(x*x) = [(y*z)*x]*x \Leftrightarrow (G,*,0)$ is an F_{52}-algebra.

$(G,\cdot,0)$ is an F_{55}-algebra if and only if $[(y\cdot z)\cdot x]x = [y\cdot(z\cdot x)]\cdot x \Leftrightarrow [(y*z)\delta*x]\delta*x = [y*(z*x)\varepsilon]\delta*x \Leftrightarrow z\mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\delta\delta\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_y\varepsilon\delta\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_y\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x \Leftrightarrow [(y*z)*x]*x = [y*(z*x)]*x \Leftrightarrow (G,*,0)$ is an F_{55}-algebra.

$(G,\cdot,0)$ is an F_{56}-algebra if and only if $[(y\cdot z)\cdot x]\cdot x = y\cdot[(z\cdot x)\cdot x] \Leftrightarrow [(y\delta*z\varepsilon)\delta*x\varepsilon]\delta*x\varepsilon = y\delta*[(z\delta*x\varepsilon)\delta*x\varepsilon]\varepsilon \Leftrightarrow [(y*z)\delta*x]\delta*x = y*[(z*x)\delta*x]\varepsilon \Leftrightarrow z\mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x = z\mathbb{R}_x\delta\mathbb{R}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x = \mathbb{R}_x\delta\mathbb{R}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_y\mathbb{R}_x\delta^2\mathbb{R}_x = \mathbb{R}_x\mathbb{R}_x\delta\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_y\mathbb{R}_x\mathbb{R}_x = \mathbb{R}_x\mathbb{R}_x\mathbb{L}_y \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\mathbb{R}_x = z\mathbb{R}_x\mathbb{R}_x\mathbb{L}_y \Leftrightarrow [(y*z)*x]*x = y*[(z*x)*x] \Leftrightarrow (G,*,0)$ is an F_{56}-algebra.

$(G,\cdot,0)$ is an F_{59}-algebra if and only if $[y\cdot(z\cdot x)]\cdot x = y\cdot[z\cdot(x\cdot x)] \Leftrightarrow [y\delta*(z\delta*x\varepsilon)\varepsilon]\delta*x\varepsilon = y\delta*[z\delta*(x\delta*x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y*(z*x)\varepsilon]\delta*x = y*[z*(x*x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_{(z*x)\varepsilon}\delta\mathbb{R}_x = y\mathbb{R}_{[z*(x*x)\varepsilon]\varepsilon} \Leftrightarrow \mathbb{R}_{(z*x)}\varepsilon\delta\mathbb{R}_x = \mathbb{R}_{[z*(x*x)]}\varepsilon^2 \Leftrightarrow \mathbb{R}_{(z*x)}\mathbb{R}_x = \mathbb{R}_{[z*(x*x)]} \Leftrightarrow [y*(z*x)]*x = y*[z*(x*x)] \Leftrightarrow (G,*,0)$ is an F_{59}-algebra.

Corollary 12 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that any of the following is true:*

1. $AC^{-1} \in \mathcal{P}(G,*) \cap \Lambda(G,*)$;
2. $AC^{-1} \in \mathcal{P}(G,*) \cap \Phi(G,*)$ with $\delta' = \delta \in \Psi(G,*)$;
3. $AC^{-1} \in \Lambda(G,*) \cap \Phi(G,*)$ with $(AC^{-1})' = AC^{-1} \in \Psi(G,*)$;

where $(G,*)$ is a principal isotope of $(G,\cdot)$ with $0C = 0'$. Then $(G,\cdot,0)$ is an F_i-algebra if and only if $(H,\diamond,0')$ is an F_i-algebra; where $i = 3,5,8,19,21,29,39,42,46,52,55,56,59$.

Proof. This follows from Theorem 20 and Theorem 14.

Theorem 21 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that* $\delta \in \Lambda(G,*)$ *and* $|\delta| = 2$. *Then* $(G,\cdot,0)$ *is an* F_{56}*-algebra if and only if* $(G,*,0)$ *is an* F_{56}*-algebra.*

Proof. By Theorem 13, $\delta = \varepsilon$.
$(G,\cdot,0)$ is an F_{56}-algebra if and only if $[(y \cdot z) \cdot x] \cdot x = y \cdot [(z \cdot x) \cdot x] \Leftrightarrow [(y * z)\delta * x]\delta * x = y * [(z * x)\delta * x]\varepsilon \Leftrightarrow z\mathbb{L}_y\delta\mathbb{R}_x\delta\mathbb{R}_x = z\mathbb{R}_x\delta\mathbb{R}_x\varepsilon\mathbb{L}_y \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\delta\delta\mathbb{R}_x = z\mathbb{R}_x\mathbb{R}_x\delta\varepsilon\mathbb{L}_y \Leftrightarrow z\mathbb{L}_y\mathbb{R}_x\mathbb{R}_x = z\mathbb{R}_x\mathbb{R}_x\mathbb{L}_y \Leftrightarrow [(y * z) * x] * x = y * [(z * x) * x] \Leftrightarrow (G,*,0)$ is an F_{56}-algebra.

Corollary 13 *Let* $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(H,\diamond,0')$ *is a BCI-algebra such that* $AC^{-1} \in \Lambda(G,*)$ *and* $|AC^{-1}| = 2$. *Then* $(G,\cdot,0)$ *is an* F_{56}*-algebra if and only if* $(H,\diamond,0')$ *is an* F_{56}*-algebra.*

Proof. This follows from Theorem 21 and Theorem 14.

Theorem 22 *Let* $(G,\cdot,0) \xrightarrow{(\delta,\varepsilon,I)} (G,*,0)$ *where* $(G,\cdot,0)$ *is a BCI-algebra and* $(G,*,0)$ *is a BCI-algebra such that* $\delta \in \mathcal{P}(G,*)$ *and* $|\delta| = 2$. *Then* $(G,\cdot,0)$ *is an* F_i*-algebra if and only if* $(G,*,0)$ *is an* F_i*-algebra; where* $i = 8,19,29,39,46,59$.

Proof. By Theorem 13, $\delta = \varepsilon$.
$(G,\cdot,0)$ is an F_8-algebra if and only if $[x \cdot (y \cdot z)] \cdot x = x \cdot [y \cdot (z \cdot x)] \Leftrightarrow [x\delta * (y\delta * z\varepsilon)\varepsilon]\delta * x\varepsilon = x\delta * [y\delta * (z\delta * x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x * (y * z)\varepsilon]\delta * x = x * [y * (z * x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_z\varepsilon\mathbb{L}_x\delta\mathbb{R}_x = y\mathbb{R}_{(z*x)\varepsilon}\varepsilon\mathbb{L}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\varepsilon\delta\mathbb{R}_x = \mathbb{R}_{(z*x)}\varepsilon^2\mathbb{L}_x \Leftrightarrow \mathbb{R}_z\mathbb{L}_x\mathbb{R}_x = \mathbb{R}_{(z*x)}\mathbb{L}_x \Leftrightarrow [x * (y * z)] * x = x * [y * (z * x)] \Leftrightarrow (G,*,0)$ is an F_8-algebra.

$(G,\cdot,0)$ is an F_{19}-algebra if and only if $[x \cdot (y \cdot x)] \cdot z = x \cdot [y \cdot (x \cdot z)] \Leftrightarrow [x\delta * (y\delta * x\varepsilon)\varepsilon]\delta * z\varepsilon = x\delta * [y\delta * (x\delta * z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x * (y * x)\varepsilon]\delta * z\varepsilon = x * [y * (x * z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z = y\mathbb{R}_{(x*z)\varepsilon}\varepsilon\mathbb{R}_x \Leftrightarrow \mathbb{R}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z = \mathbb{R}_{(x*z)}\varepsilon^2\mathbb{R}_x \Leftrightarrow \mathbb{R}_x\mathbb{L}_x\mathbb{R}_z = \mathbb{R}_{x*z)}\mathbb{R}_x \Leftrightarrow [x * (y * x)] * z = x * [y * (x * z)] \Leftrightarrow (G,*,0)$ is an F_{19}-algebra.

$(G,\cdot,0)$ is an F_{29}-algebra if and only if $[y \cdot (x \cdot z)] \cdot x = y \cdot [x \cdot (z \cdot x)] \Leftrightarrow [y\delta * (x\delta * z\varepsilon)\varepsilon]\delta * x\varepsilon = y\delta * [x\delta * (z\delta * x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y * (x * z)\varepsilon]\delta * x = y * [x * (z * x)\varepsilon]\varepsilon \Leftrightarrow z\mathbb{L}_x\varepsilon\mathbb{L}_y\delta\mathbb{R}_x = z\mathbb{R}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_y\varepsilon\delta\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_x\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_x\mathbb{L}_y\mathbb{R}_x = z\mathbb{R}_x\mathbb{L}_x\mathbb{L}_y \Leftrightarrow [y * (x * z)] * x = y * [x * (z * x)] \Leftrightarrow (G,*,0)$ is an F_{29}-algebra.

$(G,\cdot,0)$ is an F_{39}-algebra if and only if $[y \cdot (x \cdot x)] \cdot z = y \cdot [x \cdot (x \cdot z)] \Leftrightarrow [y\delta * (x\delta * x\varepsilon)\varepsilon]\delta * z\varepsilon = y\delta * [x\delta * (x\delta * z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y * (x * x)\varepsilon]\delta * z = y * [x * (x * z)\varepsilon]\varepsilon \Leftrightarrow z\mathbb{L}_{[y*(x*x)\varepsilon]\delta} = z\mathbb{L}_x\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_y \Leftrightarrow$

$\mathbb{L}_{[y*(x*x)\varepsilon\delta]} = \mathbb{L}_x^2\varepsilon^2\mathbb{L}_y \Leftrightarrow \mathbb{L}_{[y*(x*x)]} = \mathbb{L}_x^2\mathbb{L}_y \Leftrightarrow [y*(x*x)]*z = y*[x*(x*z)] \Leftrightarrow (G,*,0)$ is an F_{39}-algebra.

$(G,\cdot,0)$ is an F_{46}-algebra if and only if $[x\cdot(x\cdot y)]\cdot z = x\cdot[x\cdot(y\cdot z)] \Leftrightarrow [x\delta*(x\delta*y\varepsilon)\varepsilon]\delta*z\varepsilon = x\delta*[x\delta*(y\delta*z\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [x*(x*y)\varepsilon]\delta*z = x*[x*(y*z)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{L}_x\varepsilon\mathbb{L}_x\delta\mathbb{R}_z = y\mathbb{R}_z\varepsilon\mathbb{L}_x\varepsilon\mathbb{L}_z \Leftrightarrow \mathbb{L}_x\mathbb{L}_x\varepsilon\delta\mathbb{R}_z = \mathbb{R}_z\mathbb{L}_x\varepsilon^2\mathbb{L}_z \Leftrightarrow [x*(x*y)]*z = x*[x*(y*z)] \Leftrightarrow (G,\cdot,0)$ is an F_{46}-algebra.

$(G,\cdot,0)$ is an F_{59}-algebra if and only if $[y\cdot(z\cdot x)]\cdot x = y\cdot[z\cdot(x\cdot x)] \Leftrightarrow [y\delta*(z\delta*x\varepsilon)\varepsilon]\delta*x\varepsilon = y\delta*[z\delta*(x\delta*x\varepsilon)\varepsilon]\varepsilon \Leftrightarrow [y*(z*x)\varepsilon]\delta*x = y*[z*(x*x)\varepsilon]\varepsilon \Leftrightarrow y\mathbb{R}_{(z*x)\varepsilon}\delta\mathbb{R}_x = y\mathbb{R}_{[z*(x*x)\varepsilon]\varepsilon} \Leftrightarrow \mathbb{R}_{(z*x)}\varepsilon\delta\mathbb{R}_x = \mathbb{R}_{[z*(x*x)]}\varepsilon^2 \Leftrightarrow \mathbb{R}_{(z*x)}\mathbb{R}_x = \mathbb{R}_{[z*(x*x)]} \Leftrightarrow [y*(z*x)]*x = y*[z*(x*x)] \Leftrightarrow (G,*,0)$ is an F_{59}-algebra.

Corollary 14 *Let $(G,\cdot,0) \xrightarrow{(A,B,C)} (H,\diamond,0')$ be an isotopism; where $(G,\cdot,0)$ is a BCI-algebra and $(H,\diamond,0')$ is a BCI-algebra such that $AC^{-1} \in \mathcal{P}(G,*)$ and $|AC^{-1}| = 2$, where $(G,*)$ is a principal isotope of $(G,\cdot)$ with $0C = 0'$. Then, $(G,\cdot,0)$ is an F_i-algebra if and only if $(H,\diamond,0')$ is an F_i-algebra; where $i = 8,19,29,39,46,59$.*
Proof. This follows from Theorem 22 and Theorem 14.

Remark 6 *Note that those F_i identities which do not appear in Corollaries 12,13,14 will trivially obey these corollaries because they imply associativity in BCI-algebra with no condition(s) placed on the isotopy.*

3. Summary, Conclusion and Future Studies

We shall now highlight the theoretical and practical implications of this research, discuss our research findings, highlight practical advantages and research limitations, and then suggest some future studies.

Comparing the characterization of the permutation in the isotopy for the isotopic invariance of quasi-associativity (a measure of weak associativity) in Theorem 17 and the characterization of the permutation in the isotopy for the isotopic invariance of the 13 non-associative F_i algebras in Theorem 20, the three are the same. This is a new contribution to the fact that isotopy in BCI-algebras and quasi-associativity can be measured with 14 non-associative F_i identities.

In loop theory, all the 30 Fenyves identities that are equivalent to associativity are isotopic invariant for any isotopy and some of the other 30 Fenyves identities that are non-associative (e.g. Moufang, Bol, Extra) are also isotopic invariant for any isotopy, while the others (e.g. LC, RC, C) are not. From our results in this work, all the 46 F_i identities that are equivalent to associativity in BCI-algebras are isotopic invariant for any isotopy, while for the 14 Fenyves identities that are non-associative in BCI-algebras; they are isotopic invariant for special isotopies including some well known identities (e.g. left Bol, LC and RC). Thus, it can be concluded that the isotopy of Fenyves identities that are non-associative in BCI-algebras is of better advantage over Fenyves identities that are non-associative in loops. But, there is limitation on the isotopy of all the 46 F_i identities that are equivalent to associativity in BCI-algebras.

Those 46 Fenyves identities that are equivalent to associativity in BCI-algebras as well as F_{54} which of course are isotopic invariant under any isotopy are denoted by √ in the fourth and fifth columns of Table 1 and Table 2. While the 13 Fenyves identities that are equivalent to associativity in BCI-algebras excluding F_{54} which are isotopic invariant under special isotopies are identified by the symbol '‡' in the fourth and fifth columns of Table 1 and Table 2. Theoretically and practically, this research implies the isotopic study of 120 particular types of the 540 varieties of Fenyves quasi neutrosophic triplet loops (FQNTLs) discovered in Jaiyéolá et al. [36] (cf. Figure 1).

For future studies, based on the philosophy of representing disease-victim(s) by neutrosophic algebraic structures, some of the 14 Fenyves identities that are non-associative in BCI-algebras (quasi neutrosophic loops) can be judiciously selected with good and appropriate choice of special isotopies for which such are isotopic invariant in order to study and understand the effects of diseases and possible treatment of a patient.

FENYVES IDENTITY	$\boldsymbol{F_i \equiv ASS}$ IN A LOOP	$\boldsymbol{F_i}$ ISO INVAR IN A LOOP	$\boldsymbol{F_i}$ ISO INVAR IN BCI ALG	$\boldsymbol{F_i + BCI \Rightarrow ASS}$
F_1	√		√	√
F_2		√	√	√
F_3	√		‡	‡
F_4		√	√	√
F_5	√		‡	‡
F_6		√	√	√
F_7	√		√	√
F_8	√		‡	‡
F_9		√	√	√
F_{10}	√		√	√
F_{11}	√		√	√
F_{12}	√		√	√
F_{13}		√	√	√
F_{14}	√		√	√
F_{15}		√	√	√
F_{16}	√		√	√
F_{17}		√	√	√
F_{18}	√		√	√
F_{19}		√	‡	‡
F_{20}	√		√	√
F_{21}	√		‡	‡
F_{22}		√	√	√
F_{23}	√		√	√
F_{24}	√		√	√
F_{25}	√		√	√
F_{26}		√	√	√
F_{27}		√	√	√
F_{28}	√		√	√
F_{29}	√		‡	‡
F_{30}		√	√	√
F_{31}	√		√	√
F_{32}	√		√	√
F_{33}	√		√	√
F_{34}	√		√	√
F_{35}		√	√	√

Table 1: Characterization of the Isotopy of Fenyves Identities in Loops and BCI-Algebras

FENYVES IDENTITY	$\boldsymbol{F_i \equiv ASS}$ IN A LOOP	$\boldsymbol{F_i}$ ISO INVAR IN A LOOP	$\boldsymbol{F_i}$ ISO INVAR IN BCI ALG	$\boldsymbol{F_i + BCI \Rightarrow ASS}$
F_{36}		√	√	√
F_{37}		√	√	√
F_{38}		√	√	√
F_{39}		√	‡	‡
F_{40}		√	√	√
F_{41}		√	√	√
F_{42}		√	‡	‡
F_{43}		√	√	√
F_{44}	√		√	√
F_{45}		√	√	√
F_{46}		√	‡	‡
F_{47}	√		√	√
F_{48}		√	√	√
F_{49}	√		√	√
F_{50}	√		√	√
F_{51}		√	√	√
F_{52}	√		‡	‡
F_{53}		√	√	√
F_{54}		√	√	‡
F_{55}	√		‡	‡
F_{56}		√	‡	‡
F_{57}		√	√	√
F_{58}	√		√	√
F_{59}	√		‡	‡
F_{60}		√	√	√

Table 2: Characterization of the Isotopy of Fenyves Identities in Loops and BCI-Algebras

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References

[1] Abdel Basset M., El-hoseny M., Gamal A. and Smarandache F. (2019), *A Novel Model for Evaluation Hospital Medical Care Systems based on Plithogenic Sets*, Artificial Intelligence in Medicine, 101710.

[2] Abdel-Basset M., Manogaran G., Gamal A. and Chang V. (2019), *A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT.*, IEEE Internet of Things Journal.

[3] Abdel-Basset M., Mohammed R., Zaied A. E. N. and Smarandache F. (2019), *A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics*, Symmetry, 11(7), 903.

[4] Abdel-Basset M., Nabeeh N. A., El-Ghareeb H. A. and Aboelfetouh A. (2019), *Utilising Neutrosophic Theory to Solve Transition Difficulties of IoT-Based Enterprises*, Enterprise Information Systems, 1-21.

[5] Artzy R. (1959), *Crossed inverse and related loops*, Trans. Amer. Math. Soc. 91, 3, 480-492.

[6] Burn R.P. (1978), Finite Bol loops, *Math. Proc. Camb. Phil. Soc.* 84, 377-385.

[7] Burn R.P. (1981), Finite Bol loops II, *Math. Proc. Camb. Phil. Soc.* 88, 445-455.

[8] Burn R.P. (1985), Finite Bol loops III, *Math. Proc. Camb. Phil. Soc.* 97, 219-223.

[9] Dudek W. A. (1988), On group-like BCI-algebras, *Demonstratio Math.* 21, 2, 369-376.

[10] Dudek W. A. (1986), On some BCI-algebras with the condition (S), *Math. Japon.* 31, 1, 25-29.

[11] Dudek W. A. (1987), On medial BCI-algebras, *Prace Nauk. WSP, Czestochowa, Matematyka* 1, 25-33.

[12] Fenyves F. (1968), Extra loops I, *Publ. Math. Debrecen* 15, 235-238.

[13] Fenyves F. (1969), Extra Loops II, *Publ. Math. Debrecen* 16, 187-192.

[14] Hwang Y.S. and Ahn S.S. (2014), Soft q-ideals of soft BCI-algebras, *J. Comput. Anal. Appl.*, 16, 3, 571-582.

[15] Ilojide E., Jaiyéolá T. G. and Olatinwo M. O. (2019), *On Holomorphy of Fenyves BCI-Algebras*, Journal of the Nigerian Mathematical Society, Vol. 38, No. 2, 139-155.

[16] Imai Y. and Iseki K. (1966), On axiom systems of propositional calculi, XIV *Proc. Japan Academy* 42, 19-22.

[17] Iseki K (1966), *An algebra related with a propositional calculus*, Proc. Jpn. Acad. Ser. A Math. Sci. 42, 26-29. http://dx.doi.org/10.3792/pja/1195522171

[18] Iseki K. (1977), On BCK-Algebras with condition (S), *Math. Seminar Notes* 5, 215-222.

[19] Iseki K. (1979), BCK-Algebras with condition (S), *Math. Japon.* 24, 107-119.

[20] Jaiyéolá T.G. (2005), An isotopic study of properties of central loops, *M.Sc. dissertation*, University of Agriculture, Abeokuta.

[21] Jaiyéolá T.G. (2008), The study of the universlity of Osborn loops, *Ph.D. thesis*, University of Agriculture, Abeokuta.

[22] Jaiyéolá T. G. (2009), *On the universality of central loops*, Acta Universitatis Apulensis Mathematics-Informatics, 19, 113-124.

[23] Jaiyéolá T. G. (2012), *Osborn loops and their universality*, Scientific Annals of "Al.I. Cuza" University of Iasi., Tomul LVIII, f.2, 437-452.

[24] Jaiyéolá T. G. (2013), *New identities in universal Osborn loops II*, Algebras, Groups and Geometries, Vol. 30, No. 1, 111-126

[25] Jaiyéolá T. G. (2014), *On some simplicial complexes of universal Osborn loops*, Analele Universitatii De Vest Din Timisoara, Seria Matematica-Informatica, Vol. 52, No.1, 65-79. DOI: 10.2478/awutm-2014-0005

[26] Jaiyéolá T. G. and Adéníran J. O. (2006), *On the derivatives of central loops*, Advances in Theoretical and Applied Mathematics, 1(3), 233-244.

[27] Jaiyéolá T. G. and Adéníran J. O. (2008), *On some autotopisms of non-Steiner central loops*, Journal Of Nigerian Mathematical Society, 27, 53-68.

[28] Jaiyéolá T. G. and Adéníran J. O. (2009), *On isotopic characterization of central loops*, Creative Mathematics and Informatics, 18(1), 39-45.

[29] Jaiyéolá T. G. and Adéníran J. O. (2009), *New identities in universal Osborn loops*, Quasigroups And Related Systems, Vol. 17, No. 1, 55-76.

[30] Jaiyéolá T. G. and Adéníran J. O. (2009), *Not every Osborn loop is universal*, Acta Mathematica Academiae Paedagogiace NyÃ-regyhÃ¡ziensis, Vol. 25, No. 2, 189-190.

[31] Jaiyéolá T. G. and Adéníran J. O. (2009), *New identities in universal Osborn loops*, Quasigroups And Related Systems, Vol. 17, No. 1, 55-76.

[32] Jaiyéolá T. G. and Adéníran J. O. (2011), *Loops that are isomorphic to their Osborn loop isotopes(G-Osborn loops)*, Octogon Mathematical Magazine, Vol. 19, No. 2, 328-348.

[33] Jaiyéolá T. G. , Adéníran J. O. and Sòlárìn A. R. T. (2011), *The universality of Osborn loops*, Acta Universitatis Apulensis Mathematics-Informatics, Vol. 26, 301-320.

[34] Jaiyéolá, T.G. and Smarandache F. (2017), *Some Results on Neutrosophic Triplet Group and Their Applications*, Symmetry, 10, 202. http://dx.doi.org/10.3390/sym10060202

[35] Jaiyéolá, T.G. and Smarandache F. (2018), *Inverse Properties in Neutrosophic Triplet Loop and their Application to Cryptography*, Algorithms, 11, 32. http://dx.doi.org/10.3390/a11030032

[36] Jaiyéolá T.G., Ilojide E., Olatinwo M.O. and Smarandache F. (2018), On the Classification of Bol-Moufang Type of Some Varieties of Quasi Neutrosophic Triplet Loop (Fenyves BCI-Algebras), *Symmetry*, 10, 10, 427. https://doi.org/10.3390/sym10100427.

[37] Kepka T., Kinyon M. K. and Phillips J. D. (2007), *The structure of F-quasigroups*, J. Alg., 317, 435-461.

[38] Kepka T., Kinyon M. K. and Phillips J. D. (2008), *F-quasigroups and generalised modules*, Commentationes Mathematicae Universitatis Carolinae, 49, 2, 249-257.

[39] Kepka T., Kinyon M. K. and Phillips J. D. (2010), *F-quasigroups isotopic to groups*, Comment. Math. Univ. Carolin. 51, 2, 267-277.

[40] Kinyon M.K. and Kunen K. (2004), *The structure of extra loops*, Quasigroups and Related Systems 12, 39-60.

[41] Lee K.J. (2013), A new kind of derivations in BCI-algebras, *Appl. Math. Sci (Ruse)*, 7, 81-84.

[42] Osborn J. M.(1961), *Loops with the weak inverse property*, Pac. J. Math. 10, 295-304.

[43] Pflugfelder H.O. (1990), *Quasigroups and loops: Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 147pp.

[44] Phillips J.D. and Vojtěchovský P. (2005), *The varieties of loops of Bol-Moufang type*, Alg. Univ., 54, 259-271. http://dx.doi.org/10.1007/s00012-005-1941-1

[45] Phillips J.D. and Vojtěchovský P. (2005), *The varieties of quasigroups of Bol-Moufang type: An equational reasoning approach*, J. Alg. 293, 17-33. http://dx.doi.org/10.1016/j.jalgebra.2005.07.011

[46] Phillips J.D. and Vojtěchovský P. (2006), *C-loops; An introduction*, Publ. Math. Derbrecen 68, 1-2, 115-137.

[47] Robinson D.A. (1964), Bol-loops, *Ph.D Thesis, University of Wisconsin* Madison.

[48] Smarandache F. and Ali M. (2018), *Neutrosophic triplet group*, Neural Comput. Appl., 29, 595-601. http://dx.doi.org/10.1007/s00521-016-2535-x
[49] Syrbu P. N. (1996), *On loops with universal elasticity*, Quasigroups and Related Systems, 3, 41-54.
[50] Walendziak A. (2015), Pseudo-BCH-Algebras, *Discussiones Mathematicae, General Algebra and Applications*. 35, 5-19; doi:10.7151/dmgaa.1233.
[51] Yisheng H. (2006), BCI-Algebra, *Science Press*, Beijing, 356pp.
[52] Zhang X., Wu X., Smarandache F. and Hu M. (2018), *Left (Right)-Quasi Neutrosophic Triplet Loops (Groups) and Generalized BE-Algebras*, Symmetry, 10(7), 241; https://doi.org/10.3390/sym10070241
[53] Zhang X., Wang X., Smarandache F., Jaiyéolá, T. G. and Lian T. (2019), *Singular neutrosophic extended triplet groups and generalized groups*, Cognitive Systems Research, 57, 32-40; https://doi.org/10.1016/j.cogsys.2018.10.009
[54] Zhang X., Smarandache F. and Liang X. (2017), *Neutrosophic Duplet Semi-Group and Cancellable Neutrosophic Triplet Groups*, Symmetry, 9, 275. http://dx.doi.org/10.3390/sym9110275
[55] Zhang, X., Hu Q., Smarandache F. and An X. (2018), *On Neutrosophic Triplet Groups: Basic Properties, NT-Subgroups, and Some Notes*, Symmetry, 10, 289. http://dx.doi.org/10.3390/sym10070289

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Multi-Aspect Decision-Making Process in Equity Investment Using Neutrosophic Soft Matrices

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Abstract: Neutrosophic theory alleviates the ambiguity situation more effectively than fuzzy sets. Neutrosophic soft set deals with the combination of truth, indeterminacy and falsity membership. This provides a space for the convention with multi-aspect decision-making (MADM) problems that involve these combinations. The main aim of this paper is to provide a unique ranking for the alternatives to overcome the existing drawbacks in the said environment. Initially, a new score function and the weighted neutrosophic vector are discussed. Secondly, to show the supremacy of the proposed score function a comparison analysis is discussed between the existing score method and the proposed approach. Thirdly, algorithm and flowchart are discussed for the case study. Lastly, a new technique for ranking the alternatives is discussed which enables us to determine the unique highest score. The working model is illustrated with suitable examples to authenticate the tool and to demonstrate the effectiveness of the planned approach.

Keywords: Single valued neutrosophic sets, Neutrosophic soft matrix (NSM), weighted neutrosophic vector, Score and value function, Multi-aspect decision-analysis.

1. Introduction

Our world is complex and rapid changes keep occurring in the field of engineering, medical science, banking, modern education, social, economic, and various other fields. Complexity generally arises from ambiguity and to overcome these situations in day to day life, Zadeh (1965) introduced a fuzzy set (FS) [14] and an interval-valued fuzzy set (IVFS) [15]. Atanassov (1986) proposed the concept of intuitionistic fuzzy set (IFS) [1] and interval-valued intuitionistic fuzzy set [2] a combination of membership and non-membership functions. However, both fuzzy and intuitionistic fuzzy sets cannot treat the indeterminacy part in the day to day problems. To deal with indeterminacy situations, Smarandache (1998) grounded the neutrosophic set (NS) [10] theory which is an overview of FS and IFS. In plithogenic set (PS) elements are characterized by the attribute values. It was introduced by Smarandache [27] as a generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets.

FS, IVFS, IFS, NS, PS and hybrid of these sets are used in various decision-making problems. Decision making plays a significant role in today's social, scientific and economic endeavor. Most of the decision-making process is based on an objective to reduce the cost, reduce the production time, and increase the profit for the organization. However, considering today's environment the decision should include various objective sources to deal with uncertainty. It weighs the provided information and chooses the best criteria for subsequent action. The information provided in a complex world is likely ambiguous, hence the outcomes are vague, irrespective of the decision made on the criteria chosen. To explain this scenario, consider the criteria of taking a loan from a bank. The outcome can be ambiguous with the possibility of a loan getting approved or declined or undetermined. The primary issues in MADM are to rank the relative importance of each of the objectives. Despite our vast knowledge and experience in handling these objectives, we come across violations in our everyday life. A bank manager makes a decision in this complex environment and figures out that his/her decision becomes weird. We have come across many situations where the loan applicant fails to repay the loan amount despite following the scrutiny process. The said problem could be due to the change in information and condition according to the situation. The outcomes of these situations have nothing to do with the quality of the decisions made. The best we can do with our knowledge is that in the long run the `good decisions' will outplay the `bad decisions'.

Most of the researchers utilize NS as a significant tool to analyze MADM problems with the help of aggregation operators, information measures, score functions and machine learning algorithms. Abhishek et al. [28] developed a parametric divergence measure and initiated the concept of pattern recognition and medical diagnosis problem for neutrosophic sets. Abdel-Basset et al. [18] proposed a hybrid combination between analytical hierarchical process and neutrosophic theory to solve the uncertainty involved in the technology of the internet of things. Abhisek and Rakesh [29] proposed a notion for finding the threshold value in decision-making problems when the qualitative and quantitative information is outsized. Abdel-Basset et al. [20] proposed the concept of type 2 neutrosophic number TOPSIS method to deal with real case decision problems. Edalatpanah and Smarandache [30] found a new method to solve the data envelopment analysis using the weighted arithmetic average operator in neutrosophic sets. Abdel-Basset et al. [19] initiated a neutrosophic approach for evaluating green supply chain management to aid managers and decision-makers. Vakkas et al. [33] proposed a novel ranking method for decision-making problems in the bipolar neutrosophic environment. Pandy and Trinita [31] constructed a new approach to represent gray-scale (medical) images in the bipolar neutrosophic domain. Shazia et al. [32] presented the concept of the plithogenic hypersoft matrix and discussed some of its theoretical properties. Abdel-Basset et al. [17] developed the combination of quality function deployment with plithogenic operations and analyzed the case study of Thailand's sugar industry and also developed a novel evaluation approach to handle the hospital medical care systems based on plithogenic sets [16]. Azeddine et al. [34] introduced an improved method to map machine learning algorithms from crisp number to Neutrosophic environment. Wang and Smarandache (2010) focused on single-valued neutrosophic set [13] to magnetize on MADM problems. Chinnadurai et al., (2016) [3] discussed some of its theoretical properties. Smarandache and Teodorescu (2014) introduced the fusion of fuzzy data to neutrosophic data [11] with case studies. Garg and Nancy (2018) developed the neutrosophic Muirhead mean operators [5] for an aggregating single-valued neutrosophic set to solve MADM problems among the ambiguity. Gulistan et al., (2019) studied on neutrosophic cubic soft matrices [6] using max-min operations. Jun et al. presented elucidation to handle actual data which consists of crisp values using the neutrosophic analytic hierarchy process. Abdel-Basset et.al.

[12] developed the concept of Neutrosophic AHP-SWOT Analysis for MADM problems by analyzing a real case study.

The advantage of this proposed method is that it shortens the computation process and provides a better solution in decision-making. To establish the superiority of our improved score function a comparison study is illustrated with suitable examples. From the presented references [21, 22, 23, 24, 25, 26] it is clear that there are limitations in providing unique ranking using score function in neutrosophic MADM methods. The fact that we would like to enlighten in this manuscript is that there could always be a possibility of equal ranking among the alternatives. Hence, to our knowledge, a simple but effective way to determine the unique highest score for each object in a MADM is by including additional criteria from the parameter set which is not been discussed in any of the related literature works.

In this paper, we aim to discuss the weighted neutrosophic vector and value function of a neutrosophic soft matrix to combine the different components of truth, indeterminacy and falsity membership into a single membership value. An application of this matrix in MADM is also given by presenting the method, algorithm and numerical illustrations.

The structure of the manuscript is as follows. In section 2, some of the basic neutrosophic definitions are specified. In section 3, the notions of weighted neutrosophic vector and value functions are introduced. In section 4, an algorithm with a flowchart of NSM to MADM is developed. In section 5, case studies are presented to illustrate the working of the algorithm. This manuscript is concluded in section 6.

2. Preliminaries

In this section first we review some basic concepts and definitions.

Definition 2.1[9] Let U be the universal set and E be a set of parameters. The parameters represent some selected properties or characteristics of the elements of U. Let P(U) denote the power set of U. A pair (F, E) is called a **soft set** over U where F is a mapping $F: E \to P(U)$. It is clear that a soft set is a parameterized family of subsets of U.

Definition 2.2 [13] Let U be the universal set, then a set $\mathbb{A} = \{\langle x, T^{\mathbb{A}}(x), I^{\mathbb{A}}(x), F^{\mathbb{A}}(x)\rangle : x \in U\}$ is termed as **neutrosophic set** where $T^{\mathbb{A}}, I^{\mathbb{A}}, F^{\mathbb{A}}: X \to [0,1]$ with $0 \leq T^{\mathbb{A}}(x) + I^{\mathbb{A}}(x) + F^{\mathbb{A}}(x) \leq 3$ and the functions $T^{\mathbb{A}}, I^{\mathbb{A}}, F^{\mathbb{A}}$ are truth, indeterminacy and falsity membership degrees respectively.

Definition 2.3 [8] Let U be the universal set and E be a set of parameters. Consider $\mathbb{A} \subseteq E$. Let $NS(U)$ denote the set of all neutrosophic sets of U. The collection $(F, \mathbb{A})$ is termed to be the **neutrosophic soft set** (NSS) over U, where F is a mapping given by $F: \mathbb{A} \to NS(U)$.

Definition 2.4 [4] Let $(N^{\mathbb{A}}, E)$ be a NSS over the universe U and E be a set of parameters and $\mathbb{A} \subseteq E$. Then a subset of $U \times E$ is uniquely defined by the relation $\{(x, e): e \in \mathbb{A}, x \in N^{\mathbb{A}}(e)\}$ and denoted by $R_{\mathbb{A}} = (N^{\mathbb{A}}, E)$. The relation $R_{\mathbb{A}}$ is characterized by truth function $T^{\mathbb{A}}: U \times E \to [0,1]$, indeterminacy $I^{\mathbb{A}}: U \times E \to [0,1]$ and the falsity function $F^{\mathbb{A}}: U \times E \to [0,1]$. $R_{\mathbb{A}}$ is represented as $R_{\mathbb{A}} = \{(T^{\mathbb{A}}(x, e), I^{\mathbb{A}}(x, e), F^{\mathbb{A}}(x, e)): 0 \leq T^{\mathbb{A}} + I^{\mathbb{A}} + F^{\mathbb{A}} \leq 3, (x, e) \in U \times E\}$. Now if the set of universe $U = \{x_1, x_2, \ldots, x_m\}$ and the set of parameters $E = \{e_1, e_2, \ldots, e_n\}$, then $R_{\mathbb{A}}$ can be represented by a matrix as follows:

$$R_{\mathbb{A}} = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

where $a_{ij} = (T^{\mathbb{A}}(x,e), I^{\mathbb{A}}(x,e), F^{\mathbb{A}}(x,e)) = (T_{ij}^{\mathbb{A}}, I_{ij}^{\mathbb{A}}, F_{ij}^{\mathbb{A}})$.

The above matrix is called a **neutrosophic soft matrix** (NSM) of order $m\times$ n corresponding to the neutrosophic set $(N^{\mathbb{A}}, E)$ over U.

3. NSM theory in decision making

In this section, we define the concepts of weighted neutrosophic vector, score function and total score for a neutrosophic soft matrix. Later these notions will be used in MADM process.

Definition: 3.1 Let $\mathcal{M}$ be the collection of all neutrosophic values and $N = (n_1, n_2, \dots, n_n)$ be neutrosophic vector with components from $\mathcal{M}$. Thus the components of N are $N = ((n_1^T, n_1^I, n_1^F), (n_2^T, n_2^I, n_3^F), \dots, (n_n^T, n_n^I, n_n^F))$. Let $W = (w_1, w_2, \dots, w_n)$ be a weight vector associated with N. w_i can be considered as the significance attached to n_i; $i = 1,2,\dots,n$ with $w_i \in [0,1]$, $\sum_{i=1}^{n} w_i = 1$. Then the **weighted neutrosophic vector** corresponding to N and W denoted by WN is defined as $WN = (w_1 n_1, w_2 n_2, \dots, w_n n_n) = ((w_1 n_1^T, w_1 n_1^I, w_1 n_1^F), (w_2 n_2^T, w_2 n_2^I, w_2 n_2^F), \dots, (w_n n_n^T, w_n n_n^I, w_n n_n^F))$

Example:3.1 Let $N = ((0.4,0.3,0.6), (0.2,0.6,0.7), (0.7,0.1,0.5), (0.4,0.2,0.3))$ and $W = (0.1,0.4,0.2,0.3)$. Then $WN = ((0.04,0.03,0.06), (0.08,0.24,0.28), (0.14,0.02,0.10), (0.12,0.06,0.09))$

Definition: 3.2 Score function of a neutrosophic matrix helps to integrate the neutrosophic value into a single real number in order to bring out the importance of truth, indeterminacy and falsity membership values.

Let $A = [a_{ij}] = (T_{ij}^A, I_{ij}^A, F_{ij}^A)$. Then the score function for the element a_{ij} is defined as

$$s(a_{ij}) = s_{ij} = \frac{(T_{ij}^A + I_{ij}^A)}{2} + F_{ij}^A \ \forall\, i,j$$

Thus the **score function** for the NSM, $A = [a_{ij}]$ is given by

$$S_F(A) = \left[\frac{(T_{ij}^A + I_{ij}^A)}{2} + F_{ij}^A\right] = [s_{ij}].$$

$S_F(A)$ is also an $m \times n$ matrix, having the same dimension as A and has non-negative entries.

Definition 3.3 Let $N = [s_{ij}]$ be the matrix of score functions of a NSM N. The quantity $T_i = \sum_{j=1}^{n} s_{ij}$; $i = 1,2,\dots,m$ gives the **total of the score function** values for the i^{th} row of NSM. T_i represent the total value for the element x_i with representation to all the characteristics under consideration.

3.1 Comparison analysis with existing and proposed score functions

In this subsection, we compare and analyze the method developed in this paper with six of the recently developed score functions and methods. The below cited **Table 1** highlights the ranking difficulty of an existing score function in the neutrosophic environment. It also shows that the new score function can compute the rank of the alternatives even when the existing score function is unable to rank the alternatives.

Table 1. Comparison analysis of score values.

Neutrosophic environment	Existing & Proposed methods	Score value	Remarks
N_1 =(0.6,0.2,0.6) & N_2 =(0.6,0.4,0.2)	Sahin [25]	$S(N_1) = 0.3$ & $S(N_2) = 0.3$	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	**$S(N_1) = 1$** & $S(N_2) = 0.7$	**$S(N_1) > S(N_2)$** able to rank
N_1 =(0.7,0.3,0.1) & N_2 =(0.9,0.4,0.2)	Peng et.al., [24]	$S(N_1) = 0.1$ & $S(N_2) = 0.1$	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	$S(N_1) = 0.60$ & **$S(N_2) = 0.85$**	**$S(N_2) > S(N_1)$** able to rank
N_1 =(0.9,0.6,0.3) & N_2 =(0.6,0.4,0.2)	Garg and Nancy [23]	$S(N_1) = 0.26$ & $S(N_2) = 0.26$	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	**$S(N_1) = 1.05$** & $S(N_2) = 0.7$	**$S(N_1) > S(N_2)$** able to rank
N_1 =(0.4,0.2,0.6) & N_2 =(0.7,0.6,0.7)	Arockiarani [21]	$S(N_1) = 0.28$ & $S(N_2) = 0.28$	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	$S(N_1) = 0.9$ & **$S(N_2) = 1.35$**	**$S(N_2) > S(N_1)$** able to rank
N_1 =(0.5,0.7,0.4) & N_2 =(0.4,0.6,0.3)	Ye [26]	$S(N_1) = 0.55$ & $S(N_2) = 0.55$	$S(N_1) = S(N_2)$ unable to rank
	Proposed method	**$S(N_1) = 1$** & $S(N_2) = 0.8$	**$S(N_1) > S(N_2)$** able to rank
N_1 =(0.8,0.3,0.2) & N_2=(0.6,0.3,0.7) N_3 =(0.9,0.4,0.5) & N_4 =(0.8,0.5,04)	Mondal [22]	$S(N_p) = 0.65$, where p = 1,2 & $S(N_q) = 0.65$ where q = 3,4	$S(N_p) = S(N_q)$ unable to rank
	Proposed method	$S(N_p) = 0.95$, where p = 1,2 & **$S(N_q) = 1.1$** where q = 3,4	**$S(N_q) > S(N_p)$** able to rank

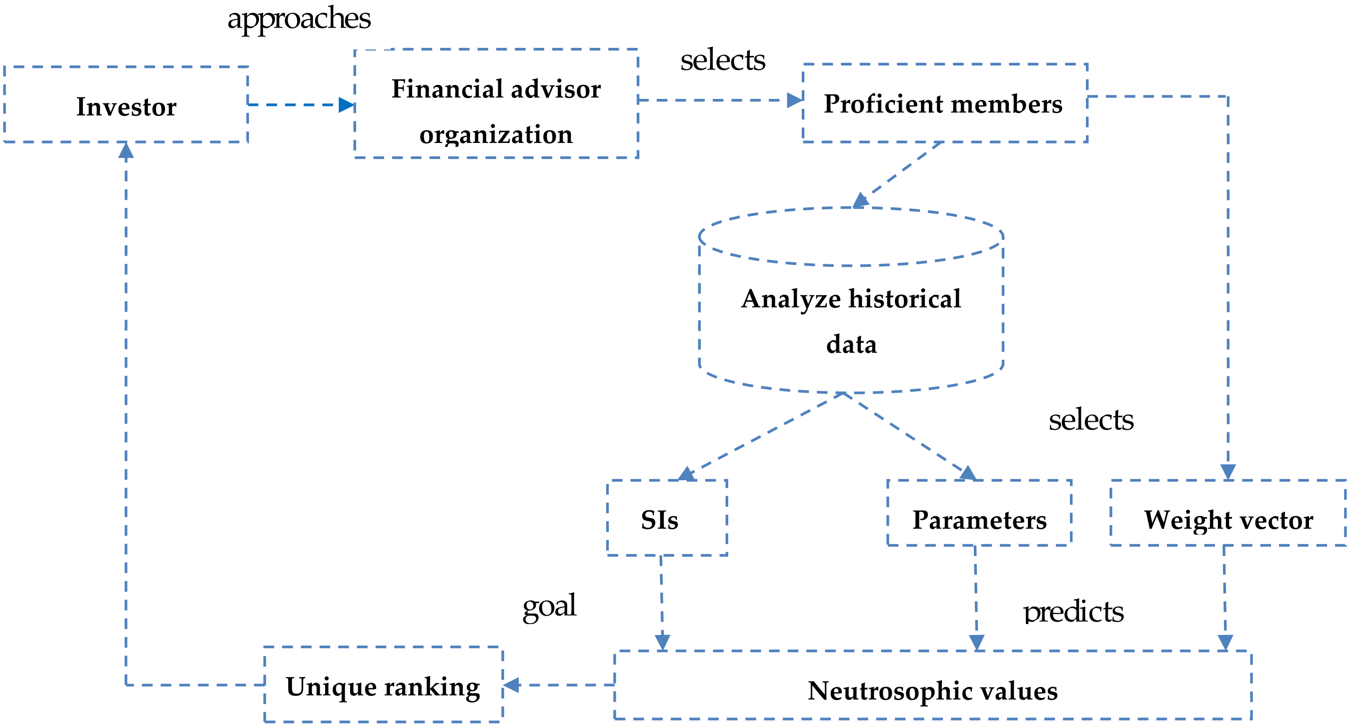
4. Application of NSM to MADM environment

In this section an application of NSM in MADM is explained. An algorithm is developed and the working of the same is illustrated with suitable examples.

4.1. Statement of the problem

Suppose a person is in the progression of stock investment (SI) in the equity market. Let's assume that person seeks the help of a financial advisor organization (FAO). FAO has a panel of highly-trained professionals to provide value-added services to the investors to ensure higher proficiency, consistency of charges and superior forecast of SI in equity market by analyzing the historical data. The FAO, in turn, selects a group of proficient members $P = \{p_1, p_2, \ldots, p_k\}$ to proceed with the same. Now according to the group let $C = \{c_1, c_2, \dots, c_p\}$ be the list of selected SIs based on historical data analysis . Let $E = \{e_1, e_2, \dots, e_q\}$ be the set of selected parameters based on which the SIs selection is to be finalized. Assume that weights are assigned for each criterion. Let $W = (w_1, w_2, \dots, w_q)$ and $\sum_{i=1}^{q} w_i = 1$. Let's assume that the group assesses the SI based on a subset of the parameter set. Let $A = \{e_1, e_2, \dots, e_l\}$ be the subset of the parameter set *E*, so that $l \leq q$. Each of the personnel verifies the listed SI historical records based on the parameter set A and presents his forecast result in the form of neutrosophic soft matrices. The respective NSM's are denoted by $N^1, N^2, \dots, N^K$. The crisis is to convert the NSM's into significant matrices which enables them to select the best SI for the investor. **Figure 1** illustrates the conceptual structure of the problem.

Figure 1. Conceptual structure of the statement



4.2. *Methodology*

Let's assume that the proficient members evaluate the SIs independently without any bias. Let $N^1, N^2, \dots, N^K$ be the NSMs obtained from the members. Using **Definition 3.1**, and weight vector W the weighted neutrosophic matrices are calculated. The resultant of weighted neutrosophic matrices are denoted by $N_w^1, N_w^2, \dots, N_w^k$ i.e., $N_w^r = WN^r = [n_{ij}^r]$ where $r = 1,2,\dots,k$. Using **Definition 3.2**, convert each of the weighted neutrosophic matrix N_w^r value into corresponding score function as $S_F[N_w^r] = [s_{ij}^r] = \left[\frac{(T_{ij}^{rA} + I_{ij}^{rA})}{2} + F_{ij}^{rA}\right]$. Then using the **Definition 3.3** the score function for the i^{th} SI as evaluated by the r^{th} expert is calculated by adding the values of the i^{th} row of the score function matrix, ie., the i^{th} row of the weighted neutrosophic matrix N_w^r. Let us denote this sum by the symbol T_i^r. The total score ST_i for the i^{th} SI is obtained by summing T_i^r over r. That is the total score for the i^{th} SI $ST_i = \sum_{r=1}^{k} T_i^r = T_i^1 + T_i^2 + \dots + T_i^k$. The total score is evaluated for all the SIs, $i = 1,2,\dots,p$. Arrange the ST_i values in decreasing order. The SI with highest ST_i value is the most suitable one for the investor. If more than one SI are there with equal highest ST_i value, the entire process is repeated by adding one more parameter into the set *A*. This process is repeated until a unique SI with highest ST_i value is identified.

4.3. Algorithm

The algorithm for ranking the alternatives of MADM problem based on NSM is given below:

Step 1: Identify the list of SIs and the list of parameters.

Step 2: Select a subset of the parameter set.

Step 3: Present the result in the form of NSMs $(N^1, N^2, \ldots, N^K)$.

Step 4: Compute the weight order for the NSMs $\left(N_W^1, N_W^2, \ldots, N_W^{\mathrm{k}}\right)$.

Step 5: Calculate the score function matrix $S_F[N_w^r] = \left[s_{ij}^r\right]$

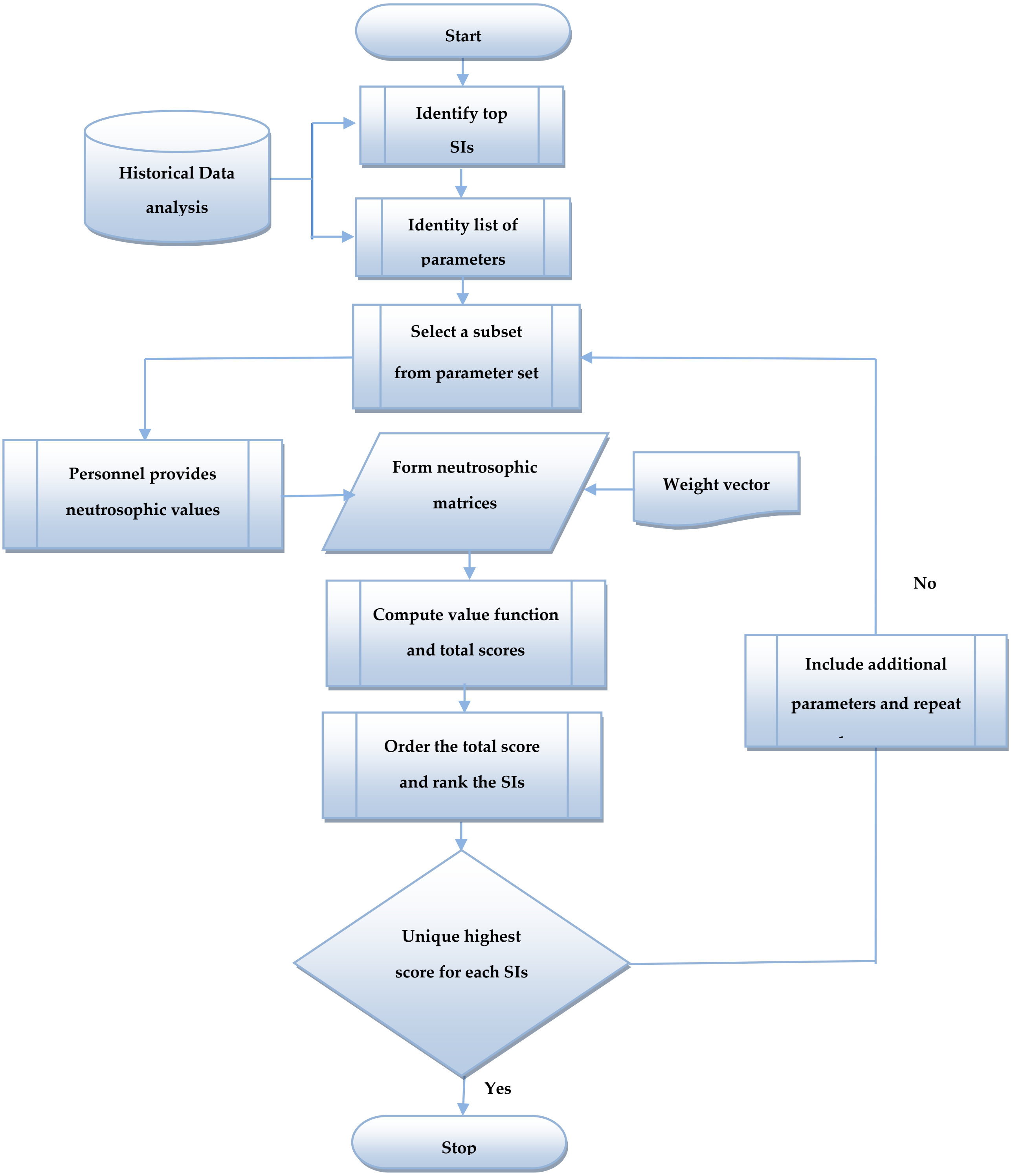
Step 6: Calculate the total value T_i^r from each of the $S_F[N_w^{\mathrm{r}}]$ matrices.

Step 7: Evaluate the ST_i for each SI.

Step 8: Order the ST_i values and select the SI with highest ST_i value as the most suitable one.

Step 9: If there are more than one SI with equal highest ST_i value, repeat the process by including another parameter into the set *A*. Continue the process until a unique SI with highest ST_i is identified.

4.4. Flowchart



5. Case studies

In this section we present two case studies to illustrate the working of the algorithm. In 5.1 we present an example where the ranking of the SIs are unique and processed based on a subset of the criteria set. In 5.2 an example is given where the initially selected set of parameters does not provide unique ranking and there are more than one SIs with equal highest total score. Addition of another parameter yields a clear ranking and the selection is performed by repeating some of the steps with enlarged parameter set.

5.1. Case study I

A person is in the process of selecting a suitable SI.

1. Let $C = (c_1, c_2, \ldots, c_7)$ be the set of listed SIs.

2. Let $E = (e_1, e_2, e_3, e_4)$ be the set of parameters which form the criteria for selection.

Here, e_1 = financial profitability projection, e_2 = asset-utilization, e_3 = conservative capital structure and e_4 = earnings momentum.

3. Let the personnel present his forecast result in the form of NSM- N^1, N^2 and N^3 for the subset of the criteria set (e_1, e_2, e_3) as

$$N^1 = \begin{bmatrix} (0.245,0.456,0.721) & (0.457,0.421,0.431) & (0.415,0.821,0.211) \\ (0.348,0.156,0.627) & (0.345,0.653,0.543) & (0.618,0.712,0.514) \\ (0.546,0.765,0.429) & (0.765,0.753,0.632) & (0.415,0.521,0.416) \\ (0.267,0.321,0.321) & (0.552,0.893,0.723) & (0.314,0.612,0.518) \\ (0.428,0.416,0.891) & (0.452,0.213,0.413) & (0.231,0.923,0.916) \\ (0.456,0.932,0.217) & (0.569,0.236,0.247) & (0.416,0.378,0.612) \\ (0.324,0.634,0.816) & (0.367,0.456,0.912) & (0.482,0.231,0.712) \end{bmatrix}$$

$$N^2 = \begin{bmatrix} (0.245,0.348,0.546) & (0.456,0.156,0.765) & (0.721,0.627,0.429) \\ (0.457,0.345,0.765) & (0.421,0.653,0.753) & (0.431,0.543,0.632) \\ (0.415,0.618,0.415) & (0.821,0.712,0.521) & (0.211,0.514,0.416) \\ (0.238,0.416,0.467) & (0.734,0.817,0.926) & (0.518,0.456,0.267) \\ (0.314,0.231,0.916) & (0.753,0.893,0.213) & (0.213,0.765,0.457) \\ (0.753,0.893,0.213) & (0.618,0.415,0.314) & (0.451,0.233,0.532) \\ (0.412,0.824,0.218) & (0.614,0.425,0.324) & (0.546,0.267,0.428) \end{bmatrix} \text{and}$$

$$N^3 = \begin{bmatrix} (0.238,0.734,0.518) & (0.765,0.345,0.734) & (0.345,0.457,0.347) \\ (0.416,0.817,0.456) & (0.429,0.653,0.817) & (0.456,0.892,0.821) \\ (0.467,0.926,0.267) & (0.156,0.543,0.926) & (0.673,0.452,0.342) \\ (0.914,0.316,0.912) & (0.245,0.431,0.211) & (0.345,0.763,0.821) \\ (0.928,0.419,0.745) & (0.348,0.345,0.618) & (0.543,0.821,0.721) \\ (0.211,0.518,0.213) & (0.245,0.456,0.721) & (0.436,0.417,0.556) \\ (0.156,0.653,0.712) & (0.348,0.345,0.618) & (0.529,0.673,0.719) \end{bmatrix}$$

4. Let the weight order of neutrosophic soft sets be $W_1 = 0.3, W_2 = 0.4, W_3 = 0.3$. Using **Definition 3.1** the results are obtained as

$$N_w^1 = \begin{bmatrix} (0.074,0.137,0.216) & (0.183,0.168,0.172) & (0.125,0.246,0.063) \\ (0.104,0.047,0.188) & (0.138,0.261,0.217) & (0.185,0.214,0.154) \\ (0.164,0.230,0.129) & (0.306,0.301,0.253) & (0.125,0.156,0.125) \\ (0.080,0.096,0.096) & (0.221,0.357,0.289) & (0.094,0.184,0.155) \\ (0.128,0.125,0.267) & (0.181,0.085,0.165) & (0.069,0.277,0.275) \\ (0.137,0.280,0.065) & (0.228,0.094,0.099) & (0.125,0.113,0.184) \\ (0.097,0.190,0.245) & (0.147,0.182,0.365) & (0.145,0.069,0.214) \end{bmatrix},$$

$$N_w^2 = \begin{bmatrix} (0.074,0.104,0.164) & (0.182,0.062,0.306) & (0.216,0.188,0.129) \\ (0.137,0.104,0.230) & (0.168,0.261,0.301) & (0.129,0.163,0.190) \\ (0.125,0.185,0.125) & (0.328,0.285,0.208) & (0.063,0.154,0.125) \\ (0.071,0.125,0.140) & (0.294,0.327,0.370) & (0.155,0.137,0.080) \\ (0.094,0.069,0.275) & (0.301,0.357,0.085) & (0.064,0.230,0.137) \\ (0.226,0.268,0.064) & (0.247,0.166,0.126) & (0.135,0.070,0.160) \\ (0.124,0.247,0.065) & (0.246,0.170,0.130) & (0.164,0.080,0.128) \end{bmatrix} and$$

$$N_w^3 = \begin{bmatrix} (0.071,0.220,0.155) & (0.306,0.138,0.294) & (0.104,0.137,0.104) \\ (0.125,0.245,0.137) & (0.172,0.261,0.327) & (0.137,0.268,0.246) \\ (0.140,0.278,0.080) & (0.062,0.217,0.370) & (0.202,0.136,0.103) \\ (0.274,0.095,0.274) & (0.098,0.172,0.084) & (0.104,0.229,0.246) \\ (0.278,0.126,0.224) & (0.139,0.138,0.247) & (0.163,0.246,0.216) \\ (0.063,0.155,0.064) & (0.098,0.182,0.288) & (0.131,0.125,0.167) \\ (0.047,0.196,0.214) & (0.139,0.138,0.247) & (0.159,0.202,0.216) \end{bmatrix}$$

5. Using **Definition 3.2** the score function matrices are obtained as

$$S_F(N_w^1) = \begin{bmatrix} 0.321 & 0.348 & 0.249 \\ 0.264 & 0.417 & 0.354 \\ 0.325 & 0.556 & 0.265 \\ 0.185 & 0.578 & 0.294 \\ 0.394 & 0.298 & 0.448 \\ 0.273 & 0.260 & 0.303 \\ 0.389 & 0.529 & 0.321 \end{bmatrix} S_F(N_w^2) = \begin{bmatrix} 0.253 & 0.428 & 0.331 \\ 0.350 & 0.516 & 0.336 \\ 0.279 & 0.515 & 0.234 \\ 0.238 & 0.681 & 0.226 \\ 0.357 & 0.414 & 0.284 \\ 0.311 & 0.332 & 0.262 \\ 0.251 & 0.337 & 0.250 \end{bmatrix} S_F(N_w^3) =$$

$$\begin{bmatrix} 0.301 & 0.516 & 0.224 \\ 0.322 & 0.543 & 0.449 \\ 0.289 & 0.510 & 0.271 \\ 0.458 & 0.220 & 0.413 \\ 0.426 & 0.386 & 0.421 \\ 0.173 & 0.429 & 0.295 \\ 0.335 & 0.386 & 0.396 \end{bmatrix}$$

6. Applying **Definition 3.3** the total of the score functions are calculated as

$$T_i^1 = \begin{bmatrix} 0.918 \\ 1.034 \\ 1.147 \\ 1.057 \\ 1.140 \\ 0.836 \\ 1.238 \end{bmatrix}, T_i^2 = \begin{bmatrix} 1.012 \\ 1.202 \\ 1.028 \\ 1.145 \\ 1.055 \\ 0.905 \\ 1.839 \end{bmatrix} and\ T_i^3 = \begin{bmatrix} 1.041 \\ 1.313 \\ 1.071 \\ 1.090 \\ 1.232 \\ 0.897 \\ 1.117 \end{bmatrix}$$

7. The total value for each candidate is calculated and presented as

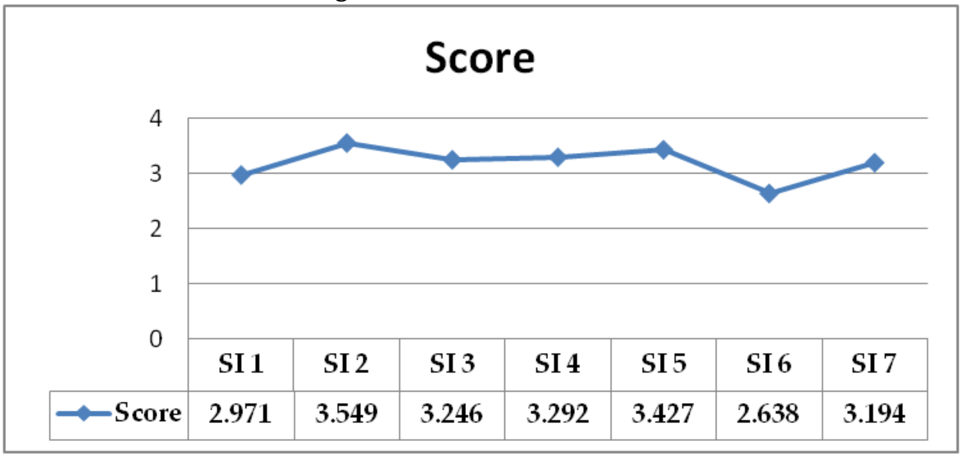
$$ST_i = \begin{bmatrix} 2.971 \\ 3.549 \\ 3.246 \\ 3.292 \\ 3.427 \\ 2.638 \\ 3.194 \end{bmatrix}$$

8. Arranging the SIs according to their total score values we obtain the ranking of the SIs as

Table 2. Tabular representation of SI's total score values.

c_i	Score	Rank
c_2	**3.549**	**1**
c_5	3.427	2
c_4	3.292	3
c_3	3.246	4
c_7	3.194	5
c_1	2.971	6
c_6	2.638	7

Figure 2. Score values of SIs.



From **Table 2** and **Figure 2**, we obtain the ranking of SIs as $c_2 > c_5 > c_4 > c_3 > c_7 > c_1 > c_6$. The SI c_2 ranks first and it is the most suitable SI for the investor.

5.2. Case study II

Consider the same example as in 5.1. A person would like to select the best SI.

1. Let $C = (c_1, c_2, \ldots, c_7)$ be the set of top listed SIs.

2. Let $E = (e_1, e_2, e_3, e_4)$ be the set of parameters which form the criteria for selection. Here, e_1 = financial profitability projection, e_2 = asset-utilization, e_3 = conservative capital structure and e_4 = earnings momentum of the SI.

3. Let the personnel present his forecast result in the form of NSM- N^1, N^2 and N^3 for the subset of the criteria set (e_1, e_2, e_3) as

$$N^1 = \begin{bmatrix} (0.245,0.456,0.721) & (0.457,0.421,0.431) & (0.415,0.821,0.211) \\ (0.247,0.156,0.547) & (0.345,0.653,0.543) & (0.618,0.712,0.614) \\ (0.546,0.765,0.429) & (0.765,0.753,0.632) & (0.415,0.521,0.416) \\ (0.567,0.552,0.521) & (0.652,0.682,0.723) & (0.313,0.412,0.568) \\ (0.429,1.000,0.891) & (0.452,0.219,0.407) & (0.231,0.922,0.916) \\ (0.456,0.932,0.217) & (0.569,0.236,0.247) & (0.416,0.378,0.612) \\ (0.324,0.634,0.816) & (0.367,0.456,0.912) & (0.482,0.231,0.712) \end{bmatrix},$$

$$N^2 = \begin{bmatrix} (0.245,0.348,0.546) & (0.456,0.156,0.765) & (0.721,0.627,0.429) \\ (0.457,0.345,0.765) & (0.421,0.653,0.753) & (0.431,0.543,0.632) \\ (0.415,0.618,0.415) & (0.821,0.712,0.521) & (0.211,0.514,0.416) \\ (0.638,0.516,0.467) & (0.734,0.817,0.926) & (0.518,0.456,0.467) \\ (0.314,0.231,0.916) & (0.753,0.893,0.213) & (0.213,0.765,0.457) \\ (0.753,0.893,0.213) & (0.618,0.415,0.314) & (0.451,0.233,0.532) \\ (0.412,0.824,0.218) & (0.614,0.425,0.324) & (0.546,0.267,0.428) \end{bmatrix} and$$

$$N^3 = \begin{bmatrix} (0.238,0.734,0.518) & (0.765,0.345,0.734) & (0.345,0.457,0.347) \\ (0.416,0.817,0.456) & (0.429,0.753,0.817) & (0.456,0.892,0.821) \\ (0.467,0.926,0.267) & (0.156,0.543,0.926) & (0.673,0.452,0.342) \\ (0.714,0.716,0.912) & (0.245,0.431,0.211) & (0.345,0.763,0.821) \\ (0.928,0.419,0.745) & (0.348,0.345,0.616) & (0.543,0.821,0.721) \\ (0.211,0.518,0.213) & (0.245,0.456,0.721) & (0.436,0.417,0.556) \\ (0.156,0.653,0.712) & (0.348,0.345,0.618) & (0.529,0.673,0.719) \end{bmatrix}$$

4. Let the weight order of neutrosophic soft sets be $W_1 = 0.3, W_2 = 0.4, W_3 = 0.3$. Using Definition 3.1 the results are obtained as

$$N_w^1 = \begin{bmatrix} (0.074,0.137,0.216) & (0.183,0.168,0.172) & (0.125,0.246,0.063) \\ (0.074,0.047,0.164) & (0.138,0.261,0.217) & (0.184,0.214,0.184) \\ (0.164,0.230,0.129) & (0.306,0.301,0.253) & (0.125,0.156,0.125) \\ (0.070,0.166,0.156) & (0.261,0.273,0.289) & (0.094,0.124,0.170) \\ (0.129,0.300,0.267) & (0.181,0.088,0.163) & (0.069,0.277,0.275) \\ (0.137,0.280,0.065) & (0.228,0.094,0.099) & (0.125,0.113,0.184) \\ (0.097,0.190,0.245) & (0.147,0.182,0.365) & (0.145,0.069,0.213) \end{bmatrix},$$

$$N_w^2 = \begin{bmatrix} (0.074,0.104,0.164) & (0.182,0.062,0.306) & (0.216,0.188,0.129) \\ (0.137,0.104,0.230) & (0.168,0.261,0.301) & (0.129,0.163,0.190) \\ (0.125,0.185,0.125) & (0.328,0.285,0.208) & (0.063,0.154,0.125) \\ (0.091,0.155,0.140) & (0.294,0.327,0.370) & (0.155,0.137,0.140) \\ (0.094,0.069,0.275) & (0.301,0.357,0.085) & (0.064,0.230,0.137) \\ (0.226,0.268,0.064) & (0.247,0.166,0.126) & (0.135,0.070,0.160) \\ (0.124,0.247,0.065) & (0.246,0.170,0.130) & (0.164,0.080,0.128) \end{bmatrix} and$$

$$N_w^3 = \begin{bmatrix} (0.071,0.220,0.155) & (0.306,0.138,0.294) & (0.104,0.137,0.104) \\ (0.125,0.245,0.137) & (0.172,0.301,0.327) & (0.137,0.268,0.246) \\ (0.140,0.278,0.080) & (0.062,0.217,0.370) & (0.202,0.136,0.103) \\ (0.214,0.215,0.274) & (0.098,0.172,0.084) & (0.104,0.229,0.246) \\ (0.278,0.126,0.224) & (0.139,0.138,0.246) & (0.163,0.246,0.216) \\ (0.063,0.155,0.064) & (0.098,0.182,0.288) & (0.131,0.125,0.167) \\ (0.047,0.196,0.214) & (0.139,0.138,0.247) & (0.159,0.202,0.216) \end{bmatrix}$$

5. Using **Definition 3.2** the score function matrices are obtained as

$$V_F(N_w^1) = \begin{bmatrix} 0.321 & 0.348 & 0.249 \\ 0.225 & 0.417 & 0.384 \\ 0.325 & 0.556 & 0.265 \\ 0.324 & 0.556 & 0.279 \\ 0.482 & 0.297 & 0.448 \\ 0.273 & 0.260 & 0.303 \\ 0.389 & 0.529 & 0.321 \end{bmatrix} V_F(N_w^2) = \begin{bmatrix} 0.253 & 0.428 & 0.331 \\ 0.350 & 0.516 & 0.336 \\ 0.279 & 0.515 & 0.234 \\ 0.313 & 0.681 & 0.286 \\ 0.357 & 0.414 & 0.284 \\ 0.311 & 0.332 & 0.262 \\ 0.251 & 0.337 & 0.250 \end{bmatrix} V_F(N_w^3) = \begin{bmatrix} 0.301 & 0.516 & 0.224 \\ 0.322 & 0.563 & 0.449 \\ 0.289 & 0.510 & 0.271 \\ 0.488 & 0.220 & 0.413 \\ 0.426 & 0.385 & 0.421 \\ 0.173 & 0.429 & 0.295 \\ 0.335 & 0.386 & 0.396 \end{bmatrix}$$

6. Applying **Definition 3.3** the total of the score functions are calculated as

$$T_i^1 = \begin{bmatrix} 0.918 \\ 1.025 \\ 1.147 \\ 1.159 \\ 1.226 \\ 0.836 \\ 1.238 \end{bmatrix}, T_i^2 = \begin{bmatrix} 1.012 \\ 1.202 \\ 1.028 \\ 1.280 \\ 1.055 \\ 0.905 \\ 1.839 \end{bmatrix}, T_i^3 = \begin{bmatrix} 1.041 \\ 1.333 \\ 1.071 \\ 1.120 \\ 1.231 \\ 0.897 \\ 1.117 \end{bmatrix},$$

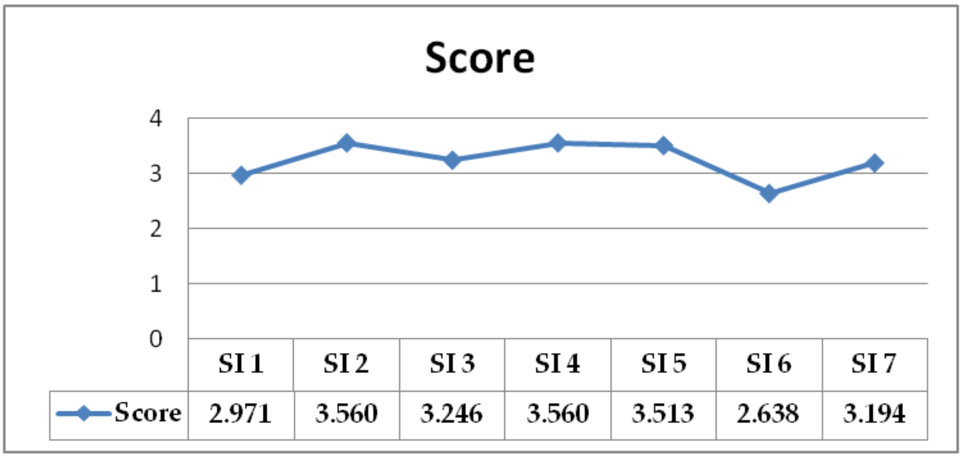
7. The total value for each SI is calculated and presented as

$$ST_i = \begin{bmatrix} 2.971 \\ 3.560 \\ 3.246 \\ 3.560 \\ 3.513 \\ 2.638 \\ 3.194 \end{bmatrix}$$

Table 3. Tabular representation of SI's total score values.

c_i	Score	Rank
$\boldsymbol{c_2}$	**3.560**	**1**
$\boldsymbol{c_4}$	**3.560**	**1**
c_5	3.513	3
c_3	3.246	4
c_7	3.194	5
c_1	2.971	6
c_6	2.638	7

Figure 3. Score values of SIs



From **Table 3** and **Figure 3**, we obtain the ranking of SIs as $c_2 = c_4 > c_5 > c_3 > c_7 > c_1 > c_6$. As there are more than one SI $(c_2$ and $c_4)$ with the same ranking we add one more parameter e_4 in the list and repeat the process.

$$N^1 = \begin{bmatrix} (0.245,0.456,0.721) & (0.457,0.421,0.431) & (0.415,0.821,0.211) & (0.536,0.665,0.129) \\ (0.247,0.156,0.547) & (0.345,0.653,0.543) & (0.618,0.712,0.614) & (0.547,0.451,0.321) \\ (0.546,0.765,0.429) & (0.765,0.753,0.632) & (0.415,0.521,0.416) & (0.357,0.451,0.631) \\ (0.567,0.552,0.521) & (0.652,0.682,0.723) & (0.313,0.412,0.568) & (0.375,0.753,0.243) \\ (0.429,1.000,0.891) & (0.452,0.219,0.407) & (0.231,0.922,0.916) & (0.251,0.562,0.726) \\ (0.456,0.932,0.217) & (0.569,0.236,0.247) & (0.416,0.378,0.612) & (0.426,0.478,0.512) \\ (0.324,0.634,0.816) & (0.367,0.456,0.912) & (0.482,0.231,0.712) & (0.416,0.252,0.317) \end{bmatrix},$$

$$N^2 = \begin{bmatrix} (0.245,0.348,0.546) & (0.456,0.156,0.765) & (0.721,0.627,0.429) & (0.546,0.765,0.429) \\ (0.457,0.345,0.765) & (0.421,0.653,0.753) & (0.431,0.543,0.632) & (0.567,0.551,0.521) \\ (0.415,0.618,0.415) & (0.821,0.712,0.521) & (0.211,0.514,0.416) & (0.457,0.421,0.431) \\ (0.638,0.516,0.467) & (0.734,0.817,0.926) & (0.518,0.456,0.467) & (0.345,0.653,0.543) \\ (0.314,0.231,0.916) & (0.753,0.893,0.213) & (0.213,0.765,0.457) & (0.231,0.922,0.916) \\ (0.753,0.893,0.213) & (0.618,0.415,0.314) & (0.451,0.233,0.532) & (0.416,0.378,0.612) \\ (0.412,0.824,0.218) & (0.614,0.425,0.324) & (0.546,0.267,0.428) & (0.456,0.932,0.217) \end{bmatrix}$$

$$N^3 = \begin{bmatrix} (0.238,0.734,0.518) & (0.765,0.345,0.734) & (0.721,0.627,0.429) & (0.546,0.765,0.429) \\ (0.416,0.817,0.456) & (0.429,0.753,0.817) & (0.431,0.543,0.632) & (0.567,0.551,0.521) \\ (0.467,0.926,0.267) & (0.156,0.543,0.926) & (0.211,0.514,0.416) & (0.457,0.421,0.431) \\ (0.714,0.716,0.912) & (0.245,0.431,0.211) & (0.518,0.456,0.467) & (0.345,0.653,0.543) \\ (0.928,0.419,0.745) & (0.348,0.345,0.616) & (0.213,0.765,0.457) & (0.231,0.922,0.916) \\ (0.211,0.518,0.213) & (0.245,0.456,0.721) & (0.451,0.233,0.532) & (0.416,0.378,0.612) \\ (0.156,0.653,0.712) & (0.348,0.345,0.618) & (0.546,0.267,0.428) & (0.456,0.932,0.217) \end{bmatrix},$$

4. Let the weight order of neutrosophic soft sets be $W_1 = 0.3, W_2 = 0.4, W_3 = 0.15$ and $W_4 = 0.15$. Using **Definition 3.1** the resultant are obtained as

$$N_w^1 = \begin{bmatrix} (0.074,0.137,0.216) & (0.183,0.168,0.172) & (0.062,0.123,0.032) & (0.080,0.100,0.019) \\ (0.074,0.047,0.164) & (0.138,0.261,0.217) & (0.093,0.107,0.092) & (0.082,0.068,0.048) \\ (0.164,0.230,0.129) & (0.306,0.301,0.253) & (0.062,0.078,0.062) & (0.054,0.068,0.095) \\ (0.070,0.166,0.156) & (0.261,0.273,0.289) & (0.047,0.062,0.085) & (0.056,0.113,0.036) \\ (0.129,0.300,0.267) & (0.181,0.088,0.163) & (0.035,0.138,0.137) & (0.038,0.084,0.109) \\ (0.137,0.280,0.065) & (0.228,0.094,0.099) & (0.062,0.057,0.092) & (0.064,0.072,0.077) \\ (0.097,0.190,0.245) & (0.147,0.182,0.365) & (0.072,0.035,0.107) & (0.062,0.038,0.048) \end{bmatrix}$$

$$N_w^2 = \begin{bmatrix} (0.074,0.104,0.164) & (0.182,0.062,0.306) & (0.108,0.094,0.064) & (0.082,0.115,0.064) \\ (0.137,0.104,0.230) & (0.168,0.261,0.301) & (0.065,0.081,0.095) & (0.085,0.083,0.078) \\ (0.125,0.185,0.125) & (0.328,0.285,0.208) & (0.032,0.077,0.062) & (0.069,0.063,0.065) \\ (0.091,0.155,0.140) & (0.294,0.327,0.370) & (0.078,0.068,0.070) & (0.052,0.098,0.081) \\ (0.094,0.069,0.275) & (0.301,0.357,0.085) & (0.032,0.115,0.069) & (0.035,0.138,0.137) \\ (0.226,0.268,0.064) & (0.247,0.166,0.126) & (0.068,0.035,0.080) & (0.062,0.057,0.092) \\ (0.124,0.247,0.065) & (0.246,0.170,0.130) & (0.082,0.040,0.064) & (0.068,0.140,0.033) \end{bmatrix}$$

$$N_w^3 = \begin{bmatrix} (0.071,0.220,0.155) & (0.306,0.138,0.294) & (0.052,0.069,0.052) & (0.082,0.115,0.064) \\ (0.125,0.245,0.137) & (0.172,0.301,0.327) & (0.068,0.134,0.123) & (0.085,0.083,0.078) \\ (0.140,0.278,0.080) & (0.062,0.217,0.370) & (0.101,0.068,0.051) & (0.069,0.063,0.065) \\ (0.214,0.215,0.274) & (0.098,0.172,0.084) & (0.052,0.114,0.123) & (0.052,0.098,0.081) \\ (0.278,0.126,0.224) & (0.139,0.138,0.246) & (0.081,0.123,0.108) & (0.035,0.138,0.137) \\ (0.063,0.155,0.064) & (0.098,0.182,0.288) & (0.065,0.063,0.083) & (0.062,0.057,0.092) \\ (0.047,0.196,0.214) & (0.139,0.138,0.247) & (0.079,0.101,0.108) & (0.068,0.140,0.033) \end{bmatrix}$$

5. Using **Definition 3.2** the score function matrices are obtained as

$$V_F(N_w^1) = \begin{bmatrix} 0.321 & 0.348 & 0.124 & 0.109 \\ 0.225 & 0.417 & 0.192 & 0.123 \\ 0.325 & 0.556 & 0.133 & 0.155 \\ 0.324 & 0.556 & 0.140 & 0.121 \\ 0.482 & 0.297 & 0.224 & 0.170 \\ 0.273 & 0.260 & 0.151 & 0.145 \\ 0.389 & 0.529 & 0.160 & 0.098 \end{bmatrix}, V_F(N_w^2) = \begin{bmatrix} 0.253 & 0.428 & 0.165 & 0.163 \\ 0.350 & 0.516 & 0.168 & 0.162 \\ 0.279 & 0.515 & 0.117 & 0.131 \\ 0.313 & 0.681 & 0.143 & 0.156 \\ 0.357 & 0.414 & 0.142 & 0.224 \\ 0.311 & 0.332 & 0.131 & 0.151 \\ 0.251 & 0.337 & 0.125 & 0.137 \end{bmatrix}$$

$$V_F(N_w^3) = \begin{bmatrix} 0.301 & 0.516 & 0.112 & 0.163 \\ 0.322 & 0.563 & 0.224 & 0.162 \\ 0.289 & 0.510 & 0.136 & 0.131 \\ 0.488 & 0.220 & 0.206 & 0.156 \\ 0.426 & 0.385 & 0.210 & 0.224 \\ 0.173 & 0.429 & 0.147 & 0.151 \\ 0.335 & 0.386 & 0.198 & 0.137 \end{bmatrix}$$

6. Applying **Definition 3.3** the total of the score functions are calculated as

$$T_i^1 = \begin{bmatrix} 0.903 \\ 0.995 \\ 1.170 \\ 1.141 \\ 1.130 \\ 0.829 \\ 1.176 \end{bmatrix}, T_i^2 = \begin{bmatrix} 1.009 \\ 1.196 \\ 1.293 \\ 1.137 \\ 0.925 \\ 0.850 \end{bmatrix} and T_i^3 = \begin{bmatrix} 1.092 \\ 1.271 \\ 1.065 \\ 1.070 \\ 1.245 \\ 0.901 \\ 1.055 \end{bmatrix}$$

7. The total value for each SI is calculated and presented as

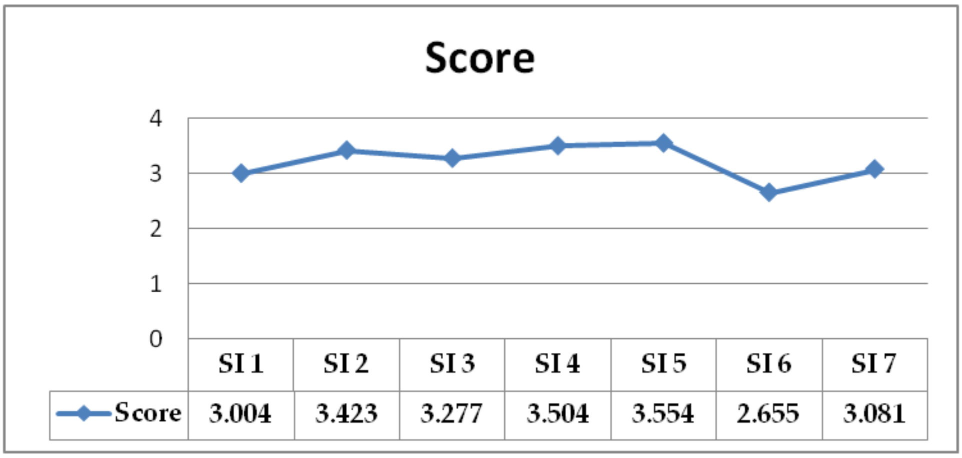
$$ST_i = \begin{bmatrix} 3.004 \\ 3.423 \\ 3.277 \\ 3.504 \\ 3.554 \\ 2.655 \\ 3.081 \end{bmatrix}$$

8. Arranging the SIs according to their total score values we obtain the ranking of the SIs as

Table 4. Tabular representation of SI's total score values.

c_i	Score	Rank
c_5	**3.554**	**1**
c_4	3.504	2
c_2	3.423	3
c_3	3.277	4
c_7	3.081	5
c_1	3.004	6
c_6	2.655	7

Figure 4. Score values of SIs



From **Table 4** and **Figure 4**, we obtain the ranking of SIs as $c_5 > c_4 > c_2 > c_3 > c_7 > c_1 > c_6$. The SI c_5 ranks first and it is the most suitable SI for the investor.

6. Conclusions

The proposed NSM computational solution supports decision-makers in solving the complex decision-making problem faced in today's ambiguity situation. In this paper, the weight vector and score function are introduced with illustrative examples. By applying the score function we solve the MADM problems in the neutrosophic environment and transforming the values of truth, indeterminacy and falsity into a single membership value to obtain a more precise, efficient, and realistic solution. An application of NSM in MADM is also explained. An algorithm is developed for this purpose and two examples are provided to illustrate the working of the algorithm. Our future work is to extend the concept of MADM problems in real-life psychology applications by using standard or hybrid neutrosophic and plithogenic tools.

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References

1. Atanassov, K. Intuitionistic fuzzy sets. Fuzzy sets and System **1986,** 20, 87-96.
2. Atanassov, K.; Gargov, G. Interval-valued intuitionistic fuzzy sets. Fuzzy Sets and Systems **1989**, 31 , 343-349.
3. Chinnadurai, V.; Swaminathan, A.; Anu, B. Some properties of Neutrosophic cubic soft set. International Journal of Computational Research and Development **2016** , 01, 113-119.
4. Deli, I.; Broumi, S. Neutrosophic soft sets and neutrosophic soft matrices based on decision making. Journal of Intelligent & Fuzzy Systems **2014**, 28, 1-28.
5. Garg, H.; Nancy. Multi-Criteria Decision-Making Method Based on Prioritized Muirhead Mean Aggregation Operator under Neutrosophic Set Environment. Symmetry **2018**, 10, 280.
6. Gulistan, M.; Beg, I.; Yaqoob, N. A new approach in decision making problems under the environment of neutrosophic cubic soft matrices. Journal of Intelligent & Fuzzy Systems **2019,** 36, 295-307.
7. Jun Yi Tey, D. et al., A Novel Neutrosophic Data Analytic Hierarchy Process for Multi-Criteria Decision Making Method: A Case Study in Kuala Lumpur Stock Exchange. IEEE Access 2019, 7, 53687-53697.
8. Maji, P.K. Neutrosophic soft sets. Annals of Fuzzy Mathematics and Informatics **2013**, 5, 157-168.
9. Molodstov, D. A. Soft set theory. Computers and Mathematics with Applications **1999**, 37, 19-31.
10. Smarandache, F. A unifying field in logics. Neutrosophy: neutrosophic probability, set and logic. Rehoboth, American Research Press, **1998**.
11. Smarandache, F.; Teodorescu, M. From Linked Data Fuzzy to Neutrosophic Data Set Decision Making in Games vs. Real Life. New Trends in Neutrosophic Theory and Applications **2016**, 115-125.
12. Abdel-Basset,M.; Mohamed,M.; Smarandache,F. An extension of neutrosophic AHP-SWOT analysis for strategic planning and decision making. Symmetry **2018**, 10, 116.
13. Wang, H.; Smarandache, F. Single valued Neutrosophic sets. Multispace and Multistructure **2010**, 4, 410-413.
14. Zadeh, L.A. Fuzzy sets. Information and Control **1965**, 8, 338-353.
15. Zadeh, L.A. The concept of a linguistic variable and its application to approximate reasoning-I. Information Science **1975**, 8, 199-249.
16. Abdel-Basset, M.; El-hoseny, M.; Gamal, A.; Smarandache, F. A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine **2019**, 100, 101710.
17. Abdel-Basset, M.; Mohamed, R., Zaied, A. E. N. H.; Smarandache, F. A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry **2019**, 11(7), 903.
18. Abdel-Basset, M.; Nabeeh, N. A.; El-Ghareeb, H. A.; Aboelfetouh, A. Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises. Enterprise Information Systems **2019**, 1-21.
19. Abdel-Baset, M.; Chang, V.; Gamal, A. Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry **2019**, 108, 210-220.
20. Abdel-Basset, M.; Saleh, M.; Gamal, A.; Smarandache, F. An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing **2019**, 77, 438-452.
21. Arockiarani, I. A fuzzy neutrosophic soft set model in medical diagnosis. IEEE Conference on Norbert Wiener **2014**, 1-8.
22. Mondal, Kalyan.; Pramanik, Surapati.; Giri, Bibhas. Role of neutrosophic logic in data mining. New Trends in Neutrosophic Theory and Applications **2016**, 15-23.

23. Nancy.; Garg, H. An improved score function for ranking neutrosophic sets and its application to decision-making process. International Journal for Uncertainty Quantification **2016**, 6, 377-385.
24. Peng, J.J.; Wang, J.Q.; Wang, J.; Zhang, H.Y.; Chen, X.H. Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. Int. J. Syst. Sci. **2016**, 47, 2342–2358.
25. Sahin, R. Multi-criteria neutrosophic decision making method based on score and accuracy functions under neutrosophic environment. arXiv preprint **2014**, arXiv:1412.5202
26. Ye, J. Multiple attribute group decision-making method with completely unknown weights based on similarity measures under single valued neutrosophic environment. Journal of Intelligent & Fuzzy Systems **2014**, 27(6), 2927–2935.
27. Smarandache, F. Plithogeny, Plithogenic Set, Logic, Probability, and Statistics; Infinite Study, **2017**, 141.
28. Guleria, Abhishek.; Srivastava, Saurabh.; Bajaj, Rakesh. On Parametric Divergence Measure of Neutrosophic Sets with its Application in Decision-making Models. Neutrosophic Sets and Systems, **2019**, 29, 101-120.
29. Abhishek,Guleria.; RakeshKumar,Bajaj.; Technique for Reducing Dimensionality of Data in Decision-Making Utilizing Neutrosophic Soft Matrices Neutrosophic Sets with its Application in Decision-making Models. Neutrosophic Sets and Systems **2019**, 29, 129-141.
30. Edalatpanah, S.A.; Smarandache, F. Data Envelopment Analysis for Simplified Neutrosophic Sets. Neutrosophic Sets and Systems **2019**, 29, 215-226.
31. Pandy, Arul.; Pricilla, Trinita. Reduction of indeterminacy of gray-scale image in bipolar neutrosophic domain. Neutrosophic Sets and Systems **2019**, 28, 1-12.
32. Rana, Shazia.; Qayyum, Madiha.; Saeed, Muhammad.; Smarandache, Florentin.;Khan, Bakhtawar. Plithogenic Fuzzy Whole Hypersoft Set, Construction of Operators and their Application in Frequency Matrix Multi Attribute Decision Making Technique. Neutrosophic Sets and Systems **2019**, 28, 34-50.
33. Uluçay, Vakkas.; Kılıç, Adil .; Yıldız, İsmet.; Şahin, Memet. A new approach for multi-attribute decision-making problems in bipolar neutrosophic sets. Neutrosophic Sets and Systems 2018, 23, 142-159.
34. Azeddine,Elhassouny.; Soufiane, Idbrahim.; Smarandache, F. Machine learning in Neutrosophic Environment: A Survey. Neutrosophic Sets and Systems **2019**, 28, 58-68.

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Structural Equivalence between Electrical Circuits via Neutrosophic Nano Topology Induced by Digraphs

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Abstract: The purpose of the present work was to study the real life problems using neutrosophic nano topological graph theory. Most real-life situations need some sort of approximation to fit mathematical models. The beauty of using neutrosophic nano topology in approximation is achieved via approximation for qualitative sub graphs without coding or using assumption. By certain nano equivalence relation, we are formalizing the structural equivalence of basic circuit of the LED light from the graphs and their corresponding neutrosophic nano topologies generated by them.

Keywords: Neutrosophic nano topology; Neutrosophic nano neighborhood; Neutrosophic nano continuous; Neutrosophic nano homeomorphism; Neutrosophic nano isomorphism.

1. Introduction

There are several reasons for the acceleration of interest in graph theory. It has become fashionable to mention that there are applications of graph theory in some areas of Physics, Chemistry, Communication Science and Computer Technology. The theory is also intimately related to many branches of Mathematics, including Group Theory, Matrix Theory, Numerical Analysis, Probability, Topology and Combinatorics.

A graph (resp., directed graph or digraph) [21], $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G)$ of un-ordered (resp., ordered) pairs of elements of $V(G)$. To avoid ambiguities, we assume that the vertex and edge sets are disjoint. We say that two vertices v and w of a graph (resp., digraph) G are adjacent if there is an edge of the form $\overline{vw}$ (resp., $\overline{vw}$ or $\overline{wv}$) joining them, and the vertices v and $w$are then incident with such an edge. A sub graph of a graph G is a graph, each of whose vertices belong to $V(G)$ and each of whose edges belongs to $E(G)$. Many theories like, Theory of Fuzzy sets [22], Theory of Intuitionistic fuzzy sets [7], Theory of Neutrosophic sets [20] and The Theory of Interval Neutrosophic sets can be considered as tools for dealing with uncertainties. However, all of these theories have their own difficulties which are pointed out. In 1965, Zadeh [22] introduced fuzzy set theory as a mathematical tool for dealing with uncertainties where each element had a degree of membership. Later on fuzzy topology was introduced by Chang [10] in 1986. The Intuitionistic fuzzy set was introduced by Atanassov [7] in 1983 as a generalization of fuzzy set, where besides the degree of membership and the degree of non-membership of each element. After this intuitionistic fuzzy topology was introduced by Coker [11].

The neutrosophic set was introduced by Smarandache [20] as a generalization of intuitionistic fuzzy set. In 2012, Salama and Alblowi [18] introduced the concept of Neutrosophic topological spaces as a generalization of intuitionistic fuzzy topological space and a neutrosophic set besides the degree of membership, the degree of indeterminacy and the degree of non-membership of each element. In 2014 Salama, Smarandache and Valeri [19] introduced the concept of neutrosophic closed sets and neutrosophic continuous functions. Smarandache's neutrosophic concept have wide range of real time applications for the fields of [1-6] Information Systems, Computer Science, Artificial Intelligence, Applied Mathematics, decision making. Mechanics, Electrical & Electronic, Medicine and Management Science etc, Rough set theory is introduced by Pawlak [17] as a new mathematical tool for representing reasoning and decision-making dealing with vagueness and uncertainty.

This theory provides the approximation of sets by means of equivalence relations and is considered as one of the first non-statistical approaches in data analysis. A rough set can be described by a pair of definable sets called lower and upper approximations. The lower approximation is the greatest definable set contained in the given set of objects while the upper approximation is the smallest definable set that contains the given set. Rough set concept can be defined quite generally by means of topological operations, interior and closure, called approximations. In 2013, a new topology called Nano topology was introduced by Lellis Thivagar [13] which is an extension of rough set theory. He also introduced Nano topological spaces which were defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it. The elements of a Nano topological space are called the Nano open sets and its complements are called the Nano closed sets. Nano means something very small. Nano topology thus literally means the study of very small surface. The fundamental ideas in Nano topology are those of approximations and indiscernibility relation.

Some properties of nano topology induced by graph were investigated by Arafa Nasef [8] et al. single valued neutrosophic graphs were introduced by Said Broumi [9] et al. in which they defined degree, order, size and neighborhood of single valued neutrosophic graph. The aim of this paper is to deal with some practical problems by utilizing neutrosophic nano topology. Nano homeomorphism [14] between two nano topological spaces are said to be topologically equivalent. Using this concept, we are formalizing the structural equivalence of basic circuit of the LED light from the graphs and their corresponding neutrosophic nano topologies generated by them.

2. Preliminaries

Definition 2.1. [13] Let $\mathcal{U}$ be a non-empty finite set of objects called the universe and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ named as indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(\mathcal{U}, \mathcal{R})$ is said to be the approximation space. Let$\mathcal{X} \subseteq \mathcal{U}$.

(i) The lower approximation of $\mathcal{X}$ with respect to $\mathcal{R}$ is the set of all objects, which can be for certain classified as $\mathcal{X}$ with respect to $\mathcal{R}$ and is denoted by $\mathcal{L}_{\mathcal{R}}(\mathcal{X})$. That is, $\mathcal{L}_{\mathcal{R}}(\mathcal{X}) = \mathcal{U}_{x\in\mathcal{U}}\{\mathcal{R}(x) : \mathcal{R}(x) \subseteq \mathcal{X}\}$ where $\mathcal{R}(x)$ denotes the equivalence class determined by $\mathbf{x}$.

(ii) The upper approximation of $\mathcal{X}$ with respect to $\mathcal{R}$ is the set of all objects, which can be possibly classified as $\mathcal{X}$ with respect to $\mathcal{R}$ and is denoted by $\mathcal{U}_{\mathcal{R}}(\mathcal{X})$. That is, $\mathcal{U}_{\mathcal{R}}(\mathcal{X}) = \mathcal{U}_{x\in\mathcal{U}}\{\mathcal{R}(x) : \mathcal{R}(x) \cap \mathcal{X} \neq \varphi\}$.

(iii) The boundary region of $\mathcal{X}$ with respect to $\mathcal{R}$ is the set of all objects which can be classified neither as $\mathcal{X}$ nor as not $\mathcal{X}$ with respect to $\mathcal{R}$ and it is denoted by $B_R(X)$. That is, $\mathcal{B}_{\mathcal{R}}(\mathcal{X}) = \mathcal{U}_{\mathcal{R}}(\mathcal{X}) - \mathcal{L}_{\mathcal{R}}(\mathcal{X})$.

Definition 2.2. [20] A neutrosophic set $\mathcal{S}$ is an object of the following form $\mathcal{A} = \{\langle s, \mathcal{P}_{\mathcal{A}}(s), \mathcal{Q}_{\mathcal{A}}(s), \mathcal{R}_{\mathcal{A}}(s) : s \in \mathcal{S}\rangle\}$ where $\mathcal{P}_{\mathcal{A}}(s)$, $\mathcal{Q}_{\mathcal{A}}(s)$ and $\mathcal{R}_{\mathcal{A}}(s)$ denote the degree of membership, the degree of indeterminacy and the degree of non-membership for each element $s \in \mathcal{S}$ to the set $\mathcal{A}$, respectively.

Definition 2.3. [18] A neutrosophic topology in a nonempty set $\mathcal{X}$ is a family $\mathfrak{J}$ of neutrosophic sets in $\mathcal{X}$ satisfying the following axioms:

(i) $0_N, 1_N \in \mathfrak{J}$;

(ii) $\mathcal{A} \cap \mathcal{B} \in \mathfrak{J}$ for any $\mathcal{A}, \mathcal{B} \in \mathfrak{J}$;

(iii) $\cup (\mathcal{A})_i$ for any arbitrary family $(\mathcal{A})_i : i \in J \subseteq \mathfrak{J}$.

Definition 2.4. [15] Let $\mathcal{U}$ be a universe and $\mathcal{R}$ be an equivalence relation on $\mathcal{U}$ and Let $\mathcal{S}$ be a neutrosophic subset of $\mathcal{U}$. Then the neutrosophic nano topology is defined by $\tau_{\mathcal{N}}(\mathcal{S}) = \{0_{\mathcal{N}}, 1_{\mathcal{N}}, \overline{\mathcal{N}}(\mathcal{S}), \underline{\mathcal{N}}(\mathcal{S}), \mathcal{B}_{\mathcal{N}}(\mathcal{S})\}$, where

(i). $\underline{\mathcal{N}}(\mathcal{S}) = \{\langle y, \mathcal{M}_{\underline{\mathcal{R}}(y)}, \mathcal{I}_{\underline{\mathcal{R}}(y)}, \mathcal{N}_{\underline{\mathcal{R}}(y)}\rangle / z \in [y]\mathcal{R}, y \in \mathcal{U}\}$.

(ii) $\overline{\mathcal{N}}(\mathcal{S}) = \{\langle y, \mathcal{M}_{\overline{\mathcal{R}}(y)}, \mathcal{I}_{\overline{\mathcal{R}}(y)}, \mathcal{N}_{\overline{\mathcal{R}}(y)}\rangle / z \in [y]\mathcal{R}, y \in \mathcal{U}\}$.

(iii) $\mathcal{B}_{\mathcal{N}}(\mathcal{S}) = \underline{\mathcal{N}}(\mathcal{S}) - \overline{\mathcal{N}}(\mathcal{S})$, where $\mathcal{M}_{\underline{\mathcal{R}}(y)} = \wedge_{z \in [y]\mathcal{R}} \mathcal{M}_{\mathcal{S}}(z)$, $\mathcal{I}_{\underline{\mathcal{R}}(y)} = \wedge_{z \in [y]\mathcal{R}} \mathcal{I}_{\mathcal{S}}(z)$, $\mathcal{N}_{\underline{\mathcal{R}}(y)} = \vee_{z \in [y]\mathcal{R}} \mathcal{N}_{\mathcal{S}}(z)$, $\mathcal{M}_{\overline{\mathcal{R}}(y)} = \vee_{z \in [y]\mathcal{R}} \mathcal{M}_{\mathcal{S}}(z)$, $\mathcal{I}_{\overline{\mathcal{R}}(y)} = \vee_{z \in [y]\mathcal{R}} \mathcal{I}_{\mathcal{S}}(z)$, $\mathcal{N}_{\overline{\mathcal{R}}(y)} = \wedge_{z \in [y]\mathcal{R}} \mathcal{N}_{\mathcal{S}}(z)$.

Definition 2.5. [8] Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ and $(\mathcal{U}, \tau_{\mathcal{R}'}(y))$ be a neutrosophic nano topological spaces, then the mapping $g: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X})) \to (\mathcal{U}, \tau_{\mathcal{R}'}(y))$ is said to be a neutrosophic nano continuous if the inverse image of every neutrosophic nano closed set in $\mathcal{V}$ is neutrosophic nano closed in $\mathcal{U}$.

Definition 2.6. [14] Let $(\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X}))$ and $(\mathcal{U}, \tau_{\mathcal{R}'}(y))$ be a neutrosophic nano topological spaces, then the mapping $g: (\mathcal{U}, \tau_{\mathcal{R}}(\mathcal{X})) \to (\mathcal{U}, \tau_{\mathcal{R}'}(y))$ is said to be a neutrosophic nano homeomorphism if

(i) g is one to one and onto.

(ii) g is neutrosophic nano continuous.

(iii) g is neutrosophic nano open.

Definition 2.7. [14] Let $\mathcal{G}$ and $\mathcal{G}'$ be any two graphs. They are isomorphic if there exist a neutrosophic nano homeomorphism $\varphi: [\mathcal{V}(\mathcal{G}), \tau(\mathcal{V}(\mathcal{H}))] \to [\mathcal{V}(\mathcal{G}), \tau(f(\mathcal{V}(\mathcal{H})))]$ for every sub graph $\mathcal{H}$ of $\mathcal{G}$.

Definition 2.8. [14] $\mathcal{N}[v]$ is said to be neutrosophic nano neighborhood of v if it is defined by $\mathcal{N}[v] = \{w \in \mathcal{V} : w$ is a neutrosophic nano neighborhood of $v\} \cup \{v\}$.

Definition 2.9. [14] Let $\mathcal{G}$ be a neutrosophic nano graph, $\mathcal{N}(v)$ a neutrosophic nano neighborhood of v in $\mathcal{V}$ and $\mathcal{H}$ a neutrosophic nano sub graph of $\mathcal{G}$, then $\tau(\mathcal{V}(\mathcal{H}))$ is a neutrosophic nano topology induced by graph $[\mathcal{V}(\mathcal{G}), \tau(\mathcal{V}(\mathcal{H}))]$. It is denoted by $\tau(\mathcal{V}(\mathcal{H})) = \{\varphi, \mathcal{V}(\mathcal{G}), \overline{\mathcal{N}}[\mathcal{V}(\mathcal{G})], \underline{\mathcal{N}}[\mathcal{V}(\mathcal{G})], \mathcal{B}_{\mathcal{N}}[\mathcal{V}(\mathcal{G})]\}$

Definition 2.10. [9] A single valued neutrosophic digraph $\mathcal{B}$is of the form $\mathcal{D} = (\mathcal{V}_{\mathcal{D}}, \mathcal{A}_{\mathcal{D}})$ where, $\mathcal{V}_{\mathcal{D}} = \{v_1, v_2, \ldots, v_n\}$ and the functions $t_{\mathcal{V}_{\mathcal{D}}}: \mathcal{V}_{\mathcal{D}} \to [0,1]$, $i_{\mathcal{V}_{\mathcal{D}}}: \mathcal{V}_{\mathcal{D}} \to [0,1]$, $f_{\mathcal{V}_{\mathcal{D}}}: \mathcal{V}_{\mathcal{D}} \to [0,1]$ denote the truth-membership function, a indeterminacy-membership function and falsity-membership function of the element $v_i \in \mathcal{V}_{\mathcal{D}}$, respectively and $0 \le t_{\mathcal{V}_{\mathcal{D}}}(v_i) + i_{\mathcal{V}_{\mathcal{D}}}(v_i) + f_{\mathcal{V}_{\mathcal{D}}}(v_i) \le 3$, $\forall v_i \in \mathcal{V}_{\mathcal{D}}$, $i = 1,2,\ldots,n$.

$\mathcal{A}_{\mathcal{D}} = \{(v_i, v_j) : (v_i, v_j) \in \mathcal{V}_{\mathcal{D}} \times \mathcal{V}_{\mathcal{D}}\}$ provided that $0 < \mathcal{E}(v_i)\mathcal{E}(v_j) \leq 0.5$ and the functions $t_{\mathcal{A}_{\mathcal{D}}} : \mathcal{A}_{\mathcal{D}} \to [0,1]$, $i_{\mathcal{A}_{\mathcal{D}}} : \mathcal{A}_{\mathcal{D}} \to [0,1]$, $f_{\mathcal{A}_{\mathcal{D}}} : \mathcal{A}_{\mathcal{D}} \to [0,1]$ are defined by

$t_{\mathcal{A}_{\mathcal{D}}}(v_i, v_j) \leq \min\,[t_{\mathcal{V}_{\mathcal{D}}}(v_i), t_{\mathcal{V}_{\mathcal{D}}}(v_j)]$

$i_{\mathcal{A}_{\mathcal{D}}}(v_i, v_j) \geq \max\,[i_{\mathcal{V}_{\mathcal{D}}}(v_i), i_{\mathcal{V}_{\mathcal{D}}}(v_j)]$

$f_{\mathcal{A}_{\mathcal{D}}}(v_i, v_j) \geq \max\,[f_{\mathcal{V}_{\mathcal{D}}}(v_i), f_{\mathcal{V}_{\mathcal{D}}}(v_j)]$

Where $t_{\mathcal{A}_{\mathcal{D}}}, i_{\mathcal{A}_{\mathcal{D}}}, f_{\mathcal{A}_{\mathcal{D}}}$ denote the truth-membership function, a indeterminacy membership function and falsity-membership function of the $arc(v_i, v_j) \in \mathcal{A}_{\mathcal{D}}$ respectively, where $0 \leq t_{\mathcal{A}_{\mathcal{D}}}(v_i, v_j) + i_{\mathcal{A}_{\mathcal{D}}}(v_i, v_j) + f_{\mathcal{A}_{\mathcal{D}}}(v_i, v_j) \leq 3, \forall (v_i, v_j) \in \mathcal{A}_{\mathcal{D}},\ i, j \in \{1, 2, \ldots n\}$.

Definition 2.11. [14] If $\mathcal{G}$ is a directed graph and $u, v \in \mathcal{V}$, then: u is in-vertex of v if $\overline{uv} \in \mathcal{E}(\mathcal{G})$. u is out-vertex of v if $\overline{vu} \in \mathcal{E}(\mathcal{G})$. The in-degree of a vertex v is the number of vertices u such that $\overline{uv} \in \mathcal{E}(\mathcal{G})$. The out-degree of a vertex v is the number of vertices u such that $\overline{vu} \in \mathcal{E}(\mathcal{G})$. Throughout this paper the word graph means directed simple graph.

3. Identifying Structural equivalence between LED light via neutrosophic nano topology

Definition 3.1. Let $\mathcal{G}$ be a neutrosophic nano graph, $v \in \mathcal{V}(\mathcal{G})$. Then we define the neutrosophic nano neighborhood of v as follows $\mathcal{N}[v] = \{u \in \mathcal{V}(\mathcal{G}) : \overline{vu} \in \mathcal{E}(\mathcal{G})\} \cup \{v\}$

Definition 3.2. Let $\mathcal{G}$ be a neutrosophic nano graph, $\mathcal{H}$ a neutrosophic nano sub graph of $\mathcal{G}$ and $\mathcal{N}(v)$ a neutrosophic nano neighborhood of v in $\mathcal{V}$. Then we define,

The lower approximation operation as follows: $\mathcal{L} : \mathcal{P}[\mathcal{V}(\mathcal{G})] \to \mathcal{P}[\mathcal{V}(\mathcal{G})]$ such that $\mathcal{N}_{\mathcal{L}}[\mathcal{V}(\mathcal{H})] = \cup_{v \in \mathcal{V}(\mathcal{G})} \{v : \mathcal{N}(v) \subseteq \mathcal{V}(\mathcal{H})\}$.

The upper approximation operation as follows: $\mathcal{U} : \mathcal{P}[\mathcal{V}(\mathcal{G})] \to \mathcal{P}[\mathcal{V}(\mathcal{G})]$ such that $\mathcal{N}_{\mathcal{U}}[\mathcal{V}(\mathcal{H})] = \cup_{v \in \mathcal{V}(\mathcal{G})} \{\mathcal{N}(v) : v \in \mathcal{V}(\mathcal{H})\}$.

(iii) The boundary region is defined as $\mathcal{N}_{\mathcal{B}}[\mathcal{V}(\mathcal{H})] = \mathcal{N}_{\mathcal{L}}[\mathcal{V}(\mathcal{H})] - \mathcal{N}_{\mathcal{U}}[\mathcal{V}(\mathcal{H})]$

Algorithm

Step:1 Taken two different electrical circuits of LED light denoted as $\mathcal{E}1$ and $\mathcal{E}2$.

Step:2 Convert the electrical circuits $\mathcal{E}1$ and $\mathcal{E}2$ to $\mathcal{N}_{\mathcal{G}1}$ and $\mathcal{N}_{\mathcal{G}2}$.

Step:3 Check whether $\mathcal{N}_{\mathcal{G}1}$ and $\mathcal{N}_{\mathcal{G}2}$ are homeomorphism corresponding neutrosophic nano topologies induced from their vertices.

Step:4 Check whether $\mathcal{N}_{\mathcal{G}1}$ is isomorphic to $\mathcal{N}_{\mathcal{G}2}$ and $[\mathcal{N}_{\mathcal{V}(\mathcal{G}1)}, \tau(\mathcal{N}_{\mathcal{V}(\mathcal{H}1)})]$ is isomorphic to $[\mathcal{N}_{\mathcal{V}(\mathcal{G}2)}, \tau(\mathcal{N}_{\mathcal{V}(\mathcal{H}2)})]$ then both graphs are isomorphic.

Step:5 Otherwise, we conclude that both the electrical circuits are entirely different.

Remark 3.3. Using the above algorithm to check that two electrical circuits are structurally equivalent.

Step:1 Consider the following basic circuit of the LED light. Using the above algorithm, we can prove whether these two circuits have functional similarities via neutrosophic nano topology induced by the vertices of its neutrosophic nano sub graphs (Figure 1).

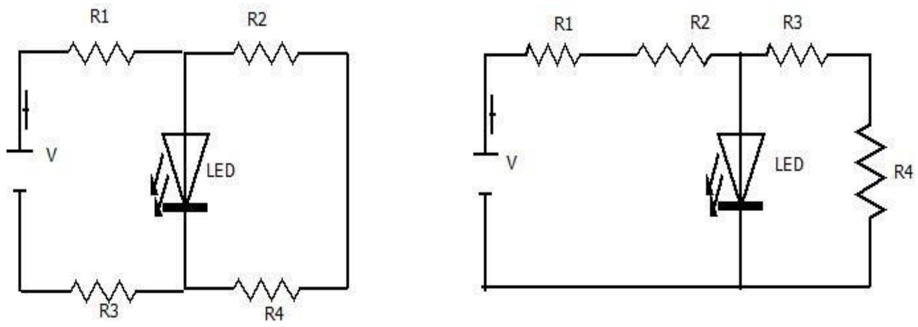
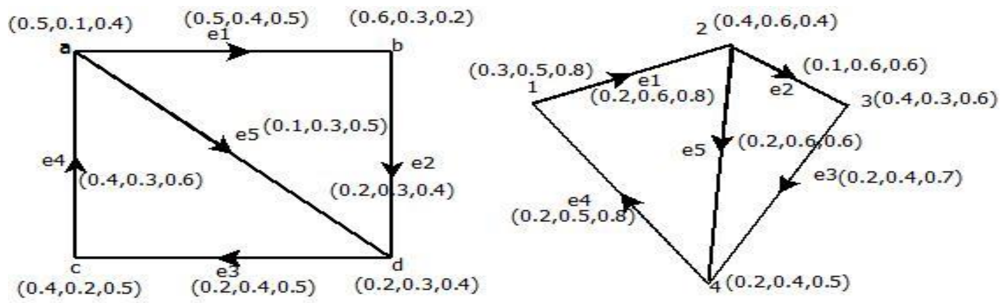


Figure 1

E1 E2

Step:2 Convert the basic circuit $\mathcal{E}1$ and $\mathcal{E}2$ into neutrosophic nano graphs $\mathcal{N}_{\mathcal{G}1}$ and $\mathcal{N}_{\mathcal{G}2}$ respectively. (Figure 2).



NG1 Figure 2 NG2

Step:3 Let $\mathcal{N}_{\mathcal{G}1}$ and $\mathcal{N}_{\mathcal{G}2}$ be two neutrosophic nano graphs.

Then $\mathcal{N}_{\mathcal{V}(\mathcal{G}1)} = \{a, b, c, d\}$ and $\mathcal{N}_{\mathcal{V}(\mathcal{G}2)} = \{1,2,3,4\}$, then the neighborhood of both graphs are $\mathcal{N}_n[d] = \{c, d\}$, $\mathcal{N}_n[c] = \{a, c\}$, $\mathcal{N}_n[b] = \{b, d\}$, $\mathcal{N}_n[a] = \{a, b, d\}$, and , $\mathcal{N}_n[1] = \{1,2\}$

Then the one to one mapping is defined as follows: .$\mathcal{N}_n[4] = \{1,4\}$, $\mathcal{N}_n[3] = \{3,4\}$, $\mathcal{N}_n[2] = \{2,3,4\}$

$f(a) = 2, f(b) = 3, f(c) = 1, f(d) = 4.$

Here f is a bijection between every pair of vertices $\mathcal{N}_{\mathcal{G}1}$ and $\mathcal{N}_{\mathcal{G}2}$, the path between every pair of vertices are equal.

Now, we prove that f is open map. Let us consider the two vertices, $\mathcal{V}(\mathcal{H}) = \{a, c\}$ and $\mathcal{V}(f(\mathcal{H})) = \{1,2\}$, then the neutrosophic nano topology of these two vertices are $\tau_{\mathcal{N}}\big(\mathcal{V}(\mathcal{H})\big) = \{\mathcal{V}(\mathcal{G}1), \varphi, \{a, c\}, \{b, d\}\}$and $\tau_{\mathcal{N}}\big(\mathcal{V}(\mathcal{H})\big) = \{\mathcal{V}(\mathcal{G}2), \varphi, \{1,2\}, \{3,4\}\}$. Hence the function are homeomorphism. Then the function f

$\varphi: [\mathcal{V}(\mathcal{G}1), \tau_{\mathcal{N}}(\mathcal{V}(\mathcal{H}))] \to [\mathcal{V}(\mathcal{G}2), \tau_{\mathcal{N}}(\mathcal{V}(\mathcal{H}))]$ is a neutrosophic nano homeomorphism. This holds for every sub graph $\mathcal{H}$ of $\mathcal{G}$.

Step:4 From the above given neutrosophic nano topology, it is concluded that all the sub graphs are neutrosophic nano homeomorphism. Hence the two different graphs are isomorphic, that is structural equivalence from the table 3.

Step:5 **Observation:** If all the sub graphs are neutrosophic nano homeomorphism then the two graphs are called neutrosophic nano isomorphism, which are structural equivalence. Using the above structural equivalence technique, we can check whether two circuits are equivalent and we can also extend our theory to many industrial products.

Table:1 Possible sub graph of $\mathcal{N}_{G1}$

$\mathcal{V}(\mathcal{H}1)$	$\mathcal{N}_L[\mathcal{V}(\mathcal{H}1)]$	$\mathcal{N}_U[\mathcal{V}(\mathcal{H}1)]$	$\mathcal{N}_B[\mathcal{V}(\mathcal{H}1)]$	$\tau_{\mathcal{N}}[\mathcal{V}(\mathcal{H}1)]$
$\{a\}$	φ	$\{a,b,d\}$	$\{a,b,d\}$	$\{V(G1),\ \varphi,\ \{a,b,d\}\}$
$\{b\}$	φ	$\{b,d\}$	$\{b,d\}$	$\{V(G1),\ \varphi,\ \{b,d\}\}$
$\{c\}$	φ	$\{a,c\}$	$\{a,c\}$	$\{V(G1),\ \varphi,\ \{a,c\}\}$
$\{d\}$	φ	$\{c,d\}$	$\{c,d\}$	$\{V(G1),\ \varphi,\ \{c,d\}\}$
$\{a,b\}$	φ	$\{a,b,d\}$	$\{a,b,d\}$	$\{V(G1),\ \varphi,\ \{a,b,d\}\}$
$\{b\ ,c\}$	φ	$V(G1)$	$V(G1)$	$\{V(G1),\ \varphi\}$
$\{c,d\}$	$\{c,d\}$	$\{a,c,d\}$	$\{a\}$	$\{V(G1),\ \varphi,\ \{a\},\ \{c,d\},\{a,c,d\}\}$
$\{a,d\}$	φ	$V(G1)$	$V(G1)$	$\{V(G1),\ \varphi\}$
$\{a,c\}$	$\{a,c\}$	$V(G1)$	$\{b,d\}$	$\{V(G1),\ \varphi,\ \{a,c\},\ \{b,d\}\}$
$\{b,d\}$	$\{b,d\}$	$\{b,c,d\}$	$\{c\}$	$\{V(G1),\ \varphi,\{c\},\ \{b,d\},\ \{b,c,d\}\}$
$\{a,b,c\}$	$\{a,c\}$	$V(G1)$	$\{b,d\}$	$\{V(G1),\ \varphi,\ \{a,c\},\ \{b,d\}\}$
$\{a,b,d\}$	$\{a,b,d\}$	$V(G1)$	$\{c\}$	$\{V(G1),\ \varphi,\ \{c\},\ \{a,b,d\}\}$
$\{b,c,d\}$	$\{b,c,d\}$	$V(G1)$	$\{a\}$	$\{V(G1),\ \varphi,\ \{a\},\ \{b,c,d\}\}$
$\{a,c,d\}$	$\{a,c,d\}$	$V(G1)$	$\{b\}$	$\{V(G1),\ \varphi,\ \{b\},\ \{a,c,d\}\}$
$V(G1)$	$V(G1)$	$V(G1)$	φ	$\{V(G1),\ \varphi\}$
φ	φ	φ	φ	$\{V(G1),\ \varphi\}$

Table:2 Possible sub graph of $\mathcal{N}_{G2}$

$\mathcal{V}(\mathcal{H}2)$	$\mathcal{N}_L[\mathcal{V}(\mathcal{H}2)]$	$\mathcal{N}_U[\mathcal{V}(\mathcal{H}2)]$	$\mathcal{N}_B[\mathcal{V}(\mathcal{H}2)]$	$\tau_{\mathcal{N}}[\mathcal{V}(\mathcal{H}2)]$
$\{1\}$	φ	$\{1,2\}$	$\{1,2\}$	$\{V(G2),\ \varphi,\ \{1,2\}\}$
$\{2\}$	φ	$\{2,3,4\}$	$\{2,3,4\}$	$\{V(G2),\ \varphi,\ \{2,3,4\}\}$
$\{3\}$	φ	$\{3,4\}$	$\{3,4\}$	$\{V(G2),\ \varphi,\ \{3,4\}\}$
$\{4\}$	φ	$\{1,4\}$	$\{1,4\}$	$\{V(G2),\ \varphi,\ \{1,4\}\}$
$\{1,2\}$	$\{1,2\}$	$V(G2)$	$\{3,4\}$	$\{V(G2),\ \varphi,\ \{1,2\},\ \{3,4\}\}$
$\{2,3\}$	φ	$\{2,3,4\}$	$\{2,3,4\}$	$\{V(G2),\ \varphi,\ \{2,3,4\}\}$
$\{3,4\}$	$\{3,4\}$	$\{1,3,4\}$	$\{1\}$	$\{V(G2),\ \varphi,\ \{1\},\ \{3,4\},\{1,3,4\}\}$
$\{1,4\}$	$\{1,4\}$	$\{1,2,4\}$	$\{2\}$	$\{V(G2),\ \varphi,\{2\},\ \{1,4\},\ \{1,2,4\}\}$
$\{1,3\}$	φ	$V(G2)$	$V(G2)$	$\{V(G2),\ \varphi\}$
$\{2,4\}$	φ	$\{2,3,4\}$	$\{2,3,4\}$	$\{V(G2),\ \varphi,\ \{2,3,4\}\}$
$\{1,2,3\}$	$\{1,2\}$	$V(G2)$	$\{3,4\}$	$\{V(G2),\ \varphi,\ \{1,2\},\ \{3,4\}\}$
$\{1,2,4\}$	$\{1,2,4\}$	$V(G2)$	$\{3\}$	$\{V(G2),\ \varphi,\ \{3\},\ \{1,2,4\}\}$
$\{2,3,4\}$	$\{2,3,4\}$	$V(G2)$	$\{1\}$	$\{V(G2),\ \varphi,\ \{1\},\ \{2,3,4\}\}$
$\{1,3,4\}$	$\{1,3,4\}$	$V(G2)$	$\{2\}$	$\{V(G2),\ \varphi,\ \{2\},\ \{1,3,4\}\}$
$V(G2)$	$V(G2)$	$V(G2)$	φ	$\{V(G2),\ \varphi\}$
φ	φ	φ	φ	$\{V(G2),\ \varphi\}$

Table:3 Neutrosophic Nano Isomorphic Table

$\mathcal{V}(\mathcal{H})$	$\tau_{\mathcal{N}}[\mathcal{V}(\mathcal{H})]$	$\mathcal{V}[f(\mathcal{H})]$	$\tau_{\mathcal{N}}[\mathcal{V}[f(\mathcal{H})]]$
$\{a\}$	$\{V(G1),\ \varphi,\ \{a,b,d\}\}$	$\{2\}$	$\{V(G2),\ \varphi,\ \{2,3,4\}\}$
$\{b\}$	$\{V(G1),\ \varphi,\ \{b,d\}\}$	$\{3\}$	$\{V(G2),\ \varphi,\ \{3,4\}\}$
$\{c\}$	$\{V(G1),\ \varphi,\ \{a,c\}\}$	$\{1\}$	$\{V(G2),\ \varphi,\ \{1,2\}\}$
$\{d\}$	$\{V(G1),\ \varphi,\ \{c,d\}\}$	$\{4\}$	$\{V(G2),\ \varphi,\ \{1,4\}\}$
$\{a,b\}$	$\{V(G1),\ \varphi,\ \{a,b,d\}\}$	$\{2,3\}$	$\{V(G2),\ \varphi,\ \{2,3,4\}\}$
$\{b\ ,c\}$	$\{V(G1),\ \varphi\}$	$\{1,3\}$	$\{V(G2),\ \varphi\}$
$\{c,d\}$	$\{V(G1),\ \varphi,\ \{a\},\ \{c,d\},\{a,c,d\}\}$	$\{1,4\}$	$\{V(G2),\ \varphi,\ \{2\},\ \{1,4\},\ \{1,2,4\}\}$
$\{a,d\}$	$\{V(G1),\ \varphi\}$	$\{2,4\}$	$\{V(G2),\ \varphi\}$
$\{a,c\}$	$\{V(G1),\ \varphi,\ \{a,c\},\ \{b,d\}\}$	$\{1,2\}$	$\{V(G2),\ \varphi,\ \{1,2\},\ \{3,4\}\}$
$\{b,d\}$	$\{V(G1),\ \varphi,\{c\},\ \{b,d\},\ \{b,c,d\}\}$	$\{3,4\}$	$\{V(G2),\ \varphi,\ \{1\},\ \{3,4\},\{1,3,4\}\}$
$\{a,b,c\}$	$\{V(G1),\ \varphi,\ \{a,c\},\ \{b,d\}\}$	$\{1,2,3\}$	$\{V(G2),\ \varphi,\ \{1,2\},\ \{3,4\}\}$
$\{a,b,d\}$	$\{V(G1),\ \varphi,\ \{c\},\ \{a,b,d\}\}$	$\{2,3,4\}$	$\{V(G2),\ \varphi,\ \{1\},\ \{2,3,4\}\}$
$\{b,c,d\}$	$\{V(G1),\ \varphi,\ \{a\},\ \{b,c,d\}\}$	$\{1,3,4\}$	$\{V(G2),\ \varphi,\ \{2\},\ \{1,3,4\}\}$
$\{a,c,d\}$	$\{V(G1),\ \varphi,\ \{b\},\ \{a,c,d\}\}$	$\{1,2,4\}$	$\{V(G2),\ \varphi,\ \{3\},\ \{1,2,4\}\}$
$V(G1)$	$\{V(G1),\ \varphi\}$	$V(G2)$	$\{V(G2),\ \varphi\}$
φ	$\{V(G1),\ \varphi\}$	φ	$\{V(G2),\ \varphi\}$

Conclusion:

The purpose of the present work was to make headway for the application of neutrosophic nano topology via graph theory. We believe that neutrosophic nano topological graph structure will be an important base for modification of knowledge extraction and processing.

The aim of this paper was to generate neutrosophic nano topological structure on the power set of vertices of simple neutrosophic digraphs, by using new definition neutrosophic neighbourhood. Based on the neutrosophic neighborhood, we define the approximations of the subgraphs of a graph. A new neutrosophic nano topological graph have been used to analyze the symbolic circuit in this paper. By means of structural equivalence on neutrosophic nano topology induced by graph we have framed an algorithm for detecting patent infringement suit.

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References

1. Abdel-Basset, M., El-hoseny, M., Gamal, A., & Smarandache, F. (2019). A novel model for evaluation Hospital medical care systems based on plithogenic sets. Artificial intelligence in medicine, 100, 101710.
2. Abdel-Basset, M., Manogaran, G., Gamal, A., & Chang, V. (2019). A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT. IEEE Internet of Things Journal.
3. Abdel-Basset, M., Mohamed, R., Zaied, A. E. N. H., & Smarandache, F. (2019). A hybrid plithogenic decision-making approach with quality function deployment for selecting supply chain sustainability metrics. Symmetry, 11(7), 903.

4. Abdel-Basset, M., Nabeeh, N. A., El-Ghareeb, H. A., & Aboelfetouh, A. (2019). Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises. Enterprise Information Systems, 1-21.
5. Abdel-Baset, M., Chang, V., & Gamal, A. (2019). Evaluation of the green supply chain management practices: A novel neutrosophic approach. Computers in Industry, 108, 210-220.
6. Abdel-Basset, M., Saleh, M., Gamal, A., & Smarandache, F. (2019). An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number. Applied Soft Computing, 77, 438-452.
7. Atanassov K., Intuitionstic fuzzy sets. Fuzzy Sets Systems, 1986, 20, p.87-96.
8. Arafa Nasef and Abd El Fattah El Atik, Some properties on nano topology induced by graphs, AASCIT Journal of Nano science, 2017, Vol 3(4), p.19-23.
9. Broumi. S, Mohamed Talea, Smarandache, F., and Bakali, A., Single valued neutrosophic graphs Degree, Order and Size, IEEE international conference on fuzzy system, 2016.
10. Chang, C.L., Fuzzy Topological Spaces, J. Math. Anal. Appl. 1968, 24, p.182-190.Coker. B., An introduction to intuitionistic fuzzy topological spaces, Fuzzy Sets and Systems, 1997, Vol 88, No.1, p.81-89.
11. Kalyan Sinha and Pinaki Majumda, Entropy based Single Valued Neutrosophic Digraph and its applications, Neutrosophic Sets and Systems, 2018, Vol. 19, 119-126.
12. Lellis Thivagar, M., Carmel Richard, On nano forms of weakly open sets, International journal of mathematics and statistics invention, 2013, Volume 1, Issue 1, p.31-37.
13. Lellis Thivagar, M., Paul Manuel and V. Sudhadevi, A detection for patent infringement suit via nano topology induced by graph, Cogent mathematics, 2016, Vol 3.
14. Lellis Thivagar, M., Jafari, S., Sudhadevi, V., and Antonysamy, V., A novel approach to nano topology via neutrosophic sets, Neutrosophic sets and systems, Vol 20, 2018.
15. Lupianez, F.G., On Neutrosophic sets and topology, Kybernetes, 2008, 37, p.797-800.
16. Pawlak, Z., Rough sets, Int.J.Comput. Inf. Sci. 1982, 11 (5), p.341-356.
17. Salama, A.A., and Alblowi, S.A., Neutrosophic set and neutrosophic topological spaces, IOSR-Journal of Mathematics, 2012, 3, p.31-35.
18. Salama, A.A., Samarandache, F., and Valeri, K., Neutrosophic closed set and neutrosophic continuous functions, Neutrosophic Sets Systems, 2014, 4, p.4-8.
19. Smarandache, F., A unifying eld in logics neutrosophic probability, set and logic, Rehoboth Ameriacan Research Press 1999.
20. Wilson, R.J., Introduction to graph theory, Fourth Edition, Longmon Maleysia, 1996.
21. Zadeh, L.A., Fuzzy Sets. Inf. Control, 1965, 8, p.338-353.

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Neutrosophic Fixed Point Theorems and Cone Metric Spaces

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Abstract. The intention of this paper is to give the general definition of cone metric space in the context of the neutrosophic theory. In this relation, we obtain some fundamental results concerting fixed points for weakly compatible mapping.

Keywords: neutrosophic theory, neutrosophic Fixed Point, neutrosophic topology, neutrosophic cone metric space, neutrosophic metric space.

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1. **Introduction**

Zadeh [13] introduced the notion of fuzzy sets. After that there have been a number of generalizations of this fundamental concept. The study of fuzzy topological spaces was first initiated by Chang [6] in the year 1968. Atanassov [12] introduced the notion of intuitionistic fuzzy sets. This notion was extended to intuitionistic L-fuzzy setting by Atanassov and Stoeva [20], which currently holds the name "intuitionistic L-topological spaces". Using the notion of intuitionistic fuzzy sets, Coker [7] introduced the notion of intuitionistic fuzzy topological space. The concept of generalized fuzzy closed set was introduced by G. Balasubramanian and P. Sundaram [11]. In various recent papers, F. Smarandache generalizes intuitionistic fuzzy sets (IFSs) and other kinds of sets to neutrosophic sets (NSs). F. Smarandache and A. Al Shumrani also defined the notion of neutrosophic topology on the non-standard interval [2, 9, 14, 16]. Also, ([8, 15, 17]) introduced the metric topology and neutrosophic geometric and studied various properties. Recently, Wadei Al-Omeri and Smarandache [18, 19] introduce and study the concepts of neutrosophic open sets and its complements in neutrosophic topological space, continuity in neutrosophic topology, and obtain some characterizations concerning neutrosophic connectedness and neutrosophic mapping.

This paper is arranged as follows. In Section 2, we will recall some notions which will be used throughout this paper. In Section 3, neutrosophic Cone Metric Space and investigate its basic properties. In Section 4, we study the neutrosophic Fixed Point Theorems and study some of their properties. Finally, Banach contraction theorem and some fixed point results on neutrosophic cone metric space are stated and proved.

2. Preliminaries

Definition 2.1. [4] Let Σ be a non-empty fixed set. A neutrosophic set (briefly NS) B is an object having the form $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$, where $\xi_B(r), \varrho_B(r)$, and $\eta_B(r)$ which represent the degree of membership function (namely $\xi_B(r)$), the degree of indeterminacy (namely $\varrho_B(r)$), and the degree of non-membership (namely $\eta_B(r)$) respectively, of each element $r \in \Sigma$ to the set B.

A neutrosophic set $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$ can be identified to an ordered triple $\langle \xi_B(r), \varrho_B(r)$
$, \eta_B(r)\rangle$ in $\rfloor 0^-, 1^+ \lfloor$ on Σ.

Remark 2.1. [4] For the sake of simplicity, we shall use the symbol $B = \{r, \xi_B(r),$
$\varrho_B(r), \eta_B(r)\}$ for the NS $B = \{\langle r, \xi_B(r), \varrho_B(r), \eta_B(r)\rangle : r \in \Sigma\}$.

Definition 2.2. [5] Let $B = \langle \xi_B(r), \varrho_B(r), \eta_B(r)\rangle$ be an NS on Σ. The complement of B(briefly$C(B)$), are defined as three types of complements

(1) $C(B) = \{\langle r, \eta_B(r), 1 - \varrho_B(r), \xi_B(r)\rangle : r \in \Sigma\}$,
(2) $C(B) = \{\langle r, 1 - \xi_B(r), 1 - \eta_B(r)\rangle : r \in \Sigma\}$
(3) $C(B) = \{\langle r, \eta_B(r), \varrho_B(r), \xi_B(r)\rangle : r \in \Sigma\}$

We have the following NSs (see [4]) which will be used in the sequel:

(1) $0_N = \{\langle r, 0, 0, 1\rangle : r \in \Sigma\}$ or
(2) $0_N = \{\langle r, 0, 1, 1\rangle : r \in \Sigma\}$ or
(3) $0_N = \{\langle r, 0, 0, 0\rangle : r \in \Sigma\}$ or
(4) $0_N = \{\langle r, 0, 1, 0\rangle : r \in \Sigma\}$

2- 1_N may be defined as four types:

(1) $1_N = \{\langle r, 1, 1, 1\rangle : r \in \Sigma\}$ or
(2) $1_N = \{\langle r, 1, 0, 0\rangle : r \in \Sigma\}$ or
(3) $1_N = \{\langle r, 1, 1, 0\rangle : r \in \Sigma\}$ or

(4) $1_N = \{\langle r, 1, 0, 1\rangle : r \in \Sigma\}$

Definition 2.3. [4] Let $x \neq \emptyset$, and generalized neutrosophic sets ($GNSs$) B and Γ be in the form $B = \{r, \xi_B(r), \varrho_B(r), \eta_B(r)\}$, $\Gamma = \{r, \xi_\Gamma(r), \varrho_\Gamma(r), \eta_\Gamma(r)\}$. We think of two possible definitions $A \subseteq \Gamma$.

(1) $B \subseteq \Gamma \Leftrightarrow \xi_B(r) \leq \xi_\Gamma(r), \varrho_B(r) \geq \varrho_\Gamma(r), \text{and}\, \eta_B(r) \leq \eta_\Gamma(r)$

(2) $B \subseteq \Gamma \Leftrightarrow \xi_B(r) \leq \xi_\Gamma(r), \varrho_B(r) \geq \varrho_\Gamma(r), \text{and}\, \eta_B(r) \geq \eta_\Gamma(r)$.

Definition 2.4. [4] Let $\{B_j : j \in J\}$ be an arbitrary family of an NSs in Σ. Then

(1) $\cap B_j$ defined as two types:

- $\cap B_j = \langle r, \underset{j\in J}{\wedge} \xi_{Bj}(r), \underset{j\in J}{\wedge} \varrho_{Bj}(r), \underset{j\in J}{\vee} \eta_{Bj}(r)\rangle < \text{Type } 1 >$
- $\cap B_j = \langle r, \underset{j\in J}{\wedge} \xi_{Bj}(r), \underset{j\in J}{\vee} \varrho_{Bj}(r), \underset{j\in J}{\vee} \eta_{Bj}(r)\rangle < \text{Type } 2 >$.

(2) $\cup B_j$ defined as two types:

- $\cup B_j = \langle r, \underset{j\in J}{\vee} \xi_{Bj}(r), \underset{j\in J}{\vee} \varrho_{Bj}(r), \underset{j\in J}{\wedge} \eta_{Bj}(r)\rangle < \text{Type } 1 >$
- $\cup B_j = \langle r, \underset{j\in J}{\vee} \xi_{Bj}(r), \underset{j\in J}{\wedge} \varrho_{Bj}(r), \underset{j\in J}{\wedge} \eta_{Bj}(r)\rangle < \text{Type } 2 >$

Definition 2.5. [3] A neutrosophic topology (briefly NT) and a non empty set Σ is a family Υ of neutrosophic subsets of Σ satisfying the following axioms

(1) $0_N, 1_N \in \Upsilon$

(2) $S_1 \cap S_2 \in \Upsilon$ for any $S_1, S_2 \in \Upsilon$

(3) $\cup S_i \in \Upsilon, \forall \{S_i | i \in I\} \subseteq \Upsilon$.

The pair (Σ, Υ) is called a neutrosophic topological space (briefly NTS) and any neutrosophic set in Υ is defined as neutrosophic open set (NOS for short) in Σ. The elements of Υ are called open neutrosophic sets. A neutrosophic set S is closed if f its $C(S)$ is neutrosophic open. For any NTS A in (Σ, Υ) ([21]), we have $Int(A^c) = [Cl(A)]^c$ and $Cl(A^c) = [Int(A)]^c$.

Definition 2.6. A subset ω of Ω is called a cone if

(1) For non-empty ω is closed, and $\omega \neq 0$,

(2) If both $u \in \omega$ and $-u \in \omega$ then $u = 0$,

(3) If $u,\, v \in S,\, u,\, v \geq 0$ and $x,\, y \in \omega$ then $ux + vy \in \omega$.

Throughout this paper, we assume that all cones have non-empty interior. For any cone, $x \prec y$ will stand for $x \preccurlyeq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in Int(\omega)$. a partial ordering $\preccurlyeq$ on Ω via ω is defined by $x \preccurlyeq y$ iff $y - x \in \omega$.

Definition 2.7. A cone metric space (briefly CMS) an ordered (Σ, d), where Σ is any set and $d : \Sigma \times \Sigma \longmapsto \Omega$ is a mapping satisfying:

(1) $d(s_1, s_2) = d(s_2, s_1)$ for all $s_1,\, s_2 \in \Sigma$,

(2) $d(s_1, s_2) = 0$ iff $s_1 = s_2$,

(3) $0 \preccurlyeq d(s_1, s_2)$ for all $s_1,\ s_2 \in \Sigma$,

(4) $d(s_1, s_3) \preccurlyeq d(s_1, s_2) + d(s_2, s_3)$ for all $s_1,\ s_2,\ s_3 \in \Sigma$.

Definition 2.8. Let (Σ, d) be a CMS. Then, for each $c_1 \gg 0$ and $c_2 \gg 0$, $c_1,\ c_2 \in \Omega$, there exists $c \gg 0$, $c \in \Omega$ such that $c \ll c_1$ and $c \ll c_2$.

Definition 2.9. A binary operation $\bigotimes : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if $\bigotimes$ satisfies the following conditions:

(1) $\bigotimes$ is continuous,

(2) $\bigotimes$ is commutative and associative,

(3) $m_1 \bigotimes m_2 \leq m_3 \bigotimes m_4$ whenever $m_1 \leq m_3$ and $m_2 \leq m_4$ $\forall m_1,\ m_2,\ m_3,\ m_4 \in [0,1]$,

(4) $m_1 \bigotimes 1 = m_1$ $\forall m_1 \in [0,1]$.

Definition 2.10. A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-conorm if $\diamond$ satisfies the following conditions:

(1) $\diamond$ is continuous,

(2) $\diamond$ is commutative and associative,

(3) $m_1 \diamond m_2 \leq m_3 \diamond m_4$ whenever $m_1 \leq m_3$ and $m_2 \leq m_4$ $\forall m_1,\ m_2,\ m_3,\ m_4 \in [0,1]$,

(4) $m_1 \diamond 1 = m_1$ $\forall m_1 \in [0,1]$.

Definition 2.11. Let Σ be a non-empty set. The mappings $\mathcal{G} : \Sigma \times \Sigma \longrightarrow \Sigma$ and $\mathcal{H} : \Sigma \longrightarrow \Sigma$ are called commutative if $\mathcal{H}(\mathcal{G}(x,y)) = \mathcal{G}(\mathcal{H}(x),\ \mathcal{H}(y))$ $\forall x,\ y \in \Sigma$.

Definition 2.12. Let $\Sigma \neq \emptyset$. An element $x \in \Sigma$ is called a common fixed point of mappings $\mathcal{G} : \Sigma \times \Sigma \longrightarrow \Sigma$ and $\mathcal{H} : \Sigma \longrightarrow \Sigma$ if $x = \mathcal{H}(x) = \mathcal{G}(x,x)$.

Definition 2.13. If U and V are two maps then, a pair of maps is called weakly compatible (briefly WCP) pair if they commute at (CP).

Definition 2.14. Let Σ be a set, $\mathcal{G}$, $\mathcal{H}$ self maps of Σ. A point x in Σ is called a coincidence point (briefly CP) of $\mathcal{G}$ and $\mathcal{H}$ if and only if $\mathcal{G}(x) = \mathcal{H}(x)$. We call $w = \mathcal{G}(x) = \mathcal{H}(x)$ a point of coincidence of $\mathcal{G}$ and $\mathcal{H}$.

Definition 2.15. Two self maps $\mathcal{G}$ and $\mathcal{H}$ of a set Σ are sporadically weakly compatible of Σ. If $\mathcal{G}$ and $\mathcal{H}$ have a unique point of coincidence, $z = \mathcal{G}(u) = \mathcal{H}(v)$, then z is the unique common fixed point of $\mathcal{G}$ and $\mathcal{H}$.

Lemma 2.2. *Two self maps $\mathcal{G}$ and $\mathcal{H}$ of a set Σ are sporadically weakly compatible of Σ. then z is the unique common fixed point of $\mathcal{G}$ and $\mathcal{H}$, if $z = \mathcal{G}(u) = \mathcal{H}(u)$ $\mathcal{G}$ and $\mathcal{H}$ have a unique point of coincidence.*

Definition 2.16. A pair of maps $\mathcal{G}$ and $\mathcal{H}$ which $\mathcal{G}$ and $\mathcal{H}$ commute of a set Σ are sporadically weakly compatible iff there is a point x in Σ which is a coincidence point of $\mathcal{G}$ and $\mathcal{H}$.

3. neutrosophic Cone Metric Space

Definition 3.1. A 3-tuple $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is said to be a neutrosophic CMS if ω is a neutrosophic cone metric (briefly NCMS) of Ω, Σ is an arbitrary set, $\diamond$ is a neutrosophic continuous t-conorm , $\bigotimes$ is a neutrosophic continuous t-norm, $\forall \epsilon_1, \epsilon_2, \epsilon_3 \in \Sigma$ and m, $n \in Int(\omega)$ (that is $n \gg 0_\Theta$, $s \gg 0_\Theta$), and Ξ, Θ are neutrosophic set on $\Sigma^2 \times Int(\omega)$ satisfying the following conditions:

(1) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) + \Theta(\epsilon_1, \epsilon_2, \epsilon_3) \leq 1_\Theta$;
(2) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) > 0_\Theta$;
(3) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) = 1$ iff $\epsilon_1 = \epsilon_2$;
(4) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) = \Xi(\epsilon_2, \epsilon_1, m)$;
(5) $\Xi(\epsilon_1, \epsilon_2, \epsilon_3) \bigotimes \Xi(\epsilon_2, \epsilon_3, n) \leq \Xi(\epsilon_1, \epsilon_3, m + n)$;
(6) $\Xi(\epsilon_1, \epsilon_2, .) : Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic continuous;
(7) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) < 0_\Theta$;
(8) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) = 0_\Theta$ if and only if $\epsilon_1 = \epsilon_2$;
(9) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) = \Theta(\epsilon_2, \epsilon_3, r)$;
(10) $\Theta(\epsilon_1, \epsilon_2, \epsilon_3) \diamond \Theta(\epsilon_2, \epsilon_3, n) \geq \Theta(\epsilon_1, \epsilon_3, m + n)$;
(11) $\Theta(\epsilon_1, \epsilon_2, .) : Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic continuous.

Then (Ξ, Θ) is called a neutrosophic cone metric on Σ. The functions $\Theta(\epsilon_1, \epsilon_2, m)$ and $\Xi(\epsilon_1, \epsilon_2, m)$ denote the degree of non-nearness and the degree of nearness between ϵ_1 and ϵ_2 with respect to n, respectively.

Example 3.2. Let $\Omega = R$, $\omega = [0, \infty)$ and $a \diamond b = max\{a, b\}$, $a \bigotimes b = min\{a, b\}$, then every neutrosophic metric space (Σ, Ξ, Θ) becomes a $NCMS$.

Example 3.3. If we take ω be an any cone, $a \bigotimes b = min\{a, b\}$, $\Sigma = \Theta$, $\Xi, \Theta : \Sigma^2 \times Int(\omega) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ defined by

$$\Xi(\epsilon_1, \epsilon_2, t) = \begin{cases} \dfrac{\epsilon_1}{\epsilon_2}, & if \quad \epsilon_1 \leq \epsilon_2, \\ \dfrac{\epsilon_1}{\epsilon_2}, & if \quad \epsilon_2 \leq \epsilon_1, \end{cases}$$

$$\Theta(\epsilon_1, \epsilon_2, t) = \begin{cases} \dfrac{\epsilon_2 - \epsilon_1}{\epsilon_2}, & if \quad \epsilon_1 \leq \epsilon_2, \\ \dfrac{\epsilon_1 - \epsilon_2}{\epsilon_2}, & if \quad \epsilon_2 \leq \epsilon_1, \end{cases}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $r \gg 0_\Theta$. Then $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a $NCMS$.

Definition 3.4. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$, $\{\epsilon_{1n}\}$ be a sequence in Σ and $\epsilon_1 \in \Sigma$. Then $\{\epsilon_{1n}\}$ is said to converge to ϵ_1 if for any $s \in (0,1)$ and any $m \gg 0_\Theta$ $\exists$ a natural number n_0 such that $\Xi(\epsilon_{1n}, x, m) > 1 - s, \Theta(\epsilon_{1n}, \epsilon_1, m) \leq s$ for all $n \geq n_0$. We denote this by $\lim_{\epsilon_{1n} \to \infty} = \epsilon_1$ or $\epsilon_{1n} \to \epsilon_1$ as $\to \infty$.

Definition 3.5. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. For $m \gg 0_\Theta$, the open ball $\Gamma(x, s, m)$ with radius $s \in (0,1)$ and center ϵ_1 is defined by $\Gamma(\epsilon_1, s, m) = \{\epsilon_2 \in \Sigma : \Xi(\epsilon_1, \epsilon_2, m) > 1 - s, \Theta(\epsilon_1, \epsilon_2, m) < s\}$.

Definition 3.6. The neutrosophic cone metric CMS $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is called complete neutrosophic CMS if every Cauchy sequence in $NCMS$ (Σ, Ξ, Θ) is convergent.

Definition 3.7. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. A subset P of Σ is said to be $\mathcal{FC}$-bounded if $\exists$ $s \in (0,1)$ and $m \gg \theta$ such that $\Xi(\epsilon_1, \epsilon_2, t) > 1 - m, \Theta(\epsilon_1, \epsilon_2, m) < s$ for all $\epsilon_1, \epsilon_2 \in P$.

Definition 3.8. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic CMS and $h : \Sigma \to \Sigma$ is a self mapping. Then h is said to be neutrosophic cone contractive if there exists $c \in (0,1)$ such that
$\frac{1}{\Xi(h(\epsilon_1), h(\epsilon_2), m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_2, m)} - 1)$
$\Theta(h(\epsilon_1), h(\epsilon_2), m) \leq c\Theta(\epsilon_1, \epsilon_2, m)$
for each $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$. The constant c is called the contractive constant of h.

Lemma 3.9. *If for two points $\epsilon_1, \epsilon_2 \in \Sigma$ and $c \in (0,1)$ such that $\Xi(\epsilon_1, \epsilon_2, cm) \geq \Xi(\epsilon_1, \epsilon_2, m)$, $\Theta(\epsilon_1, \epsilon_2, cm) \geq \Theta(\epsilon_1, \epsilon_2, m)$ then $\epsilon_1 = \epsilon_2$.*

Theorem 3.10. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$. Define $\mathcal{T} = \{K \subseteq \Sigma : \epsilon_1 \in K \text{iff there exists} s \in (0,1) \text{and} m \gg 0_\Theta \text{such that} L(\epsilon_1, s, m) \subseteq K\}$, then $\mathcal{T}$ is a neutrosophic topology on Σ.*

Proof. If ϵ_1 is empty, then $\emptyset = L(\epsilon_1, s, m) \subseteq \emptyset$. Hence the empty set belong to $\mathcal{T}$ Since for any $\epsilon_1 \in \Sigma$, any $s \in (0,1)$ and any $m \gg 0_\Theta$, $L(\epsilon_1, s, m) \subseteq \Sigma$, then $\Sigma \in \mathcal{T}$.
Let $K, L \in \mathcal{T}$ and $\epsilon_1 \in K \cap L$. Then $\epsilon_1 \in K$ and $\epsilon_1 \in L$, so there exist $m_1 \gg 0_\Theta$; $m_2 \gg 0_\Theta$ and $m_1, m_2 \in (0,1)$ such that $L(\epsilon_1, s_1, m_1) \subseteq K$ and $L(\epsilon_1, s_2, m_2) \subseteq L$.
By Proposition 2.8, for $m_1 \gg 0$; $m_2 \gg 0$, there exists $m \gg 0_\Theta$ such that $m \gg m_1$; $r \gg m_2$ and take $s = min\{m_1, m_2\}$. Then $L(\epsilon_1, s, m) \subseteq \Sigma$ $L(\epsilon_1, s_1, m_1) \cap L(\epsilon_1, s_2, m_2) \subseteq K \cap L$. Thus $K \cap L \in \mathcal{T}$. Let $K_i \in \mathcal{T}$ for each $i \in I$ and $\epsilon_1 \in \cup_{i \in I} K_i$. Then there exists $i_0 \in I$ such that $\epsilon_1 \in K_{i0}$. So, there exist $r \gg 0_\Theta$ and $s \in (0,1)$ such that $L(\epsilon_1, s, m) \subseteq K_{i_0}$. Since$K_{i_0} \subseteq \cup_{i \in I} K_i$, $L(\epsilon_1, s, m) \subseteq \cup_{i \in I} K_i$. Thus $\cup_{i \in I} K_i \in \mathcal{T}$. Hence, $\mathcal{T}$ is a neutrosophic topology on Σ. □

Theorem 3.11. *If $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is a $NCMS$, then the neutrosophic topology $(\Sigma, \mathcal{T})$ is Hausdorff.*

Proof. Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a neutrosophic CMS. Let ϵ_1, ϵ_2 be two distinct points of Σ. Then $0 < \Xi(\epsilon_1, \epsilon_2, m) < 1_\Theta$ and $0 < \Theta(\epsilon_1, \epsilon_2, m) < 1_\Theta$. Let $\Xi(\epsilon_1, \epsilon_2, m) = s_1$, $\Theta(\epsilon_1, \epsilon_2, m) = s_2$ and $s = max\{s_1, s_2\}$. Then for each $s_0 \in (s, 1)$, there exists s_3 and s_4 such that $s_3 \bigotimes s_3 \geq s_0$ and $(1_\Theta - s_4) \diamond (1_\Theta - s_4) \leq (1_\Theta - s_0)$. Put $s_4 = max\{s_3, s_4\}$ and consider the open balls $L(\epsilon_1, 1_\Theta - s_5, m/2)$ and $L(\epsilon_2, 1_\Theta - s_5, m/2)$.
Then clearly $L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1 - s_5, m/2) = \emptyset$
. Suppose that $L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1 - s_5, m/2) \neq \emptyset$. Then there exists $\epsilon_3 \in L(x, 1_\Theta - s_5, m = 2) \cap L(\epsilon_2, 1_\Theta - s_5, m/2)$.

$$\begin{aligned} s_1 =& \Xi(\epsilon_1, \epsilon_2, m) \\ \geq& \Xi(\epsilon_1, \epsilon_3, m/2) \bigotimes \Xi(\epsilon_3, \epsilon_2, m/2) \\ \geq& s_5 \bigotimes s_5 \\ \geq& s_3 \bigotimes s_3 \\ \geq& s_0 > s_1 \end{aligned}$$

and

$$\begin{aligned} s_2 =& n(\epsilon_1, \epsilon_2, m) \\ \geq& n(\epsilon_1, \epsilon_3, m/2) \bigotimes n(\epsilon_3, \epsilon_2, m/2) \\ \geq& (1_\Theta - s_5) \diamond (1_\Theta - s_5) \\ \geq& (1_\Theta - s_4) \diamond (1_\Theta - s_4) \\ \leq& 1_\Theta - s_0 < s_2 \end{aligned}$$

This is a contradiction. Hence $((\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ is Hausdorff. □

Theorem 3.12. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a $NCMS$, $\epsilon_1 \in \Sigma$ and (ϵ_{1n}) a sequence in Σ. Then (ϵ_{1n}) converges to ϵ_1 if and only if $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) \to 0$ as $n \to 1_\Theta$, for each $m \gg 0_\Theta$.*

Proof. Let $(\epsilon_{1n}) \to \epsilon_1$. Then, for each $m \gg 0_\Theta$ and $s \in (0, 1)$, there exists a natural number n_0 such that $\Xi(\epsilon_{1n}, \epsilon_1, m) > 1_\Theta - s$, $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ for all $n \gg n_0$. We have $1 - \Xi(\epsilon_{1n}, \epsilon_1, m) < m$ and $\Xi(\epsilon_{1n}, \epsilon_1, m) < m$. Hence $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) \to 0$ as $n \to 1$. Conversely, Suppose that $\Xi(\epsilon_{1n}, \epsilon_1, m) \to 1_\Theta$ as $n \to 1_\Theta$. Then, for each $m \gg 0_\Theta$ and $s \in (0, 1)$, there exists a natural number n_0 such that $1_\Theta - \Xi(\epsilon_{1n}, \epsilon_1, m) < s$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ for all $n \geq n_0$. In that case, $\Xi(\epsilon_{1n}, \epsilon_1, m) > 1_\Theta - s$ and $\Theta(\epsilon_{1n}, \epsilon_1, m) < s$ Hence $(\epsilon_{1n}) \to \epsilon_1$ as $n \to 1_\Theta$. □

4. **Neutrosophic Fixed Point Theorems**

Theorem 4.1. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete NCMS in which neutrosophic cone contractive sequences are Cauchy. Let $\mathcal{H}$ a neutrosophic cone contractive mapping. Then $\mathcal{H}$ has a unique fixed point. Where $\mathcal{H} : \Sigma \to \Sigma$ with c as the contractive constant.*

Proof. Let $\epsilon_1 \in \Sigma$ and fix $\epsilon_{1n} = \mathcal{H}^n(x), n \in \Theta$ For $m \gg 0_\Theta$, we have

$$\frac{1}{\Xi(\mathcal{H}(\epsilon_1), \mathcal{H}^2(\epsilon_1), m)} - 1_\Theta \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_{11}, m)} - 1_\Theta),$$

$$\Theta(\mathcal{H}(\epsilon_1), \mathcal{H}^2(\epsilon_1), m) \leq c\Theta(\epsilon_1, \epsilon_{11}, m).$$

And by induction

$$\frac{1}{\Xi(\epsilon_{1n+1}, \epsilon_{1n+2}, m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_1, \epsilon_{1n+1}, m)} - 1)$$

,

$$\Theta(\epsilon_{1n+1}, \epsilon_{1n+2}, m) \leq c\Theta(\epsilon_1, \epsilon_{1n+1}, m) \text{ for all } n \in \Theta.$$

Then (ϵ_{1n}) is a neutrosophic contractive sequence, by assumptions (ϵ_{1n}) converges to ϵ_2 and it is a Cauchy sequence, for some $\epsilon_2 \in \Sigma$. By Theorem 3.12, we have

$$\frac{1}{\Xi(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m)} - 1 \leq c(\frac{1}{\Xi(\epsilon_2, \epsilon_{1n}, m)} - 1) \to 0$$

$$\Theta(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) \leq c\Theta(\epsilon_2, \epsilon_{1n}, m) \to o$$

as $n \to 1$. Then for each $m \gg 0_\Theta$,

$$\lim_{n\to\infty} \Xi(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) = 1, \lim_{n\to\infty} \Theta(\mathcal{H}(\epsilon_2), \mathcal{H}(\epsilon_{1n}), m) = 0_\Theta,$$

and hence $\lim_{n\to\infty} \mathcal{H}(\epsilon_{1n}) = \mathcal{H}(\epsilon_2)$, i.e.,$\lim_{n\to\infty} \epsilon_{1n+1} = \mathcal{H}(\epsilon_2)$ and $\mathcal{H}(\epsilon_2) = \epsilon_2$. To show uniqueness. Let $\mathcal{H}(kkk) = \epsilon_3$ for some $\epsilon_3 \in W$. For $m \gg 0_\Theta$, we have

$$\begin{aligned}
\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1 =& \frac{1}{\Xi(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)} - 1\\
\leq \quad & c(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1)\\
=& c(\frac{1}{\Xi(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)} - 1)\\
\leq& c^2(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1)\\
\leq& ... \leq c^n(\frac{1}{\Xi(\epsilon_2,\epsilon_3,m)} - 1) \to 0 \,\text{as}\, n \to \infty. \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
\Theta(\epsilon_2,\epsilon_3,m) =& \Theta(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)\\
\leq& c(\Theta(\epsilon_2,\epsilon_3,m)\\
=& c\Theta(\mathcal{H}(\epsilon_2),\mathcal{H}(\epsilon_3),m)\\
\leq& c^2\Theta(\epsilon_2,\epsilon_3,m)\\
\leq& ... \leq c^n\Theta(\epsilon_2,\epsilon_3,m) \to 0 \,\text{as}\, n \to \infty. \qquad (4.2)
\end{aligned}$$

Hence $\Xi(\epsilon_2,\epsilon_3,m) = 1_\Theta$ and $\Theta(\epsilon_2,\epsilon_3,m) = 0_\Theta$ and $\epsilon_2 = \epsilon_3$. □

Theorem 4.2. *Let $(\Sigma,\Xi,\Theta,\bigotimes,\diamond)$ be a complete NCMS, for $\mathcal{G}$ be self mappings of Σ and let K,L,G. Let $\{K,G\}$ and $\{L,\mathcal{G}\}$ are pairs be sporadically weakly compatible. If there exists $c \in (0,1)$ such that*

$$\begin{aligned}
\Xi(K_{\epsilon_1},L_{\epsilon_2},c(m)) \geq min\{&\Xi(G(\epsilon_1),\mathcal{G}(\epsilon_2),m),\Xi(G(\epsilon_1),K(\epsilon_1),m)\\
&\Xi(L(\epsilon_2),\mathcal{G}(\epsilon_2),m),\Xi(K(\epsilon_1),\mathcal{G}(\epsilon_2),m),\Xi(L(\epsilon_2),G(\epsilon_1),m)\}.
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\Theta(K_{\epsilon_1},L_{\epsilon_2},c(m)) \leq max\{&\Theta(G(\epsilon_1),\mathcal{G}(\epsilon_2),r),\Theta(G(\epsilon_1),K(\epsilon_1),m)\\
&\Theta(L(\epsilon_2),\mathcal{G}(\epsilon_2),m),\Theta(K(\epsilon_1),\mathcal{G}(\epsilon_2),r),\Theta(L(\epsilon_2),G(\epsilon_1),m)\}.
\end{aligned} \tag{4.4}$$

for all $\epsilon_1,\epsilon_2 \in \Sigma$ and for all $r \gg 0_\Theta$, there exists a unique point $z \in \Sigma$ such that $K(z) = G(z) = z$ and a unique point $y \in \Sigma$ such that $L(y) = \mathcal{G}(y) = y$. Moreover $y = z$, so that there is a unique common fixed point of K,L,G and $\mathcal{G}$.

Proof. Let the pairs $\{K,G\}$ and $\{L,\mathcal{G}\}$ be sporadically weakly compatible, so there are points $\epsilon_1,\epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequality 4.3,

$$\begin{aligned}\Xi(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \geq & min\{\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)\}\\ = & min\{\Xi(K(\epsilon_1), L(\epsilon_2), r), \Xi(K(\epsilon_1), K(\epsilon_1), m),\\ & \Xi(L(\epsilon_2), L(\epsilon_2), m), \Xi(K(\epsilon_1), L(\epsilon_2), r), L(L(\epsilon_2), K(\epsilon_1), m)\}\\ = & \Xi(K_{\epsilon_1}, L_{\epsilon_2}, m).\end{aligned} \tag{4.5}$$

$$\begin{aligned}\Theta(K_{\epsilon_1}, L_{\epsilon_2}, c(m)) \leq & max\{\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m),\\ & \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)\}\\ = & max\{\Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(K(\epsilon_1), K(\epsilon_1), m),\\ & \Theta(L(\epsilon_2), L(\epsilon_2), m), \Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(L(\epsilon_2), K(\epsilon_1), m)\}\\ = & \Theta(K_{\epsilon_1}, L_{\epsilon_2}, m).\end{aligned} \tag{4.6}$$

By Lemma 3.9, $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = L(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is another point y such that $K(y) = G(y)$ and by 4.3, we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Thus $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = G(\epsilon_1)$ is the unique point of coincidence of K and G. By Lemma 2.2, z is the unique common fixed point of K and G. Similarly there is a only point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Assume that $z \neq y$, we have

$$\begin{aligned}\Xi(z, y, c(m)) = & \Xi(K(z), L(y), c(m))\\ \geq & min\{\Xi(G(z), \mathcal{G}(y), r), \Xi(G(z), K(y), m), \Xi(L(y), \mathcal{G}(y), m)\\ & \Xi(K(z), \mathcal{G}(y), m), \Xi(L(y), G(z), m)\}\\ = & min\{\Xi(z, y, m), \Xi(z, y, m), \Xi(y, y, m), \Xi(z, y, m), \Xi(y, z, m)\}\\ = & \Xi(z, y, m).\end{aligned} \tag{4.7}$$

$$\begin{aligned}\Theta(z, y, c(r)) = & \Theta(K(z), L(y), c(m))\\ \geq & min\{\Theta(G(z), \mathcal{G}(y), m), \Theta(G(z), K(y), m), \Theta(L(y), \mathcal{G}(y), m)\\ & \Theta(K(z), \mathcal{G}(y), r), \Theta(L(y), G(z), m)\}\\ = & min\{\Theta(z, y, m), \Theta(z, y, m), \Theta(y, y, m), \Theta(z, y, m), \Theta(y, z, m)\}\\ = & \Theta(z, y, m).\end{aligned} \tag{4.8}$$

by Lemma 2.2 and y is a common fixed point of K, L, G and $\mathcal{G}$. Then we have $y = z$. The uniqueness of the fixed point come from 4.6. □

Theorem 4.3. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete $NCMS$ and K, L, G and $\mathcal{G}$ be self-mappings of Σ. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ be sporadically weakly compatible. If there exists $c \in (0,1)$ such that*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi[min\{&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m)\\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)\}].\end{aligned} \tag{4.9}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta[max\{&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m)\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)\}].\end{aligned} \tag{4.10}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $\phi, \zeta : \rfloor 0^-, 1^+ \lfloor \to \rfloor 0^-, 1^+ \lfloor$ such that $\zeta(m) < m$, $\phi(m) > m$, for all $0_\Theta \ll r < 1_\Theta$, thus there is a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. The proof follows from Theorem 4.4 □

Theorem 4.4. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete $NCMS$ and K, L, G and $\mathcal{G}$ be self-mappings of Σ. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are pairs be sporadically weakly compatible. If $\exists c \in (0,1)$ such that*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \phi(&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m)\\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m)),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta(&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m)\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m)).\end{aligned} \tag{4.12}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $\phi, \zeta : \rfloor 0^-, 1^{+^5} \lfloor \to \rfloor 0^-, 1^+ \lfloor$ such that $\phi(r, 1_\Theta, 1_\Theta, m, m) > m$, $\zeta(m, 0_\Theta, 0_\Theta, m, m) < m$ for all $0 \ll m < 1$ then there exists a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are pairs be sporadically weakly compatible. There are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$.
We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.11) and (4.12), we have

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq{}& \phi(\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(G(\epsilon_1), K(\epsilon_1), m),\\ &\Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), mr), \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Xi(L(\epsilon_2), G(\epsilon_1), m))\\ ={}&\phi(\Xi(K(\epsilon_1), L(\epsilon_2), m), \Xi(K(\epsilon_1), K(\epsilon_1), m),\\ &\Xi(L(\epsilon_2), L(\epsilon_2), m), \Xi(K(\epsilon_1), L(\epsilon_2), r), L(L(\epsilon_2), K(\epsilon_1), m))\\ ={}&\phi((\Xi(K(\epsilon_1), L(\epsilon_2), m), 1_\Theta, 1_\Theta, \Xi(K(\epsilon_1), L(\epsilon_1), m), \Xi(L(\epsilon_2), K(\epsilon_2), m))\\ >{}&\Xi(K(\epsilon_1), L(\epsilon_2), m).\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \zeta(&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(G(\epsilon_1), K(\epsilon_1), m),\\ &\Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m), \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m), \Theta(L(\epsilon_2), G(\epsilon_1), m))\\ =&\zeta(\Theta(K(\epsilon_1), L(\epsilon_2), m), \Theta(K(\epsilon_1), K(\epsilon_1), m),\\ &\Theta(L(\epsilon_2), L(\epsilon_2), m), \Theta(K(\epsilon_1), L(\epsilon_2), m), L(L(\epsilon_2), K(\epsilon_1), m))\\ =&\zeta((\Theta(K(\epsilon_1), L(\epsilon_2), m), 0_\Theta, 0_\Theta, \Theta(K(\epsilon_1), L(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), m))\\ <&\Theta(K(\epsilon_1), L(\epsilon_2), m).\end{aligned}$$

a contradiction, therefore $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = G(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is a another point y such that $K(y) = G(y)$. Then by 4.11 we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$, so $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = \mathcal{G}(\epsilon_1)$ is the unique point of coincidence. z is a unique common fixed point of K and G, by Lemma 2.2. Similarly, for K and G there is a unique point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Thus for K, L, G, y is a common fixed point and $\mathcal{G}$. For the uniqueness fixed point holds from (4.11). □

Theorem 4.5. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete $NCMS$ and K, L, G and $\mathcal{G}$ be self-mappings of Σ. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ be sporadically weakly compatible. If there exists $c \in (0,1)$ for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $m \gg 0_\Theta$ satisfying*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m)\\ \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.13}$$

$$\begin{aligned}\Xi\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Theta(K(\epsilon_1), G(\epsilon_1), m)\\ \bigotimes \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \bigotimes \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.14}$$

then there exists a unique common fixed point of K, L, G and $\mathcal{G}$.

Proof. Let the pairs $\{K, G\}$ and $\{L, \mathcal{G}\}$ are sporadicallyweakly compatible, there are points $\epsilon_1, \epsilon_2 \in \Sigma$ such that $K(\epsilon_1) = G(\epsilon_1)$ and $L(\epsilon_2) = \mathcal{G}(\epsilon_2)$.
We claim that $K(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.13) and (4.14), we have

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq \Xi(G(\epsilon_1), L(\epsilon_2), m) &\bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m)\\ &\bigotimes \Xi(L(\epsilon_2), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ &= \Xi(K(\epsilon_1), L(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), K(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), L(\epsilon_2), m)\\ &\bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ &\geq \Xi(K(\epsilon_1), L(\epsilon_2), m) \bigotimes 1_\Theta \bigotimes 1_\Theta \bigotimes \Xi(K(\epsilon_1), L(\epsilon_2), m)\\ &\geq \Xi(K(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m) \\ =&\Theta(K(\epsilon_1), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), K(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), L(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m) \\ \leq&\Theta(K(\epsilon_1), L(\epsilon_2), m) \diamond 0_\Theta \diamond 0_\Theta \diamond \Theta(K(\epsilon_1), L(\epsilon_2), m) \\ \leq&\Theta(K(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

By Lemma 3.9, we have $K(\epsilon_1) = L(\epsilon_2)$, i.e. $K(\epsilon_1) = G(\epsilon_1) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Suppose that there is a another point y such that $K(y) = G(y)$. Then by (4.13, 4.14), we have $K(y) = G(y) = L(\epsilon_2) = \mathcal{G}(\epsilon_2)$. Thus $K(\epsilon_1) = K(y)$ and $z = K(\epsilon_1) = G(\epsilon_1)$ is the unique point of coincidence of K and G. Then there is a unique point $y \in \Sigma$ such that $y = L(y) = \mathcal{G}(y)$. Thus z is a common fixed point of K, L, G and $\mathcal{G}$. □

Theorem 4.6. *Let $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ be a complete neutrosophic CMS and $\mathcal{G}$ and K, L, G be self-mappings of Σ. Let $\{K, G\}$ and $\{L, \mathcal{G}\}$ are the pairs be sporadically weakly compatible. If $\exists c \in (0,1)$ for all $\epsilon_1, \epsilon_2 \in \Sigma$ and $r \gg 0_\Theta$ satisfying*

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \bigotimes \Xi(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.15}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), r) \bigotimes \Theta(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \bigotimes \Theta(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned} \tag{4.16}$$

then for K, L, G and $\mathcal{G}$ there exists a unique common fixed point.

Proof. We have,

$$\begin{aligned}\Xi(K(\epsilon_1), L(\epsilon_2), c(m)) \geq\ & \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \bigotimes \Xi(L(\epsilon_2), G(\epsilon_2), 2m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m) \\ =&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \bigotimes \Xi(G(\epsilon_1), \mathcal{G}(\epsilon_1), m) \bigotimes \Xi(\mathcal{H}(\epsilon_1), L(\epsilon_1), m) \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m) \\ \geq&\Xi(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \bigotimes \Xi(K(\epsilon_1), G(\epsilon_1), m) \bigotimes \Xi(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \bigotimes \Xi(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(K(\epsilon_1), L(\epsilon_2), c(m)) \leq\ & \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \diamond \Theta(L(\epsilon_2), G(\epsilon_2), 2m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m) \\ =&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \\ & \diamond \Theta(G(\epsilon_1), \mathcal{G}(\epsilon_1), m) \diamond \Theta(\mathcal{H}(\epsilon_1), L(\epsilon_1), m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m) \\ \leq&\Theta(G(\epsilon_1), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), G(\epsilon_1), m) \diamond \Theta(L(\epsilon_2), \mathcal{G}(\epsilon_2), m) \diamond \Theta(K(\epsilon_1), \mathcal{G}(\epsilon_2), m)\end{aligned}$$

and therefore by Theorem 4.5, K, L, G and $\mathcal{G}$ have a common fixed point. □

Theorem 4.7. *Let* $(\Sigma, \Xi, \Theta, \bigotimes, \diamond)$ *be a complete neutrosophic* CMS *and* K*,* L *be self-mappings of* Σ*. Let* K *and* L *be sporadically weakly compatible. If* $\exists$ *a point* $c \in (0,1)$ *for all* $\epsilon_1, \epsilon_2 \in \Sigma$ *and* $r \gg 0_\Theta$

$$\begin{aligned}\Xi(L(\epsilon_1), L(\epsilon_2), c(m)) \geq a\,\Xi(K(\epsilon_1), K(\epsilon_2), m) + b\,min\{&\Xi(K(\epsilon_1), K(\epsilon_2), m),\\ &\Xi(L(\epsilon_1), K(\epsilon_1), m), \Xi(L(\epsilon_2), K(\epsilon_2), m)\}\end{aligned} \tag{4.17}$$

$$\begin{aligned}\Theta(L(\epsilon_1), L(\epsilon_2), c(m)) \leq a\,\Theta(K(\epsilon_1), K(\epsilon_2), m) + b\,max\{&\Theta(K(\epsilon_1), K(\epsilon_2), m),\\ &\Theta(L(\epsilon_1), K(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), m)\}\end{aligned} \tag{4.18}$$

for all $\epsilon_1, \epsilon_2 \in \Sigma$*, where* $a, b > 0_\Theta$*,* $a + b > 1_\Theta$*. Then* K *and* L *have a unique common fixed point.*

Proof. Let the pairs $\{K, L\}$ be sporadicallyweakly compatible, so there is a point $\epsilon_1 \in \Sigma$ such that $K(\epsilon_1) = L(\epsilon_1)$. Suppose that there exists another point $\epsilon_2 \in \Sigma$ for which $K(\epsilon_2) = L(\epsilon_2)$. We claim that $G(\epsilon_1) = L(\epsilon_2)$. By inequalities (4.17) and (4.18), we have

$$\begin{aligned}\Xi(L(\epsilon_1), L(\epsilon_2), c(m)) \geq\,& a\,\Xi(K(\epsilon_1), K(\epsilon_2), m) + b\,min\{\Xi(K(\epsilon_1), K(\epsilon_2), m),\\ &\Xi(L(\epsilon_1), K(\epsilon_1), r), \Xi(L(\epsilon_2), K(\epsilon_2), m)\}\\ =& a\Xi(L(\epsilon_1), L(\epsilon_2), m) + b\,min\{\Xi(L(\epsilon_1), L(\epsilon_2), m),\\ &\Xi(L(\epsilon_1), L(\epsilon_1), m), \Xi(L(\epsilon_2), L(\epsilon_2), m), \}\\ =& a + b\Xi(L(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

$$\begin{aligned}\Theta(L(\epsilon_1), L(\epsilon_2), c(m)) \leq\,& a\,\Theta(K(\epsilon_1), K(\epsilon_2), m) + b\,max\{\Theta(K(\epsilon_1), K(\epsilon_2), m),\\ &\Theta(L(\epsilon_1), K(\epsilon_1), m), \Theta(L(\epsilon_2), K(\epsilon_2), r)\}\\ =& a\Theta(L(\epsilon_1), L(\epsilon_2), m) + b\,max\{\Theta(L(\epsilon_1), L(\epsilon_2), m),\\ &\Theta(L(\epsilon_1), L(\epsilon_1), m), \Theta(L(\epsilon_2), L(\epsilon_2), m), \}\\ =& a + b\Theta(L(\epsilon_1), L(\epsilon_2), m)\end{aligned}$$

a contradiction, since $a + b > 1_\Theta$. Therefore $L(\epsilon_1) = L(\epsilon_2)$. Therefore $K(\epsilon_1) = K(\epsilon_2)$ and $K(\epsilon_1)$ is unique. From Lemma 2.2, K and L have a unique fixed point. □

5. **Conclusion**

In this paper, the concept of neutrosophic CMS is introduced. Some fixed point theorems on neutrosophic CMS are stated and proved.

6. **Conflict of Interests**

Regarding this manuscript, the authors declare that there is no conflict of interests.

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References

1. A. K., Stoeva, intuitionistic L-fuzzy, R. Trpple, Ed., Cybernetic and System Research Elsevier, Amsterdam, 1984. Vol 2: p.539-540.
2. A. Saha, S. Broumi,New operators on interval valued neutrosophic sets, Neutrosophic Sets and Systems, 2019. 28: p.128-137. doi:10.5281/zenodo.3382525.
3. A. A. Salama, F. Smarandache, V. Kroumov. Neutrosophic closed set and neutrosophic continuous functions, Neutrosophic Sets and Systems, 2014. 4: p.4-8.
4. A. A. Salama, S. Broumi, S. A. Alblowi. Introduction to neutrosophic topological spatial region, possible application to gis topological rules, I.J. Information Engineering and Electronic Business, 2014. 6 : p.15-21.
5. A. Salama, S. Alblowi. Generalized neutrosophic set and generalized neutrousophic topological spaces, Journal computer Sci. Engineering, 2012; 2 (7): p.29-32.
6. C. L. Chang. Fuzzy topological spaces, Journal of Mathematical Analysis and Applications,1968. 24 (1): p.182-190.
7. D. Coker. An introduction to intuitionistic fuzzy topological space, Fuzzy Sets and Systems, 1997. 88 (1):p.81-89.
8. F. Smarandache, (t, i, f)-neutrosophic structures and i-neutrosophic structures (revisited), Neutrosophic Sets and Systems, 2015. 8: p.3-9. doi:10.5281/zenodo.571239.
9. F. Smarandache, Neutrosophic Probability, Set, and Logic, ProQuest Information, Learning, Ann Arbor, Michigan, USA, 1998, p: 105.
10. G. Balasubramanian and P. Sundaram, On some generalizations of fuzzy continuous functions, Fuzzy Sets and Systems 1997.86: p.93-100.
11. G. Balasubramanian, P. Sundaram. On some generalizations of fuzzy continuous functions, Fuzzy Sets and Systems 1997.86: p.93-100.
12. K. Atanassov. Intuitionistic fuzzy sets, VII ITKRs Session, Publishing House: Sofia, Bulgaria, 1983.
13. L. Zadeh. Fuzzy sets, Inform. and Control, 1965. 8: p.338-353.
14. M. Sahin, A. Kargn, Neutrosophic triplet metric topology, Neutrosophic Sets and Systems, 2019.27: p.154-162. doi:10.5281/zenodo.3275557.
15. M. L. Thivagar, S. Jafari, V. Antonysamy, V. S. Devi, The ingenuity of neutrosophic topology via n-topology, Neutrosophic Sets and Systems, 2018. 19:p. 91-100. doi:10.5281/zenodo.1235315.
16. M. A. A. Shumrani, F. Smarandache, Introduction to non-standard neutrosophic topology, Symmetry, 2019.11 (0): p.1-14, basel, Switzerland. doi:10.3390/sym11050000.
17. H. E. Khalid, F. Smarandache, A. K. Essa, The basic notions for (over, off, under) neutrosophic geometric programming problems, Neutrosophic Sets and Systems, 2018. 22: p.50-62. doi:10.5281/zenodo.2160622.
18. W. F. Al-Omeri, F. Smarandache. New Neutrosophic Sets via Neutrosophic Topological Spaces, Neutrosophic Operational Research; F. Smarandache and S. Pramanik (Editors), Pons Editions, Brussels, Belgium, 2017. Volume I: p. 189-209.
19. W. F. Al-Omeri. Neutrosophic crisp sets via neutrosophic crisp topological spaces, Neutrosophic Sets and Systems, 2016. 13: p.96-104.

20. K. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets and Systems, 1986. (20): p.87-96.
21. W. F. Al-Omeri, S. Jafari, On Generalized Closed Sets and Generalized Pre-Closed Sets in Neutrosophic Topological Spaces, Mathematics, 2019.(7)1: p.1-12. doi:doi.org/10.3390/math7010001.

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Neutrosophic quadruple a-ideals

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Abstract. The notion of neutrosophic quadruple a-ideal is introduced, and related properties are investigated. Relations between a neutrosophic quadruple p-ideal, a neutrosophic quadruple q-ideal, a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal are discussed. Conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal are provided.

Keywords: Neutrosophic quadruple BCK/BCI-number, neutrosophic quadruple BCK/BCI-algebra, neutrosophic quadruple (closed) ideal, neutrosophic quadruple $p(q, a)$-ideal.

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1. Introduction

Neutrosophic sets (NSs) proposed by (Smarandache, 1998, 1999, 2002, 2005, 2006, 2010), which is a generalization of fuzzy sets and intuitionistic fuzzy set, is a powerful tool to deal with incomplete, indeterminate and inconsistent information which exist in the real world (see [28–30]). Recently, this concept has been applied more actively to many areas (see [1], [2], [3], [4]). Neutrosophic algebraic structures in BCK/BCI-algebras are discussed in the papers [6–11, 15–18, 20, 23, 27, 32]. Smarandache [31] considered an entry (i.e., a number, an idea, an object etc.) which is represented by a known part (a) and an unknown part (bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and a, b, c, d are real or complex numbers, and then he introduced the concept of neutrosophic quadruple numbers. Jun et al. [19] used neutrosophic quadruple numbers based on a set, and constructed neutrosophic quadruple BCK/BCI-algebras. They investigated several properties, and considered (closed, positive implicative) ideal in neutrosophic quadruple BCI-algebra. Given subsets A and B of a BCK/BCI-algebra, they considered the set $NQ(A, B)$ which consists of neutrosophic quadruple

BCK/BCI-numbers with a condition. They provided conditions for the set $NQ(A, B)$ to be a (closed, positive implicative) ideal of a neutrosophic quadruple BCK/BCI-algebra. Muhiuddin et al. [24] introduced the concept of implicative neutrosophic quadruple BCK-algebras, and investigated several properties. Muhiuddin et al. [25, 26] discuss neutrosophic quadruple p-ideals and neutrosophic quadruple q-ideals.

In this paper, we consider the neutrosophic quadruple version of an a-ideal in a BCI-algebra. We discuss relations between a neutrosophic quadruple p-ideal, a neutrosophic quadruple q-ideal, a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal. We provide conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki (see [13] and [14]) and was extensively investigated by several researchers.

By a *BCI-algebra*, we mean a set X with a special element 0 and a binary operation $*$ that satisfies the following conditions:

(I) $(\forall x, y, z \in X)\ (((x * y) * (x * z)) * (z * y) = 0)$,
(II) $(\forall x, y \in X)\ ((x * (x * y)) * y = 0)$,
(III) $(\forall x \in X)\ (x * x = 0)$,
(IV) $(\forall x, y \in X)\ (x * y = 0,\ y * x = 0\ \Rightarrow\ x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)\ (0 * x = 0)$,

then X is called a *BCK-algebra.* Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{1}$$

$$(\forall x, y, z \in X)\,(x \leq y\ \Rightarrow\ x * z \leq y * z,\ z * y \leq z * x)\,, \tag{2}$$

$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{3}$$

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions (see [12]):

$$(\forall x, y \in X)(x * (x * (x * y)) = x * y), \tag{5}$$

$$(\forall x, y \in X)(0 * (x * y) = (0 * x) * (0 * y)), \tag{6}$$

$$(\forall x, y \in X)(0 * (0 * (x * y)) = (0 * y) * (0 * x)). \tag{7}$$

A nonempty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$. A subset I of a BCK/BCI-algebra X is called

- an *ideal* of X if it satisfies:

$$0 \in I, \tag{8}$$

$$(\forall x \in X)\,(\forall y \in I)\,(x * y \in I \;\Rightarrow\; x \in I)\,. \tag{9}$$

- a *closed ideal* of X (see [12]) if it is an ideal of X which satisfies:

$$(\forall x \in X)(x \in I \;\Rightarrow\; 0 * x \in I). \tag{10}$$

- a *p-ideal* of X (see [33]) if it satisfies (8) and

$$(\forall x, y, z \in X)(y \in I, (x * z) * (y * z) \in I \;\Rightarrow\; x \in I). \tag{11}$$

- a *q-ideal* of X (see [21]) if it satisfies (8) and

$$(\forall x, y, z \in X)(x * (y * z) \in I,\, y \in I \;\Rightarrow\; x * z \in I). \tag{12}$$

- an *a-ideal* of X (see [21]) if it satisfies (8) and

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in I,\, z \in I \;\Rightarrow\; y * x \in I). \tag{13}$$

Note that a subset of a BCI-algebra is a closed ideal if and only if it is both an ideal and a subalgebra.

Recall that a subset I of a BCI-algebra X is a p-ideal of X if and only if I is an ideal of X which satisfies the following condition:

$$(\forall x \in X)(0 * (0 * x) \in I \;\Rightarrow\; x \in I). \tag{14}$$

We refer the reader to the books [12, 22] for further information regarding BCK/BCI-algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

We consider neutrosophic quadruple numbers based on a set instead of real or complex numbers.

Let X be a set. A *neutrosophic quadruple X-number* is an ordered quadruple (a, xT, yI, zF) where $a, x, y, z \in X$ and T, I, F have their usual neutrosophic logic meanings (see [5]).

The set of all neutrosophic quadruple X-numbers is denoted by $N_q(X)$, that is,

$$N_q(X) := \{(a, xT, yI, zF) \mid a, x, y, z \in X\},$$

and it is called the *neutrosophic quadruple set* based on X. If X is a BCK/BCI-algebra, a neutrosophic quadruple X-number is called a *neutrosophic quadruple BCK/BCI-number* and we say that $N_q(X)$ is the *neutrosophic quadruple BCK/BCI-set*.

Let X be a BCK/BCI-algebra. We define a binary operation $\boxdot$ on $N_q(X)$ by

$$(a, xT, yI, zF) \boxdot (b, uT, vI, wF) = (a * b, (x * u)T, (y * v)I, (z * w)F)$$

for all (a, xT, yI, zF), $(b, uT, vI, wF) \in N_q(X)$. Given $a_1, a_2, a_3, a_4 \in X$, the neutrosophic quadruple BCK/BCI-number (a_1, a_2T, a_3I, a_4F) is denoted by $\tilde{a}$, that is,

$$\tilde{a} = (a_1, a_2T, a_3I, a_4F),$$

and the zero neutrosophic quadruple BCK/BCI-number $(0, 0T, 0I, 0F)$ is denoted by $\tilde{0}$, that is,

$$\tilde{0} = (0, 0T, 0I, 0F).$$

Then $(N_q(X); \boxdot, \tilde{0})$ is a BCK/BCI-algebra (see [19]), which is called *neutrosophic quadruple BCK/BCI-algebra*, and it is simply denoted by $N_q(X)$.

We define an order relation "$\ll$" and the equality "$=$" on $N_q(X)$ as follows:

$$\tilde{x} \ll \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for } i = 1, 2, 3, 4,$$
$$\tilde{x} = \tilde{y} \Leftrightarrow x_i = y_i \text{ for } i = 1, 2, 3, 4$$

for all $\tilde{x}, \tilde{y} \in N_q(X)$. It is easy to verify that "$\ll$" is an equivalence relation on $N_q(X)$.

Let X be a BCK/BCI-algebra. Given nonempty subsets A and B of X, consider the set

$$N_q(A, B) := \{(a, xT, yI, zF) \in N_q(X) \mid a, x \in A \text{ \& } y, z \in B\},$$

which is called the *neutrosophic quadruple* (A, B)*-set* (briefly, neutrosophic quadruple (A, B)-set).

The set $NQ(A, A)$ is denoted by $N_q(A)$, and it is called the *neutrosophic quadruple* A*-set* (briefly, neutrosophic quadruple A-set).

3. Neutrosophic quadruple a-ideals

Definition 3.1. Given nonempty subsets A and B of X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is an a-ideal of a neutrosophic quadruple BCI-algebra $N_q(X)$, we say $N_q(A, B)$ is a *neutrosophic quadruple* a*-ideal* of $N_q(X)$.

Example 3.2. Consider a BCI-algebra $X = \{0, a, b, c\}$ with the binary operation $*$, which is given in Table 1.

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 256 elements. Consider subsets $A = \{0, a\}$ and $B = \{0, b\}$ of X. Then

$$N_q(A, B) = \{\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2, \tilde{\beta}_3, \tilde{\beta}_4, \tilde{\beta}_5, \tilde{\beta}_6, \tilde{\beta}_7, \tilde{\beta}_8, \tilde{\beta}_9, \tilde{\beta}_{10}, \tilde{\beta}_{11}, \tilde{\beta}_{12}, \tilde{\beta}_{13}, \tilde{\beta}_{14}, \tilde{\beta}_{15}\}$$

where

$\tilde{\beta}_0 = (0, 0T, 0I, 0F)$, $\tilde{\beta}_1 = (0, 0T, 0I, bF)$, $\tilde{\beta}_2 = (0, 0T, bI, 0F)$,

TABLE 1. Cayley table for the binary operation "$*$"

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$\tilde{\beta}_3 = (0, 0T, bI, bF)$, $\tilde{\beta}_4 = (0, aT, 0I, 0F)$, $\tilde{\beta}_5 = (0, aT, 0I, bF)$,
$\tilde{\beta}_6 = (0, aT, bI, 0F)$, $\tilde{\beta}_7 = (0, aT, bI, bF)$, $\tilde{\beta}_8 = (a, 0T, 0I, 0F)$,
$\tilde{\beta}_9 = (a, 0T, 0I, bF)$, $\tilde{\beta}_{10} = (a, 0T, bI, 0F)$, $\tilde{\beta}_{11} = (a, 0T, bI, bF)$,
$\tilde{\beta}_{12} = (a, aT, 0I, 0F)$, $\tilde{\beta}_{13} = (a, aT, 0I, bF)$,
$\tilde{\beta}_{14} = (a, aT, bI, 0F)$, $\tilde{\beta}_{15} = (a, aT, bI, bF)$.

It is routine to verify that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proposition 3.3. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then the following assertions are valid.*

$$(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B) \;\Rightarrow\; \tilde{y} \boxdot (\tilde{x} \boxdot \tilde{z}) \in N_q(A, B), \tag{15}$$

$$\tilde{x} \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B) \;\Rightarrow\; \tilde{y} \boxdot \tilde{x} \in N_q(A, B) \tag{16}$$

for all $\tilde{x}, \tilde{y}, \tilde{z} \in N_q(X)$.

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ for any nonempty subsets A and B of a BCI-algebra X. Suppose that $(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B)$ for any elements $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ of $N_q(X)$. Then

$$\begin{aligned}
&((\tilde{x} \boxdot \tilde{z}) \boxdot ((\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}))) \boxdot (\tilde{0} \boxdot \tilde{y}) \\
&= ((\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y})) \boxdot ((\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y})) \\
&= \tilde{0} \in N_q(A, B).
\end{aligned}$$

Since $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, it follows that $\tilde{y} \boxdot (\tilde{x} \boxdot \tilde{z}) \in N_q(A, B)$. Finally, (16) is induced by taking $\tilde{z} = \tilde{0}$ in (15). □

Lemma 3.4 ([21])**.** *In a BCI-algebra, every a-ideal is a closed ideal.*

Lemma 3.5 ([19])**.** *If A and B are closed ideals of a BCI-algebra X, then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple closed ideal of $N_q(X)$.*

We consider relations between a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal.

Theorem 3.6. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple closed ideal of $N_q(X)$.*

Proof. Assume that $N_q(A,B)$ is a neutrosophic quadruple a-ideal of a neutrosophic quadruple BCI-algebra $N_q(X)$ where A and B are nonempty subsets of X. Since $\tilde{0} = (0, 0T, 0I, 0F) \in N_q(A,B)$, we have $0 \in A \cap B$. Let $x, y, z \in X$ be such that $(x * z) * (0 * y) \in A \cap B$ and $z \in A \cap B$. Then $(z, zT, zI, zF) \in N_q(A,B)$ and

$$\begin{aligned}
&((x, xT, xI, xF) \boxdot (z, zT, zI, zF)) \boxdot (\tilde{0} \boxdot (y, yT, yI, yF)) \\
&= (x * z, (x * z)T, (x * z)I, (x * z)F) \boxdot (0 * y, (0 * y)T, (0 * y)I, (0 * y)F) \\
&= ((x * z) * (0 * y), ((x * z) * (0 * y))T, ((x * z) * (0 * y))I, ((x * z) * (0 * y))F) \\
&\in N_q(A,B).
\end{aligned}$$

Hence

$$(y * x, (y * x)T, (y * x)I, (y * x)F) = (y, yT, yI, yF) \boxdot (x, xT, xI, xF) \in N_q(A,B),$$

that is, $y * x \in A \cap B$. Therefore A and B are a-ideals of X. Using Lemmas 3.4 and 3.5, $N_q(A,B)$ is a neutrosophic quadruple closed ideal of $N_q(X)$. □

The converse of Theorem 3.6 is not true as seen in the following example.

Example 3.7. Consider a BCI-algebra $X = \{0, 1, a\}$ with the binary operation $*$, which is given in Table 2.

TABLE 2. Cayley table for the binary operation "$*$"

$*$	0	1	a
0	0	0	a
1	1	0	a
a	a	a	0

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 81 elements. If we take $A = \{0\}$ and $B = \{0\}$, then

$$N_q(A,B) = \{\tilde{0}\}$$

which is a neutrosophic quadruple closed ideal of $N_q(X)$. But it is not a neutrosophic quadruple a-ideal of $N_q(X)$ because if we take $\tilde{1} = (0, 1T, 1I, 0F) \in N_q(X)$ then

$$(\tilde{0} \boxdot \tilde{0}) \boxdot (\tilde{0} \boxdot \tilde{1}) = \tilde{0} \in N_q(A, B),$$

but $\tilde{1} \boxdot \tilde{0} = \tilde{1} \notin N_q(A, B)$.

We provide conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal.

Theorem 3.8. *If A and B are a-ideals of a BCI-algebra X, then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Proof. Suppose that A and B are a-ideals of a BCI-algebra X. Obviously, $\tilde{0} \in N_q(A, B)$. Let $\tilde{x} = (x_1, x_2T, x_3I, x_4F)$, $\tilde{y} = (y_1, y_2T, y_3I, y_4F)$ and $\tilde{z} = (z_1, z_2T, z_3I, z_4F)$ be elements of $N_q(X)$ be such that $(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A, B)$ and $\tilde{z} \in N_q(A, B)$. Then $z_i \in A$, $z_j \in B$ for $i = 1, 2$; $j = 3, 4$, and

$$\begin{aligned}
&(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) = ((x_1, x_2T, x_3I, x_4F) \boxdot (z_1, z_2T, z_3I, z_4F)) \boxdot \\
&((0, 0T, 0I, 0F) \boxdot (y_1, y_2T, y_3I, y_4F)) \\
&= (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F) \boxdot \\
&(0 * y_1, (0 * y_2)T, (0 * y_3)I, (0 * y_4)F) \\
&= ((x_1 * z_1) * (0 * y_1), ((x_2 * z_2) * (0 * y_2))T, \\
&((x_3 * z_3) * (0 * y_3))I, ((x_4 * z_4) * (0 * y_4))F) \\
&\in N_q(A, B),
\end{aligned}$$

that is, $(x_i * z_i) * (0 * y_i) \in A$ and $(x_j * z_j) * (0 * y_j) \in B$ for $i = 1, 2$ and $j = 3, 4$. It follows from (13) that $y_i * x_i \in A$ and $y_j * x_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Thus

$$\tilde{y} \boxdot \tilde{x} = (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in N_q(A, B),$$

and therefore $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. □

Corollary 3.9. *If A is an a-ideal of a BCI-algebra X, then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Theorem 3.10. *Let A and B be ideals of a BCI-algebra X such that*

$$(\forall x, y \in X)(x * (0 * y) \in A \cap B \;\Rightarrow\; y * x \in A \cap B). \tag{17}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. Obviously $\tilde{0} \in N_q(A,B)$. Let $\tilde{x} = (x_1,\ x_2T,\ x_3I,\ x_4F)$, $\tilde{y} = (y_1,\ y_2T,\ y_3I,\ y_4F)$ and $\tilde{z} = (z_1,\ z_2T,\ z_3I,\ z_4F)$ be elements of $N_q(X)$ be such that $(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) \in N_q(A,B)$ and $\tilde{z} \in N_q(A,B)$. Then $z_1, z_2 \in A,\ z_3, z_4 \in B$ and

$$\begin{aligned}(\tilde{x} \boxdot \tilde{z}) \boxdot (\tilde{0} \boxdot \tilde{y}) &= ((x_1, x_2T, x_3I, x_4F) \boxdot (z_1, z_2T, z_3I, z_4F))\boxdot \\ &\quad (\tilde{0} \boxdot (y_1, y_2T, y_3I, y_4F)) \\ &= (x_1 * z_1, (x_2 * z_2)T, (x_3 * z_3)I, (x_4 * z_4)F)\boxdot \\ &\quad (0 * y_1, (0 * y_2)T, (0 * y_3)I, (0 * y_4)F) \\ &= ((x_1 * z_1) * (0 * y_1), ((x_2 * z_2) * (0 * y_2))T, \\ &\quad ((x_3 * z_3) * (0 * y_3))I, ((x_4 * z_4) * (0 * y_4))F) \\ &\in N_q(A,B),\end{aligned}$$

that is, $(x_i * z_i) * (0 * y_i) \in A$ and $(x_j * z_j) * (0 * y_j) \in B$ for $i = 1,2$ and $j = 3,4$. Note that

$$(x_k * (0 * y_k)) * ((x_k * z_k) * (0 * y_k)) \leq x_k * (x_k * z_k) \leq z_k$$

for $k = 1,2,3,4$. Since $z_1, z_2 \in A$ and $z_3, z_4 \in B$, we have $x_i * (0 * y_i) \in A$ and $x_j * (0 * y_j) \in B$ for $i = 1,2$ and $j = 3,4$. It follows from (17) that $y_i * x_i \in A$ and $y_j * x_j \in B$ for $i = 1,2$ and $j = 3,4$. Hence

$$\begin{aligned}\tilde{y} \boxdot \tilde{x} &= (y_1, y_2T, y_3I, y_4F) \boxdot (x_1, x_2T, x_3I, x_4F) \\ &= (y_1 * x_1, (y_2 * x_2)T, (y_3 * x_3)I, (y_4 * x_4)F) \in N_q(A,B).\end{aligned}$$

Therefore $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. □

Corollary 3.11. *Let A be an ideal of a BCI-algebra X such that*

$$(\forall x, y \in X)(x * (0 * y) \in A \ \Rightarrow\ y * x \in A). \tag{18}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Theorem 3.12. *Let A and B be ideals of a BCI-algebra X such that*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in A \cap B \ \Rightarrow\ y * (x * z) \in A \cap B). \tag{19}$$

Then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. If we put $z = 0$ in (19) and use (1), then we can induce the condition (17). Thus $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.10. □

Corollary 3.13. *Let A be an ideal of a BCI-algebra X such that*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in A \;\Rightarrow\; y * (x * z) \in A). \tag{20}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

We discuss relations between a neutrosophic quadruple a-ideal, a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal.

Lemma 3.14 ([25])**.** *Let A and B be ideals of X such that*

$$(\forall x \in X)(0 * (0 * x) \in A \text{ (resp., } B) \Rightarrow x \in A \text{ (resp., } B)). \tag{21}$$

Then $N_q(A, B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$.

Theorem 3.15. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple p-ideal of $N_q(X)$.*

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. Then A and B are a-ideals of X (see Proof of Theorem 3.6) and $\tilde{0} \in N_q(A, B)$. For $i = 1, 2$ and $j = 3, 4$, let $x_i, x_j \in X$ be such that $0 * (0 * x_i) \in A$ and $0 * (0 * x_j) \in B$. Then

$$\begin{aligned}
&(\tilde{0} \boxdot \tilde{0}) \boxdot (\tilde{0} \boxdot \tilde{x}) = \tilde{0} \boxdot (\tilde{0} \boxdot \tilde{x}) \\
&= (0 * (0 * x_1), (0 * (0 * x_2))T, (0 * (0 * x_3))I, (0 * (0 * x_4))F) \in N_q(A, B),
\end{aligned}$$

and so

$$\begin{aligned}
(x_1, x_2T, x_3I, x_4F) &= (x_1 * 0, (x_2 * 0)T, (x_3 * 0)I, (x_4 * 0)F) \\
&= (x_1, x_2T, x_3I, x_4F) \boxdot (0, 0T, 0I, 0F) \\
&= \tilde{x} \boxdot \tilde{0} \in N_q(A, B)
\end{aligned}$$

Hence $x_i \in A$ and $x_j \in B$. It follows from Lemma 3.14 that $N_q(A, B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$. □

The following example shows that the converse of Theorem 3.15 is not true in general.

Example 3.16. Consider a BCI-algebra $X = \{0, a, b\}$ with the binary operation $*$, which is given in Table 3.

Then the neutrosophic quadruple BCI-algebra $N_q(X)$ has 81 elements. If we take $A = \{0\}$ and $B = \{0\}$, then $N_q(A, B) = \{\tilde{0}\}$ is a neutrosophic quadruple p-ideal of $N_q(X)$. For two

Table 3. Cayley table for the binary operation "$*$"

$*$	0	a	b
0	0	b	a
a	a	0	b
b	b	a	0

elements (a, aT, aI, aF) and (b, bT, bI, bF) of $N_q(X)$, we have

$$\begin{aligned}
&((a, aT, aI, aF) \boxdot (0, 0T, 0I, 0F)) \boxdot ((0, 0T, 0I, 0F) \boxdot (b, bT, bI, bF)) \\
&= (a * 0, (a * 0)T, (a * 0)I, (a * 0)F) \boxdot (0 * b, (0 * b)T, (0 * b)I, (0 * b)F) \\
&= (a, aT, aI, aF) \boxdot (a, aT, aI, aF) = \tilde{0} \in N_q(A, B).
\end{aligned}$$

But

$$\begin{aligned}
(b, bT, bI, bF) \boxdot (a, aT, aI, aF) &= (b * a, (b * a)T, (b * a)I, (b * a)F) \\
&= (a, aT, aI, aF) \notin N_q(A, B).
\end{aligned}$$

Hence $N_q(A, B)$ is not a neutrosophic quadruple a-ideal of $N_q(X)$.

Lemma 3.17 ([26]). *Let A and B be ideals of a BCI-algebra X such that*

$$(\forall x, y \in X)(x * (0 * y) \in A \cap B \;\Rightarrow\; x * y \in A \cap B). \tag{22}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$.

Theorem 3.18. *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple q-ideal of $N_q(X)$.*

Proof. Assume that $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. Then A and B are a-ideals of X (see Proof of Theorem 3.6) and $\tilde{0} \in N_q(A, B)$. For $i = 1, 2$ and $j = 3, 4$, let $x_i, y_i, x_j, y_j \in X$ be such that $x_i * (0 * y_i) \in A$ and $x_j * (0 * y_j) \in B$. Since

$$\begin{aligned}
&0 * (0 * (y_k * (0 * x_k))) * (x_k * (0 * y_k)) \\
&= ((0 * (0 * y_k)) * (0 * (0 * (0 * x_k)))) * (x_k * (0 * y_k)) \\
&= ((0 * (0 * y_k)) * (0 * x_k)) * (x_k * (0 * y_k)) \\
&\leq (x_k * (0 * y_k)) * (x_k * (0 * y_k)) = 0 \in A \cap B
\end{aligned}$$

for $k = 1, 2, 3, 4$, we have $0 * (0 * (y_i * (0 * x_i))) \in A$ and $0 * (0 * (y_j * (0 * x_j))) \in B$. Since every a-ideal is a p-ideal, it follows from (14) that $y_i * (0 * x_i) \in A$ and $y_j * (0 * x_j) \in B$. Thus

$$\begin{aligned}
\tilde{y} \boxdot (\tilde{0} \boxdot \tilde{x}) &= (y_1, y_2T, y_3I, y_4F) \boxdot ((0, 0T, 0I, 0F) \boxdot (x_1, x_2T, x_3I, x_4F)) \\
&= (y_1, y_2T, y_3I, y_4F) \boxdot (0 * x_1, (0 * x_2)T, (0 * x_3)I, (0 * x_4)F) \\
&= (y_1 * (0 * x_1), (y_2 * (0 * x_2))T, (y_3 * (0 * x_3))I, (y_4 * (0 * x_4))F) \\
&\in N_q(A, B),
\end{aligned}$$

which implies from (16) that

$$\begin{aligned}
&(x_1 * y_1, (x_2 * y_2)T, (x_3 * y_3)I, (x_4 * y_4)F) \\
&= (x_1, x_2T, x_3I, x_4F) \boxdot (y_1, y_2T, y_3I, y_4F) \\
&= \tilde{x} \boxdot \tilde{y} \in N_q(A, B),
\end{aligned}$$

that is, $x_i * y_i \in A$ and $x_j * y_j \in B$ for $i = 1, 2$ and $j = 3, 4$. Using Lemma 3.17, we know that $N_q(A, B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$. □

Corollary 3.19. *For any nonempty subset A of a BCI-algebra X, if the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then it is a neutrosophic quadruple q-ideal of $N_q(X)$.*

Consider the neutrosophic quadruple BCI-algebra $N_q(X)$ in Example 3.7. If we take $A = \{0\}$ and $B = \{0, 1\}$, then $N_q(A, B) = \{\tilde{0}, \tilde{1}, \tilde{2}, \tilde{3}\}$, where $\tilde{0} = (0, 0T, 0I, 0F)$, $\tilde{1} = (0, 0T, 0I, 1F)$, $\tilde{2} = (0, 0T, 1I, 0F)$ and $\tilde{3} = (0, 0T, 1I, 1F)$, is a neutrosophic quadruple q-ideal of $N_q(X)$. But it is not a neutrosophic quadruple a-ideal of $N_q(X)$ since

$$(\tilde{0} \boxdot \tilde{0}) \boxdot (\tilde{0} \boxdot (1, 0T, 1I, 0F)) = \tilde{0} \in N_q(A, B)$$

and $(1, 0T, 1I, 0F) \boxdot \tilde{0} = (1, 0T, 1I, 0F) \notin N_q(A, B)$. This shows that the converse of Theorem 3.18 is not be true in general.

Lemma 3.20 ([26]). *For any nonempty subsets A and B of a BCI-algebra X, if the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$, then it is both a neutrosophic quadruple subalgebra and a neutrosophic quadruple ideal of $N_q(X)$.*

Theorem 3.21. *Given nonempty subsets A and B of a BCI-algebra X, the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ if and only if $N_q(A, B)$ is both a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal of $N_q(X)$.*

Proof. If $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$, then $N_q(A,B)$ is both a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal of $N_q(X)$ by Theorems 3.15 and 3.18.

Conversely, suppose that $N_q(A,B)$ is both a neutrosophic quadruple p-ideal and a neutrosophic quadruple q-ideal of $N_q(X)$. Then $N_q(A,B)$ is a neutrosophic quadruple subalgebra of $N_q(X)$ by Lemma 3.20, and A and B are both a p-ideal and a q-ideal of X. For $i = 1,2$ and $j = 3,4$, let $x_i * (0 * y_i) \in A$ and $x_j * (0 * y_j) \in B$ for $x_i, y_i, x_j, y_j \in X$. Then $x_i * y_i \in A$ and $x_j * y_j \in B$ since A and B are q-ideals of X. Recall that

$$\begin{aligned}(0 * (y_k * x_k)) * (x_k * y_k) &= ((0 * y_k) * (0 * x_k)) * (x_k * y_k)\\ &= ((0 * (x_k * y_k)) * y_k) * (0 * x_k)\\ &= (((0 * x_k) * (0 * y_k)) * y_k) * (0 * x_k)\\ &= (0 * (0 * y_k)) * y_k = 0 \in A \cap B\end{aligned}$$

for $k = 1,2,3,4$. Hence $0 * (y_i * x_i) \in A$ and $0 * (y_j * x_j) \in B$, and so $0 * (0 * (y_i * x_i)) \in A$ and $0 * (0 * (y_j * x_j)) \in B$. Since A and B are p-ideals of X, it follows from (14) that $y_i * x_i \in A$ and $y_j * x_j \in B$. Therefore $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.10. □

Lemma 3.22 ([26]). *Let A, B, I and J be ideals of a BCI-algebra X such that $I \subseteq A$ and $J \subseteq B$. If I and J are q-ideals of X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$.*

Lemma 3.23 ([]). *If A and B are p-ideals of a BCI-algebra X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$.*

Theorem 3.24. *Let A, B, I and J be ideals of a BCI-algebra X such that $I \subseteq A$ and $J \subseteq B$. If I and J are a-ideals of X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Proof. Assume that I and J are a-ideals of X. Then I and J are both p-ideals and q-ideals of X. Thus neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple q-ideal of $N_q(X)$ by Lemma 3.22. Let $0 * (0 * x) \in A \cap B$ for $x \in X$. Then

$$(0 * (0 * (0 * x))) * (0 * x) = (0 * (0 * x)) * (0 * (0 * x)) = 0 \in I \cap J.$$

Since

$$\begin{aligned}&(0*(0*(x*(0*(0*x)))))*((0*(0*(0*x)))*(0*x))\\&=((0*(0*x))*(0*(0*(0*(0*x)))))*((0*(0*(0*x)))*(0*x))\\&\leq((0*(0*(0*x)))*(0*x))*((0*(0*(0*x)))*(0*x))\\&=0\in I\cap J,\end{aligned}$$

it follows that $0*(0*(x*(0*(0*x))))\in I\cap J$. Since I and J are p-ideals of X, we have $x*(0*(0*x))\in I\cap J\subseteq A\cap B$ by (14), and so $x\in A\cap B$. This shows that A and B are p-ideals of X, and thus $N_q(A,B)$ is a neutrosophic quadruple p-ideal of $N_q(X)$ by Lemma 3.23. Therefore $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.21. □

Corollary 3.25. *Let A and I be ideals of a BCI-algebra X such that $I\subseteq A$. If I is an a-ideal of X, then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.*

Corollary 3.26. *If the zero ideal $\{0\}$ is an a-ideal of a BCI-algebra X, then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ for every ideals A and B of X.*

Theorem 3.27. *Let A,B,I and J be ideals of a BCI-algebra X such that $I\subseteq A$, $J\subseteq B$ and*

$$(\forall x,y\in X)(x*(0*y)\in I\cap J\ \Rightarrow\ y*x\in I\cap J). \tag{23}$$

Then the neutrosophic quadruple (A,B)-set $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. Let $x,y,z\in X$ be such that $(x*z)*(0*y)\in I\cap J$ and $z\in I\cap J$. Note that

$$(x*(0*y))*((x*z)*(0*y))\leq x*(x*z)\leq z\in I\cap J.$$

Hence $x*(0*y)\in I\cap J$, and so $y*x\in I\cap J$ by (23). Thus I and J are a-ideals of X. It follows from Theorem 3.24 that $N_q(A,B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$. □

Corollary 3.28. *Let A and I be ideals of a BCI-algebra X such that $I\subseteq A$ and*

$$(\forall x,y\in X)(x*(0*y)\in I\ \Rightarrow\ y*x\in I). \tag{24}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Theorem 3.29. *Let A, B, I and J be ideals of a BCI-algebra X such that $I \subseteq A$, $J \subseteq B$ and*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in I \cap J \;\Rightarrow\; y * (x * z) \in I \cap J). \tag{25}$$

Then the neutrosophic quadruple (A, B)-set $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

Proof. If we put $z = 0$ in (25) and use (1), then (23) is valid. Therefore $N_q(A, B)$ is a neutrosophic quadruple a-ideal of $N_q(X)$ by Theorem 3.27. □

Corollary 3.30. *Let A and I be ideals of a BCI-algebra X such that $I \subseteq A$ and*

$$(\forall x, y, z \in X)((x * z) * (0 * y) \in I \;\Rightarrow\; y * (x * z) \in I). \tag{26}$$

Then the neutrosophic quadruple A-set $N_q(A)$ is a neutrosophic quadruple a-ideal of $N_q(X)$.

4. Conclusions

We have applied the notion of neutrosophic quadruple set to an a-ideal in a BCI-algebra. We have introduced the concept of neutrosophic quadruple a-ideal of neutrosophic quadruple BCI-algebras, and have investigated several properties. The notions of neutrosophic quadruple p-ideal, neutrosophic quadruple q-ideal and neutrosophic quadruple closed ideal have been introduced by Smarandache, Muhiuddin, Al-Kenani, Jun, etc. We have discussed relations between a neutrosophic quadruple p-ideal, a neutrosophic quadruple q-ideal, a neutrosophic quadruple a-ideal and a neutrosophic quadruple closed ideal. We have provided conditions for the neutrosophic quadruple (A, B)-set $N_q(A, B)$ to be a neutrosophic quadruple a-ideal. We have shown that every neutrosophic quadruple a-ideal is a neutrosophic quadruple closed ideal, and heve provided example to show that the converse is false. Using the ideas and results of this paper, we will study the structure of various algebraic systems in the future.

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References

1. M. Abdel-Basset, N.A. Nabeeh, H.A. El-Ghareeb and A. Aboelfetouh, Utilising neutrosophic theory to solve transition difficulties of IoT-based enterprises, Enterprise Information Systems, (2019)., 1–21.
2. M. Abdel-Baset, V. Chang and A. Gamal, Evaluation of the green supply chain management practices: A novel neutrosophic approach, Computers in Industry, **108** (2019), 210–220.
3. M. Abdel-Basset, M. Saleh, A. Gamal and F. Smarandache, An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number, Applied Soft Computing, **77** (2019), 438-452.

4. M. Abdel-Basset, G. Manogaran, A. Gamal and F. Smarandache, A group decision making framework based on neutrosophic TOPSIS approach for smart medical device selection, Journal of Medical Systems, 43(2), (2019), 38.
5. S.A. Akinleye, F. Smarandache and A.A.A. Agboola, On neutrosophic quadruple algebraic structures, Neutrosophic Sets and Systems, **12** (2016), 122–126.
6. A. Borumand Saeid and Y.B. Jun, Neutrosophic subalgebras of BCK/BCI-algebras based on neutrosophic points, Ann. Fuzzy Math. Inform., **14**(1) (2017), 87–97.
7. H. Bordbar, M. Mohseni Takallo, R.A. Borzooei and Y. B. Jun, BMBJ-neutrosophic subalgebra in BCK/BCI-algebras, Neutrosophic Sets and Systems, 27 (2020), to appear.
8. R.A. Borzooei, H. Farahani and M. Moniri, Neutrosophic Deductive Filters on BL-Algebras, Journal of Intel- ligent and Fuzzy Systems, 26(6), (2014), 2993-3004.
9. R.A. Borzooei, M. Mohseni Takallo, F. Smarandache nad Y.B. Jun, Positive implicative BMBJ-neutrosophic ideals in BCK-algebras, Neutrosophic Sets and Systems, 23 (2018), 148-163.
10. R.A. Borzooei, F. Smarandache and Y.B. Jun, Positive implicative generalized neutrosophic ideals in BCK-algebras, Neutrosophic Sets and Systems, 23 (2018), 126 -141.
11. R.A. Borzooei, X. Zhang, F. Smarandache and Y.B. Jun, Commutative Generalized Neutrosophic Ideals in BCK-Algebras, Symmetry, 10(8), 350 (2018), 1-15.
12. Y. Huang, BCI-algebra, Science Press, Beijing, 2006.
13. K. Iséki, On BCI-algebras, Math. Seminar Notes, **8** (1980), 125–130.
14. K. Iséki and S. Tanaka, An introduction to the theory of BCK-algebras, Math. Japon., **23** (1978), 1–26.
15. Y.B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Ann. Fuzzy Math. Inform., **14** (2017), no. 1, 75–86.
16. Y.B. Jun, S.J. Kim and F. Smarandache, Interval neutrosophic sets with applications in BCK/BCI-algebra, Axioms, 2018, 7, 23.
17. Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras, Information, 2017, 8, 128.
18. Y.B. Jun, F. Smarandache, S.Z. Song and M. Khan, Neutrosophic positive implicative $\mathcal{N}$-ideals in BCK/BCI-algebras, Axioms 2018, 7, 3.
19. Y.B. Jun, S.Z. Song, F. Smarandache and H. Bordbar, Neutrosophic quadruple BCK/BCI-algebras, Axioms, (2018), 7-41; doi:10.3390/axioms7020041
20. M. Khan, S. Anis, F. Smarandache and Y.B. Jun, Neutrosophic $\mathcal{N}$-structures and their applications in semigroups, Ann. Fuzzy Math. Inform., **14**(6) (2017), 583–598.
21. Y.L. Liu, J. Meng, X.H. Zhang and Z.C. Yue, q-ideals and a-ideals in BCI-algebras, ,Southeast Asian Bulletin of Mathematics **24** (2000), 243–253.
22. J. Meng and Y. B. Jun, BCK-algebras, Kyungmoonsa Co., Seoul, Korea 1994.
23. M. Mohseni Takallo, R. A. Borzooei and Y. B. Jun, MBJ-neutrosophic structures and its applications in BCK/BCI-algebras, Neutrosophic Sets and Systems, 23 (2018), 72- 84.
24. G. Muhiuddin, A.N. Al-Kenani, E.H. Roh and Y.B. Jun, Implicative neutrosophic quadruple BCK-algebras and ideals, Symmetry, (2019), 11, 277; doi:10.3390/sym11020277.
25. G. Muhiuddin and Y.B. Jun, p-semisimple neutrosophic quadruple BCI-algebras and neutrosophic quadruple p-ideals, (submitted).
26. G. Muhiuddin, F. Smarandache and Y.B. Jun, Neutrosophic quadruple ideals in neutrosophic quadruple BCI-algebras, Neutrosophic Sets and Systems (submitted).
27. M.A. Öztürk and Y.B. Jun, Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points, J. Inter. Math. Virtual Inst., **8** (2018), 1–17.

28. F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998. http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf (last edition online).
29. F. Smarandache, A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability, American Reserch Press, Rehoboth, NM, 1999.
30. F. Smarandache, Neutrosophic set-a generalization of the intuitionistic fuzzy set, Int. J. Pure Appl. Math., **24**(3) (2005), 287–297.
31. F. Smarandache, Neutrosophic quadruple numbers, refined neutrosophic quadruple numbers, absorbance law, and the multiplication of neutrosophic quadruple numbers, Neutrosophic Sets and Systems, **10** (2015), 96–98.
32. S.Z. Song, F. Smarandache and Y.B. Jun, , Neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras, Information, 2017, 8, 130.
33. X. Zhang, J. Hao and S.A. Bhatti, On p-ideals of a BCI-algebra, Punjab Univ. J. Math., (Lahore) 27 (1994), 121–128. Title of Site. Available online: URL (accessed on Day Month Year).

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Neutrosophic LI-ideals in lattice implication algebras

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Abstract. The notion of neutrosophic set theory is applied to lattice implication algebras, and the concept of neutrosophic LI-ideals and neutrosophic lattice ideals in a lattice implication algebra are introduced. Several properties are investigated. Relationships between a neutrosophic LI-ideal and a neutrosophic lattice ideal are established, and conditions for a neutrosophic lattice ideal to be a neutrosophic LI-ideal are provided. Characterizations of a neutrosophic LI-ideal are discussed. The properties of implication homomorphism of lattice implication algebras related to neutrosophic LI-ideals are studied.

Keywords: Lattice implication algebra; neutrosophic LI-ideals; neutrosophic lattice ideal; implication homomorphism.

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1. Introduction

Smarandache in [1, 2] introduced the notion of neutrosophic set, which is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set and interval-valued (intuitionistic) fuzzy set. Then the neutrosophic components T, I, F were introduced, which represent the membership, indeterminacy, and non-membership values respectively, where $[0, 1]$ is the non-standard unit interval, and the neutrosophic set was defined. Then some examples were given from mathematics, physics, philosophy, and applications of the neutrosophic set. Afterward, the neutrosophic set operations (complement, intersection, union, difference, Cartesian product, inclusion, and n-ary relationship) were introduced, some generalizations and comments on them, and finally, the distinctions between the neutrosophic set and the intuitionistic fuzzy set. Jun and his colleagues in [3] applied the notion of neutrosophic set theory to BCK/BCI-algebras, and their properties and relations are investigated. Then in [4], the notion of interval neutrosophic length of a range neutrosophic set was introduced. Moreover, in [5], interval neutrosophic ideals were defined, and some properties were investigated.

Then in [6], they represented different kinds of interval neutrosophic ideals and studied some features and found the relation among them.

Borzooei et al. [7–10], appliad the neutrosophic sets to logical algebras and defined the concept of a commutative generalized neutrosophic ideal in a BCK-algebra, and proved some related properties. Characterizations of a commutative generalized neutrosophic ideal are considered. Also, some equivalence relations on the family of all commutative generalized neutrosophic ideals in BCK-algebras are introduced. Also, Jun in [11] introduced the notion of LI-ideals, Li-maximal ideals and prime LI-ideals of lattice implication algebras, and investigated some properties of them and studied the relation among them. Since everything in the world is full of indeterminacy, and application of this notion in decision making and multicriteria decision-making method etc. We decide applied the notion of neutrosophic set theory to lattice implication algebras. We introduce the concept of neutrosophic LI-ideals and neutrosophic lattice ideals of a lattice implication algebra, and investigate several properties. We discuss relationship between a neutrosophic LI-ideal and a neutrosophic lattice ideal. We provide conditions for a neutrosophic lattice ideal to be a neutrosophic LI-ideal. We consider characterizations of a neutrosophic LI-ideal. We study the properties of implication homomorphism of lattice implication algebras related to neutrosophic LI-ideals.

2. **Preliminaries**

By a *lattice implication algebra* we mean a bounded lattice $(L, \vee, \wedge, 0, 1)$ with order-reversing involution " $'$ " and a binary operation " $\to$ " satisfying the following axioms:

(I1) $u \to (v \to w) = v \to (u \to w)$,

(I2) $u \to u = 1$,

(I3) $u \to v = v' \to u'$,

(I4) $u \to v = v \to u = 1 \Rightarrow u = v$,

(I5) $(u \to v) \to v = (v \to u) \to u$,

(L1) $(u \vee v) \to w = (u \to w) \wedge (v \to w)$,

(L2) $(u \wedge v) \to w = (u \to w) \vee (v \to w)$,

for all $u, v, w \in L$. A lattice implication algebra L is called a *lattice H-implication algebra* if it satisfies:

$$(\forall u, v, w \in L)(u \vee v \vee ((u \wedge v) \to w) = 1). \tag{1}$$

We can define a partial ordering $\leq$ on L by condition $u \leq v$ if and only if $u \to v = 1$.

In a lattice implication algebra L, the following conditions hold (see [20]):

(a1) $0 \to u = 1$, $1 \to u = u$ and $u \to 1 = 1$.

(a2) $u \to v \leq (v \to w) \to (u \to w)$.

(a3) $u \leq v$ implies $v \to w \leq u \to w$ and $w \to u \leq w \to v$.

(a4) $u' = u \to 0$.

(a5) $u \vee v = (u \to v) \to v$.

(a6) $((v \to u) \to v')' = u \wedge v = ((u \to v) \to u')'$.

(a7) $u \leq (u \to v) \to v$.

Let L_1 and L_2 be two lattice implication algebras. A mapping $f : L_1 \to L_2$ is called an *implication homomorphism* ([19]) if $f(u \to v) = f(u) \to f(v)$ for all $u, v \in L_1$. Moreover, if f satisfies the following conditions:

$$f(u \vee v) = f(u) \vee f(v),\ f(u \wedge v) = f(u) \wedge f(v),\ f(u') = (f(u))'$$

for all $u, v \in L_1$, then f is called a *lattice implication homomorphism.* For an implication homomorphism $f : L_1 \to L_2$, the *kernel* of f, written ker$f$, is defined as follows:

$$\ker f := \{u \in L_1 \mid f(u) = 0\}.$$

Note that if an implication homomorphism $f : L_1 \to L_2$ satisfies $f(0) = 0$, then f is a lattice implication homomorphism ([19]).

Definition 2.1 ([15]). A nonempty subset G of L is called an *LI-ideal* of L if it satisfies the following statements:

(i) $0 \in G$,

(ii) $(\forall u \in L)\ (\forall v \in G)\ ((u \to v)' \in G \implies u \in G)$.

Lemma 2.2 ([15]). *Every LI-ideal G of L satisfies the following implication:*

$$(\forall u \in G)\,(\forall v \in L)\,(v \leq u \implies v \in G).$$

Let L be a non-empty set. A *neutrosophic set* (NS) in L (see [1]) is a structure of the form:

$$A_\sim := \{\langle u; A_T(u), A_I(u), A_F(u)\rangle \mid u \in L\},$$

where $A_T : L \to [0,1]$ is a truth membership function, $A_I : L \to [0,1]$ is an indeterminate membership function, and $A_F : L \to [0,1]$ is a false membership function. For the sake of simplicity, we shall use the symbol $A_\sim = (A_T,\ A_I,\ A_F)$ for the neutrosophic set, it means

$$A_\sim := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in L\}.$$

Given a neutrosophic set $A_\sim = (A_T,\ A_I,\ A_F)$ in a lattice implication algebra L. Then we consider the following sets.

$$L(A_T; \alpha) := \{u \in L \mid A_T(u) \geq \alpha\},$$

$$L(A_I; \beta) := \{u \in L \mid A_I(u) \geq \beta\},$$

$$L(A_F; \gamma) := \{u \in L \mid A_F(u) \leq \gamma\},$$

which are called *neutrosophic level subsets* of L.

We refer the reader to the books [21] for additional details lattice implication algebras, and to the site "http://fs.gallup.unm.edu/neutrosophy.htm" for further information regarding neutrosophic set theory.

3. Neutrosophic LI-ideals

From now on, we let L as lattice implication algebra unless otherwise state.

Definition 3.1. A neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in L is called a *neutrosophic LI-ideal* of L if the following assertions are valid.

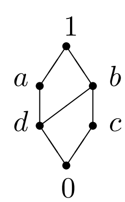
$$(\forall u \in L)\left(\ A_T(0) \geq A_T(u),\ A_I(0) \geq A_I(u),\ A_F(0) \leq A_F(u)\ \right) \tag{2}$$

and

$$(\forall x, y \in L)\left(\begin{array}{l} A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\} \\ A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\} \\ A_F(u) \leq \max\{A_F((u \to v)'), A_F(v)\} \end{array}\right) \tag{3}$$

The set of all neutrosophic LI-ideals of L is denoted by NLI(L).

Example 3.2. Let $L = \{0, a, b, c, d, 1\}$ be a poset with Hasse diagram and Cayley tables as follows:



x	x'
0	1
a	c
b	d
c	a
d	b
1	0

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Define the operations $\vee$ and $\wedge$ on L as follows:

$$u \vee v := (u \to v) \to v,\ \ u \wedge v := ((u' \to v') \to v')',$$

for all $u, v \in L$. Then L is a lattice implication algebra (see [15]). Suppose $A_\sim = (A_T, A_I, A_F)$ is a neutrosophic set in L defined by Table 1.

Table 1. Tabular representation of $A_\sim = (A_T, A_I, A_F)$

L	0	a	b	c	d	1
$A_T(u)$	0.9	0.5	0.5	0.7	0.5	0.5
$A_I(u)$	0.8	0.3	0.3	0.3	0.3	0.3
$A_F(u)$	0.2	0.4	0.6	0.6	0.4	0.6

It is routine to verify that $A_\sim = (A_T, A_I, A_F) \in$ NLI(L).

Proposition 3.3. *Every neutrosophic LI-ideal $A_{\sim} = (A_T,\, A_I,\, A_F)$ of L satisfies the following assertions.*

$$(\forall u, v \in L)\left(x \leq y \;\Rightarrow \begin{cases} A_T(u) \geq A_T(v) \\ A_I(u) \geq A_I(v) \\ A_F(u) \leq A_F(v) \end{cases} \right). \tag{4}$$

Proof. Let $A_{\sim} \in \mathrm{NLI}(L)$ and $u, v \in L$ such that $u \leq v$. Since $(u \to v)' = 0$, we have,

$$A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\} = \min\{A_T(0), A_T(v)\} = A_T(v),$$
$$A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\} = \min\{A_I(0), A_I(v)\} = A_I(v),$$
$$A_F(u) \leq \max\{A_F((u \to v)'), A_F(v)\} = \max\{A_F(0), A_F(v)\} = A_F(v).$$

□

Proposition 3.4. *Every neutrosophic LI-ideal $A_{\sim} = (A_T,\, A_I,\, A_F)$ of L satisfies the following assertions.*

$$(\forall u, v, w \in L)\left(u \leq v' \to w \;\Rightarrow \begin{cases} A_T(u) \geq \min\{A_T(v), A_T(w)\} \\ A_I(u) \geq \min\{A_I(v), A_I(w)\} \\ A_F(u) \leq \max\{A_F(v), A_F(w)\} \end{cases} \right). \tag{5}$$

Proof. Suppose $A_{\sim} \in \mathrm{NLI}(L)$ such that for all $u, v, w \in L$, $u \leq v' \to w$. Then

$$1 = u \to (v' \to w) = w' \to (u \to v) = (u \to v)' \to w,$$

and so $((u \to v)' \to w)' = 0$. By (2) and (3), we get that

$$\begin{aligned} A_T(u) &\geq \min\{A_T((u \to v)'), A_T(v)\} \\ &\geq \min\{\min\{A_T(((u \to v)' \to w)'), A_T(w)\}, A_T(v)\} \\ &= \min\{\min\{A_T(0), A_T(w)\}, A_T(v)\} \\ &= \min\{A_T(w), A_T(v)\}, \end{aligned}$$

$$\begin{aligned} A_I(u) &\geq \min\{A_I((u \to v)'), A_I(v)\} \\ &\geq \min\{\min\{A_I(((u \to v)' \to w)'), A_I(w)\}, A_I(v)\} \\ &= \min\{\min\{A_I(0), A_I(w)\}, A_I(v)\} \\ &= \min\{A_I(w), A_I(v)\}, \end{aligned}$$

and

$$\begin{aligned}A_F(u) &\geq \max\{A_F((u \to v)'), A_F(v)\}\\ &\leq \max\{\max\{A_F(((u \to v)' \to w)'), A_F(w)\}, A_F(v)\}\\ &= \max\{\max\{A_F(0), A_F(w)\}, A_F(v)\}\\ &= \max\{A_F(w), A_F(v)\}.\end{aligned}$$

Therefore, (3.4) holds. □

Definition 3.5. A neutrosophic set $A_\sim = (A_T,\ A_I,\ A_F)$ in L is called a *neutrosophic lattice ideal* of L if it satisfies (4) and

$$(\forall u, v \in L) \left(\begin{array}{l} A_T(u \vee v) \geq \min\{A_T(u), A_T(v)\} \\ A_I(u \vee v) \geq \min\{A_I(u), A_I(v)\} \\ A_F(u \vee v) \leq \max\{A_F(u), A_F(v)\} \end{array} \right) \tag{6}$$

Example 3.6. Let L be the lattice implication algebra as in Example 3.2 and $A_\sim = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in L which is defined by Table 2.

TABLE 2. Tabular representation of $A_\sim = (A_T,\ A_I,\ A_F)$

L	0	a	b	c	d	1
$A_T(u)$	0.7	0.4	0.4	0.4	0.7	0.4
$A_I(u)$	0.8	0.5	0.5	0.5	0.8	0.5
$A_F(u)$	0.3	0.6	0.6	0.6	0.3	0.6

It is easy to see that $A_\sim = (A_T,\ A_I,\ A_F)$ is a neutrosophic lattice ideal of L.

We discussthe between a neutrosophic LI-ideal and a neutrosophic lattice ideal.

Theorem 3.7. *Every neutrosophic LI-ideal is a neutrosophic lattice ideal.*

Proof. Let $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. The condition (4) is valid in Proposition 3.3. Since $((u \vee v) \to v)' = (((u \to v) \to v) \to v)' = (u \to v)' \leq (u')'$ for all $u, v \in L$, by (4) and (3), we have

$$A_T(u \vee v) \geq \min\{A_T(((u \vee v) \to v)'), A_T(v)\} \geq \min\{A_T(u), A_T(v)\},$$

$$A_I(u \vee v) \geq \min\{A_I(((u \vee v) \to v)'), A_I(v)\} \geq \min\{A_I(u), A_I(v)\},$$

and

$$A_F(u \vee v) \leq \max\{A_F(((u \vee v) \to v)'), A_F(v)\} \leq \max\{A_F(u), A_F(v)\}.$$

Therefore, $A_\sim = (A_T, A_I, A_F) \in NLI(L)$. $\square$

The converse of Theorem 3.7 is not true in general as seen in the following example.

Example 3.8. Let L be the lattice implication algebra as in Example 3.2 and $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic set in L defined by Table 3.

TABLE 3. Tabular representation of $A_\sim = (A_T, A_I, A_F)$

L	0	a	b	c	d	1
$A_T(x)$	0.8	0.4	0.4	0.4	0.8	0.4
$A_I(x)$	0.6	0.3	0.3	0.3	0.6	0.3
$A_F(x)$	0.3	0.5	0.5	0.5	0.3	0.5

Then $A_\sim = (A_T, A_I, A_F) \in L$, but $A_\sim \notin NLI(L)$ beacuse $A_T(a) = 0.4 < 0.8 = \min\{A_T((a \to d)'), A_T(d)\}$.

We investigate that under which condition, a neutrosophic lattice ideal can be a neutrosophic LI-ideal.

Theorem 3.9. *In a lattice H-implication algebra L, every neutrosophic lattice ideal is a neutrosophic LI-ideal.*

Proof. Let $A_\sim = (A_T, A_I, A_F)$ be a neutrosophic lattice ideal of a lattice H-implication algebra L. Moreover, since $0 \leq u$ for all $u \in L$, it follows from (4) that $A_T(0) \geq A_T(u)$, $A_I(0) \geq A_I(u)$ and $A_F(0) \leq A_F(u)$. Also, from $u \leq u \vee v$ for all $u, v \in L$, by (4) and (6) we get that,

$$A_T(u) \geq A_T(u \vee v) = A_T(v \vee (u' \vee v)') = A_T(v \vee (u \to v)') \geq \min\{A_T(v), A_T((u \to v)')\},$$

$$A_I(u) \geq A_I(u \vee v) = A_I(v \vee (u' \vee v)') = A_I(v \vee (u \to v)') \geq \min\{A_I(v), A_I((u \to v)')\},$$

and

$$A_F(u) \leq A_F(u \vee v) = A_F(v \vee (u' \vee v)') = A_F(v \vee (u \to v)') \leq \max\{A_F(v), A_F((u \to v)')\}.$$

Therefore, $A_\sim = (A_T, A_I, A_F) \in NLI(L)$. $\square$

We consider characterizations of a neutrosophic LI-ideal.

Theorem 3.10. *Given a neutrosophic set $A_\sim = (A_T, A_I, A_F)$ in L, the following statements are equivalent.*

(1) *$A_\sim = (A_T, A_I, A_F)$ is a neutrosophic LI-ideal of L.*
(2) *$A_\sim = (A_T, A_I, A_F)$ satisfies (5).*

(3) $A_\sim = (A_T,\ A_I,\ A_F)$ *satisfies* (4) *and*

$$(\forall u, v \in L)\left(\begin{array}{l} A_T(u' \to v) \geq \min\{A_T(u), A_T(v)\} \\ A_I(u' \to v) \geq \min\{A_I(u), A_I(v)\} \\ A_F(u' \to v) \leq \max\{A_F(u), A_F(v)\} \end{array}\right). \tag{7}$$

(4) $A_\sim = (A_T,\ A_I,\ A_F)$ *satisfies* (2) *and*

$$(\forall u, v, w \in L)\left(\begin{array}{l} A_T(u' \to w) \geq \min\{A_T((u \to v)'), A_T(v' \to w)\} \\ A_I(u' \to w) \geq \min\{A_I((x \to v)'), A_I(v' \to w)\} \\ A_F(u' \to w) \leq \max\{A_F((x \to v)'), A_F(v' \to w)\} \end{array}\right). \tag{8}$$

(5) $A_\sim = (A_T,\ A_I,\ A_F)$ *satisfies* (2) *and*

$$(\forall u, v, w \in L)\left(\begin{array}{l} A_T((u \to w)') \geq \min\{A_T((u \to v)'), A_T((v \to w)')\} \\ A_I((u \to w)') \geq \min\{A_I((u \to v)'), A_I((v \to w)')\} \\ A_F((u \to w)') \leq \max\{A_F((u \to v)'), A_F((v \to w)')\} \end{array}\right). \tag{9}$$

Proof. Suppose $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. Then $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (5) by Proposition (3.4). Let $A_\sim = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in L which satisfies the condition (3.4). Since $0 \leq u' \to u$ for all $u \in L$, we have $A_T(0) \geq \min\{A_T(u), A_T(u)\} = A_T(u)$, $A_I(0) \geq \min\{A_I(u), A_I(u)\} = A_I(u)$, and $A_F(0) \leq \max\{A_F(u), A_F(u)\} = A_F(u)$. Since $u \leq ((u \to v)')' \to v$ for all $u, v \in L$, it follows from (3.4) that $A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\}$, $A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\}$, and $A_F(u) \leq \max\{A_F((u \to v)'), A_F(v)\}$. Thus $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. Let $u, v \in L$ such that $u \leq v$. Then $u \leq v = v \vee v \leq v' \to v$, and so $A_T(u) \geq \min\{A_T(v), A_T(v)\} = A_T(v)$, $A_I(u) \geq \min\{A_I(v), A_I(v)\} = A_I(v)$, and $A_F(u) \leq \max\{A_F(v), A_F(v)\} = A_F(v)$ by (3.4). Hence $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (4). Since $u' \to v \leq u' \to v$ for all $u, v \in L$, it follows from (3.4) that $A_T(u' \to v) \geq \min\{A_T(u), A_T(v)\}$, $A_I(u' \to v) \geq \min\{A_I(u), A_I(v)\}$, and $A_F(x' \to v) \leq \max\{A_F(u), A_F(v)\}$. Hence (7) holds.

Suppose $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (4) and (7). Since $0 \leq u$ for all $u \in L$, (2) is induced by (4). Moreover, from $u \leq ((u \to v)')' \to v$ for all $u, v \in L$, we get that,

$$u' \to w \leq (((u \to v)')' \to v)' \to w = ((u \to v)')' \to (v' \to w).$$

Thus

$$A_T(u' \to w) \geq A_T(((u \to v)')' \to (v' \to w)) \geq \min\{A_T((u \to v)'), A_T(v' \to w)\},$$

$$A_I(u' \to w) \geq A_I(((u \to v)')' \to (v' \to w)) \geq \min\{A_I((u \to v)'), A_I(v' \to w)\},$$

and

$$A_F(u' \to w) \leq A_F(((u \to v)')' \to (v' \to w)) \leq \max\{A_F((u \to v)'), A_F(v' \to w)\}.$$

Hence $A_\sim = (A_T,\ A_I,\ A_F)$ satisfies (8).

Assume $A_\sim = (A_T,\, A_I,\, A_F)$ satisfies (2) and (8). Let $u, v \in L$ such that $u \leq v$. Let $w = 0$ in (8) Then

$$A_T(u) = A_T(u' \to 0) \geq \min\{A_T((u \to v)'), A_T(v' \to 0)\} = \min\{A_T(0), A_T(v)\} = A_T(v),$$

$$A_I(u) = A_I(u' \to 0) \geq \min\{A_I((u \to v)'), A_I(v' \to 0)\} = \min\{A_I(0), A_I(v)\} = A_I(v),$$

and

$$A_F(u) = A_F(u' \to 0) \leq \max\{A_F((u \to v)'), A_F(v' \to 0)\} = \max\{A_F(0), A_F(v)\} = A_F(v).$$

Therefore, $A_\sim = (A_T,\, A_I,\, A_F)$ satisfies (5).

Suppose $A_\sim = (A_T,\, A_I,\, A_F) \in NLI(L)$. Since

$$((u \to w)' \to (v \to w)')' \to (u \to v)' = (u \to v) \to ((v \to w) \to (u \to w)) = 1,$$

we have, $((u \to w)' \to (v \to w)')' \leq (u \to v)'$ for all $u, v, w \in L$. By (3) and (4), we get that

$$A_T((u \to w)') \geq \min\{A_T(((u \to w)' \to (v \to w)')'), A_T((v \to w)')\} \geq \min\{A_T((u \to v)'), A_T((v \to w)')\},$$

$$A_I((u \to w)') \geq \min\{A_I(((u \to w)' \to (v \to w)')'), A_I((v \to w)')\} \geq \min\{A_I((u \to v)'), A_I((v \to w)')\},$$

and

$$A_F((u \to w)') \leq \max\{A_F(((u \to w)' \to (v \to w)')'), A_F((v \to w)')\} \leq \max\{A_F((u \to v)'), A_F((v \to w)')\}$$

for all $u, v, w \in L$. Thus $A_\sim = (A_T,\, A_I,\, A_F)$ satisfies (9).

Let $A_\sim = (A_T,\, A_I,\, A_F)$ be a neutrosophic set in L satisfying (2) and (9). Since $(u \to 0)' = u$ for all $u \in L$, we have

$$A_T(u) = A_T((u \to 0)') \geq \min\{A_T((u \to v)'), A_T((v \to 0)')\} = \min\{A_T((u \to v)'), A_T(v)\},$$

$$A_I(u) = A_I((u \to 0)') \geq \min\{A_I((u \to v)'), A_I((v \to 0)')\} = \min\{A_I((u \to v)'), A_I(v)\},$$

and

$$A_F(u) = A_F((u \to 0)') \leq \max\{A_F((u \to v)'), A_F((v \to 0)')\} = \max\{A_F((u \to v)'), A_F(v)\}$$

for all $u, v \in L$. Therefore $A_\sim = (A_T,\, A_I,\, A_F) \in NLI(L)$. □

Theorem 3.11. *A neutrosophic set $A_\sim = (A_T,\, A_I,\, A_F)$ is a neutrosophic LI-ideal of L if and only if the nonempty neutrosophic level sets $L(A_T; \alpha)$, $L(A_I; \beta)$ and $L(A_F; \gamma)$ are LI-ideals of L for all $\alpha, \beta, \gamma \in [0, 1]$.*

Proof. Suppose $A_\sim = (A_T,\, A_I,\, A_F) \in NLI(L)$ and $\alpha, \beta, \gamma \in [0, 1]$ such that $L(A_T; \alpha)$, $L(A_I; \beta)$ and $L(A_F; \gamma)$ are nonempty. It is clear that $0 \in L(A_T; \alpha)$, $0 \in L(A_I; \beta)$ and $0 \in L(A_F; \gamma)$. Let $u, v, a, b, m, n \in L$ such that $(u \to v)' \in L(A_T; \alpha)$, $v \in L(A_T; \alpha)$, $(a \to b)' \in L(A_I; \beta)$,

$b \in L(A_I;\beta)$, $(m \to n)' \in L(A_F;\gamma)$, and $n \in L(A_F;\gamma)$. Then $A_T((u \to v)') \geq \alpha$, $A_T(v) \geq \alpha$, $A_I((a \to b)') \geq \beta$, $A_I(b) \geq \beta$, $A_F((m \to n)') \leq \gamma$, and $A_F(n) \leq \gamma$. By (2), we have

$$A_T(u) \geq \min\{A_T(u \to v)', A_T(v)\} \geq \alpha,$$

$$A_I(a) \geq \min\{A_I(a \to b)', A_I(b)\} \geq \beta,$$

and

$$A_F(m) \leq \max\{A_F(m \to n)', A_F(n)\} \leq \gamma.$$

Hence, $u \in L(A_T;\alpha)$, $a \in L(A_I;\beta)$ and $u \in L(A_F;\gamma)$. Therefore, $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are LI-ideals of L.

Conversely, let $A_\sim = (A_T,\ A_I,\ A_F)$ be a neutrosophic set in L in which the nonempty neutrosophic level sets $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are LI-ideals of L for all $\alpha,\beta,\gamma \in [0,1]$. For any $u,a,m \in L$, let $A_T(u) = \alpha$, $A_I(a) = \beta$ and $A_F(m) = \gamma$. Then $u \in L(A_T;\alpha)$, $a \in L(A_I;\beta)$ and $m \in L(A_F;\gamma)$, that is, $L(A_T;\alpha)$, $L(A_I;\beta)$ and $L(A_F;\gamma)$ are nonempty sets. Hence $0 \in L(A_T;\alpha)$, $0 \in L(A_I;\beta)$ and $0 \in L(A_F;\gamma)$ by assumption, and so $A_T(0) \geq \alpha = A_T(u)$, $A_I(0) \geq \beta = A_I(a)$ and $A_F(0) \leq \gamma = A_F(m)$. Suppose there exist $a,b \in L$ such that $A_T(a) < \min\{A_T((a \to b)'), A_T(b)\}$. Then

$$A_T(a) < \alpha_0 < \min\{A_T((a \to b)'), A_T(b)\},$$

where $\alpha_0 = \frac{1}{2}(A_T(a) + \min\{A_T((a \to b)'), A_T(b)\})$. Thus $a \notin L(A_T;\alpha_0)$, $(a \to b)' \notin L(A_T;\alpha_0)$ and $b \in L(A_T;\alpha_0)$, which is a contradiction. Hence, $A_T(u) \geq \min\{A_T((u \to v)'), A_T(v)\}$ for all $u,v \in L$. Similarly, we can verify that $A_I(u) \geq \min\{A_I((u \to v)'), A_I(v)\}$ for all $u,v \in L$. Now, suppose

$$A_F(m) > \max\{A_F((m \to n)'), A_F(n)\},$$

for some $m,n \in L$. Let $\gamma_0 := \frac{1}{2}(A_F(m) + \max\{A_F((m \to n)'), A_F(n)\})$. Then

$$A_F(m) > \gamma_0 \geq \max\{A_F((m \to n)'), A_F(n)\},$$

and so $(m \to n)' \in L(A_F;\gamma_0)$, $n \in L(A_F;\gamma_0)$, but $m \notin L(A_F;\gamma_0)$, which is a contradiction. Hence

$$A_F(m) \leq \max\{A_F((m \to n)'), A_F(n)\}$$

for all $u,v \in L$. Therefore $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$. □

Corollary 3.12. *If $A_\sim = (A_T,\ A_I,\ A_F) \in NLI(L)$, then $L(A_T;\alpha) \cap L(A_I;\beta) \cap L(A_F;\gamma)$ is an LI-ideal of L for all $\alpha,\beta,\gamma \in [0,1]$.*

Proof. Straightforward. □

Let $f : L_1 \to L_2$ be an implication homomorphisms of lattice implication algebras. For any neutrosophic set $A_\sim = (A_T,\, A_I,\, A_F)$ in L_2, we define a new neutrosophic set $A^f_\sim = (A^f_T,\, A^f_I,\, A^f_F)$ in L_1 by $A^f_T(u) = A_T(f(u))$, $A^f_I(u) = A_I(f(u))$ and $A^f_F(u) = A_F(f(u))$ for all $u \in L_1$.

Theorem 3.13. *Let $f : L_1 \to L_2$ be an implication homomorphism of lattice implication algebras with $f(0) = 0$. If $A_\sim = (A_T,\, A_I,\, A_F) \in NLI(L_2)$, then $A^f_\sim = (A^f_T,\, A^f_I,\, A^f_F) \in NLI(L_1)$.*

Proof. Let $u, v \in L_1$. Then $A^f_T(u) = A_T(f(u)) \le A_T(0) = A_T(f(0)) = A^f_T(0)$, $A^f_I(u) = A_I(f(u)) \le A_I(0) = A_I(f(0)) = A^f_I(0)$, and $A^f_F(u) = A_F(f(u)) \ge A_F(0) = A_F(f(0)) = A^f_F(0)$. Thus,

$$\begin{aligned} A^f_T(u) = A_T(f(u)) &\ge \min\{A_T((f(u) \to f(v))'), A_T(f(v))\} \\ &= \min\{A_T((f(u \to v))'), A_T(f(v))\} \\ &= \min\{A_T(f((u \to v)')), A_T(f(v))\} \\ &= \min\{A^f_T((u \to v)'), A^f_T(v)\}, \end{aligned}$$

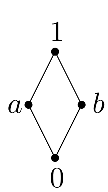
$$\begin{aligned} A^f_I(u) = A_I(f(u)) &\ge \min\{A_I((f(u) \to f(v))'), A_I(f(v))\} \\ &= \min\{A_I((f(u \to v))'), A_I(f(v))\} \\ &= \min\{A_I(f((u \to v)')), A_I(f(v))\} \\ &= \min\{A^f_I((u \to v)'), A^f_I(v)\}, \end{aligned}$$

and

$$\begin{aligned} A^f_F(u) = A_F(f(u)) &\le \max\{A_F((f(u) \to f(v))'), A_F(f(v))\} \\ &= \max\{A_F((f(u \to v))'), A_F(f(v))\} \\ &= \max\{A_F(f((u \to v)')), A_F(f(v))\} \\ &= \max\{A^f_F((u \to v)'), A^f_F(v)\}. \end{aligned}$$

Therefore, $A^f_\sim = (A^f_T,\, A^f_I,\, A^f_F) \in NLI(L_1)$. □

Example 3.14. Let $L = \{0, a, b, 1\}$ be a poset with Hasse diagram and Cayley tables as follows:



x	x'
0	1
a	b
b	a
1	0

$\to$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Defin the operations $\vee$ and $\wedge$ on L as follows:

$$u \vee v := (u \to v) \to v \ \ and \ \ u \wedge v := ((u' \to v') \to v')',$$

for all $u, v \in L$. Then L is a lattice implication algebra (see [21]). Define a function $f : L \to L$ by $f(0) = 0$, $f(a) = b$, $f(b) = a$ and $f(1) = 1$. Then f is an implication homomorphism . Let $A_{\sim} = (A_T, A_I, A_F)$ be a neutrosophic set in L defined by Table 4.

Table 4. Tabular representation of $A_{\sim} = (A_T, A_I, A_F)$

L	0	a	b	1
$A_T(x)$	0.9	0.5	0.3	0.3
$A_I(x)$	0.8	0.2	0.5	0.2
$A_F(x)$	0.2	0.7	0.4	0.7

It is routine to verify that $A_{\sim} = (A_T, A_I, A_F) \in NLI(L)$. The neutrosophic set $A^f_{\sim} = (A^f_T, A^f_I, A^f_F)$ is described by Table 5.

Table 5. Tabular representation of $A^f_{\sim} = (A^f_T, A^f_I, A^f_F)$

L	0	a	b	1
$A^f_T(x)$	0.9	0.3	0.5	0.3
$A^f_I(x)$	0.8	0.5	0.2	0.2
$A^f_F(x)$	0.2	0.4	0.7	0.7

It is routine to verify that $A^f_{\sim} = (A^f_T, A^f_I, A^f_F) \in NLI(L)$.

We give additional condition for dealing with the converse of Theorem 3.13.

Theorem 3.15. *Let $f : L_1 \to L_2$ be an implication epimorphism of lattice implication algebras with $f(0) = 0$. If $A^f_{\sim} = (A^f_T, A^f_I, A^f_F) \in NLI(L_1)$, then $A_{\sim} = (A_T, A_I, A_F) \in NLI(L_2)$.*

Proof. Let $u \in L_2$. Then there exists $a \in L_1$ such that $f(a) = u$. Hence

$$A_T(u) = A_T(f(a)) = A^f_T(a) \leq A^f_T(0) = A_T(f(0)) = A_T(0),$$

$$A_I(u) = A_I(f(a)) = A^f_I(a) \leq A^f_I(0) = A_I(f(0)) = A_I(0),$$

and

$$A_F(u) = A_F(f(a)) = A^f_F(a) \geq A^f_F(0) = A_F(f(0)) = A_F(0).$$

Let $u, v \in L_2$. Then $f(a) = u$ and $f(b) = v$ for some $a, b \in L_1$. It follows that

$$\begin{aligned} A_T(u) = A_T(f(a)) = A_T^f(a) &\geq \min\{A_T^f((a \to b)'), A_T^f(b)\} \\ &= \min\{A_T(f((a \to b)')), A_T(f(b))\} \\ &= \min\{A_T((f(a) \to f(b))'), A_T(f(b))\} \\ &= \min\{A_T((u \to v)'), A_T(v)\}, \end{aligned}$$

$$\begin{aligned} A_I(u) = A_I(f(a)) = A_I^f(a) &\geq \min\{A_I^f((a \to b)'), A_I^f(b)\} \\ &= \min\{A_I(f((a \to b)')), A_I(f(b))\} \\ &= \min\{A_I((f(a) \to f(b))'), A_I(f(b))\} \\ &= \min\{A_I((u \to v)'), A_I(v)\}, \end{aligned}$$

and

$$\begin{aligned} A_F(u) = A_F(f(a)) = A_F^f(a) &\leq \max\{A_F^f((a \to b)'), A_F^f(b)\} \\ &= \max\{A_F(f((a \to b)')), A_F(f(b))\} \\ &= \max\{A_F((f(a) \to f(b))'), A_F(f(b))\} \\ &= \max\{A_F((u \to v)'), A_F(v)\}. \end{aligned}$$

Therefore, $A_\sim = (A_T, A_I, A_F)$ is a neutrosophic LI-ideal of L_2. $\square$

4. **Conclusions**

We have applied the notion of neutrosophic set theory to lattice implication algebras. We have introduced the concepts of neutrosophic LI-ideals and neutrosophic lattice ideals of a lattice implication algebra, and investigated several properties. We have discussed the relationship between a neutrosophic LI-ideal and a neutrosophic lattice ideal, and provided conditions for a neutrosophic lattice ideal to be a neutrosophic LI-ideal. We have considered the characterizations of a neutrosophic LI-ideal. We have studied the properties of implication homomorphism of lattice implication algebras related to neutrosophic LI-ideals.

5. **Future research work**

Probing more profound, the results in this paper also provide a strong foundation for future work in logical algebric structure and in neutrosophic set. One area of future work is in combining some other kind of subalgebra like filter, implicative filter etc with neutrosophic sets. Another area is in applying the results studied here to the other algebric structures like BCI/BCK algebras. Future work will be in these two areas.

References

1. F. Smarandache, A unifying field in logics. Neutrosophy: Neutrosophic probability, set and logic, Rehoboth: American Research Press (1999).
2. F. Smarandache, Neutrosophic set, a generalization of intuitionistic fuzzy sets, International Journal of Pure and Applied Mathematics, 24(5) (2005), 287–297.
3. Y.B. Jun, Neutrosophic subalgebras of several types in BCK/BCI-algebras, Annals of Fuzzy Mathematics and Informatics, 14(1) (2017), 75–86.
4. Y.B. Jun, S.J. Kim and F. Smarandache, Interval neutrosophic sets with applications in BCK/BCI-algebra, Axioms 2018, 7, 23; doi:10.3390/axioms7020023.
5. Y.B. Jun, F. Smarandache and H. Bordbar, Neutrosophic $\mathcal{N}$-structures applied to BCK/BCI-algebras, Information 2017, 8, 128; doi:10.3390/info8040128.
6. Y.B. Jun, F. Smarandache, S.Z. Song and M. Khan, Neutrosophic positive implicative $\mathcal{N}$-ideals in BCK/BCI-algebras, Axioms 2018, 7, 3; doi:10.3390/axioms7010003.
7. H. Bordbar, M. Mohseni Takallo, R. A. Borzooei and Y. B. Jun, BMBJ-neutrosophic subalgebra in BCK/BCI-algebras, Neutrosophic Sets and Systems, 27 (2020), to appear.
8. R.A. Borzooei, X.H. Zhang, F. Smarandache and Y.B. Jun, Commutative generalized neutrosophic ideals in BCK-algebras, Symmetry 2018, 10, 350; doi:10.3390/sym10080350.
9. R. A. Borzooei, H. Farahani and M. Moniri, Neutrosophic Deductive Filters on BL-Algebras, Journal of Intel- ligent and Fuzzy Systems, 26(6), (2014), 2993-3004.
10. R.A. Borzooei, M. Mohseni Takallo, F. Smarandache nad Y.B. Jun, Positive implicative BMBJ-neutrosophic ideals in BCK-algebras, Neutrosophic Sets and Systems, 23 (2018), 148-163.
11. Y.B. Jun, On LI-ideals and prime LI-ideals of lattice implication algebras, Journal of the Korean Mathematical Society, 36(2) (1999), 369–380.
12. V. Chang, M. Abdel-Basset, M. Ramachandran, Towards a reuse strategic decision pattern framework–from theories to practices, Information Systems Frontiers, 21(1) (2019), 27-44.
13. N. A. Nabeeh, F. Smarandache, M. Abdel-Basset, H. A. El-Ghareeb, A. Aboelfetouh, An integrated neutrosophic-topsis approach and its application to personnel selection: A new trend in brain processing and analysis, IEEE Access, 7 (2019), 29734-29744.
14. N. A. Nabeeh, M. Abdel-Basset, H. A. El-Ghareeb, A. Aboelfetouh, Neutrosophic multi-criteria decision making approach for iot-based enterprises, IEEE Access, 7 (2019), 59559-59574.
15. Y.B. Jun, E.H. Roh and Y. xu, LI-ideals in lattice implication algebras, Bulletin of the Korean Mathematical Society, 35(1) (1998), 13–24.
16. M.A. Öztürk and Y.B. Jun, Neutrosophic ideals in BCK/BCI-algebras based on neutrosophic points, Journal of the International Mathematical Virtual Institute, 8 (2018), 1–17.
17. S.Z. Song, M. Khan, F. Smarandache and Y.B. Jun, A novel extension of neutrosophic sets and its application in BCK/BI-algebras,(2),(2018)308-318
18. S.Z. Song, F. Smarandache and Y.B. Jun, Neutrosophic commutative $\mathcal{N}$-ideals in BCK-algebras, Information, 8 (2017), 130; doi:10.3390/info8040130.
19. Y. Xu, Homomorphisms in lattice implication algebras, Proc. of 5th Many-Valued Logical Congress of China, (1992), 206–211.

20. Y. Xu, Lattice implication algebras, Journal of Southwest Jiaotong University, 1 (1993), 20–27.

21. Y. Xu, D. Ruan, K.Y. Qin and J. Liu, Lattice-valued logic, Studies in Fuzzyness and Soft Computing, Vol. 132, Springer-Verlag, Berlin Heidelberg, New York, 2003.

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Introduction to neutrosophic soft topological spatial region

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Abstract. Spatial information often deals with regions which are vague or incompletely determined. Understanding vagueness, indeterminacy and imprecision are the most important in GIS. Smarandache's neutrosophic set is a computational method to tackle problems involving incomplete, infinite and reliable data. The definition of soft sets was introduced by Molodtsov as a new mathematical method to tackle uncertainty. Maji presented the Neutrosophic Soft Set theory. This paper provides concepts of a neurtrosophic soft spatial region for its possible application in GIS. The notions of neutrosophic soft α-open, neutrosophic soft pre-open, neutrosophic soft semi-open and neutrosophic soft β-open sets are introduced.

Keywords: Neutrosophic soft set; neutrosophic soft topology; neutrosophic soft connected; neutrosophic soft spatial region; GIS.

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1. Introduction

Many real-life issues deal with uncertainties in economics, engineering, environment, social sciences, medical sciences, and business management. There are difficulties with classical mathematical modeling in solving the uncertainties in these data. Theories such as fuzzy set[1], rough set[2] and intuitionist fuzzy set[3] are used to prevent difficulties in dealing with uncertainty. But all of these hypotheses have some difficulties in addressing the indeterminate or contradictory data problems. Smarandache[4] described the neutrosophical set as a mathematical method for dealing with indeterminate and inaccurate problems in nature. There is a lot of use in all fields, such as IT, information systems and decision support systems.

Abdel-Basset[5] has developed a Novel Intelligent Medical Decision Support Model based on soft computing and IoT as the use of neutrosophical sets for decision-making. In[6] the researchers developed neutrosophic multi-criterion approach to help healthcare professionals predict illness. In[7] a solution is proposed to Neutrosophic Linear Fractional Programming Problem (NLFP) in the case of triangular neutrosophic number costs of the objective function, capital and engineering coefficients. In[8] the researchers suggest the method to help the patient and doctor know whether the patient is having a heart failure through neutrophic multi-criteria decision making (NMCDM).

The neutrosophical topological space theory was proposed in [9]. Further neutrosophic topological space was studied in [10]. Subsequently, the sets were added similar to the neutrosophic open and neutrosophic closed sets. Neutrosophic semi-open set[NSO] and neutrosophic semi-closed sets[NSC] have been introduced by Iswaraya et.al.[11]. Imran et.al.[12] proposed neutrosophic semi-α open sets and analysed their basic properties. Arokiarani et.al.[13] studied about neutrosophic semi-open (resp. pre-open and α-open) functions and examined their relations. Rao et.al.[14] proposed neutrosophic pre-open sets.

In [15] the researchers investigate new kind of neutrosophic continuity in neutrosophic topological spaces known as Neutrosophicαgs continuity maps and also the properties and characterization Neutrosophic αgs Irresolute Maps were examined. Anitha et.al.[16] proposed the concept of NGSR-closed sets and NGSR-open sets. NGSR continuous and NGSR-contra continuous mappings are also further studied. Dhavaseelan et.al. [17] introduced neutrosophic almost α-contra-continuous function and studied their properties. In [18] the authors introduced neutrosophic generalized b–closed sets and Neutrosophic generalized b-continuity in Neutrosophic topological spaces.

Molodstov[19] introduced the soft set theory as a computational method for tackling insecurity. Maji[20] combined the concept of soft set and neutrosophic set together by introducing the current mathematical framework called neutrosophic soft set. In[21] neutrosophic soft set was applied in making decision. Several researchers[22, 23, 24, 25, 26] applied in various mathematical systems the concept of neutrosophic soft sets. Bera[27] introduced neurosophic soft topological spaces. Neutrosophic spatial region as introduced by A.A.Salama[28]. This paper explores the theory and some of its features of neutrosophic soft topological space. The notions of neutrosophic soft α-open, neutrosophic soft pre-open, neutrosophic soft semi-open and neutrosophic soft β-open sets are introduced. Furthermore, for possible application in GIS, the simple neutrosophic soft region is introduced.

2. Preliminaries

Definition 2.1. ([19]). (F, E) is a soft set in X where $F : E \to \mathcal{P}(Y)$ is a mapping where $\mathcal{P}(Y)$ is a power set of Y. We express (F, E) by $\widetilde{F}$. $\widetilde{F} = \{(e, F(e)) : e \in E\}$.

Definition 2.2. ([4]). A neutrosophic set(NS) A on Y is defined as: $A = \{< y, T_A(y), I_A(y), F_A(y) >: y \in Y\}$ where $T, I, F : Y \longrightarrow]^{-}0, 1^{+}[$ and $-0 \leq T_A(y) + I_A(y) + F_A(y) \leq 3^{+}$

Definition 2.3. Let Y be an set and E be parameter set. Let $\mathcal{P}(Y)$ denotes the set of all neutrosophic soft set(NSS) of Y. Then (F,E) is called a NSS over Y where $F : E \to \mathcal{P}(Y)$ is a mapping. We express the NSS (F, E) by $\widetilde{F}_N$.

That is, $\widetilde{F}_N = \{(e, \{< y, T_{\widetilde{F_{N(e)}}}(y), I_{\widetilde{F_{N(e)}}}(y), F_{\widetilde{F_{N(e)}}}(y) >: y \in Y\})e \in E\}$

Definition 2.4. The complement of the NSS $\widetilde{F}_N$ is denoted by $(\widetilde{F}_N)^c$ and is defined by $\widetilde{F}_N^c = \{(e, \{< y, F_{\widetilde{F_{N(e)}}}(y), I_{\widetilde{F_{N(e)}}}(y), T_{\widetilde{F_{N(e)}}}(y) >: y \in Y\})e \in E\}$

Definition 2.5. For any two NSS $\widetilde{F}_N$ and $\widetilde{G}_N$ over Y, $\widetilde{F}_N$ is a neutrosophic soft subset of $\widetilde{G}_N$ if $T_{\widetilde{F_{N(e)}}}(y) \leq T_{\widetilde{G_{N(e)}}}(y)$; $I_{\widetilde{F_{N(e)}}}(y) \leq I_{\widetilde{G_{N(e)}}}(y)$; $F_{\widetilde{F_{N(e)}}}(y) \geq F_{\widetilde{G_{N(e)}}}(y)$; for all $e \in E$ and $y \in Y$.

Definition 2.6. A NSS $\widetilde{F}_N$ over Y is said to be null NSS if $T_{\widetilde{F_{N(e)}}}(y) = 0$; $I_{\widetilde{F_{N(e)}}}(y) = 0$; $F_{\widetilde{F_{N(e)}}}(y) = 1$; for all $e \in E$ and $y \in Y$. It is denoted by $\widetilde{\Phi}_N$.

Definition 2.7. A NSS $\widetilde{F}_N$ over Y is said to be absolute NSS if $T_{\widetilde{F_{N(e)}}}(y) = 1$; $I_{\widetilde{F_{N(e)}}}(y) = 1$; $F_{\widetilde{F_{N(e)}}}(y) = 0$; for all $e \in E$ and $y \in Y$. It is denoted by $\widetilde{Y}_N$

Definition 2.8. The union of two NSS $\widetilde{F}_N$ and $\widetilde{G}_N$ is denoted by $\widetilde{F}_N \cup \widetilde{G}_N$ and is defined by $\widetilde{H}_N = \widetilde{F}_N \cup \widetilde{G}_N$, where the truth-membership, indeterminacy-membership and falsity membership of $\widetilde{H}_N$ are as follows

$$T_{\widetilde{H_{N(e)}}}(y) = \begin{cases} T_{\widetilde{F_{N(e)}}}(y) & \text{if } e \in A - B \\ T_{\widetilde{G_{N(e)}}}(y) & \text{if } e \in B - A \\ max\{T_{\widetilde{F_{N(e)}}}(y), T_{\widetilde{G_{N(e)}}}(y)\} & \text{if } e \in A \cap B \end{cases}$$

$$I_{\widetilde{H_{N(e)}}}(y) = \begin{cases} I_{\widetilde{F_{N(e)}}}(y) & \text{if } e \in A - B \\ I_{\widetilde{G_{N(e)}}}(y) & \text{if } e \in B - A \\ \frac{I_{\widetilde{F_{N(e)}}}(y) + I_{\widetilde{G_{N(e)}}}(y)\}}{2} & \text{if } e \in A \cap B \end{cases}$$

$$F_{\widetilde{H_{N(e)}}}(y) = \begin{cases} F_{\widetilde{F_{N(e)}}}(y) & \text{if } e \in A - B \\ F_{\widetilde{G_{N(e)}}}(y) & \text{if } e \in B - A \\ min\{F_{\widetilde{F_{N(e)}}}(y), F_{\widetilde{G_{N(e)}}}(y)\} & \text{if } e \in A \cap B \end{cases}$$

Definition 2.9. The intersection of two NSS $\widetilde{F}_N$ and $\widetilde{G}_N$ is denoted by $\widetilde{F}_N \cap \widetilde{G}_N$ and is defined by $\widetilde{H}_N = \widetilde{F}_N \cap \widetilde{G}_N$, where the truth-membership, indeterminacy-membership and falsity membership of $\widetilde{H}_N$ are as follows

$$T_{\widetilde{H_{N(e)}}}(y) = min\{T_{\widetilde{F_{N(e)}}}(y), T_{\widetilde{G_{N(e)}}}(y)\},$$
$$I_{\widetilde{H_{N(e)}}}(y) = \frac{I_{\widetilde{F_{N(e)}}}(y) + I_{\widetilde{G_{N(e)}}}(y)\}}{2},$$
$$F_{\widetilde{H_{N(e)}}}(y) = may\{F_{\widetilde{F_{N(e)}}}(y), F_{\widetilde{G_{N(e)}}}(y)\}$$

3. Neutrosophic soft topological space

Definition 3.1. Let $NSS(Y, E)$ be the family of all NSS over Y and $\widetilde{\tau}_N \subset NSS(Y, E)$. Then $\widetilde{\tau}_N$ is called neutrosophic soft topology(NST) on (Y, E) if the following conditions are satisfied:

(i) $\widetilde{\Phi}_N, \widetilde{Y}_N \in \widetilde{\tau}_N$

(ii) $\widetilde{\tau}_N$ is closed under arbitrary union.

(iii) $\widetilde{\tau}_N$ is closed under finite intersection.

Then the triplet $(Y, \widetilde{\tau}_N, E)$ is called neutrosophic soft topological space(NSTS). The members of $\widetilde{\tau}_N$ are called neutrosophic soft open sets in $(Y, \widetilde{\tau}_N, E)$. A NSS $\widetilde{F}_N$ in $NSS(Y, E)$ is soft closed in $(Y, \widetilde{\tau}_N, E)$ if its complement $(\widetilde{F}_N)^c$ is neutrosophic soft open set in $(Y, \widetilde{\tau}_N, E)$.

The neutrosophic soft closure of $\widetilde{F}_N$ is the NSS, $Nscl(\widetilde{F}_N) = \cap\{\widetilde{G}_N : \widetilde{G}_N$ is neutrosophic soft closed and $\widetilde{F}_N \subseteq \widetilde{G}_N\}$.

The neutrosophic soft interior of $\widetilde{F}_N$ is the NSS, $Nsint(\widetilde{F}_N) = \cup\{\widetilde{O}_N : \widetilde{O}_N$ is neutrosophic soft closed and $\widetilde{O}_N \subseteq \widetilde{F}_N\}$.

It is easy to see that $\widetilde{F}_N$ is neutrosophic soft open if and only if $\widetilde{F}_N = Nsint(\widetilde{F}_N)$ and neutrosophic soft closed if and only if $\widetilde{F}_N = Nscl(\widetilde{F}_N)$.

Theorem 3.2. *Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS over (Y, E) and $\widetilde{F}_N$ and $\widetilde{G}_N \in NSS(Y, E)$ then*

(i) $Nsint(\widetilde{F}_N) \subset \widetilde{F}_N$ *and* $Nsint(\widetilde{F}_N)$ *is the largest open set.*

(ii) $\widetilde{F}_N \subset \widetilde{F}_N$ *implies* $Nsint(\widetilde{F}_N) \subset Nsint(\widetilde{F}_N)$

(iii) $Nsint(\widetilde{F}_N)$ *is an neutrosophic soft open set. That is* $Nsint(\widetilde{F}_N) \in \widetilde{\tau}_N$

(iv) $\widetilde{F}_N$ *is neutrosophic soft open iff* $Nsint(\widetilde{F}_N) = \widetilde{F}_N$

(v) $Nsint(Nsint(\widetilde{F}_N)) = Nsint(\widetilde{F}_N)$

(vi) $Nsint(\widetilde{\Phi}_N) = \widetilde{\Phi}_N$ *and* $Nsint(\widetilde{Y}_N) = \widetilde{Y}_N$

(vii) $Nsint(\widetilde{F}_N \cap \widetilde{G}_N) = Nsint(\widetilde{F}_N) \cap Nsint(\widetilde{G}_N)$

(viii) $Nsint(\widetilde{F}_N) \cup Nsint(\widetilde{G}_N) \subset Nsint(\widetilde{F}_N \cup \widetilde{G}_N)$

Theorem 3.3. *Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS (Y, E) and $\widetilde{F}_N$ and $\widetilde{G}_N \in NSS(Y, E)$ then*

(i) $\widetilde{F}_N \subset Nscl(\widetilde{F}_N)$ *and* $Nscl(\widetilde{F}_N)$ *is the smallest closed set*

(ii) $\widetilde{F}_N \subset \widetilde{F}_N$ *implies* $Nscl(\widetilde{F}_N) \subset Nscl(\widetilde{F}_N)$

(iii) $Nscl(\widetilde{F}_N)$ *is neutrosophic soft closed set. That is* $Nscl(\widetilde{F}_N) \in (\widetilde{\tau}_N)^c$

(iv) $\widetilde{F}_N$ *is neutrosophic soft closed iff* $Nscl(\widetilde{F}_N) = \widetilde{F}_N$

(v) $Nscl(Nscl(\widetilde{F}_N)) = Nscl(\widetilde{F}_N)$

(vi) $Nscl(\widetilde{\Phi}_N) = \widetilde{\Phi}_N$ *and* $Nscl(\widetilde{Y}_N) = \widetilde{Y}_N$

(vii) $Nscl(\widetilde{F}_N \cup \widetilde{G}_N) = Nscl(\widetilde{F}_N) \cup Nscl(\widetilde{G}_N)$

(viii) $Nscl(\widetilde{F}_N) \cap Nscl(\widetilde{G}_N) \subset Nscl(\widetilde{F}_N \cap \widetilde{G}_N)$

4. Neutrosophic soft nearly open sets

Definition 4.1. Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS and $\widetilde{F}_N$ be a neutrosophic soft open set in (Y, E), then $\widetilde{F}_N$ is called

(i) Neutrosophic soft α-open iff $\widetilde{F}_N \subseteq Nsint(Nscl(Nsint(\widetilde{F}_N)))$

(ii) Neutrosophic soft pre-open iff $\widetilde{F}_N \subseteq Nsint(Nscl(\widetilde{F}_N))$

(iii) Neutrosophic soft semi-open iff $\widetilde{F}_N \subseteq Nscl(Nsint(\widetilde{F}_N))$

(iv) Neutrosophic soft β-open iff $\widetilde{F}_N \subseteq Nscl(Nsint(Nscl(\widetilde{F}_N)))$

(v) Neutrosophic soft regular-open iff $\widetilde{F}_N = Nsint(Nscl(\widetilde{F}_N))$

Definition 4.2. Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS and $\widetilde{F}_N \in NSS(Y, E)$, then $\widetilde{F}_N$ is called

(i) Neutrosophic soft α-closed iff $Nscl(Nsint(Nscl(\widetilde{F}_N))) \subseteq \widetilde{F}_N$

(ii) Neutrosophic soft pre-closed iff $Nscl(Nsint(\widetilde{F}_N)) \subseteq \widetilde{F}_N$

(iii) Neutrosophic soft semi-clsed iff $Nsint(Nscl(\widetilde{F}_N)) \subseteq \widetilde{F}_N$

(iv) Neutrosophic soft β-closed iff $Nsint(Nscl(Nsint(\widetilde{F}_N))) \subseteq \widetilde{F}_N$

(v) Neutrosophic soft regular-closed iff $\widetilde{F}_N = Nscl(Nsint(\widetilde{F}_N))$

5. Neutrosophic soft region

Topological relationships have played a significant role during space search, analysis and reasoning through Geographical information systems (GIS) and Geospatial databases. The topological relations between smooth, unstable and fuzzy spatial regions have been developed on the basis of the nine-intersection model. In the past couple of decades a lot of emphasis has been given to the topological relationship research issue, particularly between uncertain spatial regions. Nevertheless, formal representation and calculation of topological links between unknown regions remains an open issue and needs further investigation. We discuss further definitions and proposals for a neutrosophic soft topological region, which provide an theoretical framework for the modeling of neutrosophic soft topology relations among uncertain regions.

Definition 5.1. Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS over (Y, E) and $\widetilde{F}_N \in NSS(Y, E)$. Then neutrosophic soft boundary of $\widetilde{F}_N$ is defined by $\partial\widetilde{F}_N = Nscl(\widetilde{F}_N) \cap Nscl((\widetilde{F}_N)^c)$

Definition 5.2. Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS over (Y, E). Then the neutrosophic soft exterior of $\widetilde{F}_N \in NSS(Y, E)$ is denoted by $(\widetilde{F}_N)_o$ and is defined by $(\widetilde{F}_N)_o = Nsint((\widetilde{F}_N)^c)$

Theorem 5.3. *Let $\widetilde{F}_N$ and $\widetilde{G}_N$ be two NSS over (Y, E). Then*

(i) $(\widetilde{F}_N)_o = Nsint((\widetilde{F}_N)^c)$
(ii) $(\widetilde{F}_N \cup \widetilde{G}_N)_o = (\widetilde{F}_N)_o \cap (\widetilde{G}_N)_o$
(iii) $(\widetilde{F}_N)_o \cup (\widetilde{G}_N)_o \subset (\widetilde{F}_N \cap \widetilde{G}_N)_o$

Theorem 5.4. *Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS over (Y, E) and $\widetilde{F}_N$, $\widetilde{G}_N \in NSS(Y, E)$. Then*

(i) $(\partial\widetilde{F}_N)^c = Nsint(\widetilde{F}_N) \cup Nsint((\widetilde{F}_N)^c)$
(ii) $Nscl(\widetilde{F}_N) = Nsint(\widetilde{F}_N) \cup \partial\widetilde{F}_N$
(iii) $\partial\widetilde{F}_N = Nscl(\widetilde{F}_N) \cap Nscl((\widetilde{F}_N)^c)$
(iv) $\partial\widetilde{F}_N \cap Nsint(\widetilde{F}_N) = \widetilde{\Phi}_N$
(v) $\partial(\partial(\partial(\widetilde{F}_N))) = \partial(\partial(\widetilde{F}_N))$

Definition 5.5. Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS over (Y, E). Then a pair of non-empty neutrosophic soft open sets $\widetilde{F}_N$, $\widetilde{G}_N$ is called a neutrosophic soft separation of $(Y, \widetilde{\tau}_N, E)$ if $\widetilde{Y}_N = \widetilde{F}_N \cup \widetilde{G}_N$ and $\widetilde{F}_N \cap \widetilde{G}_N = \widetilde{\Phi}_N$

Definition 5.6. A NSTS $(Y, \widetilde{\tau}_N, E)$ is said to be neutrosophic soft connected if there does not exist a neutrsophic soft separation of $(Y, \widetilde{\tau}_N, E)$. Otherwise $(Y, \widetilde{\tau}_N, E)$ is said to be neutrosophic soft disconnected.

Now we shall describe a model for basic spatial neutrosophic soft region based on neutrosophic soft connectedness.

Definition 5.7. Let $(Y, \widetilde{\tau}_N, E)$ be a NSTS. A spatial neutrosophic soft region in (Y, E) is a non empty neutrosophic soft subset $\widetilde{F}_N$ such that

(i) $Nsint(\widetilde{F}_N)$ is neutrosophic soft connected.
(ii) $\widetilde{F}_N = Nscl(Nsint(\widetilde{F}_N))$

6. **Conclusion**

The neutrosophic soft4-intersection model can be implemented as an application to GIS for neutrosophic soft topological relationships between neutrosophic soft regions with sharp neutrosophical soft boundaries and for neutrosophic soft regions with broad neutrosophical soft boundaries. These models can be used to formulate spatial database consistency constraints and can also be used in information systems such as mobile robots and route navigation systems.

References

1. Zadeh, L.A. Fuzzy sets, Information and Control, 1965, 8(3), 338 - 353.
2. Pawlak, Z. Rough sets, Int J Parallel Prog, 1982 11(5), 341 - 356.
3. Atanassov, K. Intuitionstic fuzzy sets, Fuzzy set syst, 1986,20, 87-96.
4. Smarandache, F. Neutrosophic set- a generalization of the intuitionstic fuzzy set, International Journal of pure and applied mathematics, 2005, 24(3), 287-294.
5. Abdel-Basset, M.; Manogaran, G.; Gamal, A.; Chang, V. A Novel Intelligent Medical Decision Support Model Based on Soft Computing and IoT, IEEE Internet of Things Journal, 2019.
6. Abdel-Basset, M.; Mohamed, M. A novel and powerful framework based on neutrosophic sets to aid patients with cancer, Future Generation Computer Systems, 2019, 98, 144-153.
7. Abdel-Basset, M.; Mohamed, M.; Smarandache, F. Linear fractional programming based on triangular neutrosophic numbers, International Journal of Applied Management Science, 2019, 11(1), 1-20.
8. Abdel-Basset, M.; Gamal, A.; Manogaran, G.; Long, H. V. A novel group decision making model based on neutrosophic sets for heart disease diagnosis, Multimedia Tools and Applications, 2019, 1-26.
9. Salama, A.A. ; Alblowi, S.A. Neutrosophic set and neutrosophic topological space, ISOR J. mathematics, 2012, 3(4), 31 - 35.
10. Salama, A.; Florentin Smarandache ; ValeriKroumov. Neutrosophic Closed Set and Neutrosophic Continuous Functions, Neutrosophic Sets and Systems, 2014, (4), 4-8.
11. Iswarya, P. ; Bageerathi, K. On Neutrosophic semi-open sets in Neutrosophic topological spaces, International Journal of Mathematical Trends and Technology, 2016, 37(3), 214-223.
12. Imran, Q.H.; Smarandache, F.; Riad K Al-Hamido ; Dhavaseelan, R. On Neutrosophic semi alpha open sets, Neutrosophic Sets and Systems, 2017, (18), 37-42.
13. Arokiarani, I.; Dhavaseelan, R.; Jafari, S.; Parimala, M. On some new notations and functions in neutrosophic topological Spaces, Neutrosophic Sets and Systems, 2017, 16, 16-19.
14. Venkateswara Roa, V. ; Srinivasa Rao. Neutrosophic pre-open sets and pre-closed sets in Neutrosophic topology, International Journal of Chem Tech Research, 2017, 10(10), 449-458.
15. Banu priya, V.; and Chandrasekar, S. Neutrosophic αgs Continuity And Neutrosophic αgs Irresolute Maps, Neutrosophic Sets and Systems, 2019, 28, 162-170.
16. Anitha, S.; Mohana, K.; Florentin Smarandache On NGSR Closed Sets in Neutrosophic Topological Spaces, Neutrosophic Sets and Systems, 2019, 28, 171-178.
17. Dhavaseelan,R.; Hanif PAGE Neutrosophic Almost Contra α-Continuous Functions , Neutrosophic Sets and Systems, 2019, 29, 71-77.
18. Maheswari,C.; Chandrasekar, S. Neutrosophic gb-closed Sets and Neutrosophic gb-Continuity, Neutrosophic Sets and Systems, 2019, 29, 89-100.
19. Molodtsov, D. Soft set theory - first results, Computers and Mathematics with Applications, 1999, 37, 19 - 31.
20. Maji, P. K. Neutrosophic soft set, Annals of fuzzy mathematics and Informatics,, 2013, 5(1) 157-168.
21. Faruk Karaasla. Neutrosophic soft sets with applications in decision making, International Journal of information science and Intelligent system, 2015, 4(2), 1-20.

22. Deli, I.; Broumi, S. Neutrosophic soft relations and some properties, Ann. Fuzzy Math. Inform. 2015, 9(1), 169182
23. Deli, I.; Broumi, S. Neutrosophic soft matrices and NSM-decision making, J. Intell. Fuzzy Syst. 2015, 28(5), 22332241
24. Bera, T.; Mahapatra, N.K. On neutrosophic soft function, Ann. Fuzzy Math. Inform. 2016, 12(1), 101119
25. Broumi, S;, Smarandache, F. Intuitionistic neutrosophic soft set, J. Inf. Comput. Sci., 2013, 8(2), 130140
26. Maji, P.K. An application of weighted neutrosophic soft sets in decision making problem, Springer Proc.Math.Sat., 2015, 125, 215-223.
27. Bera, T.; Mahapatra, N.K. Introduction to neutrosophic soft topological spaces, Opsearch, DOI 10.1007/s 12597-017-0308-7
28. Salama. A.A.; Said Broumi; Alblowi. S.A. Introduction to Neutrosophic topological spatial region, possible applications to GIS topological rules, I.J.Information Engineering and Electronic Business, 2014, 6, 15-21.

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Comment on "A Novel Method for Solving the Fully Neutrosophic Linear Programming Problems: Suggested Modifications"

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Abstract. Some clarifications of a previous paper with the same title are presented here to avoid any reading conflict [1]. Also, corrections of some typo errors are underlined. Each modification is explained with details for making the reader able to understand the main concept of the paper. Also, some suggested modifications advanced by Singh et al. [3] (Journal of Intelligent & Fuzzy Systems, 2019, DOI:10.3233/JIFS-181541) are discussed. It is observed that Singh et al. [3] have constructed their modifications on several mathematically incorrect assumptions. Consequently, the reader must consider only the modifications which are presented in this research.

1. Clarifications and Corrected Errors

In Section 5 and Step 3 of the proposed NLP method [1], the trapezoidal neutrosophic number was presented in the following form:
$\tilde{a}=\langle(a^l, a^{m1}, a^{m2}, a^u); T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}\rangle$,
where a^l, a^{m1}, a^{m2}, a^u are the lower bound, the first and second median values and the upper bound for trapezoidal neutrosophic number, respectively. Also, $T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}$ are the truth, indeterminacy and falsity degrees of the trapezoidal neutrosophic number. The ranking function for that trapezoidal neutrosophic number is as follows:

$$R(\tilde{a}) = \left|\left(\frac{-\frac{1}{3}(3a^l-9a^u)+2(a^{m1}-a^{m2})}{2}\right) \times (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})\right| \qquad (8)$$

The previous ranking function is only for maximization problems.
But, if NLP problem is a minimization problem, then ranking function for that trapezoidal neutrosophic number is as follows:

$$R(\tilde{a}) = \left|\left(\frac{(a^l+a^u)-3(a^{m1}+a^{m2})}{-4}\right) \times (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})\right| \qquad (9)$$

If reader deals with a symmetric trapezoidal neutrosophic number which has the following form:
$\tilde{a}=\langle(a^{m1}, a^{m2}); \alpha, \beta\rangle$,
where $\alpha = \beta,\ \alpha, \beta \geq 0$, then the ranking function for that number will be as follows:

$$R(\tilde{a}) = \left|\left(\frac{(a^{m1}+a^{m2})+2(\alpha+\beta)}{2}\right) \times (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})\right|. \qquad (10)$$

We applied Eq. (10) directly in Example 1, but we did not illustrated it in the original work [1], and this caused a reading conflict. After handling typo errors in Example 1, the crisp model of the problem will be as follows:

Maximize $Z = 18x_1 + 19x_2 + 20x_3$
Subject to
$12x_1 + 13x_2 + 12x_3 \leq 502$,
$14x_1 + 13x_3 \leq 486$,
$12x_1 + 15x_2 \leq 490$,
$x_1, x_2, x_3 \geq 0$.

The initial simplex form will be as in Table 1.

Table 1 Initial simplex form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
s_4	12	13	12	1	0	0	502
s_5	14	0	13	0	1	0	486
s_6	12	15	0	0	0	1	490
Z	-18	-19	-20	0	0	0	0

The optimal simplex form will be as in Table 2.

Table 2 Optimal form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
x_2	-12/169	1	0	1/13	-12/169	0	694/169
x_3	14/13	0	1	0	1/13	0	486/13
s_6	2208/169	0	0	-15/13	180/169	1	72400/169
Z	370/169	0	0	19/13	32/169	0	139546/169

The obtained optimal solution is $x_1 = 0, x_2 = 4.11, x_3 = 37.38$.
The optimal value of the NLPP is $\tilde{z} \approx (13,15,2,2)x_1 + (12,14,3,3)x_2 + (15,17,2,2)x_3 = (13,15,2,2) * 0 + (12,14,3,3) * 4.11 + (15,17,2,2) * 37.38 =$
$(49.32,57.54,12.33,12.33) + (560.70,635.46,74.76,74.7) = (610.02,693,87.09,87.09)$.
$\tilde{z} \approx (610.02,693,87.09,87.09)$, which is in the symmetric trapezoidal neutrosophic number form. Since the traditional form of $\tilde{a} = \langle(a^{m1}, a^{m2}); \alpha, \beta\rangle$ is:
$\tilde{a} = \langle(a^{m1} - \alpha, a^{m1}, a^{m2}, a^{m2} + \beta)\rangle$,
where $a^{m1} - \alpha = a^l$, $a^{m2} + \beta = a^u$, then the optimal value of the NLPP can also be written as $\tilde{z} \approx (522.93,610.02,693,780.09)$.

The reader must also note that one can transform the symmetric trapezoidal neutrosophic numbers from Example 1 in [1] to its traditional form, and use Eq. (8) for solving the problem, obtaining the same result. By comparing the result with other existing models mentioned in the original research [1], the proposed model is the best.
By using Eq. (8) and solving Example 2 in [1], the crisp model will be as follows:
Maximize $Z = 25x_1 + 48x_2$
Subject to

$13x_1 + 28x_2 \leq 31559$,
$26x_1 + 9x_3 \leq 16835$,
$21x_1 + 15x_2 \leq 19624$,
$x_1, x_2 \geq 0$.

The initial simplex form will be as in Table 3.

Table 3 Initial simplex form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
s_3	13	28	1	0	0	31559
s_4	26	9	0	1	0	16835
s_5	21	15	0	0	1	19624
Z	-25	-48	0	0	0	0

The optimal simplex form will be as in Table 4.

Table 4 Optimal simplex form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
x_2	0	1	7/131	0	-13/393	407627/393
s_4	0	0	67/131	1	-611/393	969250/393
x_1	1	0	-5/131	0	28/393	76087/393
Z	0	0	211/131	0	76/393	21468271/393

The optimal value of objective function is 54627.
By using Eq. (9) and solving Example 3 in [1], the crisp model will be as follows:
Minimize $Z = 6x_1+10x_2$
Subject to
$2x_1+5x_2 \geq 6$,
$3x_1+4x_2 \geq 3$,
$x_1, x_2 \geq 0$.
The optimal simplex form will be as in Table 5.

Table 5 Optimal simplex form

Basic variables	x_1	x_2	s_3	s_4	RHS
s_4	-7/5	0	-4/5	1	0
x_2	2/5	1	-1/5	0	10
Z	-2	0	-2	0	12

Hence, the optimal solution has the value of variables:
$x_1 = 0, x_2 = 1.2$, Z = 12.
The obtained result is better than Saati et al. [2] method.
By correcting typo errors which percolated in the Case study in [1], the problem formulation model will be as follows:
Maximize $\tilde{Z} = \tilde{9}x_1+\widetilde{12}x_2+\widetilde{15}x_3+\widetilde{11}x_4$
Subject to

$0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \leq \widetilde{1500}$,
$3x_1 + x_2 + 2x_3 + 3x_4 \leq \widetilde{2350}$,
$2x_1 + 4x_2 + x_3 + 2x_4 \leq \widetilde{2600}$,
$0.5x_1 + 1x_2 + 0.5x_3 + 0.5x_4 \leq \widetilde{1200}$,
$x_1 \leq \widetilde{150}$,
$x_2 \leq \widetilde{100}$,
$x_3 \leq \widetilde{300}$,
$x_4 \leq \widetilde{400}$,
$x_1, x_2, x_3, x_4 \geq 0$.

The values of each trapezoidal neutrosophic number remain the same [1].
By using Eq. (8) and solving the Case study, the crisp model will be as follows:
Maximize $\tilde{Z} = 10x_1+10x_2+12x_3+9x_4$
Subject to

$0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \leq 1225$,
$3x_1 + x_2 + 2x_3 + 3x_4 \leq 1680$,
$2x_1 + 4x_2 + x_3 + 2x_4 \leq 2030$,
$0.5x_1 + 1x_2 + 0.5x_3 + 0.5x_4 \leq 945$,
$x_1 \leq 122$,
$x_2 \leq 87$,
$x_3 \leq 227$,
$x_4 \leq 297$,
$x_1, x_2, x_3, x_4 \geq 0$.

By solving the previous model using simplex approach, the results are as follows:
$x_1 = 122, x_2 = 87, x_3 = 227, x_4 = \frac{773}{3}, Z = 7133.$

2. A Note on the modifications suggested by Singh et al. [3]

This part illustrates how Singh et al. [3] constructed their modifications of Abdel-Basset et al.'s method [1] on wrong concepts. The errors in Singh et al.'s [3] modifications reflects the misunderstanding of Abdel-Basset et al.'s method [1].

In the second paragraph of the introductory section, Singh et al. [3] assert that "in Abdel-Basset et al.'s method [1], firstly, a neutrosophic linear programming problem (NLPP) is transformed into a crisp linear programming problem (LPP) by replacing each parameter of the NLPP, represented by a trapezoidal neutrosophic number with its equivalent defuzzified crisp value". However, this is not true, since the neutrosophic linear programming problem (NLPP) is transformed into a crisp linear programming problem (LPP) by replacing each parameter of the NLPP, represented by a trapezoidal neutrosophic number with its equivalent deneutrosophic crisp value. The deneutrosophication process means transforming a neutrosophic value to its equivalent crisp value.
In Section 2, Step 1 Singh et al. [3] alleged that Abdel-Basset et al.'s method [1] for comparing two trapezoidal neutrosophic numbers is based on maximization and minimization of problem, which is again not true.
In Section 3 and Definition 4, Abdel-Basset et al. [1] illustrated that the method for comparing two trapezoidal neutrosophic numbers is as follows:

1. If $R(\tilde{A}) > R(\tilde{B})$ then $\tilde{A} > \tilde{B}$,
2. If $R(\tilde{A}) < R(\tilde{B})$ then $\tilde{A} < \tilde{B}$,
3. If $R(\tilde{A}) = R(\tilde{B})$ then $\tilde{A} = \tilde{B}$.

There is well known that if $a^l = a^{m1} = a^{m2} = a^u$, then the trapezoidal number $\tilde{a}$=$\langle (a^l, a^{m1}, a^{m2}, a^u); 1, 0, 0\rangle$ will be transformed into a real number $a = \langle (a, a, a, a); 1, 0, 0\rangle$, and hence in this case $R(a) = a$. We presented this fact to illustrate a great error in the suggested modifications of Singh et al. [3]
In the Suggested modifications section [3], the authors claimed that:

$$R\left(\sum_{i=1}^{m}\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right) = \sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle - \sum_{i=1}^{m} T_{\tilde{a}_i}$$
$$+ \textstyle\sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\le j\le n}\{T_{\tilde{c}_i}\} - max_{1\le j\le n}\{I_{\tilde{c}_i}\} - max_{1\le j\le n}\{F_{\tilde{c}_i}\} \quad (11)$$

instead of ,
$R\left(\sum_{i=1}^{m}\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right)$ =$\sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle$.

Let us consider the following example for proving the error in this suggestion [3]
Let $m = 3$, which are three trapezoidal neutrosophic numbers $\tilde{a}_1, \tilde{a}_2, \tilde{a}_3$; since $\tilde{a}_1$=$\langle (1, 1,1,1); 1, 0, 0\rangle$, $\tilde{a}_2 = \langle (2, 2,2,2); 1, 0, 0\rangle$, $\tilde{a}_3$= $\langle (3, 3,3,3); 1, 0, 0\rangle$, then,
$R\left(\sum_{i=1}^{m}\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle\right) = R(\langle (1, 1,1,1); 1, 0, 0\rangle + \langle (2, 2,2,2); 1, 0, 0\rangle + \langle (3, 3,3,3); 1, 0, 0\rangle)$ $= R(\langle (6, 6,6,6); 1, 0, 0\rangle)$, and according to the previously determined fact "if $a^l = a^{m1} = a^{m2} = a^u$ then the trapezoidal number $\tilde{a} = \langle (a^l, a^{m1}, a^{m2}, a^u); 1, 0, 0\rangle$ will be transformed into a real number $a = \langle (a, a, a, a); 1, 0, 0\rangle$ and hence in this case $R(a) = a$ ", the value of $R(\langle (6, 6,6,6); 1, 0, 0\rangle) = 6$.

And by calculating the right hand side of Eq. (11), which is $\sum_{i=1}^{m} R\,\langle a_i^l, a_i^{m1}, a_i^{m2}, a_i^u, T_{\tilde{a}_i}, I_{\tilde{a}_i}, F_{\tilde{a}_i}\rangle - \sum_{i=1}^{m} T_{\tilde{a}_i} + \sum_{i=1}^{m} I_{\tilde{a}_i} + \sum_{i=1}^{m} F_{\tilde{a}_i} + min_{1\le j\le n}\{T_{\tilde{c}_i}\} - max_{1\le j\le n}\{I_{\tilde{c}_i}\} - max_{1\le j\le n}\{F_{\tilde{c}_i}\}$, we note that,
$R\langle (1, 1,1,1); 1, 0, 0\rangle + R\langle (2, 2,2,2); 1, 0, 0\rangle + R\langle (3, 3,3,3); 1, 0, 0\rangle - 3 + 0 + 0 + 1 - 0 - 0 = \quad 1 + 2 + 3 - 3 + 0 + 0 + 1 - 0 - 0 = 4.$

And then, the left hand side of Eq. (11) does not equal the right hand side, i.e. $6 \neq 4$.
Consequently, the authors [3] built their suggestions on a wrong concept.

Beside Eq. (11), the authors [3] used the expressions $R(a) = 3a + 1$ for maximization problems, and $R(a) = -2a + 1$ for minimization problems, and this shows peremptorily that their assumptions are scientifically incorrect.

There is also a repeated error in all corrected solutions suggested by Singh et al. [3] which contradicts with the basic operations of trapezoidal neutrosophic numbers. This error is iterated in Section 7, as in Example 1, in Step 6. Singh et al. [3] illustrated that the optimal value of the NLPP is calculated using the optimal solution obtained in Step 5 as follows:
$(11,13,15,17)x_1 + (9,12,14,17)x_2 + (13,15,17,19)x_3 = (11,13,15,17) * 0 + (9,12,14,17) * 0 + (13,15,17,19) * (\frac{245}{18}) = 13\,(\frac{245}{18}) + 15\,(\frac{245}{18}) + 17\,(\frac{245}{18}) + 19\,(\frac{245}{18}) = \frac{7840}{9}$, and because the basic operation of multiplying trapezoidal neutrosophic number by a constant value is as follows:
$\gamma\tilde{a} = \begin{cases} \langle(\gamma a_1, \gamma a_2, \gamma a_3, \gamma a_4); \mathrm{T}_{\tilde{a}}, \mathrm{I}_{\tilde{a}}, \mathrm{F}_{\tilde{a}}\rangle \; if\,(\gamma \geq 0) \\ \langle(\gamma a_4, \gamma a_3, \gamma a_2, \gamma a_1); \mathrm{T}_{\tilde{a}}, \mathrm{I}_{\tilde{a}}, \mathrm{F}_{\tilde{a}}\rangle \; if\,(\gamma < 0) \end{cases}$, then the value of $(11,13,15,17) * 0 + (9,12,14,17) * 0 + (13,15,17,19) * (\frac{245}{18}) = \left(\frac{3185}{18}, \frac{1225}{6}, \frac{4165}{18}, \frac{4655}{18}; 1,0,0\right)$.Then the optimal value of the NLPP is$\tilde{z} \approx = \left(\frac{3185}{18}, \frac{1225}{6}, \frac{4165}{18}, \frac{4655}{18}\right)$.

The same error appears in Example 4, where the optimal value of the NLPP is calculated by Singh et al. [3] using the optimal solution obtained in Step 5 as follows:
$(6,8,9,12)x_1(9,10,12,14)x_2 + (12,13,15,17)x_3 + (8,9,11,13)x_4 = (6,8,9,12)(\frac{3700}{21}) + (9,10,12,14)(0) + (12,13,15,17)(\frac{6200}{7}) + (8,9,11,13)(0) = 6(\frac{3700}{21}) + 8\left(\frac{3700}{21}\right) + 9\left(\frac{3700}{21}\right) + 12(\frac{3700}{21}) + 12\left(\frac{6200}{7}\right) + 13\,(\frac{6200}{7}) + 15(\frac{6200}{7}) + 17(\frac{6200}{7}) = \frac{1189700}{21}$, which is scientifically incorrect and reflects only the weak background of the authors in the neutrosophic field.
Therefore, we concluded that it is scientifically incorrect to use Singh et al.'s modifications [3].

3. Conclusions

Clarifications and corrections of some typo errors are presented here to avoid any reading conflict. Also, the correct results of NLPPs are presented. By using three modified functions for ranking process which were presented by Abdel-Basset et al. [1], the reader will be able to solve all types of linear programming problems with trapezoidal and symmetric trapezoidal neutrosophic numbers. Also, the mathematically incorrect assumptions used by Singh et al. [3] are discussed and rejected.

Conflict of interest

We declare that we do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted.

References

[1] Abdel-Basset, M., Gunasekaran, M., Mohamed, M., & Smarandache, F. (2019). A novel method for solving the fully neutrosophic linear programming problems. Neural Computing and Applications, 31(5), 1595-1605.
[2] Saati, S., Tavana, M., Hatami-Marbini, A., & Hajiakhondi, E. (2015). A fuzzy linear programming model with fuzzy parameters and decision variables. International Journal of Information and Decision Sciences, 7(4), 312-333.
[3] Singh, A., Kumar, A., & Appadoo, S. S. (2019). A novel method for solving the fully neutrosophic linear programming problems: Suggested modifications. Journal of Intelligent & Fuzzy Systems, (Preprint), 1-12.

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