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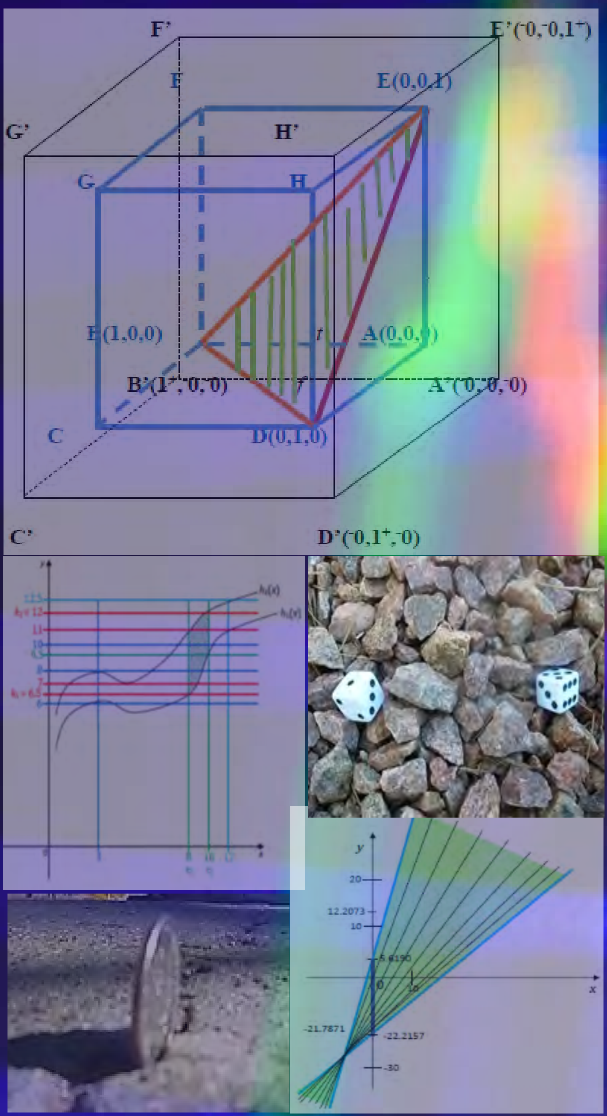

<A> <neutA> <antiA>

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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Algebraic Properties of Quasigroup Under $Q-$neutrosophic Soft Set

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Abstract. The novel concept called neutrosophic set was launched to take care of indeterminate factors in real-life data. The hybrid model of neutrosophic set and soft set has been widely studied in different areas of algebra, especially in associative structures such as fields, groups, rings, and modules. In this current paper, the novel concept is further introduce to a non-associative structure termed $Q-$neutrosophic soft quasigroup ($Q-$NS$\hat{G}$) and investigate its different algebraic properties of the quasigroups. We shown the conditions for the sets of $\alpha-$level cut of $Q-$NS$\hat{G}$ to be subquasigroups, the condition for each set of subquasigroups of a quasigroup to be $Q-$level cut neutrosophic soft subquasigroup were established. It was shown that $Q-$NS$\hat{G}$ obeys alternative property and flexible law. In addition, We defined $Q-$neutrosophic soft loop and investigate some of its characteristics. In particular, it was shown that $Q-$neutrosophic soft loop obeys inverse, weak inverse and cross inverse properties. We established the condition for a $Q-$neutrosophic soft loop to obey anti-automorphic inverse, semi-automorphic inverse and super anti-automorphic inverse properties. The necessary and sufficient condition for $Q-$neutrosophic soft set under a loop $(\hat{G}, \circ, /, \backslash)$ to be a $Q-$neutrosophic soft loop was also established.

Keywords: Q- set; Soft set; Neutrosophic set; Quasigroup; Loop.

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1. **Introduction**

Let $\hat{G}$ be a non-empty set and $(\circ)$ be an operation on $\hat{G}$. If $w \circ t \in \hat{G}$ for all $w, t \in G$, then $(\hat{G}, \circ)$ is called a *groupoid*. A groupoid $(\hat{G}, \circ)$ is called quasigroup, if there exist $a, b \in \hat{G}$ such that each of the equations:

$$a \circ w = b \qquad \text{and} \qquad t \circ a = b$$

have unique solution w, t respectively. Furthermore, the quasigroup is called a loop if there is a unique element $e \in \hat{G}$ called the *identity element* such that $\forall\ w \in G$,

$$w \circ e = e \circ w = w$$

In what follows, wt is written instead of $w \circ t$, which stipulates that $\circ$ has lower priority than juxtaposition amongst factors to be multiplied. For example we write, $p \circ qr$ stands for $p(qr)$.

Suppose that w is a fixed element in the groupoid $(\hat{G}, \circ)$, a translation map of $w \in \hat{G}$, called the left(right) translation maps written as L_w and R_w respectively are defined as

$$tL_w = w \circ t \qquad \text{and} \qquad tR_w = t \circ w.$$

Obviously, it implies that if the left and right translations maps are permutations, then a groupoid $(\hat{G}, \circ)$ is a quasigroup. And if the left and right translation maps of a quasigroup are bijections, it means that the inverse mappings L_w^{-1} and R_w^{-1} exist. Let

$$w\backslash t = tL_w^{-1} \qquad \text{and} \qquad w/t = wR_t^{-1}$$

and note that

$$w\backslash t = z \Leftrightarrow w \circ z = t \qquad \text{and} \qquad w/t = z \Leftrightarrow z \circ t = w.$$

Consequently, $(\hat{G}, \backslash)$ and $(\hat{G}, /)$ are also quasigroups.

A consideration of Fuzzy set was first initiated by Zadeh in [2], and the notion was designed to handle the challenges of uncertainty in real life data while the generalization of fuzzy set was considered by Atanassov in [4, 6] which is called intuitionistic fuzzy set. In 1971, Rosenfeld [5] for the time considered the concept of fuzzy set under the theoretical study of a group structure and established different properties and conditions for a subset of fuzzy set defined under a groups to be fuzzy subgroup. Since them, the concept has been extended to different field in mathematics. As away of generalizing the work in [5], the fuzzification of quasigroup was initiated by Dudek in 1998 [23] while 1999, Dudek and Jun [24] introduced fuzzy subquasigroup under norms to further the results in [23]. In 2000, the consideration of intuitionistic fuzzy set in a quasigroup was studied by Kyung et al. [27] as an extended method of fuzzy subquasigroup. In [23], research on intuitionistic fuzzy subquasigroup was furthered studied by Dudek [28] in 2005. It was revealed in [3] that each of these notions and their hybrid methods has their respective limitations and difficulties, and to address some of those difficulties, Molodtsov [3] launched the notion of soft set. It was reported that the notion of soft set theory is a better method for handling problems involving uncertainty, incompatible and incomplete data. Although, the study of soft set theory is not suitable for characterizing the degree of membership values as in the case of intuitionistic fuzzy set. Also, the notion is not capable for handling problems involving indeterminate data and as a result of that, a generalized concept called neutrosophic set was called out by Smarandache in 1998 [14,15], as a mathematical notion for dealing with indeterminacy occurrence. Neutrosophic set is more complex and the only generalized concept of the classical set theory found in the literature for dealing with problems involving indeterminate. The character of the degree values of a neutrosophic set are represented by the true membership T, indeterminate membership I and falsity membership F.

The different methods of determining the indeterminate factors of neutrosophic set in real-life data have been widely applied in different area in mathematics and its related field. For example, the work of Jidid et al. in [8] applied neutrosophy concept to handle the product quality control on inspection assignment form while Dey and Ray in 2023 [9] used the concept to characterized the separation axioms of neutrosophic topological spaces. The concept were used in the area of operation research in management in [10].

The hybrid model of neutrosophic sets, especially the neutrosophic consideration of soft set structure has been widely and sporadically flagged by algebraist in the recent past, (see the following articles [7,11,13,33]). However, it is important to mention the efforts of Muhammad et al. [20] and Mumtaz et al. [19], where set components of neutrosophy study, was replicated using groupoids, groups and bigroups. Furthermore, in 2020 Oyem et al. [29] conducted algebraic characterization of soft quasigroup while the generalization of his study was considered in [30]. Most recently, a study pattern of $Q-$fuzzy groups and their hybrid methods was called out by Solairaju et al. [16] and Thirunemi and Solairaju [17]. Then, was later escalated to $Q-$neutrosophic soft group in 2020 [18] to handle indeterminate data. The extension of $Q-$NS group to $Q-$NS quasigroup was recently announced by Oyebo et al. [25], which by tradition a generalization of the former.

In this present research, results of fuzzy quasigroup and its generalizations studied in the following articles [23, 24, 27, 28] are extended to neutrosophic soft quasigroup of two universal sets. Since the definition of $Q-$neutrosophic soft quasigroup was flagged up by Oyebo et. al [25], the question whether the concept obeys the following algebraic properties of quasigroup such as left(right) alternative property LAP(RAP), and flexible law are not yet known for the best of our searching. The result on characterization of supremum and infimum of fuzzy quasigroup studied by Dudek were extended to $Q-$neutrosophic soft quasigroup by capturing the behavior of an indeterminate factor of two universal sets that was lacking in structure of fuzzy quasigroup and intuitionistic fuzzy quasigroup. In addition, this paper is for the time introduced the concept of $Q-$neutrosophic soft loop which is a $Q-$neutrosophic soft quasigroup with an identity element without associative property. Also, the work of Dudek [23,24] and the generalized version in [25] did not shown results on the algebraic characteristics of the following class of quasigroup called left inverse property (LIP), right inverse property

(RIP), cross inverse property (CIP), weak inverse property (WIP), automorphic inverse prperty (AIP), anti-automorphic inverse property (AAIP), semi-automorphic inverse property (SAIP) and super anti- automorphic inverse property loop (SAAIP). In order to close up the gap, we investigate whether $Q-$neutrosophic soft quasigroup obey the properties of quasigroup mentioned above. In addition, we also pay attention to the necessary and sufficient condition for $Q-$neutrosophic soft set under a loop $(G, \circ)$ to be $Q-$neutrosophic soft loop.

The table below shown some set structures studied in the literature with their respective characterizations and generalizations.

Table 1. Properties of some set theories

Set structures	Membership function	uncertainty	inconsistency	indeterminacy	sum of membership ≤	independence (i)/ dependence(d)
Fuzzy	✓	✓	✓	×	1	d
intuitionistic fuzzy	✓	✓	✓	×	1	d
Soft	×	✓	✓	×	not applicable	not applicable
Rough	not applicable	✓	✓	×	not applicable	not applicable
interval fuzzy	✓	✓		×	1	d
vaque set	✓	✓	✓	×	1	d
Pythagorean fuzzy	✓	✓	✓	×	1	d
Neutrosophy	✓	✓	✓	✓	3	i
Spherical fuzzy set	✓	✓	✓	×	1	d

2. Preliminaries

Definition 2.1. A quasigroup(loop) $(\hat{G}, \circ)$ is said to have

(1) LIP if $\exists$ a map $J_\lambda : u \mapsto u^\lambda$ such that $u^\lambda \circ uv = v$ for all $u, v \in \hat{G}$
(2) RIP if $\exists$ a map $J_\rho : u \mapsto u^\rho$ such that $uv \circ u^\rho = v$ for all $u, v \in \hat{G}$,
(3) RAP if $t \circ ww = tw \circ w$ for all $w, t \in \hat{G}$,
(4) LAP if $ww \circ t = w \circ wt$ for all $w, t \in \hat{G}$,
(5) flexible if $uv \circ u = u \circ vu$ for all $u, v \in \hat{G}$,
(6) IPL if it satisfies $wt \circ w^{-1} = t$ or $w^{-1} \circ tw = t$ for all $w, t \in \hat{G}$ and
(7) WIPL if it satisfies the identity $t \circ (wt)^{-1} = w^{-1}$ for all $w, t \in \hat{G}$

Definition 2.2. The following identities hold in a loop $(\hat{G}, \circ)$ it is called:

(1) AIPL if $(wt)^{-1} = w^{-1}t^{-1} \ \ \forall w, t \in \hat{G}$,
(2) an AAIPL if $(wt)^{-1} = t^{-1}w^{-1}$ for all $w, t \in \hat{G}, \ \forall w, t \in \hat{G}$
(3) a SAAIPL if $(w \circ tz)^{-1} = z^{-1} \circ \left(t^{-1}w^{-1}\right)$, for all $w, t, z \in \hat{G}$ [see [26]]
(4) a SAIPL if $(wt \circ w)^{-1} = w^{-1}t^{-1} \circ w^{-1}$, for all $w, t \in \hat{G}$

Theorem 2.3. *[32] Let $(\hat{G}, \circ)$ be a quasigroup and $\hat{G}$ be a non empty subset of $\hat{G}$. Then, $\hat{G}$ is a subquasigroup of $(\hat{G}, \circ)$ if and only if $(\hat{G}, \circ)$, $(\hat{G}, /)$ and $(\hat{G}, \backslash)$ are groupoids*

Definition 2.4. [32] Let $(\hat{G}, \circ)$ be a quasigroup and $\emptyset \neq H \subseteq \hat{G}$. Then, H is called subquasigroup of $\hat{G}$ if $(H, \circ)$ is a quasigroup. Also, suppose that D and E are non empty subsets of $\hat{G}$, then $D \circ E = \{d \circ e \mid d \in D, e \in E\}, D/E = \{d/e \mid d \in D, e \in E\}$ and $E \backslash D = \{e \backslash d \mid d \in D, e \in E\}$

Definition 2.5. Let $M = [0, 1]$ and S be a subset of M. Then: the supremum of S denoted by $\sup S$ is a number $\beta_0 \in [0, 1]$ satisfying the conditions

(1) β_0 is an upper bound for S;
(2) for all $\epsilon > 0$, the number $\beta_0 - \epsilon$ is not an upper bound for S

the infimum of S denoted by $\inf S$ is a number $\alpha_0 \in [0, 1]$ satisfying the conditions

(1) α_0 is an upper bound for S;
(2) for all $\epsilon > 0$, the number $\alpha_0 + \epsilon$ is not a lower bound for S

Definition 2.6. [3] Given a set M and a parameter set $\mathfrak{A}$ of M. If $F : \mathfrak{A} \to P(M)$, where $P(M)$ is power set of M then the pair $(F, \mathfrak{A})$ is called a soft set .

Definition 2.7. [15] Given a set M. A neutrosophic set Φ (NS) on M is an object of the form
$\Phi = \{\langle m, (T_\Phi(m), I_\Phi(m), F_\Phi(m))\rangle : m \in M\}$ and the membership degree is described by $T_\Phi, I_\Phi, F_\Phi : W \to]^{-}0, 1^{+}[$.

Definition 2.8. [7] Given a set M and $\mathfrak{A}$ parameter sets. A neutrosophic soft set $(\Phi, \mathfrak{A})$ is described as $(\Phi, \mathfrak{A}) = \{\langle w, (T_\Phi(m), I_\Phi(m), F_\Phi(m))\rangle : m \in M\}$

Definition 2.9. [1] Let W be a universe of discourse and Q be a non-empty set and $\mathfrak{A} \subset E$ be a set of parameters. Let $\rho^l QNS(W)$ be the set of all multi-Q-NSs on W with dimension $l = 1$. A pair $(\Phi^Q, \mathfrak{A})$ is called a $Q-$neutrosophic soft set $(Q - NSS)$ denoted by $(\Phi^Q, A) = \{(a, \Phi^Q(a)) : a \in \mathfrak{A}, \Phi^Q(a) \in \rho^l QNS(W)\}$ over W, where $\Phi^Q : A \to \rho^l QNS(W)$ is a map such that $\Phi^Q(a) = \emptyset$ if $a \notin A$.

3. Results

Definition 3.1. Suppose that $(\hat{G}, \circ, \backslash, /)$ is a quasigroup and $(\Phi^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft set over $(\hat{G}, \circ, \backslash, /)$. Then, $(\Phi^Q, \mathfrak{A})$ is called a $Q - NS\hat{G}$ of $\hat{G}$ if for all $a \in \mathfrak{A}, w_1, w_2 \in \hat{G}, v \in Q$ satisfies the following conditions

(1) $T_{\Phi^Q(a)}(w_1 * w_2, v) \geq \min\{T_{\Phi^Q(a)}(w_1, v), T_{\Phi^Q(a)}(w_2, v)\}$
(2) $I_{\Phi^Q(a)}(w_1 * w_2, v) \leq \max\{I_{\Phi^Q(a)}(w_1, v), I_{\Phi^Q(a)}(w_1, v)\}$

(3) $F_{\Phi^Q(a)}(w_1 * w_2, v) \leq \max\{F_{\Phi^Q(a)}(w_1, v), F_{\Phi^Q(a)}(w_2, v)\}$

where $* \in \{\circ, /, \backslash\}$

Definition 3.2. Let $(\Lambda^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over $\hat{G}$ such that there exist $\alpha, \beta, \gamma \in [0,1]$ with the restriction that $\alpha_Q + \beta_Q + \gamma_Q \leq 3$. Then, $(\Lambda^Q, \mathfrak{A})_{(\alpha,\beta,\gamma)}$ is $Q-$level soft set defined as

$$(\Lambda_Q, \mathfrak{A})_{(\alpha,\beta,\gamma)} = \{f_1 \in \hat{G}, v \in Q : T_{\Lambda^Q(a)}(f_1, v) \geq \alpha, I_{\Lambda^Q(a)}(f_1, v) \leq \beta, F_{\Lambda^Q(a)}(f_1, v) \leq \gamma\}$$

for all $a \in \mathfrak{A}$

Suppose that $\alpha = \beta = \gamma$ for any $\alpha \in [0,1]$ with $\alpha + \alpha + \alpha \leq 3$ such that $(\Lambda^Q, \mathfrak{A})_\alpha = \{f_1 \in \hat{G}, v \in Q : T_{\Lambda^Q(a)}(f_1, v) \geq \alpha, I_{\Lambda^Q(a)}(f_1, v) \leq \alpha, F_{\Lambda^Q(a)}(f_1, v) \leq \alpha\}$, then $(\Lambda^Q, \mathfrak{A})_\alpha$ is called α -level set of Λ.

In neutrosophic soft set, the set

$$T(\Lambda^Q, \alpha) = \{f_1 \in \hat{G}, v \in Q : \Lambda^Q(a)(f_1, v) \geq \alpha\},$$
$$F(\Lambda^Q, \alpha) = \{f_1 \in \hat{G}, v \in Q : \Lambda^Q(a)(f_1, v) \leq \alpha\} \text{ and}$$
$$I(\Lambda^Q, \alpha) = \{f_1 \in \hat{G}, v \in Q : \Lambda^Q(a)(f_1, v) \leq \alpha\}$$

are respectively called the truth, falsity and indeterminacy α-levels cut of Λ

Theorem 3.3. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over $\hat{G}$. Then, the sets $U(\Lambda^Q, \alpha), I(\Lambda^Q, \alpha)$ and $L(\Lambda^Q, \alpha)$ are subquasigroups for all $\alpha \in Im(T_{\Lambda^Q(a)}(f_1, v)) \cap Im(I_{\Lambda^Q(a)}(f_1, v)) \cap Im(F_{\Lambda^Q(a)}(f_1, v))$, where Im donate the image under the map of membership degree.*

Proof: Let $\alpha \in Im(T_{\Lambda^Q(a)}(f_1, v)) \cap Im(I_{\Lambda^Q(a)}(f_1, v)) \cap Im(F_{\Lambda^Q(a)}(f_1, v)) \subseteq [0,1]$. Obviously, the sets $U(\Lambda^Q, \alpha), I(\Lambda^Q, \alpha)$ and $L(\Lambda^Q, \alpha)$ are non-empty and let $q \in Q$ and $f_1, h_1 \in U(\Lambda^Q, \alpha)$. Then, $T_{\Lambda^Q(a)}(f_1, v) \geq \alpha$ and $T_{\Lambda^Q(a)}(h_1, v) \geq \alpha$ for all $a \in \mathfrak{A}$. Using Definition 3.1, we have

$$T_{\Lambda^Q(a)}(f_1 h_1, v) \geq \min\{T_{\Lambda^Q(a)}(f_1, v), T_{\Lambda^Q(a)}(h_1, v)\} \geq \alpha \textit{ so that } f_1 \circ h_1 \in U(\Lambda^Q, \alpha)$$

Suppose that $f_1, h_1 \in I(\Lambda^Q, \alpha)$, then $I_{\Lambda^Q(a)}(f_1, v) \leq \alpha$ and $I_{\Lambda^Q(a)}(h_1, v) \leq \alpha$, by definition, we have

$$I_{\Lambda^Q(a)}(f_1 h_1, v) \leq \max\{I_{\Lambda^Q(a)}(f_1, v), I_{\Lambda^Q(a)}(h_1, v)\} \leq \alpha$$

Hence $f_1 \circ h_1 \in I(\Lambda^Q, \alpha)$.

Let $f_1, h_1 \in F(\Lambda^Q, \alpha)$, then $F_{\Lambda^Q(a)}(f_1, v) \leq \alpha$ and $F_{\Lambda^Q(a)}(h_1, v) \leq \alpha$. From definition, it follows that

$$F_{\Lambda^Q(a)}(f_1 h_1, v) \leq \max\{F_{\Lambda^Q(a)}(f_1, v), F_{\Lambda^Q(a)}(h_1, v)\} \leq \alpha$$

Hence, $f_1 \circ h_1 \in F(\Lambda^Q, \alpha)$. Thus, $U(\Lambda^Q, \alpha), I(\Lambda^Q, \alpha)$ and $L(\Lambda^Q, \alpha)$ are subquasigroups of $\hat{G}$

Theorem 3.4. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q-$NSS over $\hat{G}$ such that a nonempty set (Λ^Q, α^K) is a subquasigroup of $\hat{G}$ for all $\alpha \in [0,1]$. Then, $(\Lambda^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft subquasigroup of $\hat{G}$ for all $a \in \mathfrak{A}$*

Proof: We assume that the nonempty set (Λ^Q, α^K) is a subquasigroup of $\hat{G}$ for all $\alpha \in [0,1]$. We want to show that $(\Lambda^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft subquasigroup of $\hat{G}$ for all $f_1, h_1' \in \hat{G}, v \in Q$ and $a \in \mathfrak{A}$. On the contrary, suppose that Definition 3.1 does not hold and there exist $f_1, h_1' \in \hat{G}, v \in Q$, and $a \in \mathfrak{A}$ such that

$$\begin{cases} T_{\Psi^Q(a)}(f_1 \circ h_1', v) < \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1', v)\} \\ I_{\Psi^Q(a)}(f_1 \circ h_1', v) > \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1', v)\} \\ F_{\Psi^Q(a)}(f_1 \circ h_1', v) > \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1', v)\} \end{cases} \tag{1}$$

Let

$$T_{\Psi^Q(a)}(f_1, v) = \alpha_1, T_{\Psi^Q(a)}(h', v) = \beta_1 \text{ and } T_{\Psi^Q(a)}(f_1 \circ h_1\prime, v) = \gamma_1$$
$$I_{\Psi^Q(a)}(f_1, v) = \alpha_2, I_{\Psi^Q(a)}(h_1', v) = \beta_2 \text{ and } I_{\Psi^Q(a)}(f_1 \circ h_1\prime, v) = \gamma_2$$
$$F_{\Psi^Q(a)}(f_1, v) = \alpha_3, F_{\Psi^Q(a)}(h_1', v) = \beta_3 \text{ and } F_{\Psi^Q(a)}(f_1 \circ h_1\prime,) = \gamma_3$$

Then, its follows from equation 1

$$\gamma_1 < \min\{\alpha_1, \beta_1\}, \gamma_2 > \max\{\alpha_2, \beta_2\} \text{ and } \gamma_3 > \max\{\alpha_3, \beta_3\} \tag{2}$$

Put

$$\begin{cases} \gamma_1^* = \frac{1}{2}\left[T_{\Psi^Q(a)}(f_1 \circ h', v) + \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1', v)\}\right] \\ \gamma_2^* = \frac{1}{2}\left[I_{\Psi^Q(a)}(f_1 \circ h_1', v) + \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1', v)\}\right] \\ \gamma_3^* = \frac{1}{2}\left[F_{\Psi^Q(a)}(f_1 \circ h_1', v) + \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1', v)\}\right] \end{cases} \tag{3}$$

Therefore,
$\gamma_1^* = \frac{1}{2}\left[(\gamma_1, v) + \min\{\alpha_1, v), (\beta_1, v)\}\right]$
$\gamma_2^* = \frac{1}{2}\left[(\gamma_2, v) + \max\{\alpha_2, v), (\beta_2, v)\}\right]$
$\gamma_3^* = \frac{1}{2}\left[(\gamma_3, v) + \max\{\alpha_3, v), (\beta_3, v)\}\right]$
Then,
$\alpha_1 > \gamma_1^* = \frac{1}{2}\left[(\gamma_1, v) + \min\{\alpha_1, v), (\beta_1, v)\}\right] > \gamma_1$

$\alpha_2 < \gamma_2^* = \frac{1}{2}\Bigg[(\gamma_2, v) + \min\{\alpha_2, v), (\beta_2, v)\}\Bigg] < \gamma_3$

$\alpha_3 < \gamma_3^* = \frac{1}{2}\Bigg[(\gamma_3, v) + \min\{\alpha_3, v), (\beta_3, v)\}\Bigg] < \gamma_3$

Thus,

$$T_{\Psi^Q(a)}(f_1 \circ h_1', v) < \gamma_1^* < \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1', v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1', v) > \gamma_1^* > \min\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1', v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1', v) > \gamma_1^* > \min\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1', v)\}$$

It follows that $f_1, h_1' \in (\Lambda^Q, \alpha^K)$, but $f_1 \circ h_1' \notin (\Lambda^Q, \alpha^K)$ a contradiction base on the fact that

$$T_{\Psi^Q(a)}(f_1, v) = \alpha_1 \geq \min\{(\alpha_1, v), (\beta_1, v)\} > \gamma_1^*$$
$$I_{\Psi^Q(a)}(f_1, v) = \alpha_2 \leq \max\{(\alpha_2, v), (\beta_2, v)\} < \gamma_2^*$$
$$F_{\Psi^Q(a)}(f_1, v) = \alpha_3 \leq \max\{(\alpha_3, v), (\beta_3, v)\} < \gamma_3^*$$

this implies that $f_1, h_1' \in (\Lambda^Q, \alpha^K)$. Thus, condition 3.1 hold. The prof is complete

Theorem 3.5. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q-$NSS over $\hat{G}$. Then, each subquasigroup H of $\hat{G}$ is a $Q-$level neutrosophic soft subquasigroup for all $\alpha, \beta, \gamma \in [0, 1]$ and $a \in \mathfrak{A}$*

Proof: Let $(\Lambda^Q, \mathfrak{A})$ be defined by

$$T_{\Phi_{Q(a)}}(f_1, v) = \begin{cases} \alpha, & \text{if } f_1 \in H \\ 0, & \text{otherwise.} \end{cases}$$

$$I_{\Phi_{Q(a)}}(f_1, v) = \begin{cases} \beta, & \text{if } f_1 \in H \\ 0, & \text{otherwise.} \end{cases}$$

$$F_{\Phi_{Q(a)}}(f_1, v) = \begin{cases} \gamma, & \text{if } f_1 \in H \\ 0, & \text{otherwise.} \end{cases}$$

where $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma \leq 3$, for all $f_1 \in \hat{G}, v \in Q$ and $a \in \mathfrak{A}$

We consider the following cases to show that $(\Lambda^Q, \mathfrak{A})$ is a Q- neutrosophic soft quasigroup over $\hat{G}$.

Case 1: Suppose that $f_1, h_1 \in H$, then $f_1 \circ h_1 \in H$. So,

$$T_{\Psi^Q(a)}(f_1 \circ h_1, v) = \alpha = \min\{\alpha, \alpha\} = \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1, v) = \beta = \min\{\beta, \beta\} = \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1, v) = \beta = \min\{\beta, \beta\} = \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$$

Case 2: If $f, h \notin H$, then
$T_{\Psi^Q(a)}(f_1, v) = 0 = T_{\Psi^Q(a)}(h, v), I_{\Psi^Q(a)}(f_1, v) = 0 = I_{\Psi^Q(a)}(h, v)$ and $F_{\Psi^Q(a)}(f_1, v) = 0 = F_{\Psi^Q(a)}(h, v)$. Therefore,

$$T_{\Psi^Q(a)}(f_1 \circ h_1, v) \geq 0 = \min\{0, 0\} = \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{0, 0\} = \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{0, 0\} = \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$$

Case 3: If $f_1 \in H$ and $h_1 \notin H$, then $T_{\Psi^Q(a)}(f_1, v) = \alpha, I_{\Psi^Q(a)}(f_1, v) = \beta$ and $F_{\Psi^Q(a)}(f_1, v) = \gamma, F_{\Psi^Q(a)}(h_1, v) = 0 = T_{\Psi^Q(a)}(h_1, v) = I_{\Psi^Q(a)}(h_1, v)$. So,

$$T_{\Psi^Q(a)}(f_1 \circ h_1, v) \geq 0 = \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$$
$$I_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$$
$$F_{\Psi^Q(a)}(f_1 \circ h_1, v) \leq 0 = \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$$

Case 4: If $h_1 \in H$ and $f_1 \notin H$. It has a similar argument with case 3. This complete the proof.

Theorem 3.6. *If $(\Lambda^Q, \mathfrak{A})$ is a $Q - NS\hat{G}$ over $\hat{G}$. Then,*

(1) $T_{\Psi^Q(a)}(f_1, v) = \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha)\}$
(2) $I_{\Psi^Q(a)}(f_1, v) = \inf\{\beta \in [0, 1] : f_1 \in I(\Lambda^Q, \beta)\}$ *and*
(3) $F_{\Psi^Q(a)}(f_1, v) = \inf\{\gamma \in [0, 1] : f_1 \in L(\Lambda^Q, \gamma)\}$

for all $f_1 \in \hat{G}$ and $v \in Q$

Proof:

(1) Given $\epsilon > 0$, let $\delta = \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha)\}$. Then, $\delta - \epsilon < \alpha$ for some $\alpha \in [0, 1]$. This implies that $\delta - \epsilon < T_{\Psi^Q(a)}(f_1, v)$ so that $\delta \leq T_{\Psi^Q(a)}(f_1, v)$ for every an arbitrary ϵ and for all $v \in Q$ and $f_1 \in \hat{G}$.

Next, we show that $T_{\Psi^Q(a)}(f_1, v) \leq \delta$. If $T_{\Psi^Q(a)}(f_1, v) = \alpha_1$, then $f_1 \in U(\Lambda^Q, \alpha_1)$ and so

$$\alpha_1 \in \{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha), v \in Q\}$$

Hence,

$$T_{\Psi^Q(a)}(f_1, v) = \alpha_1 \leq \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha), v \in Q\} = \delta$$

Therefore,

$$T_{\Psi^Q(a)}(f_1, v) = \delta = \sup\{\alpha \in [0, 1] : f_1 \in U(\Lambda^Q, \alpha), v \in Q\}$$

(2) Let $\tau = \inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\}$. Then, $\inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\} < \tau + \epsilon$. For any $\epsilon > 0$, we show that $\beta < \tau + \epsilon$ for some $\beta \in [0,1]$ with $f_1 \in I(\Lambda^Q, \beta)$. Since ϵ is an arbitrary element, we have $I_{\Psi^Q(a)}(f_1, v) \leq \beta$ for any $v \in Q$. This implies that

$$I_{\Psi^Q(a)}(f_1, v) \leq \tau$$

.

To show that $I_{\Psi^Q(a)}(f_1, v) \geq \tau$, let $I_{\Psi^Q(a)}(f_1, v) = \beta_1$.
Then, $f_1 \in I(\Lambda^Q, \beta)$ and thus, $\beta_1 \in \{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\}$.
Hence,

$$\inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\} \leq \tau$$

That is $\tau \leq \beta_1 = I_{\Psi^Q(a)}(f_1, v)$ for any $v \in Q$. Consequently,

$$I_{\Psi^Q(a)}(f_1, v) = \tau = \inf\{\beta \in [0,1] : f_1 \in I(\Lambda^Q, \beta)\ \forall\ v \in Q\}$$

(3) The argument is similar with 2 above.

Theorem 3.7. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over a $(\hat{G}, \circ)$. The following hold*

(1) $T_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = T_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$, $I_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = I_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$ *and* $F_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = F_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$
(2) $T_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = T_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$, $I_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = I_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$ *and* $F_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = F_{\Psi^Q(a)}(hf_1 \circ f_1, v)$
(3) $T_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = T_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$, $I_{\Psi^Q(a)}(f_1^2 \circ h, v) = I_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = F_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$

Proof: Let $(\Phi^Q, \mathfrak{A})$ be a $Q - NS\hat{G}$ over a quasigroup $(\hat{G}, \circ)$. For all $f_1, h_1 \in \hat{G}, v \in Q$ and $a \in \mathfrak{A}$, we have

(1) Considering the LHS.

$$\begin{aligned} T_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) &\geq \min\{T_{\Psi^Q(a)}(f_1 \circ h_1, v), T_{\Psi^Q(a)}(f_1, v)\} \\ &= \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1 \circ h_1, v)\} \\ &\geq \min\left\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}\right\} \\ &= \min\left\{\min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\right\} \\ &= \min\left\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\right\} \end{aligned} \quad (4)$$

Considering the RHS.

$$
\begin{aligned}
T_{\Psi^{Q}(a)}(f_1 \circ h_1 f_1, v) &\geq \min\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(h_1 f_1, v)\} \\
&= \min\{T_{\Psi^{Q}(a)}(h_1 f_1, v), T_{\Psi^{Q}(a)}(f_1, v)\} \\
&\geq \min\Big\{\min\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\}, T_{\Psi^{Q}(a)}(f_1, v)\Big\} \\
&= \min\Big\{T_{\Psi^{Q}(a)}(h_1, v), \min\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(f_1, v)\}\Big\} \\
&= \min\Big\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\Big\} \qquad (5)
\end{aligned}
$$

Therefore, $\min\Big\{T_{\Psi^{Q}(a)}(h, v), T_{\Psi^{Q}(a)}(f_1, v)\Big\} = \min\Big\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(h, v)\Big\}$. Thus, $T_{\Psi^{Q}(a)}(f_1 h \circ f_1, v) = T_{\Psi^{Q}(a)}(f_1 \circ h f_1, v)$

$$
\begin{aligned}
I_{\Psi^{Q}(a)}(f_1 h_1 \circ f_1, v) &\leq \max\{I_{\Psi^{Q}(a)}(f_1 \circ h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\} \\
&= \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(f_1 \circ h_1, v)\} \\
&\leq \max\Big\{I_{\Psi^{Q}(a)}(f_1, v), \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(h_1, v)\}\Big\} \\
&= \max\Big\{\max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(f_1, v)\}, I_{\Psi^{Q}(a)}(h_1, v)\Big\} \\
&= \max\Big\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(h_1, v)\Big\} \qquad (6)
\end{aligned}
$$

Considering the RHS.

$$
\begin{aligned}
I_{\Psi^{Q}(a)}(f_1 \circ h_1 f_1, v) &\leq \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}(a)}(h_1 f_1, v)\} \\
&= \max\{I_{\Psi^{Q}(a)}(h_1 f_1, v), I_{\Psi^{Q}(a)}(f_1, v)\} \\
&\leq \max\Big\{\max\{I_{\Psi^{Q}(a)}(h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\}, I_{\Psi^{Q}(a)}(f_1, v)\Big\} \\
&= \max\Big\{I_{\Psi^{Q}(a)}(h_1, v), \max\{I_{\Psi^{Q}(a)}(f_1, v), I_{\Psi^{Q}}(a)(f_1, v)\}\Big\} \\
&= \max\Big\{I_{\Psi^{Q}(a)}(h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\Big\} \qquad (7)
\end{aligned}
$$

Therefore, $\max\Big\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\Big\} = \max\Big\{T_{\Psi^{Q}(a)}(f_1, v), T_{\Psi^{Q}(a)}(h_1, v)\Big\}$. Thus, $I_{\Psi^{Q}(a)}(f_1 h_1 \circ f_1, v) = I_{\Psi^{Q}(a)}(f_1 \circ h_1 f_1, v)$. The result for falsity membership is obtain in similar procedure.

(2) Let $f_1, h_1 \in \hat{G}, a \in \mathfrak{A}$ and $v \in Q$, we want show that $T_{\Psi^{Q}(a)}(h_1 \circ f_1^2, v) = T_{\Psi^{Q}(a)}(h_1 f_1 \circ f_1, v)$ for true membership.

Considering the RHS,

$$T_{\Psi^Q(a)}(h_1f_1 \circ f_1, v) \geq \min\{T_{\Psi^Q(a)}(h_1f_1, v), T_{\Psi^Q(a)}(f_1, v)\}$$
$$\geq \min\left\{ \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} \tag{8}$$

Considering the LHS.

$$T_{\Psi^Q(a)}(h_1 \circ f_1^2, v) \geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1^2, v)\}$$
$$= \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1 \circ f_1, v)\}$$
$$\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\}$$
$$= \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} \tag{9}$$

That is, $\min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\} = \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\}$

(3) Follows in the similar result of 2.

Corollary 3.8. *Let $(\Phi^Q, \mathfrak{A})$ be a $Q-$NSS over a quasigroup $(\hat{G}, \circ)$. Then, the following are equivalent.*

(1) $(\Phi^Q, \mathfrak{A})$ *is a* $Q - NS\hat{G}$

(2) $T_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = T_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$, $I_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = I_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$ *and* $F_{\Psi^Q(a)}(f_1h_1 \circ f_1, v) = F_{\Psi^Q(a)}(f_1 \circ h_1f_1, v)$

(3) $T_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = T_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$, $I_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = I_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$ *and* $F_{\Psi^Q(a)}(h_1 \circ f_1^2, v) = F_{\Psi^Q(a)}(h_1f_1 \circ f_1, v)$

(4) $T_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = T_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$, $I_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = I_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^2 \circ h_1, v) = F_{\Psi^Q(a)}(f_1 \circ f_1h_1, v)$

Proof: It following from Theorem 3.7.

Definition 3.9. Let $(\Psi^Q, \mathfrak{A})$ be a $Q-$NSS defined over a loop $(\hat{L}, \circ, /, \backslash)$. Then $(\Psi^Q, \mathfrak{A})$ is called a $Q-$neutrosphis soft loop $(Q - NS\hat{L})$ over $\hat{L}$ if for all $a \in \mathfrak{A}, f_1, h_1 \in \hat{L}$, and $v \in Q$ satisfies the following conditions

(1) $T_{\Psi^Q(a)}(f_1 * h_1, v) \geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}$,
$I_{\Psi^Q(a)}((f_1 * h_1), v) \leq \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}$
$F_{\Psi^Q(a)}(f_1 * h_1, v) \leq \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(h_1, v)\}$

(2) $T_{\Psi^Q(a)}(f_1^{-1}, v) \geq T_{\Psi^Q(a)}(f_1, v)$,
$I_{\Psi^Q(a)}(f_1^{-1}, v) \leq I_{\Psi^Q(a)}(f_1, v)$,
$F_{\Psi^Q(a)}(f_1^{-1}, v) \leq F_{\Psi^Q(a)}(f_1, v)$

(3) $T_{\Psi^Q(a)}(^{1-}f_1, q) \geq T_{\Psi^Q(a)}(f_1, v)$,
$I_{\Psi^Q(a)}(^{1-}f_1, v) \leq I_{\Psi^Q(a)}(f_1, v)$,
$F_{\Psi^Q(a)}(^{1-}f_1, v) \leq F_{\Psi^Q(a)}(f_1, v)$

where f^{-1} and ^{1-}f are right inverse and left inverse in $\hat{L}$ and $* \in \{\circ, /, \backslash\}$.

Theorem 3.10. *Let $(\Lambda^Q, \mathfrak{A})$ be a $Q-$NSS over a loop $\hat{L}$. Then, $(\Lambda^Q, \mathfrak{A})$ is a $Q-$neutrosophic soft subloop of $\hat{L}$ if and only if the nonempty $Q-$level soft set $(\Lambda_{Q(\alpha,\beta,\gamma)}, \mathfrak{A})$ is a soft subloop for all $\alpha, \beta, \gamma \in [0, 1]$ and $a \in \mathfrak{A}$*

Proof: The proof is follow from Theorem 3.4 with definition 3.9.

Lemma 3.11. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q - NS\hat{L}$ over a loop $(\hat{L}, \circ)$. Then, for all $f_1 \in \hat{L}, v \in Q$ the following hold*

(1) $T_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) = T_{\Psi^Q(a)}(f_1, v)$, $I_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) = I_{\Psi^Q(a)}(f_1, v)$ $aF_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) = F_{\Psi^Q(a)}(f_1, v)$

Proof. Follows from Definition 3.9. □

Theorem 3.12. *Let $(\Psi^Q, \mathbf{A})$ be a $Q - NS\hat{L}$ over a loop $(\hat{L}, \circ, /, \backslash)$. Then, for all $a \in \mathfrak{A}, f \in \hat{L}, v \in Q$ the following hold*

(1) $T_{\Psi^Q(a)}(f_1^{-1}, v) \geq T_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1^{-1}, v) \leq I_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(f_1^{-1}, v) \leq F_{\Psi^Q(a)}(f_1, v)$

(2) $T_{\Psi^Q(a)}(^{1-}f_1, v) \geq T_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(^{1-}f_1, v) \leq I_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(^{1-}f_1, v) \leq F_{\Psi^Q(a)}(f_1, v)$

(3) $T_{\Psi^Q(a)}(e, q) \geq T_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(e, v) \leq I_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q(a)}(e, v) \leq F_{\Psi^Q(a)}(f_1, v)$

Proof:

(1) Let $(\Psi^Q,)$ be a $Q - NS\hat{L}$ over loop $(G, \circ, /, \backslash)$, then for all $a \in \mathfrak{A}, f_1 \in \hat{L}, v \in Q$ we have

$$T_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) \geq T_{\Phi_{Q(a)}}(f_1, v)$$
$$I_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) \leq I_{\Phi_{Q(a)}}(f_1, v)$$
$$F_{\Psi^Q(a)}((f_1^{-1})^{-1}, v) \leq F_{\Phi_{Q(a)}}(f_1, v)$$

(2) it is similar with (1)

(3) Let $(\Psi^Q, \mathfrak{A})$ be a $Q-NS\hat{L}$ over loop $(\hat{L}, \circ)$ with an identith element $e \in \hat{L}$. Then, for all $f_1 \in \hat{L}, v \in Q$, we have

$$\begin{aligned}
T_{\Psi^Q(a)}(e,v) &= T_{\Psi^Q(a)}((f_1^{-1} \circ f_1, v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1^{-1}, v), T_{\Phi_{Q(a)}}(f_1, v)\} \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1, v), T_{\Phi_{Q(a)}}(f_1, v)\} \\
&= T_{\Phi_{Q(a)}}(f_1, v) \\
I_{\Psi^Q(a)}(e,v) &= I_{\Psi^Q(a)}((f_1^{-1} \circ f_1, v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1, v)\} \\
&= I_{\Phi_{Q(a)}}(f_1, v) \\
F_{\Psi^Q(a)}(e,v) &= F_{\Psi^Q(a)}((f_1^{-1} \circ f_1, v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1, v)\} \\
&= F_{\Phi_{Q(a)}}(f_1, v)
\end{aligned}$$

$$\begin{aligned}
T_{\Psi^Q(a)}(e,v) &= T_{\Psi^Q(a)}((f_1 \circ f_1^{-1}, v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1, v), T_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1, v), T_{\Phi_{Q(a)}}(f_1, v)\} \\
&= T_{\Phi_{Q(a)}}(f_1, v) \\
I_{\Psi^Q(a)}(e,v) &= I_{\Psi^Q(a)}((f_1 \circ f_1^{-1}, v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1, v), I_{\Phi_{Q(a)}}(f_1, v)\} \\
&= I_{\Phi_{Q(a)}}(f_1, v) \\
F_{\Psi^Q(a)}(e,v) &= F_{\Psi^Q(a)}((f_1 \circ f_1^{-1}, v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1^{-1}, v)\} \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1, v), F_{\Phi_{Q(a)}}(f_1, v)\} \\
&= F_{\Phi_{Q(a)}}(f_1, v)
\end{aligned}$$

$$\begin{aligned}
T_{\Psi^Q(a)}(e,v) &= T_{\Psi^Q(a)}((f_1/f_1,v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1,v), T_{\Phi_{Q(a)}}(f_1,v)\} \\
&= T_{\Phi_{Q(a)}}(f_1,v) \\
I_{\Psi^Q(a)}(e,v) &= I_{\Psi^Q(a)}((f_1/f_1,v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1,v), I_{\Phi_{Q(a)}}(f_1,v)\} \\
&= I_{\Phi_{Q(a)}}(f_1,v) \\
F_{\Psi^Q(a)}(e,v) &= F_{\Psi^Q(a)}((f_1/f_1,v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1,v), F_{\Phi_{Q(a)}}(f_1,v)\} \\
&= F_{\Phi_{Q(a)}}(f_1,v)
\end{aligned}$$

$$\begin{aligned}
T_{\Psi^Q(a)}(e,v) &= T_{\Psi^Q(a)}((f_1\backslash f_1,v) \\
&\geq \min\{T_{\Phi_{Q(a)}}(f_1,v), T_{\Phi_{Q(a)}}(f_1,v)\} \\
&= T_{\Phi_{Q(a)}}(f_1,v) \\
I_{\Psi^Q(a)}(e,v) &= I_{\Psi^Q(a)}((f_1\backslash f_1,v) \\
&\leq \max\{I_{\Phi_{Q(a)}}(f_1,v), I_{\Phi_{Q(a)}}(f_1,v)\} \\
&= I_{\Phi_{Q(a)}}(f_1,v) \\
F_{\Psi^Q(a)}(e,v) &= F_{\Psi^Q(a)}((f_1\backslash f_1,v) \\
&\leq \max\{F_{\Phi_{Q(a)}}(f_1,v), F_{\Phi_{Q(a)}}(f_1,v)\} \\
&= F_{\Phi_{Q(a)}}(f_1,v)
\end{aligned}$$

The proof is compete.

Theorem 3.13. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q-NS\hat{L}$ over a loop $(\hat{L}, \circ, /, \backslash)$. Then, for all $a \in \mathfrak{A}, f_1 \in \hat{L}, v \in Q$ the following hold*

(1) $T_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = T_{\Psi^Q(a)}(h_1, v)$, $I_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = I_{\Psi^Q(a)}(h_1, v)$, *and* $F_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = F_{\Psi^Q(a)}(h_1, v)$,

(2) $T_{\Psi^Q(a)}(f_1^{-1} \circ f_1 h_1, v) = T_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1^{-1} \circ f_1 h_1, v) = I_{\Psi^Q(a)}(h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^{-1} \circ f_1 h_1, v) = F_{\Psi^Q(a)}(h_1, v)$

(3) $T_{\Psi^Q(a)}(f_1^{-1} \circ h_1 f_1, v) = T_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1^{-1} \circ h_1 f_1, v) = I_{\Psi^Q(a)}(h_1, v)$ *and* $F_{\Psi^Q(a)}(f_1^{-1} \circ h_1 f_1, v) = F_{\Psi^Q(a)}(h_1, v)$

(4) $T_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) = T_{\Psi^Q(a)}(f_1^{-1}, v), I_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) = I_{\Psi^Q(a)}(f_1^{-1}, v)$ *and* $F_{\Psi^Q(a)}(h_1 \circ (f_1 h_1)^{-1}, v) = F_{\Psi^Q(a)}(f_1^{-1}, v)$

Proof:

(1) Let $(\Psi^Q, \mathfrak{A})$ be $Q-NS\hat{L}$ over a loop $(\hat{L}, \circ, /, \backslash)$. Then, we shall show that for all $a \in \mathfrak{A}, f_1, h_1 \in \hat{L}, v \in Q$

$$\begin{aligned}
&T_{\Psi^Q(a)}(h_1 f \circ f^{-1}, v) = T_{\Psi^Q(a)}(h_1, v)\\
\Rightarrow &T_{\Psi^Q(a)}(h_1 f_1 \circ f^{-1}, v) \geq \min\{T_{\Psi^Q(a)}(h_1 \circ f_1, v), T_{\Psi^Q(a)}(f_1^{-1}, v)\}\\
&\geq \min\{T_{\Psi^Q(a)}(h_1 \circ f, v), T_{\Psi^Q(a)}(f_1, v)\}\\
&\geq \min\left\{\min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\right\}\\
&= \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\}\\
&\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\}
\end{aligned} \tag{10}$$

On the other hand,

$$\begin{aligned}
&T_{\Psi^Q(a)}(h_1, v) = T_{\Psi^Q(a)}((h_1/f_1) \circ f, v)\\
&\geq \min\{T_{\Psi^Q(a)}((h_1/f_1), v), T_{\Psi^Q(a)}(f_1, v)\}\\
&\geq \min\left\{\min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\right\}\\
&= \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}\right\}\\
&\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(f_1, v)\right\}
\end{aligned} \tag{11}$$

And

$$\begin{aligned}
&I_{\Psi^Q(a)}(h_1 f_1 \circ f_1^{-1}, v) = I_{\Psi^Q(a)}(h_1, v)\\
\Rightarrow &I_{\Psi^Q(a)}(h_1 f_1 \circ f^{-1}, v) \leq \max\{I_{\Psi^Q(a)}(h_1 \circ f_1, v), I_{\Psi^Q(a)}(f_1^{-1}, v)\}\\
&\leq \max\{I_{\Psi^Q(a)}(h_1 \circ f_1, v), I_{\Psi^Q(a)}(f_1, v)\}\\
&\leq \max\left\{\max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\}, I_{\Psi^Q(a)}(f_1, v)\right\}\\
&= \max\left\{I_{\Psi^Q(a)}(h_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1, v)\}\right\}\\
&\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\right\}
\end{aligned} \tag{12}$$

On the other hand,

$$
\begin{aligned}
I_{\Psi^Q(a)}(h_1, v) &= I_{\Psi^Q(a)}((h_1/f_1) \circ f_1, v) \\
&\leq \max\{I_{\Psi^Q(a)}((h_1/f_1), v), I_{\Psi^Q(a)}(f_1, v)\} \\
&\leq \max\left\{ \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\}, I_{\Psi^Q(a)}(f_1, v)\right\} \\
&= \max\left\{I_{\Psi^Q(a)}(h_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1, v)\}\right\} \\
&\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(f_1, v)\right\} \qquad (13)
\end{aligned}
$$

Similarly, we can use the identity $T_{\Psi^Q(a)}(h_1, v) = T_{\Psi^Q(a)}((f_1 \circ f_1 \backslash h_1), v)$. Result for falsity is argued the same way.

(2) Use the same argument of 1

(3) Similar argument with 2

(4) We shall show that $T_{\Psi^Q(a)}(h_1 \circ (f_1h_1)^{-1}, v) = T_{\Psi^Q(a)}(f^{-1}, v)$.
Considering the LHS,

$$
\begin{aligned}
T_{\Psi^Q(a)}(h_1 \circ (fh_1)^{-1}, v) &\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}((fh_1)^{-1}, v)\} \\
&\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\underbrace{\{T_{\Psi^Q(a)}(f^{-1}, v), T_{\Psi^Q(a)}(h_1^{-1}, v)\}}_{AIP}\right\} \\
&\geq \min\left\{T_{\Psi^Q(a)}(h_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}\right\} \\
&= \min\left\{ \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\right\} \\
&= \min\left\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(h_1, v)\}\right\} \\
&\geq \min\left\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\right\} \qquad (14)
\end{aligned}
$$

On the other hand,

$$T_{\Psi^Q(a)}(f_1^{-1}, v) \geq T_{\Psi^Q(a)}(f_1, v) \qquad (15)$$

Note that $T_{\Psi^Q(a)}(f_1, v) = T_{\Psi^Q(a)}((f_1/h_1) \circ h_1, v)$. Then, using the last equality in (18), we get

$$\begin{aligned} T_{\Psi^Q(a)}(f_1, v) &\geq \min\{T_{\Psi^Q(a)}((f_1/h_1), v), T_{\Psi^Q(a)}(h_1, v)\} \\ &\geq \min\left\{ \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\right\} \\ &= \min\left\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q(a)}(h_1, v)\}\right\} \\ &\geq \min\left\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\right\} \end{aligned} \tag{16}$$

Considering the LHS, for the indeterminate membership

$$\begin{aligned} I_{\Psi^Q(a)}(h_1 \circ (f_1h_1)^{-1}, v) &\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}((f_1h_1)^{-1}, v) \\ &\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), \max\{\underbrace{I_{\Psi^Q(a)}(f_1^{-1}, v), I_{\Psi^Q(a)}(h_1^{-1}, v)\}}_{AIP}\right\} \\ &\leq \max\left\{I_{\Psi^Q(a)}(h_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}\right\} \\ &= \max\left\{ \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\right\} \\ &= \max\left\{I_{\Psi^Q(a)}(f_1, v), \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(h_1, v)\}\right\} \\ &\leq \max\left\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\right\} \end{aligned} \tag{17}$$

On the other hand,

$$I_{\Psi^Q(a)}(f_1^{-1}, v) \leq I_{\Psi^Q(a)}(f_1, v) \tag{18}$$

Note that $I_{\Psi^Q(a)}(f_1, v) = I_{\Psi^Q(a)}((f_1/h_1) \circ h_1, v)$. Then, using the last equalith in (18), we get

$$\begin{aligned} I_{\Psi^Q(a)}(f_1, v) &\leq \max\{I_{\Psi^Q(a)}((f_1/h_1), v), I_{\Psi^Q(a)}(h_1, v)\} \\ &\leq \max\left\{ \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\right\} \\ &= \max\left\{I_{\Psi^Q(a)}(f_1, v), \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q(a)}(h_1, v)\}\right\} \\ &\leq \max\left\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\right\} \end{aligned} \tag{19}$$

The result for falsity membership is similar with the result for indeterminate membership obtained.

Theorem 3.14. *Let $(\Psi^Q, \mathfrak{A})$ be $Q - NS\hat{L}$ over a loop $(\hat{L}, \circ)$. Then, for all $a \in \mathfrak{A}, f_1, h_1, z_1 \in \hat{L}, v \in Q$ the following hold:*

(1) $T_{\Psi^{Q}(a)}((f_1 \circ h_1)^{-1}, v) = T_{\Psi^{Q}(a)}(h_1^{-1} \circ f_1^{-1}, v)$, $I_{\Psi^{Q}(a)}((f_1 \circ h_1)^{-1}, v) = I_{\Psi^{Q}(a)}(h_1^{-1} \circ f_1^{-1}, v)$, *and* $F_{\Psi^{Q}(a)}((f_1 \circ h_1)^{-1}, v) = F_{\Psi^{Q}(a)}(h_1^{-1} \circ f_1^{-1}, v)$,

(2) $T_{\Psi^{Q}(a)}((f_1 h_1 \circ f_1)^{-1}, v) = T_{\Psi^{Q}(a)}(f_1^{-1} h_1^{-1} \circ f_1^{-1}, v), I_{\Psi^{Q}(a)}((f_1 h_1 \circ f_1)^{-1}, v) = I_{\Psi^{Q}(a)}(f_1^{-1} h_1^{-1} \circ f_1^{-1}, v)$ *and* $F_{\Psi^{Q}(a)}((f_1 h_1 \circ f_1)^{-1}, v) = F_{\Psi^{Q}(a)}(f_1^{-1} h_1^{-1} \circ f_1^{-1}, v)$

(3) $T_{\Psi^{Q}(a)}((f_1 \circ h_1 z)^{-1}, v) = T_{\Psi^{Q}(a)}(z^{-1} \circ h_1^{-1} f_1^{-1}, v), I_{\Psi^{Q}(a)}((f_1 \circ h_1 z)^{-1}, v) = I_{\Psi^{Q}(a)}(z^{-1} \circ h_1^{-1} f_1^{-1}, v)$ *and* $F_{\Psi^{Q}(a)}((f_1 \circ h_1 z)^{-1}, v) = F_{\Psi^{Q}(a)}(z^{-1} \circ h_1^{-1} f_1^{-1}, v)$

Proof:

(1) Let $(\Psi^Q, \mathfrak{A})$ be $Q-$neutrosophic soft loop over a loop $(\hat{L}, \circ)$. Then, we shall show that for all $a \in \mathfrak{A}, f_1, h_1 \in \hat{L}, v \in Q$

$$\begin{aligned} T_{\Psi^{Q}(a)}(f_1 \circ h_1)^{-1}, v) &= T_{\Psi^{Q}(a)}(f_1^{-1} \circ h_1^{-1}, v) \\ &\geq \min\{T_{\Psi^{Q}(a)}(f_1^{-1}, v), T_{\Psi^{Q}(a)}(h_1^{-1}, v)\} \\ &\geq \min\Big\{T_{\Psi^{Q}(a)}(f_1, v)\}, T_{\Psi^{Q}(a)}(h_1, v)\Big\} \end{aligned} \tag{20}$$

On the other hand

$$\begin{aligned} T_{\Psi^{Q}(a)}(h_1^{-1} \circ f_1^{-1}, v) &\geq \min\{T_{\Psi^{Q}(a)}(h_1^{-1}, v), T_{\Psi^{Q}(a)}(f_1^{-1}, v)\} \\ &\geq \min\{T_{\Psi^{Q}(a)}(h_1, v), T_{\Psi^{Q}(a)}(f_1, v)\} \end{aligned} \tag{21}$$

$$\begin{aligned} I_{\Psi^{Q}(a)}(f_1 \circ h_1)^{-1}, v) &= I_{\Psi^{Q}(a)}(f_1^{-1} \circ h_1^{-1}, v) \\ &\leq \max\{I_{\Psi^{Q}(a)}(f_1^{-1}, v), I_{\Psi^{Q}(a)}(h_1^{-1}, v)\} \\ &\leq \max\Big\{I_{\Psi^{Q}(a)}(f_1, v)\}, I_{\Psi^{Q}(a)}(h_1, v)\Big\} \end{aligned} \tag{22}$$

RHS

$$\begin{aligned} I_{\Psi^{Q}(a)}(h_1^{-1} \circ f_1^{-1}, v) &\geq \max\{I_{\Psi^{Q}(a)}(h_1^{-1}, v), I_{\Psi^{Q}(a)}(f_1^{-1}, v)\} \\ &\leq \max\{I_{\Psi^{Q}(a)}(h_1, v), I_{\Psi^{Q}(a)}(f_1, v)\} \end{aligned} \tag{23}$$

The result for falsity membership is similar with the result of indeterminate membership.

(2) For the true membership,

$$\begin{aligned}
T_{\Psi^Q(a)}((f_1h_1 \circ f_1)^{-1}, v) &= \underbrace{T_{\Psi^Q(a)}((f_1 \circ h_1)^{-1} \circ f_1^{-1}), v)}_{AIP} \\
&\geq \min\{T_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}), v), T_{\Psi^Q(a)}(f_1^{-1}, v)\} \\
&\geq \min\{\min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}, T_{\Psi^Q(a)}(f_1, v)\} \\
&= \min\{T_{\Psi^Q(a)}(f_1, v), \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}\} \\
&= \min\{\min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(f_1, v)\}, T_{\Psi^Q(a)}(h_1, v)\} \\
&\geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\}
\end{aligned} \tag{24}$$

Similarly, we obtain

$$T_{\Psi^Q(a)}(f_1^{-1}h_1^{-1} \circ f_1^{-1}, v) \geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q(a)}(h_1, v)\} \tag{25}$$

$$\begin{aligned}
I_{\Psi^Q(a)}((f_1h_1 \circ f_1)^{-1}, v) &= \underbrace{I_{\Psi^Q(a)}((f_1 \circ h_1)^{-1} \circ f_1^{-1}), v)}_{AIP} \\
&\leq \max\{I_{\Psi^Q(a)}((f_1 \circ h_1)^{-1}), v), I_{\Psi^Q(a)}(f_1^{-1}, v)\} \\
&\leq \max\{\max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}, I_{\Psi^Q(a)}(f_1, v)\} \\
&= \max\{I_{\Psi^Q(a)}(f_1, v), \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}\} \\
&= \max\{\max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(f_1, v)\}, I_{\Psi^Q(a)}(h_1, v)\} \\
&\leq \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\}
\end{aligned} \tag{26}$$

Also, we obtain the indeterminate membership for the RHS

$$I_{\Psi^Q(a)}(f_1^{-1}h_1^{-1} \circ f_1^{-1}, v) \geq \min\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q(a)}(h_1, v)\} \tag{27}$$

Using similar approach to obtain result for indeterminate membership.

(3) The proof is similar with the result obtained for 2

Theorem 3.15. *Let $(\Psi^Q, \mathfrak{A})$ be a $Q-$NSS over a loop $(\hat{L}, \circ, /, \backslash)$. Then, $(\Psi^Q, \mathfrak{A})$ is $Q - NS\hat{L}$ if and if only for all $f_1, h_1 \in \hat{L}, v \in Q$*

(1) $T_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\}$
$I_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1, v)\}$,
$F_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1, v)\}$

(2) $T_{\Psi^Q(a)}(^{1-}f_1 * h_1, v) \geq \min\{T_{\Psi^Q(a)}(f_1, v), T_{\Psi^Q}(a)(h_1, v)\}$,
$I_{\Psi^Q(a)}(^{1-}f_1 * h_1, v) \leq \max\{I_{\Psi^Q(a)}(f_1, v), I_{\Psi^Q}(a)(h_1, v)\}$
$F_{\Psi^Q(a)}(^{1-}f_1 * h_1, v) \leq \max\{F_{\Psi^Q(a)}(f_1, v), F_{\Psi^Q}(a)(h_1, v)\}$

(1) Suppose that $(\Psi^Q, \mathfrak{A})$ is a $Q-NS\hat{L}$ over $(\hat{L}, \circ, /, \backslash)$. Let $* \in \{\circ, /, \backslash\}$ then we show that $(\Psi^Q, \mathfrak{A})$ satisfies 3.9 for all $f_1, h_1 \in \hat{L}$, and $v \in Q$, we have

$$T_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1^{-1}, v)\}$$
$$\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\}$$
$$I_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1^{-1}, v)\}$$
$$\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1, v)\}$$
$$F_{\Psi^Q(a)}(h_1 * f_1^{-1}, v) \leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1^{-1}, v)\}$$
$$\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1, v)\}$$

Conversely, suppose equality (1) hold, then for all $f_1, h_1, \in G, v \in Q$, and $a \in \mathfrak{A}$, we show $(\Psi^Q, \mathfrak{A})$ is $Q-$neutrosophic soft subquasigroup over quasigroup $(\hat{L}, \circ, /, \backslash)$. Thus,

$$\begin{aligned} T_{\Psi^Q(a)}(h_1 * f_1, v) &= T_{\Psi^Q(a)}(h_1 * (f_1^{-1})^{-1}, v) \\ &\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1^{-1}, v) \\ &\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1^{-1}, v)\} \\ &\geq \min\{T_{\Psi^Q(a)}(h_1, v), T_{\Psi^Q}(a)(f_1, v)\} \end{aligned}$$

Next:

$$\begin{aligned} I_{\Psi^Q(a)}(h_1 * f_1, v) &= I_{\Psi^Q(a)}(h_1 * (f_1^{-1})^{-1}, v) \\ &\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1^{-1}, v)\} \\ &\leq \max\{I_{\Psi^Q(a)}(h_1, v), I_{\Psi^Q}(a)(f_1, v)\} \end{aligned}$$

Finally:

$$\begin{aligned} F_{\Psi^Q(a)}(h_1 * f_1, v) &= F_{\Psi^Q(a)}(h_1 * (f_1^{-1})^{-1}, v) \\ &\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1^{-1}, v)\} \\ &\leq \max\{F_{\Psi^Q(a)}(h_1, v), F_{\Psi^Q}(a)(f_1, v)\} \end{aligned}$$

(2) it is similar to (1)

4. **Conclusion**

In this study, it was found that $Q-NS\hat{G}$ obeys LIP, RIP, LAP, RAP and flexible law. With the help AIP, it was shown that $Q-NS\hat{G}$ obey AAIP, SAIP, SAAIP. $Q-NS\hat{L}$ were also defined, and the definition was used to shown when is $Q-$NSS under loop said to be $Q-NS\hat{G}$. Furthermore, this research revealed that left and right inverse elements of $Q-NS\hat{L}$ coincided. In future research, Definitions 3.1 and 3.9 will be use to study the structure of isotopy theory of quasigroup.

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References

1. M. Abu Qamar and N. Hassan, $Q-$neutrosophic soft relation and its application in decision making, Entropy, 20, 172, pp. 1-14, (2018)
2. L. A. Zadeh, Fuzzy Sets, Inform. Control, 8, pp. 338-353, (1965)
3. D. Molodtsov. Soft set theory-first results, Comput. Math. App., 37, pp. 19–31, (1999)
4. K. T. Atanassov, Intuitionistic fuzzy sets, Fuzzy sets Syst. 20, 87–96,(1986)
5. A. Rosenfeld, fuzzy groups, J. math. Anal.Appl. 35, PP. 512-517, (1971)
6. K. T. Atanassov, Intuitionistic Fuzzy Sets, Physica-Verlag, Heidelberg, N.Y., 1999.
7. P. K. Maji, Neutrosophic soft set, Ann. Fuzzy Math. Inform., 5, pp.157–168, (2013)
8. M. Jdid, F. Smarandache and S. Broumi, Inspection Assignment Form for Product Quality Control Using Neutrosophic Logic, Neutrosophic Systems with Applications, 1, 4–13,(2023) (Doi: https://doi.org/10.5281/zenodo.8171135).
9. S. Dey and G. C. Ray, Separation Axioms in Neutrosophic Topological Spaces, Neutrosophic Systems with Applications, 2(2023), 38–54. (Doi: https://doi.org/10.5281/zenodo.8195851);
10. M. Jdid and F. Smarandache, The Use of Neutrosophic Methods of Operation Research in the Management of Corporate Work, Neutrosophic Systems with Applications, 3 (2023), 1–16. (Doi: https://doi.org/10.5281/zenodo.8196397);
11. M. Abdel-Basset, M. Saleh, A. Gamal, and F. Smarandache, An approach of TOPSIS technique for developing supplier selection with group decision making under type-2 neutrosophic number, Applied Soft Computing, 77, pp.438-452, (2019)
12. R. H. Bruck, Contribution to the theory of quasigroups, Trans. Amer. Math. Soc., Vol. 60, pp. 245-354, (1946)
13. M. Abdel-Baset and V. Chang, and A. Gamal, Evaluation of the green supply chain management practices: A novel neutrosophic approach, Computers in Industry, 108, pp. 210-220, (2019)
14. F. Smarandache, Neutrosophy. Neutrosophic Probability, Set and Logic, American Research Press: Rehoboth, IL, USA, (1998).
15. F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets. Int. J. Pure Appl. Math., 24, pp. 287–297, (2005)
16. A. Solairaju and R. Nagarajan, A new structure and construction of Q-fuzzy groups, Advances in Fuzzy Mathematics, 4, pp.23-29, (2009)
17. S. Thiruveni and A. Solairaju, Neutrosophic Q-fuzzy subgroups, Int. J. Math. And Appl., 6, pp. 859-866, (2018)
18. M. Abu-Qamar, and N. Hassan, Characterization of group theory under Q-Neutrosophic soft Environment, Neutrosophic Sets and system, Vol. 27, pp. 114-131, (2019).
19. Muhammad Shabi, Mumtaz Ali, Munazza Naz, and Florentin Smarandache, Soft Neutrosophic Group, Neutrosophic Sets and Systems, Vol. 1, pp. 13-25, (2013)
20. M. Ali, F. Smarandache, M. Shabir, Soft Neutrosophic Groupoids and Their Generalization, Neutrosophic Sets and Systems, Vol. 6, pp. 62-81, (2014)
21. B. V. N. Prasad and J. Venkateswara Rao, Characterization of Quasigroups and Loops. International Journal of Scientific and Innovative Mathematical Research (IJSIMR) 1, 2, pp. 95-102, (2013)

22. Davvaz .B, Dudek W. A and Yun. Y. B, Intuitionistic fuzzy sub hyper quasi groups of hyper quasi groups, Information sciences, 170, pp. 251-262, (2005)
23. W. A. Dudek, Fuzzy subquasigroups, Quasigroups and Related Systems 5, pp. 81-98, (1998)
24. W. A. Dudek and Y. B. Jun, Fuzzy subquasigroups over a t-norm, Quasigroups and Related Systems 6, pp. 87-89, (1999)
25. Y. T.Oyebo, B. Osoba and A. O. Abdulkareem, Distributive Properties of $Q-$neutrosophic Soft Quasigroups, Neutrosophic Set Systems.58, pp. 448-524, (2023)
26. B. Osoba and Y. T. Oyebo, *On the Core of Second Smarandach Bol Loops*, International Journal of Mathematical Combinatorics, 2, pp. 18–30, (2023). http://doi.org/10.5281/zenodo.32303.
27. K. H, W. A. Dudek and Y. B. Jun, Intuitionistic Fuzzy subquasigroups of quasigroups, Quasigroups and Related Systems 7, pp. 15-28,(2000)
28. W. A. Dudek, Intuitionistic Fuzzy approach to n ary systems, Quasigroups and Related Systems 13, pp. 213-228, (2005)
29. Oyem. A, Olaluru. J. O, Jaiyeola. T. G and Akewe. H, Soft quasigroup, International Journal of mathematical Sci. Opt.: Theory and appl. 2, pp. 834-846,(2020)
30. A. Oyem, T. G. Jaiyeola, J. O. Olaleru and B. Osoba, Soft Neutrosophic quasigroups, Neutrosophic Set Systems. 50, 488–503, (2022)
31. A. I. Mal'tsev, Algebraic Systems. Nauka, Moscow, 1976 (in Russian)
32. O. Hala. Pflugfelder, Quasigroups and loops: introduction, Sigma Series in Pure Mathematics Volume 7, (1990)
33. Jaiyeola, Temitope Gbolahan; Kehinde Adam Olurode; and Benard Osoba, Some Neutrosophic Triplet Subgroup Properties and Homomorphism Theorems in Singular Weak Commutative Neutrosophic Extended Triplet Group, Neutrosophic Sets and Systems 45, 1, (2021).

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Secondary k-column symmetric Neutrosophic Fuzzy Matrices

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Abstract: **Objective:** The objective of this study is to establish the results of secondary k- column symmetric (CS) Neutrosophic fuzzy matrices. **Methods and Findings:** We have applied CS condition in neutrosophic environment to find the relation between s-k CS, s- CS, k- CS and CS. **Novelty**: We establish the necessary and sufficient criteria for s-k CS Neutrosophic fuzzy matrices and various g-inverses of an s – k CS Neutrosophic fuzzy matrices to be an s – k CS. The generalized inverses of an s – k CS P corresponding to the sets P{1, 2}, P{1, 2, 3} and P{1, 2, 4} are characterized.

Keywords: Neutrosophic fuzzy matrices (NFM), s-column symmetric, k-column symmetric, column symmetric.

1. Introduction

Zadeh [1] has studied fuzzy set (FS). Atanassov [2] introduced intuitionistic FSs. Smarandache [3] has discussed the concept of neutrosophic sets. Khan, Shyamal, and Pal [4] have studied intuitionistic fuzzy matrices (IFMs) for the first time. Atanassov [5,6] has discussed IFS and Operations over IV IFS. Hashimoto [7] has studied Canonical form of a transitive matrix. Kim and Roush [8] have studied generalized fuzzy matrices. Lee [9] has studied Secondary Skew Symmetric, Secondary Orthogonal Matrices. Hill and Waters [10] have analyzed On k-Real and k-Hermitian matrices. Meenakshi [11] has studied Fuzzy Matrix: Theory and Applications.

Anandhkumar [12,13] has studied Pseudo Similarity of NFM and On various Inverse of NFM. Punithavalli and Anandhkumar [14] have studied Reverse Sharp and Left-T And Right- T Partial Ordering on IFM. Pal and Susanta Kha [15] have studied IV Intuitionistic Fuzzy Matrices. Vidhya and Irene Hepzibah [16] have discussed on Interval Valued NFM. Anandhkumar et.al [17,18] has focused on Reverse Sharp and Left-T Right-T Partial Ordering on NFM and IFM. Anandhkumar,et.al have studied [19] Partial orderings, Characterizations and Generalization of k-idempotent NFM. Here, we introduce the Secondary k-CS NFM and introduce some basic operators on NFMs.

1.1 Literature Review

Meenakshi and Jaya Shree [20] have studied On k-kernel symmetric matrices. Meenakshi and Krishanmoorthy [21] have characterized On Secondary k-Hermitian matrices. Meenakshi and Jaya Shree [22] have studied On k -range symmetric matrices. Jaya shree [23] has studied Secondary κ-Kernel Symmetric Fuzzy Matrices. Shyamal and Pal [24] Interval valued Fuzzy matrices. Meenakshi and Kalliraja [25] have studied Regular Interval valued Fuzzy matrices. Anandhkumar [26] has studied Kernal and k-kernal Intuitionistic Fuzzy matrices. Jaya Shree [27] has discussed Secondary κ-range symmetric fuzzy matrices. Anandhkumar et.al.,[28] have studied Generalized Symmetric NFM. Kaliraja and Bhavani [29] have studied Interval Valued Secondary κ-Range Symmetric Fuzzy Matrices,

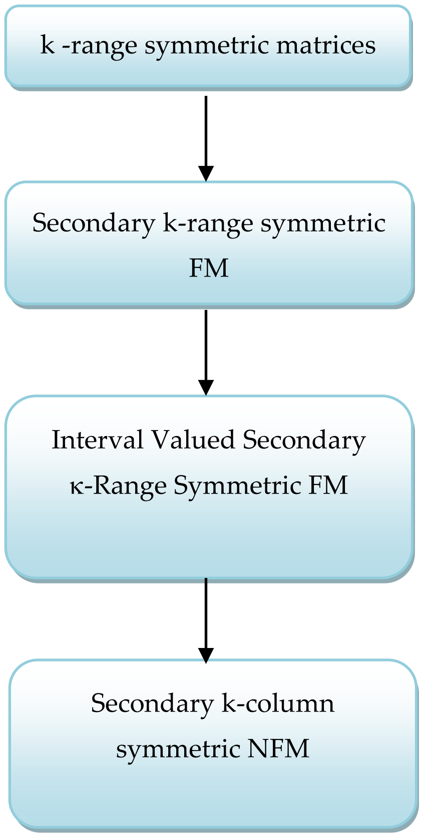
Let P be any fuzzy matrix, $P^{\dagger}$ occurs then this will coincides with the transpose of the matrix (P^T). The fuzzy matrix P belongs to Fn is known to be kernel symmetric matrix, then this shows that $N(P) = N(P^T)$ which does not implies $R[P] = R[P^T]$. But the converse is true. Symmetric matrices are established in the field of complex entries for the theory of k - hermitian matrices. This idea make use of the development of k - EP matrices in the generalization of k - hermitian matrices and also EP matrices. Hill and Waters [30] have initiated the study on κ - real and κ - Hermitian matrices. The concept of Theorems on products of EPr matrices introduced by Baskett and Katz [30]. It is commonly known that for complex matrices, the concepts of range and kernel symmetric are equivalent. But this is fails for Interval valued fuzzy matrices.

The concept of interval valued s - k Hermitian and interval valued kernel symmetric matrices for fuzzy matrices. We also expanded many basic conclusions on these two types of matrices. An Interval valued secondary s - k kernel symmetric fuzzy matrix can be described. Suitable standards for determining g - inverses of an Interval valued secondary s - k - kernel symmetric fuzzy matrices are interval valued secondary s - k - kernel symmetric are found. We establish the necessary and sufficient canditions for an interval valued s - k kernel symmetric fuzzy matrices. Meenakshi, Krishnamoorthy and Ramesh [31] have studied on s - k - EP matrices. Meenakshi and Krishnamoorthy [32] have introduced the idea of s - k hermitian matrices.

Shyamal and Pal [33] have studied Interval valued Fuzzy matrices. The definition of k-symmetric matrices was introduced by the following authors Ann Lec [34] has studied Secondary symmetric and skew symmetric secondary orthogonal matrices. . Anandhkumar et.al [35] have discussed Interval Valued Secondary k-Range Symmetric NFM.

Table:1 Extension of Neutrosophic Fuzzy Matrices based on previous works

References	**Extension of Neutrosophic Fuzzy Matrices from Fuzzy Matrices**	**Year**
[20]	On k-kernel symmetric matrices	2009
[22]	On k -range symmetric matrices	2009
[23]	Secondary k-Kernel Symmetric Fuzzy Matrices	2014
[27]	Secondary k-range symmetric FM	2018
[29]	Interval Valued Secondary κ-Range Symmetric Fuzzy Matrices	2022
Proposed	Secondary k-column symmetric Neutrosophic Fuzzy Matrices	2023



From Table 1 and process flow, it is observed that the previous studies are on k-Kernel, K-range, Secondary k-Kernel and Secondary k- range using fuzzy matrices. It is evident that there is a research gap of these studies in Neutrosophic environment. So, based on the above observation, we have established the results of K-column and Secondary k- column in neutrosophic fuzzy matrices.

Notations:

P^T = Transpose of the matrix P

P^+ = Moore-penrose inverse of P

CS = Column symmetric

C(P) = Column space of P

2. Generalized Symmetric NFM

Definition: 2.1 Let P be a NFM, if $C[P] = C[P^T]$ then P is said to be CS.

Example:2.1 Let us consider $P = \begin{bmatrix} <0.3,0.5,0.4> & <0,0,1> & <0.7,0.2,0.5> \\ <0,0,1> & <0,0,1> & <0,0,1> \\ <0.7,0.2,0.5> & <0,0,1> & <0.3,0.2,0.4> \end{bmatrix}$,

The following NFM are not CS

$$P=\begin{bmatrix}<1,1,0> & <1,1,0> & <0,0,1>\\ <0,0,1> & <1,1,0> & <1,1,0>\\ <0,0,1> & <0,0,1> & <1,1,0>\end{bmatrix},\ P^T=\begin{bmatrix}<1,1,0> & <0,0,1> & <0,0,1>\\ <1,1,0> & <1,1,0> & <0,0,1>\\ <0,0,1> & <1,1,0> & <1,1,0>\end{bmatrix},$$

$$\left(<1,1,0> \quad <0,0,1> \quad <0,0,1>\right)^T \in C(\mathrm{P}), \qquad \left(<1,1,0> \quad <0,0,1> \quad <0,0,1>\right)^T \notin C(\mathrm{P}^T)$$

$$\left(<1,1,0> \quad <1,1,0> \quad <0,0,1>\right)^T \in C(\mathrm{P}), \qquad \left(<1,1,0> \quad <1,1,0> \quad <0,0,1>\right)^T \in C(\mathrm{P}^T)$$

$$\left(<0,0,1> \quad <1,1,0> \quad <1,1,0>\right)^T \in C(\mathrm{P}), \qquad \left(<0,0,1> \quad <1,1,0> \quad <1,1,0>\right)^T \in C(\mathrm{P}^T)$$

$$C(\mathrm{P}) \notin C(\mathrm{P}^T)$$

Definition 2.2: A NFM P $\in$ F_n is s-symmetric NFM $\Leftrightarrow$ $P = VP^TV$.

Example:2.2 Let us consider $P=\begin{bmatrix}<0.4,0.3,0.2> & <0,0,1> & <0.5,0.4,0.3>\\ <0,0,1> & <0,0,1> & <0,0,1>\\ <0.5,0.4,0.3> & <0,0,1> & <0.3,0.2,0.4>\end{bmatrix},$

$$V=\begin{bmatrix}<0,0,0> & <0,0,0> & <1,1,0>\\ <0,0,0> & <1,1,0> & <0,0,0>\\ <1,1,0> & <0,0,0> & <0,0,0>\end{bmatrix}$$

Definition 2.3: A NFM P $\in$ F_n is s-CS NFM $\Leftrightarrow$ $C(P) = C(VP^TV)$.

Example:2.3 Let us consider $P=\begin{bmatrix}<0.7,0.4,0.5> & <0,0,1> & <0.8,0.2,0.1>\\ <0,0,1> & <0,0,1> & <0,0,1>\\ <0.8,0.2,0.1> & <0,0,1> & <0.5,0.7,0.3>\end{bmatrix},$

$$V=\begin{bmatrix}<0,0,0> & <0,0,0> & <1,1,0>\\ <0,0,0> & <1,1,0> & <0,0,0>\\ <1,1,0> & <0,0,0> & <0,0,0>\end{bmatrix}$$

Definition 2.4: A NFM P $\in$ F_n is s-k-CS NFM $\Leftrightarrow$ $C(P) = C(KVP^TVK)$.

Example:2.4 Let us consider $P=\begin{bmatrix}<0.7,0.3,0.4> & <0.5,0.3,0.4>\\ <0.5,0.3,0.4> & <0.7,0.3,0.5>\end{bmatrix}$,

$$\mathrm{K}=\begin{bmatrix}<1,1,0> & <0,0,0>\\ <0,0,0> & <1,1,0>\end{bmatrix},\ V=\begin{bmatrix}<0,0,0> & <1,1,0>\\ <1,1,0> & <0,0,0>\end{bmatrix},$$

Preliminary: 2.1 Let V is a permutation NFM its satisfies the conditions

(i) $VV^T = V^TV = I_n$

(ii) $V^T = V$

(iii) C(P) = C(VP)

(iv) C(P) = C(KP).

Remark 2.1: We notice that P = KVP^TVK implies that C(P) = C(KVP^TVK)

This is illustrating the following example

Example 2.5. Consider a NFM , $V=\begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix}$,

$$P=\begin{bmatrix} <0.7,0.3,0.4> & <0.5,0.3,0.4> \\ <0.5,0.3,0.4> & <0.7,0.3,0.5> \end{bmatrix}, K=\begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix},$$

$$KVP^TVK=\begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}\begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix}\begin{bmatrix} <0.7,0.3,0.4> & <0.5,0.3,0.4> \\ <0.5,0.3,0.4> & <0.7,0.3,0.5> \end{bmatrix}$$
$$\begin{bmatrix} <0,0,0> & <1,1,0> \\ <1,1,0> & <0,0,0> \end{bmatrix}\begin{bmatrix} <1,1,0> & <0,0,0> \\ <0,0,0> & <1,1,0> \end{bmatrix}$$

$$KVP^TVK=\begin{bmatrix} <0.7,0.3,0.4> & <0.5,0.3,0.4> \\ <0.5,0.3,0.4> & <0.7,0.3,0.5> \end{bmatrix}=P$$

Therefore, C(P) = C(KVP^TVK)

Example 2.6. Consider a NFM

$$K=\begin{bmatrix} <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,1,0> \end{bmatrix}, V=\begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \end{bmatrix}$$

$$P=\begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0.5,0.3,0.4> & <1,1,0> & <0,0,0> \\ <0.4,0.2,0.6> & <0.5,0.3,0.4> & <0,0,0> \end{bmatrix}$$

$$KV=\begin{bmatrix} <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,1,0> \end{bmatrix}\begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \end{bmatrix}$$

$$KV = \begin{bmatrix} <0,1,0> & <0,1,0> & <0,1,0> \\ <0,1,0> & <0,1,0> & <1,1,0> \\ <1,1,0> & <0,1,0> & <0,1,0> \end{bmatrix}$$

$$VK = \begin{bmatrix} <0,0,0> & <0,0,0> & <1,1,0> \\ <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \end{bmatrix} \begin{bmatrix} <0,0,0> & <1,1,0> & <0,0,0> \\ <1,1,0> & <0,0,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,1,0> \end{bmatrix}$$

$$VK = \begin{bmatrix} <0,1,0> & <0,1,0> & <1,1,0> \\ <1,1,0> & <0,1,0> & <0,1,0> \\ <0,1,0> & <1,1,0> & <0,1,0> \end{bmatrix}$$

$$P^T VK = \begin{bmatrix} <0.5,0.8,0.4> & <0.4,0.8,0.6> & <0,0,0.4> \\ <0,0.7,0> & <0.5,0.7,0> & <0,0.7,0> \\ <0,0,0> & <0,0,0> & <1,0,0> \end{bmatrix}$$

$$KVP^T VK = \begin{bmatrix} <0,1,0> & <0,1,0> & <0,1,0> \\ <0,1,0> & <0,1,0> & <1,1,0> \\ <1,1,0> & <0,1,0> & <0,1,0> \end{bmatrix} \begin{bmatrix} <0.5,0.8,0.4> & <0.4,0.8,0.6> & <0,0,0.4> \\ <0,0.7,0> & <0.5,0.7,0> & <0,0.7,0> \\ <0,0,0> & <0,0,0> & <1,0,0> \end{bmatrix}$$

$$KVP^T VK = \begin{bmatrix} <0,0,0> & <0,0.2,0> & <0,0,0> \\ <0,0,0> & <0,0,0> & <1,0,0> \\ <0.5,0,0> & <0.4,0,0> & <0,0,0> \end{bmatrix} \neq P$$

$P \neq KVP^T VK$ is not s- κ –symmetric iff not s- κ-CS.

Theorem 2.1: For NFM P $\in F_n$, the subsequent are equivalent :

(i) C(P) = C (P^T).

(ii) P^T = PH= KP for several IFM H, K and ρ (P) = r.

Lemma 2.1: For NFM P $\in$ F_n and a PM K, C(P) = C(Q) iff C(KPK^T) = C(KQK^T)

Theorem 2.2 . For NFM P $\in F_n$ the subsequent are equivalent

(i) C (P) = C($KVP^T VK$)

(ii) C (KVP) = C($(KVP)^T$)

(iii) C(PKV) = C($(PKV)^T$)

(iv) C(VP) = C($K(VP)^T K$)

(v) C(PK)= C($V(PK)^T V$)

(vi) $C(P^T) = C(KV(P)VK)$

(vii) $C(P) = C(P^T VK)$

(viii) $C(P^T) = C(PKV)$

(ix) $P = VKP^T VKH_1$ for $H_1 \in F_n$

(x) $P = H_1 KVP^T VK$ for $H_1 \in F_n$

(xi) $P^T = KVPVKH$ for $H \in F_n$

(xii) $P^T = HKVPKV$ for $H \in F_n$

Proof: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv)

$\Leftrightarrow$ P is s- κ- Cs

$\Leftrightarrow C(P) = C(KVP^T VK)$

$\Leftrightarrow C(KVP) = C((KVP)^T$ [Preliminary 2.1]

$\Leftrightarrow$ KVP is Column symmetric

$\Leftrightarrow$ VP is κ- Column symmetric

So , (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) hence.

(i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v)

P is s- κ – CS

$\Leftrightarrow C(P) = C(KVP^T VK)$ [By Definition 2.4]

$\Leftrightarrow C(KVP) = C((KVP)^T)$ [Preliminary 2.1]

$\Leftrightarrow C(PKV) = C((PKV)^T)$

$\Leftrightarrow$ PKV is Column symmetric

$\Leftrightarrow$ PK is s- Column symmetric

So , (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (v) hence.

(ii) $\Leftrightarrow$ (vii)

KVP is Column symmetric $\Leftrightarrow C(KVP) = C((KVP)^T)$

$\Leftrightarrow C(P) = C((KVP)^T)$ [Preliminary 2.1]

$\Leftrightarrow C(P) = C(P^T VK)$

So , (ii) $\Leftrightarrow$ (vii) hence.

(iii) $\Leftrightarrow$ (viii):

PVK is Column symmetric $\Leftrightarrow$ $C(PVK) = C((PVK)^T)$

$\Leftrightarrow C(PVK) = C(P^T)$ [Preliminary 2.1]

So ,(iii) $\Leftrightarrow$ (viii) hence.

(i) $\Leftrightarrow$ (vi)

P is s- κ- Column symmetric $\Leftrightarrow C(P) = C(KVP^TVK)$

$\Leftrightarrow C(KVP) = C((KVP)^T)$ [Preliminary 2.1]

$\Leftrightarrow (KVP)^T$ is Column symmetric

$\Leftrightarrow$ P^TVK is Column symmetric

$\Leftrightarrow$ P^T is s- κ - Column symmetric

So , (i) $\Leftrightarrow$ (vi) hence.

(i) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (x)

P is s- κ- Column symmetric $\Leftrightarrow$ $C(P) = C(KVP^TVK)$

$\Leftrightarrow C(P^T) = C(KVPVK)$

$\Leftrightarrow$ $P^T = KVPVKH$ [By Theorem 2.1]

$\Leftrightarrow P = H_1KV P^TVK$ for $H_1 \in F_n$

So , (i) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (x) hence.

(ii) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (ix)

KVP is Column symmetric $\Leftrightarrow$ VP is κ- Column symmetric

$\Leftrightarrow C(VP) = C(K(VP)^TK)$

$\Leftrightarrow$ $C(P) = C(P^TVK)$ [Preliminary 2.1]

$\Leftrightarrow C(P^T) = C(KVP)$

$\Leftrightarrow$ $P^T = HKVP$ for $H \in F_n$ [By Theorem 2.1]

$\Leftrightarrow$ $P^T = HKVPKV$

$\Leftrightarrow$ $P = VKP^TVKH_1$ for $H_1 \in F_n$

So , (ii) $\Leftrightarrow$ (xii) $\Leftrightarrow$ (ix) hence.

Corollary 2.1: For NFM $P \in F_n$ the subsequent are equivalent:

(i) $C(P) = C(VP^TV)$

(ii) $C(VP) = C(VP)^T$

(iii) $C(PV) = C(PV)^T$

(iv) P is s-CS

(v) $C(P^T) = C(VPV)$

(vi) $C(P) = C(P^TV)$

(vii) $C(P^T) = C(PV)$

(viii) $C(KVP) = C((VP)^T)$

(ix) $P = VP^TVH_1$ for $H_1 \in F_n$

(x) $P = H_1VP^TV$ for $H_1 \in F_n$

(xi) $P^T = VPVH$ for $H \in F_n$

(xii) $P^T = HVPV$ for $H \in F$

Theorem 2.3: For NFM $P \in F_n$. Then any two of the subsequent imply the other one:

(i) $C(P) = C(KP^TK)$

(ii) $C(P) = C(VKP^TKV)$

(iii) $C(P^T) = C((VKP)^T)$

Proof: (i) & (ii) $\Leftrightarrow$ (iii)

P is s- κ – Cs

$\Rightarrow C(P) = C(P^TVK)$

$\Rightarrow C(KPK) = C(KP^TK)$ [By Lemma 2.1]

Hence (i) & (ii) $\Rightarrow$ $C(P^T) = C((VPK)^T)$

So,(iii) hence.

(i) & (iii) $\Leftrightarrow$ (ii)

P is κ- Column symmetric $\Rightarrow C(P) = C(KP^TK)$

$\Rightarrow C(KPK) = C(P^T)$ [By Lemma 2.1]

Hence (i) & (iii)

$\Rightarrow$ $C(KPK)= C((VPK)^T)$

$\Rightarrow C(P) = C(P^TVK)$

$\Rightarrow$ $C(P) = C((KVP)^T)$

$\Rightarrow$ P is s- κ CS [By Theorem 2.2]

So, (ii) hence.

(iii) & (ii) implies (i)

P is s- κ – Cs

$\Rightarrow$ $C(P)= C(P^TVK)$

$\Rightarrow C(KPK) = C(KP^TV)$ [Preliminary 2.1]

Hence (ii) & (iii) $\Rightarrow$ $C(KPK) = C(P^T)$

$\Rightarrow C(P) = C(KP^TK)$ [By Lemma 2.1]

$\Rightarrow$ P is κ – Column symmetric

Therefore,(i) hold. Hence the Theorem

3.s- κ-Column Symmetric Regular NFM

In this section, it was discovered that there are various generalized inverses of matrices in NFM. The comparable standards for different g-inverses of s-k CS NFM to be s-k CS are also established. The generalized inverses of an s – κ CS P corresponding to the sets P{1, 2}, P{1, 2, 3} and P{1, 2, 4} are characterized.

Theorem 3.1: Let $\in$ F_n, Z $\in$ P {1,2} and PZ, ZP, are s- κ-CS NFM. Then P is s- κ - CS NFM ⇔ Z is s- κ – CS NFM.

Proof: $C(KVP) = C(KVPZP) \subseteq C(ZP)$ [since P = PZP]

$= C(ZVVP) = N(ZVKKVP) \subseteq C(KVP)$

Hence, $C(KVP) = C(ZP)$

$= C(KV(ZP)^T VK)$

$= C(P^T Z^T VK)$

$= C(Z^T VK)$

$= C((KVZ)^T)$

$C((KVP)^T) = C(P^T VK)$

$= C(Z^T P^T VK)$

$= C((KVPZ)^T)$

$= C(KVPZ)$

$= C(KVZ)$

KVZ is column symmetric $\Leftrightarrow C(KVP) = N((KVP)^T)$

$\Leftrightarrow C((KVZ)^T) = N(KVZ)$

$\Leftrightarrow$ KVZ is CS

$\Leftrightarrow$ Z is s- κ- CS

Theorem 3.2: Let $P \in F_n$, $Z \in P\{1,2,3\}$, $C(KVP) = C((KVZ)^T)$.Then P is s-κ-CS NFM ⇔ Z is s- κ - CS NFM.

Proof: Given $Z \in P\{1,2,3\}$, we have $PZP = P, ZPZ = Z, (PZ)^T = PZ$

$C((KVP)^T) = C(Z^T P^T VK)$ [By using PZP = P]

$= C(KV(PZ)^T)$

$= C((PZ)^T)$ [Preliminary 2.1]

$= C(PZ)$ [$(PZ)^T = PZ$]

$= C(Z)$ [By using Z = ZPZ]

$= C(KVZ)$ [Preliminary 2.1]

KVP is column symmetric NFM $\Leftrightarrow C(KVP) = C((KVP)^T)$

$\Leftrightarrow C((KVZ)^T) = C(KVZ)$

$\Leftrightarrow$ KVZ is column symmetric

$\Leftrightarrow$ Z is s- κ - column symmetric.

Theorem 3.3: Let $P \in F_n$, $Z \in P\{1,2,4\}$, $C((KVP)^T) = C(KVZ)$. Then P is s- κ- CS NFM ⇔ Z is s- κ- CS NFM.

Proof: Given $Z \in P\{1, 2, 4\}$,

$PZP = P, ZPZ = Z, (ZP)^T = ZP$

$C(KVP) = C(P)$ [Preliminary 2.1]

$= C(ZP)$ $[ZPZ = Z, PZP = P] = N((ZP)^T)$ $[(ZP)^T = ZP]$

$= C(P^T Z^T)$

$= C(Z^T)$

$= C((KVZ)^T)$. [Preliminary 2.1]

KVP is column symmetric NFM $\Leftrightarrow C(KVP) = C((KVP)^T$

$\Leftrightarrow C((KVZ)^T) = C(KVZ)$

$\Leftrightarrow$ KVZ is CS NFM

$\Leftrightarrow$ Z is s- κ – CS NFM.

4.Conclusion:

Firstly, we present equivalent characterizations of an k- CS, CS, s- CS, s-k CS NFM. Also,we give the example of s-k-symmetric NFM is s-k- CS Neutrosophic fuzzy matrix the opposite isn't always true. We discussed various generalized inverses of NFM and generalized inverses of an s – k CS P corresponding to the sets P{1, 2}, P{1, 2, 3} and P{1, 2, 4} are characterized.Finally, to conclude we have introduced the concept of secondary k-CS neutrosophic fuzzy matrices. In future we will work on interval valued secondary k-CS neutrosophic fuzzy matrices.

REFERENCES

[1]Zadeh L.A., Fuzzy Sets, Information and control.,(1965),8, pp. 338-353.
[2] K.Atanassov, Intuitionistic Fuzzy Sets: Theory and Applications, Physica-Verlag, 1999.
[3] Smarandache,F, Neutrosophic set, a generalization of the intuitionistic fuzzy set. Int J Pure Appl Math.; .,(2005),.24(3):287–297.
[4] M.Pal, S.K.Khan and A.K.Shyamal, Intuitionistic fuzzy matrices, Notes on Intuitionistic Fuzzy Sets, 8(2) (2002), 51-62.
[5] K.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 ,(1986), 87-96.
[6] K.Atanassov, Operations over interval-valued intuitionistic fuzzy sets, Fuzzy Sets and Systems, 64, (1994) ,159-174.
[7] H.Hashimoto, Canonical form of a transitive matrix, Fuzzy Sets and Systems, 11 (1983),157-162.
[8] K.H.Kim and F.W.Roush, Generalised fuzzy matrices, Fuzzy Sets and Systems, 4, (1980) ,293-315.
[9] A.Lee, Secondary Symmetric, Secondary Skew Symmetric, Secondary Orthogonal Matrices, Period Math, Hungary, 7, (1976) ,63-76.
[10] R.D. Hill and S.R.Waters, On k-Real and k-Hermitian matrices, Linear Algebra and its Applications, 169, (1992), 17-29.
[11] AR.Meenakshi, Fuzzy Matrix: Theory and Applications, MJP Publishers, Chennai, 2008.

[12] M. Anandhkumar. V.Kamalakannan, S.M.Chitra, and Said Broumi, Pseudo Similarity of Neutrosophic Fuzzy matrices, International Journal of Neutrosophic Science, Vol. 20, No. 04, PP. 191-196, 2023 .

[13]M.Anandhkumar. B.Kanimozhi, V.Kamalakannan, S.M.Chitra, and Said Broumi, On various Inverse of Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, Vol. 21, No. 02, PP. 20-31, 2023.

[14] G.Punithavalli and M.Anandhkumar "Reverse Sharp And Left-T And Right- T Partial Ordering on Intuitionistic Fuzzy matrices" Accepted in TWMS Journal 2023.

[15] M Pal and Susanta K. Khan Interval-Valued Intuitionistic Fuzzy Matrices, NIFS 11 (2005), 1, 16-27.

[16] R. Vidhya And R. Irene Hepzibah On Interval Valued Neutrosophic Fuzzy Matrices, Advances and Applications in Mathematical Sciences Volume 20, Issue 4, February 2021, Pages 561-57.

[17] M. Anandhkumar ,T. Harikrishnan, S. M. Chithra , V. Kamalakannan , B. Kanimozhi , Broumi Said ,Reverse Sharp and Left-T Right-T Partial Ordering on Neutrosophic Fuzzy Matrices, International Journal of Neutrosophic Science, Vol. 21, No. 04, PP. 135-145, 2023.

[18] M.Anandhkumar, B.Kanimozhi, S.M. Chithra, V.Kamalakannan, .Reverse Tilde (T) and Minus Partial Ordering on Intuitionistic Fuzzy Matrices, Mathematical Modelling of Engineering Problems, 2023, 10(4), pp. 1427–1432.

[19] M. Anandhkumar ,T. Harikrishnan,S. M. Chithra,V. Kamalakannan,B. Kanimozhi. "Partial orderings, Characterizations and Generalization of k-idempotent Neutrosophic fuzzy matrices." International Journal of Neutrosophic Science, Vol. 23, No. 2, 2024 ,PP. 286-295.

[20] AR.Meenakshi and D.Jaya Shree, On k-kernel symmetric matrices, International Journal of Mathematics and Mathematical Sciences, 2009, Article ID 926217, 8 Pages.

[21] AR.Meenakshi and S.Krishanmoorthy, On Secondary k-Hermitian matrices, Journal of Modern Science, 1 (2009) 70-78.

[22] AR.Meenakshi and D.Jaya Shree, On K -range symmetric matrices, Proceedings of the National conference on Algebra and Graph Theory, MS University, (2009), 58- 67.

[23] D.Jaya shree , Secondary κ-Kernel Symmetric Fuzzy Matrices, Intern. J. Fuzzy Mathematical Archive Vol. 5, No. 2, 2014, 89-94 ISSN: 2320 –3242 (P), 2320 –3250 , Published on 20 December 2014.

[24] A. K. Shyamal and M. Pal, Interval valued Fuzzy matrices, Journal of Fuzzy Mathematics 14(3) (2006), 582-592.

[25] A. R. Meenakshi and M. Kalliraja, Regular Interval valued Fuzzy matrices, Advance in Fuzzy Mathematics 5(1) (2010), 7-15.

[26] G.Punithavalli and M.Anandhkumar "Kernal and k-kernal Intuitionistic Fuzzy matrices" Accepted in TWMS Journal 2022.

[27]D. Jaya Shree, Secondary κ-range symmetric fuzzy matrices, Journal of Discrete Mathematical Sciences and Cryptography 21(1):1-11,2018.

[28] M. Anandhkumar; G.Punithavalli; T.Soupramanien; Said Broumi, Generalized Symmetric Neutrosophic Fuzzy Matrices, Neutrosophic Sets and Systems, Vol. 57,2023, 57, pp. 114–12.

[29]M. Kaliraja And T. Bhavani, Interval Valued Secondary κ-Range Symmetric Fuzzy Matrices, Advances and Applications in Mathematical Sciences Volume 21, Issue 10, August 2022, Pages 5555-5574.
[30]Baskett T. S., and Katz I. J., (1969), Theorems on products of EPr matrices," Linear Algebra and its Applications, 2, 87–103.
[31]Meenakshi AR., Krishnamoorthy S., and Ramesh G., (2008) on s-k-EP matrices", Journal of Intelligent System Research, 2, 93-100.
[32] Meenakshi AR., and Krishanmoorthy S.,(2009), on Secondary k-Hermitian matrices, Journal of Modern Science, 1, 70-78.
[33]Shyamal A. K., and Pal. M., (2006) , Interval valued Fuzzy matrices, Journal of Fuzzy Mathematics 14(3), 582-592.
[34]Ann Lec.,(1976), Secondary symmetric and skew symmetric secondary orthogonal matrices (i) Period, Math Hungary, 7, 63-70.
[35]Anandhkumar, M.; G. Punithavalli; R. Jegan; and Said Broumi.(2024) "Interval Valued Secondary k-Range Symmetric Neutrosophic Fuzzy Matrices." Neutrosophic Sets and Systems 61, 1 .

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A Neutrosophic Study for the Transmission of Infection with Pathogenic Fungi from Males of Olive Fly Insects to Their Females

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Abstract: This paper presents the study of the effectiveness of horizontal transfer of local isolates of the pathogenic fungus Beauveria bassiana (Balsamo) on adults of olive fruit fly Bactrocera oleae (Rossi) at a concentration of 10^6 spores/ml in laboratory conditions (this work was carried out in specialized scientific laboratories). In addition, it is not possible to reach the desired results in such experiments effectively when the data and observations of the study are not clear and accurate. For this reason, in this paper, experimental data will be presented with inaccurate or uncertain observations using neutrosophic statistics. The purpose is to know the success of males contaminated with pathogenic isolates in the transmission of infection to females. In laboratory conditions through a neutrosophic reading of the study data. This proposed presentation provides greater accuracy, flexibility, and applicability than the classic experimental design in the case of uncertainty.

Keywords: Beauveria bassiana, bactrocera oleae, horizontal transmission, neutrosophic logic.

1. Introduction

Entomopathogenic fungi are the most common and easiest to distinguish insect pathogens. It is characterized by its superiority in terms of species, the wide range of its terrestrial and aquatic hosts, and its ability to form spores with which to resist unsuitable environmental conditions. In addition, it would have been possible, through these characteristics, to reach the epidemiological level if it were not for its close association with environmental conditions such as humidity and heat [9,15,18].

Early symptoms of infection begin with the host insect stopping feeding and losing balance with slow movement [13]. By penetrating the fungus' hyphal cells, we notice dark black spots resulting from the deposition of melanin at the hyphal penetration sites. The mycelium outside the insect is the most prominent manifestation of infection [18].

Infection of insects with fungal diseases goes through four successive steps:

1. Contact between the host and the sites of germination.
2. Adhesion and germination of the sporophyte tube.
3. Penetration and invasion of the fungus into the tissues and organs of the host under anaerobic conditions.
4. The death of the host Balsamo to the natural obstruction of the alimentary canal, bronchi, and circulatory systems, poisoning or physiological starvation) and the production of blastospores which are contagious, and the transition to the throwing state that ends with sporulation on the surface of the host's body [21].

The time required for the pathogenic fungus to kill the insect varies according to several factors, including the stage of the insect, humidity, and the pathogenic fungus itself. Most pathogenic fungi need 3-12 days from infection until the insect dies [18]. The fungus secretes a group of secondary metabolites and mycotoxins that are chemically diverse and vary according to the genetic strain of the fungus. These toxins are Beauvericin, Bas-sianin, Beauverolides, Bassianolide and Tenellin, which kill the host by destroying its tissues and degrading its cells, in addition, the growth of the fungus impedes the path of the blood fluid. In addition, by feeding the fungus, it depletes the nutrients present in the host's body, and thus the body organs of the insects infected with it die [11, 20].

These toxins can weaken and kill the insect before the mycelium fully developed inside the insect's body [20]. The pathogenic fungus can also kill the insect through its entry into the Gut of the larvae, killing them from starvation [12].

Many studies have proven the ability of pathogenic fungi to infect insects and cause death to them. Therefore, this research was conducted to study the possibility of transmission of infection from males treated with pathogenic fungi to females, from a neutrosophic point of view. This opens the way for dealing with issues surrounding study data that are not precisely defined.

Neutrosophic means the study of ideas and concepts that are neither right nor wrong, but between that, and this means (neutrality, indeterminacy, ambiguity, contradiction, and others), and that every field of knowledge and experience has its neutrosophic part, that part that contains indeterminacy. The first to lay the foundations of the neutrosophic was the American philosopher and mathematician, "Florentin Smarandache", who presented neutrosophic logic in 1995 as a generalization of fuzzy logic [1,2]. As an extension of this, Ahmed Salama presented the theory of classical neutrosophic sets as a generalization of the theory of classical sets [3,4]. The neutrosophic has grown significantly in recent years. Many researchers have worked in the neutrosophic field of science around the world such as Huda E. Khalid et al [16,17]. Because it formed a real revolution in science through its application in many disciplines and scientific and practical fields [5-8]. In this research, we highlight the application of neutrosophic logic to the study data so that we have three cases (dead, indefinite, injured) instead of two cases as in the classical logic that does not recognize the existence of uncertain cases.

2. Research Materials and Methods:

1-Obtaining olive fruit fly adults: olive fruit fly larvae and pupae collected from the dissection of infected fruits collected from olive trees Taken from [14].

2-Isolation of the pathogenic fungus: An isolation of the fungus Beauveria bassiana approved, which follows the scientific classification. According to [19].

Kingdom of fungi, Department of Ascetic Fungi, Row: Sordariomycetes

Order: Hypocreales, Family: Clavicipitaceae, Genus: Beauveria

Genre: (Balsmo.criv.) (vuill,1912) B. bassiana.

3. Search Objective

Studying the role of male olive fruit fly in transmitting infection with the fungus Beauveria bassiana to females. Under laboratory conditions through a neutrosophic viewpoint. (That allows us to obtain incomplete or unclear information about the transmission of infection or the emergence of symptoms).

4. The Method of Work

The concentrations of 10^6 spores/ml of the sporophyte suspension of the pathogenic fungus B. bassiana were tested at a rate of 5 replications. 4 males were sprayed with the sporophyte suspension at a rate of 1 ml of the tested concentration at the age of 0-24 hours, after placing them in a glass tube and in the refrigerator at a temperature of 4ºC for a period 2-5 minutes to reduce the movement of flies. Then the contaminated males were added to 4 females aged 0-24 hours in plastic containers with a diameter of 10 cm, and a height of 20 cm. 5 replicates were approved for each concentration. Males and females were monitored, and the possibility of pathogenic fungi transmitting to females by mating or attempting to mate in laboratory conditions was recorded, while the control males were treated with distilled water. The plastic containers were placed in the incubator at a temperature of 25±2ºC, a humidity of 60± 5%, and an illumination of 12:12 (dark: light). The death rates were recorded every 48 hours starting from the fourth day (when the insects had matured sexually and became able to mate) for 8 days after treatment.

5. Results and Discussion

The males contaminated with the pathogenic fungus by spraying the sporophyte suspension in the laboratory achieved success in transmitting the infection to the females. The death of females started on the sixth day of treatment, while the males started on the fourth day of treatment. In addition, the following study shows us in days (4-6-8) the Corrected death rates and infection rates, as well as the unspecified percentages that range between the healthy and the injured who have not yet shown symptoms.

On the fourth day. The death rate of males from the treatment was 45%, and 25% of the males had symptoms that ranged from simple to severe symptoms such as slow motion or even stopping movement and going up to the top of the breeding box. In addition, there are 30% (unspecified percentage) of Males did not show any symptoms. but this does not mean that these males are healthy, as they may be carriers of spores and are able to transmit them to females even if they are resistant to them. As for females, no death rate was recorded, and 20% of them showed some symptoms of the disease, such as slow movement and lack of nutrition. Therefore, 80% of females are not determined if they are healthy or infected, but they have not yet shown symptoms of the disease.

time (day)						isolation B. bassiana	spore concentration/ml 10^6
4							
Female			Male				
Undefined "He showed no symptoms"	A patient who has symptoms	Corrected death rate	undefined "He showed no symptoms"	A patient who has symptoms	Corrected death rate		
80%	20%	%0	30%	25%	45%		

Table (1): Corrected Death Rates and Infestation of Adult Olive Fruit Fly (When Males were Treated with an Isolate of the Pathogenic Fungus B. Bassiana in Vitro)

On the sixth day of treatment. The death rate of males reached 77.8%, as the fungus spores on it, and its secretion of toxic toxins affected the males greatly. In addition, 15% of the infected males showed symptoms ranging from mild to severe, and therefore 7.2% of the males were not determined whether they were healthy or infected and did not show symptoms yet.

As for females, the death rate was 20%, and 15% of the females' showed symptoms of infection ranging from mild to severe, and therefore 65% of females are not determined whether they are healthy or infected and have not shown symptoms yet.

time (day)						Isolation B. bassiana	spore concentration/ml 10^6
6							
Female			Male				
A patient who has symptoms	Corrected death rate	A patient who has symptoms	Corrected death rate	A patient who has symptoms	Corrected death rate		
65%	15%	%20	7.2%	15%	77.8%		

Table (2): Corrected Death Rates and Infestations for Adult Olive Fruit Flies (When Males were Treated with an Isolate of the Pathogenic Fungus B. Bassiana in the Laboratory).

On the eighth day of treatment. the death rate of males reached 90%, and 7% of the males showed symptoms ranging from mild to severe, and therefore 3% of the males were unspecified (if they were completely healthy or infected, the symptoms did not appear yet).

The death rate of females reached 35%, 20% of infected people showed symptoms ranging from mild to severe, and 45% were unspecified (healthy or injured, no symptoms appeared yet).

time (day)						isolation *B. bassiana*	spore concentration/ml 10^6
8							
Female			Male				
A patient who has symptoms	Corrected death rate	A patient who has symptoms	Corrected death rate	A patient who has symptoms	Corrected death rate		
45%	20%	%35	3%	7%	90%		

Table (3): Corrected Death Rates and Infestation for Adult Olive Fruit Fly (When Males were Treated with an Isolate of the Pathogenic Fungus B. Bassiana in the Laboratory).

6. Conclusion and Results

This paper concludes that studying the role of male olive fruit fly in transmitting infection with the fungus Beauveria bassiana to females under laboratory conditions through a neutrosophic point of view provides a more general and clear view (In the transmission of infection between insects). One of the well-known classic methods ends with the insect being infected or healthy only and eliminating the idea and the state of the existence of uncertainty. That is, it is possible that there is an unspecified case that appears healthy (and did not show any symptoms of infection), yet it is a carrier of the disease and causes infection. The results of the research either indicate that males are carriers of the disease, clearly and explicitly, or infected with no symptoms yet, but they can transmit it to females in both cases. Where the males are carriers of spores and transmit them to the females through mating or attempting to mate, and this is the aim of the study. Thus, according to our study, the chance of transmitting the disease from males to females becomes higher. This provides a correct view of the shortest possible time to achieve the goal, which is the largest possible infection rate and therefore the highest death rates, and we get rid of this insect and its damage to olive fruits as soon as possible. We look forward soon to generalizing this study to other types of insects.

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References

[1] Smarandache, F, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA, 2002.

[2] Smarandache, F. A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Neutrosophic Probability. American Research Press, Rehoboth, NM, 1999.

[3] A. A. Salama, F. Smarandache Neutrosophic Crisp Set Theory, Educational. Education Publishing 1313 Chesapeake, Avenue, Columbus, Ohio 43212, (2015).

[4] A. A. Salama and F. Smarandache. "Neutrosophic crisp probability theory & decision-making process." Critical Review: A Publication of Society for Mathematics of Uncertainty, vol. 12, p. 34-48, 2016.

[5] R. Alhabib, M. Ranna, H. Farah and A. A. Salama, ''Foundation of Neutrosophic Crisp Probability Theory'', Neutrosophic Operational Research, Volume III, Edited by Florentin Smarandache, Mohamed Abdel-Basset and Dr. Victor Chang (Editors), pp.49-60, 2017.

[6] R. Alhabib, A. A Salama, ''Using Moving Averages To Pave The Neutrosophic Time Series'', International Journal of Neutrosophic Science (IJNS), Volume III, Issue 1, PP: 14-20, 2020.

[7] Jdid .M, Alhabib.R, and A. A. Salama, The static model of inventory management without a deficit with Neutrosophic logic, International Journal of Neutrosophic Science (IJNS), Volume 16, Issue 1, PP: 42-48, 2021.

[8] Jdid .M, Alhabib. R and Salama. A. A, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic Sets and Systems, 49, 2022

[9] Araujo, J.P. and Hughes, D.P. Diversity of entomopathoogens Fungi: which groups conquered the insect body. Bio rxiv, p 3756. 2014.

[10] Alhabeeb, A. F. ,Nammour, D.H and Ali .Y.A . The pathogenicity of local isolates of Beauveria bassiana (Balsmo) Vuill for larvae of Bactrocera oleae (Rossi. 1790), Journal Albaath University, and volume.39. 2017 (in Arabic).

[11] Baverstock, J., Roy, H.E., Clark, S.J. and Bell. Effect of fungal infection on the reproductive potential of aphids and their progeny, Presented at Soc, Inverteberpathol. Helsinki, 17:63-66. 2004.

[12] Cheung ,P.Y.K.,E,A. Grula. In Vivo, events associated with Entomopathology of Beauveria bassiana for the corn earworm (Heliothis zea) J Econ Entomol. 39,303-313. 1982.

[13] Ekesi,S. Pathogenicity and antifeedant activity of entomopathogenic hyphomycetes to the cowpea leaf beetle, Otheca mutabilis Shalberg .insect science and its application 31:55-60. 2001.

[14] Genc , H., Nation JL. Survival and development of Bactrocera oleae Gmelin .Diptera: Tephritidae. Immature stages at four temperatures in the laboratory. Afr. J. Biotechnol. 7:2495–500. 2008.

[15] Hajek, A.E.and St. Leger, R.J. Interactions between fungal pathogens and insect hosts. Annual Review of Entomology, 39(1): 293-322. 1994.

[16] Huda E. Khalid, Güngör, G.D., and Zainal, M.A.N. ''Neutrosophic SuperHyper Bi-Topological Spaces: Original Notions and New Insights'', Neutrosophic Sets and Systems, 2022, 51, pp. 33–45.

[17] Salama, A.A., Huda E. Khalid, Elagamy, H.A., ''Neutrosophic Fuzzy Pairwise Local Function and Its Application'' Neutrosophic Sets and Systems, 2022, 49, pp. 19–31

[18] Macleod, D.M. and Muller.K. Entomophthora species with pear-shaped to almost spherical conidia (Entomophthorales. Entomophthoraceae), Mycologica. In: Entomogenous fungi. Mycologica: 823-893. 1973.

[19] Roy, H. E., Steinkraus, D. C., Eilenberg G, J., Hajek, A.E. and Pell, J. K. Bizarre interactions and endgame: Entomopathogenic fungi and their arthropod hosts, Annual.Review. Entomology, 51: 331-375. 2006.

[20] Roberts, D.W. .Toxins of entomopathogenic fungi. In: Burges HD, editor. Microbial control of pests and plant diseases 1970-1980. London: Academic Press .pp 441-464. 1981.

[21] Wraight, S.P. Carruthers, R. I., Bradly, C. A., Jaronski, S. T., Lacey, L. A., Wood. P. ,& Galaini-Wraight, S. Pathogenicity of the entomopathogenic fungi Paecilomyces spp. And Beauveria bassiana against the silverleaf whighefly, Bemisia argentifolii. J Inverterbrate Pathology, 71(3), 217-226. 1998.

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The neutrosophic quaternions numbers

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Abstract: This article aims to study the neutrosophic quaternion numbers, where we defined the neutrosophic quaternions numbers and the two equal neutrosophic quaternions numbers, also, the neutrosophic quaternions numbers algebra were introduced by studying addition, multiplication, division and conjugate of a neutrosophic quaternions number. In addition, we have discussed how to calculate the absolute value of a neutrosophic quaternions number and its inverted.

Keywords: neutrosophic; quaternion numbers; division; multiplication; the absolute value of a neutrosophic quaternions number.

1. Introduction and Preliminaries

In an attempt to replace the current logics, Smarandache introduced the neutrosophic logic to illustrate a mathematical model of redundancy, uncertainty, contradiction, unknown, ambiguity, undefined, inconsistency, vagueness, imprecision, and incompleteness. Smarandache defined neutrosophic real number [2-4], probabilities according to neutrosophic logic [3-5-13], the neutrosophic statistics [4][6], he has also introduced the concept of integration and differentiation in neutrosophic [1-8]. Madeleine Al- Taha presented results on single valued neutrosophic (weak) polygroups [9]. Chakraborty utilized pentagonal neutrosophic number in networking problems, and Shortest Path Problems [11-12]. Yaser Alhasan probed the concepts of neutrosophic in the complex numbers [7-14-10].

Paper consists of 3 sections. In 1th section, provides an introduction, in which neutrosophic science review has given. In 2th section, frames the neutrosophic quaternion numbers. In 3th section, a conclusion to the paper is given.

2. Main Discussion

The neutrosophic quaternions numbers

Definition1

We call the numbers that take the form:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

the neutrosophic quaternions numbers, denoted by symbol H_N; where $\acute{c}, \acute{d}, \acute{c}_1, \acute{d}_1, \acute{c}_2, \acute{d}_2, \acute{c}_3, \acute{d}_3$ are real numbers, while $I =$ indeterminacy and $\breve{\imath}, \breve{\jmath}, \breve{k}$ are units such that:

$$\breve{\imath}^2 = \breve{\jmath}^2 = k^2 = \breve{\imath}\breve{\jmath}\breve{k} = -1$$
$$\breve{\imath}\breve{\jmath} = \breve{k} = -\breve{\jmath}\breve{\imath}$$
$$\breve{\jmath}\breve{k} = \breve{\imath} = -\breve{k}\breve{\jmath}$$
$$\breve{k}\breve{\imath} = \breve{\jmath} = -\breve{\imath}\breve{k}$$

We can noted that every neutrosophic quaternions number has two parts, a neutrosophic real (scalar) part and a neutrosophic vector part, where:

$\acute{c} + \acute{d}I$ is the neutrosophic real (scalar) part and $(\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$ is the neutrosophic vector part

Example 1

1) $q_I = 3 + 7I + (-4 + 8I)\breve{\imath} + (7 - 3I)\breve{\jmath} - (5 + 9I)\breve{k}$
2) $q_I = 4I\breve{\imath} + (5 + I)\breve{\jmath} + (-1 + 2I)\breve{k}$
3) $q_I = 3I + (2 + 3I)\breve{\imath} + (-1 + 2I)\breve{k}$
4) $q_I = 6 + I + (9 - 4I)\breve{\imath}$

Note:

- ✓ $0_{H_N} = 0 + 0I + (0 + 0I)\breve{\imath} + (0 + 0I)\breve{\jmath} + (0 + 0I)\breve{k}$
- ✓ $0_{H_N} = 1 + 0I + (0 + 0I)\breve{\imath} + (0 + 0I)\breve{\jmath} + (0 + 0I)\breve{k}$

Definition2
Let $q_I,\ p_I \in H_N$ where:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

$$p_I = \acute{\alpha} + \acute{b}I + \acute{u}_I = \acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}$$

then: $q_I = p_I$ if and only if:

$$\acute{c} = \acute{\alpha} \ \text{and} \ \acute{v}_I = \acute{u}_I$$

hence:

$$\acute{c}_1 + \acute{d}_1 I = \acute{\alpha}_1 + \acute{b}_1 I \quad \Rightarrow \quad \acute{c}_1 = \acute{\alpha}_1 \ \text{and} \ \acute{d}_1 = \acute{b}_1$$

$$\acute{c}_2 + \acute{d}_2 I = \acute{\alpha}_2 + \acute{b}_2 I \quad \Rightarrow \quad \acute{c}_2 = \acute{\alpha}_2 \ \text{and} \ \acute{d}_2 = \acute{b}_2$$

$$\acute{c}_3 + \acute{d}_3 I = \acute{\alpha}_3 + \acute{b}_3 I \quad \Rightarrow \quad \acute{c}_3 = \acute{\alpha}_3 \ \text{and} \ \acute{d}_3 = \acute{b}_3$$

2.1 The neutrosophic quaternions numbers algebra

2.1.1 Addition of the neutrosophic quaternions numbers

Let $q_I,\ p_I \in H_N$ where:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{i} + (\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$
$$p_I = \acute{\alpha} + \acute{b}I + \acute{u}_I = \acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}$$

then:

$$q_I + p_I = (\acute{c} + \acute{d}I + \acute{v}_I) + (\acute{\alpha} + \acute{b}I + \acute{u}_I)$$

$$= \left((\acute{c} + \acute{\alpha}) + (\acute{d} + \acute{b})I\right) + \left((\acute{c}_1 + \acute{\alpha}_1) + (\acute{b}_1 + \acute{d}_1)I\right)\breve{i} + \left((\acute{c}_2 + \acute{\alpha}_2) + (\acute{b}_2 + \acute{d}_2)I\right)\breve{j} + \left((\acute{c}_3 + \acute{\alpha}_3) + (\acute{b}_3 + \acute{d}_3)I\right)\breve{k}$$

Example 2

Let $q_I = 8 + 7I + (-5 + 8I)\breve{i} + (7 - 4I)\breve{j} - (5 + 9I)\breve{k}$ *and* $p_I = 2I + (2 - 3I)\breve{i} + (3 - I)\breve{j} + (-1 + 2I)\breve{k}$

then:

$$q_I + p_I = \left(8 + 7I + (-5 + 8I)\breve{i} + (7 - 4I)\breve{j} - (5 + 9I)\breve{k}\right) + \left(2I + (2 - 3I)\breve{i} + (3 - I)\breve{j} + (-1 + 2I)\breve{k}\right)$$
$$= (8 + 9I) + (-3 + 5I)\breve{i} + (10 - 5I)\breve{j} + (-6 - 7I)\breve{k}$$

Note:

- ✓ Clearly, zero is neutral for addition.
- ✓ For every number $q_I \in H_N$, its additive counterpart is:

$$-q_I = -\acute{c} - \acute{d}I - \acute{v}_I = -\acute{c} - \acute{d}I - (\acute{c}_1 + \acute{d}_1 I)\breve{i} - (\acute{c}_2 + \acute{d}_2 I)\breve{j} - (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

2.1.2 Multiplication of the neutrosophic quaternions numbers

Let $q_I,\ p_I \in H_N$ where:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{i} + (\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$
$$p_I = \acute{\alpha} + \acute{b}I + \acute{u}_I = \acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

then:

$$q_I . p_I = (\acute{c} + \acute{d}I + \acute{v}_I)(\acute{\alpha} + \acute{b}I + \acute{u}_I)$$

$$= \left[\acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{i} + (\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}\right]\left[\acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}\right]$$

$$\begin{aligned} = {} & (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) + (\acute{c} + \acute{d}I)(\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{c} + \acute{d}I)(\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} + (\acute{c} + \acute{d}I)(\acute{\alpha}_3 + \acute{b}_3 I)\breve{k} \\ & + (\acute{\alpha} + \acute{b}I)(\acute{c}_1 + \acute{d}_1 I)\breve{i} + (\acute{c}_1 + \acute{d}_1 I)\breve{i}(\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{c}_1 + \acute{d}_1 I)\breve{i}(\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} \\ & + (\acute{c}_1 + \acute{d}_1 I)\breve{i}\,(\acute{\alpha}_3 + \acute{b}_3 I)\breve{k} + (\acute{\alpha} + \acute{b}I)(\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{i}(\acute{c}_2 + \acute{d}_2 I)\breve{j} \\ & + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{j}(\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}(\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{\alpha} + \acute{b}I)(\acute{c}_3 + \acute{d}_3 I)\breve{k} \\ & + (\acute{c}_3 + \acute{d}_3 I)\breve{k}(\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}(\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}(\acute{\alpha}_3 + \acute{b}_3 I)\breve{k} \end{aligned}$$

$$\begin{aligned} = {} & (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) + (\acute{c} + \acute{d}I)(\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{c} + \acute{d}I)(\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} + (\acute{c} + \acute{d}I)(\acute{\alpha}_3 + \acute{b}_3 I)\breve{k} \\ & + (\acute{\alpha} + \acute{b}I)(\acute{c}_1 + \acute{d}_1 I)\breve{i} - (\acute{c}_1 + \acute{d}_1 I)(\acute{\alpha}_1 + \acute{b}_1 I) + (\acute{c}_1 + \acute{d}_1 I)\breve{i}(\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} \\ & + (\acute{c}_1 + \acute{d}_1 I)\breve{i}\,(\acute{\alpha}_3 + \acute{b}_3 I)\breve{k} + (\acute{\alpha} + \acute{b}I)(\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{i}(\acute{c}_2 + \acute{d}_2 I)\breve{j} \\ & - (\acute{\alpha}_2 + \acute{b}_2 I)(\acute{c}_2 + \acute{d}_2 I) + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}(\acute{c}_2 + \acute{d}_2 I)\breve{j} + (\acute{\alpha} + \acute{b}I)(\acute{c}_3 + \acute{d}_3 I)\breve{k} \\ & + (\acute{c}_3 + \acute{d}_3 I)\breve{k}(\acute{\alpha}_1 + \acute{b}_1 I)\breve{i} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}(\acute{\alpha}_2 + \acute{b}_2 I)\breve{j} - (\acute{c}_3 + \acute{d}_3 I)(\acute{\alpha}_3 + \acute{b}_3 I) \end{aligned}$$

$$= (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) - [(\acute{c}_1 + \acute{d}_1I)(\acute{\alpha}_1 + \acute{b}_1I) + (\acute{\alpha}_2 + \acute{b}_2I)(\acute{c}_2 + \acute{d}_2I) + (\acute{c}_3 + \acute{d}_3I)(\acute{\alpha}_3 + \acute{b}_3I)]$$
$$+ (\acute{c} + \acute{d}I)[(\acute{\alpha}_1 + \acute{b}_1I)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2I)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3I)\breve{k}]$$
$$+ (\acute{\alpha} + \acute{b}I)[(\acute{c}_1 + \acute{d}_1I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3I)\breve{k}] + (\acute{c}_1 + \acute{d}_1I)(\acute{\alpha}_2 + \acute{b}_2I)\breve{k}$$
$$- (\acute{c}_1 + \acute{d}_1I)(\acute{\alpha}_3 + \acute{b}_3I)\breve{\jmath} - (\acute{c}_2 + \acute{d}_2I)(\acute{\alpha}_1 + \acute{b}_1I)\breve{k} + (\acute{c}_2 + \acute{d}_2I)(\acute{\alpha}_3 + \acute{b}_3I)\breve{\imath}$$
$$+ (\acute{c}_3 + \acute{d}_3I)(\acute{\alpha}_1 + \acute{b}_1I)\breve{\jmath} - (\acute{c}_3 + \acute{d}_3I)(\acute{\alpha}_2 + \acute{b}_2I)\breve{\imath}$$

we can write it by the form:

$$q_I.p_I = (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) - \acute{v}_I.\acute{u}_I + (\acute{c} + \acute{d}I)\acute{u}_I + (\acute{\alpha} + \acute{b}I)\acute{v}_I + \acute{v}_I \times \acute{u}_I$$

where:

$$\acute{v}_I.\acute{u}_I = (\acute{c}_1 + \acute{d}_1I)(\acute{\alpha}_1 + \acute{b}_1I) + (\acute{c}_2 + \acute{d}_2I)(\acute{\alpha}_2 + \acute{b}_2I) + (\acute{c}_3 + \acute{d}_3I)(\acute{\alpha}_3 + \acute{b}_3I)$$

$$\acute{v}_I \times \acute{u}_I = \begin{vmatrix} \breve{\imath} & \breve{\jmath} & \breve{k} \\ \acute{c}_1 + \acute{d}_1I & \acute{c}_2 + \acute{d}_2I & \acute{c}_3 + \acute{d}_3I \\ \acute{\alpha}_1 + \acute{b}_1I & \acute{\alpha}_2 + \acute{b}_2I & \acute{\alpha}_3 + \acute{b}_3I \end{vmatrix}$$

Result1:

Multiplication of the neutrosophic quaternions numbers is not commutative because:

$$\acute{v}_I \times \acute{u}_I \neq \acute{u}_I \times \acute{v}_I$$

Example 3

Let $q_I = 2 + I + (1 - 4I)\breve{\imath} + (8 - 3I)\breve{\jmath} + (6 + 4I)\breve{k}$ *and* $p_I = 7I + (3 - 3I)\breve{\imath} + (2 - 5I)\breve{\jmath} + (-4 + 2I)\breve{k}$

then:

$$q_I.p_I = (2 + I + (1 - 4I)\breve{\imath} + (8 - 3I)\breve{\jmath} + (6 + 4I)\breve{k})(7I + (3 - 3I)\breve{\imath} + (2 - 5I)\breve{\jmath} + (-4 + 2I)\breve{k})$$
$$= 14I + 7I - [3 - 3I - 12I + 12I + 16 - 40I - 6I + 15I - 24 + 12I - 16I + 8I]$$
$$+ [(6 - 6I + 3I - 3I)\breve{\imath} + (4 - 10I + 2I - 5I)\breve{\jmath} + (-8 + 4I - 4I + 2I)\breve{k}]$$
$$+ [(7I - 28I)\breve{\imath} + (56I - 2I)\breve{\jmath} + (42I + 28I)\breve{k}] + \begin{vmatrix} \breve{\imath} & \breve{\jmath} & \breve{k} \\ 1 - 4I & 8 - 3I & 6 + 4I \\ 3 - 3I & 2 - 5I & -4 + 2I \end{vmatrix}$$

$$= 21I - (-5 - 30I) + (6 - 6I)\breve{\imath} + (4 - 13I)\breve{\jmath} + (-8 + 2I)\breve{k} + 21I\breve{\imath} + 54I\breve{\jmath} + 70I\breve{k} + (-45 + 64I)\breve{\imath} + (22 - 28I)\breve{\jmath} + (22 + 34I)\breve{k}$$

$$= 5 + 57I + (39 + 79I)\breve{\imath} + (26 - 13I)\breve{\jmath} + (12 + 106I)\breve{k}$$

Result2:

1) The neutrosophic quaternions numbers H_N is closed in relation to the addition operation, as the product of adding two neutrosophic quaternions numbers is a neutrosophic quaternions numbers, its real part is $(\acute{c} + \acute{\alpha}) + (\acute{d} + \acute{b})I$, and its vector part is:

   $$\left((\acute{c}_1 + \acute{\alpha}_1) + (\acute{b}_1 + \acute{d}_1)I\right)\breve{\imath} + \left((\acute{c}_2 + \acute{\alpha}_2) + (\acute{b}_2 + \acute{d}_2)I\right)\breve{\jmath} + \left((\acute{c}_3 + \acute{\alpha}_3) + (\acute{b}_3 + \acute{d}_3)I\right)\breve{k}.$$
2) The neutrosophic quaternions numbers H_N is closed in relation to the multiplication operation, as the product of multipl two neutrosophic quaternions numbers is a neutrosophic quaternions numbers, its real part is $(\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) - \acute{v}_I.\acute{u}_I$, and its vector part is $(\acute{c} + \acute{d}I)\acute{v}_I + (\acute{\alpha} + \acute{b}I)\acute{u}_I + \acute{v}_I \times \acute{u}_I$**.**
3) Multiplication accepts distribution on addition from the right and the left, so if we have three neutrosophic quaternions numbers $q_I, p_I, r_I \in H_N$, then:

$$q_I(p_I + r_I) = q_I.p_I + q_I.r_I$$

$$(p_I + r_I)q_I = p_I.q_I + r_I.q_I$$

4) The neutrality of multiplying numbers is $1 + 0I$

2.2 The neutrosophic quaternions numbers conjugate

Definition3

Let $q_I \in H_N$, where $q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$. The neutrosophic quaternions number conjugate define by the following form:

$$\overline{q_I} = \acute{c} + \acute{d}I - \acute{v}_I = \acute{c} + \acute{d}I - (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} - (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} - (\acute{c}_3 + \acute{d}_3 I)\breve{k}.$$

Example 4

i. $q_I = 28 + 4I + (14 - 17I)\breve{\imath} + (17 - 3I)\breve{\jmath} - (77 - 45I)\breve{k}$
$\Rightarrow \overline{q_I} = 28 + 4I - (14 - 17I)\breve{\imath} - (17 - 3I)\breve{\jmath} + (77 - 45I)\breve{k}$

ii. $q_I = (1 - 13I)\breve{\jmath} + (9 - I)\breve{k} \quad \Rightarrow \quad \overline{q_I} = -(1 - 13I)\breve{\jmath} - (9 - I)\breve{k}$

Result3:

1. The neutrosophic quaternions number conjugate of $\overline{q_I}$ is the same The neutrosophic quaternions number q_I.

$$\overline{(\overline{q_I})} = q_I$$

Proof:

Let $q_I \in H_N$, where $q_I = \acute{c} + \acute{d}I + \acute{v}_I$, then:

$$\overline{q_I} = \acute{c} + \acute{d}I - \acute{v}_I$$

$$\overline{(\overline{q_I})} = \overline{(\acute{c} + \acute{d}I - \acute{v}_I)} = \acute{c} + \acute{d}I + \acute{v}_I = q_I$$

2. If $q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$

then:

- $q_I + \overline{q_I} = 2(\acute{c} + \acute{d}I) = Re(q_I)$
- $q_I - \overline{q_I} = 2\acute{v}_I = 2(\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + 2(\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + 2(\acute{c}_3 + \acute{d}_3 I)\breve{k} = V(q_I)$

where $Re(q_I)$ is the neutrosophic real part (scalar) of the complex number and $V(q_I)$ is the neutrosophic vector part.

3. The neutrosophic quaternions number is real (scalar) if and only if $q_I = \overline{q_I}$, and it is vector if and only if $q_I = -\overline{q_I}$.

Remarks1:

$$\overline{q_{I_1} + q_{I_2}} = \overline{q_{I_1}} + \overline{q_{I_2}}$$

Proof:

Let $q_{I_1}, q_{I_2} \in H_N$, where

$$q_{I_1} = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

$$q_{I_2} = \acute{\acute{c}} + \acute{\acute{d}}I + \acute{\acute{v}}_I = \acute{\acute{c}} + \acute{\acute{d}}I + (\acute{\acute{c}}_1 + \acute{\acute{d}}_1 I)\breve{\imath} + (\acute{\acute{c}}_2 + \acute{\acute{d}}_2 I)\breve{\jmath} + (\acute{\acute{c}}_3 + \acute{\acute{d}}_3 I)\breve{k}$$

then:

$$q_{I_1} + q_{I_2} = \left(\acute{c} + \acute{d}I + \acute{\acute{c}} + \acute{\acute{d}}I\right) + \left(\acute{c}_1 + \acute{d}_1 I + \acute{\acute{c}}_1 + \acute{\acute{d}}_1 I\right)\breve{\imath} + \left(\acute{c}_2 + \acute{d}_2 I + \acute{\acute{c}}_2 + \acute{\acute{d}}_2 I\right)\breve{\jmath} + \left(\acute{c}_3 + \acute{d}_3 I + \acute{\acute{c}}_3 + \acute{\acute{d}}_3 I\right)\breve{k}$$

$$\overline{q_{I_1} + q_{I_2}} = \left(\acute{c} + \acute{d}I + \acute{\acute{c}} + \acute{\acute{d}}I\right) - \left(\acute{c}_1 + \acute{d}_1 I + \acute{\acute{c}}_1 + \acute{\acute{d}}_1 I\right)\breve{\imath} - \left(\acute{c}_2 + \acute{d}_2 I + \acute{\acute{c}}_2 + \acute{\acute{d}}_2 I\right)\breve{\jmath} - \left(\acute{c}_3 + \acute{d}_3 I + \acute{\acute{c}}_3 + \acute{\acute{d}}_3 I\right)\breve{k}$$

$$= \acute{c} + \acute{d}I - (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} - (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} - (\acute{c}_3 + \acute{d}_3 I)\breve{k} + \acute{\acute{c}} + \acute{\acute{d}}I + (\acute{\acute{c}}_1 + \acute{\acute{d}}_1 I)\breve{\imath} + (\acute{\acute{c}}_2 + \acute{\acute{d}}_2 I)\breve{\jmath} + (\acute{\acute{c}}_3 + \acute{\acute{d}}_3 I)\breve{k}$$

$$= \overline{q_{I_1}} + \overline{q_{I_2}}$$

Theorem1

The conjugate of multiplication two neutrosophic quaternions numbers is equal to the multiplication of their two conjugates.

$$\overline{q_I . p_I} = \overline{p_I} . \overline{q_I}$$

where $q_I,\ p_I \in H_N$

Proof:

Let $q_I,\ p_I \in H_N$ where:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

$$p_I = \acute{\alpha} + \acute{b}I + \acute{u}_I = \acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

then:

$$q_I . p_I = (\acute{c} + \acute{d}I + \acute{v}_I)(\acute{\alpha} + \acute{b}I + \acute{u}_I)$$

$$= \left[\acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}\right]\left[\acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}\right]$$

$$\begin{aligned} = (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) &- \left[(\acute{c}_1 + \acute{d}_1 I)(\acute{\alpha}_1 + \acute{b}_1 I) + (\acute{\alpha}_2 + \acute{b}_2 I)(\acute{c}_2 + \acute{d}_2 I) + (\acute{c}_3 + \acute{d}_3 I)(\acute{\alpha}_3 + \acute{b}_3 I)\right] \\ &+ (\acute{c} + \acute{d}I)\left[(\acute{\alpha}_1 + \acute{b}_1 I)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{\jmath} + (\acute{\alpha}_3 + \acute{b}_3 I)\breve{k}\right] \\ &+ (\acute{\alpha} + \acute{b}I)\left[(\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}\right] + (\acute{c}_1 + \acute{d}_1 I)(\acute{\alpha}_2 + \acute{b}_2 I)\breve{k} \\ &- (\acute{c}_1 + \acute{d}_1 I)(\acute{\alpha}_3 + \acute{b}_3 I)\breve{\jmath} - (\acute{c}_2 + \acute{d}_2 I)(\acute{\alpha}_1 + \acute{b}_1 I)\breve{k} + (\acute{c}_2 + \acute{d}_2 I)(\acute{\alpha}_3 + \acute{b}_3 I)\breve{\imath} \\ &+ (\acute{c}_3 + \acute{d}_3 I)(\acute{\alpha}_1 + \acute{b}_1 I)\breve{\jmath} - (\acute{c}_3 + \acute{d}_3 I)(\acute{\alpha}_2 + \acute{b}_2 I)\breve{\imath} \end{aligned}$$

we can write it by the form:

$$q_I . p_I = (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) - \acute{v}_I . \acute{u}_I + (\acute{c} + \acute{d}I)\acute{u}_I + (\acute{\alpha} + \acute{b}I)\acute{v}_I + \acute{v}_I \times \acute{u}_I$$

where:

$$\acute{v}_I . \acute{u}_I = (\acute{c}_1 + \acute{d}_1 I)(\acute{\alpha}_1 + \acute{b}_1 I) + (\acute{c}_2 + \acute{d}_2 I)(\acute{\alpha}_2 + \acute{b}_2 I) + (\acute{c}_3 + \acute{d}_3 I)(\acute{\alpha}_3 + \acute{b}_3 I)$$

$$\acute{v}_I \times \acute{u}_I = \begin{vmatrix} \breve{\imath} & \breve{\jmath} & \breve{k} \\ \acute{c}_1 + \acute{d}_1 I & \acute{c}_2 + \acute{d}_2 I & \acute{c}_3 + \acute{d}_3 I \\ \acute{\alpha}_1 + \acute{b}_1 I & \acute{\alpha}_2 + \acute{b}_2 I & \acute{\alpha}_3 + \acute{b}_3 I \end{vmatrix}$$

then:

$$\overline{q_I . p_I} = (\acute{c} + \acute{d}I)(\acute{\alpha} + \acute{b}I) - \acute{v}_I . \acute{u}_I - (\acute{c} + \acute{d}I)\acute{u}_I - (\acute{\alpha} + \acute{b}I)\acute{v}_I - \acute{v}_I \times \acute{u}_I$$

$$\overline{p_I} . \overline{q_I} = (\acute{\alpha} + \acute{b}I - \acute{u}_I)(\acute{c} + \acute{d}I - \acute{v}_I)$$

$$= [\acute{a} + \acute{b}I - (\acute{a}_1 + \acute{b}_1 I)\breve{\imath} - (\acute{a}_2 + \acute{b}_2 I)\breve{\jmath} - (\acute{a}_3 + \acute{b}_3 I)\breve{k}][\acute{c} + \acute{d}I - (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} - (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} - (\acute{c}_3 + \acute{d}_3 I)\breve{k}]$$

$$= (\acute{a} + \acute{b}I)(\acute{c} + \acute{d}I) - (\acute{a} + \acute{b}I)(\acute{c}_1 + \acute{d}_1 I)\breve{\imath} - (\acute{a} + \acute{b}I)(\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} - (\acute{a} + \acute{b}I)(\acute{c}_3 + \acute{d}_3 I)\breve{k} - (\acute{c} + \acute{d}I)(\acute{a}_1 + \acute{b}_1 I)\breve{\imath} - (\acute{a}_1 + \acute{b}_1 I)(\acute{c}_1 + \acute{d}_1 I) + (\acute{a}_1 + \acute{b}_1 I)(\acute{c}_2 + \acute{d}_2 I)\breve{k} - (\acute{a}_1 + \acute{b}_1 I)(\acute{c}_3 + \acute{d}_3 I)\breve{\jmath} - (\acute{c} + \acute{d}I)(\acute{a}_2 + \acute{b}_2 I)\breve{\jmath} - (\acute{a}_2 + \acute{b}_2 I)(\acute{c}_1 + \acute{d}_1 I)\breve{k} - (\acute{a}_2 + \acute{b}_2 I)(\acute{c}_2 + \acute{d}_2 I) - (\acute{a}_2 + \acute{b}_2 I)(\acute{c}_3 + \acute{d}_3 I)\breve{\imath} - (\acute{c} + \acute{d}I)(\acute{a}_3 + \acute{b}_3 I)\breve{k} + (\acute{a}_3 + \acute{b}_3 I)(\acute{c}_1 + \acute{d}_1 I)\breve{\jmath} - (\acute{a}_3 + \acute{b}_3 I)(\acute{c}_2 + \acute{d}_2 I)\breve{\imath} - (\acute{a}_3 + \acute{b}_3 I)(\acute{c}_3 + \acute{d}_3 I)$$

$$\begin{aligned} = (\acute{a} + \acute{b}I)(\acute{c} + \acute{d}I) &- (\acute{a}_1 + \acute{b}_1 I)(\acute{c}_1 + \acute{d}_1 I) - (\acute{a}_2 + \acute{b}_2 I)(\acute{c}_2 + \acute{d}_2 I) - (\acute{a}_3 + \acute{b}_3 I)(\acute{c}_3 + \acute{d}_3 I) \\ &- (\acute{c} + \acute{d}I)(\acute{a}_1 + \acute{b}_1 I)\breve{\imath} - (\acute{c} + \acute{d}I)(\acute{a}_2 + \acute{b}_2 I)\breve{\jmath} - (\acute{c} + \acute{d}I)(\acute{a}_3 + \acute{b}_3 I)\breve{k} \\ &- (\acute{a} + \acute{b}I)(\acute{c}_1 + \acute{d}_1 I)\breve{\imath} - (\acute{a} + \acute{b}I)(\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} - (\acute{a} + \acute{b}I)(\acute{c}_3 + \acute{d}_3 I)\breve{k} \\ &+ (\acute{a}_1 + \acute{b}_1 I)(\acute{c}_2 + \acute{d}_2 I)\breve{k} - (\acute{a}_1 + \acute{b}_1 I)(\acute{c}_3 + \acute{d}_3 I)\breve{\jmath} - (\acute{a}_2 + \acute{b}_2 I)(\acute{c}_1 + \acute{d}_1 I)\breve{k} \\ &- (\acute{a}_2 + \acute{b}_2 I)(\acute{c}_3 + \acute{d}_3 I)\breve{\imath} + (\acute{a}_3 + \acute{b}_3 I)(\acute{c}_1 + \acute{d}_1 I)\breve{\jmath} - (\acute{a}_3 + \acute{b}_3 I)(\acute{c}_2 + \acute{d}_2 I)\breve{\imath} \end{aligned}$$

$$= (\acute{c} + \acute{d}I)(\acute{a} + \acute{b}I) - \acute{v}_I . \acute{u}_I - (\acute{c} + \acute{d}I)\acute{u}_I - (\acute{a} + \acute{b}I)\acute{v}_I - \acute{v}_I \times \acute{u}_I$$

$$\Rightarrow \quad \overline{q_I . p_I} = \overline{p_I} . \overline{q_I}$$

2.3 The absolute value of a neutrosophic quaternions number

Let $q_I \in H_N$ where: $q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$, the absolute value of a neutrosophic quaternions numbers defined by the following form:

$$|q_I| = \sqrt{(\acute{c} + \acute{d}I)^2 + (\acute{c}_1 + \acute{d}_1 I)^2 + (\acute{c}_2 + \acute{d}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2}$$

Example 5

Let $q_I = 1 - 4I + I\breve{\imath} + 2I\breve{\jmath} - I\breve{k}$, then:

$$|q_I| = \sqrt{(\acute{c} + \acute{d}I)^2 + (\acute{c}_1 + \acute{d}_1 I)^2 + (\acute{c}_2 + \acute{d}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2}$$

$$= \sqrt{(1 - 4I)^2 + (I)^2 + (2I)^2 + (I)^2}$$

$$= \sqrt{1 - 8I + 16I + I + 4I + I}$$

$$= \sqrt{1 + 14I}$$

$$\sqrt{1 + 14I} \equiv x + yI$$

$$1 + 14I \equiv x^2 + 2xyI + y^2$$

by identifying we get:

$$\begin{cases} x^2 = 1 \\ y^2 + 2xy = 14 \end{cases}$$

Since the absolute value is positive, we take: $x = 1$
then:

$$y^2 + 2y = 14 \implies y^2 + 2y - 14 = 0$$

$$y = \frac{-2 + 2\sqrt{15}}{2} = -1 + \sqrt{15} \approx 2.9$$

Therefore,

$$|q_I| = \sqrt{(1-4I)^2 + (I)^2 + (2I)^2 + (I)^2} = 1 + 2.9I$$

Theorem2

Let $q_I \in H_N$ where: $q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$, multiplication the absolute value of q_I by its conjugate equals to square of the absolute value of q_I.

$$q_I . \overline{q_I} = |q_I|^2$$

Proof:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

$$\Rightarrow \quad q_I = \acute{c} + \acute{d}I - \acute{v}_I = \acute{c} + \acute{d}I - (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} - (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} - (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

$$q_I . \overline{q_I} = (\acute{c} + \acute{d}I)^2 (\acute{c} + \acute{d}I - \acute{v}_I)$$

$$= (\acute{c} + \acute{d}I)^2 - (\acute{c} + \acute{d}I)\acute{v}_I + (\acute{c} + \acute{d}I)\acute{v}_I - \acute{v}_I . \acute{v}_I$$

$$= (\acute{c} + \acute{d}I)^2 - \acute{v}_I . \acute{v}_I$$

$$= (\acute{c} + \acute{d}I)^2 + (\acute{c}_1 + \acute{d}_1 I)^2 + (\acute{c}_2 + \acute{d}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2 = |q_I|^2$$

$$\Rightarrow \quad q_I . \overline{q_I} = |q_I|^2$$

Example 6

Let $q_I = 2 - 6\mathrm{I} + 3I\breve{\imath} + (1 + 2I)\breve{\jmath} - 5\breve{k}$, then:

$$q_I . \overline{q_I} = |q_I|^2$$

$$= (2 - 6\mathrm{I})^2 + 9\mathrm{I} + (1 + 2I)^2$$

$$= 4 - 24I + 36I + 9I + 1 + 4I + 4I$$

$$= 5 + 29I$$

Remarks2:

Let $q_I \in H_N$, then:

1) $|q_I| = |\bar{q}_I| = |-q_I|$
2) $|q_I . p_I| = |q_I| . |p_I|$

Proof (2):

$$|q_I . p_I|^2 = q_I . p_I \overline{(q_I . p_I)} = q_I . p_I . \overline{p_I} . \bar{q}_I = q_I . |p_I|^2 . \bar{q}_I = q_I . \bar{q}_I . |p_I|^2 = |q_I|^2 . |p_I|^2$$

2.4 Division of neutrosophic quaternions numbers

Let $q_I,\ p_I \in H_N$ where:

$$q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$
$$p_I = \acute{\alpha} + \acute{b}I + \acute{u}_I = \acute{\alpha} + \acute{b}I + (\acute{\alpha}_1 + \acute{b}_1 I)\breve{\imath} + (\acute{\alpha}_2 + \acute{b}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$$

then:

$$\frac{q_I}{p_I} = \frac{\acute{c} + \acute{d}I + \acute{v}_I}{\acute{\alpha} + \acute{b}I + \acute{u}_I}$$

multiply the numerator and denominator by conjugate of p_I we get:

$$\frac{q_I}{p_I} = \frac{(\acute{c} + \acute{d}I + \acute{v}_I)(\acute{\alpha} + \acute{b}I - \acute{u}_I)}{(\acute{\alpha} + \acute{b}I + \acute{u}_I)(\acute{\alpha} + \acute{b}I - \acute{u}_I)}$$

$$=\frac{\left(\acute{c}+\acute{d}I+\acute{v}_I\right)\left(\acute{\alpha}+\acute{b}I-\acute{u}_I\right)}{\left(\acute{\alpha}+\acute{b}I\right)^2-(\acute{u}_I)^2}$$

$$=\frac{\left(\acute{c}+\acute{d}I\right)\left(\acute{\alpha}+\acute{b}I\right)-\acute{v}_I.\acute{u}_I+\left(\acute{c}+\acute{d}I\right)\acute{v}_I+\left(\acute{\alpha}+\acute{b}I\right)\acute{u}_I+\acute{v}_I\times\acute{u}_I}{\left(\acute{\alpha}+\acute{b}I\right)^2-(\acute{u}_I)^2}$$

where:

$$\acute{v}_I.\acute{u}_I=\left(\acute{c}_1+\acute{d}_1I\right)\left(\acute{\alpha}_1+\acute{b}_1I\right)+\left(\acute{c}_2+\acute{d}_2I\right)\left(\acute{\alpha}_2+\acute{b}_2I\right)+\left(\acute{c}_3+\acute{d}_3I\right)\left(\acute{c}_3+\acute{d}_3I\right)$$

$$\acute{v}_I\times\acute{u}_I=\begin{vmatrix}\breve{\imath} & \breve{\jmath} & \breve{k}\\ \acute{c}_1+\acute{d}_1I & \acute{c}_2+\acute{d}_2I & \acute{c}_3+\acute{d}_3I\\ \acute{\alpha}_1+\acute{b}_1I & \acute{\alpha}_2+\acute{b}_2I & \acute{\alpha}_3+\acute{b}_3I\end{vmatrix}$$

and $\left(\acute{\alpha}+\acute{b}I\right)^2-(\acute{u}_I)^2=\left(\acute{\alpha}+\acute{b}I\right)^2+\left(\acute{\alpha}_1+\acute{b}_1I\right)^2+\left(\acute{\alpha}_2+\acute{b}_2I\right)^2+\left(\acute{c}_3+\acute{d}_3I\right)^2$

Example 7

Let $q_I=2+(1-4I)\breve{\imath}-3I\breve{\jmath}+(6+4I)\breve{k}$ *and* $p_I=7I-2I\breve{\imath}+(2-5I)\breve{\jmath}+4\breve{k}$

then:

$$\frac{q_I}{p_I}=\frac{2+(1-4I)\breve{\imath}-3I\breve{\jmath}+(6+4I)\breve{k}}{7I-2I\breve{\imath}+(2-5I)\breve{\jmath}+4\breve{k}}$$

$$=\frac{\left(2+(1-4I)\breve{\imath}-3I\breve{\jmath}+(6+4I)\breve{k}\right)\left(7I+2I\breve{\imath}-(2-5I)\breve{\jmath}-4\breve{k}\right)}{\left(7I-2I\breve{\imath}+(2-5I)\breve{\jmath}+4\breve{k}\right)\left(7I+2I\breve{\imath}-(2-5I)\breve{\jmath}-4\breve{k}\right)}$$

$$=\frac{\left(2+(1-4I)\breve{\imath}-3I\breve{\jmath}+(6+4I)\breve{k}\right)\left(7I+2I\breve{\imath}-(2-5I)\breve{\jmath}-4\breve{k}\right)}{(7I)^2-\left(-2I\breve{\imath}+(2-5I)\breve{\jmath}+4\breve{k}\right)^2}$$

$$=\frac{24+45I-18I\breve{\imath}+(-4+11I)\breve{\jmath}+(8+28I)\breve{k}+(12-30I)\breve{\imath}+(4+4I)\breve{\jmath}+(-2-I)\breve{k}}{49I-(-4I-(2-5I)^2-16)}$$

$$=\frac{24+45I+(12-48I)\breve{\imath}+15I\breve{\jmath}+(6+27I)\breve{k}}{20+9I}$$

$$=\frac{24+45I}{20+9I}+\frac{12-48I}{20+9I}\breve{\imath}+\frac{15I}{20+9I}\breve{\jmath}+\frac{6+27I}{20+9I}\breve{k}$$

$$=\frac{6}{5}+\frac{171}{145}I+\left(\frac{3}{5}+\frac{267}{145}I\right)\breve{\imath}+\left(\frac{15}{29}I\right)\breve{\jmath}+\left(\frac{3}{10}+\frac{243}{290}I\right)\breve{k}$$

2.5 Inverted Neutrosophic quaternions numbers

Definition4

We define Inverted $q_I\in H_N$ as $q_I{}^{-1}\in H_N$, whereas:

$$q_I.q_I{}^{-1}=q_I{}^{-1}.q_I=1_{H_N}$$

whereas: $q_I\neq 0_{H_N}$

Remark3:

$$|q_I|^2 = q_I.\overline{q_I} \quad \Longrightarrow \quad q_I = \frac{|q_I|^2}{\overline{q_I}} \quad \Longrightarrow \quad {q_I}^{-1} = \frac{\overline{q_I}}{|q_I|^2}$$

Proof:

Let $q_I \in H_N$ where: $q_I = \acute{c} + \acute{d}I + \acute{v}_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}$, then:

$${q_I}^{-1} = \frac{1}{q_I} = \frac{1}{\acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + (\acute{c}_2 + \acute{d}_2 I)\breve{\jmath} + (\acute{c}_3 + \acute{d}_3 I)\breve{k}}$$

$$= \frac{\acute{c} + \acute{d}I}{(\acute{a} + \acute{b}I)^2 + (\acute{a}_1 + \acute{b}_1 I)^2 + (\acute{a}_2 + \acute{b}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2} - \frac{(\acute{c}_1 + \acute{d}_1 I)}{(\acute{a} + \acute{b}I)^2 + (\acute{a}_1 + \acute{b}_1 I)^2 + (\acute{a}_2 + \acute{b}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2}\breve{\imath} - \frac{(\acute{c}_2 + \acute{d}_2 I)}{(\acute{a} + \acute{b}I)^2 + (\acute{a}_1 + \acute{b}_1 I)^2 + (\acute{a}_2 + \acute{b}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2}\breve{\jmath} - \frac{(\acute{c}_3 + \acute{d}_3 I)}{(\acute{a} + \acute{b}I)^2 + (\acute{a}_1 + \acute{b}_1 I)^2 + (\acute{a}_2 + \acute{b}_2 I)^2 + (\acute{c}_3 + \acute{d}_3 I)^2}\breve{k}$$

Example 8

$$\frac{1}{2 + I + (1 - 4I)\breve{\imath} + (8 - 3I)\breve{\jmath} + (6 + 4I)\breve{k}} = \frac{2 + I}{105 + 38I} - \frac{(1 - 4I)}{105 + 38I}\breve{\imath} - \frac{(8 - 3I)}{105 + 38I}\breve{\jmath} - \frac{(6 + 4I)}{105 + 38I}\breve{k}$$

$$= \frac{2}{105} - \frac{29}{15015}I + \left(-\frac{1}{105} + \frac{458}{15015}I\right)\breve{\imath} + \left(-\frac{8}{105} + \frac{619}{15015}I\right)\breve{\jmath} + \left(-\frac{6}{105} - \frac{192}{15015}I\right)\breve{k}$$

Remark4:

$$(p_I q_I)^{-1} = {q_I}^{-1}.{p_I}^{-1} \qquad \text{, whereas: } p_I.q_I \neq 0_{H_N}$$

Remark5:

Since any neutrosophic complex number $q_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath}$ can be written in the form:

$$q_I = \acute{c} + \acute{d}I + (\acute{c}_1 + \acute{d}_1 I)\breve{\imath} + 0\breve{\jmath} + 0\breve{k}$$

then:

$$R_N \subseteq C_N \subseteq H_N$$

5. Conclusions

In this paper, we introduced the neutrosophic quaternions numbers, where all algebraic operations were studied on it. Also, we studied the absolute value of a neutrosophic quaternions number and its inverted.

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References

1. Smarandache. F; "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability", Sitech-Education Publisher, Craiova – Columbus, 2013.
2. Smarandache. F; "Finite Neutrosophic Complex Numbers, by W. B. Vasantha Kandasamy", Zip Publisher, Columbus, Ohio, USA, PP:1-16, 2011.
3. Smarandache. F; "Neutrosophy. / Neutrosophic Probability, Set, and Logic", American Research Press, Rehoboth, USA, 1998.

4. Smarandache. F; "Introduction to Neutrosophic statistics", Sitech-Education Publisher, PP:34-44, 2014.
5. Smarandache. F; "A Unifying Field in Logics: Neutrosophic Logic, Preface by Charles Le, American Research Press, Rehoboth, 1999, 2000. Second edition of the Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics", University of New Mexico, Gallup, 2001.
6. Smarandache. F; "Proceedings of the First International Conference on Neutrosophy", Neutrosophic Set, Neutrosophic Probability and Statistics, University of New Mexico, 2001.
7. Alhasan. Y; "Concepts of Neutrosophic Complex Numbers", International Journal of Neutrosophic Science, Volume 8, Issue 1, pp. 9-18, 2020.
8. Smarandache. F; "Neutrosophic Precalculus and Neutrosophic Calculus", book, 2015.
9. Madeleine Al- Tahan, "Some Results on Single Valued Neutrosophic (Weak) Polygroups", International Journal of Neutrosophic Science, Volume 2, Issue 1, PP: 38-46, 2020.
10. Alhasan. Y; "The neutrosophic integrals and integration methods", Neutrosophic Sets and Systems, Volume 49, pp. 357-374, 2022.
11. Chakraborty. A; "A New Score Function of Pentagonal Neutrosophic Number and its Application in Networking Problem", International Journal of Neutrosophic Science, Volume 1, Issue 1, PP: 40-51, 2020
12. Chakraborty. A; "Application of Pentagonal Neutrosophic Number in Shortest Path Problem", International Journal of Neutrosophic Science, Volume 3, Issue 1, PP: 21-28, 2020.
13. Smarandache. F; "Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy", Neutrosophic Logic, Set, Probability, and Statistics, University of New Mexico, Gallup, NM 87301, USA 2002.
14. Alhasan.y; "The General Exponential form of a Neutrosophic Complex Number", International Journal of Neutrosophic Science, Volume 11, Issue 2, pp. 100-107, 2020.

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A Comprehensive Analysis of Neutrosophic Bonferroni Mean Operator

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Abstract

The Neutrosophic Bonferroni operator is a novel operator that we provide in this paper. Then the arithmetic operations for Neutrosophic Bonferroni operator is developed which tells the existence of Neutrosophic Bonferroni operator. Then its properties were discussed with special cases. To group decision-making issues with several attributes, arithmetic ranking operations and the Neutrosophic approach are used. The result is compared with the existing methodology. The suggested approach will more accurately give the decision maker the ideal attribute than the existing system does. Neutrophic multicriteria is a method of decision-making that makes use of ambiguity to integrate various criteria or factors—often with imprecise or ambiguous data—to reach a result. The neutrosophic multicriteria analysis enables the assessment of subjective and qualitative factors, which can assist in resolving conflicting goals and preferences. In Neutrosophic Multi-Attribute Group Decision Making (NMAGDM) problems, all the data supplied by the decision makers (DMs) is expressed in single-value Neutrosophic triangular and trapezoidal numbers, which are studied in this work and can improve the flexibility and precision of capturing uncertainty and aggregating preferences. Studying this operator is crucial because it can be utilised to resolve multi-attribute

Keywords: Group decision making in multi-attributes using Neutrosophic(NMAGDM),Neutrosophic Bonferroni operator, weighted Neutrosophic Bonferroni operator, Neutrosophic operator.

1.Introduction

[1] was first introduced the fuzzy set theory. This theory was used in many areas which is explained in [2]as the essential ideas in fuzzy set theory are covered in Fundamentals of Fuzzy Sets. Its four-part structure makes it simple

to reference both more recent and earlier findings in the subject, In [3] the definitions of the axioms pertaining to the fundamental relationships between the entropy, distance, and similarity metrics of fuzzy collections are discussed, [4] as Using probabilistic data, we created a novel decision-making model and aggregated the data using the instantaneous probability idea. This kind of probability introduces the decision maker's attitude, which changes the objective probability and in [5] as the theory and procedure of decision making are provided by the grey relational degree-based decision making approach. The above all can deviate in various situation which was simplified by various fuzzy members like [6] used interval valued fuzzy members produced by fuzzy disjunctive and conjunctive normal forms, serve as a type II fuzzy set model to depict the second order semantic uncertainty achieved by the linguistic connectives that combine two or more fuzzy, ambiguous ideas, [7] used vague sets, [8] used intuitionistic fuzzy sets, [9] used interval type 2 fuzzy sets, [9] used fuzzy multisets. This application was clearly explained in [10] as a method for handling several qualities The suggested aggregation operators are used to make decisions in an intuitionistic fuzzy environment, and an illustration is given to show the practicality and accuracy of the recommended approach, [11] and [12]as generalization of a fuzzy set is a membership function and a non-membership function define an intuitionistic fuzzy set. In this study, we first present a technique based on the accuracy and score functions for comparing two intuitionistic fuzzy values. In [13], Xia et al. recently presented an intuitionistic multiplicative preference relation to characterize the preference information provided by a decision maker over a set of objects. Next, we develop some aggregation operators for aggregating intuitionistic fuzzy values, such as the intuitionistic fuzzy ordered weighted averaging operator, intuitionistic fuzzy hybrid aggregation operator, and intuitionistic fuzzy weighted averaging operator, and establish various properties of these operators. The intuitionistic multiplicative preference relation is made up of all the 2-tuples, which can simultaneously express how much one thing is prior to another and how much it is not. Compared to the conventional multiplicative preference relation, the 2-tuples can more fully reflect the decision maker's preferences over objects because each component derives its value from the closed interval [1/9, 9]. Finding a way to extract the object's priority weights from an intuitionistic multiplicative preference relation is a key topic of research for decision making with such information.

The intricacy of the problem has increased along with the introduction of various sorts of fuzzy members. Consequently, the Bonferroni operator was introduced as a new operator. [14] introduced the aggregation operator for mean for the first time. With the aid of the OWA operator, this was made more generic, and [20] provides the Choquet integral. The above-mentioned generalised approach is also provided by [21]. The Bonferroni mean(BM) operator of interval type-2 is defined in [15]. Additionally, [16] applies this Bonferroni mean as the Bonferroni geometric mean, which is a generalisation of the Bonferroni mean and geometric mean and can reflect the correlations of the combined arguments. To more correctly define the uncertainty and fuzziness, membership, non-membership, and uncertainty information could be taken into consideration using an intuitionistic fuzzy set. We go on building the intuitionistic fuzzy geometric Atanassov To collect the intuitionistic fuzzy information of Atanassov, define the interdependence between arguments using the Bonferroni mean. A few characteristics and unique circumstances of this mean are also looked at [17], since it is a desired feature if the BM can capture the correlations between the input arguments. It seems, nevertheless, that the existing literature only discusses using the BM to aggregate crisp numbers—it does not handle other types of reasoning. In this work, we investigate the BM in intuitionistic fuzzy environments. We construct an intuitionistic fuzzy BM (IFBM) and discuss possible specific cases for it. Next, using fuzzy multi-attribute group decision making (FMAGDM) scenarios in which the decision makers' (DMs') input is represented as trapezoidal interval type-2 fuzzy sets (IT2 FS), the weighted IFBM is used to multicriteria decision making. This is done in [18]. We introduce the idea of interval possibility mean value and provide a new method for calculating the possibility degree of two trapezoidal IT2 FS. The type-2 fuzzy geometric Bonferroni mean operator for trapezoidal intervals and the type-2 fuzzy weighted geometric Bonferroni mean operator for trapezoidal intervals (TIT2FWGBM) are the two aggregation techniques that we then develop and the Bonferroni mean (BM) is a crucial aggregation operator in decision-making, as stated in [19]. A useful aspect of the BM is its capacity to record the relationship between the individual attributes or the aggregation arguments. Proposed by Jin et al. in 2016, the extensions of the BM consist of the optimum weighted geometric Bonferroni mean (OWGBM) and the generalised optimised weighted geometric Bonferroni mean (GOWGBM). However, the OWGBM and GOWGBM lack both reducibility and boundedness, which may lead to unsuitable and irrational aggregation outputs as well as poor decision-making. To overcome these existing limitations, we propose two new measures: the generalised normalised weighted geometric Bonferroni mean (GNWGBM) and the normalised weighted geometric Bonferroni mean (NWGBM), which are based on the GOWGBM and the normalised weighted Bonferroni mean (NWBM).

Now, this can be expanded upon in this paper. The aggregating operations of a suggested Neutrosophic Bonferroni operator are defined. [22] using Bonferroni power aggregation operator but the evaluation process is limited in satisfying sum squares of non-membership and membership value. By using the above operators there will be flaws in final calculation and that can be overcome by a proposed operator Neutrosophic Bonferroni operator which satisfying some required properties and theorems and it is extended to weighted Neutrosophic Bonferroni operator with its properties and theorems. Determining the concepts of neutrosophic possibility mean value and the degree of neutrosophic possibility of two and three trapezoidal and triangular neutrosophic sets is the aim of this work. The neutrosophic Bonferroni mean operator in triangular and trapezoidal arrangements [23].This essay attempted to give a summary of every method that may be used to address the traffic issue [24]. It also applies the given approach to a profit analysis decision-making problem in [25]

Thus, the paper is formulated as follows in Section 2, the basic definitions and theorems with proof of Neutrosophic Bonferroni mean operator and theorem is given. In section 3, the properties of Neutrosophic Bonferroni operator will be explained. In section 4, the weighted Neutrosophic Bonferroni operator is given with properties and theorems are given. In section 5, the conclusion is given.

2. Neutrosophic Bonferroni operators:

Definition 2.1:

Let

$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$

$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$

represent the collection of Neutrosophic members, and we define the Neutrosophic Bonferroni mean for s,t≥0 as

$$NBM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\right) = \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m} \left(sTN_i \oplus tTN_j\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m} \left(sIN_i \oplus tIN_j\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m} \left(sFN_i \oplus tFN_j\right)\right)^{\frac{1}{m(m-1)}}\right) \quad (1)$$

<u>Theorem 2.1:</u>

Let

$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$

$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$

represent the set of Neutrosophic members, and in the case where s,t≥0, the aggregation operation on (1) is likewise a Neutrosophic member, as shown by

$NBM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\right) = (TN, IN, FN) =$
$\left((TN^U, IN^U, FN^U), (TN^L, IN^L, FN^L)\right)$where

$$(\mathrm{TN^U}, \mathrm{IN^U}, \mathrm{FN^U}) = \left(\begin{array}{c} \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i1}^{U} \oplus tTN_{j1}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i1}^{U} \oplus tIN_{j1}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i1}^{U} \oplus tFN_{j1}^{U}\right)\right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i2}^{U} \oplus tTN_{j2}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i2}^{U} \oplus tIN_{j2}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i2}^{U} \oplus tFN_{j2}^{U}\right)\right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i3}^{U} \oplus tTN_{j3}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i3}^{U} \oplus tIN_{j3}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i3}^{U} \oplus tFN_{j3}^{U}\right)\right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i4}^{U} \oplus tTN_{j4}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i4}^{U} \oplus tIN_{j4}^{U}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i4}^{U} \oplus tFN_{j4}^{U}\right)\right)^{\frac{1}{m(m-1)}} \right) \\ \min_{i=1,2,3,\ldots,m\left(Th_i^U, Ih_i^U, Fh_i^U\right)} \end{array} \right) \tag{2}$$

And

$$(TN^L, IN^L, FN^L) = \left(\begin{array}{c} \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i1}^{L} \oplus tTN_{j1}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(sIN_{i1}^{L} \oplus tIN_{j1}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{n}\left(sFN_{i1}^{L} \oplus tFN_{j1}^{L}\right)\right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i2}^{L} \oplus tTN_{j2}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i2}^{L} \oplus tIN_{j2}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i2}^{L} \oplus tFN_{j2}^{L}\right)\right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i3}^{L} \oplus tTN_{j3}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i3}^{L} \oplus tIN_{j3}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i3}^{L} \oplus tFN_{j3}^{L}\right)\right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sTN_{i4}^{L} \oplus tTN_{j4}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sIN_{i4}^{L} \oplus tIN_{j4}^{L}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes_{\substack{i,j=1\\ i\neq j}}^{m}\left(sFN_{i4}^{L} \oplus tFN_{j4}^{L}\right)\right)^{\frac{1}{m(m-1)}} \right) \\ min_{i=1,2,3,\ldots,m\left(Th_i^L, Ih_i^L, Fh_i^L\right)} \end{array} \right) \tag{3}$$

The proof of the above theorem is dome by mathematical induction,

Proof:

We start the proof by proving

$$\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN_i\oplus tTN_j\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN_i\oplus tIN_j\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN_i\oplus tFN_j\right)\right)\right)$$

$$= \left(\begin{array}{c}
\left(\begin{array}{c}
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^U_{i1}\oplus tTN^U_{j1}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^U_{i1}\oplus tIN^U_{j1}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^U_{i1}\oplus tFN^U_{j1}\right)\right)\right),\\
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^U_{i2}\oplus tTN^U_{j2}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^U_{i2}\oplus tIN^U_{j2}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^U_{i2}\oplus tFN^U_{j2}\right)\right)\right),\\
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^U_{i3}\oplus tTN^U_{j3}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^U_{i3}\oplus tIN^U_{j3}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^U_{i3}\oplus tFN^U_{j3}\right)\right)\right),\\
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^U_{i4}\oplus tTN^U_{j4}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^U_{i4}\oplus tIN^U_{j4}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^U_{i4}\oplus tFN^U_{j4}\right)\right)\right)\\
min_{i=1,2,3,\ldots,m\left(Th^U_i,Ih^U_i,Fh^U_i\right)}
\end{array}\right),\\
\left(\begin{array}{c}
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^L_{i1}\oplus tTN^L_{j1}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^L_{i1}\oplus tIN^L_{j1}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^L_{i1}\oplus tFN^L_{j1}\right)\right)\right),\\
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^L_{i2}\oplus tTN^L_{j2}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^L_{i2}\oplus tIN^L_{j2}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^L_{i2}\oplus tFN^L_{j2}\right)\right)\right),\\
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^L_{i3}\oplus tTN^L_{j3}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^L_{i3}\oplus tIN^L_{j3}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^L_{i3}\oplus tFN^L_{j3}\right)\right)\right),\\
\left(\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sTN^L_{i4}\oplus tTN^L_{j4}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sIN^L_{i4}\oplus tIN^L_{j4}\right)\right),\left(\bigotimes^{m}_{\substack{i,j=1\\ i,j\text{ is not}\\ \text{same}}}\left(sFN^L_{i4}\oplus tFN^L_{j4}\right)\right)\right)\\
min_{i=1,2,3,\ldots,m\left(Th^L_i,Ih^L_i,Fh^L_i\right)}
\end{array}\right)
\end{array}\right)$$

(4)

Then by arithmetic operations on Neutrosophic we get the following equations

$$\left(\left(\left(sTN_i \oplus tTN_j\right)\right), \left(\left(sIN_i \oplus tIN_j\right)\right), \left(\left(sFN_i \oplus tFN_j\right)\right)\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\left(\left(sTN_{i1}^U \oplus tTN_{j1}^U\right)\right), \left(\left(sIN_{i1}^U \oplus tIN_{j1}^U\right)\right), \left(\left(sFN_{i1}^U \oplus tFN_{j1}^U\right)\right)\right), \\ \left(\left(\left(sTN_{i2}^U \oplus tTN_{j2}^U\right)\right), \left(\left(sIN_{i2}^U \oplus tIN_{j2}^U\right)\right), \left(\left(sFN_{i2}^U \oplus tFN_{j2}^U\right)\right)\right), \\ \left(\left(\left(sTN_{i3}^U \oplus tTN_{j3}^U\right)\right), \left(\left(sIN_{i3}^U \oplus tIN_{j3}^U\right)\right), \left(\left(sFN_{i3}^U \oplus tFN_{j3}^U\right)\right)\right), \\ \left(\left(\left(sTN_{i4}^U \oplus tTN_{j4}^U\right)\right), \left(\left(sIN_{i4}^U \oplus tIN_{j4}^U\right)\right), \left(\left(sFN_{i4}^U \oplus tFN_{j4}^U\right)\right)\right) \\ min_{i=1,2,3,\ldots,m\left(Th_i^U, Ih_i^U, Fh_i^U\right)} \end{array} \right), \\ \left(\begin{array}{c} \left(\left(\left(sTN_{i1}^L \oplus tTN_{j1}^L\right)\right), \left(\left(sIN_{i1}^L \oplus tIN_{j1}^L\right)\right), \left(\left(sFN_{i1}^L \oplus tFN_{j1}^L\right)\right)\right), \\ \left(\left(\left(sTN_{i2}^L \oplus tTN_{j2}^L\right)\right), \left(\left(sIN_{i2}^L \oplus tIN_{j2}^L\right)\right), \left(\left(sFN_{i2}^L \oplus tFN_{j2}^L\right)\right)\right), \\ \left(\left(\left(sTN_{i3}^L \oplus tTN_{j3}^L\right)\right), \left(\left(sIN_{i3}^L \oplus tIN_{j3}^L\right)\right), \left(\left(sFN_{i3}^L \oplus tFN_{j3}^L\right)\right)\right), \\ \left(\left(\left(sTN_{i4}^L \oplus tTN_{j4}^L\right)\right), \left(\left(sIN_{i4}^L \oplus tIN_{j4}^L\right)\right), \left(\left(sFN_{i4}^L \oplus tFN_{j4}^L\right)\right)\right) \\ min_{i=1,2,3,\ldots,m\left(Th_i^L, Ih_i^L, Fh_i^L\right)} \end{array} \right) \end{array} \right)$$

(a) for $m = 2$,

$$\left(\left(\bigotimes^2{}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN_i \oplus tTN_j\right)\right), \left(\bigotimes^2{}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN_i \oplus tIN_j\right)\right), \left(\bigotimes^2{}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN_i \oplus tFN_j\right)\right)\right) =$$

$$\left(\left((sTN_1 \oplus tTN_2) \otimes (sTN_2 \oplus tTN_1)\right), \left((sIN_1 \oplus tIN_2) \otimes (sIN_2 \oplus tIN_1)\right), \left((sFN_1 \oplus tFN_2) \otimes (sFN_2 \oplus tFN_1)\right)\right)$$

$$
= \left(\begin{array}{c}
\left(\begin{array}{c}
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{U}_{i1} \oplus tTN^{U}_{j1} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{U}_{i1} \oplus tIN^{U}_{j1} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{U}_{i1} \oplus tFN^{U}_{j1} \right) \right) \right), \\
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{U}_{i2} \oplus tTN^{U}_{j2} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{U}_{i2} \oplus tIN^{U}_{j2} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{U}_{i2} \oplus tFN^{U}_{j2} \right) \right) \right), \\
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{U}_{i3} \oplus tTN^{U}_{j3} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{U}_{i3} \oplus tIN^{U}_{j3} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{U}_{i3} \oplus tFN^{U}_{j3} \right) \right) \right), \\
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{U}_{i4} \oplus tTN^{U}_{j4} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{U}_{i4} \oplus tIN^{U}_{j4} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{U}_{i4} \oplus tFN^{U}_{j4} \right) \right) \right) \\
min\left(\left(Th^{U}_{1}, Ih^{U}_{1}, Fh^{U}_{1} \right), \left(Th^{U}_{2}, Ih^{U}_{2}, Fh^{U}_{2} \right) \right)
\end{array} \right), \\
\left(\begin{array}{c}
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{L}_{i1} \oplus tTN^{L}_{j1} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{L}_{i1} \oplus tIN^{L}_{j1} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{L}_{i1} \oplus tFN^{L}_{j1} \right) \right) \right), \\
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{L}_{i2} \oplus tTN^{L}_{j2} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{L}_{i2} \oplus tIN^{L}_{j2} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{L}_{i2} \oplus tFN^{L}_{j2} \right) \right) \right), \\
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{L}_{i3} \oplus tTN^{L}_{j3} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{L}_{i3} \oplus tIN^{L}_{j3} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{L}_{i3} \oplus tFN^{L}_{j3} \right) \right) \right), \\
\left(\left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sTN^{L}_{i4} \oplus tTN^{L}_{j4} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sIN^{L}_{i4} \oplus tIN^{L}_{j4} \right) \right), \left(\bigotimes^{2}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(sFN^{L}_{i4} \oplus tFN^{L}_{j4} \right) \right) \right) \\
min\left(\left(Th^{L}_{1}, Ih^{L}_{1}, Fh^{L}_{1} \right), \left(Th^{L}_{2}, Ih^{L}_{2}, Fh^{L}_{2} \right) \right)
\end{array} \right)
\end{array} \right)
$$

Therefore, for $m = 2$, (4) is right

Suppose we assume that (4) is true for $m = k$, which is given by the following equations

$$\left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_i \oplus tTN_j\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_i \oplus tIN_j\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_i \oplus tFN_j\right)\right)\right)$$

$$= \left(\begin{array}{c}\left(\begin{array}{c}\left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i1}^{U} \oplus tTN_{j1}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i1}^{U} \oplus tIN_{j1}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i1}^{U} \oplus tFN_{j1}^{U}\right)\right)\right), \\ \left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i2}^{U} \oplus tTN_{j2}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i2}^{U} \oplus tIN_{j2}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i2}^{U} \oplus tFN_{j2}^{U}\right)\right)\right), \\ \left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i3}^{U} \oplus tTN_{j3}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i3}^{U} \oplus tIN_{j3}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i3}^{U} \oplus tFN_{j3}^{U}\right)\right)\right), \\ \left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i4}^{U} \oplus tTN_{j4}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i4}^{U} \oplus tIN_{j4}^{U}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i4}^{U} \oplus tFN_{j4}^{U}\right)\right)\right) \\ min_{i=1,2,3,\ldots,k\left(Th_i^{U},Ih_i^{U},Fh_i^{U}\right)}\end{array}\right), \\ \left(\begin{array}{c}\left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i1}^{L} \oplus tTN_{j1}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i1}^{L} \oplus tIN_{j1}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i1}^{L} \oplus tFN_{j1}^{L}\right)\right)\right), \\ \left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i2}^{L} \oplus tTN_{j2}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i2}^{L} \oplus tIN_{j2}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i2}^{L} \oplus tFN_{j2}^{L}\right)\right)\right), \\ \left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i3}^{L} \oplus tTN_{j3}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i3}^{L} \oplus tIN_{j3}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i3}^{L} \oplus tFN_{j3}^{L}\right)\right)\right), \\ \left(\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sTN_{i4}^{L} \oplus tTN_{j4}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sIN_{i4}^{L} \oplus tIN_{j4}^{L}\right)\right),\left(\bigotimes^{k}{}_{\substack{i,j=1\\ i,j\ is\ not\\ same}}\left(sFN_{i4}^{L} \oplus tFN_{j4}^{L}\right)\right)\right) \\ min_{i=1,2,3,\ldots,k\left(Th_i^{L},Ih_i^{L},Fh_i^{L}\right)}\end{array}\right)\end{array}\right) \tag{5}$$

Now we have to prove for $m = k + 1$

$$\left(\left(\bigotimes_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}}^{k+1} (sTN_i \oplus tTN_j)\right), \left(\bigotimes_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}}^{k+1} (sIN_i \oplus tIN_j)\right), \left(\bigotimes_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}}^{k+1} (sFN_i \oplus tFN_j)\right)\right)$$

$$= \left(\left(\bigotimes_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}}^{k} (sTN_i \oplus tTN_j)\right), \left(\bigotimes_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}}^{k} (sIN_i \oplus tIN_j)\right), \left(\bigotimes_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}}^{k} (sFN_i \oplus tFN_j)\right)\right)$$
$$\otimes \left(\left(\otimes_{i=1}^{k} (sTN_i \oplus tTN_{k+1})\right), \left(\otimes_{i=1}^{k} (sIN_i \oplus tIN_{k+1})\right), \left(\otimes_{i=1}^{k} (sFN_i \oplus tFN_{k+1})\right)\right)$$
$$\otimes \left(\left(\otimes_{j=1}^{k} (sTN_{k+1} \oplus tTN_j)\right), \left(\otimes_{j=1}^{k} (sIN_{k+1} \oplus tIN_j)\right), \left(\otimes_{j=1}^{k} (sFN_{k+1} \oplus tFN_j)\right)\right)$$

(6)

Using the arithmetic operations defined for Neutrosophic member, we get

$$\left(\left(\otimes_{i=1}^{k} (sTN_i \oplus tTN_{k+1})\right), \left(\otimes_{i=1}^{k} (sIN_i \oplus tIN_{k+1})\right), \left(\otimes_{i=1}^{k} (sFN_i \oplus tFN_{k+1})\right)\right)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left(\left(\otimes_{i=1}^{k} (sTN_{i1}^{U} \oplus tTN_{(k+1)1}^{U})\right), \left(\otimes_{i=1}^{k} (sIN_{i1}^{U} \oplus tIN_{(k+1)1}^{U})\right), \left(\otimes_{i=1}^{k} (sFN_{i1}^{U} \oplus tFN_{(k+1)1}^{U})\right)\right), \\ \left(\left(\otimes_{i=1}^{k} (sTN_{i2}^{U} \oplus tTN_{(k+1)2}^{U})\right), \left(\otimes_{i=1}^{k} (sIN_{i2}^{U} \oplus tIN_{(k+1)2}^{U})\right), \left(\otimes_{i=1}^{k} (sFN_{i2}^{U} \oplus tFN_{(k+1)2}^{U})\right)\right), \\ \left(\left(\otimes_{i=1}^{k} (sTN_{i3}^{U} \oplus tTN_{(k+1)3}^{U})\right), \left(\otimes_{i=1}^{k} (sIN_{i3}^{U} \oplus tIN_{(k+1)3}^{U})\right), \left(\otimes_{i=1}^{k} (sFN_{i3}^{U} \oplus tFN_{(k+1)3}^{U})\right)\right), \\ \left(\left(\otimes_{i=1}^{k} (sTN_{i4}^{U} \oplus tTN_{(k+1)4}^{U})\right), \left(\otimes_{i=1}^{k} (sIN_{i4}^{U} \oplus tIN_{(k+1)4}^{U})\right), \left(\otimes_{i=1}^{k} (sFN_{i4}^{U} \oplus tFN_{(k+1)4}^{U})\right)\right) \\ \min_{i=1,2,\ldots,k} \left((Th_i^U, Ih_i^U, Fh_i^U), (Th_{k+1}^U, Ih_{k+1}^U, Fh_{k+1}^U)\right) \end{array}\right), \\ \left(\begin{array}{c} \left(\left(\otimes_{i=1}^{k} (sTN_{i1}^{L} \oplus tTN_{(k+1)1}^{L})\right), \left(\otimes_{i=1}^{k} (sIN_{i1}^{L} \oplus tIN_{(k+1)1}^{L})\right), \left(\otimes_{i=1}^{k} (sFN_{i1}^{L} \oplus tFN_{(k+1)1}^{L})\right)\right), \\ \left(\left(\otimes_{i=1}^{k} (sTN_{i2}^{L} \oplus tTN_{(k+1)2}^{L})\right), \left(\otimes_{i=1}^{k} (sIN_{i2}^{L} \oplus tIN_{(k+1)2}^{L})\right), \left(\otimes_{i=1}^{k} (sFN_{i2}^{L} \oplus tFN_{(k+1)2}^{L})\right)\right), \\ \left(\left(\otimes_{i=1}^{k} (sTN_{i3}^{L} \oplus tTN_{(k+1)3}^{L})\right), \left(\otimes_{i=1}^{k} (sIN_{i3}^{L} \oplus tIN_{(k+1)3}^{L})\right), \left(\otimes_{i=1}^{k} (sFN_{i3}^{L} \oplus tFN_{(k+1)3}^{L})\right)\right), \\ \left(\left(\otimes_{i=1}^{k} (sTN_{i4}^{L} \oplus tTN_{(k+1)4}^{L})\right), \left(\otimes_{i=1}^{k} (sIN_{i4}^{L} \oplus tIN_{(k+1)4}^{L})\right), \left(\otimes_{i=1}^{k} (sFN_{i4}^{L} \oplus tFN_{(k+1)4}^{L})\right)\right) \\ \min_{i=1,2,\ldots,k} \left((Th_i^L, Ih_i^L, Fh_i^L), (Th_{k+1}^L, Ih_{k+1}^L, Fh_{k+1}^L)\right) \end{array}\right) \end{array}\right)$$

And

$$\left(\left(\otimes_{j=1}^{k} (sTN_{k+1} \oplus tTN_j)\right), \left(\otimes_{j=1}^{k} (sIN_{k+1} \oplus tIN_j)\right), \left(\otimes_{j=1}^{k} (sFN_{k+1} \oplus tFN_j)\right)\right)$$

$$= \begin{pmatrix} \begin{pmatrix} \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)1}^{U} \oplus tTN_{j1}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)1}^{U} \oplus tIN_{j1}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)1}^{U} \oplus tFN_{j1}^{U}\right)\right)\right), \\ \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)2}^{U} \oplus tTN_{j2}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)2}^{U} \oplus tIN_{j2}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)2}^{U} \oplus tFN_{j2}^{U}\right)\right)\right), \\ \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)3}^{U} \oplus tTN_{j3}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)3}^{U} \oplus tIN_{j3}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)3}^{U} \oplus tFN_{j3}^{U}\right)\right)\right), \\ \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)4}^{U} \oplus tTN_{j4}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)4}^{U} \oplus tIN_{j4}^{U}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)4}^{U} \oplus tFN_{j4}^{U}\right)\right)\right) \\ \min_{j=1,2,\dots,k}\left(\left(Th_{(k+1)}^{U}, Ih_{(k+1)}^{U}, Fh_{(k+1)}^{U}\right), \left(Th_{j}^{U}, Ih_{j}^{U}, Fh_{j}^{U}\right)\right) \end{pmatrix}, \\ \begin{pmatrix} \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)1}^{L} \oplus tTN_{j1}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)1}^{L} \oplus tIN_{j1}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)1}^{L} \oplus tFN_{j1}^{L}\right)\right)\right), \\ \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)2}^{L} \oplus tTN_{j2}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)2}^{L} \oplus tIN_{j2}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)2}^{L} \oplus tFN_{j2}^{L}\right)\right)\right), \\ \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)3}^{L} \oplus tTN_{j3}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)3}^{L} \oplus tIN_{j3}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)3}^{L} \oplus tFN_{j3}^{L}\right)\right)\right), \\ \left(\left(\otimes_{j=1}^{k}\left(sTN_{(k+1)4}^{L} \oplus tTN_{j4}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sIN_{(k+1)4}^{L} \oplus tIN_{j4}^{L}\right)\right), \left(\otimes_{j=1}^{k}\left(sFN_{(k+1)4}^{L} \oplus tFN_{j4}^{L}\right)\right)\right) \\ \min_{j=1,2,\dots,k}\left(\left(Th_{(k+1)}^{L}, Ih_{(k+1)}^{L}, Fh_{(k+1)}^{L}\right), \left(Th_{j}^{L}, Ih_{j}^{L}, Fh_{j}^{L}\right)\right) \end{pmatrix} \end{pmatrix}$$

The above two equations and equation (5) will applied in (6). The resulting equation will gives

$$\left(\left(\otimes_{\substack{i,j=1 \\ i\neq j}}^{k+1}\left(sTN_{i} \oplus tTN_{j}\right)\right), \left(\otimes_{\substack{i,j=1 \\ i\neq j}}^{k+1}\left(sIN_{i} \oplus tIN_{j}\right)\right), \left(\otimes_{\substack{i,j=1 \\ i\neq j}}^{k+1}\left(sFN_{i} \oplus tFN_{j}\right)\right)\right)$$

$$= \begin{pmatrix} \begin{pmatrix} \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i1}^{U} \oplus tTN_{j1}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i1}^{U} \oplus tIN_{j1}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i1}^{U} \oplus tFN_{j1}^{U}\right)\right)\right), \\ \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i2}^{U} \oplus tTN_{j2}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i2}^{U} \oplus tIN_{j2}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i2}^{U} \oplus tFN_{j2}^{U}\right)\right)\right), \\ \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i3}^{U} \oplus tTN_{j3}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i3}^{U} \oplus tIN_{j3}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i3}^{U} \oplus tFN_{j3}^{U}\right)\right)\right), \\ \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i4}^{U} \oplus tTN_{j4}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i4}^{U} \oplus tIN_{j4}^{U}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i4}^{U} \oplus tFN_{j4}^{U}\right)\right)\right) \\ min_{i=1,2,3,\dots,k+1\left(Th_{i}^{U}, Ih_{i}^{U}, Fh_{i}^{U}\right)} \end{pmatrix}, \\ \begin{pmatrix} \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i1}^{L} \oplus tTN_{j1}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i1}^{L} \oplus tIN_{j1}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i1}^{L} \oplus tFN_{j1}^{L}\right)\right)\right), \\ \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i2}^{L} \oplus tTN_{j2}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i2}^{L} \oplus tIN_{j2}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i2}^{L} \oplus tFN_{j2}^{L}\right)\right)\right), \\ \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i3}^{L} \oplus tTN_{j3}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i3}^{L} \oplus tIN_{j3}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i3}^{L} \oplus tFN_{j3}^{L}\right)\right)\right), \\ \left(\left(\otimes_{i=1}^{k+1}\left(sTN_{i4}^{L} \oplus tTN_{j4}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sIN_{i4}^{L} \oplus tIN_{j4}^{L}\right)\right), \left(\otimes_{i=1}^{k+1}\left(sFN_{i4}^{L} \oplus tFN_{j4}^{L}\right)\right)\right) \\ min_{i=1,2,3,\dots,k+1\left(Th_{i}^{L}, Ih_{i}^{L}, Fh_{i}^{L}\right)} \end{pmatrix} \end{pmatrix}$$

Next we prove (1) is true,

By the arithmetic operations defined for Neutrosophic member and equation),

It is verified that the below equation (1) is true for any n.

$$NBM^{(s,t)}\big((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_n, IN_n, FN_n)\big) = \left(\frac{1}{s+t}\left(\otimes^{m}{}_{\substack{i,j=1\\ i,j \text{ is not}\\ \text{same}}} \left(sTN_i \oplus tTN_j\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^{m}{}_{\substack{i,j=1\\ i,j \text{ is not}\\ \text{same}}} \left(sIN_i \oplus tIN_j\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^{m}{}_{\substack{i,j=1\\ i,j \text{ is not}\\ \text{same}}} \left(sFN_i \oplus tFN_j\right)\right)^{\frac{1}{m(m-1)}}\right)$$

Now we prove some important property for Neutrosophic Bonferroni mean(NBM)

3.Neutrosophic Bonferroni properties:

Property 3. 1:

This property is also called as idempotency on NBM.

Let

$$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$$
$$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$ be the and $s, t \geq 0$. If every $(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$ are equal for all i.

(i.e) $(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right) = (TN_0, IN_0, FN_0)$;

$((TN_0, IN_0, FN_0) = \left((TN_0^U, IN_0^U, FN_0^U), (TN_0^L, IN_0^L, FN_0^L)\right)$ then

$$NBM^{(s,t)}\big((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_n, IN_n, FN_n)\big) = ((TN_0, IN_0, FN_0) = \left((TN_0^U, IN_0^U, FN_0^U), (TN_0^L, IN_0^L, FN_0^L)\right) \quad (7)$$

Property 3.2:

This property is also called as boundedness on NBM.

$$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$$
$$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$ be the set of members of the Neutrosophic and for $s, t \geq 0$ and also we have $\left((TN_-^U, IN_-^U, FN_-^U), (TN_-^L, IN_-^L, FN_-^L)\right) =$

$$\left(\begin{pmatrix} \left(\min_i Ta_{i1}^U, \min_i Ia_{i1}^U, \min_i Fa_{i1}^U\right), \left(\min_i Ta_{i2}^U, \min_i Ia_{i2}^U, \min_i Fa_{i2}^U\right), \left(\min_i Ta_{i3}^U, \min_i Ia_{i3}^U, \min_i Fa_{i3}^U\right), \\ \left(\min_i Ta_{i4}^U, \min_i Ia_{i4}^U, \min_i Fa_{i4}^U\right), \left(\min_i Th_i^U, \min_i Ih_i^U, \min_i Fh_i^U\right) \end{pmatrix}, \begin{pmatrix} \left(\min_i Ta_{i1}^L, \min_i Ia_{i1}^L, \min_i Fa_{i1}^L\right), \left(\min_i Ta_{i2}^L, \min_i Ia_{i2}^L, \min_i Fa_{i2}^L\right), \left(\min_i Ta_{i3}^L, \min_i Ia_{i3}^L, \min_i Fa_{i3}^L\right), \\ \left(\min_i Ta_{i4}^L, \min_i Ia_{i4}^L, \min_i Fa_{i4}^L\right), \left(\min_i Th_i^L, \min_i Ih_i^L, \min_i Fh_i^L\right) \end{pmatrix}\right)$$

And$\left((TN_+^U, IN_+^U, FN_+^U), (TN_+^L, IN_+^L, FN_+^L)\right) =$

$$\left(\begin{pmatrix} \left(\max_i Ta_{i1}^U, \max_i Ia_{i1}^U, \max_i Fa_{i1}^U\right), \left(\max_i Ta_{i2}^U, \max_i Ia_{i2}^U, \max_i Fa_{i2}^U\right), \left(\max_i Ta_{i3}^U, \max_i Ia_{i3}^U, \max_i Fa_{i3}^U\right), \\ \left(\max_i Ta_{i4}^U, \max_i Ia_{i4}^U, \max_i Fa_{i4}^U\right), \left(\max_i Th_i^U, \max_i Ih_i^U, \max_i Fh_i^U\right) \end{pmatrix}, \begin{pmatrix} \left(\max_i Ta_{i1}^L, \max_i Ia_{i1}^L, \max_i Fa_{i1}^L\right), \left(\max_i Ta_{i2}^L, \max_i Ia_{i2}^L, \max_i Fa_{i2}^L\right), \left(\max_i Ta_{i3}^L, \max_i Ia_{i3}^L, \max_i Fa_{i3}^L\right), \\ \left(\max_i Ta_{i4}^L, \max_i Ia_{i4}^L, \max_i Fa_{i4}^L\right), \left(\max_i Th_i^L, \max_i Ih_i^L, \max_i Fh_i^L\right) \end{pmatrix}\right)$$

Then we have,

$$(TN_-, IN_-, FN_-) \leq NBM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\right) \leq (TN_+, IN_+, FN_+)$$
(8)

<u>Property 3.3:</u>

This property is also called as monotonicity on NBM.

$$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$$
$$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$ and for $s, t \geq 0$ and

$$(TM_i, IM_i, FM_i) = \left((TM_i^U, IM_i^U, FM_i^U), (TM_i^L, IM_i^L, FM_i^L)\right)$$
$$= (\left((Tb_{i1}^U, Ib_{i1}^U, Fb_{i1}^U), (Tb_{i2}^U, Ib_{i2}^U, Fb_{i2}^U), (Tb_{i3}^U, Ib_{i3}^U, Fb_{i3}^U), (Tb_{i4}^U, Ib_{i4}^U, Fb_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$$

$((Tb_{i1}^L, Ib_{i1}^L, Fb_{i1}^L), (Tb_{i2}^L, Ib_{i2}^L, Fb_{i2}^L), (Tb_{i3}^L, Ib_{i3}^L, Fb_{i3}^L), (Tb_{i4}^L, Ib_{i4}^L, Fb_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$ and for $s, t \geq 0$ and also $\left((Ta_{ik}^U \leq Tb_{ik}^U), (Ia_{ik}^U \leq Ib_{ik}^U), (Fa_{ik}^U \leq Fb_{ik}^U)\right)$ and $\left((Ta_{ik}^L \leq Tb_{ik}^L), (Ia_{ik}^L \leq Ib_{ik}^L), (Fa_{ik}^L \leq Fb_{ik}^L)\right)$

(9)

Property 4:

This property is also called as commutivity on NBM.

$$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$$
$$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$ and for $s, t \geq 0$ and

$$(TN'_i, IN'_i, FN'_i) = \left((TN_i'^U, IN_i'^U, FN_i'^U), (TN_i'^L, IN_i'^L, FN_i'^L)\right)$$
$$= (\left((Ta_{i1}'^U, Ia_{i1}'^U, Fa_{i1}'^U), (Ta_{i2}'^U, Ia_{i2}'^U, Fa_{i2}'^U), (Ta_{i3}'^U, Ia_{i3}'^U, Fa_{i3}'^U), (Ta_{i4}'^U, Ia_{i4}'^U, Fa_{i4}'^U), (Th_i'^U, Ih_i'^U, Fh_i'^U)\right),$$

$((Ta_{i1}'^L, Ia_{i1}'^L, Fa_{i1}'^L), (Ta_{i2}'^L, Ia_{i2}'^L, Fa_{i2}'^L), (Ta_{i3}'^L, Ia_{i3}'^L, Fa_{i3}'^L), (Ta_{i4}'^L, Ia_{i4}'^L, Fa_{i4}'^L), (Th_i'^L, Ih_i'^L, Fh_i'^L)))(i = 1,2, \dots, m)$ be the permutation number of above Neutrosophic member and for $s, t \geq 0$. Then,

$$NBM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\right) =$$
$$NBM^{(s,t)}\left((TN'_1, IN'_1, FN'_1), (TN'_2, IN'_2, FN'_2), \dots, (TN'_m, IN'_m, FN'_m)\right) \quad (10)$$

By giving parameters s, t different values, we will get different values.

4.Neutrosophic weighted Bonferroni operator:

Definition 4.1:

Let

$$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$$
$$= (\left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right),$$

$((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)))(i = 1,2, \dots, m)$ and for $s, t \geq 0$ and $(Tw, Iw, Fw) = \left((Tw_1, Iw_1, Fw_1), (Tw_2, Iw_2, Fw_2) \dots (Tw_m, Iw_m, Fw_m)\right)$ be the weight vector for $(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$, where $(Tw_i \geq 0, Iw_i \geq 0, Fw_i \geq 0)$ and $\sum_{i=0}^{n} Tw_i + \sum_{i=0}^{m} Iw_i + \sum_{i=0}^{m} Fw_i = 1$, then the Neutrosophic weighted Bonferroni operator is defined as

$$NWBM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\right) = \left(\frac{1}{s+t}\left(\bigotimes^{m}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(s(TN_i)^{w_i} \oplus t(TN_j)^{w_j}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\bigotimes^{m}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(s(IN_i)^{w_i} \oplus t(IN_j)^{w_j}\right)\right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\bigotimes^{m}_{\substack{i,j=1 \\ i,j \text{ is not} \\ \text{same}}} \left(s(FN_i)^{w_i} \oplus t(FN_j)^{w_j}\right)\right)^{\frac{1}{m(m-1)}}\right) \quad (14)$$

Theorem 4.1:

Let

$$(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$$
$$= \begin{pmatrix} \left((Ta_{i1}^U, Ia_{i1}^U, Fa_{i1}^U), (Ta_{i2}^U, Ia_{i2}^U, Fa_{i2}^U), (Ta_{i3}^U, Ia_{i3}^U, Fa_{i3}^U), (Ta_{i4}^U, Ia_{i4}^U, Fa_{i4}^U), (Th_i^U, Ih_i^U, Fh_i^U)\right), \\ \left((Ta_{i1}^L, Ia_{i1}^L, Fa_{i1}^L), (Ta_{i2}^L, Ia_{i2}^L, Fa_{i2}^L), (Ta_{i3}^L, Ia_{i3}^L, Fa_{i3}^L), (Ta_{i4}^L, Ia_{i4}^L, Fa_{i4}^L), (Th_i^L, Ih_i^L, Fh_i^L)\right) \end{pmatrix}$$

$(i = 1,2, \dots, n)$ and for s,t≥0 are $(Tw, Iw, Fw) = \left((Tw_1, Iw_1, Fw_1), (Tw_2, Iw_2, Fw_2) \dots (Tw_m, Iw_m, Fw_m)\right)$ be the weight vector for $(TN_i, IN_i, FN_i) = \left((TN_i^U, IN_i^U, FN_i^U), (TN_i^L, IN_i^L, FN_i^L)\right)$, where $(Tw_i \geq 0, Iw_i \geq 0, Fw_i \geq 0)$ and $\sum_{i=0}^{m} Tw_i + \sum_{i=0}^{m} Iw_i + \sum_{i=0}^{m} Fw_i = 1$. Additionally, a Neutrosophic member, so we have

$$NWBM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_n, IN_n, FN_n)\right) = (TN_w, IN_w, FN_w) = \left((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\right) \quad (15)$$

Where

$$(TN_w^U, IN_w^U, FN_w^U) = \begin{pmatrix} \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i1}^U\right)^{w_i} \oplus t\left(TN_{j1}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i1}^U\right)^{w_i} \oplus t\left(IN_{j1}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i1}^U\right)^{w_i} \oplus t\left(FN_{j1}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i2}^U\right)^{w_i} \oplus t\left(TN_{j2}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i2}^U\right)^{w_i} \oplus t\left(IN_{j2}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i2}^U\right)^{w_i} \oplus t\left(FN_{j2}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i3}^U\right)^{w_i} \oplus t\left(TN_{j3}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i3}^U\right)^{w_i} \oplus t\left(IN_{j3}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i3}^U\right)^{w_i} \oplus t\left(FN_{j3}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i4}^U\right)^{w_i} \oplus t\left(TN_{j4}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i4}^U\right)^{w_i} \oplus t\left(IN_{j4}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i4}^U\right)^{w_i} \oplus t\left(FN_{j4}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right) \\ min_{i=1,2,3,\dots,m\left(Th_i^U, Ih_i^U, Fh_i^U\right)} \end{pmatrix} \tag{16}$$

And

$$(TN_w^L, IN_w^L, FN_w^L) = \begin{pmatrix} \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i1}^L\right)^{w_i} \oplus t\left(TN_{j1}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i1}^L\right)^{w_i} \oplus t\left(IN_{j1}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i1}^L\right)^{w_i} \oplus t\left(FN_{j1}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i2}^L\right)^{w_i} \oplus t\left(TN_{j2}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i2}^L\right)^{w_i} \oplus t\left(IN_{j2}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i2}^L\right)^{w_i} \oplus t\left(FN_{j2}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i3}^L\right)^{w_i} \oplus t\left(TN_{j3}^U\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i3}^L\right)^{w_i} \oplus t\left(IN_{j3}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i3}^L\right)^{w_i} \oplus t\left(FN_{j3}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right), \\ \left(\frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(TN_{i4}^L\right)^{w_i} \oplus t\left(TN_{j4}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(IN_{i4}^L\right)^{w_i} \oplus t\left(IN_{j4}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}}, \frac{1}{s+t}\left(\otimes^m_{\substack{i,j=1 \\ i,j \text{ is not} \\ same}} \left(s\left(FN_{i4}^L\right)^{w_i} \oplus t\left(FN_{j4}^L\right)^{w_j}\right) \right)^{\frac{1}{m(m-1)}} \right) \\ min_{i=1,2,3,\dots,n\left(Th_i^L, Ih_i^L, Fh_i^L\right)} \end{pmatrix} \tag{17}$$

Now we define the property for Neutrosophic weighted Bonferroni operator

Property 4.1:

This property is also called as idempotency on NBWM.

Let

$(TN_w, IN_w, FN_w) = \left((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\right)$

$$= \begin{pmatrix} \left(\left((Ta_{i1}^U)^w, (Ia_{i1}^U)^w, (Fa_{i1}^U)^w\right), \left((Ta_{i2}^U)^w, (Ia_{i2}^U)^w, (Fa_{i2}^U)^w\right), \left((Ta_{i3}^U)^w, (Ia_{i3}^U)^w, (Fa_{i3}^U)^w\right), \left((Ta_{i4}^U)^w, (Ia_{i4}^U)^w, (Fa_{i4}^U)^w\right), (Th_i^U, Ih_i^U, Fh_i^U)\right), \\ \left(\left((Ta_{i1}^L)^w, (Ia_{i1}^L)^w, (Fa_{i1}^L)^w\right), \left((Ta_{i2}^L)^w, (Ia_{i2}^L)^w, (Fa_{i2}^L)^w\right), \left((Ta_{i3}^L)^w, (Ia_{i3}^L)^w, (Fa_{i3}^L)^w\right), \left((Ta_{i4}^L)^w, (Ia_{i4}^L)^w, (Fa_{i4}^L)^w\right), (Th_i^L, Ih_i^L, Fh_i^L)\right) \end{pmatrix},$$

$(i = 1,2, \dots, m)$ and if every $(TN_w, IN_w, FN_w) = \left((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\right)$ are equal for all.

(i.e) $(TN_w, IN_w, FN_w) = \left((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\right) = (TN_0, IN_0, FN_0)$;

$((TN_0, IN_0, FN_0) = \left((TN_0^U, IN_0^U, FN_0^U), (TN_0^L, IN_0^L, FN_0^L)\right)$ then

$$NBWM^{(s,t)}\left((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\right) = ((TN_0, IN_0, FN_0)$$
$$= \left((TN_0^U, IN_0^U, FN_0^U), (TN_0^L, IN_0^L, FN_0^L)\right)$$

Property4.2:

This property is also called as boundedness on NBWM.

$(TN_w, IN_w, FN_w) = \left((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\right) =$

$$\begin{pmatrix} \left(\left((Ta_{i1}^U)^w, (Ia_{i1}^U)^w, (Fa_{i1}^U)^w\right), \left((Ta_{i2}^U)^w, (Ia_{i2}^U)^w, (Fa_{i2}^U)^w\right), \left((Ta_{i3}^U)^w, (Ia_{i3}^U)^w, (Fa_{i3}^U)^w\right), \left((Ta_{i4}^U)^w, (Ia_{i4}^U)^w, (Fa_{i4}^U)^w\right), (Th_i^U, Ih_i^U, Fh_i^U)\right), \\ \left(\left((Ta_{i1}^L)^w, (Ia_{i1}^L)^w, (Fa_{i1}^L)^w\right), \left((Ta_{i2}^L)^w, (Ia_{i2}^L)^w, (Fa_{i2}^L)^w\right), \left((Ta_{i3}^L)^w, (Ia_{i3}^L)^w, (Fa_{i3}^L)^w\right), \left((Ta_{i4}^L)^w, (Ia_{i4}^L)^w, (Fa_{i4}^L)^w\right), (Th_i^L, Ih_i^L, Fh_i^L)\right) \end{pmatrix} (i = 1,2, \dots, m)$$ and

for $s, t \geq 0$ and also we have $(TN_-, IN_-, FN_-) = \left((TN_-^U, IN_-^U, FN_-^U), (TN_-^L, IN_-^L, FN_-^L)\right) =$

$$\left(\begin{pmatrix} \left(\min_i (Ta_{i1}^U)^{w_i}, \min_i (Ia_{i1}^U)^{w_i}, \min_i (Fa_{i1}^U)^{w_i}\right), \left(\min_i (Ta_{i2}^U)^{w_i}, \min_i (Ia_{i2}^U)^{w_i}, \min_i (Fa_{i2}^U)^{w_i}\right), \left(\min_i (Ta_{i3}^U)^{w_i}, \min_i (Ia_{i3}^U)^{w_i}, \min_i (Fa_{i3}^U)^{w_i}\right), \\ \left(\min_i (Ta_{i4}^U)^{w_i}, \min_i (Ia_{i4}^U)^{w_i}, \min_i (Fa_{i4}^U)^{w_i}\right), \left(\min_i (Th_i^U)^{w_i}, \min_i (Ih_i^U)^{w_i}, \min_i (Fh_i^U)^{w_i}\right) \end{pmatrix}, \begin{pmatrix} \left(\min_i (Ta_{i1}^L)^{w_i}, \min_i (Ia_{i1}^L)^{w_i}, \min_i (Fa_{i1}^L)^{w_i}\right), \left(\min_i (Ta_{i2}^L)^{w_i}, \min_i (Ia_{i2}^L)^{w_i}, \min_i (Fa_{i2}^L)^{w_i}\right), \left(\min_i (Ta_{i3}^L)^{w_i}, \min_i (Ia_{i3}^L)^{w_i}, \min_i (Fa_{i3}^L)^{w_i}\right), \\ \left(\min_i (Ta_{i4}^L)^{w_i}, \min_i (Ia_{i4}^L)^{w_i}, \min_i (Fa_{i4}^L)^{w_i}\right), \left(\min_i Th_i^L, \min_i Ih_i^L, \min_i Fh_i^L\right) \end{pmatrix}\right)$$

And

$(TN_+, IN_+, FN_+) = \left((TN_+^U, IN_+^U, FN_+^U), (TN_+^L, IN_+^L, FN_+^L)\right) =$

$$\left(\begin{pmatrix} \left(\max_i (Ta_{i1}^U)^{w_i}, \max_i (Ia_{i1}^U)^{w_i}, \max_i (Fa_{i1}^U)^{w_i}\right), \left(\max_i (Ta_{i2}^U)^{w_i}, \max_i (Ia_{i2}^U)^{w_i}, \max_i (Fa_{i2}^U)^{w_i}\right), \left(\max_i (Ta_{i3}^U)^{w_i}, \max_i (Ia_{i3}^U)^{w_i}, \max_i (Fa_{i3}^U)^{w_i}\right), \\ \left(\max_i (Ta_{i4}^U)^{w_i}, \max_i (Ia_{i4}^U)^{w_i}, \max_i (Fa_{i4}^U)^{w_i}\right), \left(\max_i (Th_i^U)^{w_i}, \max_i (Ih_i^U)^{w_i}, \max_i (Fh_i^U)^{w_i}\right) \end{pmatrix}, \begin{pmatrix} \left(\max_i (Ta_{i1}^L)^{w_i}, \max_i (Ia_{i1}^L)^{w_i}, \max_i (Fa_{i1}^L)^{w_i}\right), \left(\max_i (Ta_{i2}^L)^{w_i}, \max_i (Ia_{i2}^L)^{w_i}, \max_i (Fa_{i2}^L)^{w_i}\right), \left(\max_i (Ta_{i3}^L)^{w_i}, \max_i (Ia_{i3}^L)^{w_i}, \max_i (Fa_{i3}^L)^{w_i}\right), \\ \left(\max_i (Ta_{i4}^L)^{w_i}, \max_i (Ia_{i4}^L)^{w_i}, \max_i (Fa_{i4}^L)^{w_i}\right), \left(\max_i Th_i^L, \max_i Ih_i^L, \max_i Fh_i^L\right) \end{pmatrix}\right)$$

Then we have,

$$(TN_{-}, IN_{-}, FN_{-}) \leq NBWM^{(s,t)}\big((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_n, IN_n, FN_n)\big) \leq (TN_{+}, IN_{+}, FN_{+})$$

Property 4.3:

This property is also called as monotonicity on NBWM.

$(TN_w, IN_w, FN_w) = \big((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\big) =$
$\begin{pmatrix} \big(((Ta_{i1}^U)^w, (Ia_{i1}^U)^w, (Fa_{i1}^U)^w), ((Ta_{i2}^U)^w, (Ia_{i2}^U)^w, (Fa_{i2}^U)^w), ((Ta_{i3}^U)^w, (Ia_{i3}^U)^w, (Fa_{i3}^U)^w), ((Ta_{i4}^U)^w, (Ia_{i4}^U)^w, (Fa_{i4}^U)^w), (Th_i^U, Ih_i^U, Fh_i^U)\big), \\ \big(((Ta_{i1}^L)^w, (Ia_{i1}^L)^w, (Fa_{i1}^L)^w), ((Ta_{i2}^L)^w, (Ia_{i2}^L)^w, (Fa_{i2}^L)^w), ((Ta_{i3}^L)^w, (Ia_{i3}^L)^w, (Fa_{i3}^L)^w), ((Ta_{i4}^L)^w, (Ia_{i4}^L)^w, (Fa_{i4}^L)^w), (Th_i^L, Ih_i^L, Fh_i^L)\big) \end{pmatrix} (i = 1,2, \dots, m)$ and for $s, t \geq 0$

and

$(TM_w, IM_w, FM_w) = \big((TM_w^U, IM_w^U, FM_w^U), (TM_w^L, IM_w^L, FM_w^L)\big) =$
$\begin{pmatrix} \big(((Tb_{i1}^U)^w, (Ib_{i1}^U)^w, (Fb_{i1}^U)^w), ((Tb_{i2}^U)^w, (Ib_{i2}^U)^w, (Fb_{i2}^U)^w), ((Tb_{i3}^U)^w, (Ib_{i3}^U)^w, (Fb_{i3}^U)^w), ((Tb_{i4}^U)^w, (Ib_{i4}^U)^w, (Fb_{i4}^U)^w), (Th_i^U, Ih_i^U, Fh_i^U)\big), \\ \big(((Tb_{i1}^L)^w, (Ib_{i1}^L)^w, (Fb_{i1}^L)^w), ((Tb_{i2}^L)^w, (Ib_{i2}^L)^w, (Fb_{i2}^L)^w), ((Tb_{i3}^L)^w, (Ib_{i3}^L)^w, (Fb_{i3}^L)^w), ((Tb_{i4}^L)^w, (Ib_{i4}^L)^w, (Fb_{i4}^L)^w), (Th_i^L, Ih_i^L, Fh_i^L)\big) \end{pmatrix} (i = 1,2, \dots, m)$ and for $s, t \geq 0$ and also $\big(((Ta_{ik}^U)^w \leq (Tb_{ik}^U)^w), ((Ia_{ik}^U)^w \leq (Ib_{ik}^U)^w), ((Fa_{ik}^U)^w \leq (Fb_{ik}^U)^w)\big)$ and $\big(((Ta_{ik}^L)^w \leq (Tb_{ik}^L)^w), ((Ia_{ik}^L)^w \leq (Ib_{ik}^L)^w), ((Fa_{ik}^L)^w \leq (Fb_{ik}^L)^w)\big)$

Then we have

$$NBWM^{(s,t)}\big((TN_1, IN_1, FN_1), (TN_2, IN_2, FN_2), \dots, (TN_m, IN_m, FN_m)\big) \\ \leq NBWM^{(s,t)}\big((TM_1, IM_1, FM_1), (TM_2, IM_2, FM_2), \dots, (TM_m, IM_m, FM_m)\big)$$

Property 4.4:

This property is also called as commutivity on NBWM.

$(TN_w, IN_w, FN_w) = \big((TN_w^U, IN_w^U, FN_w^U), (TN_w^L, IN_w^L, FN_w^L)\big) =$
$\begin{pmatrix} \big(((Ta_{i1}^U)^w, (Ia_{i1}^U)^w, (Fa_{i1}^U)^w), ((Ta_{i2}^U)^w, (Ia_{i2}^U)^w, (Fa_{i2}^U)^w), ((Ta_{i3}^U)^w, (Ia_{i3}^U)^w, (Fa_{i3}^U)^w), ((Ta_{i4}^U)^w, (Ia_{i4}^U)^w, (Fa_{i4}^U)^w), (Th_i^U, Ih_i^U, Fh_i^U)\big), \\ \big(((Ta_{i1}^L)^w, (Ia_{i1}^L)^w, (Fa_{i1}^L)^w), ((Ta_{i2}^L)^w, (Ia_{i2}^L)^w, (Fa_{i2}^L)^w), ((Ta_{i3}^L)^w, (Ia_{i3}^L)^w, (Fa_{i3}^L)^w), ((Ta_{i4}^L)^w, (Ia_{i4}^L)^w, (Fa_{i4}^L)^w), (Th_i^L, Ih_i^L, Fh_i^L)\big) \end{pmatrix} (i = 1,2, \dots, n)$ and for $s, t \geq 0$

By giving parameters s, t different values, we have some different result.

5. Conclusion:

The classical Bonferroni mean operator and possibility degree have been extended in the trapezoidal and triangular neutrosophic environment to better organise and model the uncertainties and indeterminacy inside multi-attribute decision analysis. In FMAGDM, the neutrosophic Bonferroni operator can combine several decisions or evaluations from multiple decision-makers. Neutropphic surroundings, as opposed to trapezoidal and triangular contexts, are able to convey the decision-makers' ambiguity, indecision, and uncertainty. Based on the neutrosophic possibility degree and the TITRNWBM operator, we have introduced a novel approach for NMAGDM. Numerous difficult multiple-attribute decision-making issues can be resolved with the help of the suggested Neutrosophic Bonferroni operator and weighted Neutrosophic Bonferroni operator, both of which meet the necessary criteria and theorems. Therefore, we see this as a starting point for future research using this operator for solving multiple attributes decision making problems.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. *Information and control*, *8*(3), 338-353.

[2] Dubois, D., & Prade, H. (Eds.). (2012). *Fundamentals of fuzzy sets* (Vol. 7). Springer Science & Business Media.

[3] Xuecheng, L. (1992). Entropy, distance measure and similarity measure of fuzzy sets and their relations. *Fuzzy sets and systems*, *52*(3), 305-318.

[4Merigó, J. M. (2010). Fuzzy decision making with immediate probabilities. *Computers & Industrial Engineering*, *58*(4), 651-657.

[5]Zhao. H, Xu Z. S., Ni. M. F., and Cui. F., "Hybrid fuzzy multiple attributedecision making," Inform.-Tokyo, vol. 12, no. 5, pp. 1033–1044,2009.

[6] Turksen, I. B. (1986). Interval valued fuzzy sets based on normal forms. *Fuzzy sets and systems*, *20*(2), 191-210.

[7]Gau, W. L., & Buehrer, D. J. (1993). Vague sets. *IEEE transactions on systems, man, and cybernetics*, *23*(2), 610-614.

[8] Atanassov, K, "Intuitionistic fuzzy sets," Fuzzy Set Syst., vol. 20, pp. 87–96, 1986.

[9] Yager, R. R. (1986). On the theory of bags. Int. J. *Jour. of General Systems*, *13*.

[10] He, Y., Chen, H., Zhou, L., Liu, J., & Tao, Z. (2014). Intuitionistic fuzzy geometric interaction averaging operators and their application to multi-criteria decision making. *Information Sciences*, *259*, 142-159.

[11]Xia, M., Xu, Z., & Liao, H. (2012). Preference relations based on intuitionistic multiplicative information. *IEEE Transactions on Fuzzy Systems*, *21*(1), 113-133.

[12] Xu. Z., "Intuitionistic fuzzy aggregation operations," IEEE Trans. Fuzzy Syst., vol. 15, no. 6, pp. 1179–1187, Dec. 2007.

[13] Xu, Z. (2012). Priority weight intervals derived from intuitionistic multiplicative preference relations. *IEEE Transactions on Fuzzy Systems*, *21*(4), 642-654.

[14] Beliakov, G., & James, S. (2013). On extending generalized Bonferroni means to Atanassov orthopairs in decision making contexts. *Fuzzy Sets and Systems*, *211*, 84-98.

[15] Xia, M., Xu, Z., & Zhu, B. (2012). Generalized intuitionistic fuzzy Bonferroni means. *International Journal of Intelligent Systems*, *27*(1), 23-47.

[16] Xia, M., Xu, Z., & Zhu, B. (2013). Geometric Bonferroni means with their application in multi-criteria decision making. *Knowledge-Based Systems*, *40*, 88-100.

[17] Xu, Z., & Yager, R. R. (2010). Intuitionistic fuzzy Bonferroni means. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, *41*(2), 568-578.

[18] Gong, Y., Hu, N., Zhang, J., Liu, G., & Deng, J. (2015). Multi-attribute group decision making method based on geometric Bonferroni mean operator of trapezoidal interval type-2 fuzzy numbers. *Computers & Industrial Engineering*, *81*, 167-176.

[19] Bonferroni, C. (1950). Sullemedie multiple di potenze. *Bollettino dell'UnioneMatematicaItaliana*, *5*(3-4), 267-270.

[20] Yager, R. R. (2009). On generalized Bonferroni mean operators for multi-criteria aggregation. *International Journal of Approximate Reasoning*, *50*(8), 1279-1286.

[21] Beliakov, G., James, S., Mordelová, J., Rückschlossová, T., & Yager, R. R. (2010). Generalized Bonferroni mean operators in multi-criteria aggregation. *Fuzzy Sets and Systems*, *161*(17), 2227-2242.

[22] Wei, G., & Zhang, Z. (2019). Some single-valued neutrosophic Bonferroni power aggregation operators in multiple attribute decision making. *Journal of Ambient Intelligence and Humanized Computing*, *10*(3), 863-882.

[23]Nagarajan, D., Kanchana, A., Jacob, K., Kausar, N., Edalatpanah, S. A., & Shah, M. A. (2023). A novel approach based on neutrosophic Bonferroni mean operator of trapezoidal and triangular neutrosophic interval environments in multi-attribute group decision making. *Scientific reports*, *13*(1), 10455.

[24]Kanchana, A., Nagarajan, D., &Broumi, S. (2023). Multi-attribute group decision-making based on the Neutrosophic Bonferroni mean operator. *Neutrosophic Sets and Systems*, *57*(1), 8.

[25] Nagarajan, D., Broumi, S., Smarandache, F., & Kavikumar, J. (2021). Analysis of neutrosophic multiple regression. *Neutrosophic Sets and Systems*, *43*, 44-53.

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Study of the impact of cardiovascular exercises and their functional responses on diabetic peripheral neuropathy using neutrosophic statistics

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Abstract. The main purpose of this paper is to determine cardiovascular exercise strategies that improve functional responses in patients with diabetic peripheral neuropathy. The methodology used is the 6-minute walk test (6MWT) which is recognized in the field of treatment of health problems associated with diabetes. The sample included 34 patients of different ages. One of the difficulties we encounter in the study of these diseases is the indeterminacy and uncertainty of the diagnosis of the disease and the treatment. They are complex diseases that require family, institutional, and medical support, in addition to the patient's total cooperation. The processed data corresponds to laboratory tests that must be evaluated with an interpretation of normal or non-normal depending on values given in an interval form instead of a crisp value. That is why the authors of the article decided to process the data using Neutrosophic Statistics, where traditional methods are extended to the framework of intervals instead of crisp numbers. Finally, we obtained a linear equation in interval form to link the measurement of "dyspnea" with the "distance of meters traveled."

Keywords: neuropathy peripheral diabetic, diabetes mellitus, neutrosophic numbers, neutrosophic statistics, t-test, neutrosophic least square method.

1 Introduction

Today there are approximately 382 million people who have diabetes mellitus (DM) worldwide and the projection towards 2030 is not encouraging at all, since the WHO considers that this disease will become in the leading cause of death worldwide, which indicates a complex health panorama shortly, furthermore that at least 10% of diabetic patients present diabetic peripheral neuropathy, a figure that reaches up to 50% in patients who have had the disease for at least 10 years, and at least 75% develops a very high risk of amputations.

In Ecuador the prevalence is very high since it is estimated that at ages between 20-79 years old, it reaches up to 8.5% of the population; It is even shortly proposed that diabetes and associated neuropathies would be the second cause of deaths in general.

Diabetic peripheral neuropathy is a very common complication of type II diabetes mellitus. It is usually characterized by significant deficits in tactile sensitivity, the sense of vibration, and proprioception of the lower extremities. Performing cardiovascular exercises is one of the most effective and beneficial strategies to reduce the symptoms of diabetic peripheral neuropathy.

So, the main objective of this research is to develop cardiovascular exercise strategies that allow improving functional responses in patients with diabetic peripheral neuropathy; to verify its effectiveness. The procedure begins with a 6-minute walk test called 6MWT, to answer the question posed in this work: whether the development of strategies based on cardiovascular exercises can improve functional responses in patients with diabetic peripheral neuropathy.

For the study, we decided to use neutrosophic models since the clinical problems and treatment of diseases such as diabetes and its complications can only be studied if the uncertainty and indeterminacy of both the diagnosis and the treatment are taken into account. In this type of disease converge a series of biological, as well as sociological and psychological factors. The patient must be educated to live with his (or her) illness and avoid complications. In addition, health parameters do not correspond to a single value, but to a range of values.

Due to the aforementioned, Neutrosophic Statistics is the tool that we proposed to apply in the study of the strategies to follow in the improvement of patients who suffer from diabetic peripheral neuropathy, since Neutrosophic Statistics is the generalization of classical statistics to situations where data or parameters exist in the form of intervals, also where the size of the population cannot be precisely defined [1-3]. In our case, due to the nature of the problem that we set out to study, where an exact normal value of heart rate or laboratory test results cannot be determined, it is, therefore, necessary to use values in the form of interval or neutrosophic numbers. In this way, a greater number of measurement situations for each individual are taken into account, beyond the specific situation in which the study is carried out, which increases the reliability of the experiment. In addition, we obtained an equation to determine the number of meters that the patient can travel concerning their state of dyspnea. For this end, we use the neutrosophic least square method ([4]). In this way, the patient can determine his (her) physical condition for walking by measuring the state of his dyspnea. This can be extended to other variables.

This paper is divided into a preliminary section, where we present the main concepts of neutrosophic numbers and Neutrosophic Statistics. Section 3 contains the results of the study carried out. The last section is to give the conclusions.

2 Preliminaries

This section contains the fundamental concepts about neutrosophic numbers and neutrosophic statistics.

Neutrosophic statistics refers to a set of data, such that the data or a part of it is indeterminate to some degree, and to the methods used to analyze these data ([1]).

In classical statistics all data are determined, this is the distinction between neutrosophic statistics and classical statistics. In many cases, when the indeterminacy is zero, the neutrosophic statistics coincide with the classical statistics. The neutrosophic measurement can be used to measure indeterminate data. Neutrosophic statistical methods will allow us to interpret and organize neutrosophic data (data that may have some indeterminacies) to reveal underlying patterns. Many approaches can be used in neutrosophic statistics.

In *neutrosophic probability*, indeterminacy is different from randomness. While classical statistics refers only to randomness, neutrosophic statistics refers to both randomness and especially indeterminacy.

Neutrosophic descriptive statistics is composed of all techniques for summarizing and describing the characteristics of neutrosophic numerical data. Since neutrosophic numerical data contain indeterminacies, *neutrosophic line graphs* and *neutrosophic histograms* are represented in 3D spaces, rather than 2D spaces as in classical statistics. The third dimension, in addition to the Cartesian system XOY, is that of indeterminacy (I). From unclear graphic data, we can extract neutrosophic (unclear) information.

Neutrosophic inferential statistics consist of methods that allow the generalization of neutrosophic sampling to a population from which the sample was selected.

Neutrosophic data are data that contain some indeterminacy. In a similar way to classical statistics, it can be classified as:

- *Discrete neutrosophic data*, if the values are isolated points; for example: $7 + \mathrm{i}_1$, where $\mathrm{i}_1 \in [0,1]$, 2, $38 + \mathrm{i}_2$, where $\mathrm{i}_2 \in [10,12]$;

- and *Continuous neutrosophic data*, if the values form one or more intervals, for example [0.05, 0.1] or [0.9, 1.0] (i.e., not sure which one).

Other classification:

- *Quantitative* (*numerical*) *neutrosophic data*;

For example: a number in the interval [3, 8] (we do not know exactly), or; 50, 53, 58, or 61 (we do not know exactly);

- and *Qualitative* (*categorical*) *neutrosophic data*; for example: blue or red (we do not know exactly), white, black or green or yellow (we do not know exactly). Additionally, we can have:

- *Neutrosophic data univariate*, i.e. neutrosophic data consisting of observations on a single neutrosophic attribute;

- and *Multivariate neutrosophic data*, that is neutrosophic data consisting of observations on two or more attributes. In particular cases, we mention the *bivariate neutrosophic data* and the *trivariate neutrosophic data*.

A *neutrosophic statistical number* N has the following form:

$N = a + bI$, where a is the determinate (known) part of N, and bI is the indeterminate (unknown) part of N ([1]).

The arithmetic operations between these numbers are summarized below ([5-8]):

Given $N_1 = a_1 + b_1I$ and $N_2 = a_2 + b_2I$ two neutrosophic numbers, some operations between them are defined as follows:

- $N_1 + N_2 = a_1 + a_2 + (b_1 + b_2)I$ (Addition);
- $N_1 - N_2 = a_1 - a_2 + (b_1 - b_2)I$ (Difference),
- $N_1 \times N_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2)I$ (Product),
- $\frac{N_1}{N_2} = \frac{a_1+b_1I}{a_2+b_2I} = \frac{a_1}{a_2} + \frac{a_2b_1-a_1b_2}{a_2(a_2+b_2)}I$ (Division).

For example, $a = 4 + I$, where $I \in [0, 0.5]$, is equivalent to $a \in [4, 4.5]$, so for sure $a \geq 4$ (meaning that the determinate part of a is 4), while the indeterminate part $i \in [0, 0.5]$ means the possibility that the number a is a little greater than 4. For example, if we have the following neutrosophic data: $3 + I_1$ with $I_1 \in (0, 0.1)$; $5 + I_2$ with $I_2 \in [4, 6]$; $5 + I_3$, with $I_3 \in [0,1]$; $10 + I_4$, with $I_4 \in [1.1, 1.5)$; $9 + I_1$.

A *neutrosophic sample* is a selected subset of a population, a subset that contains some indeterminacy: either concerning several of its individuals (who may not belong to the population we study or may only partially belong to it) or concerning the subset as a whole.

While classical samples provide precise information, neutrosophic samples provide vague or incomplete information. By abuse of language, it can be said that any sample is a neutrosophic sample since its determination can be considered equal to zero.

The results of the *neutrosophic survey* are survey results that contain some indeterminacy. A *neutrosophic population* is a population that is not well determined at the level of membership (i.e., it is not certain whether some individuals belong or do not belong to the population). For example, as in the neutrosophic set, a generic $x(t, i, f) \in M$ element i% the belonging of x to M is indeterminate (unknown, unclear, neutral: neither in the population nor outside).

A *simple random neutrosophic sample* of size N from a classical or neutrosophic population is a sample of N individuals such that at least one of them has some indeterminacy.

A *neutrosophic normal distribution* of a continuous variable σ^2, for example, μ, or σ, or both can be set with two or more elements. The most common distributions are when μ, σ, or both are intervals.

The formula for the *neutrosophic frequency function* is the same, except replaced μ_N by μ and σ_N by σ:

$X_N \sim N_N(\mu_N, \sigma_N^2) = \frac{1}{\sigma_N\sqrt{2\pi}}\exp(-\frac{(x-\mu_N)^2}{2\sigma_N^2})$, where X_N means $N_N(\cdot,\cdot)$ that instead of one bell-shaped curve for X, we can have two or more bell-shaped curves that have common and uncommon regions between them and are above the x-axis. Each of them is symmetrical concerning the vertical line passing through the mean ($x = \mu$).

Let us illustrate this with a neutrosophic example for the normal distribution, let us consider a normal distribution with $\mu = 0$ and $\sigma = [1, 2]$. Therefore, the standard deviation is indeterminate.

"Within one standard deviation of the mean" is translated in this example by $\mu \pm \sigma = 10 \pm [2, 3] = [10 - 3, 10 + 3] = [7, 13]$, or approximately 68% of the values are in $x \in [7, 13]$.

"Within two times the standard deviations of the mean" translates to $\mu \pm 2\sigma = 10 \pm 2 \cdot [2, 3] = 10 \pm [4, 6] = [10 - 6, 10 + 6] = [4, 16]$, or approximately 95.4% of the values are in $x \in [4,16]$. We could also calculate the last interval as: $[7, 13] \pm \sigma = [7, 13] \pm [2, 3] = [7 - 3, 13 + 3] = [4, 16]$.

Similar to classical statistics, a *neutrosophic null hypothesis*, denoted by NH_0, is the statement that is initially assumed to be true. The *alternative neutrosophic hypothesis*, denoted by NH_a, is the other hypothesis.

When carrying out a test of NH_0 versus NH_a there are two possible conclusions: to reject NH_0 (if the sample evidence strongly suggests that NH_0 is false) or do not reject NH_0 (if the sample does not support evidence against NH_0).

Examples:

$NH_0 : \mu \in [90,100]$ $NH_a : \mu < 90$,

$NH_a: \mu > 100$, $NH_a: \mu \notin [90, 100]$, where μ represents the classical average Intelligence Quotient of all children born since January 1, 2001.

For reading applications of Neutrosophic Statistics, see [9-14].

3 Results

The study was carried out in the province of Tungurahua, Canton of Ambato–Cevallos, Ecuador. With a group of patients suffering from diabetic peripheral neuropathy. Table 1 contains a summary of the distribution in terms of gender, age, and obesity, among other characteristics of the subjects studied.

		% of the Total number of board
Age	Elderly	70.6%
	Adults	29.4%
Sex	Male	32.4%
	Female	67.6%
Index of massbodily	Low weight	2.9%
	Normal	32.4%
	Overweight	26.5
	Obesity	29.4

Table 1. Distribution of the patients under study according to their age, gender, and body mass index.

For a patient to be included in the study, the following inclusion criteria were used:

- Patients from Atahualpa and Cevallos who present diabetic peripheral neuropathy, to whom fractional exercises will be applied, during the period from April to September 2022.
- Patients with ages ranging from 30 to 80 years old who present diabetic peripheral neuropathy.
- Patients who freely express their participation in the study by signing the informed consent.

The exclusion criteria used were the following:

- Patients who present pathologies other than diabetic peripheral neuropathy.
- Patients with recent surgeries.
- Patients who, due to associated comorbidities, cannot comply with the protocol in its entirety.
- Patients with severe cardiovascular disease.

To determine the sample, probabilistic sampling was selected, a method that is characterized by seeking with great dedication to obtain qualitatively representative samples, through the inclusion of apparently typical groups, that is, they meet characteristics of interest to the researcher with patients who are located in Atahualpa and Cevallos, who present diabetic peripheral neuropathy, to whom intense cardiovascular exercises 70% to 80% will be applied, during the period October 2022-January 2023, after evaluation and from whom the data and information required to the development of the study.

In the study, an initial and final evaluation was developed with the implementation of an accessible cardiovascular exercise protocol for patients with diabetic peripheral neuropathy, 15-minute cardiovascular exercises were selected. A form was used to record the information on the 6-minute walk test data for each patient who applied the test and completed it in its entirety.

The 6-minute walk test is a variety of the Cooper test, which aims to measure the maximum distance that a person can walk for 6 minutes. The speed at which the patient walks will determine the distance in meters, that is, a test that evaluates, in an integrated manner, the response of the respiratory, cardiovascular, metabolic, skeletal muscle, and neurosensory systems that the individual develops during exercise.

The 6-minute test is a valid and reliable method to evaluate functional capacity in a population with cardiovascular problems in phase II/III. This test is very valuable for smaller healthcare facilities that wish to document functional improvements but do not have access to conventional treadmill testing.

Table 2 shows a summary of the other diseases associated with the patients under study.

Background Pathological Personal		
Disease	**Frequency**	**Percentage**
None	6	35.3
Hypertension	2	11.8
Hypothyroidism	6	35.3
Respiratory	3	17.6

Table 2. Percentage of person pathological history with diseases not directly related to diabetes and its complications.

To perform numerical calculations we use neutrosophic numbers to represent the collected data. For each aspect to be measured, the symbolic value I represents the normal range of what is measured, and in numerical calculations it is replaced by the equivalent range. For example, for oxygen saturation the normal range is between 95-100% resting, which is why we take $I = [95, 100]$, this guarantees that if a patient has a resting saturation equal to 96 before the study and after the study he (she) has 98%, then the difference is 0. In this case, a reference is obtained from what is normal to what is not normal. Although this means loss of precision, in reality, it is quite the opposite, since at the time of the study the patient may have a certain saturation that may vary at another time, for example, upon awakening and when the interval I is considered instead of a specific value is taking into account, then more situations are studied than the only one in which the experiment is carried out, and in this way the dynamic behavior of these indicators is more accurately reflected.

Before performing the statistical t Test, we checked that the data satisfied the normality condition; in this case, it was a non-parametric test of normality with the help of the Kolmogorov-Smirnov test adapted to data in the form of intervals. Although the test resulted in the data not being distributed normally, as the sample is large enough $n = 34 \geq 30$, it is well known that the t-test is robust enough to give reliable results in the case of large samples, greater than 30 [15,16].

Table 3 shows the results obtained for heart rate, it reflects the average of the initial evaluation and the final evaluation, after having the training with the exercises.

	Average initial evaluation	**Average final evaluation**	**Difference average**	**p**
Frequency cardiac resting	$-5.1 + I$	$-4.6 + I$	0.76	0.683
Frequency cardiac 3min	$-0.34 + I$	$2.41 + I$	2.75	0.561
Frequency cardiac 6min	$-0.12 + I$	$1.41 + I$	1.53	0.644

Table 3. Results regarding average heart rate.

The following table (Table 4) shows the results regarding the saturation levels during the initial and final evaluation process regarding the execution of the exercises.

	Average initial evaluation	**Average final evaluation**	**Difference average**	**p**
Saturation resting	$-1.81 + I$	$-1.53 + I$	0.28	0.988
Saturation 3 min	$-2.59 + I$	$-1.24 + I$	1.35	0.921
Saturation 6 min	$-0.76 + Yo$	$0.41 + Yo$	1.17	0.371

Table 4. Levels of average saturation.

Table 5 shows the difference in the degree of fatigue between the execution of exercises between the initial evaluation and the final evaluation.

	Average initial evaluation	**Average final evaluation**	**Difference average**	**p**
Fatigue resting	$0.02 + I$	$-0.1 + I$	-0.12	0.253
Fatigue 3 min	$2.78 + I$	$2.25 + I$	-0.53	0.253
Fatigue 6 min	$5.66 + I$	$4.78 + I$	-0.88	0.121

Table 5. Fatigue average.

Table 6 below shows the results regarding dyspnea between the initial and final evaluation regarding the execution of the exercise.

	Average initial evaluation	**Average final evaluation**	**Difference average**	**p**
Dyspnea resting	$0.02 + I$	$-0.1 + I$	-0.12	0.09
Dyspnea 3 min	$0.31 + I$	$-0.1 + I$	-0.41	0.14
Dyspnea 6 min	$0.84 + I$	$0.66 + I$	-0.18	0.34

Table 6. Dyspnea average.

Table 7 summarizes the results regarding blood pressure.

	Average initial evaluation	**Average final evaluation**	**Difference average**	**p**
Pressure arterial systolic resting	$7.35 + I$	$10.12 + I$	2.77	0.002

Pressure arterial diastolic resting	$-0.24 + I$	$0.12 + I$	2.36	0.013
Pressure arterial systolic 6 min	$16.18 + I$	$16.41 + I$	0.23	0.010
Pressure arterial diastolic 6 min	$2.88 + I$	$2.76 + I$	-0.12	0.501

Table 7. Arterial Pressure average.

Table 8 shows the final results regarding the distance traveled in meters and the volume of oxygen used. In the results of the table, we do not consider neutrosophic numbers, since the measured parameters have no limits of what is a normal parameter.

	Average initial evaluation	**Average final evaluation**	**Difference average**	**p**
Meters traveled	325.71	336.18	10.47	0.00
VO2 max	22.89	23.55	0.66	0.00

Table 8. Distance traveled and average VO2.

Finally, in the following, we consider the neutrosophic method least squares that appear in ([4]). This is an extension of the well-known statistical method. The objective is to link the variable dyspnea as a dependent variable with the variable meters traveled as an independent variable. To do this, we carry out a statistical approximation using a linear function. The Equations are the following represented by intervals.

First, we wish to obtain the coefficients of the following linear equation:

$$\hat{y}_N = a_N + b_N x_N \quad (1)$$

Where $\hat{y}_N$ is the approximation in interval form (or its equivalent in neutrosophic number) of the dependent variable, $a_N, and\ b_N$ are the coefficients in numbers within intervals of the linear equation, while x_N is the data of the independent variable given in the form of intervals/neutrosophic numbers.

The approximation of the first coefficient is obtained from Equation 2.

$$\bar{a}_N = \bar{y}_N - b_N \bar{x}_N \quad (2)$$

Where $\bar{a}_N \in [\bar{a}_L, \bar{a}_U]$ and for the approximation of b_N Equation 3 is used.

$$\bar{b}_N = \frac{n_N(\sum x_N y_N) - (\sum x_N)(\sum y_N)}{n_N(\sum x_N^2)(\sum x_N)^2} \quad (3)$$

$\bar{b}_N \in [\bar{b}_L, \bar{b}_U]$ and n_N is the number of elements in the sample.

In this way, we obtained the equation $I_{dist} = 338.601 - 77.0 I_{dys}$.

Conclusion

In this article, we carry out a study of the effectiveness of applying the 6-minute walk test (6MWT) in patients suffering from diabetic peripheral neuropathy in Ecuador. The study was carried out with 34 patients who suffer from this complication due to diabetes. Measurements were made of the results of different medical indicators applied before and after the training of the patients with the 6MWT, they are namely, "heart rate", "oxygen saturation", "fatigue", "dyspnea", "blood pressure", "distance traveled" and "condition physical cardiorespiratory (VO2max)". Within the study, we realized that the data collected are crisp and respond to the patient's state at a precise moment of measurement, although these parameters change over time and moment, it is for this reason that we converted the data from crisp to neutrosophic numbers and we apply neutrosophic statistics methods. Additionally, we found a statistical relationship between dyspnea and the number of meters traveled by patients with diabetic peripheral neuropathy, based on the neutrosophic least square method.

Specifically, about the results of the method, we conclude the following:

- It is concluded that from the state of health and physical condition of diabetic patients, the vast majority suffer from high blood pressure that affects their health condition and is a critical factor in diabetic patients if it is not adequately controlled.
- At the end of the application of the accessible cardiovascular exercise protocol for patients with diabetic peripheral neuropathy, which was designed with information from the initial diagnosis of the patient's clinical histories that allowed us to know the patient's pathologies, it was determined that the test of the 6 minutes is effective to evaluate the maximum travel distance and VO2 level in the effort between

distance and walking, to apply 15-minute cardiovascular exercises working at an intense intensity of 70% to 80% that were carried out in 8 weeks.

- At the end of the intervention, the results in older adults with diabetic peripheral neuropathy after having trained in cardiovascular exercises established that the development of the intervention favors systolic blood pressure resting and during the 6-minute exercise because it adapts the need and physical activity favors the health of patients when it is developed in a planned manner and based on their needs, regular exercise can contribute to diabetic patients by improving their quality of life.

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References

[1]. Smarandache, F. (2014). Introduction to Neutrosophic Statistics. Craiova: Sitech and Education Publisher.

[2]. Smarandache, F. (2022) Neutrosophic Statistics is an extension of Interval Statistics, while Plithogenic Statistics is the most general form of statistics (second version). International Journal of Neutrosophic Science, 19, 148-165.

[3]. Meléndez-Carballido, R., Paronyan, H., Alfaro-Matos, M., and Santillán-Molina, A. L. (2019) Neutrosophic statistics applied to demonstrate the importance of humanistic and higher education components in students of legal careers. Neutrosophic Sets and Systems, 26, 174-180.

[4]. Aslam, M., and Albassam, M. (2022) Forecasting of Wind Speed Using an Interval-Based Least Square Method. Frontiers in Energy Research, 10, 1-7.

[5]. Moore, E. R. (1966). Interval Analysis. Englewood Cliffs: Prentice Hall.

[6]. W.B., V., Kandasamy, I., Smarandache, F. (2018) Algebraic Structure of Neutrosophic Duplets in Neutrosophic Rings <ZUI>, <QUI>, and <RUI>, Neutrosophic Sets and Systems, 23, 85-95.

[7]. Abobala, M. (2020) On Some Special Substructures of Neutrosophic Rings and Their Properties. International Journal of Neutrosophic Science, 4, 72-81.

[8]. Abobala, M. (2021) Foundations of Neutrosophic Number Theory. Neutrosophic Sets and Systems, 39, 120-132.

[9]. Mena-Silva, P. A., Romero-Fernández, A., and Granda-Macías, L. A. (2020) Neutrosophic Statistics to Analyze Prevalence of Dental Fluorosis. Neutrosophic Sets and Systems, 37, 160-168.

[10]. Cadena-Piedrahita, D., Helfgott-Lerner, S., Drouet-Candel, A., Cobos-Mora, F., and Rojas-Jorge, N. (2021) Herbicides in the Irrigated Rice Production System in Babahoyo, Ecuador, Using Neutrosophic Statistics. Neutrosophic Sets and Systems, 39, 153-163.

[11]. Shahzadi, I., Aslam, M., and Aslam, H. (2021) Neutrosophic Statistical Analysis of Income of YouTube Channels. Neutrosophic Sets and Systems, 39, 101-106.

[12]. Valencia-Cruzaty, L. E., Reyes-Tomalá, M., Castillo-Gallo, C. M., and Smarandache, F. (2020) A Neutrosophic Statistic Method to Predict Tax Time Series in Ecuador. Neutrosophic Sets and Systems, 34, 33-39.

[13]. Molina-Manzano, A. D., Vildoso-Villegas, J. Y., Ochoa-Escobar, L. M., and Toapanta-Jiménez, L. (2020) Neutrosophic Statistical Analysis of the Incidence of the Facultative Vote of Young People between 16 and 18 Years Old in the Electoral Process of Ecuador. Neutrosophic Sets and Systems, 37, 355-360.

[14]. Marcia Esther, E. H., Robert Alcides, F. H. and Rene Estalin, P.P. (2022) Neutrosophic Statistics for Social Science. International Journal of Neutrosophic Science, 19, 250-259.

[15]. Zimmerman, D. W. (1997) A Note on Interpretation of the Paired-Samples t Test. Journal of Educational and Behavioral Statistics, 22, 349–360.

[16]. David, H. A., and Gunnink, J. L. (1997) The Paired t Test Under Artificial Pairing. The American Statistician, 51, 9-12.

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Determination of the degree of relationship between Activity Cost and Financial Management in beef cattle production in a region of Peru, based on Indeterminate Likert Scale and Neutrosophic Similarity

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Abstract. Activity Cost and Financial Management are two variables of vital importance in livestock production. This paper aims to measure the relationship existing between these two variables within the production of beef cattle in the Coto-Coto Chilca Livestock Fair in Peru. To do this, we selected 141 ranchers from the area to give their opinions regarding the behavior of these two variables. The data were represented with the help of an Indeterminate Likert Scale, to capture the uncertainty and indeterminacy of the respondents' opinion. Survey results were compared for the two variables using a measure of neutrosophic similarities. Neutrosophic similarities are used to measure the degree of similarity between two neutrosophic sets measured in certain aspects.

Keywords: Activity costs, financial management, profitability, resource optimization, Indeterminate Likert Scale, neutrosophic similarity measure, triple refined indeterminate neutrosophic set, refined neutrosophic set.

1 Introduction

The cost due to economic activities refers to the identification and analysis of the different activities for the allocation of the corresponding costs. Likewise, the cost by activities has the purpose of optimizing resources by identifying unnecessary activities and the efficient use of time, which makes it a great tool for making timely decisions and proposing policies that improve financial indicators to generate a competitive advantage, [1]. So, the cost by activities has the purpose of identifying highly relevant activities to assign a good cost to those that are generating a good performance for the organization.

Financial management consists of a process that seeks to plan, organize, direct, and control the economic activities and cash flows of organizations, to be able to make decisions regarding investment and financing issues in addition to stabilizing the relationship between risk and profitability, [2]. Financial management refers to the way of planning, organizing, directing, and controlling the economic movements of an organization to make financial decisions that benefit its profitability.

At an international level, in terms of production, livestock systems have evolved towards mixed agricultural-livestock and dairy production systems, among other changes. The rapid increase in per capita consumption of meat and milk has been accompanied by a change in dietary patterns. However, the benefits of expanding this activity must be carefully weighed against growing concerns about unintended consequences (particularly environmental damage and disease outbreaks).

At the national level in Peru, livestock farming has been a primary activity for the consumption and marketing of meat and milk, to provide income to livestock farmers through production. Therefore, farmers must know about breeding, feeding, and caring for animals, as well as the use of medications, supplements, and modifications to increase production.

Due to the volatile nature of the business world, reliable cost data is essential to make well-informed strategic decisions. Since accurate information is the cornerstone of any good choice, a flawed pricing system is a serious failure. Therefore, companies want reliable data to make important decisions. The expenses incurred by the company provide most of the data necessary for decision-making.

This article is focused on studying the variables Activity cost and Financial management, they are concepts of vital importance in the study of business sciences, the topic deals with the measurement of their behaviors and focusing them on our problem, for benefiting the beef cattle producers that gather at the Coto-Coto Sunday fair, our purpose is to determine the relationship between the Activities cost and Financial management in Beef Cattle Producers at this Livestock Fair.

The research design is included from a non-experimental level, the sample of study was made up of 141 beef cattle producers participating in the fair, to whom a questionnaire was applied under the Indeterminate Likert Scale as the measurement scale, [3-5].

To measure opinion, which is subjective, it is necessary to measure the uncertainty and at the same time the indeterminacy of the criteria given by the interviewee. That is why an indeterminate Likert scale is used to quantify the degree of agreement-disagreement of the interviewee with the item on which they are asked to give their opinion; in this case, how the two variables Activities cost and Financial management are appropriate. The Indeterminate Likert Scale is based on the triple refined indeterminate neutrosophic sets (TRINS) [6], which are part of the refined neutrosophic sets, where the component of indeterminacy is split into three other subcomponents, to obtain greater accuracy, [7-9].

To determine the degree of relationship that exists between the two variables, we apply a measure of neutrosophic similarity. Neutrosophic similarity is an extension of the concept of fuzzy similarity, where the degree of similarity between two elements belonging to different fuzzy sets is measured using the degree of uncertainty about a certain aspect [10-11]. In the case of neutrosophic sets, specifically Single-Valued Neutrosophic Sets, we have two additional components that are indeterminacy and falsity, which increase accuracy compared to fuzzy sets and similarities. In this case, we adapt the similarity formulas to the TRINS, which contains two additional components, five in total.

In this article, we divide the presentation into a Materials and Methods section, where we present the fundamental concepts of the Indeterminate Likert Scale and Neutrosophic Similarity. This is followed by a Results section where the details of the study carried out are presented. We finish with the Conclusions.

2 Materials and Methods

This section summarizes the main theoretical contents that we used in the study. First, we offer the basic notions about the Indeterminate Likert Scale. The second subsection is dedicated to remembering the basic concepts of Neutrosophic Similarity.

2.1. Indeterminate Likert Scale

Definition 1 ([6]). The *Single-Valued Neutrosophic Set* (SVNS) N over U is $A = \{< x;\ T_A(x), I_A(x), F_A(x) > : x \in U\}$, where $T_A: U \to [0, 1]$, $I_A: U \to [0, 1]$, and $F_A: U \to [0, 1]$, $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2 ([7]). The *refined neutrosophic logic* is defined such that: a truth T is divided into several types of truths: $T_1, T_2, \dots, T_p$, I into various indeterminacies: $I_1, I_2, \dots, I_r$ and F into various falsities: $F_1, F_2, \dots, F_s$, where all $p, r, s \geq 1$ are integers, and $p + r + s = n$.

Definition 3 ([6]). A *triple refined indeterminate neutrosophic set* (TRINS) A in X is characterized by positive $P_A(x)$, indeterminacy $I_A(x)$, negative $N_A(x)$, positive indeterminacy $I_{P_A}(x)$ and negative indeterminacy $I_{N_A}(x)$ membership functions. Each of them has a weight $w_m \in [0, 1]$ associated with it. For each $x \in X$, there are $P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x) \in [0,1]$, $w_P^m(P_A(x)), w_{I_P}^m(I_{P_A}(x)), w_I^m(I_A(x)), w_{I_N}^m(I_{N_A}(x)), w_N^m(N_A(x)) \in [0, 1]$ and $0 \leq P_A(x) + I_{P_A}(x) + I_A(x) + I_{N_A}(x)(x) + N_A(x) \leq 5$. Therefore, a TRINS A can be represented by $A = \{\langle x;\ P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x)\rangle | x \in X\}$.

Let A and B be two TRINS in a finite universe of discourse, $X = \{x_1, x_2, \dots, x_n\}$, which are denoted by:

$A = \{\langle x;\ P_A(x), I_{P_A}(x), I_A(x), I_{N_A}(x), N_A(x)\rangle | x \in X\}$ and $B = \{\langle x;\ P_B(x), I_{P_B}(x), I_B(x), I_{N_B}(x), N_B(x)\rangle | x \in X\}$, Where $P_A(x_i), I_{P_A}(x_i), I_A(x_i), I_{N_A}(x_i), N_A(x_i), P_B(x_i), I_{P_B}(x_i), I_B(x_i), I_{N_B}(x_i), N_B(x_i) \in [0, 1]$, for every $x_i \in X$. Let $w_i(i = 1,2,...,n)$ be the weight of an element x_i $(i = 1,2,...,n)$, with $w_i \geq 0$ $(i = 1,2,...,n)$ and $\sum_{i=1}^{n} w_i = 1$.

The *generalized TRINS weighted distance* is ([6, 12]):

$$d_\lambda(A,B) = \left\{\frac{1}{5}\sum_{i=1}^{n} w_i \left[|P_A(x_i) - P_B(x_i)|^\lambda + |I_{P_A}(x_i) - I_{P_B}(x_i)|^\lambda + |I_A(x_i) - I_B(x_i)|^\lambda + |I_{N_A}(x_i) - I_{N_B}(x_i)|^\lambda + |N_A(x_i) - N_B(x_i)|^\lambda\right]\right\}^{1/\lambda} \quad (1)$$

Where $\lambda > 0$.

The Indeterminate Likert Scale is formed by the following five elements:
– Negative membership,
– Indeterminacy leaning towards negative membership,
– Indeterminate membership,
– Indeterminacy leaning towards positive membership,
– Positive membership.

These values substitute the classical Likert scale with values:
–Strongly disagree,
– Disagree,
–Neither agree or disagree,
– Agree,
–Strongly agree.

Respondents are asked to give their opinion on a scale of 0-5 about their agreement in each of the possible degrees, which are "Strongly disagree", "Disagree", "Neutral", "Agree", "Strongly agree", for this end, they were provided with a visual scale like the one shown in Figure 1.

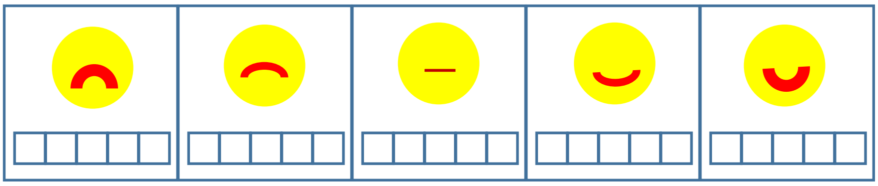

Figure 1. Graphic representation of the proposed Indeterminate Likert Scale.

2.2. Some Notions on Neutrosophic Similarity

Definition 4: ([10-11, 13-16]) The *degree of similarity* between two single-valued neutrosophic sets A and B is a mapping $S: \mathcal{N}(X) \times \mathcal{N}(X) \to [0,1]^3$, where $\mathcal{N}(X)$ is the set of all single-valued neutrosophic sets in $X = \{x_1, x_2, \cdots, x_n\}$, such that $S(A,B) = (S_T(A,B), S_I(A,B), S_F(A,B))$ satisfies conditions (S1)-(S4).

(S1) $S(A,B) = S(B,A), \forall A,B \in \mathcal{N}(X)$,

(S2) $S(A,B) = \underline{1} = (1,0,0)$ if and only if $A = B$,

(S3) $S_T(A,B) \geq 0, S_I(A,B) \geq 0, S_F(A,B) \geq 0, \forall A,B \in \mathcal{N}(X)$,

(S4) If $A \subseteq B \subseteq C$, then $S(A,B) \geq S(A,C)$ and it satisfies $S(B,C) \geq S(A,C)$.

Definition 5: ([10-11]) Let $A,B \in \mathcal{N}(X)$ in $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity between A and B is calculated by $S(A,B) = (S_T(A,B), S_I(A,B), S_F(A,B))$, where $S_T(A,B)$ is the degree of similarity of truthfulness, $S_I(A,B)$ is the degree of similarity of indeterminacy, and $S_F(A,B)$ is the degree of similarity of falsity. The formulas for similarity are the following:

$$S_T(A,B) = \left(\sum_{i=1}^{n}\left[\frac{min\ (T_A(x_i),T_B(x_i))}{max\ (T_A(x_i),T_B(x_i))}\right]\right)/n \quad (2a)$$

$$S_I(A,B) = 1 - \left(\sum_{i=1}^{n}\left[\frac{min\ (I_A(x_i),I_B(x_i))}{max\ (I_A(x_i),I_B(x_i))}\right]\right)/n \quad (2b)$$

$$S_F(A,B) = 1 - \left(\sum_{i=1}^{n}\left[\frac{min\ (F_A(x_i),F_B(x_i))}{max\ (F_A(x_i),F_B(x_i))}\right]\right)/n \quad (2C)$$

$\forall x_i \in X$.

Definition 6: ([10-11]) Suppose that for each $x_i \in X = \{x_1, x_2, \cdots, x_n\}$ a weight $w_i \in [0,1]$ is associated such that $\sum_{i=1}^{n} w_i = 1$. Let $A,B \in \mathcal{N}(X)$, then a weighted similarity measure between A and B is calculated by

$S_w(A,B) = \left(S_w^T(A,B), S_w^I(A,B), S_w^F(A,B)\right)$, where $S_w^T(A,B)$ is the degree of similarity of truthfulness, $S_w^I(A,B)$ is the degree of similarity of indeterminacy, and $S_w^F(A,B)$is the degree of similarity of the falsehood. The formulas for similarity are the following:

$$S_w^T(A,B) = \sum_{i=1}^n w_i \left[\frac{min\ (T_A(x_i),T_B(x_i))}{max\ (T_A(x_i),T_B(x_i))}\right] \quad (3a)$$

$$S_w^I(A,B) = 1 - \sum_{i=1}^n w_i \left[\frac{min\ (I_A(x_i),I_B(x_i))}{max\ (I_A(x_i),I_B(x_i))}\right] \quad (3b)$$

$$S_w^F(A,B) = 1 - \sum_{i=1}^n w_i \left[\frac{min\ (F_A(x_i),F_B(x_i))}{max\ (F_A(x_i),F_B(x_i))}\right] \quad (3c)$$

$\forall x_i \in X$.

Definition 7: ([10-11]) Let $A, B \in \mathcal{N}(X)$ in $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity between A and B is calculated by $L(A,B) = \left(L_T(A,B), L_I(A,B), L_F(A,B)\right)$, where $L_T(A,B)$ is the degree of similarity of truthfulness, $L_I(A,B)$ is the degree of similarity of indeterminacy, and $L_F(A,B)$ is the degree of similarity of falsity. The formulas for similarity are the following:

$$L_T(A,B) = 1 - \frac{\sum_{i=1}^n |T_A(x_i) - T_B(x_i)|}{\sum_{i=1}^n |T_A(x_i) + T_B(x_i)|} \quad (4a)$$

$$L_I(A,B) = \frac{\sum_{i=1}^n |I_A(x_i) - I_B(x_i)|}{\sum_{i=1}^n |I_A(x_i) + I_B(x_i)|} \quad (4b)$$

$$L_F(A,B) = \frac{\sum_{i=1}^n |F_A(x_i) - F_B(x_i)|}{\sum_{i=1}^n |F_A(x_i) + F_B(x_i)|} \quad (4c)$$

$\forall x_i \in X$.

Definition 8: ([10-11]) Let $A, B \in \mathcal{N}(X)$ in $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity between A and B is calculated by $M(A,B) = \left(M_T(A,B), M_I(A,B), M_F(A,B)\right)$, where $M_T(A,B)$ is the degree of similarity of truthfulness, $M_I(A,B)$ is the degree of similarity of indeterminacy, and $M_F(A,B)$ is the degree of similarity of falsity. The formulas for similarity are the following:

$$M_T(A,B) = \frac{1}{n}\sum_{i=1}^n \left(1 - \frac{|T_A(x_i) - T_B(x_i)|}{2}\right) \quad (5a)$$

$$M_I(A,B) = \frac{1}{n}\sum_{i=1}^n \left(\frac{|I_A(x_i) - I_B(x_i)|}{2}\right) \quad (5b)$$

$$M_F(A,B) = \frac{1}{n}\sum_{i=1}^n \left(\frac{|F_A(x_i) - F_B(x_i)|}{2}\right) \quad (5c)$$

$\forall x_i \in X$.

Definition 9: ([10-11]) Let $A, B \in \mathcal{N}(X)$ where $X = \{x_1, x_2, \cdots, x_n\}$, then a measure of similarity based on the distance between A and B is calculated by:

$$S^1(A,B) = \frac{1}{1 + d(A,B)} \quad (6)$$

Such that $d(A,B)$ is a distance function between the two single-valued neutrosophic sets.

Let us recall that the distance function satisfies the following axioms $\forall A, B, C \in \mathcal{N}(X)$:

(1) $d(A,B) \geq 0$ and $d(A,B) = 0$ if and only if $A = B$,

(2) $d(A,B) = d(B,A)$,

(3) $d(A,C) \leq d(A,B) + d(B,C)$.

3 Results

First of all, we establish the similarity formula that we use in data processing. We start with the generalized Triple Refined Indeterminate Neutrosophic weighted distance with the help of Equation 1. $\lambda = 1,2$ are the two values that define the Hamming and Euclidean distances, respectively.

We define the neutrosophic similarity on the TRINS using formula 6 combined with the distance in (1).

To collect the data, 141 cattle farmers participating in the Coto-Coto Livestock Fair in Peru were asked to give their opinions on Activity costs per and Financial management.

The survey must be evaluated for each question for each of the possible evaluations on a scale of 0-5 as shown in Figure 1. 0 indicates that the given evaluation grade is not accepted and 5 means the maximum grade for such evaluation, this step must be done on every possible evaluation. Figure 2 shows an example to rely on.

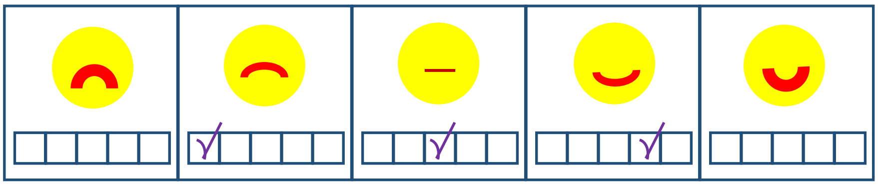

Figure 2. Example of the graphic use of the proposed Indeterminate Likert Scale.

In the example in Figure 2, it can be seen that the respondent expressed grade 0 of "strongly disagree", grade 1 of "disagree", grade 3 of "neutral", grade 4 of "agree", and grade 0 of "strongly agree". This allows us for obtaining greater precision in capturing the opinion and feelings of the respondent since there is not always a single possibility of agreement-disagreement with what is asked, rather in general there is a mixture.

The steps to follow are those:

1. Evaluate at all levels of opinion the degree of agreement-disagreement that an appropriate "Activity cost" is being applied in local livestock farming.
2. Evaluate at all levels of the opinion of the degree of agreement-disagreement that adequate "Financial management" is being applied in local livestock farming.
3. Each grade selected for each agreement-disagreement is associated with a value of 0.2. In the example in Figure 2, it is true that "Strongly disagree" has a value of $0(0.2) = 0$, "Disagree" has a value of $1(0.2) = 0.2$, "Neutral" is $3(0.2) = 0.6$, and so on. Finally, in the example, we have a TRINS equal to $(0,0.2,0.6,0.8,0)$.
4. Each of the 141 ranchers is consulted about their opinion. The data is collected and converted into the form of TRINS. Let $C(X)$ be the TRINS on "Activity cost" and $M(X)$ denotes the TRINS on "Financial management", for each of the respondents $X = \{x_1, x_2, \dots, x_{141}\}$.
5. It is calculated $d_2(C, M)$ (Equation 1) with $\omega_i = \frac{1}{141}$ $\forall x_i \in X$, and then the degree of similarity (Equation 6). This last index is the one required to determine the relationship between one variable and another.

Figures 3 and 4 contain the bar graphs with the degree of satisfaction-dissatisfaction for each of the two variables.

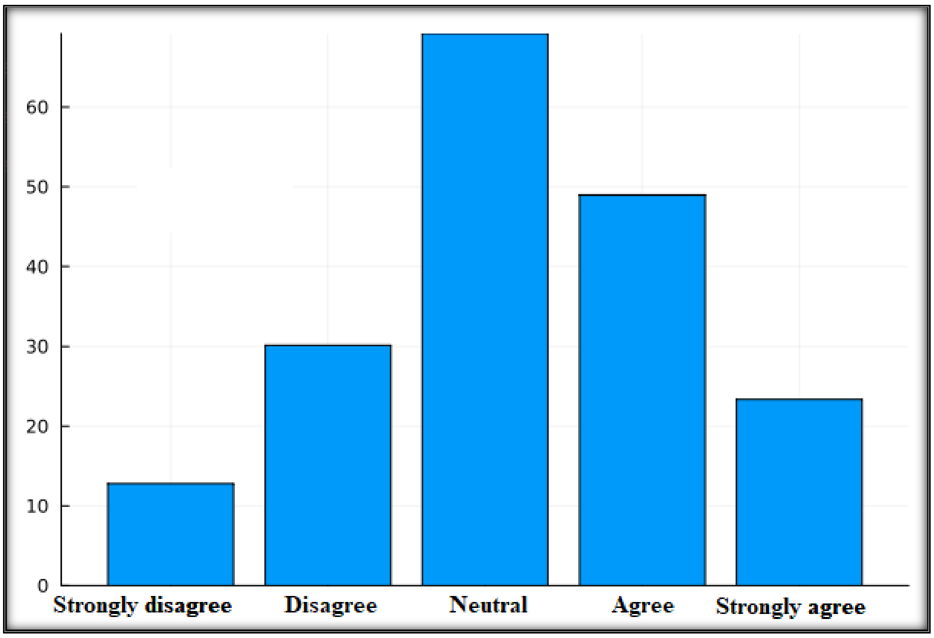

Figure 3. Bar chart on the degree of agreement-disagreement regarding "Adequate cost for activities" in percentage.

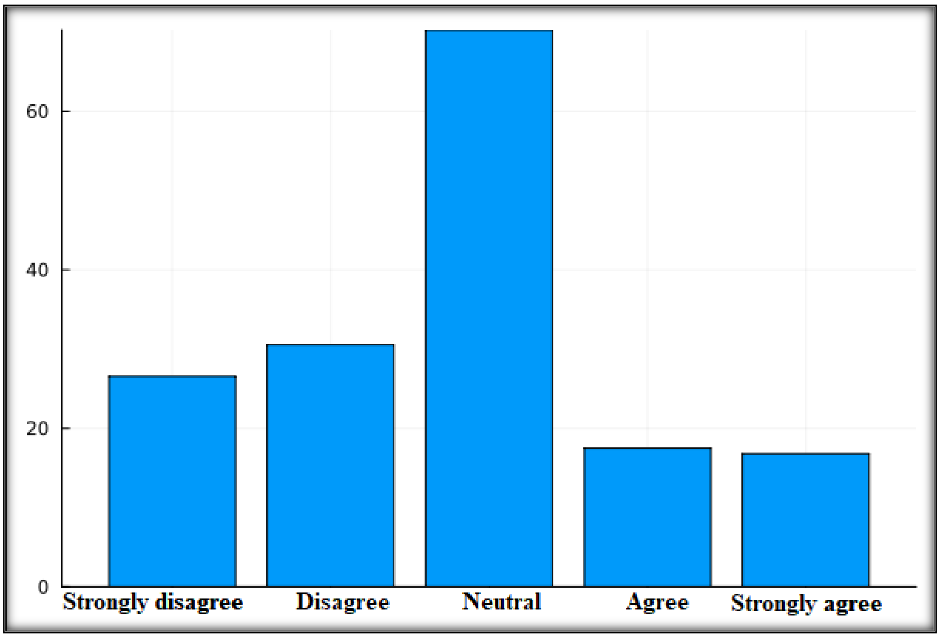

Figure 4. Bar chart on the degree of agreement-disagreement regarding "Appropriate financial management" in percentage.

The graphs in Figures 3 and 4 do not add up to 100% of the respondents. This is because the percentage of each of the opinions is calculated in terms of what each respondent thinks, who may have contradictory opinions when $T_A(x_i) + I_{T_A}(x_i) + I_A(x_i) + I_{F_A}(x_i) + F_A(x_i) > 1$.

Specifically, the degree of "Strongly agree" was calculated by $\sum_{i=1}^{141} T_A(x_i)$, the degree of "Agree" by $\sum_{i=1}^{141} I_{T_A}(x_i)$, the degree of "Neutral" by $\sum_{i=1}^{141} I_A(x_i)$, the degree of "Disagree" by $\sum_{i=1}^{141} I_{F_A}(x_i)$, and the degree of "Strongly disagree" by $\sum_{i=1}^{141} F_A(x_i)$.

Each of these values was divided by 141 and multiplied by 100 and this is how the percentages shown in both figures were obtained.

We have gotten the distance $d_2(C, M) = 0.433504$, and therefore the degree of similarity is equal to $S^1(C, M) = \frac{1}{1+0.433504} = 0.69759$.

This is interpreted as there is a degree of similarity over the average. Thus, there is a relationship between both measured variables.

Conclusion

Livestock activity has great cultural, nutritional, economic, and social importance in the rural populations of all or almost all countries. It is a source of food in terms of meat and milk, it is also a source of employment, and it maintains a traditional trade. That is why in modern times, with such high population growth, it is essential to correctly measure and manage the economic variables that are part of the production of beef and milk. Two of them are Activity cost and Financial management. In this work, we set out to study the behavior of these two variables in the town of Coto-Coto, Chilca, Peru, surveying 141 ranchers who participate in the local livestock fair. We are determined to have the greatest possible accuracy with the objective of obtaining the result that most closely resembles reality. We also accept that opinions have biases that are based on vagueness, uncertainty, and indeterminacy. The tool chosen was an Indeterminate Likert Scale that satisfies all these requirements. Additionally, we compare the individual results of each rancher's opinion on each of the variables using a measure of neutrosophic similarity, in this case, adapted to the TRINS. The results show a tendency towards neutrality regarding whether or not there is adequate behavior in both variables. Moreover, "Activity cost" shows a neutral behavior toward the positive, and "Financial management" has a neutral behavior toward the negative. The similarity between both was approximately 0.7, which is interpreted as that there is a positive relationship between both variables, therefore the improvement of one of them will imply the improvement in the other. It is recommended as a strategy to improve these variables, one and the other to produce better conjoint results. These are only previous results; in a future work we revisit this problem with more detail.

References

[1]. Tiepermann, R., and Porporato, M. (2021). Costos Basados en las Actividades (ABC): aplicación de una herramienta para la gestión estratégica en empresas de ser vicios (In Spanish). Cuadernos Latinoamericanos de Administración, 17, 1-39.

[2]. Párraga, S., Pinargote, N., García, C., and Zamora, J. (2022). Indicadores de gestión financiera en pequeñas y medianas empresas en Iberoamérica: una revisión sistemática (In Spanish). Revista Dilemas Contemporáneos: Educación, Política y Valores, 8, 1-24.

[3]. Likert, R. (1932) A technique for the measurement of attitudes. Archives of Psychology, 22, 5–55.

[4]. Kandasamy, I., Vasantha-Kandasamy, W. B., Obbineni, J. M., and Smarandache, F. (2020), Indeterminate Likert scale: feedback based on neutrosophy, its distance measures and clustering algorithm. Soft Computing, 24, 7459-7468.

[5]. South, L., Saffo, D., Vitek, O., Dunne, C., and Borkin, M. A. (2022) Effective use of Likert scales in visualization evaluations: a systematic review. Computer Graphics Forum, 41, 43-55.

[6]. Kandasamy, I., and Smarandache, F. (2016) Triple refined indeterminate neutrosophic sets for personality classification. Paper presented at the 2016 IEEE Symposium Series on Computational Intelligence (SSCI).

[7]. Smarandache, F. (2013). n-valued refined neutrosophic logic and its applications in physics: Infinite Study.

[8]. Sayyadi-Tooranloo, H., Mahmood-Zanjirchi, S., and Tavangar, M. (2020) ELECTRE Approach for Multi-attribute Decision-making in Refined Neutrosophic Environment. Neutrosophic Sets and Systems, 31, 101-119.

[9]. Agboola, A. A. A. (2015) On Refined Neutrosophic Algebraic Structures. Neutrosophic Sets and Systems, 10, 99-101.

[10]. Chatterjee, R., Majumdar, P. and Samanta, S. K. (2019). Similarity Measures in Neutrosophic Sets-I in Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets, Switzerland, Springer Nature, pp. 249-294.

[11]. Chatterjee, R., Majumdar, P. and Samanta, S. K. (2019). Similarity Measures in Neutrosophic Sets-II in Fuzzy Multi-criteria Decision-Making Using Neutrosophic Sets, Switzerland, Springer Nature, pp. 295-326.

[12]. Leonor, M. M., Easud, G. S., and Fernando, P. P. (2022) Indeterminate Likert Scale in Social Sciences Research. International Journal of Neutrosophic Science, 19, 289-297.

[13]. Guo, Y., Sengür, A., and Tian, J.W. (2016) A novel breast ultrasound image segmentation algorithm based on neutrosophic similarity score and level set. Computer Methods and Programs in Biomedicine, 123, 43-53.

[14]. Ulucay, V., Deli, I., and Sahin, M. (2018) Similarity measures of bipolar neutrosophic sets and their application to multiple criteria decision making, Neural Computing and Applications, 29, 739-748.

[15]. Chai, J.S.,·Selvachandran, G.,·Smarandache, F., Gerogiannis, V.C., Son, L.H.,·Bui, Q.T., and Vo, B. (2021) New similarity measures for single-valued neutrosophic sets with applications in pattern recognition and medical diagnosis problems. Complex & Intelligent Systems, 7, 703–723.

[16]. W. B., V., Kandasamy, I., Smarandache, F., Devvrat V., and Ghildiyal, S. (2020) Study of Imaginative Play in Children Using Single-Valued Refined Neutrosophic Sets. Symmetry, 12, 402.

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An Evidence-Based Approach to Set Theory Paradoxism: From Set Boundary to Mixed-Fermion-Boson Condensate Hypothesis

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Abstract: Physics thrives on precision, but paradoxes in set theory reveal limitations in our understanding of well-defined boundaries. Neutrosophic logic, challenging the *excluded middle* principle, introduces the concept of "betweenness" and partial belonging. This article explores among other things several possible avenues to resolve set theory paradoxism, including potential applications of neutrosophic logic in cosmology and particles, from set boundary, to the hypothetical "cosmosphere" boundary, to mixed fermion-boson condensate hypothesis. Embracing indeterminacy and fuzzy boundaries paves the way for a more holistic understanding of the universe's complexity.

Keywords: set theory paradoxism; boundaries of set; Neutrosophic Logic; betweenness; partial belonging; Lakoff & Nunez

1. Introduction

The concept of infinity has captivated mathematicians and philosophers for centuries, leading to groundbreaking discoveries and perplexing paradoxes. One such paradox stands tall: Russell's Paradox, a logical contradiction within set theory that threatened the very foundations of mathematics.[1] But what if the solution lay not in more complex axioms, but in a shift in our approach, moving from abstract symbol manipulation to an "*evidence-based*" framework grounded in human cognition?

Traditionally, attempts to resolve set theory paradoxism focused on constructing intricate axiom systems, like Zermelo-Fraenkel with Choice (ZFC). While successful in formalizing mathematics, these systems often feel removed from our intuitive understanding of infinity. Enter George Lakoff and Rafael Núñez, pioneers in cognitive science who propose a new perspective. They argue that mathematics, including set theory, is not an abstract, disembodied language, but rather a product of our embodied experience and conceptual knowledge. So, how can this cognitive lens help us tackle Russell's Paradox? Let's revisit the crux of the paradox: it arises when we consider the set of all sets that do not contain themselves. Does this set contain itself? If it does, it violates its own defining property. If it doesn't, then it contains all sets that don't contain themselves, including itself, leading to a contradiction.

From a cognitive perspective, the issue resides in our attempt to apply a single, uniform definition of "set" to all possible collections. In reality, our brains categorize and reason about different types of collections differently. Lakoff and Núñez propose that instead of a single "set" concept, we consider diverse conceptual categories like collections of physical objects, abstract ideas, or potential actions. Each category comes with its own inherent constraints and logic, shaping how we reason about its members.

Applying this framework, we might recognize that the problematic "set of all sets that do not contain themselves" belongs to a category that is fundamentally self-referential and unbounded. Such a category may not be amenable to the same logical rules as collections of physical objects or finite sets. Recognizing this cognitive limitation, we can avoid the paradoxism by simply excluding such self-referential categories from our formal set theory, focusing instead on well-defined, grounded collections.

This "*evidence-based*" approach does not negate the value of formal systems like ZFC. Instead, it complements them by acknowledging the cognitive underpinnings of mathematics and emphasizing the importance of aligning formal structures with our intuitive understanding of the world. This can lead to a more "healthy" and accessible mathematics, less prone to paradoxes and closer to how humans naturally reason about quantity and infinity.[3]

Furthermore, this shift can open doors to exploring alternative set theories that better reflect different cognitive perspectives. Imagine set theories inspired by spatial reasoning, probability judgments, or even social interactions. Such explorations could not only enrich our understanding of infinity but also provide valuable insights into the cognitive diversity of mathematical thinking.

While Lakoff and Núñez's approach is in its early stages, it offers a promising avenue for addressing long-standing mathematical challenges. By embracing the evidence of our embodied cognition, we can move beyond abstract symbol manipulation and develop a more natural, "*evidence-based*" approach to mathematics, paving the way for a more inclusive and vibrant understanding of infinity.

2. Materials and Methods

The method used here is analysis and analogy with well-defined problems such as cell biology, diffusion and osmosis etc. toward rethinking of set theory paradoxism [6], especially in light of evidence-based physics and evidence-based mathematics. Recent literature which are relevant to the theme of this article have been cited.

3. Results

3.1. *Approach #1:* ***Cell model as boundary to any set, diffusion-osmosis interpretation to set paradox***

The shadow of Russell's Paradox(-ism) looms large over set theory, its logical contradiction threatening the very foundations of mathematics. Traditional solutions focus on complex axiom systems, but what if the answer resides not in abstract symbols, but in the concrete world of living cells? This article proposes a novel "cell model" inspired by Lakoff and Núñez's cognitive science approach, utilizing the concepts of diffusion and osmosis to shed light on the paradox.

Imagine a set as a living cell, bounded by a semipermeable membrane. This membrane regulates what enters and exits the set, just as a cell membrane controls the flow of molecules. Elements within the set are like nutrients inside the cell, while the surrounding "soup" represents potential members waiting to join.[1][2]

Now consider Russell's problematic set – the set of all sets that do not contain themselves. According to the cell model, this set's defining characteristic acts as a selective membrane. It allows sets that don't contain themselves to "diffuse" in (like nutrients), but it should also allow itself in, as it doesn't contain itself, leading to contradiction.

Here's where the concept of **osmosis** comes in. Osmosis describes the spontaneous movement of molecules across a semipermeable membrane to equalize concentrations. Applied to our cell model, osmosis represents the inherent tendency of sets to be consistent and avoid paradoxes. When the problematic set tries to "diffuse" in, due to its self-referential nature, it triggers an "*osmotic pressure*" within the set. This pressure, analogous to the corrective force in osmosis, prevents the paradox by pushing the problematic set back out. Essentially, the set's defining characteristic itself acts as a barrier, preventing its own inclusion and maintaining consistency.

This model aligns with cognitive principles. Our minds naturally categorize and reason about collections differently, understanding physical objects differently from abstract ideas. The cell model reflects this, treating different types of sets as distinct "cells" with unique membranes and osmotic pressures.

This approach offers several advantages. It provides a more intuitive understanding of set boundaries and avoids complex formal machinery. It emphasizes the inherent limitations of self-referential sets, aligning with our cognitive constraints. Furthermore, it suggests alternative ways to think about set theory, inspired by biological processes like osmosis and cellular dynamics. While the cell model is a first step, further development is needed. Refining the analogy, exploring implications for different set types, and formalizing the osmotic pressure concept are crucial next steps.

In conclusion, the cell model and its osmosis interpretation offer a promising evidence-based approach to the set theory paradoxism. By grounding abstract concepts in the familiar world of living cells, we gain a new perspective, highlighting the importance of cognitive limitations and inherent dynamics within sets. This approach opens doors to a more intuitive and inclusive understanding of infinity, enriching both mathematics and our understanding of human thought.

3.2. Approach #2: ***Exploring Cognitive Constraints: A Categorical Approach to Set Theory Paradox***

This section explores how our cognitive limitations influence our understanding of infinity and contribute to paradoxes like Russell's paradox(-ism); cf. [3]. Drawing on the work of Lakoff and Núñez, you could:

- Analyze the cognitive categories we use to reason about collections, highlighting differences between physical objects, abstract ideas, and potentially unbounded sets.

- Explore how these categories shape our intuition and logic, leading to potential contradictions when applied to specific set definitions like "the set of all sets not containing themselves."

- Propose alternative set theories that respect these cognitive constraints, potentially by limiting self-referential definitions or introducing category-specific rules.

- Connect this approach to existing research in cognitive science and philosophy of mathematics, showcasing its evidence-based foundation.

This approach offers several advantages:

- Grounds the solution in evidence: It builds upon established research in cognitive science and avoids relying on unproven concepts like morphic fields.

- Addresses the root cause: It focuses on how our cognitive limitations contribute to the paradoxism, offering a deeper understanding of the issue.

- Connects to broader discussions: It aligns with ongoing research on embodied cognition and its impact on mathematics.

- Offers concrete solutions: It suggests alternative set theories grounded in cognitive constraints, contributing to the overall development of set theory.

It shall be kept in mind that, the goal of an *evidence-based approach* is to provide solutions supported by robust evidence and aligned with established scientific principles. By exploring well-researched areas like cognitive science and applying their insights to mathematical problems, we can contribute to a more robust and inclusive understanding of infinity and mathematics as a whole.[3]

3.3. *Approach #3: **Beyond the Excluded Middle: Exploring Neutrosophic Frontiers in Fermion-Boson systems***

Traditional physics excels in clear-cut definitions, but what if nature itself defies rigidity? This article explores the potential of neutrosophic logic, which goes beyond the "in" or "out" paradigm, to describe fuzzy boundaries and indeterminate states in the cosmos. We delve into intriguing possibilities like a partially defined border for the universe and hybrid particles exhibiting characteristics of both fermions and bosons. Accepting such "*betweenness*" challenges established paradigms and offers exciting avenues for future discoveries, leading us closer to a more nuanced picture of reality.

It is known, that aidst the elegance of its equations, paradoxes lurk, whispering of hidden complexities. One such riddle is the question of set theory paradoxism, where seemingly logical axioms lead to contradictory results. Traditionally, these paradoxes are resolved by upholding the "*excluded middle*" principle - every element either belongs to a set or doesn't. However, what if reality itself defies such crisp classifications?

This is where the intriguing idea of **Neutrosophic logic** emerges. It dares to challenge the rigidity of the excluded middle, introducing the notion of **partial belongingness** and **indeterminacy**. Instead of a traditional "in" or "out," elements can reside in a fuzzy "betweenness," exhibiting characteristics of both sets simultaneously. This opens a fascinating gateway to explore realms where traditional physics might reach its limits.

One intriguing application lies in the vast unknown beyond our familiar solar system. We know of the *heliosphere*, a thick "wall" of charged particles marking the boundary of our Sun's influence, as observed for instance by Voyager. Ref. [4] Could there be an analogous "***cosmosphere*,**" a boundary to the observable Universe? If future observations reveal such a barrier, Neutrosophic logic could elegantly describe its nature. Objects within this boundary might exhibit degrees of both "inside-ness" and "outside-ness," existing in a state of partial belongingness.

Another promising avenue resides in the subatomic realm. The fundamental distinction between fermions and bosons, particles with differing statistics, forms a cornerstone of quantum mechanics. But what if, as hinted by **mixed fermion-boson statistics**, there also exist particles exhibiting **both fermionic and bosonic characteristics**? The prospect of observing a condensate corresponding to such a hybrid entity, transcending the traditional Bose-Einstein condensate, would challenge and even transform our very understanding of particle classification. That particular condensate may be called *mixed fermion-boson condensate* **(MFBC).** Ref. [5]

Exploring these Neutrosophic frontiers necessitates that we must embrace the possibility of **fuzzy boundaries, indeterminate states, and partial memberships**. This doesn't negate the value of traditional physics, but rather acknowledges its limitations in the face of the universe's inherent complexity.

The road ahead may be riddled with conceptual and experimental challenges. Yet, venturing into the uncharted territory of Neutrosophic physics holds immense potential. It could reshape our understanding of the cosmos, from its grandest boundaries to its quantum enigmas, leading us closer to a truly holistic picture of reality. So, let us cast aside the shackles of the excluded middle and embrace the vibrant "betweenness" whispered by Neutrosophic logic. The Universe, in all its enigmatic splendor, awaits.

4. Applications and Concluding Remark

These authors have outlined several possible approaches to solve the set theory paradoxism, for instance by defining certain real boundary system to a given set, let say cell system. Such an *evidence-based physics* approach will allow us to figure out what can happen actually when certain entity goes in or goes out of any given cell through the boundary layer. Alternatively, we can figure out how Voyager space vehicle which goes through the boundary or thick wall of in the outer side of the Solar System, or it is often termed heliosphere, faces the edge or boundary of outer Solar System.[4] Similarly, we can hypothesize there is good likelihood that there is certain thick boundary in the foremost edge of the Universe, which may be termed *Cosmosphere*. Therefore in such a way, the meaning of set which contains all sets, i.e. the outer layer of the Universe that contains everything else inside the Universe can be figured out in astrophysics term.

We hope that more discussions of evidence-based physics approach to set theory paradoxism can be expected.

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Shortlisted References

1. Agustín Rayo with contributions from Damien Rochford. *On the brink of a paradox.* Cambridge: The MIT Press, 2019.
2. Ka-Sing Lau and Sze-Man Ngai. Dimensions of the Boundaries of Self-Similar Sets. *Experimental Mathematics*, Vol. 12 (2003), No. 1
3. Douglas C. Gill. The Mechanism of Paradox in the Structures of Logic, Mathematics, and Physics. *Open Journal of Philosophy*, 13, 155-170. https://doi.org/10.4236/ojpp.2023.132010
4. Konstantinos Dialynas et al. The Structure of the Global Heliosphere as Seen by In-Situ Ions from the Voyagers and Remotely Sensed ENAs from Cassini. *Space Science Reviews* (2022) 218:21 https://doi.org/10.1007/s11214-022-00889-0
5. C. J. Bolech et al 2023. Condensate States of Atomic Bose-Fermi Gas Mixtures. *J. Phys.: Conf. Ser.* 2494 012015
6. C. Le (editor). pARadOXisM – the Last Literary, Artistic, Philosophic and Scientific Vanguard of the Second Millennium, https://fs.unm.edu/a/paradoxism-en.htm (1980).

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MADM Model Using Einstein Aggregation Operators of Sine Single-Valued Neutrosophic Values and Its Application

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Abstract: In various fuzzy multiple attribute decision making (MADM) applications, different information descriptions and aggregation operators (AOs) play a crucial role. However, both the Einstein sum and product can include their typical algebraic operation features, but they lack the characteristics of periodicity operations. To fill the research gap of Einstein AOs for single-valued neutrosophic values (SvNVs), this article aims to propose Einstein AOs of sine SvNVs and their MADM model as their new extension. In this study, we first define a new sine SvNV, which integrates sine functions into the membership functions of indeterminacy, falsehood, and truth, and the Einstein operation laws of sine SvNVs. Then, we present the sine SvNV Einstein weighted average and geometric AOs and their properties. Furthermore, we develop a MADM model based on the proposed Einstein AOs in a SvNV circumstance. Lastly, we apply the developed MADM model to a site selection example of a hydrogen power plant as the verification of its application in a SvNV circumstance. The decision results reveal the rationality and validity of the developed model with respect to the comparison of the related models.

Keywords: decision making problem; sine single-valued neutrosophic value; einstein operation law; sine single-valued neutrosophic value Einstein aggregation operator

1. Introduction

Recently, various fuzzy multiple attribute/criteria decision making (MADM/MCDM) theories and approaches have become research hotspots in uncertain decision applications. In MADM applications, various fuzzy information descriptions and operations/aggregations imply their importance and necessity. Fuzzy sets (FSs) [1] contain only membership degrees, but lack non-membership degrees. Then, intuitionistic or interval-valued FSs (IFSs/IvFSs) [2, 3] can contain both membership and non-membership degrees and the dependent relationship of both, but cannot reflect the independent relationship of indeterminate, false, and true membership degrees in inconsistent and uncertain situations. As a general framework of different FSs, a neutrosophic set (NS) [4] can reflect them. Regarding the subsets of NS, some scholars presented single-valued or interval-valued or simplified neutrosophic values (SvNVs/IvNVs/SNVs) [5-7] and their operation laws and aggregation operators (AOs) to effectively meet scientific and engineering applications. Zhang et al. proposed the improved AOs of IvNVs for MADM [8]. Then, Yang and Li [9] introduced the power AOs of SvNVs for MADM. Liu et al. [10] presented the power Muirhead mean AOs of SvNVs for group decision making (GDM). Liu [11] also introduced the Archimedean AOs of SvNVs for MADM. Garg [12] proposed the Frank norm AOs of SvNVs for MADM. Deli and Subas [13] introduced a

SvNV sorting method for MADM. Liu and Liu [14] put forward a generalized weighted power averaging operator of SvNVs for GDM. Karaaslan and Hayat [15] presented some operations of interval-valued neutrosophic matrices and applied them to GDM. Garg [16] used the multiplicative preference relation of SvNVs for MADM. Giri and Roy [17] introduced a neutrosophic programming approach to solve the transportation problem of green four-dimensional fixed charges. Therefore, the SNVs (SvNVs and/or IvNVs) have also revealed their merits in inconsistent and uncertain MADM applications [18]. Consequently, many scholars have further developed SNV (SvNV and/or IvNV) AOs, such as ordinary weighted arithmetic and geometric AOs, Einstein AOs, generalized AOs [19], Bonferroni mean AOs [20], Hamacher AOs [21], exponential AOs [22], subtraction and division AOs [23], Frank AOs [24], prioritized interactive AOs [25], and fairly AOs [26] for SNVs (SvNVs and/or IvNVs).

Recently, some scholars [27, 28] have proposed logarithmic SvNV operation laws and logarithmic SvNV Einstein AOs for GDM in view of t-conorm and t-norm. However, they reflect some limitations, for example, logx(y) cannot be defined when x = 1 or y = 0. Due to the periodicity feature of the sine function, it implies some merit that satisfies the multiple periodicity MADM needs in real problems. Therefore, the operation laws and AOs of sine SvNVs (S-SvNVs) [29, 30] have been introduced in MADM applications. Then, there are the defects of some membership functions that belong to the range of [0, 0.46) instead of the whole range of [0, 1] in S-SvNV [29, 30]. To overcome this deficiency, AOs of tangent SvNVs (T-SvNVs), where the three membership functions belong to the whole range of [0, 1], were presented for MADM [31]. Based on cosine, sine, arccosine, and arcsine operations, Ye et al. [32] first proposed the single-valued neutrosophic credibility value trigonometric AOs for MADM. However, the Einstein sum and product can include their typical algebraic operation merits [19], but lack periodicity operation features. Furthermore, no Einstein operation laws and AOs of S-SvNVs are presented in the existing literature. Therefore, it is necessary to develop them for MADM issues with S-SvNV information to fill this gap. Motivated by the new ideas, this article will propose the Einstein AOs of S-SvNVs and their MADM model as a new extension to address the defects and research gaps in the existing S-SvNV operation laws and AOs [29, 30]. In this study, the objectives of this paper are to: (1) define a suitable S-SvNV including three membership degrees belonging to the whole range of [0, 1] and Einstein operation laws (EOLs) of S-SvNVs, (2) establish the S-SvNV Einstein weighted average (S-SvNVEWA) and geometric (S-SvNVEWG) AOs, (3) develop a MADM model using the S-SvNVEWA and S-SvNVEWG AOs, and (4) apply the proposed MADM model to a site selection example of a hydrogen power plant (HPP) in a SvNV circumstance. However, the comparison results with the existing related models indicate the rationality and validity of the proposed model.

The remainder of this article is composed of these parts. Section 2 simply reviews the preliminaries of single-valued NSs (SvNSs), including the operation laws and AOs of SvNVs and S-SvNVs. In view of the integration of sine functions into indeterminate, false, and true membership functions, Section 3 defines a new S-SvNV and the EOLs of S-SvNVs, and then presents the S-SvNVEWA and S-SvNVEWG AOs and their properties. In Section 4, a MADM model is developed in terms of the S-SvNVEWA and S-SvNVEWG AOs. Section 5 applies the developed MADM model to a site selection example of HPP in a SvNV circumstance. The comparative results of the existing related models reveal the validity of the developed model. Conclusions and future research are summarized in Section 6.

2. Preliminaries of SvNSs

2.1. Operation laws and AOs of SvNVs

Set X_C as a fixed universe set. Then, the SvNS Φ_N in X_C is represented as $\Phi_N = \{<x_c, \varphi_{Nt}(x_c), \varphi_{Nu}(x_c), \varphi_{Nv}(x_c)> | x_c \in X_C\}$ [6], where $\varphi_{Nv}(x_c)$, $\varphi_{Nu}(x_c)$, $\varphi_{Nt}(x_c) \in [0, 1]$ are the membership functions of falsehood, indeterminacy, and truth subject to $0 \le \varphi_{Nt}(x_c) + \varphi_{Nu}(x_c) + \varphi_{Nv}(x_c) \le 3$ for any $x_c \in X_C$. Then, $<x_c, \varphi_{Nt}(x_c), \varphi_{Nu}(x_c), \varphi_{Nv}(x_c)>$ in Φ_N is represented as the SvNV $\varphi_N = <\varphi_{Nt}, \varphi_{Nu}, \varphi_{Nv}>$ for simplicity.

Set two SvNVs as $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ ($s = 1, 2$) with $\mu_w > 0$. Then, their operation relationships are presented below [7, 19]:

(1) $\varphi_{N1} \supseteq \varphi_{N2} \Leftrightarrow \varphi_{Nt(1)} \geq \varphi_{Nt(2)}$, $\varphi_{Nu(1)} \leq \varphi_{Nu(2)}$, and $\varphi_{Nv(1)} \leq \varphi_{Nv(2)}$;

(2) $\varphi_{N1} = \varphi_{N2} \Leftrightarrow \varphi_{N1} \subseteq \varphi_{N2}$ and $\varphi_{N1} \supseteq \varphi_{N2}$;

(3) $\varphi_{N1} \cup \varphi_{N2} = \left\langle \varphi_{Nt(1)} \vee \varphi_{Nt(2)}, \varphi_{Nu(1)} \wedge \varphi_{Nu(2)}, \varphi_{Nv(1)} \wedge \varphi_{Nv(2)} \right\rangle$;

(4) $\varphi_{N1} \cap \varphi_{N2} = \left\langle \varphi_{Nt(1)} \wedge \varphi_{Nt(2)}, \varphi_{Nu(1)} \vee \varphi_{Nu(2)}, \varphi_{Nv(1)} \vee \varphi_{Nv(2)} \right\rangle$;

(5) $(\varphi_{N1})^c = \left\langle \varphi_{Nv(1)}, 1-\varphi_{Nu(1)}, \varphi_{Nt(1)} \right\rangle$ (Complement of φ_{N1});

(6) $\varphi_{N1} \oplus \varphi_{N2} = \left\langle \varphi_{Nt(1)} + \varphi_{Nt(2)} - \varphi_{Nt(1)}\varphi_{Nt(2)}, \varphi_{Nu(1)}\varphi_{Nu(2)}, \varphi_{Nv(1)}\varphi_{Nv(2)} \right\rangle$;

(7) $\varphi_{N1} \otimes \varphi_{N2} = \left\langle \varphi_{Nt(1)}\varphi_{Nt(2)}, \varphi_{Nu(1)} + \varphi_{Nu(2)} - \varphi_{Nu(1)}\varphi_{Nu(2)}, \varphi_{Nv(1)} + \varphi_{Nv(2)} - \varphi_{Nv(1)}\varphi_{Nv(2)} \right\rangle$;

(8) $\mu_w \cdot \varphi_{N1} = \left\langle 1-(1-\varphi_{Nt(1)})^{\mu_w}, \varphi_{Nu(1)}^{\mu_w}, \varphi_{Nv(1)}^{\mu_w} \right\rangle$;

(9) $\varphi_{N1}^{\mu_w} = \left\langle \varphi_{Nt(1)}^{\mu_w}, 1-(1-\varphi_{Nu(1)})^{\mu_w}, 1-(1-\varphi_{Nv(1)})^{\mu_w} \right\rangle$.

Set $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ ($s = 1, 2, \ldots, q$) as a collection of SvNVs with their weight vector $\boldsymbol{\mu}_W = (\mu_{w1}, \mu_{w2}, \ldots, \mu_{wq})$ subject to $0 \leq \mu_{ws} \leq 1$ and $\sum_{s=1}^{q} \mu_{ws} = 1$. Then, SvNVWA and SvNVWG are denoted as the SvNV weighted average and geometric AOs and expressed by the two formulae [19]:

$$SvNVWA(\varphi_{N1}, \varphi_{N2}, \ldots, \varphi_{Nq}) = \sum_{s=1}^{q} \mu_{ws}\varphi_{Ns} = \left\langle 1-\prod_{s=1}^{q}\left(1-\varphi_{Nt(s)}\right)^{\mu_{ws}}, \prod_{s=1}^{q}\left(\varphi_{Nu(s)}\right)^{\mu_{ws}}, \prod_{s=1}^{q}\left(\varphi_{Nv(s)}\right)^{\mu_{ws}} \right\rangle, \quad (1)$$

$$SvNVWG(\varphi_{N1}, \varphi_{N2}, \ldots, \varphi_{Nq}) = \prod_{s=1}^{q}\left(\varphi_{Ns}\right)^{\mu_{ws}} = \left\langle \prod_{s=1}^{q}\left(\varphi_{Nt(s)}\right)^{\mu_{ws}}, 1-\prod_{s=1}^{q}\left(1-\varphi_{Nu(s)}\right)^{\mu_{ws}}, 1-\prod_{s=1}^{q}\left(1-\varphi_{Nv(s)}\right)^{\mu_{ws}} \right\rangle. \quad (2)$$

Set two SvNVs as $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ ($s = 1, 2$) with $\mu_w > 0$. Then, their EOLs are presented as follows [19]:

(1) $\varphi_{N1} \oplus_E \varphi_{N2} = \left\langle \dfrac{\varphi_{Nt(1)} + \varphi_{Nt(2)}}{1+\varphi_{Nt(1)}\varphi_{Nt(2)}}, \dfrac{\varphi_{Nu(1)}\varphi_{Nu(2)}}{1+(1-\varphi_{Nu(1)})(1-\varphi_{Nu(2)})}, \dfrac{\varphi_{Nv(1)}\varphi_{Nv(2)}}{1+(1-\varphi_{Nv(1)})(1-\varphi_{Nv(2)})} \right\rangle$;

(2) $\varphi_{N1} \otimes_E \varphi_{N2} = \left\langle \dfrac{\varphi_{Nt(1)}\varphi_{Nt(2)}}{1+(1-\varphi_{Nt(1)})(1-\varphi_{Nt(2)})}, \dfrac{\varphi_{Nu(1)} + \varphi_{Nu(2)}}{1+\varphi_{Nu(1)}\varphi_{Nu(2)}}, \dfrac{\varphi_{Nv(1)} + \varphi_{Nv(2)}}{1+\varphi_{Nv(1)}\varphi_{Nv(2)}} \right\rangle$;

(3) $\mu_w \cdot \varphi_{N1} = \left\langle \dfrac{(1+\varphi_{Nt(1)})^{\mu_w} - (1-\varphi_{Nt(1)})^{\mu_w}}{(1+\varphi_{Nt(1)})^{\mu_w} + (1-\varphi_{Nt(1)})^{\mu_w}}, \dfrac{2\varphi_{Nu(1)}^{\mu_w}}{(2-\varphi_{Nu(1)})^{\mu_w} + \varphi_{Nu(1)}^{\mu_w}}, \dfrac{2\varphi_{Nv(1)}^{\mu_w}}{(2-\varphi_{Nv(1)})^{\mu_w} + \varphi_{Nv(1)}^{\mu_w}} \right\rangle$;

(4) $\varphi_{N1}^{\mu_w} = \left\langle \dfrac{2\varphi_{Nt(1)}^{\mu_w}}{(2-\varphi_{Nt(1)})^{\mu_w} + \varphi_{Nt(1)}^{\mu_w}}, \dfrac{(1+\varphi_{Nu(1)})^{\mu_w} - (1-\varphi_{Nu(1)})^{\mu_w}}{(1+\varphi_{Nu(1)})^{\mu_w} + (1-\varphi_{Nu(1)})^{\mu_w}}, \dfrac{(1+\varphi_{Nv(1)})^{\mu_w} - (1-\varphi_{Nv(1)})^{\mu_w}}{(1+\varphi_{Nv(1)})^{\mu_w} + (1-\varphi_{Nv(1)})^{\mu_w}} \right\rangle$.

For a collection of SvNVs $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ ($s = 1, 2, \ldots, q$) with their weight vector $\boldsymbol{\mu}_W = (\mu_{w1}, \mu_{w2}, \ldots, \mu_{wq})$ subject to $0 \leq \mu_{ws} \leq 1$ and $\sum_{s=1}^{q} \mu_{ws} = 1$, SvNVEWA and SvNVEWG are denoted as the SvNV Einstein weighted average and geometric AOs and introduced by the two formulae [19]:

$$SvNVEWA(\varphi_{N1}, \varphi_{N2}, \ldots, \varphi_{Nq}) = \sum_{s=1}^{q}{}_E\, \mu_{ws}\varphi_{Ns} = \begin{pmatrix} \dfrac{\prod_{s=1}^{q}\left(1+\varphi_{Nt(s)}\right)^{\mu_{ws}} - \prod_{s=1}^{q}\left(1-\varphi_{Nt(s)}\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\varphi_{Nt(s)}\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\varphi_{Nt(s)}\right)^{\mu_{ws}}}, \\ \dfrac{2\prod_{s=1}^{q}\left(\varphi_{Nu(s)}\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(2-\varphi_{Nu(s)}\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(\varphi_{Nu(s)}\right)^{\mu_{ws}}}, \dfrac{2\prod_{s=1}^{q}\left(\varphi_{Nv(s)}\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(2-\varphi_{Nv(s)}\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(\varphi_{Nv(s)}\right)^{\mu_{ws}}} \end{pmatrix}, \quad (3)$$

$$SvNVEWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq})=\prod_{s=1}^{q}{}_{E}\left(\varphi_{Ns}\right)^{\mu_{ws}}=\left\langle\begin{array}{c}\dfrac{2\prod_{s=1}^{q}(\varphi_{Nt(s)})^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\varphi_{Nt(s)})^{\mu_{ws}}+\prod_{s=1}^{q}(\varphi_{Nt(s)})^{\mu_{ws}}},\\ \dfrac{\prod_{s=1}^{q}(1+\varphi_{Nu(s)})^{\mu_{ws}}-\prod_{s=1}^{q}(1-\varphi_{Nu(s)})^{\mu_{ws}}}{\prod_{s=1}^{q}(1+\varphi_{Nu(s)})^{\mu_{ws}}+\prod_{s=1}^{q}(1-\varphi_{Nu(s)})^{\mu_{ws}}},\\ \dfrac{\prod_{s=1}^{q}(1+\varphi_{Nv(s)})^{\mu_{ws}}-\prod_{s=1}^{q}(1-\varphi_{Nv(s)})^{\mu_{ws}}}{\prod_{s=1}^{q}(1+\varphi_{Nv(s)})^{\mu_{ws}}+\prod_{s=1}^{q}(1-\varphi_{Nv(s)})^{\mu_{ws}}}\end{array}\right\rangle. \quad (4)$$

To sort SvNVs φ_{Ns} = <$\varphi_{Nt(s)}$, $\varphi_{Nu(s)}$, $\varphi_{Nv(s)}$> (s = 1, 2), the score and accuracy equations of SvNVs [19] are presented below:

$$U(\varphi_{Ns})=(2+\varphi_{Nt(s)}-\varphi_{Nu(s)}-\varphi_{Nv(s)})/3 \text{ for } U(\varphi_{Ns})\in[0,1], \quad (5)$$

$$V(\varphi_{Ns})=\varphi_{Nt(s)}-\varphi_{Nv(s)} \text{ for } V(\varphi_{Ns})\in[-1,1]. \quad (6)$$

In terms of the score and accuracy equations, a sorting order of two SvNVs is defined by the following rules:

(1) $\varphi_{N1} > \varphi_{N2}$ for $U(\varphi_{N1}) > U(\varphi_{N2})$;

(2) $\varphi_{N1} > \varphi_{N2}$ for $U(\varphi_{N1}) = U(\varphi_{N2})$ and $V(\varphi_{N1}) > V(\varphi_{N2})$;

(3) $\varphi_{N1} \cong \varphi_{N2}$ for $U(\varphi_{N1}) = U(\varphi_{N2})$ and $V(\varphi_{N1}) = V(\varphi_{N2})$.

2.2 Operation laws and AOs of S-SvNVs

Set SvNV as φ_N = <φ_{Nt}, φ_{Nu}, φ_{Nv}>. Then, S-SvNV is presented by $\sin(\varphi_N)=\left\langle\sin(0.5\varphi_{Nt}\pi),1-\sin(0.5\pi-\varphi_{Nu}),1-\sin(0.5\pi-\varphi_{Nv})\right\rangle$ [29, 30], where the membership degrees of the indeterminacy, falsehood, and truth are $1-\sin(0.5\pi-\varphi_{Nu})\in[0,0.46)$, $1-\sin(0.5\pi-\varphi_{Nv})\in[0,0.46)$, and $\sin(0.5\varphi_{Nt}\pi)\in[0,1]$, respectively.

Let $\sin(\varphi_{Ns})=\left\langle\sin(0.5\varphi_{Nt(s)}\pi),1-\sin(0.5\pi-\varphi_{Nu(s)}),1-\sin(0.5\pi-\varphi_{Nv(s)})\right\rangle$ for s = 1, 2 be two S-SvNVs with μ_w > 0. Then, their operation laws are introduced below [29, 30]:

(1) $$\sin(\varphi_{N1})\oplus\sin(\varphi_{N2})=\left\langle\begin{array}{l}1-(1-\sin(0.5\varphi_{Nt(1)}\pi))(1-\sin(0.5\varphi_{Nt(2)}\pi)),\\ (1-\sin(0.5\pi-\varphi_{Nu(1)}))(1-\sin(0.5\pi-\varphi_{Nu(2)})),\\ (1-\sin(0.5\pi-\varphi_{Nv(1)}))(1-\sin(0.5\pi-\varphi_{Nv(2)}))\end{array}\right\rangle;$$

(2) $$\sin(\varphi_{N1})\otimes\sin(\varphi_{N2})=\left\langle\begin{array}{l}\sin(0.5\varphi_{Nt(1)}\pi)\sin(0.5\varphi_{Nt(2)}\pi),\\ 1-\sin(0.5\pi-\varphi_{Nu(1)})\sin(0.5\pi-\varphi_{Nu(2)}),\\ 1-\sin(0.5\pi-\varphi_{Nv(1)})\sin(0.5\pi-\varphi_{Nv(2)})\end{array}\right\rangle;$$

(3) $$\mu_w\cdot\sin(\varphi_{N1})=\left\langle\begin{array}{l}1-(1-\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w},\\ (1-\sin(0.5\pi-\varphi_{Nu(1)}))^{\mu_w},\\ (1-\sin(0.5\pi-\varphi_{Nv(1)}))^{\mu_w}\end{array}\right\rangle;$$

(4) $$(\sin(\varphi_{N1}))^{\mu_w}=\left\langle(\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w},1-(\sin(0.5\pi-\varphi_{Nu(1)}))^{\mu_w},1-(\sin(0.5\pi-\varphi_{Nv(1)}))^{\mu_w}\right\rangle.$$

For a group of SvNVs $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ (s = 1, 2, …, q) with their weight vector $\mu_w = (\mu_{w1}, \mu_{w2}, \ldots, \mu_{wq})$ subject to $0 \le \mu_{ws} \le 1$ and $\sum_{s=1}^{q} \mu_{ws} = 1$, S-SvNVWA and S-SvNVWG are denoted as the S-SvNV weighted average and geometric AOs and introduced by the two equations [29, 30]:

$$S-SvNVWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) = \sum_{s=1}^{q} \mu_{ws} \sin(\varphi_{Ns}) = \left(1-\prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}, \prod_{s=1}^{q}\left(1-\sin\left(0.5\pi-\varphi_{Nu(s)}\right)\right)^{\mu_{ws}}, \prod_{s=1}^{q}\left(1-\sin\left(0.5\pi-\varphi_{Nv(s)}\right)\right)^{\mu_{ws}} \right), \quad (7)$$

$$S-SvNVWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) = \prod_{s=1}^{q}\left(\sin(\varphi_{Ns})\right)^{\mu_{ws}} = \left(\prod_{s=1}^{q}\left(\sin(0.5\varphi_{Nt(s)}\pi)\right)^{\mu_{ws}}, 1-\prod_{s=1}^{q}\left(\sin(0.5\pi-\varphi_{Nu(s)})\right)^{\mu_{ws}}, 1-\prod_{s=1}^{q}\left(\sin(0.5\pi-\varphi_{Nv(s)})\right)^{\mu_{ws}} \right). \quad (8)$$

2.3 Operation laws and AOs of T-SvNVs

Set SvNV as $\varphi_N = <\varphi_{Nt}, \varphi_{Nu}, \varphi_{Nv}>$. Then, T-SvNV is presented by $\tan(\varphi_N) = \langle \tan(0.25\varphi_{Nt}\pi), 1-\tan(0.25\pi(1-\varphi_{Nu})), 1-\tan(0.25\pi(1-\varphi_{Nv})) \rangle$ [31], where the membership degrees of the indeterminacy, falsehood, and truth are $1-\tan(0.25\pi(1-\varphi_{Nu})) \in [0,1]$, $1-\tan(0.25\pi(1-\varphi_{Nv})) \in [0,1]$, and $\tan(0.25\varphi_{Nt}\pi) \in [0,1]$, respectively.

Let $\tan(\varphi_{Ns}) = \langle \tan(0.25\varphi_{Nt(s)}\pi), 1-\tan(0.25\pi(1-\varphi_{Nu(s)})), 1-\tan(0.25\pi(1-\varphi_{Nv(s)})) \rangle$ for s = 1, 2 be two T-SvNVs with $\mu_w > 0$. Then, their operation laws are introduced below [31]:

(1) $$\tan(\varphi_{N1}) \oplus \tan(\varphi_{N2}) = \left\langle \begin{array}{l} 1-(1-\tan(0.25\varphi_{Nt(1)}\pi))(1-\tan(0.25\varphi_{Nt(2)}\pi)), \\ (1-\tan(0.25\pi(1-\varphi_{Nu(1)})))(1-\tan(0.25\pi(1-\varphi_{Nu(2)}))), \\ (1-\tan(0.25\pi(1-\varphi_{Nv(1)})))(1-\tan(0.25\pi(1-\varphi_{Nv(2)}))) \end{array} \right\rangle;$$

(2) $$\tan(\varphi_{N1}) \otimes \tan(\varphi_{N2}) = \left\langle \begin{array}{l} \tan(0.25\varphi_{Nt(1)}\pi)\tan(0.25\varphi_{Nt(2)}\pi), \\ 1-\tan(0.25\pi(1-\varphi_{Nu(1)}))\tan(0.25\pi(1-\varphi_{Nu(2)})), \\ 1-\tan(0.25\pi(1-\varphi_{Nv(1)}))\tan(0.25\pi(1-\varphi_{Nv(2)})) \end{array} \right\rangle;$$

(3) $$\mu_w \tan(\varphi_{N1}) = \left\langle \begin{array}{l} 1-(1-\tan(0.25\varphi_{Nt(1)}\pi))^{\mu_w}, \\ (1-\tan(0.25\pi(1-\varphi_{Nu(1)})))^{\mu_w}, \\ (1-\tan(0.25\pi(1-\varphi_{Nv(1)})))^{\mu_w} \end{array} \right\rangle;$$

(4) $$(\tan(\varphi_{N1}))^{\mu_w} = \left\langle \begin{array}{l} (\tan(0.25\varphi_{Nt(1)}\pi))^{\mu_w}, \\ 1-(\tan(0.25\pi(1-\varphi_{Nu(1)})))^{\mu_w}, \\ 1-(\tan(0.25\pi(1-\varphi_{Nv(1)})))^{\mu_w} \end{array} \right\rangle.$$

For a group of SvNVs $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ (s = 1, 2, …, q) with their weight vector $\mu_w = (\mu_{w1}, \mu_{w2}, \ldots, \mu_{wq})$ subject to $0 \le \mu_{ws} \le 1$ and $\sum_{s=1}^{q} \mu_{ws} = 1$, T-SvNVWA and T-SvNVWG are denoted as the T-SvNV weighted average and geometric AOs and introduced by the two equations [31]:

$$T-SvNVWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) = \sum_{s=1}^{q} \mu_{ws} \tan(\varphi_{Ns}) = \left(1-\prod_{s=1}^{q}\left(1-\tan\left(0.25\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}, \prod_{s=1}^{q}\left(1-\tan\left(0.25\pi(1-\varphi_{Nu(s)})\right)\right)^{\mu_{ws}}, \prod_{s=1}^{q}\left(1-\sin\left(0.25\pi(1-\varphi_{Nv(s)})\right)\right)^{\mu_{ws}} \right), \quad (9)$$

$$T-SvNVWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq})=\prod_{s=1}^{q}\left(\tan(\varphi_{Ns})\right)^{\mu_{ws}}$$
$$=\left(\prod_{s=1}^{q}\left(\tan(0.25\varphi_{Nt(s)}\pi)\right)^{\mu_{ws}},1-\prod_{s=1}^{q}\left(\tan(0.25\pi(1-\varphi_{Nu(s)}))\right)^{\mu_{ws}},1-\prod_{s=1}^{q}\left(\tan(0.25\pi(1-\varphi_{Nv(s)}))\right)^{\mu_{ws}}\right). \quad (10)$$

3. EOLs and Einstein AOs of S-SvNVs

This part presents several EOLs and Einstain AOs of S-SvNVs and their properties.

First, we give a new S-SvNV definition below.

Definition 1. If φ_N = <φ_{Nt}, φ_{Nu}, φ_{Nv}> is SvNV, then S-SvNV is defined by $\sin(\varphi_N)=\left\langle \sin(0.5\varphi_{Nt}\pi),1-\sin(0.5(1-\varphi_{Nu})\pi),1-\sin(0.5(1-\varphi_{Nv})\pi)\right\rangle$, where the membership degrees of indeterminacy, falsehood, and truth are $1-\sin(0.5(1-\varphi_{Nu})\pi)\in[0,1]$, $1-\sin(0.5(1-\varphi_{Nv})\pi)\in[0,1]$, and $\sin(0.5\varphi_{Nt}\pi)\in[0,1]$, respectively.

Definition 2. Let φ_{Ns} = <$\varphi_{Nt(s)}$, $\varphi_{Nu(s)}$, $\varphi_{Nv(s)}$> (s = 1, 2) be two SvNVs and μ_w > 0. Then, the EOLs of S-SvNVs are defined below:

(1) $$\sin(\varphi_{N1})\oplus_E\sin(\varphi_{N2})=\begin{pmatrix}\dfrac{\sin(0.5\varphi_{Nt(1)}\pi)+\sin(0.5\varphi_{Nt(2)}\pi)}{1+\sin(0.5\varphi_{Nt(1)}\pi)\sin(0.5\varphi_{Nt(2)}\pi)},\\ \dfrac{(1-\sin(0.5(1-\varphi_{Nu(1)})\pi))(1-\sin(0.5(1-\varphi_{Nu(2)})\pi))}{1+\sin(0.5(1-\varphi_{Nu(1)})\pi)\sin(0.5(1-\varphi_{Nu(2)})\pi)},\\ \dfrac{(1-\sin(0.5(1-\varphi_{Nv(1)})\pi))(1-\sin(0.5(1-\varphi_{Nv(2)})\pi))}{1+\sin(0.5(1-\varphi_{Nv(1)})\pi)\sin(0.5(1-\varphi_{Nv(2)})\pi)}\end{pmatrix};$$

(2) $$\sin(\varphi_{N1})\otimes_E\sin(\varphi_{N2})=\begin{pmatrix}\dfrac{\sin(0.5\varphi_{Nt(1)}\pi)\sin(0.5\varphi_{Nt(2)}\pi)}{1+(1-\sin(0.5\varphi_{Nt(1)}\pi))(1-\sin(0.5\varphi_{Nt(2)}\pi))},\\ \dfrac{1-\sin(0.5(1-\varphi_{Nu(1)})\pi)+1-\sin(0.5(1-\varphi_{Nu(2)})\pi)}{1+(1-\sin(0.5(1-\varphi_{Nu(1)})\pi))(1-\sin(0.5(1-\varphi_{Nu(2)})\pi))},\\ \dfrac{1-\sin(0.5(1-\varphi_{Nv(1)})\pi)+1-\sin(0.5(1-\varphi_{Nv(2)})\pi)}{1+(1-\sin(0.5(1-\varphi_{Nv(1)})\pi))(1-\sin(0.5(1-\varphi_{Nv(2)})\pi))}\end{pmatrix};$$

(3) $$(\sin(\varphi_{N1}))^{\mu_w}=\begin{pmatrix}\dfrac{2(\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w}}{(2-\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w}+(\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w}},\\ \dfrac{(2-\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_w}-(\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_w}}{(2-\sin(0.5(1+\varphi_{Nu(1)})\pi))^{\mu_w}+(\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_w}},\\ \dfrac{(2-\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w}-(\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w}}{(2-\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w}+(\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w}}\end{pmatrix};$$

$$(4)\quad \mu_w \cdot \sin(\varphi_{N1}) = \left(\begin{array}{c} \frac{(1+\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w} - (1-\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w}}{(1+\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w} + (1-\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_w}}, \\ \frac{2(1-\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_w}}{(1+\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_w} + (1-\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_w}}, \\ \frac{2(1-\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w}}{(1+\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w} + (1-\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_w}} \end{array} \right).$$

In view of EOLs of S-SvNVs, we define the S-SvNVEWA and S-SvNVEWG AOs.

Definition 3. If $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ (s = 1, 2, …, q) are a collection of SvNVs with their weight vector $\mu_W = (\mu_{w1}, \mu_{w2}, \ldots, \mu_{wq})$ for $0 \le \mu_{ws} \le 1$ and $\sum_{s=1}^{q} \mu_{ws} = 1$, the S-SvNVEWA and S-SvNVEWG AOs can be defined as follows:

$$S-SvNVEWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) = \mu_{w1}\sin(\varphi_{N1}) \oplus_E \mu_{w2}\sin(\varphi_{N2}) \oplus_E ... \oplus_E \mu_{wq}\sin(\varphi_{Nq}) = \sum_{s=1}^{q}{}_E\, \mu_{ws}\sin(\varphi_{Ns}), \quad (11)$$

$$S-SvNVWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) = \left(\sin(\varphi_{N1})\right)^{\mu_{w1}} \otimes_E \left(\sin(\varphi_{N2})\right)^{\mu_{w2}} \otimes_E ... \otimes_E \left(\sin(\varphi_{Nq})\right)^{\mu_{wq}} = \prod_{s=1}^{q}{}_E \left(\sin(\varphi_{Ns})\right)^{\mu_{ws}}. \quad (12)$$

Theorem 1. If $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ (s = 1, 2, …, q) are a collection of SvNVs with their weight vector $\mu_W = (\mu_{w1}, \mu_{w2}, \ldots, \mu_{wq})$ for $0 \le \mu_{ws} \le 1$ and $\sum_{s=1}^{q} \mu_{ws} = 1$, the aggregated result of the S-SvNVEWA AO is still S-SvNV, which is yielded by the equation:

$$S-SvNVEWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) = \sum_{s=1}^{q}{}_E\, \mu_{ws}\sin(\varphi_{Ns})$$

$$= \left(\begin{array}{c} \frac{\prod_{s=1}^{q}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}} - \prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}, \\ \frac{2\prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}, \\ \frac{2\prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}} \end{array} \right). \quad (13)$$

Proof. In terms of mathematical induction and Definition 2, we can give the proof of Theorem 1.

For q = 2, the operational results are given below:

$$\mu_{w1} \cdot \sin(\varphi_{N1}) = \left(\begin{array}{c} \frac{(1+\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_{w1}} - (1-\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_{w1}}}{(1+\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_{w1}} + (1-\sin(0.5\varphi_{Nt(1)}\pi))^{\mu_{w1}}}, \\ \frac{2(1-\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_{w1}}}{(1+\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_{w1}} + (1-\sin(0.5(1-\varphi_{Nu(1)})\pi))^{\mu_{w1}}}, \\ \frac{2(1-\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_1}}{(1+\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_1} + (1-\sin(0.5(1-\varphi_{Nv(1)})\pi))^{\mu_1}} \end{array} \right),$$

$$\mu_{w2}\cdot\sin(\varphi_{N2})=\left(\begin{array}{c}\dfrac{(1+\sin(0.5\varphi_{Nt(2)}\pi))^{\mu_{w2}}-(1-\sin(0.5\varphi_{Nt(2)}\pi))^{\mu_{w2}}}{(1+\sin(0.5\varphi_{Nt(2)}\pi))^{\mu_{w2}}+(1-\sin(0.5\varphi_{Nt(2)}\pi))^{\mu_{w2}}},\\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nu(2)})\pi))^{\mu_{w2}}}{(1+\sin(0.5(1-\varphi_{Nu(2)})\pi))^{\mu_{w2}}+(1-\sin(0.5(1-\varphi_{Nu(2)})\pi))^{\mu_{w2}}},\\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nv(2)})\pi))^{\mu_{w2}}}{(1+\sin(0.5(1-\varphi_{Nv(2)})\pi))^{\mu_{w2}}+(1-\sin(0.5(1-\varphi_{Nv(2)})\pi))^{\mu_{w2}}}\end{array}\right).$$

Then, there is the following result:

$$S-SvNVEWA(\varphi_{N1},\varphi_{N2})=\sum_{s=1}^{2}{}_{E}\,\mu_{ws}\sin(\varphi_{Ns})$$
$$=\left(\begin{array}{c}\dfrac{\prod_{s=1}^{2}\left(1+\sin\left(0.5\varphi_{t(s)}\pi\right)\right)^{\mu_{ws}}-\prod_{s=1}^{2}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{2}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{2}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{2}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{2}\left(1+\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{2}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{2}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{2}\left(1+\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{2}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}\end{array}\right). \quad (14)$$

Suppose that Eq. (13) holds for $q = p$. Then, there is the equation:

$$S-SvNVEWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Np})=\sum_{s=1}^{p}{}_{E}\,\mu_{ws}\sin(\varphi_{Ns})$$
$$=\left(\begin{array}{c}\dfrac{\prod_{s=1}^{p}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}-\prod_{s=1}^{p}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p}\left(1+\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p}\left(1+\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}\end{array}\right). \quad (15)$$

Based on Eqs. (14) and (15) for $q = p$+1, we have

$$S-SvNVEWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Np},\varphi_{Np+1})=\sum_{s=1}^{p+1}{}_E\,\mu_{ws}\sin(\varphi_{Ns})$$

$$=\left(\begin{array}{c}\dfrac{\prod_{s=1}^{p}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}-\prod_{s=1}^{p}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p}\left(1+\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{p}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p}\left(1+\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p}\left(1-\sin\left(\frac{\pi}{2}(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}\end{array}\right)$$

$$\oplus_E\left(\begin{array}{c}\dfrac{(1+\sin(0.5\varphi_{Nt(p+1)}\pi))^{\mu_{wp+1}}-(1-\sin(\frac{\pi}{2}\varphi_{Nt(p+1)}\pi))^{\mu_{wp+1}}}{(1+\sin(\frac{\pi}{2}\varphi_{Nt(p+1)}\pi))^{\mu_{wp+1}}+(1-\sin(\frac{\pi}{2}\varphi_{Nt(p+1)}\pi))^{\mu_{wp+1}}},\\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nu(p+1)})\pi))^{\mu_{wp+1}}}{(1+\sin(0.5(1-\varphi_{Nu(p+1)})\pi))^{\mu_{wp+1}}+(1-\sin(0.5(1-\varphi_{Nu(p+1)})\pi))^{\mu_{wp+1}}},\\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nv(p+1)})\pi))^{\mu_{wp+1}}}{(1+\sin(0.5(1-\varphi_{Nv(p+1)})\pi))^{\mu_{wp+1}}+(1-\sin(0.5(1-\varphi_{Nv(p+1)})\pi))^{\mu_{wp+1}}}\end{array}\right)$$

$$=\left(\begin{array}{c}\dfrac{\prod_{s=1}^{p+1}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}-\prod_{s=1}^{p+1}\left(1-\sin\left(0.5\varphi_{t(s)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p+1}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p+1}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{p+1}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p+1}\left(1+\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p+1}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}},\\ \dfrac{2\prod_{s=1}^{p+1}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{p+1}\left(1+\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}+\prod_{s=1}^{p+1}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}\end{array}\right).$$

Since Eq. (13) can hold for $q = p+1$, it can exist for all q.

Then, this S-SvNVEWA AO reveals the features below.

Theorem 2. The S-SvNVEWA AO reveals some features in view of the sine function below.

(1) Idempotency: If $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}> = <\varphi_{Nt}, \varphi_{Nu}, \varphi_{Nv}> = \varphi_N$ (s = 1, 2, …, q), then there is $S-SvNVEWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq})=\sin(\varphi_N)$.

(2) Boundedness: Set $\varphi_N^- = \left\langle \min_s(\varphi_{Nt(s)}), \max_s(\varphi_{Nu(s)}), \max_s(\varphi_{Nv(s)}) \right\rangle$ and $\varphi_N^+ = \left\langle \max_s(\varphi_{Nt(s)}), \min_s(\varphi_{Nu(s)}), \min_s(\varphi_{Nv(s)}) \right\rangle$ as the minimum and maximum SvNVs. Then, there exists $\sin(\varphi_N^-) \le S-SvNVEWA(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq}) \le \sin(\varphi_N^+)$.

(3) Monotonicity: Let $\varphi_{Ns} = <\varphi_{Nt(s)}, \varphi_{Nu(s)}, \varphi_{Nv(s)}>$ and $\varphi_{Ns}^{*} = \left\langle \varphi_{Nt(s)}^{*}, \varphi_{Nu(s)}^{*}, \varphi_{Nv(s)}^{*} \right\rangle$ (s = 1, 2, …, q) be two collections of SvNVs. Then $S-SvNVEWA(\varphi_{N1}, \varphi_{N2}, \cdots, \varphi_{Nq}) \le S-SvNVEWA(\varphi_{N1}^{*}, \varphi_{N2}^{*}, \cdots, \varphi_{Nq}^{*})$ exists if $\varphi_{Ns} \le \varphi_{Ns}^{*}$.

Proof. (1) Applying Eq. (13) for $\varphi_{Ns} = \varphi_N$, we obtain

$$S-SvNVEWA(\varphi_{N1}, \varphi_{N2}, ..., \varphi_{Nq}) = \sum_{s=1}^{q}{}_{E}\, \mu_{ws} \sin(\varphi_{Ns})$$

$$= \begin{pmatrix} \dfrac{\prod_{s=1}^{q}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}} - \prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(s)}\pi\right)\right)^{\mu_{ws}}}, \\ \dfrac{2\prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nu(s)})\pi\right)\right)^{\mu_{ws}}}, \\ \dfrac{2\prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nv(s)})\pi\right)\right)^{\mu_{ws}}} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{(1+\sin(0.5\varphi_{Nt}\pi))^{\sum_{s=1}^{q}\mu_{ws}} - (1-\sin(0.5\varphi_{Nt}\pi))^{\sum_{s=1}^{q}\mu_{ws}}}{(1+\sin(0.5\varphi_{Nt}\pi))^{\sum_{s=1}^{q}\mu_{ws}} + (1-\sin(0.5\varphi_{Nt}\pi))^{\sum_{s=1}^{q}\mu_{ws}}}, \\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nu})\pi))^{\sum_{s=1}^{q}\mu_{ws}}}{(1+\sin(0.5(1-\varphi_{Nu})\pi))^{\sum_{s=1}^{q}\mu_{ws}} + (1-\sin(0.5(1-\varphi_{Nu})\pi))^{\sum_{s=1}^{q}\mu_{ws}}}, \\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nv})\pi))^{\sum_{s=1}^{q}\mu_{ws}}}{(1+\sin(0.5(1-\varphi_{Nv})\pi))^{\sum_{s=1}^{q}\mu_{ws}} + (1-\sin(0.5(1-\varphi_{Nv})\pi))^{\sum_{s=1}^{q}\mu_{ws}}} \end{pmatrix}$$

$$= \begin{pmatrix} \dfrac{(1+\sin(0.5\varphi_{Nt}\pi)) - (1-\sin(0.5\varphi_{Nt}\pi))}{(1+\sin(0.5\varphi_{Nt}\pi)) + (1-\sin(0.5\varphi_{Nt}\pi))}, \\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nu})\pi))}{(1+\sin(0.5(1-\varphi_{Nu})\pi)) + (1-\sin(0.5(1-\varphi_{Nu})\pi))}, \\ \dfrac{2(1-\sin(0.5(1-\varphi_{Nv})\pi))}{(1+\sin(0.5(1-\varphi_{Nv})\pi)) + (1-\sin(0.5(1-\varphi_{Nv})\pi))} \end{pmatrix} = \sin(\varphi_N).$$

(2) For $\varphi_N^{-} \le \varphi_{Ns} \le \varphi_N^{+}$, $\sin(\varphi_N^{-}) \le \sin(\varphi_{Ns}) \le \sin(\varphi_N^{+})$ exists since sin(z) for 0 ≤ z ≤ π/2 is an increasing function. Then, $\sum_{s=1}^{q}{}_{E}\, \mu_{ws} \sin(\varphi_N^{-}) \le \sum_{s=1}^{q}{}_{E}\, \mu_{ws} \sin(\varphi_{Ns}) \le \sum_{s=1}^{q} \mu_{ws} \sin(\varphi_N^{+})$ is held. In view of the feature (1), $\sin(\varphi_N^{-}) \le S-SvNVEWA(\varphi_{N1}, \varphi_{N2}, ..., \varphi_{Nq}) \le \sin(\varphi_N^{+})$ can be also held.

(3) For $\varphi_{Ns} \le \varphi_{Ns}^{*}$, $\sin(\varphi_{Ns}) \le \sin(\varphi_{Ns}^{*})$ is held since sin(z) for 0 ≤ z ≤ π/2 is an increasing function. $\sum_{s=1}^{q} \mu_{ws} \sin(\varphi_{Ns}) \le \sum_{s=1}^{q} \mu_{ws} \sin(\varphi_{Ns}^{*})$ can be held in view of the feature (2). Thus, $S-SvNVEWA(\varphi_{N1}, \varphi_{N2}, \cdots, \varphi_{Nq}) \le S-SvNVEWA(\varphi_{N1}^{*}, \varphi_{N2}^{*}, \cdots, \varphi_{Nq}^{*})$ can also hold.

Example 1. Suppose that three SvNVs are φ_{N1} = <0.6, 0.2, 0.3>, φ_{N2} = <0.8, 0.1, 0.1>, and φ_{N3} = <0.7, 0.3, 0.3> with their weight vector $\boldsymbol{\mu}_W$ = (0.4, 0.3, 0.3). Using Eq. (13), we give the following aggregation result of the S-SvNVEWA AO:

$$S-SvNVEWA(\varphi_{N1},\varphi_{N2},\varphi_{N3})=\sum_{s=1}^{3}{}_E\,\mu_{ws}\sin(\varphi_{Ns})$$

$$=\left(\begin{array}{c}\dfrac{\left(\begin{array}{l}\left(1+\sin\left(0.5\times0.6\pi\right)\right)^{0.4}\left(1+\sin\left(0.5\times0.8\pi\right)\right)^{0.3}\left(1+\sin\left(0.5\times0.7\pi\right)\right)^{0.3}\\-\left(1-\sin\left(0.5\times0.6\pi\right)\right)^{0.4}\left(1-\sin\left(0.5\times0.8\pi\right)\right)^{0.3}\left(1-\sin\left(0.5\times0.7\pi\right)\right)^{0.3}\end{array}\right)}{\left(\begin{array}{l}\left(1+\sin\left(0.5\times0.6\pi\right)\right)^{0.4}\left(1+\sin\left(0.5\times0.8\pi\right)\right)^{0.3}\left(1+\sin\left(0.5\times0.7\pi\right)\right)^{0.3}\\+\left(1-\sin\left(0.5\times0.6\pi\right)\right)^{0.4}\left(1-\sin\left(0.5\times0.8\pi\right)\right)^{0.3}\left(1-\sin\left(0.5\times0.7\pi\right)\right)^{0.3}\end{array}\right)},\\ \dfrac{2\left(1-\sin\left(0.5(1-0.2)\pi\right)\right)^{0.4}\left(1-\sin\left(0.5(1-0.1)\pi\right)\right)^{0.3}\left(1-\sin\left(0.5(1-0.3)\pi\right)\right)^{0.3}}{\left(\begin{array}{l}\left(1+\sin\left(0.5(1-0.2)\pi\right)\right)^{0.4}\left(1+\sin\left(0.5(1-0.1)\pi\right)\right)^{0.3}\left(1+\sin\left(0.5(1-0.3)\pi\right)\right)^{0.3}\\+\left(1-\sin\left(0.5(1-0.2)\pi\right)\right)^{0.4}\left(1-\sin\left(0.5(1-0.1)\pi\right)\right)^{0.3}\left(1-\sin\left(0.5(1-0.3)\pi\right)\right)^{0.3}\end{array}\right)},\\ \dfrac{2\left(1-\sin\left(0.5(1-0.3)\pi\right)\right)^{0.4}\left(1-\sin\left(0.5(1-0.1)\pi\right)\right)^{0.3}\left(1-\sin\left(0.5(1-0.3)\pi\right)\right)^{0.3}}{\left(\begin{array}{l}\left(1+\sin\left(0.5(1-0.3)\pi\right)\right)^{0.4}\left(1+\sin\left(0.5(1-0.1)\pi\right)\right)^{0.3}\left(1+\sin\left(0.5(1-0.3)\pi\right)\right)^{0.3}\\+\left(1-\sin\left(0.5(1-0.3)\pi\right)\right)^{0.4}\left(1-\sin\left(0.5(1-0.1)\pi\right)\right)^{0.3}\left(1-\sin\left(0.5(1-0.3)\pi\right)\right)^{0.3}\end{array}\right)}\end{array}\right)$$

$$=<0.8918,\ 0.0414,\ 0.0573>.$$

Theorem 3. Set φ_{Ns} = <$\varphi_{Nt(s)}$, $\varphi_{Nu(s)}$, $\varphi_{Nv(s)}$> (s = 1, 2, …, q) as a collection of SvNVs with their weight vector $\boldsymbol{\mu}_W$ = (μ_{w1}, μ_{w2}, …, μ_{wq}) for $0 \le \mu_{ws} \le 1$ and $\sum_{s=1}^{q}\mu_{ws}=1$. Then, the aggregated result of the S-SvNVEWG AO is still S-SvNV, which is yielded by the equation:

$$S-SvNVEWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq})=\prod_{s=1}^{q}{}_E\left(\sin(\varphi_{Ns})\right)^{\mu_{ws}}$$

$$=\left\langle\begin{array}{c}\dfrac{2\prod_{s=1}^{q}(\sin(0.5\varphi_{Nt(s)}\pi))^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\sin(0.5\varphi_{Nt(s)}\pi))^{\mu_{ws}}+\prod_{s=1}^{q}(\sin(0.5\varphi_{Nt(s)}\pi))^{\mu_{ws}}},\\ \dfrac{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nu(s)})\pi))^{\mu_{ws}}-\prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nu(s)})\pi))^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nu(s)})\pi))^{\mu_{ws}}+\prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nu(s)})\pi))^{\mu_{ws}}},\\ \dfrac{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nv(s)})\pi))^{\mu_{ws}}-\prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nv(s)})\pi))^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{v(s)})\pi))^{\mu_{ws}}+\prod_{s=1}^{q}(\sin(0.5(1-\varphi_{v(s)})\pi))^{\mu_{ws}}}\end{array}\right\rangle. \quad (16)$$

However, the proof of Theorem 3 can be given based on a similar proof of Theorem 1, which is omitted.

Similarly, the S-SvNVEWG AO also indicates some features by the following theorem.

Theorem 4. The S-SvNVEWG AO reveals some features in view of the sine function below:

(1) Idempotency: If φ_{Ns} = <$\varphi_{Nt(s)}$, $\varphi_{Nu(s)}$, $\varphi_{Nv(s)}$> = <φ_{Nt}, φ_{Nu}, φ_{Nv}> = φ_N (s = 1, 2, …, q), then there is $S-SvNVEWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq})=\sin(\varphi_N)$.

(2) Boundedness: Set $\varphi_N^-=\left\langle \min_s(\varphi_{Nt(s)}),\max_s(\varphi_{Nu(s)}),\max_s(\varphi_{Nv(s)})\right\rangle$ and $\varphi_N^+=\left\langle \max_s(\varphi_{Nt(s)}),\min_s(\varphi_{Nu(s)}),\min_s(\varphi_{Nv(s)})\right\rangle$ as the minimum and maximum SvNVs. Then, there exists $\sin(\varphi_N^-)\le S-SvNVEWG(\varphi_{N1},\varphi_{N2},...,\varphi_{Nq})\le \sin(\varphi_N^+)$.

(3) Monotonicity: Let φ_{Ns} = <$\varphi_{Nt(s)}$, $\varphi_{Nu(s)}$, $\varphi_{Nv(s)}$> and $\varphi_{Ns}^*=\left\langle \varphi_{Nt(s)}^*,\varphi_{Nu(s)}^*,\varphi_{Nv(s)}^*\right\rangle$ (s = 1, 2, …, q) be two collections of SvNVs. Then, $S-SvNVEWG(\varphi_{N1},\varphi_{N2},\cdots,\varphi_{Nq})\le S-SvNVEWG(\varphi_{N1}^*,\varphi_{N2}^*,\cdots,\varphi_{Nq}^*)$ exists if $\varphi_{Ns}\le\varphi_{Ns}^*$.

However, the proof of Theorem 4 can also be given in terms of a similar proof of Theorem 2, which is omitted.

Example 2. Suppose that three SvNVs are φ_{N1} = <0.8, 0.3, 0.1>, φ_{N2} = <0.7, 0.2, 0.2>, and φ_{N3} = <0.9, 0.4, 0.4> with their weight vector $\boldsymbol{\mu}_W$ = (0.5, 0.3, 0.2). Using Eq. (16), we give the following aggregation result of the S-SvNVEWG AO:

$$S-SvNVEWG(\varphi_{N1},\varphi_{N2},\varphi_{N3})=\prod_{s=1}^{3}{}_E\left(\sin(\varphi_{Ns})\right)^{\mu_{ws}}$$

$$=\left\langle \begin{array}{l} \dfrac{2\times(\sin(0.5\times0.8\pi))^{0.5}(\sin(0.5\times0.7\pi))^{0.3}(\sin(0.5\times0.9\pi))^{0.2}}{\begin{pmatrix}(2-\sin(0.5\times0.8\pi))^{0.5}(2-\sin(0.5\times0.7\pi))^{0.3}(2-\sin(0.5\times0.9\pi))^{0.2}\\ +(\sin(0.5\times0.8\pi))^{0.5}(\sin(0.5\times0.7\pi))^{0.3}(\sin(0.5\times0.9\pi))^{0.2}\end{pmatrix}}, \\ \dfrac{\begin{pmatrix}(2-\sin(0.5(1-0.3)\pi))^{0.5}(2-\sin(0.5(1-0.2)\pi))^{0.3}(2-\sin(0.5(1-0.4)\pi))^{0.2}\\ -(\sin(0.5(1-0.3)\pi))^{0.5}(\sin(0.5(1-0.2)\pi))^{0.3}(\sin(0.5(1-0.4)\pi))^{0.2}\end{pmatrix}}{\begin{pmatrix}(2-\sin(0.5(1-0.3)\pi))^{0.5}(2-\sin(0.5(1-0.2)\pi))^{0.3}(2-\sin(0.5(1-0.4)\pi))^{0.2}\\ +(\sin(0.5(1-0.3)\pi))^{0.5}(\sin(0.5(1-0.2)\pi))^{0.3}(\sin(0.5(1-0.4)\pi))^{0.2}\end{pmatrix}}, \\ \dfrac{\begin{pmatrix}(2-\sin(0.5(1-0.1)\pi))^{0.5}(2-\sin(0.5(1-0.2)\pi))^{0.3}(2-\sin(0.5(1-0.4)\pi))^{0.2}\\ -(\sin(0.5(1-0.1)\pi))^{0.5}(\sin(0.5(1-0.2)\pi))^{0.3}(\sin(0.5(1-0.4)\pi))^{0.2}\end{pmatrix}}{\begin{pmatrix}(2-\sin(0.5(1-0.1)\pi))^{0.5}(2-\sin(0.5(1-0.2)\pi))^{0.3}(2-\sin(0.5(1-0.4)\pi))^{0.2}\\ +(\sin(0.5(1-0.1)\pi))^{0.5}(\sin(0.5(1-0.2)\pi))^{0.3}(\sin(0.5(1-0.4)\pi))^{0.2}\end{pmatrix}} \end{array}\right\rangle$$

$$=<0.9403,\ 0.1077,\ 0.0595>.$$

4. MADM model

This part develops a MADM model in view of the S-SvNVEWA and S-SvNVEWG AOs in the circumstance of SvNVs.

A MADM issue commonly includes a set of several alternatives Y_H = {Y_{H1}, Y_{H2}, …, Y_{Hp}} and a set of several attributes X_C = {x_{c1}, x_{c2}, …, x_{cq}}. In the MADM process, the alternatives must meet the requirements of the attributes, and then their SvNV assessment results are represented as their decision matrix $Q_N = (\varphi_{Nrs})_{p\times q}$, where φ_{Nrs} (r = 1, 2, …, p; s = 1, 2, …, q) are SvNVs provided by decision makers (DMs) according to the satisfactory assessment of an alternative Y_{Hr} over attributes x_{cs}. The weight vector of the attributes is specified by $\boldsymbol{\mu}_W$ = (μ_{w1}, μ_{w2}, …, μ_{wq}) for 0 ≤ μ_{ws} ≤ 1 and $\sum_{s=1}^{q}\mu_{ws}=1$. Thus, the algorithm of the MADM model in the circumstance of SvNVs is described in detail below.

Step 1: The aggregated values of φ_{Nr} for Y_{Hr} (r = 1, 2, …, p) are yielded by one of the S-SvNVEWA and S-SvNVEWG AOs:

$$\varphi_{Nr} = S-SvNVEWA(\varphi_{Nr1}, \varphi_{Nr2}, ..., \varphi_{Nrq}) = \sum_{s=1}^{q}{}_{E}\, \mu_{ws} \sin(\varphi_{Nrs})$$

$$= \left(\begin{array}{c} \dfrac{\prod_{s=1}^{q}\left(1+\sin\left(0.5\varphi_{Nt(rs)}\pi\right)\right)^{\mu_{ws}} - \prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(rs)}\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5\varphi_{Nt(rs)}\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5\varphi_{Nt(rs)}\pi\right)\right)^{\mu_{ws}}}, \\ \dfrac{2\prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nu(rs)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5(1-\varphi_{Nu(rs)})\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nu(rs)})\pi\right)\right)^{\mu_{ws}}}, \\ \dfrac{2\prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nv(rs)})\pi\right)\right)^{\mu_{ws}}}{\prod_{s=1}^{q}\left(1+\sin\left(0.5(1-\varphi_{Nv(rs)})\pi\right)\right)^{\mu_{ws}} + \prod_{s=1}^{q}\left(1-\sin\left(0.5(1-\varphi_{Nv(rs)})\pi\right)\right)^{\mu_{ws}}} \end{array} \right), \tag{17}$$

$$\varphi_{Nr} = S-SvNVEWG(\varphi_{Nr1}, \varphi_{Nr2}, ..., \varphi_{Nrq}) = \prod_{s=1}^{q}{}_{E} \left(\sin(\varphi_{Nrs})\right)^{\mu_{ws}}$$

$$= \left\langle \begin{array}{c} \dfrac{2\prod_{s=1}^{q}(\sin(0.5\varphi_{Nt(rs)}\pi))^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\sin(0.5\varphi_{Nt(rs)}\pi))^{\mu_{ws}} + \prod_{s=1}^{q}(\sin(0.5\varphi_{Nt(rs)}\pi))^{\mu_{ws}}}, \\ \dfrac{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nu(rs)})\pi))^{\mu_{ws}} - \prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nu(rs)})\pi))^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nu(rs)})\pi))^{\mu_{ws}} + \prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nu(rs)})\pi))^{\mu_{ws}}}, \\ \dfrac{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nv(rs)})\pi))^{\mu_{ws}} - \prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nv(rs)})\pi))^{\mu_{ws}}}{\prod_{s=1}^{q}(2-\sin(0.5(1-\varphi_{Nv(rs)})\pi))^{\mu_{ws}} + \prod_{s=1}^{q}(\sin(0.5(1-\varphi_{Nv(rs)})\pi))^{\mu_{ws}}} \end{array} \right\rangle. \tag{18}$$

Step 2: The score (accuracy) values of $U(\varphi_{Nr})$ ($V(\varphi_{Nr})$) (r = 1, 2, …, p) are yielded by Eq. (5) (Eq. (6)).

Step 3: Alternatives are sorted in the descending order of the score values (the accuracy values), and then the best alternative is decided.

Step 4: End.

5. MADM application

5.1 Site selection example of HPP

Since hydrogen is one of the most efficient and clean energy sources, its share of world energy has increased significantly. Then, it is important to choose the most suitable location for a HPP project, which is influenced by many factors, such as social, environmental, and economic factors. Hence, the site selection problem of HPP is a MADM problem. To apply the proposed MADM model to the actual MADM problem, this section adopts a site selection example of HPP in [30] for convenient comparison.

In this site selection example of HPP, experts and DMs preliminarily provide five potential locations, which are represented as a set of the five alternatives $Y_H = \{Y_{H1}, Y_{H2}, Y_{H3}, Y_{H4}, Y_{H5}\}$. Then they must satisfy the five main factors/attributes: the economic factor (x_{c1}), the technical factor (x_{c2}), the social factor (x_{c3}), the location factor (x_{c4}), and the environmental factor (x_{c5}). The weight vector of the five factors is given by μ_W = (0.22, 0.2, 0.15, 0.15, 0.28). In terms of the satisfactory degrees of each location corresponding to the five main factors, experts/DMs provide the SvNVs, which are composed of the indeterminate, false, and true degrees due to incompleteness, inconsistency, and uncertainty, including the judgements/opinions of the experts/DMs, and then their decision matrix of SvNVs $Q_N = (\varphi_{Nrs})_{5\times5}$ is presented as follows [30]:

$$Q_N = \begin{bmatrix} <0.3,0.2,0.4> & <0.2,0.2,0.6> & <0.5,0.3,0.4> & <0.3,0.3,0.4> & <0.5,0.2,0.3> \\ <0.6,0.4,0.2> & <0.6,0.3,0.2> & <0.7,0.1,0.3> & <0.7,0.1,0.2> & <0.7,0.2,0.3> \\ <0.5,0.1,0.3> & <0.6,0.1,0.2> & <0.5,0.3,0.4> & <0.6,0.4,0.3> & <0.6,0.2,0.4> \\ <0.5,0.2,0.2> & <0.4,0.5,0.2> & <0.7,0.3,0.2> & <0.4,0.5,0.4> & <0.7,0.2,0.2> \\ <0.4,0.3,0.6> & <0.4,0.1,0.5> & <0.4,0.1,0.3> & <0.3,0.2,0.4> & <0.5,0.1,0.2> \end{bmatrix}.$$

In this site selection problem of HPP, we give its MADM algorithm below.

Step 1: Applying Eq. (17) or Eq. (18), the aggregated results of the S-SvNVEWA or S-SvNVEWG AO are given below:

φ_{N1} = <0.5540, 0.0624, 0.1928>, φ_{N2} = <0.8616, 0.0520, 0.0693>, φ_{N3} = <0.7756, 0.0384, 0.1191>, φ_{N4} = <0.7792, 0.1055, 0.0604>, and φ_{N5} = <0.6073, 0.0247, 0.1594>.

Or φ_{N1} = <0.5174, 0.0670, 0.2154>, φ_{N2} = <0.8563, 0.0817, 0.0748>, φ_{N3} = <0.7707, 0.0642, 0.1326>, φ_{N4} = <0.7393, 0.1455, 0.0705>, and φ_{N5} = <0.5979, 0.0392, 0.2126>.

Step 2: By Eq. (5), the score values of $U(\varphi_{Nr})$ are yielded below:

$U(\varphi_{N1})$ = 0.7663, $U(\varphi_{N2})$ = 0.9134, $U(\varphi_{N3})$ = 0.8727, $U(\varphi_{N4})$ = 0.8711, and $U(\varphi_{N5})$ = 0.8077.

Or $U(\varphi_{N1})$ = 0.7450, $U(\varphi_{N2})$ = 0.8999, $U(\varphi_{N3})$ = 0.8579, $U(\varphi_{N4})$ = 0.8411, and $U(\varphi_{N5})$ = 0.7820.

Step 4: The sorting order of the five selection locations is $Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$ and then the best one is Y_{H2}.

It is obvious that the sorting orders corresponding to the S-SvNVEWA and S-SvNVEWG AOs are the same.

5.2 Comparative analysis

In view of the above example, this part conducts a comparative investigation with existing related MADM models in the circumstances of SvNVs.

Based on the decision making methods of the existing MADM models [19, 30, 31], we can obtain all the decision results by different AOs of Eqs. (1)–(4) and Eqs. (7)–(10) and the score function of Eq. (5), which are tubulated in Table 1. For easy comparison, the decision results of the proposed MADM model are also shown in Table 1.

Table 1. Decision results corresponding to different AOs

AO	Sorting result	Optimal location
SvNVEWA [19]	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}
SvNVEWG [19]	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}
S-SvNVWA [30]	$Y_{H2} > Y_{H4} > Y_{H3} > Y_{H5} > Y_{H1}$	Y_{H2}
S-SvNVWG [30]	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}
T-SvNVWA [31]	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}
T-SvNVWG [31]	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}
Proposed S-SvNVEWA	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}
Proposed S-SvNVEWG	$Y_{H2} > Y_{H3} > Y_{H4} > Y_{H5} > Y_{H1}$	Y_{H2}

In the decision results of Table 1, we see that the sorting results based on the proposed S-SvNVEWA and S-SvNVEWG AOs are the same as those based on the SvNVEWA and SvNVEWG AOs [19], the T-SvNVWA and T-SvNVWG AOs [31], and the S-SvNVWG AO [30], but different from the ranking order based on the S-SvNVWA AO [30]. Then, the optimal site selection is always Y_{H2} among the five alternatives. In addition, in the representation of decision information, the newly defined S-SvNVs can overcome the flaws of the existing S-SvNVs [30], which include the [0, 0.46) range of membership degrees. In the aggregation operations of decision information, the proposed S-SvNVEWA and S-SvNVEWG AOs can overcome the defects of the existing SvNVEWA and SvNVEWG AOs without including periodicity features [19]. In terms of algebraic operation performance, the proposed S-SvNVEWA and S-SvNVEWG AOs are superior to the existing T-SvNVWA and T-SvNVWG AOs. Hence, the proposed model can satisfy the real needs of DMs in the multi-stage decision process. In general, the proposed model reveals obvious superiority over the existing models [19, 30, 31]. The decision results reveal the validity and rationality of the proposed MADM model and can help us to find the best solution in the practical decision application.

The obvious advantages of this study are presented below:

The defined S-SvNV concept contains the superiority of the membership functions belonging to [0, 1], which can overcome the defects in the existing S-SvNV concept with the membership functions belonging to [0, 0.46) [29, 30].

The proposed EOLs and Einstein AOs of S-SvNVs can reflect their typical algebraic operations and compensate for the insufficiencies of the existing AOs [19, 30, 31].

(c) The developed MADM model using the proposed S-SvNVEWA and S-SvNVEWG AOs reveals its superiority over the existing MADM models using the SvNVEWA and SvNVEWG AOs [19], the S-SvNVWA and S-SvNVWG AOs [30], and the T-SvNVWA and T-SvNVWG AOs [31].

6. Conclusions

In this study, the defined S-SvNV EOLs and the proposed S-SvNVEWA and S-SvNVEWG AOs based on the monotonic membership functions of indeterminacy, falsehood, and truth can overcome the insufficiencies of the existing S-SvNV representation, operation laws, and AOs. In view of the presented S-SvNVEWA and S-SvNVEWG AOs, the developed MADM model can effectively improve the MADM models based on the existing SvNVEWA, SvNVEWG, S-SvNVWA, S-SvNVWG, and T-SvNVWA, and T-SvNVWG AOs in the SvNV circumstance. Then, the validity of the developed model was investigated by the actual site selection example of HPP and examined by comparative analysis with the existing related MADM models in the setting of SvNVs.

In this paper, the presented S-SvNVEWA and S-SvNVEWG AOs and their MADM model were used only for single-valued neutrosophic aggregations and MADM problems, which shows their limitations. Furthermore, the presented S-SvNVEWA and S-SvNVEWG AOs are only based on EOLs of S-SvNVs, but cannot imply the trigonometric EOLs of SvNVs based on trigonometric Einstein t-norm and t-conorm, which show their disadvantages. Therefore, in the future work, we need to develop the trigonometric EOLs and AOs of SNVs (SvNVs and IvNVs) and their MADM models. Then, the developed models will be used for decision making problems in the fields of engineering management, economic management, and medical management.

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References

1. Zadeh, L. A. Fuzzy sets. *Information and Control*, **1965**, 8(3), 338–353.
2. Atanassov, K. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, **1986**, 20, 87–96.
3. Atanassov, K.; Gargov, G. Interval-valued intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, **1989**, 31, 343–349.
4. Smarandache, F. *Neutrosophy: neutrosophic probability, set, and logic*. American Research Press, Rehoboth, USA,

1998.
5. Wang, H.; Smarandache, F.; Zhang, Y. Q.; Sunderraman, R. Single valued neutrosophic sets. *Multispace Multistructure*, **2010**, 4, 410–413.
6. Wang, H.; Smarandache, F.; Zhang, Y. Q.; Sunderraman, R. *Interval neutrosophic sets and logic: Theory and applications in computing*. Hexis, Phoenix, AZ, **2005**.
7. Ye, J. A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets. *Journal of Intelligent & Fuzzy Systems*, **2014**, 26, 2459–2466.
8. Zhang, H. Y.; Wang, J. Q., Chen, X. H. Interval neutrosophic sets and their application in multicriteria decision making problems. *The Scientific World Journal*, **2014**, 2014, 645953.
9. Yang, L. H.; Li, B. L. A multi-criteria decision-making method using power aggregation operators for single-valued neutrosophic sets. *International Journal of Database Theory and Application*, **2016**, 9(2), 23–32.
10. Liu, P.; Khan, Q.; Mahmood, T. Some single-valued neutrosophic power Muirhead mean operators and their application to group decision-making. *Journal of Intelligent & Fuzzy Systems*, **2019**, 37, 2515–2537.
11. Liu, P. The aggregation operators based on Archimedean t-conorm and t-norm for single-valued neutrosophic numbers and their application to decision-making. *International Journal of Fuzzy Systems*, **2016**, 18, 849–863.
12. Garg, H. Novel single-valued neutrosophic aggregated operators under Frank norm operation and its application to decision-making process. *International Journal of Uncertainty and Quantification*, **2016**, 6, 361–375.
13. Deli, I.; Subas, Y. A ranking method of single valued neutrosophic numbers and its applications to multi-attribute decision making problems. *International Journal of Machine Learning and Cybernetics*, **2017**, 8, 1309–1322.
14. Liu, P.; Liu, X. The neutrosophic number generalized weighted power averaging operator and its application in multiple attribute group decision-making. *International Journal of Machine Learning and Cybernetics*, **2018**, 9, 347–358.
15. Karaaslan F.; Hayat, K. Some new operations on single-valued neutrosophic matrices and their applications in multi-criteria group decision making. *Applied Intelligence*, **2018**, 48, 4594–4614.
16. Garg, H. SVNMPR - A new single-value neutrosophic multiplicative preference relation and their application to decision-making process. *International Journal of Intelligent Systems*, **2022**, 37(3), 2089–2130.
17. Giri, B. K.; Roy, S. K. Neutrosophic multi-objective green four-dimensional fixed-charge transportation problem. *International Journal of Machine Learning and Cybernetics*, **2022**, 13, 3089–3112.
18. Peng, X.; Dai, J. A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017. *Artificial Intelligence Review*, **2020**, 53, 199–255.
19. Peng, J. J.; Wang, J. Q.; Wang, J.; Zhang, H. Y.; Chen, X. H. Simplified neutrosophic sets and their applications in multi-criteria group decision-making problems. *International Journal of Systems Science*, **2016**, 47(10), 2342–2358.
20. Liu, P. D.; Wang, Y. M. Multiple attribute decision making method based on single-valued neutrosophic normalized weighted Bonferroni mean. *Neural Computing and Applications*, **2014**, 25(7-8), 2001–2010.
21. Liu, P.; Chu, Y.; Li, Y.; Chen, Y. Some generalized neutrosophic number Hamacher aggregation operators and their application. *International Journal of Fuzzy Systems*, **2014**, 16(2), 242–255.
22. Ye, J. Exponential operations and aggregation operators of interval neutrosophic sets and their decision making methods. *SpringerPlus*, **2016**, 5, 1488.

23. Ye, J. Subtraction and division operations of simplified neutrosophic sets. *Information*, **2017**, 8, 51.
24. Zhou, L. P.; Dong, J. Y.; Wan, S. P. Two new approaches for multi-attribute group decision-making with interval-valued neutrosophic Frank aggregation operators and incomplete weights. *IEEE Access*, **2019**, 7, 102727–102750.
25. Riaz, M.; Farid, H. M. A.; Antucheviciene, J.; Demir, G. Efficient decision making for sustainable energy using single-valued neutrosophic prioritized interactive aggregation operators. *Mathematics*, **2023**, 11(9), 2186.
26. Riaz, M.; Farid, H. M. A.; Ashraf, S.; Kamacı, H. Single-valued neutrosophic fairly aggregation operators with multi-criteria decision-making. *Computational and Applied Mathematics*, **2023**, 42, 104.
27. Garg, H. New logarithmic operational laws and their applications to multiattribute decision-making for single-valued neutrosophic numbers. *Cognitive Systems Research*, **2018**, 52, 931–946.
28. Ashraf, S.; Abdullah, S.; Smarandache, F. Logarithmic hybrid aggregation operators based on single valued neutrosophic sets and their applications in decision support systems. *Symmetry*, **2019**, 11, 364.
29. Garg, H. Decision making analysis based on sine trigonometric operational laws for single-valued neutrosophic sets and their applications. *Applied and Computational Mathematics*, **2020**, 19(2), 255–276.
30. Ashraf, S.; Abdullah, S.; Zeng, S.; Jin, H.; Ghani, F. Fuzzy decision support modeling for hydrogen power plant selection based on single valued neutrosophic sine trigonometric aggregation operators. *Symmetry*, **2020**, 12, 298.
31. Zhao, M.; Ye, J. MADM technique using tangent trigonometric SVNN aggregation operators for the teaching quality assessment of teachers. *Neutrosophic Sets and Systems*, **2022**, 50, 651–662.
32. Ye, J.; Du, S.; Yong, R. Multi-criteria decision-making model using trigonometric aggregation operators of single-valued neutrosophic credibility numbers. *Information Sciences*, **2023**, 644, 118968.

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Evaluation of Shortest path on multi stage graph problem using Dynamic approach under neutrosophic environment

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Abstract

The shortest path problem is a classic optimization problem in graph theory and computer technology. It involves identifying the shortest path between two nodes in a graph, where each edge has a numerical weight. In this paper, we put our effort into examining the use of the dynamic programming method to evaluate the shortest path (SP) between the two specified nodes in a multistage network where the parameter is a multi-value neutrosophic number (MVNN). Firstly, we propose an algorithm based on the forward and backward approach in an uncertain environment and also implement our approach in the Python-3 programming language. Furthermore, a numerical illustration has been provided to showcase the effectiveness and robustness of the novel model.

Keyword: dynamic programming approach; multistage graph; neutrosophic multi-value number; shortest path problem

1. Introduction:

The shortest path problem represents one of the primary network issues in graph theory, with numerous applications in computer science and several real-life applications such as transportation networks, communication networks, and pipeline distribution systems. In this paper, we propose a new idea for evaluating the shortest path of a multistage graph using fuzzy multivalued neutrosophic numbers as arc length.

A fuzzy set (FS) is used to identify and solve a wide range of real-world issues that involve uncertainty and improbability. Lotfi Aliasker Zadeh first suggested the fuzzy set [1], and then Atanassov (1988) [2] proposed intuitionistic fuzzy sets (IFs), which are the extended concept of fuzzy sets. Then Smarandache (1995) [3] first established the theme of the new idea of neutrosophic sets (NS). The NS is a collection of three parameters, namely fuzzy membership degree, fuzzy indeterminate degree, and fuzzy non-membership degree, with the addition of their weights being less than or equal to 3. The field of neutrosophic numbers extends beyond crisp numbers. Numerous research papers have addressed the computation of the fuzzy shortest path (FSP) in a single-stage network. For example, Wang proposed the idea of IVNS (2018) by generalizing SVN (2010) [4]. The IVNS [5] is a database that generalizes the idea of various types of sets in terms of intervals to denote the truth T, falsity F, and indeterminacy I of membership degrees. Many researchers have proposed various papers on neutrosophic environments (Basset (2018), Abdel-Basset (2018), and Dey (2019)) [6–14].

Many researchers have proposed new approaches for finding SPP in uncertain environments. Das and De (2015) [15] solved FSP using Bellman's dynamic programming method with intuitionistic fuzzy trapezoidal numbers as parameters. Bhincher and De (2011) [16] investigated the FSP in a connected network in which they used triangular and trapezoidal fuzzy numbers as parameters in two distinct approaches, namely the influential programming approach and the multi-objective linear programming approach. Kumar (2015) [17] developed a technique for determining the SP of a connected network using an interval intuitionistic trapezoidal number. Kaliraja and Meenakshi (2012) [18] used interval-based parameters and proposed a method to identify the SPP and model an interval-valued FSPP.

Said Broumi (2016) [19] proposed a new idea of evaluating the shortest path using the parameters SV-triangular and SV-trapezoidal fuzzy neutrosophic numbers. Then again, Said Broumi (2017) [20] suggested a new idea to evaluate the FSPP of a given connected network with neutrosophic trapezoidal numbers. Said Broumi (2017) [21] suggested an innovative method for formulating the SPP in which they use the parameters, which are bipolar neutrosophic numbers. Deivanayagam Pillai, N (2020) [22], solved the NSPP by using the score function, where the parameters are interval-valued neutrosophic trapezoidal and neutrosophic triangular numbers. Said Broumi (2019) [23] solved the SPP in a neutrosophic environment (NS) using the Bellman-Ford approach, where the parameter is interval-valued neutrosophic numbers(IVNNs).

The primary aim of this study is to determine the shortest path between the source node and the destination node using multi-value neutrosophic numbers, along with identifying the minimum cost between the source and destination nodes. The contents of the next parts of the paper are arranged in the following manner: Section-2, highlights the motivation and contribution of this paper. Section-3 highlights some definitions of some of the existing terminologies. Section-4 highlights the algorithm, i.e., the multistage network, for multi-valued neutrosophic numbers (MVNNs). Section-5 highlights a numerical example. Section-6 gives an implementation of our algorithm with the Python programming language. Section-7 provides a summary of the conclusions drawn from the study and offers recommendations for further research endeavors.

2. Motivation:

There are various algorithms and various parameters that are used to evaluate the SPP in uncertain circumstances. The key points are as follows:

- There are many methods used to solve the single-stage network, but our method is used to solve the multistage network in NSP.
- In this paper, a dynamic programming method is used to evaluate the shortest path (SP) between the two specified nodes in a multistage network, where the parameter is a multi-value neutrosophic number (MVNN). Firstly, we propose an algorithm based on the forward and backward approach in an uncertain environment and also implement our approach in the Python-3 programming language.

- In this paper, we are finding the minimum cost between the source node and the destination node.
- Moreover, here we illustrate one algorithm with the help of a numerical example.

3. Preliminaries

This section encompasses the review of literature concerning the fundamental concepts and definitions of fuzzy sets (FSs), neutrosophic sets (NSs), and MVNSs.

3.1 Fuzzy set (FS):

If $\breve{Z}$ is a generalised form of crisp set and $\breve{z}$ is a member of $\breve{Z}$, then fuzzy set $\breve{A}$ on $\breve{Z}$ is defined by a membership value $\mu_{\breve{A}}(\breve{z})$, which identifies the function that maps from every element to the interval [0, 1] and can be defined as $A = \{(\breve{z}, \mu_{\breve{A}}(\breve{z})), \breve{z} \in \breve{Z},\}$

and $\mu_{\breve{Z}}(\breve{z}): \breve{z} \rightarrow [0,1]$

3.2 Neutrosophic set (NS):

If $\breve{X}$ is a set and $\breve{x}$ is one of its elements in $\breve{X}$; then neutrosophic set $\breve{A}$ has the form

$$\breve{A} = \{< \breve{x}: \breve{T}_{\breve{A}}(x), \breve{I}_{\breve{A}}(x), \breve{F}_{\breve{A}}(x) > \breve{x} \epsilon \breve{X}\} ----------(1)$$

Where $\breve{T}$ denotes the truth degree, $\breve{I}$ denotes the indeterminacy degree and $\breve{F}$ denotes the falsity membership degree of the element $\breve{x} \epsilon \breve{X}$

$$0^- \leq \{\breve{T}_{\breve{A}}(x) + \breve{I}_{\breve{A}}(x) + \breve{F}_{\breve{A}}(x)\} \leq 3^+$$

Now $\breve{T}_{\breve{A}}(x), \breve{I}_{\breve{A}}(x), \breve{F}_{\breve{A}}(x)$ are denotes subsets of the interval $[0^-, 1^+]$.

3.3 Multi-valued neutrosophic set (MVNs):

If $\breve{X}$ is a set and $\breve{x}$ is one of its elements in $\breve{X}$. Then the multi-valued neutrosophic (MVN) set is represented as.

$\breve{A} = \breve{x}, \breve{T}_{\breve{A}}(x), \breve{I}_{\breve{A}}(x), \breve{F}_{\breve{A}}(x), \breve{x}\epsilon\breve{X}$

Then $\breve{T}_{\breve{A}}(x), \breve{I}_{\breve{A}}(x)$ and $\breve{F}_{\breve{A}}(x)$ are the membership function differentiate $\breve{A}$ in $\breve{X}$.

Where $\breve{T}_{\breve{A}}(x), \breve{I}_{\breve{A}}(x)$ and $\breve{F}_{\breve{A}}(x) \in [0,1]$ and the condition is

$0 \leq \alpha, \beta, \gamma \leq 1, 0 \leq \alpha^+, \beta^+, \gamma^+ \leq 3, \alpha \in \breve{T}_{\breve{A}}(x), \beta \in \breve{I}_{\breve{A}}(x), \gamma \in \breve{F}_{\breve{A}}(x).$

$\alpha^+ = \text{Sup}\breve{T}_{\breve{A}}(x), \beta^+ = \text{Sup}\,\breve{I}_{\breve{A}}(x), \gamma^+ = \text{Sup}\breve{F}_{\breve{A}}(x) - - - - - - - - -(2)$

The multi-valued neutrosophic (MVN) are called as single valued neutrosophic (SVN) sets if $\breve{A} = \{\breve{T}_{\breve{A}}(x), \breve{I}_{\breve{A}}(x), \breve{F}_{\breve{A}}(x)\}$ has just one value.

3.4 Operations of Neutrosophic number:

Assume that $\breve{A}_1 = \{\breve{T}_{\breve{A}_1}(x), \breve{I}_{\breve{A}_1}(x), \breve{F}_{\breve{A}_1}(x)\}$ and $\breve{A}_2 = \{\breve{T}_{\breve{A}_2}(x), \breve{I}_{\breve{A}_2}(x), \breve{F}_{\breve{A}_2}(x)\}$ are represent two sets of neutrosophic numbers with multiple values. Subsequently, the functions for SVNNs are defined as follows:

(a) $(\breve{A}_1 + \breve{A}_2) = \{\breve{T}_{\breve{A}_1}(x) + \breve{T}_{\breve{A}_2}(x) - \breve{T}_{\breve{A}_1}(x)\breve{T}_{\breve{A}_2}(x), \breve{I}_{\breve{A}_1}(x)\breve{I}_{\breve{A}_2}(x), \breve{F}_{\breve{A}_1}(x)\breve{F}_{\breve{A}_2}(x)\}$

(b)

$(\breve{A}_1 \times \breve{A}_2) = \{\breve{T}_{\breve{A}_1}(x)\breve{T}_{\breve{A}_2}(x), \breve{I}_{\breve{A}_1}(x) + \breve{I}_{\breve{A}_2}(x) - \breve{I}_{\breve{A}_1}(x)\breve{I}_{\breve{A}_2}(x), \breve{F}_{\breve{A}_1}(x) + \breve{F}_{\breve{A}_2}(x) - \breve{F}_{\breve{A}_1}(x)\breve{F}_{\breve{A}_2}(x)\}$

(c) $\lambda\breve{A}_1 = \{1(1 - \breve{T}_{\breve{A}_1}(x))^\lambda, \breve{I}_{\breve{A}_1}(x)^\lambda, \breve{F}_{\breve{A}_1}(x)^\lambda\}$

(d) ${\breve{A}_1}^\lambda = \{\breve{T}_{\breve{A}_1}(x)^\lambda, 1 - (1 - \breve{I}_{\breve{A}_1}(x))^\lambda, 1 - (1 - \breve{I}_{\breve{A}_1}(x))^\lambda\}$

With $\lambda > 0$

3.5 Fuzzy Graded mean Integration:

If the fuzzy triangular number $\breve{A} = (\breve{l}, \breve{m}, \breve{n})$ .Then the Fuzzy graded mean integration is expressed as:

$G(\breve{A}) = \frac{1}{6}(\breve{l}_1 + 4\breve{m} + \breve{n}) - - - - - - - - - - - - (3)$

If $\breve{A} = (\breve{l}_1, \breve{m}_1, \breve{n}_1)$ and $\breve{B} = (\breve{l}_2, \breve{m}_2, \breve{n}_2)$ is two fuzzy triangular numbers.

Then the graded mean integration representation is defined as

$$G(\breve{A}) = \frac{1}{6}(\breve{l}_1 + 4\breve{m}_2 + \breve{n}_3)$$

$$G(\breve{B}) = \frac{1}{6}(\breve{l}_1 + 4\breve{m}_2 + \breve{n}_3)$$

If $\breve{A}$ and $\breve{B}$ are two fuzzy triangular numbers then its addition is expressed as:

$$G(\breve{A} + \breve{B}) = \frac{1}{6}(\breve{l}_1 + 4\breve{m}_1 + \breve{n}_1) + \frac{1}{6}(\breve{l}_2 + 4\breve{m}_2 + \breve{n}_2) - - - - - - - - -(4)$$

If $\breve{A}$ and $\breve{B}$ are two fuzzy triangular numbers then its multiplication is expressed as:

$$G(\breve{A} \times \breve{B}) = \frac{1}{6}(\breve{l}_1 + 4\breve{m}_1 + \breve{n}_1) \times \frac{1}{6}(\breve{l}_2 + 4\breve{m}_2 + \breve{n}_2) - - - - - - - - -(5)$$

4. Algorithm: Multistage Network Utilizing Multi-Valued Neutrosophic Numbers (MVNNs)

- Step 1: Select a source and destination vertex within the provided multistage network.
- Step 2: Convert the arc length values from multi-valued neutrosophic numbers to single-value neutrosophic numbers using the fuzzy simplicity method (equation-2).
- Step 3: Convert it from single-value neutrosophic numbers to a real number using graded mean integration (definition-3).
- Step 4: Then, using a dynamic approach, i.e., a forward and backward computation approach.

Backward Approach Algorithm:

```
#Algorithm for Backward Approach

 Algorithm BGraph (G, K, n, p)
# some function as FGraph
{
B cost [1] = 0.0;
For j = 2 to n do
{
# compute b cost [j].
Let r be such that  is an edge of
G and b cost [r] + c [r, j];
D [j] = r;
}
# find a minimum cost path
P [1] = 1;                p [k] = n;
For j = k-1 to 2 do p[j] = d [p (j+1)];
}
```

Forward Approach Algorithm:

```
#Algorithm for Forward Approach

F graph (graph G, int K, int n, int p[])
{
Float cost [max size], int d [max size], r;
Cost [n] = 0.0
For (int j = n-1; j>= 1; j--)
{
Let r be a vertex such that  is an edge of G and C[j][r] + cost[r] is minimum;
Cost [j] = C[j][r] + Cost[r]
D [j] = r
}
#Find a minimum-cost path
P [1] = 1 ,  P[k] = n
For (j = 2 ; j <= K-1; j++)
P[j] = d[P(j-1)];
}
```

➢ **Step-5:** After applying the dynamic approach, i.e., forward and backward approach if both techniques produce the equal minimum values and the shortest path, then the path yielded in the process is called the optimal path or shortest path of a network.

5. Numerical Example:

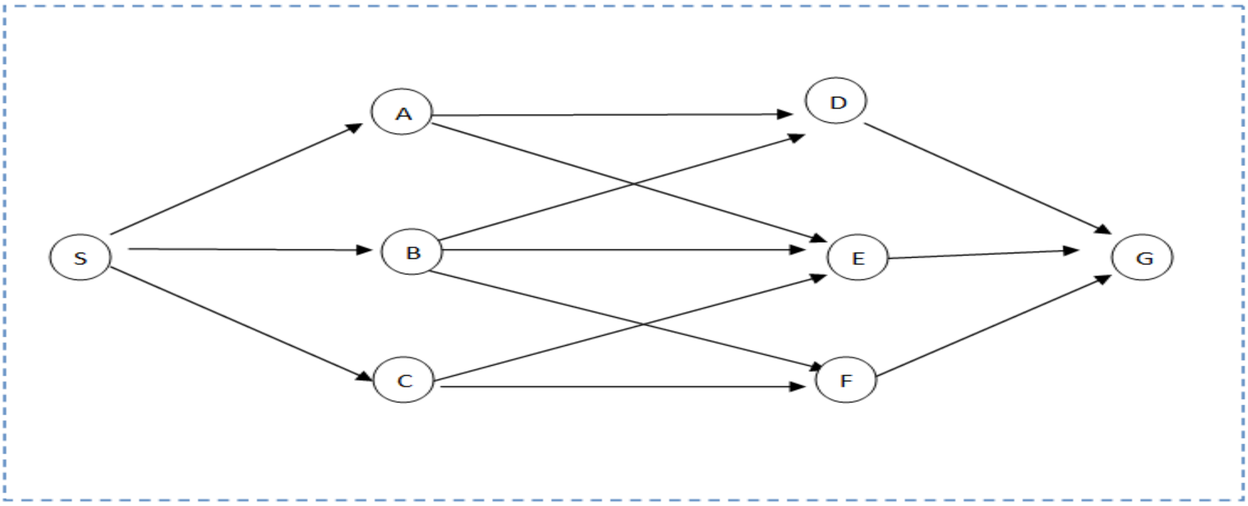

Fig- 1: Network

Arc	**Multi-membership value**
S→A	<[0.2,0.4,0.5],[0.3,0.5,0.6],[0.6,0.8,0.9]>
S→B	<[0.1,0.3,0.4],[0.3,0.4,0.7],[0.5,0.7,0.9]>
S→C	<[0.1,0.3,0.5],[0.3,0.5,0.7],[0.4,0.5,0.8]>
A→D	<[0.3,0.4,0.5],[0.4,0.5,0.6],[0.5,0.7,0.9]>

A→E	<[0.2,0.3,0.6],[0.3,0.4,0.8],[0.4,0.5,0.9]>
B→D	<[0.1,0.2,0.4],[0.3,0.4,0.6],[0.4,0.5,0.6]>
B→E	<[0.3,0.4,0.6],[0.3,0.4,0.7],[0.4,0.7,0.9]>
B→F	<[0.1,0.2,0.5],[0.2,0.4,0.5],[0.5,0.6,0.9]>
C→E	<[0.4,0.2,0.5],[0.6,0.5,0.8],[0.5,0.6,0.8]>
C→F	<[0.1,0.2,0.3],[0.2,0.5,0.6],[0.5,0.7,0.9]>
D→G	<[0.4,0.5,0.9],[0.6,0.7,0.8],[0.5,0.6,0.9]>
E→G	<[0.1,0.2,0.5],[0.2,0.4,0.5],[0.5,0.7,0.9]>
F→G	<[0.2,0.6,0.7],[0.2,0.5,0.8],[0.6,0.8,0.9]>

Table- 1: Arc weight in Multi-membership value

Implementation of Algorithm

Step-1:

From fig-1 assume that source node is S and destination node is G

Step-2:

Arc	Single Membership value
S→A	<[0.5, 0.6, 0.9]>
S→B	<[0.4,0.7,0.9]>
S→C	<[0.5,0.7,0.8]>
A→D	<[0.5,0.6, 0.9]>
A→E	<[0.6,0.8,0.9]>
B→D	<[0.4,0.6,0.6]>
B→E	<[0.6,0.7,0.9]>
B→F	<[0.5,0.5,0.9]>
C→E	<[0.5,0.8,0.8]>
C→F	<[0.3,0.6,0.9]>

D→G	<[0.9,0.8,0.9]>
E→G	<[0.5,0.5,0.9]>
F→G	<[0.7,0.8,0.9]>

Table- 2: Single Membership value

Step-3:

Converting the Single membership value into a real value by using Graded mean integration (definition-3.5)

Here $(\breve{l} = 0.5, \breve{m} = 0.6, \breve{n} = 0.9)$

$$G(\breve{A}) = \frac{1}{6}(\breve{l} + 4\breve{m} + \breve{n})$$

$$G(\breve{A}) = \frac{1}{6}(0.5 + 4 \times 0.6 + 0.9)$$

$$= 0.63$$

Similarly to find all the edge's value in Crisp number

Arc	**Single Membership value**
S→A	0.63
S→B	0.68
S→C	0.68
A→D	0.63

A→E	0.78
B→D	0.56
B→E	0.71
B→F	0.56
C→E	0.75
C→F	0.60
D→G	0.83
E→G	0.56
F→G	0.80

Table- 3: Membership value in crisp number

Step 4:

Backward Approach

In backward approach we start from source vertex, so the distance from source (S) to destination vertex (T) is (S, T) is given by

$$dis(S,G) = min\{0.63 + dis(A,G), 0.68 + dis(B,G), 0.68 + dis(C,G)\}$$

$$---------2.0$$

Now to calculate the distance (A to G), distance (B to G) and distance (C to G).

$$dis(A,G) = min\{0.63 + dis(D,G), 0.78 + dis(E,G)\}$$

$$dis(A,G) = min\{0.63 + 0.83, 0.78 + 0.56\}$$

$dis(A,G) = min\{1.46, 1.34\}$

$dis(A,G) = 1.34 --------2.1$

$dis(B,G) = min\{0.56 + dis(D,G), 0.71 + dis(E,G), 0.56 + dis(F,G)\}.$

$d(B,G) = min\{0.56 + 0.83, 0.71 + 0.56, 0.56 + 0.80\}$

$dis(B,G) = min\{1.39, 1.27, 1.36\}$

$d(B,G) = 1.27 --------2.2$

$(C,G) = min\{0.75 + dis(E,G), 0.60 + dis(F,G)\}$

$(C,G) = min\{0.75 + 0.56, 0.60 + 0.80\}$

$(C,G) = min\{1.31, 1.40\}$

$(C,G) = 1.31 --------2.3$

Now Putting all this values in equation 2.0

$dis(S,G) = \min\{0.63 + dis(A,G), 0.68 + dis(B,G), 0.68 + dis(C,G)\}$

$dis(S,G) = min\{0.63 + 1.34, 0.68 + 1.27, 0.68 + 1.31\}$

$dis(S,G) = min\{1.97, 1.95, 1.99\}$

$dis(S,G) = 1.95(\mathrm{S} - \mathrm{B} - \mathrm{E} - \mathrm{G})$

Forward approach

Here $dis(S,A) = 0.63$

$dis(S,B) = 0.68$

$dis(S,C) = 0.68$

$dis(S,D) = min\{0.63 + dis(A,D), 0.68 + dis(B,D)\}$

$dis(S,D) = min\{0.63 + 0.63, 0.68 + 0.56\}$

$dis(S,D) = min\{1.26, 1.24\}$

$dis(S,D) = 1.24$

$dis(S,E) = min\{0.63 + dis(A,E), 0.68 + dis(B,E), 0.68 + dis(C,E)\}$

$dis(S,E) = min\{0.63 + 0.78, 0.68 + 0.71, 0.68 + 0.75\}$

$dis(S,E) = min\{1.41, 1.39, 1.43\}$

$$dis(S,E) = 1.39$$

$$dis(S,F) = min\{0.63 + dis(B,F), 0.68 + dis(C,F)\}$$

$$dis(S,F) = min\{0.63 + 0.56, 0.68 + 0.60\}$$

$$dis(S,F) = min\{1.19, 1.28\}$$

$$dis(S,F) = 1.19$$

$$dis(S,G) = min\{dis(S,D) + dis(D,G), dis(S,E) + dis(E,G), dis(S,F) + dis(F,G)\}$$

$$dis(S,G) = min\{1.24 + 0.83, 1.39 + 0.56, 1.19 + 0.80\}$$

$$dis(S,G) = \{2.07, 1.95, 1.99\}$$

$$dis(S,G) = 1.95(\mathrm{S} \rightarrow \mathrm{B} \rightarrow \mathrm{E} \rightarrow \mathrm{G})$$

Implementation of Our algorithm with Python Programming Language

```python
#Implementation of our work with multistage graph in Python3 Compiler (shortest path).
import sys
#function for finding shortest distance inmulti-stage network
def Source_to_Destination(SG):
#list for storing shortest distance from particular node to N-1 node
    Distance=[0]*n
#finding the shortest paths
    for x in range(n-2, -1, -1):
        Distance[x]=infinity
#Checking nodes from next stages
        for y in range(n):
#condition when no edge exists
            if SG[x][y]==infinity:
                continue
#finding minimum distances
            Distance[x]=min(SG[x][y]+Distance[y],Distance[x])
    return Distance[0]
# Driver code
n=8
infinity=sys.maxsize
#Adjacency matrix for graph
SG=[[infinity, 0.63, 0.68, 0.68, infinity, infinity, infinity, infinity],
    [infinity, infinity, infinity,infinity, 0.63, 0.78, infinity, infinity],
    [infinity, infinity, infinity,infinity,0.56, 0.71, 0.56, infinity],
    [infinity, infinity, infinity, infinity, infinity,  0.75, 0.60, infinity],
    [infinity, infinity, infinity, infinity, infinity, infinity, infinity, 0.83],
    [infinity, infinity, infinity, infinity, infinity, infinity, infinity, 0.56],
    [infinity, infinity, infinity, infinity, infinity, infinity, infinity, 0.80]]
D=Source_to_Destination(SG)
print("SHORTEST PATH FROM SOURCE TO DESTINATION IS :",D)
```

Output Code:

```
SHORTEST PATH FROM SOURCE TO DESTINATION IS : 1.9500000000000002


...Program finished with exit code 0
Press ENTER to exit console.
```

Step-5:

In the dynamic approach, i.e., both forward and backward approaches have an equal minimum path value 1.95 and an equal path S-B-E-G, so this is the SP connecting the source vertex to destination vertex of this given Network.

7. Conclusion

In this paper, we find the shortest path (SP) on the multistage network by using the dynamic approach, i.e., the forward and backward approach, and then we implement our result in the Python programming language, and finally, we get the shortest path. The minimum cost between the source vertex and the destination vertex is 1.95. The most important objective of this research is to determine a new algorithm for solving multistage graphs. Right here, we propose a mathematical instance to show our new suggested method. I'm hoping that this paper will help new researchers find the SPP in multistage graphs.

References

[1] Zadeh, L. A. (1965). Fuzzy sets. Information and control, 8(3), 338-353.

[2] Atanassov, K. (1988). Review and new results on intuitionistic fuzzy sets. preprint Im-MFAIS-1-88, Sofia, 5(1).

[3] Smarandache, F., Abdel-Basset, M., & Broumi, S. (Eds.). (2021). Neutrosophic Sets and Systems, vol. 40, 2021. Infinite Study.

[4]Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). Single valued neutrosophic sets. Infinite study.

[5]Wang, H., Smarandache, F., Sunderraman, R., & Zhang, Y. Q. (2005). interval neutrosophic sets and logic: theory and applications in computing: Theory and applications in computing (Vol. 5). Infinite Study.

[6] Basset, M. A., Mohamed, M., Sangaiah, A. K., & Jain, V. (2018). An integrated neutrosophic AHP and SWOT method for strategic planning methodology selection. Benchmarking: An International Journal.

[7] Abdel-Basset, M., Mohamed, M., & Smarandache, F. (2018). A hybrid neutrosophic group ANP-TOPSIS framework for supplier selection problems. Symmetry, 10(6), 226..

[8] Abdel-Basset, M., Mohamed, M., & Smarandache, F. (2018). An extension of neutrosophic AHP–SWOT analysis for strategic planning and decision-making. Symmetry, 10(4), 116.

[9] Abdel-Basset, M., Mohamed, M., Smarandache, F., & Chang, V. (2018). Neutrosophic association rule mining algorithm for big data analysis. Symmetry, 10(4), 106.

[10]Abdel-Basset, M., Zhou, Y., Mohamed, M., & Chang, V. (2018). A group decision making framework based on neutrosophic VIKOR approach for

e-government website evaluation. Journal of Intelligent & Fuzzy Systems, 34(6), 4213-4224.

[11] Abdel-Basset, M., Mohamed, M., Zhou, Y., & Hezam, I. (2017). Multi-criteria group decision making based on neutrosophic analytic hierarchy process. Journal of Intelligent & Fuzzy Systems, 33(6), 4055-4066.

[12] Abdel-Basset, M., & Mohamed, M. (2018). The role of single valued neutrosophic sets and rough sets in smart city: Imperfect and incomplete information systems. Measurement, 124(10.1016).

[13] Abdel-Basset, M., Manogaran, G., Gamal, A., & Smarandache, F. (2018). A hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. Design Automation for Embedded Systems, 22(3), 257-278.

[14] Dey, A., & Pal, A. (2019). Computing the shortest path with words. International Journal of Advanced Intelligence Paradigms, 12(3-4), 355-369.

[15] Vaidyanathan, R., Das, S., & Srivastava, N. (2015). Query expansion strategy based on pseudo relevance feedback and term weight scheme for monolingual retrieval. arXiv preprint arXiv:1502.05168.

[16] De, P. K., & Bhincher, A. (2011). Dynamic programming and multi objective linear programming approaches. Appl Math Inf Sci, 5(2), 253-263.

[17] Kumar, G., Bajaj, R. K., & Gandotra, N. (2015). Algorithm for shortest path problem in a network with interval-valued intuitionistic trapezoidal fuzzy number. Procedia Computer Science, 70, 123-129.

[18] Meenakshi, A. R., & Kaliraja, M. (2012). Determination of the shortest path in interval valued fuzzy networks. Int J Math Arch, 3(6), 2377-2384.

[19] Broumi, S., Bakali, A., Talea, M., Smarandache, F., & Vladareanu, L. (2016, November). Computation of shortest path problem in a network with SV-trapezoidal neutrosophic numbers. In 2016 International Conference on Advanced Mechatronic Systems (ICAMechS) (pp. 417-422). IEEE.

[20]Book

Broumi, S., Bakali, A., Talea, M., & Smarandache, F. (2017). Shortest path problem under trapezoidal neutrosophic information. Infinite Study.

[21] Broumi, S., Bakali, A., Talea, M., Smarandache, F., & ALi, M. (2017). Shortest path problem under bipolar neutrosphic setting. In Applied Mechanics and Materials (Vol. 859, pp. 59-66). Trans Tech Publications Ltd.

[22] DeivanayagamPillai, N., Malayalan, L., Broumi, S., Smarandache, F., & Jacob, K. (2020). Application of Floyd's Algorithm in Interval Valued

Neutrosophic Setting. In Neutrosophic Graph Theory and Algorithms (pp. 77-106). IGI Global..

[23] Broumi, S., Dey, A., Talea, M., Bakali, A., Smarandache, F., Nagarajan, D., ... & Kumar, R. (2019). Shortest path problem using Bellman algorithm under neutrosophic environment. Complex & intelligent systems, 5(4), 409-416.

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Generating Neutrosophic Random Variables Following the Poisson Distribution Using the Composition Method
(The Mixed Method of Inverse Transformation Method and Rejection Method)

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Abstract:
Simulation is a numerical technique used to perform tests on a numerical computer, and involves logical and mathematical relationships interacting with each other to describe the behavior and structure of a complex system in the real world over a period of time. Analysis using simulation is a "natural" and logical extension of the mathematical analytical models inherent in operations research, because most operations research methods depend on building mathematical models that closely approximate the real-world environment and we obtain the optimal solution for them using algorithms appropriate to the type of these models. The importance of the simulation process comes In all branches of science, there are many systems that cannot be studied directly, due to the great difficulty that we may encounter when studying, and the high cost, in addition to the fact that some systems cannot be studied directly. The simulation process depends on generating a series of numbers. Randomness subject to a uniform probability distribution over the domain $[0,1]$, then converting these numbers into random variables subject to the law of probability distribution by which the system to be simulated works, using known transformation methods. In previous research, we presented a neutrosophical vision of the reverse transformation method and the method of rejection and acceptance. Which are used to transform random numbers into random variables that follow probability distributions such as: uniform distribution, exponential distribution, beta distribution..., In this research, we present a neutrosophical vision of the Composition method (the mixed method of inverse transformation method and rejection method), used to generate random variables that follow... To some Poisson distribution, the aim is to obtain neutrosophic random variables that we use when simulating systems that operate according to this distribution in order to obtain more accurate simulation results.

key words:
Simulation; neutrosophic logic; generating neutrosophic random numbers; converting neutrosophic random numbers into neutrosophic random variables; synthesis method (mixed method).

Introduction:

To keep pace with the great scientific development that our contemporary world is witnessing, it was necessary to reformulate operations research methods according to the basic concepts of neutrosophic logic, because the margin of freedom enjoyed by neutrosophic values gives more accurate results, which has prompted many researchers to prepare many researches in various fields of science. Especially in the field of mathematics and its applications [1-19], when performing the simulation process for any system according to classical logic, we begin by generating random numbers that follow a regular probability distribution over the domain $[0,1]$ using one of the known methods, and then we convert these random numbers into variables. Randomness follows the probability distribution in which the system to be simulated operates. The simulation process we conduct produces specific results that do not take into account changes that may occur in the system's operating environment. To obtain more accurate results, we have presented, in previous research, a neutrosophical vision of the following topics:

In the paper [20] we generated neutrosophic random numbers that follow a uniform neutrosophic distribution over the domain $[0,1]$.In research [21] we used the inverse transformation method to convert neutrosophic random numbers into neutrosophic random variables that follow a uniform distribution over the domain $[0,1]$.In research [22], we used the inverse transformation method to convert random numbers into random variables that follow the neutrosophic exponential distribution. In research [23] we used the rejection method to transform random numbers into random variables that follow the probability distribution according to which the system to be simulated operates. In the research [24] using the rejection method to generate random numbers that follow the beta distribution.

In this research, we present a neutrosophical study of transforming random numbers into random variables that follow the Poisson distribution using the composition method (The mixed method of inverse transformation method and rejection method), a distribution that has many uses in practical life. Such as inventory control, queueing theory, quality control, traffic flow, and many other fields of management science.

Discussion:

Classic Composition method: [25-26]

The Composition method is based on the inverse transformation method and the rejection and acceptance method is special for generating random variables that follow complex probability distributions.

Using the conditional distribution of the variable x, we assume that $f(x)$ is the law of the probability distribution to be simulated, and that $g(x|y)$ is the conditional distribution of If y belongs to the cumulative distribution $H(y)$ and $P(x,y)$ is the joint distribution of (x,y), then:

$$P(x,y) = h(y)g(x|y)$$

Thus, we find:

$$f(x) = \int_{-\infty}^{+\infty} P(x,y)dy = \int_{-\infty}^{+\infty} h(y)g(x|y)dy$$

When the time periods between possible events are distributed exponentially, the number of events that occur in one period of time has a Poisson distribution given by the following probability density function:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; x = 0,1,2, \dots , \infty$$

Where λ is the number of expected occurrences in one time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda}$. Using the relationship between the exponential distribution and the Poisson distribution, we can generate random variables that follow the Poisson distribution.

The neutrosophic vision of the method of installation:

The mixed method is based on the neutrosophic countertransference method [21-22] and the neutrosophic rejection and acceptance method [23].

Using the conditional distribution of the variable x, we assume that $f_N(x)$ is the law of the probability distribution to be simulated, and that $g_N(x|y)$ is the conditional distribution of If y belongs to the cumulative distribution $H_N(y)$ and $P_N(x,y)$ is the joint distribution of (x,y), then:

$$P_N(x,y) = h_N(y)g_N(x|y)$$

Thus, we find:

$$f_N(x) = \int_{-\infty}^{+\infty} P_N(x,y)dy = \int_{-\infty}^{+\infty} h_N(y)g_N(x|y)dy$$

When the time periods between possible events are distributed exponentially, the number of events that occur in one period of time has a neutrosophic Poisson distribution given by the following probability density function:

Where λ_N is a neutrosophic value from reference [27]. We find that what is meant by neutrosophic data are completely indeterminate values written in the following standard formula $N = a + bI$ where $a\, and\, b$ are real or complex coefficients, a represents the specified part and bI the indeterminate part (indeterminacy). For the number N, it could be $[\lambda_{1\,,}\ \lambda_2]$ or $\{\lambda_{1\,,}\ \lambda_1\}$ or...otherwise it is any set close to the real value a, expressing the number of expected occurrences in One time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda_N}$. Using the relationship between the neutrosophic exponential distribution and the neutrosophic Poisson distribution, we can generate neutrosophic random variables that follow the Poisson distribution.

Here we distinguish three cases:

First case: the random numbers are neutrosophic and the probability distribution is classical.

Second case: classical random numbers and neutrosophic probability distribution.

Third case: neutrosophic random numbers and neutrosophic probability distribution.

We start with the first case; the random numbers are neutrosophic and the probability distribution is classical:

In this case the probability density function of the Poisson distribution takes the following form:

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad ; x = 0,1,2, \dots, \infty$$

Where λ is the number of expected occurrences in one time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda}$, since the random numbers must be neutrosophic to obtain them, we follow the following steps:

a. Using the mean square method given by the following relation:

$$R_{i+1} = Mid[R_i^2] \, ; i = 0,1,2,3, - - \qquad (1)$$

Where Mid symbolizes the middle four ranks of R_i^2, and R_i is chosen, any fractional random number composed of four places (called a seed) and does not contain a zero in any of its four places, [25-26], we generate a series of random numbers that follow the distribution regular over the domain $[0,1]$, we get the following series:

$$R_1, R_2, R_3 \dots R_m, \dots \qquad (2)$$

b. Using the study given in reference [20] we convert these random numbers into neutrosophic random numbers and here we distinguish three forms of the field [0,1] with margin of indeterminacy, in the three forms we have $\varepsilon \in [0, n]$ and $0 < n < 1$

The first form: $[0 + \varepsilon, 1]$ indeterminacy at the minimum of the field we find:

$$R_{Ni} \in \left[R_i, \frac{R_i - n}{1 - n}\right] \quad ; \quad 0 < n < 1$$

It is calculated from the following relation:

$$R_{Ni} = \frac{R_i - \varepsilon}{1 - \varepsilon} \quad ; \quad \varepsilon \in [0, n]$$

The second form: $[0, 1 + \varepsilon]$ indeterminacy at the upper limit of the field we find:

$$R_{Ni} \in \left[R_i, \frac{R_i}{1 + n}\right] \quad ; \quad 0 < n < 1$$

It is calculated from the following relation:

$$R_{Ni} = \frac{R_i}{1 + \varepsilon} \quad ; \quad \varepsilon \in [0, n]$$

The third form: $[0 + \varepsilon, 1 + \varepsilon]$ Indeterminacy at the upper and lower limits of the field we find:

$$R_{Ni} \in [R_i, R_i - n] \quad ; \quad 0 < n < 1$$

It is calculated from the following relation:

$$R_{Ni} = R_i - \varepsilon \ ; \quad \varepsilon \in [0, n]$$

From each of the previous forms we get the following series of neutrosophic random numbers:

$$R_{N1}, R_{N2}, R_{N3} \dots R_{Nm}, \dots \quad (3)$$

c. Using the study mentioned in References [21-22], we convert these neutrosophic random numbers into neutrosophic random numbers that follow the exponential distribution defined by the following relation:

$$h(y) = \lambda. e^{-\lambda y}$$

Using the relation:

$$y_{Ni} = -\frac{1}{\lambda} ln R_{Ni}$$

We obtain the series of neutrosophic random numbers that follow the following exponential distribution:

$$y_{N1}, y_{N2}, y_{N3} \dots y_{Nm}, \dots \quad (4)$$

We apply the accept-reject method given in reference [23]:

We take the cumulative sum of these numbers if the following inequality:

$$\sum_{i=1}^{x} y_{Ni} \leq 1 \pm \varepsilon \leq \sum_{i=1}^{x+1} y_{Ni+1}$$

Then we consider the number x to be subject to the Poisson distribution, where x is the number of random numbers y_{Ni} that are subject to the exponential distribution.

$h(y) = \lambda. e^{-\lambda y}$, the sum of which we took and the number did not exceed $1 \pm \varepsilon$, but if we added another number y_{Ni+1} the sum became greater than $1 \pm \varepsilon$. However, if the inequality is not met, we return to Step (a), we repeat the work until we obtain the required number of random numbers that follow the Poisson distribution.

The second case: classical random numbers and Poisson-Neutrosophic distribution.

The probability density function of the neutrosophic Poisson distribution is defined by the following relationship: [28]

$$f_N(x) = \frac{\lambda_N^x e^{-\lambda_N}}{x!} \quad ; x = 0,1,2, \dots, \infty$$

Where λ_N is a neutrosophic value that expresses the number of expected occurrences in one time, this indicates that the time period between events is exponentially distributed with an average of $\frac{1}{\lambda_N}$.

a. Since classical random numbers follow a uniform distribution over the interval [0,1], we take the sequence of random numbers from relation (2).

b. Using the inverse transformation method, we convert these random numbers into neutrosophic random numbers that follow the neutrosophic exponential distribution defined by the following relation:

$$h_N(y) = \lambda_N . e^{-\lambda_N y}$$

c. Using the conversion relation:

$$y'_{Ni} = -\frac{\ln R_i}{\lambda_N}$$

We obtain the series of neutrosophic random numbers that follow the following exponential distribution:

$$y'_{N1}, y'_{N2}, y'_{N3}, \dots, y'_{Nm}, \dots \qquad (4)$$

We take the cumulative sum of these numbers if the following inequality:

$$\sum_{i=1}^{x} y'_{Ni} \leq 1 \pm \varepsilon \leq \sum_{i=1}^{x+1} y'_{Ni+1}$$

Then we consider the number x to be subject to the Poisson distribution, where x is the number of random numbers y'_{Ni} which is subject to the exponential distribution.

$h(y) = \lambda . e^{-\lambda y}$ whose sum we took and the number did not exceed $1 \pm \varepsilon$, but if we added another number y'_{Ni+1} the sum became greater than $1 \pm \varepsilon$, but if the inequality is not satisfied we return to step (a), repeating the work until we obtain the required number of random numbers that follow the Poisson distribution.

The third case: neutrosophic random numbers and neutrosophic probability distribution.

From the study in the first case, we obtain the series of neutrosophic random numbers as in (3).

Poisson Neutrosophic distribution, i.e., the probability density function is defined as it is in the second case. To convert neutrosophic random numbers $R_{N1}, R_{N2}, R_{N3} \dots R_{Nm}, \dots$, into random numbers that follow the exponential distribution, we use the following relation:

$$y''_{Ni} = -\frac{\ln R_{Ni}}{\lambda_N}$$

We obtain the series of neutrosophic random numbers that follow the following exponential distribution:

$$y''_{N1}, y''_{N2}, y''_{N3}, \dots, y''_{Nm}, \dots \qquad (5)$$

We take the cumulative sum of these numbers if the following inequality:

$$\sum_{i=1}^{x} y''_{Ni} \leq 1 \pm \varepsilon \leq \sum_{i=1}^{x+1} y''_{Ni+1}$$

achieved, then we consider the number x to be subject to the Poisson distribution, where x is the number of random numbers $y^{"}_{Ni}$ that obey the neutrosophic exponential distribution $h_N(y) = \lambda_N . e^{-\lambda_N y}$. The sum of which we took and the number did not exceed $1 \pm \varepsilon$, but if we added another number $y^{"}_{Ni+1}$, the sum became greater than $1 \pm \varepsilon$. However, if the inequality is not met, we return to step (a), and we repeat the work until we obtain the number the required random numbers follow a Poisson distribution.

Conclusion and results:

In this research, we presented a neutrosophic vision of the composition method used to generate random numbers that follow complex probability distributions from simple distributions. Random numbers that follow them can be generated using the neutrosophic inverse transformation or the neutrosophic rejection and acceptance method, using the relations provided by students and researchers in the field of mathematical statistics that link the probability distributions. Complex with simple probability distributions, such as the following relation between the Poisson distribution and the exponential distribution: When the time periods between possible events are exponentially distributed, the number of events that occur in one period of time has a Poisson distribution, which is relied upon to generate neutrosophic random numbers that follow the distribution Poisson, which has many uses in practical life, Such as inventory control, queueing theory, quality control, traffic flow, and many other fields of management science.

References:

1- Florentin Smarandache, Maissam Jdid, On Overview of Neutrosophic and Plithogenic Theories and Applications, **Doi :**https://doi.org/10.54216/PAMDA.020102.

2- Florentin Smarandache, Maissam Jdid, Research on the topics of neutrosophic operations research, Volume (1), https://fs.unm.edu/NeutrosophicOperationsResearch.pdf, Biblio Publishing, 2023.

3- Abdulrahman Astambli, Mohamed Bisher Zeina and Yasin Karmouta, On Some Estimation Methods of Neutrosophic Continuous Probability Distributions Using One-Dimensional AH-Isometry, Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 641-652. DOI: 10.5281/zenodo.7536101

4- Yaser Ahmad Alhasan and Raja Abdullah Abdulfatah, Division of refined neutrosophic numbers Neutrosophic Sets and Systems, Vol. 60, 2023, pp. 1-5. DOI: 10.5281/zenodo.10224078

5- Maissam Jdid, NEUTROSOPHIC TRANSPORT AND ASSIGNMENT ISSUES, Publisher: Global Knowledge's, ISBN, 978_1 _59973_770_6, (Arabic version). USA,2023.

6- Marwah Yahya Mustafa , Zakariya Yahya Algamal, Neutrosophic inverse power Lindley distribution: A modeling and application for bladder cancer patients, **Doi :**https://doi.org/10.54216/IJNS.210218

7- Renee Miriam M., Nivetha Martin , Aleeswari A. , Said Broumi, Rework Warehouse Inventory Model for Product Distribution with Quality Conservation in Neutrosophic Environment, **Doi :**https://doi.org/10.54216/IJNS.210215
8- Maissam Jdid, Neutrosophic linear models and algorithms to find their optimal solution, Biblio Publishing, ISBN, 978_1 _59973_778_2, (Arabic version). USA,2023.
9- Maissam Jdid, Florentin Smarandache, A Study of Systems of Neutrosophic Linear Equations, https://doi.org/10.54216/IJNS.230202.
10- Maissam Jdid, Florentin Smarandache, Neutrosophic Treatment of the Modified Simplex Algorithm to find the Optimal Solution for Linear Models https://doi.org/10.54216/IJNS.230110.
11- Maissam Jdid, Basel Shahin , Fatima Al Suleiman ,Important Neutrosophic Rules for Decision-Making in the Case of Uncertain Data , https://doi.org/10.54216/IJNS.1803014
12- Maissam Jdid, Hla Hasan, The State of Risk and Optimum Decision According to Neutrosophic Rules ,https://doi.org/10.54216/IJNS.200107
13- Zahraa A. Khalaf, Fatimah M. Mohammed, Weakly Generalized M-Closed and Strongly M-Generalized Closed Sets in Fuzzy Neutrosophic Topological Spaces, **Doi :**https://doi.org/10.54216/IJNS.210201
14- Hamiyet Merkepci , Ahmed Hatip, Algorithms for Computing Pythagoras Triples and 4-Tiples in Some Neutrosophic Commutative Rings, **Doi :**https://doi.org/10.54216/IJNS.200310
15- Najla M. Alnaqbi, Samira A. Alnuaimi , M. Elhoseny, A Neutrosophic AHP Analysis for Using Video Conferences in Smart Learning: A Systematic Review, **Doi :**https://doi.org/10.54216/IJNS.200307
16- Noura Metawa , Rhada Boujlil , Maha Metawea, Multi-Valued Neutrosophic Sets for Forecasting Cryptocurrency Volatility, **Doi:**https://doi.org/10.54216/IJNS.200306
17- Faisal Al-Sharqi , Abd Ghafur Ahmad , Ashraf Al-Quran, Mapping on Interval Complex Neutrosophic Soft Sets, **Doi :**https://doi.org/10.54216/IJNS.190406
18- Nahia Mourad, Assessment of Structural Cracks in Aircraft Using a Decision-Making Approach Based on Enhanced Entropy and Single-Valued Neutrosophic Sets, **Doi :**https://doi.org/10.54216/IJNS.190404
19- Maissam Jdid, Studying Transport Models with the Shortest Time According to the Neutrosophic Logic, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 631-638. DOI: 10.5281/zenodo.8404545
20- Maissam Jdid, Rafif Alhabib and A. A. Salama, Fundamentals of Neutrosophical Simulation for Generating Random Numbers Associated with Uniform Probability Distribution, Neutrosophic Sets and Systems, Vol. 49, 2022, pp. 92-102. DOI: 10.5281/zenodo.6426375
21- Maissam Jdid, A. Salama, Using the Inverse Transformation Method to Generate Random Variables that follow the Neutrosophic Uniform Probability Distribution, **Doi :**https://doi.org/10.54216/JNFS.060202

22- Maissam Jdid, Rafif Alhabib and A. A. Salama, The Basics of Neutrosophic Simulation for Converting Random Numbers Associated with a Uniform Probability Distribution into Random Variables Follow an Exponential Distribution, Neutrosophic Sets and Systems, Vol. 53, 2023, pp. 358-366. DOI: 10.5281/zenodo.7536049
23- Maissam Jdid, Said Broumi, Neutrosophical Rejection and Acceptance Method for the Generation of Random Variables, Neutrosophic Sets and Systems, Vol. 56, 2023, pp. 153-166. DOI: 10.5281/zenodo.8194749
24- Maissam Jdid and Nada A. Nabeeh, Generating Random Variables that follow the Beta Distribution Using the Neutrosophic Acceptance-Rejection Method, Neutrosophic Sets and Systems, Vol. 58, 2023, pp. 139-148. DOI: 10.5281/zenodo.8404445
25- Bugaha J.S, Mualla.W, and others -Operations Research Translator into Arabic, The Arab Center for Arabization, Translation, Authoring and Publishing, Damascus,1998. (Arabic version).
26- Alali. Ibrahim Muhammad, Operations Research. Tishreen University Publications, 2004. (Arabic version).
27- Smarandache, F, Neutrosophy and Neutrosophic Logic, First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM 87301, USA,2002.

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Edge Regular Complex Neutrosophic Graph Structure and it is Application

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Abstract. A modified version of a Neutrosophic Set (NS), a Complex Neutrosophic Set (CNS) offers a more accurate description of ambiguous situations than established fuzzy sets (FSs). It is widely applied in uncertain control. This study offers the idea of Single-Valued Complex Neutrophilic Graph Structure (SVCNGS). Further research is done into the relationship between an $\eta_J - edge$ regular SVCNGS degree and the η_J-degree of a vertex. Also, we introduce the notions of totally $\eta_J - edge$ regular and regular $\eta_J - edge$ SVCNGS. There is an explanation of the conditions in which $\eta_J - edge$ regular SVCNGS and totally $\eta_J - edge$ regular SVCNGS are same. Moreover, this study several $\eta_J - edge$ regular and totally $\eta_J - edge$ regular SVCNGS properties using an example, and we have discussed their application in SVCNGS. Finally, we develop an algorithm that explains the fundamental workings of our application.

Keywords: SVCNGS, $\eta_J - edge$ regular, totally $\eta_J - edge$ regular, application

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1. **Introduction**

The phrase FSs it initially used by L.A. Zadeh [48] in 1965 to describe a way to show the ambiguity of FSs. The business sector is vital to our daily lives because it helps us see ambiguities and identify them in most fields of science and medicine. T. Atanassov [4] suggested that Intuitionistic FSs (IFSs) may be created by deriving a new component, degree of membership and non-membership, based on the features of the FS. As a result, it can explain more accurately and completely than a FS. However, it can only handle partial and ambiguous information; it cannot manage the ambiguous and contradictory informationthat frequently occurs in real-life situations. It can only handle partial and ambiguous information, not the ambiguous and indeterminate information that often occurs in real-life situations. Therefore, the terms NS, a unifying field in logics and a generalization of the IFSs are introduced by F. Smarandache [[27], [28], [29], [30], [31]] and is used in many different fields to deal with ambiguous and contradictory data. If the total of these values in the NS is between 0 and 3, the terms of truth membership, indeterminacy membership, and false membership are all done separately, and the indeterminacy value is directly quantified. Neutrosophy: Neutral Probability, Neutral Set, and Neutral Logic Introduce the idea of NS, N probability, and logic in more detail. Due to the broad range of description situations it covers, the NS has quickly drawn the attention of many scholars. This new set also helps to manage the vagueness brought on by the N scope. Furthermore, a thorough evaluation of Xindong Peng and Jingguo Dai [40] citation is provided. A bibliometric analysis of the neutrosophic collection is presented, covering the period from 1998 to 2017. Ramot [18] created the idea of a Complex FS (CFS) in 2012 by changing the range of the membership function for the amount disc for complex and real integers. A helpful generalisation of FS is the membership grade of this concept, which is expressed as $re^{i\theta}$, where r stands for the amplitude term and θ for the phase term. Only values from the complex plane's unit circle are permitted. The phase term of CFS matters because it can handle cyclical problems or persistently troubling circumstances more successful. Given that this term is a part of CFS, there will undoubtedly be circumstances in which another dimension is required. In contrast to every other type of information that is currently available, CFS is described in this phrase. A detailed investigation of CFS's [43] was performed by Yazdanbakhsh and Dick. Alkouri and Salleh [2] first introduced the ideas of CIFSs in 2012. It is important to familiarize out with the novel forms, such as CIFSs, which significantly expand upon CFSs; useful details regarding these kinds of structures can be discovered in [[19], [20]]. Recently, Prem Kumar Singh developed the equation of complex vague set idea lattice and its features in his paper [16]. K. Ullah and T. Mahmood [39] presented the concept of CPFSs in 2019 in addition to expanding the range of existing distance measures to take into consideration CPF values. Mumtaz Ali and Florentin Smarandache developed the concept of a CNS in 2016 [32]. A complex-valued NS is one whose real-valued amplitude terms for truth, indeterminacy, and falsehood, along with the phase terms that go along with them, are combined to form its complex-valued membership functions. The NS is expanded upon by the CNS. Further, the establishment of Hypersoft Set Hybrids with CFS, CIFS, and CNS are introduced in 2020 by Atiqe U. R., Muhammad.S, Florentin Smarandache, and Muhammad R. A [6]. In 1975, Rosenfeld [21] developed fuzzy graph theory. Examined the Fuzzy Graphs (FG) for which Kauffmann created the fundamental concept in 1973. He explored a number of basic concepts in graph theory and developed some of their characteristics. In his remarks on FGs, Bhattacharya [7] demonstrated that the conclusions drawn from (crisp) graph theory are not necessarily applicable to FGs. In 1994, Shannon and Atanassov proposed the ideas of IF relations and IFGs. Rashmanlou [15] studied FGs with irregular interval values. Additionally, they defined FGs [17], various features of very irregular interval-valued FGs [17]. The Edge Regular IFG was first proposed by M.G. Karunambigai and K. Palanivel [10] in 2015. CFGs were developed by Thirunavukarasu et al. [38] to manage ambiguous and uncertain relationships with periodic nature. CIFGs were defined by Yaqoob et al. [44]. They looked into the homomorphisms of CIFG and demonstrated a CIFG usage among cellular network supplier companies to test their proposed approach. CNGs were introduced by Yaqoob and Akram to expand the idea of NGs and CIFGs [45]. They addressed various fundamental CNG operations and described them using specific instances. They also demonstrated CNGs' energy. Anam Luqman, Muhammad Akram, and Florentin Smarandache [1] further elaborate on the idea of CN Hypergraphs: New Social Network Models in 2019. Two voting processes are the best instances and source of inspiration for CNS and the example is provided in their introduction to prove the applicability of their suggested model. The research papers Applications of graph's total degree with bipolar fuzzy information and Estimation of most effected cycles and busiest network route based on complexity function of graph in fuzzy environment in 2022 by Soumitra Poulik and Ganesh Ghorai [[33], [34], [35]] is worth being referred to for more information. Also, in 2021 proposed the idea of Determining the order of journeys based on a graph's Wiener absolute index using bipolar fuzzy information. Sampathkumar [23] introduced Graph Structures (GSs) in 2006 to be a generalization of signed graphs and graphs with colored or labeled edges. The idea of a FGS was first presented by Dinesh [8], and also discussed some relevant properties. Recently, the notions of Operations on IFGSs were introduced by Muhammad Akram [[12], [13], [14]]. Also, introduce the ideas of simplified Interval-Valued PFGs with applications and a novel decision-making approach under CPF environments further. Later, the idea of complex Pythagorean fuzzy planar graphs was created.

1.1. *Framework of this research*

This concept can be restated in an abstract form then applied in SVCNGS. The organization of this work is as follows:

- This study introduces the idea of SVCNGS. In regular SVCNGS, the relationship between vertex degree and edge degree is further investigated.
- We also define total $\eta_J - edge$ regular SVCNGS and $\eta_J - edge$ regular SVCNGS. It is described under which conditions $\eta_J - edge$ regular SVCNGS and total $\eta_J - edge$ regular SVCNGS are comparable.

- Furthermore, Applications and algorithm explaining for SVCNGS were also covered. Finally, an explanation of all these studies is provided in Conclusion and future works.

2. Preliminaries

The construction of the research studies will be aided by the discussion of some fundamental definitions and properties in this section.

Definition 2.1. [32] An object with the form of a SVCNS Q on a non-void set X

$$Q = \{j, T_Q(j)e^{i\alpha_Q(j)}, I_Q(j)e^{i\beta_Q(j)}, F_Q(j)e^{i\gamma_Q(j)} : j \in X\}$$

where $i = \sqrt{-1}$, amplitude terms $T_Q(j), I_Q(j), F_Q(j) \in [0,1]$ and phase terms $\alpha_Q(j), \beta_Q(j), \gamma_Q(j) \in [0, 2\pi]$.

Definition 2.2. [39] Let $\chi = \{j, T_\chi(j)e^{i\alpha_\chi(j)}, I_\chi(j)e^{i\beta_\chi(j)}, F_\chi(j)e^{i\gamma_\chi(j)} : j \in X\}, \eta = \{j, T_\eta(j)e^{i\alpha_\eta(j)}, I_\eta(j)e^{i\beta_\eta(j)}, F_\eta(j)e^{i\gamma_\eta(j)} : j \in X\}$ be the two SVCNS in X, then

- $\chi \subseteq \eta$ if and only if $T_\chi(j) \leq T_\eta(j),\ I_\chi(j) \leq I_\eta(j)$ and $F_\chi(j) \leq F_\eta(j)$ for amplitude terms and $\alpha_\chi(j) \leq \alpha_\eta(j),\ \beta_\chi(j) \leq \beta_\eta(j)$ and $\gamma_\chi(j) \leq \gamma_\eta(j)$ for phase terms, for all $j \in X$;
- $\chi = \eta$ if and only if $T_\chi(j) = T_\eta(j),\ I_\chi(j) = I_\eta(j)$ and $F_\chi(j) = F_\eta(j)$ for amplitude terms and $\alpha_\chi(j) = \alpha_\eta(j),\ \beta_\chi(j) = \beta_\eta(j)$ and $\gamma_\chi(j) = \gamma_\eta(j)$ for phase terms, for all $j \in X$;

For simplicity, the $(j, T(j)e^{i\alpha(j)}, I(j)e^{i\beta(j)}, F(j)e^{i\gamma(j)} : j \in X)$ is called the SVCN Number (SVCNN), where $T(j), I(j), F(j) \in [0,1]$ such that $0 \leq T(j) + I(j) + F(j) \leq 3$ and $\alpha, \beta, \gamma \in [0, 2\pi]$.

Definition 2.3. [13] On a non-empty set X, a SVCNG is a pair $G = (\chi, \eta)$, where χ and η are SVCNSs on X and a SVCN relation on X, respectively, such that:

$$\begin{aligned} T_\eta(rs)e^{i\alpha_\eta(rs)} &\leq \min\{T_\chi(r), T_\chi(s)\}e^{i\min\{\alpha_\chi(r),\alpha_\chi(s)\}} \\ I_\eta(rs)e^{i\beta_\eta(rs)} &\leq \max\{I_\chi(r), I_\chi(s)\}e^{i\max\{\beta_\chi(r),\beta_\chi(s)\}} \\ F_\eta(rs)e^{i\gamma_\eta(rs)} &\leq \max\{F_\chi(r), F_\chi(s)\}e^{i\max\{\gamma_\chi(r),\gamma_\chi(s)\}} \end{aligned}$$

$0 \leq T_\eta(rs) + I_\eta(rs) + F_\eta(rs) \leq 3$ for all $r, s \in X$. We call χ and η the SVCN vertex set and the SVCN edge set of G, respectively.

3. Some Result on SVCNGS

The concept of SVCNGS is introduced in this section, along with definitions that are useful in understanding the main findings. With examples, we further analyse several SVCNGS characteristics.

Definition 3.1. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ is referred to as an SVCNGS of GS $\tau^* = (M, W_1, W_2, ..., W_k)$ if $\gamma = \{r, \gamma_1(r)e^{i\alpha_1(r)}, \gamma_2(r)e^{i\alpha_2(r)}, \gamma_3(r)e^{i\alpha_3(r)}\}$ is an SVCN set on Q and $\eta_J = \{rs, \eta_{1J}(rs)e^{i\beta_{1J}(rs)}, \eta_{2J}(rs)e^{i\beta_{2J}(rs)}, \eta_{3J}(rs)e^{i\beta_{3J}(rs)}\}$ are SVCN sets on M and W_J such that

$$\begin{aligned}
\eta_{1J}(r,s)e^{i\beta_{1J}(rs)} &\leq \min\{\gamma_1(r), \gamma_1(s)\}e^{i\min\{\alpha_1(r),\alpha_1(s)\}},\\
\eta_{2J}(r,s)e^{i\beta_{2J}(rs)} &\leq \max\{\gamma_2(r), \gamma_2(s)\}e^{i\max\{\alpha_2(r),\alpha_2(s)\}},\\
\eta_{3J}(r,s)e^{i\beta_{3J}(rs)} &\leq \max\{\gamma_3(r), \gamma_3(s)\}e^{i\max\{\alpha_3(r),\alpha_3(s)\}}
\end{aligned}$$

such that $0 \leq \eta_{1J}(r,s) + \eta_{2J}(r,s) + \eta_{3J}(r,s) \leq 3$ and $\beta_{1J}(rs), \beta_{2J}(rs), \beta_{3J}(rs) \in [0, 2\pi]$ for all $(r,s) \in R_J, J = 1, 2, ..., k$.

Example 3.2. An SVCNGS $\tau = (\gamma, \eta_1, \eta_2)$ of a GS $\tau^* = (M, W_1, W_2)$ given figure-1 is a SVCNGS $\tau = (\gamma, \eta_1, \eta_2)$ such that $\gamma = \{u_1(.4e^{i1.6\pi}, .6e^{i1.2\pi}, .3e^{i1.4\pi})$, $u_2(.5e^{i1.0\pi}, .6e^{i.8\pi}, .4e^{i1.6\pi}),\ u_3(.5e^{i.8\pi}, .4e^{i1.0\pi}, .6e^{i1.4\pi}),\ u_4(.3e^{i.6\pi}, .5e^{i1.8\pi}, .4e^{i1.6\pi})\}$.

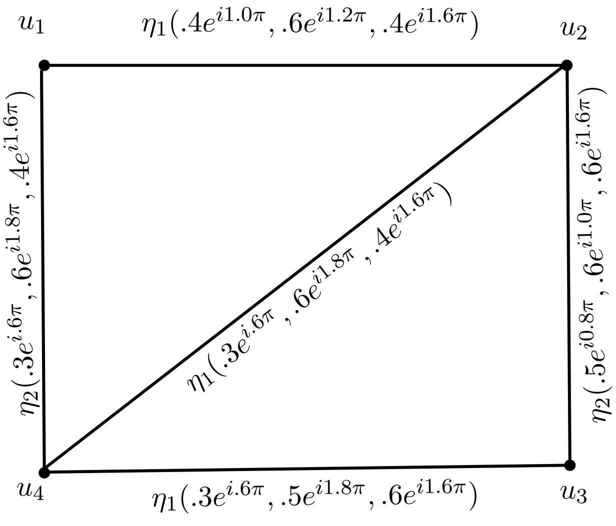

FIGURE 1. $\tau = (\gamma, \eta_1, \eta_2)$ is SVCNGS of τ^*

Definition 3.3. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then the vertex $\eta_J - degree$ is defined as

$$\begin{aligned}
&d_{\eta_J}(f) = (d_{\eta_{1J}}(f), d_{\eta_{2J}}(f), d_{\eta_{3J}}(f)),\\
&d_{\eta_{1J}}(f) = \sum_{(f,v)\in W_J} \eta_{1J}(f,v)e^{i\sum_{(f,v)\in R_J}\beta_{1J}(f,v)},\\
&d_{\eta_{2J}}(f) = \sum_{(f,v)\in W_J} \eta_{2J}(f,v)e^{i\sum_{(f,v)\in R_J}\beta_{2J}(f,v)},\\
&d_{\eta_{3J}}(f) = \sum_{(f,v)\in W_J} \eta_{3J}(f,v)e^{i\sum_{(f,v)\in W_J}\beta_{3J}(f,v)},\\
&\forall J = 1, 2, ..., k.
\end{aligned}$$

Definition 3.4. A SVCNGS $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ is $\eta_J - strong$ if

$$\begin{aligned}
\eta_{1J}(r,s)e^{i\beta_{1J}(rs)} &= \min\{\gamma_1(r), \gamma_1(s)\}e^{i\min\{\alpha_1(r),\alpha_1(s)\}},\\
\eta_{2J}(r,s)e^{i\beta_{2J}(rs)} &= \max\{\gamma_2(r), \gamma_2(s)\}e^{i\max\{\alpha_2(r),\alpha_2(s)\}},\\
\eta_{3J}(r,s)e^{i\beta_{3J}(rs)} &= \max\{\gamma_3(r), \gamma_3(s)\}e^{i\max\{\alpha_3(r),\alpha_3(s)\}} \text{ for all } (r,s) \in W_J, J = 1, 2, ..., k.
\end{aligned}$$

Example 3.5. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS. Next, for every $J = 1, 2$, the degree of a $\eta_J - strong$ vertex is shown in figure-1. The $\eta_1 - strong$ degree of vertex u_i, i=1,2,3,4 is

$$\begin{aligned}
d_{\eta_1}(u_1) &= (d_{\eta_{11}}(u_1), d_{\eta_{21}}(u_1), d_{\eta_{31}}(u_1))\\
d_{\eta_1}(u_1) &= (.4e^{i1.0\pi}, .6e^{i1.2\pi}, .4e^{i1.6\pi}),\\
d_{\eta_1}(u_2) &= (.7e^{i1.6\pi}, 1.2e^{i3.0\pi}, .8e^{i3.2\pi}),\\
d_{\eta_1}(u_3) &= (.3e^{i.6\pi}, .5e^{i1.8\pi}, .6e^{i1.6\pi}),\\
d_{\eta_1}(u_4) &= (.6e^{i1.2\pi}, 1.1e^{i3.6\pi}, 1.0e^{i3.2\pi})
\end{aligned}$$

The $\eta_2 - strong$ degree of vertex u_i, i=1,2,3,4 is

$$\begin{aligned}
d_{\eta_2}(u_1) &= (d_{\eta_{12}}(u_1), d_{\eta_{22}}(u_1), d_{\eta_{32}}(u_1))\\
d_{\eta_2}(u_1) &= (.3e^{i.6\pi}, .6e^{i1.8\pi}, .4e^{i1.6\pi}),\\
d_{\eta_2}(u_2) &= (.5e^{i.8\pi}, .6e^{i1.0\pi}, .6e^{i1.6\pi}),\\
d_{\eta_2}(u_3) &= (.5e^{i.8\pi}, .6e^{i1.0\pi}, .6e^{i1.6\pi}),\\
d_{\eta_2}(u_4) &= (.3e^{i.6\pi}, .6e^{i1.8\pi}, .4e^{i1.6\pi})
\end{aligned}$$

Theorem 3.6. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then $\sum_{i=1}^{n} d_{\eta_J}(u_i) = (2\sum_{(u_i,v)\in W_J} \eta_{1J}(u_i,v)e^{i2\sum_{(u_i,v)\in W_J}\beta_{1J}(u_i,v)}, 2\sum_{(u_i,v)\in W_J} \eta_{2J}(u_i,v)$ $e^{i2\sum_{(u_i,v)\in W_J}\beta_{2J}(u_i,v)}, 2\sum_{(u_i,v)\in W_J} \eta_{3J}(u_i,v)e^{i2\sum_{(u_i,v)\in W_J}\beta_{3J}(u_i,v)})$ is $\eta_J - strong$ SVCNGS for all $J = 1, 2, ..., k$.*

Example 3.7. Next, we demonstrate the above theorem's example - 3.6. Let us Consider a SVCNGS $\tau = (\gamma, \eta_1, \eta_2)$ as shown in figure-1. Then $\sum_{i=1}^{4} d_{\eta_J}(u_i) =$
$(2\sum_{(u_i,v)\in W_J} \eta_{1J}(u_i,v)e^{i2\sum_{(u_i,v)\in W_J}\beta_{1J}(u_i,v)}, 2\sum_{(u_i,v)\in W_J} \eta 2J(u_i,v)e^{i2\sum_{(u_i,v)\in W_J}\beta_{2J}(u_i,v)},$
$2\sum_{(u_i,v)\in W_J} \eta_{3J}(u_i,v)e^{i2\sum_{(u_i,v)\in W_J}\beta_{3J}(u_i,v)})$ is $\eta_J - strong$ SVCNGS for all $J = 1, 2$.
Twice the degree of sum of $\eta_1 - edges$ in τ is

$$\begin{aligned}
2\sum_{(u_i,v)\in W_1} \eta_{11}(u_i,v)e^{i2\sum_{(u_i,v)\in W_1}\beta_{11}(u_i,v)} &= 2(\eta_{11}(u_1,u_2) + \eta_{11}(u_2,u_4) + \eta_{11}(u_3,u_4)) \\
&\quad e^{i2(\beta_{11}(u_1,u_2)+\beta_{11}(u_2,u_4)+\beta_{11}(u_3,u_4))} \\
&= 2(.4 + .3 + .3)e^{i2(1.0\pi+.6\pi+.6\pi)} \\
&= 2.0e^{i4.4\pi}
\end{aligned}$$

$$\begin{aligned}
2\sum_{(u_i,v)\in W_1} \eta_{21}(u_i,v)e^{i2\sum_{(u_i,v)\in W_1}\beta_{21}(u_i,v)} &= 2(\eta_{21}(u_1,u_2) + \eta_{21}(u_2,u_4) + \eta_{21}(u_3,u_4)) \\
&\quad e^{i2(\beta_{21}(u_1,u_2)+\beta_{21}(u_2,u_4)+\beta_{21}(u_3,u_4))} \\
&= 2(.6 + .6 + .5)e^{i2(1.2\pi+1.8\pi+1.8\pi)} \\
&= 3.4e^{i9.6\pi}
\end{aligned}$$

$$\begin{aligned}
2\sum_{(u_i,v)\in W_1} \eta_{31}(u_i,v)e^{i2\sum_{(u_i,v)\in W_1}\beta_{31}(u_i,v)} &= 2(\eta_{31}(u_1,u_2) + \eta_{31}(u_2,u_4) + \eta_{31}(u_3,u_4)) \\
&\quad e^{i2(\beta_{31}(u_1,u_2)+\beta_{31}(u_2,u_4)+\beta_{31}(u_3,u_4))} \\
&= 2(.4 + .4 + .6)e^{i2(1.6\pi+1.6\pi+1.6\pi)} \\
&= 2.8e^{i9.6\pi}
\end{aligned}$$

Degree of $\eta_1 - strong$ vertices in SVCNGS is given by Example-3.5.

$$\begin{aligned}
\sum_{i=1}^{4} d_{\eta_1}(u_i) &= (\sum_{i=1}^{4} d_{\eta_{1J}}(u_i), \sum_{i=1}^{4} d_{\eta_{2J}}(u_i), \sum_{i=1}^{4} d_{\eta_{3J}}(u_i)) \\
&= (2.0e^{i4.4\pi}, 3.4e^{i9.6\pi}, 2.8e^{i9.6\pi})
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{4} d_{\eta_1}(u_i) &= (2\sum_{(u_i,v)\in W_1} \eta_{11}(u_i,v)e^{i2\sum_{(u_i,v)\in W_1}\beta_{11}(u_i,v)}, 2\sum_{(u_i,v)\in W_1} \eta_{21}(u_i,v) \\
&\quad e^{i2\sum_{(u_i,v)\in W_1}\beta_{21}(u_i,v)}, 2\sum_{(u_i,v)\in W_1} \lambda_{31}(u_i,v)e^{i2\sum_{(u_i,v)\in W_1}\beta_{31}(u_i,v)})
\end{aligned}$$

Similarly, we calculate

$$\begin{aligned}\sum_{i=1}^{4} d_{\eta_2}(u_i) &= (2\sum_{(u_i,v)\in W_2}\eta_{12}(u_i,v)e^{i2\sum_{(u_i,v)\in W_2}\beta_{12}(u_i,v)}, 2\sum_{(u_i,v)\in W_2}\eta_{22}(u_i,v)\\ &\quad e^{i2\sum_{(u_i,v)\in W_2}\beta_{22}(u_i,v)}, 2\sum_{(u_i,v)\in W_2}\eta_{32}(u_i,v)e^{i2\sum_{(u_i,v)\in W_2}\beta_{32}(u_i,v)})\\ &= (1.6e^{i2.8\pi}, 2.4e^{i5.6\pi}, 2.0e^{i6.4\pi})\end{aligned}$$

Definition 3.8. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. If $d_{\eta_J}(u_i) = (a, b, c)$ for all $u_i \in Q$, then for every vertex with a degree of $\eta_{1J} - degree$, there is an equal degree of a; similarly, for every vertex with a degree of $\eta_{2J} - degree$, there is an equal degree of b; and for every vertex with a degree of $\eta_{3J} - degree$, there is an equal degree of c. For all $J = 1, 2, ..., k$, τ is then considered to be η_J regular SVCNGS.

Definition 3.9. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. The total degree of η_J vertex is defined as

$$td_{\eta_J}(f) = (td_{\eta_{1J}}(f), td_{\eta_{2J}}(f), td_{\eta_{3J}}(f))$$

$$\begin{aligned} td_{\eta_{1J}}(f) &= (\sum_{(f,v)\in W_J}\eta_{1J}(f,v) + \gamma_1(f))e^{i\sum_{(f,v)\in W_J}\beta_{1J}(f,v)+\alpha_1(f)},\\ td_{\eta_{2J}}(f) &= (\sum_{(f,v)\in W_J}\eta_{2J}(f,v) + \gamma_2(f))e^{i\sum_{(f,v)\in W_J}\beta_{2J}(f,v)+\alpha_2(f)},\\ td_{\eta_{3J}}(f) &= (\sum_{(f,v)\in W_J}\eta_{3J}(f,v) + \gamma_3(f))e^{i\sum_{(f,v)\in W_J}\beta_{3J}(f,v)+\alpha_3(f)}\end{aligned}$$

The total degree of every vertex in η_{1J} has the same degree. n_1 and the total degree of each vertex in η_{2J} has the same degree. n_2, and the total degree of each vertex in η_{3J} has the same degree n_3. For all $J = 1, 2, ..., k$, τ is then considered to be totally η_J regular SVCNGS.

4. **Edge Regular in SVCNGS**

This section introduces the idea of $\eta_J - edge$ regular SVCNGS. Moreover, some properties of the $\eta_J - edge$ regular SVCNGS are explained with examples.

Definition 4.1. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$ and let $e_{ij} \in W_J$ be an edge in τ. Then the degree of an $\eta_J - edge$ $e_{ij} \in W_J$ is defined as

$$d_{\eta_J}(e_{ij}) = (d_{\eta_{1J}}(e_{ij}), d_{\eta_{2J}}(e_{ij}), d_{\eta_{3J}}(e_{ij}))$$

$$\begin{aligned} d_{\eta_{1J}}(e_{ij}) &= d_{\eta_{1J}}(u_i) + d_{\eta_{1J}}(u_j) - 2\eta_{1J}(u_i,u_j)e^{i2\beta_{1J}(u_i,u_j)} \ (or) \\ d_{\eta_{1J}}(e_{ij}) &= (\sum_{\ell^r}\eta_{1J}(u_i,u_k) + \sum_{\ell^s}\eta_{1J}(u_k,u_j)) \\ &\quad e^{i\sum_{\ell^r}\beta_{1J}(u_i,u_k)+\sum_{\ell^s}\beta_{1J}(u_k,u_j)} \end{aligned}$$

$$\begin{aligned} d_{\eta_{2J}}(e_{ij}) &= (d_{\eta_{2J}}(u_i) + d_{\eta_{2J}}(u_j) - 2\eta_{2J}(u_i,u_j)e^{i2\beta_{2J}(u_i,u_j)} \ (or) \\ d_{\eta_{2J}}(e_{ij}) &= \sum_{\ell^r}\eta_{2J}(u_i,u_k) + \sum_{\ell^s}\eta_{2J}(u_k,u_j)) \\ &\quad e^{i\sum_{\ell^r}\beta_{2J}(u_i,u_k)+\sum_{\ell^s}\beta_{2J}(u_k,u_j)} \end{aligned}$$

$$\begin{aligned} d_{\eta_{3J}}(e_{ij}) &= (d_{\eta_{3J}}(u_i) + d_{\eta_{3J}}(u_j) - 2\eta_{3J}(u_i,u_j)e^{i2\beta_{3J}(u_i,u_j)} \ (or) \\ d_{\eta_{3J}}(e_{ij}) &= (\sum_{\ell^r}\eta_{3J}(u_i,u_k) + \sum_{\ell^s}\eta_{3J}(u_k,u_j)) \\ &\quad e^{i\sum_{\ell^r}\beta_{3J}(u_i,u_k)+\sum_{\ell^s}\beta_{3J}(u_k,u_j)}, \\ &\quad \forall\ \ell^r = (u_i,u_k) \in W_J, k \neq j, \ell^s = (u_k,u_j) \in W_J, k \neq i \text{and } J = 1,2,...,k. \end{aligned}$$

Notation: An $\eta_J - edge$ of an SVCNGS is denoted by $e_{ij} \in W_J$ or $u_iu_j \in W_J$.

Note:

$$\begin{aligned} d_{\eta_{lJ}}(e_{ij}) &= (\sum_{(u_i,u_j)\in W_J}\eta_{1J}(u_i,u_j) + \sum_{(u_j,u_k)\in W_J}\eta_{1J}(u_j,u_k) - 2\eta_{1J}(u_i,u_j)) \\ &\quad e^{i\sum_{(u_i,u_j)\in W_J}\beta_{lJ}(u_i,u_j)+\sum_{(u_j,u_k)\in W_J}\beta_{1J}(u_j,u_k)-2\beta_{1J}(u_i,u_j)},\ l = 1,2,3. \text{ and } J = 1,2,...,k. \end{aligned}$$

Definition 4.2. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. The minimum $\eta_J - edge$ degree of τ is $\delta_{\eta_J}(G) = (\delta_{\eta_{1J}}(G), \delta_{\eta_{2J}}(G), \delta_{\eta_{3J}}(G))$, where

$$\begin{aligned} \delta_{\eta_{1J}}(G) &= \wedge\{d_{\eta_{1J}}(e_{ij})/e_{ij} \in W_J\} \\ \delta_{\eta_{2J}}(G) &= \wedge\{d_{\eta_{2J}}(e_{ij})/e_{ij} \in W_J\} \\ \delta_{\eta_{3J}}(G) &= \wedge\{d_{\eta_{3J}}(e_{ij})/e_{ij} \in W_J\},\ \forall\ J = 1,2,...k. \end{aligned}$$

The maximum $\eta_J - edge$ degree of τ is $\Delta_{\eta_J}(G) = (\Delta_{\eta_{1J}}(G), \Delta_{\eta_{2J}}(G), \Delta_{\eta_{3J}}(G))$, where

$$\begin{aligned} \Delta_{\eta_{1J}}(G) &= \vee\{d_{\eta_{1J}}(e_{ij})/e_{ij} \in W_J\} \\ \Delta_{\eta_{2J}}(G) &= \vee\{d_{\eta_{2J}}(e_{ij})/e_{ij} \in W_J\} \\ \Delta_{\eta_{3J}}(G) &= \vee\{d_{\eta_{3J}}(e_{ij})/e_{ij} \in W_J\},\ \forall\ J = 1,2,...k. \end{aligned}$$

Definition 4.3. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $G^* = \{M, W_1, W_2, ..., W_k\}$ and let $e_{ij} \in W_J$ be an edge in τ. Then the total degree of an $\eta_J - edge$ $e_{ij} \in W_J$ is defined as

$$td_{\eta_J}(e_ij) = (td_{\eta_{1J}}(e_{ij}), td_{\eta_{2J}}(e_{ij}), td_{\eta_{3J}}(e_{ij})),$$

$$
\begin{aligned}
td_{\eta_{1J}}(e_{ij}) &= \sum_{\ell^r} \eta_{1J}(u_i,u_k) + \sum_{\ell^s} \eta_{1J}(u_k,u_j) + \eta_{1J}(e_{ij}) \\
&\quad e^{i\sum_{\ell^r}\beta_{1J}(u_i,u_k)+\sum_{\ell^s}\beta_{1J}(u_k,u_j)+\beta_{1J}(e_{ij})}, \\
td_{\eta_{2J}}(e_{ij}) &= \sum_{\ell^r} \eta_{2J}(u_i,u_k) + \sum_{\ell^s} \eta_{2J}(u_k,u_j) + \eta_{2J}(e_{ij}) \\
&\quad e^{i\sum_{\ell^r}\beta_{2J}(u_i,u_k)+\sum_{\ell^s}\beta_{2J}(u_k,u_j)+\beta_{2J}(e_{ij})}, \\
td_{\eta_{3J}}(e_{ij}) &= \sum_{\ell^r} \eta_{3J}(u_i,u_k) + \sum_{\ell^s} \eta_{3J}(u_k,u_j) + \eta_{3J}(e_{ij}) \\
&\quad e^{i\sum_{\ell^r}\beta_{3J}(u_i,u_k)+\sum_{\ell^s}\beta_{3J}(u_k,u_j)+\beta_{3J}(e_{ij})}, \\
&\quad \forall\ \ell^r = (u_i,u_k) \in W_J, k \neq j, \ell^s = (u_k,u_j) \in W_J, k \neq i \text{and } J = 1,2,...,k.
\end{aligned}
$$

Definition 4.4. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. If $d_{\eta_J}(e_{ij}) = (p, q, r)$ for all $e_{ij} \in W_J$ for each edge of η_{1J} has the same degree p and for each edge of η_{2J} has the same degree q and for each edge of η_{3J} has the same degree r. Then τ is said to be $\eta_J - edge$ regular SVCNGS for all $J = 1, 2, ..., k$.

Example 4.5. Consider an SVCNGS
$tau = (\gamma, \eta_1, \eta_2)$ of GS $\tau^* = (M, W_1, W_2)$ given Figure-2 is $\eta_J - edge$ regular SVCNGS such that $\gamma = \{u_1(.4e^{i.5\pi}, .3e^{i.4\pi}, .5e^{i.6\pi})$,
$u_2(.4e^{i.5\pi}, .4e^{i.5\pi}, .6e^{i.6\pi}), u_3(.5e^{i.5\pi}, .3e^{i.4\pi}, .5e^{i.6\pi}),\ u_4(.4e^{i.5\pi}, .4e^{i.5\pi}, .6e^{i.6\pi})\}$. The degree of

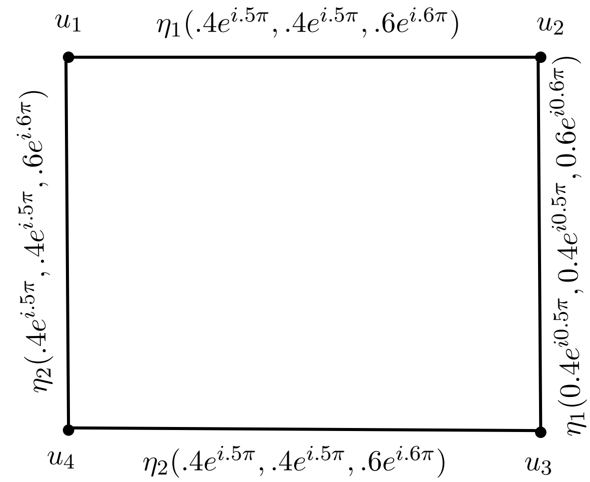

FIGURE 2. $\tau = (\gamma, \eta_1, \eta_2)$ is regular SVCNGS of τ^*

an $\eta_1 - edge$.

$$d_{\eta_1}(e_{12}) = (d_{\eta_{11}}(e_{12}), d_{\eta_{21}}(e_{12}), d_{\eta_{31}}(e_{12}))$$

$$\begin{aligned}
d_{\eta_{11}}(e_{12}) &= d_{\eta_{11}}(u_1) + d_{\eta_{11}}(u_2) - 2\eta_{11}(u_1,u_2)e^{i2\beta_{11}(u_1,u_2)} \ (or) \\
d_{\eta_{11}}(e_{12}) &= (\sum_{(u_2,u_4)\in W_1, u_4\neq u_1} \eta_{11}(u_2,u_4))e^{i\sum_{(u_2,u_4)\in W_1,u_4\neq u_1}\beta_{11}(u_2,u_4)} \\
&= (.4+.8-2(.4))e^{i(.5\pi+1.0\pi-2(.5\pi))} \\
&= .4e^{i.5}
\end{aligned}$$

$$\begin{aligned}
d_{\eta_{21}}(e_{12}) &= d_{\eta_{21}}(u_1) + d_{\eta_{21}}(u_2) - 2\eta_{21}(u_1,u_2)e^{i2\beta_{21}(u_1,u_2)} \\
&= (.4+.8-2(0.4))e^{i(.5\pi+1.0\pi-2(.5\pi))} \\
&= .4e^{i.5}
\end{aligned}$$

$$\begin{aligned}
d_{\eta_{31}}(e_{12}) &= d_{\eta_{31}}(u_1) + d_{\eta_{31}}(u_2) - 2\eta_{31}(u_1,u_2)e^{i2\beta_{31}(u_1,u_2)} \\
&= (.6+1.2-2(.6))e^{i(.6\pi+1.2\pi-2(.6\pi))} \\
&= .6e^{i.6\pi} \\
d_{\eta_1}(e_{12}) &= (.4e^{i.5\pi}, .4e^{i.5\pi}, .6e^{i.6\pi})
\end{aligned}$$

Similarly, we calculate
$d_{\eta_1}(e_{12}) = d_{\eta_1}(e_{23}) = d_{\eta_1}(e_{34}) = d_{\eta_1}(e_{14}) = (.4e^{i.5\pi}, .4e^{i.5\pi}, .6e^{i.6\pi})$
The degree of an $\eta_2 - edge$.
$d_{\eta_2}(e_{12}) = d_{\eta_2}(e_{23}) = d_{\eta_2}(e_{34}) = d_{\eta_2}(e_{14}) = (.4e^{i.5\pi}, .4e^{i.5\pi}, .6e^{i.6\pi})$
In the above example-4.5 is $\eta_J - edge$ regular SVCNGS for all $J = 1, 2$.

Definition 4.6. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. If $td_{\eta_J}(e_{ij}) = (x, y, z)$ for all $e_{ij} \in W_J$ for each edge of η_{1J} has the same total degree x and for each edge of η_{2J} has the same total degree y and for each edge of η_{3J} has the same total degree z. Then τ is said to be totally $\eta_J - edge$ regular SVCNGS for all $J = 1, 2, ..., k$.

Example 4.7. Consider an SVCNGS $\tau = (\gamma, \eta_1, \eta_2)$ of GS $\tau^* = (M, W_1, W_2)$ is given Figure-2 in example-4.5 is totally $\eta_J - edge$ regular SVCNGS for all $J = 1, 2$. The total degree of an $\eta_1 - edge$ is
$td_{\eta_1}(e_{12}) = td_{\eta_1}(e_{23}) = td_{\eta_1}(e_{34}) = td_{\eta_1}(e_{14}) = (.8e^{i1.0\pi}, .8e^{i1.0\pi}, 1.2e^{i1.2\pi})$
The total degree of an $\eta_2 - edge$ is
$td_{\eta_2}(e_{12}) = td_{\eta_2}(e_{23}) = td_{\eta_2}(e_{34}) = td_{\eta_2}(e_{14}) = (.8e^{i1.0\pi}, .8e^{i1.0\pi}, 1.2e^{i1.2\pi})$
Hence, τ is totally $\eta_J - edge$ regular SVCNGS for all $J = 1, 2$.

Theorem 4.8. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$ and τ^* is a cycle. Then $\sum_{i=1}^{n} d_{\eta_J}(u_i) = \sum_{i=1}^{n} d_{\eta_J}(e_{ij})$ for all $J = 1, 2, ..., k$ and $j = i + 1$.*

Proof. Given that $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$ and τ^* is a cycle $u_1 u_2 u_3 ... u_n$. Then

$$\sum_{i=1}^{n} d_{\eta_J}(e_{ij}) = (\sum_{i=1}^{n} d_{\eta_{1J}}(e_{ij}), \sum_{i=1}^{n} d_{\eta_{2J}}(e_{ij}), \sum_{i=1}^{n} d_{\eta_{3J}}(e_{ij}))$$
$$\forall\ J = 1, 2, ..., k \text{ and } j = i + 1.$$

Consider

$$\begin{aligned}
&\sum_{i=1}^{n} d_{\eta_{1J}}(e_{ij})\\
&= d_{\eta_{1J}}(e_{12}) + d_{\eta_{1J}}(e_{23}) + ... + d_{\eta_{1J}}(e_{n1}), \text{ where } u_{n+1} = u_1\\
&= d_{\eta_{1J}}(u_1) + d_{\eta_{1J}}(u_2) - 2\eta_{1J}(u_1, u_2)e^{i2\beta_{1J}(u_1,u_2)} + d_{\eta_{1J}}(u_2) + d_{\eta_{1J}}(u_3)\\
&\quad -2\eta_{1J}(u_2, u_3)e^{i2\beta_{1J}(u_2,u_3)} + ... + d_{\eta_{1J}}(u_n) + d_{\eta_{1J}}(u_1) - 2\eta_{1J}(u_n, u_1)e^{i2\beta_{1J}(u_n,u_1)}\\
&= 2d_{\eta_{1J}}(u_1) + 2d_{\eta_{1J}}(u_2) + ... + 2d_{\eta_{1J}}(u_n)\\
&\quad -2(\eta_{1J}(u_1, u_2)e^{i2\beta_{1J}(u_1,u_2)} + \eta_{1J}(u_2, u_3)e^{i2\beta_{1J}(u_2,u_3)} + ... + \eta_{1J}(u_n, u_1)e^{i2\beta_{1J}(u_n,u_1)})\\
&= 2\sum_{u_i \in M} d_{\eta_{1J}}(u_i) - 2\sum_{i=1}^{n} \eta_{1J}(u_i, u_{i+1})e^{i2\sum_{i=1}^{n}\beta_{1J}(u_i,u_{i+1})}\\
&= \sum_{u_i \in M} d_{\eta_{1J}}(u_i) + \sum_{u_i \in M} d_{\eta_{1J}}(u_i) - 2\sum_{i=1}^{n} \eta_{1J}(u_i, u_{i+1})e^{i2\sum_{i=1}^{n}\beta_{1J}(u_i,u_{i+1})}\\
&= \sum_{u_i \in M} d_{\eta_{1J}}(u_i) + 2\sum_{i=1}^{n} \eta_{1J}(u_i, u_{i+1})e^{i2\sum_{i=1}^{n}\beta_{1J}(u_i,u_{i+1})} - 2\sum_{i=1}^{n} \eta_{1J}(u_i, u_{i+1})e^{i2\sum_{i=1}^{n}\beta_{1J}(u_i,u_{i+1})}\\
&= \sum_{u_i \in M} d_{\eta_{1J}}(u_i)
\end{aligned}$$

Similarly, we derive the equation
$\sum_{i=1}^{n} d_{\eta_{2J}}(e_{ij}) = \sum_{u_i \in M} d_{\eta_{2J}}(u_i)$,
$\sum_{i=1}^{n} d_{\eta_{3J}}(e_{ij}) = \sum_{u_i \in M} d_{\eta_{3J}}(u_i)$.
Hence, $\sum_{i=1}^{n} d_{\eta_J}(u_i) = \sum_{i=1}^{n} d_{\eta_J}(e_{ij})$ for all $J = 1, 2, ..., k$ and $j = i + 1$. □

Theorem 4.9. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a crisp graph τ^S of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then $\sum_{e_{ij} \in W_J} d_{\eta_J}(e_{ij}) = (\sum_{e_{ij} \in W_J} d^*_{\eta_J}(e_{ij})\eta_{1J}(u_i u_j)e^{i\beta_{1J}(u_i u_j)}$, $\sum_{e_{ij} \in W_J} d^*_{\eta_J}(e_{ij})\eta_{2J}(u_i u_j)e^{i\beta_{2J}(u_i u_j)}, \sum_{e_{ij} \in W_J} d^*_{\eta_J}(e_{ij})\eta_{3J}(u_i u_j)e^{i\beta_{3J}(u_i u_j)})$ where $d^*_{\eta_J}(e_{ij}) = d^*_{\eta_J}(u_i) + d^*_{\eta_J}(u_j) - 2$ for all $e_{ij} \in W_J$ and $J = 1, 2, ..., k$.*

Proof. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a crisp graph τ^S of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. We know that $d_{\eta_J}(e_{ij}) = (d_{\eta_{1J}}(e_{ij}), d_{\eta_{2J}}(e_{ij}), d_{\eta_{3J}}(e_{ij}))$.

Therefore, in $\sum_{e_{ij}\in W_J} d_{\eta_{1J}}(e_{ij})$, every $\eta_{1J}e^{i\beta_{1J}} - edge$ contributes it's truth membership values exactly number of $\eta_{1J}e^{i\beta_{1J}} - edges$ adjacent to that $\eta_{1J}e^{i\beta_{1J}} - edge$ times.
Thus, in $\sum_{e_{ij}\in W_J} d_{\eta_{1J}}(e_{ij})$, each $\eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)}$ appears $d^*_{\eta_{1J}}(e_{ij})$ times.

$$\text{Hence, } \sum_{e_{ij}\in W_J} d_{\eta_{1J}}(e_{ij}) = \sum_{e_{ij}\in W_J} d^*_{\eta_{1J}}(e_{ij})\eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)}$$

Similarly, we solve the equation

$$\begin{aligned}
\sum_{e_{ij}\in W_J} d_{\eta_{2J}}(e_{ij}) &= \sum_{e_{ij}\in W_J} d^*_{\eta_{2J}}(e_{ij})\eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)} \\
\sum_{e_{ij}\in W_J} d_{\eta_{3J}}(e_{ij}) &= \sum_{e_{ij}\in W_J} d^*_{\eta_{3J}}(e_{ij})\eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)}
\end{aligned}$$

Hence, $\sum_{e_{ij}\in W_J} d_{\eta_J}(e_{ij}) = (\sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)}$, $\sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)}, \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)})$ □

Theorem 4.10. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a η_J regular crisp graph τ^S of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then $\sum_{e_{ij}\in W_J} d_{\eta_J}(e_{ij}) = ((k-1)\sum_{u_i\in M} d_{\eta_{1J}}(u_i), (k-1)\sum_{u_i\in M} d_{\eta_{2J}}(u_i), (k-1)\sum_{u_i\in M} d_{\eta_{3J}}(u_i))$*

Proof. By Theorem-4.9,

$$\begin{aligned}
\sum_{e_{ij}\in W_J} d_{\eta_J}(e_{ij}) &= (\sum_{e_{ij}\in W_J} d^*_{\eta_1}(e_{ij})\eta_{1J}(u_iu_j), \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{2J}(u_iu_j), \\
&\quad \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{3J}(u_iu_j)) \\
&= (\sum_{u_i,u_j\in W_J} (d^*_{\eta_J}(u_i) + d^*_{\eta_J}(u_j) - 2)\eta_{1J}(u_i,u_j)e^{i\beta_{1J}(u_i,u_j)}, \\
&\quad \sum_{u_i,u_j\in W_J} (d^*_{\eta_J}(u_i) + d^*_{\eta_J}(u_j) - 2)\eta_{2J}(u_i,u_j)e^{i\beta_{2J}(u_i,u_j)}, \\
&\quad \sum_{u_i,u_j\in W_J} (d^*_{\eta_J}(u_i) + d^*_{\eta_J}(u_j) - 2)\eta_{3J}(u_i,u_j)e^{i\beta_{3J}(u_i,u_j)})
\end{aligned}$$

□

Since, τ^S is a η_J regular crisp graph of GS, $d^*_{\eta_J}(u_i) = k$ for all $u_i \in M$.

$$\begin{aligned}\sum_{e_{ij}\in W_J} d_{\eta_J}(e_{ij}) &= ((k+k-2)\sum_{u_i,u_j\in W_J}\lambda_{1J}(u_i,u_j)e^{i\sum_{u_i,u_j\in W_J}\lambda_{1J}(u_i,u_j)},\\ &\quad (k+k-2)\sum_{u_i,u_j\in W_J}\eta_{2J}(u_i,u_j)e^{i\sum_{u_i,u_j\in W_J}\eta_{2J}(u_i,u_j)},\\ &\quad (k+k-2)\sum_{u_i,u_j\in W_J}\eta_{3J}(u_i,u_j)e^{i\sum_{u_i,u_j\in W_J}\eta_{3J}(u_i,u_j)})\\ &= (2(k-1)\sum_{u_i,u_j\in W_J}\eta_{1J}(u_i,u_j)e^{i\sum_{u_i,u_j\in W_J}\eta_{1J}(u_i,u_j)},\\ &\quad 2(k-1)\sum_{u_i,u_j\in W_J}\eta_{2J}(u_i,u_j)e^{i\sum_{u_i,u_j\in W_J}\eta_{2J}(u_i,u_j)},\\ &\quad 2(k-1)\sum_{u_i,u_j\in W_J}\eta_{3J}(u_i,u_j)e^{i\sum_{u_i,u_j\in W_J}\eta_{3J}(u_i,u_j)})\\ &= ((k-1)\sum_{u_i\in M} d_{\eta_{1J}}(u_i), (k-1)\sum_{u_i\in M} d_{\eta_{2J}}(u_i),\\ &\quad (k-1)\sum_{u_i\in Q} d_{\eta_{3J}}(u_i))\end{aligned}$$

Theorem 4.11. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a crisp graph τ^S of GS $\tau^* = \{Q, R_1, R_2, ..., R_k\}$. Then* $\sum_{e_{ij}\in W_J} td_{\eta_J}(e_{ij}) = (\sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)} + \sum_{u_iu_j\in W_J}\eta_{1J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{1J}(u_iu_j)}, \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)} + \sum_{u_iu_j\in R_J}\eta_{2J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{2J}(u_iu_j)}, \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)} + \sum_{u_iu_j\in R_J}\eta_{3J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{3J}(u_iu_j)})$

Proof. By Definition-4.3 of total degree of $\eta_J - edge$ of $\tilde{G}$.

$$td_{\eta_J}(e_ij) = (td_{\eta_{1J}}(e_{ij}), td_{\eta_{2J}}(e_{ij}), td_{\eta_{3J}}(e_{ij}))$$

$$\begin{aligned}
\sum_{e_{ij}\in W_J} td_{\eta_J}(e_ij) &= (\sum_{u_iu_j\in W_J} td_{\eta_{1J}}(e_{ij}), \sum_{u_iu_j\in W_J} td_{\eta_{2J}}(e_{ij}), \sum_{u_iu_j\in W_J} td_{\eta_{3J}}(e_{ij}))\\
&= (\sum_{e_{ij}\in W_J} (d_{\eta_{1J}}(e_{ij}) + \eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)}),\\
&\quad \sum_{e_{ij}\in W_J} (d_{\eta_{2J}}(e_{ij}) + \eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)}),\\
&\quad \sum_{e_{ij}\in W_J} (d_{\eta_{3J}}(e_{ij}) + \eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)}))\\
&= (\sum_{e_{ij}\in W_J} d_{\eta_{1J}}(e_{ij}) + \sum_{u_iu_j\in W_J} \eta_{1J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{1J}(u_iu_j)},\\
&\quad \sum_{e_{ij}\in W_J} d_{\eta_{2J}}(e_{ij}) + \sum_{u_iu_j\in W_J} \eta_{2J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{2J}(u_iu_j)},\\
&\quad \sum_{e_{ij}\in W_J} d_{\eta_{3J}}(e_{ij}) + \sum_{u_iu_j\in W_J} \eta_{3J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{3J}(u_iu_j)})
\end{aligned}$$

By Theorem-4.9, we get

$$\begin{aligned}
\sum_{e_{ij}\in W_J} td_{\eta_J}(e_{ij}) &= (\sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)} + \sum_{u_iu_j\in W_J} \eta_{1J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{1J}(u_iu_j)},\\
&\quad \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)} + \sum_{u_iu_j\in W_J} \eta_{2J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{2J}(u_iu_j)},\\
&\quad \sum_{e_{ij}\in W_J} d^*_{\eta_J}(e_{ij})\eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)} + \sum_{u_iu_j\in W_J} \eta_{3J}(u_iu_j)e^{i\sum_{u_iu_j\in W_J}\beta_{3J}(u_iu_j)})
\end{aligned}$$

□

Theorem 4.12. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. If and only if the subsequent statements are equivalent, then η_J is a constant functional.*
(i) τ is an $\eta_J - edge$ regular SVCNGS.
(ii) τ is a totally $\eta_J - edge$ regular SVCNGS.

Proof. Let us assume that η_J is a function that is constant. Then $\eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)} = c_1$, $\eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)} = c_2$ and $\eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)} = c_3$ for every $u_iu_j \in W_J$, where c_1, c_2, c_3 are constants. (1)
Assume that τ is $\eta_J - edge$ regular SVCNGS. Then $d_{\eta_J}(e_{ij}) = (p, q, r)$ for all $e_{ij} \in W_J$. (2)
Consider

$$\begin{aligned}
td_{\eta_J}(e_ij) &= (d_{\eta_{1J}}(e_{ij}) + \eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)},\\
&\quad d_{\eta_{2J}}(e_{ij}) + \eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)},\\
&\quad d_{\eta_{3J}}(e_{ij}) + \eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)})\\
&= (p + c_1, q + c_2, r + c_3) \text{ for all } u_iu_j \in W_J \text{ by (1) and (2)}
\end{aligned}$$

which implies τ is totally $\eta_J - edge$ regular SVCNGS.

Therefore, $(i) \Rightarrow (ii)$.

Let τ be totally $\eta_J - edge$ regular SVCNGS. Then $td_{\eta_J}(e_{ij}) = (x, y, z)$ for all $e_{ij} \in W_J$.

$$\begin{aligned} td_{\eta_J}(e_ij) &= (d_{\eta_{1J}}(e_{ij}) + \eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)}, d_{\eta_{2J}}(e_{ij}) + \eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)}, \\ &\quad d_{\eta_{3J}}(e_{ij}) + \eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)}). \end{aligned}$$

Now,

$$\begin{aligned} d_{\eta_J}(e_{ij}) &= (d_{\eta_{1J}}(e_{ij}), d_{\eta_{2J}}(e_{ij}), d_{\eta_{3J}}(e_{ij})) \\ &= (x - \eta_{1J}(u_iu_j)e^{i\beta_{1J}(u_iu_j)}, y - \eta_{2J}(u_iu_j)e^{i\beta_{2J}(u_iu_j)}, z - \eta_{3J}(u_iu_j)e^{i\beta_{3J}(u_iu_j)}) \\ &= (x - c_1, y - c_2, z - c_3) \text{ by}(1) \end{aligned}$$

Hence, τ is $\eta_J - edge$ regular SVCNGS.

Thus, $(ii) \Rightarrow (i)$.

Conversely, suppose that (i) and (ii) are equivalent.

As a resultτ is $\eta_J - edge$ regular SVCNGS if and only if τ is totally $\eta_J - edge$ regular SVCNGS.

We have to prove that η_J is a constant function.

Let us assume that η_J is not a constant function. (3)

Then

$\eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)} = \eta_{1J}(u_r, u_s)e^{i\eta_{1J}(u_r,u_s)}$, $\eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)} = \lambda_{2J}(u_r, u_s)e^{i\lambda_{2J}(u_r,u_s)}$ and $\lambda_{3J}(u_i, u_j)e^{i\beta_{3J}(u_i,u_j)} = \eta_{3J}(u_r, u_s)e^{i\eta_{3J}(u_r,u_s)}$ for at least one pair of $u_iu_j, u_ru_s \in R_J$. Let τ is $\eta_J - edge$ regular SVCNGS. Then $d_{\eta_J}(e_{ij}) = d_{\eta_J}(e_{rs}) = (p, q, r)$ (4)

$$\begin{aligned} \Rightarrow td_{\eta_J}(e_{ij}) &= (d_{\eta_{1J}}(e_{ij}) + \eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)}, d_{\eta_{2J}}(e_{ij}) + \eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)}, \\ &\quad d_{\eta_{3J}}(e_{ij}) + \eta_{3J}e^{i\beta_{1J}(u_i,u_j)}) \\ &= (p + \eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)}, q + \eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)}, \\ &\quad r + \eta_{3J}(u_i, u_j)e^{i\beta_{3J}(u_i,u_j)}) \ \forall \ u_iu_j \in W_J. \end{aligned}$$

and

$$\begin{aligned} td_{\eta_J}(e_{rs}) &= (d_{\eta_{1J}}(e_{rs}) + \eta_{1J}(u_ru_s)e^{i\beta_{1J}(u_ru_s)}, d_{\eta_{2J}}(e_{rs}) + \eta_{2J}(u_ru_s)e^{i\beta_{2J}(u_ru_s)}, \\ &\quad d_{\eta_{3J}}(e_{rs}) + \eta_{3J}(u_ru_s)e^{i\beta_{3J}(u_ru_s)}) \\ &= (p + \eta_{1J}(u_ru_s)e^{i\beta_{1J}(u_ru_s)}, q + \eta_{2J}(u_ru_s)e^{i\beta_{2J}(u_ru_s)}, \\ &\quad r + \eta_{3J}(u_ru_s)e^{i\beta_{3J}(u_ru_s)}), \ \forall \ u_ru_s \in W_J. \end{aligned}$$

Since,

$\eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)} \neq \eta_{1J}(u_r, u_s)e^{i\beta_{1J}(u_r,u_s)}$, $\eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)} \neq \eta_{2J}(u_r, u_s)e^{i\beta_{2J}(u_r,u_s)}$ and $\eta_{3J}(u_i, u_j)e^{i\beta_{3J}(u_i,u_j)} \neq \eta_{3J}(u_r, u_s)e^{i\beta_{3J}(u_r,u_s)}$

$\Rightarrow td_{\eta_J}(e_{ij}) \neq td_{\eta_J}(e_{rs})$

$\Rightarrow$ Not all of τ is a totally $\eta_J - edge$ regular SVCNGS
$\Rightarrow$ it is a contradiction.
Hence, η_J is a constant function. □

Theorem 4.13. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS is both $\eta_J - edge$ regular and totally $\eta_J - edge$ regular of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then η_J is a constant function.*

Proof. The result is trivial according to Theorem-4.12.
Note: The above theorem-4.12 does not hold in its converse. □

Theorem 4.14. *Let η_J be constant functions in an SVCNGS $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$ and if τ is η_J regular, Then totally $\eta_J - edge$ regular.*

Proof. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be a η_J regular SVCNGS. Then $d_{\eta_J}(u_i) = (a, b, c)$ for all $u_i \in M$. Given that η_J are constants. That is, $\eta_J(u_i, u_j) = (c_1, c_2, c_3)$ for all $u_iu_j \in W_J$ where c_1, c_2, c_3 are constant
We have to prove that τ is totally $\eta_J - edge$ regular SVCNGS.
By Definition-4.3 of totally $\eta_J - edge$ degree, we have

$$td_{\eta_J}(e_ij) = (td_{\eta_{1J}}(e_{ij}), td_{\eta_{2J}}(e_{ij}), td_{\eta_{3J}}(e_{ij}))$$

where

$$\begin{aligned} td_{\eta_{1J}}(e_{ij}) &= d_{\eta_{1J}}(u_i) + d_{\eta_{1J}}(u_j) - \eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)}, \ \forall\ u_iu_j \in W_J \\ &= a + a - c_1, \ \forall\ u_iu_j \in W_J \ (\therefore\ \tau \text{ is regular}) \\ &= 2a + c_1 = constant, \ \forall\ u_iu_j \in W_J. \end{aligned}$$

Similarly, we solve the equation

$$\begin{aligned} td_{\eta_{2J}}(e_{ij}) &= d_{\eta_{2J}}(u_i) + d_{\eta_{2J}}(u_j) - \eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)}, \ \forall\ u_iu_j \in W_J \\ &= b + b - c_2, \ \forall\ u_iu_j \in W_J \ (\therefore\ \tau \text{ is regular}) \\ &= 2b + c_2 = constant, \ \forall\ u_iu_j \in W_J. \end{aligned}$$

$$\begin{aligned} td_{\eta_{3J}}(e_{ij}) &= d_{\eta_{3J}}(u_i) + d_{\eta_{3J}}(u_j) - \eta_{3J}(u_i, u_j)e^{i\beta_{3J}(u_i,u_j)}, \ \forall\ u_iu_j \in W_J \\ &= c + c - c_3, \ \forall\ u_iu_j \in W_J \ (\therefore\ \tau \text{ is regular}) \\ &= 2c + c_2 = \text{constant}, \ \forall\ u_iu_j \in W_J. \end{aligned}$$

(ie) $td_{\eta_J}(e_ij) = (2a + c_1, 2b + c_2, 2c + c_3)$
$\Rightarrow$ τ is a totally $\eta_J - edge$ regular SVCNGS. □

Theorem 4.15. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a regular crisp graph τ^S of GS $\tau^* = \{Q, R_1, R_2, ..., R_k\}$. Then η_J is a constant if and only if τ is both η_J regular and $\eta_J - edge$ regular SVCNGS.*

Proof. Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a regular crisp GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$.
Assume that η_J are constant functions, that is $\eta_J(u_i, u_j) = (c_1, c_2, c_3)$ for all $u_iu_j \in W_J$ where c_1, c_2, c_3 are constant
To prove: τ is both η_J regular and totally $\eta_J - edge$ regular SVCNGS. By Definition-3.4 of $\eta_J - degree$ of a vertex,

$$\begin{aligned} d_{\eta_J}(u_i) &= (d_{\eta_{1J}}(u_i), d_{\eta_{2J}}(u_i), d_{\eta_{3J}}(u_i)) \\ &= (\sum_{(u_i,v_j)\in W_J} \eta_{1J}(u_i, v_j) e^{i\sum_{(u_i,v_j)\in W_J} \beta_{1J}(u_i,v_j)}, \sum_{(u_i,v_j)\in W_J} \eta_{2J}(u_i, v_j) e^{i\sum_{(u_i,v_j)\in W_J} \beta_{2J}(u_i,v_j)}, \\ &\quad \sum_{(u_i,v_j)\in W_J} \eta_{3J}(u_i, v_j) e^{i\sum_{(u_i,v_j)\in W_J} \beta_{3J}(u_i,v_j)}), \ \forall\ u_i \in M \\ &= (\sum_{(u_i,v_j)\in W_J} c_1, \sum_{(u_i,v_j)\in W_J} c_2, \sum_{(u_i,v_j)\in W_J} c_3) \\ &= (xc_1, yc_2, zc_3) \end{aligned}$$

Hence, τ is η_J regular SVCNGS.
Now,

$$td_{\eta_J}(e_{ij}) = (td_{\eta_{1J}}(e_{ij}), td_{\eta_{2J}}(e_{ij}), td_{\eta_{3J}}(e_{ij})), \text{where}$$

$$\begin{aligned} td_{\eta_{1J}}(e_{ij}) &= \sum_{u_iu_k\in W_J, k\neq j} \eta_{1J}(u_i, u_k) e^{i\sum_{u_iu_k\in W_J, k\neq j} \beta_{1J}(u_i,u_k)} + \\ &\quad \sum_{u_ku_j\in W_J, k\neq i} \eta_{1J}(u_k, u_j) e^{i\sum_{u_ku_j\in W_J, k\neq i} \beta_{1J}(u_k,u_j)} + \eta_{1J}(u_i, u_j) e^{i\beta_{1J}(u_i,u_j)} \\ &= \sum_{u_iu_k\in W_J, k\neq j} c_1 + \sum_{u_ku_j\in W_J, k\neq i} c_1 + c_1 \\ &= c_1(x-1) + c_1(x-1) + c_1, \ \forall\ u_iu_j \in W_J \\ &= c_1(2x-1), \ \forall\ u_iu_j \in W_J. \end{aligned}$$

Similarly, we solve the equation

$$\begin{aligned} td_{\lambda_{2J}}(e_{ij}) &= c_2(2y-1), \ \forall\ u_iu_j \in W_J \\ td_{\lambda_{3J}}(e_{ij}) &= c_3(2z-1), \ \forall\ u_iu_j \in W_J \end{aligned}$$

Hence, τ is also totally η_J regular SVCNGS.
Conversely, assume that τ is both η_J regular and $\eta_J - edge$ regular SVCNGS.
To prove: η_J is a constant function.

Since, τ is η_J regular, $d_{\eta_J}(u_i) = (a, b, c),\ \forall\ u_i \in M.$
Also, τ is totally $\eta_J - edge$ regular.
Then $td_{\eta_J}(e_ij) = (x, y, z,),\ \forall\ u_iu_j \in W_J.$
By Definition-4.3 of totally $\eta_J - edge$ degree,

$$td_{\eta_J}(e_ij) = (td_{\eta_{1J}}(e_{ij}), td_{\eta_{2J}}(e_{ij}), td_{\eta_{3J}}(e_{ij})), \text{where}$$

$$\begin{aligned} td_{\eta_J}(e_{ij}) &= d_{\eta_{1J}}(u_i) + d_{\eta_{1J}}(u_j) - \eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)},\ \forall\ u_iu_j \in W_J \\ x &= a + a - \eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)},\ \forall\ u_iu_j \in W_J \\ \eta_{1J}(u_i, u_j)e^{i\beta_{1J}(u_i,u_j)} &= 2a - x,\ \forall\ u_iu_j \in W_J. \end{aligned}$$

Similarly, we solve the equation

$$\begin{aligned} \eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)} &= 2b - y,\ \forall\ u_iu_j \in W_J. \\ \eta_{3J}(u_i, u_j)e^{i\beta_{3J}(u_i,u_j)} &= 2c - z,\ \forall\ u_iu_j \in W_J. \end{aligned}$$

Hence, η_J is a constant function. □

Theorem 4.16. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be an SVCNGS on a crisp graph τ^S of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. If η_J is constant functions, then τ is an $\eta_J - edge$ regular SVCNGS if and only if τ^S is an $\eta_J - edge$ regular.*

Proof. Given that η_J is constant functions. That is, $\eta_J(u_i, u_j)e^{i\beta_J(u_i,u_j)} = (c_1, c_2, c_3)$ for all $u_iu_j \in W_J$ where c_1, c_2, c_3 are constants.
Assume that τ is an $\eta_J - edge$ regular.
To Prove: τ^S is an $\eta_J - edge$ regular.
Suppose that τ^S is not an $\eta_J - edge$ regular. Then $d_{\eta_J}(e_{ij}) \neq d_{\eta_J}(e_{rs})$ for at least one pair of $e_{ij}, e_{rs} \in W_J$.
By Definition-4.1 of an $\eta_J - edge$ degree of an SVCNGS,

$$d_{\eta_J}(e_{ij}) = (d_{\eta_{1J}}(e_{ij}), d_{\eta_{2J}}(e_{ij}), d_{\eta_{3J}}(e_{ij})),$$

where

$$\begin{aligned} d_{\eta_{1J}}(e_{ij}) &= \sum_{u_iu_k\in W_J,k\neq j} \eta_{1J}(u_i,u_k)e^{i\sum_{u_iu_k\in W_J,k\neq j}\beta_{1J}(u_i,u_k)} + \\ &\quad \sum_{u_ku_j\in W_J,k\neq i} \eta_{1J}(u_k,u_j)e^{i\sum_{u_ku_j\in W_J,k\neq i}\beta_{1J}(u_k,u_j)} \\ &= \sum_{u_iu_k\in W_J,k\neq j} c_1 + \sum_{u_ku_j\in W_J,k\neq i} c_1 \\ &= c_1(d^*_{\eta_J}(u_i)-1)+c_1(d^*_{\eta_J}(u_j)-1), \\ &= c_1(d^*_{\eta_J}(u_i)+d^*_{\eta_{1J}}(u_j)-2) \\ &= c_1(d^*_{\eta_J}(e_{ij})) \end{aligned}$$

Similarly, we solve the equation

$$\begin{aligned} d_{\eta_{2J}}(e_{ij}) &= c_2(d^*_{\eta_J}(e_{ij})) \\ d_{\eta_{3J}}(e_{ij}) &= c_3(d^*_{\eta_J}(e_{ij})) \end{aligned}$$

$$\begin{aligned} \therefore d_{\eta_J}(e_{ij}) &= (c_1(d^*_{\eta_J}(e_{ij})),c_2(d^*_{\eta_J}(e_{ij})),c_3(d^*_{\eta_J}(e_{ij}))), \\ d_{\eta_J}(e_{jk}) &= (c_1(d^*_{\eta_J}(e_{jk})),c_2(d^*_{\eta_J}(e_{jk})),c_3(d^*_{\eta_J}(e_{jk}))) \end{aligned}$$

Since, $d^*_{\eta_J}(e_{ij}) \neq d^*_{\eta_J}(e_{jk}) \Rightarrow d_{\eta_J}(e_{ij}) \neq d_{\eta_J}(e_{jk})$. Thus, τ is not an $\eta_J - edge$ regular. Our assumption is contradicted by this.
Hence, τ^S is an $\eta_J - edge$ regular.
Conversely, assume that η_J are constant functions and τ^S is an $\eta_J - edge$ regular.
To prove that: τ is an $\eta_J - edge$ regular SVCNGS.
Suppose that τ is not an $\eta_J - edge$ regular SVCNGS. Then $d_{\eta_J}(e_{ij}) \neq d_{\eta_J}(e_{rs})$ for at least one pair of $u_iu_j,\ u_ru_s \in R_J$

$$(d_{\eta_{1J}}(e_{ij}),d_{\eta_{2J}}(e_{ij}),d_{\eta_{3J}}(e_{ij})) \neq (d_{\eta_{1J}}(e_{rs}),d_{\eta_{2J}}(e_{rs}),d_{\eta_{3J}}(e_{rs}))$$

Now,

$$d_{\eta_{1J}}(e_{ij}) \neq d_{\eta_{1J}}(e_{rs}),$$

$$\sum_{u_iu_k\in W_J,k\neq j} \eta_{1J}(u_i,u_k)e^{i\sum_{u_iu_k\in W_J,k\neq j}\beta_{1J}(u_i,u_k)} + \sum_{u_ku_j\in W_J,k\neq i} \eta_{1J}(u_k,u_j)e^{i\sum_{u_ku_j\in W_J,k\neq i}\beta_{1J}(u_k,u_j)} \neq$$
$$\sum_{u_ru_t\in W_J,t\neq s} \eta_{1J}(u_r,u_t)e^{i\sum_{u_ru_t\in W_J,t\neq s}\eta_{1J}(u_r,u_t)} + \sum_{u_tu_s\in W_J,t\neq r} \eta_{1J}(u_t,u_s)e^{i\sum_{u_tu_s\in W_J,t\neq r}\beta_{1J}(u_t,u_s)},$$

$$c_1(d_{\eta_{1J}}(u_i)-1)+c_1(d_{\eta_{1J}}(u_j)-1) \neq c_1(d_{\eta_{1J}}(u_r)-1)+c_1(d_{\eta_{1J}}(u_s)-1),$$
$$c_1(d_{\eta_{1J}}(u_i)+d_{\eta_{1J}}(u_j)-2) \neq c_1(d_{\eta_{1J}}(u_r)+d_{\eta_{1J}}(u_s)-2),$$
$$c_1 d_{\eta_{1J}}(e_{ij}) \neq c_1 d_{\eta_{1J}}(e_{rs})$$
$$d_{\eta_{1J}}(e_{ij}) \neq d_{\eta_{1J}}(e_{rs}).$$

Similarly, we solve the equation.

$$d_{\eta_{2J}}(e_{ij}) \neq d_{\eta_{2J}}(e_{rs}),$$
$$d_{\eta_{3J}}(e_{ij}) \neq d_{\eta_{3J}}(e_{rs})$$

$$\therefore (d_{\eta_{1J}}(e_{ij}), d_{\eta_{2J}}(e_{ij}), d_{\eta_{3J}}(e_{ij})) \neq (d_{\eta_{1J}}(e_{rs}), d_{\eta_{2J}}(e_{rs}), d_{\eta_{3J}}(e_{rs}))$$

Our assumption is contradicted by this. τ^S is an $\eta_J - edge$ regular.
Hence, τ is an $\eta_J - edge$ regular SVCNGS. □

Theorem 4.17. *Let $\tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be a η_J regular SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then τ is an $\eta_J - edge$ regular SVCNGS if and only if η_J is constant functions.*

Proof. Let
$tau = (\gamma, \eta_1, \eta_2, ..., \eta_k)$ be a η_J regular SVCNGS of GS $\tau^* = \{M, W_1, W_2, ..., W_k\}$. Then $d_{\eta_J}(u_i) = (a, b, c)$ for all $u_i \in M$. Assume that η_J is constant functions, that is $\eta_J(u_i, u_j) = (c_1, c_2, c_3)$, $\forall\ u_i u_j \in W_J$ where c_1, c_2, c_3 are constants.
By Definition-4.1 of an $\eta_J - edge$ degree,

$$d_{\eta_J}(e_{ij}) = (d_{\eta_{1J}}(e_{ij}), d_{\eta_{2J}}(e_{ij}), d_{\eta_{3J}}(e_{ij})),$$

$$\begin{aligned} d_{\eta_{1J}}(e_{ij}) &= d_{\eta_{1J}}(u_i) + d_{\eta_{1J}}(u_j) - 2\eta_{1J}(u_i, u_j) e^{i2\beta_{1J}(u_i,u_j)} \\ &= a + a - 2c_1 \\ &= 2(a - c_1) \end{aligned}$$

Similarly, we solve the equation

$$d_{\eta_{2J}}(e_{ij}) = 2(b - c_2)$$
$$d_{\eta_{3J}}(e_{ij}) = 2(c - c_3)$$

$\therefore d_{\eta_J}(e_{ij}) = (2(a - c_1), 2(b - c_2), 2(c - c_3))$.
Hence, τ is an $\eta_J - edge$ regular SVCNGS.
Conversely, we assume that τ is an $\eta_J - edge$ regular SVCNGS.
To prove that η_J is constant functions.

$d_{\eta_J}(e_{ij}) = (p, q, r)$ for all $e_{ij} \in W_J$

Now,

$$\begin{aligned} d_{\eta_{1J}}(e_{ij}) &= d_{\eta_{1J}}(u_i) + d_{\eta_{1J}}(u_j) - 2\eta_{1J}(u_i, u_j)e^{i2\beta_{1J}(u_i,u_j)} \\ p &= a + a - 2\eta_{1J}(u_i, u_j)e^{i2\beta_{1J}(u_i,u_j)} \\ \eta_{1J}(u_i, u_j)e^{i2\beta_{1J}(u_i,u_j)} &= \frac{(2a-p)}{2} \end{aligned}$$

Similarly, we solve the equation

$$\begin{aligned} \eta_{2J}(u_i, u_j)e^{i\beta_{2J}(u_i,u_j)} &= \frac{(2b-q)}{2} \\ \eta_{3J}(u_i, u_j)e^{i\beta_{3J}(u_i,u_j)} &= \frac{(2c-p)}{2} \end{aligned}$$

$\therefore \lambda_J$ is constant functions. □

5. Application

Applications are used in this article to find ambiguities in all facets of human existence. This article discusses the developments in all countries around the world, as well as the reasons for their growth. We will compute the growth and value of fundamental needs across the nations of the world. We will determine the value of a country based on how much education its citizens have access to and how much the government helps the country's poor residents. The medical facilities provided by the government for its citizens as well as the contribution it provides to global health, are also taken into account. Through the contribution of military security in that country, we can ascertain the level of security that the people get. We can also find out how much both the government and the inhabitants of that country contribute to the development of its economy. A country's government measures its progress based on how well it upholds the country's laws and works in the best interests of the people. We can determine a country's progress and strength using all the aforementioned variables. We regard a country's strength and development to be calculated as $\gamma_1 e^{i\alpha_1}$, its weakness and underdevelopment to be calculated as $\gamma_3 e^{i\alpha_3}$, and we consider a country's strength and weakness that we cannot predict, ie., indeterminacy to be calculated as $\gamma_2 e^{i\alpha_2}$. We're going to use an ambiguous value to quantify it. A set M is considered to show nations with the highest rates of strength and development. M={United States, China, Russia, Germany, United Kingdom, Japan, France, South Korea}. We can determine the development correlation between the United States and other countries using our definition-3.1 (see Table-2). We can determine the development correlation between Japan and other countries (see Table-3). We can determine the development correlation between China and other countries (see Table-4). We can determine the development correlation between Russia and other countries (see Table-5). We can determine the

TABLE 1

Country	$\gamma_1 e^{i\alpha_1}$	$\gamma_2 e^{i\alpha_2}$	$\gamma_3 e^{i\alpha_3}$
United States (US)	$.8e^{i.7\pi}$	$.4e^{i.6}$	$.6e^{i0.71\pi}$
Japan (J)	$.7e^{i.6\pi}$	$.60e^{i.7\pi}$	$.51e^{i.6\pi}$
China (C)	$.8e^{i.8\pi}$	$.52e^{i.4\pi}$	$.4e^{i.5\pi}$
Russia (R)	$.62e^{i.5\pi}$	$.5e^{i.4\pi}$	$.5e^{i.6\pi}$
Germany (G)	$.5e^{i.6\pi}$	$.6e^{i.7\pi}$	$.6e^{i.7\pi}$
United Kingdom (U)	$.7e^{i.5\pi}$	$.5e^{i.6\pi}$	$.4e^{i.7\pi}$
France (F)	$.6e^{i.4\pi}$	$.7e^{i.5\pi}$	$.5e^{i.7\pi}$
South Korea (S)	$.7e^{i.6\pi}$	$.4e^{i.7\pi}$	$.6e^{i.5\pi}$

TABLE 2. United States and other countries

□	(US, C)	(US,G)	(US,S)
□	$(.8e^{i.7\pi}, .4e^{i.5\pi}, .4e^{i.5\pi})$	$(.5e^{i.6\pi}, .6e^{i.7\pi}, .6e^{i.7\pi})$	$(.7e^{i.5\pi}, .4e^{i.7\pi}, .6e^{i.7\pi})$
□	$(.7e^{i.6\pi}, .4e^{i.6\pi}, .6e^{i.7\pi})$	$(.5e^{i.6\pi}, .4e^{i.5\pi}, .5e^{i.6\pi})$	$(.6e^{i.5\pi}, .4e^{i.6\pi}, .5e^{i.5\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.5\pi})$	$(.5e^{i.6\pi}, .5e^{i.6\pi}, .4e^{i.5\pi})$	$(.7e^{i.6\pi}, .3e^{i.5\pi}, .3e^{i.5\pi})$
□	$(.7e^{i.7\pi}, .5e^{i.5\pi}, .5e^{i.6\pi})$	$(.4e^{i.5\pi}, .5e^{i.6\pi}, .4e^{i.5\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .4e^{i.5\pi})$
□	$(.7e^{i.7\pi}, .5e^{i.6\pi}, .6e^{i.5\pi})$	$(.5e^{i.5\pi}, .5e^{i.6\pi}, .4e^{i.5\pi})$	$(.7e^{i.6\pi}, .4e^{i.6\pi}, .5e^{i.6\pi})$

TABLE 3. Japan and other countries

□	(J, R)	(J,U)	(J,F)
□	$(.6e^{i.3\pi}, .6e^{i.7\pi}, .5e^{i.6\pi})$	$(.7e^{i.5\pi}, .5e^{i.5\pi}, .4e^{i.5\pi})$	$(.6e^{i.4\pi}, .7e^{i.7\pi}, .5e^{i.7\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.3\pi})$	$(.6e^{i.4\pi}, .5e^{i.5\pi}, .5e^{i.7\pi})$	$(.6e^{i.3\pi}, .6e^{i.6\pi}, .5e^{i.3\pi})$
□	$(.5e^{i.4\pi}, .4e^{i.6\pi}, .4e^{i.5\pi})$	$(.6e^{i.5\pi}, .5e^{i.6\pi}, .5e^{i.4\pi})$	$(.5e^{i.3\pi}, .6e^{i.5\pi}, .5e^{i.2\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.5\pi}, .6e^{i.5\pi})$	$(.4e^{i.5\pi}, .5e^{i.6\pi}, .4e^{i.7\pi})$	$(.6e^{i.4\pi}, .4e^{i.3\pi}, .4e^{i.5\pi})$
□	$(.6e^{i.5\pi}, .4e^{i.5\pi}, .4e^{i.3\pi})$	$(.5e^{i.5\pi}, .5e^{i.6\pi}, .5e^{i.5\pi})$	$(.6e^{i.4\pi}, .7e^{i.6\pi}, .5e^{i.4\pi})$

TABLE 4. China and other countries

□	(C, G)	(C, U)	(C, S)
□	$(.5e^{i.5\pi}, .6e^{i.7\pi}, .6e^{i.7\pi})$	$(.6e^{i.5\pi}, .5e^{i.6\pi}, .4e^{i.7\pi})$	$(.7e^{i.6\pi}, .4e^{i.5\pi}, .4e^{i.7\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.5\pi})$	$(.7e^{i.5\pi}, .4e^{i.5\pi}, .3e^{i.7\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .6e^{i.5\pi})$
□	$(.4e^{i.4\pi}, .5e^{i.6\pi}, .4e^{i.7\pi})$	$(.6e^{i.5\pi}, .5e^{i.5\pi}, .4e^{i.5\pi})$	$(.6e^{i.3\pi}, .5e^{i.5\pi}, .5e^{i.4\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.5\pi}, .6e^{i.5\pi})$	$(.4e^{i.5\pi}, .5e^{i.6\pi}, .4e^{i.7\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.7\pi})$
□	$(.5e^{i.6\pi}, .3e^{i.4\pi}, .3e^{i.7\pi})$	$(.5e^{i.5\pi}, .5e^{i.6\pi}, .4e^{i.4\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.7\pi})$

TABLE 5. Russia and other countries

□	(R, G)	(R, F)	(R, S)
□	$(.5e^{i.5\pi}, .6e^{i.7\pi}, .6e^{i.7\pi})$	$(.6e^{i.4\pi}, .7e^{i.5\pi}, .5e^{i.7\pi})$	$(.6e^{i.3\pi}, .5e^{i.7\pi}, .6e^{i.5\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.7\pi})$	$(.5e^{i.4\pi}, .5e^{i.5\pi}, .5e^{i.5\pi})$	$(.6e^{i.5\pi}, .4e^{i.5\pi}, .4e^{i.7\pi})$
□	$(.4e^{i.4\pi}, .5e^{i.6\pi}, .6e^{i.5\pi})$	$(.6e^{i.4\pi}, .4e^{i.3\pi}, .4e^{i.5\pi})$	$(.6e^{i.3\pi}, .5e^{i.5\pi}, .5e^{i.7\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.5\pi}, .6e^{i.5\pi})$	$(.4e^{i.4\pi}, .5e^{i.5\pi}, .5e^{i.4\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.5\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.7\pi})$	$(.5e^{i.4\pi}, .5e^{i.4\pi}, .5e^{i.7\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .6e^{i.4\pi})$

development correlation between United Kingdom and other countries (see Table-5).

Using these SVCNGS, we illustrate the severity of the development between each pair of

TABLE 6. United Kingdom and other countries

□	(U, G)	(U, F)	(U, S)
□	$(.5e^{i.5\pi}, .6e^{i.7\pi}, .6e^{i.7\pi})$	$(.6e^{i.4\pi}, .7e^{i.6\pi}, .5e^{i.7\pi})$	$(.7e^{i.5\pi}, .4e^{i.5\pi}, .4e^{i.7\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .6e^{i.5\pi})$	$(.6e^{i.2\pi}, .6e^{i.5\pi}, .5e^{i.5\pi})$	$(.6e^{i.4\pi}, .5e^{i.7\pi}, .6e^{i.7\pi})$
□	$(.4e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.7\pi})$	$(.5e^{i.3\pi}, .6e^{i.5\pi}, .5e^{i.5\pi})$	$(.6e^{i.3\pi}, .5e^{i.5\pi}, .5e^{i.7\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.5\pi}, .6e^{i.7\pi})$	$(.6e^{i.4\pi}, .4e^{i.5\pi}, .4e^{i.3\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.5\pi})$
□	$(.5e^{i.4\pi}, .5e^{i.6\pi}, .5e^{i.6\pi})$	$(.5e^{i.4\pi}, .5e^{i.4\pi}, .5e^{i.7\pi})$	$(.6e^{i.4\pi}, .5e^{i.6\pi}, .6e^{i.7\pi})$

nations. On set M, numerous relations can be defined. Let's explain the relationships on M as follows: W_1=education, W_2=medical science, R_3= military, W_4= economic growth, W_5 = effected government, such that $\tau^* = (Q, R_1, R_2, R_3, R_4, R_5)$ is a GS. Each element of the relationship exemplifies a certain stage of growth between those two countries. Due to the fact that the GS is $\tau^* = (M, W_1, W_2, W_3, W_4, W_5)$, only one relationship can exist between two countries. Thus, it would be considered a part of that relationship, whose false membership amount is relatively low in comparison to various other relationships, and whose truth-membership amount is relatively high in comparison to other connections. When measured against other relationships, its truth-membership the amount is relatively high, while its indeterminacy-membership amount is relatively low. Using the previously provided data, the SVCNGS on W_1, W_2, W_3, W_4, W_5 are formed by matching items in relations with the truth-membership, indeterminacy, and false-membership. They are $\eta_1, \eta_2, \eta_3, \eta_4, \eta_5$, respectively, of these SVCNGS.

W_1={(US, C), (J, U),(C, S),(U, S)}, W_2={(C, U),(R, S)}, W_3= {(US, S),(R, F)},
W_4={(J, F),(U, F)}, W_5={(J, R),(C, G)}.

The Corresponding SVCNGS are follows:

$\eta_1 = \{(US, C)(.8e^{i.7\pi}, .4e^{i.5\pi}, .4e^{i.5\pi}), (J, U)(.7e^{i.5\pi}, .5e^{i.5\pi}, .4e^{i.5\pi}),$
$(C, S)(.7e^{i.6\pi}, .4e^{i.5\pi}, .4e^{i.7\pi}), (U, S)(.7e^{i.5\pi}, .4e^{i.5\pi}, .4e^{i.7\pi})\},$

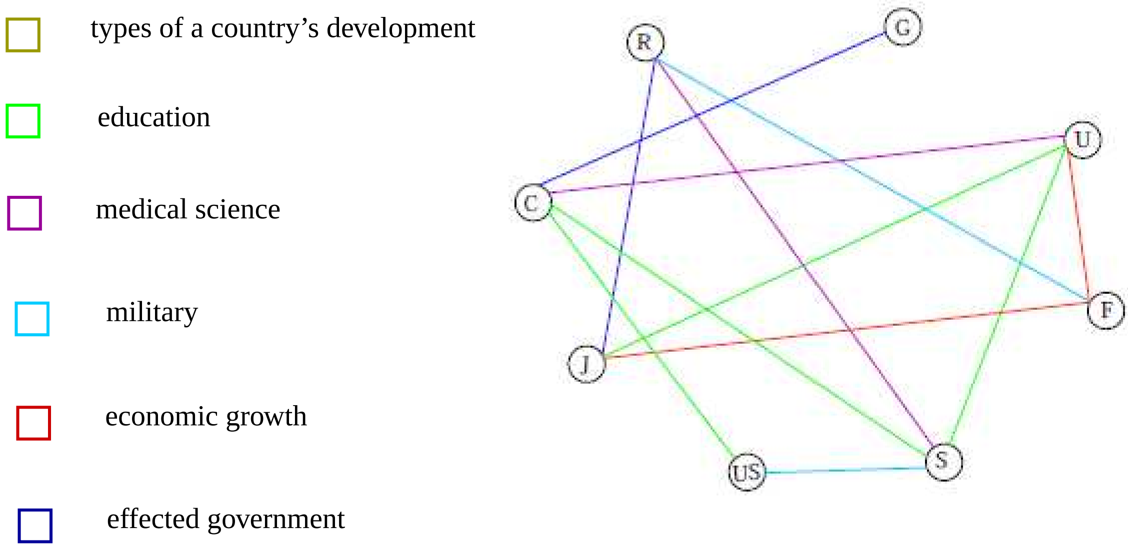

FIGURE 3. SVCNGS

$\eta_2 = \{(C, U)(.7e^{i.5\pi}, .4e^{i.5\pi}, .3e^{i.7\pi}), (R, S)(.6e^{i.5\pi}, .4e^{i.5\pi}, .4e^{i.7\pi})\}$,
$\eta_3 = \{(US, S)(.7e^{i.6\pi}, .3e^{i.5\pi}, .3e^{i.5\pi}), (R, F)(.6e^{i.4\pi}, .4e^{i.3\pi}, .4e^{i.5\pi})\}$,
$\eta_4 = \{(J, F)(.6e^{i.4\pi}, .4e^{i.3\pi}, .4e^{i.5\pi}), (U, F)(.6e^{i.4\pi}, .4e^{i.5\pi}, .4e^{i.3\pi})\}$,
$\eta_5 = \{(J, R)(.6e^{i.5\pi}, .4e^{i.5\pi}, .4e^{i.3\pi}), (C, G)(.5e^{i.6\pi}, .3e^{i.4\pi}, .3e^{i.7pi})\}$.
Therefore, the SVCNGS are represented in Figure-3 is $(\gamma, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5)$. The country with the greatest level of development is represented by each edge of the SVCNGS in Figure-3. As an illustration, the expansion of education, with values for truth-membership, indeterminacy-membership, and false-membership of $.8e^{i.7\pi}$, $.4e^{i.5\pi}$ and $.4e^{i.5\pi}$, respectively, is what contributes to the most powerful and developing relationship between the United States and China. It should be noted that the United States has the lowest vertex degree of indeterminacy-membership, false-membership, and the highest vertex degree of truth-membership for the relation proliferation of education. This shows that the United States has a proliferation of education and is developing alongside other countries. The purpose of this is article is to identify the most developed nations in the world by examining the growth and development of every nation in the world. This opens the way for the growth of all the nations in the world.

5.1. *Algorithm*

We now present the stepwise for calculation of our method which is used in this application in the following algorithm.

Algorithm

step 1: Input the set $Q = \{q_1, q_2, ..., q_n\}$ of countries (vertices) and put the membership values $\gamma = (\gamma_1 e^{i\alpha_1}, \gamma_2 e^{i\alpha_2}, \gamma_3 e^{i\alpha_3})$ of the nodes q_i's, $i = 1, 2, ..., n$, $\gamma_1, \gamma_2, \gamma_3 \in [0, 1]$ and $\alpha_1, \alpha_2, \alpha_3 \in [0, 2\pi]$.

step 2: Input the membership values $\eta_J = (\eta_{1J}(q_iq_j)e^{i\beta_{1J}(q_iq_j)}, \eta_{2J}(q_iq_j)e^{i\beta_{2J}(q_iq_j)}, \eta_{3J}(q_iq_j)e^{i\beta_{3J}(q_iq_j)})$ of the edges $q_iq_j \in W_J$ such that

$$\begin{aligned}
\eta_{1J}(q_iq_j)e^{i\beta_{1J}(q_iq_j)} &\leq \min\{\gamma_1(q_i), \gamma_1(q_j)\}e^{i\min\{\alpha_1(q_i),\alpha_1(q_j)\}},\\
\eta_{2J}(q_iq_j)e^{i\beta_{2J}(q_iq_j)} &\leq \max\{\gamma_2(q_i), \gamma_2(q_j)\}e^{i\max\{\alpha_2(q_i),\alpha_2(q_j)\}},\\
\eta_{3J}(q_iq_j)e^{i\beta_{3J}(q_iq_j)} &\leq \max\{\gamma_3(q_i), \gamma_3(q_j)\}e^{i\max\{\alpha_3(q_i),\alpha_3(q_j)\}}
\end{aligned}$$

such that $0 \leq \eta_{1J}(q_iq_j) + \eta_{2J}(q_iq_j) + \eta_{3J}(q_iq_j) \leq 3$ and $\beta_{1J}(q_iq_j), \beta_{2J}(q_iq_j), \beta_{3J}(q_iq_j) \in [0, 2\pi]$ for all $(q_iq_j) \in W_J, J = 1, 2, ..., k$.

step 3: Develop mutually disjoint, irreflexive and symmetric relations $W_1, W_2, ..., W_k$ on the set of countries M and give the name each relation as exemplifies a certain stage of growth between those two countries.

step 4: Select a countries as greatest level of development from one countries to other, whose membership value is superior to that of other nations.

step 5: Construct a graph structure on set of countries with relations, select those pairs of countries having same kind of the highest level of development as elements of same relation.

step 6: Write all elements of resulting relations $\eta_1, \eta_2, ..., \eta_k$ are CNSs on $W_1, W_2, ..., W_k$, respectively and $(\gamma, \eta_1, \eta_2, ..., \eta_k)$ is a SVCNGS.

step 7: Draw the SVCNGS, each of whose edges indicates the best level of development for the related Countries.

6. Conclusion and future works

The idea of an SVCNGS has been developed in this study article by the authors. In comparison to traditional fuzzy sets, the Set SVCNS, an extension of the NS, provides a more realistic description of uncertainty. Through fuzzy control, it can be used in a variety of ways. In this research study, the idea of SVCNGS is introduced. Further research is done on the relationship between the degree of a vertex and the degree of an $\eta_J - edge$ in regular SVCNGS. We also define totally $\eta_J - edge$ regular SVCNGS and $\eta_J - edge$ regular SVCNGS. It is described under what conditions $\eta_J - edge$ regular SVCNGS and totally $\eta_J - edge$ regular SVCNGS are comparable. We also investigated various $\eta_J - edge$ regular and totally $\eta_J - edge$ regular SVCNGS properties using an example. Furthermore, we have presented an application of SVCNGS in decision-making, that is, identification of best level of development Countries.

There are several potential areas for future research in this area, if it is possible to use the adjacency matrix SVCNGS. Further, for developing future solutions, analyze the isomorphic adjacency matrix, edge regular adjacency matrix, totally edge regular adjacency matrix, etc. Future research areas include Complex Pythagorean fuzzy graph structures, Complex bipolar fuzzy graph structures, and Complex bipolar neutrosophic graph structures, all of which are based on the various properties of the nodes and edges in GS. The following are some of this work's limitations:

- This research and related network systems were mostly focused on SVCNGS.
- This approach can only be used when there are symmetric, irreflexive, and mutually disjoint relations on the CNS.
- The SVCNGS idea is not relevant if the membership values of the characters are provided in distinct environments.
- Sometimes it may not be possible to get real data.

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References

[1] Anam Luqman, Muhammad Akram, and Florentin Smarandache, Complex Neutrosophic Hypergraphs: New SocialNetwork Models *Algorithms*, 2019,12, 234; doi:10.3390/a12110234.

[2] A. Alkouri, A. Salleh, *Complex intuitionistic fuzzy sets. AIP Conf. Proc,*(2012), 14, 464470.

[3] S. Arumugam and S. Velammal, Edge domination in graphs, *Taiwanese J. math,*(1998), 2(2), 177-196.

[4] K.T.Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets Syst,(1986), 20(1), 87-96.

[5] K.T. Atanassove and G. Gargov, Interval valued intuitionistic *fuzzy sets, Fuzzy Sets and Systems*, (1989), 31(3), 343349.

[6] Atiqe Ur Rahman, Muhammad Saeed, Florentin Smarandache and Muhammad Rayees Ahmad, Development of Hybrids of Hypersoft Set with Complex Fuzzy Set, Complex Intuitionistic Fuzzy set and Complex Neutrosophic Set, *Neutrosophic Sets and Systems*,Vol. 38, 2020.

[7] Bhattacharya, P. Some remarks on fuzzy graphs, *Pattern Recognit*Lett, 1987,6, 297302.

[8] T.Dinesh, T.V.Ramakrishnan, Generalised Fuzzy Graph Structures *Applied Mathematical Sciences*,(2011), Vol. 5, no. 4, 173 -180

[9] S.Greenfield, F.Chiclana, S.Dick, Interval-valued complex fuzzy logic, *In Proceedings of the IEEE International Conference on Fuzzy Systems, Vancouver, BC, Canada,*July (2016) 2429 ; pp. 16.

[10] M.G. Karunambigai, K.Palanivel and S. Sivasankar, Edge Regular Intuitionistic Fuzzy Graph *Advances in Fuzzy sets and Systems*, (2015) , Volume 20, No 1, 25-46.

[11] D.Milo, R.Fiedman, M. Kandel, Complex fuzzy sets, *IEEE Trans. Fuzzy Syst,*(2002), 10, 171186.

[12] Muhammad Akram and Rabia Akmal, operations on Intuitionistic Fuzzy Graph Structures,*Fuzzy Inf.Eng*, (2016), 8,389-410.

[13] Muhammad Akram and Sumera Naz, A Novel Decision-Making Approach under Complex Pythagorean Fuzzy Environment, *Math. Comput. Appl*, (2019), 24, 73.

[14] Muhammad Akram, Ayesha Bashir, Sovan Samanta, Complex Pythagorean Fuzzy Planar Graphs, *Int. J. Appl. Comput. Math*(2020) 6-58.

[15] M. Pal and H. Rashmanlou, Irregular interval-valued fuzzy graphs, *Annals of Pure and Applied Mathematics*, (2013), 3(1), 5666.

[16] Prem Kumar Singh, Complex vague set based concept lattice, *Chaos, Solitons and Fractals*, March 2017 Volume 96, Pages 145-153.

[17] H. Rashmanlou and M. Pal, Some properties of highly irregular interval-valued fuzzy graphs, *World Applied Sciences Journal*, (2013), 27(12), 17561773.

[18] D. Ramot, M. Friedman, G.A. Langholz, Kandel, Complex fuzzy logic, *IEEE Trans. Fuzzy Syst*, (2003), 11, 450461.

[19] D. Rani, H. Garg, Distance measures between the complex intuitionistic fuzzy sets and its applications to the decision-making process,*Int. J. Uncertain. Quantif,*(2017), 7, 423439.

[20] D. Rani, H. Garg, Complex intuitionistic fuzzy power aggregation operators and their applications in multi-criteria decision making, *Expert Syst*, (2018), 35, e12325.

[21] A. Rosenfeld, Fuzzy graphs, L.A. Zadeh, K.S. Fu, M. Shimura (Eds.), Fuzzy Sets and their Applications, *Academic Press, New York,*, (1975), pp.77-95.

[22] Saba Siddique, Uzma Ahmad, Muhammad Akram, A study on generalized graphs representations of complex neutrosophic information, *Journal of Applied Mathematics and Computing*, (2020), https://doi.org/10.1007/s12190-020-01400-0.

[23] E. Sampathkumar, Generalized graph structures, *Bulletin of Kerala Mathematics Association*, (2006), vol. 3, no. 2, pp. 65-123.

[24] S. Satham hussain, Said broumi, Young bae jun and Durga Nagarajan, Intuitionistic Bipolar neutrosophic set and its Application to Intuitionistic Bipolar neutrosophic Graph, *Annals of Communications in Mathematics*,(2019), Volume 2, Number 2, 121-140 ISSN: 2582-0818.

[25] S. Satham Hussain, Isnaini Rosyida, Hossein Rashmanlou, F. Mofidnakhaei, Interval intuitionistic neutrosophic sets with its applications to interval intuitionistic neutrosophic graphs and climatic analysis, *Computational and Applied Mathematics* , (2021), 40:121 https://doi.org/10.1007/s40314-021-01504-8.

[26] A. shannon, K.T. Atanassov. A first stap to a theory of the Intuitionistic fuzzy graph, *Proceeding of FUBEST(D. Lako, ED.) sofia, sept* , (1994), 28-30, pp.59-61.

[27] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *International Journal of Pure and Applied Mathematics*,(2010), 24, 289-297.

[28] F. Smarandache, Neutrosophic Graphs, in his book Symbolic Neutrosophic Theory, *Europa, Nova*.

[29] F. Smarandache, Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, *Inter. J.Pure Appl. Math.*(2005), 24, 287-297.

[30] F. Smarandache, A Unifying Field in Logics. Neutrosophy: Neutrosophic Probability, Set and Logic. Rehoboth: *American Research Press*, 1999.

[31] F. Smarandache, Neutrosophy, Neutrosophic Probability, Set, and Logic, *Amer. Res. Press, Rehoboth, USA*, 105 pages, 1998; http://fs.gallup.unm.edu/eBookneutrosophics4.pdf(4th edition).

[32] F. Smarandache, M. Ali, Complex neutrosophic set, *Neural Comput. APPl.*(2017), 28(7), 1817-1834. https://doi.org/10.1007/s00521-015-2154-y.

[33] Soumitra Poulik, Ganesh Ghorai, Applications of graphs complete degree with bipolar fuzzy information, *Complex and Intelligent Systems (2022) 8:11151127.*

[34] Soumitra Poulik, Ganesh Ghorai, Estimation of most effected cycles and busiest network route based on complexity function of graph in fuzzy environment Artificial Intelligence Review*(2022) 55:45574574.*

[35] Soumitra Poulik, Ganesh Ghorai, Determination of journeys order based on graphs Wienerabsolute index with bipolar fuzzy information Information Sciences*545 (2021) 608619*

[36] Soumitra Poulik, Ganesh Ghorai, Randic index of bipolar fuzzy graphs and its application innetwork systems Journal of Applied Mathematics and Computing *, August 2021, https://doi.org/10.1007/s12190-021-01619-5.*

[37] Soumitra Poulik, Ganesh Ghorai, Connectivity Concepts in Bipolar Fuzzy Incidence Graphs, Thai Journal of Mathematics,*(2022),Volume 20 Number 4 Pages 16091619.*

[38] Thirunavukarasu, P. Suresh, R. Viswanathan K.K, Energy of a complex fuzzy graph, Int. J. Math. Sci.Eng*, Appl, 2016, 10, 243248.*

[39] K. Ullah, T. Mahmood, Z. Ali, N. Jan, On some distance measures of complex Pythagorean fuzzy sets and their applications in pattern recognition, Complex Intell. Syst*, (2019).*

[40] Xindong Peng, Jingguo Dai, A bibliometric analysis of neutrosophic set: two decadesreview from 1998 to 2017, Artificial Intelligence Review*August 2018;https://doi.org/10.1007/s10462-018-9652-0.*

[41] R.R.Yager, Abbasov, A.M. Pythagorean membership grades, complex numbers, and decision making. Int. J. Intell. Syst *, (2013), 28, 436452.*

[42] R.R.Yager, Pythagorean membership grades in multi-criteria decision making, IEEE Trans. Fuzzy Syst*, (2014), 22, 958965.*

[43] O. Yazdanbakhsh, S. Dick, A systematic review of complex fuzzy sets and logic, Fuzzy Sets Syst,*(2018), 338, 122.*

[44] Yaqoob .N, Gulistan M, Kadry S,Wahab H, Complex intuitionistic fuzzy graphs with application incellular network provider companies, Mathematics*2019, 7, 35.*

[45] Yaqoob .N, Akram .M, Complex neutrosophic graphs, Bull. Comput. Appl. Math*2018,6, 85109.*

[46] L. A. Zadeh, Similarity Relations and fuzzy Ordering, Inf. Control*, (1971), 3, 177-200.*

[47] L. A. Zadeh, The concept of a linguistic and application to approximate reasoning I, Inform. Sci*,(1975), 8, 199-249.*

*[48] L. A. Zadeh, Fuzzy sets,*Inf. Control*, (1965), 8, 338-353.*

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A TreeSoft Set with Interval Valued Neutrosophic Set in the era of Industry 4.0

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Abstract The introduction of Industry 4.0 has brought about a significant shift in the manufacturing and supply chain management sectors, requiring supplier selection procedures to be adjusted to this rapidly changing technical environment. This study aims to improve supplier selection in Industry 4.0. This selection contains various criteria, so multi-criteria decision-making (MCDM) is used to deal with these criteria. The interval-valued neutrosophic sets (IVNSs) are used to deal with uncertainty in the evaluation process. The IVNSs are integrated with the TreeSoft Set. The TOPSIS method is an MCDM method used to rank the alternatives. The results show that the economic criterion is the most important, and supplier 7 is the best.

Keywords: TreeSoft Set; Interval Valued Neutrosophic Set; Industry 4.0; Supplier Selection; Multi-Criteria Decision Making.

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1. Introduction

Known as the fourth industrial revolution, or industry 4.0, supply networks have been significantly and extensively impacted. This change has profoundly impacted how companies plan, run, and maximize their supply chain processes. Fundamentally, Industry 4.0 denotes a paradigm change in how businesses use digital technology to improve supply chain responsiveness, productivity, and visibility. Integrating cutting-edge technology is one of Industry 4.0's most significant effects on supply chains. They include automation, big data analytics, machine learning, artificial intelligence (AI), and the Internet of Things (IoT)[1].

These advances make it possible to monitor supply chain operations in real-time, which makes data-driven decision-making easier. Massive volumes of data are gathered by Internet of Things (IoT) sensors installed on assets like machinery, vehicles, and products. These sensors provide essential insights into the whereabouts and condition of items at every stage of the supply chain. AI and machine learning algorithms may process this data to estimate demand, optimize routes, and even carry out anticipatory equipment repairs, lowering expenses and downtime[2], [3].

The idea of a "smart" and networked supply chain ecosystem is another critical component of Industry 4.0's effects on supply chains. Businesses may use digital twins to build virtual versions of their supply chains, enabling detailed modeling and simulation. This helps companies to find bottlenecks, test different scenarios, and create more effective plans. These digital twins may also be utilized for real-time process monitoring and control, allowing quick modifications to minimize interruptions or take advantage of opportunities. An essential component of supply chains enabled by Industry 4.0 is automation and robots. The physical transportation of items is more accurate and efficient when autonomous robots, drones, and automated material handling systems are used[4], [5].

Warehouses and distribution centers increasingly use automated solutions to expedite order fulfillment and lower mistake rates. Furthermore, localized and on-demand production made possible by 3D printing in manufacturing might substantially change the supply chain by lowering the need for large stocks and long-distance shipping[6], [7].

Industry 4.0 encourages openness and cooperation through the supply chain. Blockchain technology builds secure, unchangeable ledgers of goods movements and transactions. This improves transparency and confidence, especially in sectors like food and pharmaceuticals with intricate, multi-tiered supply chains. By providing a common source of truth, stakeholders can guarantee the safety and legitimacy of the product. Moreover, Industry 4.0 makes it possible to transition from a linear supply chain model to a more sustainable and circular one. Businesses may minimize waste, maximize resource utilization, and lessen their environmental impact using data-driven insights[8], [9]. Supply chain strategies are starting to place a greater emphasis on sustainability and corporate social responsibility, and Industry 4.0 technologies facilitate the achievement of these goals.

This study used the concept of multi-criteria decision-making[10]–[12] for supplier selection in Industry 4.0 under an interval-valued neutrosophic set, and TreeSoft set. The MCDM has various applications[13]–[16].

It may be inferred from the literature that most studies have focused on quantitative, quantifiable attributes when examining supplier selection in Industry 4.0. Although fuzzy set theories (FSs) are useful for handling uncertain issues, they are only suitable for handling inconsistent or ambiguous data. As a result, in the context of supplier selection in Industry 4.0, sophisticated computational methods like Neutrosophic Sets (NSs) [17], a generalization of FSs, and Intuitionistic Fuzzy Sets (IFSs) have gained significance[18], [19].

Up to this point, no research has yet addressed the Industry 4.0 challenge of supplier selection utilizing the IVNS subset of NSs[20]–[22]. One key feature of Zhang et al.'s IVNS theory is that the membership, non-membership, and indeterminacy functions are considered as intervals rather than a single precise value. Furthermore, resolving intricate scientific and technical issues in IVNS takes into account the viewpoints of several experts with varying degrees of training, expertise, and interest. As a result, to evaluate performance in a group MCDM issue[23]–[26], the views of many experts must be combined.

2. Materials and Methods

This section introduces two parts: the interval-valued neutrosophic sets (IVNSs)[27] and the TreeSoft (TS) with IVNSs (TSIVNS).

Definition 1.

The neutrosophic set k can be defined with three membership functions as truth, indeterminacy, and falsity membership degrees $T_k(u_i), I_k(u_i), and\ F_k(u_i)$,

$$k = \{(T_k(u_i), I_k(u_i), F_k(u_i)) |\ u_i \in U\} \quad (1)$$

Where U refers to the universal numbers including components u_i.

$$0^- \leq \sup T_k(u_i) + \sup I_k(u_i) + \sup F_k(u_i) \leq 3^+ \quad (2)$$

Definition 2.

We can define an IVNS as:

$$T_k(u_i) = [\inf T_k(u\ ), \sup T_k(u\ )] \quad (3)$$

$$I_k(u_i) = [\inf I_k(u\ ), \sup I_k(u\ )] \quad (4)$$

$$F_k(u_i) = [\inf F_k(u\ ), \sup F_k(u\ )] \quad (5)$$

$$k = \{[\inf T_k(u\ ), \sup T_k(u\ )\ ], [\inf I_k(u\ ), \sup I_k(u\ )\ ], [\inf F_k(u\ ), \sup F_k(u\ )\ ] |\ u \in U\} \quad (6)$$

Definition 3.

We can present some IVNS operations as:

Let interval-valued neutrosophic numbers as: $k_1 = [\inf T_{k_1}(u\ ), \sup T_{k_1}(u\ )\ ], [\inf I_{k_1}(u\ ), \sup I_{k_1}(u\ )\ ], [\inf F_{k_1}(u\ ), \sup F_{k_1}(u\ )\ ]$ and $k_2 = [\inf T_{k_2}(u\ ), \sup T_{k_2}(u\ )\ ], [\inf I_{k_2}(u\ ), \sup I_{k_2}(u\ )\ ], [\inf F_{k_2}(u\ ), \sup F_{k_2}(u\ )\ ]$

$$k_1 + k_2 = \begin{bmatrix} \begin{bmatrix} \inf T_{k_1}(u) + \inf T_{k_2}(u) - \inf T_{k_1}(u) \cdot \inf T_{k_2}(u), \\ \sup T_{k_1}(u\ ) + \sup T_{k_2}(u\ ) - \sup T_{k_1}(u\ ) \cdot \sup T_{k_2}(u\ ) \end{bmatrix}, \\ [\inf I_{k_1}(u\ ) \cdot \inf I_{k_2}(u\ ), \sup I_{k_1}(u\ ) \cdot \sup I_{k_2}(u\ )\ ], \\ [\inf F_{k_1}(u\ ) \cdot \inf F_{k_2}(u\ ), \sup F_{k_1}(u\ ) \cdot \sup F_{k_2}(u\ )\ ] \end{bmatrix} \quad (7)$$

$$k_1 \cdot k_2 = \begin{bmatrix} [\inf T_{k_1}(u) \cdot \inf T_{k_2}(u), \sup T_{k_1}(u\ ) \cdot \sup T_{k_2}(u\ )] \\ \begin{bmatrix} \inf I_{k_1}(u) + \inf I_{k_2}(u) - \inf I_{k_1}(u) \cdot \inf I_{k_2}(u), \\ \sup I_{k_1}(u\ ) + \sup I_{k_2}(u\ ) - \sup I_{k_1}(u\ ) \cdot \sup I_{k_2}(u\ ) \end{bmatrix}, \\ \begin{bmatrix} \inf F_{k_1}(u) + \inf F_{k_2}(u) - \inf F_{k_1}(u) \cdot \inf F_{k_2}(u), \\ \sup F_{k_1}(u\ ) + \sup F_{k_2}(u\ ) - \sup F_{k_1}(u\ ) \cdot \sup F_{k_2}(u\ ) \end{bmatrix} \end{bmatrix} \quad (8)$$

$$s \cdot k_1 = \begin{bmatrix} [1 - (1 - \inf T_{k_1}(u))^s, 1 - (1 - \sup T_{k_1}(u))^s\ ], \\ [(\inf I_{k_1}(u))^s, (\sup I_{k_1}(u))^s\ ], \\ [(\inf F_{k_1}(u))^s, (\sup F_{k_1}(u))^s] \end{bmatrix} \quad (9)$$

$$k_1^s = \begin{bmatrix} [(\inf T_{k_1}(u))^s, (\sup T_{k_1}(u))^s], \\ [1 - (1 - \inf I_{k_1}(u))^s, 1 - (1 - \sup I_{k_1}(u))^s\ ], \\ [1 - (1 - \inf F_{k_1}(u))^s, 1 - (1 - \sup F_{k_1}(u))^s\ ] \end{bmatrix} \quad (10)$$

2.1 TreeSoft [28]

Let U be a universe disclosure and H a non-empty subset of U, with $P(H)$ $be\ a\ powerset\ of\ H$.

Let TSR be a set of attributes of the problem (criteria),

$$TSR = \{TSR_1, TSR_2, \dots, TSR_\mathrm{n}\}, n \geq 1 \quad (11)$$

Where $TSR_1, TSR_2, \dots, TSR_\mathrm{n}$ are criteria of the first level of the tree.

Each attribute $TSR_1, 1 \leq i \leq n, is\ formed\ by\ sub - attributes$:

$TSR_1 = \{TSR_{1,1}, TSR_{1,2}. \dots,\}$

$TSR_2 = \{TSR_{2,1}, TSR_{2,2}. \dots,\}$

.

.

$TSR_\mathrm{n} = \{TSR_{\mathrm{n},1}, TSR_{\mathrm{n},2}. \dots,\}$

Where $TSR_{\mathrm{i,j}}$ are sub-attributes.

The TreeSoft set can be formed by:

$$F: P\big(Tree(TSR)\big) \rightarrow P(H) \quad (12)$$

$Tree(TSR)$ is the set of all nodes and leaves from level 1 to level m and $P(Tree(TSR))$ is the power set of the $Tree(TSR)$.

$$Tree(TSR) = \{TSR_i |\ i_1 = 1,2,3, \dots\} \cup \{TSR_i |\ i_1, i_2 = 1,2,3, \dots\} \cup \{TSR_i |\ i_1, i_2, i_3 = 1,2,3, \dots\} \cup \dots \cup \{TSR_i | i_1, i_2, \dots\ i_m = 1,2,3, \dots\} \quad (13)$$

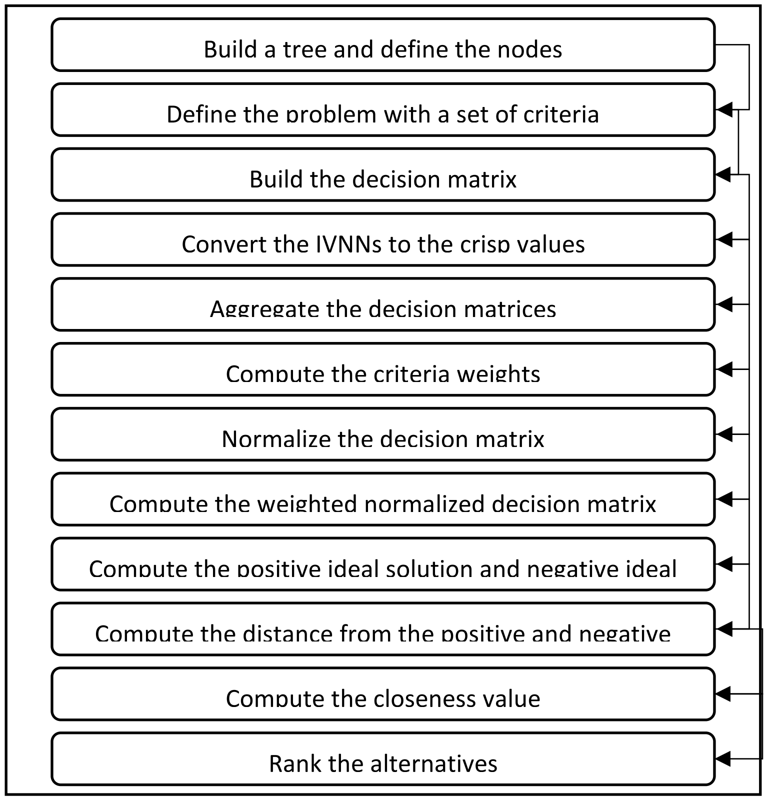

Figure 1. The steps of the TSIVNS TOPSIS method.

2.2 TSIVNS TOPSIS Method

This part introduces the TOPSIS method[29]–[31] with the TSIVNS. Figure 1 shows the research framework:

Step 1. Build a tree and define the nodes.

The tree has more than one level, in the first level, the main criteria and introduced as $TSR_1, TSR_2 \dots, TSR_n$

In the second level, the sub-criteria are introduced as $TSR_{1,1}, TSR_{1,2}, \dots.$ And $TSR_{2,1}, TSR_{2,2}, \dots.$

Step 2. Define the problem with a set of criteria.

The main, and sub-criteria are defined in this step by problem definition.

Step 3. Build the decision matrix[27].

The decision matrix is built based on the IVNSs

$$A = \begin{pmatrix} \begin{bmatrix} [\inf T_{k_{11}}(u), \sup T_{k_{11}}(u)], \\ [[\inf I_{k_{11}}(u), \sup I_{k_{11}}(u)]], \\ [[\inf F_{k_{11}}(u), \sup F_{k_{11}}(u)]] \end{bmatrix} & \cdots & \begin{bmatrix} [\inf T_{k_{1n}}(u), \sup T_{k_{1n}}(u)], \\ [[\inf I_{k_{1n}}(u), \sup I_{k_{1n}}(u)]], \\ [[\inf F_{k_{1n}}(u), \sup F_{k_{1n}}(u)]] \end{bmatrix} \\ \vdots & \ddots & \vdots \\ \begin{bmatrix} [\inf T_{k_{m1}}(u), \sup T_{k_{m1}}(u)], \\ [[\inf I_{k_{m1}}(u), \sup I_{k_{m1}}(u)]], \\ [[\inf F_{k_{m1}}(u), \sup F_{k_{m1}}(u)]] \end{bmatrix} & \cdots & \begin{bmatrix} [\inf T_{k_{mn}}(u), \sup T_{k_{mn}}(u)], \\ [[\inf I_{k_{mn}}(u), \sup I_{k_{mn}}(u)]], \\ [[\inf F_{k_{mn}}(u), \sup F_{k_{mn}}(u)]] \end{bmatrix} \end{pmatrix} \quad (13)$$

Step 4. Convert the IVNNs to the crisp values[27].

The IVNNs are converted by the score function to crisp values, then the decision matrix as:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad (14)$$

Step 5. Aggregate the decision matrices

The decision matrices are combined using the average method.

Step 6. Compute the criteria weights.

The criteria weights are computed by the average method.

Step 7. Normalize the decision matrix.

$$T_{ij} = \frac{a_{ij}}{\sqrt{\sum_{i=1}^{n} a_{ij}^2}} \quad (15)$$

Step 8. Compute the weighted normalized decision matrix

$$R_{ij} = T_{ij} \cdot w_j \quad (16)$$

Step 9. Compute the positive ideal solution and negative ideal solution.

$$P_j^+ = \left\{ \begin{matrix} \max\limits_{i=1,\dots,n} r_{ij} \ \textit{positive criteria} \\ \min\limits_{i=1,\dots,n} r_{ij} \ \textit{negative criteria} \\ . \end{matrix} \right\} \quad (17)$$

$$P_j^- = \left\{ \begin{matrix} \min\limits_{i=1,\dots,n} r_{ij} \ \textit{positive criteria} \\ \max\limits_{i=1,\dots,n} r_{ij} \ \textit{negative criteria} \\ . \end{matrix} \right\} \quad (18)$$

Step 10. Compute the distance between the positive and negative ideal solutions.

$$d^+(a_i) = \sqrt{\sum_{j=1}^{m} \left(r_{ik} - p_j^+\right)^2} \quad (19)$$

$$d^{-}(a_i) = \sqrt{\sum_{j=1}^{m}\left(r_{ik} - p_j^{-}\right)^2} \quad (20)$$

Step 11. Compute the closeness value.

$$S_i = \frac{d^{-}(a_i)}{d^{-}(a_i) + d^{+}(a_i)} \quad (21)$$

Step 12. Rank the alternatives.

3. Application

This section introduces the results of TRIVNS with the TOPSIS method to select the best supplier in industry 4.0. There are three experts are invited to evaluate the criteria and alternatives in this study.

Step 1. Build a tree and define the nodes.

We built the tree nodes with one level with ten nodes. The ten nodes present the main criteria in this study as shown in Figure 2.

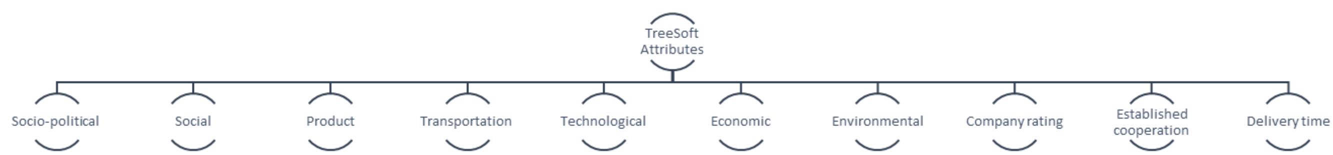

Figure 2. Level 1 of nodes.

Step 2. Define the problem with a set of criteria.

There are ten criteria used in this study.

Step 3. Build the decision matrix.

We built three decision matrices by the opinions of experts in Eq. (13) as shown in Table 1.

Table 1. Three decision matrices by IVNNs.

	TRS_1	TRS_2	TRS_3	TRS_4	TRS_5	TRS_6	TRS_7	TRS_8	TRS_9	TRS_{10}
TRA_1	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])
TRA_2	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])

TRA_3	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])
TRA_4	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])
TRA_5	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])
TRA_6	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA_7	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA_8	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])
TRA_9	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])
TRA_{10}	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])
	TRS_1	TRS_2	TRS_3	TRS_4	TRS_5	TRS_6	TRS_7	TRS_8	TRS_9	TRS_{10}
TRA_1	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])
TRA_2	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA_3	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])

TRA$_4$	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])
TRA$_5$	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA$_6$	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])
TRA$_7$	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA$_8$	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])
TRA$_9$	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA$_{10}$	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])
	TRS$_1$	TRS$_2$	TRS$_3$	TRS$_4$	TRS$_5$	TRS$_6$	TRS$_7$	TRS$_8$	TRS$_9$	TRS$_{10}$
TRA$_1$	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])
TRA$_2$	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA$_3$	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])
TRA$_4$	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])

TRA_5	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])
TRA_6	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA_7	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])
TRA_8	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])
TRA_9	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.4,0.6], [0.4,0.5], [0.3,0.4])
TRA_{10}	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.1,0.2], [0.5,0.6], [0.7,0.8])	([0.2,0.4], [0.5,0.6], [0.5,0.6])	([0.4,0.6], [0.4,0.5], [0.3,0.4])	([0.6,0.8], [0.3,0.4], [0.2,0.4])	([0.7,0.9], [0.2,0.3], [0.1,0.2])	([0.6,0.8], [0.3,0.4], [0.2,0.4])

Step 4. Convert the IVNNs to the crisp values.

The IVNNs are converted by the score function to crisp values, then the decision matrix as:

Step 5. Aggregate the decision matrices

Step 6. Compute the criteria weights as shown in Figure 3. The economy has the highest importance and society has the lowest importance.

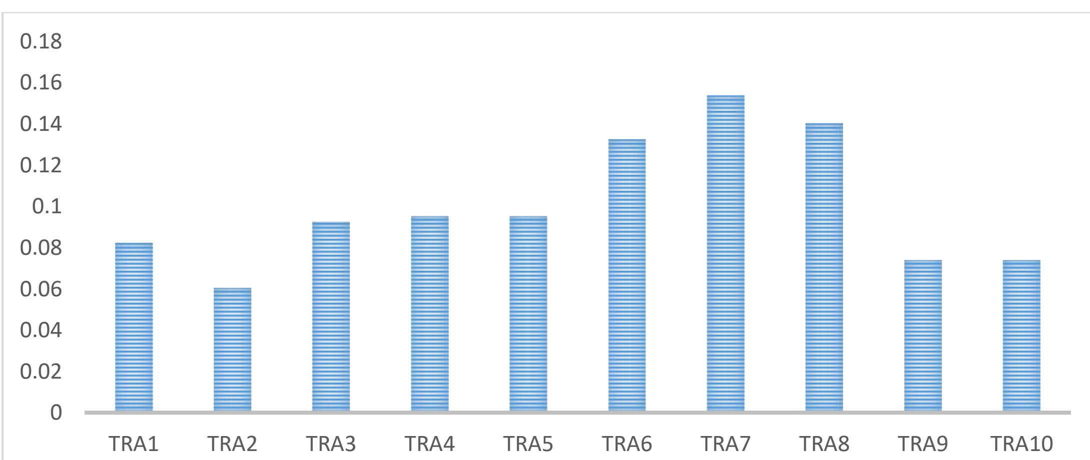

Figure 3. The criteria weights.

The criteria weights are computed by the average method.

Step 7. Normalize the decision matrix by using Eq. (15) as shown in Table 2.

Table 2. Normalization decision matrix.

	TRS$_1$	**TRS$_2$**	**TRS$_3$**	**TRS$_4$**	**TRS$_5$**	**TRS$_6$**	**TRS$_7$**	**TRS$_8$**	**TRS$_9$**	**TRS$_{10}$**
TRA$_1$	0.165528	0.185954	0.444651	0.16747	0.182788	0.378925	0.382012	0.357618	0.143203	0.207747
TRA$_2$	0.340355	0.143451	0.199523	0.182524	0.141008	0.274478	0.248961	0.218545	0.185633	0.160262
TRA$_3$	0.205515	0.262107	0.199523	0.446902	0.450877	0.309294	0.298753	0.316227	0.247511	0.276996
TRA$_4$	0.435208	0.143451	0.16912	0.440316	0.362095	0.296339	0.284061	0.23179	0.375687	0.46298
TRA$_5$	0.195286	0.376336	0.322087	0.182524	0.3325	0.281765	0.311814	0.245035	0.457896	0.411538
TRA$_6$	0.386852	0.458688	0.266031	0.487358	0.318574	0.34411	0.171416	0.302982	0.323532	0.261168
TRA$_7$	0.315247	0.480825	0.492157	0.344349	0.220216	0.226707	0.382012	0.38742	0.337676	0.160262
TRA$_8$	0.245502	0.143451	0.444651	0.27849	0.407356	0.419409	0.443232	0.344373	0.17149	0.292825
TRA$_9$	0.150649	0.143451	0.199523	0.197578	0.407356	0.378925	0.276714	0.316227	0.233368	0.276996
TRA$_{10}$	0.504953	0.480825	0.199523	0.197578	0.141008	0.170031	0.276714	0.38742	0.479995	0.46298

Step 8. Compute the weighted normalized decision matrix by using Eq. (16) as shown in Table 3.

Table 3. Weighted normalized decision matrix.

	TRS$_1$	**TRS$_2$**	**TRS$_3$**	**TRS$_4$**	**TRS$_5$**	**TRS$_6$**	**TRS$_7$**	**TRS$_8$**	**TRS$_9$**	**TRS$_{10}$**
TRA$_1$	0.013631	0.011252	0.041053	0.015932	0.017389	0.050232	0.058744	0.050196	0.010586	0.015357
TRA$_2$	0.028027	0.00868	0.018421	0.017364	0.013415	0.036386	0.038284	0.030675	0.013723	0.011847
TRA$_3$	0.016923	0.015861	0.018421	0.042516	0.042894	0.041001	0.045941	0.044386	0.018297	0.020477
TRA$_4$	0.035837	0.00868	0.015614	0.041889	0.034448	0.039284	0.043681	0.032534	0.027772	0.034225
TRA$_5$	0.016081	0.022773	0.029737	0.017364	0.031632	0.037352	0.047949	0.034394	0.033849	0.030422
TRA$_6$	0.031856	0.027756	0.024562	0.046364	0.030307	0.045617	0.026359	0.042527	0.023917	0.019307
TRA$_7$	0.025959	0.029095	0.045439	0.032759	0.02095	0.030053	0.058744	0.054379	0.024962	0.011847
TRA$_8$	0.020216	0.00868	0.041053	0.026494	0.038753	0.055598	0.068158	0.048337	0.012677	0.021647
TRA$_9$	0.012405	0.00868	0.018421	0.018796	0.038753	0.050232	0.042551	0.044386	0.017251	0.020477
TRA$_{10}$	0.041581	0.029095	0.018421	0.018796	0.013415	0.02254	0.042551	0.054379	0.035483	0.034225

Step 9. Compute the positive ideal solution and negative ideal solution by using Eqs. (17 and 18).

Step 10. Compute the distance from the positive and negative ideal solutions by using Eqs. (19 and 20).

Step 11. Compute the closeness value by using Eq. (21) as shown in Figure 4.

Step 12. Rank the alternatives. Alternative 7 is the best and alternative 2 is the worst.

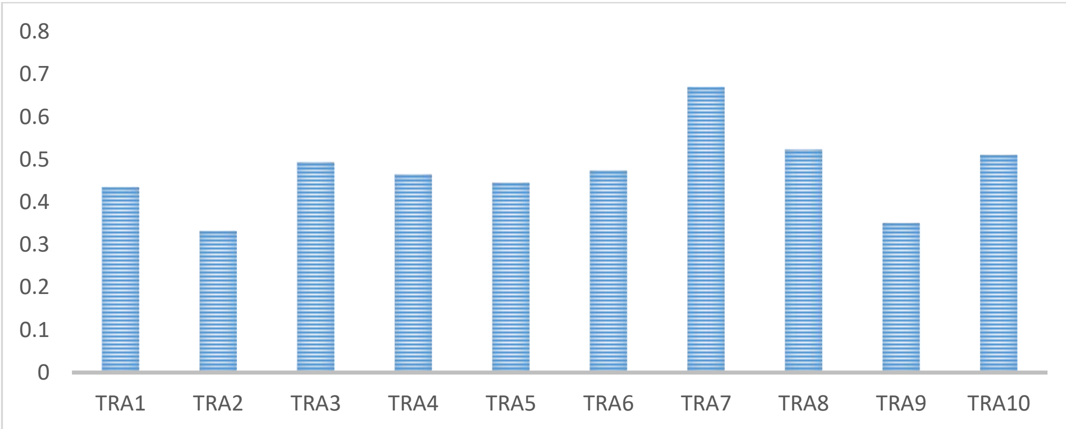

Figure 4. The closeness values.

4. Conclusions

This paper used the MCDM methodology for the supplier section in Industry 4.0. This study used the TOPSIS method as an MCDM method for ranking alternatives and used the best one. The TOPSIS method is integrated with the IVNSs and TreeSoft Set. Three experts are invited to evaluate the criteria and options in this study. This study used ten criteria and ten suppliers. The decision matrices are built by the opinions of experts. Then, we use the IVNNs to evaluate the requirements and alternatives. The IVNNs are converted to the crisp values. The results show that the economic criterion has the highest weight, and the social criterion has the lowest. Alternative 7 is the best, and alternative 2 is the worst.

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References

[1] S. K. Sahoo, S. S. Goswami, and R. Halder, "Supplier Selection in the Age of Industry 4.0: A Review on MCDM Applications and Trends," *Decision Making Advances*, vol. 2, no. 1, pp. 32–47, 2024.

[2] L. S. Dalenogare, G. B. Benitez, N. F. Ayala, and A. G. Frank, "The expected contribution of Industry 4.0 technologies for industrial performance," *International Journal of production economics*, vol. 204, pp. 383–394, 2018.

[3] H. Lasi, P. Fettke, H.-G. Kemper, T. Feld, and M. Hoffmann, "Industry 4.0," *Business & information systems engineering*, vol. 6, pp. 239–242, 2014.

[4] A. G. Frank, L. S. Dalenogare, and N. F. Ayala, "Industry 4.0 technologies: Implementation patterns in manufacturing companies," *International journal of production economics*, vol. 210, pp. 15–26, 2019.

[5] D. Gorecky, M. Schmitt, M. Loskyll, and D. Zühlke, "Human-machine-interaction in the industry 4.0 era," in *2014 12th IEEE international conference on industrial informatics (INDIN)*, Ieee, 2014, pp. 289–294.

[6] M. Ghobakhloo, "Industry 4.0, digitization, and opportunities for sustainability," *Journal of cleaner production*, vol. 252, p. 119869, 2020.

[7] M. Sony and S. Naik, "Key ingredients for evaluating Industry 4.0 readiness for organizations: a literature review," *Benchmarking: An International Journal*, vol. 27, no. 7, pp. 2213–2232, 2020.

[8] A. Raj, G. Dwivedi, A. Sharma, A. B. L. de Sousa Jabbour, and S. Rajak, "Barriers to the adoption of industry 4.0 technologies in the manufacturing sector: An inter-country comparative perspective," *International Journal of Production Economics*, vol. 224, p. 107546, 2020.

[9] C. Bai, P. Dallasega, G. Orzes, and J. Sarkis, "Industry 4.0 technologies assessment: A sustainability perspective," *International journal of production economics*, vol. 229, p. 107776, 2020.

[10] Mohamed Abouhawwash, Nitin Mittal, Asiye Yilmaz Adkinson, "Single Valued Neutrosophic Set with Multi-Criteria Decision Making Methodology for Wind Turbine Development," Neutrosophic Optimization and Intelligent Systems (NOIS), vol. 1, pp. 31–38, 2024, DOI: https://doi.org/10.61356/j.nois.2024.16189.

[11] I. Mahdavi, N. Mahdavi-Amiri, A. Heidarzade, and R. Nourifar, "Designing a model of fuzzy TOPSIS in multiple criteria decision making," *Applied Mathematics and Computation*, vol. 206, no. 2, pp. 607–617, 2008.

[12] Nada A. Nabeeh, Karam M. Sallam, "A Combined Compromise Solution (CoCoSo) of MCDM Problems for Selection of Medical Best Bearing Ring," Neutrosophic Optimization and Intelligent Systems (NOIS), vol. 1, pp. 1–13, 2024, DOI: https://doi.org/10.61356/j.nois.2024.16089.

[13] M. Mohamed, "BHARAT Decision Making Model: Harness an Innovative MCDM Methodology for Recommending Beneficial E-Commerce Website," *Multicriteria Algorithms with Applications*, vol. 2, pp. 53–64, 2024.

[14] S. N. S. Bathusha, S. Jayakumar, and S. A. K. Raj, "The Energy of Interval-Valued Complex Neutrosophic Graph Structures: Framework, Application and Future Research Directions," *Neutrosophic Systems with Applications*, vol. 13, pp. 67–101, 2024.

[15] Z. Mohamed, M. M. Ismail, and A. Abd El-Gawad, "Sustainable supplier selection using neutrosophic multi-criteria decision making methodology," *Sustain. Mach. Intell. J*, vol. 3, 2023.

[16] M. Jameel and S. Tanwar, "A Deeper Monitoring and Evaluation of the Nature of Barriers to Climate Change Adaptation Planning under Fuzzy Multi-Criteria Decision Making Methodology,"

Multicriteria Algorithms with Applications, vol. 2, pp. 43–52, 2024.

[17] F. Smarandache, "A unifying field in Logics: Neutrosophic Logic.," in *Philosophy*, American Research Press, 1999, pp. 1–141.

[18] F. Smarandache, *Introduction to neutrosophic statistics*. Infinite Study, 2014.

[19] F. Smarandache, *Introduction to neutrosophic measure, neutrosophic integral, and neutrosophic probability*. Infinite Study, 2013.

[20] F. Smarandache, *Symbolic neutrosophic theory*. Infinite Study, 2015.

[21] F. Smarandache *et al.*, "Introduction to neutrosophy and neutrosophic environment," in *Neutrosophic Set in Medical Image Analysis*, Elsevier, 2019, pp. 3–29.

[22] F. Smarandache and M. Jdid, "An Overview of Neutrosophic and Plithogenic Theories and Applications," 2023.

[23] Muhammad Saeed, Imrana Shafique, "Relation on Fermatean Neutrosophic Soft Set with Application to Sustainable Agriculture," HyperSoft Set Methods in Engineering (HSSE), vol. 1, pp. 21–33, 2024, DOI: https://doi.org/10.61356/j.hsse.2024.18250.

[24] J.-J. Wang, Y.-Y. Jing, C.-F. Zhang, and J.-H. Zhao, "Review on multi-criteria decision analysis aid in sustainable energy decision-making," *Renewable and sustainable energy reviews*, vol. 13, no. 9, pp. 2263–2278, 2009.

[25] D. Jato-Espino, E. Castillo-Lopez, J. Rodriguez-Hernandez, and J. C. Canteras-Jordana, "A review of application of multi-criteria decision making methods in construction," *Automation in construction*, vol. 45, pp. 151–162, 2014.

[26] Darvin Manuel Ramírez Guerra, Osmanys Pérez Peña, John Alex Torres Yanez, "Neutrosophic Evaluation of Ethical Factors in Remote Medical Care," HyperSoft Set Methods in Engineering (HSSE), vol. 1, pp. 11–20, 2024, DOI: https://doi.org/10.61356/j.hsse.2024.17950.

[27] R. S. U. Haq, M. Saeed, N. Mateen, F. Siddiqui, and S. Ahmed, "An interval-valued neutrosophic based MAIRCA method for sustainable material selection," *Engineering Applications of Artificial Intelligence*, vol. 123, p. 106177, 2023.

[28] F. Smarandache, *Practical applications of IndetermSoft Set and IndetermHyperSoft Set and introduction to TreeSoft Set as an extension of the MultiSoft Set*. Infinite Study, 2022.

[29] C. M. Brugha, "Structure of multi-criteria decision-making," *Journal of the Operational research Society*, vol. 55, no. 11, pp. 1156–1168, 2004.

[30] M. Panda and A. K. Jagadev, "TOPSIS in multi-criteria decision making: a survey," in *2018 2nd International Conference on Data Science and Business Analytics (ICDSBA)*, IEEE, 2018, pp. 51–54.

[31] E. Roszkowska, "Multi-criteria decision making models by applying the TOPSIS method to crisp and interval data," *Multiple Criteria Decision Making/University of Economics in Katowice*, vol. 6, no. 1, pp. 200–230, 2011.

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On Neutrosophic $\alpha_{(\gamma,\beta)}$-Continuous Functions, Neutrosophic $\alpha_{(\gamma,\beta)}$-Open (Closed) Functions in Neutrosophic Topological Spaces

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Abstract: In the present script, we explain the neutrosophic $\alpha_{(\gamma,\beta)}$ -continuous function in the neutrosophic topological spaces. We analyze their behaviour; study the various relations and properties existing among them.

Further we like to extend the study to neutrosophic $\alpha_{(\gamma,\beta)}$ -open function, $\alpha_{(\gamma,\beta)}$ -closed function, neutrosophic (γ_{ne},β_{ne}) - irresolute functions and neutrosophic $\alpha_{(\gamma,\beta)}$ - homeomorphism in the neutrosophic topological spaces. The relationship among them can be studied in detail. The neutrosophic $(\alpha_{\gamma},\beta_{ne})$ -continuous function, neutrosophic $(\gamma_{ne},\alpha_{\beta})$ -open(closed) function, neutrosophic α_{γ}-limit point, neutrosophic α_{γ}-derived set and neutrosophic α_{γ}-neighbourhood point are explained and utilized to obtain various remarkable properties. They are explored through the specified examples.

Key Words: neutrosophic γ -open set (closed set), neutrosophic α_{γ}-open set (closed set), neutrosophic $\alpha_{(\gamma,\beta)}$ -continuous function, neutrosophic $\alpha_{(\gamma,\beta)}$ -open(closed) function, neutrosophic $\alpha_{(\gamma,\beta)}$ -homeomorphism.

1.Introduction and motivation

The following notations have been used throught this paper: neut- neutrosophic, neut topo spa- neutrosophic topological space, neutro- neutrosophy, topo spa- topological space, se-set, topol-topology, ses-sets, spa-space, spas-spaces, fuz-fuzzy

Many philosophies have been raised for vagueness counting the argument of probability, the approach of fuz ses, the ideology of intuitionistic fuz ses, the idea of rough ses, and so on. Even though numerous novel approaches have been extended as a development of these concepts there are still various problems, main complications arise due to the insufficiency of parameters.

Lugojan [11] studied the generalized topology during the year 1982.The concept of fuz topo spas was dealt by Chang [5]. Neut se is a generalization of a classic se, a fuz se and a Intuitionstic fuzzy se. The word Neutro means skill on neutrals. Neut method is derived from Fuzzy logic or Intuitionstic fuz logic. In 1965, Zadeh [21] familiarized the fuz se and in 1983 Atanassov [2-4] presented the Intuitionstic fuz se. A novel division of the idea named Neutro was introduced by Smarandache [15-18] in 1999 by totaling a self-governing indeterminacy-affiliation function.

In a neut set, the indeterminacy is calibrated obviously. The certainty-affiliation function, indeterminacy- affiliation function, and erroneous- affiliation function is ultimately self-regulating. Wang [19] described the lone valued neut set and then offered the se-formularized process and a variety of resources of lone valued neu ses.

Salama [12-14], initiated neut topo spas which was a generality of Intuitionistic fuz topo spa exposed by Coker [6] also a neut se as well the degree of membership, the degree of indeterminacy then the degree of non- membership of respective element.

N. Kalaivani and G. Sai Sundara Krishnan [9] introduced the α_γ-open sets, α_γ- Ti spaces with the help of α-open sets and γ-open sets. Their properties were studied. They also introduced $\alpha_{(\gamma,\beta)}$-continuous functions, $\alpha_{(\gamma,\beta)}$-irresolute functions, [8] $\alpha_{(\gamma,\beta)}$-open(closed) functions [7] and studied them in detail.

To fill up the gap existing in the neu theory, now we want to introduce γ-open ses, neu α_γ-open ses, γT_i spas, α_γ- Ti spas, $\alpha_{(\gamma,\beta)}$-continuous functions and $\alpha_{(\gamma,\beta)}$-open (closed) functions in neutrosophic Topological Spaces.

In our earlier article [10] the insight of neu γ-open ses, neu α_γ-open ses, that are created through neu α_γ-open ses are deliberated besides few of their fundamental properties were discovered. The connection among these neu γTi spas in addition neu α_γ- Ti spas were incarnated over drawings besides evaluated their behaviour.

In third chapter we investigate neut $\alpha_{(\gamma,\beta)}$-continuous functions and analyze their properties. The neut $\alpha_{(\gamma,\beta)}$ -open functions, neut $\alpha_{(\gamma,\beta)}$ -closed functions and neut $\alpha_{(\gamma,\beta)}$ -homeomorphism are introduced and analyzed in the fourth, fifth and sixth chapters respectively. The rapport amongst them is examined in the sixth chapter.

All over this study, consider that $\mathcal{Z}_{ne}$ designate the neut topo spa $(\mathcal{Z}_{ne}, \tau_{ne})$ and $\gamma: \tau_{ne} \to P(\mathcal{Z}_{ne})$ be an operation on τ_{ne}.

2.Preliminaries

The theory of neut ses which is a tool for dealing with uncertainties was exposed by Smarandache [15-18]. Salama [12-14], Alblowi [1] familiarized the thought of neut topo spas.

The neut se, its complement, inclusion relation, union, intersection, neut topology, neut open set, neut closed se were introduced by Salama et al [12-14]. The neut functions was revealed by Turnali and Coker [20].

Definition 2.1.[10] Let $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut topo spa. A maneuver γ on the topo τ_{ne} is a charting from τ_{ne} into the power set $P\ (\mathcal{Z}_{ne}\ )$ of $\mathcal{Z}_{ne}$ such that $M \subseteq M^\gamma$ for each $M \in \tau_{ne}$. M^γ designates the charge of γ at M . It is symbolized by $\gamma: \quad \tau_{ne} \to P\ (\mathcal{Z}_{ne}\ )$.

Definition 2.2. [10] A neut subse B of a neut topo spac is supposed to remain a neut γ-open se contingent upon for individual $z \in B$, there prevails a neut open se V,aforesaid that $z \in V$ along with $V^\gamma \subseteq B$. $\tau_{(ne)\gamma}$designates the set of all neut γ-open ses.

Definition 2.3. [10] A neut subse B of a neut topo spa is thought to remain a neut γ-closed se in $(\mathcal{Z}_{ne}, \tau_{ne})$ in case that $\mathcal{Z}_{ne} - B$ is a neut γ -open se in $(\mathcal{Z}_{ne}, \tau_{ne})$.

Definition 2.4. [10] An operation γ is thought to be neut open if, for every single neut open neighbourhood V of $\mathcal{Z}_{ne}$, there occurs a neut γ-open se P akin that $z \in P$ besides $P \subseteq V^{\gamma}$.

Definition 2.5.[10] Agree $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut topo spa in addition B be a neut subse of $(\mathcal{Z}_{ne}, \tau_{ne})$. Formerly neut γ -interior of B is the congregation of entire neut γ -open ses encompassed in B in addition it is indicated through $\tau_{(ne)\gamma}$-int (B).

$\tau_{(ne)\gamma}$-int (B) = $\{V\colon V$ is a neut γ -open se then $V \subseteq B\}$.

Definition 2.6. [10] Agree $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut topo spa also B be a neut subse of $(\mathcal{Z}_{ne}, \tau_{ne})$. Formerly neut γ -closure of B is the intersection of entire neut γ -closed ses contained in B in addition it is symbolized by means of $\tau_{(ne)\gamma}$-int (B).

$\tau_{(ne)\gamma}$-cl (B) = $\{M\colon M$ is a neut γ -closed se besides $B \subseteq M\}$.

Definition 2.7. [10] A spa $(\mathcal{Z}_{ne}, \tau_{ne})$ is termed a neut $\gamma\, T_0$ spa if for each distinct point $s, t \in \mathcal{Z}_{ne}$ there lies a neut γ -open se M alike that $s \in M$ and $t \notin M^{\gamma}$ or $t \in M$ besides $s \notin M^{\gamma}$.

Definition 2.8. [10] A spa $(\mathcal{Z}_{ne}, \tau_{ne})$ is labeled a neut $\gamma\, T_1$ spa if for each distinct point $s, t \in \mathcal{Z}_{ne}$ there endures neut γ -open ses M, N containing s, t respectively aforesaid that $t \notin M^{\gamma}$ and $s \notin N^{\gamma}$.

Definition 2.9. [10] A spa $(\mathcal{Z}_{ne}, \tau_{ne})$is termed a neut $\gamma\, T_2$spa if for each distinct point $s, t \in \mathcal{Z}_{ne}$ there occurs a neut γ -open ses M, N like that $s \in M$, $t \in N$ and $M^{\gamma} \cap N^{\gamma} = \emptyset$.

Definition 2.10. [10] Let $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut top spa. Formerly a neut subse member H of $\mathcal{Z}_{ne}$is aforesaid to be a neut γ generalized closed se (ne $\gamma\, g$-closed set) if $\tau_{(ne)\gamma}$ -cl $(H) \subseteq M$ whensoever $H \subseteq M$ besides M is a neut γ -open se in $(\mathcal{Z}_{ne}, \tau_{ne})$.

Remark 2.1. [10] After the definition 4.4, Each one neut γ -closed se in $(\mathcal{Z}_{ne}, \tau_{ne})$ is a neut γ generalized closed se. Yet, the conflicting statement need not be exact.

Definition 2.11. [10] A neut top spa$(\mathcal{Z}_{ne}, \tau_{ne})$ is labeled as a neut $\gamma\, T_{\frac{1}{2}}$ spa in case that each single neut γ generalized -closed se belonging to the $(\mathcal{Z}_{ne}, \tau_{ne})$ is a neut γ -closed se.

Definition 2.12. [10] A neut conventional se H in a neut top spa $(\mathcal{Z}_{ne}, \tau_{ne})$ is concluded as a neut α_γ -open se contingent upon $H \subseteq \tau_{(ne)\gamma}$ -int($\tau_{(ne)\gamma}$ -cl $(\tau_{(ne)\gamma}$ -int $(H)))$.

Theorem 2.1. [10] Let $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut top spa and $\{A\kappa{:}\,\kappa \in J\}$ be the group of neut α_γ - open ses in $(\mathcal{Z}_{ne}, \tau_{ne})$.Formerly $\cup_{\kappa\in J} A\kappa$ is also a neut α_γ - open se.

Definition 2.13. [10] Agree $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut top spa in addition P be a subse of $\mathcal{Z}_{ne}$. At that time P is supposed to be neut α_γ- closed se on condition that $\mathcal{Z}_{ne}$–P is a neut α_γ- open se.

Definition 2.14. [10] A subse member M of $\mathcal{Z}_{ne}$ is noted to be a neut α_γ - closed se in the event that $\mathcal{Z}_{ne} - M$ is a neut α_γ -open se, which is unvaryingly, Agree $(\mathcal{Z}_{ne}, \tau_{ne})$ be a neut top spa then γ - an activity on τ_{ne} in addition M be a subse member of $\mathcal{Z}_{ne}$. Formerly M is a neut α_γ - closed se subject to $M \supseteq \tau_{(ne)\gamma}$ -cl $(\tau_{(ne)\gamma}$ -int$(\tau_{(ne)\gamma}$ -cl $(M)))$.

Definition 2.15. [10] Endorse $(\mathcal{Z}_{ne}, \tau_{ne})$ as a top spa along with Q as a neut subse of $(\mathcal{Z}_{ne}, \tau_{ne})$. Then neut $\tau_{(ne)\alpha_\gamma}$-interior of Q is the unification of all neut α_γ -open ses accommodated within Q and it is symbolized by $\tau_{(ne)\alpha_\gamma}$ -int(Q).

$\tau_{(ne)\alpha_\gamma}$ -int$(Q) = \cup\{U : U$ is a neut α_γ - open se and $U \subseteq Q\}$.

Definition 2.16. [10] Confer $(\mathcal{Z}_{ne}, \tau_{ne})$ to be a top spa along with C be a neut subse of $(\mathcal{Z}_{ne}, \tau_{ne})$. At that time, $\tau_{(ne)\alpha_\gamma}$ -closure of C is the intersection of all neut α_γ -closed ses consisting of C and it is indicated by $\tau_{(ne)\alpha_\gamma}$ -cl(C).

$\tau_{(ne)\alpha_\gamma}$ -cl $(C) = \cap\ \{F{:}\ F$ is a neut α_γ -closed se and $C \subseteq F\}$.

Remark 2.2. [10] (i) If M is a neut subse of $(\mathcal{Z}_{ne}, \tau_{ne})$. Then $\tau_{(ne)\alpha_\gamma}$ -cl (M) is a neutrosophic α_γ-closed se containing M.

(ii) M is a neut α_γ-closed se in the event $\tau_{(ne)\alpha_\gamma}$ -cl$(M) = M$.

Definition 2.17. [10] A top spa$(\mathcal{Z}_{ne}, \tau_{ne})$ is entitled a neut α_γT0 spa if for each different points p, $q \in \mathcal{Z}_{ne}$ nearby exists a α_γ - open set, P like that p $\in P$ and q $\notin P$ or $q \in P$ besides $p \notin P$.

Definition 2.18. [10]A top spa $(\mathcal{Z}_{ne}, \tau_{ne})$is termed a neut α_γ T1 spa if for each dissimilar points p, $q \in \mathcal{Z}_{ne}$ nearby exists neut α_γ -open ses, P,Q enclosing p and q commonly alike that $q \notin P$

and $p \notin Q$.

Definition 2.19. [10] A top spa$(\mathcal{Z}_{ne}, \tau_{ne})$is described a neut α_γ T2 spa if for each distinctive points p, $q \in \mathcal{Z}_{ne}$ nearby exists neut α_γ -open ses, P , Q akin that $p \in P$,$q \in Q$ and $P \cap Q = \emptyset$.

Definition 2.20. [10] Agree $(\mathcal{Z}_{ne}, \tau_{ne})$be a top spa. Then a neut subse member M of $\mathcal{Z}_{ne}$ is forenamed to be neut α_γg- closed se if $\tau_{(ne)\alpha_\gamma}$ -cl $(M) \subseteq P$ whenever $M \subseteq P$ and P is a neut α_γ -closed se in $(\mathcal{Z}_{ne}, \tau_{ne})$.

Remark 2.3. [10] After the definition 4.4 explanation, Individual neut α_γ -closed conventional se of $(\mathcal{Z}_{ne}, \tau_{ne})$ is a neut α_γg-closed se. Nevertheless, the antipode need not be appropriate.

Definition 2.21. [10] A top spa$(\mathcal{Z}_{ne}, \tau_{ne})$is termed as a neut α_γ T ½ spa supposing that individual neut α_γ g- closed se of $(\mathcal{Z}_{ne}, \tau_{ne})$is a neut α_γ- closed se.

Theorem 2.2. [10] The top spa$(\mathcal{Z}_{ne}, \tau_{ne})$is a neut α_γ T ½ spa on condition that for every single $m \in \mathcal{Z}_{ne}$, $\{m\}$ is a neut α_γ- closed se or a neut α_γ- open se in $(\mathcal{Z}_{ne}, \tau_{ne})$.

Theorem 2.3. [10] Agree$(\mathcal{Z}_{ne}, \tau_{ne})$ be a top spa also $M \subseteq \mathcal{Z}_{ne}$. At that time the succeeding statements hold:

(i) $\tau_{(ne)\alpha_\gamma}$ -int $(\mathcal{Z}_{ne} - M) = \mathcal{Z}_{ne} - \tau_{(ne)\alpha_\gamma}$ -cl (M)

(ii)$\tau_{(ne)\alpha_\gamma}$ -cl $(\mathcal{Z}_{ne} - M) = \mathcal{Z}_{ne} - \tau_{(ne)\alpha_\gamma}$ -int(M)

Theorem 2.4. [10] Accredit $(\mathcal{Z}_{ne}, \tau_{ne})$be a top spa. Supposing that a neut subse member M of $\mathcal{Z}_{ne}$ is assumed to be a neut α_γ g- closed se, thereupon $\tau_{(ne)\alpha_\gamma}$ -cl $(M) - M$ does not enclose a non-void neut α_γ- closed se.

Theorem 2.5. [10] The top spa$(\mathcal{Z}_{ne}, \tau_{ne})$is a neut α_γ T ½ spa on condition that for every single $m \in \mathcal{Z}_{ne}$, $\{m\}$ is a neut α_γ- closed se or a neut α_γ- open se in $(\mathcal{Z}_{ne}, \tau_{ne})$.

3.Neutrosophic $\alpha_{(\gamma,\beta)}$-continuous functions

In this chapter we investigate neutrosophic $\alpha_{(\gamma,\beta)}$-continuous functions as well analyze their properties.

3.1.(i) Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is aforesaid to be a neut $\alpha_{(\gamma,\beta)}$ -continuous function given for respective neut α_β- open set U of $\mathcal{Y}_{ne}$, the contrary image $f_{ne}^{-1}(U)$ is a neut α_γ -open se in $\mathcal{Z}_{ne}$.

(ii) Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is supposed to be a neut $(\alpha_\gamma, \beta_{ne})$-continuous function with the condition that the converse appearance of apiece neut β_{ne}-open se in $(\mathcal{Y}_{ne}, \sigma_{ne})$

abides to be a neut α_{γ}-open se of $(\mathcal{Z}_{ne}, \tau_{ne})$.

(iii) Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is aforesaid to be a neut $(\gamma_{ne}, \beta_{ne})$-continuous function in case that the reverse icon of every single neut β_{ne}-open se of $(\mathcal{Y}_{ne}, \sigma_{ne})$ continues to be a neut γ_{ne}-open se of $(\mathcal{Z}_{ne}, \tau_{ne})$.

3.2. Example Given $\mathcal{Z}_{ne} = \{h_1, h_2, h_3\}$,$\tau_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, F_1, F_3, F_4, F_5\}$,$\mathcal{Y}_{ne} = \{g_1, g_2, g_3\}$and $\sigma_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, H_1, H_3, H_4, H_5\}$ were

$F_1 = \{z, (0.2,0.6,0.1), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,

$F_3 = \{z, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.3)\}$,

$F_4 = \{z, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,

$F_5 = \{z, (0.7,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$

$H_1 = \{y, (0.2,0.6,0.1), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,

$H_3 = \{y, (0.7,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.3)\}$,

$H_4 = \{y, (0.7,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,

$H_5 = \{y, (0.8,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$

Define an operation γ on τ_{ne} such that$(U)^{\gamma} = \begin{cases} U \cup \{h_3\} if U \neq \{h_1\} \\ U \text{ if } U = \{h_1\} \end{cases}$

Then $\tau_{(ne)\alpha_{\gamma}} = \{1_{ne}, \mathcal{Z}_{ne}, F_1, F_3, F_5\}$.

Define an operation β on σ_{ne} such that$(U)^{\gamma} = \begin{cases} U \cup \{g_3\} if U \neq \{g_1\} \\ U \text{ if } U = \{g_1\} \end{cases}$

Then $\tau_{(ne)\alpha_{\gamma}} = \{1_{ne}, \mathcal{Z}_{ne}, H_1, H_3, H_5\}$

Define $f_{ne}: \mathcal{Z}_{ne} \to \mathcal{Y}_{ne}$as $f_{ne}(h_1) = g_1$, $f_{ne}(h_2) = g_2$ and $f_{ne}(h_3) = g_3$ Formerly the overturned copy of restricted neut α_{β}-open se acts as a neut α_{γ}- open se beneath f_{ne}.Thence f_{ne} endures to be a neut $\alpha_{(\gamma,\beta)}$- continuous function.

The subsequent 3.3. Remark and 3.4. Remark display that the thought of neut $\alpha_{(\gamma,\beta)}$-continuous functions and neut $(\gamma_{ne}, \beta_{ne})$ -irresolute functions are self-governing nevertheless after $\mathcal{Z}_{ne}$ is a neutrosophic γ -regular spa and $\mathcal{Y}_{ne}$ is a neut β-regular spa together the thoughts concur.

3.3. Remark The insights of neut $\alpha_{(\gamma,\beta)}$-continuous functions besides neut $(\gamma_{ne}, \beta_{ne})$ -irresolute functions are self-regulating.

Given $\mathcal{Z}_{ne} = \{h_1, h_2, h_3\}$,$\tau_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, F_1, F_2, F_3, F_4, F_5\}$,$\mathcal{Y}_{ne} = \{g_1, g_2, g_3\}$ and σ_{ne}

$= \{1_{ne}, \mathcal{Z}_{ne}, H_1, H_3, H_{4,} H_5\}$ were

$F_1 = \{z, (0.2,0.6,0.3), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,

$F_2 = \{z, (0.8,0.5,0.1), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$,

$F_3 = \{z, (0.8,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$,

$F_4 = \{z, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,

$F_5 = \{z, (0.7,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$

$H_1 = \{y, (0.2,0.6,0.1), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,

$H_3 = \{y, (0.7,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.3)\}$,

$H_4 = \{y, (0.7,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,

$H_5 = \{y, (0.8,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$

Define an operation γ on τ_{ne} such that $(U)^{\gamma} = \begin{cases} U \cup \{h_3\} if U \neq \{h_1\} \\ U \text{ if } U = \{h_1\} \end{cases}$

Then $\tau_{(ne)\alpha_\gamma} = \{1_{ne}, \mathcal{Z}_{ne}, F_1, F_3, F_5\}$.

Define an operation β on σ_{ne} such that $(V)^{\beta} = \begin{cases} V \text{ if } g_2 \notin V \\ cl(V) \text{ if } g_2 \in V\} \end{cases}$

Define f_{ne}: $\mathcal{Z}_{ne} \rightarrow \mathcal{Y}_{ne}$ as $f_{ne}(h_1) = g_1$, $f_{ne}(h_2) = g_2$ and $f_{ne}(h_3) = g_3$

Formerly f_{ne} is a neut $(\gamma_{ne}, \beta_{ne})$ irresolute function. But $f_{ne}^{-1}(\{g_1, g_2\}) = \{h_1, h_2\}$, is not a neut α_γ-open se under f_{ne}. Henceforth f_{ne} is not a neut $\alpha_{(\gamma,\beta)}$-continuous function.

3.4. Remark If $\mathcal{Z}_{ne}$ is a neut γ-regular spa and $\mathcal{Y}_{ne}$ is a neut β- regular spa, at that moment the conception of neut (γ, β)- irresoluteness in addition neut $\alpha_{(\gamma,\beta)}$-continuity concur.

3.5. Definition A neut member H of $\mathcal{Z}_{ne}$ is supposed to be a neut α_γ-neighbourhood of a point $t \in \mathcal{Z}_{ne}$ if there befalls a neut α_γ- open se G like that $t \in G \subseteq H$.

3.6. Theorem A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-continuous function designed for each r of $\mathcal{Z}_{ne}$, the opposing statement of each neut α_β-neighbourhood of f_{ne}(r) is a neut α_γ-neighbourhood of r.

Proof. Assume r $\in \mathcal{Z}_{ne}$in addition B be a neut α_{γ}-neighbourhood of f_{ne}(r). By the assertion of the 3.5. Definition there ensues a $V \in \sigma_{(ne)\alpha_{\beta}}$ such that f_{ne} (r) $\in V \subseteq B$. This deduces that r $\in f_{ne}^{-1}(V) \subseteq f_{ne}^{-1}(B)$. Since f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-continuous function,$f_{ne}^{-1}(V) \in \tau_{(ne)\alpha_{\gamma}}$. Henceforward $f_{ne}^{-1}(B)$ is a neut α_{γ}-neighbourhood of r.

Conversely, let $B \in \sigma_{(ne)\alpha_{\beta}}$, A=$f_{ne}^{-1}(B)$ and r $\in A$. Later by the announcement of the 3.5. Definition, there arises a set $A_r \in \tau_{(ne)\alpha_{\gamma}}$similar that r$\in A_r \subseteq A$. This supposes that $A = \cup_{r \in A} A_r$. Through the assertion of the 2.1. Theorem, A is a neut α_{γ} -open se of $\mathcal{Z}_{ne}$,. Accordingly, f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-continuous function.

3.7. Theorem A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-continuous function provided that for separate point m of $\mathcal{Z}_{ne}$ besides respective neutrosophic α_{β} -neighbourhood B of f_{ne}(m),there is a neutrosophic α_{γ}-neighbourhood A of m alike that $f_{ne}(A) \subseteq B$.

Proof. Let n$\in \mathcal{Z}_{ne}$ also B acts as a neut α_{β}- neighbourhood of f_{ne}(n). Later in there lies a set $O_{f(n)} \in \sigma_{(ne)\alpha_{\beta}}$such that f_{ne} (n) $\in O_{f(n)} \subseteq B$. It follows that m $\in f_{ne}^{-1}(O_{f(n)}) \subseteq f_{ne}^{-1}(B)$. By hypothesis, $f_{ne}^{-1}(O_{f(n)}) \subseteq \tau_{(ne)\alpha_{\gamma}}$. Let $A = f_{ne}^{-1}(B)$. Then it trails that A is a neut α_{γ}-neighbourhood of n and $f_{ne}(A) = f_{ne}(f_{ne}^{-1}(B)) \subseteq B$.

Conversely, let $U \in \sigma_{(ne)\alpha_{\beta}}$. Let W =$f_{ne}^{-1}(U)$. Let n $\in W$. Formerly f_{ne}(n)$\in U$. Thus U is a neut α_{γ}- neighbourhood of f_{ne}(n) and hence there exists a neut α_{γ}- neighbourhood $V_n$of n akin that $f_{ne}(V_n) \subseteq U$ .Accordingly it trails that n$\in V_n \subseteq f_{ne}^{-1}(f_{ne}(V_n)) \subseteq f_{ne}^{-1}(U) = W$. Since V_n is a neut α_{γ}-neighbourhood of n, which implies that there exists a $W_n \in \tau_{(ne)\alpha_{\gamma}}$like that n$\in W_n \subseteq W$.

This implies that $W \ = \cup_{n \in W} W_n$. Through 2.1. Theorem Statement, W is a neut α_γ-open se of $\mathcal{Z}_{ne}$. Consequently f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- continuous function.

3.8. Theorem Accredit $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ remain a function. Formerly the ensuing assertions are comparable:

(i) $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$is a neut $\alpha_{(\gamma,\beta)}$- continuous function;

(ii) $f_{ne}(\tau_{(ne)\alpha_\gamma}$-cl $(D)) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(D))$ holds for every member D belonging to $\mathcal{Z}_{ne}$;

(iii) For respective single neut α_β-closed se $V$of $\mathcal{Y}_{ne}$, $f_{ne}^{-1}(V)$is a neut α_γ-closed se belonging to $\mathcal{Z}_{ne}$.

Validation. (i)$\Rightarrow$ (ii) Given s $\in f_{ne}(\tau_{(ne)\alpha_\gamma}$ -cl $(D))$ besides V be a neut α_β-open se comprising s. By dint of the 3.7. Theorem, at this juncture ensues a point y $\in \mathcal{Z}_{ne}$ along with a neut α_γ-open se Ucomparable that y $\in U$ with f_{ne}(y) = s and $f_{ne}(U) \subseteq V$. Subsequently y $\in \tau_{s\alpha_\gamma}$-cl (D), $U \cap D \neq \emptyset$ besides henceforth $\emptyset \neq f_{ne}(U \cap D) \subseteq f_{ne}(U) \cap f_{ne}(D) \subseteq V \cap f_{ne}(D)$. This implies that s$\in \sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(D))$. Therefore $f_{ne}(\tau_{(ne)\alpha_\gamma}$-cl $(D)) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(D))$.

(ii) $\Rightarrow$ (iii) Given V be a neut α_β- closed se in $\mathcal{Y}_{ne}$. Then $\sigma_{(ne)\alpha_\beta}$-cl $(V) = V$. Through (ii), $f_{ne}(\tau_{(ne)\alpha_\gamma}$- cl $(f_{ne}^{-1}(V))) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(f_{ne}^{-1}(V))) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(V) = V$ holds. Consequently $\tau_{(ne)}$-cl $(f_{ne}^{-1}(A)) \subseteq f_{ne}^{-1}(V)$ and $f_{ne}^{-1}(V) = \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(V))$. Henceforth$f_{ne}^{-1}(V)$ is a neut α_γ-closed se in $\mathcal{Z}_{ne}$.

(iii)$\Rightarrow$ (i) Contemplate B as a neut α_β-open se in $\mathcal{Y}_{ne}$. Deliberate $V = \mathcal{Y}_{ne} - B$. Thereupon V is a neut α_β- closed se in $\mathcal{Y}_{ne}$. Through (iii) $f_{ne}^{-1}(V)$ is a neut α_γ-closed se in $\mathcal{Z}_{ne}$. Later$f_{ne}^{-1}(B) = \mathcal{Z}_{ne} - f_{ne}^{-1}(\mathcal{Y}_{ne} - B) = \mathcal{Z}_{ne} - f_{ne}^{-1}(V)$ is a neut α_γ-open se of $\mathcal{Z}_{ne}$. Hereafter f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- continuous function.

3.9. Theorem Agree $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ be a neut $\alpha_{(\gamma,\beta)}$- continuous also injective function. Assuming that $\mathcal{Y}_{ne}$ is a neut $\alpha_\beta T_2$ spa (respectively neut $\alpha_\beta T_1$spa), formerly $\mathcal{Z}_{ne}$ is a neut $\alpha_\gamma T_2$ spa (respectively neut $\alpha_\gamma T_1$spa).

Proof. Suppose $\mathcal{Y}_{ne}$ a neut $\alpha_\beta T_2$ spa. Given i and j be two distinct points of $\mathcal{Z}_{ne}$. Formerly, there presents dual neut α_β-open ses U , V akin that f_{ne}(i) $\in U$, f_{ne}(j) $\in V$ in addition $U \cap V = \emptyset$. Meanwhile f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- continuous function, for U along with V , in view there occurs two neut α_γ- open ses I and J such that i $\in I$ and j $\in J$, $f_{ne}(I) \subseteq U$ and $f_{ne}(J) \subseteq V$, infers that $I \cap J = \emptyset$. Henceforth $\mathcal{Z}_{ne}$is a neutrosophic $\alpha_\gamma T_2$ spa. In the similar technique it can be evinced that $\mathcal{Z}_{ne}$ is a neut $\alpha_\gamma T_1$ spa whenever $\mathcal{Y}_{ne}$ is a neut $\alpha_\beta T_1$ spa.

3.10. Theorem Accredit $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ and $g_{Ma}: (\mathcal{Y}_{ne}, \sigma_{ne}) \to (\mathcal{X}_{ne}, \delta_{ne})$ be two functions. Supposing that f_{ne} is a neut $\alpha_{(\gamma,\beta)}$ -continuous function further g_{ne} is a neut $\alpha_{(\beta,\delta)}$ -continuous function, previously $g_{ne} \circ f_{ne}: (\mathcal{X}_{ne}, \delta_{ne}) \to (\mathcal{Z}_{ne}, \tau_{ne})$ is a neut $\alpha_{(\gamma,\delta)}$ -continuous function.

Proof. Manifestation trails from the 3.1. Definition.

3.11. Definition Approve D be a neut subse of $\mathcal{Z}_{ne}$ then z be any point in $\mathcal{Z}_{ne}$. At that time z is called a neut α_γ-limit point of D suppose that $U \cap (D - \{z\}) \neq \emptyset$, for any neut α_γ-open set U encompassing z. The collection of all neut α_γ-limit points of D is commanded as a neut α_γ-derived set of A as well it is indicated as $d_{(ne)\alpha_\gamma}(D)$.

3.12. Remark Agree L , M be some subsets of $\mathcal{Z}_{ne}$. At that time,

(i) if $L \subseteq M$, then $d_{(ne)\alpha_\gamma}(L) \subseteq d_{(ne)\alpha_\gamma}(M)$.

(ii) r $\in d_{(ne)\alpha_\gamma}(L)$ if and only if r $\in \tau_{(ne)\alpha_\gamma}(L)$-cl $(L - \{r\})$.

Proof. Confirmation tracks after the declaration of 3.11. Definition.

3.13. Theorem Accept L and M are any two neutrosophic subses of $\mathcal{Z}_{ne}$. At that moment the ensuing information hold good.

(i) $L \cup d_{(ne)\alpha_\gamma}(L) \subseteq \tau_{(ne)\alpha_\gamma}$-cl (L);

(ii) $d_{(ne)\alpha_\gamma}(L \cup M) = d_{(ne)\alpha_\gamma}(L) \cup d_{(ne)\alpha_\gamma}(M)$;

(iii)$\cup_i d_{(ne)\alpha_\gamma}(L_i) = d_{(ne)}(\cup_i(L_i))$;

(iv) $d_{(ne)\alpha_\gamma}(d_{(ne)\alpha_\gamma}(L)) \subseteq d_{(ne)\alpha_\gamma}(L)$;

(v)$\tau_{(ne)\alpha_\gamma}$ -cl $(d_{(ne)}(L)) = d_{(ne)\alpha_\gamma}(L)$

Proof. (i) In case $l \in L \cup d_{(ne)\alpha_\gamma}(L)$, then to establish that $l \in \tau_{(ne)\alpha_\gamma}$-cl (L). If $l \in L$ formerly $l \in \tau_{(ne)\alpha_\gamma}$-cl (L). If $l \notin L$, at that time to indicate that $l \in \tau_{(ne)\alpha_\gamma}$-cl (L). Or else, at that time there is a neut α_γ- closed se C comprising L but not encompassing l. Then $l \in \mathcal{Z}_{ne} - C$, which is a neut α_γ-open se furthermore $U \cap L = \emptyset$. This suggests that $l \notin d_{(ne)}(L)$. This strangeness displays that x $\in \tau_{(ne)}$-cl(L). Henceforth $L \cup d_{(ne)\alpha_\gamma}(L) \subseteq \tau_{(ne)\alpha_\gamma}$- cl (L).

(ii) Let $l \in d_{(ne)\alpha_\gamma}(L \cup M)$. Through the 3.11.Definition, $\emptyset \neq U \cap ((L \cup M) - \{l\}) = U \cap [(L - \{l\}) \cap (M - \{l\})] = [U \cap (L - \{l\})] \cup [U \cap (M - \{l\})]$ and hence either $l \in d_{(ne)\alpha_\gamma}(L)$ or $d_{(ne)\alpha_\gamma}$

(M).Therefore $d_{(ne)\alpha_\gamma}$ $(L\cup M) \subseteq d_{(ne)\alpha_\gamma}$ $(L)\cup$ $d_{(ne)\alpha_\gamma}$ (M). The outcome$d_{(ne)\alpha_\gamma}$ $(L)\cup d_{(ne)\alpha_\gamma}$ (M) $\subseteq d_{(ne)\alpha_\gamma}$ $(L\cup M)$, tracks by the 3.12. (i) Remark outcome.

(iii) Validation trails on or after the 3.12. Remark and (ii) outcome.

(iv) Assuming that $l \notin d_{(ne)\alpha_\gamma}$ (L). Then $l \notin \tau_{(ne)\alpha_\gamma}$-cl $(L-\{l\})$. At this moment, there ensues a neut α_γ- open set U like that $l \in U$ and $U\cap(L-\{l\}) = \emptyset$. To attest that $l \notin d_{(ne)\alpha_\gamma}$ $(d_{(ne)\alpha_\gamma}$ $(L))$. Supposing on the conflict that $l \in d_{(ne)\alpha_\gamma}$ $(d_{(ne)\alpha_\gamma}$ $(L))$. Then $l \in \tau_{(ne)\alpha_\gamma}$-cl $(d_{(ne)\alpha_\gamma}$ (L) $-\{l\})$. Since $l \in U$, $U\cap$ $(d_{(ne)}$ (L) $-\{l\}) \neq \emptyset$ .Subsequently there is a q $\neq l$ so that q $\in U\cap$ $(d_{(ne)\alpha_\gamma}$ $(L))$. This suggests that q$\in$ $(U-\{l\})\cap$ $(d_{(ne)\alpha_\gamma}$ $(L)-\{l\})$. Later $((U-\{l\})\cap$ $(d_{(ne)\alpha_\gamma}$ (L) $-\{l\}))\neq\emptyset$, an illogicality to the datum that $U\cap$ $(d_{(ne)\alpha_\gamma}$ (L) $-\{l\}) =\emptyset$. This hints that $l \notin d_{(ne)\alpha_\gamma}$ $(d_{(ne)\alpha_\gamma}$ $(L))$ and after this $d_{(ne)\alpha_\gamma}$ $(d_{(ne)\alpha_\gamma}$ $(L))\subseteq$ $d_{(ne)\alpha_\gamma}$ (L).

(v) This trails after the 3.11. Definition.

3.14. Theorem A function $f_{ne}:(\mathcal{Z}_{ne},\tau_{ne}) \to (\mathcal{Y}_{ne},\sigma_{ne})$is a neut $\alpha_{(\gamma,\beta)}$-continuous function contingent upon $f_{ne}(d_{(ne)\alpha_\gamma}(A)) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(A))$, for entire $A\subseteq \mathcal{Z}_{ne}$.

Proof. Given $f_{ne}:(\mathcal{Z}_{ne},\tau_{ne}) \to (\mathcal{Y}_{ne},\sigma_{ne})$is a neut $\alpha_{(\gamma,\beta)}$- continuous function. Agree that $A\subseteq \mathcal{Z}_{ne}$, and z$\in d_{(ne)\alpha_\gamma}(A)$. Assume that f_{ne}(z) $\notin f_{ne}(A)$ and let Vdenote a neut α_β- neighbourhood of f_{ne}(z). Meanwhile f_{ne} is a neutrosophic $\alpha_{(\gamma,\beta)}$-continuous function, by means of 3.6. Theorem, there arises a neut α_γ- neighbourhood U of z like that $f_{ne}(U)\subseteq$ V. From z$\in d_{(ne)\alpha_\gamma}(A)$, it trails that $U\cap$ A $\neq\emptyset$. There arises at least a factor c$\in$ $U\cap$ A, suggests that f_{ne}(c)$\in$ $f_{ne}(A)$ and f_{ne}(c)$\in$ V. Meanwhile f_{ne}z)$\notin$ $f_{ne}(A)$ andf_{ne}(c)$\neq$ f_{ne}(z). Therefore, each neut α_β- neighbourhood of f_{ne}(z) encompasses a component f_{ne}(c) $\in f_{ne}(A)$ unalike from f_{ne}(z). Henceforward,f_{ne}(z)$\in$ $d_{(ne)\alpha_\beta}(f_{ne}(A$)). By dint of the assertion of 3.13. (i) Theorem $f_{ne}(d_{(ne)\alpha_\gamma}(A)) \subseteq \sigma_{(ne)\alpha_\beta}$- cl ($f_{ne}(A)$).

On the contrary, assume that f_{ne} is not a neut $\alpha_{(\gamma,\beta)}$-continuous function. Then via 3.7. Theorem, there occurs z$\notin \mathcal{Z}_{ne}$ and a neut α_β- neighbourhood V of f_{ne}(z) so that every single neut α_γ-neighbourhood $U$of z covers at least one member c$\in$ U, for which f_{ne}(c)$\notin$ V. Let A= {c $\in \mathcal{Z}_{ne}$:f_{ne}(c)$\notin$ V}. Since f_{ne}(z)$\in$ V, therefore z $\notin$ A and hence f_{ne}(z) $\notin$ $f_{ne}(A)$. Since $f_{ne}(A) \cap$ (V − f_{ne}(z)) = $\emptyset$, therefore f_{ne}(z) $\notin$ $d_{(ne)\alpha_\beta}$(($f_{ne}(A)$)). It surveys that f_{ne}(z) $\in$ f_{ne}($d_{(ne)\alpha_\beta}$((A)) − ($f_{ne}(A) \cup d_{s\alpha_\beta}$(($f_{ne}(A)$))$\neq \emptyset$, which is a flaw to the given condition. Henceforth f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-continuous function.

3.15. Theorem Let f_{ne}: $(\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$be a neut one-to-one function. Then f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- continuous function on condition that $f_{ne}(d_{(ne)\alpha_\gamma}(A)) \subseteq d_{(ne)\alpha_\beta}(f_{ne}(A))$, for all that $A \subseteq \mathcal{Z}_{ne}$.

Proof. Given $A \subseteq \mathcal{Z}_{ne}$, z$\in$ $d_{(ne)\alpha_\gamma}(A)$ and $V$is a neut α_β-neighbourhood of $f_{ne}(z)$. By the reason of$f_{ne}$is a neut $\alpha_{(\gamma,\beta)}$-continuous function by the announcement of 3.7. Theorem, at this juncture befalls a neut α_γ- neighbourhood U of z similar that $f_{ne}(U) \subseteq V$. Nevertheless z$\in$ $d_{(ne)\alpha_\gamma}(A)$ stretches there happens a component c$\in U \cap A$ corresponding that c≠z, f_{ne}(c) $\in$ $f_{ne}(A)$ then by the cause of f_{ne} is one-to-one, $f_{ne}(c) \neq f_{ne}(z)$. Consequently, every neut α_β-neighbourhood V of f_{ne}(z) comprises a component $f_{ne}(c)$ of $f_{ne}(A)$ dissimilar from f_{ne}(z). Accordingly,f_{ne}(z)$\in$ $d_{(ne)\alpha_\beta}(f_{ne}(A))$.

Thus f_{ne} $(d_{(ne)\alpha_\gamma}(A)) \subseteq d_{(ne)\alpha_\beta}$ $(f_{ne}(A))$, for altogether $A \subseteq \mathcal{Z}_{ne}$. Contrary portion trails from the 3.14. Theorem.

4. Neutrosophic $\alpha_{(\gamma,\beta)}$-open functions

4.1. Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is said to be a neut $\alpha_{(\gamma,\beta)}$-open function akin that for entire neut α_γ-open se $M \in \tau_{(ne)\alpha_\gamma}$, the image $f_{ne}(M) \in \sigma_{(ne)\alpha_\beta}$.

4.2. Example Given

$\mathcal{Z}_{ne} = \{h_1, h_2, h_3\}$,$\tau_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, F_1, F_3, F_4, F_5\}$, $\mathcal{Y}_{ne} = \{g_1, g_2, g_3\}$and $\sigma_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, H_2, H_5\}$ were

$F_1 = \{z, (0.2,0.6,0.1), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,
$F_3 = \{z, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.3)\}$,
$F_4 = \{z, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,
$F_5 = \{z, (0.7,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$
$H_2 = \{y, (0.2,0.6,0.1), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,
$H_5 = \{y, (0.8,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$
Construe an operation γ on τ_{ne} analogous that $(U)^\gamma = cl(U)$
Delineate an operation β on σ_{ne} alike that $(V)^\beta = cl(B)$

Define $f_{ne}: \mathcal{Z}_{ne} \to \mathcal{Y}_{ne}$as $f_{ne}(h_1) = g_1$, $f_{ne}(h_2) = g_3$ and $f_{ne}(h_3) = g_2$ .Then the image of each one neut α_γ-open se is a neut α_β-open se under f_{ne}.

Later f_{Ma} is a neut $\alpha_{(\gamma,\beta)}$-open function.

4.3. Theorem Suppose that $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-open function in addition supposes that $g_{ne}: (\mathcal{Y}_{ne}, \sigma_{ne}) \to (\mathcal{X}_{ne}, \delta_{ne})$ is a neut $\alpha_{(\beta,\delta)}$ -open function, at that juncture the composition $g_{ne} \circ f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{X}_{ne}, \delta_{ne})$is a neut $\alpha_{(\gamma,\delta)}$- open function.

Proof. Validation trails after the 4.1. Definition statement.

4.4. Theorem A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-open function in situation that for entire $z \in \mathcal{Z}_{ne}$, and for every $A \in \tau_{(ne)\alpha_{\gamma}}$ akin that $z \in A$, there ensues a $B \in \sigma_{(ne)\alpha_{\beta}}$ such that $f_{ne}(z) \in B$ and $B \subseteq f_{ne}(A)$.

Proof. Consider A as a neut α_{γ}- open se and $z \in \mathcal{Z}_{ne}$. At that juncture $f_{ne}(z) \in f_{ne}(A)$. Then $f_{ne}(A)$ is a neut α_{β}-neighbourhood of $f_{ne}(z)$ in $\mathcal{Y}_{ne}$ Formerly by the declaration of 3.7. Theorem there present a neut α_{γ}-open neighbourhood $B \in \sigma_{(ne)\alpha_{\beta}}$ akin that $f_{ne}(z) \in B \subseteq f_{ne}(A)$.

In reverse let $A \in \tau_{(ne)\alpha_{\gamma}}$ alike that $z \in A$. At that moment by belief, there is a $B \in \sigma_{(ne)\alpha_{\beta}}$ alike that $f_{ne}(z) \in B \subseteq f_{ne}(A)$. So $f_{ne}(A)$ is a neut α_{β}- neighbourhood of $f_{ne}(z)$ in $\mathcal{Y}_{ne}$ and this infers that $f_{ne}(A) = \cup_{f_{ne(z)} \in f_{ne(A)}} B$. Formerly through 2.1. Theorem $f_{ne}(A)$ is a neut α_{β}-open se in $\mathcal{Y}_{ne}$. Henceforward f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-open function.

4.5. Theorem A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$ -open function in the event that for entire $z \in \mathcal{Z}_{ne}$, in addition for every single neut α_{γ}- neighbourhood U of $z \in \mathcal{Z}_{ne}$,there present a neut α_{β} -neighbourhood V of $f_{ne}(z)$ alike that $V \subseteq f_{ne}(U)$.

Proof. Given U be a neut α_{γ}- neighbourhood of $z \in \mathcal{Z}_{ne}$. At that time via the statement of 3.1. Definition there occurs a neut α_{γ}-open se W such that $z \in W \subseteq U$. This mentions that $f_{ne}(z) \in f_{ne}(W) \subseteq f_{ne}(U)$. Then f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-open function, $f_{ne}(W)$ is a neut α_{β}- open se. Henceforward $V = f_{ne}(W)$ is a neut α_{β}-neighbourhood of $f_{ne}(z)$ and $V \subseteq f_{ne}(U)$.

Conversely, let $U \in \tau_{(ne)\alpha_\gamma}$ and x$\in U$. Then U is a neut α_γ-neighbourhood of z also thence, there prevails a neut α_β- neighbourhood V of $f_{ne}(z)$ akin that $f_{ne}(z) \in V \subseteq f_{ne}$ (U). That is, $f_{ne}(U)$ is a neut α_β-neighbourhood of $f_{ne}(z)$. Thus $f_{ne}(U)$ is a neut α_β -neighbourhood to each of its points. Accordingly, $f_{ne}(U)$ is a neut α_β-open se. Thence $f_{ne}$is a neut $\alpha_{(\gamma,\beta)}$- open function.

4.6. Theorem A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-open function in case that if $f_{ne}(\tau_{(ne)\alpha_\gamma}$-int $(A)) \subseteq \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(A))$, for each $A \subseteq \mathcal{Z}_{ne}$.

Proof. Let $z \in \tau_{(ne)\alpha_\gamma}$-int (A). Then there occurs a $U \in \tau_{(ne)\alpha_\gamma}$ alike that $z \in U \subseteq A$. So f_{ne} (z) $\subseteq f_{ne}(U) \subseteq f_{ne}(A)$. Meanwhile $f_{ne}$is a neut $\alpha_{(\gamma,\beta)}$ - open function, $f_{ne}(U)$ is a neut α_β-open set in $\mathcal{Y}_{ne}$. Later $f_{ne}(z) \in \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(A))$. Thus $f_{ne}(\tau_{(ne)\alpha_\gamma}$-int $(A)) \subseteq \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(A))$.

Contrariwise, let $U \in \tau_{(ne)\alpha_\gamma}$ and hereafter $f_{ne}(U) = f_{ne}$ $(\tau_{(ne)\alpha_\gamma}$-int $(U)) \subseteq \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(U)) \subseteq f_{ne}(U) \subseteq \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(U)) \subseteq f_{ne}(U)$. This implies that $f_{ne}(U)$ is a neut α_β-open se. Thusly f_{ne} is a neut $\alpha_{(\gamma,\beta)}$ -open function.

4.7. Theorem A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-open function on the supposition that $\tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}$ $(B)) \subseteq f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-int $(B))$, for each $B \subseteq \mathcal{Y}_{ne}$.

Proof. Given B be a neut subse of $\mathcal{Y}_{ne}$. Apparently $\tau_{(ne)\alpha_\gamma}$-int $(f_{Ma}^{-1}(B))$ is a neut α_γ-open se belonging to $\mathcal{Z}_{ne}$. Also f_{ne} $(\tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}$ $(B)) \subseteq f_{ne}$ $(f_{ne}^{-1}(B)) \subseteq B$. Subsequently f_{ne} is a neut $\alpha_{(\gamma,\beta)}$ -open function besides by 4.6. Theorem f_{ne} $(\tau_{(ne)\alpha_\gamma}$ -int $(f_{ne}^{-1}(B)) \subseteq \sigma_{(ne)\alpha_\beta}$ -int (B).

Hence $\tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}(B)) \subseteq f_{ne}^{-1}(f_{ne}(\tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}(B))))$. This implies that $\tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}(B))$ $\subseteq f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-int $(B))$ for all $B \subseteq \mathcal{Y}_{ne}$.

Contrarywise, accredit that $A \subseteq \mathcal{Z}_{ne}$,at that time $\tau_{(ne)\alpha_\gamma}$-int $(A) \subseteq \tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}(f_{ne}(A)) \subseteq f_{ne}^{-1}$ $(\sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(A)))$. This implies that $f_{ne}(\tau_{(ne)\alpha_\gamma}$-int $(A)) \subseteq f_{ne}(\tau_{s\alpha_\gamma}$int $(f_{ne}^{-1}(f_{ne}(A)))) \subseteq f_{ne}$ $(f_{ne}^{-1}(\sigma_{s\alpha_\beta}$ -int $(f_{ne}(A)))) \subseteq \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(A))$. Consequently $f_{ne}(\tau_{(ne)\alpha_\gamma}$-int $(A)) \subseteq \sigma_{(ne)\alpha_\beta}$-int $(f_{ne}(A))$, for all $A \subseteq \mathcal{Z}_{ne}$. By 4.6. Theorem,$f_{ne}$ is a neut $\alpha_{(\gamma,\beta)}$- open function.

4.8. Theorem A function $f_{ne}:(\mathcal{Z}_{ne},\tau_{ne}) \rightarrow (\mathcal{Y}_{ne},\sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$ -open function on the supposition that $f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-cl $(D)) \subseteq \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(D))$, for all $D \subseteq \mathcal{Y}_{ne}$.

Proof. Agree D be a neut subse of $\mathcal{Y}_{ne}$. Through 4.7. Theorem, $\tau_{(ne)\alpha_\gamma}$ -int $(f_{ne}^{-1}(\mathcal{Y}_{ne} - D))$ $\subseteq f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-int $(\mathcal{Y}_{ne} - D))$. Then $\tau_{(ne)\alpha_\gamma}$-int $(\mathcal{Z}_{ne} - f_{ne}^{-1}(D)) \subseteq f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-int $(\mathcal{Y}_{ne} - D))$. As $\sigma_{(ne)\alpha_\beta}$-int $(D) = \mathcal{Y}_{ne} - \sigma_{(ne)\alpha_\beta}$- cl $(\mathcal{Y}_{ne} - D))$, therefore $\mathcal{Z}_{ne} - \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(D)) \subseteq f_{ne}^{-1}(\mathcal{Y}_{ne} - \sigma_{(ne)\alpha_\beta}$-cl $(D))$ or $\mathcal{Z}_{ne} - \tau_{(ne)\alpha_\gamma}$- cl $(f_{ne}^{-1}(D)) \subseteq \mathcal{Z}_{ne} - f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-cl $(D))$. Hence $f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-cl $(B))$ $\subseteq \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(D))$.

Conversely, let $D \subseteq \mathcal{Y}_{ne}$ and hence, $f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$- cl $(\mathcal{Y}_{ne} - D)) \subseteq \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(\mathcal{Y}_{ne} - D))$. Then $\mathcal{Z}_{ne} - \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(\mathcal{Y}_{ne} - D)) \subseteq \mathcal{Z}_{ne} - f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$- cl $(\mathcal{Y}_{ne} - D))$. Hence $\mathcal{Z}_{ne} - \tau_{(ne)\alpha_\gamma}$-cl $(\mathcal{Z}_{ne}$ $- f_{ne}^{-1}(D)) \subseteq f_{ne}^{-1}(\mathcal{Y}_{ne} - \sigma_{(ne)\alpha_\beta}$- cl $(\mathcal{Y}_{ne} - D))$. This gives that $\tau_{(ne)\alpha_\gamma}$-int $(f_{ne}^{-1}(D)) \subseteq f_{ne}^{-1}(\sigma_{(ne)\alpha_\beta}$-int $(D))$. Using 4.5. Theorem, it follows that f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-open function.

4.9. Theorem Let $f_{ne}:(\mathcal{Z}_{ne},\tau_{ne}) \to (\mathcal{Y}_{ne},\sigma_{ne})$ and $g_{ne}:(\mathcal{Y}_{ne},\sigma_{ne}) \to (\mathcal{X}_{ne},\delta_{ne})$ be two functions such that $g_{ne} \circ f_{ne}:(\mathcal{Z}_{ne},\tau_{ne}) \to (\mathcal{X}_{ne},\delta_{ne})$is a neut $\alpha_{(\gamma,\delta)}$- continuous function. Formerly

(i) Assuming that g_{ne} is a neut $\alpha_{(\beta,\delta)}$ -open injection at that time f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-continuous function.

(ii) Supposing that $f_{ne}$is a neut $\alpha_{(\gamma,\beta)}$ -open surjection at that time g_{ne} is a neut $\alpha_{(\beta,\delta)}$- continuous function.

Proof. (i) Approve $U \in \sigma_{(ne)\alpha_\beta}$.Meanwhile g_{ne} is a neut $\alpha_{(\beta,\delta)}$ -open function, at that time $g_{ne}(U) \in \zeta_{(ne)\alpha_\delta}$. Subsequently g_{ne} is injective besides $g_{ne} \circ f_{ne}$ is a neut $\alpha_{(\gamma,\delta)}$-continuous function, $(g_{ne} \circ f_{ne})^{-1}$ (g_{ne} $(U)) = (f_{ne}{}^{-1} \circ g_{ne}{}^{-1})($ $(g_{ne}(U))$ $=f_{ne}^{-1}(g_{ne}^{-1}(g_{ne}$ $(U$ $))$ $=f_{ne}^{-1}(U)$ is a neut α_γ- open function of $\mathcal{Z}_{ne}$. This demonstrates that $f_{ne}$is a neut $\alpha_{(\gamma,\beta)}$- continuous function.

(ii) Accredit $V \in \zeta_{(ne)\alpha_\delta}$. Meanwhile $g_{ne} \circ f_{ne}$ is a neut $\alpha_{(\gamma,\delta)}$ - continuous function, at that time $(g_{ne} \circ f_{ne})^{-1}(V) \in \tau_{(ne)\alpha_\gamma}$. As well f_{ne} is a neut $\alpha_{(\gamma,\beta)}$ - open function, accordingly f_{ne} $((g_{ne} \circ f_{ne})^{-1}(V))$ is a neut α_β- open set prevailing in $\mathcal{Y}_{ne}$. By reason of f_{ne} is surjective, we obtain $(f_{ne} \circ (g_{ne} \circ f_{ne})^{-1})(V)$ $= (f_{ne} \circ (f_{ne}{}^{-1} \circ g_{ne}{}^{-1}))(V) = ((f_{ne} \circ f_{ne}^{-1}) \circ g_{ne}{}^{-1})(V) = g_{ne}{}^{-1}(V)$. It trails that $g_{ne}{}^{-1}(V) \in \sigma_{(ne)\alpha_\beta}$. This evidences that g_{ne} is a neut $\alpha_{(\beta,\delta)}$-continuous function.

5.Neutrosophic $\alpha_{(\gamma,\beta)}$-Closed Functions

5.1.(i) Definition A function $f_{ne}:(\mathcal{Z}_{ne},\tau_{ne}) \to (\mathcal{Y}_{ne},\sigma_{ne})$ is supposed to be a neut $\alpha_{(\gamma,\beta)}$- closed function provided that the image se $f_{ne}(A)$ is a neut α_β-closed se for entire neutrosophic α_γ- closed subse A of $\mathcal{Z}_{ne}$.

(ii) Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is supposed to be a neut $(\gamma_{ne}, \alpha_{\beta})$-open(closed) function supposing that the icon of each neut β_{ne} -open(closed) se prevailing in $\mathcal{Z}_{ne}$ is a neut α_{β}-open(closed) se prevailing in $\mathcal{Y}_{ne}$.

(iii) Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ is thought to be a neut $(\gamma_{ne}, \beta_{ne})$-irresolute function in case that $f_M^{-1}(P)$ is a neut γ_{ne}-open se be present in $\mathcal{Z}_{ne}$ for each β_{ne}-open set P survives in $\mathcal{Y}_{ne}$.

5.2. Example Given $\mathcal{Z}_{ne} = \{h_1, h_2, h_3\}$,$\tau_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, F_1, F_3, F_4, F_5\}$

, $\mathcal{Y}_{ne} = \{g_1, g_2, g_4\}$and $\sigma_{ne} = \{1_{ne}, \mathcal{Z}_{ne}, H_1, H_4, H_5, H_6\}$ were

$F_1 = \{z, (0.2,0.6,0.3), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,
$F_3 = \{z, (0.8,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$,
$F_4 = \{z, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,
$F_5 = \{z, (0.7,0.6,0.3), (0.1,0.1,0.1), (0.1,0.1,0.1)\}$
$H_1 = \{y, (0.2,0.6,0.1), (0.1,0.1,0.1), (0.3,0.1,0.1)\}$,
$H_4 = \{y, (0.7,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.3)\}$,
$H_5 = \{y, (0.8,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$,
$H_6 = \{y, (0.7,0.6,0.3), (0.4,0.5,0.1), (0.1,0.1,0.1)\}$

Characterize an operation γ on τ_{ne} alike that$(U)^{\gamma} = \begin{cases} U \cup \{h_3\} if U \neq \{h_1\} \\ U \text{ if } U = \{h_1\} \end{cases}$

Specify an operation β on σ_{ne} aforesaid that$(V)^{\beta} = \begin{cases} V \text{ if } g_2 \notin V \\ cl(V) \text{ if } g_2 \in V \end{cases}$

Define $f_{ne}: \mathcal{Z}_{ne} \rightarrow \mathcal{Y}_{ne}$as $f_{ne}(h_1) = g_1$, $f_{ne}(h_2) = g_2$ and $f_{ne}(h_3) = g_3$.

Then the appearance of each one neut α_{γ}-closed se is a neut α_{β}-closed se under f_{ne}.

Henceforth f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-closed function.

5.3. Theorem Accredit $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ be a neut $\alpha_{(\gamma,\beta)}$-closed function, previously the succeeding declarations hold good.

(i) Assume $g_{ne}: (\mathcal{Y}_{ne}, \sigma_{ne}) \rightarrow (\mathcal{X}_{ne}, \delta_{ne})$ is a neut $\alpha_{(\beta,\delta)}$ -closed function, then $g_{ne} \circ f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{X}_{ne}, \delta_{ne})$ is a neut $\alpha_{(\gamma,\delta)}$- closed function;

(ii) $\sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(A)) \subseteq f_{ne}(\tau_{(ne)\alpha_\gamma}$-cl $(A))$, for each subset A of $\mathcal{Z}_{ne}$;

(iii) $\sigma_{(ne)\alpha_\beta}$-cl $(\sigma_{(ne)\alpha_\beta}$-int $(\sigma_{(ne)\alpha_\beta}$ -cl $(f_{ne}(A)))) \subseteq f_{ne}(\tau_{(ne)\alpha_\gamma}$-cl $(A))$, for all neut subset A of $\mathcal{Z}_{ne}$;

(iv) for all neut subset B of $\mathcal{Y}_{ne}$ and to each neut α_γ- open set A of $\mathcal{Z}_{ne}$ encompassing $f_{ne}^{-1}(B)$, there occurs a neut α_β-open set C in $\mathcal{Y}_{ne}$ comprising B alike that $f_{ne}^{-1}(C) \subseteq A$.

Proof. Confirmations are alike to the proofs of the 4.3.,4.4.,4.5 and 4.6. Theorems.

5.4. Theorem Let $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \rightarrow (\mathcal{Y}_{ne}, \sigma_{ne})$ be a neut bijective function. Previously the ensuing assertions are analogous:

(i) f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- closed function;

(ii) f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- open function;

(iii) f_{ne}^{-1} is a neut $\alpha_{(\beta,\gamma)}$-continuous function.

Proof. (i)$\Longrightarrow$ (ii) Substantiation trails after the declarations of 4.1. Definition and 5.1. Definition.

(ii)$\Longrightarrow$ (iii) Specify that A is a neut α_γ-closed se in $\mathcal{Z}_{ne}$. Formerly $\tau_{(ne)\alpha_\gamma}$-cl $(A) = A$. By the effect of (ii) and 4.8.Theorem, f_{ne}^{-1} $(\sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(A))) \subseteq \tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(f_{ne}(A)))$ infers that $\sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}$ $(A)) \subseteq f_{ne}(\tau_{(ne)\alpha_\gamma}$-cl $(A))$. Consequently $\sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}^{-1})^{-1}(A)) \subseteq (f_{ne}^{-1})^{-1}(A)$, aimed at each single subse A of $\mathcal{Z}_{ne}$, it trails that f_{ne}^{-1} is a neut $\alpha_{(\gamma,\beta)}$- continuous function.

(iii)$\Longrightarrow$ (i) Specify that A is a neut α_γ-closed se of $\mathcal{Z}_{ne}$. Formerly $\mathcal{Z}_{ne} - A$ is a neut α_γ- open se lying in $\mathcal{Z}_{ne}$. By reason of f_{ne}^{-1} is an $\alpha_{(\gamma,\beta)}$-continuous function,($f_{ne}^{-1})^{-1}$ $(\mathcal{Z}_{ne} - A)$ is a neut α_β- open se in

$\mathcal{Y}_{ne}$. Nevertheless $(f_{ne}^{-1})^{-1}(\mathcal{Z}_{ne}- A)$ = $f_{ne}(\mathcal{Z}_{ne} - A)$ = $\mathcal{Y}_{ne} - f_{ne}(A)$. Consequently $f_{ne}(A)$ is a neut α_{β}-closed se lying in $\mathcal{Y}_{ne}$. This one demonstrates that the function f_{Ma} is a neut $\alpha_{(\gamma,\beta)}$- closed function.

5.5. Definition Let $id: \tau \to P(X)$ remain as the identity maneuver. A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is held to be a neut $\alpha_{(id,\beta)}$ - closed func if designed at any neut α-closed se F of $\mathcal{Z}_{ne}$, $f_{ne}(F)$ is a neut α_{β}-closed se in $\mathcal{Y}_{ne}$.

5.6. Theorem Supposing that $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is a bijective function also $f_{ne}^{-1}: (\mathcal{Y}_{ne}, \sigma_{ne}) \to (\mathcal{Z}_{ne}, \tau_{ne})$ is a neut $\alpha_{(id,\beta)}$ -continuous function, formerly f_{ne} is a neut $\alpha_{(id,\beta)}$-closed function.

Proof. Authentication tracks next to the descriptions of 5.1. Definition besides 5.5. Definition.

5.7. Theorem Supposing that f_{ne} is a neut $\alpha_{(\gamma,\beta)}$-continuous function. Formerly

(i) Suppose that A is a neut $\alpha_{\gamma}\, g$ -closed se in $(\mathcal{Z}_{ne}, \tau_{ne})$, later the image $f_{ne}(A)$ is a neut $\alpha_{\beta} g$ -closed se.

(ii) Given B be a neut $\alpha_{\beta}\ g$ - closed se of $(\mathcal{Y}_{ne}, \sigma_{ne})$, later the set $f_{ne}^{-1}(B)$ is a neut $\alpha_{\gamma}\ g$ -closed se.

Proof. (i) Contemplate V as a neut α_{β}-open se prevailing in $\mathcal{Y}_{ne}$alike that $f_{ne}(A)$⊆ V. By means of 3.8. Theorem statement, $f_{ne}^{-1}(V)$ is a neut α_{γ}-open se encompassing A. By postulation $\tau_{(ne)\alpha_{\gamma}}$- cl $(A) \subseteq f_{ne}^{-1}$ (V), so f_{ne} $(\tau_{(ne)\alpha_{\gamma}}$-cl $(A)))\subseteq$ V. Since f_{ne} is a neut $\alpha_{(\gamma,\beta)}$- closed function, f_{ne} $(\tau_{(ne)\alpha_{\gamma}}$- cl

$(A))$ is a neut α_β-closed se comprising $f_{ne}(A)$entails that $\sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(A)) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(f_{ne}(\tau_{(ne)\alpha_\gamma}$- cl $(A))) = f_{ne}(\tau_{(ne)\alpha_\gamma}$- cl $(A)) \subseteq V$. Hence $f_{ne}(A)$ is a neut α_β g-closed se.

(ii) Assume U be a neut α_γ- open se of $\mathcal{Z}_{ne}$ akin that $f_{ne}^{-1}(B) \subseteq U$ for any subse B in $\mathcal{Y}_{ne}$. Put F = $\tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(B)) \cap (\mathcal{Z}_{ne} - U)$. It trails from the 2.2.(ii) Remark and 2.3. Theorem, that $F$is a neut α_γ-closed se in $\mathcal{Z}_{ne}$. Meanwhile $f_{ne}$is a neut $\alpha_{(\gamma,\beta)}$- closed function, $f_{ne}(F)$is a neut $\alpha_{(\gamma,\beta)}$- closed se in$\mathcal{Y}_{ne}$. By the 2.4 Theorem declaration then by the 3.8.(ii) Theorem declaration in addition the subsequent insertion $f_{ne}(F) \subseteq \sigma_{(ne)\alpha_\beta}$-cl $(B) - B$, it is gained that $f_{ne}(F) = \emptyset$, and henceforth $F = \emptyset$. This infers that $\tau_{(ne)\alpha_\gamma}$-cl $(f_{ne}^{-1}(B)) \subseteq U$. Therefore $f_{ne}^{-1}(B)$is a neut $\alpha_\gamma g$-closed se.

5.8. Theorem Given $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-continuous and neut $\alpha_{(\gamma,\beta)}$- closed function. Then

(i) With the condition that f_{ne} is a neut injective function moreover $\mathcal{Y}_{ne}$ is a neut $\alpha_\beta T_{\frac{1}{2}}$at that juncture $\mathcal{Z}_{ne}$is a neut $\alpha_\gamma T_{\frac{1}{2}}$ space.

(ii) Conceding that f_{ne} is a surjective function besides $\mathcal{Z}_{ne}$ is a neut $\alpha_\gamma T_{\frac{1}{2}}$ formerly $\mathcal{Y}_{ne}$ is a neut $\alpha_\beta T_{\frac{1}{2}}$ space.

Proof. (i) Accept A be a neut α_γ g-closed se of $\mathcal{Z}_{ne}$. Formerly via 5.7. Theorem statement (i), $f_{ne}(A)$ is a neutrosophic $\alpha_\beta g$- closed se. Accordingly, by postulation A is a neut α_γ -closed se in $\mathcal{Z}_{ne}$. So $\mathcal{Z}_{ne}$ is a neut α_γ $T_{\frac{1}{2}}$ space.

(ii) Contemplate B as a neut α_β g-closed se lying in $\mathcal{Y}_{ne}$. At that moment it surveys after the 5.7.(ii) Proposition and the supposition that f_{ne}^{-1} (B) is a neut α_γ-closed se. Hence f_{ne} is a neut

$\alpha_{(\gamma,\beta)}$-closed function, implies that $f_{ne}(f_{ne}^{-1}(B)) = B$ is a neut α_β-closed se in $\mathcal{Y}_{ne}$. Therefore $\mathcal{Y}_{ne}$ is a neut $\alpha_\beta T_{\frac{1}{2}}$ space.

5.9. Remark Each neut $(\gamma_{ne}, \beta_{ne})$-irresolute function is a neut $(\alpha_\gamma, \beta_{ne})$-continuous function. Nonetheless, the conflicting statement need not be factual.

6.Neutrosophic $\alpha_{(\gamma,\beta)}$- Homeomorphism

6.1. Definition A function $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ is a neut $\alpha_{(\gamma,\beta)}$-homeomorphism, if f_{ne} is a bijective, neut $\alpha_{(\gamma,\beta)}$-continuous function and f_{ne}^{-1} is a neut $\alpha_{(\beta,\gamma)}$-continuous function.

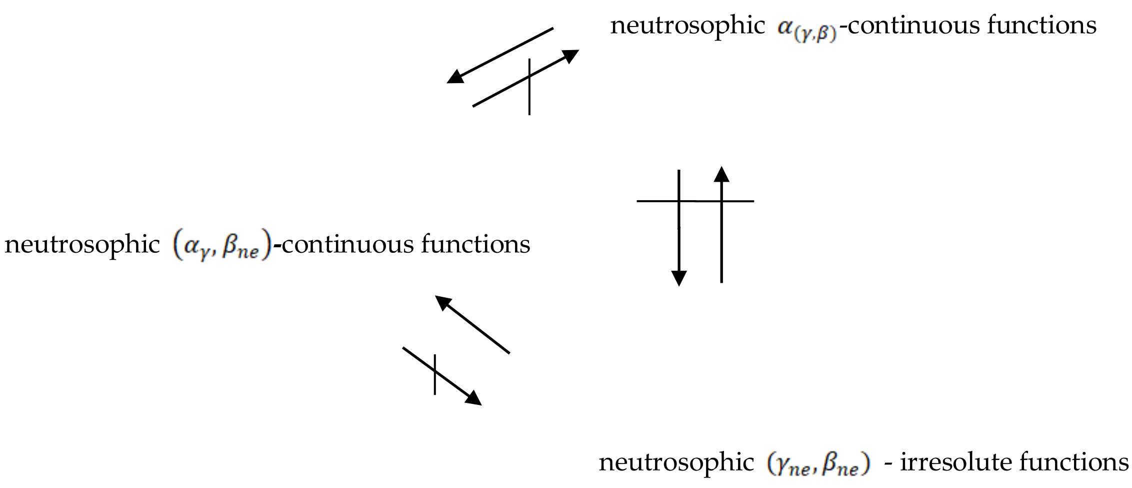
6.2. Remark Each bijective, neut $\alpha_{(\gamma,\beta)}$-continuous and neut $\alpha_{(\gamma,\beta)}$- closed function is a neut $\alpha_{(\gamma,\beta)}$-homeomorphism.

6.3. Theorem Let $f_{ne}: (\mathcal{Z}_{ne}, \tau_{ne}) \to (\mathcal{Y}_{ne}, \sigma_{ne})$ be a neut $\alpha_{(\gamma,\beta)}$-homeomorphism. If $\mathcal{Z}_{ne}$ is a neut $\alpha_\gamma T_{\frac{1}{2}}$space then $\mathcal{Y}_{ne}$is a neut $\alpha_\beta T_{\frac{1}{2}}$ space.

Proof. Assume $\{y\}$ as a singleton se of $\mathcal{Y}_{ne}$. Then there befalls a point z of $\mathcal{Z}_{ne}$ alike that $y = f_{ne}(z)$.Via 2.5. Theorem Announcement, it trails that the singleton se $\{y\}$ is furthermore a neut α_β-open se or else a neut α_β-closed se. Accordingly $\mathcal{Y}_{ne}$is a neut $\alpha_\beta T_{\frac{1}{2}}$ spa.

6.4. Remark Each neut $\alpha_{(\gamma,\beta)}$-open (closed) function is a neut $(\gamma_{ne}, \alpha_\beta)$-open (closed) function. Nonetheless, the opposing statement must not be exact. Then the succeeding comment shows the connotation amongst the neut $\alpha_{(\gamma,\beta)}$-open (closed) functions, neut $(\gamma_{ne}, \alpha_\beta)$-open (closed) functions and neut (γ_{ne}, β_s)- open (closed) functions.

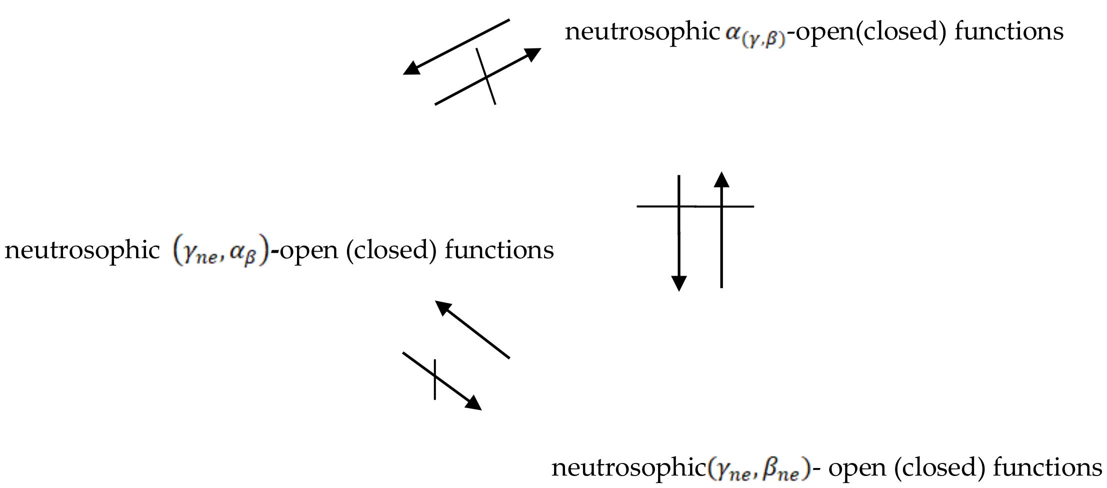
6.5. Remark From the 6.1.(i), (ii), (iii) Definitions 6.3. and 6.4. Remarks the subsequent illustrative inferences 2. Figure is attained:



$A \rightarrow B$ represents Ainfer B, $A \nrightarrow B$ represents A does not infer B.

2. Figure: Relationship between neutrosophic continuous functions

6.6. Remark As of the 4.1.,5.1. (i), (ii), (iii) Definitions and 5.9. Remark the ensuing pictorial inferences 3. Figure is gained:



$A \rightarrow B$ symbolizes A indicates at B, $A \nrightarrow B$ signifies A does not indicate at B.

3. Figure: Association between neutrosophic open (closed) functions

Conclusion and Future study

In this article the observation of neut $\alpha_{(\gamma,\beta)}$-continuous functions that are created over neut α_{γ}-open ses are considered and many of their basic properties are detailed. Also, the neut $\alpha_{(\gamma,\beta)}$-open (closed) functions are declared besides inspected their rudimentary properties. Neut α_{γ}-derived se, neut α_{γ}-frontier besides neut α_{γ}-kernel are described also experienced to create the ideas of several neut continuous functions and neut open(closed) functions. The connection amongst these neut $\alpha_{(\gamma,\beta)}$-continuous functions, neut $\alpha_{(\gamma,\beta)}$-open functions, neut $\alpha_{(\gamma,\beta)}$-closed functions are illustrated. Further the concept of the neut $(\gamma_{ne}, \alpha_{\beta})$-open(closed) function, neut $(\gamma_{ne}, \beta_{ne})$-irresolute function neut $(\alpha_{\gamma}, \beta_{ne})$ -continuous function and neut $(\gamma_{ne}, \beta_{ne})$ -continuous function, neut α_{γ}-neighbourhood of a point, neut α_{γ}-limit point, Composition of functions, neut α_{γ}-derived set and neut $\alpha_{(\gamma,\beta)}$- continuous, injective function are detailed and utilized for deriving numerous highly significant results. Contra neut $\alpha_{(\gamma,\beta)}$-continuous functions, contra neut $(\alpha_{\gamma}, \beta_{ne})$-continuous function and contra neut $(\gamma_{ne}, \beta_{ne})$-continuous function can be studied as a future work.

References

1. Alblowi, S. A., Salama, A. A., & Eisa, M. (2014). *New concepts of neutrosophic sets*. Infinite Study.
2. Atanassov, K. T. (1999). Intuitionistic fuzzy sets. In *Intuitionistic fuzzy sets* (pp. 1-137). Physica, Heidelberg.
3. Atanassov, K. (1988). Review and new results on intuitionistic fuzzy sets. *preprint Im-MFAIS-1-88, Sofia*, *5*, l.
4. Atanassov, K. T. (1992). Remarks on the intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, *51*(1), 117-118.
5. Chang, C. L. (1968). Fuzzy topological spaces. *Journal of mathematical Analysis and Applications*, *24*(1), 182-190.
6. Coker, D. (2000). An introduction to intuitionistic topological spaces. *Busefal*, *81*(2000), 51-56.
7. Kalaivani, N., & Krishnan, G. S. S. (2012). Operation approaches on α-(γ, β)-open (closed) mappings and γ generalized α-open sets. *International Journal of Scientific and Engineering Research*, *3*(8), 358-373.
8. Kalaivani, N., Saravanakumar, D., & Krishnan, G. S. S. (2012). On α- γ -irresolute functions and α - γ

-continuous functions in Topological spaces. In *Proceedings of the Jangjeon Mathematical Society* (Vol. 15, No. 4, pp. 465-476).

9. Kalaivani, N., & Krishnan, G. S. S. (2013). Operation approaches on α-γ-open sets in topological spaces. *International Journal of Mathematical Analysis*, *7*(10), 491-498.

10. N. Kalaivani, E. Chandrasekaran, Certain Characterizations of Neutrosophic γ -Open sets, Neutrosophicα_{γ}- Open sets in Neutrosophic Topological Spaces (submitted).

11. Lugojan, S. (1982). Generalized topology. *Stud. Cerc. Mat*, *34*, 348-360.

12. Salama, A. A., Smarandache, F., & Kromov, V. (2014). *Neutrosophic closed set and neutrosophic continuous functions*. Infinite Study.

13. Salama, A. A., & Alblowi, S. A. (2012). *Generalized neutrosophic set and generalized neutrosophic topological spaces*. Infinite Study.

14. Salama, A. A., & Alblowi, S. A. (2012). Neutrosophic set and neutrosophic topological spaces. *IOSR Journal of Mathematics*, *3*(4), 31-35.

15. Smarandache, F. (2010). Neutrosophic set–a generalization of the intuitionistic fuzzy set. *Journal of Defense Resources Management (JoDRM)*, *1*(1), 107-116.

16. Smarandache, F. (2005). Neutrosophic set-a generalization of the intuitionistic fuzzy set. *International journal of pure and applied mathematics*, *24*(3), 287.

17. Smarandache, F. (2002). Neutrosophy and neutrosophic logic. In *First International Conference on Neutrosophy, Neutrosophic Logic, Set, Probability, and Statistics University of New Mexico, Gallup, NM* (Vol. 87301, pp. 338-353).

18.Smarandache, F. (1999). A unifying field in Logics: Neutrosophic Logic. In *Philosophy* (pp.1-141). American Research Press.

19. Wang, H., Smarandache, F., Zhang, Y., & Sunderraman, R. (2010). *Single valued neutrosophic sets*. Infinite study.

20. Turanli, N., & Çoker, D. (2000). Fuzzy connectedness in intuitionistic fuzzy topological spaces. *Fuzzy sets and systems*, *116*(3), 369-375.

21. Zadeh, L. A. (1965). Fuzzy Sets, Inform. and Control. 8. (1965), pp. 338–353.

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On Solving Bi-objective Interval Valued Neutrosophic Assignment Problem

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Abstract: The assignment problem (AP) is a well-researched combinatorial optimization problem in which the overall assignment cost or time is minimized by assigning multiple items (tasks) to several entities (workers). Today's optimization challenges cannot be adequately addressed by a single-objective AP, hence the bi-objective AP (BOAP) is taken into consideration. This problem frequently occurs in practical applications with ambiguous parameters in real life. Henceforth, in this article the uncertain parameters are presented as interval valued neutrosophic numbers. In the present study, we formulate bi-objectives assignment problem (BOAP) having cost and time parameters as an interval valued neutrosophic numbers. We proposed interactive left-width method to solve the interval valued neutrosophic BOAP (IVNBOAP). In this method interval valued neutrosophic numbers is reduced to interval numbers using score function. Then, the bi-objective interval assignment problem (BOIAP) is reduced to a deterministic BOAP using the left-width attributes on each objective function. The reduced deterministic objective function is separated and constructed as a multi-objective AP. In the solution procedure, the global weighted sum method is adopted to convert the multi-objective AP into a single objective problem (SOP) and solved using Lingo 18.0 software. Finally, numerical examples are illustrated to clarify the steps involved in the proposed method and results are compared with the other existing methods.

Keywords: Interval Assignment Problem, Interval-valued Neutrosophic Numbers, Interactive Left-Width Method, Optimal Compromise Solution, Global Weighted Sum Method.

1. Introduction

In the AP, the objective is to distribute several tasks to an equal number of machines, people or facilities with optimal decision parameters. From the existing literature, it can be seen that several researchers have come up with different methods to resolve AP[1–3]. In all these studies, it is noted that actual deterministic numbers are used for effectual matrices of the relevant AP. In real-life situations, the elements of the effectual matrices of AP are an imprecise number than deterministic, due to the limited knowledge of personnel on problem domain, lack of data, inaccurate estimates, etc. This inexact information on decision parameters is expressed by interval numbers or fuzzy numbers or intuitionistic numbers or neutrosophic numbers. In recent years, numerous experts [4–7] have conducted thorough studies on the interval AP. When the boundary of this interval is ambiguous, that interval is a fuzzy set. In 1965, Lotfi Zadeh introduced fuzzy set theory, which was developed to provide formalized techniques for addressing imprecision through varying degrees of membership and to mathematically

describe uncertainty. Various researchers such as Gupta et al.[8] , Thorani and Shankar [9], Baidya and Bera [10] , Buvaneshwari and Anuradha [11] have utilized the various methods for solving fuzzy TP/AP to determine the optimal/ optimal compromise solution.
The fuzzy set (FS) deals with uncertainty, but hesitation is also taken into consideration in a real-life problem. Atanassov [12] has extended FS to intuitionistic FS (IFS) by including hesitation as a non-membership degree. An IFS can be a realistic and relevant tool in dealing with problems having both uncertainty and hesitation. An accuracy function was applied by Ebrahimnejad and Verdegay [13] and Mahmoodirad et al. [14] to solve the intuitionistic fuzzy transportation problem (IFTP). Roy et al. [15] proposed the intuitionistic fuzzy programming approach (IFPA) and goal programming approach (GPA) to solve the intuitionistic fuzzy multi-objective transportation problem (IFMOTP). Bharati [16] has discussed TP with interval-valued IFS influence of a new ranking. Mahajan and Gupta [29] utilized a variety of membership functions (MFs) to solve fully IFMOTP. Ahmadini and Ahmad [17] proposed the different MFs for solving the intuitionistic fuzzy multi-objective linear programming problem. IFS contemplate both the degree of MF and non-MF, but it cannot deal with reality's inherent indeterminacy. To tackle these problems, Smarandache [18] introduced a theory of the neutrosophic set (NS), which is the degree of indeterminacy as well as the degree of truth MF and falsity MF while making decisions. Das and Roy [19] developed novel method named computational algorithm for handling the multi-objective non-linear minimization programming problem in the neutrosophic environment. Risk Allah et al.[20] proposed the neutrosophic compromise programming approach to solve the MO transportation problem under neutrosophic environment and it is verified by applying the TOPSIS technique to measure the ranking degree. Broumi and Smarandaache [21] presented innovative approaches for harmonic, geometrical, and arithmetic means for interval neutrosophic sets. Khalifa et al.[22] proposed the approach for optimality conditions to the interval valued neutrosophic TP and it is solved by Weighting Tchebycheff method. Saini et.al [23] introduced a novel approach namely minimum row column method for interval-valued trapezoidal neutrosophic transportation problem. Khalil et al. [24] discussed on the aspirations levels for interval-valued true, interval-valued falsity, and interval-valued indeterminacy, which are dependent only on the algebra of interval neutrosophic sets and confluence criteria.

The contributions of this paper are as follows:

We proposed interactive left-width method to solve the interval valued neutrosophic BOAP (IVNBOAP). The IVNBOAP is first reduced to a BOIAP using the score function and it is reduced to deterministic bi-objective assignment problem using the left-width attributes on each objective function. Then, construct the multi-objective problem by splitting each objective function. The reduced multi-objective problem cannot be solved explicitly. Also, the managers are always keen on minimizing the cost and time of AP. The global weighted sum method (GWSM) is used to transform the deterministic multi-objective AP into the single-objective AP. Using the Lingo 18.0 software, the reduced problem is solved to obtain the optimal compromise solution of the IVNBOIAP.

The construction of this paper is as follows: In Section 2, basic concepts and preliminaries are presented. Section 3 describes the problem formulation of IVNBOAP and Section 4 briefly proposed the interactive left-width method. Section 5 illustrates the proposed method implementation using the numerical examples and its computational results. In Section 6, the results and discussion part have been included and Section 7 discusses sensitivity analysis and finally the conclusion and future scope of this paper.

2. Preliminaries

The fundamental concepts of arithmetic operations, partial ordering of closed bounded intervals, interval optimal solutions, and optimal compromise solutions are found in [25].

Definition 2.1[26] An interval number is a number whose precise value is unknown, but the range in which it lies is known. An interval number with lower and upper boundaries as $A=[a^L, a^U]$, where $a^L \leq a^U$. The mid and width of the interval are similarly shown as $A=<a^m, a^w>=\{a: a^L-a^U \leq a \leq a^L+a^U, a \in R\}$, where $a^m=\frac{(a^L+a^U)}{2}$ and $a^w=(a^U-a^L)$ respectively.

Definition 2.2 [26] The order relation $\leq_{LU}$ between A $=[a^L, a^U]$ and B $=[b^L, b^U]$.

$A \leq_{LU} B$ *iff* $a_L \leq b_L$ and $a_U \leq b_U$,

$A <_{LU} B$ *iff* $A \leq_{LU} B$ and $A \neq B$.

This order relation $\leq_{LU}$ represents the decision maker's (DMs) preference for the alternative with lower minimum and maximum cost, that is, if $A \leq_{LU} B$, then A is preferred to B.

Definition 2.3 [26] The order relation $\leq_{LU}$ between A $=[a^m, a^w]$ and B $=[b^m, b^w]$.

$A \leq_{mw} B$ *iff* $a_m \leq b_m$ and $a_w \leq b_w$,

$A <_{mw} B$ *iff* $A \leq_{mw} B$ and $A \neq B$.

This order relation $\leq mw$ represents the DMs preference for the alternative with lower minimum and maximum cost, that is, if $A \leq mw\ B$, then A is preferred to B. To compare interval numbers, the total of each element in the interval number is utilised as a scale. The total of all the components of the interval number that equals zero is the zero interval.

Definition 2.4 (Neutrosophic set [27]) Let X be a universe. A neutrosophic set F over X is defined by $\tilde{\mathrm{N}}^N=\left\{\left\langle x, P^N(x), Q^N(x), R^N(x)\right\rangle : x \in X\right\}$ where $P^N, Q^N, R^N : X \to \left]0^-, 3^+\right[$ are called the truth, indeterminacy and falsity MF of the element $x \in X$ to the set $\bar{D}^N$ with $0^- \leq P^N(x)+Q^N(x)+R^N(x) \leq 3^+$.

Definition 2.6 (Interval-valued neutrosophic set [21]). Let X be a nonempty set. Then an interval valued neutrosophic (IVN) set of X is defined as:

$$\tilde{\mathrm{N}}^{IVN}=\left\{\left\langle x,[P_L^{IVN}(x), P_U^{IVN}(x)],[Q_L^{IVN}(x), Q_U^{IVN}(x)],[R_L^{IVN}(x), R_U^{IVN}(x)]\right\rangle : x \in X\right\}, \text{ where}$$

$$\left([P_L^N(x), P_U^N(x)],[Q_L^N(x), Q_U^N(x)],[R_L^N(x), R_U^N(x)]\right) \in [0,1].$$

The neutrosophic numbers, trapezoidal neutrosophic numbers and its arithmetic operation are referred in [28].

Definition 2.7 Let $\tilde{f}^{IVN}$ be the TrNNs and it can be evaluated using the score function and accuracy function as follows:

i. Score function $SC\left(\tilde{f}^{IVN}\right)=\left(\frac{1}{16}\right)\left[r+s+t+u\right]*\left[P^{IVN}+(1-Q^{IVN})+(1-R^{IVN})\right]$

ii. Accuracy function $AC\left(\tilde{f}^{IVN}\right)=\left(\frac{1}{16}\right)\left[r+s+t+u\right]*\left[\lambda_{\tilde{p}^N}+(1-\delta_{\tilde{p}^N})+(1+\sigma_{\tilde{p}^N})\right]$

3. Description and Problem formulation

This section defines the model assumption, indices, formulation of interval valued neutrosophic bi-objective assignment problem.

3.1 Mathematical Model of Interval valued Neutrosophic Bi-Objective Assignment Problem

We consider n skilled workers in agencies and the n companies want the workers to process their jobs. Each worker has to be associated with one and only one company. A penalty $\tilde{c}_{ij}^{IVN}=\left(c_{ij}^1,c_{ij}^2,c_{ij}^3,c_{ij}^4;[P_L^{IVN},P_U^{IVN}],[Q_L^{IVN},Q_U^{IVN}],[R_L^{IVN},R_U^{IVN}]\right)$ is the cost of transport and $\tilde{t}_{ij}^{N}=\left(t_{ij}^1,t_{ij}^2,t_{ij}^3,t_{ij}^4;[P_L^{IVN},P_U^{IVN}],[Q_L^{IVN},Q_U^{IVN}],[R_L^{IVN},R_U^{IVN}]\right)$ is the total time to reach the companies, which is incurred when companies j (j =1,2,...,n) is processed by the workers i (i =1,2,...,n) respectively. Let $\tilde{x}_{ij}^{N}=\left(x_{ij}^{1},x_{ij}^{2},x_{ij}^{3},x_{ij}^{4};[P_L^{IVN},P_U^{IVN}],[Q_L^{IVN},Q_U^{IVN}],[R_L^{IVN},R_U^{IVN}]\right)$ denote the assignment of j^{th} company to i^{th} worker. Our aim is to determine the worker-to-company assignment at a minimum assignment cost and time to the companies.

Now, the mathematical model of the above IVNBOAP is given as detailed below.

(A) $$Minimize\ \tilde{Z}_1^{IVN}(x)=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{c}_{ij}^{IVN}\tilde{x}_{ij}^{IVN}, \quad (1)$$

$$Minimize\ \tilde{Z}_2^{IVN}(x)=\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{t}_{ij}^{IVN}\tilde{x}_{ij}^{IVN}, \quad (2)$$

subject to the constraints

$$\sum_{j=1}^{n}\tilde{x}_{ij}^{IVN}=1^{IVN}, i=1,2,...,n, \quad (3)$$

$$\sum_{i=1}^{n}\tilde{x}_{ij}^{IVN}=1^{IVN}, j=1,2,...,n, \quad (4)$$

$$\tilde{x}_{ij}^{IVN}=0^{IVN}\ or\ 1^{IVN}\ \text{for all i and j}. \quad (5)$$

Using score function (Definition 2.7) the problem (A) is reduced to bi-objective interval AP (B). Now, the mathematical model of the BOIAP is given as detailed below.

(B) $$Minimize[Z_1^L,Z_1^U]=\sum_{i=1}^{n}\sum_{j=1}^{n}\left[c_{ij}^L,c_{ij}^U\right]x_{ij} \quad (6)$$

$$Minimize[Z_2^L, Z_2^U] = \sum_{i=1}^{n}\sum_{j=1}^{n}\left[t_{ij}^L, t_{ij}^U\right]x_{ij} \quad (7)$$

subject to the constraints

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1,2,\ldots n \quad (8)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1,2,\ldots n \quad (9)$$

x_{ij} = 0 or 1 for all i and j. (10)

Ishibuchi and Tanaka[26] state that the expected value and interval uncertainty can be attributed to an interval's midpoint and width. Since the objective function (6) and (7) of Problem (B) is the cost and time function which is to be minimized simultaneously and our aim is to obtain optimal compromise solution with minimum ambiguity. We can express the problem (B) in terms of expected cost and time using definition (2.1). Since any two of the four characteristics of an interval—left limit, right limit, mid-value, and width—can be used to represent it. Finally, the objective function of BOIAP (6) and (7) can be reduced to a left and width objective value problem (M) by employing left and width attributes.

$$\text{(M)} \quad Minimize < Z_1^L, Z_1^w > = \sum_{i=1}^{n}\sum_{j=1}^{n} < c_{ij}^L, c_{ij}^w > x_{ij} \quad (11)$$

$$Minimize < Z_2^L, Z_2^w > = \sum_{i=1}^{n}\sum_{j=1}^{n} < t_{ij}^L, t_{ij}^w > x_{ij} \quad (12)$$

subject to the constraints (8) to (10).

Construct the multi objective problem (N) by splitting the left and width of each objective function (11) and (12).

$$\text{(N)} \quad Minimize \ Z_1^L = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^m x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^w x_{ij} \quad (13)$$

$$Minimize \ Z_1^w = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}^w x_{ij} \quad (14)$$

$$Minimize \ Z_2^L = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}^m x_{ij} - \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}^w x_{ij} \quad (15)$$

$$Minimize \ Z_2^w = \sum_{i=1}^{n}\sum_{j=1}^{n} t_{ij}^w x_{ij} \quad (16)$$

subject to the constraints (8) to (10).

The width of the cost coefficient of Z_1, $c_{ij}^w = \left(\dfrac{c_{ij}^U - c_{ij}^L}{2}\right)$,

The mid-point of the cost coefficient of Z_1, $c_{ij}^m = \left(\dfrac{c_{ij}^U + c_{ij}^L}{2}\right)$,

The width of the cost coefficient of Z_2, $t_{ij}^w = \left(\dfrac{t_{ij}^U - t_{ij}^L}{2}\right)$, The mid-point of the cost coefficient of Z_2, $t_{ij}^{m} = \left(\frac{t_{ij}^{U} + t_{ij}^{L}}{2}\right)$.

4. Interactive Left-Width Method (ILWM)

Step 1: Construct the problem (B) from the problem (A) using the Score function.
Step 2: Using left and width attributes, the objective function of (B) can be reduced into a left and width value problem (M).
Step 3: Construct the multi objective problem (N) by splitting the left and width objective value problem (M).
Step 4: Reduce the problem (N) into single objective problem (G) using global weighted sum method [29].
Step 5: Using step 4, the optimal compromise solution for (G) is obtained. Also, the optimal compromise solution for the problem (A) is obtained from each x_{ij} through proposed method.

4.1 Working Methodology

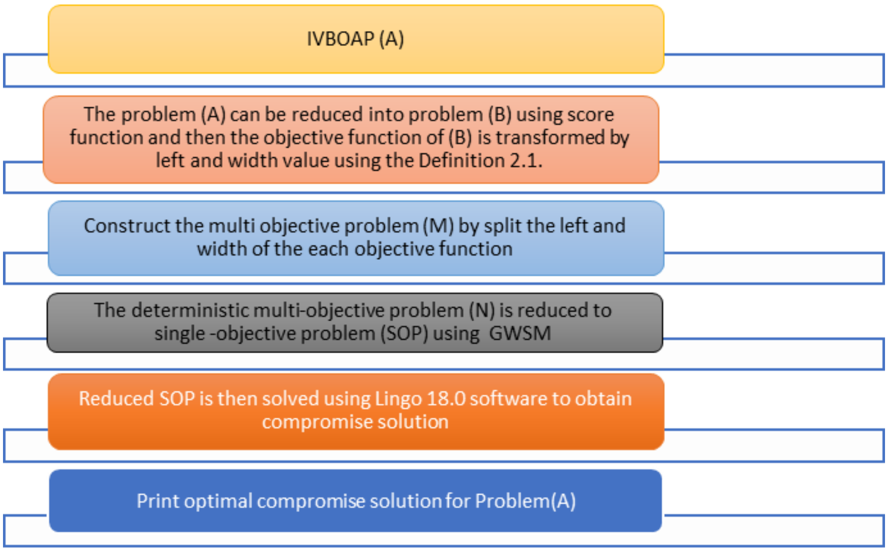

Figure 1: Working methodology of BOIAP

5. Application Example

In this section, two application examples are provided to illustrate our proposed method.
Example 5.1 A labour agency must arrange the distribution of three distinct skilled workers to three distinct companies in three different locations. Consider that there are two objectives to be considered: (i) Determine the distribution that reduces the overall cost of transferring workers to companies. (ii) Reduce the overall travel time (in hours) to the companies. We typically can't get this information precisely because the allocation schedule has been prepared in advance. The typical method for obtaining interval data for this condition is to rate the experience. Consider the following IVNBOAP, which is shown in the Table 1.

Table 1: The bi- objective interval valued neutrosophic AP.

		D1	D2	D3
c_{ij}/ t_{ij}	S1	$[\tilde{c}_{11}]^{IVN}$ $[\tilde{t}_{11}]^{IVN}$	$[\tilde{c}_{12}]^{IVN}$ $[\tilde{t}_{12}]^{IVN}$	$[\tilde{c}_{13}]^{IVN}$ $[\tilde{t}_{13}]^{IVN}$
	S2	$[\tilde{c}_{21}]^{IVN}$ $[\tilde{t}_{21}]^{IVN}$	$[\tilde{c}_{22}]^{IVN}$ $[\tilde{t}_{22}]^{IVN}$	$[\tilde{c}_{23}]^{IVN}$ $[\tilde{t}_{23}]^{IVN}$
	S3	$[\tilde{c}_{31}]^{IVN}$ $[\tilde{t}_{31}]^{IVN}$	$[\tilde{c}_{33}]^{IVN}$ $[\tilde{t}_{32}]^{IVN}$	$[\tilde{c}_{33}]^{IVN}$ $[\tilde{t}_{33}]^{IVN}$

$[\tilde{c}_{11}]^{IVN}$= ((10, 11, 12, 13);[0.1,0.7],[0.8,0.8],[0.8,0.9]); $[\tilde{t}_{11}]^{IVN}$ = ((18, 27, 29, 30);[0.1,0.5],[0.8,0.8],[0.9,0.9])

$[\tilde{c}_{12}]^{IVN}$ = ((28,29, 31, 38);[0.1,0.6],[0.7,0.7],[0.8,0.8]); $[\tilde{t}_{12}]^{IVN}$ = ((10, 12, 25, 26);[0.1,0.6],[0.8,0.9],[0.9,0.9])

$[\tilde{c}_{13}]^{IVN}$ = ((23, 25, 31, 38); [0.1,0.6], [0.7,0.7],[0.8,0.8]); $[\tilde{t}_{13}]^{IVN}$= ((10, 11, 12, 13);[0.1,0.9], [0.7,0.7],[0.9,0.6])

$[\tilde{c}_{21}]^{IVN}$ =((14, 17, 21, 28);[0.2,0.9], [0.2,0.3],[0.6,0.6]); $[\tilde{t}_{21}]^{IVN}$= ((23, 25, 31, 38);[0.1,0.5], [0.7,0.8],[0.8,0.9])

$[\tilde{c}_{22}]^{IVN}$ =((18, 19, 21, 22);[0.1,0.9], [0.8,0.8],[0.9,0.9]); $[\tilde{t}_{22}]^{IVN}$= ((14, 17, 21, 28);[0.2,0.8], [0.2,0.2],[0.6,0.6])

$[\tilde{c}_{23}]^{IVN}$ =((18,27, 29, 30);[0.1,0.5], [0.8,0.8],[0.9,0.9]); $[\tilde{t}_{23}]^{IVN}$= ((15, 17, 21, 28);[0.2,0.9], [0.1,0.2],[0.4,0.6])

$[\tilde{c}_{31}]^{IVN}$ =((14, 17, 21, 28);[0.2,0.9], [0.2,0.2],[0.6,0.6]); $[\tilde{t}_{31}]^{IVN}$ =((23, 25, 31, 38);[0.1,0.6], [0.7,0.7],[0.8,0.8])

$[\tilde{c}_{32}]^{IVN}$ =((18, 27, 29, 30);[0.1,0.5], [0.8,0.8],[0.9,0.9]); $[\tilde{t}_{32}]^{IVN}$= ((23, 25, 31, 38);[0.1,0.5], [0.8,0.8],[0.9,0.9])

$[\tilde{c}_{33}]^{IVN}$ =((28, 29, 31, 38);[0.1,0.6], [0.7,0.8],[0.8,0.9]); $[\tilde{t}_{33}]^{IVN}$= ((10, 11, 12, 13);[0.1,0.5], [0.8,0.9],[0.8,0.9])

Using Step 1, the problem (A) is reduced to problem (B) using the score function (definition 2.7) as shown in Table 2.

Table 2: The bi- objective interval assignment problem.

		Labour Agencies		
		L_1	L_2	L_3
Workers	W_1	[1,3] [3,5]	[5,9] [2,4]	[4,8] [1,5]
	W_2	[7,10] [4,6]	[2,6] [7,10]	[3,5] [9,11]
	W_3	[7,11] [4,8]	[3,5] [3,6]	[5,7] [1,2]

Using Step 2, construct the problem (M) from the problem (B) by using the definition (2.1) as shown in Table 3.

Table 3: The bi- objective left-width assignment problem(M).

		Labour Agencies		
		L_1	L_2	L_3
Workers	W_1	<1,1> <3,1>	<5,2> <2,1>	<4,2> <1,2>
	W_2	<7,1.5> <4,1>	<2,2> <7,1.5>	<3,1> <9,1>
	W_3	<7,2> <4,2>	<3,1> <3,1.5>	<5,1> <1,0.5>

By Step 3, split the problem (M) into four objectives (z_1^L, z_1^W, z_2^L, z_2^W) by left and width objective function, which is shown in below Table 4.

Table 4: Multi- objective assignment problem(N).

		Labour Agencies											
		Problem (z_1^L)			Problem (z_1^W)			Problem(z_2^L)			Problem (z_2^W)		
		L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3	L_1	L_2	L_3
Workers	W_1	1	5	4	1	2	2	3	2	1	1	1	2
	W_2	7	2	3	1.5	2	1	4	7	9	1	1.5	1
	W_3	7	3	5	2	1	1	4	3	1	2	1.5	0.5

By Step 4, the problem (M) is reduced into single objective problem (SOP) using global weighted sum method as follows. In problem (M), solve each objective function individually with the constraints (8-10) using the Hungarian algorithm. Optimal solution for (z_1^L) = 7, $x_{11}=x_{23}=x_{32}=1$, (z_1^W)=3, $x_{11}=x_{23}=x_{32}=1$, (z_2^L) =7, $x_{12}=x_{21}=x_{33}=1$, and (z_2^W)=2.5, $x_{11}=x_{23}=x_{32}=1$. Then, create the pay-off matrix is as shown in Table 5.

Table 5: Pay-off matrix

	z_1^L	z_1^W	z_2^L	z_2^W
x1	**7**	3	15	3.5
x2	7	**3**	15	3.5
x3	17	4.5	**7**	2.5
x4	17	4.5	7	**2.5**

We find the lower and upper bound of each objective function that is L1=7, L2=3, L3=7, L4=2.5, U1=17, U2=4.5, U3=15, U4=3.5. Formulate the deterministic model with the weights 0.25 to each objective,

(G)Minimize η;

subject to the constraints (9) to (11), $\eta \geq 0$;.

$$\eta = \left\{ 0.25\left(\frac{Z_1^L - 7}{17-7}\right)^2 + 0.25\left(\frac{Z_1^w - 3}{4.5-3}\right)^2 + 0.25\left(\left[\frac{Z_2^L - 7}{15-7}\right]\right)^2 + 0.25\left(\frac{Z_2^w - 2.5}{3.5-2.5}\right)^2 \right\}^{\frac{1}{2}};$$

Then, using Step 5, Lingo 18.0 software is employed to solve the problem (G) to obtain optimal allocation is $x_{11}=x_{22}=x_{33}=1$. Replace the optimal allocation of the problem (G) to the problem (B) is Z=([8,16],[11,17]) and for problem (A) is $\tilde{Z}^{IVN}$ ={((56,59,64,73);[0.7,0.9],[0.7,0.8],[0.8,0.9]), ((42,55,62,71);[0.8,0.9], [0.2,0.2],[0.6,0.6])}.

Example 5.2 The bi- objective interval- valued Neutrosophic AP model is considered in order to confirm the method's efficacy: Three separate skilled workers must be allocated among three various branches of businesses in four different locations, according to an automobile manufacturing corporation. Consider that there are two goals to consider: (i) Identify the distribution that minimizes the overall cost of hiring new personnel. (ii) Shorten the distance travelled (in hours) between the companies. Typically, the allocation plan has been created in advance, thus we are unable to obtain this information precisely. The standard method is to rate the experience to gather interval data for this circumstance. Consider the following bi-objective interval valued neutrosophic assignment problem is shown in Table 6.

Table 6: The bi- objective interval valued neutrosophic AP.

c_{ij}/ t_{ij}		D1	D2	D3	D4
	S1	$[\tilde{c}_{11}]^{IVN}$ $[\tilde{t}_{11}]^{IVN}$	$[\tilde{c}_{12}]^{IVN}$ $[\tilde{t}_{12}]^{IVN}$	$[\tilde{c}_{13}]^{IVN}$ $[\tilde{t}_{13}]^{IVN}$	$[\tilde{c}_{14}]^{IVN}$ $[\tilde{t}_{14}]^{IVN}$
	S2	$[\tilde{c}_{21}]^{IVN}$ $[\tilde{t}_{21}]^{IVN}$	$[\tilde{c}_{22}]^{IVN}$ $[\tilde{t}_{22}]^{IVN}$	$[\tilde{c}_{23}]^{IVN}$ $[\tilde{t}_{23}]^{IVN}$	$[\tilde{c}_{24}]^{IVN}$ $[\tilde{t}_{24}]^{IVN}$
	S3	$[\tilde{c}_{31}]^{IVN}$ $[\tilde{t}_{31}]^{IVN}$	$[\tilde{c}_{33}]^{IVN}$ $[\tilde{t}_{32}]^{IVN}$	$[\tilde{c}_{33}]^{IVN}$ $[\tilde{t}_{33}]^{IVN}$	$[\tilde{c}_{34}]^{IVN}$ $[\tilde{t}_{34}]^{IVN}$

$[\tilde{c}_{11}]$= ((14, 17, 23, 28);[0.3,0.8], [0.2,0.3],[0.1,0.2]); $[\tilde{t}_{11}]$= ((14, 17, 21, 28);[0.4,0.9], [0.1,0.3],[0.5,0.5])

$[\tilde{c}_{12}]$= ((26,27, 30, 33);[0.4,0.9], [0.2,0.3],[0.2,0.4]); $[\tilde{t}_{12}]$= ((26, 27, 30, 33);[0.6,0.9], [0.2,0.3],[0.2,0.3])

$[\tilde{c}_{13}]$= ((49, 50, 55, 57);[0.5,0.9], [0.4,0.5],[0.5,0.6]); $[\tilde{t}_{13}]$= ((49, 50, 55, 57);[0.5,0.9], [0.4,0.5],[0.5,0.6])

$[\tilde{c}_{14}]$=((49, 52, 55, 57);[0.4,0.9], [0.4,0.5],[0.4,0.5]); $[\tilde{t}_{14}]$= ((26, 27, 30, 33);[0.6,0.9], [0.2,0.2],[0.2,0.2])

$[\tilde{c}_{21}]$=((48, 49, 50, 51);[0.1,0.9], [0.5,0.5],[0.4,0.4]); $[\tilde{t}_{21}]$= ((17, 19, 23, 28);[0.2,0.8], [0.3,0.3],[0.2,0.2])

$[\tilde{c}_{22}]$=((53, 56, 57, 58);[0.1,0.9], [0.5,0.6],[0.9,0.9]); $[\tilde{t}_{22}]$= ((51, 56, 57, 58);[0.1,0.9], [0.5,0.6],[0.6,0.9])

$[\tilde{c}_{23}]$=((14,17, 21, 28);[0.4,0.9], [0.1,0.3],[0.5,0.5]); $[\tilde{t}_{23}]$= ((26, 27, 30, 33);[0.6,0.9], [0.2,0.2],[0.2,0.2])

$[\tilde{c}_{24}]$=((60, 61, 65, 69);[0.3,0.7], [0.5,0.7],[0.7,0.8]); $[\tilde{t}_{24}]$= ((60, 61, 65, 69);[0.4,0.6], [0.5,0.7],[0.6,0.7])

$[\tilde{c}_{31}]$=((49, 52, 56, 58);[0.4,0.9], [0.4,0.5],[0.5,0.5]); $[\tilde{t}_{31}]$= ((30, 34, 38, 45);[0.1,0.9], [0.6,0.6],[0.5,0.5])

$[\tilde{c}_{32}]$=((28, 31, 35, 38);[0.1,0.9], [0.6,0.6],[0.3,0.3]); $[\tilde{t}_{32}]$= ((49, 50, 52, 53);[0.5,0.9], [0.5,0.5],[0.4,0.4])

$[\tilde{c}_{33}]$=((48, 49, 50, 51);[0.5,0.9], [0.5,0.5],[0.4,0.4]); $[\tilde{t}_{33}]$= ((59, 65, 80, 83);[0.5,0.6], [0.7,0.7],[0.4,0.4])

$[\tilde{c}_{34}]$=((49, 52, 56, 58);[0.2,0.6], [0.6,0.6],[0.5,0.5]); $[\tilde{t}_{34}]$= ((72, 82, 83, 84);[0.4,0.6], [0.6,0.7],[0.4,0.5])

By Step 1, using the score function (definition 2.7) the problem (A) is reduced to problem (B) as shown in Table 7.

Table 7: The bi- objective interval unbalanced assignment problem.

c_{ij}/ t_{ij}		D1	D2	D3	D4
	S1	[10,12]	[15,16]	[21,24]	[21,25]
		[9,11]	[16,17]	[21,24]	[16,18]
	S2	[15,25]	[10,20]	[9,11]	[18,19]
		[9,13]	[14,19]	[16,18]	[19,20]
	S3	[20,26]	[10,17]	[20,25]	[15,20]
		[9,17]	[20,26]	[25,27]	[28,29]

Using Step 2, construct the problem (N) by using the equations (6-17) which is shown in Table 8.

Table 8: The bi- objective left-width unbalanced assignment problem.

c_{ij}/ t_{ij}		D1	D2	D3	D4
	S1	<10,1>	<15,0.5>	<21,1.5>	<21,2>
		<9,1>	<16,0.5>	<21,1.5>	<16,1>
	S2	<15,5>	<10,5>	<9,1>	<18,0.5>
		<9,2>	<14,2.5>	<16,1>	<19,0.5>
	S3	<20,3>	<10,3.5>	<20,2.5>	<15,2.5>
		<9,4>	<20,3>	<25,1>	<28,0.5>

By Step 3, split the problem (N) into four objective using left and width values of the function. Table 9 and Table 10 shows that the multi- objective unbalanced assignment problem (MOUBAP) for cost and time.

Table 9: The multi- objective unbalanced assignment problem for cost.

		Labour Agencies (z_1^L)				**Labour Agencies (z_1^W)**			
		D1	D2	D3	D4	D1	D2	D3	D4
Workers	S1	10	15	21	21	1	0.5	1.5	2
	S2	15	10	9	18	5	5	1	0.5
	S3	20	10	20	15	3	3.5	2.5	2.5

Table 10: The multi- objective unbalanced assignment problem for time.

		Labour Agencies (z_2^L)				**Labour Agencies (z_2^W)**			
		D1	D2	D3	D4	D1	D2	D3	D4
Workers	S1	9	16	21	16	1	0.5	1.5	1
	S2	9	14	16	19	2	2.5	1	0.5
	S3	9	20	25	28	4	3	1	0.5

Now, using Step 4, reduce the MOUBAP into SOP using global weighted sum method. Then, formulate the deterministic model with the weights 0.25 to each objective function. Using Step 5, obtain the optimal compromise solution for the problem (B) is $x_{11}=x_{23}=x_{34}=x_{42}=1$, Z=([34,43],[53,58]) and for the problem (A) is $\tilde{Z}^{IVN}$ = {((77, 86, 100, 114);[0.4,0.9], [0.1,0.3],[0.1,0.2]), ((112, 126, 134, 143);[0.6,0.9], [0.1,0.2],[0.2,0.2])}.

6. Result and Discussion

The numerical examples are used to investigate the efficacy of the proposed interactive left-width method to obtain the optimal compromise solution. Table 11 and Table 12 displays the comparison between the optimal compromise solution for the problem (B) with different existing solution methods. Table 11 demonstrates that optimal compromise solution for example 1, which is obtained by our proposed method is same to Global criteria method (GCM) [30] and obtain minimum value to the Fuzzy programming approach [31], Weighted sum method [32]. Table 12 demonstrates that optimal compromise solution for example 2, which is obtained by our proposed method is same to Global criteria method and obtain minimum value to the Fuzzy programming approach, Weighted sum method. To show the effectiveness, the same is plotted in the Figure 2 and Figure 3. The optimal compromise solution for our proposed approach is minimum by taking average to the interval. Overall, the proposed strategy is better suited to problems involving multi-criteria in structures.

Table 11 Optimal compromise solution for different approaches-Example 1

Approaches	Allocations	Optimal compromise solution
Fuzzy programming approach[31]	$x_{11}=x_{23}=x_{32}=1$	Z=([7,13],[15,22])
Weighted sum method[32]	$x_{11}=x_{23}=x_{32}=1$	Z=([7,13],[15,22])
Global criteria method[30]	$x_{11}=x_{22}=x_{33}=1$	Z=([8,16],[11,17])
Proposed interactive left-width method	$x_{11}=x_{22}=x_{33}=1$	Z=([8,16],[11,17])

Table 12 Optimal compromise solution for different approaches-Example 2

Approaches	Allocations	Optimal compromise solution
Fuzzy programming approach	$x_{14}=x_{23}=x_{32}=x_{41}=1$	Z=[(40,53);(52,62)]
Weighted sum method	$x_{14}=x_{23}=x_{32}=x_{41}=1$	Z=[(40,53);(52,62)]
Global criteria method	$x_{11}=x_{23}=x_{34}=x_{42}=1$	Z=([34,43],[53,58])
Proposed interactive left-width method	$x_{11}=x_{23}=x_{34}=x_{42}=1$	Z=([34,43],[53,58])

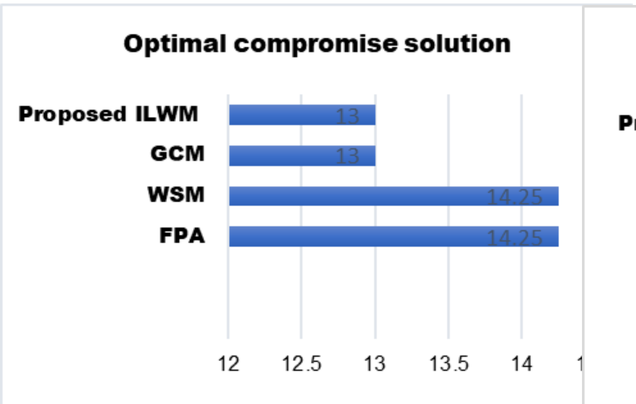

Figure 2: Comparison for Example 1

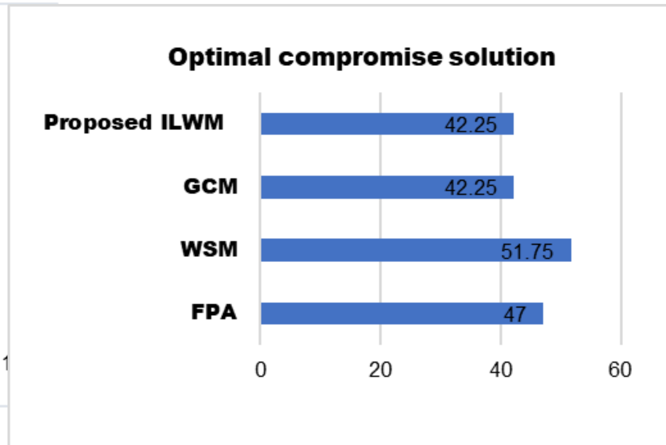

Figure 3: Comparison for Example 2

7. Sensitivity analysis

In this section, sensitivity analysis (SA) is performed for the optimality in terms of cost coefficients for the problem (B). First, we perform the SA of first objective problem having the interval cost $\left[c_{ij}^L, c_{ij}^U\right]$ and then for the second objective having the interval time $\left[t_{ij}^L, t_{ij}^U\right]$. We split interval cost $\left[c_{ij}^L, c_{ij}^U\right]$ and time $\left[t_{ij}^L, t_{ij}^U\right]$ as lower bound IAP c_{ij}^L , t_{ij}^L and upper bound IAP c_{ij}^U , t_{ij}^U . Analyse the sensitivity of (i,j)th cost of upper and the lower bound of interval for the problem (B). Using GWSM the optimal compromise solution obtained for problem (B) is $x_{11}=x_{22}=x_{33}=1$. Therefore, the basic cells of the given problem (B) are (1,1), (2,2) and (3,3). Now, analyse the sensitivity range of (i,j)th cost for the c_{ij}^L to the problem (B). Replace the (i,j)th cost value c_{ij}^L by $c_{ij}^L + \boldsymbol{\lambda}$ in which the parameter $\boldsymbol{\lambda}$ may vary. We find the modi indices ui and vj to calculate $(c_{ij}^L + \boldsymbol{\lambda}) - (ui + vj) \geq 0$ for all i and j. Then, we compute the minimum and maximum range of λ (i.e) [λ^*, λ^{**}], so that the optimal basis to the problem c_{ij}^L is not changed. Hence, the sensitivity ranges is [$c_{ij}^L + \lambda^*$, $c_{ij}^L + \lambda^{**}$]. Similarly, we can calculate for c_{ij}^U , t_{ij}^L and t_{ij}^U .

Now, we preform the SA for each cell in the c_{ij}^L which is a basic/ non-basic variable cell.

Case (Ia): Now, we consider the SA of the c_{ij}^L in the basic cell (1, 1) and compute the ranges of non-basic variables, $(c_{ij}^L + \boldsymbol{\lambda}) - (ui + vj) \geq 0$ for all i and j.

Table 13

u_i / v_j	$v_1 = 1 + \lambda$	$v_2 = 5$	$v_3 = 4$
$u_1 = 0$	$1 + \lambda$	5	4
$u_2 = 0$	7	2	3
$u_3 = 0$	7	3	5

Compute the ranges of non-basic variables, $7 - (0 + 1 + \lambda) \geq 0$. Then, $\boldsymbol{\lambda}$ varies from -∞ to 6. Therefore, sensitivity range of c_{ij}^L varies from -∞ to 7..

Case (Ib): We consider the SA of the c_{11}^U in the basic cell (1, 1).

Table 14

u_i / v_j	$v_1 = 3$	$v_2 = 6$	$v_3 = 5$
$u_1 = 0$	$3 + \lambda$	9	8
$u_2 = 0$	10	6	5

u_3=0	11	5	7

Compute the ranges of non-basic variables (i.e) 10 − (0 +3+λ) ≥ 0 and 11 − (0 +3+λ) ≥ 0. Then, choose the minimum range of λ that varies from -∞ to 7. Therefore, c_{11}^U varies from -∞ to 10. Thus, the cell (1,1) interval cost, $\left[c_{11}^L, c_{11}^U\right]$ varies from (-∞, -∞) to [7, 10].

Similarly, we can do for the second objective function. Then, the lower **δ** varies from -∞ to 1 and t_{11}^L varies from -∞ to 4. Then, upper **δ** varies from -∞ to 1 and t_{11}^U varies from -∞ to 6. Therefore, $\left[t_{ij}^L, t_{ij}^U\right]$ varies from [-∞, -∞] to [4, 6].

Next, we consider the SA of the c_{ij}^L in the cell (1, 2) which is a non-basic cell.

Case (IIa): We consider the SA of the lower bound TP in the cell (1, 2).

Table 15

u_i / v_j	**v_1 =1**	**v_2 =2**	**v_3 =5**
u_1 =0	1	5+λ	4
u_2=0	7	2	3
u_3=0	7	3	5

Then, $\boldsymbol{\lambda}_{12}$ varies from -3 to ∞. Thus, c_{12} varies from 2 to ∞.

Case (IIb): We consider the SA of the upper bound TP in the cell (1, 2).

Table 16

u_i / v_j	**v_1 =3**	**v_2 =6**	**v_3 =5**
u_1 =0	3	9+λ	8
u_2=0	10	6	5
u_3=0	11	5	7

Then, **λ** varies from -∞ to -3. Thus, c_{12} varies from -∞ to 6. Similarly, we can do for the second objective function. Then, lower **δ** varies from 5 to ∞ and t_{12} varies from 10 to ∞. Then, upper **δ** varies from 6 to ∞ and t_{12} varies from 10 to ∞. Therefore, $\left[t_{ij}^L, t_{ij}^U\right]$ varies from [7,10] to [∞, ∞]. Similarly, we can find the sensitivity ranges of costs in the problem (B) which is shown in Table 17 and Table 18.

Table 17 SA for First objective problem(B)

Limit for c_{ij}^{L}	**Limit for** c_{ij}^{U}	**Limit for** $\left[c_{ij}^{L}, c_{ij}^{U}\right]$
$-\infty \le c_{11} \le 7$	$-\infty \le c_{11} \le 10$	$[-\infty,-\infty] \le c_{11} \le [7,10]$
$2 \le c_{12} \le \infty$	$6 \le c_{12} \le \infty$	$[2,6] \le c_{12} \le [\infty,\infty]$
$5 \le c_{13} \le \infty$	$7 \le c_{13} \le \infty$	$[5,7] \le c_{13} \le [\infty,\infty]$
$1 \le c_{21} \le \infty$	$3 \le c_{21} \le \infty$	$[1,3] \le c_{21} \le [\infty,\infty]$
$-\infty \le c_{22} \le 3$	$-\infty \le c_{22} \le 3$	$[-\infty,-\infty] \le c_{22} \le [3,3]$
$5 \le c_{23} \le \infty$	$7 \le c_{23} \le \infty$	$[5,7] \le c_{23} \le [\infty,\infty]$
$1 \le c_{31} \le \infty$	$3 \le c_{31} \le \infty$	$[1,3] \le c_{31} \le [\infty,\infty]$
$2 \le c_{32} \le \infty$	$6 \le c_{32} \le \infty$	$[2,6] \le c_{32} \le [\infty,\infty]$
$-\infty \le c_{33} \le 3$	$-\infty \le c_{33} \le 5$	$[-\infty,-\infty] \le c_{11} \le [3,5]$

Table 18 SA for Second objective problem(B)

Limit for t_{ij}^{L}	**Limit for** t_{ij}^{U}	**Limit for** $\left[t_{ij}^{L}, t_{ij}^{U}\right]$
$-\infty \le t_{11} \le 4$	$-\infty \le t_{11} \le 6$	$[-\infty,-\infty] \le t_{11} \le [4,6]$
$7 \le t_{12} \le \infty$	$10 \le t_{12} \le \infty$	$[7,10] \le t_{12} \le [\infty,\infty]$
$1 \le t_{13} \le \infty$	$2 \le t_{13} \le \infty$	$[1,2] \le t_{13} \le [\infty,\infty]$
$3 \le t_{21} \le \infty$	$5 \le t_{21} \le \infty$	$[3,5] \le t_{21} \le [\infty,\infty]$
$-\infty \le t_{22} \le 2$	$-\infty \le t_{22} \le 4$	$[-\infty,-\infty] \le t_{22} \le [2,4]$
$1 \le t_{23} \le \infty$	$2 \le t_{23} \le \infty$	$[1,2] \le t_{23} \le [\infty,\infty]$
$3 \le t_{31} \le \infty$	$5 \le t_{31} \le \infty$	$[3,5] \le t_{31} \le [\infty,\infty]$
$7 \le t_{32} \le \infty$	$10 \le t_{32} \le \infty$	$[7,10] \le t_{32} \le [\infty,\infty]$
$-\infty \le t_{33} \le 1$	$-\infty \le t_{33} \le 5$	$[-\infty,-\infty] \le t_{33} \le [1,5]$

Table 17 and Table 18, show that the sensitivity of the interval cost parameter is used to examine how uncertainties in a parameter affect the overall uncertainty of the problem (B). This helps the DM to change the variables within models, based on information specific to a certain scenario to understand the outcome of a real-life situation.

8. Concluding remarks and future research directions

This study proposed a novel solution methodology interactive left-width technique for the interval valued neutrosophic BOAP. In this methodology the problem is first reduced to BOIAP using score function and it is reduced to a deterministic bi-objective AP using the left-width technique on each objective function. Then, each objective function of left-width problem is separated along with the constraints and multi-objective AP is constructed. Global weighted sum method is adopted to convert the multi-objective AP into SOP and then reduced problem is solved using Lingo 18.0 software to obtain the optimal compromise solution. This article demonstrates the effectiveness of the proposed interactive left-width method in problem and obtains the following improved results for the same case study: (i) illustrate the reliability and transparency of our proposed method and (ii) less assignment costs and shorter total allocation time. Applying nonlinear membership functions requires a significant amount of computational effort, which is the primary limitation of the proposed method. In future research, evolutionary computation may be used to effectively address multi-objective interval nonlinear problems in uncertain parameters. Further future research endeavors could potentially employ the solution method to address other supply chain planning problems such as inventory management, vendor selection, production distribution planning, and procurement-production-distribution planning.

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References

1. Spivey, M.Z.; Powell, W.B. The dynamic assignment problem. *Transp. Sci.* **2004**, *38*, 399–419, doi:10.1287/trsc.1030.0073.
2. Lotfi, V. A labeling algorithm to solve the assignment problem. *Comput. Oper. Res.* **1989**, *16*, 397–408, doi:10.1016/0305-0548(89)90028-2.
3. Basirzadeh, H. Ones assignment method for solving assignment problems. *Appl. Math. Sci.* **2012**, *6*, 2345–2355.
4. Majumdar, S. Interval linear assignment problems. *Univers. J. Appl. Math.* **2013**, *1*, 14–16, doi:10.13189/ujam.2013.010103.
5. Salehi, K. An approach for solving multi-objective assignment problem with interval parameters. *Manag. Sci. Lett.* **2014**, *4*, 2155–2160, doi:10.5267/j.msl.2014.7.031.
6. Sobana, V.E.; Anuradha, D.; Kaspar, S. Solving two stage fully interval integer transportation problems. *Int. J. Sci. Technol. Res.* **2020**, *9*, 2405–2412.
7. Buvaneshwari, T.K.; Anuradha, D. Solving bi-objective interval assignment problem using genetic approach. *Adv. Math. Sci. J.* **2021**, *10*, 759–768, doi:10.37418/amsj.10.2.7.
8. Prabha, S.K.; Devi, S.; Deepa, S. An efficient algorithm to obtain the optimal solution for fuzzy transportation problems. *Int. J. Comput. Organ. Trends* **2014**, *4*, 32–39, doi:10.14445/22492593/ijcot-v4p309.
9. Thorani, Y.L.P.; Shankar, N.R. Application of fuzzy assignment problem. *Adv. Fuzzy Math.* **2017**, *12*, 911–939.
10. Baidya, A.; Bera, U.K. Solid transportation problem under fully fuzzy environment. *Int. J. Math. Oper. Res.* **2019**, *15*, 498–539, doi:10.1504/IJMOR.2019.102997.
11. Buvaneshwari, T.K.; Anuradha, D. Fuzzy transportation problem with additional constraint in different environments. *J. Appl. Math. Informatics Vol.* **2022**, *40*, 933–947.
12. Atanassov, K.T. Intuitionistic fuzzy sets. *Fuzzy Sets Syst.* **1986**, *20*, 87–96, doi:10.1016/S0165-0114(86)80034-3.
13. Ebrahimnejad, A.; Verdegay, J.L. A new approach for solving trapezoidal intuitionistic fuzzy transportation problem. *Fuzzy Optim. Decis. Mak.* **2018**, *17*, 447–474,

doi:https://doi.org/10.1007/s10700-017-9280-1.

14. Mahmoodirad, A.; Allahviranloo, T.; Niroomand, S. A new effective solution method for fully intuitionistic fuzzy transportation problem. *Soft Comput.* **2019**, *23*, 4521–4530, doi:10.1007/s00500-018-3115-z.
15. Mondal, A.; Roy, S.K.; Midya, S. Intuitionistic fuzzy sustainable multi-objective multi-item multi-choice step fixed-charge solid transportation problem. *J. Ambient Intell. Humaniz. Comput.* **2023**, *14*, 6975–6999, doi:10.1007/s12652-021-03554-6.
16. Bharati, S.K. Transportation problem with interval-valued intuitionistic fuzzy sets: impact of a new ranking. *Prog. Artif. Intell.* **2021**, *10*, 129–145, doi:10.1007/s13748-020-00228-w.
17. Ahmadini, A.A.H.; Ahmad, F. Solving intuitionistic fuzzy multiobjective linear programming problem under neutrosophic environment. *AIMS Math.* **2021**, *6*, 4556–4580, doi:10.3934/math.2021269.
18. Farooq, S.; Hamza, A.; Smarandache, F. Linear and non-linear decagonal neutrosophic numbers: alpha cuts, representation, and solution of large MCDM problems. *Int. J. Neutrosophic Sci.* **2021**, *14*, 24–41, doi:10.54216/ijns.140102.
19. Das, P.; Roy, T.K. Multi-objective non-linear programming problem based on neutrosophic optimization technique and its application in riser design problem. *Neutrosophic Sets Syst.* **2015**, *9*, 88–95.
20. Rizk-Allah, R.M.; Hassanien, A.E.; Elhoseny, M. A multi-objective transportation model under neutrosophic environment. *Comput. Electr. Eng.* **2018**, *69*, 705–719, doi:10.1016/j.compeleceng.2018.02.024.
21. Broumi, S.; Florentin Smarandache. New operations on interval neutrosophic sets. *J. new theory* **2015**, *1*, 24–37.
22. Khalifa, H.A.E.W.; Kumar, P. A novel method for neutrosophic assignment problem by using interval-valued trapezoidal neutrosophic number. *Neutrosophic Sets Syst.* **2020**, *36*, 24–36, doi:10.5281/zenodo.4065363.
23. Saini, R.K.; Sangal, A.; Ahirwar, A. A novel approach by using interval-valued trapezoidal neutrosophic numbers in transportation problem. *Neutrosophic Sets Syst.* **2022**, *51*, 234–253, doi:10.5281/zenodo.7135283.
24. Khalil, S.; Kousar, S.; Freen, G.; Imran, M. Multi-objective interval-valued neutrosophic optimization with application. *Int. J. Fuzzy Syst.* **2022**, *24*, 1343–1355, doi:10.1007/s40815-021-01192-w.
25. Pandian, P.; Natarajan, G. A new method for finding an optimal solution of fully interval integer transportation problems. *Appl. Math. Sci.* **2010**, *4*, 1819–1830.
26. Ishibuchi, H.; Tanaka, H. Multiobjective programming in optimization of the interval objective function. *Eur. J. Oper. Res.* **1990**, *48*, 219–225, doi:10.1016/0377-2217(90)90375-L.
27. Smarandache, F. *A Unifying Field in Logics: Neutrosophic Logic, Neutrosophy, Neutrosophic Set, Neutrosophic Probability and Statistics*; 1998; American Research Press Rehoboth.
28. Broumi, S.; Smarandache, F.; Talea, M.; Bakali, A. Operations on interval valued neutrosophic graphs. *New Trends Neutrosophic Theory Appl.* **2016**, *12*, 5–29.
29. Buvaneshwari, T.K.; Anuradha, D. Solving bi-objective bi-item solid transportation problem with fuzzy stochastic constraints involving normal distribution. *AIMS Math.* **2023**, *8*, 21700–21731, doi:10.3934/math.20231107.

30. Zelany, M. A concept of compromise solutions and the method of the displaced ideal. *Comput. Oper. Res.* **1974**, *1*, 479–496, doi:10.1016/0305-0548(74)90064-1.
31. Zimmermann, H.J. Fuzzy programming and linear programming with several objective functions. *Fuzzy Sets Syst.* **1978**, *1*, 45–55, doi:10.1016/0165-0114(78)90031-3.
32. Stanimirovi´c, I.P.; Zlatanovi´c, M.L.; Petkovi´c, M.D. On the linear weighted sum method for multi-objective optimization. *Sci. J. Facta Univ.* **2011**, *26*, 49–63.

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