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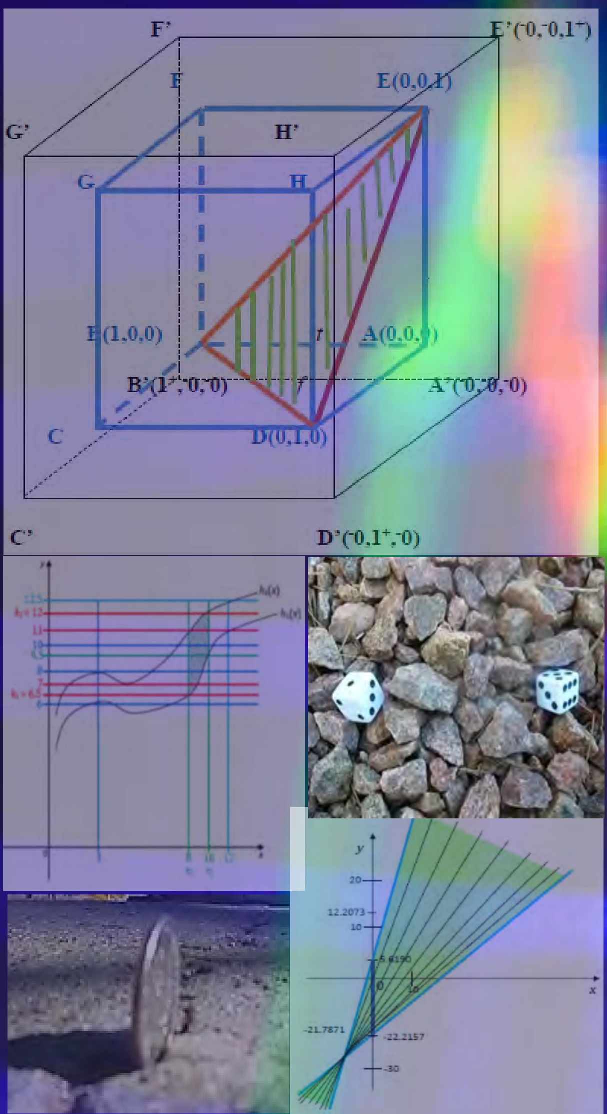

<A> <neutA> <antiA>

Florentin Smarandache . Mohamed Abdel-Basset . Said Broumi
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*"*Neutrosophic Sets and Systems*"* has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Contents

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Neutrosophy means: Common Parts to Uncommon Things and Uncommon Parts to Common Things

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Abstract: Let <*A*> be an item, concept, idea, proposition, school of thought, current, theory, etc. and <anti*A*> be the opposite of <*A*>. Analogously for <*B*> and its opposite <anti*B*>.

Neutrosophy means to find:

(i) common parts to uncommon things (that is, <*A*> and <anti*A*> have something in common, or their intersection <*A*>∩<anti*A*> is not empty), and vice versa:

(ii) (ii) uncommon parts to common things (the two equal items <*A*>=<*B*> have also uncommon parts, either <*A*>∩<anti*B*> is not empty, or <anti*A*>∩<*B*> is not empty).

Both, the *Common Parts to Uncommon Things*, and the *Uncommon Parts to Common Things*, end up being parts of indeterminacy / neutrality situated between the opposites: denoted by <neut*A*>, which means neither <*A*> nor <anti*A*>, but in between them; and respectively by <neut*B*>, which similarly means neither <*B*> nor <anti*B*>, but in between them.

Keywords: Neutrosophy, Paradoxism, Dialectics, Yin Yang, Soft Sciences, Capitalism, Socialism, Psychoanalysis, Analytical Psychology, Democracy, Representative Democracy, Alchemy, Science, Dialectics, Structuralism, Post-Structuralism, Social Systems Theory, Paradoxical Intention, Pro-choice, Pro-life, Cognitive-Behavioral Therapy, Psychodynamic Therapy.

1. Introduction

Neutrosophy, a philosophical framework that I developed more than two decades ago [Smarandache 1998], explores the relationships and interactions between opposites and their neutralities/indeterminacies, seeking to find commonalities between them and identifying uncommon elements within similar entities, emphasizing the complexity and interconnectedness of concepts. Neutrosophy transcends traditional binary thinking by examining the interplay between opposites and the neutralities/indeterminacies between them.

Neutrosophy is an extension of the movement called *Paradoxism* [Smarandache 1980], in literature/arts/science/philosophy, *Dialectics* [Hegel], [Marx], and *Yin Yang Ancient Chinese Philosophy* [1] – because the last three schools took into consideration the dynamics between the opposites only, while omitting their neutralities/indeterminacies that play an important role in the balance between opposites.

[1] Britannica, The Editors of Encyclopaedia. "yinyang". Encyclopedia Britannica, 12 Feb. 2024, https://www.britannica.com/topic/yinyang. Accessed 30 May 2024.

By identifying and studying common parts in uncommon things and uncommon parts in common things, neutrosophy reveals the complexity and interconnectedness inherent in various ideas and phenomena.

This article shortly presents the two core principles of neutrosophic research in soft sciences, then exemplifies these neutrosophic principles, and suggests applications, illustrating how neutrosophy provides a nuanced understanding of the world.

2. Two Neutrosophic Core Principles

2.1. Searching for Common Parts in Uncommon Things

This principle posits the study of opposing concepts that share (some) common elements. For example:

- **Political Ideologies**: *Capitalism*[2] [Hickel] and *Socialism*[3] [Cole] both pretend their aim to be the improvement of societal welfare and economic prosperity, though their proposed methods differ.
- **Psychological Theories**: Freud's *Psychoanalysis*[4] [Freud] and Jung's *Analytical Psychology*[5] [Jung] both focus on the unconscious mind but having distinct theoretical foundations and methodologies.

2.2. Searching for Uncommon Parts in Common Things

This principle propounds the research of similar or equivalent concepts containing elements that are distinct or oppositional.

- **Forms of Governance**: *Democracy*[6] and *Representative Democracy*[7] both emphasize the role of the people, yet representative democracy involves elected officials, while direct democracy involves direct citizen participation. [Landemore]
- **Forms of Thinking:** *Empiricism* [Gupta] *vs. Alchemy* [Ferguson]: Both seek understanding of the world, but *alchemy*[8] lacks the methodology and empirical support of *science*[9].

3. Investigation in Soft Sciences

3.1. Hegelian Dialectics

[2] Britannica, The Editors of Encyclopaedia. "What is capitalism?". Encyclopedia Britannica, 24 Feb. 2023, https://www.britannica.com/question/What-is-capitalism. Accessed 27 May 2024.

[3] Britannica, The Editors of Encyclopaedia. "What is socialism?". Encyclopedia Britannica, 11 Apr. 2022, https://www.britannica.com/question/What-is-socialism. Accessed 27 May 2024.

[4] Jay, Martin Evan. "Sigmund Freud". Encyclopedia Britannica, 20 May. 2024, https://www.britannica.com/biography/Sigmund-Freud. Accessed 28 May 2024.

[5] Fordham, Frieda. "Carl Jung". Encyclopedia Britannica, 18 Mar. 2024, https://www.britannica.com/biography/Carl-Jung. Accessed 28 May 2024.

[6] Shapiro, Ian , Dahl, Robert A. and Froomkin, David. "democracy". Encyclopedia Britannica, 6 May. 2024, https://www.britannica.com/topic/democracy. Accessed 29 May 2024.

[7] Raikar, Sanat Pai. "representative democracy". Encyclopedia Britannica, 29 Feb. 2024, https://www.britannica.com/topic/representative-democracy. Accessed 29 May 2024.

[8] Britannica, The Editors of Encyclopaedia. "alchemy (pseudoscience)". Encyclopedia Britannica, 29 Mar. 2024, https://www.britannica.com/topic/alchemy. Accessed 29 May 2024.

[9] Britannica, The Editors of Encyclopaedia. "science". Encyclopedia Britannica, 25 May. 2024, https://www.britannica.com/science/science. Accessed 29 May 2024.

In Hegelian *Dialectics* [10], ideas and reality develop through the interaction of opposites. An initial idea (*thesis*) is countered by an opposing idea (*antithesis*), leading to a resolution (*synthesis*) that integrates elements of both. [Hegel]

3.2. Structuralism and Post-Structuralism

These theories explore the so-called construction and de-construction of meaning.

- *Structuralism*: Lévi-Strauss examines how elements within a culture or system are interrelated. [Lévi-Strauss]
- *Post-Structuralism*: Derrida focuses on the instability of these structures, emphasizing the gaps or the contradiction in meaning and interpretation. [Derrida]

3.3. Social Systems Theory

Systems Theory in Sociology looks at how different parts of a system interact and form a whole. For example, Luhmann views society as a complex set of communications and interactions, where even opposing elements are part of the system's overall functioning. [Luhmann]

3.4. Paradoxical Intention

In Psychology, the *Paradoxical Thinking* [Frankl] involves recognizing and integrating contradictory thoughts or behaviors. The method encourages patients to actively engage in the very behaviors they fear, in order to reduce the anxiety associated with those.

4. Applications in Contemporary Contexts

4.1. Migrations: Acculturation

In today's Western multicultural societies [Vani, Mangan], shared values of locals coexist with the unique and different cultural practices of migrants, leading to various individual outcomes. [Berry] *Acculturation*,[11] which results from intercultural contact, offers four different outcomes in varying degrees: integration, assimilation, separation, and marginalization.

Let us apply a neutrosophic framework. *Integration* (two-way exchange process) involves engaging with and identifying with both cultures. *Assimilation* (one-way exchange process) entails adopting the host country's culture while rejecting one's heritage culture. *Separation* (zero-way exchange process) involves identifying with one's heritage culture and interacting solely with one's own group. *Marginalization* (minus-way exchange process) represents a (quasi)total lack of identification with both cultures.

Considering the growing significance of global migrations, and the pivotal role workplace integration plays in adaptation, there's a pressing need for focus on the socialization process[12] of migrants within organizations, not only in culture.[13]

One could expand upon Benson's approach [Benson], which is rooted in a dynamic understanding that an organization is shaped by historical processes of social construction, which is

[10] Knox, T. Malcolm. "Georg Wilhelm Friedrich Hegel". Encyclopedia Britannica, 23 Apr. 2024, https://www.britannica.com/biography/Georg-Wilhelm-Friedrich-Hegel. Accessed 29 May 2024.
[11] Berry J. W. (1990). "Psychology of acculturation." In J. Berman (Ed.), "Cross-cultural perspectives: Nebraska Symposium on Motivation", Vol. 37, 201–234. Lincoln: University of Nebraska Press.
[12] McGahan A. M. (2020). "Immigration and impassioned management scholarship." *Journal of Management Inquiry* 29(1), 111-114. https://doi.org/10.1177/1056492619877617
[13] Omanović, V., & Langley, A. (2023). "Assimilation, Integration or Inclusion? A Dialectical Perspective on the Organizational Socialization of Migrants." *Journal of Management Inquiry* 32(1), 76-97. https://doi.org/10.1177/10564926211063777

in constant move. Benson formalized[14] four key principles of analysis (social construction, context, contradiction, and praxis), which - with neutrosophy as tool - can each be further subdivided into various facets.

In any cross-cultural process, a meta-analysis of adjustment processes is necessary,[15] and an insightful instrument is provided by a neutrosophic standpoint that could uncover various factors, in the given case - factors that could alleviate the inequalities migrants encounter in socialization.

4.2. Ethics: Pro-choice vs. Pro-life

The Pro-choice and the Pro-life options are two contradictory viewpoints, mostly reduced to a total opposition. A neutrosophic view over their shared goals might suggest a possible foundation for constructive dialogue and collaborative efforts.

4.2.1. Pro-choice movement

The *Pro-choice* movement can be traced back to the early 20th century, with activists like Margaret Sanger advocating for women's access to contraception, and founding the American Birth Control League[16] in 1921. Around 1970, significant legal challenges and efforts emerged to reform restrictive abortion laws in the United States. Organizations such as the National Association for the Repeal of Abortion Laws [NARAL], founded in 1969, were instrumental in advocating for women's right to choose. The pivotal moment for the Pro-choice movement came with the Supreme Court case Roe v. Wade[17], which extended the constitutional right to privacy to a woman's decision to have an abortion, decision that galvanized Pro-choice and Pro-life activists. Today, the Pro-choice movement often intersects with other social movements, including LGBTQ+ rights.

4.2.2. Pro-life movement

The *Pro-life* movement originated in Catholic opposition to Pro-choice movement in the 1950s [Munson]. In response to the growing feminist movement and efforts to liberalize abortion laws, Pro-life groups formed structured organizations, e.g. National Right to Life Committee in 1968 [NRLC]. In the years following Roe v. Wade trial, the Pro-life movement expanded its strategies to include lobbying for restrictive abortion laws, dismantled for a moment by a new Supreme Court decision in the case Planned Parenthood v. Casey[18]. The Pro-life movement gained lately significant political influence. While primarily focused on abortion, the Pro-life movement often intersects with other conservative causes, including opposition to euthanasia.

4.2.3. In search of a common ground

Finding common ground between the *Pro-choice* and *Pro-life* positions[19] can be challenging due to their fundamentally opposing views on abortion. However, both sides can agree on the following statements: [Spitzer et al.]

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[14] Benson J. K. (1983). “A dialectical method for the study of organizations.” In Morgan G. (Ed.), “Beyond method: Strategies for social research”, Sage Publications, 331-346.

[15] Nguyen, A.-M. T.D., Benet-Martínez, V. (2013). “Biculturalism and adjustment: A meta-analysis.” *Journal of Cross-Cultural Psychology* 44(1), 122–159. DOI: 10.1177/0022022111435097.

[16] Moses, Theodora R. "American Birth Control League". Encyclopedia Britannica, 20 Mar. 2023, https://www.britannica.com/topic/American-Birth-Control-League. Accessed 29 May 2024.

[17] Roe v. Wade, 410 U.S. 113 (1973). https://supreme.justia.com/cases/federal/us/410/113/.

[18] Planned Parenthood of Southeastern Pa. v. Casey, 505 U.S. 833 (1992), https://supreme.justia.com/cases/federal/us/505/833/. Also, https://www.oyez.org/cases/1991/91-744.

[19] Britannica, The Editors of Encyclopaedia. "Pro and Con: Abortion". Encyclopedia Britannica, 27 Dec. 2021, https://www.britannica.com/story/pro-and-con-abortion. Accessed 29 May 2024.

- Both can support measures to reduce unintended pregnancies, such as comprehensive sex education and increased access to contraception.
- Both can encourage adoption as a viable option for women who do not wish to or cannot raise a child,
- Bot can approve making the adoption process easier, affordable, less stigmatized.
- Both can agree on the importance of supporting women and families, meaning: better access to healthcare, longer parental leave, more affordable childcare.
- Both can agree on the importance of protecting women's safety.
- Both can work towards improving educational and economic opportunities for women, to empower choices about their reproductive lives.

4.3. Psychology: Cognitive-Behavioral Therapy vs. Psychodynamic Therapy

Cognitive-Behavioral Therapy [20] (CBT) focuses on present thoughts and behaviors, while *Psychodynamic Therapy* [21] (PDT) explores unconscious processes and past experiences — both approaches aiming to alleviate psychological distress.

Although both therapies appear effective in addressing mental health concerns, they diverge in their theoretical orientations, goals, techniques, and duration. However, they can also complement each other when used together.

Let us engage in a quick comparison:

4.3.1. Basics

- Rooted in the cognitive model, CBT focuses on the relationship between thoughts, feelings, and behaviors. It emphasizes identifying and challenging negative or maladaptive thought patterns and replacing them with more adaptive ones. *CBT is present-focused and goal-oriented.*
- Based on psychoanalytic principles, psychodynamic therapy explores how unconscious conflicts and early life experiences influence current thoughts, feelings, and behaviors. It aims to bring unconscious material into conscious awareness to promote insight and healing. *Psychodynamic therapy tends to be exploratory and insight-oriented.*

4.3.2. Therapeutic Techniques

- Techniques in CBT include cognitive restructuring[22], behavioral experiments, exposure therapy [23], and skill-building exercises such as relaxation and problem-solving techniques.
- PDT techniques include free association[24], dream analysis, interpretation of transference and countertransference, and exploring childhood experiences and relationships with significant others.

4.3.3. Therapeutic Relationship

[20] Moulds, M., Grisham, J., & Graham, B. (2022). "Cognitive Behavioral Therapy for Anxiety." Oxford Research Encyclopedia of Psychology. Retrieved 29 May. 2024, from https://oxfordre.com/psychology/view/10.1093/acrefore/9780190236557.001.0001/acrefore-9780190236557-e-331.

[21] Crits-Christoph, P. (1992). "The efficacy of brief dynamic psychotherapy: A meta-analysis." *American Journal of Psychiatry* 149(2):151–158.

[22] Identifying and challenging negative thoughts.

[23] Gradual exposure to feared stimuli.

[24] Encouraging clients to speak freely without censorship.

- The therapist and patient work together to set specific goals, and actively engage in homework assignments between sessions in CBT.
- In PDT, the therapist serves as a neutral interpreter and guide, facilitating exploration of deeper emotions and conflicts.

4.3.4. Duration and Focus

- CBT is typically *short-term* (from a few weeks to several months) and focused on addressing specific symptoms or problems.
- PDT is usually *longer-term* (lasting several months to years) and focuses on exploring underlying emotional issues and patterns.

5. Conclusion

Neutrosophy offers a framework for understanding the wide interactions and the inébranlable connections of some concepts traditionally seen as opposites and their neutralities/indeterminacies. By applying this framework to the soft sciences, one gain deeper insights into the nuanced relationships between ideas, theories, and practices. This approach can lead to an integrative understanding of human knowledge and experience.

Neutrosophy transcends the limits and finds, in any field of knowledge, common ideas to uncommon schools of thought, and reciprocally: uncommon ideas to common schools of thoughts.

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On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2 \leq n \leq 3$

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Abstract:

The objective of this paper is to study the two-fold fuzzy algebra based on n-refined neutrosophic rings for some different special values of n, where we study some of the special elements in the case of two-fold 2-refined neutrosophic ring and 3-refined neutrosophic rings such as units, idempotents, and nilpotent elements. Also, we present the concept of two-fold ring homomorphism with its elementary properties.

Keywords: two-fold algebra, 2-refined neutrosophic ring, 3-refined neutrosophic ring, nilpotent

Introduction

The concept of fuzzy algebraic structure is considered a direct application of fuzzy sets and fuzzy mappings [1-2, 4, 6-8], where a fuzzy mapping with truth and falsity values is used to build many algebraic structures.

Also, the concept of neutrosophic set was used by many different authors to generalize classical algebraic structures by using logical conditions instead of algebraic elements [13], where we can see neutrosophic rings, neutrosophic matrices, and neutrosophic mappings [5, 9-12]. The concept of n-refined neutrosophic rings was defined in [18], and then it was studied by many authors in [19], where ideals, Diophantine equations and other related structures were classified and provided [20-21].

Recently, Smarandache in [14] has defined two-fold neutrosophic algebras as novel algebraic structures, and this new concept has been used in [16] to define two-fold fuzzy algebra by combining the standard fuzzy number theoretical system defined in [15], with the concept of two-fold algebraic structure, and many interesting theorems and examples were illustrated about this topic.

On the other hand, Hatip et.al [17], have combined real vector spaces, complex vector spaces, and algebraic modules with a fuzzy well-defined mapping to define and study two-fold fuzzy vector spaces and two-fold fuzzy modules, where they studied many elementary properties of these new structures.

Main Discussion

Definition:

Let $f: \mathbb{R} \to [0,1]$ with: $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$, then f is called a fuzzy mapping.

We use this definition of fuzzy mappings, that is because the property $\begin{matrix} f(0) = 0 \\ f(1) = 1 \end{matrix}$ is very useful in algebraic structures and operations.

Example:

To understand the concept of fuzzy mapping, we will illustrate two different fuzzy mappings defined on the real field R.

Define: $f, g, h: \mathbb{R} \to [0,1]$ such that:

$$f(x) = \begin{cases} \min(x^2, \frac{1}{x^2}) & if \quad x \neq 0, x \neq -1 \\ 0 & if \quad x = 0 \\ 0.9 & if \quad x = -1 \end{cases}, \quad g(x) = \begin{cases} |x^3| & if -1 < x \leq 1 \\ \frac{1}{|x^3|} & if \quad x > 1 \quad or \quad x < -1 \\ 0.1 & if \quad x = -1 \end{cases}$$

We can see that f and g lie in the closed interval [0,1], with f$(0) = g(0) = 0, f(1) = g(1) = 1$.

Definition:

Let $R_2(I) = \left\{a_0 + \sum_{i=1}^{2} a_i I_i \quad ; a_i \in \mathbb{R} \ . I_i I_j = I_{\min(i.j)} \right\}$

be 2-refined commutative neutrosophic ring with unity, let $f: \mathbb{R} \to [0,1]$ be any fuzzy mapping such that $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$, we define:

$f_2 : R_2(I) \to [0,1] \quad ; \ f_2 \ (a_0 + \sum_{i=1}^{2} a_i I_i \ ) = \max(f(a_i))$, and

$[R_2(I)]_{f_2} = \left\{(a_0 + \sum_{i=1}^{2} a_i I_i \ )_{f_n \ (a_0 + \sum_{i=1}^{2} a_i I_i \ )} \quad ; \ a_i \in \mathbb{R}\right\}$, is called the two-fold fuzzy 2-Refined neutrosophic ring.

Definition:

Operations on $[R_2(I)]_{f_2}$ are define as follows:

*: $[R_2(I)]_{f_2} \times [R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$

$\circ$:$[R_2(I)]_{f_2} \times [R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$

Such that: $\begin{cases} X_{f_2(X)} * Y_{f_2(Y)} = (X + Y)_{f_2(X+Y)} \\ X_{f_2(X)} \circ Y_{f_2(Y)} = (X \cdot Y)_{f_2(XY)} \end{cases}$

Definition:

Let P an ideal of $R_2(I)$, we define the corresponding two-fold fuzzy 2-refined neutrosophic ideal as follows:

$P_{f_2} = \left\{X_{f_2(X)} \quad ; X \in P\right\}$

Definition:

Let P_{f_2} be a two-fold fuzzy 2-refined neutrosophic ideal, we define the two-fold fuzzy 2-refined factor as:

$[R_2(I)]_{f_2} / P_{f_2} = XP_{f_2} \quad ; \quad X \in [R_2(I)]_{f_2}$.

Definition:

Let $h : R_2(I) \to R_2(I)$ be a ring homomorphism, we define:

H_n: $[R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$ such that:

$H_2\big(X_{f_2(X)}\big) = (h(X))_{f_2(h(X))}$.

The mapping (H_2) is called two-fold fuzzy 2-refined neutrosophic homomorphism.

The kernel $k_{er}(H_2)$is:

$k_{er}(H_n) = \{X \in [R_2(I)]_{f_2} \quad ; \quad H_2\big(X_{f_2(X)}\big) = 0_0\} = \ (k_{er}(h))_{f_2}$.

The direct image $I_m(H_2)$ is:

$I_m(H_2) = \ (I_m(h))_{f_2}$.

Definition:

Let $H_2 \ .G_2 \ : \ [R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$ be two homomorphisms, then: $H_2 \times G_2 \ :$ $[R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$ with:

$(H_2 \times G_2)\big(X_{f_2(X)}\big) = H_2(G_2\big(X_{f_2(X)}\big))$.

Theorem (1):

1] *, ○ are commutative.

2] *, ○ are associative.

3] (○) is distributive on (*).

4] *, ○ has identities.

5] (*) is invertible, i.e any element$X_{f_2(X)} \in [R_2(I)]_{f_2}$ has an iverse with respect to (*).

Theorem (2):

Let P_{f_2} be a two-fold ideal of$[R_2(I)]_{f_2}$, then:

$$\begin{cases} X_{f_2(X)} * Y_{f_2(Y)} \in P_{f_2} \\ r_{f_2(r)} \cdot X_{f_2(X)} \in P_{f_2} \end{cases} \quad ; \quad X.Y \in P \quad .r \in R_2(I)$$

Theorem (3):

$[R_2(I)]_{f_2} / P_{f_2}$ is a commutative ring with unity.

Theorem (4):

Let H_2: $[R_2(I)]_{f_2} \rightarrow [R_2(I)]_{f_2}$ be a homomorphism, then:

1] $H_2\left(X_{f_n(X)} * Y_{f_n(Y)}\right) = H_2\left(X_{f_2(X)}\right) * H_2\left(Y_{f_2(Y)}\right)$

2] $H_2\left(X_{f_2(X)} \circ Y_{f_2(Y)}\right) = H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right)$

3] $k_{er}(H_2)$ is an ideal of $[R_2(I)]_{f_2}$.

4] $I_m(H_2)$ is a subring of $[R_2(I)]_{f_2}$.

5] $[R_2(I)]_{f_2} / k_{er}(H_2) \cong I_m(H_2)$.

6] If P_{f_2} is an ideal of $[R_2(I)]_{f_2}$, then $H_2\left(P_{f_2}\right)$ is an ideal.

7] $H_2(0_0) = 0_0 \quad .H_2(1_1) = 1_1$

Theorem (5):

Let $H_2 \ .G_2 : [R_2(I)]_{f_2} \rightarrow [R_2(I)]_{f_2}$ be two homomorphisms, then:

1] $H_2\left(-X_{f_2(X)}\right) = -H_2\left(X_{f_2(X)}\right)$.

2] $H_2\left(X^{-1}_{f_2(X^{-1})}\right) = [H_2\left(X_{f_2(X)}\right)]^{-1}$, if X is invertible.

3] $H_2 \times G_2$ is a homomorphism.

Definition:

Let $X_{f_2(X)} \in [R_2(I)]_{f_2}$, then:

1] $X_{f_2(X)}$ is idempotent if $X_{f_2(X)} \circ X_{f_2(X)} = X_{f_2(X)}$.

2] $X_{f_2(X)}$ is a nilpotent if there exists $m \in \mathbb{N}$ such that: $X_{f_2(X)} \circ X_{f_2(X)} \circ \ldots \circ X_{f_2(X)} \ (m - times) = 0_0$

3] $X_{f_2(X)}$ is a zero divisor if there exists $Y_{f_2(Y)}$ such that: $X_{f_2(X)} \circ Y_{f_2(Y)} = 0_0$

Theorem (6):

Let $X_{f_2(X)} \in [R_2(I)]_{f_2}$, then we have:

1] $X_{f_2(X)}$ is idempotent if and only if X is idempotent in $R_2(I)$.

2] $X_{f_2(X)}$ is nilpotent if and only if X is nilpotent in $R_2(I)$.

3] $X_{f_2(X)}$ is a zero divisor if and only if X is a zero diviso in $R_2(I)$.

Theorem (7):

Let H_2: $[R_2(I)]_{f_2} \rightarrow [R_2(I)]_{f_2}$, then:

1] If $X_{f_2(X)} \in [R_2(I)]_{f_2}$ is idempotent, then $H_2\left(X_{f_2(X)}\right)$ is idempotent.

2] If $X_{f_2(X)}$is nilpotent, then $H_2\left(X_{f_2(X)}\right)$ is nilpotent.

3] If $X_{f_2(X)}$is a zero divisor, then $H_2\left(X_{f_2(X)}\right)$ is a zero divisor.

4] If $X_{f_2(X)}$is a unit, then $H_2\left(X_{f_2(X)}\right)$ is a unit.

Proof of theorem (1):

1] $X_{f_2(X)} * Y_{f_2(Y)} = (X+Y)_{f_2(X+y)} = (Y+X)_{f_2(Y+X)} = Y_{f_2(Y)} * X_{f_2(X)}$.

$X_{f_2(X)} \circ Y_{f_2(Y)} = (XY)_{f_2(Xy)} = (YX)_{f_2(YX)} = Y_{f_2(Y)} \circ X_{f_2(X)}$.

2] $X_{f_2(X)} * Y_{f_2(Y)} * Z_{f_{2n}(Z)}) = X_{f_2(X)} * (Y+Z)_{f_2(Y+Z)} = (X+Y+Z)_{f_2(X+Y+Z)} =$

$(X+Y)_{f_n(X+Y)} * Z_{f_2(Z)} = (X_{f_2(X)} * Y_{f_2(Y)}) * Z_{f_2(Z)}$.

$X_{f_2(X)} \circ Y_{f_2(Y)} \circ Z_{f_2(Z)}) = (XYZ)_{f_2(XYZ)} = (XY)_{f_2(XY)} \circ Z_{f_2(Z)} = (X_{f_2(X)} \circ$

$Y_{f_n(Y)}) \circ Z_{f_2(Z)}$.

3] $X_{f_2(X)} \circ (Y_{f_2(Y)} * Z_{f_2(Z)}) = (XY+XZ)_{f_2(XY+XZ)} = (XY)_{f_2(Xy)} * (XZ)_{f_2(XZ)} = (X_{f_2(X)} \circ$

$Y_{f_2(Y)}) * (X_{f_2(X)} \circ Z_{f_2(Z)})$

4] $X_{f_2(X)} * 0_0 = (X+0)_{f_2(X+0)} = X_{f_2(X)}$.

$X_{f_2(X)} \circ 1_1 = (X \cdot 1)_{f_2(X \cdot 1)} = X_{f_2(X)}$.

5] For $X_{f_n(X)}$, we have $(-X)_{f_2(-X)}$ such that:

$$X_{f_2(X)} * (-X)_{f_2(-X)} = (\mathrm{X} - X)_{f_2(X-X)} = 0_0.$$

Proof of theorem (2):

$X_{f_2(X)} * Y_{f_2(Y)} = (X+Y)_{f_2(X+y)} \in P_{f_2}$, that is because $X+Y \in P$.

$r_{f_2(r)} \circ X_{f_2(X)} = (rX)_{f_2(rX)} \in P_{f_2}$, that is because $rX \in P$.

Proof of theorem (3):

Define: $*' : \left([R_2(I)]_{f_2}/P_{f_2} \right) \times \left([R_2(I)]_{f_n} /P_{f_2} \right) \to [R_2(I)]_{f_2}/P_{f_2}$

$$\circ' : ([R_2(I)]_{f_2}/P_{f_2}) \times ([R_2(I)]_{f_n} /P_{f_2}) \to [R_2(I)]_{f_2}/P_{f_2}$$

Such that:

$(X_{f_2(X)}P_{f_2}) *' \left(Y_{f_2(Y)}P_{f_2}\right) = \left(X_{f_2(X)} * Y_{f_2(Y)}\right) P_{f_2}$

$(X_{f_2(X)}P_{f_2}) \circ' \left(Y_{f_2(Y)}P_{f_2}\right) = (X_{f_2(X)} \circ Y_{f_2(Y)})P_{f_2}$

We have:

$(X_{f_2(X)}P_{f_2}) *' \left(0_0 P_{f_n}\right) = X_{f_2 \ (X)} \ P_{f_2}$,

$(X_{f_2(X)}P_{f_2}) \circ' \left(1_1 P_{f_2}\right) = X_{f_2(X)} \ P_{f_2}$,

$(X_{f_2(X)}P_{f_2}) *' \left((-X)_{f_2(-X)}P_{f_2}\right) = 0_0 \ P_{f_2}$,

$(X_{f_2(X)}P_{f_2}) *' \left[\left(Y_{f_2(Y)}P_{f_2}\right) *' \left(Z_{f_2(Z)}P_{f_2}\right)\right] = \left((X_{f_2(X)}P_{f_2}\right) *' \left[(Y * Z) \ P_{f_2}\right] = [X * Y * Z] \ P_{f_n} =$

$[(X_{f_2(X)}P_{f_2}) *' \left(Y_{f_2(Y)}P_{f_2}\right)] *' \left(Z_{f_2(Z)}P_{f_2}\right)$,

$(X_{f_n(X)}P_{f_n}) \circ' \left[\left(Y_{f_2(Y)}P_{f_2}\right) \circ' \left(Z_{f_2(Z)}P_{f_2}\right)\right] = \left(X \ P_{f_2}\right) \circ' \left[(Y \circ Z) \ P_{f_2}\right] = [X \circ Y \circ Z] \ P_{f_n} =$

$[(X_{f_n(X)}P_{f_2}) \circ' \left(Y_{f_2(Y)}P_{f_2}\right)] \circ' \left(Z_{f_2(Z)}P_{f_2}\right)$.

$\left((X_{f_2(X)}P_{f_2}\right) \circ' \left[\left(Y_{f_2(Y)}P_{f_2}\right) *' \left(Z_{f_2(Z)}P_{f_2}\right)\right] = [X \circ (Y * Z)]\ \ P_{f_2} = [(X \circ Y) * (X \circ Z)]\ \ P_{f_2}$

$$= (X_{f_2(X)}P_{f_2}) \circ' \left(Y_{f_2(Y)}P_{f_2}\right)] *' [(X_{f_2(X)}P_{f_2}) \circ' \left(Z_{f_2(Z)}P_{f_2}\right)]$$

Thus, our proof is complete.

Proof of theorem (4):

1] $H_2(X * Y) = (h(X+Y))_{f_2(h(X+Y))} = (h(X)+h(Y))_{f_2((h(X)+h(Y)))} = H_2\left(X_{f_2(X)}\right) * H_2\left(Y_{f_2(Y)}\right)$.

2] $H_2(X \circ Y) = (h(XY))_{f_2(h(XY))} = (h(X)h(Y))_{f_2((h(X)h(Y)))} = H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right)$.

3] since $k_{er}(H_2) = [k_{er}(h)]_{f_2}$, and $k_{er}(h)$ is an ideal of $R_2(I)$, we get: $k_{er}(H_2)$ is an ideal of $[R_2(I)]_{f_2}$.

4] It can be proved by the same.

5] We have that:

$R_2(I)/k_{er}(h) \quad \cong \quad I_m(h)$, thus:

$[R_2(I)]_{f_2}/\ [k_{er}(h)]_{f_2} \quad \cong \quad [I_m(h)]_{f_2}$, therefor:

$[R_2(I)]_{f_2}/\ k_{er}(H_n) \quad \cong \quad I_m(H_n)$.

6] $H_n\left(P_{f_2}\right) = \{\ [h(P)]_{f_2}\}$, and $h(P)$ is an ideal of $R_2(I)$, thus $H_2\left(P_{f_2}\right)$ is an ideal of $[R_2(I)]_{f_2}$.

7] $\begin{cases} H_n(0_0) = (h(0))_{f_2(h(0))} = 0_0 \\ H_n(1_1) = (h(1))_{f_2(h(1))} = 1_1 \end{cases}$.

Proof of theorem (5):

1] $H_2\left(-X_{f_2(X)}\right) = (h(-X))_{f_2(h(-X))} = [-h(X)]_{f_2(-h(X))} = -H_2\left(X_{f_2(X)}\right)$.

2] $H_2\left(X^{-1}_{f_2(X^{-1})}\right) = (h(X^{-1}))_{f_2\left(h(X^{-1})\right)} = [(h(X))^{-1}]_{f_2[(h(X))^{-1}]} = [H_2\left(X_{f_2(X)}\right)]^{-1}$.

3]$(\ H_2 \ \times G_2)\left[X_{f_2(X)} * Y_{f_2(Y)}\right] = (\ H_2 \ \times G_2)[X+Y]_{f_2(X+Y)} = [(h \circ g)(X+Y)]_{f_2[(h\circ g)(X+Y)]}$

$= [(h \circ g)(X) + (h \circ g)(Y)]_{f_2[(h \circ g)(X)+(h \circ g)(Y)]} = (H_2 \times G_2)(X_{f_2(X)}) * (H_2 \times G_2)(Y_{f_2(Y)})$.

$(H_2 \times G_2)\left[X_{f_2(X)} \circ Y_{f_2(Y)}\right] = [(h \circ g)(XY)]_{f_2[(h \circ g)(XY)]} = \left[\left(h \circ g(X)\right)\left(h \circ g(Y)\right)\right]_{f_2[(h \circ g(X))(h \circ g(Y))]}$

$= (H_2 \times G_2)(X_{f_2(X)}) \circ (H_2 \times G_2)(Y_{f_2(Y)})$.

Proof of theorem (6):

1] $X_{f_2(X)} \circ X_{f_2(X)} = X_{f_2(X)} \Leftrightarrow (X^2)_{f_2(X^2)} = X_{f_n(X)} \Leftrightarrow X^2 = X$, and X is idempotent in $R_2(I)$.

2] $X_{f_2(X)} \circ \dots \circ X_{f_2(X)}$ $(m - times) = 0_0 \Leftrightarrow (X^m)_{f_2\ (X^m)} = 0_0$, thus $X^m = 0$, and X is nilpotent in $R_2(I)$.

3] Its proof is similar to 1 and 2.

Proof of theorem (7):

1] $H_2\left(X_{f_2(X)}\right) \circ H_2\left(X_{f_2(X)}\right) = [(h(X))^2]_{f_2((h(X))^2)} = (h(X^2))_{f_2(h(X^2))} = (h(X))_{f_2(h(X))} = H_2\left(X_{f_2(X)}\right)$.

2] $H_2\left(X_{f_2(X)}\right)^m = (h(X^m))_{f_2(h(X^m))} = [h(0)]_{f_2(0)} = 0_0$.

3] If $X_{f_2(X)} \circ Y_{f_2(Y)} = 0_0$, then:

$H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right) = (h(XY))_{f_2(h(XY))} = 0_0$.

4] If $X_{f_2(X)} \circ Y_{f_2(Y)} = 1_1$, then:

$H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right) = (h(XY))_{f_2(h(XY))} = (h(1))_{f_2(h(1))} = 1_1$.

Definition:

Let $R_3(I) = \{a_0 + \sum_{i=1}^{3} a_i I_i \quad ; a_i \in \mathbb{R}\ . I_i I_j = I_{\min(i.j)}\ \}$

be 3-refined commutative neutrosophic ring with unity, let $f: \mathbb{R} \to [0.1]$

Such that $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$, we define:

$f_3 \ : \ R_3(I) \rightarrow [0.1] \quad ; \ f_3 \ (a_0 + \sum_{i=1}^{3} a_i I_i \ ) = \max(f(a_i))$, and

$[R_3(I)]_{f_3} \ = \left\{(a_0 + \sum_{i=1}^{3} a_i I_i \ )_{f_n \ (a_0 + \sum_{i=1}^{3} a_i I_i \ )} \quad ; \ a_i \in \mathbb{R}\right\}$, is called the two-fold fuzzy 3-Refined neutrosophic ring.

Definition:

Operations on $[R_3(I)]_{f_3}$ are define as follows:

$*: \ [R_3(I)]_{f_3} \ \times [R_3(I)]_{f_3} \ \rightarrow [R_3(I)]_{f_3}$

$\circ: [R_3(I)]_{f_3} \ \times [R_3(I)]_{f_3} \ \rightarrow [R_3(I)]_{f_3}$

Such that: $\begin{cases} X_{f_3 \ (X)} * Y_{f_3 \ (Y)} = (X + Y)_{f_3 \ (X+Y)} \\ X_{f_3 \ (X)} \circ Y_{f_3 \ (Y)} = (X \cdot Y)_{f_3 \ (XY)} \end{cases}$

Definition:

Let P an ideal of $R_3(I)$, we define the corresponding two-fold fuzzy 3-refined neutrosophic ideal as follows:

$P_{f_3} \ = \{X_{f_3 \ (X)} \quad ; X \in P\}$

Definition:

Let P_{f_3} be a two-fold fuzzy 3-refined neutrosophic ideal, we define the two-fold fuzzy 3-refined factor as:

$[R_3(I)]_{f_3} \ / P_{f_3} \ = X P_{f_3} \quad ; \ X \in [R_3(I)]_{f_3}$.

Definition:

Let $h : R_3(I) \rightarrow R_3(I)$ be a ring homomorphism, we define:

H_n: $[R_3(I)]_{f_3} \ \rightarrow [R_3(I)]_{f_3}$ such that:

$H_2(X_{f_3 \ (X)}) = (h(X))_{f_3 \ (h(X))}$.

The mapping (H_3) is called two-fold fuzzy 3-refined neutrosophic homomorphism.

The kernel $k_{er}(H_3)$is:

$k_{er}(H_n) = \{X \in [R_3(I)]_{f_3} \quad ; \; H_3(X_{f_3\ (X)}) = 0_0\} = \; (k_{er}(h))_{f_3}$.

The direct image $I_m(H_3)$ is:

$I_m(H_3) = \; (I_m(h))_{f_3}$.

Definition:

Let $H_3 \;. G_3 \;:\; [R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$ be two homomorphisms, then: $H_3 \;. G_3 \;:$ $[R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$ with:

$(H_3 \times G_3)(X_{f_3(X)}) = H_3(G_3(X_{f_3(X)}))$.

Theorem (8):

1] *, ○ are commutative.

2] *, ○ are associative.

3] (○) is distributive on (*).

4] *, ○ has identities.

5] (*) is invertible, i.e any element$X_{f_2(X)} \in [R_2(I)]_{f_2}$ has an iverse with respect to (*).

Theorem (9):

Let P_{f_3} be a two-fold ideal of$[R_3(I)]_{f_3}$, then:

$$\begin{cases} X_{f_3(X)} * Y_{f_3(Y)} \in P_{f_3} \\ r_{f_3(r)} \cdot X_{f_3(X)} \in P_{f_3} \end{cases} \quad ; \; X.Y \in P \;\; .r \in R_3(I)$$

Theorem (10):

$[R_3(I)]_{f_3}/P_{f_3}$ is a commutative ring with unity.

Theorem (11):

Let H_3: $[R_3(I)]_{f_3} \to [R_2(I)]_{f_3}$be a homomorphism, then:

1] $H_2(X_{f_3(X)} * Y_{f_3(Y)}) = \; H_2(X_{f_3(X)}) * H_2\left(Y_{{f_3}_2(Y)}\right)$

2] $H_2(X_{f_3(X)} \circ Y_{f_3(Y)}) = \; H_2(X_{f_3(X)}) \circ H_2(Y_{f_3(Y)})$

3] $k_{er}(H_3)$ is an ideal of $[R_3(I)]_{f_3}$.

4] $I_m(H_3)$ is a subring of $[R_3(I)]_{f_3}$.

5] $[R_3(I)]_{f_3}/\ k_{er}(H_3) \cong I_m(H_3)$.

6] If P_{f_3} is an ideal of $[R_3(I)]_{f_3}$, then $H_3(P_{f_3})$ is an ideal.

7] $H_3(0_0) = 0_0$ $. H_3(1_1) = 1_1$

Theorem (12):

Let $H_3\ , G_3\ :\ [R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$ be two homomorphisms, then:

1] $H_3\left(-X_{f_3(X)}\right) = -H_3\left(X_{f_3(X)}\right)$.

2] $H_3\left(X^{-1}_{f_3(X^{-1})}\right) = [H_3\left(X_{f_3(X)}\right)]^{-1}$, if X is invertible.

3] $H_3 \times G_3$ is a homomorphism.

Definition:

Let $X_{f_3(X)} \in [R_2(I)]_{f_2}$, then:

1] $X_{f_3(X)}$ is idempotent if $X_{f_3(X)} \circ X_{f_3(X)} = X_{f_3(X)}$.

2] $X_{f_3(X)}$ is a nilpotent if there exists $m \in \mathbb{N}$ such that: $X_{f_3(X)} \circ X_{f_3(X)} \circ \dots \circ X_{f_3(X)}\ (m - times) = 0_0$

3] $X_{f_3(X)}$ is a zero divisor if there exists $Y_{f_3(Y)}$ such that: $X_{f_3(X)} \circ Y_{f_3(Y)} = 0_0$

Theorem (13):

Let $X_{f_3(X)} \in [R_3(I)]_{f_3}$, then we have:

1] $X_{f_3(X)}$ is idempotent if and only if X is idempotent in $R_3(I)$.

2] $X_{f_3(X)}$ is nilpotent if and only if X is nilpotent in $R_3(I)$.

3] $X_{f_3(X)}$ is a zero divisor if and only if X is a zero diviso in $R_3(I)$.

Theorem (14):

Let H_3: $[R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$, then:

1] If $X_{f_3(X)} \in \ [R_3(I)]_{f_2}$ is idempotent, then $H_3\big(X_{f_3(X)}\big)$ is idempotent.

2] If $X_{f_3(X)}$is nilpotent, then $H_3\big(X_{f_3(X)}\big)$ is nilpotent.

3] If $X_{f_3(X)}$is a zero divisor, then $H_3\big(X_{f_3(X)}\big)$ is a zero divisor.

4] If $X_{f_3(X)}$is a unit, then $H_3\big(X_{f_3(X)}\big)$ is a unit.

Proof of theorem (8):

1] $X_{f_3(X)} * Y_{f_3(Y)} = (X+Y)_{f_3(X+y)} = (Y+X)_{f_3(Y+X)} = Y_{f_3(Y)} * X_{f_3(X)}$.

$X_{f_3(X)} \circ Y_{f_3(Y)} = (XY)_{f_3(Xy)} = (YX)_{f_3(YX)} = Y_{f_3(Y)} \circ X_{f_3(X)}$.

2] $X_{f_3(X)} * Y_{f_3(Y)} * Z_{f_3(Z)}) = X_{f_3(X)} * (Y+Z)_{f_3(Y+Z)} = (X+Y+Z)_{f_3(X+Y+Z)} =$

$(X+Y)_{f_3(X+Y)} * Z_{f_3(Z)} = (X_{f_3(X)} * Y_{f_3(Y)}) * Z_{f_3(Z)}$.

$X_{f_3(X)} \circ Y_{f_2(Y)} \circ Z_{f_2(Z)}) = (XYZ)_{f_3(XYZ)} = (XY)_{f_3(XY)} \circ Z_{f_3(Z)} = (X_{f_3(X)} \circ$

$Y_{f_3(Y)}) \circ Z_{f_3(Z)}$.

3] $X_{f_3(X)} \circ (Y_{f_3(Y)} * Z_{f_3(Z)}) = (XY+XZ)_{f_3(XY+XZ)} = (XY)_{f_2(Xy)} * (XZ)_{f_2(XZ)} = (X_{f_2(X)} \circ$

$Y_{f_2(Y)}) * (X_{f_2(X)} \circ Z_{f_2(Z)})$

4] $X_{f_3(X)} * 0_0 = (X+0)_{f_3(X+0)} = X_{f_3(X)}$.

$X_{f_3(X)} \circ 1_1 = (X \cdot 1)_{f_3(X \cdot 1)} = X_{f_3(X)}$.

5] For $X_{f_3(X)}$, we have $(-X)_{f_3(-X)}$ such that:

$$X_{f_3(X)} * (-X)_{f_3(-X)} = (\mathrm{X} - X)_{f_3(X-X)} = 0_0.$$

Proof of theorem (9):

$X_{f_3(X)} * Y_{f_3(Y)} = (X+Y)_{f_3(X+y)} \ \in P_{f_3}$, that is because $X+Y \ \in P$.

$r_{f_3(r)} \circ X_{f_3(X)} = (rX)_{f_3(rX)} \in P_{f_3}$, that is because $rX \ \in P$.

Proof of theorem (10):

Define: $*' : \left([R_3(I)]_{f_3}/P_{f_3}\right) \times \left([R_3(I)]_{f_3}/P_{f_3}\right) \to [R_3(I)]_{f_3}/P_{f_{3_2}}$

$$\circ' : \left([R_3(I)]_{f_3}/P_{f_3}\right) \times \left([R_3(I)]_{f_3}/P_{f_3}\right) \to [R_3(I)]_{f_3}/P_{f_3}$$

Such that:

$(X_{f_3(X)}P_{f_3}) *' \left(Y_{f_3(Y)}P_{f_3}\right) = \left(X_{f_2(X)} * Y_{f_3(Y)}\right) \; P_{f_3}$

$(X_{f_3(X)}P_{f_3}) \circ' \left(Y_{f_3(Y)}P_{f_3}\right) = (X_{f_3(X)} \circ Y_{f_3(Y)})P_{f_3}$

We have:

$(X_{f_3(X)}P_{f_3}) *' \left(0_0 P_{f_n}\right) = X_{f_2 \;\;(X)} \; P_{f_2}$,

$(X_{f_3(X)}P_{f_3}) \circ' \left(1_1 P_{f_2}\right) = X_{f_2(X)} \; P_{f_2}$,

$(X_{f_3(X)}P_{f_3}) *' \left((-X)_{f_2(-X)}P_{f_2}\right) = 0_0 \; P_{f_2}$,

$(X_{f_3(X)}P_{f_3}) *' \left[\left(Y_{f_3(Y)}P_{f_3}\right) *' \left(Z_{f_3(Z)}P_{f_3}\right)\right] = \left((X_{f_3(X)}P_{f_3}\right) *' \left[(Y * Z) \; P_{f_3}\right] = [X * Y * Z] \; P_{f_3} =$

$(X_{f_3(X)}P_{f_3}) *' \left(Y_{f_3(Y)}P_{f_3}\right)] *' \left(Z_{f_3(Z)}P_{f_3}\right)$,

$(X_{f_3(X)}P_{f_3}) \circ' \left[\left(Y_{f_3(Y)}P_{f_3}\right) \circ' \left(Z_{f_3(Z)}P_{f_3}\right)\right] = \left(X \;\; P_{f_3}\right) \circ' \left[(Y \circ Z) \; P_{f_3}\right] = [X \circ Y \circ Z] \; P_{f_3} =$

$(X_{f_3(X)}P_{f_3}) \circ' \left(Y_{f_3(Y)}P_{f_3}\right)] \circ' \left(Z_{f_3(Z)}P_{f_3}\right)$.

$\left((X_{f_3(X)}P_{f_3}\right) \circ' \left[\left(Y_{f_3(Y)}P_{f_3}\right) *' \left(Z_{f_3(Z)}P_{f_3}\right)\right] = [X \circ (Y * Z)] \; P_{f_3} = [(X \circ Y) * (X \circ Z)] \; P_{f_3}$

$$= (X_{f_3(X)}P_{f_3}) \circ' \left(Y_{f_3(Y)}P_{f_3}\right)] *' [(X_{f_3(X)}P_{f_3}) \circ' \left(Z_{f_3(Z)}P_{f_3}\right)]$$

Thus, our proof is complete.

Proof of theorem (11):

1] $H_3(X * Y) = (h(X + Y))_{f_3(h(X+Y))} = (h(X) + h(Y))_{f_3((h(X)+h(Y)))} = H_3\left(X_{f_3(X)}\right) * H_3\left(Y_{f_{3_2}(Y)}\right)$.

2] $H_3(X \circ Y) = (h(XY))_{f_3(h(XY))} = (h(X)h(Y))_{f_3((h(X)h(Y)))} = H_3\left(X_{f_3(X)}\right) \circ H_3\left(Y_{f_3(Y)}\right)$.

3] since $k_{er}(H_3) = [k_{er}(h)]_{f_2}$, and $k_{er}(h)$ is an ideal of $R_3(I)$, we get: $k_{er}(H_3)$ is an ideal of $[R_3(I)]_{f_2}$.

4] It can be proved by the same.

5] We have that:

$R_3(I)/k_{er}(h) \cong I_m(h)$, thus:

$[R_3(I)]_{f_3}/ [k_{er}(h)]_{f_2} \cong [I_m(h)]_{f_2}$, therefor:

$[R_3(I)]_{f_3}/ k_{er}(H_n) \cong I_m(H_n)$.

6] $H_3(P_{f_3}) = \{ [h(P)]_{f_3}\}$, and $h(P)$ is an ideal of $R_3(I)$, thus $H_3(P_{f_3})$ is an ideal of $[R_3(I)]_{f_3}$.

7] $\begin{cases} H_3(0_0) = (h(0))_{f_3(h(0))} = 0_0 \\ H_3(1_1) = (h(1))_{f_3(h(1))} = 1_1 \end{cases}$.

Proof of theorem (12):

It is similar to that of theorem 5.

Proof of theorem (13):

It holds by a similar argument of theorem 6.

Proof of theorem (14):

It is similar to that of theorem 7.

Conclusion

In this paper we studied the two-fold fuzzy algebra based on n-refined neutrosophic rings for some different special values of n, where we studied some of special elements in the case of two-fold 2-refined neutrosophic ring and 3-refined neutrosophic ring such as units, idempotenets and nilpotent elements. Also, we presented the concept of two-fold ring homomorphism with its elementary properties.

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On The Special Gamma Function Over The Complex Two-Fold Algebras

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Abstract:

The concept of special functions plays an important role in mathematical analysis and physics as well. In this paper, we study some different types of the special Gamma function defined on the two-fold fuzzy complex field, where we combine the classical Gamma function with the two-fold fuzzy algebra defined on complex numbers. On the other hand, many elementary properties of this new special function will be determined in terms of theorems and proofs.

Keywords: Gamma function, two-fold fuzzy algebra, complex field, special function.

Introduction

The theory of special functions is considered one of the most comprehensive and important theories in mathematics due to its wide applications in various fields of knowledge and physics [1-3]. The gamma function is one of the most famous functions in mathematics that plays a central role in number theory, probability, and the calculation of random processes [6-7].

Neutrosophic logic as a good generalization of fuzzy logic plays a central role in modern studies that are related to algebra and analysis [4-5], with very wide applications in decision-making and geometry [8-9].

In [10], Smarandache proposed the concept of two-fold algebras, and then these idea was used in the study of fuzzy number theoretical relations [11], and in module theory [12].

In this work, we are motivated to use the two-fold fuzzy complex algebra with Gamma functions to generate a new analytical structure and to study its properties. This study may be very helpful in the future because it opens a wide door to use of two-fold algebra in defining and presenting some new types of special functions that can be applied in other fields of knowledge.

Main Discussion

Definition:

Let $\mathbb{C}$ be the complex field, $\mu: \mathbb{R} \times \mathbb{R} \to [0.1]$, we define the complex twofold fuzzy algebra as follows.

$\mathbb{C}_f = \left\{(a+bi)_{\mu(x)} \quad ; a.b.x \quad \in \mathbb{R} \ . \quad i^2 = -1\right\}$,

Binary operation:

$(*): C_f \times C_f \to C_f$ such that:

$(a+bi)_{\mu(x)} * (c+di)_{\mu(y)} = [(a+c)+(b+d)i]_{\max(\mu(x).\mu(y))}$

Theorem1:

Let $(C_f.*)$ be the two fold complex fuzzy algebra defined previously, then:

1] $(*)$ is well defined.

2] $(*)$ is commutative.

3] $(*)$ is associative.

4] For each $(a+bi)_{\mu(x)} \in C_f$, there exists $o_{\mu(x)} \in C_f$ such that:

$(a+bi)_{\mu(x)} * o_{\mu(x)} = (a+bi)_{\mu(x)}$.

Example:

For $: \mathbb{R} \times \mathbb{R} \to [0.1] \quad ; \quad \mu(x) = \begin{cases} e^x & ; x \le 0 \\ e^{\frac{1}{x}} & ; x \ge 0 \end{cases}$, we have:

$A = (3+2i)_{\mu(3)} \quad . B = (2-5i)_{\mu(-2)}$, then:

$A * B = \ (5 - 3i)_{\max\left(\frac{1}{e^3}, \frac{1}{e^2}\right)} = \ (5 - 3i)_{\frac{1}{e^2}}.$

Definition:

Let $A = (a + bi)_{\mu(x)} \in C_f$, we define:

1] $\bar{A} = (a - bi)_{\mu(x)}$

2] $|A| = (\sqrt{a^2 + b^2})_{\mu(x)}$

Theorem 2:

For $A = (a + bi)_{\mu(x)} \in C_f$. $B = (c + di)_{\mu(y)} \in C_f$, we have:

1] $A * \bar{A} = (2a)_{\mu(x)}$

2] $\overline{A * B} = \bar{A} * \bar{B}$

Example:

Take $A = \ (1 + 2i)_{\mu(5)}$. $B = \ (3 - i)_{\mu(-10)}$; $\mu(x) = \begin{cases} e^x & ; x \leq 0 \\ e^{\frac{1}{x}} & ; x > 0 \end{cases}$

$\bar{A} = \ (1 - 2i)_{\mu(5)} = \ (1 - 2i)_{e^{-5}}$. $\bar{B} = \ (3 + i)_{\mu(-10)} = \ (3 + i)_{e^{-10}}$

$A * B = \ (4 + i)_{e^{-5}}$. $\bar{A} * \bar{B} = \ (4 - i)_{e^{-5}}$. $\overline{A * B} = \ (4 - i)_{e^{-5}}.$

$|A| = \sqrt{5}_{e^{-5}}$. $|A| = \sqrt{10}_{e^{-10}}$

Definition:

We define the following binary operation on C_f:

$$\circ: C_f \times C_f \to C_f : \quad (a + bi)_{\mu(x)} \circ \ (c + di)_{\mu(y)}$$
$$= [ac - bd + (ad + bc)i]_{\min(\mu(x).\mu(y))}$$

Theorem 3:

1] ($\circ$) is well defined.

2] ($\circ$) is commutative.

3] ($\circ$) is associative.

4] ($\circ$) is distributive on ($*$).

5] For each $(a + bi)_{\mu(x)} \in C_f$, there exists $1_{\mu(x)} \in C_f$ such that:

$(a + bi)_{\mu(x)} \circ 1_{\mu(x)} = \ (a + bi)_{\mu(x)}.$

Theorem 4:

Let $A = (a + bi)_{\mu(x)}$. $B = (c + di)_{\mu(y)} \in C_f$, then:

1] $|A \circ B| = |A| \circ |B|$

2] $|\bar{A}| = |A|$

3]$A \circ \bar{A} = |A^2|$

Definition:

Let $A = (a + bi)_{\mu(x)} \in C_f$, we define the first special function Gamma on C_f as follows:

$\Gamma_1\left(A_{\mu(x)}\right) = (\Gamma(A))_{\mu(\Gamma(x))}$; $a.x > 0$.

The second Gamma function on C_f is defined as follows:

$\Gamma_2\left(A_{\mu(x)}\right) = (\Gamma(A))_{\mu(\Gamma(x))}$; $a > 0$. $x \in \mathbb{R}$.

The third Gamma function on C_f is defined as follows:

$\Gamma_3\left(A_{\mu(x)}\right) = (A)_{\mu(\Gamma(x))}$; $x > 0$. $A \in C_f$

Theorem5:

Consider $\Gamma_1.\Gamma_2.\Gamma_3$ the three types of special Gamma functions defined over C_f , then:

1] $A_{\mu(x)} \circ \Gamma_2\left(A_{\mu(x)}\right) = \Gamma_2[(A + 1)_{\mu(x)}]$

2] $A_{\mu(x)} \circ \Gamma_1\left(A_{\mu(x)}\right) = \Gamma_1[(A + 1)_{\mu(x)}]$

3] for $A = (a)_{\mu(x)} \in C_f$; $a \in \mathbb{R}^+$, we have:

$$\begin{cases} \lim\limits_{a \to 0^+} A_{\mu(x)} \circ \Gamma_2\left(A_{\mu(x)}\right) = 1_{\mu(x)} \\ \lim\limits_{a \to 0^+} A_{\mu(x)} \circ \Gamma_1\left(A_{\mu(x)}\right) = 1_{\mu(x)} \end{cases}$$

4] for $A = a_{\mu(x)} \in C_f$; $a.x \in \mathbb{R}^+$, we have:

$$\begin{cases} \Gamma_1\left(A_{\mu(x)}\right) = \left(2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt\right)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt\right)} \\ \Gamma_2\left(A_{\mu(x)}\right) = \left(2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt\right)_{\mu(x)} \\ \Gamma_3\left(A_{\mu(x)}\right) = (a)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\, dt\right)} \end{cases}$$

Definition:

The two fold neutrosophic complex algebra is defined as follows:

$$C_N = \{ \ (a+bi)_{(t.j.f)} \quad ; \quad a.b \in \mathbb{R} \quad . \quad t.j.f \in [0.1] \ . \ i^2 = -1 \ \}$$

We define the following binary operations:

$$*: C_N \times C_N \to C_N \quad ; \quad (a+bi)_{(t_1.j_1.f_1)} * (c+di)_{(t_2.j_2.f_2)}$$
$$= [a+c+(d+b)i]_{(t.j.f)}$$

Where: $\begin{cases} t = \max(t_1.t_2) \\ f = \min(f_1.f_2) \\ j = \min(j_1.j_2) \end{cases}$

$$\circ: C_N \times C_N \to C_N \quad ; \quad (a+bi)_{(t_1.j_1.f_1)} \circ (c+di)_{(t_2.j_2.f_2)}$$
$$= [ac - bd + (ad+bc)i]_{(t.j.f)}$$

Where: $\begin{cases} t = \min(t_1.t_2) \\ j = \max(j_1.j_2) \\ f = \max(f_1.f_2) \end{cases}$

Example:

Let $A = (2+5i)_{\left(\frac{1}{2}.\frac{1}{3}.\frac{1}{5}\right)}$ $.B = (1-i)_{\left(0.\frac{1}{2}.\frac{1}{3}\right)}$, we have:

$A * B = (3+4i)_{\left(\frac{1}{2}.\frac{1}{3}.\frac{1}{5}\right)}$ $.A \circ B = (7+3i)_{\left(0.\frac{1}{2}.\frac{1}{3}\right)}$

Definition:

Let $A = (a+bi)_{(t.j.f)} \in C_N$, we define:

1] $\bar{A} = (a-bi)_{(t.j.f)}$

2] $|A| = (\sqrt{a^2+b^2})_{(t.j.f)}$

Theorem 6:

Let $(C_N.\circ \ .*)$ be the two fold neutrosophic complex algebra, then:

1] $(*)$ is well defined.

2] $(\circ)$ is well defined.

3] $(*).(\circ)$ are commutative.

4] $(*).(\circ)$ are associative.

5] $(\circ)$ is distributive on $(*)$.

6] for each $A = (a + bi)_{(t.j.f)} \in C_N$, there exists: $0_{(t.j.f)}.1_{(t.j.f)}$

Such that $\begin{cases} A * 0 = A \\ A \circ 1 = A \end{cases}$

Theorem 7:

Let $(C_N.\circ\ .*)$ be the neutrosophic two fold algebra, then:

For $A\ .B \in C_N$, we have:

1] $\overline{A * B} = \bar{A} * \bar{B}$

2] $\overline{A \circ B} = \bar{A} \circ \bar{B}$

3] $A \circ \bar{A} = |A^2|$

4] $|A \circ B| = |A| \circ |B|$

Definition:

Let $(C_N.\circ\ .*)$ be the neutrosophic two fold algebra, we define the following types of Gamma special function:

$$\Gamma_1\big(A_{(t.j.f)}\big) = (\Gamma(A))_{(\Gamma(t).\Gamma(j).\Gamma(f))} \qquad ; a > 0.$$

$$\Gamma_2\big(A_{(t.j.f)}\big) = (\Gamma(A))_{(t.j.f)} \qquad ; a > 0.$$

$$\Gamma_3\big(A_{(t.j.f)}\big) = (A)_{(\Gamma(t).\Gamma(j).\Gamma(f))}$$

Theorem 8:

Consider $\Gamma_1.\Gamma_2.\Gamma_3$ the three types of Gamma functions over $(C_N.\circ\ .*)$, we have:

1] $A_{(t.j.f)}\ \Gamma_2\big(A_{(t.j.f)}\big) = \Gamma_2(A+1)_{(t.j.f)}$

2] $\Gamma_1\big(A_{(t.j.f)}\big) = (L)_{(a_1.b_1.c_1)} \quad ; where: \begin{cases} L = 2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt \\ a_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2(t)-1}\, dt \\ b_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2j-1}\, dt \\ c_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2f-1}\, dt \end{cases}$

3] For $A = a \in \mathbb{R}^+ \quad ; \quad \Gamma_2\big(A_{(t.j.f)}\big) = (L)_{(t.j.f)} \quad .where\ \ L = 2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\, dt.$

4] For any $A \in C_N$. $\Gamma_3\big(A_{(t.j.f)}\big) = (A)_{(a_1.b_1.c_1)}$; $where \begin{cases} a_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2(t)-1}\, dt \\ b_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2j-1}\, dt \\ c_1 = 2\int_0^\infty e^{-t^2} \cdot t^{2f-1}\, dt \end{cases}$

Proof of theorem (1):

1] assume that: $\begin{cases} (a+bi)_{\mu(x)} = (a'+b'i)_{\mu(x')} \\ (c+di)_{\mu(y)} = (c'+d'i)_{\mu(y')} \end{cases}$

Then: $\begin{cases} a = a' \ .b = b' \\ c = c' \ .d = d' \\ \mu(y) = \mu(y') \ .\mu(x) = \mu(x') \end{cases}$

Hence: $(a+bi)_{\mu(x)} * (c+di)_{\mu(y)} = [a+c+(d+b)i]_{\max(\mu(x).\mu(y))}$

$= [a'+c'+(d'+b')i]_{\max\left(\mu(x').\mu(y')\right)} = (a'+b'i)_{\mu(x')} * (c'+d'i)_{\mu(y')}$.

2] $(a+bi)_{\mu(x)} * (c+di)_{\mu(y)} = [a+c+(b+d)i]_{\max(\mu(x).\mu(y))} = (c+di)_{\mu(y)} * (a+bi)_{\mu(x)}$.

3] Let $X = (a+bi)_{\mu(x)}$ $.Y = (c+di)_{\mu(y)}$ $.Z = (m+ni)_{\mu(z)}$, then:

$X * (Y * Z) = X * [c+m+i(d+n)]_{\max(\mu(y).\mu(z))}$

$= [a+c+m+i(b+d+n)]_{\max(\mu(x).\mu(y).\mu(z))}$

$= (a+c+i(b+d))_{\max(\mu(x).\mu(y))} * (m+ni)_{\mu(z)} = (X*Y)*Z$.

4] It is clear that:

$(a+bi)_{\mu(x)} * o_{\mu(x)} = (a+0+ib)_{\max(\mu(x).\mu(x))} = (a+bi)_{\mu(x)}$.

Proof of theorem (2):

1] $A * \bar{A} = (a+bi)_{\mu(x)} * (a-bi)_{\mu(x)} = (2a)_{\mu(x)}$.

2] $\overline{A*B} = \overline{(a+c+\imath(d+b))}_{\max(\mu(x).\mu(y))} = (a+c-i(d+b))_{\max(\mu(x).\mu(y))} = (a-ib)_{\mu(x)} * (c-id)_{\mu(y)} = \bar{A} * \bar{B}$.

Proof of theorem (3):

1] assume that: $\begin{cases} (a+bi)_{\mu(x)} = & (a'+b'i)_{\mu(x')} \\ (c+di)_{\mu(y)} = & (c'+d'i)_{\mu(y')} \end{cases}$

Then: $\begin{cases} a = a' & .b = b' \\ c = c' & .d = d' \\ \mu(x) = \mu(x') & .\mu(y) = \mu(y') \end{cases}$

So that: $(a+bi)_{\mu(x)} \circ (c+di)_{\mu(y)} = [ac - bd + i(ad+bc)]_{\min(\mu(x).\mu(y))}$

$= [a'c' - b'd' + i(a'd' + b'c')]_{\min(\mu(x').\mu(y'))} = (a'+b'i)_{\mu(x')} \circ$

$(c'+d'i)_{\mu(y')}$.

2] $(a+bi)_{\mu(x)} \circ (c+di)_{\mu(y)} = [ac - bd + i(ad+bc)]_{\min(\mu(x).\mu(y))} = [ca - db +$

$i(da+cb)]_{\min(\mu(x).\mu(y))} = (c+di)_{\mu(y)} \circ (a+bi)_{\mu(x)}$.

3] Let $X = (a+bi)_{\mu(x)}$. $Y = (c+di)_{\mu(y)}$ $.Z =$ $(m+ni)_{\mu(z)}$, then:

$X \circ (Y \circ Z) = X \circ [(c+di)(m+ni)]_{\min(\mu(y).\mu(z))} = [(a+bi)(c+di)(m+$

$ni)]_{\min(\mu(x).\mu(y).\mu(z))}$

$= [(a+bi)(c+di)]_{\min(\mu(x).\mu(y))} \circ (m+ni)_{\mu(z)} = (X \circ Y) \circ Z.$

4] $X \circ (Y * Z) = (a+bi)_{\mu(x)} \circ [(c+m) + i(d+n)]_{\max(\mu(y).\mu(z))} = [(a+bi)(c+$

$di) + (a+bi)(m+ni)]_{\min(\mu(x).\max(\mu(y).\mu(z)))}$

Also $(X \circ Y) * (X \circ Z) = [(a+bi)(c+di)]_{\min(\mu(x).\mu(y))} * [(a+bi)(m+$

$ni)]_{\min(\mu(x).\mu(z))}$

$= [(a+bi)(c+di) + (a+bi)(m+ni)]_{\max(\min(\mu(x).\mu(y)).\min(\mu(x).\mu(z)))} = X \circ (Y * Z)$

5] it holds directly from the definition.

Proof of theorem (4):

1] $|A \circ B| = [|(a+bi)(c+di)|]_{\min(\mu(x).\mu(y))} = [\sqrt{a^2+b^2} \cdot \sqrt{c^2+d^2}]_{\min(\mu(x).\mu(y))} =$

$[\sqrt{a^2+b^2}]_{\mu(x)} \circ [\sqrt{c^2+d^2}]_{\mu(y)} = |A| \circ |B|$

2] $|\bar{A}| = (\sqrt{a^2+b^2})_{\mu(x)} = |A|$.

3]$A \circ \bar{A} = [(a+bi)(a-bi)]_{\min(\mu(x).\mu(x))} = (a^2+b^2)_{\mu(x)} = |A^2|.$

Proof of theorem (5):

Before the proof get started, it will be useful to write the formulas of Gamma function:

1] $\Gamma(Z) = 2\int_0^\infty e^{-t} \cdot t^{z-1}\,dt \quad ; \quad \mathbb{R}_e(Z) > 0 \ .z \in \mathbb{C}$

2] $\Gamma(Z+1) = Z \ \ \Gamma(Z)$

3] $\lim\limits_{x\to 0^+} x\Gamma(x) = 1 \quad ; \quad x \in \mathbb{R}^+$

4] $\Gamma(x) = 2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\,dx \quad ; 0 < x < \infty$

Now we prove the first part:

$\boxed{1}$ $A_{\mu(x)} \circ \Gamma_2(A_{\mu(x)}) = (A\Gamma(A))_{\mu(x)} = (\Gamma(A+1))_{\mu(x)} = \Gamma_2[(A+1)_{\mu(x)}]$

$\boxed{2}$ $A_{\mu(x)} \circ \Gamma_1(A_{\mu(x)}) = (A\Gamma(A))_{\mu(\Gamma(x))} = (\Gamma(A+1))_{\mu(\Gamma(x))} = \Gamma_1[(A+1)_{\mu(x)}]$

$\boxed{3}$ $\lim\limits_{a\to 0^+} A_{\mu(x)} \circ \Gamma_2(A_{\mu(x)}) = \lim\limits_{a\to 0^+}[a\Gamma(a)]_{\mu(x)} = 1_{\mu(x)}$

$\lim\limits_{a\to 0^+} A_{\mu(x)} \circ \Gamma_1(A_{\mu(x)}) = \lim\limits_{a\to 0^+}[A\Gamma(A)]_{\mu(\Gamma(x))} = \lim\limits_{a\to 0^+}[a\Gamma(a)]_{\mu(\Gamma(x))} = 1_{\mu(x)}$

$$\boxed{4} \ \ since \ \begin{cases} \Gamma(A) = 2\int\limits_0^\infty e^{-t} \cdot t^{2a-1}\,dt \\ \Gamma(x) = 2\int\limits_0^\infty e^{-t^2} \cdot t^{2x-1}\,dt \end{cases}$$

$$\text{We get: } \begin{cases} \Gamma_1(A_{\mu(x)}) = (2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\,dt)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\,dt\right)} \\ \Gamma_2(A_{\mu(x)}) = (2\int_0^\infty e^{-t^2} \cdot t^{2a-1}\,dt)_{\mu(x)} \\ \Gamma_3(A_{\mu(x)}) = (a)_{\mu\left(2\int_0^\infty e^{-t^2} \cdot t^{2x-1}\,dt\right)} \end{cases}$$

Proof of theorem (6):

1] Assume that: $(a+bi)_{(t_1.j_1.f_1)} = (a'+b'i)_{(t_1'.j_1'.f_1')}$. $(c+di)_{(t_2.j_2.f_2)} = (c'+d'i)_{(t_2'.j_2'.f_2')}$.

Then: $\begin{cases} a = a' \ .b = b' \ .c = c' \ \ .d = d' \\ t_1 = t_1' \ .t_2 = t_2' \\ j_1 = j_1' \ .j_2 = j_2' \\ f_1 = f_1' \ . \ \ f_2 = f_2' \end{cases}$

$$(a+bi)_{(t_1.j_1.f_1)} * \ (c+di)_{(t_2.j_2.f_2)} = [a+c+(b+d)i]_{(t_3.j_3.f_3)}$$

$$= [a'+c'+(b'+d')i]_{(t_4.j_4.f_4)}$$

Where $\begin{cases} t_3 = \max(t_1.t_2) = t_4 = \max(t_1' \ .t_2') \\ f_3 = \min(f_1.f_2) = f_4 = \min(f_1' \ .f_2') \\ j_3 = \min(j_1.j_2) = j_4 = \min(j_1' \ .j_2') \end{cases}$

2] Assume that: $\begin{cases} (a+bi)_{(t_1.j_1.f_1)} = (a'+b'i)_{(t_1'.j_1'.f_1')} \\ (c+di)_{(t_2.j_2.f_2)} = (c'+d'i)_{(t_2'.j_2'.f_2')} \end{cases}$

We get: $\begin{cases} a = a' \ .b = b' \ .c = c' \ \ .d = d' \\ t_1 = t_1' \ .t_2 = t_2' \\ j_1 = j_1' \ .j_2 = j_2' \\ f_1 = f_1' \ . \ \ f_2 = f_2' \end{cases}$

Hence: $(a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)} = [ac-bd+i(ad+bc)]_{(t_3.j_3.f_3)}$

$$= [a'c' \ -b'd'+i(a'd'+b'c')]_{(t_3.j_3.f_3)} = (a'+b'i)_{(t_1'.j_1'.f_1')} \circ (c'+d'i)_{(t_2'.j_2'.f_2')}$$

Where $\begin{cases} t_3 = \min(t_1.t_2) = \min(t_1' \ .t_2') \\ j_3 = \max(j_1.j_2) = \max(j_1' \ .j_2') \\ f_3 = \max(f_1.f_2) = \max(f_1' \ .f_2') \end{cases}$

3]$(a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)} = [ac-bd+i(ad+bc)]_{(t_3.j_3.f_3)}$

$$= [ca-db+i(da+cb)]_{(t_3.j_3.f_3)} = \ (c+di)_{(t_2.j_2.f_2)} \circ (a+bi)_{(t_1.j_1.f_1)}$$

, Where $\begin{cases} t_3 = \min(t_1.t_2) = \min(t_2.t_1) \\ j_3 = \max(j_1.j_2) = \max(j_2.j_1) \\ f_3 = \max(f_1.f_2) = \max(f_2.f_1) \end{cases}$

$(a+bi)_{(t_1.j_1.f_1)} * \ (c+di)_{(t_2.j_2.f_2)} = \ (c+di)_{(t_2.j_2.f_2)} * \ (a+bi)_{(t_1.j_1.f_1)}$ by a similar argument.

4] $(a+bi)_{(t_1.j_1.f_1)} \circ [\ (c+di)_{(t_2.j_2.f_2)} \circ \ (m+ni)_{(t_3.j_3.f_3)}] = [(a+bi)(c+di)(m+ni)]_{(t.j.f)} = L$

Where $\begin{cases} t = \min(t_1.t_2.t_3) \\ j = \max(j_1.j_2.j_3) \\ f = \max(f_1.f_2.f_3) \end{cases}$

So that: $L = [(a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)}] \circ \ (m+ni)_{(t_3.j_3.f_3)}$, hence $(\circ)$ is associative.

The associativity of (*) can be proved by the same.

5] $(a+bi)_{(t_1.j_1.f_1)} \circ [\ (c+di)_{(t_2.j_2.f_2)} * \ (m+ni)_{(t_3.j_3.f_3)}] = [(a+bi)[(c+di)+(m+ni)]]_{(t.j.f)} = L.$

Where $\begin{cases} t = \min(t_1.\max(t_2.t_3)) \\ j = \max(j_1.\min(j_2.j_3)) \\ f = \max(f_1.\min(f_2.f_3)) \end{cases}$

Thus: $L = [(a+bi)_{(t_1.j_1.f_1)} \circ \ (c+di)_{(t_2.j_2.f_2)}] * \ [(a+bi)_{(t_1.j_1.f_1)} \circ (m+ni)_{(t_3.j_3.f_3)}].$

6] $\begin{cases} A * 0 = (A+0)_{(t.j.f)} = A_{(t.j.f)} \\ A \circ 1 = (A \cdot 1)_{(t.j.f)} = A_{(t.j.f)} \end{cases}$

Proof of theorem (7):

1] $\overline{A * B} = \overline{(A+B)}_{(t.j.f)} = (\bar{A}+\bar{B})_{(t.j.f)} = \bar{A}_{(t.j.f)} * \bar{B}_{(t.j.f)}.$

2] $\overline{A \circ B} = \overline{(A \cdot B)}_{(t.j.f)} = (\bar{A} \cdot \bar{B})_{(t.j.f)} = \bar{A}_{(t.j.f)} \circ \bar{B}_{(t.j.f)}.$

$\boxed{3}$. $\boxed{4}$ hold directly from the definition.

Proof of theorem (8):

It can be proved by a similar argument to that of theorem 5.

Conclusion

In this paper, we defined some different types of the special Gamma function on the two-fold fuzzy complex field, where we combined the classical Gamma function with the two-fold fuzzy algebra defined on complex numbers. On the other hand, many elementary properties of this new special function are determined and presented.

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The limits of 2- refined neutrosophic

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Abstract: this paper aims to present the limits of 2- refined neutrosophic, where we studied the of the neutrosophic factorization method and the neutrosophic rationalization method of the limits of 2- refined neutrosophic, we verified the results of these methods using the L'Hôpital's rule. Also We introduced some special limits and 2- refined neutrosophic trigonometric limits. In addition to clarifying this by solving appropriate numerical examples.

Keywords: indeterminacy; trigonometric; neutrosophic; factorization; limits.

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1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \ldots, b_nI_n)$ where $a, b_1, b_2, \ldots, b_n \in R\ \ or\ \ C$ [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8-9-10].

Alhasan and Abdulfatah also presented the division of refined neutrosophic numbers [11], where:

$$\frac{a_1 + b_1I_1 + c_1I_2}{a_2 + b_2I_1 + c_2I_2} \equiv \frac{a_1}{a_2} + \left[\frac{{a_2}^2 b_1 + a_2b_1c_2 - a_1a_2b_2 - a_2b_2c_1}{a_2(a_2 + c_2)(a_2 + b_2 + c_2)}\right] I_1 + \left[\frac{a_2c_1 - a_1c_2}{a_2(a_2 + c_2)}\right] I_2$$

where: $a_2 \neq 0$, $a_2 \neq -c_2$ and $a_2 \neq -b_2 - c_2$

This paper addressed many topics, following the introduction and preliminary material presented in the first part, the limits of 2-refined neutrosophic were discussed in the main discussion section. The final section contained the conclusion.

2. Main Discussion

2.1 The neutrosophic factorization method of the limits of 2- refined neutrosophic

Let $\frac{f(x,I_1,I_2)}{g(x,I_1,I_2)}$ is rational 2- refined neutrosophic function, if $f(x,I_1,\ I_2), g(x,I_1,\ I_2)$ contains some common factors, then we can eliminate out the common factors from the numerator and denominator and after that we calculate the limit.

Example 1

Evaluate:

$$\lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2}$$

Solution:

$$\lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2} = \frac{0}{0}$$

Method1:

$$x^2-9+9I_1+7I_2 = (x+3+I_1+I_2)(x-3-I_1-I_2)$$

$$\lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2} = \lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{(x+3+I_1+I_2)(x-3-I_1-I_2)}$$

$$= \lim_{x\to 3+I_1+I_2} \frac{1}{x+3+I_1+I_2} = \frac{1}{6+2I_1+2I_2} = \frac{1}{6} - \frac{1}{40}I_1 - \frac{1}{24}I_2$$

Method2:

by using L'Hôpital's rule

$$\Longrightarrow \quad \lim_{x\to 3+I_1+I_2} \frac{x-3-I_1-I_2}{x^2-9+9I_1+7I_2} = \lim_{x\to 3+I_1+I_2} \frac{1}{2x}$$

$$= \frac{1}{2(3+I_1+I_2)} = \frac{1}{6+2I_1+2I_2} = \frac{1}{6} - \frac{1}{40}I_1 - \frac{1}{24}I_2$$

2.2 The neutrosophic rationalization method of the limits of 2- refined neutrosophic

Example 2

Evaluate:

$$\lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x} - \sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x}$$

Solution:

$$\lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x} - \sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x} = \frac{0}{0}$$

Method1:

$$\Rightarrow \lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x}-\sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{\left(\sqrt{1-(1+I_1+2I_2)x}-\sqrt{1+(1+I_1+2I_2)x}\right)\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}{(2+3I_1-I_2)x\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{1-(1+I_1+2I_2)x-[1+(1+I_1+2I_2)x]}{(2+3I_1-I_2)x\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{-(2+2I_1+4I_2)x}{(2+3I_1-I_2)x\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{-(2+2I_1+4I_2)}{(2+3I_1-I_2)\left(\sqrt{1-(1+I_1+2I_2)x}+\sqrt{1+(1+I_1+2I_2)x}\right)}$$

$$= \frac{-1-I_1-2I_2}{2+3I_1-I_2} = -\frac{1}{2}+2I_1-\frac{5}{2}I_2$$

Method2:

by using L'Hôpital's rule

$$\Rightarrow \lim_{x\to 0+0I_1+0I_2} \frac{\sqrt{1-(1+I_1+2I_2)x}-\sqrt{1+(1+I_1+2I_2)x}}{(2+3I_1-I_2)x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{\frac{-(1+I_1+2I_2)}{2\sqrt{1-(1+I_1+2I_2)x}}-\frac{(1+I_1+2I_2)}{2\sqrt{1+(1+I_1+2I_2)x}}}{2+3I_1-I_2}$$

$$= \frac{\frac{-(1+I_1+2I_2)}{2\sqrt{1-0}}-\frac{(1+I_1+2I_2)}{2\sqrt{1+0}}}{2+3I_1-I_2}$$

$$= \frac{-1-I_1-2I_2}{2+3I_1-I_2} = -\frac{1}{2}+2I_1-\frac{5}{2}I_2$$

Example 3

Evaluate:

$$\lim_{x\to 6+2I_1-3I_2} \frac{1-\sqrt{x-5-2I_1+3I_2}}{x-5-2I_1+3I_2}$$

Solution:

$$\lim_{x\to 6+2I_1-3I_2} \frac{1-\sqrt{x-5-2I_1+3I_2}}{x-5-2I_1+3I_2} = \frac{0}{0}$$

Method1:

$$\Rightarrow \quad \lim_{x\to 6+2I_1-3I_2} \frac{\left(1-\sqrt{x-5-2I_1+3I_2}\right)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}{(x-5-2I_1+3I_2)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x\to 6+2I_1-3I_2} \frac{1-(x-5-2I_1+3I_2)}{(x-5-2I_1+3I_2)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x\to 6+2I_1-3I_2} \frac{-x+5+2I_1-3I_2}{(x-5+I)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x\to 6+2I_1-3I_2} \frac{-(x-5-2I_1+3I_2)}{(x-5-2I_1+3I_2)\left(1+\sqrt{x-5-2I_1+3I_2}\right)}$$

$$= \lim_{x\to 6+2I_1-3I_2} \frac{-1}{\left(1+\sqrt{x-5-2I_1+3I_2}\right)} = \frac{-1}{2}$$

Method2:

by using L'Hôpital's rule

$$\Rightarrow \quad \lim_{x\to 6+2I_1-3I_2} \frac{1-\sqrt{x-5-2I_1+3I_2}}{x-5-2I_1+3I_2}$$

$$= \lim_{x\to 6+2I_1-3I_2} \frac{\frac{-1}{2\sqrt{x-5-2I_1+3I_2}}}{1}$$

$$= \lim_{x\to 6+2I_1-3I_2} \frac{-1}{2\sqrt{x-5-2I_1+3I_2}} = \frac{-1}{2}$$

Example 4

Evaluate:

$$\lim_{x\to a+bI_1+cI_2} \frac{\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}}{\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}}$$

Solution:

$$\lim_{x\to a+bI_1+cI_2} \frac{\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}}{\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}} = \frac{0}{0}$$

$$\Rightarrow \quad \lim_{x\to a+bI_1+cI_2} \frac{\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}}{\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}}$$

$$= \lim_{x\to a+bI_1+cI_2} \frac{\left(\sqrt{a+bI_1+cI_2+2x}-\sqrt{3x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}{\left(\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x\to a+bI_1+cI_2} \frac{a+bI_1+cI_2+2x-3x}{\left(\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{a+bI_1+cI_2-x}{\left(\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)} = \frac{0}{0}$$

then:

$$\lim_{x \to a+bI_1+cI_2} \frac{a+bI_1+cI_2-x}{\left(\sqrt{15a+15bI_1+15cI_2+x}-4\sqrt{x}\right)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)} \cdot \frac{\sqrt{15a+15bI_1+15cI_2+x}+4\sqrt{x}}{\sqrt{15a+15bI_1+15cI_2+x}+4\sqrt{x}}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{(a+bI_1+cI_2-x)\left(\sqrt{15a+15bI_1+15cI_2+x}+4\sqrt{x}\right)}{(15a+15bI_1+15cI_2+x-16x)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{(a+bI_1+cI_2-x)\left(\sqrt{15a+15bI_1+15cI_2+x}+4\sqrt{x}\right)}{15(a+bI_1+cI_2-x)\left(\sqrt{a+bI_1+cI_2+2x}+\sqrt{3x}\right)}$$

$$= \lim_{x \to a+bI_1+cI_2} \frac{\left(\sqrt{15a+15bI_1+15cI_2+x}+4\sqrt{x}\right)}{15\left(\sqrt{a+bI+2x}+\sqrt{3x}\right)}$$

$$= \frac{\left(\sqrt{15a+15bI_1+15cI_2+a+bI_1+cI_2}+4\sqrt{a+bI_1+cI_2}\right)}{15\left(\sqrt{a+bI_1+cI_2+2(a+bI_1+cI_2)}+\sqrt{3(a+bI_1+cI_2)}\right)}$$

$$= \frac{\sqrt{16(a+bI_1+cI_2)}+2\sqrt{a+bI_1+cI_2}}{15\left(\sqrt{3(a+bI_1+cI_2)}+\sqrt{3(a+bI_1+cI_2)}\right)} = \frac{4\sqrt{a+bI_1+cI_2}}{30\sqrt{3(a+bI_1+cI_2)}}$$

$$= \frac{2\sqrt{a+bI_1+cI_2}}{15\sqrt{3(a+bI_1+cI_2)}} = \frac{2}{15\sqrt{3}}$$

2.3 2- Refined neutrosophic trigonometric limits

1) $\lim_{x \to 0} sin(a+bI_1+cI_2)\,x = 0$

2) $\lim_{x \to 0} cos(a+bI_1+cI_2)\,x = 1$

3) $\lim_{x \to 0} \frac{sin(a+bI_1+cI_2)x}{x} = a+bI_1+cI_2$

Proof (3):

Put $(a+bI_1+cI_2)x = y$ $\Rightarrow$ $x = \frac{1}{a+bI_1+cI_2}y$

When $x \longrightarrow 0$ then: $y \longrightarrow 0$

$$\Rightarrow \quad \lim_{x \to 0} \frac{sin(a+bI_1+cI_2)\,x}{x} = \lim_{y \to 0} \frac{siny}{\frac{1}{a+bI_1+cI_2}y}$$

$$= (a+bI_1+cI_2) \lim_{y \to 0} \frac{siny}{y} = a+bI_1+cI_2$$

4) $\lim_{x\to 0}\frac{x}{\sin(a+bI_1+cI_2)\,x}=\frac{1}{a+bI_1+cI_2}$

$$=\frac{1}{a}+\left[\frac{-b}{(a+c)(a+b+c)}\right]I_1-\left[\frac{c}{a(a+c)}\right]I_2$$

Where a , b, c are real coefficients, $a\neq 0$, $a\neq -c$ and $a\neq -b-c$

Proof (4):

Put $(a+bI_1+cI_2)x=y$ $\Longrightarrow$ $x=\frac{1}{a+bI_1+cI_2}y$

When $x\longrightarrow 0$ then: $y\longrightarrow 0$

$$\Longrightarrow \quad \lim_{x\to 0}\frac{x}{sin(a+bI_1+cI_2)\,x}=\lim_{y\to 0}\frac{\frac{1}{a+bI_1+cI_2}y}{siny}$$

$$=\frac{1}{a+bI_1+cI_2}\lim_{y\to 0}\frac{y}{siny}$$

$$=\frac{1}{a+bI_1+cI_2}$$

$$=\frac{1}{a}+\left[\frac{-b}{(a+c)(a+b+c)}\right]I_1-\left[\frac{c}{a(a+c)}\right]I_2$$

5) $\lim_{x\to 0}\frac{tan(a+bI_1+cI_2)x}{x}=a+bI_1+cI_2$

6) $\lim_{x\to 0}\frac{x}{tan(a+bI_1+cI_2)x}=\frac{1}{a+bI_1+cI_2}=\frac{1}{a}+\left[\frac{-b}{(a+c)(a+b+c)}\right]I_1-\left[\frac{c}{a(a+c)}\right]I_2$

where a , b, c are real coefficients, $a\neq 0$, $a\neq -c$ and $a\neq -b-c$. We can prove 5 and 6 by the same method in 3, 4

Example 5

1) $\lim_{x\to 0+0I_1+0I_2}\frac{sin(2+I_1+3I_2)\,x}{(1-4I_1+I_2)x}=\frac{2+I_1+3I_2}{1-4I_1+I_2}\lim_{x\to 0+0I_1+0I_2}\frac{sin(2+I_1+3I_2)\,x}{(2+I_1+3I_2)x}$

$$=\frac{2+I_1+3I_2}{1-4I_1+I_2}=2-\frac{11}{2}I_1+\frac{1}{2}I_2$$

2) $\lim_{x\to 0+0I_1+0I_2}\frac{x}{sin(1+5I_1-4I_2)\,x}=\frac{1}{1+5I_1-4I_2}\lim_{x\to 0+0I_1+0I_2}\frac{(1+5I_1-4I_2)x}{sin(1+5I_1-4I_2x}$

$$=\frac{1}{1+5I_1-4I_2}=1+\frac{5}{6}I_1-\frac{4}{3}I_2$$

3) $$\lim_{x\to 0+0I_1+0I_2} \frac{\sin(3+4I_1-4I_2)\,x}{\tan(2-8I_1-4I_2)\,x} = \lim_{x\to 0+0I} \frac{\frac{\sin(3+4I_1-4I_2)\,x}{x}}{\frac{\tan(2-8I_1-4I_2)\,x}{x}}$$

$$= \frac{\lim_{x\to 0+0I_1+0I_2} \frac{\sin(3+4I_1-4I_2)\,x}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{\tan(2-8I_1-4I_2)\,x}{x}}$$

$$= \frac{3+4I_1-4I_2}{2-8I_1-4I_2} = \frac{3}{2} + 4I_1 - I_2$$

4) $$\lim_{x\to 0+0I_1+0I_2} \frac{1-\cos(1+4I_1-I_2)\,x}{x^2} = \lim_{x\to 0+0I_1+0I_2} \frac{2sin^2(1+4I_1-I_2)x}{x^2}$$

$$= 2 \lim_{x\to 0+0I_1+0I_2} \left(\frac{sin(1+4I_1-I_2)x}{x}\right)^2$$

$$= 2(1+4I_1-I_2)^2 = 2 + 32I_1 - 2I_2$$

5) $$\lim_{x\to 0+0I_1+0I_2} \frac{(2+I_1+I_2)x - sin(1+I_1+I_2)\,x}{(2+I_1+2I_2)x} = \lim_{x\to 0+0I_1+0I_2} \left(\frac{(2+I_1+I_2)x}{(2+I_1+2I_2)x} - \frac{sin(1+I_1+I_2)\,x}{(2+I_1+2I_2)x}\right)$$

$$= \lim_{x\to 0+0I_1+0I_2} \left(\frac{2+I_1+I_2}{2+I_1+2I_2} - \frac{sin(1+I_1+I_2)x}{(2+I_1+2I_2)x}\right)$$

$$= \frac{2+I_1+I_2}{2+I_1+2I_2} - \left(\frac{1+I_1+I_2}{2+I_1+2I_2}\right)$$

$$= 1 + \frac{1}{20}I_1 - \frac{1}{4}I_2 - \left(\frac{1}{2} + \frac{1}{10}I_1 + 0I_2\right)$$

$$= \frac{1}{2} + \frac{3}{20}I_1 - \frac{1}{4}I_2$$

2.4 Some special limits

1) $$\lim_{x\to 0+0I_1+0I_2} e^{(a+bI_1+cI_2)x} = 1$$

2) $$\lim_{x\to 0+0I_1+0I_2} \frac{e^{(a+bI_1+cI_2)x}-1}{x} = a + bI_1 + cI_2$$

Proof (2):

put $(a+bI_1+cI_2)x = y$ $\Longrightarrow$ $x = \frac{1}{a+bI_1+cI_2}y$

when $x \longrightarrow 0+0I_1+0I_2$ then: $y \longrightarrow 0+0I_1+0I_2$

$$\Longrightarrow \lim_{x\to 0+0I_1+0I_2} \frac{e^{(a+bI_1+cI_2)x}-1}{x} = \lim_{x\to 0+0I_1+0I_2} \frac{e^y - 1}{\frac{1}{a+bI_1+cI_2}y}$$

$$= (a + bI_1 + cI_2) \lim_{x\to 0+0I_1+0I_2} \frac{e^y - 1}{y} = (a + bI_1 + cI_2)(1) = a + bI_1 + cI_2$$

3) $\lim_{x\to 0+0I_1+0I_2} \frac{ln(1 + (a + bI_1 + cI_2)x)}{x} = a + bI_1 + cI_2$

Proof (3):

Put $(a + bI_1 + cI_2)x = y \quad \Rightarrow \quad x = \frac{1}{a+bI_1+cI_2} y$

When $x \longrightarrow 0 + 0I_1 + 0I_2$ then: $y \longrightarrow 0 + 0I_1 + 0I_2$

$$\Rightarrow \lim_{x\to 0+0I_1+0I_2} \frac{ln(1 + (a + bI_1 + cI_2)x)}{x} = \lim_{x\to 0+0I_1+0I_2} \lim_{y\to 0+0I} \frac{ln(1 + y)}{\frac{1}{a + bI_1 + cI_2} y}$$

$$= (a + bI_1 + cI_2) \lim_{x\to 0+0I_1+0I_2} \frac{ln(1 + y)}{y} = (a + bI_1 + cI_2)(1)$$

$$= a + bI_1 + cI_2$$

4) $\lim_{x\to 0+0I_1+0I_2} \frac{(a + bI_1 + cI_2)^x - 1}{x} = ln(a + bI_1 + cI_2)$

$$= ln\, a + [ln(a + b + c) - ln(a + c)]I_1 + [ln(a + c) - ln\, a]I_2$$

where: $a > 0$, $a + b > 0$, $a + b + c > 0$

Proof (4):

Put $(a + bI_1 + cI_2)^x - 1 = y \quad \Rightarrow \quad (a + bI_1 + cI_2)^x = y + 1$

$$ln(a + bI_1 + cI_2)^x = ln(1 + y)$$

$$x \; ln(a + bI_1 + cI_2) = ln(1 + y)$$

$$x = \frac{1}{ln(a + bI_1 + cI_2)} ln(1 + y)$$

When $x \longrightarrow 0 + 0I_1 + 0I_2$ then: $y \longrightarrow 0 + 0I_1 + 0I_2$

$$\Rightarrow \lim_{x\to 0+0I_1+0I_2} \frac{(a + bI_1 + cI_2)^x - 1}{x} = \lim_{x\to 0+0I_1+0I_2} \frac{y}{\frac{1}{ln(a + bI_1 + cI_2)} ln(1 + y)}$$

$$= ln(a + bI_1 + cI_2) \lim_{x\to 0+0I_1+0I_2} \frac{y}{ln(1 + y)} = ln(a + bI_1 + cI_2) \; (1) = ln(a + bI_1 + cI_2)$$

$$= ln\, a + [ln(a + b + c) - ln(a + c)]I_1 + [ln(a + c) - ln\, a]I_2$$

Corollary 1

$$\lim_{x\to 0+0I_1+0I_2} \frac{(a+bI_1+cI_2)^x-1}{(r+sI_1+tI_2)^x-1} = \frac{ln(a+bI_1+cI_2)}{ln(r+sI_1+tI_2)}$$

$$= \frac{ln\, a + [ln(a+b+c) - ln(a+c)]I_1 + [ln(a+c) - ln\, a]I_2}{ln\, r + [ln(r+s+t) - ln(r+t)]I_1 + [ln(r+t) - ln\, r]I_2}$$

where: $a > 0$, $a+b > 0$, $a+b+c > 0$ *and* $r > 0$, $r+s > 0$, $r+s+t > 0$

Proof:

$$\lim_{x\to 0+0I_1+0I_2} \frac{\frac{(a+bI_1+cI_2)^x-1}{x}}{\frac{(r+sI_1+tI_2)^x-1}{x}} = \frac{\lim_{x\to 0+0I_1+0I_2} \frac{(a+bI_1+cI_2)^x-1}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{(r+sI_1+tI_2)^x-1}{x}}$$

$$= \frac{ln(a+bI_1+cI_2)}{ln(r+sI_1+tI_2)}$$

$$= \frac{ln\, a + [ln(a+b+c) - ln(a+c)]I_1 + [ln(a+c) - ln\, a]I_2}{ln\, r + [ln(r+s+t) - ln(r+t)]I_1 + [ln(r+t) - ln\, r]I_2}$$

Example 6

1) $\lim_{x\to 0+0I_1+0I_2} e^{(9+12I_1-15I_2)x} = 1$

2) $\lim_{x\to 0+0I_1+0I_2} \frac{e^{(-5+13I_1-I_2)x}-1}{x} = -5+13I_1-I_2$

3) $\lim_{x\to 0+0I_1+0I_2} \frac{(2+4I_1+7I_2)^x-1}{x} = ln(2+4I_1+7I_2)$

$$= ln\, 2 + [ln\, 13 - ln\, 6]I_1 + [ln\, 9 - ln\, 2]I_2$$

$$= ln\, 2 + [ln\, 13 - ln\, 6]I_1 + [2\, ln\, 3 - ln\, 2]I_2$$

4) $\lim_{x\to 0+0I_1+0I_2} \frac{(1+I_1+I_2)^x-1}{e^{(1+2I_1+3I_2)x}-1} = \lim_{x\to 0+0I_1+0I_2} \frac{\frac{(1+I_1+I_2)^x-1}{x}}{\frac{e^{(1+2I_1+3I_2)x}-1}{x}}$

$$= \frac{\lim_{x\to 0+0I_1+0I_2} \frac{(1+I_1+I_2)^x-1}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{e^{(1+2I_1+3I_2)x}-1}{x}} = \frac{ln(1+I_1+I_2)}{1+2I_1+3I_2}$$

$$= \left(1 - \frac{1}{12}I_1 - \frac{3}{4}I_2\right)([ln\, 3 - ln\, 2]I_1 + [ln\, 2]I_2)$$

5) $\lim_{x\to 0+0I_1+0I_2} \frac{(2+3I_1+4I_2)^x-1}{(2+I_1+I_2)^x-1} = \frac{ln(2+3I_1+4I_2)}{ln(2+I_1+I_2)} = \frac{ln\, 2 + [ln\, 9 - ln\, 6]I_1 + [ln\, 6 - ln\, 2]I_2}{ln\, 2 + [ln\, 4 - ln\, 3]I_1 + [ln\, 3 - ln\, 2]I_2}$

$$= \frac{ln\, 2 + \left[ln\frac{3}{2}\right]I_1 + [ln\, 3]I_2}{ln\, 2 + \left[ln\frac{4}{3}\right]I_1 + \left[ln\frac{3}{2}\right]I_2}$$

$$= 1 + \left[\frac{ln\,3 + ln\frac{9}{4} - ln\frac{2}{3} - ln\,4}{ln\,3\,.\,ln\,4}\right] I_1 + \left[\frac{ln\,6 - ln\,3}{ln\,2\,.\,ln\,3}\right] I_2$$

$$= 1 + \left[\frac{ln\frac{81}{32}}{ln\,3\,.\,ln\,4}\right] I_1 + \left[\frac{1}{ln\,3}\right] I_2$$

6) $$\lim_{x\to 0+0I_1+0I_2} \frac{(6+5I_1-2I_2)^x - 1}{x} = ln(6+5I_1-2I_2)$$

$$= ln\,6 + [ln\,9 - ln\,4]I_1 + [ln\,4 - ln\,6]I_2$$

$$= ln\,6 + [3\,ln\,2 - 2\,ln\,2]I_1 + [2\,ln\,2 - ln\,6]I_2$$

7) $$\lim_{x\to 0+0I_1+0I_2} \frac{ln(1+(6+6I_1-6I_2)x)}{x} = 6 + 6I_1 - 6I_2$$

8) $$\lim_{x\to 0+0I_1+0I_2} \frac{e^{(3+I_1-I_2)x} - 1}{sin(3+2I_1+I_2)\,x} = \lim_{x\to 0+0I_1+0I_2} \frac{\frac{e^{(3+I_1-I_2)x} - 1}{x}}{\frac{sin(3+2I_1+I_2)\,x}{x}}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{\frac{e^{(3+I_1-I_2)x} - 1}{x}}{\frac{sin(3+2I_1+I_2)\,x}{x}} = \frac{\lim_{x\to 0+0I_1+0I_2} \frac{e^{(3+I_1-I_2)x} - 1}{x}}{\lim_{x\to 0+0I_1+0I_2} \frac{sin(3+2I_1+I_2)\,x}{x}}$$

$$= \frac{3+I_1-I_2}{3+2I_1+I_2} = 1 + 0I_1 - \frac{1}{2}I_2$$

9) $$\lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - (6+3I_1+I_2)^x}{x} = \lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - (6+3I_1+I_2)^x - 1 + 1}{x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - 1 - ((6+3I_1+I_2)^x - 1)}{x}$$

$$= \lim_{x\to 0+0I_1+0I_2} \frac{(7+I_1+4I_2)^x - 1}{x} - \lim_{x\to 0+0I_1+0I_2} \frac{(6+3I_1+I_2)^x - 1}{x}$$

$$= ln(7+I_1+4I_2) - ln(6+3I_1+I_2) = ln\left(\frac{7+I_1+4I_2}{6+3I_1+I_2}\right)$$

$$= ln\left(\frac{7}{6} - \frac{13}{35}I_1 + \frac{17}{42}I_2\right) = ln\frac{7}{6} + \left[ln\frac{6}{5} - ln\frac{11}{7}\right] I_1 + \left[ln\frac{11}{7} - ln\frac{7}{6}\right] I_2$$

3. Conclusions

One of the key concepts in calculus is limits. Its focus is on the study of derivation by studying the fundamental ideas of infinitesimal quantities. This was the goal of putting forward the idea of The limits of 2- refined neutrosophic in this paper. Several methods for solving The limits of 2- refined neutrosophic were discussed, in addition to presenting special rules to facilitate finding these limits. Also, We obtained the same results by solving the examples in different ways, such as L'Hôpital's rule.

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Combined TOPSIS Technique for MAGDM Based on the Distance Measures and CRITIC under Single-Valued Neutrosophic Sets and Applications to Quality Evaluation of Whole Process Engineering Consulting Service Modes

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Abstract: According to survey data released by the National Bureau of Statistics, from 2011 to 2022, the total output value of China's construction industry showed an increasing trend year by year. From 2018 to 2021, infrastructure investment has always maintained a positive growth, indicating that the current construction industry is in a rapid development stage. In order to improve the development quality of the construction industry as much as possible, it is of great significance to develop the whole Process engineering consulting service model. The quality evaluation of whole process engineering consulting service modes is MAGDM. The single-valued neutrosophic sets (SVNSs) is useful tool to cope with uncertain information during the quality evaluation of whole process engineering consulting service modes. In this paper, the single-valued neutrosophic number combined TOPSIS (SVNN-CTOPSIS) model based on single-valued neutrosophic number Hamming distances (SVNNHD) and single-valued neutrosophic number Euclidean distance (SVNNED) is formed to cope with the MAGDM. The CRITIC model is utilized to obtain the weight numbers in light with the SVNNHD and SVNNED under SVNSs. Finally, numerical example and comparative analysis for quality evaluation of whole process engineering consulting service modes is utilized to verify SVNN-CTOPSIS model. The main contributions of this study are formed: (1) the CRITIC model is formed to obtain the weight numbers in light with SVNNHD and SVNNED;

(2) the SVNN-CTOPSIS model is formed in light with SVNNHD and SVNNED under SVNNs; (3) Finally, numerical example and comparative analysis for quality evaluation of whole process engineering consulting service modes is employed to verify SVNN-CTOPSIS model.

Keywords: MAGDM; SVNSs; TOPSIS; CRITIC model; performance evaluation

1. Introduction

Against the backdrop of rapid development in the construction industry, the investment entities of engineering projects are developing in a diversified direction[1, 2]. Traditional "fragmented" consulting services are no longer able to effectively meet the investment needs of investors. In the face of such situations, the Opinions on Promoting the Sustainable and Healthy Development of the Construction Industry (Guo Ban Fa [2017] No. 19) put forward for the first time the idea of cultivating the whole process engineering consulting, which has promoted the transformation of China's engineering consulting work from the past professional division of labor model to the whole process, cross stage integration model[3-5]. In the practical application of the consortium consulting service model[6, 7], a project will be handed over to two or more consulting units to jointly carry out the whole process engineering consulting services of engineering projects, and the lead party will be responsible for the coordination of the consulting business during the service development process. In the current process engineering consulting service development process, there are relatively many factors to apply the consortium consulting service model. Common factors include large project scale, complex work content, and lack of professional service qualifications of some consulting units. Considering that the consortium consulting service model needs to unite multiple consulting units in the application process, in order to improve the reliability of consulting services and the quality of the division of responsibilities, the management relationship and responsibilities of each consulting unit need to be noted in the contract before the application of the consortium consulting service model, so as to lay a good foundation for the orderly development of the whole process engineering consulting services. The integrated whole process consulting service model is a work mode in which a consulting unit is responsible for coordinating the entire process consulting work of engineering projects[8-10]. Due to the comprehensive range of consulting services covered by the service model, only a few consulting units in the current consulting service industry have the qualifications and experience to apply the consulting service model. Therefore, compared to other consulting service techniques, The application experience of the integrated whole process consulting service model is relatively limited. However, due to the fact that the consulting service model is mainly managed by one unit, the difficulty of consulting service management is relatively low, which can effectively improve the integration level of consulting services[11-13]. The "1+N" part of the combined consulting service mode in the whole Process engineering consulting service work is mainly composed of one consulting unit, which distributes a number of consulting businesses to different consulting units in the form of contracting or combination to complete the consulting services. In this process, the consulting unit is responsible for coordinating the whole Process engineering consulting business. In the current work process of consulting companies, the

"1+N" combined consulting service model is mainly applied to project research and decision-making, bidding agency, survey and design, engineering supervision, cost decision-making, and other work[14-16]. The business combination techniques mainly include the combination of design units and cost analysis units, design units and supervision units, supervision units and bidding agency units Combining design units with survey and supervision units, etc. For the current construction activities of engineering projects, the "1+N" part combination consulting service model is highly similar to the traditional single consulting contract business[14-18]. Therefore, this work model has been widely used in the early stage of the whole process engineering consulting service. In a word, in the process of gradually advancing the supply side structural reform, the construction industry is carrying out the organizational model reform represented by the whole Process engineering consulting. In order to make the whole Process engineering consulting service model better meet the needs of the current development of the construction industry, based on clarifying the shortcomings of the traditional professional engineering consulting service model, the actual situation of the project is analyzed, the development of a more complete and reliable process engineering consulting service model can provide strong support for the smooth implementation of project management[19-21].

With rapid development of GDM issues, MAGDM techniques have greatly attracted academic attention [22-30]. In order to put forward the objective things through employing precise numbers. Zadeh [31] creatively put forward the fuzzy sets (FSs) theory. Atanassov [32] creatively put forward intuitionistic fuzzy sets (IFSs). However, IFSs didn't put forward uncertain membership. In order to manage a more efficient technique, Smarandache [33] creatively put forward the neutrosophic sets(NSs). Zenat, Mahmoud and Amal [34] put forward the TOPSIS model for green supply chain practices under SVNSs. Ahmed, Nehal and Ibrahim [35] put forward the CRITIC model for coping with the product design in virtual reality under SVNSs. Abduallah et al. [36] put forward the AHP-VIKOR model for coping with the Supply chain (SC) networks with neutrosophic theory. Karam et al. [37] put forward the TOPSIS for assessment quality of suppliers under SVNSs. M. Sabry [38] put forward the CRITIC-EDAS model for urban energy internet assessment by type 2 neutrosophic numbers (T2NNs). Abduallah et al. [39] put forward the MEREC-CoCoSo for coping with the autonomous vehicles and distributed resources using type-2 neutrosophic numbers (T2NN). The quality evaluation of whole process engineering consulting service modes is the real-life MAGDM [40-44]. The SVNSs [45] is useful technique to cope with uncertain information during the quality evaluation of whole process engineering consulting service modes. Furthermore, many techniques employed the TOPSIS model [46-49] and CRITIC model [50-54] separately to solve the MAGDM. Unfortunately, few valuable existing works were managed the combined TOPSIS based on SVNNHD and SVNNED under SVNSs. The main objective of this study is to cope with MAGDM through employing the SVNN-CTOPSIS model with SVNNHD and SVNNED model. Finally, numerical example and comparative analysis for quality evaluation of whole process engineering consulting service modes is utilized to verify SVNN-CTOPSIS model. The main research goals and motivation of this study are formed: (1) the CRITIC model is formed to obtain the weight numbers in light with SVNNHD and SVNNED; (2) the SVNN-CTOPSIS model is formed in light with SVNNHD and SVNNED under SVNNs; (3) Finally, numerical example and comparative analysis for quality evaluation of whole process engineering consulting service modes is employed to verify SVNN-CTOPSIS model.

The remaining sections is formed. The SVNSs is formed in Sect 2. The SVNN-CTOPSIS model is formed for MAGDM in Sect. 3. The quality evaluation of whole process engineering consulting service modes and some comparative analyses is formed to verify the SVNN-CTOPSIS model in Sect. 4. The conclusion is formed in Sect. 5.

2. Preliminaries

Wang et al. [45] formed the SVNSs.

Definition 1 [45]. The SVNSs is formed:

$$DA=\left\{\left(\vartheta, DT_A\left(\vartheta\right), DI_A\left(\vartheta\right), DF_A\left(\vartheta\right)\right)\middle|\vartheta\in\Theta\right\} \tag{1}$$

where $DT_A\left(\vartheta\right), DI_A\left(\vartheta\right), DF_A\left(\vartheta\right)$ depicts truth membership, indeterminacy membership and falsity membership, $DT_A\left(\vartheta\right), DI_A\left(\vartheta\right), DF_A\left(\vartheta\right)\in\left[0,1\right]$, $0\le DT_A\left(\vartheta\right)+DI_A\left(\vartheta\right)+DF_A\left(\vartheta\right)\le 3$.

Definition 2 [55]. The score value information (SVI) of $DA=\left(DT_A, DI_A, DF_A\right)$ and $DB=\left(DT_B, DI_B, DF_B\right)$ is formed:

$$SVI\left(DA\right)=\frac{\left(2+DT_A-DI_A-DF_A\right)}{3}, SVI\left(DA\right)\in\left[0,1\right]. \tag{2}$$

$$SVI\left(DB\right)=\frac{\left(2+DT_B-DI_B-DF_B\right)}{3}, SVI\left(DB\right)\in\left[0,1\right]. \tag{3}$$

Definition 3 [55]. The accuracy value information (AVI) of $DA=\left(DT_A, DI_A, DF_A\right)$ and $DB=\left(DT_B, DI_B, DF_B\right)$ is formed:

$$AVI\left(DA\right)=\frac{1+DT_A-DF_A}{2},\ AVI\left(DA\right)\in\left[0,1\right]. \tag{4}$$

$$AVI\left(DB\right)=\frac{1+DT_B-DF_B}{2},\ AVI\left(DB\right)\in\left[0,1\right]. \tag{5}$$

Peng et al. [55] formed the order between two SVNNs.

Definition 4[55]. Let $DA=\left(DT_A,DI_A,DF_A\right)$ and $DB=\left(DT_B,DI_B,DF_B\right)$, let $SVI\left(DA\right)=\frac{\left(2+DT_A-DI_A-DF_A\right)}{3}$ and $SVI\left(DB\right)=\frac{\left(2+DT_B-DI_B-DF_B\right)}{3}$, and let $AVI\left(DA\right)=\frac{1+DT_A-DF_A}{2}$ and $AVI\left(DB\right)=\frac{1+DT_B-DF_B}{2}$, if $SVI\left(DA\right)<SVI\left(DB\right)$, then: $DA<DB$; if $SVI\left(DA\right)=SVI\left(DB\right)$, then: (1)if $AVI\left(DA\right)=AVI\left(DB\right)$, then $DA=DB$; (2) if $AVI\left(DA\right)>AVI\left(DB\right)$, then: $DA<DB$.

Definition 5[45]. Let $DA=\left(DT_A,DI_A,DF_A\right)$ and $DB=\left(DT_B,DI_B,DF_B\right)$ be SVNNs, the following operations are formed:

(1) $DA\oplus DB=\left(DT_A+DT_B-DT_ADT_B,DI_ADI_B,DF_ADF_B\right)$;

(2) $DA\otimes DB=\left(DT_ADT_B,DI_A+DI_B-DI_ADI_B,DF_A+DF_B-DF_ADF_B\right)$;

(3) $\pi DA=\left(1-\left(1-DT_A\right)^{\pi},\left(DI_A\right)^{\pi},\left(DF_A\right)^{\pi}\right),\pi>0;$

(4) $\left(DA\right)^{\pi}=\left(\left(DT_A\right)^{\pi},\left(DI_A\right)^{\pi},1-\left(1-DF_A\right)^{\pi}\right),\pi>0.$

Definition 6 [56]. Let $DA=\left(DT_A,DI_A,DF_A\right)$ and $DB=\left(DT_B,DI_B,DF_B\right)$, then SVNN Hamming distance (SVNNHD) and SVNN Euclidean distance (SVNNED) between $DA=\left(DT_A,DI_A,DF_A\right)$ and $DB=\left(DT_B,DI_B,DF_B\right)$ is formed:

$$SVNNHD\left(DA,DB\right)=\frac{1}{3}\left(\left|DT_A-DT_B\right|+\left|DI_A-DI_B\right|+\left|DF_A-DF_B\right|\right) \tag{6}$$

$$SVNNED\left(DA,DB\right)=\sqrt{\frac{1}{3}\left(\left|DT_A-DT_B\right|^2+\left|DI_A-DI_B\right|^2+\left|DF_A-DF_B\right|^2\right)} \tag{7}$$

The SVNNWA & SVNNWG model are formed.

Definition 8 [55]. If $DA_j=\left(DT_j,DI_j,DF_j\right)$, the SVNNWA operator is formed:

$$\begin{aligned}\text{SVNNWA}_{dw}\left(DA_1,DA_2,\cdots,DA_n\right)&=\overset{n}{\underset{j=1}{\oplus}}\left(dw_jDA_j\right)\\&=\left(1-\prod_{j=1}^{n}\left(1-DT_j\right)^{dw_j},\prod_{j=1}^{n}\left(DI_j\right)^{dw_j},\prod_{j=1}^{n}\left(DF_j\right)^{dw_j}\right)\end{aligned} \tag{8}$$

with weight $dw = \left(dw_1, dw_2, ..., dw_n\right)^T$, $\sum_{j=1}^{n} dw_j = 1.$

Definition 9 [55]. If $DA_j = \left(DT_j, DI_j, DF_j\right)$, the SVNNWG model is formed::

$$\begin{aligned} &\mathrm{SVNNWG}_{dw}\left(DA_1, DA_2, \cdots, DA_n\right) = \underset{j=1}{\overset{n}{\otimes}}\left(DA_j\right)^{dw_j} \\ &= \left(\prod_{j=1}^{n}\left(DT_j\right)^{dw_j}, 1-\prod_{j=1}^{n}\left(1-DI_j\right)^{dw_j}, 1-\prod_{j=1}^{n}\left(1-DF_j\right)^{dw_j}\right) \end{aligned} \tag{9}$$

with weight $dw = \left(dw_1, dw_2, ..., dw_n\right)^T$, $\sum_{j=1}^{n} dw_j = 1.$

3.

4. SVNN-CTOPSIS FOR MAGDM IN LIGHT WITH SVNNHD AND SVNNED

Then, the SVNN-CTOPSIS model is formed for MAGDM. Let $DY = \left(DY_1, DY_2, \ldots, DY_m\right)$ be alternatives. Let $DZ = \left(DZ_1, DZ_2, \ldots, DZ_n\right)$ be attributes, $dw = \left\{dw_1, dw_2, \ldots, dw_n\right\}$ be weight for $DZ = \left(DZ_1, DZ_2, \ldots, DZ_n\right)$, where $dw_j \in [0,1], \sum_{j=1}^{n} dw_j = 1$. Assume DMs $DX = \left\{DX_1, DX_2, \ldots, DX_l\right\}$ with weight $d\omega = \left\{d\omega_1, d\omega_2, \ldots, d\omega_l\right\}$, $d\omega_k \in [0,1]$, $\sum_{k=1}^{l} d\omega_k = 1$. And $DR^{(k)} = \left(DR_{ij}^{(k)}\right)_{m\times n} = \left(DT_{ij}^{(k)}, DI_{ij}^{(k)}, DF_{ij}^{(k)}\right)_{m\times n}$ is called as group SVNN-matrix. The calculating procedures are formed (See Figure 1).

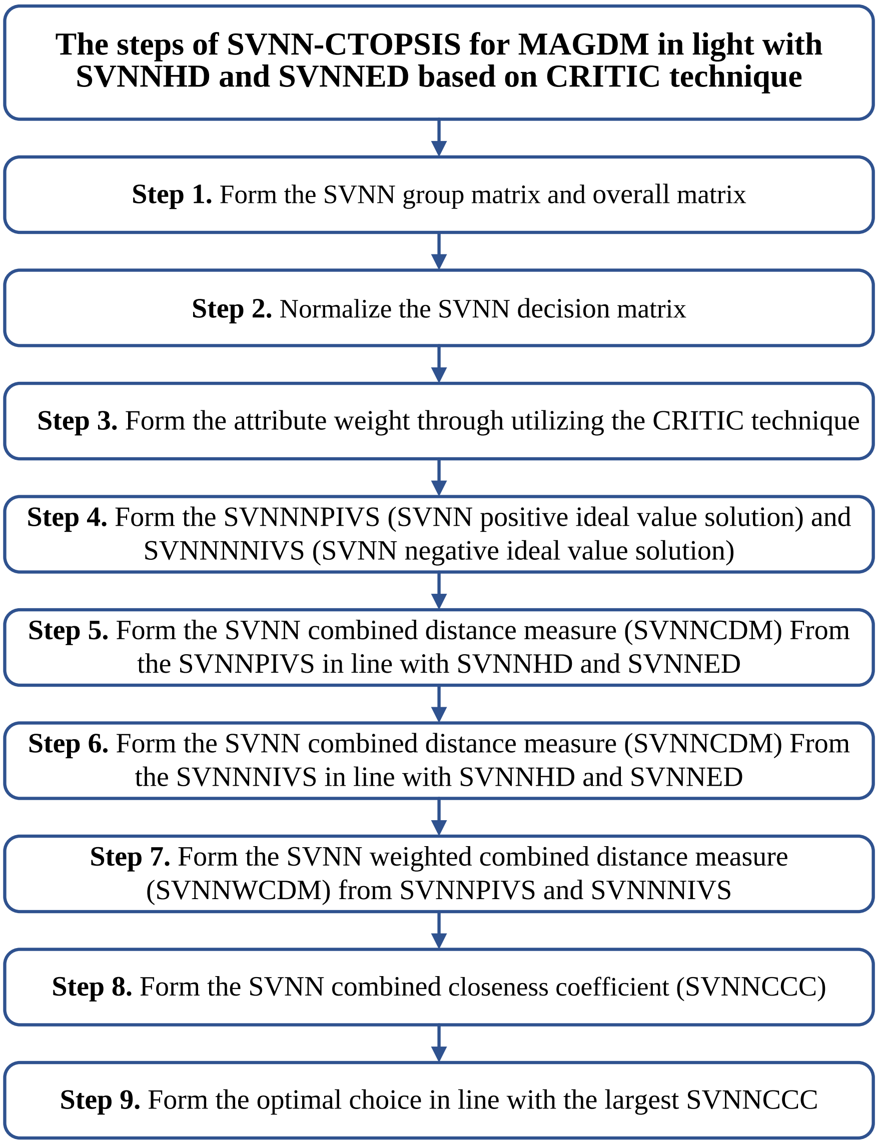

Figure 1. SVNN-CTOPSIS for MAGDM in light with SVNNHD and SVNNED based on CRITIC technique

Step 1. Form group SVNN-matrix $DR^{(k)} = \left(DR_{ij}^{(k)}\right)_{m\times n} = \left(DT_{ij}^{(k)}, DI_{ij}^{(k)}, DF_{ij}^{(k)}\right)_{m\times n}$ and single SVNN-matrix $DR = \left(DR_{ij}\right)_{m\times n}$ through SVNNWG technique.

$$DR^{(k)} = \left[DR_{ij}^{(k)}\right]_{m\times n} = \begin{bmatrix} DR_{11}^{(k)} & DR_{12}^{(k)} & \dots & DR_{1n}^{(k)} \\ DR_{21}^{(k)} & DR_{22}^{(k)} & \dots & DR_{2n}^{(k)} \\ \vdots & \vdots & \vdots & \vdots \\ DR_{m1}^{(k)} & DR_{m2}^{(k)} & \dots & DR_{mn}^{(k)} \end{bmatrix} \tag{10}$$

$$DR = \left[DR_{ij} \right]_{m\times n} = \begin{bmatrix} DR_{11} & DR_{12} & \dots & DR_{1n} \\ DR_{21} & DR_{22} & \dots & DR_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ DR_{m1} & DR_{m2} & \dots & DR_{mn} \end{bmatrix} \tag{11}$$

$$\begin{aligned} DR_{ij} &= \left(DT_{ij}, DI_{ij}, DF_{ij} \right) \\ &= \left(1-\prod_{k=1}^{l}\left(1-DT_{ij}^{(k)}\right)^{d\omega_k}, \prod_{k=1}^{l}\left(DI_{ij}^{(k)}\right)^{d\omega_k}, \prod_{k=1}^{l}\left(DF_{ij}^{(k)}\right)^{d\omega_k} \right) \end{aligned} \tag{12}$$

Step 2. Form normalized $DR^N = \left[DR_{ij}^N \right]_{m\times n}$ in line with $DR = \left[DR_{ij} \right]_{m\times n}$

$$\begin{aligned} DR_{ij}^N &= \left(DT_{ij}^N, DI_{ij}^N, DF_{ij}^N \right) \\ &= \begin{cases} \left(DT_{ij}, DI_{ij}, DF_{ij} \right), & DZ_j \ is \text{ the } benefit\ attribute \\ \left(DF_{ij}, DI_{ij}, DT_{ij} \right), & DZ_j \ is \text{ the cost } attribute \end{cases} \end{aligned} \tag{13}$$

Step 3. Form the weight information through utilizing the CRITIC technique.

The CRITIC [57] is utilized to put forward the weights information.

(1) The SVNN correlation decision coefficient (SVNNCDC) is formed.

$$SVNNCDC_{jt} = \frac{\sum_{i=1}^{m}\left(\varphi\left(DSVNN_{ij}\right)-\varphi\left(DSVNN_{j}\right)\right)\left(\varphi\left(DSVNN_{it}\right)-\varphi\left(DSVNN_{t}\right)\right)}{\sqrt{\sum_{i=1}^{m}\left(\varphi\left(DSVNN_{ij}\right)-\varphi\left(DSVNN_{j}\right)\right)^2}\sqrt{\sum_{i=1}^{m}\left(\varphi\left(DSVNN_{it}\right)-\varphi\left(DSVNN_{t}\right)\right)^2}},$$

$$j,t = 1,2,\dots,n, \tag{14}$$

where

$$\varphi\left(DSVNN_j\right) = \frac{1}{2m}\sum_{i=1}^{m}\left(SVI\left(DR_{ij}^N\right)+AVI\left(DR_{ij}^N\right)\right),$$

$$\varphi\left(DSVNN_t\right) = \frac{1}{2m}\sum_{i=1}^{m}\left(SVI\left(DR_{it}^N\right)+AVI\left(DR_{it}^N\right)\right),$$

$$\varphi\left(DSVNN_{ij}\right) = \frac{1}{2}\left(SVI\left(DR_{ij}^N\right)+AVI\left(DR_{ij}^N\right)\right),$$

$$\varphi\left(DSVNN_{it}\right) = \frac{1}{2}\left(SVI\left(DR_{it}^N\right)+AVI\left(DR_{it}^N\right)\right).$$

(2) Form the SVNN standard deviation numbers (SVNNSDN).

$$SVNNSDN_j = \sqrt{\frac{1}{m-1}\sum_{i=1}^{m}\left(\varphi\left(DSVNN_{ij}\right)-\varphi\left(DSVNN_j\right)\right)^2} \tag{15}$$

(3) Form the attribute weight information.

$$dw_j = \frac{SVNNSDN_j \sum_{t=1}^{n}\left(1-SVNNCDC_{jt}\right)}{\sum_{j=1}^{n}\left(SVNNSDN_j \sum_{t=1}^{n}\left(1-SVNNCDC_{jt}\right)\right)} \tag{16}$$

Step 4. Form the SVNNNPIVS (SVNN positive ideal value solution) and SVNNNNIVS (SVNN negative ideal value solution):

$$SVNNPIVS = \left(SVNNPIVS_1, SVNNPIVS_2, \cdots, SVNNPIVS_n\right) \tag{17}$$

$$SVNNNIVS = \left(SVNNNIVS_1, SVNNNIVS_2, \cdots, SVNNNIVS_n\right) \tag{18}$$

$$SVNNPIVS_j = \left(DT_j^{N+}, DI_j^{N+}, DF_j^{N+}\right) \tag{19}$$

$$SVNNNIVS_j = \left(DT_j^{N-}, DI_j^{N-}, DF_j^{N-}\right) \tag{20}$$

$$SVI\left(SVNNPIVS_j\right) = \max_i SVI\left(DR_{ij}^N\right) = \max_i SVI\left(DT_{ij}^N, DI_{ij}^N, DF_{ij}^N\right) \tag{21}$$

$$SVI\left(SVNNNIVS_j\right) = \min_i SVI\left(DR_{ij}^N\right) = \min_i SVI\left(DT_{ij}^N, DI_{ij}^N, DF_{ij}^N\right) \tag{22}$$

Step 5. Construct the SVNN combined distance measure (SVNNCDM) between $DR_{ij}^N = \left(DT_{ij}^N, DI_{ij}^N, DF_{ij}^N\right)$ and $SVNNPIVS_j = \left(DT_j^{N+}, DI_j^{N+}, DF_j^{N+}\right)$ in line with SVNNHD and SVNNED.

$$\begin{aligned} & SVNNCDM\left(DR_{ij}^N, SVNNPIVS_j\right) \\ & = \frac{1}{2}\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N+}\right| + \left|DI_{ij}^N - DI_j^{N+}\right| + \left|DF_{ij}^N - DF_j^{N+}\right|\right) \\ + \sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N+}\right|^2 + \left|DI_{ij}^N - DI_j^{N+}\right|^2 + \left|DF_{ij}^N - DF_j^{N+}\right|^2\right)} \end{array}\right) \end{aligned} \tag{23}$$

Step 6. Construct the SVNN combined distance measure (SVNNCDM) between $DR_{ij}^N = \left(DT_{ij}^N, DI_{ij}^N, DF_{ij}^N\right)$ and $SVNNNIVS_j = \left(DT_j^{N-}, DI_j^{N-}, DF_j^{N-}\right)$ in line with SVNNHD and SVNNED.

$$
\begin{aligned}
&SVNNCDM\left(DR_{ij}^{N}, SVNNNIVS_j\right)\\
&=\frac{1}{2}\left(\begin{array}{l}\frac{1}{3}\left(\left|DT_{ij}^{N}-DT_j^{N-}\right|+\left|DI_{ij}^{N}-DI_j^{N-}\right|+\left|DF_{ij}^{N}-DF_j^{N-}\right|\right)\\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^{N}-DT_j^{N-}\right|^2+\left|DI_{ij}^{N}-DI_j^{N-}\right|^2+\left|DF_{ij}^{N}-DF_j^{N-}\right|^2\right)}\end{array}\right)
\end{aligned} \tag{24}
$$

Step 7. Construct the SVNN weighted combined distance measure (SVNNWCDM) between $DR_{ij}^{N}=\left(DT_{ij}^{N}, DI_{ij}^{N}, DF_{ij}^{N}\right)$ and $SVNNPIVS_j=\left(DT_j^{N+}, DI_j^{N+}, DF_j^{N+}\right)$ in line with SVNNHD and SVNNED and SVNN weighted combined distance measure (SVNNWCDM) between $DR_{ij}^{N}=\left(DT_{ij}^{N}, DI_{ij}^{N}, DF_{ij}^{N}\right)$ and $SVNNNIVS_j=\left(DT_j^{N-}, DI_j^{N-}, DF_j^{N-}\right)$ in line with SVNNHD and SVNNED.

$$
\begin{aligned}
&SVNNWCDM\left(DR_i^{N}, SVNNPIVS\right)\\
&=\sum_{j=1}^{n}\left(dw_j SVNNCDM\left(DR_{ij}^{N}, SVNNPIVS_j\right)\right)\\
&=\frac{1}{2}\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l}\frac{1}{3}\left(\left|DT_{ij}^{N}-DT_j^{N+}\right|+\left|DI_{ij}^{N}-DI_j^{N+}\right|+\left|DF_{ij}^{N}-DF_j^{N+}\right|\right)\\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^{N}-DT_j^{N+}\right|^2+\left|DI_{ij}^{N}-DI_j^{N+}\right|^2+\left|DF_{ij}^{N}-DF_j^{N+}\right|^2\right)}\end{array}\right)\right)
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
&SVNNWCDM\left(DR_i^{N}, SVNNNIVS\right)\\
&=\sum_{j=1}^{n}\left(dw_j SVNNCDM\left(DR_{ij}^{N}, SVNNNIVS_j\right)\right)\\
&=\frac{1}{2}\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l}\frac{1}{3}\left(\left|DT_{ij}^{N}-DT_j^{N-}\right|+\left|DI_{ij}^{N}-DI_j^{N-}\right|+\left|DF_{ij}^{N}-DF_j^{N-}\right|\right)\\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^{N}-DT_j^{N-}\right|^2+\left|DI_{ij}^{N}-DI_j^{N-}\right|^2+\left|DF_{ij}^{N}-DF_j^{N-}\right|^2\right)}\end{array}\right)\right)
\end{aligned} \tag{26}
$$

Step 8. Form the SVNN combined closeness coefficient (SVNNCCC):

$$SVNNCCC_i = \frac{SVNNWCDM\left(DR_i^N, SVNNNIVS\right)}{\left(\begin{array}{l} SVNNWCDM\left(DR_i^N, SVNNNIVS\right) \\ +SVNNWCDM\left(DR_i^N, SVNNPIVS\right) \end{array}\right)}$$

$$= \frac{\sum_{j=1}^{n}\left(dw_j SVNNCDM\left(DR_{ij}^N, SVNNNIVS_j\right)\right)}{\left(\begin{array}{l} \sum_{j=1}^{n}\left(dw_j SVNNCDM\left(DR_{ij}^N, SVNNNIVS_j\right)\right) \\ +\sum_{j=1}^{n}\left(dw_j SVNNCDM\left(DR_{ij}^N, SVNNPIVS_j\right)\right) \end{array}\right)}$$

$$= \frac{\frac{1}{2}\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right| + \left|DI_{ij}^N - DI_j^{N-}\right| + \left|DF_{ij}^N - DF_j^{N-}\right|\right) \\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right|^2 + \left|DI_{ij}^N - DI_j^{N-}\right|^2 + \left|DF_{ij}^N - DF_j^{N-}\right|^2\right)} \end{array}\right)\right)}{\left(\begin{array}{l} \frac{1}{2}\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right| + \left|DI_{ij}^N - DI_j^{N-}\right| + \left|DF_{ij}^N - DF_j^{N-}\right|\right) \\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right|^2 + \left|DI_{ij}^N - DI_j^{N-}\right|^2 + \left|DF_{ij}^N - DF_j^{N-}\right|^2\right)} \end{array}\right)\right) \\ +\frac{1}{2}\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N+}\right| + \left|DI_{ij}^N - DI_j^{N+}\right| + \left|DF_{ij}^N - DF_j^{N+}\right|\right) \\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N+}\right|^2 + \left|DI_{ij}^N - DI_j^{N+}\right|^2 + \left|DF_{ij}^N - DF_j^{N+}\right|^2\right)} \end{array}\right)\right) \end{array}\right)}$$

$$= \frac{\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right| + \left|DI_{ij}^N - DI_j^{N-}\right| + \left|DF_{ij}^N - DF_j^{N-}\right|\right) \\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right|^2 + \left|DI_{ij}^N - DI_j^{N-}\right|^2 + \left|DF_{ij}^N - DF_j^{N-}\right|^2\right)} \end{array}\right)\right)}{\left(\begin{array}{l} \sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right| + \left|DI_{ij}^N - DI_j^{N-}\right| + \left|DF_{ij}^N - DF_j^{N-}\right|\right) \\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N-}\right|^2 + \left|DI_{ij}^N - DI_j^{N-}\right|^2 + \left|DF_{ij}^N - DF_j^{N-}\right|^2\right)} \end{array}\right)\right) \\ +\sum_{j=1}^{n}\left(dw_j\left(\begin{array}{l} \frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N+}\right| + \left|DI_{ij}^N - DI_j^{N+}\right| + \left|DF_{ij}^N - DF_j^{N+}\right|\right) \\ +\sqrt{\frac{1}{3}\left(\left|DT_{ij}^N - DT_j^{N+}\right|^2 + \left|DI_{ij}^N - DI_j^{N+}\right|^2 + \left|DF_{ij}^N - DF_j^{N+}\right|^2\right)} \end{array}\right)\right) \end{array}\right)} \quad (27)$$

Step 9. Form the optimal choice in line with the largest SVNNCCC.

4. Illustrative example and comparative analysis

4.1 Illustrative example

Considering that the promotion time of the whole process consulting service model in the consulting service industry in China is relatively short, some consulting service units have insufficient understanding of this work model. In actual work, the whole process consulting service model cannot achieve satisfactory work results for owners. In order to improve the application effect of the whole Process engineering consulting service model, in the current development process of the consulting service industry, consulting service units can improve their own working ability by optimizing their own ideas, implementing corresponding rules and policies, building talent teams, carrying out resource integration work, etc., and provide impetus for the implementation of the whole process consulting service projects. Although the National Development and Reform Commission and the Ministry of Housing and Urban Rural Development jointly issued the Guiding Opinions on Promoting the Development of Process engineering consulting Services, the housing construction management departments in most regions have insufficient awareness of the policies and the degree of implementation is not deep enough, so that some government public resource service platforms have not opened businesses related to bidding transactions, In the process of engineering project promotion, only the segmented engineering consulting service transaction module has been established, which has hindered the development of the whole process engineering consulting service. At this stage, in order to effectively solve the above problems and achieve the smooth implementation of relevant policies, the competent construction departments, construction owners and consulting service enterprises in various regions need to fully recognize the advantages of the whole Process engineering consulting service model in the application process for engineering project survey, design, supervision, management and cost services, and the way to open the whole Process engineering consulting service module by the local government's public resource service platform, Ensure the smooth implementation of the whole process engineering consulting project. Then, in the process of carrying out subsequent consulting services, the coherence of engineering consulting services is improved by constructing the entire industry chain, providing support for the improvement of the quality of subsequent engineering construction activities. Although the consulting service enterprises, owners and regulators can carry out their work based on the Guiding Opinions on Promoting the Development of process engineering consulting Services in the actual work process, due to the relatively short application time of the whole process engineering consulting service model in China, each entity may lack a mature contract model in the process of carrying out the whole process engineering consulting services, and disputes arise. In order to effectively solve the above problems, in the current process of engineering consulting services, consulting service units can strengthen communication and negotiation between various entities based on the actual needs of the engineering project and traditional engineering consulting contracts, and clarify the rights and responsibilities of different participating units. Specifically, in order to minimize the probability of conflicts and disputes, during the contract signing process, each participating entity needs to refine the issues of defining the rights and responsibilities of each entity, the requirements for clause performance, and dispute resolution clauses, in order to achieve effective division of the rights and responsibilities of each participating entity. At the same time, when negotiating related business, consulting service units can, on the basis of clarifying relevant systems and policies, complete consulting fee negotiation activities that can meet the requirements of the owner based on factors such as the scale and complexity of the project and the scope of the engineering consultation. Then, by recording the negotiation results in the contract, economic disputes can be avoided. The quality evaluation of whole process engineering consulting service modes is MAGDM. In this work, the quality evaluation of whole process engineering consulting service modes is formed through SVNN-CTOPSIS technique. There are five whole process engineering consulting service modes $DY_i\left(i=1,2,3,4,5\right)$ which are evaluated through three experts $DX=\left\{DX_1,DX_2,DX_3\right\}$ with equal weight values in light with four attributes: ① DZ_1 is the resource integration for whole process engineering consulting service; ② DZ_2 is the talent team construction for whole process engineering consulting service; ③ DZ_3 is the work ability for whole process engineering consulting service; ④ DZ_4 is the management cost for whole process engineering consulting service. The DZ_4 is cost type. Then, the SVNN-CTOPSIS model is formed to achieve the optimal whole process engineering consulting service mode.

Step 1. Form group SVNN-matrix $DR^{(k)}=\left(DR_{ij}^{(k)}\right)_{5\times 4}\left(k=1,2,3\right)$ in light with linguistic scales (See Table 1) as in Table 2-4. The single SVNN-matrix is achieved in Table 5.

Table 1. Linguistic scale and SVNNs

Linguistic scales	SVNNs
Exceedingly Terrible-DET	(0.0000, 1.0000, 1.0000)
Very Terrible-DVT	(0.1000, 0.9000, 0.9000)
Terrible-DT	(0.3000, 0.7000, 0.7000)
Medium-DM	(0.5000, 0.5000, 0.5000)
Well-DW	(0.7000, 0.3000, 0.3000)
Very Well-DVW	(0.9000, 0.1000, 0.1000)
Exceedingly Well-DEW	(1.0000, 0.0000, 0.0000)

Table 2. SVNN-matrix from DX_1

	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	DT	DVW	DVT	DM
DY_2	DVW	DW	DM	DVT
DY_3	DW	DM	DT	DVW
DY_4	DM	DW	DVW	DVT
DY_5	DVT	DVT	DVW	DM

Table 3. SVNN-matrix from DX_2

	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	DVW	DW	DVT	DM
DY_2	DM	DW	DVT	DVW
DY_3	DM	DT	DVW	DW
DY_4	DVT	DM	DVW	DT
DY_5	DT	DW	DM	DVW

Table 4. SVNN-matrix from DX_3

	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	DVW	DVT	DT	DW
DY_2	DW	DT	DM	DM
DY_3	DVW	DT	DM	DW
DY_4	DVT	DM	DW	DVT
DY_5	DT	DVW	DW	DM

Table 5. $DR=\left[DR_{ij}\right]_{5\times4}$

	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	(0.45, 0.49, 0.41)	(0.47, 0.36, 0.48)	(0.43, 0.28, 0.32)	(0.48, 0.36, 0.39)
DY_2	(0.40, 0.28, 0.34)	(0.69, 0.31, 0.38)	(0.39, 0.34, 0.46)	(0.56, 0.48, 0.45)
DY_3	(0.58, 0.46, 0.37)	(0.56, 0.39, 0.46)	(0.48, 0.35, 0.39)	(0.25, 0.34, 0.43)

DY_4	(0.43, 0.19, 0.31)	(0.49, 0.43, 0.52)	(0.56, 0.49, 0.47)	(0.53, 0.37, 0.49)
DY_5	(0.51, 0.28, 0.16)	(0.54, 0.29, 0.34)	(0.63, 0.26, 0.35)	(0.67, 0.43, 0.36)

Step 2. Form the $DR = \left[DR_{ij}\right]_{5\times4}$ into standardized $DR^N = \left[DR_{ij}^N\right]_{5\times4}$ (See Table 6).

Table 6. The $DR^N = \left[DR_{ij}^N\right]_{5\times4}$

	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	(0.45, 0.49, 0.41)	(0.47, 0.36, 0.48)	(0.43, 0.28, 0.32)	(0.39, 0.36, 0.48)
DY_2	(0.40, 0.28, 0.34)	(0.69, 0.31, 0.38)	(0.39, 0.34, 0.46)	(0.45, 0.48, 0.56)
DY_3	(0.58, 0.46, 0.37)	(0.56, 0.39, 0.46)	(0.48, 0.35, 0.39)	(0.43, 0.34, 0.25)
DY_4	(0.43, 0.19, 0.31)	(0.49, 0.43, 0.52)	(0.56, 0.49, 0.47)	(0.49, 0.37, 0.53)
DY_5	(0.51, 0.28, 0.16)	(0.54, 0.29, 0.34)	(0.63, 0.26, 0.35)	(0.36, 0.43, 0.67)

Step 3. Form the weight numbers in light with CRITIC (Table 7).

Table 7. The achieved weight

Attributes	DZ_1	DZ_2	DZ_3	DZ_4
weight numbers	0.2803	0.3002	0.2573	0.1622

Step 4. Form the SVNNNPIVS and SVNNNNIVS (Table 8):

Table 8. The SVNNNPIVS and SVNNNNIVS

	DZ_1	DZ_2
SVNNNPIVS	(0.58, 0.46, 0.37)	(0.69, 0.31, 0.38)
SVNNNNIVS	(0.40, 0.28, 0.34)	(0.47, 0.36, 0.48)
	DZ_3	DZ_4
SVNNNPIVS	(0.63, 0.26, 0.35)	(0.49, 0.37, 0.53)
SVNNNNIVS	(0.39, 0.34, 0.46)	(0.36, 0.43, 0.67)

Step 5. Calculate the $SVNNCDM\left(DR_{ij}^{N}, SVNNPIVS_{j}\right)$ between $DR_{ij}^{N}=\left(DT_{ij}^{N}, DI_{ij}^{N}, DF_{ij}^{N}\right)$ and $SVNNPIVS_{j}=\left(DT_{j}^{N+}, DI_{j}^{N+}, DF_{j}^{N+}\right)$ in line with SVNNHD and SVNNED (Table 9).

Table 9. $SVNNCDM\left(DR_{ij}^{N}, SVNNPIVS_{j}\right)$.

Alternatives	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	0.4601	0.2837	0.3718	0.5046
DY_2	0.5183	0.0000	0.4138	0.3371
DY_3	0.0000	0.3168	0.4446	0.3653
DY_4	0.4857	0.3760	0.6098	0.0000
DY_5	0.4208	0.4284	0.0000	0.4551

Step 6. Calculate the $SVNNCDM\left(DR_{ij}^{N}, SVNNNIVS_{j}\right)$ between $DR_{ij}^{N}=\left(DT_{ij}^{N}, DI_{ij}^{N}, DF_{ij}^{N}\right)$ and $SVNNPIVS_{j}=\left(DT_{j}^{N-}, DI_{j}^{N-}, DF_{j}^{N-}\right)$ in line with SVNNHD and SVNNED (Table 10).

Table 10. $SVNNCDM\left(DR_{ij}^{N}, SVNNNIVS_{j}\right)$.

Alternatives	DZ_1	DZ_2	DZ_3	DZ_4
DY_1	0.4514	0.0000	0.3647	0.4950
DY_2	0.0000	0.3689	0.0000	0.3307
DY_3	0.4128	0.3108	0.4059	0.4465
DY_4	0.4765	0.2783	0.5982	0.3583
DY_5	0.5085	0.4202	0.4361	0.0000

Step 7. Construct the SVNNWCDM from the SVNNPIVS and SVNNNIVS (Table 11).

Table 11. The SVNNWCDM from the SVNNPIVS and SVNNNIVS

	$SVNNWCDM\left(DR_{i}^{N}, SVNNPIVS\right)$	$SVNNWCDM\left(DR_{i}^{N}, SVNNNIVS\right)$

DY_1	0.3916	0.3006
DY_2	0.3064	0.1644
DY_3	0.2687	0.3858
DY_4	0.4059	0.4291
DY_5	0.3204	0.3809

Step 8. Form the SVNNCCC (Table 12).

Table 12. The SVNNCCC

	SVNNCCC	Order
DY_1	0.4343	4
DY_2	0.3491	5
DY_3	0.5895	1
DY_4	0.5139	3
DY_5	0.5432	2

Step 9. In light with SVNNCCC, the obtained order is: $DY_3 > DY_5 > DY_4 > DY_1 > DY_2$ and DY_3 is the optimal whole process engineering consulting service mode.

4.2. Comparative analysis

The formed SVNN-CTOPSIS model is compared with SVNNWA model [55], SVNNWG model [55], SVNWBPM operator [58], SVNWGBPM operator[58], SVNN-WASPAS technique [59] and SVNN-TODIM technique [60], SVNN-GRA method[61], SVNN-VIKOR technique[62] and SVNN-CODAS technique [63]. The sufficient comparative results are verified in Table 17 and Figure 2.

Table 17. Order for different models

	Order
SVNNWA model [55]	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$
SVNNWG model [55]	$DY_3 > DY_5 > DY_1 > DY_4 > DY_2$
SVNWBPM operator [58]	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$
SVNWGBPM operator[58]	$DY_3 > DY_5 > DY_1 > DY_4 > DY_2$
SVNN-WASPAS technique [59]	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$
SVNN-TODIM technique [60]	$DY_3 > DY_5 > DY_1 > DY_4 > DY_2$
SVNN-GRA method [61]	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$

SVNN-VIKOR technique [62]	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$
SVNN-CODAS technique [63]	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$
SVNN-CTOPSIS model	$DY_3 > DY_5 > DY_4 > DY_1 > DY_2$

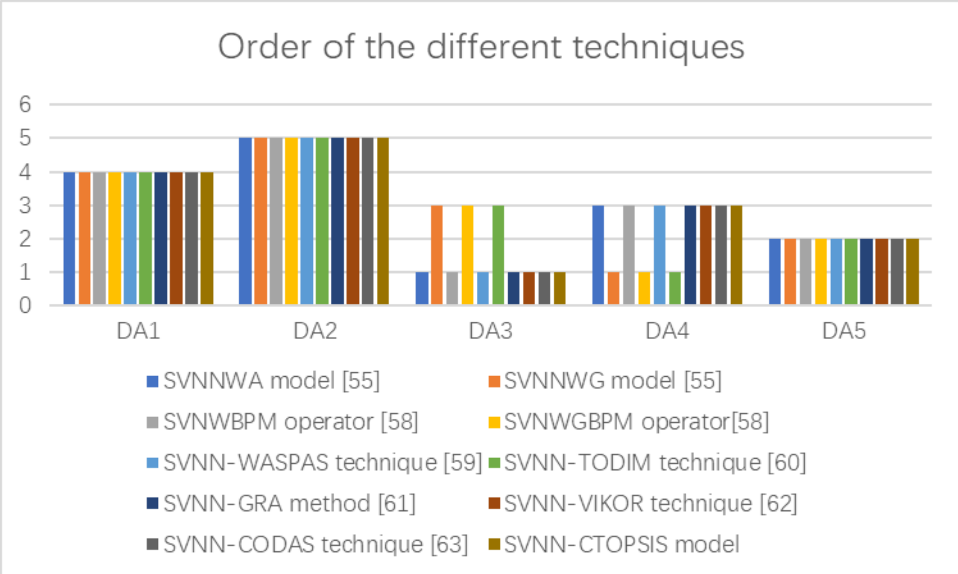

Figure 2. Order for different models

Through the above analysis, it could be seen that the order of these models is slightly different, however, these models have the same optimal whole process engineering consulting service mode and worst whole process engineering consulting service mode. This verifies the SVNN-CTOPSIS model is effective. Thus, the main advantages of the conducted SVNN-CTOPSIS model are formed: (1) the formed SVNN-CTOPSIS not only formed the uncertainty in MAGDM, but also portrays the combined distance measures from the SVNNNPIVS and SVNNNNIVS during the quality evaluation of whole process engineering consulting service modes. (2) the formed SVNN-CTOPSIS conducted the different behavior of the SVNNHD and SVNNED model as MAGDM techniques when they are combined.

5. Conclusion

The whole Process engineering consulting is a mode of providing all-round and full life cycle engineering consulting services for engineering construction projects from the perspective of the overall value appreciation of the project in order to effectively achieve the value objectives of engineering construction projects, which is also an international practice. With the rapid development of China's engineering construction industry, the scale of engineering projects continues to expand, and the complexity of engineering continues to increase. This poses serious challenges to China's engineering consulting industry. The previous fragmented engineering consulting model is no longer suitable for the new national conditions, nor can it cope with the international development of China's engineering consulting industry both domestically and internationally. Seamless integration with the international engineering consulting industry, large-scale implementation of the whole Process engineering consulting model is imperative. The quality evaluation of whole process engineering consulting service modes is MAGDM. In this paper, the SVNN-CTOPSIS model based on SVNNHD and SVNNED is formed to cope with the MAGDM. The CRITIC model is utilized to obtain the weight numbers in light with the SVNNHD and SVNNED under SVNSs. Finally, numerical example and comparative analysis for quality evaluation of whole process engineering consulting service modes is employed to verify SVNN-CTOPSIS model. The main contributions of this study are formed: (1) the CRITIC model is formed to obtain the weight numbers in light with SVNNHD and SVNNED; (2) the SVNN-CTOPSIS model is formed in light with SVNNHD and SVNNED under SVNNs; (3) Finally, numerical example and comparative analysis for quality evaluation of whole process engineering consulting service modes is utilized to verify SVNN-CTOPSIS model.

There may be some possible research limitations for quality evaluation of whole process engineering consulting service modes, which could be further conducted in our future research contents: (1) It is a worthwhile research contents to conduct prospect theory[64-70] for quality evaluation of whole process engineering consulting service modes under SVNSs; (2) It is also worthwhile research contents to conduct regret theory[71-77] for quality evaluation of whole process engineering consulting service modes under SVNSs environment; (3) In subsequent research contents, the application of SVNSs needs to be formed with consensus issues [78-83] for quality evaluation of whole process engineering consulting service modes.

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Fusing Type -2 Neutrosophic in Decision-Making Methodology for Appreciation Blockchain Capabilities in Securing Environment of Vehicles Fog: Practice Realistic Scenarios

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Abstract
Right now, Contemporary technologies have become imperative in many domains to achieve societal safety. As is practiced in transportation systems through merging information and communication technologies (ICTs) in transportation to be an intelligent sector. As well Internet of Vehicles (IoVs) for facilitating communication between vehicles for safe driving. Similarly, fog computing in Vehicular Ad Hoc Networking (VANET) contributes significantly to addressing timing and latency issues by enabling cloud services for nearby vehicles. Nonetheless, there are hazards of cyber-attacks on vehicles in VFN, which makes it uneasy to disclose personal information to unidentified fog devices. Consequently, an online criminal might target vehicles with counterfeit attacks. Herein, blockchain technology (BCT) is another technology of ICTs and is provided in this study as a handler for the problem of cyber-attacks. Due to BCT's characteristics of permanent, or immutable, peer-to-peer, decentralized, and distributed ledger technology. Thereby, this study contributes to constructing an appraiser model for appraising BCT as the secured methodology in VFN. Multi-criteria decision-making (MCDM) techniques such as entropy and weighted sum method (WSM) have been harnessed in the appraising process motivated by the uncertainty theory of Type-2 neutrosophic sets (T2NSs). The appraiser model's findings indicated that BCT 5(A5) was the optimal candidate based on its ranking. In contrast, BCT 4 (A4) is the worst one.

Keywords: Vehicular Ad-hoc Networks (VANET); Internet of Vehicles (IoVs); Vehicular fog network (VFN); Blockchain Technology (BCT); Multi-Criteria Decision Making; Type-2 Neutrosophic

1. **Introduction**

There are more accidents and problems with traffic congestion these days due to the massive growth in the number of vehicles on the road. This highlights the necessity for significant planning to guarantee traffic efficiency and road safety. Various technologies have been implemented to promote safer and more efficient driving on roads. One such technology is the Vehicular Ad-hoc Network (VANET), which allows vehicles to exchange information about their location, speed, and other road-related parameters. This increases the vehicles' awareness of the conditions of the surrounding roads and facilitates the making of more informed and timely decisions [1]. Up until recently, VANET's primary goal was to gather and share data with other drivers to improve comfort and safety for drivers in a moving vehicle environment [2]. But VANET is quickly evolving into a transportation network where intelligent cars with integrated sensors, adapters, and control units may effectively communicate with nearby cars in addition to monitoring their environment [3]. The main issues with connected Vehicles in VANET are privacy and security. Vehicle data security can be easily breached by anyone with a connection to a vehicle, such as an owner, mechanic, or member of the governmental staff. Data validation, access control, device and network security, and driver and vehicle privacy are among the potential security risks that attackers may exploit [4]. As such, creating privacy and security solutions for connected Vehicles in VANET is a more difficult task. But as today's technologies—such as cloud computing platforms, wireless technologies, sensor devices, and smart cars—develop more quickly, the demand for stronger vehicular networks has grown. Thus, the Internet of Vehicles (IoVs) emerged, able to take advantage of and integrate all these cutting-edge technologies to offer drivers and passengers of automobiles more rewarding real-time services [1]. IoVs are a next-generation wireless roadside system that is rapidly expanding [1]. A variety of vehicle interactions are now possible thanks to recent developments in sensor and communication technologies such as vehicle-to-vehicle (V2V), vehicle-to-infrastructure (V2I), vehicle-to-roadside units, vehicle-to-mobile-infrastructure, vehicle-to-sensors, and vehicle-to-personal devices [5]. IoVs idea seeks to establish a networked infrastructure for the exchange of resources and information among smart vehicles, hence enabling the advancement of the Intelligent Transportation System (ITS). IoV will enable continuous connectivity between vehicles, roadside infrastructure, and pedestrians, and will increase the number of intelligent and linked automobiles in IT [6]. IoV is developing more quickly because of ongoing advancements in intelligent vehicle technology. Data exchange and interaction in the IoV is currently a popular area of study. Road data, car-generated data, data supplied by other nodes, etc. are all included in the vehicle interaction data [7]. By supplying connected Vehicles with services like storage, infrastructure, and increased processing capacity, cloud computing enables them to be charged by their needs [5]. Numerous dangers, including identity theft, denial of service, access control, data breaches, and data loss, affect cloud computing. By employing devices that can provide cloud computing's characteristics to the necessary vehicle, fog computing extends the functionality of cloud computing to the network's edge [6]. Consequently, vehicular fog network (VFN) refers to the network that has been integrated with IoVs and fog devices. The fact that VFN stores all its data on a single, centralized cloud server creates serious security risks since if one of the entities is compromised, the entire system is at risk. The disadvantage of one entity being hacked is eliminated by Blockchain (BC) technology, which is dispersed and decentralized. To write and validate transactional data and transport all verified transactions in a block, it works with numerous connected vehicles [7].

2. Comprehensive Review of Earlier Insights

In recent years, numerous technical methods have been presented by numerous researchers to improve IoT performance. Because of its distinct qualities, IoVs is one of the main subjects of literature studies among them [8]. Vehicles in IoV process a lot of data. Additionally, they use a mesh network to directly perform V2V connection and ensure reliable data flow. The data could be about simple text messages, multimedia, or proximity to a location. Ensuring network security becomes imperative to uphold user trust [9]. According to Song et al.[10], a group of vehicles with similar average speeds and directions of travel can be formed based on navigation, and intergroup communication will keep the positions of the individual vehicles and those of other vehicles hidden. However, because of the vehicle's speed and the unpredictability of the surrounding environment, there is still a serious problem with communication between geographically independent groups of vehicles. This problem manifests itself in the form of difficult information exchange and the need to repeat the intermediate authentication process whenever a vehicle rejoins another group of vehicles [8]. To address the security concerns around the Internet of Vehicles, a novel form of BC framework has been investigated to facilitate the safe transfer of information [11]. The reliability of a node and a message were recorded in a ledger on a local public BC that the researchers built for this purpose. Authors in [12] have identified problems with passing alert messages without disclosing the sender's identity as well as a lack of imagination in cars to do so. Their proposal was for an effective incentive announcement network built on BC technology that protects anonymity, enables vehicles to operate in the network anonymously, and provides incentives for their efforts. The researchers in [13] used consortium BC and smart contract technologies in order to enable the safe exchange and storage of data within in-vehicle edge networks. These technologies work to prevent information from being shared illegally. The researchers also developed a reputation-based data-sharing strategy to guarantee that the vehicles continuously provided high-quality data. The authors in [14] built software-defined fault tolerance and quality-of-service-aware IoT-based vehicular networks using edge computing made possible by BC. This resulted in a reduction in overall communication time, message failure fault tolerance, and safe service delivery for VANET. The ability for vehicles to exchange messages is what VANET is there for. The difficulty here is that such messages must be stored and forwarded by a reliable party. An additional obstacle is that the vehicle can refuse to take part in the creation and dissemination of announcement messages unless doing so benefits it. Authors have proposed a BC-enabled safe data-sharing system for the Internet of Vehicles (IoVs) that uses a parent and auxiliary BC to store the messages by various organizations from various places in order to address this issue and provide secure communication [15]. In order to address timeliness and latency difficulties, vehicular fog networking integrates fog computing and vehicular ad hoc networking to offer cloud services to neighboring automobiles [16]. Security and privacy concerns plague vehicular fog computing [17]. Another issue is that, even though the cloud and fog service providers are reliable organizations, automobiles in VFN frequently feel uneasy disclosing private information to unidentified fog devices [18]. The internet connection of vehicles in VFN is another major factor contributing to cyberattacks. BC, a distributed, decentralized, immutable, consensus-based network, may be a useful way to address VFN's issues with cyberattacks, latency, and timeliness [19]. Despite all its benefits, BC technology is still in its infancy and many firms still have reservations. According to a PWC (PricewaterhouseCoopers) poll, the main obstacles to BC adoption include regulatory ambiguity (48%), a lack of confidence (45%), and the question of whether the BC network can be connected (44%) [20]. Therefore, are all BC services appropriate for every firm at this point? Perhaps not the answer. Each type of BCT—private, public, and community—has pros and cons of its own. Therefore, by considering both economics and other pertinent criteria, enterprises should use a scientific decision-making tool to determine which BC service provider is more appropriate [21].

2.1 Blockchain (BCT)

BC is a viable method to address challenges. BC consists of a group of interconnected blocks that are connected by certain cryptographic procedures to form a chain. The blocks store information such as records, queries, and transactions. The digital ledger, which is updated by every network member, records every new block that is created and added to the chain. For this reason, another name for BC technology is distributed ledger technology (DLT) [22]. A BC can be classified as either public (completely dispersed and permissionless), private (permissioned, belonging to a particular organization), or consortium (federated, resembling a private BC) [23].

- Public BC: This BC is entirely decentralized, distributed, and permissionless, allowing any connected autonomous vehicles (CAVs) to connect to the network and view its contents. Take cryptocurrency networks like Ethereum and Bitcoin, for instance. It costs a lot of computer power to publish a new block. A processing charge is required to store a transaction on the BC.
- Private BC: This is a single organization-created, fully permissioned BC. The authority organization is aware of every member of the organization and does not impose any fees for transaction processing.
- Consortium BC: This kind of BC is comparable to a private BC, but it spans several organizations (several authorities) as opposed to just one.

BC technology eliminates the need for a central authority by enabling everyone to create and approve transactions in a peer-to-peer network, greatly lowering the time and money associated with the middleman [1].

2.2 Vehicular Fog Networking (VFN)

Cisco was coining the phrase "fog computing." The fog offers decentralized distributed computing capabilities at the edge of the network, in contrast to the cloud, which is a centralized server. Fog provides a more effective way around the restrictions of cloud computing by utilizing this feature [24]. Any device that can share resources on rent and is referred to as a fog device can provide fog functionality. Applications that need a quick reaction and are time-sensitive are the greatest candidates for fog computing [25]. One of the major uses of fog computing is the Internet IoVs; this integration is called a Vehicular Fog Network (VFN)[15]. Because vehicles do not need to send data to the cloud, a VFN has the advantages of low latency, reduced network bandwidth requirements, security, and increased reliability. Any dynamic node, such as a vehicle, or any static node, such as a router, switch, base station, or RSU, could function as a fog device in a VFN. A fog device can be hired out to the necessary cars for computation and storage because it has an underutilized infrastructure. In addition, data segregation, forwarding, and real-time decision-making for vehicular communication are all impacted by fog [26]. Even while the fog sends all the data it needs for analysis later, it communicates only the data that is needed.

The BC idea is used with VFN to increase security by storing reward point values and vehicle reliability in a traffic scenario. Furthermore, the combination of fog computing with the BC idea may be able to address the main security issues in an IoVs environments [5].

2.3 BCT in Vehicular Fog Network

Vehicular fog computing, a novel vehicular network architecture, is introduced with the BC security framework. BC security transactions are accelerated by vehicle network design and fog computing, which together offer cloud computing capabilities at the network's edge. VFN is the name of this system. Applying the BC concept to VFN increases its security by storing reward point values and vehicle trustworthiness in a traffic scenario. Additionally, fog computing and the BC idea have the potential to address the main security issues in an IoVs environments [25]. In complex road traffic scenarios where vehicles lack confidence, BCT is well suited for decentralized application environments with distributed consensus features. Data is secure against easy manipulation by adversaries because of BC technology. Multiple service providers may be able to collaboratively manage the user's account information with the help of this encryption feature [8]. To accomplish the full identity authentication process across several servers, a user simply needs to keep track of their account details on the ledger, potentially increasing efficiency. Nevertheless, in contrast to other Internet of Things, IoVs based on BC technology allows for energy consumption to be met directly by the vehicle, avoiding the drawback of high energy consumption of the BC network [8].

BC technology is also having a significant impact on businesses that we never would have predicted would become unstable. It makes sense to research this kind of topic since the service provider selection problem in a BC system might undergo significant changes in the future. Furthermore, an enterprise's performance and success are directly correlated with the choice of suitable BC service providers. Enterprises seeking growth and development will collaborate with capable firms to create BC technology, viewing these firms as their own BC service providers [21].

3 Methodology: Appraising of Blockchain

In this study, the advantage of Entropy technique to determine the weights of criteria in MCDM problems is combined with WSM to evaluate and rank a set of BCT as security methodology. these techniques under the authority of T2NSs.

Phase 1: Problem Formulation

Step 1.1: Set of BCTs is determined as alternatives that contribute to the appraiser model. will where the alternatives are represented as BCTs = {BCT1, BCT 2, . . . , BCT m}.The determined alternatives are appraised based on a set of criteria as C = {C1, C2, . . . , Cn} which is mentioned in Table 1 .

Step 1.2: the panel of DMs is formed for appraising the alternatives of BCTs.

Table1: Determined criteria based on blockchain technology [1]

Criteria	Description
Decentralization: C1	BC technology demonstrates a decentralized nature in which data records are held and managed by all participating entities, in contrast to centralized storage platforms where both data storage and maintenance are handled by a trusted single node. This helps VFN settings by avoiding the single point of failure problem, reducing maintenance costs related to centralized server configurations, and reducing resource constraints.
Immutability: C2	The BC is nearly impossible to tamper with or alter since the creation and validation of new blocks of transactions must be approved by all or most of the peers using various consensus procedures before being added to the BC.
Security and privacy: C3	The adoption of digital signatures and cryptographic hash functions in BC technology can guarantee the security of transaction data as well as the privacy of users taking part on the Internet of Vehicles.
Transparency: C4	All participants have access to all timestamped BC transactions since they each maintain a copy of the public ledger. As a result, peers can transparently manage, search for, and validate transactions at any moment without the need for a

	middleman. By handling their transactions, peers are relieved, and the intermediary party's time and financial expenses are also reduced because of its self-auditability and transparency.
Automation: C5	Smart contracts, which are software programs that can be launched automatically by a triggering event or upon fulfilling a predetermined set of rules, are made possible by BC technology. This BC's automation feature can allow many VFN applications operate more efficiently and provide a range of services on their own without requiring a trusted third party.
Traceability: C6	Every transaction record, along with a timestamp indicating when it occurred and was added to the public ledger, is stored in the BC. The fact that the recordings are timestamped makes it easier to identify the events in a chronological order, improving traceability and supporting VFN non-repudiation requirement.

Phase2: Generating criteria weights

Step 2.1: Construct neutrosophic decision matrices. DMs utilized the linguistic terms presented in Table2 to assess the opinions of DMs about each criterion [27]

Step 2.2: Use the de-neutrosophic Eq. (1) for transforming neutrosophic decision matrices to the crisp matrices [27].

$$S(U_1^{\sim}) = \frac{1}{12} + (8 + \left(T_{T_{U1}}(z) + 2\left(T_{I_{U1}}(z)\right) + T_{F_{U1}}(z)\right) - \left(I_{T_{U1}}(z) + 2\left(I_{I_{U1}}(z)\right) + I_{F_{U1}}(z)\right) - \left(F_{T_{U1}}(z) + 2\left(F_{I_{U1}}(z)\right) + F_{F_{U1}}(z)\right) \quad (1)$$

Table2: Linguistic Scale

Linguistic Terms	T2N scale for $<(T_T, T_I, T_F), (I_T, I_I, I_F), (F_T, F_I, F_F)>$
Very Bad (VB)	⟨(0.20, 0.20, 0.10),(0.65, 0.80, 0.85),(0.45, 0.80, 0.70)⟩
Bad (B)	⟨(0.35, 0.35, 0.10),(0.50, 0.75, 0.80),(0.50, 0.75, 0.65)⟩
Medium Bad (MB)	⟨(0.50, 0.30, 0.50),(0.50, 0.35, 0.45),(0.45, 0.30, 0.60)⟩
Medium (M)	⟨(0.40, 0.45, 0.50),(0.40, 0.45, 0.50),(0.35, 0.40, 0.45)⟩
Medium Good (mg)	⟨(0.60, 0.45, 0.50),(0.20, 0.15, 0.25),(0.10, 0.25, 0.15)⟩
Good (G)	⟨(0.70, 0.75, 0.80),(0.15, 0.20, 0.25),(0.10, 0.15, 0.20)⟩
Very Good (VG)	⟨(0.95, 0.90, 0.95),(0.10, 0.10, 0.05),(0.05, 0.05, 0.05)⟩

Step 2.3. Eq. (2) is employed in crisp matrices to aggregate it into a single decision matrix.

$$x_{t_{ij}} = \frac{\sum_{j=1}^{N} S(U_i^{\sim})}{N} \quad (2)$$

Where: $S(U_i^{\sim})$ refers to value of criterion in matrix, N refers to number of decision makers

Step 2.4: Normalizing the aggregated decision matrix r_{ij} based on Eq.(3)

$$r_{ij} = \frac{x_{ij}}{\sum_{i=1}^{n} x_{ij}} \quad (3)$$

Where: $\sum_{i=1}^{n} x_{ij}$ represents sum of each criterion in aggregated matrix per column.

Step 2.5: Compute Entropy e_i for normalized matrix by Eq.(4)

$e_j = (-h) \sum_{i=1}^{n} r_{ij} \; ln\,(r_{ij})$ (4)

$where\ (h) = \frac{1}{\ln(n)}$; n refers to number of alternatives

Step 2.6: Calculation of variation coefficient

$d_j == |1 - e_j|$ (5)

Step 2.7: Calculation of weights

$w_j = \frac{d_i}{\sum_{i=1}^{n} d_j}$ (6)

Phase 3: Recommending the most secure BCT amongst BCTs

Step 3.1: Eq.s(3,8) are employed for normalizing the aggregated matrix from previous phase 2.

$\mathrm{N} = \frac{1}{x_{ij}}$ (7)

$\mathrm{Nor_{Aggj}} = \frac{\mathrm{N}}{\mathrm{sum(N)}}$, For Non − Benficial criteria (8)

Step 3.2: weighted decision matrix is generated based on Eq.(9)

$\delta_{\mathrm{ij}} = w_j * \mathrm{Nor_{Agg}}$ (9)

Step 3.3: Obtaining global score based on Eq.(10).

$\mathrm{V}(\delta_{\mathrm{ij}}) = \sum_{\mathrm{j=1}}^{\mathrm{n}} \delta_{\mathrm{ij}_{\mathrm{ij}}}$ (10)

Where $V(\delta_{\mathrm{ij}})$ is global score values.

4 Implementation of Appraiser Model in Realism:Case Study

To ensure the accuracy of the constructed appraiser model, we applied it to a smart city aiming for sustainable development. We are volunteering five BCTs to be candidates in this study which appraising based on six criteria have been determined in Table 1.

4.1 Weighting criteria based on entropy- T2NSs.

- Five Neutrosophic decision matrices are constructed and converted to crisp values using score function of Eq.(1).
- The de-neutrosophic matrices are combined based on Eq.(2) into a single matrix called an aggregated matrix as listed in Table 3.
- The aggregated matrix normalized according to Eq.(3) and generate normalized matrix as listed in Table 4.
- The normalized matrix is harnessed in Eq.(4) for computing entropy as in Table 5.
- Finally, Eq.(6) is applied for generating criteria weights which resulted in Table 6. Fig 1 showcases the weights of criteria where C1 has the highest value otherwise C6 has the lowest value .

Table 3: Aggregated decision matrix

	C1	C2	C3	C4	C5	C6
BCT1	0.7092	0.5617	0.5017	0.5617	0.5300	0.4725

BCT 2	0.4492	0.6008	0.6350	0.5600	0.5525	0.5067
BCT 3	0.5342	0.5175	0.6725	0.6025	0.6792	0.5967
BCT 4	0.5058	0.7208	0.5317	0.4358	0.4725	0.5400
BCT 5	0.5567	0.4617	0.5067	0.7092	0.7008	0.6183
sum	2.7550	2.8625	2.8475	2.8692	2.9350	2.7342

Table 4: Normalizing the aggregated decision matrix

	C1	C2	C3	C4	C5	C6
BCT1	0.2574	0.1962	0.1762	0.1958	0.1806	0.1728
BCT 2	0.1630	0.2099	0.2230	0.1952	0.1882	0.1853
BCT 3	0.1939	0.1808	0.2362	0.2100	0.2314	0.2182
BCT 4	0.1836	0.2518	0.1867	0.1519	0.1610	0.1975
BCT 5	0.2021	0.1613	0.1779	0.2472	0.2388	0.2262

Table 5: Compute Entropy ej for normalize

	C1	C2	C3	C4	C5	C6
BCT1	-0.3493	-0.3195	-0.3059	-0.3193	-0.3091	-0.3034
BCT 2	-0.2957	-0.3277	-0.3346	-0.3189	-0.3144	-0.3124
BCT 3	-0.3181	-0.3092	-0.3408	-0.3277	-0.3387	-0.3322
BCT 4	-0.3112	-0.3473	-0.3133	-0.2863	-0.2940	-0.3203
BCT 5	-0.3231	-0.2943	-0.3072	-0.3455	-0.3420	-0.3362

Table 6: Compute Weight Vector

	C1	C2	C3	C4	C5	C6
$\sum_{i=1}^{n} rij\ lnrij$	-1.5974	-1.5980	-1.6019	-1.5976	-1.5981	-1.6045
ej	0.9925	0.9929	0.9953	0.9926	0.9930	0.9969
dj	0.0075	0.0071	0.0047	0.0074	0.0070	0.0031
Wi	0.2030	0.1938	0.1280	0.2003	0.1911	0.0838

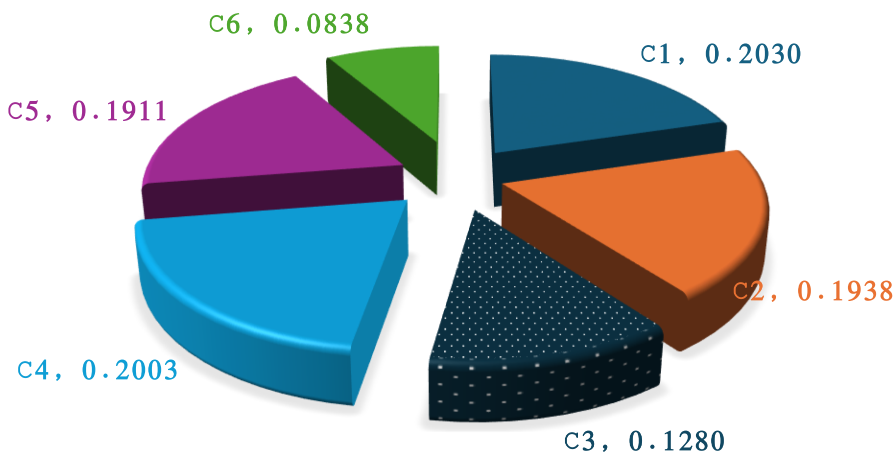

Fig 1. Final weights of criteria

4.2 Obtaining optimal secure BCT using WSM and T2NSs

- In our case, all criteria are beneficial. hence, we utilized the normalized matrix from entropy based on T2NSs for generating a weighted decision matrix by utilizing Eq(9) as in Table 7.
- Finally, the candidates of BCTs are ranked based on values of global score. The findings of BCTs ranking are represented in Fig where BCT3 is the optimal alternative whilst BCT4 is the worst alternative.

Table 7: Weighted decision matrix

	C1	C2	C3	C4	C5	C6
BCT1	0.0523	0.0380	0.0225	0.0392	0.0345	0.0145
BCT 2	0.0331	0.0407	0.0285	0.0391	0.0360	0.0155
BCT 3	0.0394	0.0350	0.0302	0.0421	0.0442	0.0183
BCT 4	0.0373	0.0488	0.0239	0.0304	0.0308	0.0166
BCT 5	0.0410	0.0312	0.0228	0.0495	0.0456	0.0190

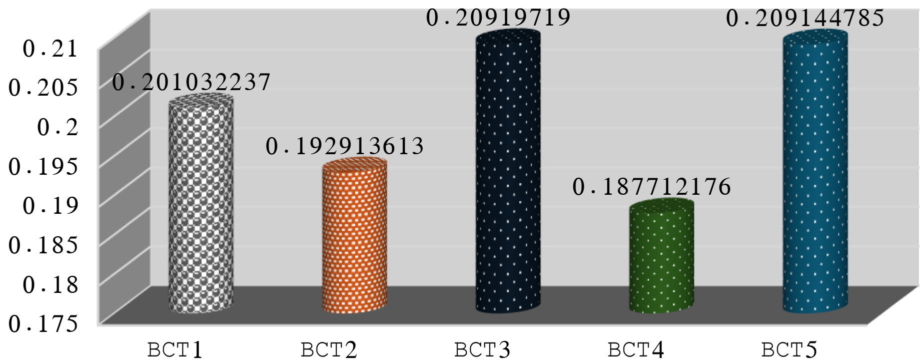

Fig 2. Final rank for alternative based on WSM-T2NSs

5 Comparative Analysis

We applied another method besides implementing our appraiser model in the real case study; we performed various scenarios for changing the criteria's weights by implementing sensitivity analysis. The objective of the sensitivity analysis process is to verify the stability of model's decision by determining how decisions are affected based on changes in the values of criteria weights.

Fig 3 illustrates the seven cases for changing the values of criteria weights besides criteria weights obtained from entropy based on T2NSs. The findings of the changed values of criteria weights are formed in Fig 4. According to this Fig the decision of the worst BCT for all cases is like the appraiser model's decision where BCT 4 is the worst. Nevertheless, the difference in the optimal BCT where the constructed appraiser model and six cases agree that BCT3 is the optimal followed by BCT5. Otherwise, case five where BCT5 is the optimal followed by BCT3.

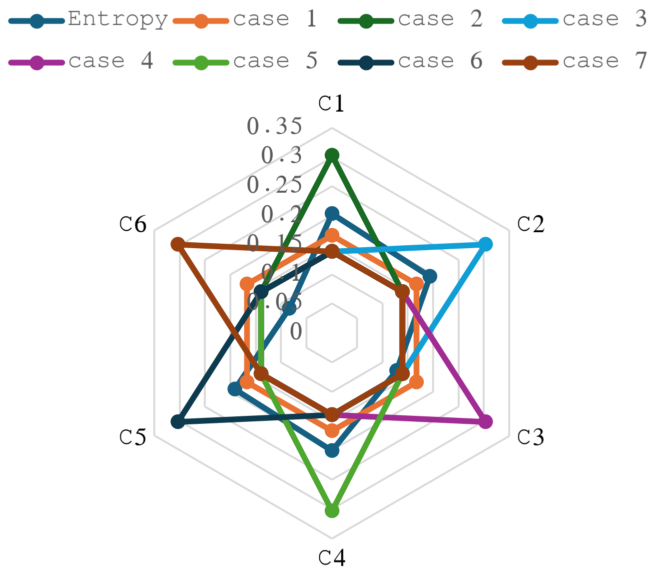

Fig 3. Changing values of criteria weights

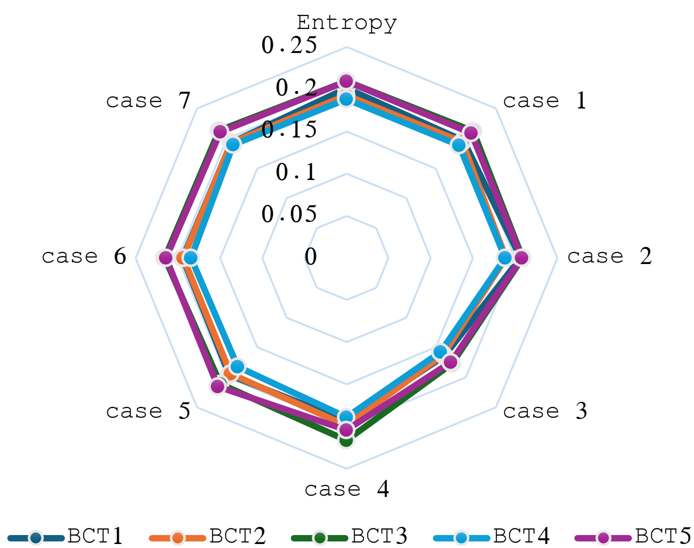

Fig 4. The decision of ranking BCTs based on various cases

6 Conclusions

This survey for prior studies demonstrated the security for both the earlier Vehicular Ad Hoc Networking (VANET) and other technologies as IoVs in intelligent transportation systems is a critical issue. Hence, Vehicular Fog Network (VFN) is constructed through integrating fog computing and VANET to provide cloud services to nearby vehicles to deal with timeliness and latency issues. There was also a focus on the capabilities of the recently developed BC technology in VFN. Making use of BCT to enable secure and efficient data trading for IoVs is becoming increasingly useful. BC technology is also having a significant impact on businesses that we never would have predicted would become unstable. It makes sense to research this kind of topic since the service provider selection problem in a BC system might undergo significant changes in the future. Furthermore, an enterprise's performance and success are directly correlated with the choice of suitable BC service providers. Enterprises seeking growth and development will collaborate with capable firms to create BC technology, viewing these firms as their own BC service providers. The problem of selecting optimal BC is represented in selection according to set of attributes. MCDM techniques are employed in BCs selection to analyze attributes and recommend the optimal BCs among set of Decision makers. Herein, the entropy technique implemented in BCTs selection to obtain attributes' weights through the preferences of experts who related to our scope. The rating is performed by applying T2NSs. The results of the implementation of entropy indicated that Decentralization (C1) is optimal attribute otherwise Traceability (C6) is the least based on the final values of its weights. After that WSM leverages the generated weights of attributes to rank BCTs candidates and recommend the best and worst BCT. In our study, there is an agreement on recommending BCT3 as the optimal candidate based on its ranking. In contrast BCT4 is the worst one. But in case five BCT5 is recommended as optimal securing methodology in VFN.

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Pentapartitioned Neutrosophic Subtraction Algebra

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Abstract: This paper aims to define the concepts of Semi-group and Pentapartitioned Neutrosophic Subtraction Algebra. We also examine a few of their fundamental characteristics. Additionally, we provide a few appropriate instances on Pentapartitioned Neutrosophic Subtraction Algebra.

Keywords: Pentapartitioned Neutrosophic Set; Subtraction Algebra, Subtraction Semi-Group.

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1. Introduction:

Schein [30] developed the basic principles of Subtraction Algebra in 1992. Afterwards, Zelinka [37] presented the idea of subtraction semi-group. In the year 2004, Kyung et al. [26] presented some notes on subtraction semi-group. Later on, Jun and Kim [23] grounded the notions of ideal in subtraction algebra. In the year 2008, Jun and Kim [24] also studied the concept of prime and irreducible ideals in subtraction algebra. The concepts of Weak Subtraction Algebras were then grounded by Lee et al. [27], who also looked at a technique for creating Weak Subtraction Algebra from a quasi-ordered set. Zadeh [36] proposed the concept of fuzzy sets for the first time in 1963. Later, in 2007, Kim et al. [25] addressed the concept of fuzzy ideals in subtraction algebras. In 1986, Atanassov [3] constructed a new idea of Intuitionistic Fuzzy Set by broadening the idea of Fuzzy Set. Ezhilarasi and Sriram [18] grounded the notion of intuitionistic fuzzy ideals of subtraction algebra. In 1998, Smarandache [31, 32] laid out the concept of Neutrosophic Set (NS) to aid in dealing with unpredictable occurrences with indeterminacy. In 2006, Vasantha Kandasamy and Smarandache [35] came up with neutrosophic algebraic structures in the context of neutrosophic set. In the year 2020, Ibrahim et al. [21] introduced the notions of Neutrosophic Subtraction Algebra and Neutrosophic Subtraction Semi-Group. Mallick and Pramanik [28] recently presented the principles of a Pentapartitioned Neutrosophic Set by broadening the theories of NS, in which any component has a total of five independent components such as: truth, contradiction, ignorance, unknown, and false membership.

In this article, we introduce the notion of Pentapartitioned Neutrosophic Subtraction Algebra and Semi-Group.

This article's remaining portion is structured as follows:

We review some pertinent definitions and findings on semi group and subtraction algebra in section-2. By extending the theory of neutrosophic subtraction algebra, we present the concepts of pentapartitioned neutrosophic subtraction algebra in section-3. We also develop some findings regarding pentapartitioned neutrosophic subtraction algebra. In section-4, we conclude this article.

2. Relevant Definitions and Results:

This section includes a few definitions and results that are fundamental to understanding the drafting of this article's primary outcomes.

Definition 2.1. [26] Let us consider a binary operation '−' on a fixed set $\tilde{\hat{A}}$. Then, the structure ($\tilde{\hat{A}}$, −) is referred to as a subtraction algebra if the following holds:

(i) $\acute{\rho} - (\underset{.}{\psi} - \acute{\rho}) = \acute{\rho}$;

(ii) $\acute{\rho} - (\acute{\rho} - \underset{.}{\psi}) = \underset{.}{\psi} - (\underset{.}{\psi} - \acute{\rho})$;

(iii) $(\acute{\rho} - \underset{.}{\psi}) - \tilde{e} = (\acute{\rho} - \tilde{e}) - \underset{.}{\psi}$; for all $\acute{\rho}, \underset{.}{\psi}, \tilde{e} \in \tilde{\hat{A}}$.

Remark 2.1. [26] Let us consider a subtraction algebra ($\tilde{\hat{A}}$, −). Then,

(i) $\acute{\rho} - 0 = \acute{\rho}$ and $0 - \acute{\rho} = 0$

(ii) $\acute{\rho} - (\acute{\rho} - \underset{.}{\psi}) \leq \underset{.}{\psi}$.

(iii) $\acute{\rho} \leq \underset{.}{\psi} \Leftrightarrow \acute{\rho} = \underset{.}{\psi} - w$ for some $w \in \tilde{\hat{A}}$.

(iv) $\acute{\rho} \leq \underset{.}{\psi} \Rightarrow \acute{\rho} - \tilde{e} \leq \underset{.}{\psi} - \tilde{e}$ and $\tilde{e} - \underset{.}{\psi} \leq \tilde{e} - \acute{\rho}$ for all $\tilde{e} \in \tilde{\hat{A}}$.

(v) $\acute{\rho} - (\acute{\rho} - (\acute{\rho} - \underset{.}{\psi})) = \acute{\rho} - \underset{.}{\psi}$.

Definition 2.2. [28] Let us consider that $\tilde{\hat{A}}$ be a fixed set. A pentapartitioned neutrosophic set (P-NS) $\tilde{N}$ over $\tilde{\hat{A}}$ is defined by:

$$\tilde{N} = \{(\acute{\rho}, T_{\tilde{N}}(\acute{\rho}), C_{\tilde{N}}(\acute{\rho}), G_{\tilde{N}}(\acute{\rho}), U_{\tilde{N}}(\acute{\rho}), F_{\tilde{N}}(\acute{\rho})): \acute{\rho} \in \tilde{\hat{A}}\},$$

where $T_{\tilde{N}}(\acute{\rho})$, $C_{\tilde{N}}(\acute{\rho})$, $G_{\tilde{N}}(\acute{\rho})$, $U_{\tilde{N}}(\acute{\rho})$, $F_{\tilde{N}}(\acute{\rho})$ ($\in$]0,1[) are the truth, contradiction, ignorance, unknown, falsity membership values of each $\acute{\rho} \in \tilde{\hat{A}}$. So $0 \leq T_{\tilde{N}}(\acute{\rho})+C_{\tilde{N}}(\acute{\rho})+G_{\tilde{N}}(\acute{\rho})+U_{\tilde{N}}(\acute{\rho})+F_{\tilde{N}}(\acute{\rho}) \leq 5$.

The null P-NS (0_{PN}) and the absolute P-NS (1_{PN}) over $\tilde{\hat{A}}$ are defined as follows:

(i) $0_{PN} = \{(\acute{\rho}, 0, 0, 1, 1, 1): \acute{\rho} \in \tilde{\hat{A}}\}$;

(ii) $1_{PN} = \{(\acute{\rho}, 1, 1, 0, 0, 0): \acute{\rho} \in \tilde{\hat{A}}\}$.

Clearly, $0_{PN} \subseteq \tilde{N} \subseteq 1_{PN}$, where $\tilde{N}$ is a P-NS over $\tilde{\hat{A}}$.

Assume that $\tilde{N} = \{(\acute{\rho}, T_{\tilde{N}}(\acute{\rho}), C_{\tilde{N}}(\acute{\rho}), G_{\tilde{N}}(\acute{\rho}), U_{\tilde{N}}(\acute{\rho}), F_{\tilde{N}}(\acute{\rho})): \acute{\rho} \in W\}$ and $\bar{H} = \{(\acute{\rho}, T_{H}(\acute{\rho}), C_{H}(\acute{\rho}), G_{H}(\acute{\rho}), U_{H}(\acute{\rho}), F_{H}(\acute{\rho})): \acute{\rho} \in W\}$ be two P-NSs over W. Then,

(i) $\tilde{N} \subseteq \bar{H} \Leftrightarrow T_{\tilde{N}}(\acute{\rho}) \leq T_{H}(\acute{\rho})$, $C_{\tilde{N}}(\acute{\rho}) \leq C_{H}(\acute{\rho})$, $G_{\tilde{N}}(\acute{\rho}) \geq G_{H}(\acute{\rho})$, $U_{\tilde{N}}(\acute{\rho}) \geq U_{H}(\acute{\rho})$, $F_{\tilde{N}}(\acute{\rho}) \geq F_{H}(\acute{\rho})$, for all $\acute{\rho} \in W$.

(ii) $\tilde{N} \cap \bar{H} = \{(\acute{\rho}, \min\{T_{\tilde{N}}(\acute{\rho}), T_{H}(\acute{\rho})\}, \min\{C_{\tilde{N}}(\acute{\rho}), C_{H}(\acute{\rho})\}, \max\{G_{\tilde{N}}(\acute{\rho}), G_{H}(\acute{\rho})\}, \max\{U_{\tilde{N}}(\acute{\rho}), U_{H}(\acute{\rho})\}, \max\{F_{\tilde{N}}(\acute{\rho}), F_{H}(\acute{\rho})\}): \acute{\rho} \in W\}$.

(iii) $\tilde{N} \cup \bar{H} = \{(\acute{\rho}, \max\{T_{\tilde{N}}(\acute{\rho}), T_{H}(\acute{\rho})\}, \max\{C_{\tilde{N}}(\acute{\rho}), C_{H}(\acute{\rho})\}, \min\{G_{\tilde{N}}(\acute{\rho}), G_{H}(\acute{\rho})\}, \min\{U_{\tilde{N}}(\acute{\rho}), U_{H}(\acute{\rho})\}, \min\{F_{\tilde{N}}(\acute{\rho}), F_{H}(\acute{\rho})\}): \acute{\rho} \in W\}$.

(iv) $\tilde{N}^c = \{(\acute{\rho}, F_{\tilde{N}}(\acute{\rho}), U_{\tilde{N}}(\acute{\rho}), 1-G_{\tilde{N}}(\acute{\rho}), C_{\tilde{N}}(\acute{\rho}), T_{\tilde{N}}(\acute{\rho})): \acute{\rho} \in W\}$ and $\bar{H}^c = \{(\acute{\rho}, F_{H}(\acute{\rho}), U_{H}(\acute{\rho}), 1-G_{H}(\acute{\rho}), C_{H}(\acute{\rho}), T_{H}(\acute{\rho})): \acute{\rho} \in W\}$.

Definition 2.3. [21] Assume that $\tilde{\hat{A}}$ be a fixed set. A set $\tilde{\hat{A}}(I) = <\tilde{\hat{A}} \cup I>$ generated by $\tilde{\hat{A}}$ and I is referred to as an neutrosophic set. The members of $\tilde{\hat{A}}(I)$ are of the form $(\acute{\rho}, yI)$, where $\acute{\rho}$ and y are elements of $\tilde{\hat{A}}$. I is referred to as an indeterminate and it has the property $I^n = I$ for all positive integer n.

Definition 2.4. [21] Let us consider a classical subtraction algebra $(\tilde{\hat{A}}, -)$, and let $\tilde{\hat{A}}(I) = <\tilde{\hat{A}} \cup I>$ be a set generated by $\tilde{\hat{A}}$ and I. Consider the neutrosophic algebraic structure $(\tilde{\hat{A}}(I), -_N)$ where for all $(\hat{a}, \tilde{e}I)$, $(\ddot{w}, \acute{\rho}I) \in (\tilde{\hat{A}}(I), -_N)$ is defined by $(\hat{a}, \tilde{e}I) -_N (\ddot{w}, \acute{\rho}I) = (\hat{a} - \ddot{w}, (\tilde{e} - \acute{\rho})I)$ $\forall \hat{a}, \tilde{e}, \ddot{w}, \acute{\rho} \in \tilde{\hat{A}}$.
We denote $(\tilde{\hat{A}}(I), -_N)$ as a neutrosophic subtraction algebra.
An element $\acute{\rho} \in \tilde{\hat{A}}$ is represented by $(\acute{\rho}, 0) \in \tilde{\hat{A}}(I)$ and $(0, 0)$ represents the constant element in $\tilde{\hat{A}}(I)$.
Definition 2.5. [21] Assume that $(\tilde{\hat{A}}(I), -_N)$ be a neutrosophic subtraction algebra. Then, a non-empty subset $Ħ(I)$ is referred to as a neutrosophic subtraction sub-algebra of $\tilde{\hat{A}}(I)$ if the following conditions hold:
(i) If $Ħ(I) \neq \emptyset$
(ii) $(\hat{a}, \tilde{e}I) -_N (\ddot{w}, \acute{\rho}I) \in Ħ(I)$ for all $(\hat{a}, \tilde{e}I), (\ddot{w}, \acute{\rho}I) \in Ħ(I)$.
(iii) $Ħ(I)$ contains a proper subset which is a subtraction algebra.
If $Ħ(I)$ does not contain a proper subset which is a subtraction algebra, then $Ħ(I)$ is referred to as a pseudo neutrosophic subtraction sub-algebra of $\tilde{\hat{A}}(I)$.

3. Pentapartitioned Neutrosophic Subtraction Algebra:

In this section we established the concept of pentapartitioned neutrosophic subtraction algebra on P-N-Ss, and established several results on it in the form of theorems, propositions, etc.

Definition 3.1. Assume that $\tilde{\hat{A}}$ be a fixed set. Then, a set $\tilde{\hat{A}}(P_N) = <\tilde{\hat{A}} \cup C \cup G \cup U>$ which is generated by $\tilde{\hat{A}}$ and P_N is referred to as a pentapartitioned neutrosophic set. The members of $\tilde{\hat{A}}(P_N)$ are of the form $(\hat{a}, \tilde{e}C, \ddot{w}G, ư U)$, where $\hat{a}, \tilde{e}, \ddot{w}$, and $ư$ are the elements of $\tilde{\hat{A}}$. Here, C, G, U are called contradiction, ignorance, unknown and it has the property $C^n = C$, $G^n = G$ and $U^n = U$ for all positive integer n.
Definition 3.2. Assume that $(\tilde{\hat{A}}, -)$ be any classical subtraction algebra. Suppose that $\tilde{\hat{A}}(P_N) = <\tilde{\hat{A}} \cup C \cup G \cup U>$ be a set generated by $\tilde{\hat{A}}$ and P_N. Consider the pentapartitioned neutrosophic algebraic structure $(\tilde{\hat{A}}(P_N), -_N)$ where for all $(\hat{a}, \tilde{e}C, \ddot{w}G, ưU)$, $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ư_1U) \in (\tilde{\hat{A}}(I), -_N)$ is defined as follows:
$(\hat{a}, \tilde{e}C, \ddot{w}G, ưU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ư_1U) = (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ư - ư_1)U)$, $\forall\ \hat{a}, \tilde{e}, \ddot{w}, ư, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ư_1 \in \tilde{\hat{A}}$.
We call $(\tilde{\hat{A}}(P_N), -_N)$ a pentapartitioned neutrosophic subtraction algebra.
An element $\acute{\alpha} \in \tilde{\hat{A}}$ is represented by $(\acute{\alpha}, 0, 0, 0) \in \tilde{\hat{A}}(P_N)$ and $(0, 0, 0, 0)$ represents the constant element in $\tilde{\hat{A}}(P_N)$.
Theorem 3.1. Every pentapartitioned neutrosophic subtraction algebra $(\tilde{\hat{A}}(P_N), -_N)$ is a subtraction algebra.
Proof. Assume that $(\tilde{\hat{A}}(P_N), -_N)$ is a subtraction algebra. Assume that $\acute{\alpha} = (\hat{a}, \tilde{e}C, \ddot{w}G, ưU)$, $y = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ư_1U)$, $z = (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ư_2U) \in \tilde{\hat{A}}(P_N)$, where $\hat{a}, \tilde{e}, \ddot{w}, ư, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ư_1, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, ư_2 \in \tilde{\hat{A}}$.
(i) We have,
$\acute{\alpha} -_N (y -_N \acute{\alpha})$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ưU) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ư_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ưU))$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ưU) -_N ((\hat{a}_1-\hat{a}), (\tilde{e}_1-\tilde{e})C, (\ddot{w}_1-\ddot{w})G, (ư_1-ư)U)$
$= (\hat{a} - (\hat{a}_1 - \hat{a}), (\tilde{e} - (\tilde{e}_1 - \tilde{e}))C, (\ddot{w} - (\ddot{w}_1 - \ddot{w}))G, (ư - (ư_1 - ư))U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, ưU)$ Since $\hat{a}, \tilde{e}, \ddot{w}, ư \in \tilde{\hat{A}}$
$= \acute{\alpha}$
(ii) We have,
$\acute{\alpha} -_N (\acute{\alpha} -_N y)$

$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U))$

$= (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (((\hat{a}_1-\hat{a}), (\tilde{e}_1-\tilde{e})C, (\ddot{w}_1-\ddot{w})G, (ự_1-ự)U))$

$= ((\hat{a} - (\hat{a} - \hat{a}_1)), (\tilde{e} - (\tilde{e} - \tilde{e}_1))C, (\ddot{w} - (\ddot{w} - \ddot{w}_1))G, (ự - (ự - ự_1))U)$

$= ((\hat{a}_1 - (\hat{a}_1 - \hat{a})), (\tilde{e}_1 - (\tilde{e}_1 - \tilde{e}))C, (\ddot{w}_1 - (\ddot{w}_1 - \ddot{w}))G, (ự_1 - (ự_1 - ự))U)$ Since $\hat{a}, \tilde{e}, \ddot{w}, ự, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ự_1 \in \tilde{\tilde{A}}$.

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N ((\hat{a}_1-\hat{a}), (\tilde{e}_1-\tilde{e})C, (\ddot{w}_1-\ddot{w})G, (ự_1-ự)U)$

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU))$

$= y -_N (y -_N \acute{\alpha})$

(iii) We have,

$(\acute{\alpha} -_N y) -_N z$

$= ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= ((\hat{a}- \hat{a}_1), (\tilde{e}- \tilde{e}_1)C, (\ddot{w}- \ddot{w}_1)G, (ự- ự_1)U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= (((\hat{a} - \hat{a}_1)-\hat{a}_2), ((\tilde{e} - \tilde{e}_1) -\tilde{e}_2)C, ((\ddot{w} - \ddot{w}_1) -\ddot{w}_2)G, ((ự - ự_1) -ự_2)U)$

$= (((\hat{a} - \hat{a}_2)-\hat{a}_1), ((\tilde{e} - \tilde{e}_2) -\tilde{e}_1)C, ((\ddot{w} - \ddot{w}_2) -\ddot{w}_1)G, ((ự - ự_2) -ự_1)U)$ Since $\hat{a}, \tilde{e}, \ddot{w}, ự, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ự_1, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, ự_2 \in \tilde{\tilde{A}}$.

$= ((\hat{a}- \hat{a}_2), (\tilde{e}- \tilde{e}_2)C, (\ddot{w}- \ddot{w}_2)G, (ự- ự_2)U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$

$= ((\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U)$

$= (\acute{\alpha} -_N z) -_N y$

From Vasantha and Smarandache [30] and Ibrahim et al. [18], we note that, if $\acute{\alpha} \le y$ then we cannot in general say $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and $\acute{\alpha}U \le yU$ it may so happen that $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and $\acute{\alpha}U \le yU$. Thus, the pentapartitioned neutrosophic order in general needs not to preserve the order. If a set $\tilde{\tilde{A}}$ is ordered under "≤" then the pentapartitioned neutrosophic part of $<\tilde{\tilde{A}} \cup C \cup G \cup U>$ may or may not have the preservations of order under ≤; i.e., if $\acute{\alpha} \le y$, $\acute{\alpha}, y \in \tilde{\tilde{A}}$ then $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and$\acute{\alpha}U \le yU$ may occur or may not occur. For the work of the partial ordering we consider suppose $\acute{\alpha}C \le yC$, $\acute{\alpha}G \le yG$, and $\acute{\alpha}U \le yU$ occur.

Theorem 3.2. For a pentapartitioned neutrosophic subtraction algebra $(\tilde{\tilde{A}}(P_N), -_N)$, the relation "≤" is a partial order relation on $\tilde{\tilde{A}}(P_N)$.

Proof. Assume that $\acute{\alpha} = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU)$, $\mu = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U)$, $\theta = (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, ự_2U) \in \tilde{\tilde{A}}(P_N)$ with $\hat{a}, \tilde{e}, \ddot{w}, ự, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, ự_1, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, ự_2 \in \tilde{\tilde{A}}$.

(i) Since $\acute{\alpha}-\acute{\alpha} = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a} - \hat{a},(\tilde{e} - \tilde{e})C, (\ddot{w} - \ddot{w})G, (ự - ự)U) = (0, 0C, 0G, 0U)$ then $\acute{\alpha}\le \acute{\alpha}$.

(ii) Suppose that $\acute{\alpha}\le \mu$ and $\mu\le \acute{\alpha}$.

Then, $(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) = (0, 0C, 0G, 0U)$ implies $(\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (ự - ự_1)U) = (0, 0C, 0G, 0U)$

and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (0, 0C, 0G, 0U)$ implies $((\hat{a}_1-\hat{a}), (\tilde{e}_1-\tilde{e})C, (\ddot{w}_1-\ddot{w})G, (ự_1-ự)U) = (0, 0C, 0G, 0U)$.

Now, we have

$(\hat{a}, \tilde{e}C, \ddot{w}G, ựU) = (\hat{a}, \tilde{e}C, \ddot{w}G, ựU) -_N (0, 0C, 0G, 0U)$

$= (\hat{a} - 0, (\tilde{e} - 0)C, (\ddot{w} - 0)G, (ự - 0)U)$

$= ((\hat{a} - (\hat{a} - \hat{a}_1)), (\tilde{e} - (\tilde{e} - \tilde{e}_1))C, (\ddot{w} - (\ddot{w} - \ddot{w}_1))G, (ự - (ự - ự_1))U)$ Since, $\hat{a} - \hat{a}_1 =0$, $\tilde{e} - \tilde{e}_1 = 0$, $\ddot{w} - \ddot{w}_1 = 0$ and $ự - ự_1 = 0$.

$= ((\hat{a}_1 - (\hat{a}_1 - \hat{a})), (\tilde{e}_1 - (\tilde{e}_1 - \tilde{e}))C, (\ddot{w}_1 - (\ddot{w}_1 - \ddot{w}))G, (ự_1 - (ự_1 - ự))U)$ Since $\tilde{\tilde{A}}$ is a subtraction algebra

$= ((\hat{a}_1 - 0), (\tilde{e}_1 - 0)C, (\ddot{w}_1 - 0)G, (ự_1 - 0)U)$ Since, $\hat{a} - \hat{a}_1 = 0$, $\tilde{e} - \tilde{e}_1 = 0$, $\ddot{w} - \ddot{w}_1 = 0$ and $ự - ự_1 = 0$.

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, ự_1U) -_N (0, 0C, 0G, 0U)$

$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)$.

(iii) Let $\acute{\alpha} \leq \mu$ and $\mu \leq \theta$. Therefore, $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) \leq (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) \leq (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)$.

Then, $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) = (0, 0C, 0G, 0U)$ implies $(\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (\underset{.}{u} - \underset{.}{u}_1)U) = (0, 0C, 0G, 0U)$

and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) = (0, 0C, 0G, 0U)$ implies $(\hat{a}_1\text{-}\hat{a}_2), (\tilde{e}_1\text{-}\tilde{e}_2)C, (\ddot{w}_1\text{-}\ddot{w}_2)G, (\underset{.}{u}_1\text{-}\underset{.}{u}_2)U) = (0, 0C, 0G, 0U)$.

Now, we have

$(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)$

$= (\hat{a} - \hat{a}_2, (\tilde{e} - \tilde{e}_2)C, (\ddot{w} - \ddot{w}_2)G, (\underset{.}{u} - \underset{.}{u}_2)U)$

$= (\hat{a} - \hat{a}_2, (\tilde{e} - \tilde{e}_2)C, (\ddot{w} - \ddot{w}_2)G, (\underset{.}{u} - \underset{.}{u}_2)U)$

$= ((\hat{a} - \hat{a}_2)\text{-}0, ((\tilde{e} - \tilde{e}_2)\text{-}0)C, ((\ddot{w} - \ddot{w}_2)\text{-}0)G, ((\underset{.}{u} - \underset{.}{u}_2)\text{-}0)U)$

$= ((\hat{a} - \hat{a}_2)\text{-}(\hat{a} - \hat{a}_1), ((\tilde{e} - \tilde{e}_2)\text{-}(\tilde{e} - \tilde{e}_1))C, ((\ddot{w} - \ddot{w}_2)\text{-} (\ddot{w} - \ddot{w}_1))G, ((\underset{.}{u} - \underset{.}{u}_2)\text{-}(\underset{.}{u} - \underset{.}{u}_1))U)$

$= ((\hat{a} \text{-}(\hat{a} - \hat{a}_1)) - \hat{a}_2), (((\tilde{e} \text{-}(\tilde{e} - \tilde{e}_1)) - \tilde{e}_2))C, (((\ddot{w} \text{-} (\ddot{w} - \ddot{w}_1)) - \ddot{w}_2))G, (((\underset{.}{u} \text{-}(\underset{.}{u} - \underset{.}{u}_1)) - \underset{.}{u}_2))U)$

$= ((\hat{a}_1\text{-}(\hat{a}_1 - \hat{a})) - \hat{a}_2), (((\tilde{e}_1\text{-}(\tilde{e}_1 - \tilde{e})) - \tilde{e}_2))C, (((\ddot{w}_1\text{-} (\ddot{w}_1 - \ddot{w})) - \ddot{w}_2))G, (((\underset{.}{u}_1\text{-}(\underset{.}{u}_1 - \underset{.}{u})) - \underset{.}{u}_2))U)$

$= ((\hat{a}_1 - \hat{a}_2)\text{-}(\hat{a}_1 - \hat{a}), ((\tilde{e}_1 - \tilde{e}_2)\text{-} (\tilde{e}_1 - \tilde{e}))C, ((\ddot{w}_1 - \ddot{w}_2)\text{-} (\ddot{w}_1 - \ddot{w}))G, ((\underset{.}{u}_1 - \underset{.}{u}_2)\text{-}(\underset{.}{u}_1 - \underset{.}{u}))U)$

$= (0\text{-}(\hat{a}_1 - \hat{a}), (0\text{-} (\tilde{e}_1 - \tilde{e}))C, (0\text{-} (\ddot{w}_1 - \ddot{w}))G, (0\text{-}(\underset{.}{u}_1 - \underset{.}{u}))U)$

$= (0, 0C, 0G, 0U)$

Hence, $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) \leq (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)$. Consequently, "$\leq$" is a partial order relation.

Proposition 3.1. Assume that $(\tilde{\hat{A}}(P_N), -_N)$ be a pentapartitioned neutrosophic subtraction algebra . If $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) \in \tilde{\hat{A}}(P_N)$, with $\hat{a}, \tilde{e}, \ddot{w}, \underset{.}{u} \in \tilde{\hat{A}}$, then the following are true*:*

(i) $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (0, 0C, 0G, 0U) = (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U)$;

(ii) $(0, 0C, 0G, 0U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) = (0, 0C, 0G, 0U)$;

(iii) $((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) = (0, 0C, 0G, 0U)$;

(iv) $((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) = (\hat{a}, \tilde{e}I) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)$;

(v) $((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U)) = (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)$;

(vi) $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))) = (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)$;

(vii) $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) \leq (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)$;

(viii) $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) \Leftrightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) = (0, 0C, 0G, 0U)$ *and* $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) = (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U)$.

Proof. (i) We have, $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (0, 0C, 0G, 0U)$

$= (\hat{a} - 0, (\tilde{e} - 0)C, (\tilde{e} - 0)G, (\tilde{e} - 0)U) = (\hat{a} - (\hat{a} - \hat{a}), (\tilde{e} - (\tilde{e} - \tilde{e}))C, (\tilde{e} - (\tilde{e} - \tilde{e}))G, (\tilde{e} - (\tilde{e} - \tilde{e}))U) = (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U)$.

(ii) $(0, 0C, 0G, 0U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) = (0\text{-}\hat{a}, (0\text{-}\tilde{e})C, (0\text{-}\ddot{w})G, (0\text{-}\underset{.}{u})U) = (0-(\hat{a}-0), (0-(\tilde{e}-0))C, (0-(\tilde{e}-0))G, (0-(\tilde{e}-0))U) = (0, 0C, 0G, 0U)$.

(iii) We have, $((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U)$

$= (\hat{a}\text{-}\hat{a}_1, (\tilde{e}\text{-}\tilde{e}_1)C, (\ddot{w}\text{-}\ddot{w}_1)G, (\underset{.}{u}\text{-}\underset{.}{u}_1)U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U)$

$= ((\hat{a} - \hat{a}_1) - \hat{a}, ((\tilde{e} - \tilde{e}_1) - \tilde{e})C, ((\ddot{w} - \ddot{w}_1) - \ddot{w})G, ((\underset{.}{u} - \underset{.}{u}_1) - \underset{.}{u})U)$

$= ((\hat{a} - \hat{a}) - \hat{a}_1, ((\tilde{e} - \tilde{e}) - \tilde{e}_1)C, ((\ddot{w} - \ddot{w}) - \ddot{w}_1)G, ((\underset{.}{u} - \underset{.}{u}) - \underset{.}{u}_1)U)$

$= (0-\hat{a}_1, (0-\tilde{e}_1)C, (0-\ddot{w}_1)G, (0-\text{ự}_1)U)$
$= (0, 0C, 0G, 0U)$.

(iv) $((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) = (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (\text{ự} - \text{ự}_1)U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.
$= (((\hat{a} - \hat{a}_1)-\hat{a}_1), ((\tilde{e} - \tilde{e}_1) - \tilde{e}_1)C, ((\ddot{w} - \ddot{w}_1) - \ddot{w}_1)G, ((\text{ự} - \text{ự}_1) - \text{ự}_1)U)$
$= ((\hat{a} - \hat{a}_1) - (\hat{a}_1 - (\hat{a} - \hat{a}_1)), ((\tilde{e} - \tilde{e}_1) - (\tilde{e}_1 - (\tilde{e} - \tilde{e}_1))C, ((\ddot{w} - \ddot{w}_1) - (\ddot{w}_1 - (\ddot{w} - \ddot{w}_1))G, ((\text{ự} - \text{ự}_1) - (\text{ự}_1 - (\text{ự} - \text{ự}_1))U$
$= (\hat{a}-\hat{a}_1, (\tilde{e}-\tilde{e}_1)C, (\ddot{w}-\ddot{w}_1)G, (\text{ự}-\text{ự}_1)U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.

(v) $((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U)) = (\hat{a}-\hat{a}_1, (\tilde{e}-\tilde{e}_1)C, (\ddot{w}-\ddot{w}_1)G, (\text{ự}-\text{ự}_1)U) -_N ((\hat{a}_1-\hat{a}), (\tilde{e}_1-\tilde{e})C, (\ddot{w}_1-\ddot{w})G, (\text{ự}_1-\text{ự})U) = ((\hat{a}-\hat{a}_1)- (\hat{a}_1-\hat{a}), ((\tilde{e}-\tilde{e}_1)- (\tilde{e}_1-\tilde{e}))C, (\ddot{w}-\ddot{w}_1)- (\ddot{w}_1-\ddot{w}))G, (\text{ự}-\text{ự}_1)- (\text{ự}_1-\text{ự}))U) = ((\hat{a} - (\hat{a}_1 - \hat{a})) - \hat{a}_1, ((\tilde{e} - (\tilde{e}_1 - \tilde{e})) - \tilde{e}_1)C, ((\ddot{w} - (\ddot{w}_1 - \ddot{w})) - \ddot{w}_1)G, ((\text{ự} - (\text{ự}_1 - \text{ự})) - \text{ự}_1)I)$
$= (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (\text{ự} - \text{ự}_1)U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.

(vi) $(\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U))) = (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a} - \ddot{w}, (\tilde{e} - \text{ự})I)) = (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (\text{ự} - \text{ự}_1)U)$.
$= (\hat{a} - (\hat{a} - (\hat{a} - \hat{a}_1)), (\tilde{e} - (\tilde{e} - (\tilde{e} - \tilde{e}_1)))C, (\ddot{w} - (\ddot{w} - (\ddot{w} - \ddot{w}_1)))G, (\text{ự} - (\text{ự} - (\text{ự} - \text{ự}_1)))U)$
$= ((\hat{a} - \hat{a}_1) - ((\hat{a} - \hat{a}_1) - \hat{a}), ((\tilde{e} - \tilde{e}_1) - ((\tilde{e} - \tilde{e}_1) - \tilde{e}))C, ((\ddot{w} - \ddot{w}_1) - ((\ddot{w} - \ddot{w}_1) - \ddot{w}))G, ((\text{ự} - \text{ự}_1) - ((\text{ự} - \text{ự}_1) - \text{ự}))U)$
Since from the properties of $\tilde{\hat{A}}$, if $\hat{a}, \hat{a}_1 \in \tilde{\hat{A}}$ then $(\hat{a} - \hat{a}_1) - \hat{a} = 0$ then we have
$((\hat{a} - \hat{a}_1) - ((\hat{a} - \hat{a}_1) - \hat{a}), ((\tilde{e} - \tilde{e}_1) - ((\tilde{e} - \tilde{e}_1) - \tilde{e}))C, ((\ddot{w} - \ddot{w}_1) - ((\ddot{w} - \ddot{w}_1) - \ddot{w}))G, ((\text{ự} - \text{ự}_1) - ((\text{ự} - \text{ự}_1) - \text{ự}))U)$
$= ((\hat{a} - \hat{a}_1) - 0, ((\tilde{e} - \tilde{e}_1) - 0)C, ((\ddot{w} - \ddot{w}_1) - 0)G, ((\text{ự} - \text{ự}_1) - 0)U)$
$= (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (\text{ự} - \text{ự}_1)U)$
$= (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.

(vi) $((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U))) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) = ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)) = (0, 0C, 0G, 0U) \Rightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)) \leq (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.

(vii) Suppose that $(\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) = (0, 0C, 0G, 0U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (0, 0C, 0G, 0U)$. Then $(\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (0, 0C, 0G, 0U)$.
$= (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N ((\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U))$
$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U))$
$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (0, 0C, 0G, 0U)$.
$= (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.
$\Rightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$.
Conversely, suppose that $(\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U)$
Then, $(\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) = (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (0, 0C, 0G, 0U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) = (0, 0C, 0G, 0U) \Rightarrow (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) = (0, 0C, 0G, 0U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \text{ự}_1U) -_N (\hat{a}, \tilde{e}C, \ddot{w}G, \text{ự}U) = (0, 0C, 0G, 0U)$.

Definition 3.3. Assume that $\tilde{\tilde{A}}_1(P_N)$ and $\tilde{\tilde{A}}_2(P_N)$ be two pentapartitioned neutrosophic subtraction algebra. Then, the direct product of $\tilde{\tilde{A}}_1(P_N)$ and $\tilde{\tilde{A}}_2(P_N)$ is denoted by $\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N)$ and defined as follows:

$\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N) = \{((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) : (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{.}{u}U) \in \tilde{\tilde{A}}_1(P_N), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) \in \tilde{\tilde{A}}_2(P_N)\}$.

Proposition 3.2. Assume that $(\tilde{\tilde{A}}_1(P_N), -_N)$ and $(\tilde{\tilde{A}}_2(P_N), -_N)$ be two pentapartitioned neutrosophic subtraction algebra. Then, $(\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N), -_N)$ is also a pentapartitioned neutrosophic subtraction algebra.

Proof. Suppose that $\acute{\alpha} = ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))$, $\mu = ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))$, $\theta = ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U)) \in \tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N)$, for all $\hat{a}_0, \tilde{e}_0, \ddot{w}_0, \underset{.}{u}_0, \hat{a}_2, \tilde{e}_2, \ddot{w}_2, \underset{.}{u}_2, \hat{a}_4, \tilde{e}_4, \ddot{w}_4, \underset{.}{u}_4 \in \tilde{\tilde{A}}_1$ and $\hat{a}_1, \tilde{e}_1, \ddot{w}_1, \underset{.}{u}_1, \hat{a}_3, \tilde{e}_3, \ddot{w}_3, \underset{.}{u}_3, \hat{a}_5, \tilde{e}_5, \ddot{w}_5, \underset{.}{u}_5 \in \tilde{\tilde{A}}_2$. Therefore,

(i) We have,

$\acute{\alpha} -_N (\mu -_N \acute{\alpha})$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))]$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [(\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)]$

$= (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U)), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N ((\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) = ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))$

$= \acute{\alpha}$.

(ii) We have,

$\acute{\alpha} -_N (\acute{\alpha} -_N \mu)$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))]$

$= ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))]$

$= (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))$

$= (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U) -_N (\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U)), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N ((\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))$

$= ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N [((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N ((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U))]$

$= \mu -_N (\mu -_N \acute{\alpha})$.

(iii) We have,

$(\acute{\alpha} -_N \mu) -_N \theta$

$= [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U)) -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))] -_N ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U))$

$= [((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U))] -_N ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U))$

$= (((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{.}{u}_0U) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{.}{u}_2U)) -_N (\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{.}{u}_4U), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{.}{u}_1U) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{.}{u}_3U)) -_N (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{.}{u}_5U))$

= $(((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{\cdot}{\mu}_0U) -_N (\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{\cdot}{\mu}_4U)) -_N (\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{\cdot}{\mu}_2U), ((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) -_N (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{\cdot}{\mu}_5U)) -_N (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{\cdot}{\mu}_3U))$

= $[((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{\cdot}{\mu}_0U) -_N (\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{\cdot}{\mu}_4U)),((\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) -_N (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{\cdot}{\mu}_5U))] -_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{\cdot}{\mu}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{\cdot}{\mu}_3U))$

=$[((\hat{a}_0, \tilde{e}_0C, \ddot{w}_0G, \underset{\cdot}{\mu}_0U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U)) -_N ((\hat{a}_4, \tilde{e}_4C, \ddot{w}_4G, \underset{\cdot}{\mu}_4U), (\hat{a}_5, \tilde{e}_5C, \ddot{w}_5G, \underset{\cdot}{\mu}_5U))]$

$-_N ((\hat{a}_2, \tilde{e}_2C, \ddot{w}_2G, \underset{\cdot}{\mu}_2U), (\hat{a}_3, \tilde{e}_3C, \ddot{w}_3G, \underset{\cdot}{\mu}_3U))$

= $(\acute{\alpha} -_N \theta) -_N \mu$.

Proposition 3.3. Assume that $\tilde{\tilde{A}}_1(P_N)$ be a pentapartitioned neutrosophic subtraction algebra. Suppose that A be a classical subtraction algebra. Then, the structure $(\tilde{\tilde{A}}_1(P_N)\times A, -_N)$ is a pentapartitioned neutrosophic subtraction algebra.

Definition 3.4. Assume that $(\tilde{\tilde{A}}(P_N), -_N)$ be a pentapartitioned neutrosophic subtraction algebra. Then, if the following criteria are met, a non-empty subset $A(P_N)$ is referred to as a neutrosophic subtraction sub-algebra of $\tilde{\tilde{A}}(P_N)$:

(i) $A(P_N) \neq \emptyset$;

(ii) $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) \in A(P_N)$, for all $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) \in A(P_N)$;

(iii) $A(P_N)$ contains a proper subset which is a subtraction algebra.

$A(P_N)$ is referred to as a pseudo pentapartitioned neutrosophic subtraction sub-algebra of $\tilde{\tilde{A}}(P_N)$ if it does not contain a proper subset that is a subtraction algebra.

Definition 3.5. Suppose that $(\tilde{\tilde{A}}, -, *)$ be any subtraction semi-group, and assume that $\tilde{\tilde{A}}(P_N) = <\tilde{\tilde{A}} \cup C \cup G \cup U>$ be a set generated by $\tilde{\tilde{A}}$ and P_N. Then, $(\tilde{\tilde{A}}(P_N), -_N, *)$ is referred to as a pentapartitioned neutrosophic subtraction semi-group. Let $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U)$ and $(\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U)$ be any two elements of $\tilde{\tilde{A}}(P_N)$ with $\hat{a}, \tilde{e}, \ddot{w}, \underset{\cdot}{\mu}, \hat{a}_1, \tilde{e}_1, \ddot{w}_1, \underset{\cdot}{\mu}_1 \in \tilde{\tilde{A}}$. Then we define the following:

$$(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U) -_N (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) = (\hat{a} - \hat{a}_1, (\tilde{e} - \tilde{e}_1)C, (\ddot{w} - \ddot{w}_1)G, (\underset{\cdot}{\mu} - \underset{\cdot}{\mu}_1)U),$$

and $(\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U) * (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) = (\hat{a}\hat{a}_1, (\tilde{e}\hat{a}_1+\hat{a}\tilde{e}_1+ \tilde{e}\tilde{e}_1)C, (\ddot{w}\hat{a}_1+\hat{a}\ddot{w}_1+ \ddot{w}\ddot{w}_1)G, (\underset{\cdot}{\mu}\hat{a}_1+\hat{a}\underset{\cdot}{\mu}_1+ \underset{\cdot}{\mu}\underset{\cdot}{\mu}_1)U)$.

Definition 3.6. The direct product of two pentapartitioned neutrosophic subtraction semi-groups $\tilde{\tilde{A}}_1(P_N)$ and $\tilde{\tilde{A}}_2(P_N)$ is denoted by $\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N)$ and defined as follows:

$\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N) = \{((\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U)) : (\hat{a}, \tilde{e}C, \ddot{w}G, \underset{\cdot}{\mu}U) \in \tilde{\tilde{A}}_1(P_N), (\hat{a}_1, \tilde{e}_1C, \ddot{w}_1G, \underset{\cdot}{\mu}_1U) \in \tilde{\tilde{A}}_2(P_N)\}$.

Proposition 3.5. Assume that $(\tilde{\tilde{A}}_1(P_N), -_N, *)$ and $(\tilde{\tilde{A}}_2(P_N), -_N, *)$ be two pentapartitioned neutrosophic subtraction semi-group. Then, $(\tilde{\tilde{A}}_1(P_N) \times \tilde{\tilde{A}}_2(P_N), -_N, *)$ is also a pentapartitioned neutrosophic subtraction semi-group.

Proposition 3.6. Assume that $(\tilde{\tilde{A}}(P_N), -_N, *)$ be a pentapartitioned neutrosophic subtraction semi-group, and suppose that $(A, -, *)$ be a classical subtraction semi-group. Then, $(\tilde{\tilde{A}}(P_N) \times A, -_N, *)$ is a pentapartitioned neutrosophic subtraction semi-group.

4. Conclusions:

The basic ideas of pentapartitioned neutrosophic subtraction semi-group and pentapartitioned neutrosophic subtraction algebra have been examined in this paper. Further, the basic properties of subtraction algebra and subtraction semi-group have been analyzed and established. We believe that numerous fresh investigations can be executed in the years to come by utilizing the concepts of pentapartitioned neutrosophic subtraction semi-group and pentapartitioned neutrosophic subtraction algebra.

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On NCT- Set Theory

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Abstract: We introduce a new class of neutrosophic crisp set, and then it presented some operations via this sets like, $NCT-$intersection, $NCT-$union and algebraic $NCT-$difference to arrive at the algebraic ring construction. Also, we introduce the $NCT-$ points, which showed that a NCT –set is $NCT-$union of its $NCT-$points, and so did we introduced the concept of the function on this sets, called NCT – function and some of their important properties. Finally, we introduced the concept of the topology and some of the concepts entrusted to it, as an introduction to those spaces that can be studied in detail in the future.

Keywords: NCT – sets; $NCT-$points; $NCT-$function and $NCT-$topological spaces.

1. Introduction

The main focus of this research is the construction of a new type of neutrosophic crisp sets, and the first to know these sets neutrosophic and neutrosophic crisp sets is the scientist Florentin between the years 1999-2005 [1-3], when we looking at these sets, we notice that they are determined within spaces $\boldsymbol{X} \times [\mathbf{0}, \mathbf{1}]^{\mathbf{3}}$ and $\boldsymbol{P}(\boldsymbol{X}) \times \boldsymbol{P}(\boldsymbol{X}) \times \boldsymbol{P}(\boldsymbol{X})$, respectively when Zadeh [4] defined the fuzzy sets in 1965 , which were identified in space $\boldsymbol{X} \times [\mathbf{0}, \mathbf{1}]$, and through this, Salama and Florentin generalized these sets , which he called neutrosophic sets [5-7]. The researcher Almohammed [8] invested in the fuzzy sets by finding a new definition of the local function in 2020. Imran et al. [9-11] provided the view of new types of weakly neutrosophic crisp continuity, new concepts of weakly neutrosophic crisp separation axioms, and new concepts of neutrosophic crisp open sets. Molodtsov [12] found a new type of sets at the $\boldsymbol{E} \times \boldsymbol{P}(\boldsymbol{X})$ (where $\boldsymbol{X}$ is universal set and $\boldsymbol{E}$ the parameters of elements of $\boldsymbol{X}$) spaces and named them soft sets, where Al-Swidi and others [13-16] invested these sets by linking them with the fuzzy sets as well as defining new points, which in turn obtained equivalents for the separation axioms. Tomma et al. [17-19] gave the view of stable neutrosophic crisp topological space, necessary and sufficient conditions for a stability of the concepts of stable interior and stable exterior via neutrosophic crisp sets, and confused crisp set stable neutrosophic topological spaces. Al-Tamimi et al. [20] provided partner sets for generalizations of multi neutrosophic sets. Sfook et al. [21] introduced neutrosophic crisp grill topological spaces. Abdulsada et al. [22,23] provided the view of separation axioms of center topological space, and Center set theory of proximity space. Finally, the senses of new types of weakly neutrosophic crisp open mappings and new types of weakly neutrosophic crisp closed functions were informed by Al-Obaidi et al. [24,25]. In this research, we introduce a new concept of neutrosophic crisp set, and then it presented some operations via this sets like, $\boldsymbol{NCT}-$intersection, $\boldsymbol{NCT}-$union and algebraic

$\boldsymbol{NCT}$ −difference to arrive at the algebraic ring construction. Also, we introduce the $\boldsymbol{NCT}-$ points, which showed that a $\boldsymbol{NCT}$ –set is $\boldsymbol{NCT}$ −union of its $\boldsymbol{NCT}$ −points, and so did we introduced the class of the function on this sets, called $\boldsymbol{NCT}$ – function and some of their important properties.

2. NCT- Sets

We presented a new class of neutrosophic crisp sets, complete with a neutrosophic crisp point, all operations are binary, a ring-building qualification, and boolean algebra.

Definition 2.1. Let $X \neq \emptyset$. A neutrosophic crisp triple set NCT_A is an object having the form $NCT_A = \langle A_1, A_2, A_3 \rangle$. Where $A_1, A_2, A_3 \subseteq X$ satisfying $A_1 \subseteq A_2$ and $A_2 \cap A_3 = \emptyset$. And $NCT(X) = \{NCT_A = \langle A_1, A_2, A_3 \rangle : A_1 \subseteq A_2$ and $A_2 \cap A_3 = \emptyset\}$ is the collection of all $NCT-$ sets on X.
From this definition we see that if $A_3 = X$, then $A_1 = A_2 = \emptyset$ and if $A_2 = X$, then $A_3 = \emptyset$, finally if $A_1 = X$, then $A_2 = X$ and $A_3 = \emptyset$.

Definition 2.2. Let $NCT_A = \langle A_1, A_2, A_3 \rangle$ and $NCT_B = \langle B_1, B_2, B_3 \rangle$ are NCT −a non-empty set X over sets. Therefore:

1. NCT_A is a $NCT-$ subset of NCT_B if $A_3 \supseteq B_3$ and $A_1 \subseteq B_1, A_2 \subseteq B_2$. We write $NCT_A \sqsubseteq NCT_B$.
2. $NCT_A = NCT_B$ iff $NCT_A \sqsubseteq NCT_B$ and $NCT_B \sqsubseteq NCT_A$.
3. The NCT −complement of $NCT-$ set NCT_A is $CNCT_A = \langle A_3, A_2^c, A_1 \rangle$.
4. $NCT_A \sqcup NCT_B = \langle A_1 \cup B_1, A_2 \cup B_2, A_3 \cap B_3 \rangle$ is a crisp triple set that is neutrosophic in union (NCT −union set).
5. $NCT_A \sqcap NCT_B = \langle A_1 \cap B_1, A_2 \cap B_2, A_3 \cup B_3 \rangle$ is the intersection a crisp triple set that is neutrosophic (NCT −intersection sets).
6. $NCT_A - NCT_B = NCT_A \sqcap CNCT_B$.
7. $NCT_A \ \triangle\ NCT_B = (NCT_A \sqcap CNCT_B) \sqcup (CNCT_A \sqcap NCT_B)$.

Now we will explain the concepts NCT −universel and NCT −null set which are among the basic concepts in our work.

Definition 2.3. Let $X \neq \emptyset$. Then:

1. $NCT_X = \langle X, X, \emptyset \rangle$ is NCT −universel set.
2. $NCT_\varphi = \langle \emptyset, \emptyset, X \rangle$ is NCT −null set. Clearly $CNCT_X = NCT_\varphi$ and $CNCT_\varphi = NCT_X$.

The three most significant relationships NCT −union, NCT −intersection and NCT −complement listed in the following properties.

Proposition 2.4. Let $NCT_S = \langle S_1, S_2, S_3 \rangle$ and $NCT_J = \langle J_1, J_2, J_3 \rangle$ are $NCT-$ a nonempty set X over sets. Then:

1. $NCT_S \sqcap NCT_S = NCT_S$.
2. $NCT_S \sqcup NCT_\varphi = NCT_S$.
3. $NCT_S \sqcup NCT_X = NCT_X$.
4. $NCT_S \sqcap NCT_\varphi = NCT_\varphi$.
5. $C(NCT_S \sqcup NCT_J) = CNCT_S \sqcap CNCT_J$.
6. $C(NCT_S \sqcap NCT_J) = CNCT_S \sqcup CNCT_J$.
7. $NCT_S \sqcup NCT_S = NCT_S$.

Proving the above proposition directly by applying Definitions 2.1, 2.2 and 2.3.
For any $NCT_A = \langle A_1, A_2, A_3 \rangle$, we get the following two properties , but the opposite is not necessarily true $NCT_\varphi \sqsubseteq NCT_A \sqcap CNCT_A$ and $NCT_A \sqcup CNCT_A \sqsubseteq NCT_X$ if $X = \{q_1, q_2, q_3\}$ and $NCT_A =$

$\langle\{q_1\},\{q_1,q_2\},\{q_3\}\rangle$,then $NCT_A \sqcap CNCT_A = \langle\emptyset,\emptyset,\{q_1,q_3\}\rangle \not\sqsubseteq NCT_\varphi = \langle\emptyset,\emptyset,X\rangle$. And, if $X = \{q_1,q_2,q_3\}$ and $NCT_A = \langle\{q_1\},\{q_1,q_2\},\{q_3\}\rangle$, then $NCT_X = \langle X,X,\emptyset\rangle \not\sqsubseteq (NCT_A \sqcup CNCT_A) = \langle\{q_1,q_3\},X,\emptyset\rangle$.

The following proposition shows the algebraic properties (associative and distributive laws) of these $NCT-$ sets via $NCT-$ union and $NCT-$intersection relations.

Proposition 2.5. Let $NCT_O = \langle O_1,O_2,O_3\rangle$, $NCT_Q = \langle Q_1,Q_2,Q_3\rangle$ and $NCT_J = \langle J_1,J_2,J_3\rangle$ over a nonempty set X, be three $NCT-$sets. Then:

1. $NCT_O \sqcap (NCT_Q \sqcup NCT_J) = (NCT_O \sqcap NCT_Q) \sqcup (NCT_O \sqcap NCT_J)$.
2. $NCT_O \sqcap (NCT_Q \sqcap NCT_J) = (NCT_O \sqcap NCT_Q) \sqcap NCT_J$.
3. $NCT_O \sqcup (NCT_Q \sqcup NCT_J) = (NCT_O \sqcup NCT_Q) \sqcup NCT_J$.
4. $NCT_O \sqcup (NCT_Q \sqcap NCT_J) = (NCT_O \sqcup NCT_Q) \sqcap (NCT_O \sqcup NCT_J)$.

New we defined the $NCT-$union and $NCT-$intersection relations on any collection of $NCT-$sets.

Definition 2.6. Let $\{NCT_{A_i} : i \in I\}$ be a $NCT-$sets in X, and $NCT_{A_i} = \langle A_{i1},A_{i2},A_{i3}\rangle$. Then

1. $\sqcup_{i\in I} NCT_{A_i} = \langle \cup_{i\in I} A_{i1}, \cup_{i\in I} A_{i2}, \cap_{i\in I} A_{i3}\rangle$.
2. $\sqcap_{i\in I} NCT_{A_i} = \langle \cap_{i\in I} A_{i1}, \cap_{i\in I} A_{i2}, \cup_{i\in I} A_{i3}\rangle$.

Proposition 2.7. Let NCT_A, NCT_B, NCT_C and $\{NCT_{A_i} : i \in I\}$ in X.Then

1. $NCT_A \sqsubseteq NCT_B$ and $NCT_B \sqsubseteq NCT_C$, implies $NCT_A \sqsubseteq NCT_C$.
2. $NCT_{A_i} \sqsubseteq NCT_B$ $\forall i \in I$, then $\sqcup_{i\in I} NCT_{A_i} \sqsubseteq NCT_B$.
3. $NCT_B \sqsubseteq NCT_{A_i}$ $\forall i \in I$, then $NCT_B \sqsubseteq \sqcap_{i\in I} NCT_{A_i}$.
4. $NCT_A \sqsubseteq NCT_B$ iff $CNCT_B \sqsubseteq CNCT_A$.

Proof: Obvious.

Now we present the definition of points $NCT-$points.

Definition 2.8. Let $X \neq \emptyset$ and $p \in X$. So $NCT-$points ($NCTP$) are structure:

1. $NCT_{\tilde{p}} = \langle\{p\},\{p\},\{p\}^c\rangle$.
2. $NCT_{\tilde{\tilde{p}}} = \langle\emptyset,\{p\},\{p\}^c\rangle$.
3. $NCT_{\tilde{\tilde{\tilde{p}}}} = \langle\emptyset,\emptyset,\{p\}^c\rangle$.

It's simple to see that $NCT-$points are $NCT-$sets. Also, the cardinal number of all $NCT-$points is $3n$, where n is the cardinal number of universal sets X.

Definition 2.9. Let $X \neq \emptyset$ and $p \in X$ and $NCT_A = \langle A_1,A_2,A_3\rangle$. Then the $NCT-$belong as follows:

1. $NCT_{\tilde{p}}$ € NCT_A iff $p \in A_1$ and $NCT_{\tilde{p}} \breve{€}$ NCT iff $p \notin A_1$.
2. $NCT_{\tilde{\tilde{p}}}$ € NCT_A iff $p \in A_2$ and $NCT_{\tilde{\tilde{p}}} \breve{€}$ NCT iff $p \notin A_2$.
3. $NCT_{\tilde{\tilde{\tilde{p}}}}$ € NCT_A iff $p \notin A_3$ and $NCT_{\tilde{\tilde{\tilde{p}}}} \breve{€}$ NCT iff $p \notin A_3$.

We now take the properties of belonging to the three points.

Proposition 2.10. Let $\{NCT_{A_i} : i \in I\}$ is $NCT-$set in X. Then

1. $NCT_{\tilde{p}}$ € $\sqcap_{i\in I} NCT_{A_i}$ iff $NCT_{\tilde{p}}$ € NCT_{A_i} for each $i \in I$.
2. $NCT_{\tilde{\tilde{p}}}$ € $\sqcap_{i\in I} NCT_{A_i}$ iff $NCT_{\tilde{\tilde{p}}}$ € NCT_{A_i} for each $i \in I$.
3. $NCT_{\tilde{\tilde{\tilde{p}}}}$ € $\sqcap_{i\in I} NCT_{A_i}$ iff $NCT_{\tilde{\tilde{\tilde{p}}}}$ € NCT_{A_i} for each $i \in I$.
4. $NCT_{\tilde{p}}$ € $\sqcup_{i\in I} NCT_{A_i}$ iff $\exists i \in I$ such that $NCT_{\tilde{p}}$ € NCT_{A_i}.
5. $NCT_{\tilde{\tilde{p}}}$ € $\sqcup_{i\in I} NCT_{A_i}$ iff $\exists i \in I$ such that $NCT_{\tilde{\tilde{p}}}$ € NCT_{A_i}.
6. $NCT_{\tilde{\tilde{\tilde{p}}}}$ € $\sqcup_{i\in I} NCT_{A_i}$ iff $\exists i \in I$ $\ni NCT_{\tilde{\tilde{\tilde{p}}}}$ € NCT_{A_i}.

Proposition 2.11. Let NCT_A and NCT_B is $NCT-$set in X. Then:

i. $NCT_A \sqsubseteq NCT_B$ iff $\forall NCT_{\tilde{P}}$ with $NCT_{\tilde{P}}$ € $NCT_A \Longrightarrow NCT_{\tilde{P}}$ € NCT_B , for each $NCT_{\tilde{\tilde{P}}}$ with $NCT_{\tilde{\tilde{P}}}$ € $NCT_A \Longrightarrow NCT_{\tilde{\tilde{P}}}$ € NCT_B and for each $NCT_{\tilde{\tilde{\tilde{P}}}}$ we have $NCT_{\tilde{\tilde{\tilde{P}}}}$ € $NCT_A \Longrightarrow NCT_{\tilde{\tilde{\tilde{P}}}}$ € NCT_B.

ii. $NCT_A = NCT_B$ iff for each $NCT_{\tilde{P}}$ we have $NCT_{\tilde{P}}$ € $NCT_A \Leftrightarrow NCT_{\tilde{P}}$ € NCT_B and for each $NCT_{\tilde{\tilde{P}}}$ we have $NCT_{\tilde{\tilde{P}}}$ € $NCT_A \Leftrightarrow NCT_{\tilde{\tilde{P}}}$ € NCT_B and for each $NCT_{\tilde{\tilde{\tilde{P}}}}$ we have $NCT_{\tilde{\tilde{\tilde{P}}}}$ € $NCT_A \Leftrightarrow NCT_{\tilde{\tilde{\tilde{P}}}}$ € NCT_B.

Proposition 2.12. Let $NCT_A = \langle A_1, A_2, A_3 \rangle$ be triple in X . Then $NCT_A = (\{NCT_{\tilde{P}} : NCT_{\tilde{P}}$ € $NCT_A\}) \sqcup \left(\{NCT_{\tilde{\tilde{P}}} : NCT_{\tilde{\tilde{P}}}\right.$ € $\left.NCT_A\}\right) \sqcup \left(\left\{NCT_{\tilde{\tilde{\tilde{P}}}} : NCT_{\tilde{\tilde{\tilde{P}}}}\right.\right.$ € $\left.\left.NCT_A\right\}\right)$.

Proposition 2.13. Let $NCT_S = \langle S_1, S_2, S_3 \rangle$, $NCT_J = \langle J_1, J_2, J_3 \rangle$ and $NCT_R = \langle R_1, R_2, R_3 \rangle$ be $NCT-$sets. Then:

1. $NCT_S \sqcup NCT_J \sqsupseteq (NCT_S \sqcap NCT_J) \sqcup (NCT_J - NCT_S) \sqcup (NCT_S - NCT_J)$.
2. $(NCT_S - NCT_J) \sqcup (NCT_S - NCT_R) = NCT_S - (NCT_J \sqcap NCT_R)$.
3. $(NCT_S \sqcup NCT_J) - (NCT_R - NCT_S) = NCT_S \sqcup (NCT_J - NCT_R)$.
4. $(NCT_S \sqcap NCT_J) - (CNCT_S \sqcup NCT_R) = NCT_S \sqcap (NCT_J - NCT_R)$.
5. Not necessary if $NCT_S \sqsubseteq NCT_J$ and $NCT_S \sqsubseteq CNCT_J$, then $NCT_S = NCT_\varphi$.
6. Not necessary if $NCT_S \sqsubseteq NCT_J$ and $CNCT_S \sqsubseteq NCT_J$ then $NCT_S = NCT_X$.
7. $(NCT_S \sqcup NCT_J) - NCT_J \sqsupseteq NCT_S - NCT_J$.
8. $NCT_S - NCT_J \sqsubseteq NCT_S - (NCT_S \sqcap NCT_J)$.
9. $\sqcap \{NCT_J \in NCT_S(X)\} = NCT_\varphi$.
10. $NCT_S \sqcap CNCT_J \sqsubseteq (NCT_S \sqcup NCT_J) \sqcap CNCT_J$.
11. $(NCT_S \sqcup NCT_J) - (NCT_J \sqcap NCT_S) \sqsupseteq (NCT_S - NCT_J) \sqcup (NCT_J - NCT_S)$.
12. $NCT_S - (NCT_J \sqcup NCT_R) = (NCT_S - NCT_J) - NCT_R$.
13. $NCT_S \ \varDelta NCT_\varphi = NCT_S$.
14. $NCT_\varphi = NCT_S \ \varDelta NCT_S$ if and only if $S_1 \cup S_3 = X$.
15. $NCT_S \ \varDelta \ NCT_J = NCT_J \ \varDelta \ NCT_S$.

Proof.

The converse of part (1) is not true in general for example, if $X = \{o_1, o_2, o_3\}$, $NCT_A = \langle \{o_1\}, \{o_1, o_2\}, \{o_3\} \rangle$ and $NCT_B = \langle \{o_2\}, \{o_2, o_3\}, \{o_1\} \rangle$, then: $NCT_A - (NCT_A \sqcap NCT_B) = \langle \{o_1\}, \{o_1\}, \{o_3\} \rangle \not\sqsubseteq NCT_A - NCT_B = \langle \{o_1\}, \{o_1\}, \{o_2, o_3\} \rangle$.

The converse of part 2 is not true generally, for instance, if $X = \{e_1, e_2, e_3\}$, $NCT_A = \langle \{e_1\}, \{e_1, e_2\}, \{e_3\} \rangle$ and $NCT_B = \langle \{o_2\}, \{o_2, o_3\}, \{o_1\} \rangle$,then: $NCT_A \sqcup NCT_B = \langle \{o_1, o_2\}, X, \emptyset \rangle \not\sqsubseteq (NCT_A \sqcap NCT_B) \sqcup (NCT_B - NCT_A) \sqcup (NCT_A - NCT_B) = \langle \{o_1\}, \{o_1, o_3\}, \emptyset \rangle$.

Part 7 , Let $X = \{o_1, o_2, o_3\}$, $NCT_A = \langle \emptyset, \emptyset, \{o_1, o_2\} \rangle$ and $NCT_B = \langle \emptyset, \emptyset, \{o_1\} \rangle$, then $NCT_A \sqsubseteq NCT_B$,$NCT_A \sqsubseteq CNCT_B$ and $NCT_A \neq NCT_\varphi$.

Part 8 , Let $X = \{o_1, o_2, o_3\}$, $NCT_A = \langle X, X, \{o_1, o_2\} \rangle$ and $NCT_B = \langle X, X, \{o_1\} \rangle$, then $NCT_A \sqsubseteq NCT_B$, $CNCT_A \sqsubseteq NCT_B$ and $NCT_A \neq NCT_X$.

Since Proposition 2.13, part 12 assures that the NCT −null set serves as an identification element for Proposition 13.2, part 13 ensures that every member of $NC^*(X) = \{NCT_A = \langle A_1, A_2, A_3 \rangle : NCT_A \ is \ NCT-set \ and \ A_1 \cup A_3 = X\}$.

Finally, the fact that part 14 has its own inverse demonstrates that the symbol is commutative. All of this lends credence to the contention that $(NC^*(X) \ , \varDelta)$ is a commutative group.

Theorem 2.14. Let X is non-null and $NC^*(X) = \{NCT_S = \langle S_1, S_2, S_3 \rangle : NCT_A \ is \ NCT-set \ and \ S_1 \cup S_3 = X\}$ on X. So $(NC^*(X), \varDelta, \sqcap)$ form a ring.

Proof.

According to propositions 2.4 and propositions 2.5, the groups $(NC^*(X),\sqcap)$ and $(NC^*(X)\ ,\varDelta)$ are semigroups and commutative groups. Only the distribution on the left must be examined $\sqcap$ operation on $\varDelta$.

$(NCT_S \sqcap NCT_W)\varDelta(NCT_S \sqcap NCT_P) = \{(NCT_S \sqcap NCT_W) \sqcap (CNCT_S \sqcup CNCT_P\ )\} \sqcup \{(NCT_S \sqcap NCT_P\ ) \sqcap (CNCT_S \sqcup CNCT_W)\} = \{[(NCT_S \sqcap NCT_W) \sqcap CNCT_S] \sqcup [(NCT_S \sqcap NCT_W) \sqcap CNCT_P\ ]\} \sqcup \{[(NCT_S \sqcap NCT_P) \sqcap CNCT_S] \sqcup [(NCT_S \sqcap NCT_P) \sqcap CNCT_W]\} = \{(NCT_S \sqcap NCT_W) \sqcap CNCT_P\ \} \sqcup \{(NCT_S \sqcap NCT_P) \sqcap CNCT_W\} = NCT_S \sqcap (NCT_W \varDelta NCT_P)$. Therefore $(NC^*(X),\sqcap\ ,\varDelta)$ is a ring.

Now we introduce the concept of NCT-function.

Definition 2.15. Let $f : X \to Y$ be a function. Define NCT –function $f_{NCT} : NCT(X) \to NCT(Y)$ by:

1. If $NCT_A = \langle A_1, A_2, A_3\rangle \in NCT_A(X)$,then $f_{NCT}(NCT_A) = \langle f(A_1), f(A_2), f-(A_3)\rangle$, where $f-(A_3) = Y - (f(X - A_3))$.
2. If $NCT_B = \langle R_1, R_2, R_3\rangle \in NCT_B(Y)$, then ${f_{NCT}}^{-1}(NCT_B) = \langle f^{-1}(R_1), f^{-1}(R_2), f^{-1}(R_3)\rangle$.

We now take the most important properties of the NCT-function that we will adopt in our research.

Proposition 2.16. Let $f_{NCT} : NCT(X) \to NCT(Y)$ be a $NCT -$ function and NCT_A, $NCT_{A_i}(i \in I) \in NCT(X), NCT_B\ , NCT_{B_j}(j \in J) \in NCT(Y)$. Then:

1. $NCT_{A_1} \sqsubseteq NCT_{A_2} \Longrightarrow f_{NCT}(NCT_{A_1}) \sqsubseteq f_{NCT}(NCT_{A_2})$.
2. $NCT_{B_1} \sqsubseteq NCT_{B_2} \Longrightarrow {f_{NCT}}^{-1}(NCT_{B_1}) \sqsubseteq {f_{NCT}}^{-1}(NCT_{B_2})$.
3. If $NCT_A \sqsubseteq {f_{NCT}}^{-1}(f_{NCT}(NCT_A))$ and f is 1-1, then $NCT_A = {f_{NCT}}^{-1}(f_{NCT}(NCT_A))$.
4. If $f_{NCT}\left({f_{NCT}}^{-1}(NCT_B)\right) \sqsubseteq NCT_B$ and f is onto, then $f_{NCT}\left({f_{NCT}}^{-1}(NCT_B)\right) = NCT_B$
5. ${f_{NCT}}^{-1}\left(\sqcup NCT_{B_j}\right) = \sqcup {f_{NCT}}^{-1}(NCT_{B_j})$.
6. ${f_{NCT}}^{-1}\left(\sqcap NCT_{B_j}\right) = \sqcap {f_{NCT}}^{-1}(NCT_{B_j})$.
7. $f_{NCT}(\sqcup NCT_{A_i}) = \sqcup f_{NCT}(NCT_{A_i})$.
8. $f_{NCT}(\sqcap NCT_{A_i}) \sqsubseteq (\sqcap f_{NCT}(NCT_{A_i}))$ and if f is 1-1, then $f_{NCT}(\sqcap NCT_{A_i}) = \sqcap f_{NCT}(NCT_{A_i})$.
9. ${f_{NCT}}^{-1}(NCT_Y) = NCT_X$.
10. ${f_{NCT}}^{-1}(NCT_\varphi) = NCT_\varphi$.
11. $f_{NCT}(NCT_X) = NCT_Y$, if f is onto.
12. $f_{NCT}(NCT_\varphi) = NCT_\varphi$.

Proof.

Let $NCT_{A_i} = \langle A_{i1}, A_{i2}, A_{i3}\rangle$, $NCT_{B_j} = \langle B_{j1}, B_{j2}, B_{j3}\rangle$, $(i \in I, j \in J)$, $NCT_A = \langle A_1, A_2, A_3\rangle$ and $NCT_B = \langle B_1, B_2, B_3\rangle$.

1. Let $NCT_{A_1} \sqsubseteq NCT_{A_2}$. Since $A_{11} \subseteq A_{21}, A_{12} \subseteq A_{22}$ and $A_{23} \subseteq A_{13}$,then $f(A_{11}) \subseteq f(A_{21}), f(A_{12}) \subseteq f(A_{22})$ and $X - A_{13} \subseteq X - A_{23} \Longrightarrow f(X - A_{13}) \subseteq f(X - A_{23}) \Longrightarrow Y - f(X - A_{23}) \subseteq Y - f(X - A_{13}) \Longrightarrow f-(A_{23}) \subseteq f-(A_{13})$. Hence $f_{NCT}(NCT_{A_1}) \sqsubseteq f_{NCT}(NCT_{A_2})$.
2. It is similar to (1.).
3. ${f_{NCT}}^{-1}(f_{NCT}(NCT_A)) = {f_{NCT}}^{-1}(f_{NCT}(\langle A_1, A_2, A_3\rangle)) = {f_{NCT}}^{-1}(\langle f(A_1), f(A_2), f-(A_3)\rangle) = \langle f^{-1}(f(A_1)), f^{-1}(f(A_2)), f^{-1}(f-(A_3))\rangle \sqsupseteq \langle A_1, A_2, A_3\rangle = NCT_A$.
4. It is similar to (3.).
5. ${f_{NCT}}^{-1}\left(\sqcup NCT_{B_j}\right) = {f_{NCT}}^{-1}(\langle \cup B_{j1}, \cup B_{j2}, \cap B_{j3}\rangle) = \langle f^{-1}(\cup B_{j1}), f^{-1}(\cup B_{j2}), f^{-1}(\cap B_{j3})\rangle = \langle \cup f^{-1}(B_{j1}), \cup f^{-1}(B_{j2}), \cap f^{-1}(B_{j3})\rangle = \sqcup {f_{NCT}}^{-1}(NCT_{B_j})$.
6. It is similar to (5.).

7. $f_{NCT}(\sqcup NCT_{A_i}) = f_{NCT}(\langle \cup A_{i1}, \cup A_{i2}, \cap A_{i3}\rangle) = \langle f(\cup A_{i1}), f(\cup A_{i2}), f-(\cap A_{i3})\rangle = \langle \cup f(A_{i1}), \cup f(A_{i2}), \cap f-(A_{i3})\rangle = \sqcup F(NCT_{A_i})$. Noties that $f-(\cap A_{i3}) = Y - f(X - \cap A_{i3}) = Y - f(\cup (X - A_{i3}) = Y - \cup f(X - A_{i3}) = \cap (Y - f(X - A_{i3})) = \cap f-(A_{i3})$.
8. $f_{NCT}(\sqcap NCT_{A_i}) = f_{NCT}(\langle \cap A_{i1}, \cap A_{i2}, \cup A_{i3}\rangle) = \langle f(\cap A_{i1}), f(\cap A_{i2}), f-(\cup A_{i3})\rangle \sqsubseteq \langle \cap f(A_{i1}), \cap f(A_{i2}), \cup f-(A_{i3})\rangle = \sqcap F(NCT_{A_i})$. Noties that $f-(\cup A_{i3}) = Y - f(X - \cup A_{i3}) = Y - f(\cap (X - A_{i3}) \supseteq Y - \cap f(X - A_{i3}) = \cup (Y - f(X - A_{i3})) = \cup f-(A_{i3})$.
9. ${f_{NCT}}^{-1}(NCT_Y) = {f_{NCT}}^{-1}(\langle Y, Y, \emptyset\rangle) = \langle f^{-1}(Y), f^{-1}(Y), f^{-1}(\emptyset)\rangle = \langle X, X, \emptyset\rangle = NCT_X$.

(10.), (11.), (12.) are like (9.).

3. Neutrosophic Crisp Triple Topological Space

In this section, we investigate some of the properties generated by NCT −sets, such as interior, exterior and boundary $NCT-$ points, which are the structure for all topological concepts, as well as closures.

Definition 3.1. The pair (NCT_X, τ^X_{NCT}) is called neutrosophic crisp triple topological space ($NCTT$) over NCT (X) , if you achieve the following:

1. $NCT_X, NCT_\varphi \in \tau^X_{NCT}$ ($\in$ is the classical belonging).
2. τ^X_{NCT} is closed under the finite NCT −intersection.
3. τ^X_{NCT} is closed under the NCT −union of every subfamily of τ_{NCT} .

Any member of τ_{NCT} is called NCT −open and the complement is called NCT −closed.

- i- For any NCT- set NCT_A the NCT – interior of NCT_A is of the form NCT −int (NCT_A) = $\sqcup\{ NCT_P ;\ \exists NCT_H \in \tau_{NCT} \ni NCT_P \in NCT_H \sqsubseteq NCT_A \}$. From this definition we can show that NCT −int (NCT_A) $= \sqcup\{ NCT_H \in \tau^X_{NCT} ;\ NCT_H \sqsubseteq NCT_A \}$.
- $NCT_A \in \tau_{NCT}$ iff NCT −int (NCT_A) $= NCT_A$.
- ii- For any NCT-set NCT_A the NCT −closure of NCT_A is of the form $NCT-cl(NCT_A) = \sqcup\{ NCT_P;\ \forall NCT_H \in \tau^X_{NCT} \ni NCT_P \in NCT_H \ni NCT_A \sqcap NCT_H \neq NCT_\varphi \}$.
- From above we can show that $NCT-cl\ (NCT_A) = \sqcap\{ NCT_F ;\ CNCT_F \in \tau_{NCT} \ni NCT_A \sqsubseteq NCT_F \}$.
- NCT_F is NCT −closed iff $NCT-cl(NCT_F) = NCT_F$.
- iii- For any NCT −set NCT_A the NCT −exterior of NCT_A is of the form NCT −ext (NCT_A) = NCT −int ($CNCT_A$).
- iv- For any NCT −set NCT_A the $NCT-$ boundary of NCT_A is of the form NCT −fr (NCT_A) = $\sqcup\{ NCT_P ;\ NCT_P$ not NCT −interior and NCT −exterior point of NCT-A}.

So, from definition and properties above we can concluded.

1- $NCT_X = NCT -$int (NCT_A) $\sqcup$ NCT − ext(NCT_A) $\sqcup$ NCT − fr (NCT_A), for any $NCT-$set NCT_A.

2- NCT −int (NCT_A) = C(NCT-cl (CNCT_A)) and $NCT-cl$(NCT_A) = C(NCT −int($CNCT_A$)).

Lemma 3.2. If NCT_H is $NCT-$open set and any $NCT-$set NCT_A , then $NCT_H \sqcap$ NCT − cl(NCT_A) $\sqsubseteq$ NCT − cl($NCT_A \sqcap NCT_H$).

Proof.

Let $NCT_P \in NCT_H \sqcap$ NCT − cl(NCT_A) , if possible, that $NCT_P \breve{\notin}$ NCT − cl($NCT_A \sqcap NCT_H$) .Then there is some $NCT-$open set NCT_K containing NCT_P and $NCT_K \sqcap NCT_A \sqcap NCT_H = NCT_\varphi$, but $NCT_K \sqcap NCT_H \in \tau_{NCT}$ and $NCT_P \in NCT_K \sqcap NCT_H$, this shows that $NCT_P \breve{\notin}$ NCT − cl(NCT_A) , which contradiction . This obligates us to $NCT_P \in$ NCT − cl($NCT_A \sqcap NCT_H$).

Proposition 3.3. Let $(NCT_X \ ,\tau_{NCT}^X)$be any $NCTT$ over $NCT(X)$,then the following properties are hold:

1- If $NCT_H \in \tau_{NCT}^X$ or $NCT_K \in \tau_{NCT}^X$, then $NCT-\mathrm{int}\ (NCT-cl(NCT_K \sqcap NCT_H)) = NCT-int(NCT-cl\ (NCT_K)) \sqcap \mathrm{NCT-int(NCT-cl}(NCT_H)))$.

2- $NCT-int\ (\ NCT-cl\ (\ NCT-int\ (NCT-c\ \mathrm{l}\ (\ NCT_A\ )))) = NCT-int\ (NCT-cl\ (NCT_A))$.

3- $NCT-cl\ (\ NCT-int\ (\ NCT-cl\ (NCT-int\ (NCT_A\ )))) = NCT-cl\ (NCT\ int\ (NCT_A))$.

4- $NCT-int\ (\ NCT-cl\ (\ NCT-int\ (NCT_A \sqcap NCT_B))) = NCT-int\ (NCT-cl\ (NCT-int\ (NCT_A))) \sqcap \mathrm{NCT-int}\ (\ \mathrm{NCT-cl(NCT-int}\ (NCT_B))$.

5- $NCT-cl\ (NCT-int\ (NCT-cl\ (NCT_A \sqcup NCT_B\ ))) = NCT-cl\ (NCT-int\ (NCT-cl\ (NCT_A))) \sqcup NCT-cl(\mathrm{NCT-int}\ (\ \mathrm{NCT-cl}\ (NCT_B)))$.

Proof.

(1) Since $NCT_H \sqsubseteq NCT-cl(NCT_H)$ and $NCT_K \sqsubseteq NCT-cl(NCT_K)$. So, $NCT-int\ (NCT-cl\ (NCT_K \sqcap NCT_H) \sqsubseteq NCT-int(NCT-cl\ (NCT_K) \sqcap \mathrm{NCT-cl}(\ NCT_H)) = NCT-int\ (NCT-cl\ (NCT_K)) \sqcap \mathrm{NCT-int(NCT-cl}(NCT_{H)})$. Conversely, If $NCT_H \in \tau_{NCT}^X$ or $NCT_K \in \tau_{NCT}^X$, then by Lemma 3.2. For if $NCT_H \in \tau_{NCT}$, $NCT_H \sqcap NCT-int\big(\ \mathrm{NCT-cl}(\ NCT_K)\big) \sqsubseteq NCT-int(NCT-cl\ (NCT_H \sqcap NCT_K))$. (1)

Imply that $NCT-int\ (NCT-cl\ (NCT_H \sqcap NCT-int(\ \mathrm{NCT-cl}(\ NCT_K)))) \sqsubseteq NCT-int(NCT-cl(NCT-int(NCT-cl\ (NCT_H \sqcap NCT_K))))$. (2)

But $NCT-int(NCT-cl(NCT_K\ )) \in \tau_{NCT}^X$, again, by Lemma 3.2, we get that $NCT-int\big(NCT-cl(NCT_H)\big) \sqcap \mathrm{NCT-int(NCT-int(NCT-cl}(NCT_K))) = NCT-int\big(NCT-cl(NCT_H)\big) \sqcap \mathrm{NCT-int(NCT-cl}(NCT_K)) \sqsubseteq NCT-int(\ NCT-cl[NCT_H \sqcap \mathrm{NCT-int}\big(\mathrm{NCT-cl}(\ NCT_K)\big)])$. (3)

From eq. (2) and (3), we get the following inequality $NCT-int\big(NCT-cl(NCT_H)\big) \sqcap \mathrm{NCT-int(NCT-cl}(NCT_K)) \sqsubseteq NCT-int(\ NCT-cl(NCT-int(NCT-cl(NCT_H \sqcap NCT_K\ ))))$. (4)

But $NCT-int\ (NCT-cl(NCT-int\ (NCT-cl(NCT_H \sqcap NCT_K)))) \sqsubseteq NCT-int(\ NCT-cl(\ NCT_H \sqcap NCT_K\ ))$. (5)

From eq. (3) and (4) we get the result.

(2) Let $NCT_H = NCT-int(NCT-cl(NCT_A))$,to show that $NCT_H = NCT-int(NCT-cl(NCT_H))$ .Since $NCT-int(NCT-cl(NCT_H)) = NCT-int(NCT-cl\big(\ NCT-int\big(NCT-cl(NCT_A)\big)\big) \sqsubseteq NCT-int(NCT-cl(NCT-cl(NCT_A)))) = NCT-int(NCT-cl(NCT_A)) = NCT_H$. (6)

But $NCT-int(NCT_H) = NCT-int(NCT-int\big(NCT-cl(NCT_A)\big)) = NCT-int\big(NCT-cl(NCT_A)\big) = NCT_H \sqsubseteq NCT-int(NCT-cl(NCT_H))$. (7)

From eq. (6) and (7) we get the result. Similarly, we can prove part (3). To proof 4 and 5, direct from part (1).

Definition 3.4. $NCT-$ function $f_{NCT}\colon(\ NCT_X\ ,\tau_{NCT}^X\ ) \longrightarrow (\ NCT_Y\ ,\tau_{NCT}^Y\ )$ is $NCT-$ open (NCT −closed) map, if the image of each set NCT −open (NCT −closed) in NCT_X is NCT −open (NCT −closed) in NCT_Y.

Example 3.5. let $X = \{1,2,3\}, Y = \{a,b,c\}$ and $f: X \longrightarrow Y$ s.t. $f\ (1) = a, f\ (2) = c$ and $f\ (3) = b$. For $\tau_{NCT}^X = \{NCT_X, NCT_\varphi, <\{1\},\{1,2\},\{3\}>\}$, $\tau_{NCT}^Y = \{NCT_Y, NCT_\varphi, <\{b\},\{a,b\},\varphi>\}$,then $f_{NCT}\colon(\ NCT_X\ ,\tau_{NCT}^X\ ) \longrightarrow (\ NCT_Y\ ,\tau_{NCT}^Y\ )$ is not $NCT-$open and not $NCT-$closed. For $\tau_{NCT}^X = \{\ NCT_X, NCT_\varphi, <\{1\},\{1,2\},\{3\}>\ \}$, $\tau_{NCT}^Y = \{\ NCT_Y, NCT_\varphi, <\{b\},\{a,c\},\{b\}>\ \}$,then $f_{NCT}\colon(\ NCT_X\ ,\tau_{NCT}^X\ ) \longrightarrow (\ NCT_Y\ ,\tau_{NCT}^Y\ )$ is NCT −open and NCT −closed.

Theorem 3.6. let $f_{NCT}\colon(\ NCT_X\ ,\tau_{NCT}^X\ ) \longrightarrow (\ NCT_Y\ ,\tau_{NCT}^Y\ )$ is NCT −closed (NCT −open) map, for any $NCT_S \sqsubseteq NCT_Y$ and any $NCT-$ open ($NCT-$ closed) NCT_U containing $f_{NCT}^{-1}(NCT_S\ ), \exists NCT$ −open (NCT −closed) NCT_V containing NCT_S s.t. $f_{NCT}^{-1}(NCT_V\ ) \sqsubseteq NCT_U$.

Proof.

Let $NCT_V = NCT_Y - f_{NCT}(\ NCT_X - \ NCT_U)$, since $f_{NCT}^{-1}(NCT_S\ ) \sqsubseteq NCT_U$, it follows that $NCT_S \sqsubseteq NCT_V$, and because f_{NCT} is $NCT-$closed map, NCT_V $NCT-$open in NCT_Y. Observing that $f_{NCT}^{-1}(NCT_V\ ) = NCT_X - f_{NCT}^{-1}(f_{NCT}(\ NCT_X - \ NCT_U)) \sqsubseteq NCT_X - (NCT_X - NCT_U))$ $=NCT_U$.

Theorem 3.7. The following four properties of $NCT-$ function $f_{NCT}: (\ NCT_X\ , \tau_{NCT}^X\ ) \longrightarrow (\ NCT_Y\ , \tau_{NCT}^Y\ )$are equivalent:

- **i-** f_{NCT} is an $NCT-$open map.
- **ii-** $f_{NCT}(\ NCT - int(NCT_A)) \sqsubseteq NCT - int(f_{NCT}(NCT_A))$ for each NCT_A in NCT_X.
- **iii-** For each $NCT-$point NCT_P and $NCT-$open NCT_U containing it, there exist NCT_V $NCT-$open in NCT_Y containing $f_{NCT}(NCT_P\ )$ s.t. $NCT_V \sqsubseteq\ f_{NCT}(NCT_U)$.

Proof.

Since $NCT - int(NCT_A)\ \sqsubseteq\ NCT_A$, by Proposition 2.16 (1), we have $f_{NCT}(NCT - int(NCT_A)) \sqsubseteq f_{NCT}(NCT_A)$, by (i) $f_{NCT}(NCT - int(NCT_A))$ is $NCT-$ open in NCT_Y and because $NCT - int(f_{NCT}(NCT_A))$ is the $NCT-$union of $NCT-$open sets contained in $f_{NCT}(NCT_A)$. We must have $f_{NCT}(NCT - int(NCT_A)) \sqsubseteq NCT - int(f_{NCT}(NCT_A))$.

ii $\longrightarrow \boldsymbol{i}$, let NCT_U is $NCT-$open in NCT_X , $NCT_U = NCT - int(NCT_U\ )$ and so $f_{NCT}(NCT_U) = f_{NCT}(NCT - int(NCT_U)) \sqsubseteq\ NCT - int\ \ (f_{NCT}(NCT_U))\ \ )) \sqsubseteq\ f_{NCT}(NCT_U)$, thus $f_{NCT}(NCT_U) = NCT - int(f_{NCT}(NCT_U))$ and therefore $f_{NCT}(NCT_U)$ is $NCT-$ open in NCT_Y , that is f_{NCT} is an $NCT-$open map.

$\boldsymbol{i} \longrightarrow \boldsymbol{iii}$, let $NCT_P^X \in\ NCT_U\ \ \in \tau_{NCT}^X$, but $f_{NCT}(\ NCT_P^X \in\ f_{NCT}(NCT_U)\ \ \in \tau_{NCT}^Y$.

$\boldsymbol{iii} \longrightarrow \boldsymbol{i}$, let $NCT_U\ \ \in \tau_{NCT}^X$, by iii, each $NCT_P^Y \in f_{NCT}(\ NCT_U)$ has $NCT-$open NCT_V in NCT_Y s.t. $NCT_V \sqsubseteq\ f_{NCT}(NCT_U) = \sqcup\{NCT_V;\ NCT_P^Y \in\ f_{NCT}(NCT_U)\}$is $NCT-$open in NCT_Y.

Proposition 3.8. $f_{NCT}: (\ NCT_X\ , \tau_{NCT}^X\ ) \longrightarrow (\ NCT_Y\ , \tau_{NCT}^Y\ )$ is $NCT-$ closed map iff $NCT - cl\ \ (f_{NCT}(NCT_A)) \sqsubseteq f_{NCT}(NCT - cl(NCT_A))$, for each NCT_A in NCT_X.

Proof.

Since $NCT - cl(NCT_A)$ is $NCT-$closed in NCT_X , and so $f_{NCT}(NCT - cl(NCT_A))$, is $NCT-$ closed in NCT_Y , since $f_{NCT}(NCT_A)\ \sqsubseteq\ f_{NCT}(NCT - cl(NCT_A))$, obtain $NCT - cl\ \ (f_{NCT}(NCT_A))\ \sqsubseteq\ NCT - cl(f_{NCT}(NCT - cl(NCT_A)))= f_{NCT}(NCT - cl(NCT_A)$. Conversely, if the condition hold and NCT_A is $NCT-$ closed in NCT_X , then $f_{NCT}(NCT_A) \sqsubseteq NCT - cl(f_{NCT}(NCT_A)) \sqsubseteq f_{NCT}(NCT - cl(NCT_A))$ $=f_{NCT}(NCT_A)$, so that $f_{NCT}(NCT_A)$ is $NCT-$closed in NCT_Y.

4. Conclusions

First, Salama and Florentin have established several types of neutrosophic crisp points, but they do not cover spaces, the reason for this is the type of neutrosophic crisp space structure with its unique characteristics to be used for life's problems and scientific problems, and then we proposed four new types of neutrosophic crisp points to enhance these spaces as follows:

- **1-** New conceptual from the NCP points as $P_{N_1} = \langle P_1\ , P_2\ , P_3 \rangle \ni P_i \neq \emptyset$ for $i = 1$ or $i = 2$ or $i = 3$ moreover, there are different focus points empty.
- **2-** $P_{N_2} = \langle P_1\ , P_2\ , P_3 \rangle\ \ni P_i \neq \emptyset$ for $i = 1$ or $i = 2$ or $i = 3$ and other points are individual, and $p_i \subseteq A_i, \forall\ \ i = 1,2,3$ iff $P_{N_i} \in A$.
- **3-** $PN3\ = <\ A1, A2, A3\ >$ is called neutrosophic crisp point, if exactly only one subset $Ai\ \neq \emptyset$, for $i = 1, 2, 3$.
- **4-** $Ai\ \ = \emptyset$ for $i = 1$ or $i = 2$ or $i = 3$ and as for the remaining two sets, the first set is any mono set and the second is its complement.

Second, we can modify most of the important mathematical concepts on $NCT-$ sets to be inputs for future research studies, and here we define the concept of ideal and filter as follows.

I- The sub collection I_{NCT} of $T(X)$ is neutrosophic crisp triple ideal ($NCT-$ideal) if fulfill that:

1. If $NCT_A \in I_{NCT}$ and $NCT_B \sqsubseteq NCT_A$, so $NCT_B \in I_{NCT}$.
2. In general, $I_{NCT}$is closed beneath the finite $NCT -$ union. In general, 2 using this definition, we are able to alter all notions, namely the results and then in the publications.

II- The sub collection $F_{NCT}$of $T(X)$ is neutrosophic crisp triple filter (NCT −filter) if:

1. If $NCT_A \in F_{NCT}$ and $NCT_A \sqsubseteq NCT_B$, so $NCT_B \in F_{NCT}$.
2. $F_{NCT}$is closed under finite NCT −intersection, additionally characterize the vicinity connection on $NCT_A(X)$, The idea of a neutrosophic crisp triple set can also be generalized to the ideas, theories and concepts.

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Assessment of deep learning techniques for bone fracture detection under neutrosophic domain

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Abstract

With the increasing strain on the health system, there is a growing need for automatic medical image diagnosis. Emerging technologies for medical diagnosis can help to achieve the goals of sustainable development. However, analyzing medical images can be challenging due to uncertain data, ambiguity, and impreciseness. To address this issue, we have developed a novel BoneNet-NS technique to classify fractures in X-ray bone images. The proposed approach is based on the power of deep learning (DL) and neutrosophic set (NS) to deal with aleatoric uncertainty. Moreover, we present two frameworks for integrating NS with DL models: BoneNet-NS1 and BoneNet-NS2. We employ various DL models, including Xception, ResNet52V2, DenseNet121, and customized CNN to evaluate both frameworks. Furthermore, 4924 X-ray bone images are utilized to distinguish between fractured and non-fractured classes. The statistical analyses demonstrate that BoneNet-NS2 performs better than BoneNet-NS1 for most DL models. Specifically, using the ResNet52V2 model, our proposed BoneNet-NS2 achieved the highest accuracy, log loss, precision, recall, F1-score, and AUC with values of 99.7%, 0.006, 99.7%, 99.7%, 99.7%, and 99.7%, respectively.

Keywords: Deep Learning; Neutrosophic Set; Bone Fracture Detection; Artificial Intelligence

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1. Introduction

Bone fractures and cracks usually result from exposure to falling, direct blows to the body, collisions, such as traffic accidents or bullet wounds, and injuries resulting from playing sports. These fractures are diagnosed through X-rays, which are considered one of the cheapest types of medical imaging modalities. Furthermore, X-rays can be used in mobile or wsearable devices for quick and accurate fracture detection, utilizing DL algorithms for diagnosis and detection [1]. X-ray images suffer some noise, fuzziness, vagueness, impreciseness, and uncertainty. These aleatoric uncertainties result in low-quality images, bad image contrast, and edge representation.

DL models performance can be affected by aleatoric uncertainty or data uncertainty. This sort of uncertainty is caused by intrinsic noise in the data, such as measurement error or imprecise annotation, which cannot be decreased by gathering more data [2].

Aleatoric uncertainty must be quantified and identified [3] to be eliminated using different techniques, including wavelet thresholding, Gaussian smoothing, and anisotropic filtering. However, these methods often result in losing some image details or creating unrealistic contrast, making it harder to identify diseases. There are numerous deblurring techniques, such as the Richardson-Lucy algorithm, Wiener filter, regularized filter, and inverse filter. These methods can lead to noise amplification, boundary artifacts, and high computational requirements. Similarly, contrast enhancement methods like normalization, histogram equalization, low and high pass, and contrast stretching, can result in abnormal brightness, unnatural appearance, and noise amplification [4].

Soft computing techniques address the widespread imprecision and ambiguity of real-world problems. Fuzzy set (FS) has been presented by Zadeh [5] to handle data uncertainty using membership degree. Many bone fracture classification studies used FS to deal with uncertainty. Vasilakakis et al. [6] introduced wavelet fuzzy phrases (WFP) for feature extraction and bone fracture diagnosis. It extracts textural information from 2D discrete wavelet transform (DWT) images using FS. The words create sentences that represent the image's contents. The approach obtains a classification accuracy of 84%, outperforming other cutting-edge methods.

Intuitionistic Fuzzy Set (IFS) was introduced in 1986 to handle uncertainty by associating each element to membership degree and non-membership degree. In this context, singh et al. [7] proposed a segmentation paradigm for brain MR images that takes into account noise, intensity inhomogeneity (IH), uncertainty, and structural complexity. The framework uses local spatial and gray level information as a local parameter-free fuzzy factor to maintain the quantity of structural features. It incorporates a novel method to IH and employs IFS theory to eliminate uncertainty in assigning membership to pixels near tissue borders. A process is devised to generate artifact-free pictures that may be compared to the original image for expert interpretation.

Soft computing approaches such as FS and IFS aim to handle the uncertainties within data, but they suffer from certain issues, such as [8]:

- In IFS, the sum of membership values is 1.
- IFS and FS can't differentiate between relative truth (truth in at least one world) and absolute truth (truth in all possible worlds).
- Elements in IFS can't be non-standard.
- IFS and FS can't deal with some contradiction paradoxes.

In 1998, Smarandache [8] introduced a neutrosophic set (NS) to deal with higher dimensions of uncertainty. NS can associate each element in the universe with three independent degrees of membership: true, indeterminacy, and false. The values of membership range from $]^{-}0,1^{+}[$ in a non-standard unit interval. NS can deal with some contradictory paradoxes. But in image analysis and processing, many studies deal with interval [0,1] because deal with image intensities in range $]^{-}0,1^{+}[$ is difficult [9].

DL methods show great results and performance in medical image classification. Can et al. [10] introduced an alternative pooling layer, named the common vector approach pooling approach, to solve the restrictions associated with average pooling in DL algorithms. The trials are carried out on a huge dataset containing twenty distinct dental diseases classified into seven groups. Our suggested technique achieved a high accuracy rate of 86.4% for recognizing dental issues across the seven oral categories. Wang et al. [11] proposed a novel intelligent defect diagnosis method based on hybrid DL for chip X-ray images. The system has four stages: image segmentation, normalization, reconstruction, defect identification, contour matching, and qualification diagnostics. The system's efficacy and resilience are tested on real-world inspection lines, with an evaluation accuracy of 92.5%.

In this study, we introduce an integrated framework between DL and NS to handle uncertainty in three degrees of membership for bone fracture classification. The two frameworks employ four different DL models, such as Xception, ResNet152V2, DenseNet121, and customized CNN in terms of accuracy, binary cross entropy loss/log loss, precision, recall, F1-score, and area under curve (AUC). Hence, the main contribution of this paper can be stated as follows:

- We apply the uncertainty handling feature to power DL models. The uncertainty was handled using three degrees of membership: true, indeterminacy, and falsity.
- Two different frameworks were proposed to combine DL with NS, and a comparison was made between them using four different DL models.
- The first framework (BoneNet-NS1) integrates an NS image (true image, indeterminacy image, false image) as input to the DL model.
- In the second framework (BoneNet-NS2), some enhancement on the NS image has been done, and then the NS image is converted to gray-scale images as input to the DL model.
- The proposed work was evaluated on a bone fracture dataset with 4924 images, classified into fractured and non-fractured images.
- The second framework (BoneNet-NS2) shows superior results compared to the first framework (BoneNet-NS1) for most DL models.
- ResNet52V2 shows the highest results using (BoneNet-NS2) in terms of accuracy, log loss, precision, recall, F1-score, and AUC with 99.7 %, 0.006, 99.7%, 99.7%, 99.7%, and 99.7, respectively.

The remainder of the paper is divided as follows. Section 2 provides the related work for this study. Section 3 presents the methodology for the NS and DL algorithms. Moreover, section 4 introduces the steps of the proposed approach. Section 5 presents experimental results. Section 6 provides the managerial Implications and section 7 illustrates the conclusions and future directions of our work.

2. Related work

In this section, we summarize some recent studies that integrate the DL with NS in the medical field. Khalifa et al. [12] investigated the influence of NS on DL models utilizing restricted COVID-19 x-ray datasets. The work used Alexnet, Googlenet, and Restnet18 DL models to transform medical images from grayscale to the NS domain, which includes True (T), Indeterminacy (I), and Falsity (F) images. Over 36 trials were completed, and the Indeterminacy (I) NS domain achieved the highest testing accuracy and performance metrics of 87.1%. Hu et al. [13] introduced the NeutSS-PLP technique for extracting polyp regions from colonoscopy images, which employs a short-connected saliency detection network with NS enhancement. The approach improves specular reflection identification in colonoscopy images by establishing local and global thresholds and defining T, I, and F functions. The approach also incorporates two-level short connections to make use of multi-level and multi-scale capabilities.

Cai et al. [14] proposed an automatic detection method for MCCs that employs NS domain transformation, similar to a standard CAD system. A DCNN1 classifier was developed to distinguish individual MCs while reducing FP MCs. A novel adaptive NRSL technique was used to accelerate the learning process. For cluster-based evaluation, the MCC detection technique achieved 92.5% sensitivity at 0.50 frames per second per image. A strong DCNN2 classifier was developed for diagnosis using automatic detection, with AUCs of 0.908 and 0.872, respectively. The results indicate that the suggested approach considerably enhances the automatic detection and classification of MCCs on FFDMs.

Yasser et al. [15, 16] introduced a reliable and intuitive diagnostic technique for automatically identifying COVID-19 using digital chest X-rays. The tool employs a hybrid architecture that combines NS approaches and ML. Classification characteristics are retrieved from X-ray images utilizing morphological features and PCA. The ML networks divide chest X-rays into positive COVID-19 patients and normal people. Guo and Ashour [17] presented a classification model consisting of two stages: multiple deep convolution neural networks (MDCNN) and NSS approach. The NMDCNN determines reinforced training numbers for each epoch using NSS and then classifies dermoscopic images as malignant or benign using incremental learning and maximum voting. The model's competence was evaluated using the International Skin Imaging Collaboration dataset.

Özyurt et al. [18] introduced the NS-EMFSE technique to classify tumor areas in brain imaging as benign or malignant. CNN characteristics were utilized to classify data, along with support vector machine (SVM) and KNN classifiers. An experimental study of 80 benign and 80 malignant tumors revealed outstanding classification performance with several classifiers, with CNN features outperforming SVM with an average success rate of 95.62%. Another contribution by Talouki et al. [19] presented a novel image completion approach that uses NS-based segmentation to fill in image holes. This strategy decreases spatial and intensity ambiguity, maintains boundaries and homogeneity, and minimizes discontinuity. The method favors outside pixels and employs extended similarity criteria to identify patches with the best match.

Guo et al. [20] presented a deep neural network (DNN) for WBC extraction from blood images, with an emphasis on object indeterminacy in the NS domain. The network uses WBC indeterminacy as a fusion component to enable segmentation into discrete areas. The model surpasses three original encoder-decoder networks, reaching high precision rates and the greatest mean segmentation accuracy (0.95301).

Table 1 summarizes all the aforementioned related works in terms of year, task, disease, modality, dataset, number of images, number of classes, model, and the obtained accuracy. From these studies, it was concluded that NS studies using DL are still growing. Also, there are no studies on bone fracture classification based on NS and DL. So, in our study, we proposed a novel approach that integrates the environment of NS and DL on X-ray images for bone fracture classification.

Table 1 Summary of previous works using NS and DL for medical image analysis

Ref.	Year	Task	Disease	Modality	Dataset	No. images	No.classes	Model	Accuracy
[12]	2021	Classification	COVID-19	X-ray	COVID-19 x-ray dataset	306	4	Alexnet, Googlenet, Restnet1	87.1% for I domain
[13]	2022	Segmentation	Colorectal polyp	-	EndoScene, Kvasir-SEG	EndoScene=912 Kvasir-SEG=1000	EndoScene =8 Kvasir-SEG =4	saliency detection network	EndoScene =0.971,
[14]	2019	Cluster, classification	Breast cancer	Mammograms	NFH dataset	676	2	DCNN	0.813
[15, 16]	2020,2022	Classification	Covid-19	X-ray	COVID-19 Dataset, healthy dataset	570	2	(MFs), (PCA)	98.46%
[17]	2019	classification	Skin cancer	dermoscopic images	c ISIC2016	1279	2	MDCNN	85.22%
[18]	2019	Segmentation, Classification	Brain tumor	MRI	TCIA	500	2	CNN	95.62%.
[19]	2024	image completion	-	-	-	-	-	-	-
[20]	2024	Segmentation	-	pathological imaging	three datasets	Varies	5	Encoder-Decoder	95.3%

3. Preliminaries

Definition 1: Neutrosophic set (NS)

In NS, The element X in the universe can be associated with three membership function $\{True\_Membership, Indeterminacy\_Membership, False\_Membership\}$ as {T, I, F} [21]. This independent membership values are ranging from zero and one, where $0 \leq T + I + F \leq 3$. The Standard and non-standard NS has interval$]\,^{-}0, 1^{+}[$. Many real studies utilize interval of $[0,1]$ instead of $]\,^{-}0, 1^{+}[$ as it is hard to use in some problems with exact values. The NS can handle indeterminacy value by introducing a 3-D of membership, as in Figure 1, in contrast of IFS that introduce only 2-D membership degrees. For each element X in NS, where X is continuous, the NS can be denoted as follows [22, 23]:

$$NS = \int_{\mathrm{X}} < T(\mathrm{x}), I(\mathrm{x}), F(\mathrm{x})/\mathrm{x}, \mathrm{x} \in \mathrm{X} > \tag{1}$$

Since X is discrete, the NS can be denoted as follows:

$$NS = \sum_{i=1}^{n} < T(\mathrm{x}_i), I(\mathrm{x}_i), F(\mathrm{x}_i)/\mathrm{x}_i, \mathrm{x}_i \in \mathrm{X} > \tag{2}$$

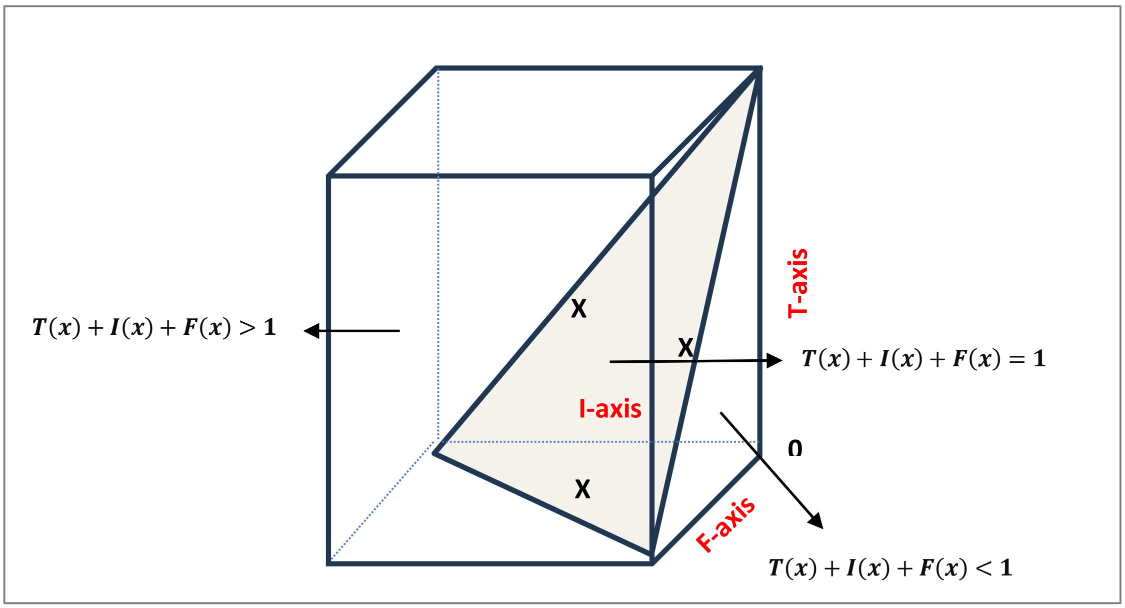


Figure 1 NS True, Indeterminacy, False membership functions [24].

Definition2: Image in NS domain

The pixel is denoted as (True, Indeterminacy, False) memberships, which can be represented as:

$$P_{NS}(a,b) = \{True(a,b), Indeterminacy(a,b), False(a,b)\} \tag{3}$$

The True, Indeterminacy, and False can be represented as follows:

$$True(x, \mathrm{y}) = \frac{\bar{g}(x,\mathrm{y}) - \bar{g}_{min}}{\bar{g}_{max} - \bar{g}_{min}} \tag{4}$$

$$\bar{g}(x,y) = (\frac{1}{window \times window}) \sum_{m=i-window/2}^{i+window/2} \sum_{n=j-window/2}^{j+window/2} g(m,n) \tag{5}$$

$$False(x,y) = 1 - True(x,y) \tag{6}$$

$$\delta(x,y) = abs(g(x,y) - \bar{g}(x,y)) \tag{7}$$

$$Indeterminacy(x,y) = \frac{\delta(x,y)-\delta_{min}}{\delta_{max}-\delta_{min}} \tag{8}$$

where $\bar{g}(x,y)$ is the local mean-value (LV) of the image, $\delta(x,y)$ is the absolute value (AV) define by the difference between intensity and LV [25].

Definition 3: NS Entropy

The entropy of an image reveals how the intensity is distributed. The high entropy value suggests that the pixel probability and uniform distribution are identical. In contrast, the minimal entropy value implies an inequality in pixel probability and a non-uniform distribution. The NS entropy is expressed as follows:

$$E_{NS} = E_{True} + E_{Indeterminacy} + E_{False} \tag{9}$$

$$E_{True} = -\sum_{i=min\{True\}}^{max\{True\}} p_{True}(i) lnP_{True}(i) \tag{10}$$

$$E_{Indeterminacy} = -\sum_{i=min\{Indeterminacy\}}^{max\{Indeterminacy\}} p_{IIndeterminacy}(i) lnP_{Indeterminacy}(i) \tag{11}$$

$$E_{False} = -\sum_{i=min\{False\}}^{max\{False\}} p_{False}(i) lnP_{False}(i) \tag{12}$$

Since E_{True} , $E_{Indeterminacy}$, and E_{False} are entropies for True, Indeterminacy, and False, respectively.

Definition 4. α-mean operation

This operation aims to minimize the IM by computing the mean value between the neighbors in NS image:

$$\bar{P}(\alpha) = Pixel(\overline{True}(\alpha), \overline{IIndeterminacy}(\alpha), \overline{False}(\alpha)) \tag{13}$$

$$\overline{True}(\alpha) = \begin{cases} True, & Indeterminacy < \alpha \\ \overline{True_{\alpha}}, & Indeterminacy \geq \alpha \end{cases} \tag{14}$$

$$\overline{True}_{\alpha}(x,y) = (\frac{1}{window \times window}) \sum_{m=i-window/2}^{i+window/2} \sum_{n=j-window/2}^{j+window/2} True(m,n) \tag{15}$$

$$\bar{F}alse(\alpha) = \begin{cases} False, & Indeterminacy < \alpha \\ \overline{False}, & Indeterminacy \geq \alpha \end{cases} \tag{16}$$

$$\bar{F}alse_{\alpha}(x,y) = \left(\frac{1}{window \times window}\right) \sum_{m=i-window/2}^{i+window/2} \sum_{n=j-window/2}^{j+window/2} False(m,n) \tag{17}$$

$$\overline{Indeterminacy}_{\alpha}(x,y) = \frac{\bar{\delta}_{True}(x,y) - \delta_{True_{min}}}{\delta_{True_{max}} - \delta_{True_{min}}} \tag{18}$$

$$\bar{\delta}_{T}(x,y) = abs(\bar{T}rue(x,y) - \overline{\overline{True}}(x,y)) \tag{19}$$

$$\bar{\bar{T}}rue(x,y) = \left(\frac{1}{window \times window}\right) \sum_{m=i-window/2}^{i+window/2} \sum_{n=j-window/2}^{j+window/2} \overline{True}(m,n) \tag{20}$$

where $\bar{\delta}_{True}$ is the AV between mean intensity and mean value.

Definition 5. Contrast intensification operator

In FS, the contrast intensification operator decreases the fuzziness of an FS A by increasing membership degree that is greater than 0.5, and reducing membership degree that is less than it [26]. In NS, the intensification is defined as part of β-enhancement operation by [27] that depending on the computed $\boldsymbol{\alpha}$ -mean operator. The intensification operator aims to enhance the truth and false degree based on the following rules:

$$\grave{T}rue_{intens}(\alpha) = \begin{cases} 2True^2, & \bar{T}rue(\alpha) \leq 0.5 \\ 1 - 2(1 - True(x,z))^2, & \overline{True}(\alpha) > 0.5 \end{cases} \tag{21}$$

$$Fa\grave{l}se_{intens}(\alpha) = \begin{cases} 2False^2, & \overline{False}(\alpha) \leq 0.5 \\ 1 - 2(1 - False(x,z))^2, & \overline{False}(\alpha) > 0.5 \end{cases} \tag{22}$$

Definition 6. $\boldsymbol{\beta}$-enhancement operation

$$\grave{P}(\beta) = P(Tr\grave{u}e(\beta), Indeter\grave{m}inacy(\beta), Fa\grave{l}se(\beta)) \tag{23}$$

$$Tr\grave{u}e(\beta) = \begin{cases} \bar{T}rue(\alpha), & \overline{Indeterminacy}_{\alpha}(x,y) < \beta \\ \grave{T}rue_{intens}(\alpha), & \overline{Indeterminacy}_{\alpha}(x,y) \geq \beta \end{cases} \tag{24}$$

$$\grave{F}(\beta) = \begin{cases} \bar{F}alse(\alpha), & Indeterminacy_{\alpha}(x,y) < \beta \\ \grave{T}_{intens}(\alpha), & \overline{Indeterminacy}_{\alpha}(x,y) \geq \beta \end{cases} \tag{25}$$

$$Indeter\grave{m}inacy_{\beta}(x,y) = \frac{\grave{\delta}_{True}(x,y) - \grave{\delta}_{True_{min}}}{\grave{\delta}_{True_{max}} - \grave{\delta}_{True_{min}}} \tag{26}$$

$$\grave{\delta}_{True}(x,y) = abs(Tr\grave{u}e(\beta) - \overline{Tr\grave{u}e}(x,y)) \tag{27}$$

$$\bar{\grave{T}}rue(x,y) = \left(\frac{1}{window \times window}\right) \sum_{m=i-window/2}^{i+window/2} \sum_{n=j-window/2}^{j+window/2} \grave{T}rue(m,n) \tag{28}$$

Since $\overline{Tr\grave{u}e}(x,y)$ is the AV between intensity and its LV after β-enhancement operation.

Definition 7. NS complement

The complement of NS is NS^c, where $True^c(\mathrm{x,y}) = False(\mathrm{x,y}), False^c(\mathrm{x,y}) = True(\mathrm{x,y}), Indeterminacy^c(\mathrm{x,y}) = 1 - Indeterminacy(\mathrm{x,y}), \mathrm{x,y} \in \mathcal{N}$.

Definition 8. Convert from NS domain

This process aims to transform form NS to crisp set [28]. The following equation is used to transform from NS domain to spatial domain

$$\widehat{\widehat{True}}(\mathcal{n}) = \bar{g}_{min} + (\bar{g}_{max} + \bar{g}_{min}.\widehat{True}(\mathcal{n})) \tag{29}$$

where $\widehat{True}(\mathcal{n})$ is the truth domain after enhancement.

4. Proposed method

In this section, we discuss two proposed frameworks BoneNet-NS1 and BoneNet-NS2 based on NS and different DL models.

4.1. BoneNet-NS1 framework

In this part, we describe the proposed BoneNet-NS1 framework which is based on NS input images to different DL models. The proposed BoneNet-NS1 framework is shown in Figure 2 and Algorithm 1. The main steps of the first proposed BoneNet-Ns1 framework can be summarized as:

Step 1: Convert RGB X-ray images to gray scale images

Each pixel in color image 24-bit is converted to gray scale 8-bit image in interval (0-255) where image size equals $W * H$.

Step 2: Compute LV of 8-bit image

The LV of a pixel in a $W * H$ gray picture may be determined by running a window across the image pixels. The window calculates the average value of nearby pixels for each pixel. In our study use a window measure 5 by 5.

Step 3: Compute the maximum and minimum values of LV.

Step 4: Compute True, Indeterminacy, and False in NS.

We calculate True, Indeterminacy, and False by Eqs. (4-8). Hence, every pixel within image will be denoted as: $\mathrm{P(a,b) = P(True(a,b), Indeterminancy(a,b), False(a,b))}$.

Step 5: Apply classification model

The NS image is an input to the DL model such as Xception, ResNet152V2, DenseNet121, and customized CNN to evaluate their performance using the NS image. The DL learning model classified the bone X-ray images as fractured or not fractured. The customized CNN is implemented as demonstrated in Table 2.

Table 2 Customized CNN architecture

Layer name	Filters	Kernel size	Activation	Pool size	Output size
Conv2d	32	(3,3)	Relu	-	222 x 222
BatchNormalization	-	-	-	-	222 x 222
Max pooling 2d	-	-	-	(2,2)	111 x 111
Conv2d	64	(3,3)	Relu	-	109 x 109
BatchNormalization	-	-	-	-	109 x 109
Max pooling 2d	-	-	-	(2,2)	54 x 54
Dropout	Dropout percentage is 0.3				54 x 54
Conv2d	128	(3,3)	Relu	-	52 x 52
BatchNormalization	-	-	-	-	52 x 52
Max pooling 2d	-	-	-	(2,2)	26 x 26
Dropout	Dropout percentage is 0.3				54 x 54
Flatten	-				86528
Dense	Dense (256)		Relu	-	256
Dropout	Dropout percentage is 0.3				256
Dense	Dense (128)		Relu	-	128
Dropout	Dropout percentage is 0.3				128
Dense	Desne (1)		sigmoid	-	1

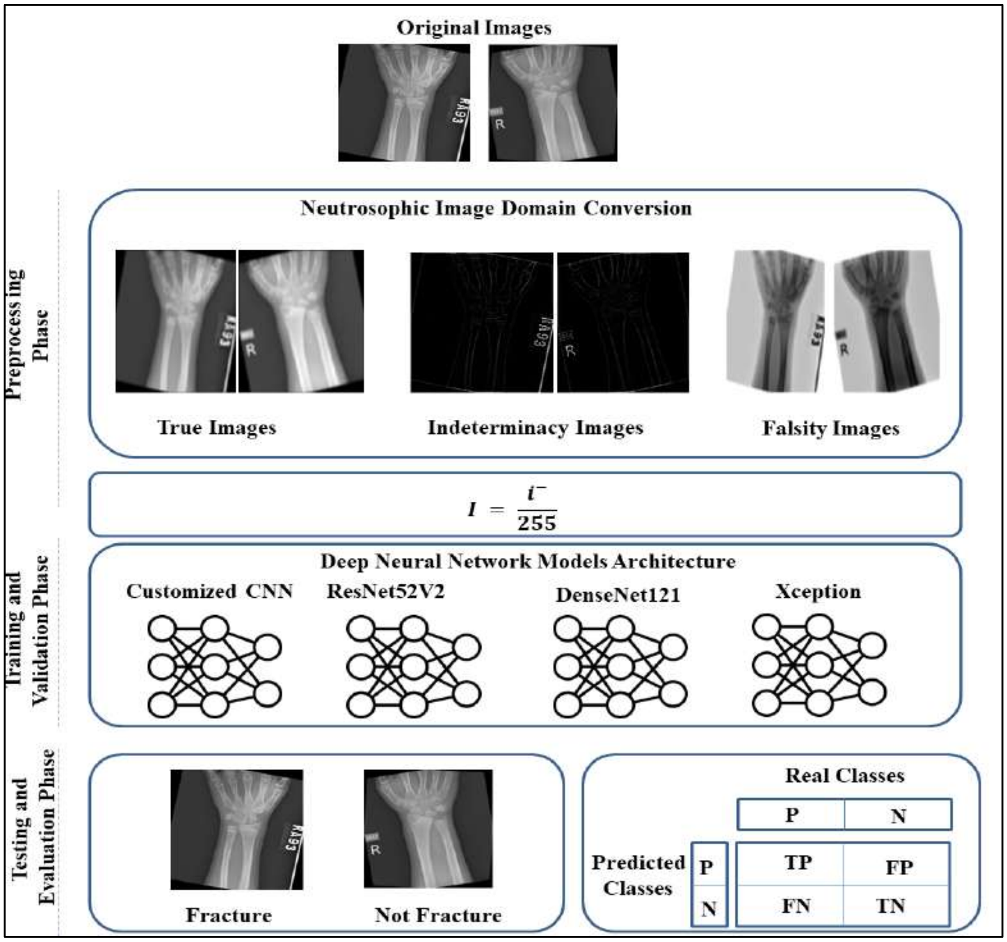


Figure 2 The BoneNet-NS1 general framework

Algorithm 1 BoneNet-NS1 for bone fracture classification approach

Input: Gray image with intensities in interval from 0 to 255**.**

For each pixels in image **do**

A 5×5 filter, to obtain the LV.

Obtain the min and max of LV.

Represent each pixel into NS image using Eqs. 4-8.

Classify the NS images using DL model**.**

End

Output: return class

4.2. BoneNet-NS2 framework

In this part, we describe the proposed BoneNet-NS2 framework which is based on NS enhancement operations on X-ray bone images and DL models. The steps from 1 to 4 in BoneNet-NS1 framework are similar in BoneNet-NS2 framework first four steps. The DL model input in the second framework is gray scale image that previously enhanced under NS domain. The proposed BoneNet-NS2 framework is shown in Figure 3 and Algorithm 2. The main steps of BoneNet-NS2 framework can be summarized as:

Step 1: Convert RGB X-ray images to gray scale images

Each pixel in color image 24-bit is converted to gray scale 8-bit image in interval (0-255) where image size equals $W * H$.

Step 2: Compute LV of 8-bit image

The LV of a pixel in a $W * H$ gray picture may be determined by running a window across the image pixels. The window calculates the average value of nearby pixels for each pixel. In our study we use window measure 5 by 5.

Step 3: Compute the maximum and minimum values of LV.

Step 4: Compute True, Indeterminacy, and False in NS.

We calculate True, Indeterminacy, and False by Eqs. (4-8). Hence, every pixel within image will be denoted as: *P(a, b) = P(True(a, b), Indeterminancy(a, b), False(a, b))*.

Step 5: Perform enhancement operation

The enhancement operation aims to minimize the indeterminacy data and enhances the truth data. This operation is obtained by α-mean, intensification, and β-enhancement operators. The α-mean operation and β-enhancement operation, are used to minimize the indeterminacy image. The mean value between neighbors can be defined by α-mean operation. The α and β parameters can represented as follows [29]

$$\alpha = \alpha_{min} + \frac{(\alpha_{max} - \alpha_{min})(EntropyI - Entropy_{min})}{(Entropy_{max} - Entropy_{min})} \tag{39}$$

$$\beta = 1 - \alpha \tag{40}$$

$$EntropyI = \sum_{i=1}^{W} \sum_{j=1}^{H} pixel(x, y) \log_2 pixel(x, y) \tag{41}$$

$$EN_{max} = -\log_2 \frac{1}{hw} \tag{42}$$

Since the W and H are the width and height of image. Our study uses a modified approach to compute α and β parameters, enhancing the results under the NS domain using Equations (39) to (42). Then, $\alpha - mean$, intensification operation, and $\beta - enhancment$ operation is calculated on the truth image based on Eqs. (13-28). The Entropy in Eqs. (9-12) calculates the alteration of the local pixels. The α-mean operation provides high entropy of indeterminacy image and uniform distribution, while the β-enhancement operation enhances the true image. This procedure improves the sensitivity of the indeterminacy image to local pixels. The NS enhancement operation is demonstrated in Figure 4.

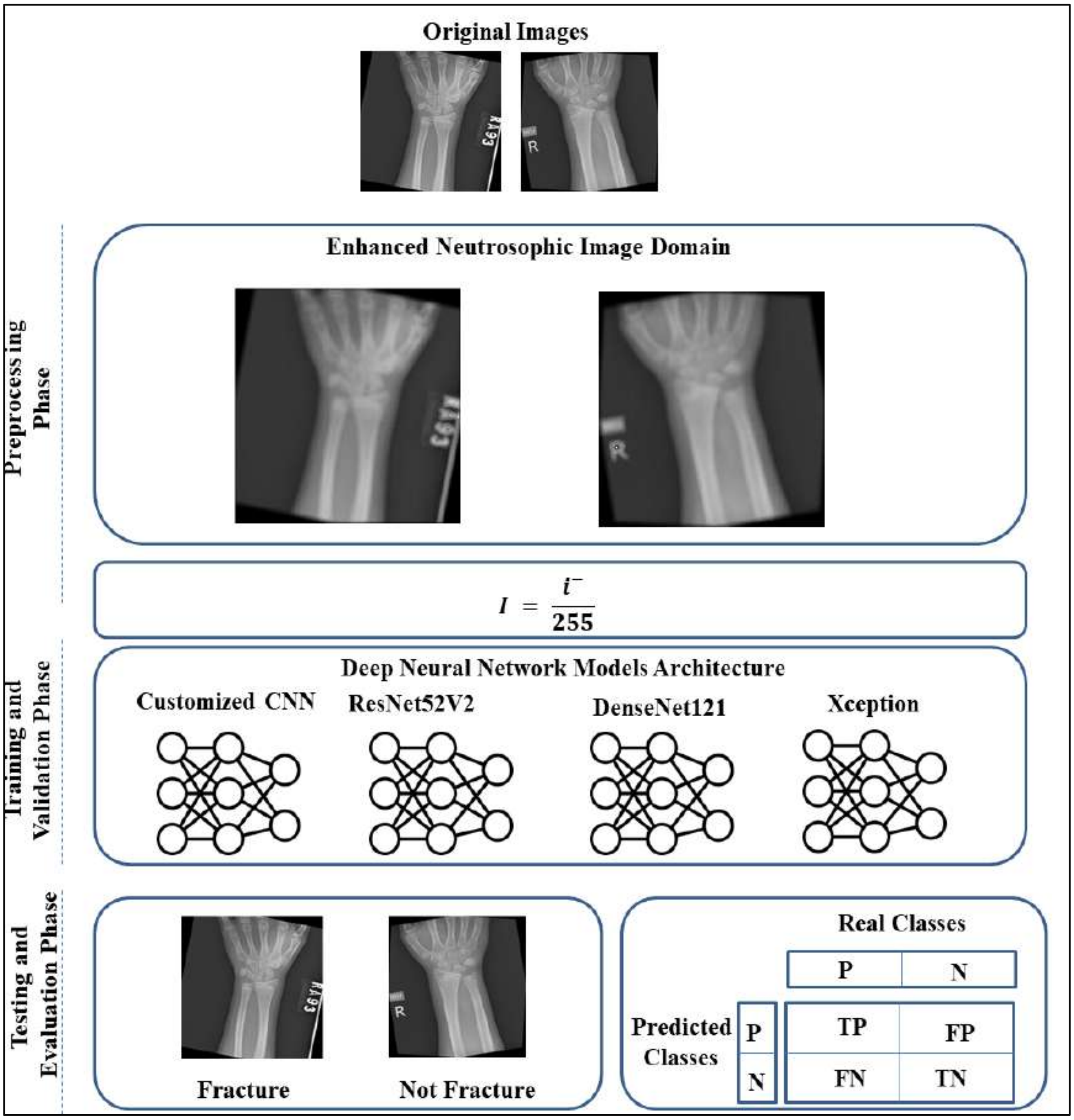


Figure 3 The BoneNet-NS2 general framework.

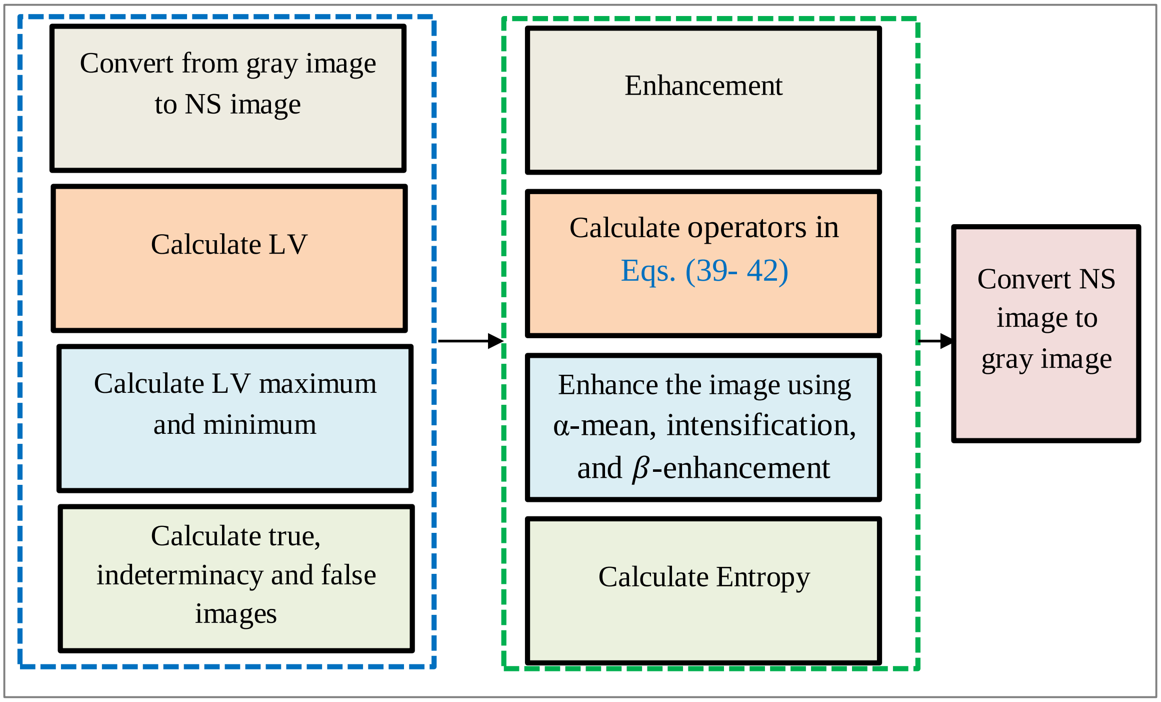


Figure 4 NS enhancement operations

Algorithm 2 BoneNet-NS2 for bone fracture classification approach

Input: Gray image with intensities in interval from 0 to 255**.**

For each pixels in image **do**

A 5×5 filter, to obtain the LV.
Obtain the min and max of LV.
Represent each pixel into NS image using Eqs. (4-8).
Calculate α, β parameters using Eqs. (39-42).
Obtain α-mean operation using Eqs. (13-20).
Obtain intensification operation using Eqs. (21-22).
Obtain β-enhancement operation Eqs. (23-28).
Calculate entropy to enhance True image Eqs. (9-12).
Convert NS image to gray image Eq. (29)
Classify the enhanced images using DL model**.**

End

Output: return class.

5. Experimental results

This section includes: the bone X-ray image dataset settings, evaluation metrics, implementation settings, visualization of proposed approach, statistical analysis related to bone X-ray image classification are illustrated.

5.1. Dataset settings

Our proposed is evaluated using 4924 bone X-ray images from Kaggle dataset [30]. The bone fractured dataset was adjusted to a resolution of 224 x 244 pixels divided into training, validation, and testing data. The data is classified into two classes (fracture and non-fractured). The bone X-ray dataset description is summarized in Table 3 and Figure 5.

Table 3 Bone X-ray dataset description.

	Fractured	Non-fractured	Total Classes
Train	2097	2020	4117
Test	200	201	401
Validation	169	237	406
Total	2466	2458	**4924**

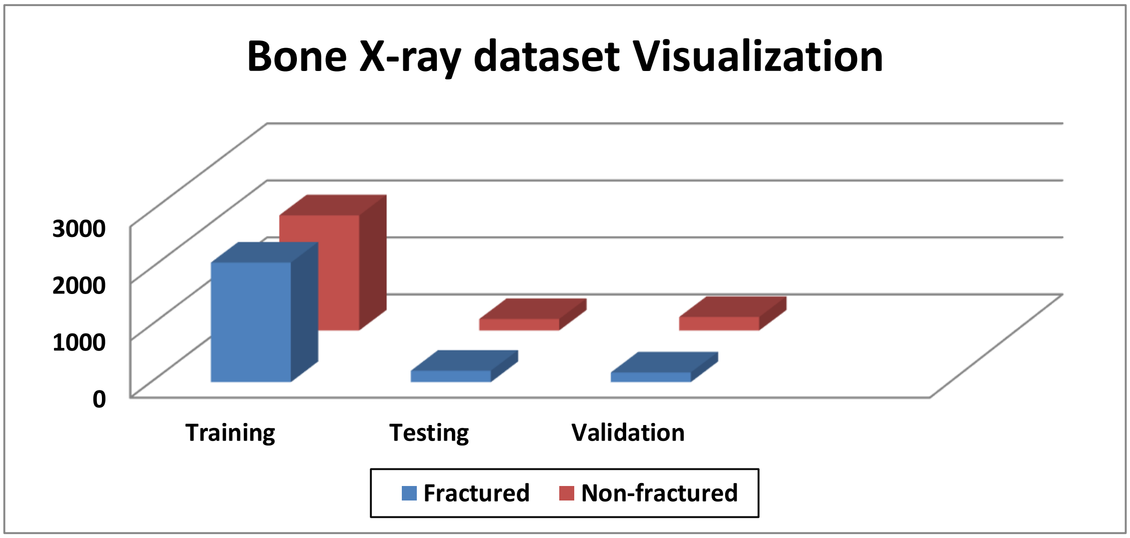


Figure 5 Bone X-ray dataset visualization.

5.2. Evaluation metrics

Our proposed work was evaluated using binary cross entropy/log loss, accuracy, precision, recall, F1-score, and AUC.

- **Binary cross entropy/log loss** is a loss function used to evaluate the change between predicted binary outcomes and actual binary labels which can be denoted as follows [31]:

$$-\frac{1}{N}\sum_{i=1}^{N} y_i \cdot \log(p(y_i)) + (1-y_i)\cdot \log(1-p(y_i)) \tag{43}$$

- To displaying a confusion matrix of proposed work, the following metrics can be computed:

$$Accuracy = \frac{TP+TN}{(TP+FP+TN+FN)} \tag{44}$$

$$Precision = \frac{TP}{(TP+FP)}) \quad (45)$$

$$Recall = \frac{TP}{(TP+FN)} \quad (46)$$

$$F1-score = 2 * \frac{Precision \cdot Recall}{(Precision + Recal)} \quad (47)$$

Where TP, FN, TN, and FP represent the number of true positive, the number of false negative, the number of true negative, and the number false positive, respectively

- **Area under the curve (AUC)** is a single measure that expresses how well a binary classification model performs overall in differentiating between positive and negative examples [32].

$$ROC \quad -AUC = \int_0^1 TPR(FPR)\mathrm{d}FPR \quad (48)$$

$$= \int_0^1 TPR(FPR^{-1}(x))\mathrm{d}x$$

Where TPR , FPR represent True Positive Rate and False Negative Rate

5.3. Implementation settings

Table 4 describes a complete detail of our implementations in consideration of parameters used training DL models.

Table 4 Implementation settings

Frameworks	Python using the Kaggle platform and keras API
Optimizer	Adam
Epochs for customized CNN	20
Epochs for other DL models	10
Batch size	32

5.4. Visualization analysis

In this section, we introduce the visual analysis for our approach. We support our experiments on radiological bone images for NS algorithm Figures 6. Figure 6 shows the two different inputs for two proposed frameworks. Figures 6 (b), (c), and (D) show the NS image, which is the input to the BoneNet-NS1. Figure 6 (e) shows the final converted gray image, which is the input to BoneNet-NS2 after performing some enhancement operations under the NS environment.

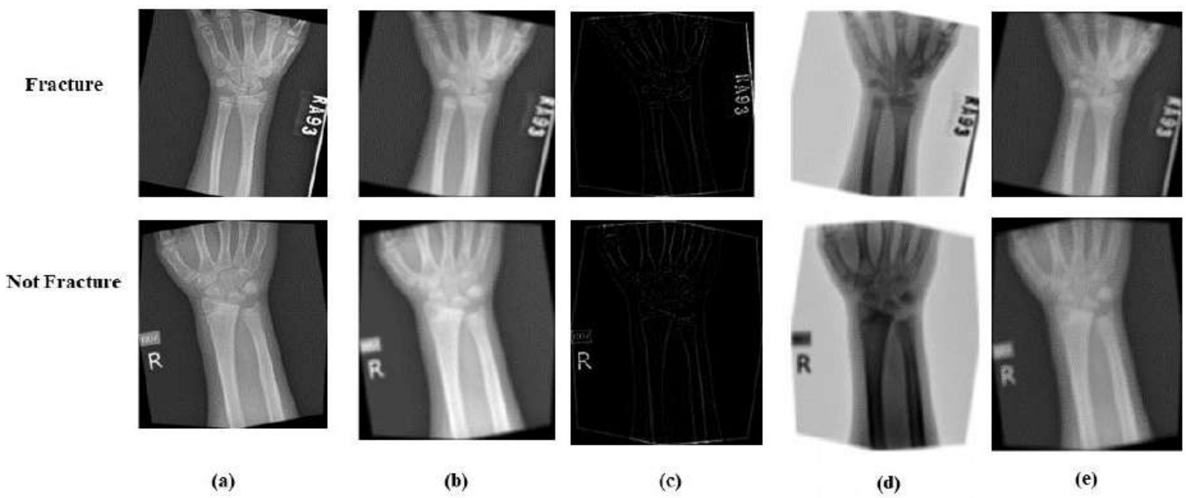


Figure 6 bone dataset under NS domain (a) original images, (b) True image, (c) Indeterminacy image, (d) Falsity image, and (e) Enhanced grayscale image.

5.5. Statistical analysis

In this section, we discuss the efficiency of our approach to bone X-ray fracture classification using the NS and DL models. We introduce two frameworks for NS and DL integration. The first framework (BoneNet-NS1) uses an NS image as an input to the DL model. The second framework (BoneNet-NS2) uses a gray image that was previously enhanced under NS domain using $\alpha - mean$ and $\beta - enhancment$ operations. Our approach was evaluated using four different DL models such as Xception, ResNet152V2, DenseNet121, and customized CNN. Table 5 and Table 8 show results of accuracy, log loss, precision, recall, F1-score, and AUC. The table shows superior results in terms of accuracy for the Xception model.

The first framework (BoneNet-NS1) was applied to Xception, ResNet152V2, DenseNet, and customized CNN models, and their results are summarized in Table 6 and Table 9. The results show superior results in True and indeterminacy domains for ResNet152V2, DenseNet121, and customized CNN. But in the false domain, the Xception and customized CNN show lower results than the results in Table 5.

The second framework (BoneNet-NS2) was applied to Xception, ResNet152V2, DenseNet, and customized CNN models, and their results are summarized in Table 7 and Table 10. The Bone-Net-NS2 shows superior results for all models than Table 5 and Table 6. ResNet52V2 shows the highest results using (BoneNet-NS2) in terms of accuracy, log loss, precision, recall, F1-score, and AUC with 99.7 %, 0.006, 99.7%, 99.7%, 99.7%, and 99.7, respectively.

Table 5 Evaluation of DL model on bone X-ray dataset before using NS

Model	Accuracy	log loss	Precision	Recall	F1-score	AUC
Xception	**0.994**	0.014	0.995	0.995	0.994	0.995
ResNet152V2	0.989	0.034	0.990	0.989	0.989	0.989
DenseNet121	0.962	0.089	0.962	0.962	0.962	0.962
Customized CNN	0.939	0.324	0.942	0.939	0.939	0.939

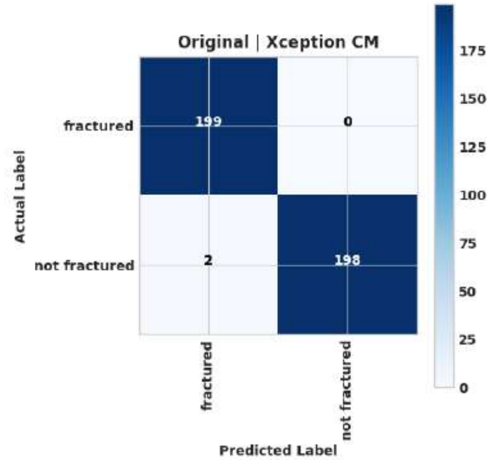
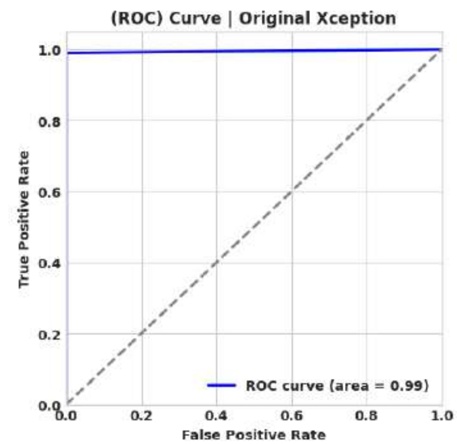
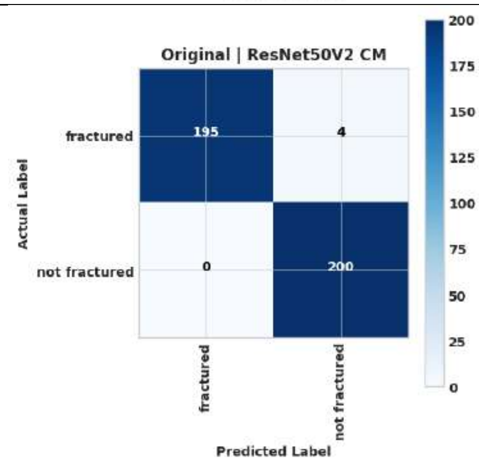
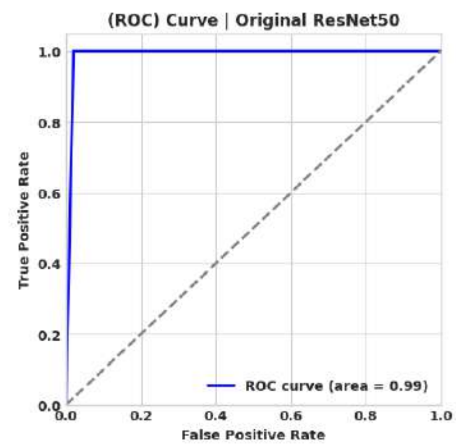
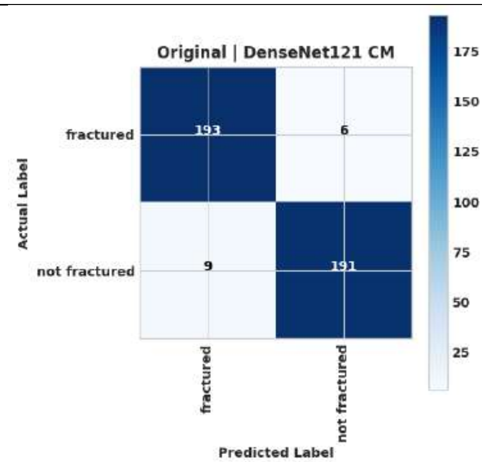
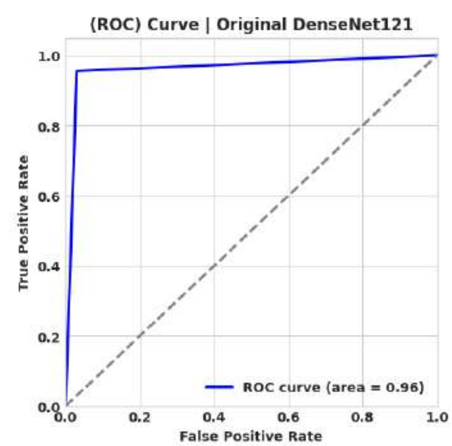
Table 6 Evaluation of DL model on bone X-ray dataset using BoneNet-NS1

True domain						
Model	**Accuracy**	**log loss**	**Precision**	**Recall**	**F1-score**	**AUC**
Xception	0.994	0.014	0.995	0.995	0.994	0.995
ResNet152V2	**0.994**	**0.013**	**0.995**	**0.995**	**0.994**	**0.995**
DenseNet121	**0.992**	**0.024**	**0.992**	**0.992**	**0.992**	**0.992**
Customized CNN	**0.944**	**0.243**	**0.946**	**0.944**	**0.944**	**0.944**
Indeterminacy domain						
Xception	0.974	0.052	0.975	0.974	0.974	0.974
ResNet152V2	**0.992**	**0.018**	**0.992**	**0.992**	**0.992**	**0.992**
DenseNet121	**0.984**	**0.050**	**0.985**	**0.984**	**0.984**	**0.984**
Customized CNN	**0.969**	**1.325**	**0.971**	**0.970**	**0.969**	**0.97**
False domain						
Xception	0.994	0.016	0.995	0.995	0.994	0.995
ResNet152V2	**0.997**	**0.005**	**0.997**	**0.997**	**0.997**	**0.997**
DenseNet121	**0.987**	**0.033**	**0.987**	**0.987**	**0.987**	**0.987**
Customized CNN	0.796	1.579	0.827	0.797	0.792	0.797

Table 7 Evaluation of DL model on bone X-ray dataset using BoneNet-NS2

Model	Accuracy	log loss	Precision	Recall	F1-score	AUC
Xception	**0.997**	**0.014**	**0.997**	**0.997**	**0.997**	**0.997**
ResNet52V2	**0.997**	**0.006**	**0.997**	**0.997**	**0.997**	**0.997**
DenseNet121	**0.972**	**0.098**	**0.972**	**0.972**	**0.972**	**0.972**
Customized CNN	**0.962**	**0.260**	**0.962**	**0.962**	**0.962**	**0.962**

Table 8 Confusion matrix and ROC of DL model on bone X-ray dataset before using NS

Model	Confusion Matrix	Roc Curve
Xception		
ResNet52V2		
DenseNet121		

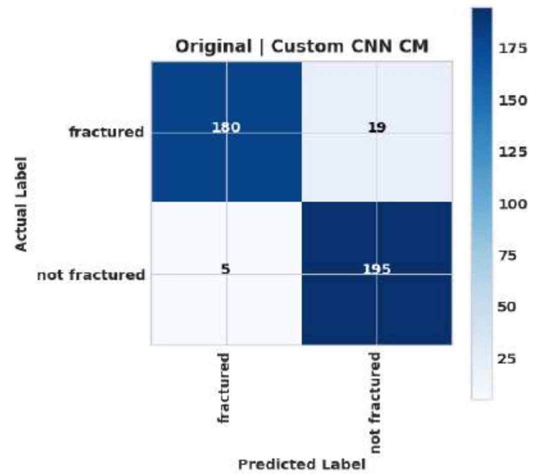
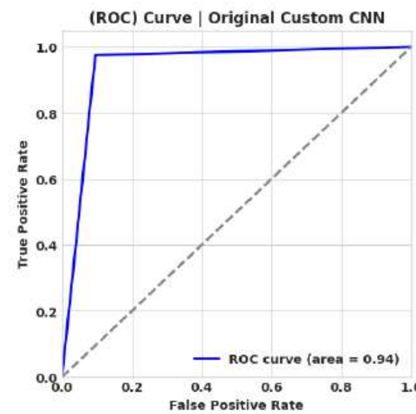
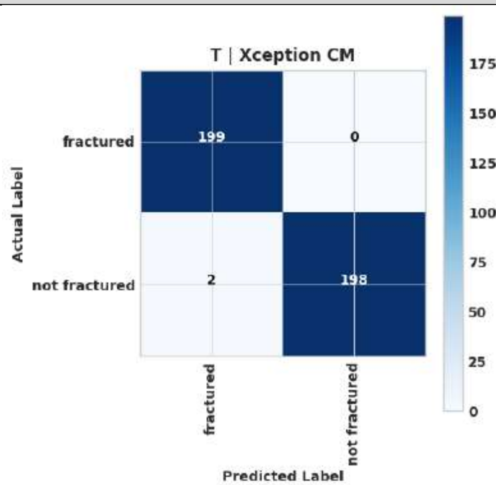
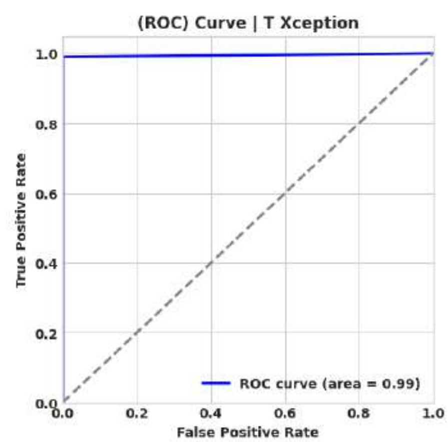
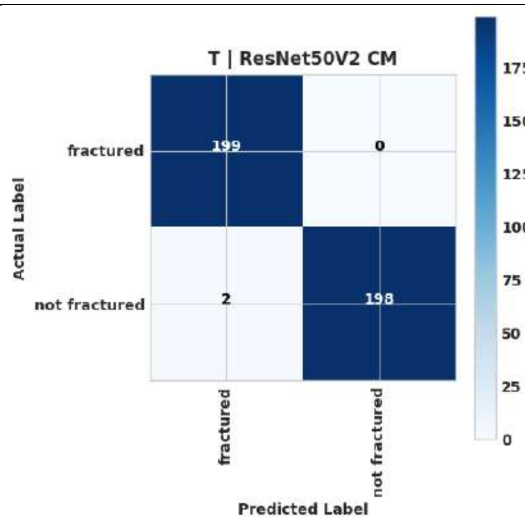
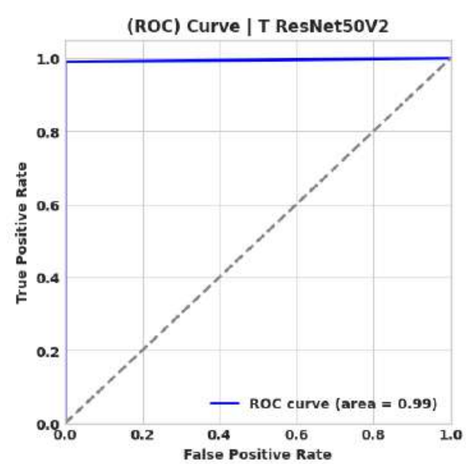
Customized CNN		

Table 9 Confusion matrix and ROC of DL model on bone X-ray dataset After using BoneNet-NS1

T Dataset		
Model	**Confusion Matrix**	**Roc Curve**
Xception		
ResNet52V2		

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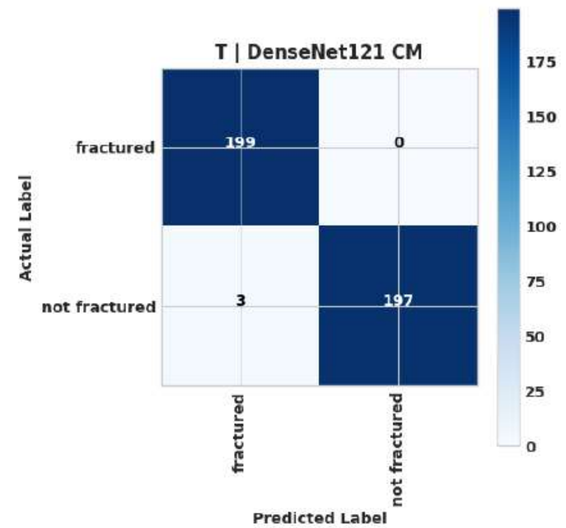

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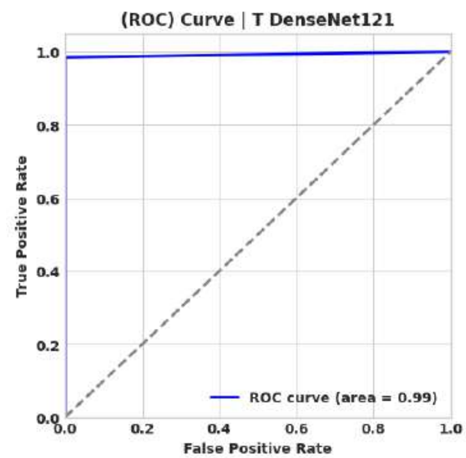

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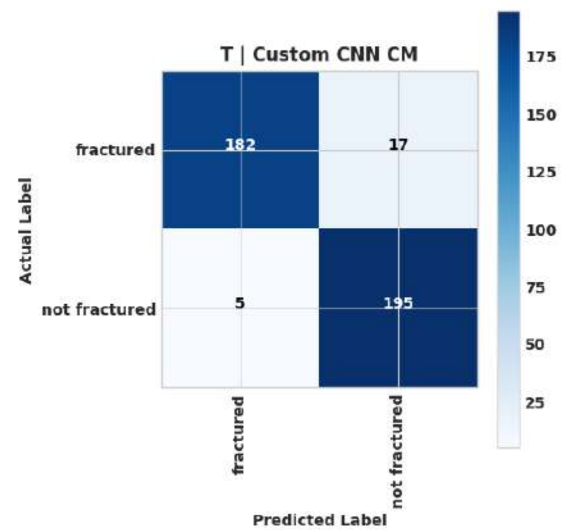

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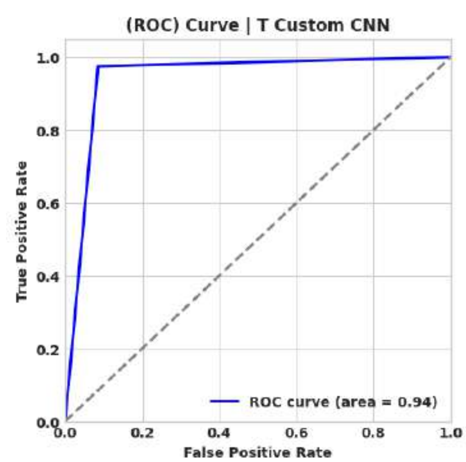

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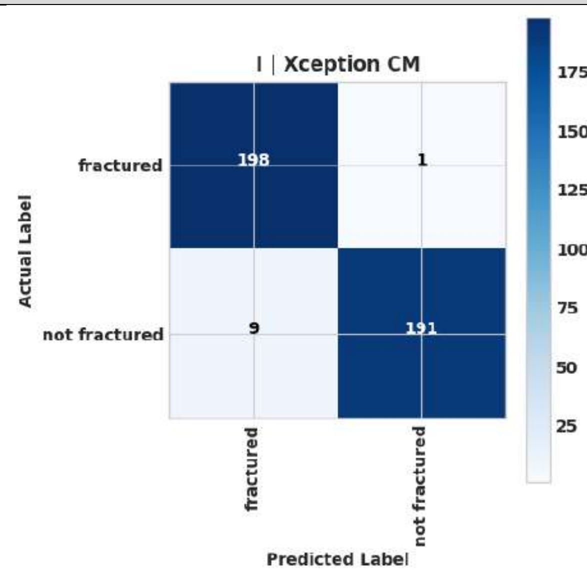

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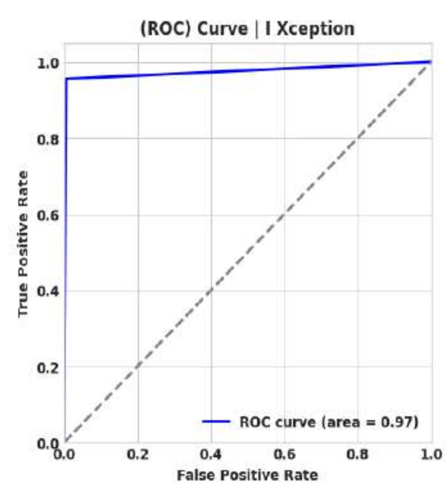

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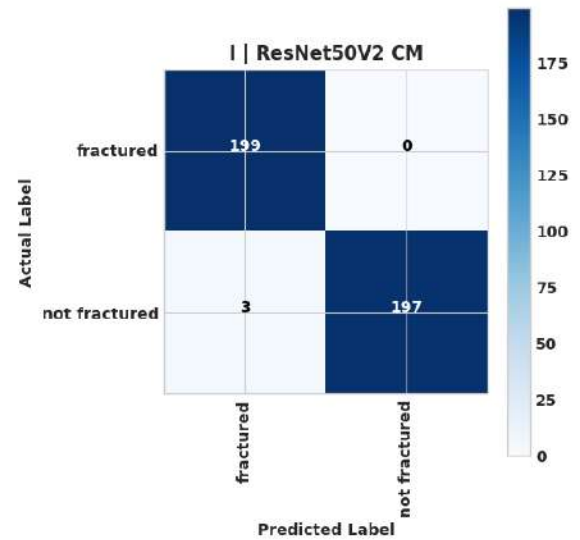
<td>
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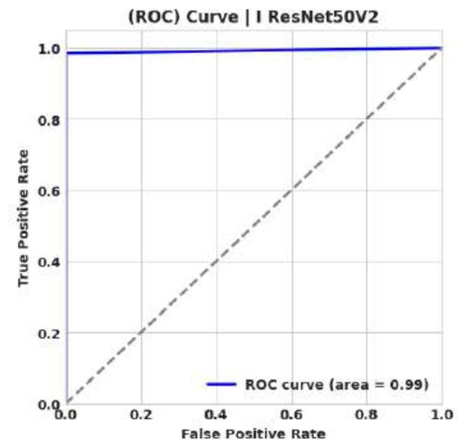
<td>
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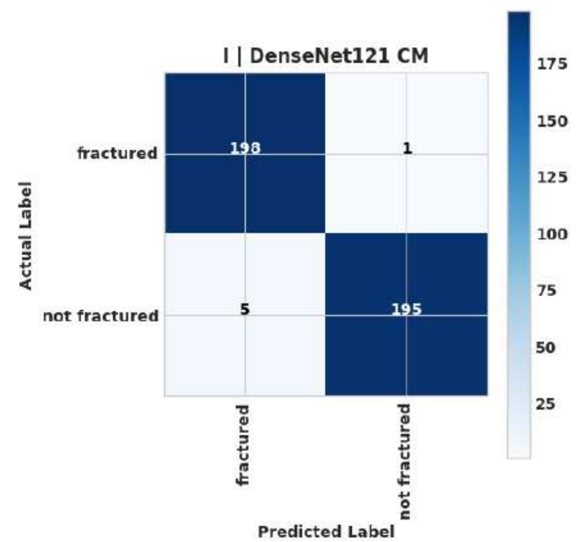
<td>
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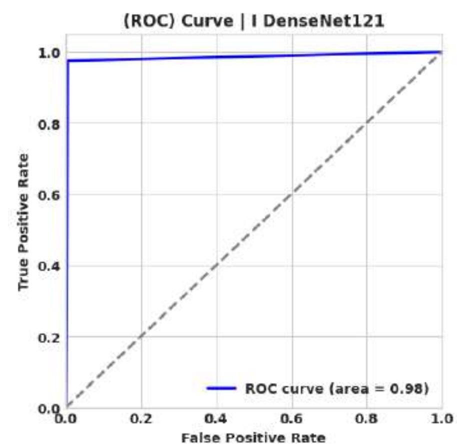
<td>
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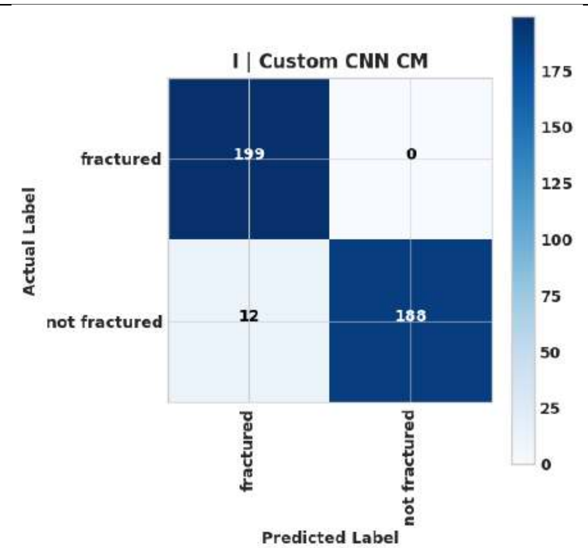
<td>
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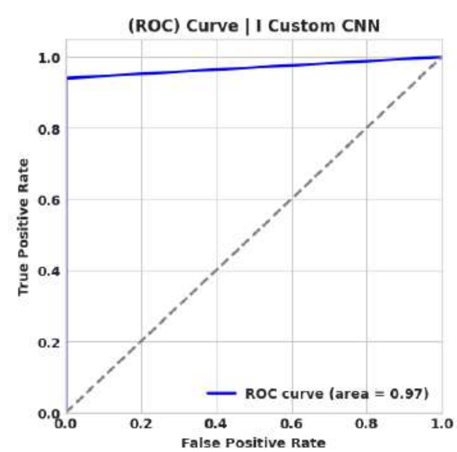
<td>
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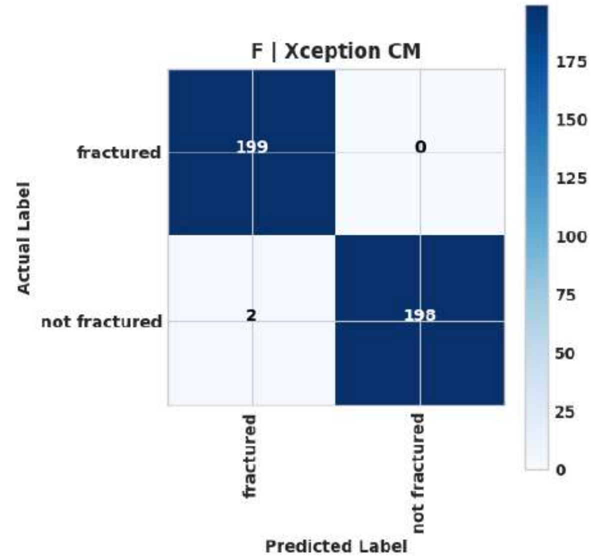
<td>
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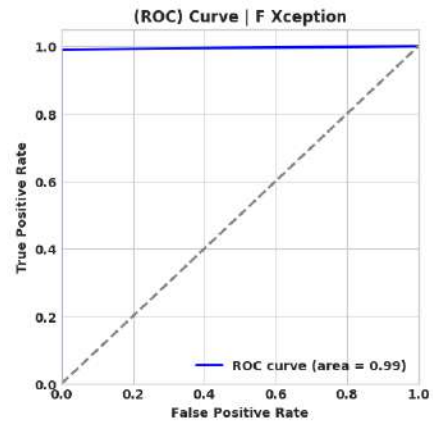
<td>
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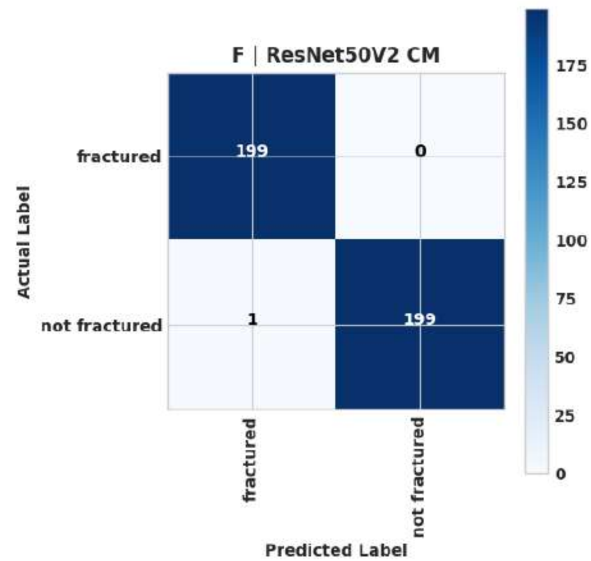
<td>
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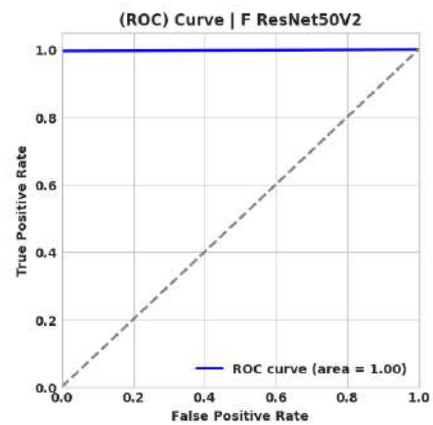
<td>
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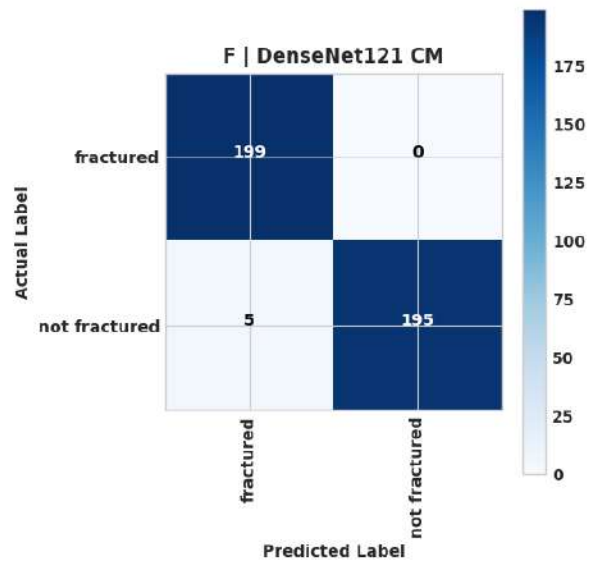
<td>
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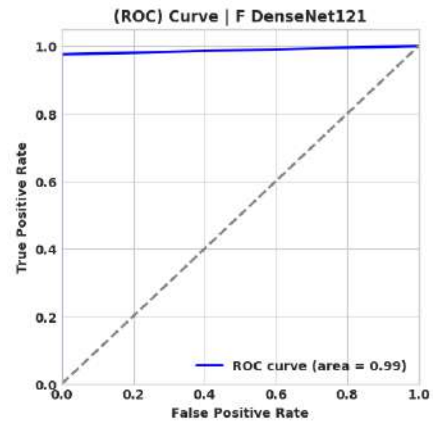
<td>
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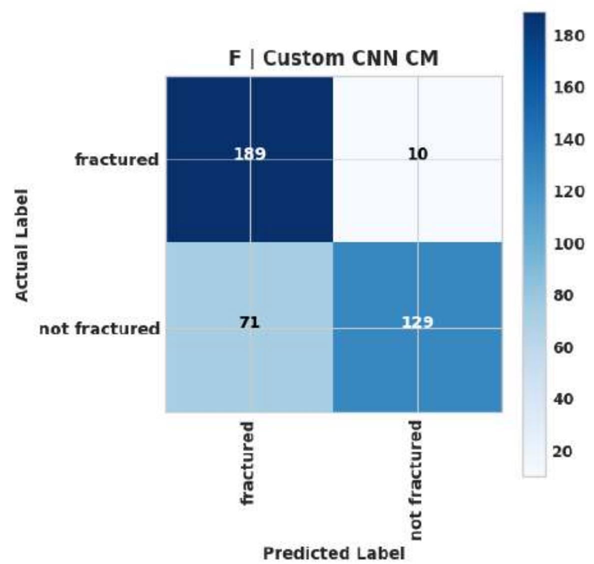
<td>
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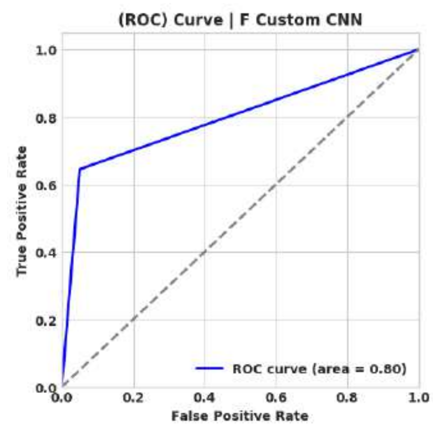
<td>
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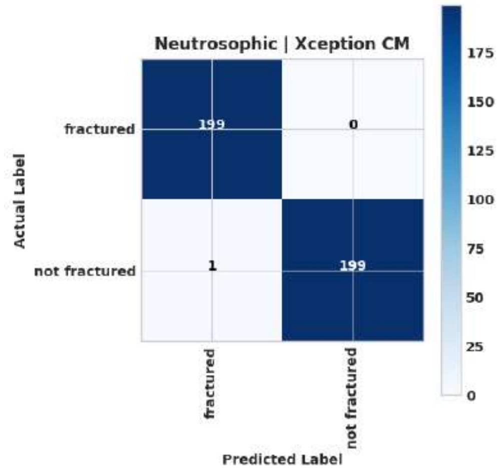
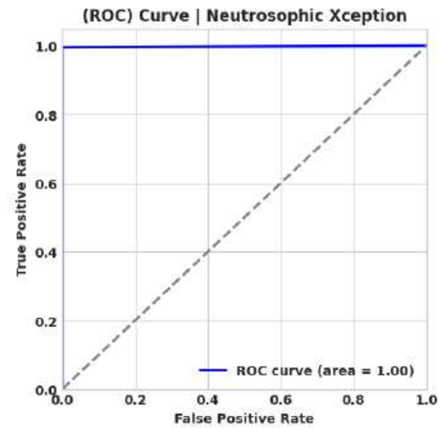
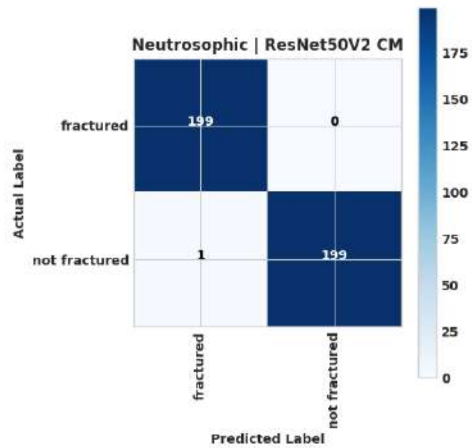
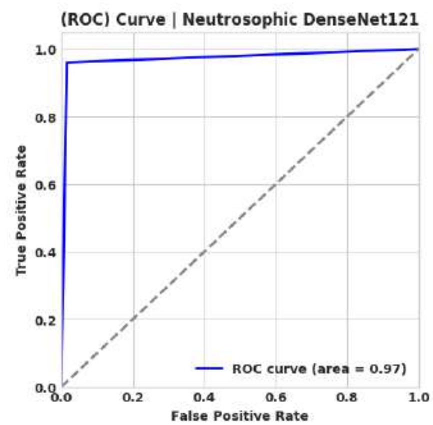
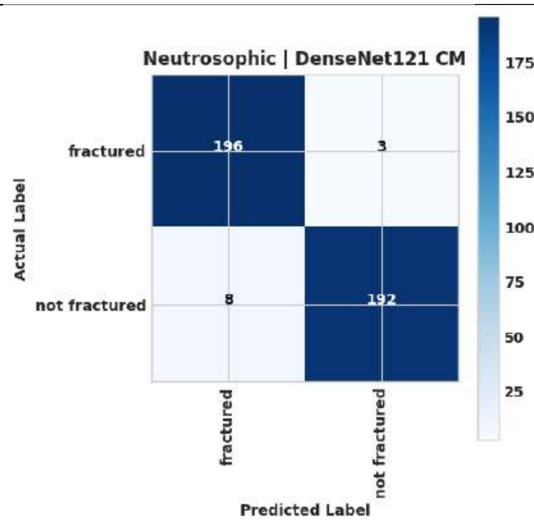
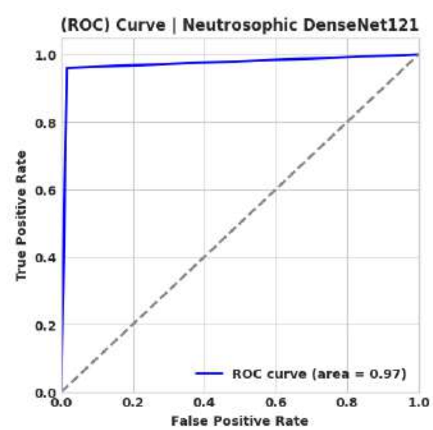
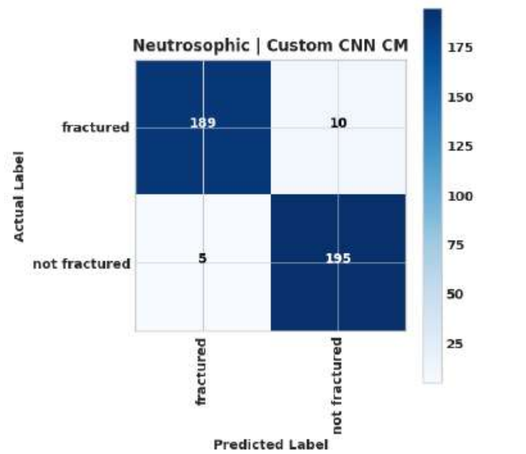
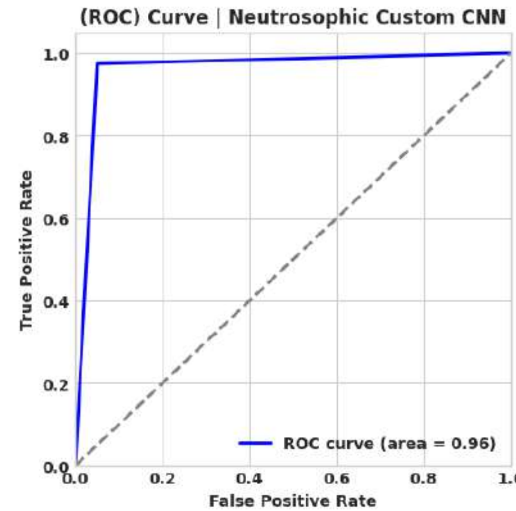
</table>

Table 10 Confusion matrix and ROC of DL model on bone X-ray dataset After using BoneNet-NS2

Model	Confusion Matrix	Roc Curve
Xception		
ResNet52V2		
DenseNet121		

<table>
<tr>
<td>Custom CNN</td>
<td>
</td>
<td>
</td>
</tr>
</table>

6. Managerial implementation

Sustainable development indicators contribute to assessing the progress of countries and institutions in achieving sustainable development goals. These indicators revolve around the recommendations of the Twenty-First Century Agenda set by the United Nations, which include appropriate health care for all members of society, especially remote and rural areas, to control endemic and epidemic diseases resulting from environmental pollution. We introduce an automatic approach for bone fracture identification using X-ray images based on DL and NS techniques. The proposed approach can reduce the increasing pressure on healthcare infrastructure.

7. Conclusion

The NS environment classification approach depends on only three degrees of membership. Using NS and DL for classification tasks can provide more ability to deal with uncertainty and increase the performance and accuracy of classification tasks. The main challenge of this study is that bone radiological image contains aleatoric uncertainty which leads to bad contrast and inconsistent boundaries. This affects the performance of bone fractured classification and identification. In our study, we introduce two frameworks: BoneNet-NS1 and BoneNet-NS2 for bone fractured classification using X-ray images. The proposed framework is based on different DL models and NS to handle uncertainty data in images. The second framework shows superior results during using -mean and enhancement operations under the NS domain and input this enhanced gray image to DL models. The proposed framework was evaluated on bone fracture X-ray dataset on 4924 images. The second framework shows superior results with most DL models. The higher results were shown in the proposed (BoneNet-NS2) on ResNet52V2 in terms of accuracy, log loss, precision, recall, F1-score, and AUC with 99.7 %, 0.006, 99.7%, 99.7%, 99.7%, and 99.7, respectively.

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Neutrosophic Spherical Sets in MCDM

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Abstract:

Decision-making plays a crucial role in achieving success across various scenarios, especially when confronted with complex issues inundated with abundant facts and information. Employing multi-criteria decision-making (MCDM) methods and techniques becomes particularly indispensable in tackling such formidable challenges. This study introduces novel neutrosophic SWGM and SWAM accuracy functions, which enhance traditional aggregation operators. Furthermore, it introduces the CODAS technique tailored for addressing Multiple Attribute Group Decision Making problems utilizing the newly defined operators. To exemplify the proposed methodology, a supplier selection problem is examined.

Keywords: Neutrosophic Spherical Set (NSS); Decision Matrix (D-Mx); Negative Ideal Solution (NIS); Positive Ideal Solution (PIS); Spherical Weighted Arithmetic Mean (SWAM); Spherical Weighted Geometric Mean (SWGM); Score function (SF); Accuracy Function (AF).

1. Introduction

Multi-Criteria Decision-Making (MCDM) is a structured approach that considers multiple criteria and attributes to assess and pinpoint the best option or resolution among a set of competing alternatives. Decision-makers are often tasked with navigating conflicting objectives or standards when selecting from available options in diverse real-life contexts. MCDM serves as a tool to assist decision-makers in achieving the most advantageous decision by carefully weighing and addressing these considerations.

The MCDM consists of criteria, a set of alternatives, and expert evaluations of the alternatives for each criterion. These sections evaluate the specialized knowledge and score the options based on suitability. These days, a vast array of MCDM approaches have been developed and applied in many different kinds of industries [1], the transportation industry [2], economics [3], health [4], energy planning [5], manufacturing [6], construction [7], supplier selection [8], and more.

A recent development in MCDM is the distance-based methodology known as Combinative Distance-based Assessment (CODAS). This methodology compares the Euclidean distance (ED) with the Taxicab distance (TD) to determine which alternatives are preferred.

Uncertainty stands as one of the pivotal factors impacting the decision-making process. Employing a Fuzzy Set (FS) offers a means to surmount this uncertainty. As a development of classical set theory, Lotfi A. Zadeh invented FSs [9] in 1965. The idea behind FSs is to represent and manipulate uncertainty more flexibly and realistically, especially in situations where traditional binary logic may not be suitable. Fuzzy MCDM techniques aim to resolve the uncertainty associated with decision-making problems [10].

Atanassov presented a broader version of fuzzy sets called intuitionistic fuzzy sets, which provide a more comprehensive treatment of ambiguity and uncertainty [11], and are formally known as Intuitionistic Fuzzy Sets (IFS). IFSs extend the capabilities of traditional fuzzy sets by encompassing notions of non-membership and hesitancy. Employing an intuitionistic fuzzy Multi-Criteria Decision-Making (MCDM) approach, Karagoz S, Deveci M, Simic V, Aydin N, and Bolukbas [12] evaluated various choices for the selection of a designated dismantling center location.

An approach utilizing CODAS technique, grounded on intuitionistic fuzzy [13] Multiple Criteria Decision Making (MCDM), is proposed to aid in waste management. The method involves employing the intuitionistic fuzzy weighted averaging operator to amalgamate the diverse viewpoints of decision-makers into a unified consensus.

Expanding on the notion of Intuitionistic Fuzzy Sets (IFS), a mathematical concept called Interval-Valued Intuitionistic Fuzzy Sets (IVIFS) incorporates intervals to represent degrees of membership and non-membership. Roy, Das, Kar, and Pamuèar (2019) extended the CODAS approach with IVIFS, offering a framework for assessing Multiple Criteria Decision Making (MCDM) challenges where only partial weight information is available. Additionally, Peng and Garg [15] introduced methodologies for addressing emergency decision-making using similarity measures, CODAS, and weighted distance-based approximation within interval-valued fuzzy soft sets.

The Pythagorean Fuzzy Set (PFS) is the foundation of a recently introduced novel structure intended to handle uncertainty in practical decision-making situations. When awarding membership, nonstandard FSs, IFSs, and IVFSs permit a degree of commitment smaller than one. A class of nonstandard Pythagorean fuzzy subsets is introduced in [16], where the membership grades are pairs (a, b) that meet the condition that $a^2 + b^2 \leq 1$. PFS is far more effective at modeling such uncertainty than an IFS.

Peng, Xindong, and Ma, Xueling investigated an algorithm for resolving MCDM problems based on CODAS and created a novel approach for handling MCDM difficulties in a Pythagorean fuzzy environment [17]. Zhang, X., Xu, Z outline some new Pythagorean Fuzzy Set (PFS) operating regulations and discuss their beneficial characteristics [18]. To successfully address the MCDM difficulties involving PFSs, it is also suggested that an enhanced strategy for order preference be similar to the optimum solution method.

The Neutrosophic Set (NS) theory extends classical sets, FSs, and IFSs that aim to manage unclear, incomplete, and contrasting facts. This approach, which resolved indeterminacy using a new type of set and allowed for a more refined representation of uncertain particulars, was given by Florentin Smarandache [19-20].

Smarandache illustrates in [21-22] that offsets and off-uninorms have applicability within digital image processing, particularly for tasks like image segmentation and edge detection. Furthermore, the paper offers algorithms and examples to elucidate these concepts.

One specific type of NS is a single-valued set which has been proposed to manage with incomplete information. [23] offers a novel method for solving multi-attribute group decision-making issues by applying the order choose by similarity technique to a single-valued neutrosophic environment. Additionally, create the TOPSIS technique for MADM in a streamlined neutrosophic setting.

Broumi, Je, and Smarandache are set to enhance the TOPSIS method [24] to accommodate interval neutrosophic uncertain linguistic information. They will introduce an extended version of the TOPSIS method tailored for resolving multiple attribute decision-making dilemmas where attribute values are expressed as interval neutrosophic uncertain linguistic variables and attribute weights remain unspecified. Broumi introduced the innovative concept of the Neutrosophic Inverse Soft Expert Set (NISES) in [25], which finds application within the Failure Mode and Effect Analysis (FMEA) framework.

H. Garg presents novel applications for combining Single-Valued Neutrosophic (SVN) data, which are applied to solve problems related to MCDM [26]. Gundogdu, F. Kutlu, and Kahraman, C. presented the idea of generalized three-dimensional Spherical Fuzzy Fets (SFSs) with a few critical distinctions from previous FSs [27]. The spherical vague distances, established with examples, provide the basis of the new kind of FS. An illustrated example of spherical SF TOPSIS, a MCDM approach, is shown.

To evaluate the obstacles to the growth of clean energy, a proposed technique based on MCDM approaches in a SFS has been mentioned in [28]. Additionally, CODAS outperformed the other approaches when the outcomes of the MOORA, COPRAS, and CODAS procedures were compared. Biswas, Chatterjee, and Majumder [29] apply a SFS to rank the statements. After calculating scores, they utilize an MCDA based on the SFS to determine the statements' relative ranking according to the judgments of a selection panel. The LOPCOW (modified SF LOgarithmic Percentage Change-driven Objective Weighting) approach is employed.

Smarandache introduced the concepts of Neutrosophic Two-Fold Algebra [30-31] along with its corresponding Neutrosophic Two-Fold Law, and explored their extensions into Fuzzy Two-Fold Algebras and Laws. Additionally, they discovered nine novel topologies while enhancing and revisiting seven previously established ones [32]. Smarandache demonstrated that the Super Hyper Function [33] serves as a broader framework encompassing classical Function, Super Function, and Hyper Function. They also pioneered the Super Hyper Soft Set and its variations, including the Fuzzy and Fuzzy Extension Super Hyper Soft Set, [34] while establishing that the Super Hyper Soft Set comprises multiple Hyper Soft Sets.

In this study, we create a novel notion, the Neutrosophic Spherical Set (NSS), by fusing the ideas of spherical measure and neutrosophic logic. The spherical fuzzy distances established in the literature are the foundation for the new class of Neutrosophic sets. The presentation includes the proofs for addition, subtraction, and multiplication arithmetic operations. Accuracy functions, scoring, and aggregation operations are constructed. An exemplary example of Spherical Neutrosophic CODAS, a MCDM process, is shown.

2. Preliminaries

Definition 2.1 [19]

Consider M be the universe. A NS $\tilde{K}$ in M is characterized by a truth $T_{\tilde{K}}$, indeterminacy $I_{\tilde{K}}$ and a falsity $F_{\tilde{K}}$ membership functions

$$\tilde{K} = \left\{ \left\langle \tilde{k}, \left(T_{\tilde{K}}\left(\tilde{k}\right)\right), \left(I_{\tilde{K}}\left(\tilde{k}\right)\right), \left(F_{\tilde{K}}\left(\tilde{k}\right)\right) \right\rangle : \tilde{k} \in M, T_K, I_K, F_K \in]^{-}0, 1^{+}[\right\}$$

then

$$0^{-} \le \left(T_{\tilde{K}}\left(\tilde{k}\right)\right) + \left(I_{\tilde{K}}\left(\tilde{k}\right)\right) + \left(F_{\tilde{K}}\left(\tilde{k}\right)\right) \le 3^{+}$$

Definition 2.2 [27]

A SFS $\tilde{S}$ of the universe of discourse Z is given by $\tilde{S} = \{< s, (T_{\tilde{S}}(s), I_{\tilde{S}}(s), F_{\tilde{S}}(s)) >| s \in Z\}$

Where $T_{\tilde{S}}(s) : Z \to [0,1]$, $I_{\tilde{S}}(s) : Z \to [0,1]$, $F_{\tilde{S}}(s) : Z \to [0,1]$ and

$$0 \le T_{\tilde{S}}^{2}(s) + I_{\tilde{S}}^{2}(s) + F_{\tilde{S}}^{2}(s) \le 1 \quad \forall s \in Z$$

For each s, the numbers $T_{\tilde{S}}(s), I_{\tilde{S}}(s)$ and $F_{\tilde{S}}(s)$ are membership, non-membership and hesitancy of s to $\tilde{A}$, respectively.

3. Neutrosophic spherical set

The squared sum of the parameters in NSSs can range 0 and $\sqrt{3}$, it is possible to define each of them individually between 0 and 1 independently. In this section, the explanation of NSS and overview of spherical distance measurement, arithmetic operation and aggregation and de-neutrosophication processes are provided.

Definition 3.1. NSS $\tilde{S}$ of the universe of discourse Z is given by

$$\tilde{S} = \{< s, (T_{\tilde{S}}(s), I_{\tilde{S}}(s), F_{\tilde{S}}(s)) >| s \in Z\} \tag{1}$$

Where,

$$T_{\tilde{S}}(s) : Z \to [0,1], \quad I_{\tilde{S}}(s) : Z \to [0,1], \; F_{\tilde{S}}(s) : Z \to [0,1]$$

and

$$0 \le T_{\tilde{S}}^{2}(s) + I_{\tilde{S}}^{2}(s) + F_{\tilde{S}}^{2}(s) \le \sqrt{3} \quad \forall s \in Z \tag{2}$$

For each s, the numbers $T_{\tilde{S}}(s), I_{\tilde{S}}(s)$ *and* $F_{\tilde{S}}(s)$ are the degree of Membership, Non-Membership, and Hesitant Membership of s to $\tilde{S}$, respectively**Error! Reference source not found.**.

Definition 3.2. Basic Operators

$$\tilde{A}\oplus\tilde{B}=\left\{\left(T_{\tilde{A}}^{\;2}+T_{\tilde{B}}^{\;2}-T_{\tilde{A}}^{\;2}T_{\tilde{B}}^{\;2}\right)^{\frac{1}{2}},\left(I_{\tilde{A}}^{\;2}+I_{\tilde{B}}^{\;2}-I_{\tilde{A}}^{\;2}I_{\tilde{B}}^{\;2}\right)^{\frac{1}{2}},\left(F_{\tilde{A}}^{\;2}+F_{\tilde{B}}^{\;2}-F_{\tilde{A}}^{\;2}F_{\tilde{B}}^{\;2}\right)^{\frac{1}{2}}\right\} \tag{3}$$

$$\tilde{A}\otimes\tilde{B}=\left\{\left(T_{\tilde{A}}T_{\tilde{B}}\right),\left(I_{\tilde{A}}I_{\tilde{B}}\right),\left(F_{\tilde{A}}F_{\tilde{B}}\right)\right\} \tag{4}$$

$$\lambda\bullet\tilde{A}=\left\{\left(1-\left(1-T_{\tilde{A}}^{\;2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-I_{\tilde{A}}^{\;2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-F_{\tilde{A}}^{\;2}\right)^{\lambda}\right)^{\frac{1}{2}}\right\} \tag{5}$$

$$\tilde{A}^{\lambda}=\left\{T_{\tilde{A}}^{\;\lambda},I_{\tilde{A}}^{\;\lambda},F_{\tilde{A}}^{\;\lambda}\right\}\lambda>0 \tag{6}$$

Definition 3.3. For these NSS $\tilde{M}=\left(T_{\tilde{M}},I_{\tilde{M}},F_{\tilde{M}}\right)$ and $\tilde{N}=\left(T_{\tilde{N}},I_{\tilde{N}},F_{\tilde{N}}\right)$, the following applies to $\lambda,\lambda_1,\lambda_2>0$.

1. $\tilde{M}\oplus\tilde{N}=\tilde{N}\oplus\tilde{M}$ (7)
2. $\tilde{M}\otimes\tilde{N}=\tilde{M}\otimes\tilde{N}$ (8)
3. $\lambda\left(\tilde{M}\oplus\tilde{N}\right)=\lambda\tilde{M}\oplus\lambda\tilde{N}$ (9)
4. $\lambda_1\tilde{M}\oplus\lambda_2\tilde{M}=\left(\lambda_1+\lambda_2\right)\tilde{M}$ (10)
5. $\left(\tilde{M}\otimes\tilde{N}\right)^{\lambda}=\tilde{M}^{\lambda}\otimes\tilde{N}^{\lambda}$ (11)
6. $\tilde{M}^{\lambda_1}\otimes\tilde{M}^{\lambda_2}=\tilde{M}^{\lambda_1+\lambda_2}$ (12)

Proof:

According to Definition 3.2, we will prove equations (7-9 and 11) since equation (10 and 12) are comparable to the corresponding proofs of equations (9 and 11),

1. $\tilde{M}\oplus\tilde{N}=\tilde{N}\oplus\tilde{M}$

$$\tilde{M}\oplus\tilde{N}=\left\{\left(T_{\tilde{M}}^{\;2}+T_{\tilde{N}}^{\;2}-T_{\tilde{M}}^{\;2}T_{\tilde{N}}^{\;2}\right)^{\frac{1}{2}},\left(I_{\tilde{M}}^{\;2}+I_{\tilde{N}}^{\;2}-I_{\tilde{M}}^{\;2}I_{\tilde{N}}^{\;2}\right)^{\frac{1}{2}},\left(F_{\tilde{M}}^{\;2}+F_{\tilde{N}}^{\;2}-F_{\tilde{M}}^{\;2}F_{\tilde{N}}^{\;2}\right)^{\frac{1}{2}}\right\}$$

$$\tilde{N}\oplus\tilde{M}=\left\{\left(T_{\tilde{N}}^{\;2}+T_{\tilde{M}}^{\;2}-T_{\tilde{N}}^{\;2}T_{\tilde{M}}^{\;2}\right)^{\frac{1}{2}},\left(I_{\tilde{N}}^{\;2}+I_{\tilde{M}}^{\;2}-I_{\tilde{N}}^{\;2}I_{\tilde{M}}^{\;2}\right)^{\frac{1}{2}},\left(F_{\tilde{N}}^{\;2}+F_{\tilde{M}}^{\;2}-F_{\tilde{N}}^{\;2}F_{\tilde{M}}^{\;2}\right)^{\frac{1}{2}}\right\}$$

Hence 1 is proved.

2. $\tilde{M}\otimes\tilde{N}=\tilde{M}\otimes\tilde{N}$

$$\tilde{M}\otimes\tilde{N}=\left\{\left(T_{\tilde{M}}T_{\tilde{N}}\right),\left(I_{\tilde{M}}I_{\tilde{N}}\right),\left(F_{\tilde{M}}F_{\tilde{N}}\right)\right\}$$

$$\tilde{N}\otimes\tilde{M}=\left\{\left(T_{\tilde{N}}T_{\tilde{M}}\right),\left(I_{\tilde{N}}I_{\tilde{M}}\right),\left(F_{\tilde{N}}F_{\tilde{M}}\right)\right\}$$

Hence 2 is proved.

3. $\lambda\left(\tilde{M}\oplus\tilde{N}\right)=\lambda\tilde{M}\oplus\lambda\tilde{N}$

$$\lambda\left(\tilde{M}\oplus\tilde{N}\right)=\lambda\left\{\left(T_{\tilde{M}}{}^{2}+T_{\tilde{N}}{}^{2}-T_{\tilde{M}}{}^{2}T_{\tilde{N}}{}^{2}\right)^{\frac{1}{2}},\left(I_{\tilde{M}}{}^{2}+I_{\tilde{N}}{}^{2}-I_{\tilde{M}}{}^{2}I_{\tilde{N}}{}^{2}\right)^{\frac{1}{2}},\left(F_{\tilde{M}}{}^{2}+F_{\tilde{N}}{}^{2}-F_{\tilde{M}}{}^{2}F_{\tilde{N}}{}^{2}\right)^{\frac{1}{2}}\right\}$$

$$=\left\{\begin{matrix}\left(1-\left(1-\left(T_{\tilde{M}}{}^{2}+T_{\tilde{N}}{}^{2}-T_{\tilde{M}}{}^{2}T_{\tilde{N}}{}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-\left(I_{\tilde{M}}{}^{2}+I_{\tilde{N}}{}^{2}-I_{\tilde{M}}{}^{2}I_{\tilde{N}}{}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}},\\ \left(1-\left(1-\left(F_{\tilde{M}}{}^{2}+F_{\tilde{N}}{}^{2}-F_{\tilde{M}}{}^{2}F_{\tilde{N}}{}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}}\end{matrix}\right\}$$

$$\lambda\tilde{M}\oplus\lambda\tilde{N}=\left\{\left(1-\left(1-T_{\tilde{M}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-I_{\tilde{M}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-F_{\tilde{M}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}}\right\}$$
$$\oplus\left\{\left(1-\left(1-T_{\tilde{N}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-I_{\tilde{N}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-F_{\tilde{N}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}}\right\}$$

$$=\left\{\begin{matrix}\left(1-\left(1-T_{\tilde{M}}^{2}\right)^{\lambda}+1-\left(1-T_{\tilde{N}}^{2}\right)^{\lambda}-\left(1-\left(1-T_{\tilde{M}}^{2}\right)^{\lambda}\right)\left(1-\left(1-T_{\tilde{N}}^{2}\right)^{\lambda}\right)\right)^{\frac{1}{2}},\\ \left(1-\left(1-I_{\tilde{M}}^{2}\right)^{\lambda}+1-\left(1-I_{\tilde{N}}^{2}\right)^{\lambda}-\left(1-\left(1-I_{\tilde{M}}^{2}\right)^{\lambda}\right)\left(1-\left(1-I_{\tilde{N}}^{2}\right)^{\lambda}\right)\right)^{\frac{1}{2}},\\ \left(1-\left(1-F_{\tilde{M}}^{2}\right)^{\lambda}+1-\left(1-F_{\tilde{N}}^{2}\right)^{\lambda}-\left(1-\left(1-F_{\tilde{M}}^{2}\right)^{\lambda}\right)\left(1-\left(1-F_{\tilde{N}}^{2}\right)^{\lambda}\right)\right)^{\frac{1}{2}}\end{matrix}\right\}$$

$$=\left\{\left(1-\left(1-T_{\tilde{M}}^{2}\right)^{\lambda}\left(1-T_{\tilde{N}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-I_{\tilde{M}}^{2}\right)^{\lambda}\left(1-I_{\tilde{N}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-F_{\tilde{M}}^{2}\right)^{\lambda}\left(1-F_{\tilde{N}}^{2}\right)^{\lambda}\right)^{\frac{1}{2}}\right\}$$

$$=\left\{\begin{matrix}\left(1-\left(1-\left(T_{\tilde{M}}^{2}+T_{\tilde{N}}^{2}-T_{\tilde{M}}^{2}T_{\tilde{N}}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}},\left(1-\left(1-\left(I_{\tilde{M}}^{2}+I_{\tilde{N}}^{2}-I_{\tilde{M}}^{2}I_{\tilde{N}}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}},\\ \left(1-\left(1-\left(F_{\tilde{M}}^{2}+F_{\tilde{N}}^{2}-F_{\tilde{M}}^{2}F_{\tilde{N}}^{2}\right)\right)^{\lambda}\right)^{\frac{1}{2}}\end{matrix}\right\}$$

Hence 3 is proved.

Since

4. $\lambda_1\tilde{M}\oplus\lambda_2\tilde{M}=\left(\lambda_1+\lambda_2\right)\tilde{M}$

5. $\left(\tilde{M}\otimes\tilde{N}\right)^{\lambda}=\tilde{M}^{\lambda}\otimes\tilde{N}^{\lambda}$

$$\left(\tilde{M}\otimes\tilde{N}\right)^{\lambda}=\left\{\left(T_{\tilde{M}}T_{\tilde{N}},I_{\tilde{M}}I_{\tilde{N}},F_{\tilde{M}}F_{\tilde{N}}\right)^{\lambda}\right\}$$
$$=\left\{T_{\tilde{M}}{}^{\lambda}T_{\tilde{N}}{}^{\lambda},I_{\tilde{M}}{}^{\lambda}I_{\tilde{N}}{}^{\lambda},F_{\tilde{M}}{}^{\lambda}F_{\tilde{N}}{}^{\lambda}\right\}$$
$$\tilde{M}^{\lambda}\otimes\tilde{N}^{\lambda}=\left\{T_{\tilde{M}}{}^{\lambda},I_{\tilde{M}}{}^{\lambda},F_{\tilde{M}}{}^{\lambda}\right\}\otimes\left\{T_{\tilde{N}}{}^{\lambda},I_{\tilde{N}}{}^{\lambda},F_{\tilde{N}}{}^{\lambda}\right\}$$
$$=\left\{T_{\tilde{M}}{}^{\lambda}T_{\tilde{N}}{}^{\lambda},I_{\tilde{M}}{}^{\lambda}I_{\tilde{N}}{}^{\lambda},F_{\tilde{M}}{}^{\lambda}F_{\tilde{N}}{}^{\lambda}\right\}$$

Hence 5 is proved.

Definition 3.4. SWAM as, $z=\left(z_1,z_2,z_3,....z_n\right)$; $z_j\in[0,1];\sum_{j=1}^{n}z_j\le\sqrt{3}$ SWAM is defined as;

$$SWAM_z\left(\tilde{A}_1,\tilde{A}_2,...\tilde{A}_n\right)=z_1\tilde{A}_1+z_2\tilde{A}_2+z_3\tilde{A}_3+....+z_n\tilde{A}_n$$

$$\left\{\left[1-\prod_{j=1}^{n}\left(1-T_{\tilde{A}}{}^{2}\right)^{z_j}\right]^{\frac{1}{2}},\left[1-\prod_{j=1}^{n}\left(1-I_{\tilde{A}}{}^{2}\right)^{z_j}\right]^{\frac{1}{2}},\left[1-\prod_{j=1}^{n}\left(1-F_{\tilde{A}}{}^{2}\right)^{z_j}\right]^{\frac{1}{2}}\right\} \tag{13}$$

Definition 3.5. SWGM as, $z=\left(z_1,z_2,z_3,....z_n\right)$; $z_j\in[0,1];\sum_{j=1}^{n}z_j\le\sqrt{3}$

SWGM is defined as; $SWGM_z\left(\tilde{A}_1,\tilde{A}_2,...\tilde{A}_n\right)=\tilde{A}_1^{z_1}+\tilde{A}_2^{z_2}+\tilde{A}_3^{z_3}+....+\tilde{A}_n^{z_n}$

$$\left\{\prod_{j=1}^{n}T_{\tilde{A}}{}^{z_j},\prod_{j=1}^{n}I_{\tilde{A}}{}^{z_j},\prod_{j=1}^{n}F_{\tilde{A}}{}^{z_j}\right\} \tag{14}$$

Definition 3.6. The SF and AF for NSS classification are defined by;

$$Score\left(\tilde{S}\right)=\left(T_{ijw}-F_{ijw}\right)^2-\left(I_{ijw}-F_{ijw}\right)^2 \tag{15}$$

$$Accuracy\left(\tilde{S}\right)=T_{\tilde{S}}{}^{2}+I_{\tilde{S}}{}^{2}+F_{\tilde{S}}{}^{2} \tag{16}$$

Note that: $\tilde{S}<\tilde{T}$ iff

1. $Score\left(\tilde{S}\right)<Score\left(\tilde{T}\right)or$
2. $Score\left(\tilde{S}\right)=Score\left(\tilde{T}\right)and\ Accuracy\left(\tilde{S}\right)<Accuracy\left(\tilde{T}\right)$ (17)

4. Neutrosophic Spherical CODAS

A D-Mx with entries that represent the assessment scores of every choice in relation to every criterion in a neutrosophic environment can be used to represent an MCDM problem. Suppose that $S=\left\{s_1,s_2,s_3,....s_m\right\}(m\ge 2)$ represents distinct collection of m possible options and

$K=\{K_1,K_2,K_3,....K_n\}$ be the weight vector derived from every requirement that meet $0\le z_j\le 1$ and $\sum_{j=1}^{n} z_j\le\sqrt{3}$.

Step 1. Let DMs use the linguistic terms (LT) listed in Table 1 to complete the assessment matrices for decisions and criteria.

Table 1. Terms used in linguistics and their associated Spherical Neutrosophic Number

LT		(T, I, F)
Probably More Significant	PMS	(.9, .6, .2)
Extremely Significant	ES	(.8, .7, .2)
High Priority	HP	(.7, .6, .5)
Relatively Greater Significance	RGS	(.6, .7, .4)
Equally Important	EI	(.5, .8, .4)
Very Minimal Significance	VMS	(.4, .6, .7)
Low Priority	LP	(.5, .7, .6)
extremely low significant	ELS	(.5, .6 .6)
Definitely Not Important	DNI	(.2, .9, .6)

Step 2. Aggregate the outcomes reached by DM.

Aggregate the outcomes reached by DM using SWAM. Aggregate the DMs' Neutrosophic Spherical linguistic judgements of the selection criteria. Assemble and neutrosophic D-Mx based on DMs' views. Indicate the Alternative's evaluation value.

$S_i\left(i=1,2,...m\right)$ with respect to criterion $K_j\left(j=1,2...n\right)$ by $K_j\left(\tilde{S}_i\right)=\left(T_{ij},I_{ij},F_{ij}\right)$ and $D=\left(K_j\left(\tilde{S}_i\right)\right)_{m\times n}$ is a Neutrosophic Spherical Decision Matrix (NS D-Mx). D-Mx for MCDM problem using NSS, $D=\left(K_j\left(\tilde{S}_i\right)\right)_{m\times n}$ must be put together as shown in equation (18).

$$D=\left(K_j\left(\tilde{S}_i\right)\right)_{m\times n}=\begin{pmatrix}\left(\tilde{T}_{11},\tilde{I}_{11},\tilde{F}_{11}\right) & \left(\tilde{T}_{12},\tilde{I}_{12},\tilde{F}_{12}\right) & \cdots & \left(\tilde{T}_{1n},\tilde{I}_{1n},\tilde{F}_{1n}\right)\\ \left(\tilde{T}_{21},\tilde{I}_{21},\tilde{F}_{21}\right) & \left(\tilde{T}_{22},\tilde{I}_{22},\tilde{F}_{22}\right) & \cdots & \left(\tilde{T}_{2n},\tilde{I}_{2n},\tilde{F}_{2n}\right)\\ \vdots & \vdots & \cdots & \vdots\\ \left(\tilde{T}_{m1},\tilde{I}_{m1},\tilde{F}_{m1}\right) & \left(\tilde{T}_{m2},\tilde{I}_{m2},\tilde{F}_{m2}\right) & \cdots & \left(\tilde{T}_{mn},\tilde{I}_{mn},\tilde{F}_{mn}\right)\end{pmatrix} \tag{18}$$

Step 3. Build the weighted aggregated NS D-Mx. Following the determination of the alternative ratings and the weights assigned to the criteria, the aggregated weighted NS D-Mx is built using multiplication equation and then the aggregated weighted NS D-Mx can be defined as follows:

$$D=\left(K_j\left(\tilde{S}_{iz}\right)\right)_{m\times n}=\begin{pmatrix} \left(T_{11z},I_{11z},F_{11z}\right) & \left(T_{12z},I_{12z},F_{12z}\right) & \cdots & \left(T_{1nz},I_{1nz},F_{1nz}\right) \\ \left(T_{21z},I_{21z},F_{21z}\right) & \left(T_{22z},I_{22z},F_{22z}\right) & \cdots & \left(T_{2nz},I_{2nz},F_{2nz}\right) \\ \vdots & \vdots & \cdots & \vdots \\ \left(T_{m1z},I_{m1z},F_{m1z}\right) & \left(T_{m2z},I_{m2z},F_{m2z}\right) & \cdots & \left(T_{mnz},I_{mnz},F_{mnz}\right) \end{pmatrix} \tag{19}$$

Step 4. Utilising Eq. (20), deneutroscope the aggregated weighted D-Mx.

$$Score\left(K_j\left(\tilde{S}_{iz}\right)\right)=\left(T_{ijz}-F_{ijz}\right)^2-\left(I_{ijz}-F_{ijz}\right)^2 \tag{20}$$

Step 5. Find the NSPIS and NSNIS according to the SF acquired in Step 4.
Regarding the NS-PIS:

$$S^*=\left\{K_j,\max_i\left\langle Score\left(K_j\left(S_{iz}\right)\right)\right\rangle \mid j=1,2,\ldots n\right\} \tag{21}$$

$$S^*=\left\{\left\langle K_1,\left(T_1^*,I_1^*,F_1^*\right)\right\rangle,\left\langle K_2,\left(T_2^*,I_2^*,F_2^*\right)\right\rangle,\ldots\left\langle K_n,\left(T_n^*,I_n^*,F_n^*\right)\right\rangle\right\}$$

Regarding the NS -NIS:

$$S^-=\left\{K_j,\min_i\left\langle Score\left(K_j\left(S_{iz}\right)\right)\right\rangle \mid j=1,2,\ldots n\right\} \tag{22}$$

$$S^-=\left\{\left\langle K_1,\left(T_1^-,I_1^-,F_1^-\right)\right\rangle,\left\langle K_2,\left(T_2^-,I_2^-,F_2^-\right)\right\rangle,\ldots\left\langle K_n,\left(T_n^-,I_n^-,F_n^-\right)\right\rangle\right\}$$

Step 6. The distances between alternative S_i, NS-PIS, and NS-NIS should be calculated, accordingly.

For the NS-NIS:

$$D\left(S_i,S^-\right)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(\left(T_{S_i}-T_{S^-}\right)^2+\left(I_{S_i}-I_{S^-}\right)^2+\left(F_{S_i}-F_{S^-}\right)^2\right)} \tag{23}$$

For the NS-NIS:

$$D\left(S_i,S^*\right)=\sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(\left(T_{S_i}-T_{S^*}\right)^2+\left(I_{S_i}-I_{S^*}\right)^2+\left(F_{S_i}-F_{S^*}\right)^2\right)} \tag{24}$$

Step 7 Calculate the minimum and maximum distances to the NS-NIS and NS-PIS, respectively.

$$D_{\max}\left(S_i,S^-\right)=\max_{i\le i\le m}\left(S_i,S^-\right) \tag{25}$$

$$D_{\min}\left(S_i,S^*\right)=\min_{i\le i\le m}\left(S_i,S^*\right) \tag{26}$$

Step 8 Compute the revised proximity ratio in Equation (27).

$$\xi\left(S_i\right)=\frac{D\left(S_i,S^-\right)}{D_{\max}\left(S_i,S^-\right)}-\frac{D\left(S_i,S^*\right)}{D_{\min}\left(S_i,S^*\right)} \tag{27}$$

Equation (27) because the subtraction's second element is at least equal to its first element, the result is zero or negative. We altered this equality from Equation (28) so that we might get zero or a result.

$$\xi(S_i) = \frac{D(S_i, S^*)}{D_{\min}(S_i, S^*)} - \frac{D(S_i, S^-)}{D_{\max}(S_i, S^-)} \tag{28}$$

Step 9. Determine the best solution by rating the alternatives in the best possible order. We organize the alternatives according to the rising closeness ratio values since we wish to use Equation (28).

5.Illustrative Example

A supplier selection issue is devised and solved by employing our recommended technique. Four vendors of air conditioners were considered count (S_1, S_2, S_3, S_4) and evaluated for their efficacy. The number of qualitative and quantitative aspects considered will determine how many different criteria are used to pick suppliers. In accordance with on the number of qualitative and quantitative factors are considered, the decision-making criteria for supplier selection may change. Several criteria and sub-criteria have been established using a comprehensive literature assessment. Four of these criteria are used in this exemplary example: price (K_1), quality (K_2), delivery (K_3) and performance (K_4). Three decision makers with experience in supply chain and logistics management (ĐM1, ĐM2, and ĐM3) take part in the procedure for evaluation. The weights of these DMs, which are, respectively, 0.4, 0.5 and 0.3, represent their various levels of experience.

First, the judgements made by the decision-makers are compiled using the language phrases listed in Table 1 with regard to the objective. A decision is rendered in Tables 2-4.

Table 2. Decisions of DM1

ĐM1	(K_1)	(K_2)	(K_3)	(K_4)
S_1	ES	HP	EI	RGS
S_2	PMS	EI	HP	EI
S_3	LP	RGS	ES	ELS
S_4	ELS	ES	LP	HP

Table 3. Decisions of DM2

ĐM2	(K_1)	(K_2)	(K_3)	(K_4)
S_1	PMS	HP	ES	PMS

S_2	VMS	ES	HP	EI
S_3	HP	RGS	RGS	RGS
S_4	ELS	EI	LP	LP

Table 4. Decisions of DM3

ĐM3	(K_1)	(K_2)	(K_3)	(K_4)
S_1	HP	ES	PMS	RGS
S_2	VMS	PMS	ES	VMS
S_3	VMS	ELS	HP	HP
S_4	LP	EI	ES	RGS

The significance levels of the DMs are considered when combining these judgements utilizing the SWAM and SWGM operators. The decision matrices shown in Tables 5 and 6 are obtained.

Table 5. NS D-Mx by using SWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	(0.873,0.682,0.340)	(0.773,0.673,0.487)	(0.821,0.764,0.311)	(0.825,0.707,0.364)
S_2	(0.743, 0.643,0.652)	(0.821,0.764,0.311)	(0.773,0.673,0.487)	(0.517,0.806,0.549)
S_3	(0.629,0.682,0.638)	(0.621,0.723,0.502)	(0.752,0.723,0.415)	(0.646,0.690,0.544)
S_4	(0.540,0.673,0.643)	(0.687,0.814,0.379)	(0.657,0.744,0.582)	(0.649,0.715,0.568)

Table 6. NS D-Mx by using SWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	(0.779,0.576,0.190)	(0.678,0.567,0.330)	(0.656,0.656,0.191)	(0.66,0.603,0.235)

S_2	(0.460,0.541,0.394)	(0.656,0.656,0.191)	(0.678,0.567,0.330)	(0.407,0.701,0.393)
S_3	(0.481,0.576,0.517)	(0.512,0.622,0.376)	(0.636,0.622,0.269)	(0.527,0.585,0.418)
S_4	(0.435,0.567,0.541)	(0.525,0.725,0.252)	(0.501,0.651,0.389)	(0.525,0.612,0.445)

Table 7 displays the important weights of the language phrases used to express the criteria determined by DMs.

Table 7. The weights assigned to each criterion

Criteria	DM1	DM2	DM3
(K_1)	LP	VMS	HP
(K_2)	RGS	EI	RGS
(K_3)	PMS	RGS	ES
(K_4)	HP	HP	VMS

The weight of each criterion is determined by the decision-makers' strategies for the criteria aggregated by the SWAM operator provided in Equation (13), which are shown in Table 8.

Table 8. Aggregation of Criteria weights according to SWAM operator

Criteria	Weights of each criterion
(K_1)	(0.576,0.682,0.672)
(K_2)	(0.605,0.791,0.434)
(K_3)	(0.834,0.715,0.330)
(K_4)	(0.751,0.691,0.535)

The aggregated weighted neutrosophic spherical choice matrices are constructed using Equation (4) once the weights assigned to the criteria and evaluations of the substitutions have been determined, as illustrated in Tables 9 and 10.

Table 9. Weighted NS D-Mx according to SWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	(0.503,0.465,0.228)	(0.468,0.532,0.211)	(0.685,0.547,0.103)	(0.620,0.488,0.195)
S_2	(0.428,0.439,0.438)	(0.497,0.604,0.135)	(0.645,0.481,0.161)	(0.388,0.557,0.293)
S_3	(0.362,0.465,0.429)	(0.376,0.571,0.218)	(0.627,0.517,0.137)	(0.485,0.477,0.291)
S_4	(0.311,0.459,0.432)	(0.415,0.643,0.165)	(0.548,0.532,0.192)	(0.487,0.494,0.304)

Table 10. Weighted NS D-Mx according to SWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	(0.449,0.393,0.128)	(0.410,0.448,0.143)	(0.548,0.469,0.063)	(0.498,0.416,0.126)
S_2	(0.265,0.369,0.265)	(0.397,0.519,0.083)	(0.566,0.405,0.109)	(0.306,0.484,0.210)
S_3	(0.277,0.393,0.348)	(0.310,0.492,0.163)	(0.531,0.445,0.089)	(0.396,0.404,0.224)
S_4	(0.251,0.387,0.364)	(0.317,0.573,0.109)	(0.418,0.466,0.128)	(0.395,0.423,0.238)

SF are calculated using Equation (19) and Tables 11 and 12, which are based on Tables 9 and 10. PIS are represented by blue values, while NIS values are represented by yellow values.

Table 11. SF according to SWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.0196191	-0.0370441	0.1424804	0.0945117
S_2	0.0001038	-0.0893889	0.1322302	-0.0605558
S_3	0.0031443	-0.1001252	0.0961946	0.0030986
S_4	0.0140321	-0.1663112	0.0111351	-0.0024082

Table 12. SF according to SWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S_1	0.033096	-0.0217872	0.0700743	0.0542487
S_2	-0.0108362	-0.0913385	0.1209456	-0.0660912
S_3	0.0029492	-0.0864905	0.0687862	-0.002763
S_4	0.0122932	-0.1718773	-0.0300535	-0.0095923

The NS-PIS and NS-NIS corresponding to the highest and worst scores are shown in Tables 13 and 14.

Table 13. NS-PIS and NS-NIS according to SWAM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S^* (Best)	(0.503,0.465,0.228)	(0.468,0.532,0.211)	(0.685,0.547,0.103)	(0.620,0.488,0.195)
S^- (Worst)	(0.428,0.439,0.438)	(0.415,0.643,0.165)	(0.548,0.532,0.192)	(0.388,0.557,0.293)

Table 14. NS-PIS and NS-NIS according to SWGM operator

Alternatives	(K_1)	(K_2)	(K_3)	(K_4)
S^* (Great)	(0.449,0.393,0.128)	(0.410,0.448,0.143)	(0.566,0.405,0.109)	(0.498,0.416,0.126)
S^- (Poor)	(0.265,0.369,0.265)	(0.317,0.573,0.109)	(0.418,0.466,0.128)	(0.306,0.484,0.210)

Based on Equations (23 and 24), the next step we can figure out how far apart option S_i is from both the NS-PIS and NS-NIS, respectively. Tables 15 and 16 provide their information.

Table 15. Distance to PIS and NIS according to SWAM operator

Alternatives	$D(S_i, S^*)$	$D(S_i, S^-)$
S_1	1.06252	0.142724324
S_2	0.132168332	0.052504021
S_3	0.113622049	0.070743093

S_4	0.138092728	0.059237447

Table 16. Distance to PIS and NIS according to SWGM operator

Alternatives	$D\left(S_i, S^*\right)$	$D\left(S_i, S^-\right)$
S_1	0.028542681	0.136333827
S_2	0.117780274	0.067011546
S_3	0.119173652	0.07645776
S_4	0.145694319	0.053394324

We calculate the maximum and minimum distances to the NS-NIS and NS-PIS, respectively, from Tables 15 and 16. The closeness ratios are computed using Equation (28), and they are shown in Tables 17 and 18.

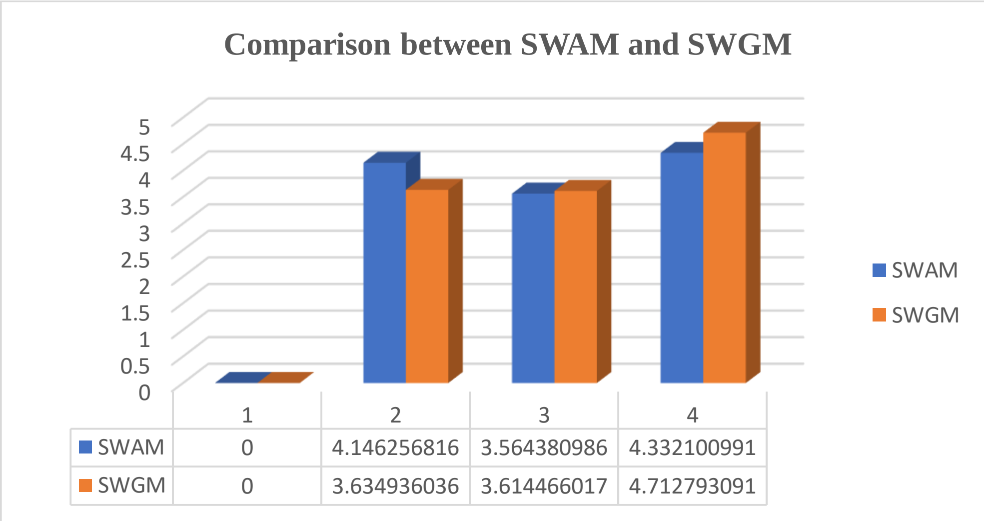
Table 17. Every alternative's closeness ratio according to the SWAM operator

Alternatives	Closeness Ratio	Rank
S_1	0	1
S_2	12438.77	3
S_3	10693.143	2
S_4	12996.303	4

Table 18. Closeness ratio of each alternative according to SWGM operator

Alternatives	Closeness Ratio	Rank
S_1	0	1
S_2	3.634936	3
S_3	3.614466	2
S_4	4.7127931	4

According to the SWAM operator, the closeness ratio for each alternative show that the best option is S_1, and over all ranking is $S_1 > S_3 > S_2 > S_4$. The closest alternative, according to the proximity ratios based on the SWGM operator, is S_1, and overall ranking is $S_1 > S_3 > S_2 > S_4$.The aggregation operators determine how the ranks differ. However, in both strategies, the best and worst options are the same.



6. Conclusions

This study introduces two novel accuracy functions, neutrosophic SWGM and SWAM, which represent significant advancements over conventional aggregation operators by integrating neutrosophic spherical sets. Through the development and application of an algorithm for the CODAS technique, we have effectively addressed the supplier selection problem. Our approach prioritizes alternatives based on distance measurements, utilizing the neutrosophic spherical CODAS approach to compute closeness ratios between criteria. Significantly, our comparison between SWAM and SWGM operators demonstrates comparable rankings and their efficacy in assessing alternatives. This research contributes to the advancement of decision-making methodologies, particularly in complex scenarios where traditional methods may fall short.

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The indefinite refined neutrosophic integrals by parts

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Abstract: this article's goal is to present the indefinite refined neutrosophic integrals by parts. All situations where integration by parts can be used are covered, including the use of rotating integrals to solve recurring and non-terminating functions like the product of trigonometric and exponential functions. Furthermore, the Tabular method has been implemented in the computation of the indefinite refined neutrosophic integrals.

Keywords: parts; integrals; neutrosophic; Tabular method; indeterminacy$\boldsymbol{I_1}, \boldsymbol{I_2}$.

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1. Introduction and Preliminaries

To describe a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, and contradiction, Smarandache suggested the neutrosophic Logic as an alternative to the current logics. Smarandache made refined neutrosophic numbers available in the following form: $(a, b_1I_1, b_2I_2, \ldots, b_nI_n)$ where $a, b_1, b_2, \ldots, b_n \in R$ *or* C [1]

Agboola introduced the concept of refined neutrosophic algebraic structures [2]. Also, the refined neutrosophic rings I was studied in paper [3], where it assumed that I splits into two indeterminacies I_1 [contradiction (true (T) and false (F))] and I_2 [ignorance (true (T) or false (F))]. It then follows logically that: [3]

$$I_1I_1 = {I_1}^2 = I_1 \qquad (1)$$

$$I_2I_2 = {I_2}^2 = I_2 \qquad (2)$$

$$I_1I_2 = I_2I_1 = I_1 \qquad (3)$$

In addition, there are many papers presenting studies on refined neutrosophic numbers [4-5-6-7-8].

Smarandache discussed neutrosophic indefinite integral (Refined Indeterminacy) [11]

Let $g: \mathbb{R} \to \mathbb{R} \cup \{I_1\} \cup \{I_2\} \cup \{I_3\}$, where I_1, I_2, and I_3 are types of sub indeterminacies,

$$g(x) = 7x - 2I_1 + x^2I_2 + 4x^3I_3$$

then:

$$F(x) = \int [7x - 2I_1 + x^2I_2 + 4x^3I_3]dx$$

$$= \frac{7x^2}{2} - 2xI_1 + \frac{x^3}{3}I_2 + x^4 I_3 + a_0 + a_1 I_1 + a_2 I_2 + a_3 I_3$$

where a_0, a_1, a_2 and a_3 are real constants.

Additionally, Alhasan gave multiple calculus presentations in which he covered neutrosophic definite and indefinite integrals. Also, he introduced the most significant uses of definite integrals in neutrosophic logic [9-10]. Several studies were also presented in the field of neutro logic in statistics and others [12-13].

2. Main Discussion

2.1 The indefinite refined neutrosophic integration by parts

Let: $f: D_f \subseteq R(I_1, I_2) \to R(I_1, I_2)$ and $g: D_g \subseteq R(I_1, I_2) \to R(I_1, I_2)$

then, for the product rule:

$$\frac{d}{dx}[f(x, I_1, I_2).g(x, I_1, I_2)] = \acute{f}(x, I_1, I_2)g(x, I_1, I_2) + f(x, I_1, I_2)\acute{g}(x, I_1, I_2)$$

integrating both sides of this equation gives us:

$$\int \frac{d}{dx}[f(x, I_1, I_2).g(x, I_1, I_2)] \ dx = \int \acute{f}(x, I_1, I_2)g(x, I_1, I_2) \ dx + \int f(x, I_1, I_2)\acute{g}(x, I_1, I_2) \ dx$$

$$\int f(x, I_1, I_2)\acute{g}(x, I_1, I_2) \ dx = f(x, I_1, I_2).g(x, I_1, I_2) - \int \acute{f}(x, I_1, I_2)g(x, I_1, I_2) \ dx$$

it is usually convenient to write this using the notation:

$$u_N = f(x, I_1, I_2) \quad \Rightarrow \quad du_N = \acute{f}(x, I_1, I_2) \ dx$$

$$dv_N = \acute{g}(x, I_1, I_2) \ dx \quad \Rightarrow \quad v_N = g(x, I_1, I_2)$$

so the neutrosophic integration by parts algorithm becomes

$$\int u_N \ dv_N = u_N.v_N - \int v_N \ du_N$$

There are four cases of the neutrosophic integration by parts:

- state1:

$$\int (a + bI_1 + cI_2)x^n \ e^{(r+sI_1+tI_2)x} \ dx$$

where a, b, c, r, s, t are real numbers, while $I_1, I_2 =$ indeterminacy), $r \neq 0$, $r \neq -t$ and $r \neq -s - t$.

$$u_N = (a + bI_1 + cI_2)x^n \quad \Rightarrow \quad du_N = n(a + bI_1 + cI_2)x^{n-1} \ dx$$

$$dv_N = e^{(r+sI_1+tI_2)x} \ dx \quad \Rightarrow \quad v_N = \frac{1}{r+sI_1+tI_2} e^{(r+sI_1+tI_2)x}$$

$$\int u_N \ dv_N = u_N.v_N - \int v_N \ du_N$$

$$\int (a+bI_1+cI_2)x^n \; e^{(r+sI_1+tI_2)x} \; dx = \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)\left(x^n e^{(r+sI_1+tI_2)x} - \int nx^{n-1}\, e^{(r+sI_1+tI_2)x} dx\right) + C$$

$$= \left(\frac{a}{r} + \left[\frac{rb+bt-as-sc}{(r+t)(r+s+t)}\right] I_1 + \left[\frac{rc-at}{r(r+t)}\right] I_2\right)\left(x^n e^{(r+sI_1+tI_2)x} - \int nx^{n-1}\, e^{(r+sI_1+tI_2)x} dx\right) + C$$

by repeated the integration, then we can find the required integral
where C is an indeterminate real constant (i.e. constant of the form $a_0 + a_1I_1 + a_2I_2$, where a_0, a_1 *and* a_2 are real numbers, while $I_1, I_2 =$ indeterminacy).

Example 1

Find:

$$\int (2+2I_1+I_2)x \; e^{(3+3I_1+2I_2)x} \; dx$$

Solution:

$$u_N = (2+2I_1+I_2)x \quad \Rightarrow \quad du_N = (2+2I_1+I_2) \; dx$$

$$dv_N = e^{(3+3I_1+2I_2)x} \; dx \quad \Rightarrow \quad v_N = \frac{1}{3+3I_1+2I_2} e^{(3+3I_1+2I_2)x}$$

$$\int u_N \; dv_N = u_N . v_N - \int v_N \; du_N$$

$$\int (2+2I_1+I_2)x \; e^{(3+3I_1+2I_2)x} \; dx = \left(\frac{2+2I_1+I_2}{3+3I_1+2I_2}\right)\left(xe^{(3+3I_1+2I_2)x} - \int e^{(3+3I_1+2I_2)x}\, dx\right)$$

$$= \left(\frac{2}{3} + \frac{18+12-12-9}{3(5)(8)} I_1 + \frac{3-4}{3(5)} I_2\right)\left(xe^{(3+3I_1+2I_2)x} - \frac{1}{3+3I_1+2I_2} e^{(3+3I_1+2I_2)x}\right)$$

$$= \left(\frac{2}{3} - \frac{9}{40} I_1 - \frac{1}{15} I_2\right)\left(x - \frac{1}{3} + \frac{3}{40} I_1 + \frac{2}{15} I_2\right) e^{(3+3I_1+2I_2)x} + C$$

- state2:

$$\int (a+bI_1+cI_2)x^n \;\; sin(r+sI_1+tI_2)x \;\; dx \quad or \quad \int (a+bI_1+cI_2)x^n \;\; cos(r+sI_1 + tI_2)x \;\; dx$$

$$u_N = (a+bI_1+cI_2)x^n \quad \Rightarrow \quad du_N = n(a+bI_1+cI_2)x^{n-1} \;\; dx$$

$$dv_N = sin(r+sI_1+tI_2)x \;\; dx \quad \Rightarrow \quad v_N = \frac{-1}{r+sI_1+tI_2} cos(r+sI_1+tI_2)x$$

$$\int u_N \; dv_N = u_N . v_N - \int v_N \; du_N$$

$$\int (a+bI_1+cI_2)x^n \;\; sin(r+sI_1+tI_2)x \;\; dx = \left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)\Bigg((a+bI_1+cI_2)x^n \;\; sin(r+sI_1+tI_2)x + \int nx^{n-1}\, cos(r+sI_1+tI_2)x \;\; dx\Bigg) + C$$

$$= \left(\frac{a}{r} + \left[\frac{rb + bt - as - sc}{(r+t)(r+s+t)}\right] I_1 + \left[\frac{rc - at}{r(r+t)}\right] I_2\right)\left((a + bI_1 + cI_2)x^n \;\; sin(r + sI_1 + tI_2)x + \int nx^{n-1} cos(r + sI_1 + tI_2)x \;\; dx\right) + C$$

By repeating the integration, we are able to find the required integral.

We calculate the second integral using the same method:

$$\int (a + bI_1 + cI_2)x^n \;\; cos(r + sI_1 + tI_2)x \;\; dx$$

Example 2

Find:

$$\int (3 + I_1 + 6I_2)x \;\; sin(1 + 2I_1 + 3I_2)x \;\; dx$$

Solution:

$$u_N = (3 + I_1 + 6I_2)x \quad \Longrightarrow \quad du_N = (3 + I_1 + 6I_2) \;\; dx$$

$$dv_N = sin(1 + 2I_1 + 3I_2)x \;\; dx \quad \Longrightarrow \quad v_N = \frac{-1}{1 + 2I_1 + 3I_2} cos(1 + 2I_1 + 3I_2)x$$

$$\int u_N \;\; dv_N = u_N . v_N - \int v_N \;\; du_N$$

$$\int (3 + I_1 + 6I_2)x \;\; sin(1 + 2I_1 + 3I_2)x \;\; dx$$

$$= \left(\frac{3 + I_1 + 6I_2}{1 + 2I_1 + 3I_2}\right)\left(-x \;\; cos(1 + 2I_1 + 3I_2)x + \int cos(1 + 2I_1 + 3I_2)x \, dx\right)$$

$$= \left(3 - \frac{7}{12} I_1 - \frac{3}{4} I_2\right)\left(-x \;\; cos(1 + 2I_1 + 3I_2)x + \frac{1}{1 + 2I_1 + 3I_2} \;\; sin(1 + 2I_1 + 3I_2)x\right)$$

$$= \left(3 - \frac{7}{12} I_1 - \frac{3}{4} I_2\right)\left(-x \;\; cos(1 + 2I_1 + 3I_2)x + \left(1 - \frac{1}{12} I_1 - \frac{3}{4} I_2\right) sin(1 + 2I_1 + 3I_2)x\right) + C$$

- state3:

$$\int e^{(a+bI_1+cI_2)x} \;\; sin(r + sI_1 + tI_2)x \;\; dx \qquad or \qquad \int e^{(a+bI_1+cI_2)x} \;\; cos(r + sI_1 + tI_2)x \;\; dx$$

$$u_N = e^{(a+bI_1+cI_2)x} \quad \Longrightarrow \quad du_N = (a + bI_1 + cI_2)e^{(a+bI_1+cI_2)x} \;\; dx$$

$$dv_N = sin(r + sI_1 + tI_2)x \;\; dx \quad \Longrightarrow \quad v_N = \frac{-1}{r + sI_1 + tI_2} cos(r + sI_1 + tI_2)x$$

$$\int u_N \;\; dv_N = u_N . v_N - \int v_N \;\; du_N$$

$$\int e^{(a+bI_1+cI_2)x} \;\; sin(r + sI_1 + tI_2)x \;\; dx$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right)e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x$$
$$+\left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)\int e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x \ \ dx \qquad (*)$$

by using integration by parts again to evaluate:

$$\int e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x \ \ dx$$

$$u_N = e^{(a+bI_1+cI_2)x} \qquad \Rightarrow \qquad du_N = (a+bI_1+cI_2)e^{(a+bI_1+cI_2)x} \ \ dx$$

$$dv_N = cos(r+sI_1+tI_2)x \ \ dx \qquad \Rightarrow \qquad v_N = \frac{1}{r+sI_1+tI_2}sin(r+sI_1+tI_2)x$$

$$\int u_N \ \ dv_N = u_N.v_N - \int v_N \ \ du_N$$

$$\int e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x \ \ dx$$

$$= \left(\frac{1}{r+sI_1+tI_2}\right)sin(r+sI_1+tI_2)x$$
$$+\left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)\int e^{(a+bI_1+cI_2)x} \ \ sin(r+sI_1+tI_2)x \ \ dx$$

by substitution in $(*)$:

$$\int e^{(a+bI_1+cI_2)x} \ \ sin(r+sI_1+tI_2)x \ \ dx$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right)e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x$$
$$+\left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)\left[\left(\frac{1}{r+sI_1+tI_2}\right)sin(r+sI_1+tI_2)x\right.$$
$$\left.+\left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)\int e^{(a+bI_1+cI_2)x} \ \ sin(r+sI_1+tI_2)x \ \ dx\right]$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right)e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{(r+sI_1+tI_2)^2}\right)sin(r+sI_1+tI_2)x$$
$$+\left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)^2\int e^{(a+bI_1+cI_2)x} \ \ sin(r+sI_1+tI_2)x \ \ dx$$

$$\Rightarrow \ \left(1-\left(\frac{a+bI_1+cI_2}{r+sI_1+tI_2}\right)^2\right)\int e^{(a+bI_1+cI_2)x} \ \ sin(r+sI_1+tI_2)x \ \ dx$$

$$= \left(\frac{-1}{r+sI_1+tI_2}\right)e^{(a+bI_1+cI_2)x} \ \ cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{(r+sI_1+tI_2)^2}\right)sin(r+sI_1+tI_2)x$$

$$\Rightarrow \ \int e^{(a+bI_1+cI_2)x} \ \ sin(r+sI_1+tI_2)x \ \ dx$$

$$= \left(\frac{(r+sI_1+tI_2)^2}{(r+sI_1+tI_2)^2-(a+bI_1+cI_2)^2}\right)\left[\left(\frac{-1}{r+sI_1+tI_2}\right)e^{(a+bI_1+cI_2)x}\ \ cos(r+sI_1+tI_2)x + \left(\frac{a+bI_1+cI_2}{(r+sI_1+tI_2)^2}\right)sin(r+sI_1+tI_2)x + C\right]$$

We calculate the second integral by using the same method:

$$\int e^{(a+bI_1+cI_2)x}\ \ cos(r+sI_1+tI_2)x\ \ dx$$

Example 3

Find:

$$\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\ \ dx$$

Solution:

$$u_N = e^{(1-I_1+I_2)x} \qquad \Longrightarrow \qquad du_N = (1-I_1+I_2)e^{(1-I_1+I_2)x}\ \ dx$$

$$dv_N = cos(1+I_1+I_2)x\ \ dx \qquad \Longrightarrow \qquad v_N = \frac{1}{1+I_1+I_2}sin(1+I_1+I_2)x$$

$$\int u_N\ \ dv_N = u_N.v_N - \int v_N\ \ du_N$$

$$\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\ \ dx$$

$$= \frac{1}{1+I_1+I_2}e^{(1-I_1+I_2)x}sin(1+I_1+I_2)x - \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\int e^{(1-I_1+I_2)x}\ \ sin(1+I_1+I_2)x\ \ dx \quad (*)$$

By using integration by parts again to evaluate:

$$\int e^{(1-I_1+I_2)x}\ \ sin(1+I_1+I_2)x\ \ dx$$

$$u_N = e^{(1-I_1+I_2)x} \qquad \Longrightarrow \qquad du_N = (1-I_1+I_2)e^{(1-I_1+I_2)x}\ \ dx$$

$$dv_N = sin(1+I_1+I_2)x\ \ dx \qquad \Longrightarrow \qquad v_N = \frac{-1}{1+I_1+I_2}cos(1+I_1+I_2)x$$

$$\int e^{(1-I_1+I_2)x}\ \ sin(1+I_1+I_2)x\ \ dx$$

$$= \frac{-1}{1+I_1+I_2}e^{(1-I_1+I_2)x}cos(1+I_1+I_2)x + \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\,dx$$

by substitution in $(*)$:

$$\int e^{(1-I_1+I_2)x}\ \ cos(1+I_1+I_2)x\ \ dx$$

$$= \frac{1}{1+I_1+I_2} e^{(1-I_1+I_2)x} sin(1+I_1+I_2)x - \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\left[\frac{-1}{1+I_1+I_2} e^{(1-I_1+I_2)x} cos(1+I_1+I_2)x + \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)\int e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\, dx \;\right]$$

$$= \frac{1}{1+I_1+I_2} e^{(1-I_1+I_2)x} sin(1+I_1-2I_2)x + \frac{1-I_1+I_2}{(1+I_1+I_2)^2} e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x - \left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)^2 \int e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\, dx$$

$$\Rightarrow \left(1+\left(\frac{1-I_1+I_2}{1+I_1+I_2}\right)^2\right)\int e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\, dx$$

$$= \frac{1}{1+I_1+I_2} e^{(1-I_1+I_2)x} sin(1+I_1+I_2)x + \frac{1-I_1+I_2}{(1+I_1+I_2)^2} e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x$$

$$\Rightarrow \left(1+\left(1-\frac{2}{3}I_1\right)^2\right)\int e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\, dx$$

$$= \frac{1}{1+I_1+I_2} e^{(1-I_1+I_2)x} sin(1+I_1+I_2)x + \frac{1-I_1+I_2}{1+5I_1+3I_2} e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x$$

$$\Rightarrow \left(1-\frac{8}{9}I_1\right)\int e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\, dx$$

$$= \frac{1}{1+I_1+I_2} e^{(1-I_1+I_2)x} sin(1+I_1+I_2)x + \frac{1-I_1+I_2}{1+5I_1+3I_2} e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x$$

$$= \frac{1}{1-\frac{8}{9}I_1}\left[\left(1-\frac{1}{6}I_1-\frac{1}{2}I_2\right) e^{(1-I_1+I_2)x} sin(1+I_1+I_2)x + \left(1-\frac{7}{18}I_1-\frac{1}{2}I_2\right) e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\right] + C$$

$$= (1+8I_1)\left[\left(1-\tfrac{1}{6}I_1-\tfrac{1}{2}I_2\right) e^{(1-I_1+I_2)x} sin(1+I_1+I_2)x + \left(1-\tfrac{7}{18}I_1-\tfrac{1}{2}I_2\right) e^{(1-I_1+I_2)x} \; cos(1+I_1+I_2)x\right] + C$$

- state4:

$$\int (a+bI_1+cI_2)x^n \;\; ln(r+sI_1+tI_2)x \;\; dx$$

$$u_N = ln(r+sI_1+tI_2)x \qquad \Rightarrow \qquad du_N = \frac{1}{x} \; dx$$

$$dv_N = (a+bI_1+cI_2)x^n \;\; dx \qquad \Rightarrow \qquad v_N = \frac{a+bI_1+cI_2}{n+1} x^{n+1}$$

$$\int u_N \quad dv_N = u_N . v_N - \int v_N \quad du_N$$

$$\int (a + bI_1 + cI_2)x^n \quad ln(r + sI_1 + tI_2)x \quad dx$$
$$= \left(\frac{a + bI_1 + cI_2}{n+1}\right)x^{n+1}. l\,n(r + sI_1 + tI_2)x - \frac{a + bI_1 + cI_2}{n+1}\int \frac{1}{x}x^{n+1} \quad dx$$

$$= \left(\frac{a + bI_1 + cI_2}{n+1}\right)x^{n+1}. l\,n(r + sI_1 + tI_2)x - \frac{a + bI_1 + cI_2}{(n+1)^2}x^{n+1} + C$$

$$= \left(\frac{a + bI_1 + cI_2}{n+1}\right)\left[x^{n+1}. l\,n(r + sI_1 + tI_2)x - \frac{1}{n+1}x^{n+1}\right] + C$$

Example 4

Find:

$$\int (1 + 3I_1 - 2I_2)x \quad ln(2 + I_1 + I_2)x \quad dx$$

Solution:

$$u_N = ln(2 + I_1 + I_2)x \quad \Longrightarrow \quad du_N = \frac{1}{x} \quad dx$$

$$dv_N = (1 + 3I_1 - 2I_2)x \quad dx \qquad \Longrightarrow \qquad v_N = \frac{1}{2}(1 + 3I_1 - 2I_2)x^2$$

$$\int u_N \quad dv_N = u_N . v_N - \int v_N \quad du_N$$

$$\int (1 + 3I_1 - 2I_2)x \quad ln(2 + I_1 + I_2)x \quad dx$$
$$= \frac{1}{2}(1 + 3I_1 - 2I_2)x^2. ln(2 + I_1 + I_2)x - \frac{1}{2}(1 + 3I_1 - 2I_2)\int xdx$$

$$= \frac{1}{2}(1 + 3I_1 - 2I_2)x^2 ln(2 + I_1 + I_2)x - \frac{1}{2}(1 + 3I_1 - 2I_2).\frac{x^2}{2}$$

$$= \left(\frac{1}{2} + \frac{3}{2}I_1 - I_2\right)\left[x^2 ln(2 + I_1 + I_2)x - \frac{1}{2}x^2\right] + C$$

Remark:

To find the following integrals:

1) $\int (a + bI_1 + cI_2)x^n \quad sin^{-1}(r + sI_1 + tI_2)x\, dx$

2) $\int (a + bI_1 + cI_2)x^n \quad cos^{-1}(r + sI_1 + tI_2)x \quad dx$

3) $\int (a + bI_1 + cI_2)x^n \quad tan^{-1}(r + sI_1 + tI_2)x\, dx$

we are following the same state 4, whereas:

$$u = \quad sin^{-1}(r + sI_1 + tI_2)x \qquad Or \qquad cos^{-1}(r + sI_1 + tI_2)x \qquad Or \qquad tan^{-1}(r + sI_1 + tI_2)x$$

$$and \quad dv = (a + bI_1 + cI_2)x^n \quad dx$$

Example 5

Find:

$$\int (2 - 2I_1 - I_2)x \;\; tan^{-1}(1 - I_1 + 2I_2)x \;\; dx$$

Solution:

$$u_N = tan^{-1}(1 - I_1 + 2I_2)x \quad \Rightarrow \quad du_N = \frac{1 - I_1 + 2I_2}{1 + (1 - 5I_1 + 8I_2)x^2} \quad dx$$

$$dv_N = (2 - 2I_1 - I_2)x \;\; dx \qquad \Rightarrow \qquad v_N = \frac{1}{2}(2 - 2I_1 - I_2)x^2$$

$$\int u_N \;\; dv_N = u_N . v_N - \int v_N \;\; du_N$$

$$\int (2 - 2I_1 - I_2)x \;\; tan^{-1}(1 - I_1 + 2I_2)x \;\; dx$$

$$= \frac{1}{2}(2 - 2I_1 - I_2)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \frac{1}{2}(2 - 2I_1 - I_2)(1 - I_1 + 2I_2)\int \frac{x^2}{1 + (1 - 5I_1 + 8I_2)x^2} \;\; dx$$

$$= \frac{1}{2}(2 - 2I_1 - I_2)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \frac{2 - 5I_1 + I_2}{2}\int \left(\frac{1}{1 - 5I_1 + 8I_2} - \frac{1}{1 - 5I_1 + 8I_2} \;\; \frac{1}{1 + (1 - 5I_1 + 8I_2)x^2}\right) \;\; dx$$

$$= \frac{1}{2}(2 - 2I_1 - I_2)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \frac{2 - 5I_1 + I_2}{2}\left(\frac{1}{1 - 5I_1 + 8I_2}x - \frac{1 - I_1 + 2I_2}{1 - 5I_1 + 8I_2} \quad tan^{-1}(1 - I_1 + 2I_2)x\right) + C$$

$$= \left(1 - I_1 - \frac{1}{2}I_2\right)x^2 \;\; tan^{-1}(1 - I_1 + 2I_2)x - \left(1 - \frac{5}{2}I_1 + \frac{1}{2}I_2\right)\left[\left(1 - \frac{5}{36}I_1 + \frac{8}{9}I_2\right)x - \left(1 - \frac{1}{6}I_1 - \frac{2}{3}I_2\right)tan^{-1}(1 - I_1 + 2I_2)x\right] + C$$

2.2 Tabular method

We use this method to find the integrals by parts in the states 1 and 2, as following:

- Differentiate the polynomial function, and we repeat that until we get to zero.

- Integrate the second function, repeat that, and stop once we reach the zero that resulted from the differentiation.
- Put the derivative products in one column and the integral products in the column that corresponds to it.
- Draw an arrow from each first-column entry to the second-column entry one row below it.
- Beginning with a +, label the arrows with alternating + and − signs.
- Compute the product of the expressions at the tip and tail of each arrow, and then multiply the result by + or −, depending on the arrow's sign.

Example 6

We can find the following integral by using tabular method:

$$\int (1+3I_1-2I_2)x^2\, e^{(2+I_1+I_2)x}\ \ dx$$

Derivation	Integration
$(+)\ \ (1+3I_1-2I_2)x^2$	$e^{(2+I_1+I_2)x}$
$(-)\ \ \ 2(1+3I_1-2I_2)x$	$\frac{1}{2+I_1+I_2}e^{(2+I_1+I_2)x}$
$(+)\ \ 2(1+3I_1-2I_2)$	$\frac{1}{4+7I_1+5I_2}e^{(2+I_1+I_2)x}$
0	$\frac{1}{16+175I_1+65I_2}e^{(2+I_1+I_2)x}$

hence:

$$\int (1+3I_1-2I_2)x^2\, e^{(2+I_1+I_2)x}\ \ dx$$

$$=\left(\frac{1+3I_1-2I_2}{2+I_1+I_2}\right)x^2e^{(2+I_1+I_2)x}-\left(\frac{4+6I_1-4I_2}{4+7I_1+5I_2}\right)xe^{(2+I_1+I_2)x}+\left(\frac{4+6I_1-4I_2}{16+175I_1+65I_2}\right)e^{(2+I_1+I_2)x}$$

$$=\left(\frac{1}{4}+\frac{5}{12}I_1-\frac{5}{12}I_2\right)x^2e^{(2+I_1+I_2)x}-\left(1+\frac{3}{8}I_1-I_2\right)xe^{(2+I_1+I_2)x}+\left(\frac{1}{4}+\frac{3}{128}I_1-\frac{1}{4}I_2\right)e^{(2+I_1+I_2)x}+C$$

Example 7

We can find the following integral by using tabular method:

$$\int (I_1+I_2)x\, cos(1+I_1+I_2)x\ \ \ dx$$

derivation	integration
$(+)\ \ (I_1+I_2)x$	$cos(1+I_1+I_2)x$
$(-)\ \ \ (I_1+I_2)$	$\frac{1}{1+I_1+I_2}sin(1+I_1+I_2)x$
$(+)\ \ \ \ \ \ \ 0$	$\frac{-1}{1+5I_1+3I_2}cos(1+I_1+I_2)x$

hence:

$$\int (I_1+I_2)x\cos(1+I_1+I_2)x \quad dx = \frac{I_1+I_2}{1+I_1+I_2}x.\sin(1+I_1+I_2)x + \frac{I_1+I_2}{1+5I_1+3I_2}\cos(1+I_1+I_2)x$$

$$= \left(\frac{1}{6}I_1+\frac{1}{2}I_2\right)x.\sin(1+I_1+I_2)x + \left(-\frac{1}{36}I_1+\frac{1}{4}I_2\right)\cos(1+I_1+I_2)x$$

3. Conclusions

This paper is an extension of the papers that were presented on the indefinite refined neutrosophic integrals. The importance of this paper lies in that it presented the indefinite refined neutrosophic integrals by parts and the Tobler method, as we found that applying the Tobler method is easier to calculate the indefinite refined neutrosophic integrals than the indefinite refined neutrosophic integrals by parts for some cases.

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A robust framework for medical diagnostics based on interval-valued Q-neutrosophic soft sets with aggregation operators

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Abstract: The best way to deal with complicated life scenarios that accompany the decision-making process is to update previous concepts constantly. Therefore, researchers must constantly discover powerful mathematical tools that suit the accompanying circumstances. In this regard, we combine both soft set, neutrosophic set, and interval setting under Q-two-dimensional universal information to introduce a new hybrid innovative model called interval valued-Q-neutrosophic soft sets. The core goal of this model is to keep the features of previous models like soft sets, neutrosophic sets, and Q-Fuzzy sets in dealing with the lack of uncertainty and neutrality associated with real-life issues. This new approach allows decision-makers to employ interval-valued form with Q-two-dimensional universal information, which provides them with more stability and feasibility in describing uncertain information more completely and accurately. Under the our propose model, we discuss effectively set-theory operations such as subset, union, intersection, complement, AND operation, and OR operation for interval valued-Q-neutrosophic soft sets, as well as some special operations like the necessity and possibility operations of an interval valued-Q-neutrosophic soft sets. In addition, we presented many properties supported by numerical examples that explain how they work. Finally, this new model has been successfully tested in dealing with one of the medical diagnostic problems based on hypothetical data for a respiratory disease. Building an algorithm based on the aggregation operator for interval valued-Q-neutrosophic soft set data solved this issue (i.e., selecting the optimal alternative).

Keywords: fuzzy set; neutrosophic set; soft set; Q- neutrosophic set, Q- neutrosophic soft set

1. Introduction

In our daily lives, numerous complicated issues contain diverse uncertainties and vagueness in human thinking. The decision-making process associated with human thinking is affected by these issues, which can have a significant impact on the effectiveness of the decision-making process, leading to suboptimal or even incorrect decisions. To address these provocations, Zadeh [1] first initiated a mathematical instrument called fuzzy set (FS) as a mathematical structure consisting of one function called the membership function (MF) or truth- MF that works on universal discourse U as a domain and close intervals [0, 1] as a codomain. But from a logical standpoint, it indicates that for every degree of judgment with a degree of truthfulness, there is another degree called the degree of falsehood or the degree of diss truthfulness. Accordingly, Atanassov [2] introduced another concept called intuitionistic fuzzy sets (IFS) by adding a second function called the nonmembership function (NMF), or falsehood-MF. It works in parallel with the truth-MF of correctness through the manifestations of falsehood-MF. Both FS and IFS show better accuracy levels in dealing with different issues in real-life applications. Later, researchers realized that the membership and non-membership values of an FS and IFS are insufficient for dealing with ambiguous indefinite, and inconsistent information in a real-world situation. Based on this need, Smarandache [3] developed another mathematical idea called the neutrosophic set (NS) as a generalization of FS and IFS. This concept is related to three functions MF, NMF and indeterminacy-membership function (IMF), each of which starts from U and rests in the closed interval [0,1]. This idea attracted the curiosity of many scholars around the world and pushed them to applied in many areas, including decision-making, machine learning, pattern recognition, medical diagnosis, market prediction, and image processing. From a scientific point of view, the degree of truth, falsity, and indeterminacy that exist in all the models mentioned above are organized into one single value. Still, sometimes in real situations, these memberships are uncertain, and it is hard for an expert

to express their certainty with a single value. To clarify this issue, consider this example: when you ask someone about the expected temperature for tomorrow, it is challenging to organize this degree immediately with a single value, but when he or she puts this expected degree in the form of an interval value, this person will find it easy to guess the desired degree. As a result, many researchers have reorganized the above models into interval form to make them more flexible and adaptable for addressing real-life problems that include uncertain, unpredictable, and incomplete information. For instance, the notion of an interval neutrosophic set (INS) has been proposed by Wang et al. [4] as an extension of an interval fuzzy set (IVFS) [5] and interval intuitionistic fuzzy set (IVIFS) [6] and they also give the set-theoretic operators of INS. The INS can independently represent the truth-membership degree, indeterminacy-membership degree, and falsity-membership degree, all of them in interval form. So, many investigators have studied it in depth and used it in many areas, such as making decisions, recognizing patterns, data mining, predicting the market, machine learning, and image processing. Molodtsov, on the other hand, pointed out that none of the above-mentioned models have good parameterization of the alternatives. This makes it hard to describe the alternatives to a problem because these parameters cannot be specified well enough. To address these difficulties, Molodtsov [7] came up with a soft set (SS) as a powerful parameter tool to deal with these problems.

This concept (SS), along with the concepts above (FS, IFS, NS), created a storm of important research work, for instance: Cagman et al. [8] introduced the fuzzy soft set (FSS) concept and provided its operations and properties. Following them, Maji [9] introduced neutrosophic soft set and its operations and properties. Deli [10] generalises the notions of SS and NS to interval-NSs under interval form. Saber et al. [11] started the research on the topological-NS information of soft sets by introducing a new approach called single-valued neutrosophic soft topological space. In complex spaces, a lot of research has been introduced [12-20].

1.1. Research gap: the fuzzy set environment and its extension lack the ability to handle two-dimensional information that is available in universal discourse U. For example, if we consider that U contains three patients, u_1, u_2, and u_3, who are suspected of being infected with a disease, it is difficult to describe their condition through a single object (one dimension). This motivates Adam and Hassan [21] to propose new strategies when they build a new model of Q-fuzzy sets (Q-FSs) to serve uncertainty and two-dimensionality simultaneously. After that, Broumi [22] extended to a Q-intuitionistic fuzzy soft set by combining IFSs and SSs by adding a two-dimensional non-membership function. These models are an extension of FSs and IFSs, so it is not feasible to deal with uncertain information that is saturated with positions of neutrality and ambiguity. To address this aspect, recently Abu Qamar and Hassan [23] established the notion of Q-neutrosophic soft sets (Q-NSSs) as a generalisation of NSSs and Q-FSs by upgrading the membership functions of NSSs to two dimensions. This approach has good capabilities compared to the works mentioned in this literature, but the outputs of this model are single values. As we mentioned previously, these values constitute an obstacle for the decision-maker and do not give him sufficient freedom to build numerical data that describes the information of the trouble to be clear up.

Moreover, in interactions process with the concepts described above and as a powerful tool, many researchers have used a technical known as Aggregation Operators (AO) to deal with various fields. This powerful tool allows us to summarize the data and data exploration emerging from the analysis of the problem using the above concepts, condense it, and extract the values with a clear meaning, thus facilitating the task of the user (decision maker) in the process of making clearly informed decisions. Xu [24] developed a new algorithm to solve the DM problem using AO for IFS environments. Chen and Ye [25] extend the Dombi Weighted AO (DWAO) for single-valued neutrosophic numbers (SVNNs) using the operations of both the Dombi T-norm and T-conorm and employ it in solving some real-life applications. Liu and Tang [26] generalised AO in interval-valued neutrosophic seting, and they showed their application to solve decision-making. Zulqarnain et al. [27] proposed the generalised aggregate operators on soft computing in a neutrosophic setting. Al-Sharqi et al. used this tool with many concepts within the fuzzy environment, such as fuzzy hypersoft [28], q-rung orthopair fuzzy neutrosophic valued [29], neutrosophic soft matrix [30], and bipolar neutrosophic hypersoft setting [31], and they employed all these concepts with AO in solving different real-life applications [32-35].

1.2. Novelity and Contributions: This manuscript aimed to suggest techniques a new idea called IV-Q-NSSs, which stands for interval-valued Q-neutrosophic soft sets. These are a more developed form of Q-NSSs, and each membership function is unique to Q-NSSs given in interval form. This format gives the user more freedom and efficiency when dealing with everyday scenarios, especially those saturated with neutral, two-dimensional uncertainty information.

The main contributions shown in this work that were made to achieve these objectives are:

i. A new technique (IV-Q-NSSs) is proposed to contain the effects of uncertainty information in two-dimensional.
ii. To demonstrate the theoretical side of this model, we presented the basic operations, supported by an illustrative numerical example. In addition to presenting the basic properties and theories of IV-Q-NSSs.
iii. On the applied side, these techniques have been added to solve one of the decision-making problems in the medical field by proposing a multi-step algorithm that works on IV-Q-NSS data.

1.3. The following diagram presents the stand down of the paper:

Figure 1: a representation of results.

2. Preliminaries

In this part, we recollect some critical consepts related to our proposed approach like FS, Q-FS, SS, and NS.

Definition 2.1. [1] Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \dots, \mathfrak{u}_n\}$ be the initial points space(non-empty universal set). Then an FS $\mathcal{F}$ on $\mathfrak{U}$ is defined by following form:

$$\mathcal{F} = \{\mathfrak{u}_j, \hat{P}^t(\mathfrak{u}_j) | \mathfrak{u}_j \in \mathfrak{U}\}$$

Where $\mathcal{F}$ is a mapping defined as $\mathcal{F}: \mathfrak{U} \to [0,1]$ such that $\hat{P}^t \in [0,1]$ and called truth membership function (TMF).

Definition 2.2. [21] Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \dots, \mathfrak{u}_n\}$ be the initial points space(non-empty universal set) and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \dots, \mathfrak{q}_n\}$ be nonempty set. Then an Q-FS $\mathcal{F}_{\mathfrak{Q}}$ on the order pair $(\mathfrak{U}, \mathfrak{Q})$is defined by following form:

$$\mathcal{F}_{\mathfrak{Q}} = \{(u, \mathfrak{q}), \hat{P}^t(\hat{u}, \check{\mathfrak{q}}) | (\hat{u}, \check{\mathfrak{q}}) \in \mathfrak{U} \times \mathfrak{Q}\}$$

Where $\mathcal{F}$ is a mapping defined as $\mathcal{F}_{\mathfrak{Q}}: \mathfrak{U} \times \mathfrak{Q} \to [0,1]$ such that $\hat{P}^t_{\mathfrak{Q}} \in [0,1]$ and called Q-truth membership function (TMF).

Definition 2.3. [3] Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \dots, \mathfrak{u}_n\}$ be the initial points space(non-empty universal set). Then an NS N on $\mathfrak{U}$ is defined by following form:

$$\mathrm{N} = \{\mathfrak{u}_j, \hat{P}^t(\mathfrak{u}_j), \hat{P}^i(\mathfrak{u}_j), \hat{P}^f(\mathfrak{u}_j) | \mathfrak{u}_j \in \mathfrak{U}\}$$

Where N is a mapping defined as $\mathrm{N}: \mathfrak{U} \to [0,1]$ such that $\hat{P}^t(\mathfrak{u}_j), \hat{P}^i(\mathfrak{u}_j), \hat{P}^f(\mathfrak{u}_j) \in [0,1]$ and named truth membership function (TMF), neutrality membership function (NMF), and falsity membership function (FMF) with stander condition $0 \le \hat{P}^t(\mathfrak{u}_j) + \hat{P}^i(\mathfrak{u}_j) + \hat{P}^f(\mathfrak{u}_j) \le 1$.

Definition 2.4. [24] Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \dots, \mathfrak{u}_n\}$ be the initial points space(non-empty universal set). Then an Q-NS N on $(\mathfrak{U} \times \mathfrak{Q})$is defined by following form:

$$\mathrm{N}_{\mathfrak{Q}} = \{\mathfrak{u}_j, \hat{P}^t_{\mathfrak{Q}}(u, \mathfrak{q}), \hat{P}^{\mathrm{i}}_{\mathfrak{Q}}(u, \mathfrak{q}), \hat{P}^{\mathrm{f}}_{\mathfrak{Q}}(u, \mathfrak{q}) | (u, \mathfrak{q}) \in \mathfrak{U} \times \mathfrak{Q}\}$$

Where $\mathrm{N}_{\mathfrak{Q}}$ is a mapping defined as $\mathrm{N}_{\mathfrak{Q}}: \mathfrak{U} \times \mathfrak{Q} \to [0,1]$ such that $\hat{P}^t_{\mathfrak{Q}}(u, \mathfrak{q}), \hat{P}^{\mathrm{i}}_{\mathfrak{Q}}(u, \mathfrak{q}), \hat{P}^{\mathrm{f}}_{\mathfrak{Q}}(u, \mathfrak{q}) \in [0,1]$ and called truth membership function (TMF), neutrality membership function (NMF), and falsity membership function (FMF) with stander condition $0 \le \hat{P}^t_{\mathfrak{Q}}(u, \mathfrak{q}) + \hat{P}^{\mathrm{i}}_{\mathfrak{Q}}(u, \mathfrak{q}) + \hat{P}^{\mathrm{f}}_{\mathfrak{Q}}(u, \mathfrak{q}) \le 1$.

Definition 2.5. [10] Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \dots, \mathfrak{u}_n\}$ be the initial points space(non-empty universal set). Then an IVNS N on $\mathfrak{U}$ is defined by following form:

$$\mathrm{N} = \{\mathfrak{u}_j, \hat{P}^t(\mathfrak{u}_j), \hat{P}^i(\mathfrak{u}_j), \hat{P}^f(\mathfrak{u}_j) | \mathfrak{u}_j \in \mathfrak{U}\}$$

Where $\hat{P}^t(\mathfrak{u}_j) = [\hat{P}^{t,l}(\mathfrak{u}_j), \hat{P}^{t,u}(\mathfrak{u}_j)]$, $\hat{P}^i(\mathfrak{u}_j) = [\hat{P}^{i,l}(\mathfrak{u}_j), \hat{P}^{i,u}(\mathfrak{u}_j)]$ and $\hat{P}^f(\mathfrak{u}_j) = [\hat{P}^{f,l}(\mathfrak{u}_j), \hat{P}^{f,u}(\mathfrak{u}_j)]$
Such that the domen of these terms is $\mathfrak{U}$ and the co-domen is $[0,1]$ and $\hat{P}^{t,l}(\mathfrak{u}_j), \hat{P}^{t,u}(\mathfrak{u}_j)$ are lower and upper of TMF, $\hat{P}^{i,l}(\mathfrak{u}_j), \hat{P}^{i,u}(\mathfrak{u}_j)$ are lower and upper of IMF and $\hat{P}^{f,l}(\mathfrak{u}_j), \hat{P}^{f,u}(\mathfrak{u}_j)$ are lower and upper of FMF, with two stander conditions $0 \le \hat{P}^{t,l}(\mathfrak{u}_j) + \hat{P}^{i,l}(\mathfrak{u}_j) + \hat{P}^{f,l}(\mathfrak{u}_j) \le 1$ and $0 \le \hat{P}^{t.u}(\mathfrak{u}_j) + \hat{P}^{i,u}(\mathfrak{u}_j) + \hat{P}^{f,u}(\mathfrak{u}_j) \le 1$.

Definition 2.6. [10] Assume that
$N_1 = \{\mathfrak{u}_j, \hat{P}^t_1(\mathfrak{u}_j), \hat{P}^i_1(\mathfrak{u}_j), \hat{P}^f_1(\mathfrak{u}_j) | \mathfrak{u}_j \in \mathfrak{U}\}$, $N_2 = \{\mathfrak{u}_j, \hat{P}^t_2(\mathfrak{u}_j), \hat{P}^i_2(\mathfrak{u}_j), \hat{P}^f_2(\mathfrak{u}_j) | \mathfrak{u}_j \in \mathfrak{U}\}$ be two INS on initial points space(non-empty universal set) $\mathfrak{U}$
where $\hat{P}^t_1(\mathfrak{u}_j) = [\hat{P}^{t,l}_1(\mathfrak{u}_j), \hat{P}^{t,u}_1(\mathfrak{u}_j)]$, $\hat{P}^i_1(\mathfrak{u}_j) = [\hat{P}^{i,l}_1(\mathfrak{u}_j), \hat{P}^{i,u}_1(\mathfrak{u}_j)]$ and $\hat{P}^f_1(\mathfrak{u}_j) = [\hat{P}^{f,l}_1(\mathfrak{u}_j), \hat{P}^{f,u}_1(\mathfrak{u}_j)]$ and $\hat{P}^t_2(\mathfrak{u}_j) = [\hat{P}^{t,l}_2(\mathfrak{u}_j), \hat{P}^{t,u}_2(\mathfrak{u}_j)]$, $\hat{P}^i_2(\mathfrak{u}_j) = [\hat{P}^{i,l}_2(\mathfrak{u}_j), \hat{P}^{i,u}_2(\mathfrak{u}_j)]$ and $\hat{P}^f_2(\mathfrak{u}_j) = [\hat{P}^{f,l}_2(\mathfrak{u}_j), \hat{P}^{f,u}_2(\mathfrak{u}_j)]$ Then,

- **i. Complement** $N_1^C = \{\mathfrak{u}_j, \hat{P}^f_1(\mathfrak{u}_j), 1 - \hat{P}^i_1(\mathfrak{u}_j), \hat{P}^t_1(\mathfrak{u}_j) | \mathfrak{u}_j \in \mathfrak{U}\}$
- **ii. Union**: $N_1 \cup N_2 = \{\mathfrak{u}_j, max[\hat{P}^t_1(\mathfrak{u}_j), \hat{P}^t_2(\mathfrak{u}_j)], \min[\hat{P}^i_1(\mathfrak{u}_j), \hat{P}^i_2(\mathfrak{u}_j)], \min[\hat{P}^f_1(\mathfrak{u}_j), \hat{P}^f_2(\mathfrak{u}_j)] | \mathfrak{u}_j \in \mathfrak{U}\}$.
- **iii. Intersection**: $N_1 \cap N_2 = \{\mathfrak{u}_j, min[\hat{P}^t_1(\mathfrak{u}_j), \hat{P}^t_2(\mathfrak{u}_j)], \max[\hat{P}^i_1(\mathfrak{u}_j), \hat{P}^i_2(\mathfrak{u}_j)], \max[\hat{P}^f_1(\mathfrak{u}_j), \hat{P}^f_2(\mathfrak{u}_j)] | \mathfrak{u}_j \in \mathfrak{U}\}$.
- **iv. Subset** $N_1 \subseteq N_2$ if $\hat{P}^t_1(\mathfrak{u}_j) \le \hat{P}^t_2(\mathfrak{u}_j)$, $\hat{P}^i_1(\mathfrak{u}_j) \ge \hat{P}^i_2(\mathfrak{u}_j)$, $\hat{P}^f_1(\mathfrak{u}_j) \ge \hat{P}^f_2(\mathfrak{u}_j)$.

Definition 2.7. [7] A pair $(\mathcal{F}, \bar{\mathrm{A}} \subseteq \mathcal{E})$ is named SSs over a non-empty universe of discourse $\mathfrak{U}$ if $\mathcal{F}\colon\ \bar{\mathrm{A}} \subseteq \mathcal{E} \longrightarrow P(\mathfrak{U})$, such that the term $P(\mathfrak{U})$ indicate the power set of $\mathfrak{U}$.

3. The Mathematical Structure of Interval Valued-Q-neutrosophic Soft Sets (IV-Q-NSSs)

This section proposes the general framework definition of our concept IV-Q-NSS with fundamental operations like empty IV-Q-NSS, absolute IV-Q-NSS, subset IV-Q-NSS, and equality between two IV-Q-NSS. Also, to clarify our model more, we will give some numerical examples.

Definition 3.1. Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \dots, \mathfrak{u}_n\}$ be the initial points space(non-empty universal set), $\mathfrak{Q} \neq \emptyset, ie \quad \mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_1, \mathfrak{q}_1, \dots, \mathfrak{q}_n\}$ and $\mathcal{E} = \{e_1, e_2, e_3, \dots, e_n\}$ be a set of attribute (parameters set). Let $\bar{\mathrm{A}} \subseteq \mathcal{E}$ be sub set of attribute set , then a duet $(\hat{P}_{\mathfrak{Q}}, \bar{\mathrm{A}})$ is called a interval-valued $\mathfrak{Q}$-neutrosophic soft set over the initial points space (non-empty universal set) $\mathfrak{U}$, where $\hat{P}_{\mathfrak{Q}}$ given as following mapping

$$\hat{P}_{\mathfrak{Q}}\colon \bar{\mathrm{A}} \to \mathfrak{Q} - IVNS(\mathfrak{U})$$

Then ,the $IV - \mathfrak{Q} - NSS\mathfrak{U})$ can be characterized by the following get form

$$\left(\hat{P}_{\mathfrak{Q}}, \bar{\mathrm{A}}\right) = \hat{P}_{\mathfrak{Q}_{\bar{\mathrm{A}}}} = \{e \in \bar{\mathrm{A}} \ , < \hat{P}^{t}_{\mathfrak{Q}}(u,\mathfrak{q})(e), \hat{P}^{\mathrm{i}}_{\mathfrak{Q}}(u,\mathfrak{q})(e), \hat{P}^{\mathrm{f}}_{\mathfrak{Q}}(u,\mathfrak{q})(e) > |(u,\mathfrak{q}) \in \mathfrak{U} \times \mathfrak{Q}\}$$

Where

$$\hat{P}^{t}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}) = [\hat{P}^{t,l}_{\mathfrak{Q}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e})]$$
$$\hat{P}^{i}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}) = [\hat{P}^{i,l}_{\mathfrak{Q}}(u,\mathfrak{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e})]$$
$$\hat{P}^{f}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}) = [\hat{P}^{f,l}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}), \hat{P}^{f,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e})]$$

Such that , the terms here $\hat{P}^{t,l}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}), \hat{P}^{t,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}), \hat{P}^{i,l}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}), \hat{P}^{i,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e})$ and $\hat{P}^{f,l}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}), \hat{P}^{f,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e})$ refer to true interval membership, indeterminacy interval membership, and falsehood interval membership of objects $(u,\mathfrak{q}) \in \ \mathfrak{U} \times \mathfrak{Q}$, with two stander conditions $0 \leq \hat{P}^{t,l}_{\mathfrak{Q}}(u,\mathfrak{q})(e) + \hat{P}^{i,l}_{\mathfrak{Q}}(u,\mathfrak{q})(e) + \hat{P}^{i,l}_{\mathfrak{Q}}(u,\mathfrak{q})(e) \leq 1$ and $0 \leq \hat{P}^{t,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}) + \hat{P}^{i,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}) + \hat{P}^{i,u}_{\mathfrak{Q}}(u,\mathfrak{q})(\mathrm{e}) \leq 1$.

Now, to shed more light on the above definition, we present below the following numerical example, which describes the mechanism of action of our approach presented in this work.

Example 3.2. Assume that we are interested in analyzing the attractiveness of three houses that one person is thinking of buying one of them. Now, let us analyze this attractiveness according to our model (IV-Q-NSS), therefore we assume that the three houses present as following universal set $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3\}$ and $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2\}$ be a set constituting two cities under consideration and $\mathcal{E} = \{e_1, e_2, e_3\}$ be a collection of

$$\hat{P}_{\mathfrak{Q}_{\bar{\mathrm{A}}}} = \left\{\left(\mathrm{e}_1, \frac{\langle[0.2,0.8],[0.1,0.7],[0.4,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_1)}, \frac{\langle[0.1,0.4],[0.5,0.8],[0.7,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_2)},\right.\right.$$
$$\frac{\langle[0.3,0.6],[0.2,0.7],[0.5,0.8]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_1)}, \frac{\langle[0.4,0.6],[0.2,0.9],[0.5,0.7]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_2)}$$
$$\left.\frac{\langle[0.1,0.5],[0.3,0.7],[0.2,0.8]\rangle}{(\mathfrak{u}_3,\mathfrak{q}_1)}, \frac{\langle[0.4,0.8],[0.4,0.6],[0.2,0.8]\rangle}{(\mathfrak{u}_3,\mathfrak{q}_2)}\right)$$

$\left(e_2, \frac{\langle[0.1,0.8],[0.5,0.7],[0.3,0.4]\rangle}{(u_1,q_1)}, \frac{\langle[0.1,0.8],[0.4,0.7],[0.2,0.6]\rangle}{(u_1,q_2)}\right.$

$\frac{\langle[0.5,0.8],[0.4,0.9],[0.2,0.7]\rangle}{(u_2,q_1)}, \frac{\langle[0.1,0.2],[0.2,0.5],[0.4,0.7]\rangle}{(u_2,q_2)}$

$\left.\frac{\langle[0.1,0.4],[0.2,0.5],[0.3,0.7]\rangle}{(u_3,q_1)}, \frac{\langle[0.1,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(u_3,q_2)}\right)$

$\left(e_3, \frac{\langle[0.7,0.9],[0.2,0.8],[0.3,0.6]\rangle}{(u_1,q_1)}, \frac{\langle[0.4,0.7],[0.2,0.5],[0.1,0.7]\rangle}{(u_1,q_2)}\right.$

$\frac{\langle[0.1,0.8],[0.1,0.4],[0.3,0.6]\rangle}{(u_2,q_1)}, \frac{\langle[0.5,0.6],[0.3,0.6],[0.2,0.7]\rangle}{(u_2,q_2)}$

$\left.\frac{\langle[0.4,0.6],[0.2,0.7],[0.3,0.6]\rangle}{(u_3,q_1)}, \frac{\langle[0.4,0.8],[0.8,0.9],[0.3,0.7]\rangle}{(u_3,q_2)}\right)\Bigg\}$

Definition 3.3 Let $\hat{P}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{A}\ , < \hat{P}^{t}_{\mathfrak{Q}}(u,q)(e), \hat{P}^{i}_{\mathfrak{Q}}(u,q)(e), \hat{P}^{f}_{\mathfrak{Q}}(u,q)(e) > |(u,q) \in \mathfrak{U} \times \mathfrak{Q}\}$ be a $IV - Q - NSS$ on initial point space (universal set). Then $\hat{P}_{Q_{\bar{A}}}$ knowing as $IV - Q - NS - nullset\ and\ refer\ as \hat{P}_{\emptyset_{(0)}} if \quad \hat{P}_{\emptyset_{(0)}}(u,q) = \{([0,0],[1,1],[1,1])\}$.

Example 2.4. The term $\hat{P}_{\emptyset_{(0)}(u_3,q_2)}(e_1) = \left(e_1, \frac{([0,0],[1,1],[1,1])}{(u_3,q_2)}\right)$ is consider $IV - Q - NS - nullset$ on $\mathfrak{U}$.

Definition 3.5 Let $\hat{P}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{A}\ , < \hat{P}^{t}_{\mathfrak{Q}}(u,q)(e), \hat{P}^{i}_{\mathfrak{Q}}(u,q)(e), \hat{P}^{f}_{\mathfrak{Q}}(u,q)(e) > |(u,q) \in \mathfrak{U} \times \mathfrak{Q}\}$ be a $IV - Q - NSS$ on initial point space (universal set). Then $\hat{P}_{Q_{\bar{A}}}$ knowing as $IV - Q - NS -$ absolute $set\ and\ refer\ as\ \hat{P}_{\emptyset_{(1)}} if \quad \hat{P}_{\emptyset_{(1)}}(u,q) = \{([1,1],[0,0],[0,0])\}$.

Example 3.6. The term $\hat{P}_{\emptyset_{(1)}(u_3,q_2)(e_1)} = \left(e_1, \frac{([1,1],[0,0],[0,0])}{(u_3,q_2)}\right)$ is consider $IV - Q - NS -$ absolute set on $\mathfrak{U}$.

Definition 3.7 Let $\hat{P}_{Q_{\bar{A}}} and \hat{P}_{Q_{\bar{B}}}$ be two $IV - Q - NSSs$ on non empty universal set (initial points space) $\mathfrak{U}$ with $\mathfrak{Q}$ .Then we say that $\hat{P}_{Q_{\bar{A}}}$ is $IV - Q - NSS\ subset\ of\ \hat{P}_{Q_{\bar{B}}}$ and refer to this relation as $\hat{P}_{Q_{\bar{A}}} \subseteq \hat{P}_{Q_{\bar{B}}}$ if fulifed the following conditions
For $A \subseteq B$ and $\hat{P}_{Q_{\bar{A}}} \subseteq \hat{P}_{Q_{\bar{B}}}$ for all $e \in A\ \ and\ \ B\ \ ,(u,q) \in \mathcal{U} \times Q$
Then
$\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(u,q) \le \hat{P}^{t,l}_{\mathfrak{Q}_{\bar{B}}}(u,q), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(u,q) \le \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{B}}}(u,q),$
$\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(u,q) \ge \hat{P}^{i,l}_{\mathfrak{Q}_{\bar{B}}}(u,q), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(u,q) \ge \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u,q),$

$\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(u,q) \ge \hat{P}^{f,l}_{\mathfrak{Q}_{\bar{B}}}(u,q), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(u,q) \ge \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{B}}}(u,q).$

Example 3.8. Assume that the two terms in example 3.2, where $B = \{e_1\}$,such that

$\hat{P}_{Q_{\bar{A}}(u_1,q_2)}(e_1) = \left(e_1, \frac{\langle[0.1,0.4],[0.5,0.8],[0.7,0.8]\rangle}{(u_1,q_2)}\right)$

$\hat{P}_{Q_{\bar{B}}(u_1,q_2)}(e_1) = \left(e_1, \frac{\langle[0.2,0.5],[0.3,0.4],[0.2,0.8]\rangle}{(u_1,q_2)}\right)$

Then, it's clear $\hat{P}_{Q_{\bar{A}}} \subseteq \hat{P}_{Q_{\bar{B}}}$.

Definition 3.9. Let $\hat{P}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{A}\ , < \hat{P}^{t}_{\mathfrak{Q}}(\hat{u},\breve{q})(e), \hat{P}^{i}_{\mathfrak{Q}}(\hat{u},\breve{q})(e), \hat{P}^{f}_{\mathfrak{Q}}(\hat{u},\breve{q})(e) > |(\hat{u},\breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\}$

$\in \mathrm{IV}-\mathrm{Q}-\mathrm{NSS}(\mathfrak{U})$.Then, its complement given as $\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}} or\ \ c\hat{P}_{\mathfrak{Q}_{\bar{A}}}$ and its defined as following:.

$$\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{\mathrm{A}}\ \ ,< P^{t\ c}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), P^{i\ c}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), P^{f\ c}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) > |(\hat{u},\breve{\mathrm{q}}) \in \mathfrak{U}\times\mathfrak{Q}\}$$

Or $c\hat{P}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{\mathrm{A}}\ \ ,< c\hat{P}^{t}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), c\hat{P}^{\mathrm{i}}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), c\hat{P}^{\mathrm{f}}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) > |(\hat{u},\breve{\mathrm{q}}) \in \mathfrak{U}\times\mathfrak{Q}\}$

Where

$$P^{t\ c}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) = \left[P^{t,l\ c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e), P^{t,u\ c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e)\right] = \left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e)\right],$$

$$P^{i\ c}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) = \left[P^{i,l\ c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e), P^{i,u\ c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e)\right] = \left[1-\hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e), 1-\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e)\right],$$

$$P^{f\ c}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) = \left[P^{f,l\ c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e), P^{f,u\ c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e)\right] = \left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u},\breve{\mathrm{q}})(e)\right],$$

Or

$$c\hat{P}^{t}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) = \left[c\hat{P}^{t,l}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), c\hat{P}^{t,u}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e)\right] = \left[\hat{P}^{f,l}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), \hat{P}^{f,u}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e)\right],$$

$$c\hat{P}^{i}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) = \left[c\hat{P}^{i,l}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), c\hat{P}^{i,u}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e)\right] = \left[1-\hat{P}^{i,u}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), 1-\hat{P}^{f,l}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e)\right],$$

$$c\hat{P}^{f}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e) = \left[c\hat{P}^{f,l}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), c\hat{P}^{f,u}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e)\right] = \left[\hat{P}^{t,l}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e), \hat{P}^{t,u}_{\mathfrak{Q}}(\hat{u},\breve{\mathrm{q}})(e)\right].$$

Based on a above definition we sat that this $\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}} or\ \ c\hat{P}_{\mathfrak{Q}_{\bar{A}}}$ is the complement of $\mathrm{IV}-\mathrm{Q}-\mathrm{NSS}(\mathfrak{U})$.

Example3.10. Assume that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2\}$ be initial point (universal set) , $\mathfrak{Q} = \{q_1, q_2\}$ and $\bar{A} = \{e_1, e_2\}$. Then

$$\hat{P}_{\mathfrak{Q}_{\bar{A}}} =$$

$$\left\{\left(\mathrm{e}_1, \frac{\langle[0.2,0.8],[0.1,0.7],[0.4,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_1)}, \frac{\langle[0.1,0.4],[0.5,0.8],[0.7,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_2)}\right.\right.$$
$$\left.\frac{\langle[0.1,0.5],[0.3,0.7],[0.2,0.8]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_1)}, \frac{\langle[0.4,0.8],[0.4,0.6],[0.2,0.8]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_2)}\right)$$
$$\left(\mathrm{e}_2, \frac{\langle[0.1,0.8],[0.5,0.7],[0.3,0.4]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_1)}, \frac{\langle[0.1,0.8],[0.4,0.7],[0.2,0.6]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_2)}\right.$$
$$\left.\left.\frac{\langle[0.1,0.4],[0.2,0.5],[0.3,0.7]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_1)}, \frac{\langle[0.1,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_2)}\right)\right\}$$

Then the complement operation defining $\hat{P}_{\mathfrak{Q}_{\bar{A}}}\ \ as\ \ P^{c}_{\mathfrak{Q}_{\bar{A}}}\ \ or\ \ cP_{\mathfrak{Q}_{\bar{A}}}$ basedon definition as following

$$\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}} = c\hat{P}_{\mathfrak{Q}_{\bar{A}}} =$$

$$\left\{\left(\mathrm{e}_1, \frac{\langle[0.4,0.8],[0.3,0.9],[0.2,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_1)}, \frac{\langle[0.7,0.8],[0.2,0.5],[0.1,0.4]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_2)}\right.\right.$$
$$\left.\frac{\langle[0.2,0.8],[0.3,0.7],[0.1,0.5]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_1)}, \frac{\langle[0.2,0.8],[0.4,0.6],[0.4,0.8]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_2)}\right)$$
$$\left(\mathrm{e}_2, \frac{\langle[0.3,0.4],[0.3,0.5],[0.1,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_1)}, \frac{\langle[0.2,0.6],[0.3,0.6],[0.1,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_2)}\right.$$
$$\left.\left.\frac{\langle[0.3,0.7],[0.5,0.8],[0.1,0.4]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_1)}, \frac{\langle[0.5,0.7],[0.5,0.6],[0.1,0.6]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_2)}\right)\right\}$$

Proposition 3.11 If $\hat{P}_{\mathfrak{Q}_{\bar{A}}} \in \mathrm{IV}-\mathrm{Q}-\mathrm{NSS}(\mathcal{X})$. Then $c\left(c\hat{P}_{\mathfrak{Q}_{\bar{A}}}\right) = \left(\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}}\right) or \left(\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}}\right)^{c} = \hat{P}_{\mathfrak{Q}_{\bar{A}}}$

Proof: From above definition, we have

$\hat{P}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{A}\ , < \hat{P}^{t}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e), \hat{P}^{i}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e), \hat{P}^{f}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e) > |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\}$

Then,

$\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{A}\ , < P^{t\,c}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e), P^{i\,c}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e), P^{f\,c}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e) > |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\}$

$= \Big\{e \in \bar{A}\ , \Big\langle \Big[P^{t,l\,c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), P^{t,u\,c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big], \Big[P^{i,l\,c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), P^{i,u\,c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big],$

$\Big[P^{f,l\,c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), P^{f,u\,c}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big]\Big\rangle : |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\Big\}$

$\left(\hat{P}^{c}_{\mathfrak{Q}_{\bar{A}}}\right)^{c} = \Big\{e \in \bar{A}\ , \Big\langle \Big[\Big(\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}, \Big(\hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}\Big], \Big[\Big(1 - \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}, \Big(1 - \hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}\Big],$

$\Big[\Big(\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}, \Big(\hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}\Big]\Big\rangle : |(u, q) \in \mathfrak{U} \times \mathfrak{Q}\Big\}$

$= \Big\{e \in \bar{A}\ , \Big\langle \Big[\Big(\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}, \Big(\hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}\Big], \Big[\Big(1 - \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}, \Big(1 - \hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}\Big],$

$\Big[\Big(\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}, \Big(\hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)^{c}\Big]\Big\rangle : |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\Big\}$

$= \Big\{e \in \bar{A}\ , \Big\langle \Big[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big], \Big[\Big(1 - \Big(1 - \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)\Big), \Big(1 - \Big(1 - \hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big)\Big)\Big],$

$\Big[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big]\Big\rangle : |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\Big\}$

$= \Big\{e \in \bar{A}\ , \Big\langle \Big[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big], \Big[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big],$

$\Big[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(\hat{u}, \breve{q})(e)\Big]\Big\rangle : |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\Big\}$

$= \hat{P}_{\mathfrak{Q}_{\bar{A}}} = \{e \in \bar{A}\ , < \hat{P}^{t}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e), \hat{P}^{i}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e), \hat{P}^{f}_{\mathfrak{Q}}(\hat{u}, \breve{q})(e) > |(\hat{u}, \breve{q}) \in \mathfrak{U} \times \mathfrak{Q}\}$

Hence we get $c\left(c\hat{P}_{\mathfrak{Q}_{\bar{A}}}\right) = \hat{P}_{\mathfrak{Q}_{\bar{A}}}$.

Definition 3.12 The union of two $IV - QNSS$ $\hat{P}_{\mathfrak{Q}_{\bar{C}}}$ and written as $\hat{P}_{\mathfrak{Q}_{\bar{A}}} \cup \hat{P}_{\mathfrak{Q}_{\bar{B}}} = \hat{P}_{\mathfrak{Q}_{\bar{C}}}$, where $\bar{C} = \bar{A} \cup \bar{B}$ and for all $c \in \bar{C}$, $(u, q) \in \mathfrak{U} \times \mathfrak{Q}$, the three $IV - Q - NSS$ member ships function given as follows :.

$$\hat{P}^{t}_{\mathfrak{Q}_{\bar{C}}}(u, q) \begin{cases} P^{t}_{\mathfrak{Q}_{\bar{A}}}(u, q) = \Big[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(u, q)(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(u, q)(e)\Big] & if\ \ c \in \bar{A} - \bar{B} \\ P^{t}_{\mathfrak{Q}_{\bar{B}}}(u, q) = \Big[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{B}}}(u, q)(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u, q)(e)\Big] & if\ \ c \in \bar{B} - \bar{A} \\ P^{t}_{\mathfrak{Q}_{\bar{\bar{C}}}}(u, q) = \max\Big[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u, q)(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u, q)(e)\Big] & if\ \ c \in \bar{A} \cap \bar{B} \end{cases}$$

$$\hat{P}^{i}_{\mathfrak{Q}_{\bar{C}}}(u, q) \begin{cases} P^{f}_{\mathfrak{Q}_{\bar{A}}}(u, q) = \Big[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(u, q)(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(u, q)(e)\Big] & if\ \ c \in \bar{A} - \bar{B} \\ P^{f}_{\mathfrak{Q}_{\bar{B}}}(u, q) = \Big[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{B}}}(u, q)(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u, q)(e)\Big] & if\ \ c \in \bar{B} - \bar{A} \\ P^{f}_{\mathfrak{Q}_{\bar{\bar{C}}}}(u, q) = \min\Big[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u, q)(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u, q)(e)\Big] & if\ \ c \in \bar{A} \cap \bar{B} \end{cases}$$

$$\hat{P}^{f}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})\begin{cases} P^{f}_{\mathfrak{Q}_{\bar{A}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} - \bar{B} \\ P^{f}_{\mathfrak{Q}_{\bar{B}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{B} - \bar{A} \\ P^{f}_{\mathfrak{Q}_{\bar{\bar{C}}}}(\mathfrak{u},\mathfrak{q}) = \min\left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} \cap \bar{B} \end{cases}$$

Where

$$\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \max\left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{t,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$$\hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \max\left[\hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$$\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \min\left[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{i,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$$\hat{P}^{i,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \min\left[\hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$$\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \min\left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{f,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$$\hat{P}^{f,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \min\left[\hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right].$$

Here. The max represents the largest value of $IV - QNSS$ and min represents the smallest value of $IV - QNSS$.

Definition 3.13 The intersection of two $IV - QNSS$ $\hat{P}_{\mathfrak{Q}_{\bar{C}}}$ and written as $\hat{P}_{\mathfrak{Q}_{\bar{A}}} \cup \hat{P}_{\mathfrak{Q}_{\bar{B}}} = \hat{P}_{\mathfrak{Q}_{\bar{C}}}$, where $\bar{C} = \bar{A} \cap \bar{B}$ and for all $c \in \bar{C}$, $(\mathfrak{u},\mathfrak{q}) \in \mathfrak{U} \times \mathfrak{Q}$,the three $IV - Q - NSS$ member ships function given as follows :.

$$\hat{P}^{t}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})\begin{cases} P^{t}_{\mathfrak{Q}_{\bar{A}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} - \bar{B} \\ P^{t}_{\mathfrak{Q}_{\bar{B}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{B} - \bar{A} \\ P^{t}_{\mathfrak{Q}_{\bar{\bar{C}}}}(\mathfrak{u},\mathfrak{q}) = \min\left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} \cap \bar{B} \end{cases}$$

$$\hat{P}^{i}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})\begin{cases} P^{f}_{\mathfrak{Q}_{\bar{A}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} - \bar{B} \\ P^{f}_{\mathfrak{Q}_{\bar{B}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{B} - \bar{A} \\ P^{f}_{\mathfrak{Q}_{\bar{\bar{C}}}}(\mathfrak{u},\mathfrak{q}) = \max\left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} \cap \bar{B} \end{cases}$$

$$\hat{P}^{f}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})\begin{cases} P^{f}_{\mathfrak{Q}_{\bar{A}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} - \bar{B} \\ P^{f}_{\mathfrak{Q}_{\bar{B}}}(\mathfrak{u},\mathfrak{q}) = \left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{B} - \bar{A} \\ P^{f}_{\mathfrak{Q}_{\bar{\bar{C}}}}(\mathfrak{u},\mathfrak{q}) = \mathrm{mix}\left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e)\right] & if\ \ c \in \bar{A} \cap \bar{B} \end{cases}$$

Where

$$\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \min\left[\hat{P}^{t,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{t,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$$\hat{P}^{t,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathfrak{q})(e) = \min\left[\hat{P}^{t,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathfrak{q})(e), \hat{P}^{t,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathfrak{q})(e)\right],$$

$\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathrm{q})(e) = \max\left[\hat{P}^{i,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathrm{q})(e), \hat{P}^{i,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathrm{q})(e)\right],$

$\hat{P}^{i,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathrm{q})(e) = \max\left[\hat{P}^{i,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathrm{q})(e), \hat{P}^{i,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathrm{q})(e)\right],$

$\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{C}}}(u,\mathrm{q})(e) = \max\left[\hat{P}^{f,l}_{\mathfrak{Q}_{\bar{A}}}(u,\mathrm{q})(e), \hat{P}^{f,l}_{\mathfrak{Q}_{\bar{B}}}(u,\mathrm{q})(e)\right],$

$\hat{P}^{f,u}_{\mathfrak{Q}_{\bar{C}}}(u,\mathrm{q})(e) = \max\left[\hat{P}^{f,u}_{\mathfrak{Q}_{\bar{A}}}(u,\mathrm{q})(e), \hat{P}^{f,u}_{\mathfrak{Q}_{\bar{B}}}(u,\mathrm{q})(e)\right].$

Here. The max represents the largest value of $IV - QNSS$ and min represents the smallest value of $IV - QNSS$.

Example 3.14. Let $\mathcal{X} = \{u_1, u_2\}$ be non-empty initial universal set, $\mathfrak{L} = \{e_1, e_2\} and\ Q = \{\mathrm{q}_1\}. then\ , if\ \ \bar{A} = \{e_1\} \subseteq \mathfrak{L}$,
$\bar{B} = \{e_1, e_2\} \subseteq \mathfrak{L}\ \ , then\ \ the\ \ two\ \ IV - Q - NSSs\ \ (\hat{P}_Q, \bar{A}), (\hat{P}_Q, \bar{B})$
Will be analyze as following

$(\hat{P}_Q, \bar{A}) =$
$\left\{\left(\mathrm{e}_1, \frac{\langle[0.2,0.8],[0.1,0.7],[0.4,0.8]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.1,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(\mathrm{u}_2,\mathrm{q}_1)}\right)\right\}$
$(\hat{P}_Q, \bar{B}) =$
$\left\{\left(\mathrm{e}_1, \frac{\langle[0.3,0.5],[0.2,0.4],[0.6,0.7]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.6,0.9],[0.5,0.8],[0.4,0.6]\rangle}{(\mathrm{u}_2,\mathrm{q}_1)}\right)\right.$
$\left.\left(\mathrm{e}_2 \frac{\langle[0.6,1],[0.7,0.9],[0.6,0.8]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.3,0.5],[0.8,0.8],[0.6,0.7]\rangle}{(\mathrm{u}_2,\mathrm{q}_1)}\right)\right\}$
Then, $(\hat{P}_{\mathfrak{Q}}, \hat{A})\ \ \cup (\hat{P}_{\mathfrak{Q}}, \hat{B}) =$
$\left\{\left(\mathrm{e}_1, \frac{\langle[0.3,0.8],[0.1,0.7],[0.4,0.7]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.6,0.9],[0.4,0.5],[0.4,0.6]\rangle}{(\mathrm{u}_1,\mathrm{q}_2)}\right)\right.$
$\left.\left(\mathrm{e}_2, \frac{\langle[0.6,1],[0.7,0.9],[0.6,0.8]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.3,0.5],[0.8,0.8],[0.6,0.7]\rangle}{(\mathrm{u}_2,\mathrm{q}_1)}\right)\right\}$
$(\hat{P}_{\mathfrak{Q}}, \hat{A})\ \ \cap (\hat{P}_{\mathfrak{Q}}, \hat{B}) =$
$\left\{\left(\mathrm{e}_1, \frac{\langle[0.2,0.5],[0.2,0.7],[0.6,0.8]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.1,0.6],[0.5,0.8],[0.5,0.7]\rangle}{(\mathrm{u}_1,\mathrm{q}_2)}\right)\right.$
$\left.\left(\mathrm{e}_2, \frac{\langle[0.6,1],[0.7,0.9],[0.6,0.8]\rangle}{(\mathrm{u}_1,\mathrm{q}_1)}, \frac{\langle[0.3,0.5],[0.8,0.8],[0.6,0.7]\rangle}{(\mathrm{u}_2,\mathrm{q}_1)}\right)\right\}$

Proposition 3. 15. Let $(\hat{P}_{\mathfrak{Q}}, \hat{A}), (\hat{P}_{\mathfrak{Q}}, \hat{B}) and (\hat{P}_{\mathfrak{Q}}, C)$ be three $IV - Q - NSSs\ \ on\ \ non-universal\ \ set\ \ U$. The, the following points are satisfied:
1. $(\hat{P}_{\mathfrak{Q}}, \hat{A}) \cup (\hat{P}_{\mathfrak{Q}}, \hat{B}) = (\hat{P}_{\mathfrak{Q}}, \hat{B}) \cup (\hat{P}_{\mathfrak{Q}}, \hat{A})$
2. $(\hat{P}_{\mathfrak{Q}}, \hat{A}) \cap (\hat{P}_{\mathfrak{Q}}, \hat{B}) = (\hat{P}_{\mathfrak{Q}}, \hat{B}) \cap (\hat{P}_{\mathfrak{Q}}, \hat{A})$

3. $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup \left((\hat{P}_{\mathfrak{Q}},\hat{B}) \cup (\hat{P}_{\mathfrak{Q}},C)\right) = \left((\hat{P}_{\mathfrak{Q}},\hat{A}),(\hat{P}_{\mathfrak{Q}},\hat{B})\right) \cup (\hat{P}_{\mathfrak{Q}},C)$

4. $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup \left((\hat{P}_{\mathfrak{Q}},\hat{B}) \cup (\hat{P}_{\mathfrak{Q}},C)\right) = \left((\hat{P}_{\mathfrak{Q}},\hat{A}),(\hat{P}_{\mathfrak{Q}},\hat{B})\right) \cup (\hat{P}_{\mathfrak{Q}},C)$

5. $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup \phi = \phi \cup (\hat{P}_{\mathfrak{Q}},\hat{A}) = (\hat{P}_{\mathfrak{Q}},\hat{A})$
6. $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cap \phi = \phi \cap (\hat{P}_{\mathfrak{Q}},\hat{A}) = \phi$
7. $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup U = U \cup (\hat{P}_{\mathfrak{Q}},\hat{A}) = U$
8. $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cap U = U \cap (\hat{P}_{\mathfrak{Q}},\hat{A}) = (\hat{P}_{\mathfrak{Q}},\hat{A})$

Proof (1). Now we will show that $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup (\hat{P}_{\mathfrak{Q}},\hat{B})$=$(\hat{P}_{\mathfrak{Q}},\hat{B}) \cup (\hat{P}_{\mathfrak{Q}},\hat{A})$based on **Definition 3.12** Also, in this case we will consider the case $c \in \bar{A} \cap \bar{B}$ and other case are trivial .

Now, take the left side $(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup (\hat{P}_{\mathfrak{Q}},\hat{B}), then$

$$(\hat{P}_{\mathfrak{Q}},\hat{A}) \cup (\hat{P}_{\mathfrak{Q}},\hat{B}) = \{< c \ \ (\max\{\hat{P}^t_{\mathfrak{Q}_{\bar{A}}}(u,t), \hat{P}^t_{\mathfrak{Q}_{\bar{B}}}(u,t),$$

$$\min\left\{\hat{P}^i_{\mathfrak{Q}_{\bar{A}}}(u,t), \hat{P}^i_{\mathfrak{Q}_{\bar{B}}}(u,q)\right\}, \min\left\{\hat{P}^f_{\mathfrak{Q}_{\bar{A}}}(u,t), \hat{P}^f_{\mathfrak{Q}_{\bar{B}}}(u,q)\right\} : (u,q) \in U \times Q >$$

$$=\left\{< c(\max\left[\max P^{t,i}_{Q_{\bar{A}}}(u,q), P^{t,l}_{Q_{\bar{B}}}(u,q)\right], \left[\max \ \ [P^{t,u}_{Q_{\bar{A}}}(u,q), P^{t,u}_{Q_{\bar{B}}}(u,q)\right]\right\}$$

$$,\min\left\{ \ \ \min\left[P^{i,l}_{Q_{\bar{A}}}(u,q), P^{i,l}_{Q_{\bar{B}}}(u,q)\right], \min\left[P^{i,u}_{Q_{\bar{A}}}(u,q), P^{i,u}_{Q_{\bar{B}}}(u,q)\right]\right\},$$

$$\min\left\{\min\left[P^{f,u}_{Q_{\bar{A}}}(u,q), P^{f,u}_{Q_{\bar{B}}}(u,q)\right], \min\left[P^{f,u}_{Q_{\bar{A}}}(u,q), P^{f,u}_{Q_{\bar{B}}}(u,q)\right] > (u,q) \in U \times Q\right\}$$

$$= \{< c \ \ (\max\{\hat{P}^t_{\mathfrak{Q}_{\bar{A}}}(u,t), \hat{P}^t_{\mathfrak{Q}_{\bar{B}}}(u,t),$$

$$\min\left\{\hat{P}^i_{\mathfrak{Q}_{\bar{A}}}(u,t), \hat{P}^i_{\mathfrak{Q}_{\bar{B}}}(u,q)\right\}, \min\left\{\hat{P}^f_{\mathfrak{Q}_{\bar{A}}}(u,t), \hat{P}^f_{\mathfrak{Q}_{\bar{B}}}(u,q)\right\} : (u,q) \in U \times Q >\}$$

$$= (\hat{P}_{\mathfrak{Q}},\hat{A}) \cup (\hat{P}_{\mathfrak{Q}},\hat{B})$$

Definition 3.16. Assume that $(\hat{P}_{\mathfrak{Q}},\hat{A})$ and $(\hat{P}_{\mathfrak{Q}},\hat{B})$ $are\ \ two\ \ IV-Q-NSSs$ on initial point space (non-empty universal set) U ,then $(\hat{P}_{\mathfrak{Q}},\hat{A})$ $AND(\hat{P}_{\mathfrak{Q}},\hat{B})$ $is\ an\ IV-Q-NSSs$ and denoted by $(\hat{P}_{\mathfrak{Q}},\hat{A})\wedge(\hat{P}_{\mathfrak{Q}},\hat{B})$ and it defined by the following formalh $(\hat{P}_{\mathfrak{Q}},\hat{A})$ $AND(\hat{P}_{\mathfrak{Q}},\hat{B}) = (\hat{P}_{\mathfrak{Q}},\bar{A}\times\hat{B}), where$

$$\hat{P}_{\mathfrak{Q}}(\bar{a},\bar{b})^{(u,q)} = \hat{P}_{\mathfrak{Q}_{(\bar{a})}}{}^{(u,q)} \cap \hat{P}_{\mathfrak{Q}_{(\bar{b})}}{}^{(u,q)}$$

For all $(\bar{a},\bar{b}) \in \bar{A}\times\bar{B}$, where $\cap\, is\ \ tha\ \ intersection$ operation of two $IV-Q-NSSs$ on initial points pace (non-empty universal set)

Now , based on the intersection definition the three $IV-Q-NSSs$ membership function defined as following

$$P^t_{Q_{(\bar{a},\bar{b})}}(u,q) = \min\{P^t_{Q_{(\bar{a})}}(u,q), P^t_{Q_{(\bar{b})}}(u,q)\} = \min\left\{\min\left[P^{t,l}_{Q_{(\bar{a})}}(u,q), P^{t,l}_{Q_{(\bar{b})}}(u,q)\right], \min\left[P^{t,u}_{Q_{(\bar{a})}}(u,q), P^{t,u}_{Q_{(\bar{b})}}(u,q)\right]\right\},$$

$$P^i_{Q_{(\bar{a},\bar{b})}}(u,q) = \text{mix}\{P^i_{Q_{(\bar{a})}}(u,q), P^i_{Q_{(\bar{b})}}(u,q)\} = \text{mix}\left\{\text{mix}\left[P^{i,l}_{Q_{(\bar{a})}}(u,q), P^{i,l}_{Q_{(\bar{b})}}(u,q)\right], \text{mix}\left[P^{i,u}_{Q_{(\bar{a})}}(u,q), P^{i,u}_{Q_{(\bar{b})}}(u,q)\right]\right\},$$

$$P^f_{Q_{(\bar{a},\bar{b})}}(u,q) = \text{mix}\{P^f_{Q_{(\bar{a})}}(u,q), P^f_{Q_{(\bar{b})}}(u,q)\} = \text{mix}\left\{\text{mix}\left[P^{f,l}_{Q_{(\bar{a})}}(u,q), P^{f,l}_{Q_{(\bar{b})}}(u,q)\right], \text{mix}\left[P^{f,u}_{Q_{(\bar{a})}}(u,q), P^{f,u}_{Q_{(\bar{b})}}(u,q)\right]\right\}.$$

Definition 3.17. Assume that $(\hat{P}_{\mathfrak{Q}}, \hat{A})$ and $(\hat{P}_{\mathfrak{Q}}, \hat{B})$ are two $IV - Q - NSSs$ on initial point space (non-empty universal set) U ,then $(\hat{P}_{\mathfrak{Q}}, \hat{A})$ $OR(\hat{P}_{\mathfrak{Q}}, \hat{B})$ is an $IV - Q - NSSs$ and denoted by $(\hat{P}_{\mathfrak{Q}}, \hat{A}) \vee (\hat{P}_{\mathfrak{Q}}, \hat{B})$ and it defined by the following formalh $(\hat{P}_{\mathfrak{Q}}, \hat{A}) OR(\hat{P}_{\mathfrak{Q}}, \hat{B}) = (\hat{P}_{\mathfrak{Q}}, \bar{A} \times \hat{B}), where$

$$\hat{P}_{\mathfrak{Q}}(\bar{a}, \bar{b})^{(u,q)} = \hat{P}_{\mathfrak{Q}_{(\bar{a})}}{}^{(u,q)} \cup \hat{P}_{\mathfrak{Q}_{(\bar{b})}}{}^{(u,q)}$$

For all $(\bar{a}, \bar{b}) \in \bar{A} \times \bar{B}$, where $\cup$ is tha $union$operation of two $IV - Q - NSSs$ on initial points pace (non-empty universal set) U.

Now , based on the $union$ definition the three $IV - Q - NSSs$ membership function defined as following

$$P^t_{Q_{(\bar{a},\bar{b})}}(u,q) = \text{mix}\{ P^t_{Q_{(\bar{a})}}(u,q), P^t_{Q_{(\bar{b})}}(u,q)\} = \text{mix}\left\{\text{mix}\left[P^{t,l}_{Q_{(\bar{a})}}(u,q), P^{t,l}_{Q_{(\bar{b})}}(u,q)\right], \text{mix}\left[P^{t,u}_{Q_{(\bar{a})}}(u,q), P^{t,u}_{Q_{(\bar{b})}}(u,q)\right]\right\},$$

$$P^i_{Q_{(\bar{a},\bar{b})}}(u,q) = \min\{ P^i_{Q_{(\bar{a})}}(u,q), P^i_{Q_{(\bar{b})}}(u,q)\} = \min\left\{\min\left[P^{i,l}_{Q_{(\bar{a})}}(u,q), P^{i,l}_{Q_{(\bar{b})}}(u,q)\right], \min\left[P^{i,u}_{Q_{(\bar{a})}}(u,q), P^{i,u}_{Q_{(\bar{b})}}(u,q)\right]\right\},$$

$$P^f_{Q_{(\bar{a},\bar{b})}}(u,q) = \min\{ P^f_{Q_{(\bar{a})}}(u,q), P^f_{Q_{(\bar{b})}}(u,q)\} = \min\left\{\min\left[P^{f,l}_{Q_{(\bar{a})}}(u,q), P^{f,l}_{Q_{(\bar{b})}}(u,q)\right], \min\left[P^{f,u}_{Q_{(\bar{a})}}(u,q), P^{f,u}_{Q_{(\bar{b})}}(u,q)\right]\right\}.$$

Example 3.18. Let $\mathcal{X} = \{u_1, u_2\}$ be non-empty initial universal set, $\mathfrak{L} = \{e_1, e_2, e_3\} and$ $Q = \{q_1\}. then$ $, if$ $\bar{A} = \{e_1\} \subseteq \mathfrak{L}$,

$\bar{B} = \{e_2, e_3\} \subseteq \mathfrak{L}$ $, then$ the two $IV - Q - NSSs$ $(\hat{P}_Q, \bar{A}), (\hat{P}_Q, \bar{B})$

Will be analyze as following

$$(\hat{P}_Q, \bar{A}) = \left\{\left(e_1, \frac{\langle[0.2,0.8],[0.1,0.7],[0.4,0.8]\rangle}{(u_1,q_1)}, \frac{\langle[0.1,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(u_2,q_1)}\right)\right\}$$

$$(\hat{P}_Q, \bar{B}) = \left\{\left(e_2, \frac{\langle[0.3,0.5],[0.2,0.4],[0.6,0.7]\rangle}{(u_1,q_1)}, \frac{\langle[0.6,0.9],[0.5,0.8],[0.4,0.6]\rangle}{(u_2,q_1)}\right)\right.$$
$$\left.\left(e_3 \frac{\langle[0.6,1],[0.7,0.9],[0.6,0.8]\rangle}{(u_1,q_1)}, \frac{\langle[0.3,0.5],[0.8,0.8],[0.6,0.7]\rangle}{(u_2,q_1)}\right)\right\}$$

Then,

$$(\hat{P}_{\mathfrak{Q}}, \hat{A})\ AND(\hat{P}_{\mathfrak{Q}}, \hat{B}) = (\hat{P}_{\mathfrak{Q}}, \bar{A} \times \hat{B}) =$$
$$\left\{\left((e_1,e_2), \frac{\langle[0.2,0.5],[0.2,0.7],[0.6,0.8]\rangle}{(u_1,q_1)}, \frac{\langle[0.1,0.3],[0.5,0.8],[0.7,0.8]\rangle}{(u_1,q_2)}\right)\right.$$
$$\left.\left((e_1,e_3), \frac{\langle[0.3,0.5],[0.7,0.9],[0.6,0.8]\rangle}{(u_2,q_1)}, \frac{\langle[0.3,0.5],[0.8,0.8],[0.6,0.7]\rangle}{(u_2,q_2)}\right)\right\}$$

And $(\hat{P}_{\mathfrak{Q}}, \hat{A})$ $OR(\hat{P}_{\mathfrak{Q}}, \hat{B}) = (\hat{P}_{\mathfrak{Q}}, \bar{A} \times \hat{B}) =$

$$\left\{\left((e_1,e_2), \frac{\langle[0.3,0.8],[0.1,0.4],[0.4,0.7]\rangle}{(u_1,q_1)}, \frac{\langle[0.6,0.9],[0.4,0.5],[0.4,0.6]\rangle}{(u_1,q_2)}\right)\right.$$

$$\left((e_1, e_3), \frac{\langle[0.6,1],[0.1,0.7],[0.4,0.8]\rangle}{(u_2, q_1)}, \frac{\langle[0.3,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(u_2, q_2)}\right)\Big\}$$

Proposition 3.19 Assume that $(\hat{P}_Q, \bar{A}), (\hat{P}_Q, \bar{B}) and\ (\hat{P}_Q, \bar{C})$ be three $IV-Q-NSSs$ no non-empty initial universal set U. Then following point (properties) will be satisfied:

1. $(\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B}) \wedge (\hat{P}_Q, \bar{C}) = \left((\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B})\right) \wedge (\hat{P}_Q, \bar{C})$

2. $(\hat{P}_Q, \bar{A}) \vee (\hat{P}_Q, \bar{B}) \vee (\hat{P}_Q, \bar{C}) = \left((\hat{P}_Q, \bar{A}) \vee (\hat{P}_Q, \bar{B})\right) \vee (\hat{P}_Q, \bar{C})$

Proof.1. Assume that $\bar{a} \in \bar{A}, \bar{b} \in \bar{B}$ and the thired one $\bar{c} \in \bar{C}$ and $(\hat{P}_Q, \bar{B}) \wedge (\hat{P}_Q, \bar{C}) = (\hat{P}_Q, \bar{B} \times \bar{C}), such\ \ that$ $\hat{P}_Q(\bar{b}, \bar{c}) = \hat{P}_Q(\bar{b}) \cap \hat{P}_Q(\bar{c})$

Now , we have $(\hat{P}_Q, \bar{A}) \wedge \left((\hat{P}_Q, \bar{B}) \wedge (\hat{P}_Q, \bar{C})\right) = (\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B} \times \bar{C}) = (\hat{P}_Q, \bar{A} \times \bar{B} \times \bar{C}),$

Such that
$(\hat{P}_Q, \bar{a} \times \bar{b} \times \bar{c}) = \hat{P}_Q(a) \cap \hat{P}_Q(b, c) = \hat{P}_Q(a) \cap \hat{P}_Q(b) \cap \hat{P}_Q(c)$
Also we have $(\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B}) = (\hat{P}_Q, \bar{A} \times \bar{B})$ such that

$$\hat{P}_Q(\bar{a}, \bar{b}) = \hat{P}_Q(\bar{a}) \cap \hat{P}_Q(\bar{b})$$

Therefor $\left((\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B})\right) \wedge (\hat{P}_Q, \bar{C}) = (\hat{P}_Q, \bar{A} \times \bar{B}) \wedge (\hat{P}_Q, \bar{C})$

$= (\hat{P}_Q, \bar{A} \times \bar{B} \times \bar{C})\ \ where\ \ (\hat{P}_Q, \bar{a} \times \bar{b} \times \bar{c}) = \hat{P}_Q(\bar{a}, \bar{b}) \cap \hat{P}_Q(\bar{c}) = \hat{P}_Q(\bar{a}) \cap \hat{P}_Q(\bar{b}) \cap \hat{P}_Q(\bar{c}).$

Hence $(\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B}) \wedge (\hat{P}_Q, \bar{C}) = \left((\hat{P}_Q, \bar{A}) \wedge (\hat{P}_Q, \bar{B})\right) \wedge (\hat{P}_Q, \bar{C})$

Proof 2. Same proof (1)

Definition 3.20. (Necessity operation (NO)). The NO define on $\mathrm{IV-Q-NSS}$ $(\hat{P}_Q, \bar{A})$ on non-empty initial universal set U and denoted ass following, for all $\bar{a} \in \bar{A}$

$$\widehat{\boxdot}\,(\hat{P}_Q, \bar{A}) = \{< \bar{a}\left[(u, q), \hat{P}^t_{Q(\bar{a})}(u, q), \hat{P}^i_{Q(\bar{a})}(u, q), 1 - \hat{P}^t_{Q(\bar{a})}(u, q)\right] : (a, q) \in U \times Q >\}$$
$$= \{< \bar{a}\ \ [(u, q)[\hat{P}^{t,l}_{Q(\bar{a})}(u, q), \hat{P}^{t,u}_{Q(\bar{a})}(u, q)], [\hat{P}^{i,l}_{Q(\bar{a})}(u, q), \hat{P}^{i,u}_{Q(\bar{a})}(u, q)], [1 - \hat{P}^{t,u}_{Q(\bar{a})}(u, q), 1 - \hat{P}^{t,l}_{Q(\bar{a})}(u, q)]; (u, q) \in U \times Q\}$$

Proposition 3.21 Assume that $(\hat{P}_Q, \bar{A})$ and $(\hat{P}_Q, \bar{B})$ be two $IV-Q-NSSs\ \ on\ \ U.$ Then

1. $\widehat{\boxdot}\left((\hat{P}_Q, \bar{A}) \cup \widehat{\boxdot}\,(\hat{P}_Q, \bar{B})\right) = \widehat{\boxdot}\,(\hat{P}_Q, \bar{A}) \cup \widehat{\boxdot}\,(\hat{P}_Q, \bar{A})$

2. $\widehat{\boxdot}\left((\hat{P}_Q, \bar{A}) \cap \widehat{\boxdot}\,(\hat{P}_Q, \bar{B})\right) = \widehat{\boxdot}\,(\hat{P}_Q, \bar{A}) \cap \widehat{\boxdot}\,(\hat{P}_Q, \bar{A})$

3. $\widehat{\boxdot}\left(\widehat{\boxdot}\,(\hat{P}_Q, \bar{A})\right) = (\hat{P}_Q, \bar{A})$

Proof. The proof of these facts is directly based on the definitions above.

Example 3.22 Reconsider the term in example 3.2, then

$$\widehat{\boxdot}\, \hat{P}_{Q(u_1,q_2)}(e_1) = \widehat{\boxdot}\left(e_1, \frac{\langle[0.1,0.4],[0.5,0.8],[0.7,0.8]\rangle}{(u_1,q_2)}\right)=\left(e_1, \frac{\langle[0.1,0.4],[0.5,0.8],[0.6,0.9]\rangle}{(u_1,q_2)}\right)$$

Definition 3.23 (Possibility operation (PO)). The PO on an $\mathrm{IV}-\mathrm{Q}-\mathrm{NSS}(\hat{P}_Q,\bar{A})$ on non empty universal set U is indicated by $\widehat{\triangle}\,(\hat{P}_Q,\bar{A})$ and given by , for all $\bar{a}\in\bar{A}$

$$\widehat{\triangle}\,(\hat{P}_Q,\bar{A}) = \{<\bar{a}[(u,q),1-\hat{P}^{f}_{Q(\bar{a})}(u,q),\hat{P}^{i}_{Q(\bar{a})}(u,q),\hat{P}^{f}_{Q(\bar{a})}(u,q)] : (a,q)\in U\times Q>\}$$

$$= \{<\bar{a}\ \ [(u,q)[1-\hat{P}^{t,u}_{Q(\bar{a})}(u,q),1-\hat{P}^{f,l}_{Q(\bar{a})}(u,q)],[\hat{P}^{i,l}_{Q(\bar{a})}(u,q),\hat{P}^{i,u}_{Q(\bar{a})}(u,q)],[\hat{P}^{t,u}_{Q(\bar{a})}(u,q),\hat{P}^{t,l}_{Q(\bar{a})}(u,q)];(u,q)\in U\times Q\}$$

Example 3.24 Reconsider the term in example 3.2, then

$$\widehat{\boxdot}\, \hat{P}_{Q(u_1,q_2)}(e_1) = \widehat{\boxdot}\left(e_1, \frac{\langle[0.1,0.4],[0.5,0.8],[0.7,0.8]\rangle}{(u_1,q_2)}\right)=\left(e_1, \frac{\langle[0.2,0.3],[0.5,0.8],[0.7,0.8]\rangle}{(u_1,q_2)}\right).$$

Proposition 3.25 Assume that $(\hat{P}_Q,\bar{A})$ and $(\hat{P}_Q,\bar{B})$ be two $IV-Q-NSSs\ \ on\ \ U$.
Then:

1.$\widehat{\triangle}\left((\hat{P}_Q,\bar{A})\cup\ \widehat{\triangle}\,(\hat{P}_Q,\bar{B})\right)=\ \widehat{\triangle}\,(\hat{P}_Q,\bar{A})\cup\widehat{\triangle}\,(\hat{P}_Q,\bar{A})$.

2.$\widehat{\triangle}\left((\hat{P}_Q,\bar{A})\cap\ \widehat{\triangle}\,(\hat{P}_Q,\bar{B})\right)=\widehat{\triangle}\,(\hat{P}_Q,\bar{A})\ \ \cap\widehat{\triangle}\,(\hat{P}_Q,\bar{B})$.

3.$\widehat{\triangle}\left(\widehat{\triangle}\,(\hat{P}_Q,\bar{A})\right)=(\hat{P}_Q,\bar{A})$.

Proof. The proof of these facts is directly based on the definitions above**.**

Proposition 3.26 Let $(\hat{P}_Q,\bar{A})$ be an $\mathrm{IV}-\mathrm{Q}-\mathrm{NSS}$ over U, then we have the following point (proportion)
1.$\widehat{\triangle}\widehat{\boxdot}\,(\hat{P}_Q,\bar{A})=\widehat{\boxdot}\,(\hat{P}_Q,\bar{A})$
2.$\widehat{\boxdot}\widehat{\triangle}\,(\hat{P}_Q,\bar{A})=\widehat{\triangle}\,(\hat{P}_Q,\bar{A})$

Proof. The proof of these facts is directly based on the definitions above**.**

4. An Application of IV-Q-NSs in Medical Field Under Uncertainty

In this section, we will show the apparatus for appealing to our put-forward model in dealing with daily life situations. By narrating an issue in the medical field and showing the mechanism for representing its data proposed by our proposed model. After that, we will work on creating an algorithm consisting of a number of sequential steps that analyze the algebraic structure of our proposed model and the data it represents. Now we will provide some definitions that will be useful to us in building the above algorithm.

Definition 4.1 Let $(\hat{P}_Q,\bar{A})$ be IV-QNSS on non-empty initial universal set U. Then, an IV-QNS aggregation operator of $(\hat{P}_Q,\bar{A})$ and denoted by $\breve{\Pi}^{agg}_Q$ is defined by

$$\breve{\Pi}^{agg}_Q = \left\{<\bar{a}[(u,q),\hat{P}^{t,agg}_{Q(\bar{a})}(u,q),\hat{P}^{i,agg}_{Q(\bar{a})}(u,q),\hat{P}^{f,agg}_{Q(\bar{a})}(u,q)] : (u,q)\in U\times Q>\right\}$$

Where $\hat{P}^{t,agg}_Q,\hat{P}^{i,agg}_Q,\hat{P}^{f,agg}_Q: U\times Q\to[0,1]$, such that

$\hat{P}_Q^{t,l,agg} = \frac{1}{|\bar{A}|}. \sum_{(u,q)\in U\times Q} \hat{P}_{Q((u,q))}^{t,l}, \hat{P}_Q^{t,u,agg} = \frac{1}{|\bar{A}|}. \sum_{(u,q)\in U\times Q} \hat{P}_{Q(u,q)}^{t,u},$

$\hat{P}_Q^{i,l,agg} = \frac{1}{|\bar{A}|}. \sum_{(u,q)\in U\times Q} \hat{P}_{Q(u,q)}^{i,l}, \hat{P}_Q^{i,u,agg} = \frac{1}{|\bar{A}|}. \sum_{(u,q)\in U\times Q} \hat{P}_{Q(u,q)}^{i,u},$

$\hat{P}_Q^{f,l,agg} = \frac{1}{|\bar{A}|}. \sum_{(u,q)\in U\times Q} \hat{P}_{Q(u,q)}^{f,l}, \hat{P}_Q^{f,u,agg} = \frac{1}{|\bar{A}|}. \sum_{(u,q)\in U\times Q} \hat{P}_{Q(u,q)}^{f,u},$

Remark: The value of $(\hat{P}_Q, \bar{A})$can be reduce to IV-QFS using the following definition .

Definition 4.2. The IV-QNS can be reduced to Interval-Valued-Q-fuzzy set (IV-QFS)

$$(\hat{P}_Q, \bar{A}) = \{< \bar{a}, [(u,q), \hat{P}_{Q(\bar{a})}^t(u,q)] : (u,q) \in U \times Q >\}$$

Where $\hat{P}_Q^t: U \times Q \to [0,1]$ such that

$\hat{P}_Q^{t,l} = \frac{1}{3}[\hat{P}_Q^{t,l} + \hat{P}_Q^{i.l} + \hat{P}_Q^{f.l}], \ \hat{P}_Q^{t,u} = \frac{1}{3}[\hat{P}_Q^{t,u} + \hat{P}_Q^{i.u} + \hat{P}_Q^{f.u}]$

Now, using the above definitions, we lever up the following algorithm for a decision medical field method:

<u>Algorithm</u>

Step 1. Put up an IV-Q-NSSs on U.

Step 2. Calculate IV-Q-NS aggregation operator

Step 3. Calculate the reduced value of the IV-Q-NS aggregation operator to IV-QFS aggregation operator.

Step 4. Convert IV-QFS aggregation operator $(\hat{P}_Q^{t,l}, \hat{P}_Q^{t,u})$ to SV-QFS aggregation operator $(\hat{P}_Q^t)$,i.e. $\hat{P}_Q^t = \frac{\hat{P}_Q^{t,l}, +_Q^{t,u}}{2}$.

Step 5. The optimal decision is the element available in M, such that $M = max_{(u,q)\in U\times Q}\{\hat{P}_Q^t\}$.

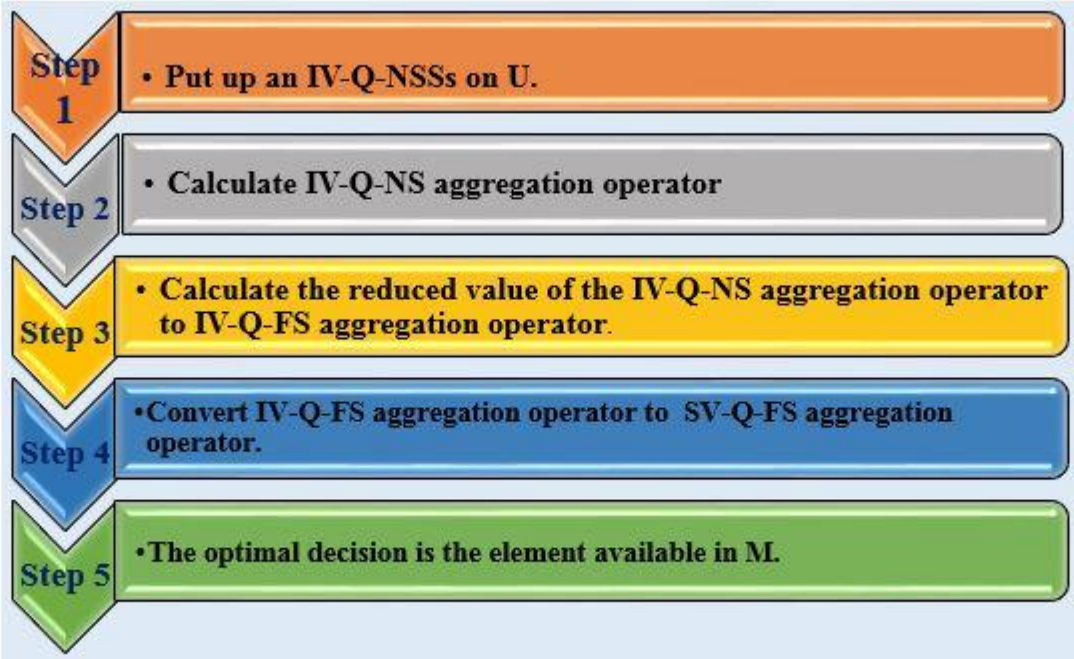

Figure 2: a representation of algorithm in an abbreviated way.

Now, we provide a case study related to the medical field for IV-Q-NSS strategic decision-making method.

On a cold winter day, many patients visited the office of a respiratory doctor to diagnose their health condition (COVID-positive or not) based on the symptoms they were experiencing. To help the doctor organize and analyze patient data based on our proposed model, we asked him to select a value between 0 and 1 that describes the severity of symptoms and their association with the disease (Covid), where the closer the ratio is to 1, the more serious the symptoms are (impact of symptoms). Therefore:
Suppose that $\mathfrak{U} = \{\mathfrak{u}_1, \mathfrak{u}_2, \mathfrak{u}_3, \mathfrak{u}_4\}$ be a patient set contains four patients, $\mathfrak{Q} = \{\mathfrak{q}_1, \mathfrak{q}_2\}$ where $\mathfrak{q}_1$ =infected and $\mathfrak{q}_2$ =uninfected,while $\bar{A} \subseteq \mathcal{E} = \{\bar{a}_1, \bar{a}_2, \bar{a}_3, \bar{a}_4\}$ be a set of symptoms contains four symptoms such that $\bar{a}_1$ =Headache, $\bar{a}_2$ =Sore throat, $\bar{a}_3$ =Muscle pain, $\bar{a}_4$ =Chest pain.

Now, after the doctor has examined each patient and set a numerical value between 0 and 1 for each of the symptoms above, our proposed model can be built in a way that is consistent with the examining doctor's report.

$$\hat{P}_{\mathfrak{Q}_{\bar{A}}} =$$
$$\left\{\left(\bar{a}_1, \frac{\langle[0.2,0.8],[0.1,0.7],[0.4,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_1)}, \frac{\langle[0.1,0.4],[0.5,0.8],[0.7,0.8]\rangle}{(\mathfrak{u}_1,\mathfrak{q}_2)}\right.\right.$$
$$\frac{\langle[0.3,0.6],[0.2,0.7],[0.5,0.8]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_1)}, \frac{\langle[0.4,0.6],[0.2,0.9],[0.5,0.7]\rangle}{(\mathfrak{u}_2,\mathfrak{q}_2)}$$
$$\frac{\langle[0.6,0.8],[0.4,0.5],[0.3,0.5]\rangle}{(\mathfrak{u}_3,\mathfrak{q}_1)}, \frac{\langle[0.3,0.7],[0.2,0.4],[0.1,0.8]\rangle}{(\mathfrak{u}_3,\mathfrak{q}_2)}$$

$$\frac{\langle[0.1,0.5],[0.3,0.7],[0.2,0.8]\rangle}{(u_4,q_1)},\frac{\langle[0.4,0.8],[0.4,0.6],[0.2,0.8]\rangle}{(u_4,q_2)}\Bigg)$$

$$\Bigg(e_2,\frac{\langle[0.1,0.8],[0.5,0.7],[0.3,0.4]\rangle}{(u_1,q_1)},\frac{\langle[0.1,0.8],[0.4,0.7],[0.2,0.6]\rangle}{(u_1,q_2)}$$

$$\frac{\langle[0.5,0.8],[0.4,0.9],[0.2,0.7]\rangle}{(u_2,q_1)},\frac{\langle[0.1,0.2],[0.2,0.5],[0.4,0.7]\rangle}{(u_2,q_2)}$$

$$\frac{\langle[0.6,0.8],[0.4,0.5],[0.3,0.5]\rangle}{(u_3,q_1)},\frac{\langle[0.3,0.7],[0.2,0.4],[0.1,0.8]\rangle}{(u_3,q_2)}$$

$$\frac{\langle[0.1,0.4],[0.2,0.5],[0.3,0.7]\rangle}{(u_4,q_1)},\frac{\langle[0.1,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(u_4,q_2)}\Bigg)$$

$$\Bigg(e_3,\frac{\langle[0.1,0.8],[0.5,0.7],[0.3,0.4]\rangle}{(u_1,q_1)},\frac{\langle[0.1,0.8],[0.4,0.7],[0.2,0.6]\rangle}{(u_1,q_2)}$$

$$\frac{\langle[0.5,0.8],[0.4,0.9],[0.2,0.7]\rangle}{(u_2,q_1)},\frac{\langle[0.1,0.2],[0.2,0.5],[0.4,0.7]\rangle}{(u_2,q_2)}$$

$$\frac{\langle[0.6,0.8],[0.4,0.5],[0.3,0.5]\rangle}{(u_3,q_1)},\frac{\langle[0.3,0.7],[0.2,0.4],[0.1,0.8]\rangle}{(u_3,q_2)}$$

$$\frac{\langle[0.1,0.4],[0.2,0.5],[0.3,0.7]\rangle}{(u_4,q_1)},\frac{\langle[0.1,0.6],[0.4,0.5],[0.5,0.7]\rangle}{(u_4,q_2)}\Bigg)$$

$$\Bigg(e_4,\frac{\langle[0.7,0.9],[0.2,0.8],[0.3,0.6]\rangle}{(u_1,q_1)},\frac{\langle[0.4,0.7],[0.2,0.5],[0.1,0.7]\rangle}{(u_1,q_2)}$$

$$\frac{\langle[0.1,0.8],[0.1,0.4],[0.3,0.6]\rangle}{(u_2,q_1)},\frac{\langle[0.5,0.6],[0.3,0.6],[0.2,0.7]\rangle}{(u_2,q_2)}$$

$$\frac{\langle[0.6,0.8],[0.4,0.5],[0.3,0.5]\rangle}{(u_3,q_1)},\frac{\langle[0.3,0.7],[0.2,0.4],[0.1,0.8]\rangle}{(u_3,q_2)}$$

$$\frac{\langle[0.4,0.6],[0.2,0.7],[0.3,0.6]\rangle}{(u_4,q_1)},\frac{\langle[0.4,0.8],[0.8,0.9],[0.3,0.7]\rangle}{(u_4,q_2)}\Bigg)\Bigg\}$$

Step 2. The IV-Q-NS aggregation operator is given as

$$\breve{\Pi}_{ivQ-NS}^{agg} =$$
$$\{\big((u_1,q_1),\langle[0.275,0.825],[0.325,0.725],[0.327,0.550]\rangle\big),$$
$$\big((u_1,q_2),\langle[0.124,0.342],[0.451,0.537],[0.463,0.643]\rangle\big),$$
$$\big((u_2,q_1),\langle\langle[0.335,0.673],[0.326,0.673],[0.421,0.568]\rangle\rangle\big),$$
$$\big((u_2,q_2),\langle\langle[0.453,0.765],[0.321,0.547],[0.322,0.629]\rangle\rangle\big),$$
$$\big((u_3,q_1),\langle\langle[0.237,0.763],[0.327,0.743],[0.382,0.639]\rangle\rangle\big),$$
$$\big((u_3,q_2),\langle\langle[0.287,0.325],[0.210,0.482],[0.238,0.734]\rangle\rangle\big),$$
$$\big((u_4,q_1),\langle\langle[0.128,0.342],[0.438,0.983],[0.364,0.754]\rangle\rangle\big),$$
$$\big((u_4,q_2),\langle\langle[0.234,0.432],[0.543,0.578],[0.334,0.749]\rangle\rangle\big)\}$$

Step 3. Calculate the reduced value of the IV-Q-NS aggregation operator to IV-QFS aggregation operator.
$$\breve{\Pi}_{ivQ-FS}^{agg} =$$

$\{((\mathfrak{u}_1,\mathfrak{q}_1),\langle[0.309,0.700]\rangle),\ ((\mathfrak{u}_1,\mathfrak{q}_2),\langle[0.346,0.507]\rangle),$
$((\mathfrak{u}_2,\mathfrak{q}_1),\langle[0.346,0.638]\rangle),((\mathfrak{u}_2,\mathfrak{q}_2),\langle[0.365,0.647]\rangle),$
$((\mathfrak{u}_3,\mathfrak{q}_1),\langle[0.285,0.715]\rangle),((\mathfrak{u}_3,\mathfrak{q}_2),\langle[0.245,0.513]\rangle),$
$((\mathfrak{u}_4,\mathfrak{q}_1),\langle[0.310,0.693]\rangle),((\mathfrak{u}_4,\mathfrak{q}_2),\langle[0.370,0.586]\rangle)\}$

Step 4. Convert IV-Q-FS aggregation operator $(\hat{P}_Q^{t,l},\hat{P}_Q^{t,u})$ to SV-QFS aggregation operator $(\hat{P}_Q^{t})$.

$\breve{\Pi}_{ivQ-FS}^{agg} =$
$\{((\mathfrak{u}_1,\mathfrak{q}_1),\langle 0.515\rangle),\ ((\mathfrak{u}_1,\mathfrak{q}_2),\langle 0.426\rangle),$
$((\mathfrak{u}_2,\mathfrak{q}_1),\langle 0.492\rangle),((\mathfrak{u}_2,\mathfrak{q}_2),\langle 0.506\rangle),$
$((\mathfrak{u}_3,\mathfrak{q}_1),\langle 0.500\rangle),((\mathfrak{u}_3,\mathfrak{q}_2),\langle 0.379\rangle),$
$((\mathfrak{u}_4,\mathfrak{q}_1),\langle 0.501\rangle),((\mathfrak{u}_4,\mathfrak{q}_2),\langle 0.478\rangle)\}$

Step 5. The optimal decision is the element available inM_i, such that

$M_1 = max_{(\mathfrak{u}_1,\mathfrak{q}_{1,2})\in U\times Q}\{0.515,0.426\} = 0.515.$
$M_2 = max_{(\mathfrak{u}_2,\mathfrak{q}_{1,2})\in U\times Q}\{0.492,0.506\} = 0.506.$
$M_3 = max_{(\mathfrak{u}_3,\mathfrak{q}_{1,2})\in U\times Q}\{0.500,0.379\} = 0.500.$
$M_4 = max_{(\mathfrak{u}_4,\mathfrak{q}_{1,2})\in U\times Q}\{0.501,0..478\} = 0.501.$

By looking at Table 1. below, which contains a comparison between the results obtained, it is clear that all patients $\mathfrak{u}_1,\mathfrak{u}_3,\ \mathfrak{u}_4$ are infected except the patient $\mathfrak{u}_2$.

Table 1: Comparison of the results obtained from the above algorithm

Patients	Degree of $(\mathfrak{u},\mathfrak{q}_1)$	Degree of $(\mathfrak{u},\mathfrak{q}_2)$	Comparison	Result
$\mathfrak{u}_1$	**0.515**	**0.426**	$\mathfrak{q}_1 > \mathfrak{q}_2$	**Yes**
$\mathfrak{u}_2$	**0.492**	**0.506**	$\mathfrak{q}_1 < \mathfrak{q}_2$	**NO**
$\mathfrak{u}_3$	**0.500**	**0.379**	$\mathfrak{q}_1 > \mathfrak{q}_2$	**Yes**
$\mathfrak{u}_4$	**0.501**	**0.478**	$\mathfrak{q}_1 > \mathfrak{q}_2$	**Yes**

5. Comparison with existing works

Now in this part, the proposed model is compared with some prevailing works like Adam and Hassan [21], Abu Qamar and Hassan [23], and Zhang et al. [36]. This comparison will focus on the structural structure of these methods compared to our method presented in this work, where the similarities and differences between these concepts were analyzed. Firstly, Abu Qamar and Hassan developed the notion of Q-NSSs as an extension of Adam and Hassan's notion, and this notion depicts decision-making data that has two diminutions in a single value, which causes some constraint for the decision maker when analysing data for the problem. Secondly, Zhang et al. defined INSs as a generalisation of FSs and IFSs and NSs to address real situations with a set of numbers in the real unit interval. This model has the ability to represent decision-making information that is characterised by uncertainty, indeterminacy, and inconsistency in one dimension (one universal set).

On the other hand, our model addresses all the complexities that appeared in the concepts referred to above, as its structural structure provides it with all the advantages that the currently prevailing methods lack. Moreover, Table 2 provides a further comparison between our proposed method and other prevailing methods based on some of the criteria fixed in the table.

Propose Methoed	TM	IM	FM	SS	TD	IV
FS	√	×	×	×	×	×
Q-FS	√	×	×	×	√	×
IFS	√	×	√	×	×	×
NS	√	√	√	×	√	×
Q-NSS	√	√	√	√	√	×
Our model IV-Q-NSS	√	√	√	√	√	√

Table 2: Comparison between our proposed method and other prevailing methods

In this table, each of (TM, IM, FM, SS, TD and IV) point out to True, Indeterminacy, Falsity, Matching with Soft set, Tow dimension, and Interval- Valued.

6. Conclusion

IV-Q-NSS is a useful tool for dealing with Q-two-dimensional universal information in interval form. It is made up of three NS membership degrees in interval form. Also, this tool was created to deal with the relationship between parameters in the SS environment when these parameters play a key role in the deep description of two-dimensional universal information. So, in this paper, we suggested an interval-valued Q-neutrosophic soft set (IV-Q-NSS) mean set theory. This theory includes special operations like the necessity and possibility operations of an IV-Q-NSS, as well as operations like the complement of an IV-Q-NSS, the union of two IV-Q-NSSs, the intersection of two IV-Q-NSSs, and the AND and OR operation of two IV-Q-NSSs. In addition, we presented many properties supported by numerical examples that explain how they work. Future, this new model has been successfully tested in dealing with one of the medical diagnostic problems based on hypothetical data for a respiratory disease when a new algorithm based on the aggregation operator for IV-Q-NSS data was built to solve this issue. Finally, directions will likely focus on improving some of the gaps in this work in the soft computing environment, as it is preferable to expand the work in this environment by integrating these tools with the hypersoft set (HSS) [37], where this environment will enable us to give a more accurate description of the parameters related to the SS environment. In addition to applying some mathematical tools, such as the similarity measure, the distance measure, or other measures on IV-Q-NSSs. In addition, this environment can be combined with other environments such as the algebraic environment [38-42] and the soft topological environment [43-46] and the use of other techniques such as techniques for measuring similarity and distance [47-49] between two objects.

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Evaluation of the shortest path based on the Traveling Salesman problem with a genetic algorithm in a neutrosophic environment

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Abstract: In The traveling salesman problem (TSP) is an essential and the most popular conventional combinatorial optimization network problem in operations research, in which the TSP evaluates the shortest route or path in a network. In TSP, every node has been visited only once, excluding the starting node. In TSP, edge lengths are usually expressed to indicate journey time and expenses instead of distance from a location. The exact arc length can't be predicted because journey times and expenses vary depending on the amount of payload, climate, highway conditions, and so on. As a result, the Neutrosophic numbers introduce a new tool for dealing with unpredictability in TSP. The present article addresses TSP on a neutrosophic network where the edge weight is a neutrosophic number rather than a real number. For solving the Neutrosophic TSP, an algorithmic technique based on the genetic algorithm (GA) is proposed. We created a new mutation and crossover for our suggested GA. We used mathematical examples to show the usefulness of the algorithm that we suggested. The results of experiments suggest that the proposed GA can find the shortest path in a TSP within a neutrosophic framework. This provides valuable insights for decision-makers dealing with real-world situations characterized by imprecise and incomplete data.

Keywords: Connected network; Neutrosophic number; Shortest path problem; Traveling Salesman problem

1. Introduction

The traveling salesman problem (TSP) is one of the most essential and extensively researched systems. TSP was primarily introduced in 1930 and became extremely popular after 1950 [1–3]. Even in the past few years, optimization problems have appeared in the field of engineering research, and many papers have been published related to optimization [4-11]. The selection maker in the TSP discovers the shortest possible route or path for the salesman who visits every single node (city) exactly once (except the initial city) and returns to the initial node (city). It's a well-known optimization problem [12]. In practice, the edge cost in a traffic network path [13–16] may have various parameters that are difficult to determine precisely (e.g., travel cost, travel time, road capacity, traffic frequency, and so on).

The decision-maker has to consider the uncertainty in travel costs and time because, depending on the climate, road conditions, expense, and time at which travel may vary, As a result, in such real-life scenarios, decision-makers cannot precisely determine the parameters (traveling time, traveling speed, and traveling cost) between two separate cities but appropriately determine the TSP. As a result, fuzzy sets can be used to deal with this type of uncertainty [17-19], allowing the decision-maker to make decisions based on uncertain data. The cost and duration of travel between two cities are primarily determined by the mode of transportation used. It generally changes daily, and experts can regard the travel costs and time of this problem as ambiguous [20- 22]. Several efforts have been made to solve the fuzzy traveling salesman problem [23–27].

The TSP [3] belongs to the complete NP problem. But it will be simple to understand but extremely difficult to solve. The computation duration for the TSP grows steadily whenever the total number of places increases. The TSP is commonly used as an example of an optimization problem to show the efficiency of a newly developed approach. Numerous heuristic and metaheuristic approaches are available to solve the TSP, including the genetic algorithm (GA), the harmony searching algorithm, and the artificial bee colony algorithms. A genetic algorithm is a type of heuristic optimization algorithm in which a chromosome describes a possible solution to a given optimization problem. The population is built from a variety of chromosomes. Chromosomes are reassembled in this algorithm to create new chromosomes. This recombination method is carried out primarily through three biological operations, i.e., selection, mutation, and crossover. The GA is used in this method to find the best solutions. The GA can solve many optimization problems [28, 29]. It is also employed in the solution of the TSP [30–35].

In this research paper, we have proposed a modified genetic algorithm for finding the shortest path using TSP in a neutrosophic environment. Initially, the GA generated by Darwin's law, primarily used for traditional TSPs when arc lengths are crisp numbers and the environment is certain, does not particularly apply to an uncertain environment. As a result, in this paper, we enhance the GA using an aspect of neutrosophic set theory. Here, the neutrosophic number represents the TSP's arc length. A mathematical model is introduced for a TSP with neutrosophic numbers as arc lengths. We present the utility of neutrosophic sets as arc lengths for TSP. We have updated our suggested GA with a new crossover as well as a mutation. We have demonstrated the effectiveness of our suggested algorithm with a numerical example. The other parts of this research paper are prepared in the following ways: Section-2 covers the fundamentals of neutrosophic sets (NSs). Section-3 proposes a GA for finding the shortest path using TSP in a neutrosophic environment. In Section-4, we present an improved pseudo-code of a GA for determining the shortest path (SP) of a connected network with respect to TSP. In Section-5 we presented a numerical example. In Section-6, we find the best route using TSP in the neutrosophic environment. In Section-7, we compare our method with other existing methods. Section-8 contains the conclusions and recommendations for additional research.

2. Preliminaries

Definition-2.1

Assume set $\ddot{X}$ is the universal set. An intuitionistic fuzzy set $\ddot{A}$ in $\ddot{X}$is written in the form: $\ddot{A} = \{\ddot{x}, \mu_{\ddot{A}}(x), \nu_{\ddot{A}}(\ddot{x})$

With $\mu_{\ddot{A}}: \ddot{X} \rightarrow [0,1]$ and $\nu_{\ddot{A}}: \ddot{X} \rightarrow [0,1]$ are the functions that define the degrees of membership non-membership of $x \in \ddot{X}$ to $\ddot{A} \in \ddot{X}$, respectively, and for every $x \in \ddot{X}$, $\mu_{\ddot{A}}(x), \nu_{\ddot{A}}(\ddot{x}) \leq 1$.

Definition-2.2

Assume $\ddot{X}$ is a set of space points (objects), and $\tilde{x}$ represents the associated generic elements in $\ddot{X}$; then the element in neutrosophic set $\widetilde{A}$ has the form

$$\widetilde{A} = \{< \ddot{x}: \ddot{T}_{\ddot{A}}(\ddot{x}), \ddot{I}_{\ddot{A}}(\ddot{x}), \ddot{F}_{\ddot{A}}(\ddot{x}) > \ddot{x} \epsilon \ddot{X}\}$$

Where three membership degree $\ddot{T}_{\ddot{A}}, \ddot{I}_{\ddot{A}}, \ddot{F}_{\ddot{A}}: \ddot{X} \rightarrow [0^-, 1^+]$ where $\ddot{T}$, $\ddot{I}$, and $\ddot{F}$ represent the truth function, the indeterminacy function, and the falsity function, respectively.

$$0^- \leq \{\ddot{T}_{\ddot{A}}(\ddot{x}) + \ddot{I}_{\ddot{A}}(\ddot{x}) + \ddot{F}_{\ddot{A}}(\ddot{x})\} \leq 3^+$$

Now $\ddot{T}_{\ddot{A}}(\ddot{x}), \ddot{I}_{\ddot{A}}(\ddot{x}), \ddot{F}_{\ddot{A}}(\ddot{x})$ are representing subsets of the interval $[0^-, 1^+]$ hence it's challenging to implement NSs to real-world situations.

Definition-2.3

Euclidean Distance: This is the most commonly used distance measure for neutrosophic numbers. It is a generalization of the Euclidean distance between two points in a multi-dimensional space. For neutrosophic numbers, you can calculate the Euclidean distance by treating each component ($\ddot{T}$, $\ddot{I}$, $\ddot{F}$) as a separate coordinate and using the standard Euclidean distance formula:

$$\text{Euclidean Distance} = \sqrt{(\ddot{T}_1 - \ddot{T}_2)^2 + (\ddot{I}_1 - \ddot{I}_2)^2 + (\ddot{F}_1 - \ddot{F}_2)^2}$$

The representation of cities as ($\ddot{T}$, $\ddot{I}$, $\ddot{F}$) in the framework of the TSP with a GA in a neutrosophic environment may have particular significance that corresponds with neutrosophic principles:

$\ddot{T}$ (truth): In a neutrosophic environment, T may represent a city's truth value. Neutrosophic is a philosophy that deals with indeterminacy and has three principles: truth ($\ddot{T}$), indeterminacy ($\ddot{I}$), and falsity ($\ddot{F}$). A city's "truth" value may indicate how precisely its location or characteristics have been identified.

$\ddot{I}$ (indeterminacy): The degree of indeterminacy or uncertainty associated with a city is represented by $\ddot{I}$. It reflects the degree of ambiguity or imprecision in the city's available information. $\ddot{I}$ denote the level of unknown or partially known information in a neutrosophic context.

$\ddot{F}$ (falsity): The degree of falsity or incorrectness associated with a city is represented by $\ddot{F}$. This value indicates how much of the available information about the city is deceptive, in error, or incorrect.

When calculating distances and making decisions in the genetic algorithm, using ($\ddot{T}$, $\ddot{I}$, $\ddot{F}$) For city representation in a TSP within a neutrosophic environment, it allows for the consideration of uncertainties and ambiguities. It recognizes that information about each city may not be entirely true or false, but may contain varying degrees of truth, indeterminacy, and falsity, making the TSP solution more adaptable to such uncertainties.

3. Proposed GA for finding the shortest path using TSP

A genetic algorithm [36, 27] is an optimization algorithm inspired by natural selection. They are used to solve difficult optimization and search problems. The following are the typical steps in a GA:

Initialization:

Begin the NTSP's population of potential solutions with neutrosophic number representations.

Selection:

Evaluate the fitness of every member of the population. The fitness function assesses how well each person eliminates the problem.Choose people from the population to be the next generation's parents. Individuals with higher fitness values are more likely to be chosen, but some diversity should also be preserved.

Crossover (Recombination):

Take two carefully chosen parents and combine their genetic information to produce one or more offspring. Crossover occurs when parts of the parent chromosomes are exchanged or combined to create new individuals. Crossover methods, such as single-point, two-point, or uniform crossover, can vary.

Mutation (Mutate (child)):

Change some of the genetic information of the offspring at random. Mutation adds diversity to the population and keeps it from becoming stuck in local optima.

Replacement (Replace Weakest (child1, child2)):

In order to create a new population, combine the parent individuals and their progeny. You can also decide to use selection mechanisms (elitism, for example) to determine which members of the previous generation are retained in the new population, ensuring that the best solutions are not lost.

Termination:

Verify the termination terms, which other factors may determine, the number of generations, or the highest level of fitness attained. The algorithm terminates if the termination condition is satisfied; if not, return to the selection stage.

Result:

Once the algorithm terminates, the best solution found in the final population is the output of the genetic algorithm.

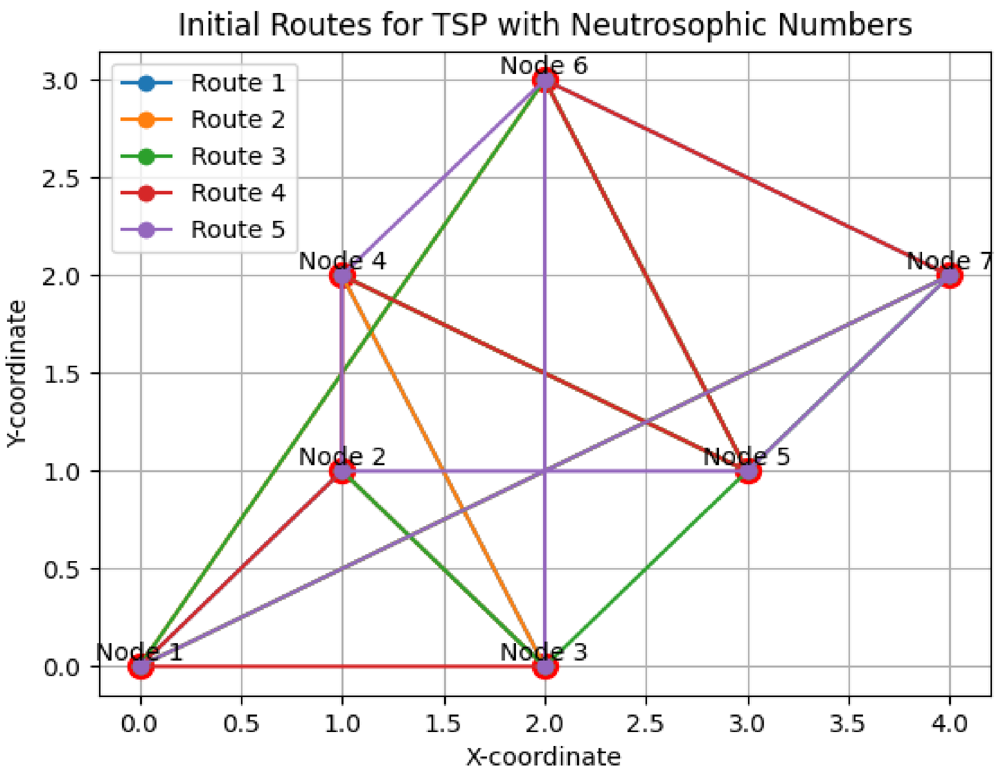
4. Pseudo code of genetic algorithm

```
Initialize Population ( )
best solution = null
best fitness = +infinity
generation = 0
```

```
while generation < max generations:
    fitness values = Evaluate Population()

    For i from 1 to population size by 2:
        parent1 = Select Parent()
        parent2 = Select Parent()
        child1, child2 = Crossover(parent1, parent2)
        child1 = Mutate(child1)
        child2 = Mutate(child2)
        Replace Weakest(child1, child2)

    best individual, best individual fitness = Get Best Individual()

    If best individual fitness < best fitness:
        best solution = best individual
        best fitness = best individual fitness

    generation = generation + 1

return the best solution
```

5. Numerical example:

In this example, we'll consider a small TSP with 7 cities, and we'll represent the cities using neutrosophic numbers in a simplified format, where each city is represented as ($\ddot{T}$, $\ddot{I}$, $\ddot{F}$) The values for $\ddot{T}$ (truth-membership), $\ddot{I}$ (indeterminacy), and $\ddot{F}$ (falsity-membership) range from 0 to 1.



Initialization:

In this step, we generate an initial population shown in Table-1 and we create an initial population of 5 routes in a 7-node network. Where each route represents a permutation of the 7 cities in Table 2. Each city is represented as a neutrosophic number (T̈, Ï, F̈).

Table-1. Here's a set of 7 cities with their neutrosophic number

City	Neutrosophic number
1	(0.7,0.2,0.1)
2	(0.6, 0.3, 0.1)
3	(0.5, 0.3, 0.2)
4	(0.2, 0.5, 0.3)
5	(0.4, 0.4, 0.2)
6	(0.3, 0.4, 0.3)
7	(0.6, 0.3, 0.1)

Table 2. All possible routes

Route no	Possible route
1	[1, 2, 3, 4, 5, 6, 7]

2	[2, 3, 4, 5, 6, 1, 7]
3	[4, 5, 6, 1, 7, 3, 2]
4	[3, 1, 2, 4, 5, 6, 7]
5	[1, 7, 5, 2, 4, 6, 3]

Step 2: Evaluation (Fitness):

We calculate the fitness of each route in the population using a fitness function. In this example, the fitness function takes into account the neutrosophic numbers and aims to minimize the total distance while considering indeterminacy:

$$\text{Fitness} = \sum \frac{(T_{\ i} * \ (distance_i) \quad)}{(I_{\ i} + \ 1)}$$

Where:

$T_{\ i}$ is the truth-membership of the node's highest neutrosophic number.

$I_{\ i}$ is the indeterminacy of the node's minimum neutrosophic number.

$(distance_i)$ is the distance between two consecutive nodes in the route.

Fitness for Route 1 :(using definition-2.3)

Distance between node	**Crisp value**
1-2	0.14
2-3	0.14
3-4	0.37
4-5	0.24
5-6	0.14
6-7	0.37

Total Distance =0.14+0.14+0.37+0.24+0.14+0.37=1.4

$$\text{Fitness} = \sum \frac{(0.7 * \quad 1.4) \quad)}{(0.1 + \quad 1)}$$

$$\text{Fitness} = \sum \frac{(0.7 * \quad 1.4) \quad)}{(0.1 + \quad 1)} = 0.89$$

Similarly find the fitness for Route 2, Route 3, Route 4, and Route 5

Fitness function of Route	Minimize distance
Route 1:	0.89
Route 2:	0.96

Route 3:	0.88
Route 4:	0.86
Route 5:	0.94

Step 3: Selection (Choose Two Routes):

Use a selection mechanism (e.g., roulette wheel selection or tournament selection) to choose routes to become parents for the next generation. Routes with lower fitness values have a higher chance of being selected.

Select the two routes with the lowest fitness values:

Route 4 (Fitness: 0.86)

Route 3 (Fitness: 0.88)

Step 4: Crossover (Order Crossover):

Apply crossover (recombination) to pairs of selected routes to create new routes (offspring). The crossover should respect the neutrosophic number representation.

Combine genetic information from Route 4 and Route 3:

Parent Route 3: [4, 5, 6, 1, 7, 3, 2]

Parent Route 4: [3, 1, 2, 4, 5, 6, 7]

Crossover Point (e.g., after Node 1):

Offspring Route: [4, 5, 6, 1| 2, 7, 3]

Complete the offspring by adding the missing nodes while avoiding duplicates:

Offspring Route: [4, 5, 6, 1, 2, 7, 3]

Step 5: Mutation (Swap Two Random Nodes):

Apply mutation operators to introduce small random changes in the routes. The mutation should also respect the neutrosophic numbers.

Let's swap nodes 1 and 7:

Mutated Offspring Route: [4, 5, 6, 7, 2, 1, 3]

Step 6: Replacement (Replace One of the Parents):

Replace some of the old routes with the newly created offspring to form the next generation.

Replace one of the parents (Route 3) with the mutated offspring:

Updated Population:

Route 1: [4, 5, 6, 3, 2, 1, 7]

Route 2: [1, 2, 3, 4, 5, 7, 6]

Route 3: [4, 5, 6, 1, 7, 3, 2]

Route 4: [4, 5, 6, 7, 2, 1, 3]

Route 5: [1, 7, 5, 2, 4, 6, 3]

Step 7: Termination (End of One Generation):

Choose a termination condition, such as reaching a certain number of generations or achieving a satisfactory fitness value. These steps would be repeated for several generations, with each generation improving the routes. Termination can be based on reaching a certain number of generations or a satisfactory fitness value. This example shows one generation of a GA for the TSP in a 7-node network with neutrosophic numbers. More sophisticated fitness functions and genetic operators would be used in practice.

6. Result: The best route found in the final population is the solution to the TSP with neutrosophic numbers.

Table 3. The Optimal result of shortest path

Solution using Linear programming In lingo software	**Solution using Proposed Genetic algorithm for finding shortest path using TSP**
Minz=0.86	The cost of NTSP=0.86 And shortest route=3 →1→ 2→ 4→ 5→ 6→7

Our suggested GAs are implemented to identify the shortest path of a neutrosophic graph's traveling salesman problem. In example 1 we take 7 nodes (Figure 1) and 5 possible paths (Table 2). We have arbitrarily chosen a possible path. Table 1 contains a description of the neutrosophic number as edge weights. This is the new concept we apply for neutrosophic numbers in the Travelling salesman problem using a genetic algorithm determining the shortest network path. In example 1, we find the optimal result of the shortest path in lingo software and our proposed method. In both cases, we got the same cost and the same shortest path in Table 3.

7. Comparison of our methodology with other methodology

In this section, we compare our methodology with some other existing methodologies and finally analyze our methodology for evaluating the shortest path based on the Traveling Salesman problem with a GA in a neutrosophic environment, which gives the optimal result. We discuss this in Table-4.

Table 4. Comparison of shortest path with the shortest path cost in crisp number

Methodology	**Shortest path**	**shortest path cost in crisp number**
Traveling Salesman problem in Neutrosophic environment [38]	3 →1→ 2→ 4→ 5→ 6→7	The cost of NMST= 1.02
Travelling salesman problem using fuzzy environment[39]	3 →1→2→4→5→ 6→7	1.86
Our proposed method in Neutrosophic environment	3 →1→2→4→5→ 6→7	0.86

8. Conclusion:

The results of our experiments suggest that the proposed GA can find the shortest path in a TSP within a neutrosophic framework. This provides valuable insights for decision-makers dealing with real-world situations characterized by imprecise and incomplete data. Furthermore, this research opens up avenues for future exploration, such as refining the GA parameters and techniques for handling larger and more complex instances of the TSP in neutrosophic environments. Additionally, evaluating the algorithm's performance on different types of neutrosophic data and its potential integration into decision support systems could be areas for further investigation. This research serves as a valuable contribution to the field of operations research and optimization, with the potential to enhance decision-making in practical, real-world applications

Conflicts of Interest: The authors declare no conflict of interest.

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An Integrated CODAS Method and Novel Surface-based Weighted Distance Measures under Neutrosophic Environment

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Abstract: Information theory provides suitable tools for solving practical problems, particularly multi-criteria decision-making (MCDM) problems under neutrosophic environments. Generally, a wide range of MCDM solution methods are constructed based on distance measures category. The defined distance measures for continuous neutrosophic numbers, especially its trapezoidal type, is very limited rather than discrete type. The main goals of this study is to come up with a way to add two new types of weighted distance measures based on meaningful surfaces: Euclidean and Hamming. To define these measures, all three components of the neutrosophic trapezoidal fuzzy number (truth, indeterminacy, and falsity membership functions) have been used simultaneously. The proof of some theorems and properties for the weighted distance measures demonstrates their validity. The CODAS algorithm is known as one of the distance-based methods for solving MCDM problems. The following represents the CODAS algorithm based on two novel distance measures. In addition, an explanatory example from the research literature is given to check the performance of the proposed hybrid algorithm. The results of this study indicate that the algorithm based on the proposed measures obtains a reasonable and appropriate ranking order between the options. Furthermore, the sensitivity parameter analysis and comparative analysis show the flexibility and accuracy of the suggested measures in the combined algorithm. The acceptable efficiency of proposed distance measures formed on the surfaces can shed light on research related to distance measures in the methodology and implicated aspects.

Keywords: Neutrosophic Set; Neutrosophic Trapezoidal Fuzzy Number; Distance Measure; Multi-Criteria Decision Making

1. Introduction

Information measure as an efficient tool for extracting the final result in the competitive environment and complex conditions of today's organizations, it is inevitable to encounter multi-criteria decision-making problems under uncertainty. Fuzzy sets (Zadeh (1965) [1]) and their innovative extensions have introduced acceptable covers to match the expression of data with the human mind. Among them, Neutrosophic Sets (NS) (Smarandache, (1999) [2]), which are considered as a multi-dimensional generalization of fuzzy sets to adequately describe the uncertainty involving the truth, indeterminacy, and falsity of decision makers' attitudes, have attracted the attention of many researchers. In general, many methods have been introduced to solve multi-criteria decision-making problems in conditions caused by uncertainty. Essential operators such as aggregating operators [3-7], preference relations [8], distance measures [9,10], similarity measures [11-17], correlation coefficient [18, 19], etc. [20-22] have a practical effect in solving MCDM problems.

Recently, Chen and Pan (2021) [23] presented a complete classification of MCDM problem-solving methods based on the overall structure of the solution techniques. This category includes numerical, distance-based, pairwise comparison, outranking methods, and so on, which can be considered for solution approaches. Most MCDM solution methods are in the group of distance-based methods such as COmplex Proportional Assessment (COPRAS) [24], Data Envelopment Analysis (DEA) [25, 26], Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [27-31], and Compromise Programming (CP) [32], VlseKriterijumska Optimizacija I KOmpromisno Resenje (VIKOR) [33-35].

The CODAS method (Combinative Distance-based Assessment), which is known as one of the new distance-based methods to solve the MCDM problem, is proposed by Keshavarz Ghorabaee et al. (2016) [36] for the first time. In this sense, Euclidean and Taxicab distances are applied to calculate the assessment results of alternatives. In this idea, degrees are stated by a threshold parameter (Keshavarz Ghorabaee et al., 2016) [36]. Since its introduction, this method has had significant expansions from a theoretical perspective, combined with other existing methods and practical models.

From the theoretical point of view, we can mention the extension of the method from deterministic data to the fuzzy CODAS method proposed by Keshavarz Ghorabaee et al. [37] to cover uncertainty in MCDM which is employed for a selection problem (market segment evaluation) under fuzzy data. Bolturk (2018) [38] developed the CODAS method into Pythagorean fuzzy sets (PFSs) to handle more flexible data, and then the proposed method is used to explain the supplier selection problems. Yeni and Özçelik (2019) [39], after using interval-valued Atanassov intuitionistic fuzzy weighted aggregation, investigated extending the traditional fuzzy CODAS for interval-valued Atanassov intuitionistic fuzzy data in personnel selection problems. Later, the novel extension of fuzzy CODAS under the Interval-Valued Intuitionistic Trapezoidal Fuzzy Set (IVITrFS) was presented by Seker (2020) [40]. Subsequently, Wang et al. (2020) [41] established the CODAS method under the 2-tuple linguistic neutrosophic information, and expressed the computing steps for multiple attribute group decision-making (MAGDM). The idea of the CODAS method was extended by Deveci et al. (2022) [42] to support the evaluation of activities in mining sites under q-rung orthopter fuzzy sets (q-ROFSs). Menekse et al. (2022) [43] talk about an interval-valued spherical fuzzy CODAS process that can help clear up problems that aren't very clear.

From the combined point of view, more and more detailed research has been done so far. For instance, two alternatives to MCDM methods, named fuzzy AHP (Fuzzy Analytical Hierarchy Process) and CODAS, are integrated by Panchal et al. (2017) [44] for the evaluation process of a selection problem in a factory. Seker and Aydin (2020) [45] combined Interval-Valued Intuitionistic Fuzzy Analytical Hierarchy Process (IVIF-AHP) and IVIF-CODAS to depict an integrated MCDM framework, then they obtained the ranking of the alternatives in public transportation service quality. A multi-criteria group decision-making (MAGDM) framework process based on a combination of the full consistency method (FUCOM) and CODAS method has appeared in Biswas (2021) [46] for the first time in the literature. An integrated SWARA (Stepwise Weight Assessment Ratio Analysis) and CODAS methods are stated for the e-scooter charging station location selection problem in Pythagorean fuzzy information by Ayyildiz (2022) [47]. Recently, Mohamed and El-Saber (2023) [48] constructed the multi-stage intelligent decision-making model (MsIDMM) based on the CODAS method with interval-valued neutrosophic sets to evaluate the renewable energy sources.

Jafarzadeh et al. (2023) [49] combined the SWARA and the CODAS algorithm is applied to evaluate the clean energy barriers under a spherical fuzzy environment as a decision-making process. Garg et al. (2023) [50] constructed a theme of the (CODAS) method and the Dombi sine weighted arithmetic aggregation operator with complex intuitionistic fuzzy data for multi-criteria group decision-making problems. Sahmutoglu et al. (2023) [51] presented an integrated AHP-CODAS under Interval-Valued neutrosophic for risk assessment methodology in the district of Turkey, which is repeatedly exposed to floods. Dorfeshan et al. (2023) [52] presented the MABACODAS method, which includes MABAC and CODAS processes for MCDM under interval type-2 fuzzy information.

From a practical point of view, the use of CODAS and its combinations can be mentioned, such as Location selection problem [53, 54], Technological system evaluation problem [55], Material selection problem [56], Personnel selection problem [39], Service quality evaluation problem [45, 57], Cloud computing technology selection problem [58], Flexible Manufacturing System (FMS) selection [59], and so on [49, 60-62].

Distance measures (DMs) are substantial research topics for describing the distinctions and differences between various kinds of objects. The application of distance measures does not only include the procedure of decision-making problems based on distance measures, it can play a broad role in clustering algorithms, pattern recognition problems, medical diagnosis and image processing under uncertainty [63-67]. It is clear that acquaintance with distance measures that examine the nature of neutrosophic trapezoidal numbers from different perspectives can play an essential role in researchers' knowledge of MCDM problems and improvement of solution methods. Therefore, our primary focus in this research is introducing two new weighted measures based on surface distance under neutrosophic trapezoidal fuzzy information. These measures are used for presentation and productivity in the CODAS algorithm. In the proposed distances, the influences of all neutrosophic trapezoidal fuzzy numbers of the components are investigated. In addition, the choice of weights based on the decision maker's preferences determines the overall impact of each component on the final result. The meaningful structure of the proposed measures, along with their logical properties and characteristics, guarantees their proper performance in combination with other algorithms.

The rest of the study is constructed as follows: in Sect. 2, the required conceptions and operations of neutrosophic sets and numbers are explained. In Sect. 3, the conceptual structure of the main idea is given, along with two suggested distance measures for neutrosophic numbers. Then, in Sect. 4, theorems and properties are proven to ensure consistent and reasonable formulations of proposed distance measures. While in Sect. 5. The CODAS algorithm based on two novel distance measures for MCDM problems presented based on two new distance measures for MCDM problems. In Sect. 6, an illustrative example is solved in comparison to other existing methods. Furthermore, a sensitive analysis of parameters for the suggested hybrid approach is given to examine the effectiveness and robustness of the results. Lastly, in Sect. 6, some conclusions and future studies are stated.

2. Preliminaries

In this section, we consider a brief required definition of neutrosophic sets and neutrosophic trapezoidal fuzzy numbers (NTraFNs), along with some essential operators which are related to the subsequent sections of our study.

Definition 1 [2]. Assume that U is a universe of discourse, then a neutrosophic set $\widetilde{N}$ in U is defined by the following representation [2]:

$$\widetilde{N} = \{\langle u, \zeta_{\widetilde{N}}(u), \eta_{\widetilde{N}}(u), \theta_{\widetilde{N}}(u)\rangle \mid 0 \le \zeta_{\widetilde{N}}(u), \eta_{\widetilde{N}}(u), \theta_{\widetilde{N}}(u) \le 1, u \in U\}, \quad (1)$$

Where $\zeta_{\widetilde{N}}: U \to [0,1]$ is truth-membership function, $\eta_{\widetilde{N}}: U \to [0,1]$ is falsity-membership function, and $\eta_{\widetilde{N}}: U \to [0,1]$ is an indeterminacy-membership function. Furthermore $0 \le \zeta_{\widetilde{N}}(u) + \eta_{\widetilde{N}}(u) + \theta_{\widetilde{N}}(u) \le 3$.

Definition 2 [11]. Assume $\tilde{n} = \langle \zeta_{\tilde{n}}(u), \eta_{\tilde{n}}(u), \theta_{\tilde{n}}(u)\rangle$ is a neutrosophic fuzzy number in the set of real numbers. Then, its truth membership function is

$$\zeta_{\tilde{n}}(u) = \begin{cases} \zeta_{\tilde{n}}^{l}(u), & \alpha_1 \le u \le \alpha_2 \\ 1, & \alpha_2 \le u \le \alpha_3 \\ \zeta_{\tilde{n}}^{r}(u), & \alpha_3 \le u \le \alpha_4 \\ 0, & o.w \end{cases} \quad (2)$$

Its falsity membership function is

$$\eta_{\tilde{n}}(u) = \begin{cases} \eta_{\tilde{n}}^{l}(u), & \beta_1 \le u \le \beta_2 \\ 1, & \beta_2 \le u \le \beta_3 \\ \eta_{\tilde{n}}^{r}(u), & \beta_3 \le u \le \beta_4 \\ 0, & o.w \end{cases} \quad (3)$$

And its indeterminacy membership function is

$$\theta_{\tilde{n}}(u) = \begin{cases} \theta_{\tilde{n}}^{l}(u), & \gamma_1 \le u \le \gamma_2 \\ 1, & \gamma_2 \le u \le \gamma_3 \\ \theta_{\tilde{n}}^{r}(u), & \gamma_3 \le u \le \gamma_4 \\ 0, & o.w \end{cases} \quad (4)$$

Where $0 \le \zeta_{\tilde{n}}(u), \eta_{\tilde{n}}(u), \theta_{\tilde{n}}(u) \le 1$ and $0 \le \zeta_{\tilde{n}}(u) + \eta_{\tilde{n}}(u) + \theta_{\tilde{n}}(u) \le 3$.

Definition 3 [2]. Let two neutrosophic fuzzy numbers be $\tilde{n}_1 = \langle \zeta_{\tilde{n}_1}(u), \eta_{\tilde{n}_1}(u), \theta_{\tilde{n}_1}(u)\rangle$ and $\tilde{n}_2 = \langle \zeta_{\tilde{n}_2}(u), \eta_{\tilde{n}_2}(u), \theta_{\tilde{n}_2}(u)\rangle$. Then,

$$\tilde{n}_1 \subseteq \tilde{n}_2 \Leftrightarrow \quad \zeta_{\tilde{n}_1}(u) \leq \zeta_{\tilde{n}_2}(u), \quad \eta_{\tilde{n}_1}(u) \geq \eta_{\tilde{n}_2}(u), \theta_{\tilde{n}_1}(u) \geq \theta_{\tilde{n}_2}(u) \quad , for \ \forall \ u \in \mathrm{U}$$

Definition 4 [11]. Assume U be a universe of discourse, $\tilde{n} = \langle(\alpha_1, \alpha_2, \alpha_3, \alpha_4), (\beta_1, \beta_2, \beta_3, \beta_4), (\gamma_1, \gamma_2, \gamma_3, \gamma_4)\rangle$ is a neutrosophic trapezoidal fuzzy number in U that its truth-membership function is defined as

$$\zeta_{\tilde{n}}(u) = \begin{cases} \dfrac{(u-\alpha_1)}{\alpha_2-\alpha_1}, & \alpha_1 \leq u \leq \alpha_2 \\ 1, & \alpha_2 \leq u \leq \alpha_3 \\ \dfrac{(\alpha_4-u)}{\alpha_4-\alpha_3},, & \alpha_3 \leq u \leq \alpha_4 \\ 0, & o.w \end{cases} \qquad (5)$$

Its falsity-membership function is defined as

$$\eta_{\tilde{n}}(u) = \begin{cases} \frac{(\beta_2-u)}{\beta_2-\beta_1}, & \beta_1 \leq u \leq \beta_2 \\ 1, & \beta_2 \leq u \leq \beta_3 \\ \frac{(u-\beta_3)}{\beta_4-\beta_3}, & \beta_3 \leq u \leq \beta_4 \\ 0, & o.w \end{cases} \qquad (6)$$

and its indeterminacy-membership function is defined as

$$\theta_{\tilde{n}}(u) = \begin{cases} \frac{(\gamma_2-u)}{\gamma_2-\gamma_1}, & \gamma_1 \leq u \leq \gamma_2 \\ 1, & \gamma_2 \leq u \leq \gamma_3 \\ \frac{(u-\gamma_3)}{\gamma_4-\gamma_3}, & \gamma_3 \leq u \leq \gamma_4 \\ 0, & o.w \end{cases} \qquad (7)$$

Where $0 \leq \zeta_{\tilde{n}}(u), \eta_{\tilde{n}}(u), \theta_{\tilde{n}}(u) \leq 1$ and $0 \leq \zeta_{\tilde{n}}(u) + \eta_{\tilde{n}}(u) + \theta_{\tilde{n}}(u) \leq 3$.

Figure 1 depicts the general representation of a neutrosophic trapezoidal fuzzy number.

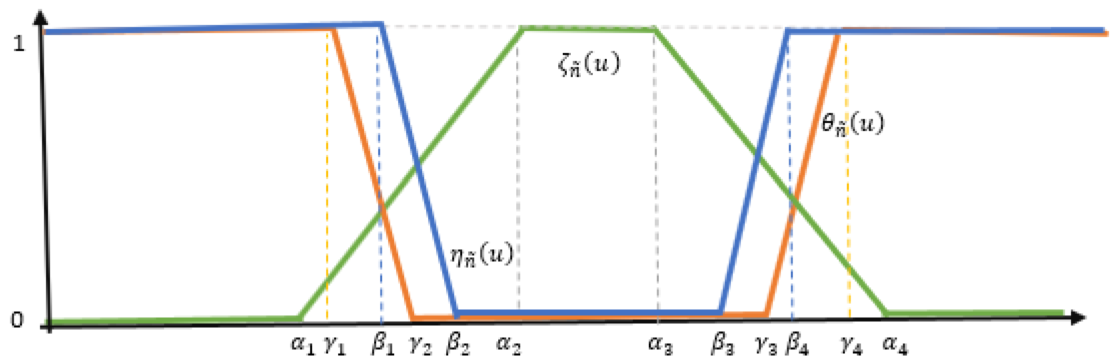

Figure 1. Trapezoidal neutrosophic fuzzy number.

Definition 5 [11]. Assume λ is a positive actual number, and consider two neutrosophic trapezoidal fuzzy numbers

$$\tilde{n}_1 = \langle(\alpha_{1\tilde{n}_1}, \alpha_{2\tilde{n}_1}, \alpha_{3\tilde{n}_1}, \alpha_{4\tilde{n}_1}), (\beta_{1\tilde{n}_1}, \beta_{2\tilde{n}_1}, \beta_{3\tilde{n}_1}, \beta_{4\tilde{n}_1}), (\gamma_{1\tilde{n}_1}, \gamma_{2\tilde{n}_1}, \gamma_{3\tilde{n}_1}, \gamma_{4\tilde{n}_1})\rangle$$
$$\tilde{n}_2 = \langle(\alpha_{1\tilde{n}_2}, \alpha_{2\tilde{n}_2}, \alpha_{3\tilde{n}_2}, \alpha_{4\tilde{n}_2}), (\beta_{1\tilde{n}_2}, \beta_{2\tilde{n}_2}, \beta_{3\tilde{n}_2}, \beta_{4\tilde{n}_2}), (\gamma_{1\tilde{n}_2}, \gamma_{2\tilde{n}_2}, \gamma_{3\tilde{n}_2}, \gamma_{4\tilde{n}_2})\rangle$$

Then, the following operations are valid.

1) $\tilde{n}_1 \oplus \tilde{n}_2 = \langle(\alpha_{1\tilde{n}_1} + \alpha_{1\tilde{n}_2} - \alpha_{1\tilde{n}_1}\alpha_{1\tilde{n}_2}, \alpha_{2\tilde{n}_1} + \alpha_{2\tilde{n}_2} - \alpha_{2\tilde{n}_1}\alpha_{2\tilde{n}_2}, \alpha_{3\tilde{n}_1} + \alpha_{3\tilde{n}_2} - \alpha_{3\tilde{n}_1}\alpha_{3\tilde{n}_2}, \alpha_{4\tilde{n}_1} + \alpha_{4\tilde{n}_2} - \alpha_{4\tilde{n}_1}\alpha_{4\tilde{n}_2}), (\beta_{1\tilde{n}_1}\beta_{1\tilde{n}_2}, \beta_{2\tilde{n}_1}\beta_{2\tilde{n}_2}, \beta_{3\tilde{n}_1}\beta_{3\tilde{n}_2}, \beta_{4\tilde{n}_1}\beta_{4\tilde{n}_2}), (\gamma_{1\tilde{n}_1}\gamma_{1\tilde{n}_2}, \gamma_{2\tilde{n}_1}\gamma_{2\tilde{n}_2}, \gamma_{3\tilde{n}_1}\gamma_{3\tilde{n}_2}, \gamma_{4\tilde{n}_1}\gamma_{4\tilde{n}_2})\ \rangle$,

2) $\tilde{n}_1 \otimes \tilde{n}_2 = \langle(\alpha_{1\tilde{n}_1}\alpha_{1\tilde{n}_2}, \alpha_{2\tilde{n}_1}\alpha_{2\tilde{n}_2}, \alpha_{3\tilde{n}_1}\alpha_{3\tilde{n}_2}, \alpha_{4\tilde{n}_1}\alpha_{4\tilde{n}_2}), (\beta_{1\tilde{n}_1} + \beta_{1\tilde{n}_2} - \beta_{1\tilde{n}_1}\beta_{1\tilde{n}_2}, \beta_{2\tilde{n}_1} + \beta_{2\tilde{n}_2} - \beta_{2\tilde{n}_1}\beta_{2\tilde{n}_2}, \beta_{3\tilde{n}_1} + \beta_{3\tilde{n}_2} - \beta_{3\tilde{n}_1}\beta_{3\tilde{n}_2}, \beta_{4\tilde{n}_1} + \beta_{4\tilde{n}_2} - \beta_{4\tilde{n}_1}\beta_{4\tilde{n}_2}), (\gamma_{1\tilde{n}_1} + \gamma_{1\tilde{n}_2} - \gamma_{1\tilde{n}_1}\gamma_{1\tilde{n}_2}, \gamma_{2\tilde{n}_1} + \gamma_{2\tilde{n}_2} - \gamma_{2\tilde{n}_1}\gamma_{2\tilde{n}_2}, \gamma_{3\tilde{n}_1} + \gamma_{3\tilde{n}_2} - \gamma_{3\tilde{n}_1}\gamma_{3\tilde{n}_2}, \gamma_{4\tilde{n}_1} + \gamma_{4\tilde{n}_2} - \gamma_{4\tilde{n}_1}\gamma_{4\tilde{n}_2})\ \rangle$,

3) $\lambda\tilde{n}_1 = \langle\left(\left(1-(1-\alpha_{1\tilde{n}_1})^{\lambda}\right),\left(1-(1-\alpha_{2\tilde{n}_1})^{\lambda}\right),\left(1-(1-\alpha_{3\tilde{n}_1})^{\lambda}\right),\left(1-(1-\alpha_{4\tilde{n}_1})^{\lambda}\right)\right), ,\left(\beta^{\lambda}_{1\tilde{n}_1},\beta^{\lambda}_{2\tilde{n}_1},\beta^{\lambda}_{3\tilde{n}_1},\beta^{\lambda}_{4\tilde{n}_1}\right),\left(\gamma^{\lambda}_{1\tilde{n}_1},\gamma^{\lambda}_{2\tilde{n}_1},\gamma^{\lambda}_{3\tilde{n}_1},\gamma^{\lambda}_{4\tilde{n}_1}\right)\rangle$

4) $\tilde{n}_1{}^{\lambda} = \langle\left(\alpha^{\lambda}_{1\tilde{n}_1},\alpha^{\lambda}_{2\tilde{n}_1},\alpha^{\lambda}_{3\tilde{n}_1},\alpha^{\lambda}_{4\tilde{n}_1}\right),\left(\left(1-(1-\beta_{1\tilde{n}_1})^{\lambda}\right),\left(1-(1-\beta_{2\tilde{n}_1})^{\lambda}\right),\left(1-(1-\beta_{3\tilde{n}_1})^{\lambda}\right),\left(1-(1-\beta_{4\tilde{n}_1})^{\lambda}\right)\right),\left(\left(1-(1-\gamma_{1\tilde{n}_1})^{\lambda}\right),\left(1-(1-\gamma_{2\tilde{n}_1})^{\lambda}\right),\left(1-(1-\gamma_{3\tilde{n}_1})^{\lambda}\right),\left(1-(1-\gamma_{4\tilde{n}_1})^{\lambda}\right)\right)\rangle$

Example 1: Assume $\lambda = 2$, and consider two neutrosophic trapezoidal fuzzy numbers

$\tilde{n}_1 = \langle(0.10,0.15,0.20,0.25),(0.05,0.10,0.30,0.35),(0.05,0.20,0.30,0.45)\rangle$

$\tilde{n}_2 = \langle(0.15,0.20,0.30,0.35),(0.20,0.30,0.40,0.50),(0.35,0.40,0.50,0.55)\rangle$

Then, according to Definition 5, we have

1) $\tilde{n}_1 \oplus \tilde{n}_2 = \langle(0.23,0.32,0.44,0.51),(0.01,0.03,0.12,0.18),(0.02,0.08,0.15,0.25)\rangle$,
2) $\tilde{n}_1 \otimes \tilde{n}_2 = \langle(0.02,0.03,0.06,0.09),(0.24,0.37,0.58,0.67),(0.38,0.52,0.65,0.75)\rangle$,
3) $2\tilde{n}_1 = \langle(0.19,0.28,0.36,0.44),(0.002,0.01,0.09,0.12),(0.002,0.04,0.09,0.20)\rangle$,
4) $\tilde{n}_1{}^2 = \langle(0.01,0.02,0.04,0.06),(0.10,0.19,0.51,0.58),(0.10,0.36,0.51,0.70)\rangle$.

3. Suggested weighted distance measures for neutrosophic trapezoidal fuzzy numbers

The distance measure concept is one of the most important theoretical and practical tools in information theorem that can be applied to evaluate the difference and distance of objects.

Here, we propose the conceptual scheme to model the distance measure between two neutrosophic trapezoidal fuzzy numbers (see Figure 2).

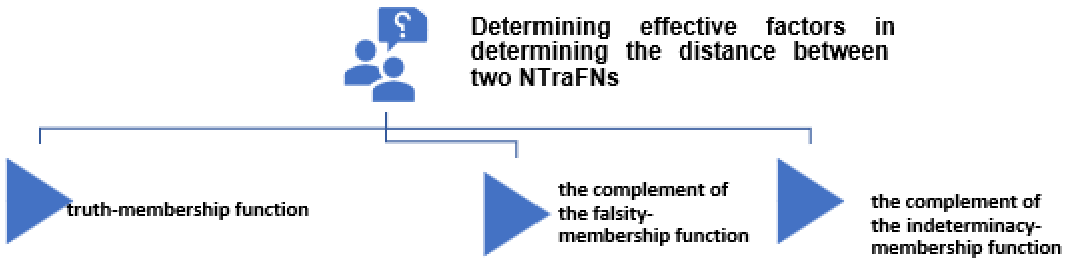

Figure 2. The main factors to determine the distance measure between NTraFNs.

Suppose $\tilde{n}_i = \langle(\alpha_{1\tilde{n}_i},\alpha_{2\tilde{n}_i},\alpha_{3\tilde{n}_i},\alpha_{4\tilde{n}_i}),(\beta_{1\tilde{n}_i},\beta_{2\tilde{n}_i},\beta_{3\tilde{n}_i},\beta_{4\tilde{n}_i}),(\gamma_{1\tilde{n}_i},\gamma_{2\tilde{n}_i},\gamma_{3\tilde{n}_i},\gamma_{4\tilde{n}_i})\rangle$ and $\tilde{n}_j = \left\langle(\alpha_{1\tilde{n}_j},\alpha_{2\tilde{n}_j},\alpha_{3\tilde{n}_j},\alpha_{4\tilde{n}_j}),(\beta_{1\tilde{n}_j},\beta_{2\tilde{n}_j},\beta_{3\tilde{n}_j},\beta_{4\tilde{n}_j}),(\gamma_{1\tilde{n}_j},\gamma_{2\tilde{n}_j},\gamma_{3\tilde{n}_j},\gamma_{4\tilde{n}_j})\right\rangle$ are two neutrosophic trapezoidal fuzzy numbers.

Step 1. Obtain the left and right line formulas corresponding to the truth-membership function;$[f^{l_{\tilde{n}_i}}(z), f^{r_{\tilde{n}_i}}(z)]$, the complement of the falsity-membership function; $[g^{l_{\tilde{n}_i}}(z), g^{r_{\tilde{n}_i}}(z)]$ and the complement of the indeterminacy-membership function; $[h^{l_{\tilde{n}_i}}(z), h^{r_{\tilde{n}_i}}(z)]$ for each trapezoidal neutrosophic fuzzy numbers with respect to the vertical lines z =0 and $z = 1$ respectively. In this sense, we can write:

$$[f^{l_{\tilde{n}_i}}(z), f^{r_{\tilde{n}_i}}(z)] = [(\alpha_{1\tilde{n}_i}) + z(\alpha_{2\tilde{n}_i} - \alpha_{1\tilde{n}_i}),(1-\alpha_{4\tilde{n}_i}) + z(\alpha_{4\tilde{n}_i} - \alpha_{3\tilde{n}_i})] \quad (8)$$

$$\left[g^{l\tilde{n}_i}(z), g^{r\tilde{n}_i}(z)\right] = \left[(\beta_{1\tilde{n}_i}) + \mathrm{z}(\beta_{2\tilde{n}_i} - \beta_{1\tilde{n}_i}), (1 - \beta_{4\tilde{n}_i}) + \mathrm{z}(\beta_{4\tilde{n}_i} - \beta_{3\tilde{n}_i})\right] \quad (9)$$

$$\left[h^{l\tilde{n}_i}(z), h^{r\tilde{n}_i}(z)\right] = \left[(\gamma_{1\tilde{n}_i}) + \mathrm{z}(\gamma_{2\tilde{n}_i} - \gamma_{1\tilde{n}_i}), (1 - \gamma_{4\tilde{n}_i}) + \mathrm{z}(\gamma_{4\tilde{n}_i} - \gamma_{3\tilde{n}_i})\right] \quad (10)$$

Step 2.a. Calculate the area of the left half of each interval function respective to the vertical axis. So, we can express

$$Q^{l\zeta}(\tilde{n}_i) = \int_0^1 \left((\alpha_{1\tilde{n}_i}) + z(\alpha_{2\tilde{n}_i} - \alpha_{1\tilde{n}_i})\right) dz = (\alpha_{1\tilde{n}_i}) + \frac{(\alpha_{2\tilde{n}_i} - \alpha_{1\tilde{n}_i})}{2} \quad (11)$$

$$Q^{l\eta}(\tilde{n}_i) = \int_0^1 \left((\beta_{1\tilde{n}_i}) + z(\beta_{2\tilde{n}_i} - \beta_{1\tilde{n}_i})\right) dz = (\beta_{1\tilde{n}_i}) + \frac{(\beta_{2\tilde{n}_i} - \beta_{1\tilde{n}_i})}{2} \quad (12)$$

$$Q^{r\theta}(\tilde{n}_i) = \int_0^1 \left((\gamma_{1\tilde{n}_i}) + z(\gamma_{2\tilde{n}_i} - \gamma_{1\tilde{n}_i})\right) dz = (\gamma_{1\tilde{n}_i}) + \frac{(\gamma_{2\tilde{n}_i} - \gamma_{1\tilde{n}_i})}{2} \quad (13)$$

b. Calculate the area of the right half of each interval function respective to the vertical axis. So, we can express

$$Q^{r\zeta}(\tilde{n}_i) = \int_0^1 \left((1 - \alpha_{4\tilde{n}_i}) + z(\alpha_{4\tilde{n}_i} - \alpha_{3\tilde{n}_i})\right) dz = (1 - \alpha_{4\tilde{n}_i}) + \frac{(\alpha_{4\tilde{n}_i} - \alpha_{3\tilde{n}_i})}{2} \quad (14)$$

$$Q^{r\eta}(\tilde{n}_i) = \int_0^1 \left((1 - \beta_{4\tilde{n}_i}) + z(\beta_{4\tilde{n}_i} - \beta_{3\tilde{n}_i})\right) dz = (1 - \beta_{4\tilde{n}_i}) + \frac{(\beta_{4\tilde{n}_i} - \beta_{3\tilde{n}_i})}{2} \quad (15)$$

$$Q^{r\theta}(\tilde{n}_i) = \int_0^1 \left((1 - \gamma_{4\tilde{n}_i}) + z(\gamma_{4\tilde{n}_i} - \gamma_{3\tilde{n}_i})\right) dz = (1 - \gamma_{4\tilde{n}_i}) + \frac{(\gamma_{4\tilde{n}_i} - \gamma_{3\tilde{n}_i})}{2} \quad (16)$$

Step 3.a. The suggested surface-based weighted hamming distance measure is introduced as

$$\mathrm{D}^{HS}(\tilde{n}_i, \tilde{n}_j) = \{ \omega^{l\zeta}\left(\left|Q^{l\zeta}(\tilde{n}_i) - Q^{l\zeta}(\tilde{n}_j)\right| + \omega^{r\zeta}\left|Q^{r\zeta}(\tilde{n}_i) - Q^{r\zeta}(\tilde{n}_j)\right|\right) + \omega^{l\eta}\left(\left|Q^{l\eta}(\tilde{n}_i) - Q^{l\eta}(\tilde{n}_j)\right| + \omega^{r\eta}\left|Q^{r\eta}(\tilde{n}_i) - Q^{r\eta}(\tilde{n}_j)\right|\right) + \omega^{l\theta}\left(\left|Q^{l\theta}(\tilde{n}_i) - Q^{l\theta}(\tilde{n}_j)\right| + \omega^{r\theta}\left|Q^{r\theta}(\tilde{n}_i) - Q^{r\theta}(\tilde{n}_j)\right|\right)\} \quad (17)$$

Where $\omega^{l\zeta}, \omega^{r\zeta}, \omega^{l\eta}, \omega^{r\eta}, \omega^{l\theta}, \omega^{r\theta} \in [0,1]$ and satisfies $\omega^{l\zeta} + \omega^{r\zeta} + \omega^{l\eta} + \omega^{r\eta} + \omega^{l\theta} + \omega^{r\theta} = 1$.
If $\omega^{l\zeta} = \omega^{r\zeta} = \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta}$, then

$$\mathrm{D}^{HS}(\tilde{n}_i, \tilde{n}_j) = \frac{1}{6}\{\left(\left|Q^{l\zeta}(\tilde{n}_i) - Q^{l\zeta}(\tilde{n}_j)\right| + \left|Q^{r\zeta}(\tilde{n}_i) - Q^{r\zeta}(\tilde{n}_j)\right|\right) + \left(\left|Q^{l\eta}(\tilde{n}_i) - Q^{l\eta}(\tilde{n}_j)\right| + \left|Q^{r\eta}(\tilde{n}_i) - Q^{r\eta}(\tilde{n}_j)\right|\right) + \left(\left|Q^{l\theta}(\tilde{n}_i) - Q^{l\theta}(\tilde{n}_j)\right| + \left|Q^{r\theta}(\tilde{n}_i) - Q^{r\theta}(\tilde{n}_j)\right|\right)\} \quad (18)$$

b. Similarly, the suggested surface-based weighted Euclidean distance measure introduced as

$$\mathrm{D}^{ES}(\tilde{n}_i, \tilde{n}_j) = \left\{ \omega^{l\zeta}\left(Q^{l\zeta}(\tilde{n}_i) - Q^{l\zeta}(\tilde{n}_j)\right)^2 + \omega^{r\zeta}\left(Q^{r\zeta}(\tilde{n}_i) - Q^{r\zeta}(\tilde{n}_j)\right)^2 + \omega^{l\eta}\left(Q^{l\eta}(\tilde{n}_i) - Q^{l\eta}(\tilde{n}_j)\right)^2 + \omega^{r\eta}\left(Q^{r\eta}(\tilde{n}_i) - Q^{r\eta}(\tilde{n}_j)\right)^2 + \omega^{l\theta}\left(Q^{l\theta}(\tilde{n}_i) - Q^{l\theta}(\tilde{n}_j)\right)^2 + \omega^{r\theta}\left(Q^{r\theta}(\tilde{n}_i) - Q^{r\theta}(\tilde{n}_j)\right)^2 \right\}^{1/2} \quad (19)$$

Where $\omega^{l\zeta}, \omega^{r\zeta}, \omega^{l_\eta}, \omega^{r_\eta}, \omega^{l\theta}, \omega^{r\theta} \in [0,1]$ and satisfies $\omega^{l\zeta} + \omega^{r\zeta} + \omega^{l_\eta} + \omega^{r_\eta} + \omega^{l\theta} + \omega^{r\theta} = 1$.
If $\omega^{l\zeta} = \omega^{r\zeta} = \omega^{l_\eta} = \omega^{r_\eta} = \omega^{l\theta} = \omega^{r\theta}$, then

$$\mathrm{D}^{ES}(\tilde{n}_i, \tilde{n}_j) = \frac{1}{\sqrt{6}}\Big\{ \left(Q^{l\zeta}(\tilde{n}_i) - Q^{l\zeta}(\tilde{n}_j)\right)^2 + \left(Q^{r\zeta}(\tilde{n}_i) - Q^{r\zeta}(\tilde{n}_j)\right)^2 + \left(Q^{l_\eta}(\tilde{n}_i) - Q^{l_\eta}(\tilde{n}_j)\right)^2 + \left(Q^{r_\eta}(\tilde{n}_i) - Q^{r_\eta}(\tilde{n}_j)\right)^2 + \left(Q^{l\theta}(\tilde{n}_i) - Q^{l\theta}(\tilde{n}_j)\right)^2 + \left(Q^{r\theta}(\tilde{n}_i) - Q^{r\theta}(\tilde{n}_j)\right)^2 \Big\}^{1/2} \tag{20}$$

Example 2: Suppose $\tilde{n}_1 = \langle(0.10,0.15,0.20,0.25),(0.05,0.10,0.30,0.35),(0.05,0.20,0.30,0.45)\rangle$ and $\tilde{n}_2 = \langle(0.15,0.20,0.30,0.35),(0.20,0.30,0.40,0.50),(0.35,0.40,0.50,0.55)\rangle$ are two neutrosophic trapezoidal fuzzy numbers. Then, according to Eqs 19 and 20, we have

$$\mathrm{D}^{HS}(\tilde{n}_1, \tilde{n}_2) = \frac{1}{6}\{(0.05 + 0.10) + (0.10 + 0.125) + (0.275 + 0.15)\} = 0.13$$

$$\mathrm{D}^{ES}(\tilde{n}_1, \tilde{n}_2) = \frac{1}{\sqrt{6}}\{\ 0.0025 + 0.01 + 0.01 + 0.016 + 0.076 + 0.0225\ \}^{1/2} = 0.15$$

4. Theorems and properties

In this part, we focus on noteworthy features of suggested surface-based weighted hamming and Euclidean distance measures.

Theorem 1: let $\tilde{n}_1$, $\tilde{n}_2$, and $\tilde{n}_3$ are three neutrosophic trapezoidal fuzzy numbers on U. We represent the distance measure between the two numbers $\tilde{n}_1$ and $\tilde{n}_2$ is denoted as $D(\tilde{n}_1, \tilde{n}_2)$. Demonstrate that equation (18) satisfies the following distance measure principles.

A 1) $0 \le D(\tilde{n}_1, \tilde{n}_2) \le 1$

A 2) $D(\tilde{n}_1, \tilde{n}_2) = 0 \Leftrightarrow \tilde{n}_1 \sim \tilde{n}_2$

A 3) $D(\tilde{n}_1, \tilde{n}_2) = D(\tilde{n}_2, \tilde{n}_1)$

A 4) If $\tilde{n}_1 \subseteq \tilde{n}_2 \subseteq \tilde{n}_3 \Longrightarrow D(\tilde{n}_1, \tilde{n}_2) \le D(\tilde{n}_1, \tilde{n}_3)$ and $D(\tilde{n}_1, \tilde{n}_2) \le D(\tilde{n}_1, \tilde{n}_3)$

Proof:

1) In every term of (18) the result of each absolute value is positive and smaller than 1, the structure of the relation and the fact that the sum of the weights is one, thematic principle 1 is valid. Therefore:

$$0 \le \mathrm{D}^{HS}(\tilde{n}_1, \tilde{n}_2) \le 1$$

2) If $\mathrm{D}^{HS}(\tilde{n}_1, \tilde{n}_2) = 0$, then:

$$Q^{l\zeta}(\tilde{n}_1) = Q^{l\zeta}(\tilde{n}_2), \quad Q^{r\zeta}(\tilde{n}_1) = Q^{r\zeta}(\tilde{n}_2), \quad Q^{l_\eta}(\tilde{n}_1) = Q^{l_\eta}(\tilde{n}_2), \quad Q^{r_\eta}(\tilde{n}_1) = Q^{r_\eta}(\tilde{n}_2)$$
$$Q^{l\theta}(\tilde{n}_1) = Q^{l\theta}(\tilde{n}_2), \quad Q^{r\theta}(\tilde{n}_1) = Q^{r\theta}(\tilde{n}_2)$$

Therefore

$$\tilde{n}_1 \sim \tilde{n}_2$$

The converse of principle 2 is also can be proven in a similar way.

3) Since each of the expressions is in absolute value. Therefore:

$$\mathrm{D}^{HS}(\tilde{n}_1, \tilde{n}_2) = \mathrm{D}^{HS}(\tilde{n}_2, \tilde{n}_1)$$

4) If $\tilde{n}_1 \subseteq \tilde{n}_2 \subseteq \tilde{n}_3$, then:

$$|Q^{l\zeta}(\tilde{n}_1)-Q^{l\zeta}(\tilde{n}_2)| \le |Q^{l\zeta}(\tilde{n}_1)-Q^{l\zeta}(\tilde{n}_3)|, \qquad |Q^{r\zeta}(\tilde{n}_1)-Q^{r\zeta}(\tilde{n}_2)| \le |Q^{r\zeta}(\tilde{n}_1)-Q^{r\zeta}(\tilde{n}_3)|$$
$$|Q^{l\eta}(\tilde{n}_1)-Q^{l\eta}(\tilde{n}_2)| \le |Q^{l\eta}(\tilde{n}_1)-Q^{l\eta}(\tilde{n}_3)|, \qquad |Q^{r\eta}(\tilde{n}_1)-Q^{r\eta}(\tilde{n}_2)| \le |Q^{r\eta}(\tilde{n}_1)-Q^{r\eta}(\tilde{n}_3)|$$
$$|Q^{l\theta}(\tilde{n}_1)-Q^{l\theta}(\tilde{n}_2)| \le |Q^{l\theta}(\tilde{n}_1)-Q^{l\theta}(\tilde{n}_3)|, \qquad |Q^{r\theta}(\tilde{n}_1)-Q^{r\theta}(\tilde{n}_2)| \le |Q^{r\theta}(\tilde{n}_1)-Q^{r\theta}(\tilde{n}_3)|$$

Hence

$$\begin{aligned}\mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_2) = \frac{1}{6}\{&(|Q^{l\zeta}(\tilde{n}_1)-Q^{l\zeta}(\tilde{n}_2)| + |Q^{r\zeta}(\tilde{n}_1)-Q^{r\zeta}(\tilde{n}_2)|)\\ &+ (|Q^{l\eta}(\tilde{n}_1)-Q^{l\eta}(\tilde{n}_2)| + |Q^{r\eta}(\tilde{n}_1)-Q^{r\eta}(\tilde{n}_2)|)\\ &+ (|Q^{l\theta}(\tilde{n}_1)-Q^{l\theta}(\tilde{n}_2)| + |Q^{r\theta}(\tilde{n}_1)-Q^{r\theta}(\tilde{n}_2)|)\}\end{aligned}$$

$$\le \frac{1}{6}\{(|Q^{l\zeta}(\tilde{n}_1)-Q^{l\zeta}(\tilde{n}_3)| + |Q^{r\zeta}(\tilde{n}_1)-Q^{r\zeta}(\tilde{n}_3)|) + (|Q^{l\eta}(\tilde{n}_1)-Q^{l\eta}(\tilde{n}_3)| + |Q^{r\eta}(\tilde{n}_1)-Q^{r\eta}(\tilde{n}_3)|) +$$

$$(|Q^{l\theta}(\tilde{n}_1)-Q^{l\theta}(\tilde{n}_3)| + |Q^{r\theta}(\tilde{n}_1)-Q^{r\theta}(\tilde{n}_3)|)\} = \mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3)$$

As a result

$$\mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_2) \le \mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3)$$

It can be shown in a similar way

$$\mathrm{D}^{HS}(\tilde{n}_2,\tilde{n}_3) \le \mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3)$$

Therefore, relation (20) satisfies all measure properties.

Theorem 2. let $\tilde{n}_1 \subseteq \tilde{n}_2 \subseteq \tilde{n}_3$ then, demonstrate

$$\mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3) \le \mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_2) + \mathrm{D}^{HS}(\tilde{n}_2,\tilde{n}_3) \qquad (21)$$

Proof: Starting from the left side of (18), we can write:

$$\begin{aligned}\mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3) = \frac{1}{6}\{&(|Q^{l\zeta}(\tilde{n}_1)-Q^{l\zeta}(\tilde{n}_2)+Q^{l\zeta}(\tilde{n}_2)-Q^{l\zeta}(\tilde{n}_3)| + |Q^{r\zeta}(\tilde{n}_1)-Q^{r\zeta}(\tilde{n}_2)+Q^{r\zeta}(\tilde{n}_2)-Q^{r\zeta}(\tilde{n}_3)|)\\ &+ (|Q^{l\eta}(\tilde{n}_1)-Q^{l\eta}(\tilde{n}_2)+Q^{l\eta}(\tilde{n}_2)-Q^{l\eta}(\tilde{n}_3)|\\ &+ |Q^{r\eta}(\tilde{n}_1)-Q^{r\eta}(\tilde{n}_2)+Q^{r\eta}(\tilde{n}_2)-Q^{r\eta}(\tilde{n}_3)|)\\ &+ (|Q^{l\theta}(\tilde{n}_1)-Q^{l\theta}(\tilde{n}_2)+Q^{l\theta}(\tilde{n}_2)-Q^{l\theta}(\tilde{n}_3)|\\ &+ |Q^{r\theta}(\tilde{n}_1)-Q^{r\theta}(\tilde{n}_2)+Q^{r\theta}(\tilde{n}_2)-Q^{r\theta}(\tilde{n}_3)|)\}\end{aligned}$$

According to the Triangular inequality property of absolute value, we can say

$$\begin{aligned}\mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3) \le \frac{1}{6}\{&(|Q^{l\zeta}(\tilde{n}_1)-Q^{l\zeta}(\tilde{n}_2)| + |Q^{r\zeta}(\tilde{n}_1)-Q^{r\zeta}(\tilde{n}_2)|)\\ &+ (|Q^{l\eta}(\tilde{n}_1)-Q^{l\eta}(\tilde{n}_2)| + |Q^{r\eta}(\tilde{n}_1)-Q^{r\eta}(\tilde{n}_2)|)\\ &+ (|Q^{l\theta}(\tilde{n}_1)-Q^{l\theta}(\tilde{n}_2)| + |Q^{r\theta}(\tilde{n}_1)-Q^{r\theta}(\tilde{n}_2)|)\}\end{aligned}$$

$$\begin{aligned}+\frac{1}{6}\{&(|Q^{l\zeta}(\tilde{n}_2)-Q^{l\zeta}(\tilde{n}_3)| + |Q^{r\zeta}(\tilde{n}_2)-Q^{r\zeta}(\tilde{n}_3)|) + (|Q^{l\eta}(\tilde{n}_2)-Q^{l\eta}(\tilde{n}_3)| + |Q^{r\eta}(\tilde{n}_2)-Q^{r\eta}(\tilde{n}_3)|)\\ &+ (|Q^{l\theta}(\tilde{n}_2)-Q^{l\theta}(\tilde{n}_3)| + |Q^{r\theta}(\tilde{n}_2)-Q^{r\theta}(\tilde{n}_3)|)\}\\ &= \mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_3) + \mathrm{D}^{HS}(\tilde{n}_2,\tilde{n}_3).\end{aligned}$$

Now, let us demonstrate some meaningful properties of the proposed measures. For this aim, consider the following NTraFNs:

$$\tilde{n}_1 = \langle (a,a,a,a),(a,a,a,a),(0,0,0,0)\rangle, \tilde{n}_2 = \langle (b,b,b,b),(0,0,0,0),(0,0,0,0)\rangle,$$
$$\tilde{n}_3 = \langle (0,0,0,0),(0,0,0,0),(0,0,0,0)\rangle, \tilde{n}_4 = \langle (1,1,1,1),(0,0,0,0),(0,0,0,0)\rangle.$$

Property 1: If $\omega^{l\zeta} = \omega^{r\zeta} = 0.5, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0, then$

$$\mathrm{D}^{HS}(\tilde{n}_1,\tilde{n}_2) = \mathrm{D}^{ES}(\tilde{n}_1,\tilde{n}_2) = |a-b|.$$

Property 2: If $\omega^{l\zeta} = \omega^{r\zeta} = 0.5, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0, then$

$$D^{HS}(\tilde{n}_3, \tilde{n}_4) = D^{ES}(\tilde{n}_3, \tilde{n}_4) = 1.$$

Property 3: If $\omega^{l\zeta} = \omega^{r\zeta} = 0.5, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0, then$

$$D^{HS}(\tilde{n}_1, \tilde{n}_3) = D^{ES}(\tilde{n}_1, \tilde{n}_3) = a.$$

Property 4: If $\omega^{l\zeta} = \omega^{r\zeta} = 0.5, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0, then$

$$D^{HS}(\tilde{n}_1, \tilde{n}_4) = D^{ES}(\tilde{n}_1, \tilde{n}_4) = 1 - a.$$

Example 3: Find D^{HS} and D^{ES} between the following numbers

$$\tilde{n}_1 = \langle(0.5,0.5,0.5,0.5), (0,0,0,0), (0,0,0,0)\rangle, \tilde{n}_2 = \langle(0.7,0.7,0.7,0.7), (0,0,0,0), (0,0,0,0)\rangle,$$

$$\tilde{n}_3 = \langle(0,0,0,0), (0,0,0,0), (0,0,0,0)\rangle, \tilde{n}_4 = \langle(1,1,1,1), (0,0,0,0), (0,0,0,0)\rangle.$$

Regard to three following cases for weights as

Case1: $\omega^{l\zeta} = \omega^{r\zeta} = 0.5, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0$

Case2: $\omega^{l\zeta} = \omega^{r\zeta} = \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 1/6$

Case3: $\omega^{l\zeta} = \omega^{r\zeta} = 0.3, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0.1$

Since the mentioned numbers in example 1 are all deterministic, we expect the proposed distance measures to verify an acceptable performance with changes in the weighting coefficients. We consider three modes for weight variation according to what was mentioned earlier. The weighted Euclidean and Hamming distance between the numbers $\tilde{n}_1, \tilde{n}_2, \tilde{n}_3$, and $\tilde{n}_4$ under states 1, 2, and 3 are given in tables 1 to 6.

Case1	$\tilde{n}_1$	$\tilde{n}_2$	$\tilde{n}_3$	$\tilde{n}_4$
$\tilde{n}_1$	0	0.2000	0.5000	0.5000
$\tilde{n}_2$	0.2000	0	0.7000	0.3000
$\tilde{n}_3$	0.5000	0.7000	0	1.0000
$\tilde{n}_4$	0.5000	0.3000	1.0000	0

Table 1. Hamming distance measure for case 1 of Ex 1.

Case1	$\tilde{n}_1$	$\tilde{n}_2$	$\tilde{n}_3$	$\tilde{n}_4$
$\tilde{n}_1$	0	0.2000	0.5000	0.5000
$\tilde{n}_2$	0.2000	0	0.7000	0.3000
$\tilde{n}_3$	0.5000	0.7000	0	1.0000
$\tilde{n}_4$	0.5000	0.3000	1.0000	0

Table 2. Euclidean distance measure for case 1 of Ex 1.

Case2	$\tilde{n}_1$	$\tilde{n}_2$	$\tilde{n}_3$	$\tilde{n}_4$
$\tilde{n}_1$	0	0.0667	0.1667	0.1667
$\tilde{n}_2$	0.0667	0	0.2333	0.1000
$\tilde{n}_3$	0.1667	0.2333	0	0.3333
$\tilde{n}_4$	0.1667	0.3333	0.3333	0

Table 3. Hamming distance measure for case 2 of Ex 1.

Case2	$\tilde{n}_1$	$\tilde{n}_2$	$\tilde{n}_3$	$\tilde{n}_4$
$\tilde{n}_1$	0	0.1155	0.2877	0.2877
$\tilde{n}_2$	0.1155	0	0.4041	0.1732
$\tilde{n}_3$	0.2877	0.4041	0	0.5774
$\tilde{n}_4$	0.2877	0.1732	0.5774	0

Table 4. Euclidean distance measure for case 2 of Ex 1.

Case3	$\tilde{n}_1$	$\tilde{n}_2$	$\tilde{n}_3$	$\tilde{n}_4$
$\tilde{n}_1$	0	0.1200	0.3000	0.3000
$\tilde{n}_2$	0.1200	0	0.4200	0.1800
$\tilde{n}_3$	0.3000	0.4200	0	0.6000

Case3	$\tilde{n}_1$	$\tilde{n}_2$	$\tilde{n}_3$	$\tilde{n}_4$
$\tilde{n}_1$	0	0.1549	0.3873	0.3873
$\tilde{n}_2$	0.1549	0	0.5422	0.2324
$\tilde{n}_3$	0.3873	0.5422	0	0.7746

$\tilde{n}_4$	0.3000	0.1800	0.6000	0

Table 5. Hamming distance measure for case 3 of Ex 1.

$\tilde{n}_4$	0.3873	0.2324	0.7746	0

Table 6. Euclidean distance measure for case 3 of Ex 1.

As expected, the results of Tables 1 and 2 in case 1 reflect the logical and uniform performance of the proposed distance measures. In addition, by increasing the weighting coefficients related to the indeterminacy and falsity-membership functions for cases 2 and 3 (Tables 3, 4, 5, and 6), smaller values for Euclidean and Hamming distances are obtained compared to case 1.

5. NTraFNs-CODAS method based on novel weighted distance measures

Keshavarz Ghorabaee et al. (2016) [36] introduced the CODAS method as one of the distance-based methods in 2016 to solve the multi-criteria decision-making problem. In this attitude, the Euclidean and Hamming distances of each option are used to determine the most desirable option from the negative ideal. Based on our current studies and knowledge from the research literature, no extension of this method has been done on neutrosophic trapezoidal fuzzy data for selection problems. Therefore, in this part, we want to present the CODAS algorithm under neutrosophic trapezoidal fuzzy information and using two surface-based weighted distance measures. Figure 3 shows the general structure of the NTraFNs-CODAS algorithm.

More precisely, the steps of the CODAS method are expressed as follows:

Step 1: Record the alternative sets $O = \{o_1, o_2, \ldots, o_s\}$, attribute sets $E = \{e_1, e_2, \ldots, e_t\}$ and relevant weight sets $\Xi = \{\xi_1, \xi_2, \ldots, \xi_t\}$. Then construct the NTraFNs decision matrix, which is denoted as $\tilde{U} = [\tilde{u}_{ij}]_{s*t}$ such that each array is given by

$$\tilde{u}_{ij} = \left\langle \left(\alpha_{1\tilde{u}_{ij}}, \alpha_{2\tilde{u}_{ij}}, \alpha_{3\tilde{u}_{ij}}, \alpha_{4\tilde{u}_{ij}}\right), \left(\beta_{1\tilde{u}_{ij}}, \beta_{2\tilde{u}_{ij}}, \beta_{3\tilde{u}_{ij}}, \beta_{4\tilde{u}_{ij}}\right), \left(\gamma_{1\tilde{u}_{ij}}, \gamma_{2\tilde{u}_{ij}}, \gamma_{3\tilde{u}_{ij}}, \gamma_{4\tilde{u}_{ij}}\right) \right\rangle \tag{22}$$

Where $i \epsilon \{1,2,\ldots,s\}$ and $i \epsilon \{1,2,\ldots,t\}$.

Step 2: Obtain the normalized and then the weighted normalized NTraFNs decision matrix, which can denote as $\tilde{U}^n = [\tilde{u}_{ij}^n]_{s*t}$ and $\tilde{U}^{wn} = [\tilde{u}_{ij}^{wn}]_{s*t}$ where

$$\tilde{u}_{ij}^{wn} = \tilde{u}_{ij}^n * \xi_j \tag{23}$$

$$\tilde{u}_{ij}^{wn} = \tilde{u}_{ij} = \left\langle \left(\alpha_{1\tilde{u}_{ij}}^{wn}, \alpha_{2\tilde{u}_{ij}}^{wn}, \alpha_{3\tilde{u}_{ij}}^{wn}, \alpha_{4\tilde{u}_{ij}}^{wn}\right), \left(\beta_{1\tilde{u}_{ij}}^{wn}, \beta_{2\tilde{u}_{ij}}^{wn}, \beta_{3\tilde{u}_{ij}}^{wn}, \alpha_{4\tilde{u}_{ij}}^{wn}\right), \left(\gamma_{1\tilde{u}_{ij}}^{wn}, \gamma_{2\tilde{u}_{ij}}^{wn}, \gamma_{3\tilde{u}_{ij}}^{wn}, \gamma_{4\tilde{u}_{ij}}^{wn}\right) \right\rangle \tag{24}$$

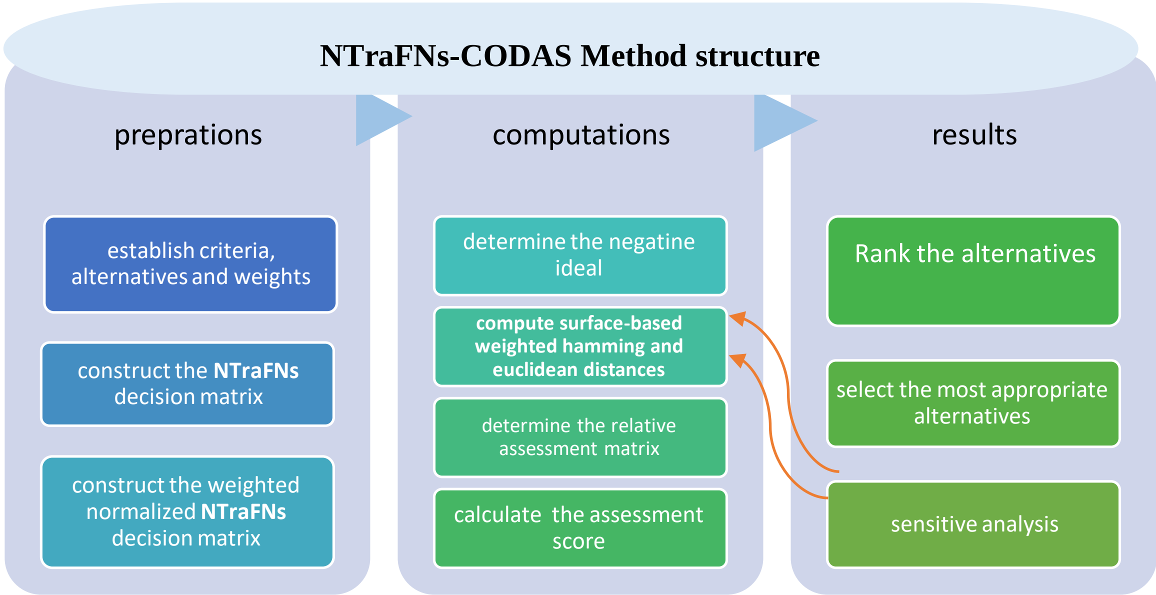

Figure 3. NTraFNs-CODAS Method structure.

Step 3: Recognize the negative ideal solution. This matrix defines by $\widetilde{NI} = [\tilde{n\iota}_{ij}]_{1*t}$ such that:

$$\tilde{n\iota}_{ij} = \min_i \tilde{u}_{ij}^{wn} \tag{25}$$

In such a way as

$$\min_i \tilde{u}_{ij}^{wn} = \left\langle \begin{array}{c} \left(\min_i(\alpha^{wn}_{1\tilde{u}_{ij}}), \min_i(\alpha^{wn}_{2\tilde{u}_{ij}}), \min_i(\alpha^{wn}_{3\tilde{u}_{ij}}), \min_i(\alpha^{wn}_{4\tilde{u}_{ij}})\right), \\ \left(\max_i(\beta^{wn}_{1\tilde{u}_{ij}}), \max_i(\beta^{wn}_{2\tilde{u}_{ij}}), \max_i(\beta^{wn}_{3\tilde{u}_{ij}}), \max_i(\alpha^{wn}_{4\tilde{u}_{ij}})\right), \\ \left(\max_i(\gamma^{wn}_{1\tilde{u}_{ij}}), \max_i(\gamma^{wn}_{2\tilde{u}_{ij}}), \max_i(\gamma^{wn}_{3\tilde{u}_{ij}}), \max_i(\gamma^{wn}_{4\tilde{u}_{ij}})\right) \end{array} \right\rangle \tag{26}$$

Step 4: Calculate the surface-based weighted Hamming and Euclidean distances between alternatives and $\widetilde{NI}$ according to Eqs (17, 19). Then D_i^{HS} and D_i^{ES} are computed as aggregated distances in Eqs (27, 28)

$$D_i^{HS} = \sum_{j=1}^{t} \mathrm{D}^{HS}\left(\tilde{u}_{ij}^{wn}, \tilde{n\iota}_{ij}\right) \tag{27}$$

$$D_i^{ES} = \sum_{j=1}^{t} \mathrm{D}^{ES}\left(\tilde{u}_{ij}^{wn}, \tilde{n\iota}_{ij}\right) \tag{28}$$

Step 5: Organize the relative assessment matrix as follows

$$RA = [p_{ik}]_{s*s} \tag{29}$$

Where each array of RA is obtained by applying Eqs 30 and 31

$$p_{ik} = \left\{(D_i^{ES} - D_k^{ES}) + \left(\gamma(D_i^{ES} - D_k^{ES}) * (D_i^{HS} - D_k^{HS})\right)\right\}, \tag{30}$$

$$\Phi(u) = \begin{cases} 1 & |u| \geq \varphi, \\ 0 & |u| \leq \varphi, \end{cases} \tag{31}$$

as a threshold value chosen by the decision-maker's opinion for the function $\Phi(u)$. In this paper, $\varphi = 0.02$ is assumed for computations.

Step 6: The assessment value for each alternative is determined as the following equation

$$AS_i = \sum_{k=1}^{s} p_{ik}. \tag{32}$$

Step 7: The highest Assessment value of step 6 indicates the most desirable choice.

5 -1 Illustrative example

In order to show the efficiency of the proposed hybrid method, we adopted the illustrative example of the material selection problem discussed by Jana and Karaaslan [68]. The customer desires to buy a tablet from the list of primarily selected five alternatives $O = \{o_1, o_2, o_3, o_4, o_5\}$. The following four attributes are considered by the customer (Figure 4):

(1) Options (e1);
(2) Hardware (e2);
(3) Affordable price (e3); and
(4) Customer support (e4).

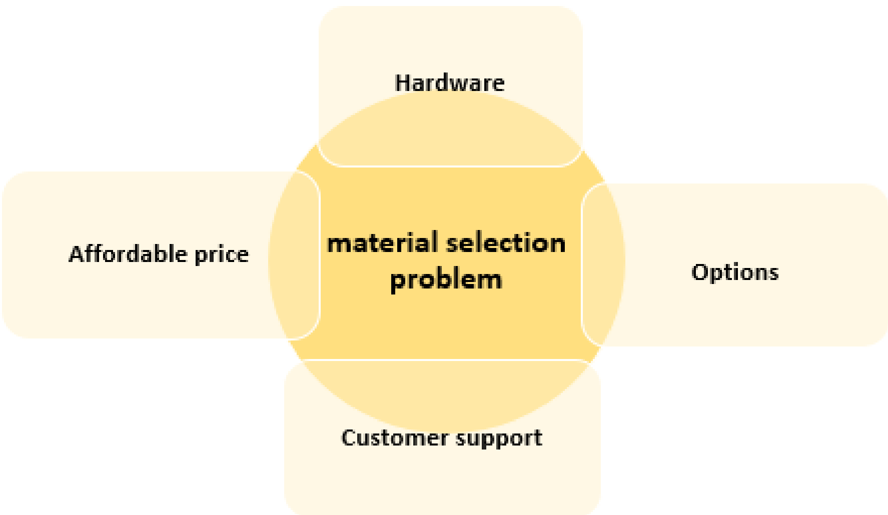

Figure 4. Criteria of material selection problem.

Assume that the weight vectors are provided by experts for the four attributes under the TrNFNs as follows:

$$\xi_1 = \langle(0.3,0.5,0.8,0.9),(0.1,0.3,0.6,0.7),(0.2,0.3,0.6,0.6)\rangle$$
$$\xi_2 = \langle(0.5,0.6,0.7,0.9),(0.3,0.5,0.6,0.8),(0.2,0.4,0.7,0.8)\rangle$$
$$\xi_3 = \langle(0.6,0.7,0.8,0.9),(0.0,0.1,0.2,0.3),(0.1,0.1,0.2,0.3)\rangle$$
$$\xi_4 = \langle(0.4,0.6,0.7,0.7),(0.2,0.3,0.4,0.5),(0.2,0.3,0.6,0.6)\rangle$$

Table 7 provides information (neutrosophic trapezoidal fuzzy numbers) on the experts' opinions of the five alternatives on the relevant criteria for the decision-making process.

Table 7. NTraFNs decision matrix.

	e_1	e_2	e_3	e_4
o_1	(0/1,0/2,0/3,0/3), ⟨(0/0,0/3,0/4,0/4),⟩ (0/2,0/5,0/6,0/7)	(0/4,0/5,0/6,0/6), ⟨(0/1,0/1,0/4,0/6),⟩ (0/2,0/5,0/6,0/7)	(0/2,0/2,0/3,0/4), ⟨(0/5,0/6,0/6,0/8),⟩ (0/0,0/2,0/2,0/5)	(0/5,0/5,0/6,0/6), ⟨(0/2,0/7,0/7,0/7),⟩ (0/2,0/3,0/3,0/3)
o_2	(0/2,0/2,0/4,0/4), ⟨(0/3,0/3,0/5,0/6),⟩ (0/1,0/2,0/2,0/5)	(0/3,0/5,0/6,0/7), ⟨(0/2,0/2,0/3,0/4),⟩ (0/4,0/5,0/8,0/9)	(0/4,0/5,0/5,0/7), ⟨(0/3,0/3,0/4,0/6),⟩ (0/2,0/3,0/4,0/5)	(0/1,0/1,0/2,0/8), ⟨(0/6,0/6,0/7,0/8),⟩ (0/0,0/1,0/2,0/4)
o_3	(0/5,0/7,0/8,0/9), ⟨(0/2,0/4,0/5,0/8),⟩ (0/3,0/3,0/5,0/5)	(0/1,0/2,0/2,0/3), ⟨(0/2,0/5,0/6,0/6),⟩ (0/1,0/2,0/3,0/4)	(0/3,0/3,0/4,0/5), ⟨(0/1,0/4,0/4,0/6),⟩ (0/2,0/2,0/3,0/7)	(0/0,0/2,0/3,0/9), ⟨(0/1,0/7,0/7,0/8),⟩ (0/6,0/7,0/7,0/8)
o_4	(0/0,0/2,0/3,0/7), ⟨(0/4,0/5,0/6,0/8),⟩ (0/4,0/5,0/5,0/9)	(0/5,0/5,0/7,0/8), ⟨(0/4,0/5,0/6,0/6),⟩ (0/5,0/6,0/7,0/8)	(0/5,0/6,0/6,0/9), ⟨(0/3,0/5,0/5,0/6),⟩ (0/1,0/5,0/5,0/6)	(0/5,0/7,0/8,0/9), ⟨(0/5,0/6,0/6,0/6),⟩ (0/2,0/3,0/3,0/3)
o_5	(0/2,0/4,0/4,0/5), ⟨(0/3,0/6,0/6,0/9),⟩ (0/0,0/2,0/3,0/5)	(0/1,0/5,0/7,0/9), ⟨(0/2,0/3,0/3,0/6),⟩ (0/6,0/7,0/7,0/9)	(0/4,0/4,0/7,0/7), ⟨(0/1,0/4,0/4,0/7),⟩ (0/2,0/4,0/4,0/6)	(0/0,0/1,0/2,0/3), ⟨(0/2,0/2,0/4,0/5),⟩ (0/1,0/1,0/3,0/4)

From step 2, the weighted normalized NTraFNs decision matrix is shown in Table 8.

Table 8. Weighted normalized NTraFNs decision matrix.

	e_1	e_2	e_3	e_4
o_1	(0/03,0/1,0/24,0/27) ⟨ (0/1,0/51,0/76,0/82) (0/36,0/65,0/84,0/88	(0/2,0/3,0/42,0/54), ⟨(0/37,0/55,0/76,0/92 (0/44,0/64,0/82,0/9)	(0/12,0/14,0/24,0/36 ⟨ (0/5,0/64,0/68,0/86) (0/1,0/28,0/36,0/65)	(0/2,0/3,0/42,0/42), ⟨(0/36,0/79,0/82,0/85 (0/28,0/44,0/51,0/58
o_2	(0/06,0/1,0/32,0/36), ⟨(0/37,0/51,0/8,0/88), (0/28,0/44,0/68,0/8)	(0/15,0/3,0/42,0/63), ⟨(0/44,0/6,0/72,0/88), (0/52,0/7,0/94,0/98)	(0/24,0/35,0/4,0/63) ⟨ (0/3,0/37,0/52,0/72) (0/28,0/37,0/52,0/65	(0/04,0/06,0/14,0/56 ⟨ (0/68,0/72,0/82,0/9) (0/1,0/28,0/44,0/64)
o_3	(0/15,0/35,0/64,0/81 ⟨ (0/28,0/58,0/8,0/94) (0/44,0/51,0/8,0/8)	(0/05,0/12,0/14,0/27 ⟨(0/44,0/75,0/84,0/92 (0/28,0/52,0/79,0/88	(0/18,0/21,0/32,0/45 ⟨ (0/1,0/46,0/52,0/72) (0/28,0/28,0/44,0/79	(0/0,0/12,0/21,0/63) ⟨ (0/28,0/79,0/82,0/9) (0/64,0/76,0/79,0/88
o_4	(0/0,0/1,0/24,0/63), ⟨(0/46,0/65,0/84,0/94 (0/52,0/65,0/8,0/96)	(0/25,0/3,0/49,0/72) ⟨(0/58,0/75,0/84,0/92 (0/6,0/76,0/91,0/96)	(0/3,0/42,0/48,0/81), ⟨ (0/3,0/55,0/6,0/72), (0/19,0/55,0/6,0/72)	(0/2,0/42,0/56,0/63) ⟨ (0/6,0/72,0/76,0/8), (0/28,0/44,0/51,0/58
o_5	(0/06,0/2,0/32,0/45) ⟨(0/37,0/72,0/84,0/97 (0/2,0/44,0/72,0/8)	(0/05,0/3,0/49,0/81) ⟨(0/44,0/65,0/72,0/92 (0/68,0/82,0/91,0/98	(0/24,0/28,0/56,0/63 ⟨ (0/1,0/46,0/52,0/79) (0/28,0/46,0/52,0/72	(0/0,0/06,0/14,0/21) ⟨(0/36,0/44,0/64,0/75 (0/19,0/28,0/51,0/64

Due to Eqs. 25 and 26 of step 3, the $\widetilde{NI}$ matrix obtained as:

$$\widetilde{NI} = \begin{bmatrix} \langle (0.00,0.10,0.24,0.27),(0.46,0.72,0.84,0.97),(0.52,0.65,0.84,0.96) \rangle \\ \langle (0.05,0.12,0.14,0.27),(0.58,0.75,0.84,0.92),(0.68,0.82,0.94,0.98) \rangle \\ \langle (0.12,0.14,0.24,0.36),(0.50,0.64,0.68,0.86),(0.28,0.55,0.60,0.79) \rangle \\ \langle (0.00,06,0.14,0.21),(0.68,0.70,0.82,0.90),(0.64,0.76,0.79,0.88) \rangle \end{bmatrix}'$$

Then, the calculations of D_i^{HS} and D_i^{ES} with regard to Eqs. 17, 19, 27, and 28 are summarized in Table 9.

Table 9. Surface-based weighted Hamming and Euclidean distances of alternatives.

Alts	o_1	o_2	o_3	o_4	o_5
D_i^{HS}	0.5180	0.5585	0.5430	0.7470	0.5980
D_i^{ES}	0.6448	0.6829	0.7504	0.8942	0.7950

Now, following steps 5 and 6, once the arrays of the relative assessment matrix have been found, Eq. 32 is used to find the assessment value of each option.

Table 10. Ranking alternatives based on RA matrix.

	Relative assessment matrix						
Alts	o_1	o_2	o_3	o_4	o_5	AS	Rank
o_1	0	-0.0382	-0.1056	-0.2483	-0.1500	-0.5421	5
o_2	0.0382	0	-0.0675	-0.2105	0.1120	-0.3517	4
o_3	0.1057	0.0675	0	-0.1432	-0.0445	-0.0145	3
o_4	0.2506	0.2121	0.1444	0	0.0995	0.7065	1
o_5	0.1505	0.1122	0.0446	-0.0989	0	0.2084	2

Finally, the highest assessment value of step 6 shows the most desirable material. The ranking order of all candidates is available in table 10.

5 -2 Sensitive analysis

As mentioned in the previous part, the evaluation scores and the ranking order corresponding to each alternative in the multi-criteria decision-making problem were obtained using the NTraFNs-CODAS method for $\varphi = 0.02$ (in step 5) and the specified weights ($\omega^{l\zeta} = \omega^{r\zeta} = 0.3, \omega^{l\eta} = \omega^{r\eta} = \omega^{l\theta} = \omega^{r\theta} = 0.1$) as parameters of the problem (Table 10). In this part, we want to investigate the effect of the sensitive analysis of φ and weights on the evaluation values and ranking of options for the material selection problem in the environment with neutrosophic trapezoidal fuzzy data.

A. Change of the parameter φ

The results of evaluating and ranking the options for different values are obtained in Table 11. As can be seen, although these changes have had a slight effect on the evaluation values, they have not had any effect on the final ranking of the options.

Table 11. Ranking alternatives based on sensitive analysis of φ values.

Alts	AS related to sensitive analysis of φ values					Rank
	0.01	0.02	0.03	0.04	0.05	
o_1	-0.5428	-0.5421	-0.5414	-0.5406	-0.5399	5
o_2	-0.3522	-0.3517	-0.3513	-0.3506	-0.3504	4
o_3	-0.0149	-0.0145	-0.0142	-0.0139	-0.0135	3
o_4	0.7051	0.7065	0.7079	0.7093	0.7108	1
o_5	0.2080	0.2084	0.2087	0.2091	0.2094	2

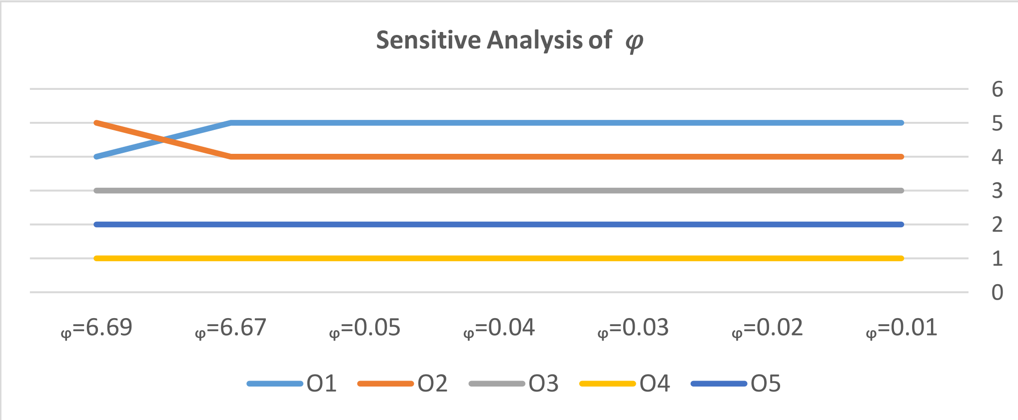

Figure 5. Ranking results based on sensitive analysis of φ.

Figure 5, clearly emphasizes the sameness of ranking results for sensitive analysis of φ.

Now, we desire to discuss on the admissible increase of φ, which does not any effect on the ranking result.

By increasing the value of φ to 6.67, the results of the evaluation options are obtained as follows:

$$\mathrm{AS}_1 = -0.0544,\ \mathrm{AS}_2 = -0.0541, \mathrm{AS}_3 = 0.2074, \mathrm{AS}_4 = 1.6446, \mathrm{AS}_5 = 0.4324$$

And for $\varphi = 6.69$, we have

$$\mathrm{AS}_1 = -0.0530, \mathrm{AS}_2 = -0.0533, \mathrm{AS}_3 = 0.2081, \mathrm{AS}_4 = 1.6474, \mathrm{AS}_5 = 0.4330$$

So, based on the analyses above, the stability and efficiency of the proposed NTraFNs-CODAS algorithm to changing of φ on $[0.01, 6.67]$ are observed.

B. Changes of the weights in proposed measures

The weights in Eqs 17 and 19 are the other parameters of the proposed NTraFNs-CODAS method, which can show the flexibility of the results. In the NTraFNs-CODAS method, the results are calculated according to the weights in the weighted Hamming and Euclidean distances. It is clear that by changing the weight coefficients related to the proposed measures Eqs (17, 19), the effectiveness of each term will be different in calculating the final assessment scores. As a result, various categories of solution are available for the decision maker. In Table 12, six cases are considered for weight variation; hence the values of evaluation and ranking of the options are obtained based on each case.

Table 12. Ranking alternatives based on sensitive analysis of different weights.

cases	alternatives				
	o_1	o_2	o_3	o_4	o_5
$Case\ \ 1{:}\,\omega^{l_\zeta} = \omega^{r_\zeta} = 0,$ $\omega^{l_\eta} = \omega^{r_\eta} = \omega^{l_\theta} = \omega^{r_\theta} = 0.25$	$0/3690$	$0/0609$	$0/3452$	$-1/2390$	$0/4766$
Ranking result	2	4	3	5	1
$Case\ \ 2{:}\,\omega^{l_\zeta} = \omega^{r_\zeta} = 0.1,$ $\omega^{l_\eta} = \omega^{r_\eta} = \omega^{l_\theta} = \omega^{r_\theta} = 0.2$	$-0/0392$	$-0/1507$	$0/2252$	$-0/4172$	$0/3841$
Ranking result	3	4	2	5	1
$Case\ \ 3{:}\,\omega^{l_\zeta} = \omega^{r_\zeta} = 0.2,$ $\omega^{l_\eta} = \omega^{r_\eta} = \omega^{l_\theta} = \omega^{r_\theta} = 0.15$	$-0/3104$	$-0/2646$	$0/1107$	$0/1667$	$0/2987$
Ranking result	5	4	3	2	1
$Case\ \ 4{:}\,\omega^{l_\zeta} = \omega^{r_\zeta} = 0.3,$ $\omega^{l_\eta} = \omega^{r_\eta} = \omega^{l_\theta} = \omega^{r_\theta} = 0.1$	$-0/5421$	$-0/3517$	$-0/0145$	$0/7065$	$0/2084$
Ranking result	5	4	3	1	2
$Case\ \ 5{:}\,\omega^{l_\zeta} = \omega^{r_\zeta} = 0.4,$ $\omega^{l_\eta} = \omega^{r_\eta} = \omega^{l_\theta} = \omega^{r_\theta} = 0.05$	$-0/7646$	$-0/4169$	$-0/1599$	$1/2569$	$0/1030$
Ranking result	5	4	3	1	2
$Case\ \ 6{:}\,\omega^{l_\zeta} = \omega^{r_\zeta} = 0.5,$ $\omega^{l_\eta} = \omega^{r_\eta} = \omega^{l_\theta} = \omega^{r_\theta} = 0$	$-1/0968$	$-0/3169$	$-0/3942$	$2/0162$	$-0/1670$
Ranking result	5	3	4	1	2

In addition, Figure 6 depicts the changes in the rank of the options concerning the variant in different modes (weights). In the suggested modes, the lowest fluctuation in the ranks has been observed for o_2 and o_5.

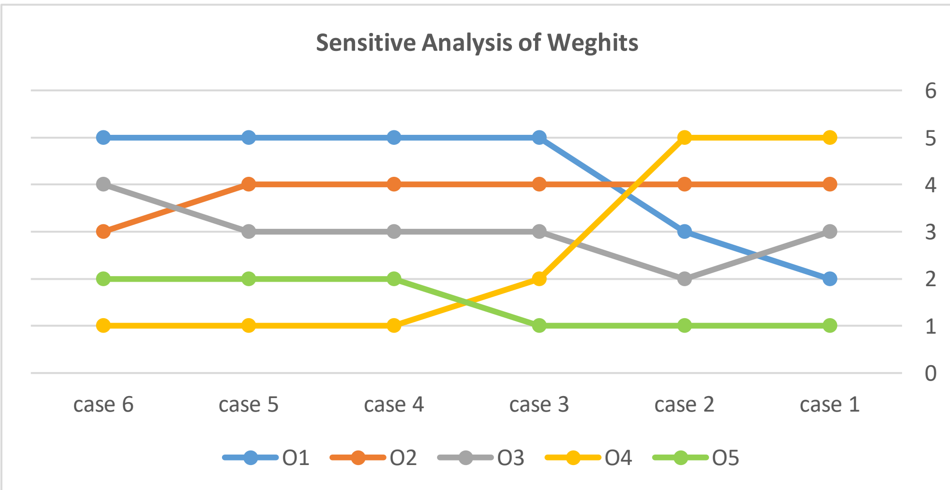

Figure 6. Ranking results based on different cases of weights.

5 -3 Comparative discussions

The performance of the proposed NTraFNs-CODAS Method is compared with some of the existing approaches (Biswas et al. [69], Pramanik et al. [70], Jana and Karaasslan [68], Suresh [71]) in this section. Researchers usually have investigated similarity measures and ranking methods to evaluate the alternatives in multi-criteria decision-making with NTraFNs. For example, Biswas et al. [69] extended the concepts of the Cosine similarity measure and weighted Cosine similarity measure according to an expected interval (EI) and expected value (EV) definitions with NtraFNs. Also, they find the desirable candidate for the MCDM problem based on this similarity measure. Later, Pramanik et al. [70] developed the TOPSIS method for MADM, where the weight information of attributes is incompletely known or completely unknown, under trapezoidal neutrosophic information for the first time. In another research, Jana and Karaasslan [68] introduced Dice and Jaccard similarity measures and weighted Dice and Jaccard similarity measures between NTraFNs for solving the MCDM method. Recently, Suresh [71] proposed a ranking strategy for MCDM under neutrosophic trapezoidal fuzzy numbers according to the Euclidean Distance measure and the centroid concept.

Differing from these studies, the proposed NTraFNs-CODAS Method is established based on two novel distance measures for the material selection problems. The results of applying these methods are summarized to the Table 13 and Figure 7.

Table 13. Ranking alternatives based on different methods.

Approaches		alternatives				
		o_1	o_2	o_3	o_4	o_5
Biswas et al.[69]	SC	$0/846$	$0/852$	$0/823$	$0/884$	$0/844$
	Ranking result	3	2	5	1	4
	SWC	$0/837$	$0/863$	$0/828$	$0/896$	$0/857$
	Ranking result	4	2	5	1	3
Pramanik et al. [70]	RCW	$0/349$	$0/433$	$0/407$	$0/784$	$0/417$
	Ranking result	5	2	4	1	3
Jana and Karaasslan[68]	SWD	$0/876$	$0/904$	$0/881$	$0/927$	$0/873$
	Ranking result	3	2	4	1	5
	SWJ	$0/728$	$0/762$	$0/719$	$0/803$	$0/722$
	Ranking result	3	2	5	1	4
Suresh [71]	R	$0/228$	$0/199$	$0/258$	$0/191$	$0/230$
	Ranking result	3	2	5	1	4
Proposed method	NTraFNs-CODAS	$-0/542$	$-0/352$	$-0/015$	$0/707$	$0/208$
	Ranking result	5	4	3	1	2

As can be seen, although the methods do not have the same performance in ranking all the options, they all choose option 2 as the best option. According to Figure 7, it can be said that the proposed method has the most similarity in the ranking results with the method presented by Pramanik et al. [70].

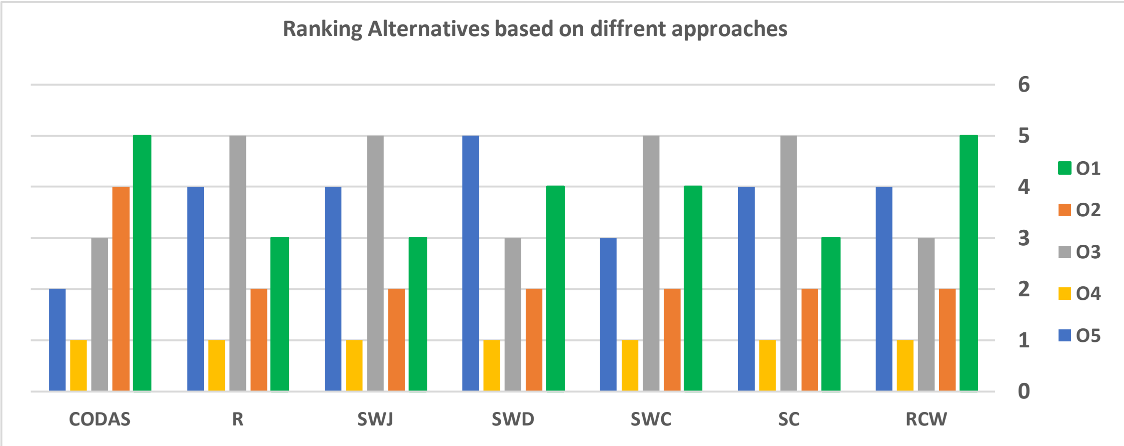

Figure 7. Ranking alternatives based on different methods.

6. Conclusions

The distance measures that can investigate the discriminationn between two neutrosophic trapezoidal fuzzy numbers do not gain indispensable comprehensiveness in the literature from the various perspectives. There is much less research in this field compared to the discrete neutersophic numbers. However, due to the greater flexibility in continuous neutrosophic numbers, examining issues under this type of numbers can be more preferred by decision-makers. Therefore, in this research, two surface-based distances were presented. The proposed weighted distance measures not only establish the basic principles of the measure but also apply to some logical properties of the measure, as shown in Example 1. Therefore, it can be properly used in distance-based decision-making algorithms. In the following, the CODAS algorithm was considered under neutrosophic trapezoidal fuzzy data for the first time in this manuscript. The effectiveness of the proposed NTraFNs-CODAS algorithm was shown for solving MCDM. According to Table 10, the ranking of the options is as follows: $o_4 \succ o_5 \succ o_3 \succ o_2 \succ o_1$. The sensitivity analysis of the threshold parameter (φ) showed that the ranking of alternatives remains constant until the value of φ is selected from the [0.01,6.65]. However, it cannot be expected that the ranking of the options will remain constant with the changes in the weighting coefficients (cases 1 to 6). Table 12 and the Figure 4 interpret the results of the impact of weight changes in the ranking of options. In addition, a comparative analysis of the NTraFNs-CODAS method with some existing methods demonstrates that the performance of our method is most similar to Pramanik et al. [70]. As suggestions for future research, the following can be considered:

1-Explore more features and properties for surface-based distance measures.

2- The conceptual structure of the method should be developed to other fuzzy extensions from a theoretical and practical point of view.

3- The suggested distance measures should be used in other decision-making methods based on distance, and its results should be compared with the method.

4- Weighted distance measures based on the area of surfaces can be used in other fields related to optimization, such as clustering, classification, medical diagnosis, and location problems.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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A Hybrid Approach to Micro Vague Topological Space via Neutrosophic Topological Space

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Abstract. Plenty of topologists are exploring and discovering multiple forms of topological spaces. The chief objective of the current enquiry is to establish and evaluate an original hybrid topological space named Neutrosophic Micro Vague Topological Space. When contrasted with distinctive fuzzy sets, Neutrosophic Micro Vague sets give more adaptive framework for dealing with uncertainties and obscurity because they allow more nuanced portrayal of the multitude of the elements of inconsistencies. Some of the basic definitions and operations on Neutrosophic Micro Vague Sets are defined and examined with numerical examples. Furthermore,some of the basic algebraic properties of Neutrosophic Micro Vague Sets are described and investigated with appropriate examples.

Keywords: Neutrosophic Micro Vague set; Neutrosophic Micro Vague Topology; Neutrosophic Micro Vague Topological Space; Absolute Neutrosophic Micro Vague set; Null Neutrosophic Micro Vague Set.

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1. **Introduction**

Fuzzy sets (FS) which were first coined by L.A. Zadeh [1] in 1965 are the most required perspectives in modern mathematics. C.L. Chang [2] pioneered a version of fuzzy topology in 1967. Atanassov [3] recommended the Intuitionistic Fuzzy Set (IFS) in 1986, which has been widely utilised in various fields of mathematics. Around 1993, Gau and Buehrer [4] identified Vague sets (VS) as a further development of FS research and they are considered as a unique instance of context-aware FS. Bustince. H along with Burillo. P [15] demonstrated that VSs are IFSs in 1996. Florentin Smarandache [5], [6] an eminent mathematician and analyst, premiered neutrosophy as a broadening of fuzzy set theory. The referrals "neutrosophy" speaks to the scrutiny of not just truth and falsehood as essential components but also indeterminacy which deals with uncertainty or unpredictability in precisely the same way as fuzzy systems operate. Mathematically a Neutrosophic set (NS) has been assigned over the universal set X and contains three subsets. 1. Truth component: this subset reflects the components that are unquestionably part of the set. 2. Indeterminacy component: this subset represents elements that are unsure whether or not they belong to the set. 3. Falsity component: this subset indicates the elements which are unambiguously not in the set. The Neutrosophic Two Fold Algebra was first presented by F. Smarandache [17] in 2024.

Implementing the concept of neutrosophic frequency and neutrosophic relative frequency distribution, Adebisi S. A. and Broumi S. [16] have examined the educational progress of a group of students in their primary disciplines. Shawkat Alkhazaleh [7] formed the Neutrosophic Vague set (NVS) idea during 2015 as an amalgamation of NS and VS. He defined the basic operations for NVSs such as Union, Intersection, complement and inclusion. These operations are intended to manage the triple membership degrees while retaining the traits of unpredictability and indeterminate nature. NVSs have specific features that set them apart from other set enhancements. In 2024, Smarandache et.al [18] conducted an evaluation of Blockchain Cybersecurity Based on Tree Soft and Opinion Weight Criteria Method under Uncertainty Climate.

M. Lellis Thivagar [9], [13] coined the Nano topology (NT) and Neutrosophic Nano Topology (NNT) in the year 2013 and 2018 respectively. NT is centred on the concepts of lower approximation, higher approximation and boundary region which was brought by Z. Pawlak [8]. S. Chandrasekar [10] constructed Micro Topology (MT) later in 2019 by employing the basic extension idea on NT. MT affords a lens that enables mathematicians and scientists to study and comprehend the multifaceted intricacies of spaces, structures and networks particularly where smaller factors are important. Emergent features at tiny sizes can be shown by MT. MT entails scale relying analysis in which the features of a space are explored at different degrees of detail or resolution. MT is also attributed to more technical topics like sheaf theory and homotopy theory which deal with local patterns and continual deviations respectively. In an environment of metric spaces, MT may entail investigating neighbourhood qualities convergence at a point and other local aspects. In 2023, Vargees Vahini T and Trinita Pricilla M [11] established the novel topological space named Micro Vague Topological Space. Mary Margaret A et.al [12] in 2021 pioneered Neutrosophic Vague Nano Topological Space and have studies some of the basic characteristics.

In this paper, we suggested an innovative topology called the Neutrosophic Micro Vague Topology. Some novel sets in Neutrosophic Micro Vague Topological Space are introduced and discussed. Additionally, using numerical examples certain fundamental definitions, operations and some of the basic algebraic characteristics on Neutrosophic Micro Vague Sets are addressed and explained. Learning about and employing these qualities enables academics and practitioners to capitalise on Neutrosophic Micro Vague sets in several types of domains contributing to more robust and flexible modelling of highly complex and volatile data. It can be useful in an assortment of domains including pure mathematics, mathematical physics and some areas of computer science where fine-grained spatial relationships are noteworthy. The particular application and methodologies used may differ depending on the situation in which Neutrosophic Micro Vague topology is used.

2. **Preliminaries**

Definition 2.1. [4] A VS $\mathfrak{F}$ in the universe of discourse Λ is characterized by a truth membership function $\vartheta_{\mathfrak{F}}$ and a falsity membership function $\lambda_{\mathfrak{F}}$ as follows: $\vartheta_{\mathfrak{F}} : \Lambda \to [0,1]$; $\lambda_{\mathfrak{F}} : \Lambda \to [0,1]$ and $\vartheta_{\mathfrak{F}} + \lambda_{\mathfrak{F}} \leq 1$ where $\vartheta_{\mathfrak{F}}(\grave{m})$ is a lower bound on the grade of membership of $\grave{m}$ derived from the evidence for $\grave{m}$ and $\lambda_{\mathfrak{F}}(\grave{m})$ is a lower bound on the grade of membership of the negation of $\grave{m}$ derived from the evidence against $\grave{m}$. The Vague set $\mathfrak{F}$ is written as $A = \{\langle \grave{m}, \vartheta_{\mathfrak{F}}(\grave{m}), 1-\lambda_{\mathfrak{F}}(\grave{m})\rangle \,|\grave{m} \in \Lambda\}$.

Definition 2.2. [6] Let Λ be a non-empty set and $\mathfrak{I}$ be the unit interval $[0,1]$. A *Neut.Set* is an object of the form $\mathfrak{D} = \{\langle \grave{m}, \pi_{\mathfrak{D}}(\grave{m}), \phi_{\mathfrak{D}}(\grave{m}), \varphi_{\mathfrak{D}}(\grave{m}); \grave{m} \in \Lambda\rangle\}$ where $\pi_{\mathfrak{D}}(\grave{m}), \phi_{\mathfrak{D}}(\grave{m}), \varphi_{\mathfrak{D}}(\grave{m}) \in [0,1]$ with $0 \leq \pi_{\mathfrak{D}}(\grave{m}) + \phi_{\mathfrak{D}}(\grave{m}) + \varphi_{\mathfrak{D}}(\grave{m}) \leq 3 \;\; \forall\, \grave{m} \in \Lambda$. Here, $\pi_{\mathfrak{D}}(\grave{m}), \phi_{\mathfrak{D}}(\grave{m})$ *and* $\varphi_{\mathfrak{D}}(\grave{m})$ are respectively denote the Degree of truth membership, Degree of indeterminacy membership and Degree of falsity membership.

Definition 2.3. [7] A Neut. Vag. Set $\mathfrak{Z}$ on the universe of discourse Λ is written as $\mathfrak{Z} = \left\{\left\langle \grave{m}; \hat{\pi}_{\mathfrak{Z}}(\grave{m}), \hat{\phi}_{\mathfrak{Z}}(\grave{m}), \hat{\varphi}_{\mathfrak{Z}}(\grave{m})\right\rangle |\grave{m} \in \Lambda\right\}$ whose truth membership, indeterminacy membership and false membership function are defined as follows:

$$\hat{\pi}_{\mathfrak{Z}}(\grave{m}) = \left[\pi^-, \pi^+\right], \; \hat{\phi}_{\mathfrak{Z}}(\grave{m}) = \left[\phi^-, \phi^+\right], \;\; \hat{\varphi}_{\mathfrak{Z}}(\grave{m}) = \left[\varphi^-, \varphi^+\right]$$

Where,

(1) $\pi^+ = 1 - \varphi^-$
(2) $\varphi^+ = 1 - \pi^-$ and
(3) ${}^-0 \leq \left(\pi^-\right)^2 + \left(\phi^-\right)^2 + \left(\varphi^-\right)^2 \leq 2^+$.

Definition 2.4. [11] Assume $(S, \sigma_Y(\mathfrak{Z}))$ a Nano.Vag. topological space. Let $\theta_Y(\mathfrak{Z}) = \{\mathfrak{H} \cup (\mathfrak{H}' \cap \theta)$: $\mathfrak{H}, \mathfrak{H}' \in \sigma_Y(\mathfrak{Z})\}$. Then $\eta_Y(\mathfrak{Z})$ is termed as Mic.Vag. topology (Shortly $\mathcal{MV}$ Topology) of $\sigma_Y(\mathfrak{Z})$ by θ where $\theta \notin \sigma_Y(X)$; Then, $\theta_Y(X)$ fulfills the criteria listed here:

(1) $0_{\mathcal{MV}}, 1_{\mathcal{MV}} \in \theta_Y(\mathfrak{Z})$
(2) Arbitrary union of any sub collection of $\theta_Y(\mathfrak{Z})$ is in $\theta_Y(\mathfrak{Z})$
(3) Finite intersection of sub collection of $\theta_Y(\mathfrak{Z})$ is in $\theta_Y(\mathfrak{Z})$.

The triplet $(S, \sigma_Y(\mathfrak{Z}), \theta_Y(\mathfrak{Z}))$ is called the Micro Vague Topological Space. The elements of $\theta_Y(\mathfrak{Z})$ are called Micro Vague open sets and the complement of a Micro Vague Open set is called a Micro Vague Closed set.

Definition 2.5. [12] Let $\acute{\mathfrak{L}}$ be a non-empty set and $\mathfrak{R}$ be an equivalence relation on $\acute{\mathfrak{L}}$. Let $\mathfrak{S}$ be a Neut.Vag. set in $\acute{\mathfrak{L}}$. If the collection $\vartheta_{\mathfrak{R}}(\mathfrak{S}) = \{0_{NV}, 1_{NV}, \underline{\mathrm{NV}(\mathfrak{S})}, \overline{\mathrm{NV}(\mathfrak{S})}, B_{NV}(\mathfrak{S})\}$ satisfies the following axioms:

(1) $0_{nv}, 1_{nv} \in \vartheta_{\mathfrak{R}}(\mathfrak{S})$.
(2) Arbitrary union of any sub collection of $\vartheta_{\mathfrak{R}}(\mathfrak{S})$ is in $\vartheta_{\mathfrak{R}}(\mathfrak{S})$.
(3) Finite intersection of sub collection of $\vartheta_{\mathfrak{R}}(\mathfrak{S})$ is in $\vartheta_{\mathfrak{R}}(\mathfrak{S})$.

then, $\vartheta_{\mathfrak{R}}(\mathfrak{S})$is called the NVNT and $(\acute{\mathfrak{L}}, \vartheta_{\mathfrak{R}}(\mathfrak{S}))$ is called the NVNTS. The elements of $\vartheta_{\mathfrak{R}}(\mathfrak{S})$ are called NVNOS and the complement of it is called NVNCS.

3. Proposed Neutrosophic Micro Vague Topological Space

Definition 3.1. Let $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}))$ be a $\mathcal{NVNTS}$. Let $\Omega_{\mathcal{Y}}(\mathfrak{S}) = \{\Phi \cup (\Phi' \cap \Omega) : \Omega \notin \Psi_{\mathcal{Y}}(\mathfrak{S})\}$. Then $\Omega_{\mathcal{Y}}(\mathfrak{S})$ is called the Neutrosophic Micro Vague Topology (*shortly* $\mathcal{NMVT}$) of $\vartheta_{\mathfrak{R}}(\mathfrak{S})$ by η if it satisfies the following axioms:

(1) $0_{\mathcal{NMV}}, 1_{\mathcal{NMV}} \in \Omega_{\mathcal{Y}}(\mathfrak{S})$.
(2) The union of the elements of any sub collection of $\Omega_{\mathcal{Y}}(\mathfrak{S})$ is in $\Omega_{\mathcal{Y}}(\mathfrak{S})$.
(3) The intersection of the elements of any finite sub collection of $\Omega_{\mathcal{Y}}(\mathfrak{S})$ is in $\Omega_{\mathcal{Y}}(\mathfrak{S})$.

The triplet $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ is called the *Neutrosophic Micro Vague Topological Space* (*denoted by* $\mathcal{NMVTS}$). The elements of $\Omega_{\mathcal{Y}}(\mathfrak{S})$ are called $\mathcal{NMVOS}$ and the complement is called as $\mathcal{NMVCS}$.

Example 3.2. Let $\acute{\mathfrak{L}} = \{\alpha, \beta, \gamma, \delta\}$ be the Universe. Let $\acute{\mathfrak{L}}/\mathcal{Y} = \{\{\alpha, \delta\}, \{\beta, \gamma\}\}$ be an equivalence relation on $\acute{\mathfrak{L}}$. Let $\mathfrak{S} = \{<\alpha, [0.3, 0.5], [0.2, 0.6], [0.8, 0.9]>, <\beta, [0.6, 0.7], [0.5, 0.7], [0.2, 0.5]>, <\gamma, [0.2, 0.5], [0.9, 0.9], [0.3, 0.4]>, <\delta, [0.6, 0.8], [0.5, 0.9], [0.3, 0.8]>\}$ be a subset of $\acute{\mathfrak{L}}$. Then, $\Psi_{\mathcal{Y}}(\mathfrak{S}) = \{0_{NV}, 1_{NV}, \{<\alpha, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.9, 0.9], [0.3, 0.5]>, <\gamma, [0.2, 0.5], [0.9, 0.9], [0.3, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}, \{<\alpha, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>, <\beta, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4]>, <\gamma, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4]>, <\delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>\}, \{<\alpha, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>, <\beta, [0.3, 0.5], [0.5, 0.7][0.2, 0.5]>, <\gamma, [0.3, 0.5], [0.5, 0.7], [0.2, 0.5]>, <\delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>\}\}$ is a $\mathcal{NVNT}$ on $\acute{\mathfrak{L}}$. Let $\Omega = \{<\alpha, [0.2, 0.7], [0.1, 0.6], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>, <\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}$. Then, $\Omega_{\mathcal{Y}}(\mathfrak{S}) = \{0_{NMV}, 1_{NMV}, \{<\alpha, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.9, 0.9], [0.3, 0.5]>, <\gamma, [0.2, 0.5], [0.9, 0.9], [0.3, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}, \{<\alpha, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>,$

$<\beta, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4]>$,$<\gamma, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4]>$, $<\delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>$},{$<\alpha, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>$, $<\beta, [0.3, 0.5], [0.5, 0.7], [0.2, 0.5]>$, $<\gamma, [0.3, 0.5], [0.5, 0.7], [0.2, 0.5]>$, $<\delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>$},{$<\alpha, [0.2, 0.7], [0.1, 0.6], [0.8, 0.9]>$,$<\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>$,$<\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>$,$<\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>$},{$<\alpha, [0.6, 0.8], [0.1, 0.6], [0.3, 0.8]>$, $<\beta, [0.3, 0.5], [0.5, 0.7], [0.2, 0.5]>$,$<\gamma, [0.3, 0.5], [0.5, 0.7], [0.2, 0.5]>$, $<\delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>$},{$<\alpha, [0.2, 0.5], [0.5, 0.9], [0.8, 0.9]>$,$<\beta, [0.2, 0.5], [0.9, 0.9], [0.3, 0.5]>$, $<\gamma, [0.1, 0.4], [0.9, 0.9], [0.3, 0.5]>$, $<\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>$},{$<\alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9]>$, $<\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>$, $<\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>$, $<\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>$}, {$<\alpha, [0.3, 0.7], [0.1, 0.6], [0.8, 0.9]>$, $<\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>$, $<\gamma, [0.2, 0.5], [0.9, 0.9], [0.2, 0.5]>$, $<\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>$}, {$<\alpha, [0.3, 0.7], [0.2, 0.6], [0.8, 0.9]>$, $<\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>$, $<\gamma, [0.2, 0.5], [0.9, 0.9], [0.2, 0.5]>$,$<\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>$},{$<\alpha, [0.6, 0.8], [0.1, 0.6], [0.3, 0.8]>$,$<\beta, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4]>$, $<\gamma, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4]>$, $<\delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]>$}} is called the $\mathcal{NMVT}$ on $\mathfrak{S}$. The triplet $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ is called the $\mathcal{NMVTS}$.

Definition 3.3. Let $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ be two $\mathcal{NMV}$ sets in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. If $\forall \nu_s \in \acute{\mathfrak{L}}$, $\widehat{\Gamma}_P(\nu_s) \leq \widehat{\Gamma}_Q(\nu_s)$, $\widehat{\Delta}_P(\nu_s) \geq \widehat{\Delta}_Q(\nu_s)$, $\widehat{\Upsilon}_P(\nu_s) \geq \widehat{\Upsilon}_Q(\nu_s)$ then the $\mathcal{NMV}$ set $P_{\mathcal{NMV}}$ is included or contained in the $\mathcal{NMV}$ set $Q_{\mathcal{NMV}}$, denoted by $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ where $1 \leq s \leq n$.

Remark 3.4. Here, the set $P_{\mathcal{NMV}} = \{ < \nu_s, \widehat{\Gamma}_P(\nu_s), \widehat{\Delta}_P(\nu_s), \widehat{\Upsilon}_P(\nu_s) > \}$ denotes the $\mathcal{NMV}$ set in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ where $\widehat{\Delta}_P(\nu_s) = \left[\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)\right]$, $\widehat{\Delta}_P(\nu_s) = \left[\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)\right]$ *and* $\widehat{\Upsilon}_P(\nu_s) = \left[\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)\right]$.

Example 3.5. Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{ < \alpha, [0.2, 0.5], [0.5, 0.9], [0.8, 0.9] >, < \beta, [0.2, 0.5], [0.9, 0.9], [0.3, 0.5] >, < \gamma, [0.1, 0.4], [0.9, 0.9], [0.3, 0.5] >, < \delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9] >\}$ and $Q_{\mathcal{NMV}} = \{ < \alpha, [0.6, 0.8], [0.1, 0.6], [0.3, 0.8] >, < \beta, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4] >, < \gamma, [0.6, 0.7], [0.5, 0.7], [0.2, 0.4] >, < \delta, [0.6, 0.8], [0.2, 0.6], [0.3, 0.8] >\}$ be two $\mathcal{NMV}$ sets. Here, $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$.

Definition 3.6. Let $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ be two $\mathcal{NMV}$ in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. If $\forall \nu_s \in \acute{\mathfrak{L}}$, $\widehat{\Gamma}_P(\nu_s) = \widehat{\Gamma}_Q(\nu_s)$, $\widehat{\Delta}_P(\nu_s) = \widehat{\Delta}_Q(\nu_s)$, $\widehat{\Upsilon}_P(\nu_s) = \widehat{\Upsilon}_Q(\nu_s)$, then the $\mathcal{NMV}$ set $P_{\mathcal{NMV}}$ is equal to the $\mathcal{NMV}$ set $Q_{\mathcal{NMV}}$, denoted by $P_{\mathcal{NMV}} = Q_{\mathcal{NMV}}$ where $1 \leq s \leq n$.

Definition 3.7. The complement of a $\mathcal{NMV}$ set $P_{\mathcal{NMV}}$ in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ denoted by $P^C_{\mathcal{NMV}}$ is defined as $P^C_{\mathcal{NMV}} = \{< \nu_s, [1 - \Gamma_P^+(\nu_s),\ 1 - \Gamma_P^-(\nu_s)], [1 - \Delta_P^+(\nu_s),\ 1 - \Delta_P^-(\nu_s)], [1 - \Upsilon_P^+(\nu_s),\ 1 - \Upsilon_P^-(\nu_s)] >\}$.

Example 3.8. Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{ < \alpha, [0.3, 0.7], [0.1, 0.6], [0.8, 0.9] >, < \beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5] >,$

$< \gamma, [0.2, 0.5], \ [0.9, 0.9], \ [0.2, 0.5] >, < \delta, \ [0.3, 0.5], [0.5, 0.9], [0.8, 0.9] >\}$ be a $\mathcal{NMV}$ set. Then the complement of P is as follows:

$$P^C_{\mathcal{NMV}} = \{ < \alpha, [0.3, 0.7], \ [0.4, 0.9], \ [0.1, 0.2] >, < \beta, \ [0.5, 0.8], \ [0.1, 0.3], \ [0.5, 0.7] >, \\ < \gamma, [0.5, 0.8], \ [0.1, 0.1], \ [0.5, 0.8] >, < \delta, \ [0.5, 0.7], [0.1, 0.5], [0.1, 0.2] >\}.$$

Definition 3.9. Let $P_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. If $\forall \nu_s \in \acute{\mathfrak{L}}$, $\widehat{\Gamma}_P(\nu_s) = [1, 1]$, $\widehat{\Delta}_P(\nu_s) = [0, 0]$ and $\widehat{\Upsilon}_P(\nu_s) = [0, 0]$, then $P_{\mathcal{NMV}}$ is called Absolute $\mathcal{NMV}$ set where $1 \le s \le n$.

Definition 3.10. Let $P_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. If $\forall \nu_s \in \acute{\mathfrak{L}}$, $\widehat{\Gamma}_P(\nu_s) = [0, 0]$, $\widehat{\Delta}_P(\nu_s) = [1, 1]$ and $\widehat{\Upsilon}_P(\nu_s) = [1, 1]$, then $P_{\mathcal{NMV}}$ is called Null $\mathcal{NMV}$ set where $1 \le s \le n$.

Definition 3.11. The Union of two $\mathcal{NMV}$ sets $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ is a $\mathcal{NMV}$ set R which is written as $R = P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$ whose $\widehat{\Gamma}_R(\nu_s), \widehat{\Delta}_R(\nu_s), \widehat{\Upsilon}_R(\nu_s)$ are defined $\forall \ \nu_s \in \acute{\mathfrak{L}}$ where $1 \le s \le n$ as follows:

$$\widehat{\Gamma}_R(\nu_s) = \left[\bigvee\left(\Gamma^-_P(\nu_s), \ \Gamma^-_Q(\nu_s)\right), \ \bigvee\left(\Gamma^+_P(\nu_s), \ \Gamma^+_Q(\nu_s)\right)\right]$$
$$\widehat{\Delta}_R(\nu_s) = \left[\bigwedge\left(\Delta^-_P(\nu_s), \ \Delta^-_Q(\nu_s)\right), \ \bigwedge\left(\Delta^+_P(\nu_s), \ \Delta^+_Q(\nu_s)\right)\right]$$
$$\widehat{\Upsilon}_R(\nu_s) = \left[\bigwedge\left(\Upsilon^-_P(\nu_s), \ \Upsilon^-_Q(\nu_s)\right), \ \bigwedge\left(\Upsilon^+_P(\nu_s), \ \Upsilon^+_Q(\nu_s)\right)\right]$$

Example 3.12. Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{ < \alpha, [0.6, 0.8], \ [0.2, 0.6], \ [0.3, 0.8] >, < \beta, \ [0.6, 0.7], \ [0.5, 0.7], \ [0.2, 0.4] >, < \gamma, [0.6, 0.7], \ [0.5, 0.7], \ [0.2, 0.4] >, < \delta, \ [0.6, 0.8], [0.2, 0.6], [0.3, 0.8] >\}$ and $Q_{\mathcal{NMV}} = \{ < \alpha, [0.6, 0.8], \ [0.1, 0.6], \ [0.3, 0.8] >, < \beta, \ [0.3, 0.5], \ [0.5, 0.7], \ [0.2, 0.5] >, < \gamma, [0.3, 0.5], \ [0.5, 0.7], \ [0.2, 0.5] >, < \delta, \ [0.6, 0.8], [0.2, 0.6], [0.3, 0.8] >\}$ be two $\mathcal{NMV}$ sets. Then the union $R_{\mathcal{NMV}} = P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$ is given as follows:

$$R_{\mathcal{NMV}} = \{ < \alpha, [0.6, 0.8], \ [0.1, 0.6], \ [0.3, 0.8] >, < \beta, \ [0.6, 0.7], \ [0.5, 0.7], \ [0.2, 0.4] >, \\ < \gamma, [0.6, 0.7], \ [0.5, 0.7], \ [0.2, 0.4] >, < \delta, \ [0.6, 0.8], [0.2, 0.6], [0.3, 0.8] >\}.$$

Definition 3.13. The Intersection of two $\mathcal{NMV}$ sets $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ is a $\mathcal{NMV}$ set $S_{\mathcal{NMV}}$ which is written as $S_{\mathcal{NMV}} = P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}$ whose $\widehat{\Gamma}_S(\nu_s), \widehat{\Delta}_S(\nu_s), \widehat{\Upsilon}_S(\nu_s)$ are defined $\forall \ \nu_s \in \acute{\mathfrak{L}}$ where $1 \le s \le n$ as follows:

$$\widehat{\Gamma}_S(\nu_s) = \left[\bigwedge\left(\Gamma^-_P(\nu_s), \ \Gamma^-_Q(\nu_s)\right), \ \bigwedge\left(\Gamma^+_P(\nu_s), \ \Gamma^+_Q(\nu_s)\right)\right]$$
$$\widehat{\Delta}_S(\nu_s) = \left[\bigvee\left(\Delta^-_P(\nu_s), \ \Delta^-_Q(\nu_s)\right), \ \bigvee\left(\Delta^+_P(\nu_s), \ \Delta^+_Q(\nu_s)\right)\right]$$
$$\widehat{\Upsilon}_S(\nu_s) = \left[\bigvee\left(\Upsilon^-_P(\nu_s), \ \Upsilon^-_Q(\nu_s)\right), \ \bigvee\left(\Upsilon^+_P(\nu_s), \ \Upsilon^+_Q(\nu_s)\right)\right]$$

Example 3.14. Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{ <\alpha, [0.6, 0.8], \ [0.2, 0.6], \ [0.3, 0.8]>, <\beta, \ [0.6, 0.7], \ [0.5, 0.7], \ [0.2, 0.4]>, <\gamma, [0.6, 0.7], \ [0.5, 0.7], \ [0.2, 0.4]>, <\delta, \ [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]> \}$ and $Q_{\mathcal{NMV}} = \{ <\alpha, [0.6, 0.8], \ [0.1, 0.6], \ [0.3, 0.8]>, <\beta, \ [0.3, 0.5], \ [0.5, 0.7], \ [0.2, 0.5]>, <\gamma, [0.3, 0.5], \ [0.5, 0.7], \ [0.2, 0.5]>, <\delta, \ [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]> \}$ be two $\mathcal{NMV}$ sets. Then the intersection $S_{\mathcal{NMV}} = P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}$ is given as follows:

$$\begin{aligned} S_{\mathcal{NMV}} = \{ & <\alpha, [0.6, 0.8], \ [0.2, 0.6], \ [0.3, 0.8]>, <\beta, \ [0.3, 0.5], \ [0.5, 0.7], \ [0.2, 0.5]>, \\ & <\gamma, [0.3, 0.5], \ [0.5, 0.7], \ [0.2, 0.5]>, <\delta, \ [0.6, 0.8], [0.2, 0.6], [0.3, 0.8]> \}. \end{aligned}$$

Definition 3.15. Let $\{P_{s_{\mathcal{NMV}}} : s \in D\}$ where $D = \{1, \ 2, \ \ldots, \ n\}$ be an arbitrary family of $\mathcal{NMV}$ sets. Then

(1) $\cup P_{s_{\mathcal{NMV}}} =$

$$\left\{ \left\langle \begin{array}{c} \grave{m}; \left[\bigvee_{s \in D} \left(\Gamma^-_{P_{s_{\mathcal{NMV}}}} \right), \bigvee_{s \in D} \left(\Gamma^+_{P_{s_{\mathcal{NMV}}}} \right) \right], \\ \left[\bigwedge_{s \in D} \left(\Delta^-_{P_{s_{\mathcal{NMV}}}} \right), \bigwedge_{s \in D} \left(\Delta^+_{P_{s_{\mathcal{NMV}}}} \right) \right], \left[\bigwedge_{s \in D} \left(\Upsilon^-_{P_{s_{\mathcal{NMV}}}} \right), \bigwedge_{s \in D} \left(\Upsilon^+_{P_{s_{\mathcal{NMV}}}} \right) \right] \end{array} \right\rangle \right\}$$

(2) $\cap P_{s_{\mathcal{NMV}}} =$

$$\left\{ \left\langle \grave{m}; \begin{array}{c} \left[\bigwedge_{s \in D} \left(\Gamma^-_{P_{s_{\mathcal{NMV}}}} \right), \bigwedge_{s \in D} \left(\Gamma^+_{P_{s_{\mathcal{NMV}}}} \right) \right], \\ \left[\bigvee_{s \in D} \left(\Delta^-_{P_{s_{\mathcal{NMV}}}} \right), \bigvee_{s \in D} \left(\Delta^+_{P_{s_{\mathcal{NMV}}}} \right) \right], \left[\bigvee_{s \in D} \left(\Upsilon^-_{P_{s_{\mathcal{NMV}}}} \right), \bigvee_{s \in D} \left(\Upsilon^+_{P_{s_{\mathcal{NMV}}}} \right) \right] \end{array} \right\rangle \right\}$$

Proposition 3.16. *Let $P_{\mathcal{NMV}}$, $Q_{\mathcal{NMV}}$, $R_{\mathcal{NMV}}$ and $S_{\mathcal{NMV}}$ be $\mathcal{NMV}$ sets in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$.*

(1) *If $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}} \subseteq S_{\mathcal{NMV}}$, then*
   (a) $(P_{\mathcal{NMV}} \cup R_{\mathcal{NMV}}) \subseteq (Q_{\mathcal{NMV}} \cup S_{\mathcal{NMV}})$
   (b) $(P_{\mathcal{NMV}} \cap R_{\mathcal{NMV}}) \subseteq (Q_{\mathcal{NMV}} \cap S_{\mathcal{NMV}})$

(2) *If $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ and $P_{\mathcal{NMV}} \subseteq R_{\mathcal{NMV}}$, then $P_{\mathcal{NMV}} \subseteq (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})$*

(3) *If $P_{\mathcal{NMV}} \subseteq R_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}} \subseteq R_{\mathcal{NMV}}$, then $(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \subseteq R_{\mathcal{NMV}}$*

(4) *If $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}} \subseteq R_{\mathcal{NMV}}$, then $P_{\mathcal{NMV}} \subseteq R_{\mathcal{NMV}}$*

(5) *If $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$, then $\overline{Q_{\mathcal{NMV}}} \subseteq \overline{P_{\mathcal{NMV}}}$*

(6) $\overline{1}_{\mathcal{NMV}} = 0_{\mathcal{NMV}}$

(7) $\overline{0}_{\mathcal{NMV}} = 1_{\mathcal{NMV}}$

Proof. Proof is obvious. □

Corollary 3.17. *Let $P_{s_{\mathcal{NMV}}}(s \in D)$ and $Q_{\mathcal{NMV}}$ be $\mathcal{NMV}$ sets in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then*

(1) $P_{s\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ *for each* $s \in D$ *implies that* $\cup(P_{s\mathcal{NMV}}) \subseteq Q_{\mathcal{NMV}}$
(2) $Q_{\mathcal{NMV}} \subseteq P_{s\mathcal{NMV}}$ *for each* $s \in D$ *implies that* $Q_{\mathcal{NMV}} \subseteq \cap(P_{s\mathcal{NMV}})$

Proof. Proof is obvious. □

Remark 3.18. (1) In $\mathcal{NMVTS}$, the boundary region cannot be empty.
(2) Let $\{\eta_i | i \in \mathrm{l}\}$ be the family of $\mathcal{NMVT}s$ on X_i, then $\bigcap_{i\in \mathrm{l}} \eta_i$ is a $\mathcal{NMVT}$ in X.
(3) Let $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ and $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \eta'_{\mathfrak{R}}(\mathfrak{S}))$ be two $\mathcal{NMVTS}$s over X. Then $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S})) \cup \eta'_{\mathfrak{R}}(\mathfrak{S}))$ need not to be a $\mathcal{NMVTS}$.

4. Properties of Neutrosophic Micro Vague Sets

Theorem 4.1. ***(Idempotent law)*** *For any non-empty* $\mathcal{NMV}$ *set* $P_{\mathcal{NMV}}$ *in* $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$*,*

(1) $P_{\mathcal{NMV}} \cup P_{\mathcal{NMV}} = P_{\mathcal{NMV}}$.
(2) $P_{\mathcal{NMV}} \cap P_{\mathcal{NMV}} = P_{\mathcal{NMV}}$.

Proof. The proof is obvious. □

Example 4.2. (1). Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{<\alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>, <\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}$ be a $\mathcal{NMV}$ set. Then, $P_{\mathcal{NMV}} \cup P_{\mathcal{NMV}} = \{<\alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>, <\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}$

(2). Similar to (1).

Theorem 4.3. ***(Identity law)*** *For any non-empty* $\mathcal{NMV}$ *sets* $P_{\mathcal{NMV}}$ *and* $Q_{\mathcal{NMV}}$ *in* $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$*, identity law holds:*

(1) $P_{\mathcal{NMV}} \cup 0_{\mathcal{NMV}} = P_{\mathcal{NMV}}$.
(2) $Q_{\mathcal{NMV}} \cap 1_{\mathcal{NMV}} = Q_{\mathcal{NMV}}$.

Proof. **(1).** Let $P_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ of the form $P_{\mathcal{NMV}} = \{<\nu_s, [\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)], [\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)], [\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)]>\}$. Let the null $\mathcal{NMV}$ set be of the form $0_{\mathcal{NMV}} = \{<\nu_s, [0, 0], [1, 1], [1, 1]>\}$. Then,

$$P_{\mathcal{NMV}} \cup 0_{\mathcal{NMV}} = \left\{ \left\langle \nu_s, \begin{array}{c} \left[\bigvee\left(\Gamma_P^-(\nu_s),\ 0\right),\ \bigvee\left(\Gamma_P^+(\nu_s),\ 0\right)\right], \\ \left[\bigwedge\left(\Delta_P^-(\nu_s),\ 1\right),\ \bigwedge\left(\Delta_P^+(\nu_s),\ 1\right)\right], \\ \left[\bigwedge\left(\Upsilon_P^-(\nu_s),1\right),\ \bigwedge\left(\Upsilon_P^+(\nu_s),1\right)\right] \end{array} \right\rangle \right\}$$

$= P_{\mathcal{NMV}}$.

Therefore, $P_{\mathcal{NMV}} \cup 0_{\mathcal{NMV}} = P_{\mathcal{NMV}}$.

(2). Let $Q_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ of the form $Q_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)], [\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)], [\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)] > \}$. Let the absolute $\mathcal{NMV}$ set be of the form $1_{\mathcal{NMV}} = \{< \nu_s, [1,1], [0,0], [0,0] >\}$. Then,

$$Q_{\mathcal{NMV}} \cap 1_{\mathcal{NMV}} = \left\{ \left\langle \nu_s, \begin{array}{c} \left[\bigwedge\left(\Gamma_Q^-(\nu_s),1\right),\ \bigwedge\left(\Gamma_Q^+(\nu_s),\ 1\right)\right], \\ \left[\bigvee\left(\Delta_Q^-(\nu_s),0\right),\ \bigvee\left(\Delta_Q^+(\nu_s),0\right)\right], \\ \left[\bigvee\left(\Upsilon_Q^-(\nu_s),0\right),\ \bigvee\left(\Upsilon_Q^+(\nu_s),0\right)\right] \end{array} \right\rangle \right\}$$

$= Q_{\mathcal{NMV}}$.

Therefore, $Q_{\mathcal{NMV}} \cap 1_{\mathcal{NMV}} = Q_{\mathcal{NMV}}$. □

Example 4.4. (1). Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{ < \alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9] >, < \beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5] >, < \gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5] >, < \delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9] >\}$ be a $\mathcal{NMV}$ set. Then, $P_{\mathcal{NMV}} \cup 0_{\mathcal{NMV}} = \{ < \alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9] >, < \beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5] >, < \gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5] >, < \delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9] >\}$

(2). Similar to (1).

Theorem 4.5. ***(Dominance law)*** *Let $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ be $\mathcal{NMV}$ subsets of the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then for the null set and the absolute set the following conditions holds:*

(1) $P_{\mathcal{NMV}} \cap 0_{\mathcal{NMV}} = 0_{\mathcal{NMV}}$.
(2) $Q_{\mathcal{NMV}} \cup 1_{\mathcal{NMV}} = 1_{\mathcal{NMV}}$.

Proof. **(1).** Let $P_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ of the form $P_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)], [\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)], [\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)] > \}$. Let the empty $\mathcal{NMV}$ set be of the form $0_{\mathcal{NMV}} = \{< \nu_s, [0,0], [1,1], [1,1] >\}$. Then,

$$P_{\mathcal{NMV}} \cap 0_{\mathcal{NMV}} = \left\{ \left\langle \nu_s, \begin{array}{c} \left[\bigwedge \left(\Gamma_P^-(\nu_s), 0\right), \ \bigwedge \left(\Gamma_P^+(\nu_s), \ 0\right)\right], \\ \left[\bigvee \left(\ \Delta_P^-(\nu_s), 1\right), \ \bigvee \left(\Delta_P^+(\nu_s), 1\right)\right], \\ \left[\bigvee \left(\ \Upsilon_P^-(\nu_s), 1\right), \ \bigvee \left(\Upsilon_P^+(\nu_s), 1\right)\right] \end{array} \right\rangle \right\}$$

$= 0_{\mathcal{NMV}}$.

Therefore, $P_{\mathcal{NMV}} \cap 0_{\mathcal{NMV}} = 0_{\mathcal{NMV}}$.

(2). Let $Q_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ of the form $Q_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)], [\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)], \ [\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)] > \}$. Let the absolute $\mathcal{NMV}$ set be of the form $1_{\mathcal{NMV}} = \{< \nu_s, [1,1], [0,0], [0,0] >\}$. Then,

$$Q_{\mathcal{NMV}} \cup 1_{\mathcal{NMV}} = \left\{ \left\langle \nu_s, \begin{array}{c} \left[\bigvee \left(\Gamma_Q^-(\nu_s), \ 1\right), \ \bigvee \left(\Gamma_Q^+(\nu_s), \ 1\right)\right], \\ \left[\bigwedge \left(\Delta_Q^-(\nu_s), \ 0\right), \ \bigwedge \left(\Delta_Q^+(\nu_s), \ 0\right)\right], \\ \left[\bigwedge \left(\Upsilon_Q^-(\nu_s), 0\right), \ \bigwedge \left(\Upsilon_Q^+(\nu_s), 0\right)\right] \end{array} \right\rangle \right\}$$

$= 1_{\mathcal{NMV}}$.

Hence $Q_{\mathcal{NMV}} \cup 1_{\mathcal{NMV}} = 1_{\mathcal{NMV}}$. □

Example 4.6. (1). Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{ < \alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9] >, < \beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5] >, < \gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5] >, < \delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9] > \}$ be a $\mathcal{NMV}$ set and $0_{\mathcal{NMV}} = \{< \alpha, [0,0], [1,1], [1,1] >, < \beta, [0,0], [1,1], [1,1] >, < \gamma, [0,0], [1,1], [1,1] >, < \delta, [0,0], [1,1], [1,1] >\}$. Then, $P_{\mathcal{NMV}} \cap 0_{\mathcal{NMV}} = 0_{\mathcal{NMV}}$

(2). Similar to (1).

Theorem 4.7. ***(Double Complement law)*** *For any $\mathcal{NMV}$ subset $P_{\mathcal{NMV}}$ in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$, $\left(\mathcal{P}_{\mathcal{NMV}}^C\right)^C = P_{\mathcal{NMV}}$.*

Proof. Let $P_{\mathcal{NMV}} = \left\{\left\langle \nu_s, \left[\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)\right], \left[\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)\right], \left[\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)\right]\right\rangle\right\}$ be a $\mathcal{NMV}$ subset in a $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then,

$$\mathcal{P}_{\mathcal{NMV}}^C = \left\{ \left\langle \nu_s, \begin{array}{c} \left[1 - \Gamma_P^+(\nu_s), 1 - \Gamma_P^-(\nu_s)\right], \left[1 - \Delta_P^+(\nu_s), 1 - \Delta_P^-(\nu_s)\right], \\ \left[1 - \Upsilon_P^+(\nu_s), 1 - \Upsilon_P^-(\nu_s)\right] \end{array} \right\rangle \right\}$$

Now,

$$\left(\mathcal{P}^{C}_{\mathcal{NMV}}\right)^{C}=\left\{\left\langle \nu_s,\ \begin{matrix}\left[1-\left(1-\Gamma^{+}_{P}(\nu_s)\right),1-\left(1-\Gamma^{-}_{P}(\nu_s)\right)\right],\\ \left[1-\left(1-\Delta^{+}_{P}(\nu_s)\right),1-\left(1-\Delta^{-}_{P}(\nu_s)\right)\right],\\ \left[1-\left(1-\Upsilon^{+}_{P}(\nu_s)\right),1-\left(1-\Upsilon^{-}_{P}(\nu_s)\right)\right]\end{matrix}\right\rangle\right\}$$

$$=\left\{\left\langle \nu_s,\left[\Gamma^{-}_{P}(\nu_s),\Gamma^{+}_{P}(\nu_s)\right],\left[\Delta^{-}_{P}(\nu_s),\Delta^{+}_{P}(\nu_s)\right],\left[\Upsilon^{-}_{P}(\nu_s),\Upsilon^{+}_{P}(\nu_s)\right]\right\rangle\right\}$$

Therefore, $\left(\mathcal{P}^{C}_{\mathcal{NMV}}\right)^{C}=P_{\mathcal{NMV}}$. □

Example 4.8. (1). Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}},\Psi_{\mathcal{Y}}(\mathfrak{S}),\Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $P_{\mathcal{NMV}} = \{<\alpha,[0.2,0.7],[0.2,0.6],[0.8,0.9]>,<\beta,[0.2,0.5],[0.7,0.9],[0.3,0.5]>,<\gamma,[0.1,0.4],[0.9,0.9],[0.2,0.5]>,<\delta,[0.3,0.5],[0.5,0.9],[0.8,0.9]>\}$ be a $\mathcal{NMV}$ set. Then,

$$\mathcal{P}^{C}_{\mathcal{NMV}}=\{<\alpha,[0.3,0.8],[0.4,0.8],[0.1,0.2]>,<\beta,[0.5,0.8],[0.1,0.3],[0.5,0.7]>,<\gamma,[0.6,0.9],[0.1,0.1],[0.5,0.8]>,<\delta,[0.5,0.7],[0.1,0.5],[0.1,0.2]>\}$$

$$\left(\mathcal{P}^{C}_{\mathcal{NMV}}\right)^{C}=\{<\alpha,[0.2,0.7],[0.2,0.6],[0.8,0.9]>,<\beta,[0.2,0.5],[0.7,0.9],[0.3,0.5]>,<\gamma,[0.1,0.4],[0.9,0.9],[0.2,0.5]>,<\delta,[0.3,0.5],[0.5,0.9],[0.8,0.9]>\}=\mathcal{P}_{\mathcal{NMV}}$$

Theorem 4.9. ***(Absorption law)*** *For any two $\mathcal{NMV}$ subsets $P_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}}$ in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}},\Psi_{\mathcal{Y}}(\mathfrak{S}),\Omega_{\mathcal{Y}}(\mathfrak{S}))$,*

(1) $Q_{\mathcal{NMV}}\cup(Q_{\mathcal{NMV}}\cap R_{\mathcal{NMV}})=Q_{\mathcal{NMV}}$
(2) $Q_{\mathcal{NMV}}\cap(Q_{\mathcal{NMV}}\cup R_{\mathcal{NMV}})=Q_{\mathcal{NMV}}$

Proof. **(1).** Let $Q_{\mathcal{NMV}}=\{<\nu_s,[\Gamma^{-}_{Q}(\nu_s),\Gamma^{+}_{Q}(\nu_s)],[\Delta^{-}_{Q}(\nu_s),\Delta^{+}_{Q}(\nu_s)],\ [\Upsilon^{-}_{Q}(\nu_s),\Upsilon^{+}_{Q}(\nu_s)]>\}$ and $R_{\mathcal{NMV}}=\{<\nu_s,[\Gamma^{-}_{R}(\nu_s),\Gamma^{+}_{R}(\nu_s)],[\Delta^{-}_{R}(\nu_s),\Delta^{+}_{R}(\nu_s)],\ [\Upsilon^{-}_{R}(\nu_s),\Upsilon^{+}_{R}(\nu_s)]>\}$ be the subsets of the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}},\Psi_{\mathcal{Y}}(\mathfrak{S}),\Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then,

$$Q_{\mathcal{NMV}}\cap R_{\mathcal{NMV}}=\left\{\left\langle \nu_s,\ \begin{matrix}\left[\left(\Gamma^{-}_{Q}(\nu_s)\wedge\Gamma^{-}_{R}(\nu_s)\right),\left(\Gamma^{+}_{Q}(\nu_s)\wedge\Gamma^{+}_{R}(\nu_s)\right)\right],\\ \left[\left(\Delta^{-}_{Q}(\nu_s)\wedge\Delta^{-}_{R}(\nu_s)\right),\left(\Delta^{+}_{Q}(\nu_s)\wedge\Delta^{+}_{R}(\nu_s)\right)\right],\\ \left[\left(\Upsilon^{-}_{Q}(\nu_s)\wedge\Upsilon^{-}_{R}(\nu_s)\right),\left(\Upsilon^{+}_{Q}(\nu_s)\wedge\Upsilon^{+}_{R}(\nu_s)\right)\right]\end{matrix}\right\rangle\right\}$$

Case (i): If $Q_{\mathcal{NMV}}\subseteq R_{\mathcal{NMV}}$, then,

$$Q_{\mathcal{NMV}}\cap R_{\mathcal{NMV}}=\left\{\left\langle \nu_s,\left[\Gamma^{-}_{Q}(\nu_s),\Gamma^{+}_{Q}(\nu_s)\right],\left[\Delta^{-}_{Q}(\nu_s),\Delta^{+}_{Q}(\nu_s)\right],\left[\Upsilon^{-}_{Q}(\nu_s),\Upsilon^{+}_{Q}(\nu_s)\right]\right\rangle\right\}$$
$$Q_{\mathcal{NMV}}\cup(Q_{\mathcal{NMV}}\cap R_{\mathcal{NMV}})$$

$$= \left\{ \left\langle \nu_s, \left[\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s) \right], \left[\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s) \right], \left[\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s) \right] \right\rangle \right\} \cup \left\{ \left\langle \nu_s, \left[\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s) \right], \left[\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s) \right], \left[\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s) \right] \right\rangle \right\}$$
$$= Q_{\mathcal{NMV}}.$$

Case (ii): If $R_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$, then,
$Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}} = \{\langle \nu_s, [\Gamma_R^-(\nu_s), \Gamma_R^+(\nu_s)], [\Delta_R^-(\nu_s), \Delta_R^+(\nu_s)], [\Upsilon_R^-(\nu_s), \Upsilon_R^+(\nu_s)]\rangle\}$

$Q_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})$

$$= \left\{ \left\langle \nu_s, \left[\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s) \right], \left[\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s) \right], \left[\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s) \right] \right\rangle \right\} \cup \left\{ \left\langle \nu_s, [\Gamma_R^-(\nu_s), \Gamma_R^+(\nu_s)], [\Delta_R^-(\nu_s), \Delta_R^+(\nu_s)], [\Upsilon_R^-(\nu_s), \Upsilon_R^+(\nu_s)] \right\rangle \right\}$$

$= Q_{\mathcal{NMV}}.$

Hence $Q_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}}) = Q_{\mathcal{NMV}}$.

(2). Proof of (2) is similar to (1). □

Example 4.10. (1). Let us consider the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined in ex.3.2. Let $Q_{\mathcal{NMV}} = \{ <\alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>, <\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}$ and $R_{\mathcal{NMV}} = \{<\alpha, [0, 0], [1, 1], [1, 1]>, <\beta, [0, 0], [1, 1], [1, 1]>, <\gamma, [0, 0], [1, 1], [1, 1]>, <\delta, [0, 0], [1, 1], [1, 1]>\}$ be $\mathcal{NMV}$ sets. Then,

$Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}}$
$$= \{ <\alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>, <\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\}.$$

$Q_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})$
$$= \{ <\alpha, [0.2, 0.7], [0.2, 0.6], [0.8, 0.9]>, <\beta, [0.2, 0.5], [0.7, 0.9], [0.3, 0.5]>, <\gamma, [0.1, 0.4], [0.9, 0.9], [0.2, 0.5]>, <\delta, [0.3, 0.5], [0.5, 0.9], [0.8, 0.9]>\} = Q_{\mathcal{NMV}}$$

(2). Similar to (1).

Theorem 4.11. ***(De-Morgan law)*** *Let $Q_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}}$ be any two subsets of $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then the following statements hold true.*

(1) $(Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})^C = (Q_{\mathcal{NMV}})^C \cap (R_{\mathcal{NMV}})^C$.
(2) $(Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})^C = (Q_{\mathcal{NMV}})^C \cup (R_{\mathcal{NMV}})^C$.

Proof. Let $Q_{\mathcal{NMV}} = \{ <\nu_s, [\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)], [\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)], [\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)]>\}$ and $R_{\mathcal{NMV}} = \{ <\nu_s, [\Gamma_R^-(\nu_s), \Gamma_R^+(\nu_s)], [\Delta_R^-(\nu_s), \Delta_R^+(\nu_s)], [\Upsilon_R^-(\nu_s), \Upsilon_R^+(\nu_s)]>\}$ be the subsets of

$\mathcal{NMVTS}\ (\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$.

1. LHS:

$$Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}} = \left\{\left\langle \nu_s, \left[\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)\right], \left[\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)\right], \left[\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)\right]\right\rangle\right\} \cup$$

$$\left\{\left\langle \nu_s, \left[\Gamma_R^-(\nu_s), \Gamma_R^+(\nu_s)\right], \left[\Delta_R^-(\nu_s), \Delta_R^+(\nu_s)\right], \left[\Upsilon_R^-(\nu_s), \Upsilon_R^+(\nu_s)\right]\right\rangle\right\}$$

$$Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}} = \left\{\left\langle \nu_s, \begin{array}{c} \left[\Gamma_Q^-(\nu_s) \vee \Gamma_R^-(\nu_s), \Gamma_Q^+(\nu_s) \vee \Gamma_R^+(\nu_s)\right], \\ \left[\Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s), \Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s)\right], \\ \left[\Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s), \Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)\right] \end{array} \right\rangle\right\}$$

$$(Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})^c = \left\{\left\langle \nu_s, \begin{array}{c} \left[1 - (\Gamma_Q^+(\nu_s) \vee \Gamma_R^+(\nu_s)), 1 - (\Gamma_Q^-(\nu_s) \vee \Gamma_R^-(\nu_s))\right], \\ \left[1 - (\Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s)), 1 - (\Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s))\right], \\ \left[1 - (\Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)), 1 - \left(\Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s)\right)\right] \end{array} \right\rangle\right\}$$

2. RHS:

$$Q_{\mathcal{NMV}}^C = \left\{\left\langle \nu_s, \begin{array}{c} \left[1 - \Gamma_Q^+(\nu_s), 1 - \Gamma_Q^-(\nu_s)\right], \\ \left[1 - \Delta_Q^+(\nu_s), 1 - \Delta_Q^-(\nu_s)\right], \\ \left[1 - \Upsilon_Q^+(\nu_s), 1 - \Upsilon_Q^-(\nu_s)\right] \end{array} \right\rangle\right\} \text{ and}$$

$$R_{\mathcal{NMV}}^C = \left\{\left\langle \nu_s, \begin{array}{c} \left[1 - \Gamma_R^+(\nu_s), 1 - \Gamma_R^-(\nu_s)\right], \\ \left[1 - \Delta_R^+(\nu_s), 1 - \Delta_R^-(\nu_s)\right], \\ \left[1 - \Upsilon_R^+(\nu_s), 1 - \Upsilon_R^-(\nu_s)\right] \end{array} \right\rangle\right\}$$

$Q_{\mathcal{NMV}}^C \cap R_{\mathcal{NMV}}^C$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[1-\Gamma_Q^+(\nu_s), 1-\Gamma_Q^-(\nu_s)\right], \\ \left[1-\Delta_Q^+(\nu_s), 1-\Delta_Q^-(\nu_s)\right], \\ \left[1-\Upsilon_Q^+(\nu_s), 1-\Upsilon_Q^-(\nu_s)\right] \end{array} \right\rangle \right\} \cap \left\{ \left\langle \nu_s, \begin{array}{c} \left[1-\Gamma_R^+(\nu_s), 1-\Gamma_R^-(\nu_s)\right], \\ \left[1-\Delta_R^+(\nu_s), 1-\Delta_R^-(\nu_s)\right], \\ \left[1-\Upsilon_R^+(\nu_s), 1-\Upsilon_R^-(\nu_s)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[1-\Gamma_Q^+(\nu_s) \wedge 1-\Gamma_R^+(\nu_s), 1-\Gamma_Q^-(\nu_s) \wedge 1-\Gamma_R^-(\nu_s)\right], \\ \left[1-\Delta_Q^+(\nu_s) \vee 1-\Delta_R^+(\nu_s), 1-\Delta_Q^-(\nu_s) \vee 1-\Delta_R^-(\nu_s)\right], \\ \left[1-\Upsilon_Q^+(\nu_s) \vee 1-\Upsilon_R^+(\nu_s), 1-\Upsilon_Q^-(\nu_s) \vee 1-\Upsilon_R^-(\nu_s)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[1-(\Gamma_Q^+(\nu_s) \vee \Gamma_R^+(\nu_s)), 1-(\Gamma_Q^-(\nu_s) \vee \Gamma_R^-(\nu_s))\right], \\ \left[1-(\Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s)), 1-(\Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s))\right], \\ \left[1-(\Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)), 1-\left(\Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

So, LHS = RHS.

2.Proof of (2) is similar as proof of (1). □

Corollary 4.12. *Let $P_{\mathcal{NMV}}$, $Q_{\mathcal{NMV}}$, $R_{\mathcal{NMV}}$ and $S_{\mathcal{NMV}}$ be $\mathcal{NMV}$ sets in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then,*

(1) $\overline{\cup(P_{i\mathcal{NMV}})} = \cap\left(\overline{P_{i\mathcal{NMV}}}\right)$

(2) $\overline{\cap(P_{i\mathcal{NMV}})} = \cup\left(\overline{P_{i\mathcal{NMV}}}\right)$

Proof. The proof is obvious from the above theorem. □

Theorem 4.13. ***(Commutative law)*** *Let $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ be $\mathcal{NMV}$ sets in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then the following statements hold true.*

(1) $P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}} = Q_{\mathcal{NMV}} \cup P_{\mathcal{NMV}}$.

(2) $P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}} = Q_{\mathcal{NMV}} \cap P_{\mathcal{NMV}}$.

Proof. Let $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ be $\mathcal{NMV}$ sets in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ of the form $P_{\mathcal{NMV}} = \{ < \nu_s, \left[\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)\right], \left[\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)\right], \left[\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)\right] > \}$ and $Q_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)], [\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)], [\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)] > \}$.

$1.P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\bigvee\left(\Gamma_P^-(\nu_s),\ \Gamma_Q^-(\nu_s)\right),\ \bigvee\left(\Gamma_P^+(\nu_s),\ \Gamma_Q^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Delta_P^-(\nu_s),\ \Delta_Q^-(\nu_s)\right),\ \bigwedge\left(\Delta_P^+(\nu_s),\ \Delta_Q^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Upsilon_P^-(\nu_s),\ \Upsilon_Q^-(\nu_s)\right),\ \bigwedge\left(\Upsilon_P^+(\nu_s),\ \Upsilon_Q^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\bigvee\left(\Gamma_Q^-(\nu_s),\Gamma_P^-(\nu_s)\right),\ \bigvee\left(\Gamma_Q^+(\nu_s),\Gamma_P^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Delta_Q^-(\nu_s),\Delta_P^-(\nu_s)\right),\ \bigwedge\left(\Delta_Q^+(\nu_s),\Delta_P^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Upsilon_Q^-(\nu_s),\Upsilon_P^-(\nu_s)\right),\ \bigwedge\left(\Upsilon_Q^+(\nu_s),\Upsilon_P^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$= Q_{\mathcal{NMV}} \cup P_{\mathcal{NMV}}$.
Therefore, $P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}} = Q_{\mathcal{NMV}} \cup P_{\mathcal{NMV}}$.
2.Proof of (2) is similar to (1). □

Theorem 4.14. ***(Associative law)*** *Following conditions are true for the $\mathcal{NMV}$ sets $P_{\mathcal{NMV}}$, $Q_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}}$ of the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$.*

(1) $(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \cup R_{\mathcal{NMV}} = P_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})$.
(2) $(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \cap R_{\mathcal{NMV}} = P_{\mathcal{NMV}} \cap (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})$.

Proof. Let $P_{\mathcal{NMV}}$, $Q_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}}$ be subsets of $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined as $P_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)], [\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)], [\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)] > \}$, $Q_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)], [\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)], [\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)] > \}$ and $R_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_R^-(\nu_s), \Gamma_R^+(\nu_s)], [\Delta_R^-(\nu_s), \Delta_R^+(\nu_s)], [\Upsilon_R^-(\nu_s), \Upsilon_R^+(\nu_s)] > \}$.

$1.P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\bigvee\left(\Gamma_P^-(\nu_s),\ \Gamma_Q^-(\nu_s)\right),\ \bigvee\left(\Gamma_P^+(\nu_s),\ \Gamma_Q^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Delta_P^-(\nu_s),\ \Delta_Q^-(\nu_s)\right),\ \bigwedge\left(\Delta_P^+(\nu_s),\ \Delta_Q^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Upsilon_P^-(\nu_s),\ \Upsilon_Q^-(\nu_s)\right),\ \bigwedge\left(\Upsilon_P^+(\nu_s),\ \Upsilon_Q^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_Q^-(\nu_s) \right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_Q^+(\nu_s) \right) \right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_Q^-(\nu_s) \right), \left(\Delta_P^+(\nu_s) \wedge \Delta_Q^+(\nu_s) \right) \right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_Q^-(\nu_s) \right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_Q^+(\nu_s) \right) \right] \end{array} \right\rangle \right\}$$

Then,

$(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \cup R_{\mathcal{NMV}}$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_Q^-(\nu_s) \right) \vee \left(\Gamma_R^-(\nu_s) \right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_Q^+(\nu_s) \right) \vee \left(\Gamma_R^+(\nu_s) \right) \right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_Q^-(\nu_s) \right) \wedge \left(\Delta_R^-(\nu_s) \right), \left(\Delta_P^+(\nu_s) \wedge \Delta_Q^+(\nu_s) \right) \wedge \left(\Delta_R^+(\nu_s) \right) \right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_Q^-(\nu_s) \right) \wedge \left(\Upsilon_R^-(\nu_s) \right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_Q^+(\nu_s) \right) \wedge \left(\Upsilon_R^+(\nu_s) \right) \right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_Q^-(\nu_s) \vee \Gamma_R^-(\nu_s) \right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_Q^+(\nu_s) \vee \Gamma_R^+(\nu_s) \right) \right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s) \right), \left(\Delta_P^+(\nu_s) \wedge \Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s) \right) \right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s) \right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s) \right) \right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \right) \vee \left(\Gamma_Q^-(\nu_s) \vee \Gamma_R^-(\nu_s) \right), \left(\Gamma_P^+(\nu_s) \right) \vee \left(\Gamma_Q^+(\nu_s) \vee \Gamma_R^+(\nu_s) \right) \right], \\ \left[\left(\Delta_P^-(\nu_s) \right) \wedge \left(\Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s) \right), \left(\Delta_P^+(\nu_s) \right) \wedge \left(\Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s) \right) \right], \\ \left[\left(\Upsilon_P^-(\nu_s) \right) \wedge \left(\Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s) \right), \left(\Upsilon_P^+(\nu_s) \right) \wedge \left(\Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s) \right) \right] \end{array} \right\rangle \right\}$$

$= P_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})$.

2. Proof of (2) is similar to proof of (1). □

Theorem 4.15. ***(Distributive law)*** *Let $P_{\mathcal{NMV}}$, $Q_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}}$ be $\mathcal{NMV}$ sets in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then distributive law holds.*

(1) $P_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}}) = (P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \cap (P_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})$.

(2) $P_{\mathcal{NMV}} \cap (Q_{\mathcal{NMV}} \cup R_{\mathcal{NMV}}) = (P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \cup (P_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})$.

Proof.

Let $P_{\mathcal{NMV}}$, $Q_{\mathcal{NMV}}$ and $R_{\mathcal{NMV}}$ be subsets of $\mathcal{NMV}$ topological space $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ defined as $P_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)], [\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)], [\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)] > \}$, $Q_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_Q^-(\nu_s), \Gamma_Q^+(\nu_s)], [\Delta_Q^-(\nu_s), \Delta_Q^+(\nu_s)], [\Upsilon_Q^-(\nu_s), \Upsilon_Q^+(\nu_s)] > \}$ and $R_{\mathcal{NMV}} = \{ < \nu_s, [\Gamma_R^-(\nu_s), \Gamma_R^+(\nu_s)], [\Delta_R^-(\nu_s), \Delta_R^+(\nu_s)], [\Upsilon_R^-(\nu_s), \Upsilon_R^+(\nu_s)] > \}$.

1. LHS: $Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}} = \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_Q^-(\nu_s) \wedge \Gamma_R^-(\nu_s)\right), \left(\Gamma_Q^+(\nu_s) \wedge \Gamma_R^+(\nu_s)\right)\right], \\ \left[\left(\Delta_Q^-(\nu_s) \vee \Delta_R^-(\nu_s)\right), \left(\Delta_Q^+(\nu_s) \vee \Delta_R^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_Q^-(\nu_s) \vee \Upsilon_R^-(\nu_s)\right), \left(\Upsilon_Q^+(\nu_s) \vee \Upsilon_R^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$

$P_{\mathcal{NMV}} \cup (Q_{\mathcal{NMV}} \cap R_{\mathcal{NMV}})$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\Gamma_P^-(\nu_s), \Gamma_P^+(\nu_s)\right], \\ \left[\Delta_P^-(\nu_s), \Delta_P^+(\nu_s)\right], \\ \left[\Upsilon_P^-(\nu_s), \Upsilon_P^+(\nu_s)\right] \end{array} \right\rangle \right\} \cup \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_Q^-(\nu_s) \wedge \Gamma_R^-(\nu_s)\right), \left(\Gamma_Q^+(\nu_s) \wedge \Gamma_R^+(\nu_s)\right)\right], \\ \left[\left(\Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s)\right), \left(\Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s)\right), \left(\Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s)\right) \vee \left(\Gamma_Q^-(\nu_s) \wedge \Gamma_R^-(\nu_s)\right), \left(\Gamma_P^+(\nu_s)\right) \vee \left(\Gamma_Q^+(\nu_s) \wedge \Gamma_R^+(\nu_s)\right)\right], \\ \left[\left(\Delta_P^-(\nu_s)\right) \vee \left(\Delta_Q^-(\nu_s) \wedge \Delta_R^-(\nu_s)\right), \left(\Delta_P^+(\nu_s)\right) \vee \left(\Delta_Q^+(\nu_s) \wedge \Delta_R^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_P^-(\nu_s)\right) \vee \left(\Upsilon_Q^-(\nu_s) \wedge \Upsilon_R^-(\nu_s)\right), \left(\Upsilon_P^+(\nu_s)\right) \vee \left(\Upsilon_Q^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

RHS:

$P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\bigvee\left(\Gamma_P^-(\nu_s),\ \Gamma_Q^-(\nu_s)\right),\ \bigvee\left(\Gamma_P^+(\nu_s),\ \Gamma_Q^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Delta_P^-(\nu_s),\ \Delta_Q^-(\nu_s)\right),\ \bigwedge\left(\Delta_P^+(\nu_s),\ \Delta_Q^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Upsilon_P^-(\nu_s),\ \Upsilon_Q^-(\nu_s)\right),\ \bigwedge\left(\Upsilon_P^+(\nu_s),\ \Upsilon_Q^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_Q^-(\nu_s)\right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_Q^+(\nu_s)\right)\right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_Q^-(\nu_s)\right), \left(\Delta_P^+(\nu_s) \wedge \Delta_Q^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_Q^-(\nu_s)\right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_Q^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$(P_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\bigvee\left(\Gamma_P^-(\nu_s),\ \Gamma_R^-(\nu_s)\right),\ \bigvee\left(\Gamma_P^+(\nu_s),\ \Gamma_R^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Delta_P^-(\nu_s),\ \Delta_R^-(\nu_s)\right),\ \bigwedge\left(\Delta_P^+(\nu_s),\ \Delta_R^+(\nu_s)\right)\right], \\ \left[\bigwedge\left(\Upsilon_P^-(\nu_s),\ \Upsilon_R^-(\nu_s)\right),\ \bigwedge\left(\Upsilon_P^+(\nu_s),\ \Upsilon_R^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_R^-(\nu_s)\right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_R^+(\nu_s)\right)\right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_R^-(\nu_s)\right), \left(\Delta_P^+(\nu_s) \wedge \Delta_R^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_R^-(\nu_s)\right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

Then, $(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \cap (P_{\mathcal{NMV}} \cup R_{\mathcal{NMV}})$

$$= \left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_Q^-(\nu_s)\right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_Q^+(\nu_s)\right)\right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_Q^-(\nu_s)\right), \left(\Delta_P^+(\nu_s) \wedge \Delta_Q^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_Q^-(\nu_s)\right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_Q^+(\nu_s)\right)\right] \end{array} \right\rangle \right\} \cap$$

$$\left\{ \left\langle \nu_s,\ \begin{array}{c} \left[\left(\Gamma_P^-(\nu_s) \vee \Gamma_R^-(\nu_s)\right), \left(\Gamma_P^+(\nu_s) \vee \Gamma_R^+(\nu_s)\right)\right], \\ \left[\left(\Delta_P^-(\nu_s) \wedge \Delta_R^-(\nu_s)\right), \left(\Delta_P^+(\nu_s) \wedge \Delta_R^+(\nu_s)\right)\right], \\ \left[\left(\Upsilon_P^-(\nu_s) \wedge \Upsilon_R^-(\nu_s)\right), \left(\Upsilon_P^+(\nu_s) \wedge \Upsilon_R^+(\nu_s)\right)\right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma^-_P(\nu_s) \vee \Gamma^-_Q(\nu_s) \right) \wedge \left(\Gamma^-_P(\nu_s) \vee \Gamma^-_R(\nu_s) \right), \left(\Gamma^+_P(\nu_s) \vee \Gamma^+_Q(\nu_s) \right) \wedge \left(\Gamma^+_P(\nu_s) \vee \Gamma^+_R(\nu_s) \right) \right], \\ \left[\left(\Delta^-_P(\nu_s) \wedge \Delta^-_Q(\nu_s) \right) \vee \left(\Delta^-_P(\nu_s) \wedge \Delta^-_R(\nu_s) \right), \left(\Delta^+_P(\nu_s) \wedge \Delta^+_Q(\nu_s) \right) \vee \left(\Delta^+_P(\nu_s) \wedge \Delta^+_R(\nu_s) \right) \right], \\ \left[\left(\Upsilon^-_P(\nu_s) \wedge \Upsilon^-_Q(\nu_s) \right) \vee \left(\Upsilon^-_P(\nu_s) \wedge \Upsilon^-_R(\nu_s) \right), \left(\Upsilon^+_P(\nu_s) \wedge \Upsilon^+_Q(\nu_s) \right) \vee \left(\Upsilon^+_P(\nu_s) \wedge \Upsilon^+_R(\nu_s) \right) \right] \end{array} \right\rangle \right\}$$

$$= \left\{ \left\langle \nu_s, \begin{array}{c} \left[\left(\Gamma^-_P(\nu_s) \right) \vee \left(\Gamma^-_Q(\nu_s) \wedge \Gamma^-_R(\nu_s) \right), \left(\Gamma^+_P(\nu_s) \right) \vee \left(\Gamma^+_Q(\nu_s) \wedge \Gamma^+_R(\nu_s) \right) \right], \\ \left[\left(\Delta^-_P(\nu_s) \right) \vee \left(\Delta^-_Q(\nu_s) \wedge \quad \Delta^-_R(\nu_s) \right), \left(\Delta^+_P(\nu_s) \right) \vee \left(\Delta^+_Q(\nu_s) \wedge \Delta^+_R(\nu_s) \right) \right], \\ \left[\left(\Upsilon^-_P(\nu_s) \right) \vee \left(\Upsilon^-_Q(\nu_s) \wedge \Upsilon^-_R(\nu_s) \right), \left(\Upsilon^+_P(\nu_s) \right) \vee \left(\Upsilon^+_Q(\nu_s) \wedge \Upsilon^+_R(\nu_s) \right) \right] \end{array} \right\rangle \right\}$$

Therefore, LHS = RHS. Hence Distributive law holds.

2. Proof of (2) is similar to (1). □

5. **Closure and Interior of $\mathcal{NMV}$ set**

Definition 5.1. Let $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ be a $\mathcal{NMVTS}$. Let $P_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set. The $\mathcal{NMV}$ interior of $P_{\mathcal{NMV}}$ is defined as the union of all $\mathcal{NMVOS}s$ contained in $P_{\mathcal{NMV}}$. (i.e) $\mathcal{NMV}int\,(P_{\mathcal{NMV}}) = \cup\{\mathrm{G} : \mathrm{G} \text{ is a } \mathcal{NMV} \text{ open set and } \mathrm{G} \subseteq P_{\mathcal{NMV}}\}$. Clearly, $\mathcal{NMV}$-int $(P_{\mathcal{NMV}})$ is the largest $\mathcal{NMV}$ open set that is contained in $P_{\mathcal{NMV}}$.

Definition 5.2. Let $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ be a $\mathcal{NMVTS}$. Let $P_{\mathcal{NMV}}$ be a $\mathcal{NMV}$ set. The $\mathcal{NMV}$ closure of $P_{\mathcal{NMV}}$ is defined as the intersection of all $\mathcal{NMVCS}s$ containing $P_{\mathcal{NMV}}$. (i.e) $\mathcal{NMV}cl\,(P_{\mathcal{NMV}}) = \cap\{\mathrm{K} : \mathrm{K} \text{ is } \mathcal{NMV} \text{ closed set and } P_{\mathcal{NMV}} \subseteq \mathrm{K}\}$. Clearly, $\mathcal{NMV}$-cl $(P_{\mathcal{NMV}})$ is the smallest $\mathcal{NMV}$ closed set that contains $P_{\mathcal{NMV}}$.

Proposition 5.3. *Let $P_{\mathcal{NVM}}$ be any $\mathcal{NMV}$ set in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. Then*

(1) $\mathcal{NMV}int\,(1 - P_{\mathcal{NMV}}) = 1 - (\mathcal{NMV}cl\,(P_{\mathcal{NMV}}))$
(2) $\mathcal{NMV}cl\,(1 - P_{\mathcal{NMV}}) = 1 - (\mathcal{NMV}int\,(P_{\mathcal{NMV}}))$

Proof. **(1)** By definition, $\mathcal{NMV}cl\,(P_{\mathcal{NMV}}) = \cap\{\mathrm{K} : \mathrm{K} \text{ is } \mathcal{NMV} \text{ closed set and } P_{\mathcal{NMV}} \subseteq \mathrm{K}\}$. Therefore, $1 - (\mathcal{NMV}cl\,(P_{\mathcal{NMV}})) = 1 - \cap\{\mathrm{K} : \mathrm{K} \text{ is } \mathcal{NMV} \text{ closed set and } P_{\mathcal{NMV}} \subseteq \mathrm{K}\}$

$$= \cup\{(1 - \mathrm{K}) : \mathrm{K} \text{ is } \mathcal{NMV} \text{ closed set and } P_{\mathcal{NMV}} \subseteq \mathrm{K}\}$$

$$= \cup\{\mathrm{G} : \mathrm{G} \text{ is a } \mathcal{NMV} \text{ open set and } \mathrm{G} \subseteq (1 - P_{\mathcal{NMV}})\}$$

$$= \mathcal{NMV}int\,(1 - P_{\mathcal{NMV}})$$

(2) The proof is similar to (1). □

Proposition 5.4. *For any two $\mathcal{NMV}$ sets $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ the following statements hold:*

(1) *$P_{\mathcal{NMV}}$ is $\mathcal{NMV}$ Closed set if and only if $\mathcal{NMV}-cl\ (P_{\mathcal{NMV}}) = P_{\mathcal{NMV}}$*
(2) *$P_{\mathcal{NMV}}$ is $\mathcal{NMV}$ Open set if and only if $\mathcal{NMV}-int\ (P_{\mathcal{NMV}}) = P_{\mathcal{NMV}}$*
(3) *$P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ implies $\mathcal{NMV}-int\ (P_{\mathcal{NMV}}) \subseteq \mathcal{NMV}-int\ (Q_{\mathcal{NMV}})$*
(4) *$P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$ implies $\mathcal{NMV}-cl\ (P_{\mathcal{NMV}}) \subseteq \mathcal{NMV}-cl\ (Q_{\mathcal{NMV}})$*
(5) *$\mathcal{NMV}-cl\ (\mathcal{NMV}-cl\ (P_{\mathcal{NMV}})) = \mathcal{NMV}-cl\ (P_{\mathcal{NMV}})$*
(6) *$\mathcal{NMV}-int\ (\mathcal{NMV}-int\ (P_{\mathcal{NMV}})) = \mathcal{NMV}-int\ (P_{\mathcal{NMV}})$*

Proof. (1) If $P_{\mathcal{NMV}}$ is a $\mathcal{NMV}$ closed set, then $P_{\mathcal{NMV}}$ is the smallest $\mathcal{NMV}$ closed set containing itself and hence $\mathcal{NMV}-cl(P_{\mathcal{NMV}}) = P_{\mathcal{NMV}}$. Conversely if $\mathcal{NMV}-cl(P_{\mathcal{NMV}}) = P_{\mathcal{NMV}}$, then $P_{\mathcal{NMV}}$ is the smallest $\mathcal{NMV}$ closed set containing itself and hence $P_{\mathcal{NMV}}$ is $\mathcal{NMV}$ closed set.

(2) Let $P_{\mathcal{NMV}}$ be a Neutrosophic Micro Vague Open set in the $\mathcal{NMVTS}$ $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$. We know that $\mathcal{NMV}-int(P_{\mathcal{NMV}})$ of any set is a subset of the set $P_{\mathcal{NMV}}$. So, $\mathcal{NMV}-int(P_{\mathcal{NMV}}) \subseteq P_{\mathcal{NMV}}$. Since, $P_{\mathcal{NMV}}$ is a Neutrosophic Micro Vague open set, we have $P_{\mathcal{NMV}} \subseteq \mathcal{NMV}-int(P_{\mathcal{NMV}})$. Therefore, $\mathcal{NMV}-int(P_{\mathcal{NMV}}) = P_{\mathcal{NMV}}$. Conversely suppose if $\mathcal{NMV}-int(P_{\mathcal{NMV}}) = P_{\mathcal{NMV}}$, then since $\mathcal{NMV}-int(P_{\mathcal{NMV}})$ is a $\mathcal{NMV}$ open set, clearly $P_{\mathcal{NMV}}$ is also a $\mathcal{NMV}$ open set.

(3) Let $P_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$, then $1-P_{\mathcal{NMV}} \subseteq 1-Q_{\mathcal{NMV}}$, this implies that $\mathcal{NMV}-cl\ (1-P_{\mathcal{NMV}}) \subseteq \mathcal{NMV}-cl\ (1-Q_{\mathcal{NMV}}) \Longrightarrow \mathcal{NMV}-int(P_{\mathcal{NMV}}) \subseteq \mathcal{NMV}-int(Q_{\mathcal{NMV}})$.

(4) Similarly, it is proved that $\mathcal{NMV}-cl(P_{\mathcal{NMV}}) \subseteq \mathcal{NMV}-cl(Q_{\mathcal{NMV}})$.

(5) Since $\mathcal{NMV}-int(P_{\mathcal{NMV}})$ is a $\mathcal{NMV}$ open set, $\mathcal{NMV}-int\ (\mathcal{NMV}-int(P_{\mathcal{NMV}})) = \mathcal{NMV}-int(P_{\mathcal{NMV}})$.

(6) Similarly, since $\mathcal{NMV}-int(P_{\mathcal{NMV}})$ is a $\mathcal{NMV}$ closed set, then $\mathcal{NMV}-cl\ (\mathcal{NMV}-cl(P_{\mathcal{NMV}})) = \mathcal{NMV}-cl(P_{\mathcal{NMV}})$.

Hence Proved. □

Proposition 5.5. *For any two $\mathcal{NMV}$ sets $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ the following statements hold:*

(1) *$\mathcal{NMV}-cl\ (P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) = \mathcal{NMV}-cl\ (P_{\mathcal{NMV}}) \cup \mathcal{NMV}-cl\ (Q_{\mathcal{NMV}})$*
(2) *$\mathcal{NMV}-cl\ (P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV}-cl\ (P_{\mathcal{NMV}}) \cap \mathcal{NMV}-cl\ (Q_{\mathcal{NMV}})$*

Proof. (1) Since $P_{\mathcal{NMV}} \subseteq P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}} \subseteq P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$, then $\mathcal{NMV} - cl(P_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$ and $\mathcal{NMV} - cl(Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$. Therefore, $\mathcal{NMV} - cl(P_{\mathcal{NMV}}) \cup \mathcal{NMV} - cl(Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$.

Conversely since $P_{\mathcal{NMV}} \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}})$ and $Q_{\mathcal{NMV}} \subseteq \mathcal{NMV} - cl(Q_{\mathcal{NMV}})$, then $P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}} \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}}) \cup \mathcal{NMV} - cl(Q_{\mathcal{NMV}})$. Besides $\mathcal{NMV} - cl(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$ is the smallest $\mathcal{NMV}$ closed set that containing $P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$. Therefore, $\mathcal{NMV} - cl\ (P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}}) \cup \mathcal{NMV} - cl(Q_{\mathcal{NMV}})$. Thus, $\mathcal{NMV} - cl\ (P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) = \mathcal{NMV} - cl(P_{\mathcal{NMV}}) \cup \mathcal{NMV} - cl(Q_{\mathcal{NMV}})$.

(2) Since, $P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}} \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}}) \cap \mathcal{NMV} - cl(Q_{\mathcal{NMV}})$ and $\mathcal{NMV} - cl(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}})$ is the smallest $\mathcal{NMV}$ closed set that containing $P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}$, then $\mathcal{NMV} - cl(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - cl(P_{\mathcal{NMV}}) \cap \mathcal{NMV} - cl(Q_{\mathcal{NMV}})$.

Hence Proved. □

Proposition 5.6. *For any two $\mathcal{NMV}$ sets $P_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}}$ in $(\acute{\mathfrak{L}}, \Psi_{\mathcal{Y}}(\mathfrak{S}), \Omega_{\mathcal{Y}}(\mathfrak{S}))$ the following statements hold:*

(1) $\mathcal{NMV} - int\ (P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}) \supseteq \mathcal{NMV} - int\ (P_{\mathcal{NMV}}) \cup \mathcal{NMV} - int\ (Q_{\mathcal{NMV}})$

(2) $\mathcal{NMV} - int\ (P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) = \mathcal{NMV} - int\ (P_{\mathcal{NMV}}) \cap \mathcal{NMV} - int\ (Q_{\mathcal{NMV}})$

Proof. (1) Since $P_{\mathcal{NMV}} \subseteq P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$ and $Q_{\mathcal{NMV}} \subseteq P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}}$, then $\mathcal{NMV} - int(P_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$ and $\mathcal{NMV} - int(Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int(P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$. Therefore, $\mathcal{NMV} - int(P_{\mathcal{NMV}}) \cup \mathcal{NMV} - int(Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int\ (P_{\mathcal{NMV}} \cup Q_{\mathcal{NMV}})$.

(2) Since $P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}} \subseteq P_{\mathcal{NMV}}$ and $P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}} \subseteq Q_{\mathcal{NMV}}$, then $\mathcal{NMV} - int(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int(P_{\mathcal{NMV}})$ and $\mathcal{NMV} - int(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int(Q_{\mathcal{NMV}})$. So, $\mathcal{NMV} - int(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int(P_{\mathcal{NMV}}) \cap \mathcal{NMV} - int(Q_{\mathcal{NMV}})$.

On the other hand, since $\mathcal{NMV} - int(P_{\mathcal{NMV}}) \subseteq P_{\mathcal{NMV}}$ and $\mathcal{NMV} - int(Q_{\mathcal{NMV}}) \subseteq Q_{\mathcal{NMV}}$, then $\mathcal{NMV} - int(P_{\mathcal{NMV}}) \cap \mathcal{NMV} - int(Q_{\mathcal{NMV}}) \subseteq P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}$. Besides $\mathcal{NMV} - int(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) \subseteq P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}$ and it is the biggest $\mathcal{NMV}$ open set that contained in $P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}$. Therefore, $\mathcal{NMV} - int(P_{\mathcal{NMV}}) \cap \mathcal{NMV} - int(Q_{\mathcal{NMV}}) \subseteq \mathcal{NMV} - int(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}})$. Thus $\mathcal{NMV} - int(P_{\mathcal{NMV}} \cap Q_{\mathcal{NMV}}) = \mathcal{NMV} - int(P_{\mathcal{NMV}}) \cap \mathcal{NMV} - int(Q_{\mathcal{NMV}})$.

Hence the proof. □

Proposition 5.7. (1) $\mathcal{NMV} - cl\ (Q^C_{\mathcal{NMV}}) = [\mathcal{NMV} - int\ (Q_{\mathcal{NMV}})]^C$

(2) $\mathcal{NMV} - int\ (Q^C_{\mathcal{NMV}}) = [\mathcal{NMV} - cl\ (Q_{\mathcal{NMV}})]^C$

Proof. Proof of the above proposition is obvious. □

6. Conclusion

By applying simple extension on Neutrosophic Vague Nano topological spaces, this research presents a brand-new topological space known as Neutrosophic Micro Vague Topological Space. Various operations on Neutrosophic Micro Vague sets such as union, intersection, inclusion and complement are defined with suitable examples. Moreover, some of the fundamental algebraic set properties for Neutrosophic Micro Vague sets have been described and evaluated with appropriate examples. Neutrosophic micro vague set encourages inventiveness in variety of domains. In dynamic contexts where information is continuously shifting or evolving, these sets are advantageous and they offer a strong foundation for managing uncertainty in a variety of applications which enhances decision-making and problem-solving skills. Utilizing this advanced framework researchers may explore applications in fields including image processing, natural language comprehension, robotics and optimization. In future, Neutrosophic Micro Vague sets can be used in association with other mathematical infrastructure to represent various characteristics of unknowns and insufficient accuracy.

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Neutrosophic Ranked Set Sampling Scheme for Estimating Population Mean: An Application to Demographic Data

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Abstract. The primary goal of this study is to address the limitations of classical statistics in handling ambiguous or indeterminate data. The best alternative to classical and fuzzy statistics for handling such data uncertainty is neutrosophic statistics, which is a generalization of both. A generalization of classical statistics, neutrosophic statistics addresses hazy, ambiguous, and unclear information. To achieve this, this manuscript recommends the neutrosophic ranked set sampling approach. We have introduced neutrosophic estimators for estimating the mean of the finite population using auxiliary information under neutrosophic ranked set sampling to address the challenges of estimation of the population mean of neutrosophic data. The proposed estimators outperform the other existing estimators and proposed estimators evaluated in this work using MSE and PRE criteria, and equations for bias and mean squared error produced for the suggested estimators up to the first order of approximation. Under neutrosophic ranked set sampling, the suggested estimator has demonstrated superiority over the class of estimators, unbiased estimators, and comparable estimators. Using the R programming language, a numerical illustration and a simulation study have been conducted to demonstrate the effectiveness of the suggested methodology. When computing results when working with ambiguous, hazy, and neutrosophic-type data, the provided estimators are particularly helpful. These estimators produce findings that are not single-valued but rather have an interval form where our population parameter may lie more frequently. Since we now have an estimated interval with the population mean's unknown value provided a minimum MSE, the estimators are more effective.

Keywords: Neutrosophic ranked set sampling; Neutrosophic Statistics; Ranked Set Sampling; Study Variable; Auxiliary Variable; Bias; Mean Squared Error.

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1. Introduction

Sampling is a crucial practice for a variety of reasons, such as cost and time constraints. In sampling theory, the goal of estimation procedures is to enhance the effectiveness of estimators for population parameters while minimizing sampling errors. To achieve this, auxiliary information is utilized to improve estimator efficiency, and this information can be incorporated at various stages of the process. When highly correlated auxiliary information is not readily available, it can be gathered from previous surveys. Estimation techniques like ratio, product, and regression are commonly employed in this context.

For instance, Sisodia and Dwivedi [1] introduced a modified ratio estimator that incorporates the coefficient of variation of auxiliary information. Pandey and Dubey [2], Bahl and Tuteja [3], Upadhyaya and Singh [4], Singh et al. [5], Kadilar and Cingi [6], and Singh et al. [7] have also proposed population parameter estimation methods using auxiliary information. However, our focus here is on ranked set sampling.

Efforts in sampling continually strive for improvements in estimator efficiency, cost-effectiveness, simplicity, and time savings. Ranked Set Sampling (RSS) offers a superior alternative to Simple Random Sampling (SRS) in various fields, including medicine, agriculture, earth sciences, statistics, and mathematics, especially when measurements are cumbersome, time-consuming, or expensive. The RSS technique was initially described for population mean estimation by McIntyre [8], and the mathematical theory behind RSS was provided by Takahashi and Wakimoto [9]. Dell and Clutter [10] demonstrated that, under both perfect and imperfect ranking scenarios, the mean estimate in RSS remains unbiased.

Numerous researchers, such as Samawi and Muttlak [11], Stokes [12], Al-Shaleh and Al-Omari [13], Bouza [14], Ganesh and Ganeslingam [15], Bouza [16], Kadilar et al. [17], Singh et al. [18], Mandowara and Mehta [19], Al-Omari and Bouza [20] have contributed to the field of ranked set sampling. For recent work, one can prefer Singh and Vishwakarma [21], Bhushan and Kumar [22], and Singh and Kumari [23].

Classical ranked set sampling deals solely with precise data, assuming no uncertainty. However, data can be uncertain and imprecise in practice, containing sets or intervals. To address such situations, fuzzy logic is a valuable tool that handles data with imprecision. Fuzzy statistics are used to analyze data with fuzzy, ambiguous, unclear, uncertain, or imprecise characteristics. Yet, they do not account for the degree of indeterminacy inherent in the data.

Neutrosophic statistics, an extension of fuzzy logic, offers a way to measure both indeterminacy and determinate aspects of uncertain or imprecise data.
When dealing with data that contains indeterminacy, neutrosophic statistics are employed. Neutrosophic statistics expand upon classical statistics and encompass fuzzy and intuitionistic statistics. Neutrosophy is applicable when observations in a population or sample lack precision, are indeterminate or are vague. Some examples of neutrosophic data include district-wise water level measurements with intervals, variations in machinery part sizes due to measurement errors, and day-wise temperature measurements resulting in interval-type data.

Atanassov [24] and Atanassov [25] defined Neutrosophic statistics is a generalization of classical statistics as well as fuzzy. The concept of neutrosophy was initially introduced by Smarandache [26-32], and extensive literature on neutrosophic sets, logic, and statistics can be found in his works. In the realm of sampling theory, Tahir et al. [33] recently addressed the estimation of population parameters under a neutrosophic environment, introducing neutrosophic ratio-type estimators for population means under SRS. One can also prefer Singh & Mishra [34] and Singh et al. [35] for neutrosophic estimators under SRS framework. However, there has been little focus on neutrosophic ranked set sampling for estimating population parameters.

Efficiency improvements in estimators are a constant objective in sampling. In this context, we propose enhanced neutrosophic ranked set sampling (NRSS) estimators for population mean estimation, with a particular emphasis on minimizing mean square error (MSE) and enhancing precision.

Our study is designed as follows: Section 1 presents an introduction, and Section 2 outlines motivation, needs, and research gaps. Section 3 outlines the NRSS method. Section 4 presents existing NRSS estimators. Section 5 presents proposed NRSS estimators, Section 6 presents an empirical study using natural growth rate data, and Section 7 offers a simulation study. Section 8 is dedicated to a discussion, and Section 9 covers a conclusion.

2. Motivation, Need and Research Gap

This article's main objective is to introduce a less explored approach known as "neutrosophic ranked set sampling" for dealing with neutrosophic or interval-type data. This method can encompass various types of NRSS, similar to classical RSS. Our study focuses on sampling theory, marking the instance of proposing an RSS technique tailored to neutrosophic data, along with the development of NRSS estimators for population mean estimation. This is a significant step in expanding the field of sampling theory and comparing these estimators with existing neutrosophic methods such as ratio, product, and generalized estimators. RSS is considered a superior alternative to SRS, making it an attractive avenue for further exploration in the context of NRSS.

Several factors drive our exploration of NRSS and its associated estimators for population parameter estimation. A primary motivation is to introduce RSS and RSS estimators in a neutrosophic setting. Previous research in survey sampling has predominantly focused on clear, well-defined data, where classical sampling methods yield precise results, albeit with potential risks of inaccuracies, overestimations, or underestimations. However, classical methods fall short when handling set-type or undetermined data, characteristic of neutrosophic data, which is more prevalent in real-world scenarios than crisp data. As such, there is a growing need for additional neutrosophic statistical techniques. Traditional statistical approaches are ill-suited to compute accurate estimates of unknown parameters when dealing with indeterminate, vague, imprecise, set-type, or interval-type data. Neutrosophic statistics serve as a suitable replacement for classical statistics in such scenarios.

Inspired by the work of Tahir et al. [33] and driven by the need to bridge the gap between classical and neutrosophic statistics, our work introduces enhanced NRSS estimators for population mean estimation. Despite thorough research in the field, we found not many prior studies in survey sampling that addressed the estimation of population means in the presence of auxiliary variables using neutrosophic data under ranked set sampling other than Singh and Vishwakarma [36]. Following Singh and Vishwakarma's work, this research represents a significant step toward filling this gap and contributes to the evolving domain of neutrosophic statistics.

It has been well-established by multiple authors that RSS is a more suitable option than SRS when dealing with cumbersome, expensive, or time-consuming measurements. The challenges associated with measurements in a neutrosophic context exacerbate these issues. Therefore, our research introduces an NRSS method to enhance the accuracy of the population mean estimators in this unique context.

3. **Sampling Methodology**

Numerous methods can be used to display the neutrosophic observations, and the neutrosophic numbers may include an unknown interval [a, b]. We are describing neutrosophic values as $Z_{rssN} \equiv Z_{rssL} + Z_{rssU} I_{rssN}$ with $I_{rssN} \in [I_{rssL}, I_{rssU}]$, N is here to represent the neutrosophic number and I_{rssN} is the degree of indeterminacy. Hence, our neutrosophic observations will lie in an interval $Z_{rssN} \in [a, b]$, where 'a'and 'b'denote the neutrosophic data's lower and upper values.

In RSS, a small subset of randomly chosen population units are measured after they have been ranked solely based on observation or past experience. Within the framework of RSS, multiple independent random sets, each comprising m units drawn from the population, are chosen. Each unit within a set has an equal probability of selection. The constituents of each random set are then ranked based on the characteristics of the auxiliary variable. Subsequently, the selection process involves choosing the smallest unit from the first ordered set, followed by the next-smallest unit from the second ordered set, and so on. This sequential selection continues until the largest rank in the m^{th} set is reached. Throughout this iterative cycle, a total of rm (= n) units are measured, and this entire process is repeated r times.

The method of NRSS consists of selecting $m_N \in [m_L, m_U]$ bivariate random samples of size $m_N \in [m_L, m_U]$ from a population of size N, and then ranking inside each sample concerning for auxiliary variable $X_N \in [X_L, X_U]$ associated with $Y_N \in [Y_L, Y_U]$. The book "Introduction to Neutrosophic Statistics"by Smarandache [32] will be the basis for the ranking of the neutrosophic number. To show the process of ranking, we are utilizing here two sets as $X_{1N} \in [X_{1L}, X_{1U}]$ and $X_{2N} \in [X_{2L}, X_{2U}]$, also their mid-points are as $X_{1midN} = [X_{1L} + X_{1U}]/2$ and $X_{2midN} = [X_{2L} + X_{2U}]/2$. The ordering of neutrosophic numbers $m_N \in [m_L, m_U]$ can be done as $X_{1N} \in [X_{1L}, X_{1U}]$ will be less than $X_{2N} \in [X_{2L}, X_{2U}]$ if $X_{1midN} \leq X_{2midN}$, also if both are same that is $X_{1midN} = X_{2midN}$ then we will compare or see by $X_{1L} \leq X_{2L}$. Further, if again $X_{1L} = X_{2L}$ then this implies $X_{1U} = X_{2U}$ and hence $X_{1N} \in [X_{1L}, X_{1U}] = X_{2N} \in [X_{2L}, X_{2U}]$, so the neutrosophic number ranking will be carried out in this manner. In the whole NRSS structure, first we count the smallest unit of the first data set size $m_N \in [m_L, m_U]$, for the first measurement unit in the entire NRSS structure, and then we scrap the remaining units. In a similar manner, we count the second-smallest observation from the second data set as the second observation and discard the remaining observations. This process counts the total $m_N \in [m_L, m_U]$ neutrosophic bivariate units for up to the m^{th} term. After r cycles of these steps, the total $n_N = m_N r \in [n_L, n_U]$ bivariate NeRSS units are obtained.

Consider a neutrosophic random sample of size $n_N \in [n_L, n_U]$ using RSS, which is acquired from a finite population of 'N' units $(U_1, U_2, ..., U_N)$. The neutrosophic study and auxiliary variable are $Y_N \in [Y_L, Y_U]$ and $X_N \in [X_L, X_U]$. Let $\overline{y}_{[n]N} \in [\overline{y}_{[n]L}, \overline{y}_{[n]U}]$ and $\overline{x}_{(n)N} \in [\overline{x}_{(n)L}, \overline{x}_{(n)U}]$ be the sample means of the neutrosophic study and auxiliary variables respectively, and also, let $\overline{Y}_N \in [\overline{Y}_L, \overline{Y}_U]$ and $\overline{X}_N \in [\overline{X}_L, \overline{X}_U]$ be the population means of the neutrosophic study and auxiliary variables, respectively. The correlation coefficient between both neutrosophic study and auxiliary variables is $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$, $C_{xN} \in [C_{xL}, C_{xU}]$ and $C_{yN} \in [C_{yL}, C_{yU}]$ be the coefficient of variation of neutrosophic variables Y_N and X_N.

Let the neutrosophic mean error terms are $\epsilon_{0N} \in [\epsilon_{0L}, \epsilon_{0U}]$ and $\epsilon_{1N} \in [\epsilon_{1L}, \epsilon_{1U}]$. To obtain the bias and MSE of the estimators, we write

$\overline{y}_{[n]N} = \overline{Y}_N(1+\epsilon_{0N})$, $\overline{x}_{(n)N} = \overline{X}_N(1+\epsilon_{1N})$

$E(\epsilon_{0N}^2) = (\eta c_{yN}^2 - D_{y[N]}^2) = V_{yrN}$

$E(\epsilon_{1N}^2) = (\eta c_{xN}^2 - D_{x[N]}^2) = V_{xrN}$

$E(\epsilon_{0N}, \epsilon_{0N}) = (\eta C_{yx} - D_{yx[N]}^2) = V_{yxrN}$

where,

$\eta_N = \frac{1}{n_N r}$,

$D_{y[N]}^2 = \frac{1}{m_N^2 r \overline{Y}_N^2} \sum_{i=1}^{m_N} (\mu_{[iyN]} - \overline{Y}_N)^2$

$D_{x[N]}^2 = \frac{1}{m_N^2 r \overline{X}_N^2} \sum_{i=1}^{m_N} (\mu_{[ixN]} - \overline{X}_N)^2$

$D_{yx[N]} = \frac{1}{m_N^2 r \overline{Y}_N \overline{X}_N^2} \sum_{i=1}^{m_N} (\mu_{[iyN]} - \overline{Y}_N)(\mu_{[ixN]} - \overline{X}_N)$

where $\mu_{[iy]}$ and $\mu_{(ix)}$ are the means of the i^{th} ranked set and are given by

$$\mu_{[iyN]} = \frac{1}{r}\sum_{j=1}^{r} y_{j[N]}, \mu_{(ixN)} = \frac{1}{r}\sum_{l=1}^{r} x_{j[N]}.$$

$\eta_N \in [\eta_L, \eta_U]$; $S_{xN} \in [S_{xL}, S_{xU}]$; $S_{yN} \in [S_{yL}, S_{yU}]$; $S_{xyN} \in [S_{xyL}, S_{xyU}]$ $e_{0N}^2 \in [e_{0L}^2, e_{0U}^2]$;$e_{1N}^2 \in [e_{1L}^2, e_{1U}^2]$;$e_{0N}e_{1N} \in [e_{0L}e_{1L}, e_{0U}e_{0U}]$;

$C_{xN} \in [C_{xL}, C_{xU}]$; $C_{yN} \in [C_{yL}, C_{yU}]$; $C_{xyN} \in [C_{xyL}, C_{xyU}]$ $D_{y[N]}^2 \in [D_{y[L]}^2, D_{y[U]}^2]$; $D_{x[N]}^2 \in [D_{x[L]}^2, D_{x[U]}^2]$; $D_{yx[N]} \in [D_{yx[L]}, D_{yx[U]}]$; $\rho_{yxN} \in [\rho_{yxL}, \rho_{yxU}]$; $V_{yrN} \in [V_{yrL}, V_{yrU}]$
$V_{xrN} \in [V_{xrL}, V_{xrU}]$; $V_{yxrN} \in [V_{yxrL}, V_{yxrU}]$; $\mu_{[iyN]} \in [\mu_{[iyL]}, \mu_{[iyU]}]$; $\mu_{[ixN]} \in [\mu_{[ixL]}, \mu_{[ixU]}]$.

4. **Existing Estimators**

Using the NRSS technique, the usual unbiased estimator for the population mean $\overline{Y}$ is provided by

$$\overline{y}_{[n]N} = \frac{1}{n_N}\sum_{i=1}^{n_N} y_{[i]N} \tag{1}$$

The variance of the estimator $\overline{y}_{[n]N}$ is given by

$$var(\overline{y}_{[n]N}) = \overline{Y}_N^2 V_{yrN} \tag{2}$$

The ratio estimator under NRSS for the population mean $\overline{Y}$

$$\overline{y}_{rN} = \overline{y}_{[n]N}\left(\frac{\overline{X}_N}{\overline{x}_{[n]N}}\right) \tag{3}$$

The MSE of the estimator $\overline{y}_{rN}$ is given by

$$MSE(\overline{y}_{rN}) = \overline{Y}_N^2(V_{yrN} + V_{xrN} - 2V_{yxrN}) \tag{4}$$

Using NRSS, the regression estimator for the population mean $\overline{Y}$ is provided by

$$\overline{y}_{regN} = \overline{y}_{[n]N} + \beta(\overline{X}_N - \overline{y}_{[n]N}) \tag{5}$$

The MSE of the estimator $\overline{y}_{regN}$ is given by

$$MSE(\overline{y}_{regN}) = \overline{Y}_N^2 \left(V_{yrN} - \frac{V_{yxrN}^2}{V_{xrN}} \right) \tag{6}$$

Using NRSS, the exponential estimator for the population mean $\overline{Y}$ is provided by

$$\overline{y}_{expN} = \overline{y}_{[n]N} \exp \left(\frac{\overline{X}_N - \overline{x}_{[n]N}}{\overline{X}_N + \overline{x}_{[n]N}} \right) \tag{7}$$

The MSE of the estimator $\overline{y}_{expN}$ is given by

$$MSE(\overline{y}_{expN}) = \overline{Y}^2 \left(V_{yrN} + \frac{V_{yrN}}{4} - V_{yxrN} \right) \tag{8}$$

Vishwakarma and Singh (2021) gave NRSS generalized class of estimators

$$\overline{y}_{vsN} = \overline{y}_{[n]N} \left(\frac{A_N\overline{X}_N + B_N}{A_N\overline{x}_{[n]N} + B_N} \right)^{\delta} \tag{9}$$

The MSE of the estimator $\overline{y}_{vsN}$ is given by

$$MSE(\overline{y}_{vsN}) = \overline{Y}_N^2 \left(V_{yrN} - \frac{V_{yxrN}^2}{V_{xrN}} \right) \tag{10}$$

5. **Proposed Estimators**

No single estimator is universally effective in all situations. Consequently, prioritizing estimators that yield minimal Mean Squared Error (MSE) and high precision is desirable. The objective of this section is to develop estimators that demonstrate effective performance across a broader range of scenarios. We have chosen to incorporate Mishra et al.'s [37] estimator within the NRSS and have additionally introduced two novel estimators for the mean of a finite population under NRSS, leveraging auxiliary variables for improved accuracy.

$$1.) P_{1N} = \overline{y}_{[n]N} \left(g_{1N} + 1\right) + g_{2N} \log \left(\frac{\overline{x}_{[n]N}}{\overline{X}_N} \right) \tag{11}$$

where the constants g_{1N} and g_{2N} ensure that the estimators' MSE is kept to a minimal. Expressing the estimator P_{1N} given in equation (11) in terms of $\epsilon's$ we get

$$P_{1N} = \overline{Y}_N \left(1 + \epsilon_{01}\right) \left(g_{1N} + 1\right) + g_{2N} log \left(\frac{\overline{X}_N \left(1 + \epsilon_{1N}\right)}{\overline{X}_N} \right) \tag{12}$$

Taking expectations by focusing on first-order approximation, we obtain MSE,

$$MSE\left(P_{1N}\right) = \overline{Y}_N^{\,2} V_{yrN} + g_{1N}^2 A_{1N} + g_{2N}^2 B_{1N} - 2g_{1N} C_{1N} - 2g_{2N} D_{1N} + 2g_{1N} g_{2N} E_{1N} \quad (13)$$

where,

$$A_{1N} = \overline{Y}_N^{\,2}(1 + V_{yrN})$$

$$B_{1N} = V_{xrN}$$

$$C_{1N} = \overline{Y}_N^{\,2} V_{yrN}$$

$$D_{1N} = \overline{Y}\ V_{yxrN}$$

$$E_{1N} = \overline{Y}\left(V_{yxrN} - \frac{1}{2} V_{xrN}\right)$$

To find out the minimum MSE for P_{1N}, we partially differentiate equation (13) w.r.t. g_{1N} & g_{2N} and equating to zero we get

$$g_{1N}{}^{*} = \frac{B_{1N} C_{1N} - D_{1N} E_{1N}}{E_{1N}^2 - A_{1N} B_{1N}} \quad (14)$$

$$g_{2N}{}^{*} = \frac{A_{1N} D_{1N} - C_{1N} E_{1N}}{E_{1N}^2 - A_{1N} B_{1N}} \quad (15)$$

Putting the optimum value of g_{1N} & g_{2N} in the equation (13), we obtain a minimum value of MSE of P_{1N} as

$$MinMSE(P_{1N}) = C_{1N} + \frac{B_{1N} C_{1N}^2 + A_{1N} D_{1N}^2 - 2C_{1N} D_{1N} E_{1N}}{E_{1N}^2 - A_{1N} B_{1N}} \quad (16)$$

where $MSE(P_{1N}) \in [MSE(P_{1L}), MSE(P_{1U})]$

$$2.)P_{2N} = g_{3N}\overline{y}_{[n]N} + g_{4N} exp\left(\frac{\overline{X}_N - \overline{x}_{[n]N}}{\overline{X}_N + \overline{x}_{[n]N}}\right)\left(1 + log\frac{\overline{x}_{[n]N}}{\overline{X}_N}\right) \quad (17)$$

Expressing P_{2N} given in equation (17) in terms of $\epsilon's$ we get

$$P_{2N} = g_{3N}\overline{Y}_N\left(1 + \epsilon_{0N}\right) + g_{4N}\ exp\left(\frac{-\epsilon_{1N}}{2 + \epsilon_{1N}}\right)\left(1 + \log\left(1 + \epsilon_{1N}\right)\ \right) \quad (18)$$

$$P_{2N} - \overline{Y}_N = \left(g_{3N} - 1\right)\overline{Y}_N + g_{3N}\overline{Y}_N \epsilon_{0N} + g_{4N}\left(1 + \frac{\epsilon_{1N}}{2} - \frac{5\epsilon_{1N}{}^2}{8}\right) \quad (19)$$

$$Bias(P_{2N}) = \overline{Y}_N\left(g_{3N} - 1\right) + g_{4N}\left[1 - \frac{5}{8} V_{xrN}\right] \quad (20)$$

CASE 1: IF SUM OF WEIGHTS IS FIXED($g_{3N} + g_{4N} = 1$)

The MSE of the estimator P_{2N} is shown as

$$MSE(P_{2N}) = \overline{Y}_N^2 \left[V_{yrN} + g_{4N}^2 V_{xrN} - 2g_{4N} V_{yxrN}\right] \tag{21}$$

To find out the minimum value of MSE for P_{2N}, we partially differentiate equation (21) w.r.t. g_{4N}, and equating to zero we get

$$g_{4N}{}^* = \frac{V_{yxrN}}{V_{xrN}} \tag{22}$$

Putting the optimum value of g_{4N} in the equation (21), we obtain a minimum MSE of P_{2N} as

$$MinMSE(P_{2N}) = \overline{Y}_N{}^2 \left(V_{yrN} - \frac{V_{yxrN}{}^2}{V_{xrN}}\right) \tag{23}$$

where $MSE(P_{2N}) \in [MSE(P_{2L}), MSE(P_{2U})]$

CASE 2: IF THE SUM OF WEIGHTS IS ADJUSTABLE ($g_{3N} + g_{4N} \neq 1$)

$$P_{2N} - \overline{Y}_N = (g_{3N} - 1)\,\overline{Y}_N + g_{3N}\overline{Y}_N \epsilon_{0N} + g_{4N}\left(1 + \frac{\epsilon_{1N}}{2} - \frac{5\epsilon_1{}^2}{8}\right) \tag{24}$$

Squaring on both sides we get

$$\begin{aligned}(P_{2N} - \overline{Y}_N)^2 = \overline{Y}_N{}^2 + \overline{Y}_N{}^2 g_{3N}^2(1+\epsilon_{01}^2) + g_{4N}^2\left(1-\epsilon_{1N}^2\right) - 2g_{3N}\overline{Y}_N{}^2 - 2g_{4N}\overline{Y}_N\left(1 - \frac{5\epsilon_{1N}^2}{8}\right) \\ + 2g_{3N}g_{4N}\overline{Y}_N\left(1 - \frac{5\epsilon_{1N}{}^2}{8} + \frac{\epsilon_{0N}\epsilon_{1N}}{2}\right)\end{aligned} \tag{25}$$

Taking expectations by focusing on first-order approximation, we obtain mean square error (MSE),

$$MSE\,(P_{2N}) = \overline{Y}_N{}^2 V_{yrN} + g_{3N}^2 A_{2N} + g_{4N}^2 B_{2N} - 2g_{3N}C_{2N} - 2g_{4N}D_{2N} + 2g_{3N}g_{4N}E_{2N} \tag{26}$$

where,

$$\begin{aligned} A_{2N} &= \overline{Y}_N{}^2(1+V_{yrN}) \\ B_{2N} &= 1 - V_{xrN} \\ C_{2N} &= \overline{Y}_N{}^2 \\ D_{2N} &= \overline{Y}_N\left(1 - \frac{5}{8}V_{xrN}\right) \\ E_{2N} &= \overline{Y}_N\left(1 - \frac{5}{8}V_{xrN} + \frac{1}{2}V_{yxrN}\right)\end{aligned}$$

To find out the minimum MSE for P_{2N}, we partially differentiate equation (26) w.r.t. g_{3N} & g_{4N} and equating to zero we get

$$g_{3N}{}^* = \frac{B_{2N}C_{2N} - D_{2N}E_{2N}}{A_{2N}B_{2N} - E_{2N}^2} \tag{27}$$

$$g_{4N}{}^{*} = \frac{A_{2N}D_{2N} - C_{2N}E_{2N}}{A_{2N}B_{2N} - E_{2N}^2} \tag{28}$$

Putting the optimum value of g_{3N} & g_{4N} in the equation (26), we obtain a minimum MSE of P_{2N} as

$$MinMSE(P_{2N}) = C_{2N} + \frac{B_{2N}C_{2N}^2 + A_{2N}D_{2N}^2 - 2C_{2N}D_{2N}E_{2N}}{E_{2N}^2 - A_{2N}B_{2N}} \tag{29}$$

where $MSE(P_{2N}) \in [MSE(P_{2L}), MSE(P_{2U})]$

$$3.)P_{3N} = g_{5N}\overline{y}_{[n]N} + \ g_{6N}\left(\frac{\overline{X}_N}{\overline{x}_{[n]N}}\right) exp\left(\frac{\overline{X}_N - \overline{x}_{[n]N}}{\overline{X}_N + \overline{x}_{[n]N}}\right) \tag{30}$$

Expressing P_{3N} given in equation (30) in terms of $\epsilon's$ we get

$$P_{3N} = g_{5N}\overline{Y}_N\left(1+\epsilon_{0N}\right) + g_{6N}(1+\epsilon_{1N})^{-1}\ exp\left(\frac{-\epsilon_1}{2+\epsilon_{1N}}\right) \tag{31}$$

$$P_{3N} - \overline{Y}_N = \left(g_{5N}-1\right)\overline{Y}_N + g_{5N}\overline{Y}_N\epsilon_{01} + g_{6N}\left(1 - \frac{3\epsilon_{1N}}{2} + \frac{15{\epsilon_{1N}}^2}{8}\right) \tag{32}$$

$$Bias(P_{3N}) = \overline{Y}_N\left(g_{5N}-1\right) + g_{6N}\left[1 + \frac{15}{8}V_{xrN}\right] \tag{33}$$

CASE 1: IF SUM OF WEIGHTS IS FIXED ($g_{5N} + g_{6N} = 1$)

The MSE of the estimator P_{3N} is shown as

$$MinMSE(P_{3N}) = {\overline{Y}_N}^2\left[V_{yrN} + g_{6N}^2V_{xrN} - 2g_{6N}V_{yxrN}\right] \tag{34}$$

To find out the minimum value of MSE for P_{3N}, we partially differentiate equation (34) w.r.t. g_{6N} and equating to zero we get

$$g_{6N}{}^{*} = \frac{V_{yxrN}}{V_{xrN}} \tag{35}$$

Putting the optimum value of g_{6N} in the equation (34), we obtain a minimum MSE of P_{3N} as

$$MinMSE(P_{3N}) = {\overline{Y}_N}^2\left(V_{yrN} - \frac{V_{yxrN}^2}{V_{xrN}}\right) \tag{36}$$

where $MSE(P_{3N}) \in [MSE(P_{3L}), MSE(P_{3U})]$

CASE 2: IF THE SUM OF WEIGHTS IS ADJUSTABLE ($g_{5N} + g_{6N} \neq 1$)

$$P_{3N} - \overline{Y}_N = \left(g_{5N}-1\right)\overline{Y}_N + g_{5N}\overline{Y}\epsilon_{0N} + g_{6N}\left(1 - \frac{3\epsilon_{1N}}{2} + \frac{15{\epsilon_{1N}}^2}{8}\right) \tag{37}$$

Squaring on both sides we get

$$\left(P_{3N}-\overline{Y}_N\right)^2=\overline{Y}_N^{\,2}+\overline{Y}_N^{\,2}g_{5N}{}^2(1+\epsilon_{0N}^2)+g_{6N}{}^2\left(1+6\epsilon_{1N}^2\right)-2g_{5N}\overline{Y}_N^{\,2}-2g_{6N}\overline{Y}_N\left(1+\frac{15\epsilon_{1N}{}^2}{8}\right)$$
$$+2g_{5N}g_{6N}\overline{Y}_N\left(1+\frac{15\epsilon_{1N}{}^2}{8}-\frac{3\epsilon_{0N}\epsilon_{1N}}{2}\right)\quad(38)$$

By utilizing first-order approximations for expectations, we can derive mean square error (MSE)

$$MSE\left(P_{3N}\right)=\overline{Y}_N^{\,2}V_{yrN}+g_{5N}^2A_{3N}+g_{6N}^2B_{3N}-2g_{5N}C_{3N}-2g_{6N}D_{3N}+2g_{5N}g_{6N}E_{3N}\quad(39)$$

where,

$$A_{3N}=\ \overline{Y}_N^{\,2}(1+V_{yrN})$$

$$B_{3N}=\ 1+6V_{xrN}$$

$$C_{3N}=\ \overline{Y}_N^{\,2}$$

$$D_{3N}=\ \overline{Y}_N\left(1+\frac{15}{8}V_{xrN}\right)$$

$$E_{3N}=\ \overline{Y}_N\left(1+\frac{15}{8}V_{xrN}-\frac{3}{2}V_{yxrN}\right)$$

To find out the minimum MSE for P_{3N}, we partially differentiate equation (39) w.r.t. g_{5N} & g_{6N} and equating to zero we get

$$g_{5N}{}^*=\frac{B_{3N}C_{3N}-D_{3N}E_{3N}}{A_{3N}B_{3N}-E_{3N}^2}\quad(40)$$

$$g_{6N}{}^*=\frac{A_{3N}D_{3N}-C_{3N}E_{3N}}{A_{3N}B_{3N}-E_{3N}^2}\quad(41)$$

Putting the optimum value of g_{5N} & g_{6N} in the equation (39), we obtain a minimum MSE of P_{3N} as

$$MinMSE((P_{3N}))=C_{3N}+\frac{B_{3N}C_{3N}^2+A_{3N}D_{3N}^2-2C_{3N}D_{3N}E_{3N}}{E_{3N}^2-A_{3N}B_{3N}}\quad(42)$$

where $MSE(P_{3N})\in[MSE(P_{3L}),MSE(P_{3U})]$
$P_{iN}\in[P_{iL},P_{iU}];i=1,2,3$, $A_{iN}\in[A_{iL},A_{iU}];i=1,2,3$, $B_{iN}\in[B_{iL},B_{iU}];i=1,2,3$
$C_{iN}\in[C_{iL},C_{iU}];i=1,2,3$, $D_{iN}\in[D_{iL},D_{iU}];i=1,2,3$, $E_{iN}\in[E_{iL},E_{iU}];i=1,2,3$

6. Numerical Illustrations

Here, we evaluate the performance of the recommended estimators in comparison to the other existing estimators considered in this paper. We have taken real-life natural growth rate data from the sample registration system (SRS) (2020). The data mentioned in the sample registration system (SRS) (2020) have four neutrosophic variables for every state, but in our research, we use birth rate vs natural growth rate. Here, the birth rate is the neutrosophic auxiliary variable $X_N \in [X_L, X_U]$ and natural growth rate is a neutrosophic study variable $Y_N \in [Y_L, Y_U]$.

Table 1: The Data of Natural Growth Rate as per SRS 2020

State	BRl	BRu	NGRl	NGRu	State	BRl	BRu	NGRl	NGRu
Andhra Pradesh	15	16	9	10.1	Uttar Pradesh	22.1	26.1	19.3	16.7
Assam	14.3	21.9	8.9	15.5	Uttarakhand	15.6	17	10.5	10.3
Bihar	21	26.2	15.7	20.7	West Bengal	11.2	16.1	10.8	5.4
Chhattisgarh	17.3	23.4	11	15	Arunachal Pradesh	15	17.8	11.8	10.6
NCT of Delhi	14.1	15.5	10.6	11.4	Goa	11.7	12.4	6.9	5.3
Gujrat	17.1	21.1	12	15.1	Himachal Pradesh	10	15.7	8.7	5.6
Haryana	17.7	21.2	12.3	14.7	Manipur	12.8	13.5	9.5	8
Jammu & Kashmir	11.1	16.1	7	11.3	Meghalaya	12.9	25.1	19.6	8.5
Jharkhand	17.6	23.4	13.1	17.9	Mizoram	11.7	16.8	13	7.1
Karnataka	15	17.5	10.2	10.5	Nagaland	11.8	12.9	9	8.4
Kerala	13.1	13.3	6.1	6.3	Sikkim	14	18.2	14.5	9.7
Madhya Pradesh	18.8	26	13.1	19.2	Tripura	10.7	13.4	8	4.2
Maharashtra	14.6	15.3	9.1	10.1	Andaman & Nicobar	10	11.5	5.4	4.7
Odisha	13.1	18.7	6.6	11.2	Chandigarh	12.8	18.1	14	9
Punjab	13.6	14.9	6.6	7.9	Dadar Nagar Haveli	18	21.4	18.1	13.3
Rajasthan	20.8	24.4	15.7	18.6	Ladakh	10.8	15.2	10	6.5

Tamil Nadu	13.6	14	6.8	8.5	Lakshadweep	13.1	19.9	12.7	8.1
Telangana	15.9	16.8	9.6	11.7	Puducherry	13.1	13.1	7	5.6

Further, we have drawn total $n_N = m_N r = 12$ samples from the given population of size 36 by utilizing the method of NRSS with set size $m_N = [3, 3]$ and replication $r = 4$. The NRSS method for the study and auxiliary variables is used to draw the NRSS sample simultaneously, as explained in Section 2. The formula for Percent Relative Efficiency (PRE) is defined, as

$$PRE\,(Estimators) = \frac{MSE(\overline{y}_{[n]N})}{MSE(estimator)} \times 100 \tag{43}$$

Table 2: The MSE and PRE of the Estimators

Estimators	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[0.51461, 0.95605]	[0, 0.46]	[100, 100]
$\overline{y}_{rN}$	[0.11421, 0.16089]	[0, 0.29]	[451, 594]
$\overline{y}_{regN}$	[0.06610, 0.11572]	[0, 0.42]	[778, 826]
$\overline{y}_{expN}$	[0.26370, 0.43450]	[0, 0.39]	[195, 220]
$\overline{y}_{vsN}$	[0.06610, 0.11572]	[0, 0.42]	[778, 826]
t_{p1}	[0.06550, 0.11467]	[0, 0.42]	[786, 834]
t_{p2}	[0.01237, 0.03204]	[0, 0.61]	[2983, 4158]
t_{p3}	[0.01551, 0.03096]	[0, 0.49]	[3088, 3317]

7. Simulation Studies

We perform simulation studies to check the recommended estimator's efficiency with other existing estimators like the conventional, ratio, regression estimator, etc. This is done via the following steps

1. It is well known that a neutrosophic normal distribution (NND) will be followed by neutrosophic random variables (NRV), i.e.$(X_N, Y_N) \sim NN[(\mu_{xN}, \sigma^2_{xN}), (\mu_{yN}, \sigma^2_{yN})]$, $X_N \in [X_L, X_U]$, $Y_N \in [Y_L, Y_U]$, $\mu_{xN} \in [(\mu_{xL}, \mu_{xU})]$, $\mu_{yN} \in [(\mu_{yL}, \mu_{yU})]$, $\sigma^2_{xN} \in [\sigma^2_{xL}, \sigma^2_{xU}]$, $\sigma^2_{yN} \in [\sigma^2_{yL}, \sigma^2_{yU}]$.

We have generated 4-variate random observations of size N=1000 from a 4-variate normal distribution with mean $(\mu_{xL},\ \mu_{yL}\ \ \mu_{xU}, \mu_{yU}) = (50, 50, 60, 60)$ and covariance matrix $\begin{bmatrix} \sigma^2_{xL} & \rho_{xyL}\sigma_{xL}\sigma_{yL} & 0 & 0 \\ \rho_{xyL}\sigma_{xL}\sigma_{yL} & \sigma^2_{yL} & 0 & 0 \\ 0 & 0 & \sigma^2_{xU} & \rho_{xyL}\sigma_{v}\sigma_{yL} \\ 0 & 0 & \rho_{xyU}\sigma_{xU}\sigma_{yU} & \sigma^2_{yU} \end{bmatrix}$, where we have $\sigma^2_{xL} = 100$, $\sigma^2_{yL} = 100$, $\sigma^2_{xU} = 121$, $\sigma^2_{yU} = 121$.

2. For this N = 1000 simulated population, the parameters were computed.

3. A sample of size n with $m_N = 3$ and $r = 4, 6, 10$ has been selected from this simulated population.

4. To find the MSE of each estimator under study, use the sample data.

5. To get MSEs, the entire step 3–4 process was repeated 10,000 times. The MSE of each population mean estimator is the average of the 10,000 values that were obtained.

6. The PRE of each estimator in relation to $\overline{y}_{[n]N}$ has been calculated using the formula.

7. It can be done for some other population with parameters $(\mu_{xL},\ \mu_{yL}\ ,\ \mu_{xU}, \mu_{yU}) = (150, 150, 200, 200)$ where we have $\sigma^2_{xL} = 625$, $\sigma^2_{yL} = 625$, $\sigma^2_{xU} = 961$, $\sigma^2_{yU} = 961$.

Table 3: MSEs and PREs of the recommended and existing estimators under NRSS for Population 1

n=12	ρ=0.9			ρ=0.8		
Estimators	MSE	I_N	PRE	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[4.74611, 8.39614]	[0, 0.43]	[100, 100]	[5.28228, 8.15711]	[0, 0.35]	[100, 100]
$\overline{y}_{rN}$	[1.45809, 1.7075]	[0, 0.14]	[326, 492]	[2.85339, 3.37538]	[0, 0.15]	[185, 242]
$\overline{y}_{regN}$	[1.27631, 1.45609]	[0, 0.12]	[372, 577]	[2.41149, 2.65902]	[0, 0.09]	[219, 307]
$\overline{y}_{expN}$	[2.07258, 2.97855]	[0, 0.30]	[229, 282]	[3.03606 , 3.68283]	[0, 0.17]	[174, 221]
$\overline{y}_{vsN}$	[1.27631, 1.45609]	[0, 0.12]	[372, 577]	[2.41149, 2.65902]	[0, 0.09]	[219, 307]
t_{p1}	[1.27201, 1.44744]	[0, 0.12]	[373, 580]	[2.40421, 2.64772]	[0, 0.09]	[220, 308]
t_{p2}	[0.62571, 0.99329]	[0, 0.37]	[759, 845]	[0.81364, 1.50894]	[0, 0.46]	[541, 649]
t_{p3}	[0.46767, 0.53091]	[0, 0.11]	[1015, 1581]	[0.9161, 1.0529]	[0, 0.12]	[577, 775]
	ρ=0.7			ρ=0.6		
	MSE	I_N	PRE	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[5.80652, 8.11845]	[0, 0.28]	[100, 100]	[6.14863, 8.47812]	[0, 0.27]	[100, 100]
$\overline{y}_{rN}$	[4.21099, 5.02332]	[0, 0.16]	[138, 162]	[5.45556, 6.86807]	[0, 0.20]	[113, 123]
$\overline{y}_{regN}$	[3.46994, 3.75404]	[0, 0.07]	[167, 216]	[4.25988, 4.95358]	[0, 0.14]	[144, 171]
$\overline{y}_{expN}$	[3.97544, 4.50379]	[0, 0.11]	[146, 180]	[4.76362, 5.58763]	[0, 0.14]	[129, 152]
$\overline{y}_{vsN}$	[3.46994, 3.75404]	[0, 0.07]	[167, 216]	[4.25988, 4.95358]	[0, 0.14]	[144, 171]
t_{p1}	[3.45901, 3.74]	[0, 0.07]	[168, 217]	[4.24612, 4.93587]	[0, 0.13]	[145, 172]
t_{p2}	[0.89482, 1.7252]	[0, 0.48]	[471, 649]	[0.92517, 1.84276]	[0, 0.49]	[460, 665]
t_{p3}	[1.34764, 1.57163]	[0, 0.14]	[431, 517]	[1.74662, 2.15988]	[0, 0.19]	[352, 393]
n=18	ρ=0.9			ρ=0.8		
	MSE	I_N	PRE	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[3.28385, 5.96403]	[0, 0.44]	[100, 100]	[3.68678, 5.79281]	[0, 0.36]	[100, 100]

$\overline{y}_{rN}$	[1.03602,1.21419]	[0, 0.14]	[317, 491]	[2.02872, 2.39945]	[0, 0.15]	[182, 241]
$\overline{y}_{regN}$	[0.9431, 1.08339]	[0, 0.12]	[348, 550]	[1.78499, 1.98643]	[0, 0.10]	[207, 292]
$\overline{y}_{expN}$	[1.45551, 2.11756]	[0, 0.31]	[226, 282]	[2.1514, 2.61751]	[0, 0.17]	[171, 221]
$\overline{y}_{vsN}$	[0.9431, 1.08339]	[0, 0.12]	[348, 550]	[1.78499, 1.98643]	[0, 0.10]	[207, 292]
t_{p1}	[0.94105, 1.07914]	[0, 0.12]	[349, 553]	[1.78141, 1.98074]	[0, 0.10]	[207, 292]
t_{p2}	[0.453, 0.74575]	[0, 0.39]	[725, 800]	[0.58369, 1.12502]	[0, 0.48]	[515, 632]
t_{p3}	[0.34604, 0.4]	[0, 0.13]	[949, 1491]	[0.67659, 0.79081]	[0, 0.14]	[545, 733]
	ρ=0.7			ρ=0.6		
$\overline{y}_{[n]N}$	[4.06532, 5.77401]	[0, 0.29]	[100, 100]	[4.32649, 6.02582]	[0, 0.28]	[100, 100]
$\overline{y}_{rN}$	[2.96637, 3.57305]	[0, 0.16]	[137, 162]	[3.85889, 4.88325]	[0, 0.20]	[112, 123]
$\overline{y}_{regN}$	[2.54775, 2.79559]	[0, 0.08]	[160, 207]	[3.14193, 3.68093]	[0, 0.14]	[138, 164]
$\overline{y}_{expN}$	[2.81002, 3.20285]	[0, 0.12]	[145, 180]	[3.38185, 3.97694]	[0, 0.14]	[128, 152]
$\overline{y}_{vsN}$	[2.54775, 2.79559]	[0, 0.08]	[160, 207]	[3.14193, 3.68093]	[0, 0.14]	[138, 164]
t_{p1}	[2.54239, 2.78846]	[0, 0.08]	[160, 207]	[3.13511, 3.67194]	[0, 0.14]	[138, 164]
t_{p2}	[0.63681, 1.28576]	[0, 0.50]	[449, 638]	[0.6594, 1.36624]	[0, 0.51]	[441, 656]
t_{p3}	[0.98574, 1.17705]	[0, 0.16]	[412, 491]	[1.28237 , 1.61483]	[0, 0.20]	[337, 373]
n=30	ρ=0.9			ρ=0.8		
	MSE	I_N	PRE	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[2.02919, 3.74983]	[0, 0.45]	[100, 100]	[2.28937, 3.65813]	[0, 0.37]	[100, 100]
$\overline{y}_{rN}$	[0.64949, 0.76453]	[0, 0.15]	[312, 490]	[1.27051, 1.51332]	[0, 0.16]	[180, 242]
$\overline{y}_{regN}$	[0.6084, 0.70205]	[0, 0.13]	[334, 534]	[1.15219, 1.29443]	[0, 0.10]	[199, 283]
$\overline{y}_{expN}$	[0.90713, 1.33037]	[0, 0.31]	[224, 282]	[1.34678, 1.65497]	[0, 0.18]	[170, 221]
$\overline{y}_{vsN}$	[0.6084, 0.70205]	[0, 0.13]	[334, 534]	[1.15219, 1.29443]	[0, 0.10]	[199, 283]
t_{p1}	[0.60762, 0.70039]	[0, 0.13]	[334, 535]	[1.1508, 1.29218]	[0, 0.10]	[199, 283]
t_{p2}	[0.28907, 0.48705]	[0, 0.40]	[702, 770]	[0.36999, 0.72973]	[0, 0.49]	[501, 619]
t_{p3}	[0.2232, 0.26151]	[0, 0.14]	[909, 1434]	[0.43531, 0.51667]	[0, 0.15]	[526, 708]
	ρ=0.7			ρ=0.6		
	MSE	I_N	PRE	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[2.53419, 3.64756]	[0, 0.30]	[100, 100]	[2.70872, 3.79625]	[0, 0.28]	[100, 100]
$\overline{y}_{rN}$	[1.85449, 2.25458]	[0, 0.17]	[137, 162]	[2.41511, 3.07899]	[0, 0.21]	[112, 123]
$\overline{y}_{regN}$	[1.63839, 1.81847]	[0, 0.09]	[155, 201]	[2.03392, 2.38552]	[0, 0.14]	[133, 159]
$\overline{y}_{expN}$	[1.76129, 2.02175]	[0, 0.12]	[144, 180]	[2.12632, 2.50541]	[0, 0.15]	[127, 152]
$\overline{y}_{vsN}$	[1.63839, 1.81847]	[0, 0.09]	[155, 201]	[2.03392, 2.38552]	[0, 0.14]	[133, 159]
t_{p1}	[1.63631, 1.81563]	[0, 0.09]	[155, 201]	[2.03124, 2.38195]	[0, 0.14]	[133, 159]
t_{p2}	[0.40188, 0.83762]	[0, 0.52]	[435, 631]	[0.41655, 0.8877]	[0, 0.53]	[428, 650]
t_{p3}	[0.63275, 0.76768]	[0, 0.17]	[401, 475]	[0.82381, 1.05238]	[0, 0.21]	[329, 361]

Table 4: MSEs and PREs of the recommended and existing estimator under NRSS for Population 2

n=12	ρ=0.9			ρ=0.8		
Estimators	MSE	I_N	PRE	MSE	I_N	PRE
$\overline{y}_{[n]N}$	[29.57415, 66.14823]	[0, 0.55]	[100, 100]	[33.01424, 64.78497]	[0, 0.49]	[100, 100]
$\overline{y}_{rN}$	[9.15248, 13.43196]	[0, 0.31]	[323, 492]	[17.79146, 26.83172]	[0, 0.33]	[186, 241]
$\overline{y}_{regN}$	[8.07017, 11.47311]	[0, 0.29]	[366, 577]	[15.07181, 21.11836]	[0, 0.28]	[219, 307]
$\overline{y}_{expN}$	[12.91778, 23.47267]	[0, 0.44]	[229, 282]	[18.97746, 29.18226]	[0, 0.34]	[174, 222]
$\overline{y}_{vsN}$	[8.07017, 11.47311]	[0, 0.29]	[366, 577]	[15.07181, 21.11836]	[0, 0.28]	[219, 307]
t_{p1}	[8.06334, 11.45174]	[0, 0.29]	[367, 578]	[15.06036, 21.09007]	[0, 0.28]	[219, 307]
t_{p2}	[3.96677, 7.94098]	[0, 0.50]	[746, 833]	[5.0761, 12.12343]	[0, 0.58]	[534, 650]
t_{p3}	[2.96821, 4.26557]	[0, 0.30]	[996, 1551]	[5.75332, 8.46915]	[0, 0.32]	[574, 765]
	ρ=0.7			ρ=0.6		
	MSE	PRE	MSE	PRE		
$\overline{y}_{[n]N}$	[36.2907, 66.5611]	[0, 0.45]	[100, 100]	[38.4289, 67.3345]	[0, 0.42]	[100, 100]
$\overline{y}_{rN}$	[26.3081, 40.3418]	[0, 0.34]	[138, 165]	[33.9908, 54.1438]	[0, 0.37]	[113, 124]
$\overline{y}_{regN}$	[21.6871, 30.9702]	[0, 0.29]	[167, 215]	[26.6243, 39.3420]	[0, 0.32]	[144, 171]
$\overline{y}_{expN}$	[24.8384, 37.1523]	[0, 0.33]	[146, 179]	[29.7571, 44,3811]	[0, 0.32]	[129, 152]
$\overline{y}_{vsN}$	[21.6871, 30.9702]	[0, 0.29]	[167, 215]	[26.6243, 39.3420]	[0, 0.32]	[144, 171]
t_{p1}	[21.67, 30.9336]	[0, 0.29]	[167, 215]	[26.6026, 39.2976]	[0, 0.32]	[144, 171]
t_{p2}	[5.6044, 13.5728]	[0, 0.58]	[490, 648]	[5.7578, 14.4769]	[0, 0.60]	[465, 667]
t_{p3}	[8.4746, 12.8287]	[0, 0.33]	[428, 519]	[10.9482, 17.1972]	[0, 0.36]	[351, 392]
n=18	ρ=0.9			ρ=0.8		
	MSE	PRE	MSE	PRE		
$\overline{y}_{[n]N}$	[20.46536, 47.09541]	[0, 0.56]	[100, 100]	[23.0424, 46.00735]	[0, 0.49]	[100, 100]
$\overline{y}_{rN}$	[6.46258, 9.56084]	[0, 0.32]	[317, 493]	[12.65581, 19.08902]	[0, 0.33]	[182, 241]
$\overline{y}_{regN}$	[5.91444, 8.54838]	[0, 0.30]	[346, 551]	[11.15622, 15.77654]	[0, 0.29]	[207, 292]
$\overline{y}_{expN}$	[9.0553, 16.73269]	[0, 0.45]	[226, 281]	[13.44981, 20.74272]	[0, 0.35]	[171, 222]
$\overline{y}_{vsN}$	[5.91444, 8.54838]	[0, 0.30]	[346, 551]	[11.15622, 15.77654]	[0, 0.29]	[207, 292]
t_{p1}	[5.91119, 8.53782]	[0, 0.30]	[346, 552]	[11.15058, 15.76229]	[0, 0.29]	[207, 292]
t_{p2}	[2.85023, 5.91261]	[0, 0.51]	[718, 797]	[3.63777, 9.02249]	[0, 0.59]	[510, 633]
t_{p3}	[2.17347, 3.19023]	[0, 0.31]	[942, 1476]	[4.23969, 6.33831]	[0, 0.33]	[543, 726]

	ρ=0.7			ρ=0.6		
	MSE	PRE	MSE	PRE		
$\overline{y}_{[n]N}$	[25.40823, 47.41032]	[0, 0.46]	[100, 100]	[27.04054, 47.85793]	[0, 0.43]	[100, 100]
$\overline{y}_{rN}$	[18.53928, 28.71605]	[0, 0.35]	[137, 165]	[24.0593, 38.54122]	[0, 0.37]	[112, 124]
$\overline{y}_{regN}$	[15.92344, 23.08036]	[0, 0.31]	[160, 205]	[19.63708, 29.23447]	[0, 0.32]	[138, 164]
$\overline{y}_{expN}$	[17.55694, 26.47846]	[0, 0.33]	[145, 179]	[21.13014, 31.59581]	[0, 0.33]	[128, 151]
$\overline{y}_{vsN}$	[15.92344, 23.08036]	[0, 0.31]	[160, 205]	[19.63708, 29.23447]	[0, 0.32]	[138, 164]
t_{p1}	[15.91501, 23.06175]	[0, 0.30]	[160, 206]	[19.62635, 29.21191]	[0, 0.32]	[138, 164]
t_{p2}	[3.98907, 10.09363]	[0, 0.60]	[470, 637]	[4.10388, 10.73474]	[0, 0.61]	[446, 659]
t_{p3}	[6.18868, 9.58314]	[0, 0.35]	[411, 495]	[8.02644, 12.83379]	[0, 0.37]	[337, 373]
n=30	ρ=0.9			ρ=0.8		
	MSE	PRE	MSE	PRE		
$\overline{y}_{[n]N}$	[12.6555, 29.7430]	[0, 0.57]	[100, 100]	[14.3085, 29.0534]	[0, 0.50]	[100, 100]
$\overline{y}_{rN}$	[4.0533, 6.0421]	[0, 0.32]	[312, 492]	[7.9305, 12.0457]	[0, 0.34]	[180, 241]
$\overline{y}_{regN}$	[3.8106, 5.5601]	[0, 0.31]	[332, 535]	[7.2011, 10.2805]	[0, 0.29]	[199, 283]
$\overline{y}_{expN}$	[5.6476, 10.5702]	[0, 0.46]	[224, 281]	[8.4200, 13.1170]	[0, 0.35]	[170, 221]
$\overline{y}_{vsN}$	[3.8106, 5.5601]	[0, 0.31]	[332, 535]	[7.2011, 10.2805]	[0, 0.29]	[199, 283]
t_{p1}	[3.8093, 5.5560]	[0, 0.31]	[332, 535]	[7.19898, 10.2749]	[0, 0.29]	[199, 283]
t_{p2}	[1.8154, 3.8495]	[0, 0.52]	[697, 773]	[2.3060, 5.8458]	[0, 0.60]	[497, 620]
t_{p3}	[1.3995, 2.0815]	[0, 0.32]	[904,1429]	[2.7238, 4.1296]	[0, 0.34]	[525, 704]
	ρ=0.7			ρ=0.6		
	MSE	PRE	MSE	PRE		
$\overline{y}_{[n]N}$	[15.83871, 29.97289]	[0, 0.47]	[100, 100]	[16.9295, 30.15039]	[0, 0.43]	[100, 100]
$\overline{y}_{rN}$	[11.59369, 18.12673]	[0, 0.36]	[137, 165]	[15.06837, 24.32259]	[0, 0.38]	[112, 124]
$\overline{y}_{regN}$	[10.23995, 14.99098]	[0, 0.31]	[155, 200]	[12.712, 18.94614]	[0, 0.32]	[133, 159]
$\overline{y}_{expN}$	[11.00532, 16.73579]	[0, 0.34]	[144, 179]	[13.28795, 19.90633]	[0, 0.33]	[127, 151]
$\overline{y}_{vsN}$	[10.23995, 14.99098]	[0, 0.31]	[155, 200]	[12.712, 18.94614]	[0, 0.32]	[133, 159]
t_{p1}	[10.23667, 14.98357]	[0, 0.31]	[155, 200]	[12.70778, 18.93718]	[0, 0.32]	[133, 159]
t_{p2}	[2.51734, 6.55744]	[0, 0.61]	[457, 629]	[2.59353, 6.98028]	[0, 0.62]	[432, 653]
t_{p3}	[3.96677, 6.23344]	[0, 0.36]	[399, 481]	[5.15061, 8.35123]	[0, 0.38]	[329, 361]

Table 5: PREs of the NRSS estimators over estimators under NSRS for Population 1

n=12	ρ=0.9	ρ=0.8	ρ=0.7	ρ=0.6
Estimators	PRE	PRE	PRE	PRE
$\overline{y}_{[n]N}$	[120, 183]	[120, 165]	[120, 149]	[120, 142]
$\overline{y}_{rN}$	[119, 122]	[119, 125]	[120, 126]	[119, 131]
$\overline{y}_{rgN}$	[120, 121]	[120, 121]	[119, 121]	[120, 121]
$\overline{y}_{expN}$	[119, 148]	[119, 131]	[119, 121]	[119, 121]
$\overline{y}_{vsN}$	[120, 121]	[120, 121]	[119, 121]	[120, 121]
t_{p1}	[120, 121]	[120, 121]	[119, 121]	[120, 121]
t_{p2}	[121, 170]	[122, 193]	[123, 205]	[122, 210]
t_{p3}	[121, 121]	[121, 125]	[122, 128]	[121, 131]
n=18	ρ=0.9	ρ=0.8	ρ=0.7	ρ=0.6
	PRE	PRE	PRE	PRE
$\overline{y}_{[n]N}$	[112, 174]	[112, 156]	[112, 142]	[112, 133]
$\overline{y}_{rN}$	[112, 115]	[112, 118]	[113, 120]	[112, 124]
$\overline{y}_{regN}$	[112, 113]	[112, 113]	[113, 113]	[112, 113]
$\overline{y}_{expN}$	[112, 139]	[112, 122]	[112, 115]	[112, 113]
$\overline{y}_{vsN}$	[112, 113]	[112, 113]	[113, 113]	[112, 113]
t_{p1}	[112, 113]	[112, 113]	[112, 113]	[112, 113]
t_{p2}	[113, 165]	[112, 188]	[114, 197]	[113, 204]
t_{p3}	[113, 114]	[113, 118]	[113, 121]	[113, 124]
n=30	ρ=0.9	ρ=0.8	ρ=0.7	ρ=0.6
	PRE	PRE	PRE	PRE
$\overline{y}_{[n]N}$	[107, 169]	[107, 150]	[107, 136]	[107, 127]
$\overline{y}_{rN}$	[106, 110]	[106, 112]	[107, 115]	[106, 118]
$\overline{y}_{regN}$	[107, 107]	[106, 107]	[107, 107]	[107, 107]
$\overline{y}_{expN}$	[107, 133]	[107, 116]	[107, 110]	[107, 107]
$\overline{y}_{vsN}$	[107, 107]	[106, 107]	[107, 107]	[107, 107]
t_{p1}	[107, 107]	[106, 107]	[107, 107]	[107, 107]
t_{p2}	[107, 159]	[107, 183]	[108, 192]	[107, 199]
t_{p3}	[107, 109]	[107, 112]	[107, 116]	[107, 119]

Table 6: PREs of the NRSS estimators over estimators under NSRS for Population 2

n=12	ρ=0.9	ρ=0.8	ρ=0.7	ρ=0.6
Estimators	PRE	PRE	PRE	PRE
$\overline{y}_{[n]N}$	[121, 183]	[120, 165]	[121, 149]	[120, 142]
$\overline{y}_{rN}$	[120, 121]	[119, 125]	[120, 126]	[119, 131]
$\overline{y}_{regN}$	[120, 122]	[120, 121]	[119, 123]	[120, 121]
$\overline{y}_{expN}$	[120, 147]	[119, 131]	[121, 121]	[119, 121]
$\overline{y}_{vsN}$	[120, 122]	[120, 121]	[119, 123]	[120, 121]
t_{p1}	[119, 122]	[120, 121]	[119, 122]	[120, 121]
t_{p2}	[120, 167]	[121, 194]	[123, 205]	[123, 210]
t_{p3}	[120, 120]	[120, 125]	[122, 127]	[121, 131]
n=18	ρ=0.9	ρ=0.8	ρ=0.7	ρ=0.6
	PRE	PRE	PRE	PRE
$\overline{y}_{[n]N}$	[112, 176]	[112, 156]	[113, 142]	[112, 133]
$\overline{y}_{rN}$	[113, 115]	[112, 118]	[113, 120]	[112, 124]
$\overline{y}_{regN}$	[113, 113]	[112, 113]	[113, 114]	[112, 113]
$\overline{y}_{expN}$	[141, 112]	[122, 113]	[115, 113]	[113, 112]
$\overline{y}_{vsN}$	[113, 113]	[112, 113]	[112, 113]	[112, 113]
t_{p1}	[113, 113]	[112, 113]	[112, 113]	[112, 113]
t_{p2}	[113, 161]	[112, 189]	[114, 197]	[144, 205]
t_{p3}	[113, 115]	[112, 118]	[113, 121]	[113, 124]
n=30	ρ=0.9	ρ=0.8	ρ=0.7	ρ=0.6
	PRE	PRE	PRE	PRE
$\overline{y}_{[n]N}$	[107, 170]	[107, 150]	[107, 136]	[107, 127]
$\overline{y}_{rN}$	[107, 109]	[106, 112]	[107, 115]	[107, 118]
$\overline{y}_{regN}$	[107, 107]	[106, 107]	[107, 107]	[107, 107]
$\overline{y}_{expN}$	[107, 135]	[107, 116]	[107, 110]	[107, 107]
$\overline{y}_{vsN}$	[107, 107]	[106, 107]	[107, 107]	[107, 107]
t_{p1}	[107, 107]	[106, 107]	[107, 107]	[106, 107]
t_{p2}	[107, 156]	[106, 183]	[108, 192]	[108, 199]
t_{p3}	[107, 109]	[106, 113]	[107, 116]	[107, 119]

Table 7: PREs of the estimators (neutrosophic vs classical) for Population 1

n=12	PRE		PRE		PRE		PRE	
	ρ=0.9		ρ=0.8		ρ=0.7		ρ=0.6	
Estimators	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical
$\overline{y}_{[n]N}$	[100, 100]	100	[100, 100]	100	[100, 100]	100	[100, 100]	100
$\overline{y}_{rN}$	[326, 492]	291	[185, 242]	167	[138, 162]	125	[113, 123]	103
$\overline{y}_{regN}$	[372, 577]	364	[219, 307]	215	[167, 216]	166	[144, 171]	142
$\overline{y}_{expN}$	[229, 282]	246	[174, 221]	179	[146, 180]	148	[129, 152]	129
$\overline{y}_{vsN}$	[372, 577]	364	[219, 307]	215	[167, 216]	166	[144, 171]	142
t_{p1}	[373, 580]	366	[220, 308]	215	[168, 217]	166	[145, 172]	143
t_{p2}	[759, 845]	559	[541, 649]	497	[471, 649]	506	[460, 665]	530
t_{p3}	[1015,1581]	923	[577, 775]	523	[431, 517]	390	[352, 393]	320
n=18	PRE		PRE		PRE		PRE	
	ρ=0.9		ρ=0.8		ρ=0.7		ρ=0.6	
	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical
$\overline{y}_{[n]N}$	[100, 100]	100	[100, 100]	100	[100, 100]	100	[100, 100]	100
$\overline{y}_{rN}$	[317, 491]	285	[182, 241]	165	[137, 162]	124	[112, 123]	103
$\overline{y}_{regN}$	[348, 550]	342	[207, 292]	203	[160, 207]	158	[138, 164]	136
$\overline{y}_{expN}$	[226, 282]	243	[171, 221]	177	[145, 180]	146	[128, 152]	128
$\overline{y}_{vsN}$	[348, 550]	342	[207, 292]	203	[160, 207]	158	[138, 164]	136
t_{p1}	[349, 553]	343	[207, 292]	204	[160, 207]	158	[138, 164]	136
t_{p2}	[725, 800]	534	[515, 632]	482	[449, 638]	496	[441, 656]	522
t_{p3}	[949, 1491]	862	[545, 733]	493	[412, 491]	371	[337, 373]	307
n=30	PRE		PRE		PRE		PRE	
	ρ=0.9		ρ=0.8		ρ=0.7		ρ=0.6	
	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical
$\overline{y}_{[n]N}$	[100, 100]	100	[100, 100]	100	[100, 100]	100	[100, 100]	100
$\overline{y}_{rN}$	[312, 490]	280	[180, 242]	163	[137, 162]	123	[112, 123]	102
$\overline{y}_{regN}$	[334, 534]	326	[199, 283]	195	[155, 201]	152	[133, 159]	131
$\overline{y}_{expN}$	[224, 282]	240	[170, 221]	175	[144, 180]	145	[127, 152]	127
$\overline{y}_{vsN}$	[334, 534]	326	[199, 283]	195	[155, 201]	152	[133, 159]	131
t_{p1}	[334, 535]	327	[199, 283]	195	[155, 201]	152	[133, 159]	132
t_{p2}	[702, 770]	521	[501, 619]	476	[435, 631]	491	[428, 650]	519
t_{p3}	[909,1434]	821	[526, 708]	474	[401, 475]	357	[329, 361]	296

Table 8: PREs of the estimators (neutrosophic vs classical) for Population 2

n=12	PRE		PRE		PRE		PRE	
	ρ=0.9		ρ=0.8		ρ=0.7		ρ=0.6	
Estimators	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical
$\overline{y}_{[n]N}$	[100, 100]	100	[100, 100]	100	[100, 100]	100	[100, 100]	100
$\overline{y}_{rN}$	[323, 492]	322	[186, 241]	183	[138, 165]	136	[113, 124]	112
$\overline{y}_{regN}$	[366, 577]	369	[219, 307]	217	[167, 215]	168	[144, 171]	144
$\overline{y}_{expN}$	[229, 282]	227	[174, 222]	173	[146, 179]	146	[129, 152]	129
$\overline{y}_{vsN}$	[366, 577]	369	[219, 307]	217	[167, 215]	168	[144, 171]	144
t_{p1}	[367, 578]	369	[219, 307]	217	[167, 215]	168	[144, 171]	144
t_{p2}	[746, 833]	757	[534, 650]	641	[490, 648]	635	[465, 667]	654
t_{p3}	[996,1551]	1013	[574, 765]	574	[428, 519]	429	[351, 392]	352
n=18	PRE		PRE		PRE		PRE	
	ρ=0.9		ρ=0.8		ρ=0.7		ρ=0.6	
	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical
$\overline{y}_{[n]N}$	[100, 100]	100	[100, 100]	100	[100, 100]	100	[100, 100]	100
$\overline{y}_{rN}$	[317, 493]	314	[182, 241]	180	[137, 165]	135	[112, 124]	112
$\overline{y}_{regN}$	[346, 551]	344	[207, 292]	205	[160, 205]	160	[138, 164]	137
$\overline{y}_{expN}$	[226, 281]	223	[171, 222]	171	[145, 179]	144	[128, 151]	128
$\overline{y}_{vsN}$	[346, 551]	344	[207, 292]	205	[160, 205]	160	[138, 164]	137
t_{p1}	[346, 552]	345	[207, 292]	205	[160, 206]	160	[138, 164]	137
t_{p2}	[718, 797]	725	[510, 633]	623	[470, 637]	622	[446, 659]	642
t_{p3}	[942,1476]	945	[543, 726]	543	[411, 495]	408	[337, 373]	337
n=30	PRE		PRE		PRE		PRE	
	ρ=0.9		ρ=0.8		ρ=0.7		ρ=0.6	
	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical	neutrosophic	classical
$\overline{y}_{[n]N}$	[100, 100]	100	[100, 100]	100	[100, 100]	100	[100, 100]	100
$\overline{y}_{rN}$	[312, 492]	308	[180, 241]	178	[137, 165]	134	[112, 124]	111
$\overline{y}_{regN}$	[332, 535]	329	[199, 283]	197	[155, 200]	154	[133, 159]	133
$\overline{y}_{expN}$	[224, 281]	221	[170, 221]	169	[144, 179]	143	[127, 151]	127
$\overline{y}_{vsN}$	[332, 535]	329	[199, 283]	197	[155, 200]	154	[133, 159]	133
t_{p1}	[332, 535]	329	[199, 283]	197	[155, 200]	154	[133, 159]	133
t_{p2}	[697, 773]	703	[497, 620]	611	[457, 629]	611	[432, 653]	634
t_{p3}	[904,1429]	900	[525, 704]	520	[399, 481]	392	[329, 361]	325

8. Discussion

The study established mathematical expressions for novel NRSS estimators, approximating up to the first order. Subsequently to examine the properties of the proposed NRSS estimators, numerical illustrations and simulation studies were conducted. The former used real-world natural growth rate data, while the latter involved two artificial neutrosophic datasets with varying correlation coefficients and sample sizes. The results were encapsulated in Tables 2, 3, and 4, showcasing MSEs and PREs for both existing and proposed neutrosophic ranked set estimators. We have computed the PREs of the NRSS estimators over estimators under NSRS and these results are displayed in Tables 5 and 6.

In Table 2, the MSEs of the existing and proposed estimators are given along with PRE. The superiority of the suggested NRSS estimators over the existing NRSS estimators is displayed in Table 2 in the bolded text. We also see the MSE and PRE of the recommended estimator are lesser and higher than other existing estimators. It is evident from the table that recommended estimators outperformed existing ones, offering lower MSEs and higher PREs, and it has been established that t_{p3} is the best estimator available.

Similarly, in Table 3 and Table 4, the MSEs of the recommended and existing estimators are given along with PRE through a simulation study based on artificial neutrosophic data for different values of the correlation coefficient and different sample sizes. Like Table 2, also in Tables 3 and 4, the superiority of the suggested NRSS estimators over the existing NRSS estimators is displayed by the bolded text. We also see the MSEs and PREs of the recommended estimators are lesser and higher, respectively than those of other existing estimators. Hence, Tables 3 and 4 mirrored these findings, with the proposed estimators continuing to outshine existing ones, demonstrating lower MSEs and higher PREs in the simulation study too.

From Tables 3 and 4, we see with the increase in values of sample sizes, and correlation coefficients, the MSE and PRE of the recommended estimator decrease and increase. Therefore, under NRSS, the suggested estimators exhibit sensitivity similar to that of classical ranked set sampling.

Tables 5 and 6 featured PRE values of the proposed NRSS estimators over NSRS counterparts. We see from Tables 5 and 6, that all PRE values exceeded 100 that is all the NRSS estimators are superior to corresponding estimators under NSRS as RSS is the best replacement for SRS. The comparison between classical RSS and NRSS using PREs is provided in Tables 7 and 8. Tables 7 and 8 demonstrate that the PREs of the suggested estimators obtained through classical RSS are lower than those obtained using NRSS, indicating that the latter method is more effective than the former.

The study highlighted that classical ranked set sampling was ill-suited for dealing with vague or indeterminate data. NRSS proved superior for estimating uncertain or interval data. The tables presented dependable results for neutrosophic data compared to classical results.

9. Conclusion

In this research paper, we've put forth some enhanced neutrosophic ranked set estimators designed for estimating population means while making use of auxiliary information. To assess their accuracy, we calculated both bias and MSE for these proposed estimators, focusing on first-order approximations. We compared our recommended estimators against existing ones, by using a natural population's data on natural growth rates and two simulated populations. Through a combination of numerical illustrations and simulated studies, we've found compelling evidence that our proposed estimators outperform existing ones within the framework of neutrosophic ranked set sampling. Among these estimators, t_{p3} emerged as the top performer. It's important to note that the sensitivity analysis of our recommended estimators under NRSS mirrors that of classical RSS.

Moreover, a comparison between the recommended estimators under NRSS and the estimators under NSRS revealed that NRSS is a more effective alternative to NSRS, much like classical RSS to classical SRS. Our study underscores the efficiency and reliability of NRSS for handling neutrosophic data, with our proposed NRSS delivering superior mean estimations compared to existing methods.

The current investigation is subject to certain constraints, notably concerning the applicability of neutrosophic ranked set sampling. This method proves to be proficient in estimating population parameters under conditions of equal allocation, perfect ranking, and adherence to a symmetric distribution. However, when these conditions are not met, the efficiency of the estimation diminishes, leading to suboptimal results.

Based on the numerical illustrations and simulation studies we've conducted, it's reasonable to recommend the use of our proposed estimators over the alternatives presented in this paper in various real-world scenarios, spanning fields like agriculture, mathematics, biology, poultry farming, economics, commerce, and the social sciences.

Furthermore, given the limited availability of neutrosophic RSS estimators, there's ample room for further exploration. Building upon this study, we can consider defining variations of neutrosophic ranked set sampling, such as unbalanced NRSS, median NRSS, extreme NRSS, double NRSS, and percentile neutrosophic ranked set sampling, akin to what exists in classical ranked set sampling. Additionally, we can explore the replacement of our proposed estimators with alternative methods or estimators.

Expanding beyond sampling theory, further research avenues in statistics, encompassing fields like control charts, inference, reliability analysis, non-parametric estimation, hypothesis testing, and some other fields of science, present promising opportunities for exploration.

Conflicts of Interest: The authors declare no conflict of interest.

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TreeSoft Approach for Refining Air Pollution Analysis: A Case Study

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Abstract: The ongoing research employs both the neutrosophic set and the Decision Making and Evaluation Method (DEMATEL) to analyze and determine the factors influencing supplier selection within the realm of Supply Chain Management (SCM). Recognized as a proactive strategy for enhancing performance and securing competitive advantages, DEMATEL guides this investigation. The study utilizes neutrosophic set Theory to refine overall assessments, introducing a novel scale to distinctly represent each value. Through a practical case study focusing on selecting the optimal supplier for a distribution company, the proposed methodology's application is illustrated. The research framework integrates a neutrosophic DEMATEL approach for data collection, incorporating surveys among experts and interviews with professionals in management, procurement, and production. In addressing application-oriented challenges characterized by multiple criteria marked by ambiguity and varying degrees of accuracy, Smarandache introduced Treesoft sets an extension of hypersoft sets to effectively navigate through ambiguous and imprecise options.

Keywords: Air pollution, DEMATEL method, Neutrosophic set, Treesoft set.

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1. Literature Review

Air pollution [1] stands as one of the most significant environmental hazards to human health. By mitigating air pollution, nations can alleviate the burden of diseases such as stroke, heart disease, lung cancer, chronic respiratory diseases, and asthma, both chronic and acute. Shockingly, in 2019, 99% of the global population inhabited areas failing to meet the air quality standards set by the World Health Organization (WHO). The combined impact of ambient air pollution and household air pollution contributes to approximately 6.7 million premature deaths annually. Specifically, outdoor air pollution was responsible for an estimated 4.2 million premature deaths worldwide in 2019, with 89% occurring in low- and middle-income countries, notably concentrated in the WHO Southeast Asia and Western Pacific Regions. Implementing policies and investments that promote cleaner transportation, energy-efficient housing, sustainable power generation, industrial practices, and enhanced municipal waste management would effectively curb major sources of outdoor air pollution. Additionally, ensuring access to clean household energy sources would significantly diminish ambient air pollution in various regions.

Decision-making encompasses a nuanced cognitive process aimed at problem-solving while considering multiple factors to achieve desired outcomes. This process may unfold rationally or irrationally, driven by implicit and explicit assumptions influenced by biological, cultural, physiological, and social dynamics. Decision-making complexity is further shaped by the level of risk and authority involved. In contemporary times, computer technologies facilitate automated calculations and estimations for decision-making conundrums, leveraging mathematical equations, diverse statistical approaches, and economic theories.

Multicriteria decision-making [6] (MCDM) endeavors to identify optimal choices by weighing several criteria throughout the selection process. These methods find application across diverse sectors like finance, engineering, and robotics[2]–[5].

In this realm, the Decision-Making Trial and Evaluation Laboratory [6] (DEMATEL) method emerges as a systematic approach for exploring cause-and-effect relationships among factors within complex systems, finding extensive utility across various domains. However, the DEMATEL method heavily relies on expert judgment, introducing a subjective element into the analysis[7], [8]. Overall, the DEMATEL method serves as a potent tool for dissecting intricate relationships and dependencies among factors within specific contexts, offering valuable insights for decision-makers[9], [10].

It's crucial to mitigate the subjectivity inherent in expert judgment, particularly in fields like aerospace where objective data are often scarce. Recent efforts have integrated methods such as Criteria Importance Through Intercriteria Correlation (CRITIC), artificial neural networks, analytic hierarchy process, and analytic network process into DEMATEL to lessen subjectivity by refining data processing techniques. However, these endeavors have not fully tackled subjectivity during the data acquisition phase.

To address this gap, fuzzy triangular numbers and the multi-criteria group-based decision-making (MCGDM) method have been extensively utilized to quantify the uncertainty surrounding expert opinions. While prior studies focused primarily on reducing subjectivity through either data processing or data collection alone, there's a notable absence of comprehensive research addressing subjectivity from both perspectives.

Thus, this study aims to establish a systematic framework for selecting influential factors and mitigating subjectivity in both the data collection and analysis processes, thereby offering a more holistic approach to addressing this issue.

The hypersoft sets, an extension of soft sets, utilize a multi-argument approximation function to address limitations in current structures for attribute-valued disjoint sets. Subsequently, Multi Soft sets were introduced to handle ambiguity in real-world scenarios, further evolving to include tree soft sets, which closely resemble hypersoft sets [11]. Treesoft sets focus on parameters, sub-parameters, and subsequent levels of granularity, whereas hypersoft sets deal with parameters and their sub-levels[12].

The TrSS model [13] is proposed to model specific criteria and elucidate their relationships, aiding in problem-solving. Treesoft sets aid in categorizing problems into functional and non-functional attributes, enhancing the clarity of the DEMATEL method. The DEMATEL method encompasses three main components: diverse criteria, a range of alternatives, and a comparison process among them.

2. Approach

This section is to propose the Neutrosophic DEMATEL method under the Tree-soft set.

2.1 Tree Soft Set

Smarandache proposes the definition of TreeSoft Set as:

Let U be the universe of discourse, and H be a non-empty subset of U, with P(H) being a power set of H.

Let A be a set of attributes (parameters, factors, etc.,), $A = \{A_1, A_2 \dots \dots . A_n\}, n \geq 1$, where $A_1, A_2 \dots \dots . A_n$ are considered attributes of the first level. Each attribute $A_i, 1 \leq i \leq n$, is formed by sub-attributes: $A1 = \{A1{,}1, A1{,}2, \dots\}$ $A2 = \{A2{,}1\ , A2{,}2\ , \dots\ \}$ $An = \{An, 1\ , An, 2\ , \dots\ \}$ where the above $A_{i,j}$ are sub-attributes (or attributes of the second level) (since they have two-digit indexes). Again, each sub-attribute $A_{i,j}$ is formed by sub-sub-attributes (attributes of the third level): $A_{i,j,k}$ and so on, as much refinement as needed into each application, up to sub-sub-…-sub-attributes (or attributes of m-level (or having m digits into the indexes).

Therefore, a graph tree is formed, which we denote as Tree(A), whose root is A (considered of level zero), then nodes of level 1, level 2, up to level m. We call the leaves of the graph-tree, all terminal nodes (nodes that have no descendants). Then the TreeSoft Set is:

$$F: P(\text{Tree}(A)) \rightarrow P(H)$$

Tree(A) is the set of all nodes and leaves (from level 1 to level m) of the graph tree, and P(Tree(A)) is the powerset of the Tree(A). All node sets of the TreeSoft Set of level m are:

$$\text{Tree}(A) = \{Ai1 \mid i1 = 1, 2, \dots\}$$

So, the Problem must be defined as the tree structure.

Example: Consider the set $A=\{A_1, A_2, A_3, A_4, A_5, A_6\}$ be the Air pollution symptoms and P(A) is the powerset of A with the corresponding attributes $T=\{T_1, T_2, T_3, T_4, T_5\}$.

Now let us assume that the classification of Air pollution and its effects are given by the following terms:

2.2 Air Pollutions

PM

T_{11} Heart Disease

T_{111} Angina

T_{112} Heart attacks

T_{113} Heart Failure

T_{12} Asthma

T_{121} Trigger Coughing

T_{122} Wheezing

T_{13} Low birth weight

T_{131} Improper Immune System

CO_2

T_{21} Headache

T_{211} Relative Humidity

T_{212} High Risk of Migraine

T_{22} Sleepiness

T_{221} Headaches

T_{222} Fatigue

T_{23} Stagnant

T_{231} Difficult to breath

T_{232} Cough

T_{233} Headache

O_3

T_{31} Chest Pain

T_{311} Shortness of breath

T_{312} Cough

T_{313} Wheezing

T_{32} Coughing

T_{321} Headaches

T_{322} Vomiting

T_{323} Dizziness

T_{33} Throat irritation

T_{331} Cough

T_{332} Tightness of the chest

NO_2

T_{41} Damage to the human respiratory tract

T_{411} Asthma

T_{412} Lung Cancer

T_{42} Asthma

T_{421} Trigger Coughing

T_{422} Wheezing

SO_2

T_{51} Damage trees and plants

T_{511} Increasing temperature

T_{512} Injury to foliage of leaf

T_{52} Inhibit plant growth

T_{521} Leaf cuticles

T_{522} Stomatal Conductance

3. Tree soft set Neutrosophic DEMATEL Approach

Step 1: Build a tree and define the nodes.

The tree has more than one level, in the first level, the main criteria and introduced as TrS_1, TrS_2…….. TrSn. In the second level, the sub-criteria are introduced as TrS_{11}, TrS_{12}…….. TrS_{1n} and TrS_{21}, TrS_{22}……

Step 2: Define a problem with a set of criteria

The main and sub-criteria are defined in this step by problem definition.

Step 3: Identifying decision goals: collecting relevant information presenting the problem.

i. Selection of experts and decision-makers that have experience in the field.

ii. Identifying the relevant criteria to the problem.

Step 4: Pairwise comparison matrices between relevant criteria.

i. Identify the criteria.

ii. Experts make pairwise comparison matrices between criteria.

iii. Experts should determine the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (β), and the minimum falsity membership degree (θ) of single-valued neutrosophic numbers.

iv. Determine the crisp value of each opinion, using the following equation:

Table 1: The pairwise comparison matrix between criteria

Criteria	A_1	A_2	….	A_n
A_1	(p_{11}, q_{11}, r_{11})	(p_{12}, q_{12}, r_{12})	….	(p_{1n}, q_{1n}, r_{1n})
A_2	(p_{21}, q_{21}, r_{21})	(p_{22}, q_{22}, r_{22})	….	(p_{2n}, q_{2n}, r_{2n})
….	….	….	….	….
A_n	(p_{n1}, q_{n1}, r_{n1})	(p_{n2}, q_{n2}, r_{n2})	….	(p_{nn}, q_{nn}, r_{nn})

Table 2: The pairwise comparison matrix between criteria with the T, F, and I values

Criteria	A_1	A_2	….	A_n
A_1	$(p_{11}, q_{11}, r_{11}; T, F, I)$	$(p_{12}, q_{12}, r_{12}\ ; T, F, I)$	….	$(p_{1n}, q_{1n}, r_{1n}\ ; T, F, I)$
A_2	$(p_{21}, q_{21}, r_{21}\ ; T, F, I)$	$(p_{22}, q_{22}, r_{22}\ ; T, F, I)$	….	$(p_{2n}, q_{2n}, r_{2n}\ ; T, F, I)$
….	….	….	….	….
A_n	$(p_{n1}, q_{n1}, r_{n1}\ ; T, F, I)$	$(p_{n2}, q_{n2}, r_{n2}\ ; T, F, I)$	….	$(p_{nn}, q_{nn}, r_{nn}\ ; T, F, I)$

Table 3: The crisp values of the comparison matrix

Criteria	A_1	A_2	$\ldots.$	A_n
A_1	CrV_{11}	CrV_{21}	$\ldots.$	CrV_{m1}
A_2	CrV_{12}	CrV_{22}	$\ldots.$	CrV_{m2}
$\ldots.$	$\ldots.$	$\ldots.$	$\ldots.$	$\ldots.$
A_n	CrV_{1n}	CrV_{2n}	$\ldots.$	CrV_{mn}

Step 5: Integration of matrices

All opinions of experts need to be integrated into one matrix presenting the average opinions of all experts about each criterion.

Table 4: Integration of the average opinions of all experts

Criteria	A_1	A_2	$\ldots.$	A_n
A_1	CrV_{11}	CrV_{21}	$\ldots.$	CrV_{m1}
A_2	CrV_{12}	CrV_{22}	$\ldots.$	CrV_{m2}
$\ldots.$	$\ldots.$	$\ldots.$	$\ldots.$	$\ldots.$
A_n	CrV_{1n}	CrV_{2n}	$\ldots.$	CrV_{mn}

Score function $S_f = \frac{1}{9}(a + b + c) \times (2 + T - F - I)$
Accuracy function $A_f = \frac{1}{9}(a + b + c) \times (2 + T - F - I)$

Step 6; Generating a direct relation matrix

An initial direct relation matrix A is a $n \times n$ matrix obtained by pairwise comparisons, $A = [A_{ij}]_{n \times n}$.

A_{ij} denotes the degree to which criterion i affects criterion j.

Step 7: Normalizing the direct relation matrix

The normalized direct relation matrix can be obtained using the equation:

$$K = \frac{1}{Max \sum_{j=1}^{n} a_{ij}}$$

$$N = K \times A$$

Step 8: Total relation matrix

A total relation matrix (T), in which (I) denotes the identity matrix, is shown as follows:

$$T = N \times (I - N)^{-1}$$

Step 9: Obtaining the sum of rows and columns

The sum of rows is denoted by (R), and the sum of columns is denoted by(C). Calculate $R + C$ and $R - C$.

$$C = \left[\sum_{i=1}^{n} a_{ij}\right]_{1\times n} = [a_j]_{n\times 1}$$

$$R = \left[\sum_{j=1}^{n} a_{ij}\right]_{1\times n} = [a_j]_{n\times 1}$$

Step 10: Draw the cause-and-effect diagram

The cause and effect diagram is in blue shade presented by $R + C$ and in orange shade presented by $R - C$ which is a degree of relation and it depicts the steps of the proposed model.

4. The proposed methodology in a case study

In this section, we describe the details of the proposed methodology of a Tree soft set approach of neutrosophic sets and the DEMATEL method of cause and effect for the air pollution criteria.

4.1. The calculation process of the Treesoft set-neutrosophic DEMATEL Approach

For collecting data, we are going to analyze the criteria of air pollution of cause and effect. The three experts determined the most important evaluation criterion to be used. The criteria symbols in this research are as follows: PM (T1), CO_2 (T2), O_3 (T3), NO_2 (T4), SO_2 (T5). The data collected from the three experts were analyzed by the Tree soft set of the Neutrosophic DEMATEL method. The steps that were conducted are the following.

Step 1: Build a tree and define the nodes.

The tree has more than one level, in the first level, the main criteria and introduced as TrS_1, TrS_2…….. TrSn. In the second level, the sub-criteria are introduced as TrS_{11}, TrS_{12}…….. TrS_{1n} and TrS_{21}, TrS_{22}……

Step 2: Define a problem with a set of criteria

The main and sub-criteria are defined in this step by problem definition.

Step 3: Identifying decision goals: collecting relevant information presenting the problem.

The first step of the Tree soft set Neutrosophic DEMATEL method is the selection of the best experts in the field of management purchasing and setup contracts. We selected three experts, to which we further refer as the first expert, the second expert, and the third expert. We sorted five evaluation criteria as selected by the team of experts, namely: PM (T1), CO_2 (T2), O_3 (T3), NO_2 (T4), SO_2 (T5)

Step 4: Pairwise comparison matrices between relevant criteria.

i. Identify the criteria.

ii. Experts make pairwise comparison matrices between criteria.

iii. Experts should determine the maximum truth-membership degree (T), the minimum indeterminacy-membership degree (I), and the minimum falsity membership degree (F) of single-valued neutrosophic numbers.

iv. Determine the crisp value of each opinion, using the following equation:

$$CrV = \frac{CrV_1 + CrV_2 + CrV_3}{3}$$

Table 5: The pairwise comparison matrix of criteria

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	(0.3,0.6,0.4)	(0.3,0.4,0.2)	(0.3, 0.7, 0.8)	(0.8,0.3,0.5)	(0.5, 0.4, 0.8)
A_2	(0.5, 0.6, 0.2)	(0.4, 0.4 ,0.5)	(0.8, 0.2 ,0.4)	(0.4, 0.6, 0.4)	(0.9, 0.7 ,0.1)
A_3	(0.9, 0.7, 0.6)	(0.5, 0.9, 0.4)	(0.3, 0.7, 0.5)	(0.8, 0.3, 0.5)	(0.7, 0.5, 0.6)
A_4	(1.0, 0.4, 0.5)	(0.5, 0.5, 0.5)	(0.4, 0.6, 0.4)	(0.4, 0.5, 0.2)	(0.1, 0.6, 0.3)
A_5	(0.5, 0.5, 0.5)	(0.2, 0.4, 0.6)	(0.3, 0.7, 0.8)	(0.7, 0.5, 0.3)	(0.4, 0.6, 0.5)

Table 6: The pairwise comparison matrix of criteria

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	(0.1, 0.5, 0.4)	(0.4, 0.5, 0.6)	(0.5, 0.7, 0.8)	(0.3, 0.2, 0.6)	(0.2, 0.4, 0.3)
A_2	(0.9, 0.6, 0.3)	(0.3, 0.6, 0.8)	(0.2, 0.7, 0.3)	(0.3, 0.4, 0.7)	(0.5, 0.3, 0.5)
A_3	(0.7, 0.5, 0.3)	(0.4, 0.2, 0.5)	(0.3, 0.4, 0.6)	(0.4, 0.2, 0.5)	(0.8, 0.6, 0.4)
A_4	(0.1, 0.5, 0.4)	(0.3, 0.5, 0.4)	(0.2, 0.3, 0.4)	(0.7, 0.5, 0.4)	(0.5, 0.7, 0.8)
A_5	(0.5, 0.5, 0.5)	(0.8, 0.5, 0.4)	(0.5, 0.5, 0.1)	(0.1, 0.5, 0.0)	(0.6, 0.4, 0.3)

Table 7: The pairwise comparison matrix of criteria

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	(0.3, 0.6, 0.5)	(0.3, 0.4, 0.2)	(0.3, 0.7, 0.1)	(0.1, 0.5, 0.6)	(0.8, 0.4, 0.5)
A_2	(1.0, 0.4, 0.5)	(0.5, 0.9, 0.4)	(0.5, 0.4, 0.5)	(0.4, 0.1, 0.6)	(0.1, 0.6, 0.3)
A_3	(0.7, 0.7, 0.6)	(0.2, 0.4, 0.6)	(0.3, 0.7, 0.5)	(0.4, 0.7, 0.3)	(0.5, 0.4, 0.8)
A_4	(0.9, 0.7, 0.6)	(0.5, 0.5, 0.5)	(0.2, 0.8, 0.3)	(0.7, 0.2, 0.6)	(0.1, 0.6, 0.3)
A_5	(0.3, 0.6, 0.4)	(0.5, 0.9, 0.0)	(0.4, 0.5, 0.7)	0.4, 0.5, 0.2	(0.9, 0.4, 0.3)

Table 8: The pairwise comparison matrix of criteria with T, F, I values

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	(0.3, 0.6, 0.4; 0.5, 0.3, 0.4)	(0.3, 0.4, 0.2; 0.7, 0.2, 0.5)	(0.3, 0.7, 0.8; 0.9, 0.4, 0.6)	(0.8, 0.3, 0.5; 0.4, 0.3, 0.5)	(0.5, 0.4, 0.8; 0.8, 0.2, 0.4)
A_2	(0.5, 0.6, 0.2; 0.5, 0.2, 0.1)	(0.4, 0.4, 0.5; 0.8, 0.5, 0.3)	(0.8, 0.2, 0.4; 0.4, 0.5, 0.6)	(0.4, 0.6, 0.4; 0.5, 0.2, 0.1)	(0.9, 0.7, 0.1; 0.7, 0.4, 0.6)
A_3	(0.9, 0.7, 0.6; 0.2, 0.4, 0.3)	(0.5, 0.9, 0.4; 0.9, 0.5, 0.4)	(0.3, 0.7, 0.5; 0.5, 0.7, 0.2)	(0.8, 0.3, 0.5; 0.7, 0.3, 0.4)	(0.7, 0.5, 0.6; 0.3, 0.4, 0.1)
A_4	(1.0, 0.4, 0.5; 0.7, 0.2, 0.4)	(0.5, 0.5, 0.5; 0.3, 0.1, 0.5)	(0.4, 0.6, 0.4; 0.9, 0.4, 0.6)	(0.4, 0.5, 0.2; 0.9, 0.1, 0.6)	(0.1, 0.6, 0.3; 0.7, 0.6, 0.5)
A_5	(0.5, 0.5, 0.5; 0.8, 0.4, 0.5)	(0.2, 0.4, 0.6; 0.4, 0.3, 0.6)	(0.3, 0.7, 0.8; 0.4, 0.2, 0.3)	(0.7, 0.5, 0.3; 0.5, 0.2, 0.9)	(0.4, 0.6, 0.5; 0.8, 0.3, 0.5)

Table 9: The pairwise comparison matrix of criteria with T, F, I values

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	(0.1, 0.5, 0.4; 0.5, 0.6, 0.2)	(0.4, 0.5, 0.6; 0.2, 0.1, 0.7)	(0.5, 0.7, 0.8; 0.5, 0.4, 0.4)	(0.3, 0.2, 0.6; 0.2, 0.5, 0.7)	(0.2, 0.4, 0.3; 0.4, 0.3, 0.6)
A_2	(0.9, 0.6, 0.3; 0.5, 0.4, 0.6)	(0.3, 0.6, 0.8; 0.5, 0.2, 0.6)	(0.2, 0.7, 0.3; 0.2, 0.5, 0.7)	(0.3, 0.4, 0.7; 0.8, 0.7, 0.5)	(0.5, 0.3, 0.5; 0.4, 0.3, 0.2)
A_3	(0.7, 0.5, 0.3; 0.2, 0.3, 0.1)	(0.4, 0.2, 0.5; 0.2, 0.3, 0.5)	(0.3, 0.4, 0.6; 0.3, 0.6, 0.8)	(0.4, 0.2, 0.5; 0.4, 0.8, 0.1)	(0.8, 0.6, 0.4; 0.5, 0.3, 0.3)
A_4	(0.1, 0.5, 0.4; 0.4, 0.3, 0.5)	(0.3, 0.5, 0.4; 0.5, 0.4, 0.3)	(0.2, 0.3, 0.4; 0.2, 0.3, 0.9)	(0.7, 0.5, 0.4; 0.5, 0.7, 0.2)	(0.5, 0.7, 0.8; 0.4, 0.7, 0.2)
A_5	(0.5, 0.5, 0.5; 0.8, 0.6, 0.3)	(0.8, 0.5, 0.4; 0.6, 0.3, 0.8)	(0.5, 0.5, 0.1; 0.2, 0.6, 0.3)	(0.1, 0.5, 0.0; 0.2, 0.5, 0.4)	(0.6, 0.4, 0.3; 0.3, 0.8, 0.4)

Table 10: The pairwise comparison matrix of criteria with T, F, I values

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	(0.3, 0.6, 0.5; 0.5, 0.4, 0.6)	(0.3, 0.4, 0.2; 0.4, 0.5, 0.8)	(0.3, 0.7, 0.1; 0.7, 0.4, 0.1)	(0.1, 0.5, 0.6; 0.3, 0.1, 0.7)	(0.8, 0.4, 0.5; 0.3, 0.5, 0.4)
A_2	(1.0, 0.4, 0.5; 0.2, 0.1, 0.3)	(0.5, 0.9, 0.4; 0.8, 0.7, 0.9)	(0.5, 0.4, 0.5; 0.8, 0.4, 0.8)	(0.4, 0.1, 0.6; 0.4, 0.5, 0.6)	(0.1, 0.6, 0.3; 0.6, 0.5, 0.2)
A_3	(0.7, 0.7, 0.6; 0.4, 0.6, 0.9)	(0.2, 0.4, 0.6; 0.8, 0.6, 0.5)	(0.3, 0.7, 0.5; 0.7, 0.5, 0.8)	(0.4, 0.7, 0.3; 0.1, 0.5, 0.4)	(0.5, 0.4, 0.8; 0.7, 0.2, 0.1)
A_4	(0.9, 0.7, 0.6; 0.6, 0.5, 0.7)	(0.5, 0.5, 0.5; 0.4, 0.3, 0.5)	(0.2, 0.8, 0.3; 0.4, 0.3, 0.5)	(0.7, 0.2, 0.6; 0.2, 0.3, 0.6)	(0.1, 0.6, 0.3; 0.7, 0.4, 0.6)
A_5	(0.3, 0.6, 0.4; 0.3, 0.2, 0.4	(0.5, 0.9, 0.0; 0.4, 0.5, 0.8)	(0.4, 0.5, 0.7; 0.7, 0.5, 0.8)	(0.4, 0.5, 0.2; 0.7, 0.4, 0.6)	(0.9, 0.4, 0.3; 0.6, 0.5, 0.2)

Table 11: Crisp value of the comparison matrix

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	0.26	0.2	0.38	0.284	0.416
A_2	0.318	0.289	0.202	0.342	0.321
A_3	0.367	0.4	0.269	0.356	0.36
A_4	0.443	0.283	0.296	0.269	0.178
A_5	0.317	0.2	0.38	0.233	0.333

Table 12: Crisp value of the comparison matrix

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	0.189	0.233	0.378	0.1222	0.15
A_2	0.3	0.321	0.133	0.249	0.274
A_3	0.3	0.171	0.13	0.183	0.38
A_4	0.178	0.24	0.1	0.284	0.333
A_5	0.317	0.283	0.159	0.087	0.159

Table 13: Crisp value of the comparison matrix

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	0.233	0.11	0.269	0.2	0.264
A_2	0.38	0.24	0.249	0.159	0.211
A_3	0.2	0.227	0.233	0.187	0.453
A_4	0.342	0.267	0.231	0.217	0.189
A_5	0.246	0.171	0.249	0.208	0.338

Step 5: Integration of matrices

All opinions of experts need to be integrated into one matrix presenting the average opinions of all experts about each criterion.

Table 14: Integration of matrices

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	0.227	0.181	0.342	0.2021	0.277
A_2	0.333	0.283	0.195	0.25	0.269
A_3	0.289	0.266	0.211	0.242	0.398
A_4	0.321	0.263	0.209	0.257	0.233
A_5	0.293	0.218	0.263	0.176	0.277

Step 6: Generating a direct relation matrix

An initial direct relation matrix A is a $n \times n$ matrix obtained by pairwise comparisons, $A = [A_{ij}]_{n \times n}$.

A_{ij} denotes the degree to which criterion i affects criterion j.

Step 7: Normalizing the direct relation matrix

The normalized direct relation matrix can be obtained using the equation:

$$K = \frac{1}{Max \sum_{j=1}^{n} a_{ij}}$$

$$N = K \times A$$

Table 15: Normalizing the direct relation matrix

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	0.185	0.147	0.278	0.164	0.225
A_2	0.250	0.213	0.147	0.188	0.202
A_3	0.206	0.189	0.150	0.172	0.283
A_4	0.25	0.205	0.163	0.200	0.182
A_5	0.239	0.178	0.214	0.143	0.226

Step 8: Total relation matrix

A total relation matrix (T), in which (I) denotes the identity matrix, is shown as follows:

$$T = N \times (I - N)^{-1}$$

Table 16: Total relation matrix

Criteria	A_1	A_2	A_3	A_4	A_5
A_1	998.99	820.45	870.41	763.33	1001.49
A_2	999.99	821.29	872.41	764.07	1002.39
A_3	999.99	821.29	871.97	764.08	1002.51
A_4	999.99	821.28	872.27	764.08	1002.37
A_5	999.99	821.26	871.79	764.03	1002.43

Step 9: Find the sum of rows and columns

The sum of rows is denoted by (R), and the sum of columns is denoted by (C). Calculate $R + C$ and $R - C$.

$$C = \left[\sum_{i=1}^{n} a_{ij}\right]_{1\times n} = \left[a_j\right]_{n\times 1}$$

$$R = \left[\sum_{j=1}^{n} a_{ij}\right]_{1\times n} = \left[a_j\right]_{n\times 1}$$

Row + Column	Row - Column
9453.62	-544.28
8465.72	354.58
8818.69	100.99

8279.58	640.4
9470.69	-551.69

Step 10: Draw the cause-and-effect diagram

The cause-and-effect diagram is in blue shade presented by $R + C$ and in orange shade presented by $R - C$ which is a degree of relation, and it depicts the steps of the proposed model.

4.2. Analysing the evaluation criteria

The research results determine the most important criterion. From this causal chart, according to the Tree soft set Neutrosophic DEMATEL Method, the importance of all criteria was established. According to experts' opinions, NO_2 (A_4) had the greatest impact and SO_2 (A_5) had a lesser impact on the selection of the cause of air pollution.

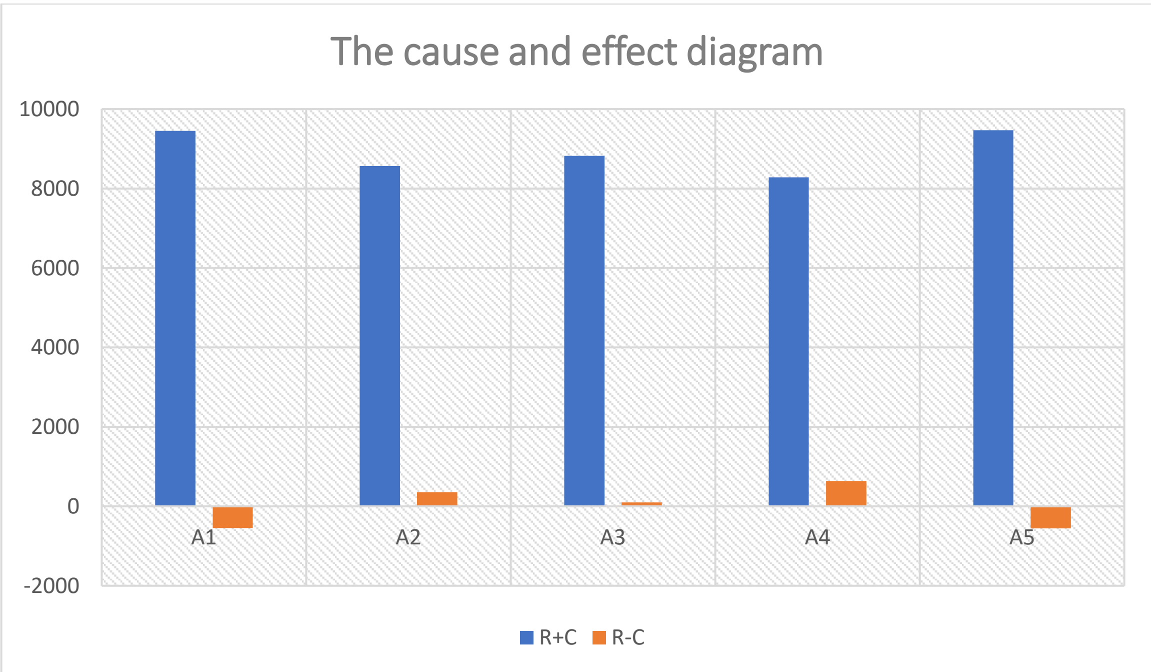

Figure 1. The cause-and-effect diagram.

5. Conclusions

Potential supply chain management practices have been developed and performed using the Tree soft set Neutrosophic DEMATEL Method to select the best standards that have a greater impact on other criteria. The proposed approach succeeded in developing the DEMATEL Method by applying to it the Neutrosophic Set Theory, using a new scale from 0 to 1 and employing the maximum truth membership degree (T), the minimum indeterminacy membership degree (I) and the minimum falsity membership degree (F) of a single-valued neutrosophic number. The opinions were collected from experts through interviews and consequently analyzed using the Neutrosophic DEMATEL Method, by comparisons of each criterion, according to each expert, and their formulation of each value according to a single-valued neutrosophic number.

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On Generalized Difference Rough Ideal Statistical Convergence in Neutrosophic Normed Spaces

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Abstract. This article's main goal is to provide and investigate a novel statistical convergence generalisation for generalized difference sequences in Neutrosophic Normed Spaces (*NNS*) called rough ideal statistical convergence. The collection of approximate optimal statistical limit points and cluster points is defined, and their algebraic and topological features are examined. Additionally, we included the relationship for generalized difference sequences in *NNS* between rough I-statistical cluster points and rough I-statistical limit points.

Keywords: Neutrosophic normed space; ideal statistical convergence; rough ideal statistical convergence; difference sequence

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1. **Introduction**

Numerous fields of analysis and number theory have found use for statistical convergence, which has been thoroughly researched to comprehend the convergence behaviour of various kinds of sequences and series. It enables mathematicians to investigate convergence in more versatile and general contexts, which may produce fresh ideas and discoveries in a variety of mathematical fields. Zygmund first introduced statistical convergence in his 1935 monograph, which was first published in Warsaw [52]. Steinhaus [50] and Fast [20] developed the idea of statistical convergence, and then later reintroduced by Schoenberg [42]. The idea of natural density serves as its foundation.

The Natural density, denoted by $\delta(K_p)$ of the set $K_p = \{p \in \mathbb{N} : p \leq n\} \subseteq \mathbb{N}$, is defined as $\delta(K_p) = \lim_{p\to\infty} \frac{|K_p|}{n}$, where $|K_p|$ denotes the cardinality of the set K_p. If, for each $\epsilon > 0$, $\delta(\{p \in \mathbb{N} : ||y_p - y_0|| \geq \epsilon\}) = 0$, then a sequence $y = \{y_p\}$ in $\mathbb{R}$ is considered as statistically convergent to $y_0 \in \mathbb{R}$.

For sequences on finite-dimensional normed linear spaces, Phu [38] first proposed rough convergence in 2001. After that, numerous authors were inspired to work on this type of convergence on various sequence-spaces, including those for double sequences [32, 33], triple sequences [14], lacunary sequences [26], ideals [33,36] etc. Despite this, it has been established in a variety of spaces, see [2, 3, 9, 14] etc. It was later expanded to infinite-dimensional normed linear spaces [39]. In 2008, Aytar [5] also worked on same and introduced new generalized convergence named rough statistical convergence.

In 2000, Kostryko *et al.* [30] proposed the concept of ideal convergence(I-convergence) by generalizing the statistical convergence with the helps of ideals. For more details we refer [6–8,29,31]. With the help of ideals, In 2011, a new generalisation named rough ideal statistical convergence in normed spaces was defined by Das, Savas and Ghosal [13]. They studied its fundamental properties.

Initially, Kizmaz [28] proposed the concept of difference sequence spaces as $\mathcal{Z}(\Delta) = \{y = (y_p) : (\Delta y_p) \in \mathcal{Z}\}$ for $\mathcal{Z} = l_\infty$(spaces of all bounded sequences), C(convergent sequences), C_0(null sequences), where $\Delta y = (\Delta y_p) = (y_p - y_{p+1})$. In particular, $l(\Delta), C(\Delta)$ and $C_0(\Delta)$ are also Banach spaces, relative to a norm induced by $||y_p||_\Delta = |y_1| + \sup_k |\Delta y_p|$ and the generalized difference sequence spaces was introduced as (see [18]): $\mathcal{Z}(\Delta^m y_p) = \{y = (y_p) : \Delta^m y_p \in \mathcal{Z}\}$ where $\Delta^m y = (\Delta^m y_p) = (\Delta^{m-1} y_p - \Delta^{m-1} y_{p+1})$ so that $\Delta^m y_p = \sum_{r=0}^{m} (-1)^r \binom{m}{r} y_{k+r}$.

Various characteristics and properties of difference sequences have been explored by researchers can be found in [17–19, 21, 35]. Demir and Gümüş [15] investigated rough convergence through difference sequences on finite dimensional normed space. In 2022, Gümüş [16] defined rough statistical convergence for (Δy_p) sequences and established some properties for the collection of approximate optimal statistical limit points. Karabacak and Or [22] also studied these generalized convergence in normed linear spaces.

In order to examine the ambigious qualitative or quantitative data, Zadeh [51] invented an extension of classical sets known as fuzzy sets. Atannassov [4] prsented a new generalization named intuitionistic fuzzy sets(I.F.s). Smarandache [44] suggested Neutrosophic sets, a new variant of I.F.s. Additionally, Neutrosophic soft linear spaces [11] and Neutrosophic metric spaces [25] are also defined using this idea. Not only these, further extensions related to super soft hyper set, Revolutionary topologies, Neutrosophic topologies, Neutrosophic algebra, Neutrosophic SuperHyperStructure and Neutrosophic numbers can be seen in [10, 45–49]. In addition to establishing a number of sequential concepts in these spaces, including convergence, Cauchy, and convexity, Bera and Mahapatra [12] introduced the term neutrosophic norm. These places are employed in situations where complicated, ambiguous, and indeterminate information must be handled at the same time. There are several applications in decision-making, control theory, and other domains where imprecision and uncertainty exist.

Neutrosophic fuzzy normed spaces are a relatively advanced and specialized topic in mathematics and are primarily used in areas that require the modeling of complex and uncertain data. More recently, statistical convergence and its features in these spaces have been expanded by Kirişci and Şimşek [25].

Notable advancements in the area of rough convergence in past few years served as inspiration for the current research work. Very recently, Kişi and Yildil [27], added the theory of rough statistical convergence via difference operator in *NNS* and Kaur and Meenakshi also [24] worked on generalized ideal convergence in same space, which are the motivation for this research paper. The significance of this work is to investigate the theory of rough I-statistical convergence(rough I-st-convergence) for generalized difference sequences in the setup of *NNS* including the algebraic and topological characteristics for the set of rough I-st-limit points and rough I-st-cluster points for generalised difference sequences. Along with investigating some features, we also established the relationship between limit points set and cluster points set. The main objective is to find the feasibility of new notion convergence with classical one convergence. Also to identify whether this convergence i.e. rough I-st-converegnce for generalized difference sequences in *NNS*, serves the theory of convergence in *NNS*.
In the next section, the basic definitions are included which are necessary for the development of this work. Section 3 and 4 includes the main results of the paper. In section 3, rough I-st-convergence for generalized difference sequences in *NNS* has been introduced. The algebraic and topological characteristics for the set of rough I-st-limit points for generalized difference sequences like convexity, closedness etc. are proved. In section 4, rough I-st-cluster points for generalized difference sequences in *NNS* have been defined and the relationship between limit points and cluster points has been explored.

2. Preliminaries

we begin this section with fundamental definitions utilised for perceptual progression, which are as follows:

Definition 2.1. [34] A t-norm is a continuous correspondance $\circledast : [0,1]\times[0,1] \to [0,1]$, which is associative, commutative and having identity 1 and $f \circledast g \leq j \circledast k$ for each whenever $f \leq j$ and $g \leq k$ for each f, g, j and $k \in [0,1]$.

Definition 2.2. [34] A t-conorm is a continuous correspondance $\diamond : [0,1]\times[0,1] \to [0,1]$, which is associative, commutative and having identity 0 and $f \diamond g \leq j \diamond k$ whenever $f \leq j$ and $g \leq k$ for each f, g, j and $k \in [0,1]$.

In veiw of these notations, Kirişci and Şimşek [25], recently defined *NNS* and worked on statistical convergence in same spaces.

Definition 2.3. [25] Consider $N = \{p, \psi(p), \eta(p), \sigma(p) : p \in \mathcal{V}\}$ be a normed space such that $N : \mathcal{V} \times \mathbb{R}^+ \to [0,1]$, $\mathcal{V}$ be a vector space and $\circledast$, $\diamond$ are continuous t-norm and continuous t-conorm respectively. Then $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is named as Neutrosophic Normed Spaces (NNS) if the following axioms hold:

For each $p, q \in \mathcal{V}$ and $\xi, \lambda > 0$ and for every $\alpha \neq 0$ we have

(1) $0 \leq \psi(p,\xi), \eta(p,\xi), \sigma(p,\xi) \leq 1$ for every $\xi \in \mathbb{R}^+$;
(2) $\psi(p,\xi) + \eta(p,\xi) + \sigma(p,\xi) \leq 3$ for $\xi \in \mathbb{R}^+$;
(3) $\psi(p,\xi) = 1, \eta(p,\xi) = 0$ and $\sigma(p,\xi) = 0$ for $\xi > 0$ iff $p = 0$;
(4) $\psi(p,\xi) = 0, \eta(p,\xi) = 1$ and $\sigma(p,\xi) = 1$ for $\xi \leq 0$;
(5) $\psi(\alpha p,\xi) = \psi(p, \frac{\xi}{|\alpha|}), \eta(\alpha p,\xi) = \eta(p, \frac{\xi}{|\alpha|})$ and $\sigma(\alpha p,\xi) = \sigma(p, \frac{\xi}{|\alpha|})$;
(6) $\psi(p,\lambda) \circledast \psi(p,\xi) \leq \psi(p+q, \lambda+\xi)$;
(7) $\psi(p, \circledast)$ is continuous non-decreasing function;
(8) $\eta(p,\lambda) \diamond \eta(q,\xi) \geq \eta(p+q, \lambda+\xi)$;
(9) $\eta(p, \diamond)$ is continuous non-decreasing function;
(10) $\sigma(p,\lambda) \diamond \sigma(q,\xi) \geq \sigma(p+q, \lambda+\xi)$;
(11) $\sigma(p, \diamond)$ is continuous non-decreasing function;
(12) $\lim_{\xi\to\infty} \psi(p,\xi) = 1, \lim_{\xi\to\infty} \eta(p,\xi) = 0$ and $\lim_{\xi\to\infty} \sigma(p,\xi) = 0$.

Here, $N(\psi, \eta, \sigma)$ is the Neutrosophic norm on $\mathcal{V}$.

Definition 2.4. [25] A sequence $\{y_p\}$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is called *statistically convergent* to $\kappa \in \mathbb{Y}$ w.r.t to norms (ψ, η, σ) if for $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{p \in \mathbb{N} : \psi(y_p - \kappa, \epsilon) \leq 1 - \lambda \text{ or } \eta(y_p - \kappa, \epsilon) \geq \lambda \text{ or } \sigma(y_p - \kappa, \epsilon) \geq \lambda\}| = 0.$$

equivalently $\delta(\mathbb{A}(\epsilon, \lambda)) = 0$ where

$\mathbb{A}(\epsilon, \lambda) = \{p \in \mathbb{N} : \psi(y_p - \kappa, \epsilon) \leq 1 - \lambda \text{ or } \eta(y_p - \kappa, \epsilon) \geq \lambda \text{ or } \sigma(y_p - \kappa, \epsilon) \geq \lambda\}$.

In 2023, Kişi and Yildil [27], defined rough statistical convergence via difference operator in NNS as follow:

Definition 2.5. [27] A sequence $\mathit{\Delta} y = \{\mathit{\Delta} y_p\}$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is called *rough statistically convergent* to $\kappa \in \mathbb{Y}$ w.r.t. norms (ψ, η, σ) for some $r > 0$ if for each $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n} |\{p \leq n : \psi(\mathit{\Delta} y_p - \kappa; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\mathit{\Delta} y_p - \kappa; r + \epsilon) \geq \lambda \text{ or } \sigma(\mathit{\Delta} y_p - \kappa; r + \epsilon) \geq \lambda\}| = 0, .$$

or

$$\delta(\{p \leq n : \psi(\mathit{\Delta} y_p - \kappa; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\mathit{\Delta} y_p - \kappa; r + \epsilon) \geq \lambda \text{ or } \sigma(\mathit{\Delta} y_p - \kappa; r + \epsilon) \geq \lambda\}) = 0, .$$

It is denoted by $\mathit{\Delta} y_p \xrightarrow{r - st_{(\psi,\eta,\sigma)}} \kappa$ or $r - st_{(\psi,\eta,\sigma)} - \lim_{p\to\infty} \mathit{\Delta} y_p = \kappa$.

Let $st_{(\psi,\eta,\sigma)} - LIM^r_{y_p}$ represents the the set of all rough st-limit points of the differenc sequence $\mathit{\Delta} y = \{\mathit{\Delta} y_p\}$.

3. Rough ideal statistical convergence for generalized difference sequences in *NNS*

In this section, we introduce the notion of Rough I-Statistical Convergence for generalized difference sequences in *NNS* and then examined some results using generalized difference sequence. Throughout the article I is an admissible ideal and $(\Delta^m y_p) = \Delta^{m-1}y_p - \Delta^{m-1}y_{p+1}$, where $m \in \mathbb{N}$, be the generalized difference sequence.

Definition 3.1. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*, $\Delta^m y = (\Delta^m y_p)$ where $m \in \mathbb{N}$, a generalized difference sequence is considered as *rough Δ^m-statistical convergent* to $\xi \in \mathbb{Y}$ w.r.t. neutrosophic norm (ψ, η, σ) for $r \geq 0$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}| = 0,$$

or

$$\delta(\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}) = 0.$$

It is denoted by $\Delta^m y_p \xrightarrow{r-st_{(\psi,\eta,\sigma)}} \xi$ or $r - st_{(\psi,\eta,\sigma)} - \lim_{p\to\infty} \Delta^m y_p = \xi$.
Let $st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ represnts the collection of all rough st-limit points of $(\Delta^m y_p)$.

Remark 3.2. For $r = 0$, the notion rough Δ^m-st-convergence is equivalent to the Δ^m- st-convergence for $(\Delta^m y_p)$ in *NNS*.

Remark 3.3. For $m = 1$, the notion rough Δ^m-statistical convergence agrees with rough Δ-st-convergence studied in [27].

The $r - st_{(\psi,\eta,\sigma)}$-limit of a generalized difference sequence may not be unique in *NNS*. So, consider the set of rough st-limit points of $(\Delta^m y_p)$ as $st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \left[\xi : \Delta^m y_p \xrightarrow{r-st_{(\psi,\eta,\sigma)}} \xi\right]$. Note that the sequence $(\Delta^m y_p)$ is $r_{(\psi,\eta,\sigma)}$-convergent if $LIM^{r(\psi,\eta,\sigma)}_{\Delta^m y_p} \neq \phi$, where

$$LIM^{r(\psi,\eta,\sigma)}_{\Delta^m y_p} = \left[\xi^* \in \mathbb{Y} : \Delta^m y_p \xrightarrow{r_{(\psi,\eta,\sigma)}} \xi^*\right].$$

Example 3.4. Let $(\mathbb{Y}, \|.\|)$ be real normed space. For every $q > 0$ and for all $\Delta^m y = (\Delta^m y_p) \in \mathbb{Y}$, define $\psi(\Delta^m y_p, q) = \frac{q}{q+||\Delta^m y_p||}$, $\eta(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$ and $\sigma(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$. Then $\mathbb{Y} = (V, N, \circledast, \diamond)$ is *NNS*. Consider a sequence $(\Delta^m y_p)_{m\in\mathbb{N}}$ such that

$$\Delta^m y_p = \begin{cases} (-1)^p, & \text{if } p \neq n^2 \\ p, & \text{if } p = n^2 \ . \end{cases}$$

Then $\Delta^m y_p = (-1, 2, 3, 1, 5, 6, 7, 8, -1.........)$ and Clearly for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\lim_{n\to\infty} \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r+\epsilon) \geq \lambda\}| = 0.$$

Also,

$$st_{(\psi,\eta,\sigma)} - LIM^{r}_{\Delta^m y_p} = \begin{cases} \phi & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

For unbounded sequences, $LIM^{r(\psi,\eta,\sigma)}_{\Delta^m y_p} = \phi$, But $st_{(\psi,\eta,\sigma)} - LIM^{r}_{\Delta^m y_p} \neq \phi$. That is the sequence might be rough st-convergent. Above example shows that a generalized sequence can be rough st-convergent but not rough convergent.

Definition 3.5. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*, $\Delta^m y = (\Delta^m y_p)$, be a generalized difference sequence in $\mathbb{Y}$ is considered to be *I-Δ^m-statistically convergent* to $\xi \in \mathbb{Y}$ w.r.t. neutrosophic norm (ψ, η, σ) if for every $\epsilon > 0$ and $\lambda \in (0, 1)$

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; \epsilon) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \xi; \epsilon) \geq \lambda \text{ or}$$
$$\sigma(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta\} \in I,$$

It is denoted by $\Delta^m y_p \xrightarrow{I-st_{(\psi,\eta,\sigma)}} \xi$. Let $I - st_{(\psi,\eta,\sigma)} - LIM_{\Delta^m y_p}$ represnts the collection of all I- statistical limit points of $(\Delta^m y_p)$.

Next, we added rough I-st-convergence for generalized differenece sequences in *NNS*.

Definition 3.6. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*, $\Delta^m y = (\Delta^m y_p)$, be a generalized difference sequence in $\mathbb{Y}$ is considered to be *rough I-Δ^m-statistically convergent* to $\xi \in \mathbb{Y}$ w.r.t. neutrosophic norm (ψ, η, σ) for some $r \geq 0$ if for every $\epsilon > 0$ and $\lambda \in (0, 1)$

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta\} \in I,$$

It is denoted by $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \xi$. Let $I - st_{(\psi,\eta,\sigma)} - LIM^{r}_{\Delta^m y_p}$ represents the collection of all rough I-st-limit points of $(\Delta^m y_p)$.

Remark 3.7. For $r = 0$, the notion *rough I-Δ^m-st-convergence* agrees with the *I-Δ^m-st-convergence* in *NNS*.

Example 3.8. Let $(\mathbb{Y}, \|.\|)$ be any real normed space. For every $q > 0$ and for all $\Delta^m y = (\Delta^m y_p) \in \mathbb{Y}$, define $\psi(\Delta^m y_p, q) = \frac{q}{q+||\Delta^m y_p||}$, $\eta(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$ and $\sigma(\Delta^m y_p, q) = \frac{\Delta^m y_p}{q+||\Delta^m y_p||}$. Then $\mathbb{Y} = (V, N, \circledast, \diamond)$ is *NNS*. If we take $I_d = \{A \in \mathbb{N} \text{ with } \delta(A) = 0\}$ then for this admissible ideal, define $(\Delta^m y_p)_{m \in \mathbb{N}}$ such that

$$\Delta^m y_p = \begin{cases} p, & \text{if } p \in A \\ (-1)^p, & p \notin A. \end{cases}$$

Then

$$I - st_{(\psi,\eta,\sigma)} - LIM^{r}_{\Delta^m y_p} = \begin{cases} \phi & r < 1 \\ [1-r, r-1] & \text{otherwise.} \end{cases}$$

Hence $I - st_{(\psi,\eta,\sigma)} - LIM^{r}_{\Delta^m y_p} \neq \phi$.

Definition 3.9. A sequence $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is $I - \Delta^m$-st bounded if $\exists\, G > 0$, for $\epsilon > 0$ and $0 < \lambda < 1$ such that

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}| \geq \delta\} \in I$$

We found the following results for generalised difference sequences using the aforementioned definitions in *NNS*.

Theorem 3.10. *Let $\mathbb{Y} = (V, N, \circledast, \diamond)$ be NNS. A sequence $(\Delta^m y_p)$ in $\mathbb{Y}$ is I-Δ^m-st-bounded iff $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$ for some $r > 0$.*

Proof. Firstly suppose the sequence $(\Delta^m y_p)$ is I-Δ^m-st-bounded in NNS $\mathbb{Y} = (V, N, \circledast, \diamond)$. Then for each $\epsilon > 0$, $\lambda \in (0, 1)$ and some $r > 0$, $\exists G > 0$ such that

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \xi; r + \epsilon) \geq \lambda\}| \geq \delta\} \in I.$$

Since I is admissible ideal , therefore $M = \mathbb{N} \setminus H$ is a non-empty set, where

$$H = \left\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}| \geq \delta\right\}.$$

Choose $p \in M$, then

$$\frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p; G) \geq \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}| < \delta$$

$$\implies \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p; G) > 1 - \lambda \text{ and } \eta(\Delta^m y_p; G) < \lambda \text{ or } \sigma(\Delta^m y_p; G) \geq \lambda\}| \geq 1 - \delta. \quad (1)$$

Let $\mathbb{K} = \{p \leq n : \psi(\Delta^m y_p; G) > 1 - \lambda$ and $\eta(\Delta^m y_p; G) < \lambda$ and $\sigma(\Delta^m y_p; G) < \lambda\}$.
Also for $p \in \mathbb{K}$,

$$\begin{aligned}\psi(\Delta^m y_p; r + G) &\geq \min\{\psi(0, r), \psi(\Delta^m y_p, G)\}\\ &= \min\{1, \psi(\Delta^m y_p; G)\}\\ &> 1 - \lambda\end{aligned}$$

$$\begin{aligned}\eta(\Delta^m y_p; r + G) &\leq \max\{\eta(0, r), \eta(\Delta^m y_p, G)\}\\ &= \max\{0, \eta(\Delta^m y_p; G)\}\\ &< \lambda\end{aligned}$$

$$\begin{aligned}\sigma(\Delta^m y_p; r + G) &\leq \max\{\sigma(0, r), \sigma(\Delta^m y_p, G)\}\\ &= \max\{0, \sigma(\Delta^m y_p; G)\}\\ &< \lambda\end{aligned}$$

Thus $\mathbb{K} \subset \{p \le n : \psi(\Delta^m y_p; r+G) > 1-\lambda, \eta(\Delta^m y_p; r+G) < \lambda, \sigma(\Delta^m y_p; r+G) < \lambda\}$.
Using (1), we have

$$1-\delta \le \frac{|\mathbb{K}|}{n} \le \frac{1}{n}\left|\{p \le n : \psi(\Delta^m y_p; r+G) > 1-\lambda, \eta(\Delta^m y_p; r+G) < \lambda, \sigma(\Delta^m y_p; r+G) < \lambda\}\right|.$$

Therefore,

$$\frac{1}{n}\left|\{p \le n : \psi(\Delta^m y_p; r+G) \le 1-\lambda \text{ or } \eta(\Delta^m y_p; r+G) \ge \lambda \text{ or } \sigma(\Delta^m y_p; r+G) < \lambda\}\right| < 1-(1-\delta) < \delta.$$

Then for $n \in \mathbb{N}$,

$$\frac{1}{n}|\{p \le n : \psi(y_p; r+G) \le 1-\lambda \text{ or } \eta(y_p; r+G) \ge \lambda \text{ or } \sigma(\Delta^m y_p; r+G) \ge \lambda\}| \ge \delta \subset H \in I.$$

Hence $0 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$. Therefore, $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y} \ne \phi$.
Conversely; for some $r > 0$, let $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \ne \phi$. Then $\exists$ some $\omega \in \mathbb{Y}$ such that $\omega \in I - st_{(\psi,\eta)} - LIM^r_{\Delta^m y_p}$.
For every $\epsilon > 0$ and $0 < \lambda < 1$, we have

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - \omega; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - \omega; r+\epsilon) \ge \lambda \text{ or}$$

,

$$\sigma(\Delta^m y_p - \omega; r+\epsilon) < \lambda\}| \ge \delta\} \in I$$

It follows that nearly all $\Delta^m y_p$'s are encircled in a ball with centre ω in *NNS*, This suggests $(\Delta^m y_p)$ is *I-Δ^m-statistically bounded* in *NNS*. □

Next, we will show that the algebraic characterization also hold for rough I-st-convergent sequences for generalized difference sequences in *NNS*.

Theorem 3.11. *Let $(\Delta^m y_p)$ and $(\Delta^m y_q)$ be two generalized difference sequences in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ with I as admissible ideal and $r \ge 0$, then the following results holds:*
(i) if $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L$ and $\beta \in \mathbb{R}$, then $\beta\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \beta L$.
(ii) if $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L_1$ and $\Delta^m y_q \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L_2$ then $(\Delta^m y_p + \Delta^m y_q) \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} (L_1 + L_2)$

Proof. (i) If $\beta = 0$ then the result is obvious. So assume $\beta \ne 0$. As $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} L$ then for given $\lambda > 0$ and $r \ge 0$,
Consider $G = \{n \in \mathbb{N} : \frac{1}{n}|\{p \le n : \psi(\Delta^m y_p - L; r+\epsilon) \le 1-\lambda$ or $\eta(\Delta^m y_p - L; r+\epsilon) \ge \lambda$ or $\sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}| \ge \delta\} \in I$. As I is admissible ideal, therefore take $M = \mathbb{N} \setminus G$ as an non-empty set. Choose $m \in M$, then

$$\frac{1}{n}\left|\{p \le n : \psi(\Delta^m y_p - L; r+\epsilon) \le 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) \ge \lambda \text{ or } \sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}\right| < \delta$$

$$\implies \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - L; r+\epsilon) > 1-\lambda \text{ or } \eta(\Delta^m y_p - L; r+\epsilon) < \lambda$$
$$\text{or } \sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}| \geq 1-\delta.$$
$$\implies \frac{1}{n}|\mathbb{K}| \geq 1-\delta. \tag{2}$$

Where

$$\mathbb{K} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - L; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - L; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - L; r+\epsilon) < \lambda\}.$$

It suffices to demonstrate that for each $\lambda > 0$ and $r \geq 0$;

$$\mathbb{K} \subset \{m \in \mathbb{N} : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda, \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda, \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}.$$

Let $k \in \mathbb{K}$, then $\psi(\Delta^m y_k - L; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_k - L; r+\epsilon) < \lambda$ and $\sigma(\Delta^m y_p - L; r+\epsilon) < \lambda$.
So;

$$\begin{aligned}
\psi(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \psi\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\geq \min \left\{\psi(\Delta^m y_p - L, r+\epsilon), \psi\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\geq \min \{\psi(\Delta^m y_p - L, r+\epsilon), 1\} \\
&= \psi(\Delta^m y_p - L, r+\epsilon) > 1-\lambda, \\
\eta(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \eta\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\leq \max \left\{\eta(\Delta^m y_p - L, r+\epsilon), \eta\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\leq \max \{\eta(\Delta^m y_p - L, r+\epsilon), 0\} \\
&= \sigma(\Delta^m y_p - L, r+\epsilon) < \lambda. \\
\text{and } \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) &= \sigma\left(\Delta^m y_p - L, \frac{r+\epsilon}{|\beta|}\right) \\
&\leq \max \left\{\sigma(\Delta^m y_p - L, r+\epsilon), \sigma\left(0, \frac{r+\epsilon}{|\beta|} - (r+\epsilon)\right)\right\} \\
&\leq \max \{\sigma(\Delta^m y_p - L, r+\epsilon), 0\} \\
&= \sigma(\Delta^m y_p - L, r+\epsilon) < \lambda.
\end{aligned}$$

which gives;

$$\mathbb{K} \subset \{m \in \mathbb{N} : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda, \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda, \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}.$$

Using (2), we have

$$1-\delta \leq \frac{|\mathbb{K}|}{n} \leq \frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) > 1-\lambda, \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda,$$
$$\sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) < \lambda\}|.$$

Therefore,

$$\frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda,$$
$$\text{or } \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda\}| < 1-(1-\delta) < \delta.$$

Then

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\beta\Delta^m y_p - \beta L; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda,$$
$$\text{or } \sigma(\beta\Delta^m y_p - \beta L; r+\epsilon) \geq \lambda\}| \geq \delta\} \subset \mathbb{G} \in I.$$

which shows that $\beta\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \beta L$.

(ii) In the similar manner, we can prove (ii) part. So, we are omitting its proof. □

In next result, we will show the set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ is closed.

Theorem 3.12. *The set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ of a generalized difference sequence $(\Delta^m y_p)$ is a closed set in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$.*

Proof. If $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \phi$ then the result is obvious as $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$ is either empty set or singleton set.

Let $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \neq \phi$.

Let $\Delta^m x = (\Delta^m x_p)$ be a convergent sequence in $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ which converges to $x_0 \in \mathbb{Y}$.

For $\lambda \in (0,1)$ and $\epsilon > 0 \quad \exists \; m_0 \in \mathbb{N}$ such that

$$\psi\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) > 1-\lambda, \eta\left(\Delta^m x_p - x_0; \frac{\epsilon}{2}\right) < \lambda, \sigma(\Delta^m x_p - x_0; \frac{\epsilon}{2}) < \lambda \text{ for all } \; p \geq m_0.$$

Let us take $\Delta^m x_{m_1} \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ with $m_1 > m_0$ such that

$$\mathbb{A} = \{p \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda,$$
$$\text{or } \sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda\}| \geq \delta\} \in I$$

Take $G = \mathbb{N} \setminus \mathbb{A}$ is a non-empty set. Choose $n \in \; G$, then

$$\frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) \geq \lambda\}| < \delta$$
$$\Rightarrow \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda,$$
$$\sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda\}| \geq 1-\delta.$$

$$\text{Let } \mathbb{B}_n = \{p \leq n : \psi(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda,$$
$$\sigma(\Delta^m y_p - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}) < \lambda\}.$$

Then for $j \in \mathbb{B}_n$, $j \geq m_0$, we have

$$\begin{aligned}
\psi(\Delta^m y_j - x_0; r+\epsilon) &\geq \min \left\{\psi\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \psi\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} \\
&> 1-\lambda, \\
\eta(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\eta\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \eta\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} \\
&< \lambda \\
\sigma(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\sigma\left(\Delta^m y_j - \Delta^m x_{m_1}; r+\frac{\epsilon}{2}\right), \sigma\left(\Delta^m x_{m_1} - x_0; \frac{\epsilon}{2}\right)\right\} \\
&< \lambda.
\end{aligned}$$

Therefore;

$$j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}.$$

Hence

$$\mathbb{B}_n \subset \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}$$

which implies

$$1-\delta \leq \frac{|\mathbb{B}_n|}{n} \leq \frac{1}{n}|\{p \leq n : \psi\left(\Delta^m y_p - x_0; r+\epsilon\right) > 1-\lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}|.$$

Therefore,

$$\frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - x_0; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda$$
$$\text{or } \sigma(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda\}| < 1-(1-\delta) = \delta.$$

Then

$$\{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m y_p - x_0; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda \text{ or}$$
$$\sigma(\Delta^m y_p - x_0; r+\epsilon) \geq \lambda\}| \geq \delta\} \subset \mathbb{A} \in I.$$

implies $x_0 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ in $(\mathbb{Y}, \psi, \eta)$. □

The convexity of the set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$ is demonstrated in the following result.

Theorem 3.13. *The set $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ of a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is a convex set for some non-negative number r.*

Proof. Let $\varphi_1, \varphi_2 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$. For convexity we have to show that

$$(1-\omega)\varphi_1 + \omega\varphi_2 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y}$$

for any real number $\omega \in (0,1)$.

Since $\varphi_1, \varphi_2 \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$, then $\exists\ p \in \mathbb{N}$ for each $\epsilon > 0$ and $\lambda \in (0,1)$ such that

$$\mathbb{A}_0 = \{p \in \mathbb{N} : \psi\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \leq 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \geq \lambda$$
$$\text{or } \sigma\left(\Delta^m y_p - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right) \geq \lambda\},$$

and

$$\mathbb{A}_1 = \{p \in \mathbb{N} : \psi\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \leq 1-\lambda \text{ or } \eta\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \geq \lambda$$
$$\text{or } \sigma\left(\Delta^m y_p - \varphi_2; \frac{r+\epsilon}{2\omega}\right) \geq \lambda\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \geq \delta\right\} \in I,$$

Now choose $\delta_1 \in (0,1)$ such that $(1-\delta_1) \in (0,\delta)$ Let

$$\mathbb{A} = \left\{n \in \mathbb{N} : \frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \geq \delta_1\right\} \in I,$$

Now for $n \notin \mathbb{A}$, $\frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_0 \cup \mathbb{A}_1\}\right| < 1-\delta_1$ or $\frac{1}{n}\left|\{p \leq n : p \notin \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \geq 1-(1-\delta_1) = \delta_1$ This implies $\{p \leq n : m \notin \mathbb{A}_0 \cup \mathbb{A}_1\} \neq \phi$

Let $m_0 \in (\mathbb{A}_0 \cup \mathbb{A}_1)^c = \mathbb{A}_0^c \cap \mathbb{A}_1^c$

Then

$$\begin{aligned}
\psi(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) &= \psi[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\geq \min\left\{\psi\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \psi\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \min\left\{\psi\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \psi\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&> 1-\lambda,
\end{aligned}$$

$$\begin{aligned}
\eta(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) &= \eta[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\leq \max\left\{\eta\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \eta\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \max\left\{\eta\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \eta\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&< \lambda,
\end{aligned}$$

$$\begin{aligned}
\sigma(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) &= \sigma[(1-\omega)(\Delta^m y_{m_0} - \varphi_1) + \omega(\Delta^m y_{m_0} - \varphi_2); r+\epsilon] \\
&\leq \max\left\{\sigma\left((1-\omega)(\Delta^m y_{m_0} - \varphi_1); \frac{r+\epsilon}{2}\right), \sigma\left(\omega(\Delta^m y_{m_0} - \varphi_2); \frac{r+\epsilon}{2}\right)\right\} \\
&= \max\left\{\sigma\left(\Delta^m y_{m_0} - \varphi_1; \frac{r+\epsilon}{2(1-\omega)}\right), \sigma\left(\Delta^m y_{m_0} - \varphi_2; \frac{r+\epsilon}{2\omega}\right)\right\} \\
&< \lambda.
\end{aligned}$$

This implies $\mathbb{A}_0^c \cap \mathbb{A}_1^c \subset \mathbb{B}^c$ where

$$\mathbb{B} = \{p \in \mathbb{N} : \psi(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \leq 1-\lambda \text{ or } \eta\,(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \geq \lambda$$
$$\text{or } \sigma\,(\Delta^m y_{m_0} - [(1-\omega)\varphi_1 + \omega\varphi_2]; r+\epsilon) \geq \lambda\}.$$

So for $n \notin \mathbb{A}$,

$$\delta_1 \leq \frac{1}{n}\left|\{p \leq n : p \notin \mathbb{A}_0 \cup \mathbb{A}_1\}\right| \leq \frac{1}{n}\left|\{p \leq n : p \notin \mathbb{B}\}\right|$$

or

$$\frac{1}{n}\left|\{p \leq n : p \in \mathbb{B}\}\right| < 1 - \delta_1 < \delta$$

Thus $\mathbb{A}^c \subset \left\{n \in \mathbb{N} : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| < \delta\right\}$. Since $\mathbb{A}^c \in \mathbb{F}(I)$, So, $\left\{n : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| < \delta\right\} \in \mathbb{F}(I)$, which implies

$\left\{n : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| \geq \delta\right\} \in I$. This implies that $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ is a convex set. □

Theorem 3.14. *A generalized difference sequence $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is rough-I-Δ^m-statistically convergent to $\rho \in \mathbb{Y}$ w.r.t. the norm (ψ, η, σ) for some $r > 0$ if there exists a sequence $\Delta^m z = (\Delta^m z_p)$ in $\mathbb{Y}$ such that $I - st_{(\psi,\eta,\sigma)} - LIM_{\Delta^m z_p} = \rho$ in $\mathbb{Y}$ and for each $\lambda \in (0,1)$ we have $\psi(\Delta^m y_p - \Delta^m z_p; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda$ for all $p \in \mathbb{N}$.*

Proof. Since $\Delta^m z = (\Delta^m z_p)$ be a generalized difference sequence in $\mathbb{Y}$, which is *I-Δ^m-statistically convergent* to $\rho \in \mathbb{Y}$ and $\psi(\Delta^m y_p - \Delta^m z_p; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \Delta^m z_p; r+\epsilon) < \lambda$ for all $p \in \mathbb{N}$ and $\lambda \in (0,1)$.
Then by definition, for any $\epsilon, \delta > 0$ and $\lambda \in (0,1)$ the set

$$\mathbb{M} = \{n \in \mathbb{N} : \frac{1}{n}|\{p \leq n : \psi(\Delta^m z_p - \rho; \epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m z_p - \rho; \epsilon) \geq \lambda$$
$$\text{or } \sigma(\Delta^m z_p - \rho; \epsilon) \geq \lambda\}| \geq \delta\} \in I.$$

Define

$$\mathbb{A}_1 = \{p \in \mathbb{N} : \psi(\Delta^m z_p - \rho; \epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m z_p - \rho; \epsilon) \geq \lambda \text{ or } \sigma(\Delta^m z_p - \rho; \epsilon) \geq \lambda\}$$

$$\mathbb{A}_2 = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m z_p; r) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - \Delta^m z_p; r) \geq \lambda \text{ or } \sigma(\Delta^m y_p - \Delta^m z_p; r) \geq \lambda\}.$$

For $\delta > 0$, we have

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta\right.\right\} \in I,$$

Now choose $\delta_1 \in (0,1)$ such that $(1-\delta_1) \in (0, \delta)$ and let

$$\mathbb{A} = \left\{n : \frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta_1\right.\right\} \in I,$$

Now for $n \notin \mathbb{A}$

$$\frac{1}{n}\left|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} < 1 - \delta_1\right.$$

$$\frac{1}{n}\left|\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2|\}\right. \geq 1-(1-\delta_1)=\delta_1$$

This implies $\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2\} \neq \phi$

Let $p \in (\mathbb{A}_1 \cup \mathbb{A}_2)^c = \mathbb{A}_1^c \cap \mathbb{A}_2^c$

Then

$$\begin{aligned}
\psi\left(\Delta^m y_p-\rho; r+\epsilon\right) &\geq \min\ \{\psi(\Delta^m y_p-\Delta^m z_p; r), \psi(\Delta^m z_p-\rho; \epsilon)\} \\
&> 1-\lambda \\
\eta\left(\Delta^m y_p-\rho; r+\epsilon\right) &\leq \max\ \{\eta(\Delta^m y_p-\Delta^m z_p; r), \eta(\Delta^m z_p-\rho; \epsilon)\} \\
&< \lambda \\
\sigma\left(\Delta^m y_p-\rho; r+\epsilon\right) &\leq \max\ \{\sigma(\Delta^m y_p-\Delta^m z_p; r), \sigma(\Delta^m z_p-\rho; \epsilon)\} \\
&< \lambda.
\end{aligned}$$

Which gives $\mathbb{A}_1^c \cap \mathbb{A}_2^c \subset \mathbb{B}^c$, where

$\mathbb{B} = \{p \in \mathbb{N} : \psi(\Delta^m y_p-\rho; r+\epsilon) \leq 1-\lambda$ or $\eta(\Delta^m y_p-\rho; r+\epsilon) \geq \lambda$ or $\sigma(\Delta^m y_p-\rho; r+\epsilon) \geq \lambda\}$.

So for $n \notin \mathbb{A}$,

$$\delta_1 \leq \frac{1}{n}|\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq \frac{1}{n}|\{p \leq n : p \notin \mathbb{B}\}|$$

or

$$\frac{1}{n}|\{p \leq n : p \in \mathbb{B}\}| < 1-\delta_1 < \delta$$

Thus $\mathbb{A}^c \subset \{n : \frac{1}{n}|p \leq n : p \in \mathbb{B}| < \delta\}$. Since $\mathbb{A}^c \in \mathbb{F}(I)$, So, $\left\{n : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| < \delta\right\} \in \mathbb{F}(I)$, which implies $\left\{n : \frac{1}{n}\left|p \leq n : p \in \mathbb{B}\right| \geq \delta\right\} \in I$.

Hence, $\Delta^m y_p \xrightarrow{r-I-st_{(\psi,\eta,\sigma)}} \rho$ in *NNS* $\mathbb{Y} = (V, N, \circledast, \diamond)$. $\square$

Theorem 3.15. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized differenece sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ then the existence of two elements $\alpha_1, \alpha_2 \in I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y}$ is not possible for $r > 0$ and $\lambda \in (0,1)$ such that $\psi(\alpha_1-\alpha_2; cr) \leq 1-\lambda$ or $\eta(\alpha_1-\alpha_2; cr) \geq \lambda$ or $\sigma(\alpha_1-\alpha_2; cr) \geq \lambda$ for $c > 2$.*

Proof. Let us assume the existence of two elements $\alpha_1, \alpha_2 \in I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y}$ is possible such that

$$\psi(\alpha_1-\alpha_2; cr) \leq 1-\lambda \text{ or } \eta(\alpha_1-\alpha_2; cr) \geq \lambda \text{ or } \sigma(\alpha_1-\alpha_2; cr) \geq \lambda \text{ for } c > 2. \tag{3}$$

Then for each $\epsilon > 0$ and $\lambda \in (0,1)$. Define,

$$\begin{aligned}
\mathbb{A}_1 = \{p \in \mathbb{N} : \psi(\Delta^m y_p-\alpha_1; r+\frac{\epsilon}{2}) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p-\alpha_1; r+\frac{\epsilon}{2}) \geq \lambda \\
\text{or } \sigma(\Delta^m y_p-\alpha_1; r+\frac{\epsilon}{2}) \geq \lambda\}
\end{aligned}$$

$$\mathbb{A}_2 = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}) \leq 1 - \lambda \text{ or } \eta(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}) \geq \lambda \\ \text{or } \sigma(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}) \geq \lambda\}.$$

Then

$$\frac{1}{n}|\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq \frac{1}{n}|\{p \leq n : p \in \mathbb{A}_1\}| + \frac{1}{n}|\{p \leq n : p \in \mathbb{A}_2\}|$$

So, by the property of I-*convergence*, we have

$$I - \lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2\}| \leq I - \lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1\}| + I - \lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_2\}| = 0$$

Thus

$$\left\{n : \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta\right\} \in I, \text{for all } \ \delta > 0$$

Now choose $0 < \delta_1 = 1/2 < 1$ such that $(1 - \delta_1) \in (0, \delta)$

Let

$$\mathbb{K} = \left\{n : \frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq \delta_1\right\} \in I,$$

Now for $n \notin \mathbb{K}$

$$\frac{1}{n} |\{p \leq n : p \in \mathbb{A}_1 \cup \mathbb{A}_2|\} < 1 - \delta_1 = 1/2$$

$$\frac{1}{n} |\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2|\} \geq 1 - (1 - \delta_1) = 1/2$$

This implies $\{p \leq n : p \notin \mathbb{A}_1 \cup \mathbb{A}_2\} \neq \phi$. Then for $p \in \mathbb{A}_1^c \cap \mathbb{A}_2^c$ we have

$$\begin{aligned}
\psi(\alpha_1 - \alpha_2; 2r + \epsilon) &\geq \min \left\{\psi\left(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}\right), \psi\left(\Delta^m y_p - \alpha_1; r + \frac{\epsilon}{2}\right)\right\} \\
&> 1 - \lambda, \\
\eta(\alpha_1 - \alpha_2; 2r + \epsilon) &\leq \max \left\{\eta\left(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}\right), \eta\left(\Delta^m y_p - \alpha_1; r + \frac{\epsilon}{2}\right)\right\} \\
&< \lambda \\
\sigma(\alpha_1 - \alpha_2; 2r + \epsilon) &\leq \max \left\{\sigma\left(\Delta^m y_p - \alpha_2; r + \frac{\epsilon}{2}\right), \sigma\left(\Delta^m y_p - \alpha_1; r + \frac{\epsilon}{2}\right)\right\} \\
&< \lambda.
\end{aligned}$$

Hence,

$$\psi(\alpha_1 - \alpha_2; 2r + \epsilon) > 1 - \lambda, \eta(\alpha_1 - \alpha_2; 2r + \epsilon) < \lambda, \sigma(\alpha_1 - \alpha_2; 2r + \epsilon) < \lambda. \tag{4}$$

Then from (4) we have

$$\psi(\alpha_1 - \alpha_2; cr) > 1 - \lambda, \eta(\alpha_1 - \alpha_2; cr) < \lambda, \sigma(\alpha_1 - \alpha_2; cr) < \lambda \text{ for } c > 2.$$

It contradicts (3). So, existence of two elements is not possible such that $\psi(\alpha_1 - \alpha_2; cr) \leq 1 - \lambda$ or $\eta(\alpha_1 - \alpha_2; cr) \geq \lambda$ or $\sigma(\alpha_1 - \alpha_2; cr) \geq \lambda$ for $c > 2$. □

4. Rough ideal statistical cluster points for generalized difference sequences in *NNS*

Definition 4.1. Let $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ be *NNS*. Then $\gamma \in \mathbb{Y}$ is called *rough I-Δ^m-statistical cluster point* of the sequence $\Delta^m y = (\Delta^m y_p)$ in $\mathbb{Y}$ w.r.t. norm (ψ, η, σ) for some $r > 0$ if for every $\epsilon > 0$ and $\lambda \in (0,1)$

$$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \gamma; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \gamma; r+\epsilon) < \lambda, \eta(\Delta^m y_p - \gamma; r+\epsilon) < \lambda\}) \neq 0$$

where $\delta_I(A) = I - \lim_{n\to\infty} \frac{1}{n} |\{p \leq n : p \in A\}|$ if exists. Here γ is known as *r-I-Δ^m-st-cluster point* of a sequence $(\Delta^m y_p)$.
Let $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ indicates the collection of all *r-I-Δ^m-st-cluster points* w.r.t. the norm (ψ, η, σ) of a sequence $(\Delta^m y_p)$ in *NNS* $\mathbb{Y} = (V, N, \circledast, \diamond)$. If $r = 0$ then the notion stands for only *I-Δ^m-st-cluster point* in *NNS* $\mathbb{Y} = (V, N, \circledast, \diamond)$. Symbolically; $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.

In the next result, we have derived the closedness of the set $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ of generalized difference sequence $(\Delta^m y_p)$ in $\mathbb{Y}$.

Theorem 4.2. *The set $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ of generalized difference sequence $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$ is closed for some $r > 0$.*

Proof. If $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \phi$, then the result is obvious. So nothing to prove.
Let us suppose $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \neq \phi$. Consider $\Delta^m x = (\Delta^m x_p)$ be a generalized difference sequence such that

$$(\Delta^m x) \subseteq \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \text{ and } \Delta^m x_p \xrightarrow{(\psi,\eta)} x_0.$$

To prove closedness, we will prove $x_0 \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
As $\Delta^m x_p \xrightarrow{(\psi,\eta,\sigma)} x_0$, so for $\lambda \in (0,1)$ and $\epsilon > 0$, $\exists$ $p_\epsilon \in \mathbb{N}$ such that
$\psi(\Delta^m x_p - x_0; \frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m x_p - x_0; \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m x_p - x_0; \frac{\epsilon}{2}) < \lambda$ for $p \geq p_\epsilon$.
Choose some $p_0 \in \mathbb{N}$ such that $p_0 \geq p_\epsilon$. Then we have

$$\psi(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m x_{p_0} - x_0; \frac{\epsilon}{2}) < \lambda.$$

Again as $\Delta^m x = (\Delta^m x_p) \subseteq \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, we have $\Delta^m x_{p_0} \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.

$$\implies \delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - c; r + \frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}); r + \frac{\epsilon}{2}) < \lambda,$$
$$\sigma(\Delta^m y_p - \Delta^m x_{p_0}); r + \frac{\epsilon}{2}) < \lambda\}) \neq 0. \tag{5}$$

Consider

$$\mathbb{G} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) < \lambda\}$$

Choose $j \in \mathbb{G}$, then we have $\psi(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda$.

Now,

$$\begin{aligned}
\psi(\Delta^m y_j - x_0; r+\epsilon) &\geq \min \left\{\psi\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \psi\left(\Delta^m x_{p_0} - x_0; r + \frac{\epsilon}{2}\right)\right\} \\
&> 1 - \lambda, \\
\eta(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\eta\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \eta\left(\Delta^m x_{p_0} - y_0; r + \frac{\epsilon}{2}\right)\right\} \\
&< \lambda \\
\sigma(\Delta^m y_j - x_0; r+\epsilon) &\leq \max \left\{\sigma\left(\Delta^m y_j - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}\right), \sigma\left(\Delta^m x_{p_0} - y_0; r + \frac{\epsilon}{2}\right)\right\} \\
&< \lambda.
\end{aligned}$$

Thus

$$j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1 - \lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, (\Delta^m y_p - x_0; r+\epsilon) < \lambda\}.$$

Hence

$$\{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) > 1-\lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) < \lambda, \sigma(\Delta^m y_p - \Delta^m x_{p_0}; r+\frac{\epsilon}{2}) < \lambda\}$$

$$\subseteq \{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1 - \lambda, \eta(y_m - x_0; r+\epsilon) < \lambda, \sigma(y_m - x_0; r+\epsilon) < \lambda\}.$$

On the other side, we have the inequality:

$$\begin{aligned}
&\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) > 1 - \lambda, \eta(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda, \\
&\qquad \sigma(\Delta^m y_p - \Delta^m x_{p_0}; r + \frac{\epsilon}{2}) < \lambda\}) \\
&\leq \delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1 - \lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \\
&\qquad \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}). \qquad (6)
\end{aligned}$$

Using (5), we conclude that

$$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - x_0; r+\epsilon) > 1 - \lambda, \eta(\Delta^m y_p - x_0; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - x_0; r+\epsilon) < \lambda\}) \neq 0,$$

as the set on L.H.S.of (6) possesses natural density more than zero.

So, $x_0 \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. □

Theorem 4.3. *Let $\Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ be the collection of all I-Δ^m-st-cluster points of $\Delta^m y = (\Delta^m y_p)$ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$. Then for any arbitrary $\nu \in \Gamma^{I}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, $\lambda \in (0,1)$ and $r \geq 0$, we have $\psi(\zeta - \nu; r) > 1 - \lambda, \eta(\zeta - \nu; r) < \lambda, \sigma(\zeta - \nu; r) < \lambda$ for all $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.*

Proof. Since $\nu \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ then for $\lambda \in (0,1)$ and $\epsilon > 0$,

$$\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu;\epsilon) < \lambda, \sigma(\Delta^m y_p - \nu;\epsilon) < \lambda\}\right) \neq 0. \tag{7}$$

Now it is sufficient to show that if any $\zeta \in \mathbb{Y}$ satisfying $\psi(\zeta - \nu;\epsilon) > 1-\lambda, \eta(\zeta - \nu;\epsilon) < \lambda, \eta(\zeta - \nu;\epsilon) < \lambda$ then $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.

Suppose $j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu;\epsilon) < \lambda, \sigma(\Delta^m y_p - \nu;\epsilon) < \lambda\}$ then $\psi(\Delta^m y_j - \nu;\epsilon) > 1-\lambda, \eta(\Delta^m y_j - \nu;\epsilon) < \lambda, \sigma(\Delta^m y_j - \nu;\epsilon) < \lambda$.

On the other side,

$$\begin{aligned}
\psi\left(\Delta^m y_j - \zeta; r+\epsilon\right) &\geq \min\left\{\psi\left(\Delta^m y_j - \nu;\epsilon\right), \psi\left(\zeta - \nu; r\right)\right\} \\
&> 1-\lambda, \\
\eta\left(\Delta^m y_j - \zeta; r+\epsilon\right) &\leq \max\left\{\eta\left(\Delta^m y_j - \nu;\epsilon\right), \eta\left(\zeta - \nu; r\right)\right\} \\
&< \lambda \\
\sigma\left(\Delta^m y_j - \zeta; r+\epsilon\right) &\leq \max\left\{\sigma\left(\Delta^m y_j - \nu;\epsilon\right), \sigma\left(\zeta - \nu; r\right)\right\} \\
&< \lambda
\end{aligned}$$

So, we have $\psi\left(\Delta^m y_j - \zeta; r+\epsilon\right) > 1-\lambda, \eta\left(\Delta^m y_j - \zeta; r+\epsilon\right) < \lambda, \sigma\left(\Delta^m y_j - \zeta; r+\epsilon\right) < \lambda$.

Thus

$j \in \{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}$

which gives the inclusion

$$\begin{aligned}
&\{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu;\epsilon) < \lambda, \sigma(\Delta^m y_p - \nu;\epsilon) < \lambda\} \\
&\subseteq \{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \nu;\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \nu;\epsilon) < \lambda, \sigma(\Delta^m y_p - \nu;\epsilon) < \lambda\}\right) \\
&\leq \delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}\right).
\end{aligned}$$

Therefore, from (7),

$$\delta_I\left(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}\right) \neq 0.$$

Hence $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. □

Theorem 4.4. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$. and $\overline{B(\rho,\lambda,r)} = \{\Delta^m y \in \mathbb{Y} : \psi(\Delta^m y - \rho; r) \geq 1-\lambda, \eta(\Delta^m y - \rho; r) \leq \lambda, \sigma(\Delta^m y - \rho; r) \leq \lambda\}$, denotes the closure of the open ball $B(\rho,\lambda,r) = \{\Delta^m y \in \mathbb{Y} : \psi(\Delta^m y - \rho; r) > 1-\lambda, \eta(\Delta^m y - \rho; r) < \lambda, \sigma(\Delta^m y - \rho; r) < \lambda\}$ for some $r > 0$ and $0 < \lambda < 1$ with $\rho \in \mathbb{Y}$ then $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \bigcup\limits_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$.*

Proof. Let $\zeta \in \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$ then there exists some $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ for $r > 0,$ and $0 < \lambda < 1$ such that $\psi(\rho-\zeta;r) > 1-\lambda, \eta(\rho-\zeta;r) < \lambda, \sigma(\rho-\zeta;r) < \lambda$.
As $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ then there exists a set

$$\mathbb{M} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \rho;\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \rho;\epsilon) < \lambda, \sigma(\Delta^m y_p - \rho;\epsilon) < \lambda\}$$

with $\delta_I(\mathbb{M}) \neq 0$. For $p \in \mathbb{M}$,

$$\begin{aligned}
\psi\left(\Delta^m y_p - \zeta; r+\epsilon\right) &\geq \ \min\ \left\{\psi\left(\Delta^m y_p - \rho;\epsilon\right), \psi\left(\rho-\zeta;r\right)\right\} \\
&> 1-\lambda, \\
\eta\left(\Delta^m y_p - \zeta; r+\epsilon\right) &\leq \ \max\ \left\{\eta\left(\Delta^m y_p - \rho;\epsilon\right), \eta\left(\rho-\zeta;r\right)\right\} \\
&< \lambda \\
\sigma\left(\Delta^m y_p - \zeta; r+\epsilon\right) &\leq \ \max\ \left\{\sigma\left(\Delta^m y_p - \rho;\epsilon\right), \sigma\left(\rho-\zeta;r\right)\right\} \\
&< \lambda.
\end{aligned}$$

This implies that
$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - \zeta; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \zeta; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \zeta; r+\epsilon) < \lambda\}) \neq 0.$
Hence $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. So, $\bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} \subseteq \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
Conversely, Take $\zeta \in \Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ if possible let $\zeta \notin \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$ i.e. $\zeta \notin \overline{B\left(\rho,\lambda,r\right)}$ for all $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$.
Then for all $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, we have $\psi(\zeta-\rho;r) \leq 1-\lambda$ or $\eta(\zeta-\rho;r) \geq \lambda$ or $\sigma(\zeta-\rho;r) \geq \lambda$.
According to theorem (4.3) for any arbitrary $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$, we have $\psi(\zeta-\rho;r) > 1-\lambda, \eta(\zeta-\rho;r) < \lambda, \sigma(\zeta-\rho;r) < \lambda$ which is contradiction to our supposition. Hence $\zeta \in \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$. Hence $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \bigcup_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$. This completes the proof. □

Theorem 4.5. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$, Then for $\lambda \in (0,1)$ and $r > 0$,*
(i) If $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ then $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \overline{B\left(\rho,\lambda,r\right)}$.
(ii) $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \left\{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B\left(\xi,\lambda,r\right)}\right\}$.

Proof. Let $\xi \in I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$ and $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$
For $\epsilon > 0$ and $\lambda \in (0,1)$,

Consider

$$\mathbb{G} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \xi; r+\epsilon) > 1-\lambda, \eta(\Delta^m y_p - \xi; r+\epsilon) < \lambda, \sigma(\Delta^m y_p - \xi; r+\epsilon) < \lambda\}$$

and

$$\mathbb{H} = \{p \in \mathbb{N} : \psi(\Delta^m y_p - \rho; \epsilon) > 1-\lambda, \eta(\Delta^m y_p - \rho; \epsilon) < \lambda, \sigma(\Delta^m y_p - \rho; \epsilon) < \lambda\}$$

with $\delta_I(\mathbb{G}^c) = 0$ and $\delta_I(\mathbb{H}) \neq 0$ respectively. Now for $p \in \mathbb{G} \cap \mathbb{H}$,

$$\begin{aligned}
\psi(\xi - \rho; r) &\geq \min\{\psi(\Delta^m y_p - \rho; \epsilon), \psi(\Delta^m y_p - \xi; r+\epsilon)\} \\
&> 1-\lambda, \\
\eta(\xi - \rho; r) &\leq \max\{\eta(\Delta^m y_p - \rho; \epsilon), \eta(\Delta^m y_p - \xi; r+\epsilon)\} \\
&< \lambda, \\
\sigma(\xi - \rho; r) &\leq \max\{\sigma(\Delta^m y_p - \rho; \epsilon), \sigma(\Delta^m y_p - \xi; r+\epsilon)\} \\
&< \lambda.
\end{aligned}$$

Thus $\xi \in \overline{B(\rho, \lambda, r)}$. Hence $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \overline{B(\rho, \lambda, r)}$.

(ii) It follows from (i) part that $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} \subseteq \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)}$

Take $y \in \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)}$ then

$\psi(y - \rho; r) \geq 1-\lambda, \eta(y-\rho; r) \leq \lambda, \sigma(y-\rho; r) \leq \lambda$ for all $\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. This implies that $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(y, \lambda, r)}$ i.e. $\bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)} \subseteq \{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\}$.

Now assume $y \notin I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$, then for $\lambda \in (0,1)$ and $\epsilon > 0$, we have

$$\delta_I(\{p \in \mathbb{N} : \psi(\Delta^m y_p - y; r+\epsilon) \leq 1-\lambda \text{ or } \eta(\Delta^m y_p - y; r+\epsilon) \geq \lambda \text{ or } \sigma(\Delta^m y_p - y; r+\epsilon) \geq \lambda\}) \neq 0.$$

It means there exists some I-statistical cluster point ρ for the sequence $\Delta^m y = (\Delta^m y_p)$ with $\psi(y - \rho; r+\epsilon) \leq 1-\lambda$ or $\eta(y-\rho; r+\epsilon) \geq \lambda$ or $\sigma(y-\rho; r+\epsilon) \geq \lambda$.

Thus $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \nsubseteq \overline{B(y, \lambda, r)}$ and $y \notin \{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\}$

Hence $\{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\} \subseteq I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$

And $\bigcap_{\rho \in \Gamma^I_{st(\psi,\eta)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)} \subseteq I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.

So, $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho, \lambda, r)} = \{\xi \in \mathbb{Y} : \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) \subseteq \overline{B(\xi, \lambda, r)}\}$. □

Theorem 4.6. *Let $\Delta^m y = (\Delta^m y_p)$ is ideal statistically convergent to ρ in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$, then $\overline{B(\rho, \lambda, r)} = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.*

Proof. Since $(\Delta^m y_p)$ is ideal statistically convergent to ρ w.r.t.the norms (ψ,η) i.e. $(\Delta^m y_p) \xrightarrow{I-st(\psi,\eta,\sigma)} \rho$, then by definition

$$\mathbb{A}=\left\{n:\frac{1}{n}\left|\{p\leq n:\psi(\Delta^m y_p-\rho;\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m y_p-\rho;\epsilon)\geq\lambda \text{ or } \sigma(\Delta^m y_p-\rho;\epsilon)\geq\lambda\}\right|>\delta\right\}\in I.$$

Take $G=\mathbb{N}\setminus\mathbb{A}$, as non-empty set then for $p\in G^c$,

$$\frac{1}{n}\left|\{p\leq n:\psi(\Delta^m y_p-\rho;\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m y_p-\rho;\epsilon)\geq\lambda \text{ or } \sigma(\Delta^m y_p-\rho;\epsilon)\geq\lambda\}\right|<\delta$$

$$\Rightarrow \frac{1}{n}\left|\{p\leq n:\psi(\Delta^m y_p-\rho;\epsilon)>1-\lambda,\eta(\Delta^m y_p-\rho;\epsilon)<\lambda,\sigma(\Delta^m y_p-\rho;\epsilon)<\lambda\}\right|\geq 1-\delta.$$

Put $\mathbb{B}_n=\{p\leq n:\psi\left(\Delta^m y_p-\rho;\epsilon\right)>1-\lambda,\eta\left(\Delta^m y_p-\rho;\epsilon\right)<\lambda,\sigma\left(\Delta^m y_p-\rho;\epsilon\right)<\lambda\}$ for $j\geq m$.

Now for $j\in\mathbb{B}_n$, we have $\psi\left(\Delta^m y_j-\rho;\epsilon\right)>1-\lambda,\eta\left(\Delta^m y_j-\rho;\epsilon\right)<\lambda,\sigma\left(\Delta^m y_j-\rho;\epsilon\right)<\lambda$.
Let $y\in\overline{B\left(\rho,\lambda,r\right)}$. We will prove $y\in I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y_p}$

$$\begin{aligned}
\psi(\Delta^m y_j-y;r+\epsilon)&\geq\min\{\psi(\Delta^m y_j-\rho,\epsilon),\psi(y-\rho,r)\}\\
&>1-\lambda\\
\eta(\Delta^m y_j-y;r+\epsilon)&\leq\max\{\eta(\Delta^m y_j-\rho,\epsilon),\eta(y-\rho,r)\}\\
&<\lambda\\
\sigma(\Delta^m y_j-y;r+\epsilon)&\leq\max\{\sigma(\Delta^m y_j-\rho,\epsilon),\sigma(y-\rho,r)\}\\
&<\lambda.
\end{aligned}$$

Hence $\mathbb{B}_n\subset\{p\in\mathbb{N}:\psi(\Delta^m y_p-y;r+\epsilon)>1-\lambda,\eta(\Delta^m y_p-y;r+\epsilon)<\lambda,\sigma(\Delta^m y_p-y;r+\epsilon)<\lambda\}$, which gives:
$1-\delta\leq\frac{|\mathbb{B}_n|}{n}\leq\frac{1}{n}|\{p\leq n:\psi(\Delta^m y_p-y;r+\epsilon)>1-\lambda,\eta(\Delta^m y_p-y;r+\epsilon)<\lambda,\sigma(\Delta^m y_p-y;r+\epsilon)<\lambda\}|$.
Therefore,

$$\begin{gathered}
\frac{1}{n}|\{p\leq n:\psi(\Delta^m y_p-y;r+\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m y_p-y;r+\epsilon)\geq\lambda\\
\text{or } \sigma(\Delta^m y_p-y;r+\epsilon)\geq\lambda\}|<1-(1-\delta)=\delta.
\end{gathered}$$

Then

$$\begin{gathered}
\{n\in\mathbb{N}:\frac{1}{n}|\{p\leq n:\psi(\Delta^m y_p-y;r+\epsilon)\leq 1-\lambda \text{ or } \eta(\Delta^m y_p-y;r+\epsilon)\geq\lambda\\
\text{or } \sigma(\Delta^m y_p-y;r+\epsilon)\geq\lambda\}|\geq\delta\}\subset\mathbb{A}\in I.
\end{gathered}$$

i.e. $y\in I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y_p}$ in NNS $\mathbb{Y}=(V,N,\circledast,\diamond)$.
Hence $\overline{B\left(\rho,\lambda,r\right)}\subseteq I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y_p}$. Also $I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y_p}\subseteq\overline{B\left(\rho,\lambda,r\right)}$
Hence, $I-st_{(\psi,\eta,\sigma)}-LIM^r_{\Delta^m y_p}=\overline{B\left(\rho,\lambda,r\right)}$. □

Theorem 4.7. *Let $\Delta^m y = (\Delta^m y_p)$ be a generalized difference sequence in NNS $\mathbb{Y} = (\mathcal{V}, N, \circledast, \diamond)$, which is ideal statistically convergent to ξ then $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.*

Proof. Firstly, Assume $y_p \xrightarrow{I-st_{(\psi,\eta,\sigma)}} \xi$, which gives $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \{\xi\}$. Then for $r > 0$ and $\lambda \in (0,1)$ by Theorem (4.4) , we have $\Gamma^{r(I)}_{st_{(\psi,\eta,\sigma)}}(\Delta^m y_p) = \overline{B(\xi,\lambda,r)}$. Also from Theorem (4.6), $\overline{B(\xi,\lambda,r)} = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.
Hence $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$.
Conversely, Assume $\Gamma^{r(I)}_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p}$, then by Theorem (4.4) and (4.5)(ii),

$$\bigcap_{\xi \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \bigcup_{\xi \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)}$$

This is possible only if either $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p) = \phi$ or $\Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$ is a singlton set. Then $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \bigcap_{\rho \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)} \overline{B(\rho,\lambda,r)} = \overline{B(\xi,\lambda,r)}$ for some $\xi \in \Gamma^I_{st(\psi,\eta,\sigma)}(\Delta^m y_p)$. Also, By Theorem (4.4), $I - st_{(\psi,\eta,\sigma)} - LIM^r_{\Delta^m y_p} = \xi$. □

Conclusions

The present work is more generalized than rough statistical convergence for difference sequences on *NNS* defined by Kişi and Yildil [27]. For this type of convergence various properties like statistical boundness, algebraic properties, closedness, convexity and relationship between limit points and cluster points have been obtained. Further this type of convergence can be investigated for double sequences, triple sequences in various sapces like Iintuitionistic fuzzy normed spaces, probablistic norm spaces or neutrosophic normed spaces.

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A Survey on Neutrosophic Principles for Inventory Management Problem

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Abstract. This paper presents a thorough assessment and classification of different uncertain environments used by researchers to analyze inventory management(IM) systems across various sectors, such as ABC analysis, Last In, First Out (LIFO), and batch tracking. Moreover, it introduces the concepts of the neutrosophic principle and fuzzy principle in inventory management. it also investigates the difficulties associated with the traditional inventory model. The primary focus of the study lies in inventory management under the Neutrosophic principle, specifically addressing uncertain demand and imprecise data. By shedding light on the potential of neutrosophic principle, this manuscript contributes valuable insights into overcoming the challenges posed by fuzzy models and enhancing decision-making in the realm of inventory control system.

Keywords: Supply chain; Trapezoidal fuzzy number(TpFN); Fuzzy economic order quantity (FEOQ); Trapezoidal Neutrosophic number(TpNN); Neutrosophic inventory management(NIM).

1. Introduction

An inventory control system is a mathematical algorithm used to optimize inventory levels for businesses and organizations. It considers factors such as carrying cost, stock out cost, demand, lead time, and ordering costs to determine the most suitable inventory levels. Ronald H. Ballou defines inventory models as quantitative models that determine the appropriate order quantities, timing of orders, and safety stock levels for specific inventory items or sets of items. Notable studies in the field of inventory control models have made significant contributions.In 1996, Song and Zipkin [1] introduced a model incorporating a Markovian representation of the supply system. Feng and Xiao [2], in 2001, focused on enhancing airline seat inventory control through a dynamic model and optimal policy approach. Levi et al. (2007) [3] presented sampling-based policies that provided provable near-optimality for stochastic inventory control models. In 2011, Che-Fu Hsueh [4] conducted research to examine inventory control policies within a manufacturing/remanufacturing system across the entire product life cycle. Lastly, Zhou et al. (2013) [5] proposed a comprehensive inventory control model that integrated multiple products and echelons, along with a joint replenishment strategy. These studies have offered valuable insights into the development and improvement of inventory control policies and systems.

After examining the problem of classical inventory, it has become clear that there are many problems that cannot be solved. That is why in 1965, Zadeh [6] introduced fuzzy logic. Fuzzy set is a mathematical framework for handling uncertainty and vagueness. In 1988, Dubois and Parade [7] proposed a model of IM that handles uncertainty. Section 3 explains how uncertainty is handled in inventory management. Table 1 and Figure 1 provide a summary of the significant contributions to understanding the FIM.

Table 1. Represents the impact of uncertainty in Inventory Control Models.

Authors	**Year**	**Application and Environment**	**Contribution**
Paksoy and Pehlivan [8]	2013	Application: Supply Chain Environment: TpFN	A fuzzy linear programming model is proposed for optimizing multistage supply chain networks by incorporating triangular and trapezoidal membership functions.
Ranganathan and Thirunavukarasu [9]	2014	Application: ICM for fixed deterioration Environment: TpFN	A fuzzy environment ICM for managing constant deterioration.
Sadeghi et al. [10]}	2016	Application: MIEPQM Environment: TpFN	Two tuned meta-heuristics for optimizing the MIEPQM with trapezoidal fuzzy demand and backordering.

Continued on next page

Table 1 – Continued from previous page

Authors	Year	Application and Environment	Contribution
Singh and Singh [11]	2016	Application: Vendor and Buyers Problem Environment: TpFN	A relationship model between vendors and buyers for deteriorating items, incorporating shortages, fuzzy trapezoidal costs, and inflation.

This manuscript displays different applications and methodologies of fuzzy inventory control in below Figure 1.

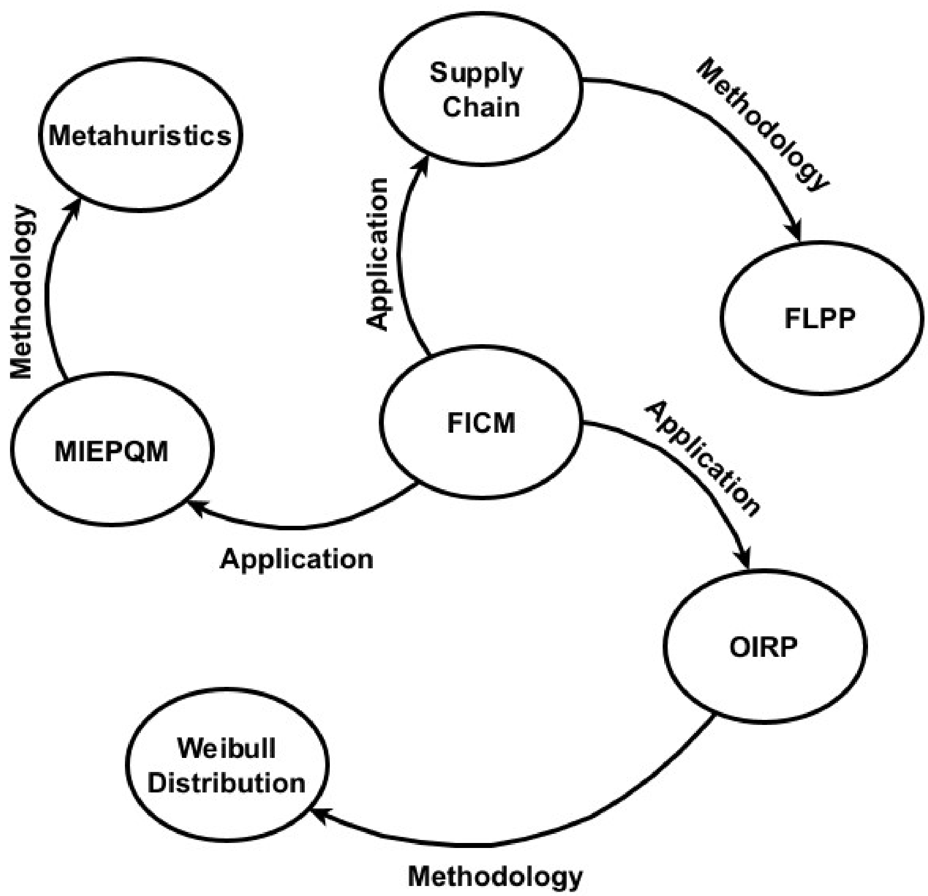

Figure 1. Represent the different applications involved in FIM

In our ongoing investigation, we examined the unique characteristics of fuzzy logic and found that some challenges are important to know. Our goal is to deliver valuable understandings of different methods that are commonly associated with IM. We will focus on important aspects of IM in fuzzy extension scenarios. We aim to assist academics in developing a deep understanding of NIM. Additionally, our manuscript surveys crucial aspects of NIM comprehensively. Moreover, we explore the existence of and challenges of IM that are associated with uncertainty. These challenges include ABC analysis [12], First In, First Out (FIFO) [13], and safety stock [14].

The introduction of this research article lays the groundwork for a comprehensive exploration of fuzzy theory and neutrosophic theory in the context of inventory management. In section 2, we established important definitions related to fuzzy theory, while section 3 shed light on the challenges of the FIM. Moving to section 4, we introduced the key concepts of neutrosophic theory, and specifically in section 4.1, we examined its practical application. In section 5, we delved into IM under the neutrosophic principle, and finally, in section 6, we arrived at the conclusion of our investigation in IM within neutrosophic environments. The forthcoming sections offer a comprehensive analysis of the application and implications of these theories in inventory management, opening new avenues for addressing uncertainties and enhancing decision-making processes in complex supply chain scenarios.

1.1. *List of Abbreviations are as follows:*

FIM stands for "Fuzzy Inventory Management".
ICM stands for "Inventory Control Model".
NTN stands for "Neutrosophic Triangular Number".
PSAOIM stands for "Particle Swarm Algorithm to optimize inventory management".
PIM stands for "Production Inventory Model".
ITFN stands for "Intuitionistic Triangular Fuzzy Number".
MIEPQM stands for "Multi-item Economic Production Quantity Model".
IVTNN stands for "Interval Valued Trapezoidal Neutrosophic Number".

2. **Some Important Definitions Related to Fuzzy Theory**

Definition 2.1. [6] Fuzzy Set: As per the Zedah's definition, The set $\widetilde{f}$ is illustrated as $\widetilde{f} = \left\{\left(\psi, \mu_{\widetilde{f}}(\psi)\right) : \psi \in f, \mu_{\widetilde{f}}(\psi) \in [0,1]\right\}$and generally denoted by the ordered pair $\left(\psi, \mu_{\widetilde{f}}(\psi)\right)$, here $\psi \in f$ be the crisp set and $\mu_{\widetilde{f}}(\psi) \in [0,1]$; such that $0 \leq \mu_{\widetilde{f}}(\psi) \leq 1$, $\widetilde{f}$is termed as the fuzzy set.

Definition 2.2. [15] Intuitionistic Fuzzy set (IFS): A set $\widetilde{IFS}$, denoted as $\widetilde{IFS} = \{\langle \delta; [\tau(\delta), \gamma(\delta)] \rangle : \delta \in \varphi\}$ can be represented graphically as a membership function where $\tau(\delta), \gamma(\delta) : \varphi \rightarrow [0,1]$the truth membership function is denoted by$\tau(\delta)$ and the false membership function is denoted by $\gamma(\delta)$. The condition for the set $\tau(\delta), \gamma(\delta)$ to satisfy $0 \leq \tau(\delta) + \gamma(\delta) \leq 1$

Definition 2.3. [16] Trapezoidal Fuzzy Number (TpFN): A trapezoidal fuzzy number $\widetilde{TF}$can be illustrated as $(j_{n_1}, j_{n_2}, j_{n_3}, j_{n_4})$shown in Figure.2 with the membership function $\mu_{\widetilde{TF}}$ as follows (ref Figure 2.)

$$\upsilon_{\widetilde{TF}}(\varphi) = \begin{cases} \dfrac{\varphi - j_{n_1}}{j_{n_2} - j_{n_1}}, \ j_{n_1} \leq \varphi \leq j_{n_2}; \\ 1, \ j_{n_2} \leq \varphi \leq j_{n_3}; \\ \dfrac{j_{n_4} - \varphi}{tf_4 - tf_3}, \ j_{n_3} \leq \varphi \leq j_{n_4}; \\ 0, \ Otherwise \end{cases}$$

$$where \quad j_{n_1}, j_{n_2}, j_{n_3}, j_{n_4} \in \mathbb{R}$$

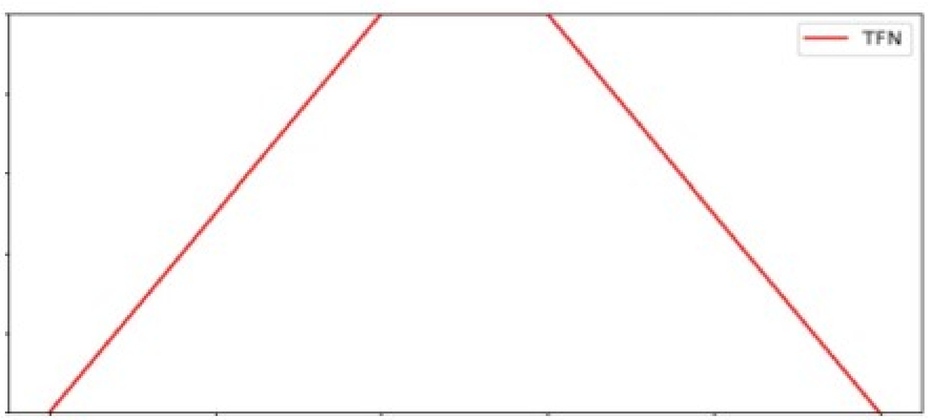

FIGURE 2. Figure 2:Trapezoidal Fuzzy Number

Definition 2.4. [17] Trapezoidal Intuitionistic Fuzzy Number (TpIFN):Let$\widetilde{TpIF} = \left\langle \left([t, u, v, w] ; \tau_{\widetilde{TpIF}} \right), \left([u_1, r, s, w_1]; \gamma_{\widetilde{TpIF}} \right) \right\rangle$ has a non-membership $\gamma_{\widetilde{TpIF}}(\varphi)$, and a membership $\tau_{\widetilde{TpIF}}(\varphi)$ functions as follows (ref Figure3):

$$\tau_{\widetilde{TpIF}}(\varphi) = \begin{cases} \dfrac{(\varphi - t)}{(n - q)} \tau_{\widetilde{TpIF}}, \ \text{t} \leq \varphi \leq u; \\ \tau_{\widetilde{TpIF}}, \ \text{u} \leq \varphi \leq v; \\ \dfrac{(w - \varphi)}{(w - v)} \tau_{\widetilde{TpIF}}, \ \text{v} < \varphi \leq w; \\ 0, \ \text{Otherwise.} \end{cases}$$

$$, \gamma_{\widetilde{TpIF}}(\varphi) = \begin{cases} \dfrac{(u-\varphi)+\nu_{\widetilde{ITF}}(\varphi-t_1)}{(u-m_1)}\gamma_{\widetilde{TpIF}}, \ \mathrm{t} \leq \varphi \leq u; \\ \gamma_{\widetilde{TpIF}}, \ \mathrm{n} \leq \varphi \leq v; \\ \dfrac{(\varphi-v)+\nu_{\widetilde{TpIF}}(w_1-\varphi)}{(w_1-o)}\gamma_{\widetilde{TpIF}}, \ \mathrm{v} < \varphi \leq w; \\ 0, \ \text{Otherwise.} \end{cases}$$

Where $0 \leq \tau_{\widetilde{TpIF}}(\delta) \leq 1; 0 \leq \gamma_{\widetilde{TpIF}}(\delta) \leq 1;$ and $\tau_{\widetilde{TpIF}} + \gamma_{\widetilde{TpIF}} \leq 1; t, u, v, w \in \mathbb{R}.$

Definition 2.5. [18] α-cut: α-cut of $\underset{-}{\tilde{A}} = (a_1^a, a_2^n, a_3^k, a_4^i)$ is $A(\alpha) = [A_L(\alpha), A_R(\alpha)] Where, A_L(\alpha) = a_1^a + (a_2^n - a_1^a)$ & , $A_R(\alpha) = a_4^i - (a_4^i - a_3^k)\alpha$

Definition 2.6. [18] Arithmetical operation in fuzzy Environment : Let $\underset{-}{\tilde{A}} = \left(\tilde{a}_1^a, \tilde{a}_2^n, \tilde{a}_3^k, \tilde{a}_4^i\right)and\underset{-}{\tilde{B}} = \left(\tilde{b}_1^a, \tilde{b}_2^n, \tilde{b}_3^k, \tilde{b}_4^i\right)$ are two Trapezoidal Neutrosophic numbers, then,

$\underset{-}{\tilde{A}} \oplus \underset{-}{\tilde{B}} = \left(\tilde{a}_1^a + \tilde{b}_1^a, \tilde{a}_2^n + \tilde{b}_2^n, \tilde{a}_3^k + \tilde{b}_3^k, \tilde{a}_4^i + \tilde{b}_4^i\right).$

$\underset{-}{\tilde{A}} \otimes \underset{-}{\tilde{B}} = \left(\tilde{a}_1^a\tilde{b}_1^a, \tilde{a}_2^n\tilde{b}_2^n, \tilde{a}_3^k\tilde{b}_3^k, \tilde{a}_4^i\tilde{b}_4^i\right)$

$\alpha \otimes \underset{-}{\tilde{A}} = \left(\alpha\tilde{a}_1^a, \alpha\tilde{a}_2^n, \alpha\tilde{a}_3^k, \alpha\tilde{a}_4^i\right), \alpha \geq 0$

$\alpha \otimes \underset{-}{\tilde{A}} = \left(\alpha\tilde{a}_4^i, \alpha\tilde{a}_3^k, \alpha\tilde{a}_2^n, \alpha\tilde{a}_1^a\right), \alpha < 0$

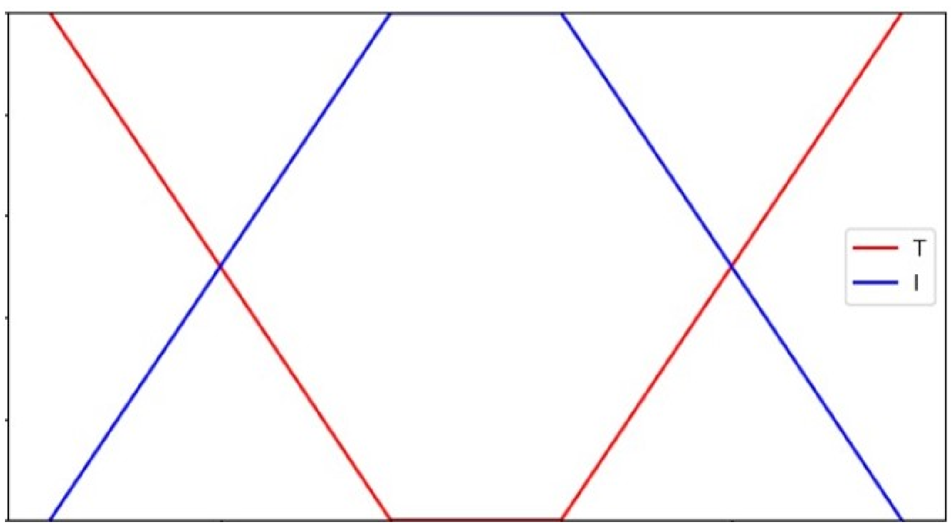

FIGURE 3. Trapezoidal Fuzzy Number

In Section 2, we presented several crucial definitions related to fuzzy theory. As we are aware, fuzzy inventory models involve various parameters, such as demand, holding cost, ordering cost, and others, which are incapable of managing uncertainties. Consequently, it becomes necessary to confront certain challenges. In the forthcoming Section 3, we intend to discuss these challenges in detail.

3. Challenges of the FIM

The FIM is a technique designed to handle the variability and uncertainty in supply and demand. Sometimes, things are not clear or certain. Fuzzy theory is a way to deal with that, but it is not enough. Fuzzy theory alone fails to solve uncertainty, imprecise and vagueness. So, some people came up with new ideas to improve fuzzy theory. These ideas are called extended fuzzy theories i.e., Intuitionistic theory by Atanassov in 1986 [15], Neutrosophic theory [19] by Smarandche in 1990, and Pythagorean theory by Yager in 2013 [20]. The next paragraph will explain more about these ideas.

4. Important Definitions, and Introduction Related to Neutrosophic Environment

This section discusses important definitions, preliminaries and applications related to Neutrosophic logic. It also highlights some applications of Neutrosophic Inventory Models (NIM).

Definition 4.1. [21] Trapezoidal Neutrosophic number (TpNNs): Let N be a TpNNs in the set of real numbers with the truth, falsity and indeterminacy membership functions are defined by

$$T_N(\omega) = \left\{ \begin{array}{l} \dfrac{(\omega-p)t_N}{q-p}, p \leq \omega < q \\ t_N \text{ ,q} \leq \omega \leq r \\ \dfrac{(s-\omega)t_N}{s-r}, r < \omega \leq s \\ 0 \text{ ,otherwise} \end{array} \right\}, \ I_N(\omega) = \left\{ \begin{array}{l} \dfrac{q-\omega+(\omega-p)i_N}{q-p}, q > \omega \geq p \\ i_N \text{ ,q} \leq \omega \leq r \\ \dfrac{\omega-c+(d-\omega)t_N}{s-r}, r < \omega \leq s \\ 0 \text{ ,otherwise} \end{array} \right\}$$

$$\text{, and } F_N(\omega) = \left\{ \begin{array}{l} \dfrac{(\omega-p)f_N+p-\omega}{q-p}, p \leq \omega < q \\ f_N \text{ ,q} \leq \omega \leq r \\ \dfrac{\omega-r+(s-\omega)f_N}{s-r}, r < \omega \leq s \\ 0 \text{ ,otherwise} \end{array} \right\}$$

Where $i_N = \left[i^L, i^U\right] \subset [0,1]$,$f_N = \left[f^L, f^U\right] \subset [0,1]$, and $t_N = \left[t^L, t^U\right] \subset [0,1]$ are interval numbers. Then the number N can be denoted by $\left([p,q,r,s]\,;[i^L,i^U],[f^L,f^U],[t^L,t^U]\right)$ and is called IVTNN.

Definition 4.2. [22] : Let $T_{\tilde{d}}, I_{\tilde{d}}, F_{\tilde{d}} \in [0,1]$, then a SVTpN number$\tilde{d} = \left\langle \left[\tilde{d}^a, \tilde{d}^s, \tilde{d}^h, \tilde{d}^o\right], (T_{\tilde{d}}, I_{\tilde{d}}, F_{\tilde{d}}) \right\rangle$ is a special Ns on the real number set R, whose truth-MF $\psi_{\tilde{d}}(x)$, falsity-MF $\zeta_{\tilde{d}}(x)$, and indeterminacy-MF $\xi_{\tilde{d}}(x)$ are given as follows:

$$\psi_{\tilde{d}}(x) = \begin{cases} \frac{T_{\tilde{d}}(x-\tilde{d}^a)}{(\tilde{d}^s-\tilde{d}^a)}, \tilde{d}^a \le x \le \tilde{d}^s \\ T_{\tilde{d}}, \tilde{d}^s \le x \le \tilde{d}^h \\ \frac{T_{\tilde{d}}(\tilde{d}^o-x)}{(\tilde{d}^o-\tilde{d}^h)}, \tilde{d}^h \le x \le \tilde{d}^o \\ 0, otherwise \end{cases}, \ \xi_{\tilde{d}}(x) = \begin{cases} \frac{(\tilde{d}^s-x+I_{\tilde{d}}(x-\tilde{d}^a))}{(\tilde{d}^s-\tilde{d}^a)}, \tilde{d}^a \le x \le \tilde{d}^s \\ I_{\tilde{d}}, \tilde{d}^s \le x \le \tilde{d}^h \\ \frac{(x-\tilde{d}^h+I_{\tilde{d}}(\tilde{d}^o-x))}{(\tilde{d}^o-\tilde{d}^h)}, \tilde{d}^h \le x \le \tilde{d}^o \\ 1, otherwise \end{cases} \text{,and}$$

$$\zeta_{\tilde{d}}(x) = \begin{cases} \frac{(\tilde{d}^s-x+F_{\tilde{d}}(x-\tilde{d}^a))}{(\tilde{d}^s-\tilde{d}^a)}, \tilde{d}^a \le x \le \tilde{d}^s \\ F_{\tilde{d}}, \tilde{d}^s \le x \le \tilde{d}^h \\ \frac{(x-\tilde{d}^h+F_{\tilde{d}}(\tilde{d}^o-x))}{(\tilde{d}^o-\tilde{d}^h)}, \tilde{d}^h \le x \le \tilde{d}^o \\ 1, otherwise \end{cases}$$

4.1. *The Application of the Neutrosophic Principle in Inventory management*

Neutrosophic principles exhibit a broad spectrum of applications spanning across diverse sectors and domains. Presented in Table 2 is a comprehensive overview of the notable progressions in Neutrosophic principle, highlighting their multifarious implementations in various fields.

Table 2. Comprehensive overview in inventory management under neutrosophic environment.

Authors	**Year**	**Environment**	**Application**	**Contribution**
Mullai and Surya [23]	2018	TpNN	Economic Order Quantity (EOQ)	To presents the development of an IM with a price break, utilizing an EOQ approach.
Sarma et al. [24]	2019	TpNN	Disaster Management	Cost minimization in disaster management under uncertainty using TpNN necessitates redistribution.

Continued on next page

Table 2 – Continued from previous page

Authors	Year	Environment	Application	Contribution
Martin et al. [25]	2020	TpNN	Production Management	To presents a revised PIM that explores the transition towards a smart production process.
Bhavani et al. [26]	2022	TpNN	PSAOIM	To introduces a restructured inventory system with a neutrosophic cost pattern, incorporating novel demand considerations such as deterioration and discounts on defective items. The proposed system employs a PSOOIM.
Sugapriya et al. [27]	2022	TpNN	Power Demand Patterns	A two-warehouse system is proposed for managing TpNN disparate and expeditious deteriorate items with power demand.

This article features Figure 3, which illustrates the vital components and processes of Neutrosophic Principle in a visually engaging manner. This graphical representation facilitates readers' understanding of the concepts discussed and offers a practical view of how Neutrosophic Principle operates. Furthermore, Figure 3 seamlessly integrates Table 2 for easy reference.

The main objective of this table 2 is to show that fuzzy isn't the only way to handle uncertainty. There are other methods, like Neutrosophic principles, that can give better and more precise solution. These methods are becoming more popular in many areas and applications. However, due to limitations, it is impractical to extensively discuss all these mentioned applications within this paper. So, we'll focus on how the Neutrosophic theory affects inventory management.

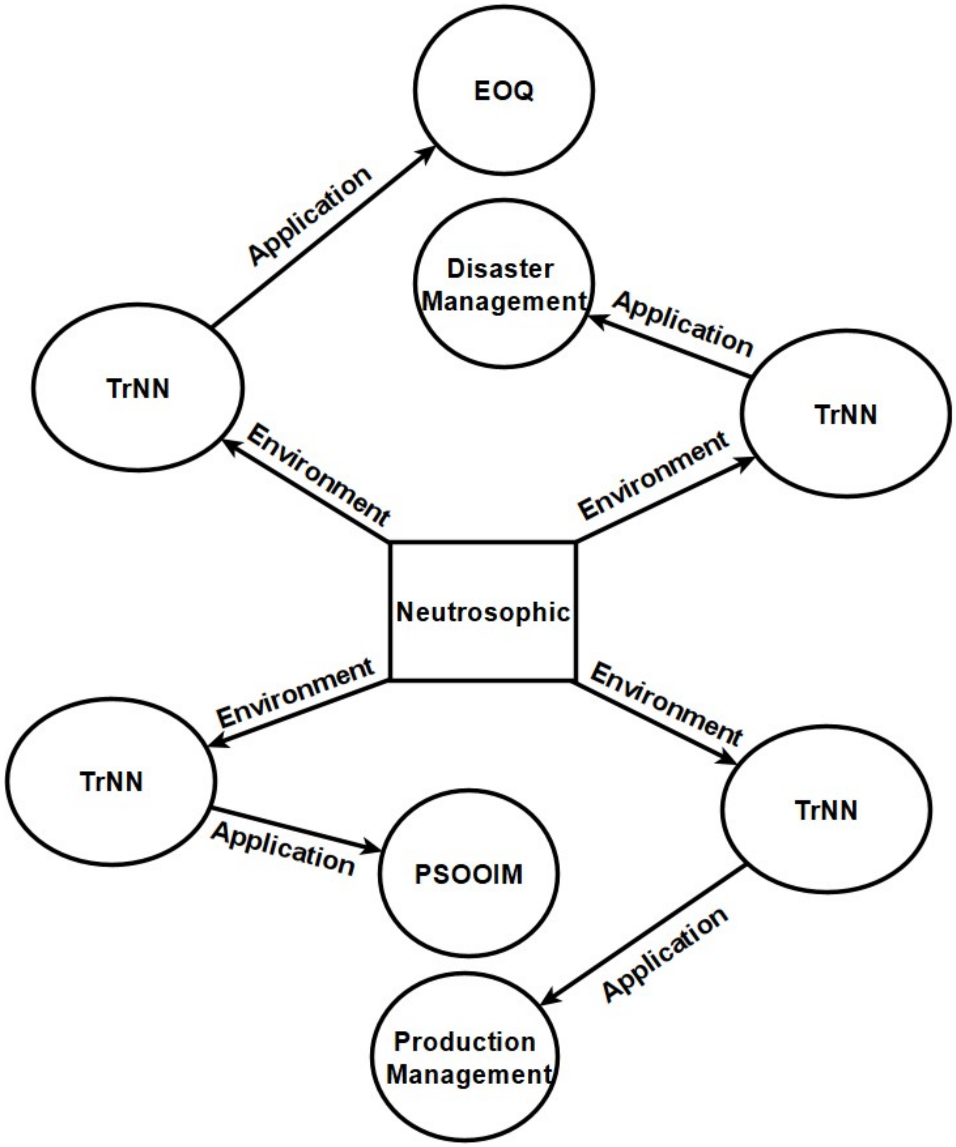

Figure 4. This illustrates the various applications and environments associated with Neutrosophic Principle.

5. The Inventory Management under the Neutrosophic Principle

Supply Chain is a complex decision-making problem with conflicting objectives in various supply chain operations and their corresponding sub-criteria. Haq et al. (2021) [28] aims to develop a model that incorporates key components of real-world supply chain planning. Haq et al. (2021) [28] propose a supply chain model that involves multiple suppliers, plants, warehouses, and distributors. This approach addresses the challenges of a complex multi-site composite supply chain problem under uncertainty by utilizing a fuzzy multi-objective model. Haq et al. (2021) primary objective is to optimize transportation cost and delivery time concurrently. To handle the ambiguity inherent in the supply chain, Haq et al. (2021) [28] employ neutrosophical set theory, using falsity, indeterminacy, and truth membership functions. Additionally, a neutrosophical compromise programming approach is employed to obtain the desired solution. To showcase the effectiveness of authors models, Haq et al. (2021) [28] present an industrial design problem. The reported findings are compared against other well-known approaches.

Suresh et al. (2021) [29] explores the application of the Euclidean Distance measure in frame centroid-based ranking for NTFN and NTpFN. The effectiveness of the model is demonstrated through an illustrative example of a multi-criteria decision-making (MCDM) problem in the Neutrosophic fuzzy environment. The proposed ranking approach offers a solution to various decision-making and optimization problems characterized by uncertainty.

Mondal et al. (2021) [30] looked at a system for managing inventory of seasonal products. These products have changing demand rates and partial backordering in a viable market. The Weibull distribution shows the seasonality and versatility of these products. Weibull distribution deterioration rates, fully permissible payment delays, and partial backordering are considered in this paper. The proposed EOQ model is optimized using the neutrosophic set which quatifies imprecise information in real-life senarios. The study suggests reducing expenses on early promotions to lessen demand fluctuations at the start of the cycle. It also shows that the best time to deplete inventory depends on the demand during shortages within a neutrosophic environment.

Lakshmi et al. (2022) [31] The purpose of this manuscript is to present a TpN approach for dealing with the logarithmic demand model involving shortage of deteriorating items. The vendor determines the order placement for customers based on stock availability. The logarithmic demand model is applied to multiple products and takes into account the shortage of items initially. Additionally, a practical example is provided to demonstrate the extraction of optimal values and the attainment of valuable and effective results.

Conclusion

This manuscript provides a comprehensive assessment and classification of uncertain situations used in IM systems across diverse areas. By recognizing the limitations of traditional inventory models, the study presents the concepts of the neutrosophic and fuzzy principle as alternative approaches in inventory management. This manuscript focused on how the Neutrosophic principle can help manage inventory, especially when dealing with uncertain demand and imprecise data. Our research highlights the potential of the neutrosophic theory to improve decision-making in inventory management by overcoming the limitations of fuzzy models. This study contributes valuable insights and paves the way for future research in this field.

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On Rough Statistical Convergence in Neutrosophic Normed Spaces

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Abstract. In this paper, we have presented rough statistical convergence of sequence on neutrosophic normed spaces as an important convergence criterion. As neutrosophication can handle partially dependent components, partially independent components and even independent components involved in real-world problems. By examining some properties related to rough convergence in these spaces we provide some useful functional tools in the situations of inconsistency and indeterminacy in the real world. Further, we have established some equivalent conditions on the set of statistical limit points as well as on the set of cluster points in these spaces for rough statistically convergent sequences.

Keywords: Neutrosophic normed space; statistical convergence; rough convergence, rough statistical convergence

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1. **Introduction**

It has been seen that new quests are revealing in the real-life problems with time. So many methods are already existing, and researchers are still investigating new variants for existing and future problems. In modern logic, the three-way decision situations like from accepting/rejecting/pending, from yes/no/not-applicable, from sports win/lose/tie etc. the standard analysis is not sufficient, which leads to employ non-standard analysis. Smarandache [36–38] provided neutrosophic sets with regard to the powerful advancement of the intuitionistic fuzzy sets, that established the concept for the classic sets, fuzzy sets, vague sets etc., for non-standard analysis. The conception of neutrosophic set can manage the indeterminate data of the problem whereas the impression of fuzzy set theory as well as intuitionistic fuzzy set theory are not able to provide solution when the relation is indeterminate. Neutrosophic sets are providing the ideas to solve real-life situations where indeterminacy occurs like Databases [4], Image processing problems [15], Control theory [1], Medical diagnosis problems [2], Decision making problems [19], and so on.

As every element of the neutrosophic set has a truth value, a false value and an indeterminacy value respectively, which lies in the non-standard unit interval. Due to this nature neutrosophic is more adjustable and efficient tool because of its ability to handle, not only the free components of information but also partially independent and dependent information. In neutrosophic set elements may have inconsistent information (*i.e.* sum of the components > 1) or incomplete information (*i.e.* sum of the components < 1) or consistent information (*i.e.* sum of the components $= 1$), and other interval-valued components (*i.e.* without any restriction on the sum of superior or inferior components).

Definition 1.1. [36] Let U be a subset of $\mathbb{X}$ (which is space of points) with $a \in \mathbb{X}$. Then set U with $\tau(a)$, $\upsilon(a)$ and $\eta(a)$ in $\mathbb{X}$ is called neutrosophic set(NS) and expressed as

$$U = \{< a, \tau(a), \upsilon(a), \eta(a) > : \ a \in \mathbb{X}, \tau(a), \upsilon(a), \eta(a) \in I\}$$

where $\tau(a)$, $\upsilon(a)$ and $\eta(a)$ denotes truth membership, indeterminacy membership and falsity membership functions respectively such that $0^- \leq \tau(a) + \upsilon(a) + \eta(a) \leq 3^+$. Also $I =]0^-, 1^+[$ represents some non-standard unit interval.

However, Wang et al. [41] and Ye [42] have customized the existing definition for neutrosophic sets by suggesting the structure of single-valued neutrosophic sets and simplified neutrosophic sets respectively using the interval $[0, 1]$, that can be utilized in engineering and scientific applications.

Later, Mahapatra and Bera [9] studied the notion of neutrosophic soft linear spaces. Kirişci and Şimşek [20] has introduced neutrosophic metric spaces and established its basic characteristic properties like open ball, compactness, completeness and nowhere dense. Further, Kirişci and Şimşek [21] presented the idea of neutrosophic normed spaces as a notable consideration of neutrosophic metric spaces. Further, required basic terms related to neutrosophic normed spaces are elaborated as below:

Definition 1.2. [35] A continuous t-norm is the mapping $\circledast : [0, 1] \times [0, 1] \to [0, 1]$ such that

(i) $\circledast$ is continuous, associative, commutative and with identity 1,

(ii) $a_1 \circledast b_1 \leq a_2 \circledast b_2$ whenever $a_1 \leq a_2$ and $b_1 \leq b_2$, for $a_1, a_2, b_1, b_2 \in [0, 1]$.

Definition 1.3. [35] A continuous t-conorm is the mapping $\odot : [0, 1] \times [0, 1] \to [0, 1]$ such that

(i) $\odot$ is continuous, associative, commutative and with identity 0,

(ii) $a_1 \odot b_1 \leq a_2 \odot b_2$ whenever $a_1 \leq a_2$ and $b_1 \leq b_2$, for $a_1, a_2, b_1, b_2 \in [0, 1]$.

Definition 1.4. [21] A neutrosophic normed space(NNS) is 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ with vector space $\mathbb{X}$, continuous t-norm $\circledast$, continuous t-conorm $\odot$ and normed space $\aleph = \{< \tau(a), \upsilon(a), \eta(a) > : a \in \mathbb{X}\}$ such that $\aleph : \mathbb{X} \times \mathbb{R}^+ \to [0,1]$, if for each $x, y \in \mathbb{X}$ and $s, t > 0$, we have

(i) $0 \leq \tau(x,t), \upsilon(x,t), \eta(x,t) \leq 1$,
(ii) $\tau(x,t) + \upsilon(x,t) + \eta(x,t) \leq 3$,
(iii) $\tau(x,t) = 1$, $\upsilon(x,t) = 0$ and $\eta(x,t) = 0$ for $t > 0$ iff $x = 0$,
(iv) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ for $t \leq 0$,
(v) $\tau(\alpha x, t) = \tau\left(x, \frac{t}{|\alpha|}\right)$, $\upsilon(\alpha x, t) = \upsilon\left(x, \frac{t}{|\alpha|}\right)$ and $\eta(\alpha x, t) = \eta\left(x, \frac{t}{|\alpha|}\right)$ for $\alpha \neq 0$,
(vi) $\tau(x, \circ)$ is continuous non-decreasing function,
(vii) $\tau(x,s) \circledast \tau(x,t) \leq \tau(x+y, s+t)$,
(viii) $\upsilon(x, \circ)$ is continuous non-increasing function,
(ix) $\upsilon(x,s) \odot \upsilon(y,t) \geq \upsilon(x+y, s+t)$,
(x) $\eta(x, \circ)$ is continuous non-increasing function,
(xi) $\eta(x,s) \odot \eta(y,t) \geq \eta(x+y, s+t)$,
(xii) $\lim_{t\to\infty} \tau(x,t) = 1$, $\lim_{t\to\infty} \upsilon(x,t) = 0$ and $\lim_{t\to\infty} \eta(x,t) = 0$.

Then (τ, υ, η) is known as neutrosophic norm.

Example 1.5. [21] Let $(\mathbb{X}, \|\,.\,\|)$ be any normed space. For every $t > 0$ and all $x \in \mathbb{X}$, take
(i) $\tau(x,t) = \frac{t}{t+\|x\|}$, $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ and $\boldsymbol{\eta}(x,t) = \frac{\|x\|}{t}$ when $t > \|x\|$,
(ii) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\boldsymbol{\eta}(x,t) = 1$ when $t \leq \|x\|$.
Also, $a \circledast b = ab$ and $a \odot b = a + b - ab \;\forall\; a, b \in [0,1]$.
Then, 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS which satisfies above mentioned conditions.

A generalized version of intuitionistic fuzzy norms is considered in neutrosophic normed spaces that help to explore the basic properties like convergence and completeness in such spaces using continuous and bounded linear operators. Omran and Elrawy [30] have discussed the continuous operators along with bounded operators in the setting of neutrosophic normed spaces. Also, Khan and Khan [16] studied the various topological characterization of such spaces. Further, Kirişci and Şimşek [21] have established sequence convergence on neutrosophic normed spaces as given below.

Definition 1.6. [21] Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $x = \{x_k\}$ in $\mathbb{X}$ is called convergent to $\xi \in X$ with respect to neutrosophic norm (τ, υ, η) if for every $\epsilon > 0$ and $t > 0$ we can find $k_0 \in \mathbb{N}$ provided $\tau(x_k - \xi, t) > 1 - \epsilon$, $\upsilon(x_k - \xi, t) < \epsilon$ and $\eta(x_k - \xi, t) < \epsilon$ for $k \geq k_0$. It is convenient to represent symbolically by (τ, υ, η)-$\lim_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{(\tau,\upsilon,\eta)} \xi$.

Remark 1.7. Let $(\mathbb{X}, \|\,.\,\|)$ be any normed space. For every $t > 0$ and all $x \in \mathbb{X}$, take
(i) $\tau(x,t) = \frac{t}{t+\|x\|}$, $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ and $\eta(x,t) = \frac{\|x\|}{t}$ when $t > \|x\|$,
(ii)$\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ when $t \leq \|x\|$.
Also, $a \circledast b = ab$ and $a \odot b = a + b - ab \ \forall \ a, b \in [0,1]$.
Then, 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS.
Also, $x_k \xrightarrow{(\tau,\upsilon,\eta)} x$ if and only if $x_k \xrightarrow{\|\cdot\|} x$.

Also Kirişci and Şimşek [21] have established the statistical convergence for sequences on neutrosophic normed spaces with natural density. Although, natural density of set A, where $A \subseteq \mathbb{N}$, has given by $\delta(A) = \lim\limits_{n\to\infty} \dfrac{1}{n} \mid \{a \leq n : a \in A\} \mid$, provided limit exists, where $\mid . \mid$ designates the order of the enclosed set. Further, sequence $x = \{x_k\}$ converges statistically to ξ, if $A(\epsilon) = \{k \in \mathbb{N} : |x_k - \xi| > \epsilon\}$ has zero natural density (see [14]).

Definition 1.8. [21] Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $x = \{x_k\}$ in $\mathbb{X}$ is called statistically convergent to $\xi \in \mathbb{X}$ with respect to neutrosophic norm (τ, υ, η) if for every $\epsilon > 0$ and $t > 0$, we have

$$\delta(\{k \in \mathbb{N} : \tau(x_k - \xi, t) \leq 1 - \epsilon \text{ or } \upsilon(x_k - \xi, t) \geq \epsilon,\ \eta(x_k - \xi, t) \geq \epsilon\}) = 0.$$

It is convenient to represent symbolically by $St_{(\tau,\upsilon,\eta)} - \lim\limits_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{St_{(\tau,\upsilon,\eta)}} \xi$.

Some remarkable results on the theory of neutrosophic normed spaces and statistical convergence on the framework of neutrosophic normed spaces have been studied in different aspects ($c.f.$ [8, 10, 16–18, 22, 23, 23, 25, 34, 39]). This convergence concept can also be further studied in different directions as this theory has a basic function in so many areas of mathematics, economics, science and technology. In this paper we associate this theory with the rough convergence of sequences.

The concept of rough convergence deals with the approximate solution of any real-life situation from the numerical point of view. It helps to verify the correctness of solutions obtained from the computer programs and to draw conclusions from scientific experiments. The rough convergence has been initially introduced by Phu [32] as an interesting generalization of usual convergence for the sequences on finite-dimensional normed linear spaces, and later suggested on infinite-dimensional normed linear spaces [33]. Apart from defining the idea of rough convergence, he also contributed towards the properties like closeness and convexity of the rough limit set.

Definition 1.9. [32] A sequence $x = \{x_k\}$ in a normed linear space $(\mathbb{X}, \|\,.\,\|)$ is called rough convergent to $\xi \in \mathbb{X}$ for some non-negative number r if for every $\epsilon > 0$ we can find $k_0 \in \mathbb{N}$ provided $\|x_k - \xi\| < r + \epsilon$ for $k \geq k_0$.

Aytar [6] developed the extensive notion of rough convergence by applying the statistical analogue about this concept as rough statistical convergence for the sequences, like usual convergence is continued to statistical convergence for sequences using natural density by Fast [13]. Moreover, Aytar [7] also examined some criteria related to convexity and closeness associated with the set of rough statistical limit points. In fact, he established some properties related to this set with the set of rough cluster points.

Definition 1.10. [6] A sequence $x = \{x_k\}$ in any normed linear space $(\mathbb{X}, \|\,.\,\|)$ is called rough statistically convergent to $\xi \in \mathbb{X}$ for some non-negative number r if for every $\epsilon > 0$ we get

$$\delta(\{k \in \mathbb{N} : \|x_k - \xi\| \geq r + \epsilon\}) = 0,$$

and ξ is identified as r-St-limit of sequence $x = \{x_k\}$.

In the literature, during last few years, considerable progress is going on the field of rough convergence ($c.f.$ [3, 5, 11, 12, 24, 26–29, 31]) in different aspects which leads us to investigate and explore rough statistical convergence on the theory of neutrosophic normed spaces. The significance of introducing rough convergence in this structure is to obtain an efficient tool of convergence for acting on various types of uncertainties and imprecision unified in real-life systems.

2. Main Results

We first mention the conception of rough statistical convergence of sequences on neutrosophic normed spaces that will be helpful in studying the major results of our work.

Definition 2.1. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough convergent to $\xi \in \mathbb{X}$ with respect to neutrosophic norm (τ, υ, η) for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0, 1)$ we can find $k_0 \in \mathbb{N}$ provided

$$\tau(x_k - \xi; r + \epsilon) > 1 - \lambda, \; \upsilon(x_k - \xi, t) < \lambda \text{ and } \eta(x_k - \xi, t) < \lambda \text{ for } k \geq k_0.$$

It is convenient to represent symbolically by $r_{(\tau,\upsilon,\eta)} - \lim\limits_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{r_{(\tau,\upsilon,\eta)}} \xi$.

Definition 2.2. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be rough statistically convergent to $\xi \in \mathbb{X}$ with respect to neutrosophic norm (τ, υ, η) for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0, 1)$,

$$\delta(\{k \in \mathbb{N} : \tau(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \upsilon(x_k - \xi, r + \epsilon) \geq \lambda, \; \eta(x_k - \xi, r + \epsilon) \geq \lambda\}) = 0.$$

It is convenient to represent symbolically by r-$St_{(\tau,\upsilon,\eta)}$-$\lim\limits_{k\to\infty} x_k = \xi$ or $x_k \xrightarrow{r\text{-}St_{(\tau,\upsilon,\eta)}} \xi$.

Remark 2.3. For $r = 0$, the impression of rough statistical convergence due to neutrosophic norm (τ, υ, η) agrees with the impression of statistical convergence due to neutrosophic norm (τ, υ, η) in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$.

The r-$St_{(\tau,\upsilon,\eta)}$-limit of a sequence may be not unique. Therefore, we take into consideration r-$St_{(\tau,\upsilon,\eta)}$-limit set of sequence $x = \{x_k\}$ as $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r = \{\xi : x_k \xrightarrow{r\text{-}St_{(\tau,\upsilon,\eta)}} \xi\}$. Moreover, sequence $x = \{x_k\}$ is $r_{(\tau,\upsilon,\eta)}$-convergent if $LIM_x^{r_{(\tau,\upsilon,\eta)}} \neq \phi$ where $LIM_x^{r_{(\tau,\upsilon,\eta)}} = \{\xi^* \in \mathbb{X} : x_k \xrightarrow{r_{(\tau,\upsilon,\eta)}} \xi^*\}$. For unbounded sequence $LIM_x^{r_{(\tau,\upsilon,\eta)}}$ is always empty.
But in rough statistical convergence on a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$, we may have $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r \neq \phi$ for unbounded sequence. To justify this the next example is given.

Example 2.4. Consider any real normed space $(\mathbb{X}, \|\,.\,\|)$. For every $t > 0$ and all $x \in \mathbb{X}$, take
(i) $\tau(x,t) = \frac{t}{t+\|x\|}$, $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ and $\eta(x,t) = \frac{\|x\|}{t}$ when $t > \|x\|$,
(ii)$\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ when $t \leq \|x\|$.
Also, $a \circledast b = ab$ and $a \odot b = a + b - ab$ $\forall$ $a, b \in [0,1]$. Then, 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS. Define a sequence

$$x_k = \begin{cases} (-1)^k & k \neq n^2 \\ k & \text{otherwise} \end{cases}$$

Then

$$St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r = \begin{cases} \phi & r < 1 \\ [1-r, r-1] & \text{otherwise} \end{cases}$$

and $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r = \phi$ for all $r \geq 0$. Thus, this sequence is divergent in ordinary sense being unbounded. Also, this sequence is not rough convergent in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ for any r.

If $x' = \{x_{k_i}\}$ be a sub-sequence of $x = \{x_k\}$ in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ then $LIM_{x_k}^{r_{(\tau,\upsilon,\eta)}} \subset LIM_{x_{k_i}}^{r_{(\tau,\upsilon,\eta)}}$. But this fact fails to hold in case of rough statistical convergence. This can be justified with the next given example.

Example 2.5. Consider any real normed space $(\mathbb{X}, \|.\|)$. For every $t > 0$ and all $x \in \mathbb{X}$, take
(i) $\tau(x,t) = \frac{t}{t+\|x\|}$, $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ and $\eta(x,t) = \frac{\|x\|}{t}$ when $t > \|x\|$,
(ii)$\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ when $t \leq \|x\|$.
Also, $a \circledast b = ab$ and $a \odot b = a + b - ab$ $\forall$ $a, b \in [0,1]$. Then, 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS. Also the sequence

$$x_k = \begin{cases} k & k \neq n^2 \\ 0 & \text{otherwise} \end{cases}$$

have $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r = [-r, r]$. And its sub-sequence $x' = \{1, 4, 9, ...\}$ have $St_{(\tau,\upsilon,\eta)}$-$LIM_{x'}^r = \phi$.

But this fact is true for non-thin sub-sequences of the rough statistical convergent sequence in a NNS which is explained by the next result.

Theorem 2.6. *If $x' = \{x_{k_i}\}$ be a non-thin sub-sequence of the sequence $x = \{x_k\}$ in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ then $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r \subset St_{(\tau,\upsilon,\eta)}$-$LIM_{x'}^r$.*

Proof. Proof of this result is trivial so we are omitting it. □

Theorem 2.7. *The set $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ of any sequence $x = \{x_k\}$ in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ is a closed set.*

Proof. We have nothing to prove as $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r = \phi$.
Let $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r \neq \phi$ for some $r > 0$ and consider $y = \{y_k\}$ be a convergent sequence in $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ with respect to neutrosophic norm (τ, υ, η) to $y_0 \in \mathbb{X}$.
For $t \in (0,1)$ take $\lambda \in (0,1)$ with $(1-\lambda) \circledast (1-\lambda) > 1-t$ and $\lambda \odot \lambda < t$. Then for $\epsilon > 0$ and $\lambda \in (0,1)$ we get $k_1 \in \mathbb{N}$ such that

$$\tau\left(y_k - y_0; \frac{\epsilon}{2}\right) > 1-\lambda, \upsilon\left(y_k - y_0; \frac{\epsilon}{2}\right) < \lambda \text{ and } \eta\left(y_k - y_0; \frac{\epsilon}{2}\right) < \lambda \text{ for all } \ k \geq k_1.$$

Let us choose $y_m \in St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ with $m > k_1$ such that

$$\delta\left(\{k \in \mathbb{N} : \tau\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \leq 1-\lambda \text{ or } \upsilon\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \geq \lambda,\ \eta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) \geq \lambda\}\right) = 0. \tag{1}$$

For $j \in \{k \in \mathbb{N} : \tau\left(x_k - y_m; r + \frac{\epsilon}{2}\right) > 1-\lambda, \upsilon\left(x_k - y_m; r + \frac{\epsilon}{2}\right) < \lambda$ and $\eta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) < \lambda\}$ we have $\tau\left(x_j - y_m; r + \frac{\epsilon}{2}\right) > 1-\lambda,\ \upsilon\left(x_j - y_m; r + \frac{\epsilon}{2}\right) < \lambda$ and $\eta\left(x_j - y_m; r + \frac{\epsilon}{2}\right) < \lambda$. Then, we have

$$\begin{aligned}
\tau(x_j - y_0; r + \epsilon) &\geq \tau\left(x_j - y_m; r + \frac{\epsilon}{2}\right) \circledast \tau\left(y_m - y_0; \frac{\epsilon}{2}\right) \\
&> (1-\lambda) \circledast (1-\lambda) \\
&> 1-t, \\
\upsilon(x_j - y_0; r + \epsilon) &\leq \upsilon\left(x_j - y_m; r + \frac{\epsilon}{2}\right) \odot \upsilon\left(y_m - y_0; \frac{\epsilon}{2}\right) \\
&< \lambda \odot \lambda \\
&< t,
\end{aligned}$$

and

$$\begin{aligned}
\eta(x_j - y_0; r + \epsilon) &\leq \eta\left(x_j - y_m; r + \frac{\epsilon}{2}\right) \odot \eta\left(y_m - y_0; \frac{\epsilon}{2}\right) \\
&< \lambda \odot \lambda \\
&< t.
\end{aligned}$$

Hence, $j \in \{k \in \mathbb{N} : \tau(x_k - y_0; r + \epsilon) > 1-t,\ \upsilon\left(x_k - y_0; r + \epsilon\right) < t$ and $\eta\left(x_k - y_0; r + \epsilon\right) < t\}$.
Now we have the following inclusion

$$\begin{aligned}
&\{k \in \mathbb{N} : \tau\left(x_k - y_m; r + \frac{\epsilon}{2}\right) > 1-\lambda,\ \upsilon\left(x_k - y_m; r + \frac{\epsilon}{2}\right) < \lambda \text{ and } \eta\left(x_k - y_m; r + \frac{\epsilon}{2}\right) < \lambda\} \\
&\quad \subseteq \{k \in \mathbb{N} : \tau(x_k - y_0; r + \epsilon) > 1-t,\ \upsilon\left(x_k - y_0; r + \epsilon\right) < t \text{ and } \eta\left(x_k - y_0; r + \epsilon\right) < t\}
\end{aligned}$$

Therefore,

$$\delta(\{k \in \mathbb{N} : \tau(x_k - y_0; r+\epsilon) \leq 1-t \text{ or } \upsilon\,(x_k - y_0; r+\epsilon) \geq t,\ \eta\,(x_k - y_0; r+\epsilon) \geq t\})$$
$$\leq \delta\left(\left\{k \in \mathbb{N} : \tau\left(x_k - y_m; r+\frac{\epsilon}{2}\right) \leq 1-\lambda \text{ or } \upsilon\left(x_k - y_m; r+\frac{\epsilon}{2}\right) \geq \lambda,\ \eta\left(x_k - y_m; r+\frac{\epsilon}{2}\right) \geq \lambda\right\}\right)$$

Using (1) we get

$$\delta(\{k \in \mathbb{N} : \tau(x_k - y_0; r+\epsilon) \leq 1-t \text{ or } \upsilon\,(x_k - y_0; r+\epsilon) \geq t,\ \eta\,(x_k - y_0; r+\epsilon) \geq t\}) = 0$$

Therefore, $y_0 \in St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$. □

In next result, we are proving the convexity for set $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$.

Theorem 2.8. *Let $x = \{x_k\}$ be any sequence in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then, rough statistical limit set $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ with respect to neutrosophic norm (τ,υ,η) is convex for some non-negative number r.*

Proof. Let $\xi_1, \xi_2 \in St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$. For the convexity of the set $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$, we have to show that $[(1-\beta)\xi_1 + \beta\xi_2] \in St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ for some $\beta \in (0,1)$.
For $t \in (0,1)$ take $\lambda \in (0,1)$ with $(1-\lambda) \circledast (1-\lambda) > 1-t$ and $\lambda \odot \lambda < t$. Now for every $\epsilon > 0$ and $\lambda \in (0,1)$, we define

$$M_1 = \{k \in \mathbb{N} : \tau\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \leq 1-\lambda \text{ or } \upsilon\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \geq \lambda,\ \eta\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \geq \lambda\},$$

$$M_2 = \{k \in \mathbb{N} : \tau\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \leq 1-\lambda \text{ or } \upsilon\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \geq \lambda,\ \eta\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \geq \lambda\}.$$

As $\xi_1, \xi_2 \in St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$, we have $\delta(M_1) = \delta(M_2) = 0$. For $k \in M_1^c \cap M_2^c$ we have

$$\begin{aligned}
\tau(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) &\geq \tau((1-\beta)(x_k - \xi_1) + \beta(x_k - \xi_2); r+\epsilon) \\
&\geq \tau\left((1-\beta)(x_k - \xi_1); \frac{r+\epsilon}{2}\right) \circledast \tau\left(\beta(x_k - \xi_2); \frac{r+\epsilon}{2}\right) \\
&\geq \tau\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \circledast \tau\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \\
&> (1-\lambda) \circledast (1-\lambda) \\
&> 1-t,
\end{aligned}$$

$$\begin{aligned}
\upsilon(x_k - [(1-\beta)\xi_1 + \beta\xi_2]; r+\epsilon) &\leq \upsilon((1-\beta)(x_k - \xi_1) + \beta(x_k - \xi_2); r+\epsilon) \\
&\leq \upsilon\left((1-\beta)(x_k - \xi_1); \frac{r+\epsilon}{2}\right) \odot \upsilon\left(\beta(x_k - \xi_2); \frac{r+\epsilon}{2}\right) \\
&\leq \upsilon\left(x_k - \xi_1; \frac{r+\epsilon}{2(1-\beta)}\right) \odot \upsilon\left(x_k - \xi_2; \frac{r+\epsilon}{2\beta}\right) \\
&< \lambda \odot \lambda \\
&< t.
\end{aligned}$$

and

$$\begin{aligned}\eta(x_k-[(1-\beta)\xi_1+\beta\xi_2];r+\epsilon) &\leq \eta((1-\beta)(x_k-\xi_1)+\beta(x_k-\xi_2);r+\epsilon)\\ &\leq \eta\left((1-\beta)(x_k-\xi_1);\frac{r+\epsilon}{2}\right)\odot\eta\left(\beta(x_k-\xi_2);\frac{r+\epsilon}{2}\right)\\ &\leq \eta\left(x_k-\xi_1;\frac{r+\epsilon}{2(1-\beta)}\right)\odot\eta\left(x_k-\xi_2;\frac{r+\epsilon}{2\beta}\right)\\ &< \lambda\odot\lambda\\ &< t.\end{aligned}$$

Thus, we get $\delta(\{k\in\mathbb{N}:\tau(x_k-[(1-\beta)\xi_1+\beta\xi_2];r+\epsilon)\leq 1-t$ or $\upsilon(x_k-[(1-\beta)\xi_1+\beta\xi_2];r+\epsilon)\geq 1-t,\ \eta(x_k-[(1-\beta)\xi_1+\beta\xi_2];r+\epsilon)\geq 1-t\})=0$.

Hence, $[(1-\beta)\xi_1+\beta\xi_2]\in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ *i.e.* $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ is a convex set. □

Theorem 2.9. *A sequence $x=\{x_k\}$ in a NNS $(\mathbb{X},\aleph,\circledast,\odot)$ is rough statistically convergent to $\xi\in\mathbb{X}$ with respect to neutrosophic norm (τ,υ,η) for some non-negative number r if a sequence $y=\{y_k\}$ exists in $\mathbb{X}$, which is statistically convergent to $\xi\in\mathbb{X}$ with respect to neutrosophic norm (τ,υ,η) and for every $\lambda\in(0,1)$ have $\tau(x_k-y_k;r)>1-\lambda$, $\upsilon(x_k-y_k;r)<\lambda$ and $\eta(x_k-y_k;r)<\lambda$ for all $k\in\mathbb{N}$.*

Proof. Let $\epsilon>0$ and $\lambda\in(0,1)$. Consider $y_k\xrightarrow{St_{(\tau,\upsilon,\eta)}}\xi$ and $\tau(x_k-y_k;r)<\lambda$, $\upsilon(x_k-y_k;r)>1-\lambda$ and $\eta(x_k-y_k;r)>1-\lambda$ for all $k\in\mathbb{N}$. For given $\lambda\in(0,1)$ take $t\in(0,1)$ with $(1-t)\circledast(1-t)>1-\lambda$ and $\lambda\odot\lambda<t$. Define

$$A=\{k\in\mathbb{N}:\tau(y_k-\xi;\epsilon)\leq 1-t \text{ or } \upsilon(y_k-\xi;\epsilon)\geq t,\ \eta(y_k-\xi;\epsilon)\geq t\}$$

$$B=\{k\in\mathbb{N}:\tau(x_k-y_k;r)\leq 1-t \text{ or } \upsilon(x_k-y_k;r)\geq t,\ \eta(x_k-y_k;r)\geq t\}$$

Clearly, $\delta(A)=0$ and $\delta(B)=0$. For $k\in A^c\cap B^c$, we have

$$\begin{aligned}\tau(x_k-\xi;r+\epsilon) &\geq \tau(x_k-y_k;r)\circledast\tau(y_k-\xi;\epsilon)\\ &> (1-t)\circledast(1-t)\\ &> 1-\lambda,\end{aligned}$$

$$\begin{aligned}\upsilon(x_k-\xi;r+\epsilon) &\leq \upsilon(x_k-y_k;r)\odot\upsilon(y_k-\xi;\epsilon)\\ &< t\odot t\\ &< \lambda,\end{aligned}$$

and

$$\begin{aligned}\eta(x_k-\xi;r+\epsilon) &\leq \eta(x_k-y_k;r)\odot\eta(y_k-\xi;\epsilon)\\ &< t\odot t\\ &< \lambda.\end{aligned}$$

Then $\tau(x_k-\xi;r+\epsilon)>1-\lambda$, $\upsilon(x_k-\xi;r+\epsilon)<\lambda$ and $\eta(x_k-\xi;r+\epsilon)<\lambda$ for all $k\in A^c\cap B^c$. This implies that $\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)\leq 1-\lambda$ or $\upsilon(x_k-\xi;r+\epsilon)\geq\lambda,\ \eta(x_k-\xi;r+\epsilon)\geq\lambda\}\subseteq A\cup B$.

Then, $\delta(\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)\leq 1-\lambda$ or $\upsilon(x_k-\xi;r+\epsilon)\geq\lambda,\ \eta(x_k-\xi;r+\epsilon)\geq\lambda\})\leq\delta(A)+\delta(B)$.

Hence, we get $\delta(\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)\leq 1-\lambda$ or $\upsilon(x_k-\xi;r+\epsilon)\geq\lambda,\ \eta(x_k-\xi;r+\epsilon)\geq\lambda\})=0$.

Therefore, $x_k\xrightarrow{r\text{-}St_{(\tau,\upsilon,\eta)}}\xi$. $\square$

Theorem 2.10. *Let $x=\{x_k\}$ be any sequence in a NNS $(\mathbb{X},\aleph,\circledast,\odot)$. There does not exist elements $y,z\in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ for some $r>0$ and every $\lambda\in(0,1)$ with $\boldsymbol{\tau}(y-z;mr)\leq 1-\lambda$ or $\upsilon(y-z;mr)\geq\lambda$, $\boldsymbol{\eta}(y-z;mr)\geq\lambda$*

Proof. We prove this result by contradiction. Assume there are elements $y,z\in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ such that

$$\tau(y-z;mr)\leq 1-\lambda \text{ or } \upsilon(y-z;mr)\geq\lambda,\ \eta(y-z;mr)\geq\lambda \text{ for } m>2 \tag{2}$$

As $y,z\in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$.

For given $\lambda\in(0,1)$ take $t\in(0,1)$ with $(1-t)\circledast(1-t)>1-\lambda$ and $\lambda\odot\lambda<t$. Then for every $\epsilon>0$ and $t\in(0,1)$ we have $\delta(K_1)=\delta(K_2)=0$ where $K_1=\{k\in\mathbb{N}:\tau\left(x_k-y;r+\frac{\epsilon}{2}\right)\leq 1-t$ or $\upsilon\left(x_k-y;r+\frac{\epsilon}{2}\right)\geq t,\ \eta\left(x_k-y;r+\frac{\epsilon}{2}\right)\geq t\}$ and $K_2=\{k\in\mathbb{N}:\tau\left(x_k-z;r+\frac{\epsilon}{2}\right)\leq 1-t$ or $\upsilon\left(x_k-z;r+\frac{\epsilon}{2}\right)\geq t,\eta\left(x_k-z;r+\frac{\epsilon}{2}\right)\geq t\}$. For $k\in K_1^c\cap K_2^c$ we have

$$\begin{aligned}\tau(y-z;2r+\epsilon)&\geq\tau\left(x_k-z;r+\frac{\epsilon}{2}\right)\circledast\tau\left(x_k-y;r+\frac{\epsilon}{2}\right)\\&>(1-t)\circledast(1-t)\\&>1-\lambda,\end{aligned}$$

$$\begin{aligned}\upsilon(y-z;2r+\epsilon)&\leq\upsilon\left(x_k-z;r+\frac{\epsilon}{2}\right)\odot\upsilon\left(x_k-y;r+\frac{\epsilon}{2}\right)\\&<t\odot t\\&<\lambda,\end{aligned}$$

and

$$\begin{aligned}\eta(y-z;2r+\epsilon)&\leq\eta\left(x_k-z;r+\frac{\epsilon}{2}\right)\odot\eta\left(x_k-y;r+\frac{\epsilon}{2}\right)\\&<t\odot t\\&<\lambda.\end{aligned}$$

Hence,

$$\tau(y-z;2r+\epsilon)>1-\lambda,\ \upsilon(y-z;2r+\epsilon)<\lambda \text{ and } \eta(y-z;2r+\epsilon)<\lambda. \tag{3}$$

Then, from (3) we have

$$\tau(y-z;mr) > 1-\lambda,\ \upsilon(y-z;mr) < \lambda \text{ and } \eta(y-z;mr) < \lambda \text{ for } m > 2.$$

which is a contradiction to (2). Therefore, there does not exists elements $y, z \in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ such that $\tau(y-z;mr) \leq 1-\lambda$ or $\upsilon(y-z;mr) \geq \lambda$, $\eta(y-z;mr) \geq \lambda$ for $m > 2$. □

Now, we are giving definition of statistically bounded sequence on NNS as follows:

Definition 2.11. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). A sequence $x = \{x_k\}$ in $\mathbb{X}$ is said to be statistically bounded with respect to neutrosophic norm (τ, υ, η) if for every $\epsilon > 0$ and $\lambda \in (0,1)$ we can find a real number $M > 0$ satisfying

$$\delta(\{k \in \mathbb{N} : \tau(x_k; M) \leq 1-\lambda \text{ or } \upsilon(x_k, M) \geq \lambda,\ \eta(x_k, M) \geq \lambda\}) = 0.$$

In view of the above definition, we obtained the next interesting result on rough statistical convergence on NNS.

Theorem 2.12. *Let $x = \{x_k\}$ be any sequence in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then $x = \{x_k\}$ is statistically bounded with respect to neutrosophic norm (τ, υ, η) if and only if $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r \neq \phi$ for some $r > 0$.*

Proof. Necessary part:
Consider sequence $x = \{x_k\}$ which is statistically bounded in NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then, for every $\epsilon > 0$, $\lambda \in (0,1)$ and some $r > 0$ we get a real number $M > 0$ satisfying

$$\delta(\{k \in \mathbb{N} : \tau(x_k; M) \leq 1-\lambda \text{ or } \upsilon(x_k, M) \geq \lambda,\ \eta(x_k, M) \geq \lambda\}) = 0.$$

Let $K = \{k \in \mathbb{N} : \tau(x_k; M) \leq 1-\lambda \text{ or } \upsilon(x_k, M) \geq \lambda,\ \eta(x_k, M) \geq \lambda\}$.
For $k \in K^c$ we have $\tau(x_k; M) > 1-\lambda$, $\upsilon(x_k, M) < \lambda$ and $\eta(x_k, M) < \lambda$.
Also

$$\begin{aligned} \tau(x_k; r+M) &\geq \tau(0;r) \circledast \tau(x_k; M) \\ &> 1 \circledast (1-\lambda) \\ &= 1-\lambda, \\ \upsilon(x_k; r+M) &< \upsilon(0;r) \odot \upsilon(x_k; M) \\ &< 0 \odot \lambda \\ &= \lambda, \end{aligned}$$

and

$$\begin{aligned} \eta(x_k; r+M) &< \eta(0;r) \odot \eta(x_k; M) \\ &< 0 \odot \lambda \\ &= \lambda. \end{aligned}$$

Hence, $0 \in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$. Therefore, $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r \neq \phi$.

Sufficient Part:

Let $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r \neq \phi$ for some $r > 0$. So some $\xi \in \mathbb{X}$ exists such that $\xi \in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$. For every $\epsilon > 0$ and $\lambda \in (0,1)$ we have

$$\delta(\{k \in \mathbb{N} : \tau(x_k - \xi; r + \epsilon) \leq 1 - \lambda \text{ or } \upsilon(x_k - \xi, r + \epsilon) \geq \lambda,\ \eta(x_k - \xi, r + \epsilon) \geq \lambda\}) = 0.$$

Therefore, almost all x_k's are contained in some ball with center ξ which implies that $x = \{x_k\}$ is statistically bounded in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. □

Further, we define statistical cluster point for the sequence on NNS and establish some results related to it.

Definition 2.13. Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm (τ, υ, η). Then $\gamma \in \mathbb{X}$ is said to be rough statistical cluster point of sequence $x = \{x_k\}$ in $\mathbb{X}$ with respect to neutrosophic norm (τ, υ, η) for some non-negative number r if for every $\epsilon > 0$ and $\lambda \in (0,1)$,

$$\delta(\{k \in \mathbb{N} : \tau(x_k - \gamma; r + \epsilon) > 1 - \lambda \text{ and } \upsilon(x_k - \gamma; r + \epsilon) < \lambda,\ \eta(x_k - \gamma; r + \epsilon) < \lambda\}) > 0,$$

i.e.

$$\delta(\{k \in \mathbb{N} : \tau(x_k - \gamma; r + \epsilon) > 1 - \lambda \text{ and } \upsilon(x_k - \gamma; r + \epsilon) < \lambda,\ \eta(x_k - \gamma; r + \epsilon) < \lambda\}) \neq 0.$$

In this case, γ is known as $r\text{-}St_{(\tau,\upsilon,\eta)}$-cluster point of a sequence $x = \{x_k\}$.

Let $\Gamma^r_{(\tau,\upsilon,\eta)}(x)$ denotes the set of all $r\text{-}St_{(\tau,\upsilon,\eta)}$-cluster points with respect to neutrosophic norm (τ, υ, η) of sequence $x = \{x_k\}$ in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. If $r = 0$ then we get ordinary statistical cluster point with respect to neutrosophic norm (τ, υ, η) in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ *i.e.* $\Gamma^r_{(\tau,\upsilon,\eta)}(x) = \Gamma_{(\tau,\upsilon,\eta)}(x)$.

Theorem 2.14. *Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS. Then, $\Gamma^r_{(\tau,\upsilon,\eta)}(x)$ which is the set of of all $r\text{-}St_{(\tau,\upsilon,\eta)}$-cluster points with respect to neutrosophic norm (τ, υ, η) of any sequence $x = \{x_k\}$ is closed for some non-negative real number r.*

Proof. (i) If $\Gamma^r_{(\tau,\upsilon,\eta)}(x) = \phi$, then we have to prove nothing.

(ii) If $\Gamma^r_{(\tau,\upsilon,\eta)}(x) \neq \phi$. Then, take sequence $y = \{y_k\} \subseteq \Gamma^r_{(\tau,\upsilon,\eta)}(x)$ such that $y_k \xrightarrow{(\tau,\upsilon,\eta)} y_*$. It is enough to show that $y_* \in \Gamma^r_{(\tau,\upsilon,\eta)}(x)$. Now for $t \in (0,1)$ take $\lambda \in (0,1)$ with $(1-\lambda) \circledast (1-\lambda) > (1-t)$ and $\lambda \odot \lambda < t$.

As $y_k \xrightarrow{(\tau,\upsilon,\eta)} y_*$, then for every $\epsilon > 0$ and $\lambda \in (0,1)$ we get $k_\epsilon \in \mathbb{N}$ such that $\tau\left(y_k - y_*; \frac{\epsilon}{2}\right) > 1 - \lambda$, $\upsilon\left(y_k - y_*; \frac{\epsilon}{2}\right) < \lambda$ and $\eta\left(y_k - y_*; \frac{\epsilon}{2}\right) < \lambda$ for $k \geq k_\epsilon$.

Now choose $k_0 \in \mathbb{N}$ such that $k_0 \geq k_\epsilon$. Then, we have $\tau\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right) > 1-\lambda$, $\upsilon\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right) < \lambda$ and $\eta\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right) < \lambda$. Again as $y = \{y_k\} \subseteq \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$, we have $y_{k_0} \in \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$. Then

$$\delta\left(\left\{k \in \mathbb{N} : \tau\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda,\ \upsilon\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda \text{ and } \eta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}\right) > 0. \tag{4}$$

Choose $j \in \left\{k \in \mathbb{N} : \tau\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda,\ \upsilon\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda \text{ and } \eta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\right\}$, then we have $\tau\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda$, $\upsilon\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda$ and $\eta\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda$.

$$\begin{aligned}
\tau(x_j - y_*; r + \epsilon) &\geq \tau\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) \circledast \tau\left(y_{k_0} - y_*; \frac{\epsilon}{2}\right) \\
&> (1-\lambda) \circledast (1-\lambda) \\
&> 1 - t,
\end{aligned}$$

$$\begin{aligned}
\upsilon(x_j - y_*; r + \epsilon) &\geq \upsilon\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) \odot \upsilon\left(y_{k_0} - y_{\odot}; \frac{\epsilon}{2}\right) \\
&< \lambda \odot \lambda \\
&< t,
\end{aligned}$$

and

$$\begin{aligned}
\eta(x_j - y_*; r + \epsilon) &\geq \eta\left(x_j - y_{k_0}; r + \frac{\epsilon}{2}\right) \odot \eta\left(y_{k_0} - y_{\odot}; \frac{\epsilon}{2}\right) \\
&< \lambda \odot \lambda \\
&< t.
\end{aligned}$$

Thus, $j \in \{k \in \mathbb{N} : \tau(x_k - y_*; r + \epsilon) > 1-t,\ \upsilon(x_k - y_*; r + \epsilon) < t \text{ and } \eta(x_k - y_*; r + \epsilon) < t\}$. Hence

$$\begin{aligned}
&\{k \in \mathbb{N} : \tau\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda,\ \upsilon\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda \text{ and } \eta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\} \\
&\qquad \subseteq \{k \in \mathbb{N} : \tau(x_k - y_*; r + \epsilon) > 1-t,\ \upsilon(x_k - y_*; r + \epsilon) < t \text{ and } \eta(x_k - y_*; r + \epsilon) < t\}.
\end{aligned}$$

Now,

$$\begin{aligned}
&\delta(\{k \in \mathbb{N} : \tau\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) > 1-\lambda,\ \upsilon\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda \text{ and } \eta\left(x_k - y_{k_0}; r + \frac{\epsilon}{2}\right) < \lambda\}) \\
&\qquad \leq \delta(\{k \in \mathbb{N} : \tau(x_k - y_*; r + \epsilon) > 1-t,\ \upsilon(x_k - y_*; r + \epsilon) < t \text{ and } \eta(x_k - y_*; r + \epsilon) < t\}).
\end{aligned} \tag{5}$$

Then using equation (4) and equation (5), we get

$$\delta(\{k \in \mathbb{N} : \tau(x_k - y_*; r + \epsilon) > 1-t,\ \upsilon(x_k - y_*; r + \epsilon) < t \text{ and } \eta(x_k - y_*; r + \epsilon) < t\}) > 0.$$

Therefore, $y_* \in \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$. $\square$

Theorem 2.15. *Let $\Gamma_{(\tau,\upsilon,\eta)}(x)$ be the set of all statistical cluster points with respect to neutrosophic norm (τ,υ,η) of sequence $x=\{x_k\}$ in a NNS $(\mathbb{X},\aleph,\circledast,\odot)$ and r be some non-negative real number. Then, for an arbitrary $\gamma\in\Gamma_{(\tau,\upsilon,\eta)}(x)$ and $\lambda\in(0,1)$ we have $\boldsymbol{\tau}(\xi-\gamma;r)>1-\lambda$ and $\upsilon(\xi-\gamma;r)<\lambda$, $\boldsymbol{\eta}(\xi-\gamma;r)<\lambda$*

Proof. For $\lambda\in(0,1)$ take $t\in(0,1)$ with $(1-t)\circledast(1-t)>1-\lambda$ and $t\odot t<\lambda$. Let $\gamma\in\Gamma_{(\tau,\upsilon,\eta)}(x)$. Then, for every $\epsilon>0$ and $t\in(0,1)$ we have

$$\delta(\{k\in\mathbb{N}:\tau(x_k-\gamma;\epsilon)>1-t,\ \upsilon(x_k-\gamma;\epsilon)<t \text{ and } \eta(x_k-\gamma;\epsilon)<t\})>0. \tag{6}$$

Now we will show that if $\xi\in\mathbb{X}$ have $\tau(\xi-\gamma;r)>1-t$, $\upsilon(\xi-\gamma;r)<t$ and $\eta(\xi-\gamma;r)<t$ then $\xi\in\Gamma^r_{(\tau,\upsilon,\eta)}(x)$.
Let $j\in\{k\in\mathbb{N}:\tau(x_k-\gamma;\epsilon)>1-t,\ \upsilon(x_k-\gamma;\epsilon)<t \text{ and } \eta(x_k-\gamma;\epsilon)<t\}$, then $\tau(x_j-\gamma;\epsilon)>1-t$, $\upsilon(x_j-\gamma;\epsilon)<t$ and $\eta(x_j-\gamma;\epsilon)<t$. Now,

$$\begin{aligned}\tau(x_j-\xi;r+\epsilon)&\geq\tau(x_j-\gamma;\epsilon)\circledast\tau(\xi-\gamma;r)\\&>(1-t)\circledast(1-t)\\&>1-\lambda,\end{aligned}$$

$$\begin{aligned}\upsilon(x_j-\xi;r+\epsilon)&\leq\upsilon(x_j-\gamma;\epsilon)\odot\upsilon(\xi-\gamma;r)\\&<t\odot t\\&<\lambda,\end{aligned}$$

and

$$\begin{aligned}\eta(x_j-\xi;r+\epsilon)&\leq\eta(x_j-\gamma;\epsilon)\odot\eta(\xi-\gamma;r)\\&<t\odot t\\&<\lambda.\end{aligned}$$

we have $\tau(x_j-\xi;r+\epsilon)>1-\lambda$, $\upsilon(x_j-\xi;r+\epsilon)<\lambda$ and $\eta(x_j-\xi;r+\epsilon)<\lambda$. Thus $j\in\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)>1-\lambda,\ \upsilon(x_k-\xi;\epsilon)<\lambda \text{ and } \eta(x_k-\xi;\epsilon)<\lambda\}$. Now the next inclusion holds.

$$\begin{aligned}&\{k\in\mathbb{N}:\tau(x_k-\gamma;\epsilon)>1-t,\ \upsilon(x_k-\gamma;\epsilon)<t \text{ and } \eta(x_k-\gamma;\epsilon)<t\}\\\subseteq&\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)>1-\lambda,\ \upsilon(x_k-\xi;r+\epsilon)<\lambda \text{ and } \eta(x_k-\xi;r+\epsilon)<\lambda\}.\end{aligned}$$

Then

$$\begin{aligned}&\delta(\{k\in\mathbb{N}:\tau(x_k-\gamma;\epsilon)>1-t,\ \upsilon(x_k-\gamma;\epsilon)<t \text{ and } \eta(x_k-\gamma;\epsilon)<t\})\\\leq&\,\delta(\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)>1-\lambda,\ \upsilon(x_k-\xi;\epsilon)<t \text{ and } \eta(x_k-\xi;r+\epsilon)<\lambda\}).\end{aligned}$$

Using equation (6) we get $\delta(\{k\in\mathbb{N}:\tau(x_k-\xi;r+\epsilon)>1-\lambda,\ \upsilon(x_k-\xi;r+\epsilon)<\lambda \text{ and } \eta(x_k-\xi;r+\epsilon)<\lambda\})>0$. Therefore, $\xi\in\Gamma^r_{(\tau,\upsilon,\eta)}(x)$. □

Theorem 2.16. *If* $\overline{B(c,\lambda,r)} = \{x \in \mathbb{X} : \varphi(x-c;r) \geq 1-\lambda,\ \vartheta(x-c;r) \leq \lambda\}$ *represents the closure of open ball* $B(c,\lambda,r) = \{x \in \mathbb{X} : \varphi(x-c;r) > 1-\lambda,\ \vartheta(x-c;r) < \lambda\}$ *for some* $r > 0$, $\lambda \in (0,1)$ *and fixed* $c \in \mathbb{X}$ *then* $\mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x) = \bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$.

Proof. For $\lambda \in (0,1)$ take $t \in (0,1)$ with $(1-t) \circledast (1-t) > 1-\lambda$ and $t \odot t < \lambda$. Let $\gamma \in \bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$ then there exists $c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)$ for some $r > 0$ and every $t \in (0,1)$ such that

$$\tau(c-\gamma;r) > 1-t,\ \upsilon(c-\gamma;r) < t \text{ and } \eta(c-\gamma;r) < t.$$

Fix $\epsilon > 0$. Since $c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)$ then we get $K = \{k \in \mathbb{X} : \tau(x_k - c;\epsilon) > 1-t,\ \upsilon(x_k - c;\epsilon) < t \text{ and } \eta(x_k - c;\epsilon) < t\}$ with $\delta(K) > 0$. Now, for $k \in K$,

$$\begin{aligned}\tau(x_k-\gamma;r+\epsilon) &\geq \tau(x_k-c;\epsilon) \circledast \tau(c-\gamma;r)\\ &> (1-t) \circledast (1-t)\\ &> 1-\lambda,\end{aligned}$$

$$\begin{aligned}\upsilon(x_k-\gamma;r+\epsilon) &\leq \upsilon(x_k-c;\epsilon) \odot \upsilon(c-\gamma;r)\\ &< t \odot t\\ &< \lambda,\end{aligned}$$

and

$$\begin{aligned}\eta(x_k-\gamma;r+\epsilon) &\leq \eta(x_k-c;\epsilon) \odot \eta(c-\gamma;r)\\ &< t \odot t\\ &< \lambda.\end{aligned}$$

This implies that $\delta(\{k \in \mathbb{N} : \varphi(x_k-\gamma;r+\epsilon) > 1-\lambda \text{ and } \vartheta(x_k-\gamma;r+\epsilon) < \lambda\}) > 0$. Hence, $\gamma \in \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$.
Therefore, $\bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} \subseteq \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$.
Conversely,
Let $\gamma \in \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$. For result, we will show that $\gamma \in \bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$.
Let if possible, $\gamma \notin \bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$ *i.e.* $\gamma \notin \overline{B(c,\lambda,r)}$ for all $c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)$.
Then $\tau(\gamma-c;r) \leq 1-\lambda$ or $\upsilon(\gamma-c;r) \geq \lambda$, $\eta(\gamma-c;r) \geq \lambda$ for every $c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)$. By Theorem 2.15 for arbitrary $c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)$ we have $\tau(\gamma-c;r) > 1-\lambda$, $\upsilon(\gamma-c;r) < \lambda$ and $\eta(\gamma-c;r) < \lambda$ for every $c \in \mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x)$ which leads to contradiction of our assumption. Therefore, $\gamma \in \bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$. Hence, $\mathit{\Gamma}^r_{(\tau,\upsilon,\eta)}(x) \subseteq \bigcup_{c \in \mathit{\Gamma}_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$. □

Theorem 2.17. *Let $x = \{x_k\}$ be any sequence in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ then for any $\lambda \in (0,1)$,*

(i) If $c \in \Gamma_{(\tau,\upsilon,\eta)}(x)$ then $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r \subseteq \overline{B(c,\lambda,r)}$.

(ii) $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r = \bigcap\limits_{c \in \Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} = \{\xi \in \mathbb{X} : \Gamma_{(\tau,\upsilon,\eta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.

Proof. (i) Let $\epsilon > 0$. For given $\lambda \in (0,1)$ take $t \in (0,1)$ with $(1-t) \circledast (1-t) > 1-\lambda$ and $t \odot t < \lambda$. Consider $\xi \in St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ and $c \in \Gamma_{(\tau,\upsilon,\eta)}(x)$.
Now for every $\epsilon > 0$ and $t \in (0,1)$ consider sets

$$A = \{k \in \mathbb{N} : \tau(x_k - \xi : r+\epsilon) > 1-t,\ \upsilon(x_k - \xi : r+\epsilon) < t \text{ and } \eta(x_k - \xi : r+\epsilon) < t\} \text{ with } \delta(A^c) = 0,$$

and

$$B = \{k \in \mathbb{N} : \tau(x_k - c;\epsilon) > 1-t,\ \upsilon(x_k - c;\epsilon) < t \text{ and } \eta(x_k - c;\epsilon) < t\} \text{ with } \delta(B) \neq 0.$$

Now for $k \in A \cap B$ we have

$$\begin{aligned} \tau(\xi - c; r) &\geq \tau(x_k - c;\epsilon) \circledast \tau(x_k - \xi; r+\epsilon) \\ &> (1-t) \circledast (1-t) \\ &> 1-\lambda. \end{aligned}$$

$$\begin{aligned} \upsilon(\xi - c; r) &\geq \upsilon(x_k - c;\epsilon) * \upsilon(x_k - \xi; r+\epsilon) \\ &< t \odot t \\ &< \lambda, \end{aligned}$$

and

$$\begin{aligned} \eta(\xi - c; r) &\geq \eta(x_k - c;\epsilon) * \eta(x_k - \xi; r+\epsilon) \\ &< t \odot t \\ &< \lambda. \end{aligned}$$

Therefore, $\xi \in \overline{B(c,\lambda,r)}$. Hence, $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r \subseteq \overline{B(c,\lambda,r)}$.

(ii) By previous part we have $St_{(\tau,\upsilon,\eta)}$-$LIM_x^r \subseteq \bigcap\limits_{c \in \Gamma_{\wp}(x)} \overline{B(c,\lambda,r)}$.
Assume $y \in \bigcap\limits_{c \in \Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$ then $\tau(y-c;r) \geq 1-\lambda$, $\upsilon(y-c;r) \leq \lambda$ and $\eta(y-c;r) \leq \lambda$ for all $c \in \Gamma_{(\tau,\upsilon,\eta)}(x)$. This implies that $\Gamma_{(\tau,\upsilon,\eta)}(x) \subseteq \overline{B(y,\lambda,r)}$, *i.e.* $\bigcap\limits_{c \in \Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} \subseteq \{\xi \in \mathbb{X} : \Gamma_{(\tau,\upsilon,\eta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$.
Further, let $y \notin St_{(\tau,\upsilon,\eta)}$-$LIM_x^r$ then for $\epsilon > 0$ we have $\delta(\{k \in \mathbb{N} : \tau(x_k - y; r+\epsilon) \leq 1-\lambda \text{ or } \upsilon(x_k - y; r+\epsilon) \geq \lambda,\ \eta(x_k - y; r+\epsilon) \geq \lambda\}) \neq 0$, which implies that a statistical cluster point c exists for $x = \{x_k\}$ with $\tau(y-c; r+\epsilon) \leq 1-\lambda$, $\upsilon(y-c; r+\epsilon) \geq \lambda$ or $\eta(y-c; r+\epsilon) \geq \lambda$. Thus, $\Gamma_{(\tau,\upsilon,\eta)}(x) \nsubseteq \overline{B(y,\lambda,r)}$ and $y \notin \{\xi \in \mathbb{X} : \Gamma_{(\tau,\upsilon,\eta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$. This implies

that $\{\xi \in \mathbb{X} : \Gamma_{(\tau,\upsilon,\eta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\} \subseteq St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ and we get $\bigcap_{c\in\Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} \subseteq St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$.

Therefore, $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r = \bigcap_{c\in\Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} = \{\xi \in \mathbb{X} : \Gamma_{(\tau,\upsilon,\eta)}(x) \subseteq \overline{B(\xi,\lambda,r)}\}$. □

Theorem 2.18. *Let $x = \{x_k\}$ be any sequence in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ which is statistically convergent to $\xi \in \mathbb{X}$ with respect to neutrosophic norm (τ,υ,η) then there exists $\lambda \in (0,1)$ such that $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r = \overline{B(\xi,\lambda,r)}$ for some $r > 0$.*

Proof. Let $\epsilon > 0$. For given $\lambda \in (0,1)$ take $t \in (0,1)$ with $(1-t) \circledast (1-t) > 1-\lambda$ and $t \odot t < \lambda$. Since $x_k \xrightarrow{St_{(\tau,\upsilon,\eta)}} \xi$ then there is a set $A = \{k \in \mathbb{N} : \tau(x_k - \xi : \epsilon) \leq 1-t$ or $\upsilon(x_k - \xi : \epsilon) \geq t,\ \eta(x_k - \xi : \epsilon) \geq t\}$ with $\delta(A) = 0$. Consider $y \in \overline{B(\xi,t,r)} = \{y \in \mathbb{X} : \tau(y-\xi;r) \geq 1-t,\ \upsilon(y-\xi;r) \leq t,\ \eta(y-\xi;r) \leq t\}$.

For $k \in A^c$

$$\begin{aligned} \tau(x_k - y; r+\epsilon) &\geq \tau(x_k - \xi;\epsilon) \circledast \tau(y-\xi;r) \\ &> (1-t) \circledast (1-t) \\ &> 1-\lambda, \\ \upsilon(x_k - y; r+\epsilon) &\leq \upsilon(x_k - \xi;\epsilon) \odot \upsilon(y-\xi;r) \\ &< t \odot t \\ &< \lambda, \end{aligned}$$

and

$$\begin{aligned} \eta(x_k - y; r+\epsilon) &\leq \eta(x_k - \xi;\epsilon) \odot \eta(y-\xi;r) \\ &< t \odot t \\ &< \lambda. \end{aligned}$$

This implies that $y \in St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$, *i.e.* $\overline{B(\xi,\lambda,r)} \subseteq St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$. Also $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r \subseteq \overline{B(\xi,\lambda,r)}$. Hence, $St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r = \overline{B(\xi,\lambda,r)}$. □

Theorem 2.19. *Let $x = \{x_k\}$ be any sequence in a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ which converges statistically with respect to neutrosophic norm (τ,υ,η) then $\Gamma^r_{(\tau,\upsilon,\eta)}(x) = St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$ for some $r > 0$.*

Proof. Necessary part:

Suppose $x_k \xrightarrow{St_{(\tau,\upsilon,\eta)}} \xi$. Then $\Gamma_{(\tau,\upsilon,\eta)}(x) = \{\xi\}$. By Theorem 2.16 for some $r > 0$ and $\lambda \in (0,1)$ we have $\Gamma^r_{(\tau,\upsilon,\eta)}(x) = \overline{B(\xi,\lambda,r)}$. Also by Theorem 2.18 we get $\overline{B(\xi,\lambda,r)} = St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$. Hence, $\Gamma^r_{(\tau,\upsilon,\eta)}(x) = St_{(\tau,\upsilon,\eta)}\text{-}LIM_x^r$.

Sufficient part:

Let $\Gamma^r_{(\tau,\upsilon,\eta)}(x) = St_{(\tau,\upsilon,\eta)}\text{-}LIM^r_x$. By using Theorem 2.16 and Theorem 2.17(ii) we get

$$\bigcup_{c\in\Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} = \bigcap_{c\in\Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)}$$

. This implies that either $\Gamma_{(\tau,\upsilon,\eta)}(x) = \phi$ or $\Gamma_{(\tau,\upsilon,\eta)}(x)$ is a singleton set. Then $St_{(\tau,\upsilon,\eta)}\text{-}LIM^r_x = \bigcap_{c\in\Gamma_{(\tau,\upsilon,\eta)}(x)} \overline{B(c,\lambda,r)} = \overline{B(\xi,\lambda,r)}$ for some $\xi \in \Gamma_{(\tau,\upsilon,\eta)}(x)$, further by Theorem 2.18 we get $St_{(\tau,\upsilon,\eta)}\text{-}LIM^r_x = \{\xi\}$. □

3. Conclusion

We have introduced the considerable convergence structure called rough statistical convergence on neutrosophic normed spaces. As neutrosophic set is an effective tools to control the inconsistent and indeterminate data, also the theory of rough convergence is a powerful mathematical technique for dealing the convergence problems with present incompleteness in data. The computational techniques with these structures may not always sufficient to produce the best results alone although the merging of two or more of them can provide much improved results. Thus, the significance of introducing rough convergence in this structure is that resultant computational techniques will give a novel mathematical tool to deal with the convergence problems that have been motivated on the basis of practical approach by factual incompleteness, indeterminacy and inconsistency of the data. Moreover, rough statistical convergence on neutrosophic normed spaces can be explored for the different setups like double sequences, triple sequences, ideals, difference sequences and many more.

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