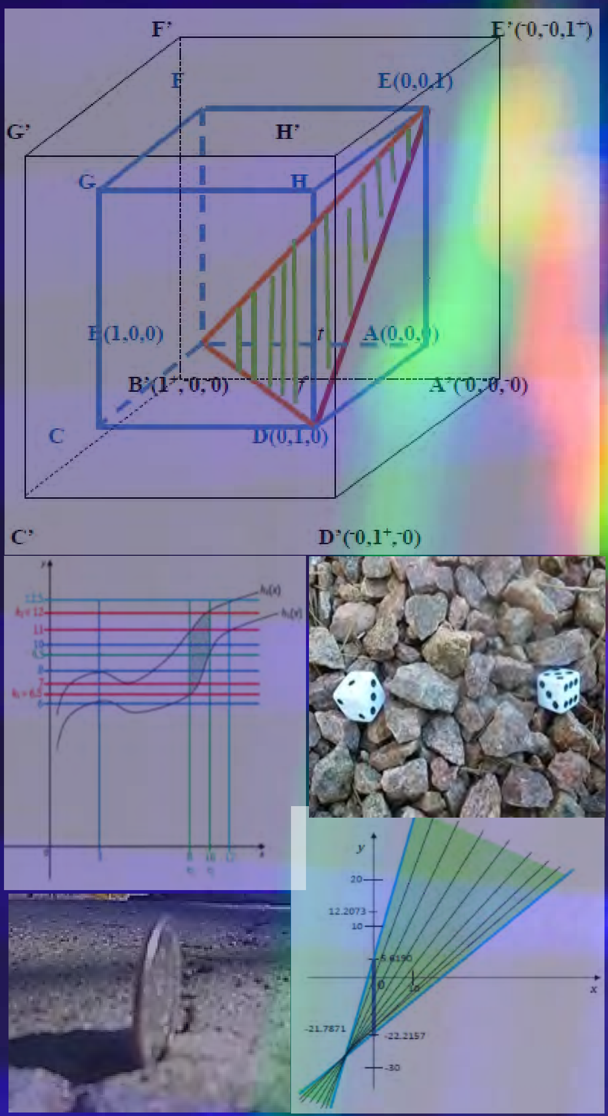
Volume 96,2026

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering



<A> <neutA> <antiA>

ISSN 2331-6055 (Print)
ISSN 2331-608X (Online)

Neutrosophic Science
International Association (NSIA)

ISSN 2331-6055 (print) *ISSN 2331-608X (online)*

Neutrosophic Sets and Systems

An International Journal in Information Science and Engineering

University of New Mexico

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"Neutrosophic Sets and Systems" has been created for publications on advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics that started in 1995 and their applications in any field, such as the neutrosophic structures developed in algebra, geometry, topology, etc.

The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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ISSN 2331-6055 (print) ISSN 2331-608X (online)

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Neutrosophic Sets and Systems

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Google Dictionaries have translated the neologisms "**neutrosophy**" (1) and"**neutrosophic**" (2), coined in 1995 for the first time, into about 100 languages.
FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

Directed Neutrosophic Graph using Morphological Operators and Its Applications

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Abstract. Connectedness provides well suited solutions for several application problems. In Graph theory, graph is connected when there exist a path between any two arbitrary vertices. Neutrosophic set theory together with Morphological operators are useful for constructing the Neutrosophic directed connected subgraphs (NDCS).Using NDCS, method to generate an ordering of subgraphs and construction of spanning trees is presented in this paper. $\mathcal{M}$ induced vertex sequences and $\mathcal{M}$ induced Edge sequences are defined. These sequences generate spanning tree filtration of the graph. Algorithms are given for obtaining such filtrations. Directed Neutrosophic Morphological Network (DNG-MN) is defined. The spanness of a Neutrosophic graph induced by Morphological dilation is given in the paper. Some applications of these concepts are also discussed in this paper.

Keywords: Connected directed graphs, Morphological dilation, erosion, $\mathcal{M}$ induced edge and vertex sequence. Neutrosophic directed graph, Directed Neutrosophic Graph Morphological Network (DNG-MN).

Literature Survey

A Fuzzy Graph is a type of graph such that each edge and vertex in the graph labeled with membership degree, representing uncertainty in the relation between vertices [18]. Uncertainty can be modeled using fuzzy set theory [20–23, 25], Intuitionistic fuzzy set theory [1], Neutrosophic sets etc. The Neutrosophic Graph is a generalization of the Fuzzy Graph. Neutrosophic Sets and graph [27] provide essential tools for analyzing ambiguous or imprecise information. Concepts like Fuzzy and Neutrosophic Sets with graph theory plays an important role in the modeling of this type of information. These models are widely examined in the literature [1–4]. This will extend classical graph theory by adopting Neutrosophic sets which will be helpful for the modeling of ambiguous and complex relationships.

Product Perspective from Fuzzy to Neutrosophic Graph Extension was studied in [34]. Bathusha et.el presented the Energy of dominating single valued Neutrosophic graph structure and was studied in [32]. Fujita et.el studied various types of Neutrosophic graphs and super hyper tree [31], [33]. Bino et.al studied about the Morphological operators and filtering of Hypergraphs [8, 9, 29]. This paper also addresses the various morphological operators [2, 4, 28] which are applied on a Neutrosophic graph framework. Abraham et.al defined the morphological operators on Intuitionistic fuzzy soft graphs [12]. Dhanya et al. Studied the algebra of morphological operators [29] on intuitionistic hypergraph and its application using Neutrosophic hypergraph [13–18]. Sujatha Ramalingam et.al applied the span integrity of fuzzy graph on brain network analysis [3]. Further, this framework has been extended into Directed Neutrosophic Graph Morphological Network (DNG-MN) framework. $\mathcal{M}$ induced edge sequences and vertex sequences are introduced in the paper. Based on these sequences, spanning tree filtrations are obtained. Algorithms for constructing spanning trees [5–7, 19, 20] are extended by considering the new framework. Practical applications are mentioned [24, 26] for further analysis which will help to increase the efficiency of the existing decision making algorithms.

1. Introduction

Graph theory, describes relationships and interconnections within the graph by considering the vertices (nodes) and edges. Graph provides essential insights into connectivity between vertices. Several application problems can be modeled using graph theory. Recently Graph theory concepts are applied in artificial intelligence. Subgraphs of a graph can be constructed using Morphological vertex dilation and erosion [2, 12, 27]. There are disjoint subgraphs such that in a connected graph there exist path between any two arbitrary vertices. So there exist common vertices and edges in these subgraphs. We are dealing with Neutrosophic graph and its underlying graph and subgraphs. Neutrosophic set theory and logic plays an important role in decision making problems. Eigen values of adjacency matrices corresponding to truth degree, indeterminacy degree and falsity degree of a Neutrosophic subgraphs obtained by taking the vertex dilation is computed. By comparing these eigen values, Neutrosophic graph modification can be performed. After applying the pairwise comparison of sum of membership degrees, direction can be assigned to edges. Algorithms are given for these constructions. The same procedure with different morphological operator like erosion will generate another modified Neutrosophic graph. This lead to the generalized framework for performing similar construction of modified Neutrosophic graph. This is given as a system namely Neutrosophic directed Morphological Network. Algorithms for finding spanning trees of this network system are given. Such a spanning tree may be optimal. Spanning tree filtrations are important in several application problems. Edge sequences and vertex sequences are obtained by considering edges and vertices which are common to the sets of subgraphs. By including these sequences to the system, it can be considered as a Generalized Directed Neutrosophic Graph Morphological Network (DNG-MN) systems. Spanning tree which satisfying optimality conditions of this system can be considered in application problems like brain diseases. Span integrity and spanness of dilation induced vertex deleted graphs are calculated. It is useful for ordering the subgraphs. Algorithms are given for the construction of such systems.

1.1. *Construction of connected subgraphs*

Consider a Neutrosophic Graph G as shown in Fig. 1, with neutrosophic weights (membership, indeterminacy, non-membership) (μ, γ, κ) for each vertex in G. The edges are also having the three weights (μ, γ, κ) where for the edge e_{ij} between vertices v_i and v_j, the membership weight of the edge represented as $\mu_{e_{ij}}$ is the max (μ_{vi}, μ_{vj}), the indeterminacy weight of the edge represented as γ_{ij} is the avg $(\gamma_{vi}, \gamma_{vj})$ and non-membership weight of the edge represented as $\kappa_{e_{ij}}$ is the min $(\kappa_{vi}, \kappa_{vj})$.

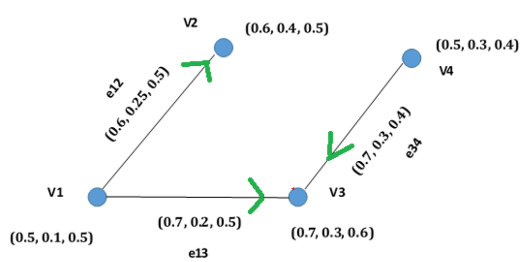


FIGURE 1. Neutrosophic graph G

Let v be any vertex in V. Find the Neutrosophic graph dilation $N_{\delta_G}(v)$ of v. Let $N\delta_G(v_i) = NG_i$ and the underlying graph be G_i. Similarly, for any v_i, we get the underlying subgraphs G_i. Fig. 2 shows the subgraph $NG_1 = \delta(v_1)$.

Find the adjacency matrix $A(NG_1)$ of the subgraphs NG_1 and then find the eigen values of $A(NG_i)$.

Let

$$A(NG_1) = (\mu_{v_iv_j}, \gamma_{v_iv_j}\kappa_{v_iv_j}) = \begin{bmatrix} (0,0,0) & (0.6, 0.25, 0.5) & (0.7, 0.2, 0.5) \\ (0.6, 0.25, 0.5) & (0,0,0) & (0,0,0) \\ (0.7, 0.2, 0.5) & (0,0,0) & (0,0,0) \end{bmatrix}$$

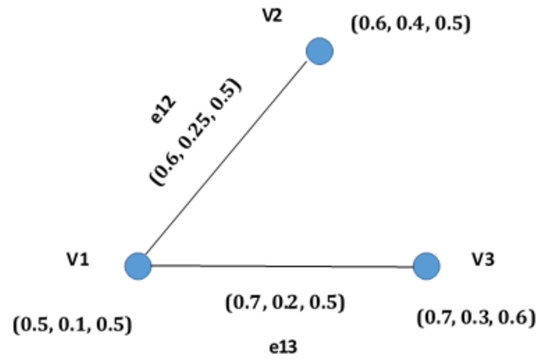


FIGURE 2. Neutrosophic subgraph NG_1

and

$$A(\mu_{v_iv_j}) = \begin{bmatrix} 0 & 0.6 & 0.7 \\ 0.6 & 0 & 0 \\ 0.7 & 0 & 0 \end{bmatrix} \quad A(\gamma_{v_iv_j}) = \begin{bmatrix} 0 & 0.25 & 0.2 \\ 0.25 & 0 & 0 \\ 0.2 & 0 & 0 \end{bmatrix}$$

$$\text{and} \quad A(\kappa_{v_iv_j}) = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0 \\ 0.5 & 0 & 0 \end{bmatrix}$$

The eigen values of adjacency matrix $A(\mu_{v_iv_j})$ of membership degree $\mu_{v_iv_j}$ are $\lambda_{1\mu} = 0$, $\lambda_{2\mu} = 0.92$ and $\lambda_{3\mu} = -0.92$. Now find the absolute maximum Max $\lambda_{i\mu}$, which is 0.92. Similarly the eigen values of the adjacency matrix $A(\gamma_{v_iv_j})$ of indeterminacy degree $\gamma_{v_iv_j}$ are $\lambda_{1\gamma} = 0$, $\lambda_{2\gamma} = 0.32$ and $\lambda_{3\gamma} = -0.32$. Now find the average absolute value Avg $\lambda_{i\gamma}$, which gives 0.43. Likewise the eigen values of the adjacency matrix $A(\kappa_{v_iv_j})$ of non-membership degree $\kappa_{v_iv_j}$ are $\lambda_{1\kappa} = 0$, $\lambda_{2\kappa} = 0.71$ and $\lambda_{3\kappa} = -0.71$. Now the absolute minimum value Min $\lambda_{i\kappa}$ results to 0. Thus Eig $(N(G_i)) = (\text{Max}\,\lambda_{i\mu}, \text{Avg}\,\lambda_{i\gamma}, \text{Min}\,\lambda_{i\kappa}) = (0.92, 0.43, 0)$ where Max $\lambda_{i\mu}$ is the absolute maximum of Eigen values of $A(\mu_{v_iv_j})$, Min $\lambda_{i\kappa}$ is the absolute minimum of Eigen values of $A(\kappa_{v_iv_j})$ and Min $\lambda_{i\gamma}$ is the absolute minimum of Eigen values of $A(\gamma_{v_iv_j})$.

Similarly the subgraph $NG_2 = \delta(v_2)$ is as shown in Fig. 3.

The membership adjacency matrix $A(\mu_{v_iv_j})$ of NG_2, the indeterminacy adjacency matrix $A(\gamma_{v_iv_j})$ and the non-membership adjacency matrix $A(\kappa_{vivj})$ are as given below.

$$A(\mu_{v_iv_j}) = \begin{bmatrix} 0 & 0.6 \\ 0.6 & 0 \end{bmatrix} \quad A(\gamma_{v_iv_j}) = \begin{bmatrix} 0 & 0.25 \\ 0.25 & 0 \end{bmatrix} \text{ and } A(\kappa_{v_iv_j}) = \begin{bmatrix} 0 & 0.5 \\ 0.5 & 0 \end{bmatrix}$$

The eigen values of adjacency matrix $A(\mu_{v_iv_j})$ of membership degree $\mu_{v_iv_j}$ are $\lambda_{1\mu} = 0.6$, $\lambda_{2\mu} = -0.6$ and absolute maximum Max $\lambda_{i\mu}$, is 0.6. Similarly the eigen values of the adjacency matrix $A(\gamma_{v_iv_j})$ of indeterminacy degree $\gamma_{v_iv_j}$ are $\lambda_{1\gamma} = 0.25$, $\lambda_{2\gamma} = -0.25$ and the average absolute value Avg $\lambda_{i\gamma}$, is 0.25. Likewise the eigen values of the adjacency matrix $A(\kappa_{v_iv_j})$

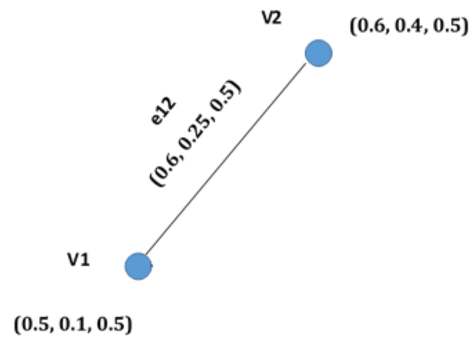


FIGURE 3. subgraph NG_2

of non-membership degree $\kappa_{v_iv_j}$ are $\lambda_{1\kappa} = 0.5$, $\lambda_{2\kappa} = -0.5$ and the absolute minimum value Min $\lambda_{i\kappa}$ results to 0.5.

Thus

$$\text{Eig}\,(N(G_2)) = (\text{Max}\,\lambda_{i\mu}, \text{Avg}\,\lambda_{i\gamma}, \text{Min}\,\lambda_{i\kappa}) = (0.6, 0.25, 0.5).$$

Considering the vertex V_3 we get the subgraph $N(G_3)$ as in Fig. 4.

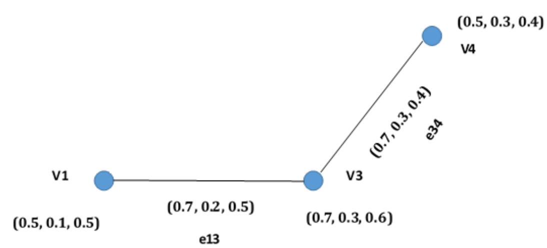


FIGURE 4. subgraph $N(G_3)$

The three adjacency matrix

$A(\mu_{v_iv_j}) = \begin{bmatrix} 0 & 0.7 & 0 \\ 0.7 & 0 & 0.7 \\ 0 & 0.7 & 0 \end{bmatrix}$ with eigen values $\lambda_{1\mu} = 0$, $\lambda_{2\mu} = 0.99$ and $\lambda_{3\mu} = -0.99$. The absolute 0 maximum Max $\lambda_{i\mu}$, is 0.99

$A(\gamma_{v_iv_j}) = \begin{bmatrix} 0 & 0.2 & 0 \\ 0.2 & 0 & 0.3 \\ 0 & 0.3 & 0 \end{bmatrix}$ with eigen values $\lambda_{1\gamma} = 0$, $\lambda_{2\gamma} = 0.36$ and $\lambda_{3\gamma} = -0.36$. The average of absolute values gives 0.24

and $A(\kappa_{v_iv_j}) = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.4 \\ 0 & 0.4 & 0 \end{bmatrix}$ with eigen values $\lambda_{1\kappa} = 0$, $\lambda_{2\kappa} = 0.64$ and $\lambda_{3\kappa} = -0.64$. The absolute minimum gives 0. Now

Eig $(N(G_3)) = (\text{Max}\,\lambda_{i\mu}, \text{Avg}\,\lambda_{i\gamma}, \text{Min}\,\lambda_{i\kappa}) = (0.99, 0.24, 0)$.

Considering the vertex V_4, we get the subgraph $N(G_4)$ as in Fig. 5.

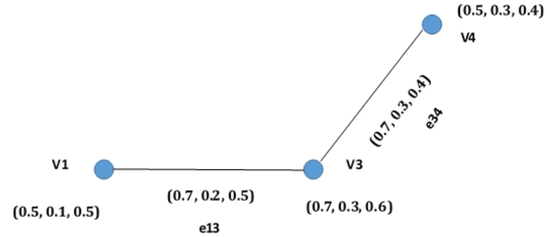


FIGURE 5. Subgraph V_4

The adjacency matrices are $A(\mu_{v_iv_j}) = \begin{bmatrix} 0 & 0.7 \\ 0.7 & 0 \end{bmatrix}$ with eigen values $\lambda_{1\mu} = 0.7$, $\lambda_{2\mu} = -0.7$ and absolute maximum Max $\lambda_{i\mu}$, is 0.7.

$A(\gamma_{v_iv_j}) = \begin{bmatrix} 0 & 0.3 \\ 0.3 & 0 \end{bmatrix}$ with eigen values $\lambda_{1\gamma} = 0.3$, $\lambda_{2\gamma} = 0.3$ and the average of absolute values gives 0.3 and $A(\kappa vivj) = \begin{bmatrix} 0 & 0.4 \\ 0.4 & 0 \end{bmatrix}$ with eigen values $\lambda_{1\kappa} = 0.4$, $\lambda_{2\kappa} = -0.4$ and the absolute minimum gives 0.4. Now

$\text{Eig}\,(N(G_4)) = (\text{Max}\,\lambda_{i\mu}, \text{Avg}\,\lambda_{i\gamma}, \text{Min}\,\lambda_{i\kappa}) = (0.7, 0.3, 0.4)$.

Let $G_{i,i=1,2,...}$ be subgraphs. Compare the Eigen values of adjacency matrices. Let the membership adjacency matrix be $A_\mu(G)$, indeterminacy membership adjacency matrix be $A_\gamma(G)$ and Non-membership matrix be $A\kappa(G)$. Apply the following rules to add any new edges to the underlying graph.

(1) Consider the subgraphs obtained and the given graph. Compare the absolute values of Eigen values of the membership adjacency matrices. Arrange these values in descending order. Consider the pairs of subgraphs with the corresponding eigen values ordered by $\lambda_i > \lambda_j$. If there exist atleast one edge which connects any two vertices of the two subgraphs then no need to add any new edges to the given graph.
(2) If the subgraphs G_i and G_j are disconnected then join any vertex in G_i to a vertex in G_j.
(3) Use any two vertices which were not connected in the previous step.

Compare the Eigen values. For $A(\mu_{v_iv_j})$, here, $0.99 > 0.92 > 0.7 > 0.6$ where $G_3 > G_1 > G_4 > G_2$. Check with all pairs of subgraphs and apply the above rule.

Consider the following pair wise connectivity for applying the above mentioned rules.

(a) (G_3, G_4)—Vertices v_3 and v_4 are common to the sub graphs G_3, G_4. So by rule 1, no modification is required.
(b) (G_3, G_1)—Vertices v_3 and v_1 are common to the sub graphs G_3, G_1. So by rule 1, no modification is required.

(c) (G_3, G_2)—Vertex v_1 is common to the sub graphs G_3, G_2. So by rule 1, no modification is required.

(d) (G_4, G_1)—Vertex v_3 is common to the sub graphs G_4, G_1. So by rule 1, no modification is required.

(e) (G_4, G_2) There is no edge between G_4 and G_2. Hence draw a new edge which connects a vertex in G_4 to a vertex in G_2. So, draw an edge between v_2 and v_3.

(f) (G_1, G_2)—Vertices v_1 and v_2 are common to the sub graphs G_3, G_1. So by rule 1, no modification is required.

Considering case (e) above, a new edge is to be added in the main graph G from subgraph G_4 to G_2. The two nodes selected for this connection will be the two nearest ones among all possible pairs in two subgraphs G_4 and G_2. The Euclidean distance between the (v_4, v_1) is 0.2236, between (v_3, v_1) is 0.3, between (v_3, v_2) is 0.1732, between (v_4, v_2) is 0.1732. Since both the pairs (v_3, v_2) and (v_4, v_2) are having the same distance, let us select any one out of it. Here we have selected the pair (v_3, v_2). The modified Neutrosophic graph is shown in Fig. 6. And the red coloured edge is a new edge.

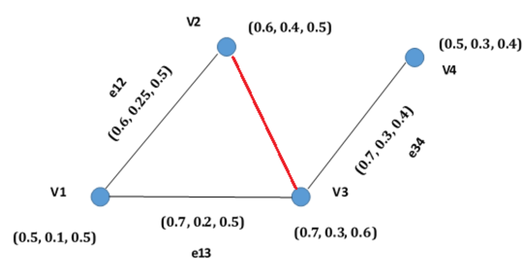


FIGURE 6. Modified neutrosophic graph. MG

Similarly compare the Eigen values of $A_\kappa(G)$ and $A_\gamma(G)$. Apply the previous steps to modify the graph. A graph renewal algorithm is given below.

Algorithm 1: Construction of the modified Neutrosophic graph:

Step 1: Let $G_i, G_j, G_k, \ldots$ be subgraphs of G.

Step 2: Calculate the Eigen values of the membership adjacency matrix $A_\mu(G)$, Indetermnacy matrix $A_\gamma(G)$ and Non membership matrix be $A_\kappa(G)$ and arrange these values in descending order.

Step 3: If $\lambda_i > \lambda_j$ then check whether the subgraphs are connected. Otherwise draw an edge from a vertex v_i in G_i to a vertex v_j in G_j.

Step 4: Repeat step 3 for the other pairs of subgraphs

Step 5: Obtain the transformed graph $G = \cup_i(G_i)$, $i = 1, 2, \ldots n$, G_is are underlying subgraphs. Keep the same membership degree in the new graph also. This represent a transformed Neutrosophic graph.

1.2. *δ-induced directed connected sub graphs*

$\delta(v_i) = G_i$, $\delta(v_j) = G_j$, $\delta(v_k) = G_k$ be distinct Neutrosophic subgraphs which satisfies the condition (1). Let C_μ and C_κ denote the sum of membership degree and indeterminacy degree of each subgraph obtained. Then $C_{\mu+\gamma} = C_\mu + C_\gamma$.

$$\text{i.e., } C_\mu = [\sum[(\mu_{G_{S_i}})]] \text{ and } C_\gamma = [\sum(\gamma_{G_{S_j}})]$$

If $C^i_{\mu+\gamma} > C^j_{\mu+\gamma}$ then G_i is a strong subgraph than G_j. There exist an edge connecting to G_i, G_j, G_k. $G_i \to Gj \Leftrightarrow \exists$ a path from $v_i \to v_j$.

Rule for converting the transformed IF graph into a directed Neutrosophic graph.

(a) Calculate $C_{\mu+\gamma}$ for each vertex v of the MG

$$\text{i.e.} C_{\mu+\gamma} = [\sum[(\mu_{(G_{S_i}, \gamma_{G_{s_j}}})]].$$

(b) If $C_{\mu+\gamma} > C^j_{\mu+\gamma}$ then v_i is a strong vertex than v_j and if $C^i_{\mu+\gamma} > C^j_{\mu+\gamma}$ then assign directions from higher valued vertex to the lower valued vertex. Here it is directed towards V_j.

Here, $C^i_{\mu+\gamma} = 2.6$, $C^2_{\mu+\gamma} = 1.6$, $C^3_{\mu+\gamma} = 1.4$, $C^4_{\mu+\gamma} = 1.9$. So let the ordering be $V_1 > V_4 > V_2 > V_3$.

Hence assign directions w. r. to this order as shown in Fig. 7.

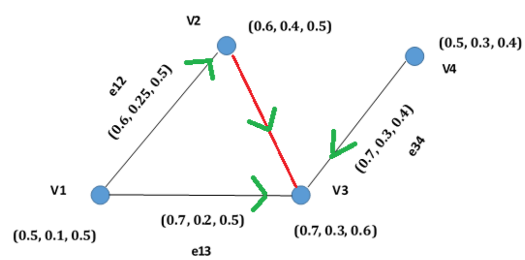


FIGURE 7. Directed Modified Neutrosophic graph

i) $V_1 > V_2$. Hence assign directions from V_1 to V_2.

ii) $V_1 > V_3$. Hence assign directions from V_1 to V_3.

iii) $V_2 > V_3$. Hence assign directions from V_2 to V_3.

iv) $V_4 > V_3$. Hence assign directions from V_4 to V_3.

Algorithm 2: Construction of δ-induced directed connected transformed Neutrosophic graphs

Step 1: Let $V_i, V_j, V_k, \ldots$ be the vertices of the transformed Neutrosophic graph.

Step 2: For each vertex , Calculate $C_{\mu+\gamma}$ which is the sum of membership degree and non membership degree.

Step 3: If $C^i_{\mu+\gamma} > C^i_{\mu+\gamma}$ then assign directions from vertex V_i to V_j.

Step 4: Repeat steps 2 to 3 until all edges are considered.

Step 5: Obtain a directed transformed Neutrosophic Graph.

2. Neutrosophic Graph Transformation

Definition: A Neutrosophic Graph can be modified by applying Morphological operators like dilation and erosion. This can be obtained with δ-induced or ϵ induced directed connected sub graphs. This is called a Neutrosophic Graph transformation. This may contain an addition of edges.

2.1. *Definition: Neutrosophic Graph Space*

$(NG, \mathcal{M})$ represent a family of Neutrosophic graph obtained by applying operators in $\mathcal{M}$, the set of Morphological operators, is called an Neutrosophic graph space. The Neutrosophic graph space is denoted by NG_M. Let the sub graphs of NG_M be $NG_{M1}, NG_{M2}, \ldots NG_{Mn}$.

2.2. *Definition: $\mathcal{M}$ induced sequence*

The set of edges of $NG_{M1}, NG_{M2}, \ldots NG_{Mn}$ formed by taking the edges which lies in the intersection of any two subgraphs form a sequence of edges. This family of sequence of edges are called $\mathcal{M}$ induced edge sequence and is denoted by $\mathfrak{E}_{M \cdot E}$ where $\mathfrak{E}_{M \cdot E} = \{\mathcal{E}_{\mathcal{M}_1}, \mathcal{E}_{\mathcal{M}_2}, \ldots \mathcal{E}_{\mathcal{M}_n}\}$ where $\mathcal{E}_{\mathcal{M}_1} = \{e_{11} \cdot e_{12}, \ldots e_{1n}\}, \ldots \mathcal{E}_{\mathcal{M}_n} = \{e_{n1}.e_{n2}, \ldots e_{nn}\}$. Similarly for the vertices which lies in the intersection of any two sub graphs form an $\mathcal{M}$ induced sequence of vertices and is denoted by $\mathfrak{E}_{\mathcal{M}_V}$.

$$\mathcal{E}_{\mathcal{M}_V} == \mathcal{V}_{\mathcal{M}_1}, \mathcal{V}_{\mathcal{M}_2}, \ldots \mathcal{V}_{\mathcal{M}_n}, \mathcal{V}_{\mathcal{M}_1} = v11 \ldots\ldots v1n \ldots . \mathcal{V}_{\mathcal{M}_n} = \{v_{n_1} \ldots v_{n_n}\}.$$

Maximal vertex Intersection Number: The Maximal intersection number of an $\mathcal{M}$ **induced vertex sequence** is the number of sets having vertices common in $\mathcal{V}_{\mathcal{M}_1}, \mathcal{V}_{\mathcal{M}_2}, \ldots \mathcal{V}_{\mathcal{M}_n}$. It is denoted by $\mathcal{M}_\tau$. These numbers are useful for identifying a path or spanning trees in NG.

Maximal edge Intersection Number: The Maximal intersection number of an $\mathcal{M}$ **induced edge sequence** is the number of sets having edges common in $\mathcal{E}_{\mathcal{M}_1}, \mathcal{E}_{\mathcal{M}_2}, \ldots \mathcal{E}_{\mathcal{M}_n}$. It is denoted by $\mathcal{M}_\sigma$.

The path or spanning trees are obtained by considering the set of edges corresponding to this number.

Algorithm 3: Construction of $\mathcal{M}$ induced sequences and directed spanning trees.

Step 1: Apply Algorithm 1 and Algorithm 2 to obtain a transformed directed NG with δ-induced or ϵ induced directed connected sub graphs.

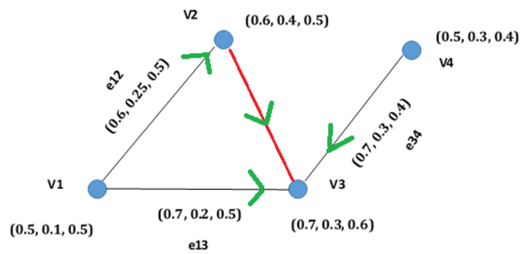
Step 2: Find the sequences $\mathfrak{E}_{\mathcal{M}_\mathcal{E}} = \{\mathcal{E}_{\mathcal{M}_1}, \mathcal{E}_{\mathcal{M}_2}, \ldots \mathcal{E}_{\mathcal{M}_n}\}$ and $\mathcal{E}_{\mathcal{M}_V} == \{\mathcal{V}_{\mathcal{M}_1}, \mathcal{V}_{\mathcal{M}_2}, \ldots \mathcal{V}_{\mathcal{M}_n}\}$.

Step 3: Find the Maximal intersection number and set of vertices.

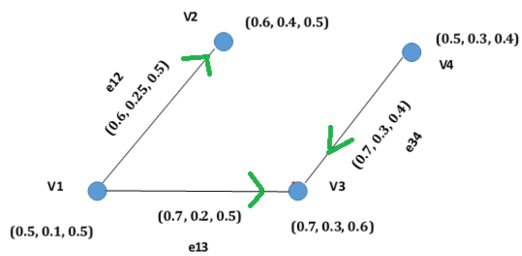
Step 4: Find all directed spanning trees passing through the vertices which are common to these sets.

Illustration of the construction of a directed spanning tree:

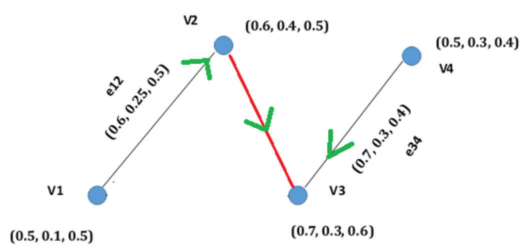
Consider an M induced vertex sequence of the Neutrosophic graph and apply the procedure to generates various paths.



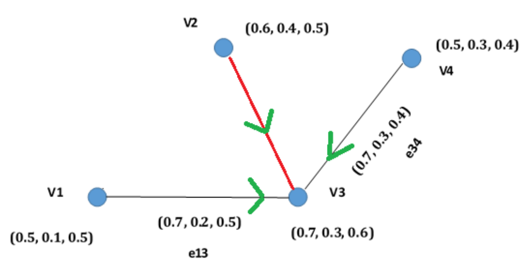
Here $\mathcal{V}_{\mathcal{M}_1} = \{v_1, v_2, v_3\},\ \mathcal{V}_{\mathcal{M}_2} = \{v_1, v_2\},\ V_{\mathcal{M}_3} = \{v_1, v_3, v_4\},\ V_{\mathcal{M}_4} = \{v_3, v_4\}$. v_1 and v_3 are vertices having intersection in three sets. So, the **maximal intersection number** $\mathcal{M}_\tau$ is 3 and the intersection vertices are v_1, v_3. The directed induced paths or directed spanning trees passing through these vertices are given below.



Spanning tree 1



Spanning tree 2



Spanning tree 3

Since any two edges in $\mathfrak{E}_{\mathcal{M}_\mathcal{E}}$ lies in the intersection of subgraphs, the corresponding union of subgraphs must contain various paths. Consider a directed NG_M. The consecutive edges from $\mathfrak{E}_{\mathcal{M}_\mathcal{E}}$ induces a directed connected path.

Similarly, Since any two vertices in $\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}$ lies in the intersection of subgraphs, the corresponding union of subgraphs must contain various paths. Consider a directed NG_M. The consecutive vertices from $\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}$ induces a directed connected path.

2.3. *Definition: Neutrosophic Graph Morphological Network*

NG_M with $\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}} = \{\mathcal{E}_{\mathcal{M}_1}, \mathcal{E}_{\mathcal{M}_2}, \ldots \mathcal{E}_{\mathcal{M}_n}\}$, $\mathcal{E}_{\mathcal{M}_V} == \{\mathcal{V}_{\mathcal{M}_1}, \mathcal{V}_{\mathcal{M}_2}, \ldots \mathcal{V}_{\mathcal{M}_n}\}$ and Morphological operators in $\mathcal{M}$ is called an Neutrosophic graph Morphological network and is denoted by $(NG_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathcal{E}_{\mathcal{M}_V}, \mathcal{M})$.

Since the collection of underlying subgraphs in NG_M form a lattice, there is an order relation between subgraphs and the related theories can be applied to $\{NG_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathcal{E}_{\mathcal{M}_V}, \mathcal{M}\}$ also. Minimum connected subgraph can be considered as infimum and the whole network which represents the supremum of the lattice. Since the underlying graph and subgraphs are connected by assumption, any two edges in $\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}$ or any two vertices in $\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}$ induces a path between any two vertices in the corresponding subgraphs. Hence $\{NG_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathcal{E}_{\mathcal{M}_V}, \mathcal{M}\}$ generates various paths between vertices in any two underlying subgraphs in NG_M.

This generates various directed spanning trees also.

2.4. *Sequence of edges and vertices*

The learning architecture of the sequence of edges in $\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}$ and vertices in $\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}$ can be obtained by considering a Morphological operator on the Neutrosophic graph Morphological network $(NG_M, , \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$.

Generalized $\mathcal{M}$ induced vertex sequences and edge sequences

Let the number of subgraphs be n and the number of subgraphs for grouping is k where $k \leq n$.

Then the **Generalized $\mathcal{M}$ induced vertex sequences and edge sequences are** $\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}} = \{\mathcal{V}^k_{\mathcal{M}_1}, \mathcal{V}^k_{\mathcal{M}_2}, \ldots \mathcal{V}^k_{\mathcal{M}n}\}$ and $\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}} = \{\mathcal{E}^k_{\mathcal{M}_1}, \mathcal{E}^k_{\mathcal{M}_2}, \ldots \mathcal{E}^k_{\mathcal{M}n}\}$.

Algorithm 4 : Construction of all possible trees

Step 1: Apply algorithm 1 and Algorithm 2 on NG

Step 2: Apply Algorithm 3 with various values of k and obtain the sequences $\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}} = \{\mathcal{V}^k_{\mathcal{M}_1}, \mathcal{V}^k_{\mathcal{M}_2}, \ldots \mathcal{V}^k_{\mathcal{M}n}\}$ and $\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}} = \{\mathcal{E}^k_{\mathcal{M}_1}, \mathcal{E}^k_{\mathcal{M}_2}, \ldots \mathcal{E}^k_{\mathcal{M}n}\}$ where $k \leq n$.

Step 3: Find the **maximal intersection number $\mathcal{M}_\tau$ and related vertices**.

Step 4: Obtain all possible trees which are passing through these vertices. There may or may not be spanning Trees. The spanning trees obtained by considering various values of $k \leq n$ are called k level spanning trees and is denoted by S^k_n.

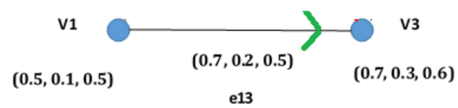
Example: Consider the case with $n = 4$, $k = 3$ and with $n = 4$, $k = 4$.

The possible combinations of subgraphs and its vertex sequences are given below:

Case 1) $n = 4$, $k = 3$.

For (G_1, G_2, G_3, G_4), $\mathcal{V}^3_{\mathcal{M}_1} = \{v_1\}$, $\mathcal{V}^3_{\mathcal{M}_2} = \{v_3\}$, $\mathcal{V}^3_{\mathcal{M}_3} = \{\}$, $\mathcal{V}^3_{\mathcal{M}_4} = \{\}$.

No vertex is common to any of these sets in the induced vertex sequence $\mathcal{E}_{\mathcal{M}_{\mathcal{V}}} = \{V^3_{\mathcal{M}_1}, V^3_{\mathcal{M}_2}, V^3_{\mathcal{M}_3} V^3_{\mathcal{M}_4}\}$. Hence there is no spanning tree for $k = 3$. But a tree can be drawn through the vertices v_1, v_3. Here $|S^k_n| = 0$.



Case 2) $n = 4$, $k = 4$

For (G_1, G_2, G_3, G_4), no vertex is common to these subgraphs. Hence there is no spanning tree. Here $|S^k_n| = 0$.

Also there is no tree.

Therefore, there is no k level spanning tree connectivity between subgraphs for $k > 2$.

Proposition: If there exist k level spanning tree connectivity for every value of $k \leq n$ in the Neutrosophic graph Morphological network $(NG_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$ then the underlying graph is the union of all spanning trees.

Proof: Since $G_i = \delta(v_i)$ and if there exist k level spanning tree connectivity for every value of $k \leq n$ then the vertex sequence contains all the vertices of G and since various spanning trees exist which implies that the given graph is the union of all spanning trees obtained.

2.5. *Directed Neutrosophic graph Morphological network*

$(NG_M, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$ with a directed Morphological network is called **Directed Neutrosophic graph Morphological network and is denoted by** $(NG_{\vec{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$, (DNG–MN). Consider $(NG_M, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$. Let $e_{11} \in \epsilon_{\mathcal{M}_1}$, $e_{12} \in \epsilon_{\mathcal{M}_1}$. Then e_{11}, e_{12} are consecutive edges means there exist a directed path containing these edges and there exist an underlying subgraph where these edges belong. Hence the consecutive edges form a directed path. It is denoted by $e_{ii} \approx e_{jj}$.

Let $e_{ii} \approx e_{jj}$, for $i = 1, 2, \ldots n$, $j = 1, 2 \ldots n$. Then $P_{\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}}$ represent the set of all such directed paths and this will form a directed network.

2.6. *Proposition*

Consider $(NG_{\vec{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$. If one edge in $\mathcal{E}_{\mathcal{M}_1}$ and one edge in $\mathcal{E}_{\mathcal{M}_2}$ are consecutive then there exist at least a path of length two.

Proof: By definition.

2.7. *Theorem*

$\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}} = \{\mathcal{E}^k_{\mathcal{M}_1}, \mathcal{E}^k_{\mathcal{M}_2}, \ldots \mathcal{E}_{\mathcal{M}n}\}$ with edges in $\mathcal{E}_{\mathcal{M}_i}$, $i = 1, 2, \ldots n$ are consecutive then these consecutive edges generates a directed network.

Proof: By Proposition 2.8.

2.8. *Theorem: Minimal spanning tree*

Consider $(NG^{\rightarrow}_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$. Then $P_{\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}}$ induces a directed Minimal spanning tree where $P_{\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}}$ is the set of all paths obtained by using edges in $P_{\mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}}$.

Proof: $P_{\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}}$ represent the set of all paths obtained by using edges in $\mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}$. Hence this contain a minimal spanning tree.

2.9. *Theorem*

Let $(NG^{\rightarrow}_{M}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$ be a Directed Neutrosophic graph Morphological network (DNG–MN). Then $\{\epsilon_{\mathcal{M}_i, e_{ij} \in \epsilon_{\mathcal{M}_i}}\}$ induces a filtration.

Proof: Let $(NG^{\rightarrow}_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$ be a Directed Neutrosophic graph Morphological network (DNG–MN). By theorem, $(NG^{\rightarrow}_{\mathcal{M}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$ induces a minimal spanning tree which represents a filtration.

2.10. *Sublevel filtration of network*

Each induced subgraphs represent sublevels. These sub graphs $\{G_i\}$ generates filtrations. Hence a network has an ordered filtrations. In such cases a starting vertex is assumed as a vertex which lies left and assume that the network levels are advancing towards right. Since the graph space is a lattice it induces a partial order relation and hence allows directions and movements in the navigation from left to right. If we choose a vertex in random then the ordering become a consensus ordering or permutation ordering. Different filtrations can be generated using different starting vertices. A Neutrosophic transformation can be applied to obtain a new vertex (a dummy vertex) so that various other ordering may be applied.

2.11. *Theorem*

$P_{\mathfrak{E}_{\mathcal{M}_1 \cdot \mathcal{E}}} \preceq P_{\mathfrak{E}_{\mathcal{M}_2 \cdot \mathcal{E}}} \preceq \cdots \preceq P_{\mathfrak{E}_{\mathcal{M}_n \cdot \mathcal{E}}}$ is a filtration of the subgraphs $G_1, \ldots . G_n$.

Proof: By applying the Decomposition theorem: $NG_M = \cup_{i=1}^{n} NG_{Mi}$ and apply proposition 2.8, theorem 2.9.

3. **Application to Path planning and Disease detection**

3.1. *Path planning in a Directed Neutrosophic graph Morphological Network (DNG–MN)*

Path planning in a graph like DNG–MN requires specific constraints. MST algorithms meet these requirements. However the method discussed in this paper gives another framework for a general search problems and path planning. Since the proposed method deals with subgraphs, the local search is also possible. Vertices which is common to any two subgraphs are important because they play a crucial role in the formulation of an optimal path in DNG–MN. These vertices are playing as linking t vertices in the sub graphs and this will generate various paths. This will be helpful for the analysis of diseases like cancer, alshimers etc.

3.2. *Analogy with the brain neural network*

Here we can consider the analogy with the brain network. Connections between brain regions do not have the same degree of adjacency. Regions can be converted to a DNG–MN $(NG^{\rightarrow}_{M}, \mathfrak{E}_{\mathcal{M}_{\mathcal{E}}}, \mathfrak{E}_{\mathcal{M}_{\mathcal{V}}}, \mathcal{M})$ and obtain a filtration of the form $P_{\mathfrak{E}_{\mathcal{M}_{1\cdot\mathcal{E}}}} \preceq P_{\mathfrak{E}_{\mathcal{M}_{2\cdot\mathcal{E}}}} \preceq \cdots \preceq P_{\mathfrak{E}_{\mathcal{M}_{n\cdot\mathcal{E}}}}$. This filtration will be helpful for the study of Brain neural network.

3.3. *Algorithm 5: Brain Disease detection algorithm*

Step 1: Consider the DNG–MN corresponding to the image of a brain network.
Step 2: Find the vertex dilation induced subgraphs $G_i = \delta(v_i)$.
Step 3: Calculate the Eigen values of $A_\mu(G)$, $A_\kappa(G)$ and $A_\nu(G)$ and arrange these values in descending order and apply algorithm 1 to obtain the transformed $N(G_i)$.
Step 4: Apply algorithm 2 to obtain a directed transformed NG.
Step 5: Apply algorithm 3 and 4 to obtain spanning trees and find a filtration in the underlying graph of DNG–MN of the form $P_{\mathfrak{E}_{\mathcal{M}_{1\cdot\mathcal{E}}}} \preceq P_{\mathfrak{E}_{\mathcal{M}_{2\cdot\mathcal{E}}}} \preceq \cdots \preceq P_{\mathfrak{E}_{\mathcal{M}_{n\cdot\mathcal{E}}}}$.
Step 6: If there is no spanning trees then there is a chance of a disease in a region corresponding to any subgraphs.

3.4. *Optimal network and flow of neurons*

Consider a filtration in the underlying graph of DNG-MN of the form $P_{\mathfrak{E}_{\mathcal{M}_{1\cdot\mathcal{E}}}} \preceq P_{\mathfrak{E}_{\mathcal{M}_{2\cdot\mathcal{E}}}} \preceq \cdots \preceq P_{\mathfrak{E}_{\mathcal{M}_{n\cdot\mathcal{E}}}}$. Let this be a filtration of the subgraphs $G_1, \ldots, G_n$. Since each filtration is a spanning tree or paths, consider the maximum spanning tree in each subgraphs and then calculate the Span integrity [3] of these spanning tree/ paths. The Vertex Span integrity of DNG–MN with respect to dilation is defined as $(\text{Max}\,\mu_{\delta(v_i)} + S\mu_{G-\delta(v_i)}, \text{Min}\,\kappa_{\delta(vi)} + S\kappa_{G-\delta(v_i)}, \text{Ave}\,\gamma_{\delta(v_i)} + S\gamma_{G-\delta(v_i)})$ where $\text{Max}\,\mu_{\delta(vi)}$ is the maximum membership degree of a vertex w.r.to dilation of v_i, $\text{Min}\,\kappa_{\delta(v_i)}$, the minimum membership degree of a vertex w.r.to dilation of v_i, Min $\gamma_{\delta(vi)}$ is the average membership degree of a vertex w.r.to dilation of v_i and $S\mu_{G-\delta_{(vi)}}$ denotes the spanness [3] of G-$\delta(v_i)$ w. r.to membership degree, $S\kappa_{G-\delta(v_i)}$ is the spanness of $G-\delta(v_i)$ w. r.to the non membership degree and $S\gamma_{G-\delta(vi)}$ denotes the spanness of $G-\delta(v_i)$ w. r.to the indeterminancy membership degree. Similarly we can define the edge span integrity. The reordering of paths can be done w. r.to the values obtained. In application point of view, the subgraph ordering will give the information about region of a brain to be prioritized.

Algorithm 6: Ordering of subgraphs using DNG–MN and span integrity

Step 1: Apply Algorithms 1,2,3 and obtain DNG–MN.

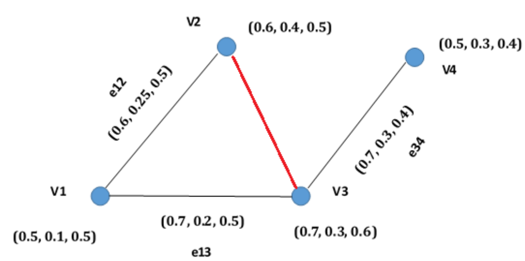
Step 2: Consider a filtration in the underlying graph of DNG–MN of the form $P_{\mathfrak{E}_{\mathcal{M}_{1\cdot\mathcal{E}}}} \preceq P_{\mathfrak{E}_{\mathcal{M}_{2\cdot\mathcal{E}}}} \preceq \cdots \preceq P_{\mathfrak{E}_{\mathcal{M}_{n\cdot\mathcal{E}}}}$.

Let this be a filtration of the subgraphs $G_1, \ldots . G_n$.

Step 3: Find the maximum spanning tree in each subgraphs and then calculate the Span integrity [3] of these spanning tree/paths.

Step 4: Order the subgraphs w. r.to the values obtained. i) μ ordering- w.r.to membership degree ii) κ ordering—w.r.to non membership degree iii) γ ordering—indeterminacy degree.

Illustration: Consider a DNG–MN



Max $\mu_{\delta(v_1)}$, Min $\kappa_{\delta(v1)}$, Ave $\gamma_{\delta(v_i)} = (0.7, 0.1, 0.5)$, $S_{G-\delta(v_1)} = (1.4, 0.2)$. So the span integrity w.r.to the dilation of v_1 is $(2.1, 0.3)$. Similarly calculate the Span integrity for other vertices. This will give an idea about the strength of regions. So, this method is useful for the prediction diseases like brain diseases.

Subgraph	Max $\mu_{\delta(v_i)}$, Min $\kappa_{\delta(v_i)}$, Ave $\gamma_{\delta(v_i)}$	$S_{G-\delta(v_i)}$	Span Integrity
G_1	$(0.7, 0.1, 0.5)$	$(0.5, 0.3, 0.4)$	$(1.2, 0.4, 0.9)$
G_2	$(0.6, 0.1, 0.53)$	$(1.2, 0.6, 1.0)$	$(1.8, 0.7, 1.53)$
G_3	$(0.7, 0.1, 0.5)$	$(0.6, 0.4, 0.5)$	$(1.3, 0.5, 1.0)$
G_4	$(0.7, 0.3, 0.5)$	$(1.1, 0.5, 1.0)$	$(1.8, 0.8, 1.5)$

μ ordering—w.r.to membership degree: $G_2 = G_4 > G_3 > G_1$

κ ordering—w.r.to non membership degree: $G_1 < G_3 < G_2 < G_4$

γ ordering—w.r.to indeterminacy degree: $G_1 < G_3 < G_2 < G_4$.

4. Discussions

4.1. *Data representation*

Several computational complexities arise while implementing the morphological operators like dilation, erosion etc. on neutrosophic graphs [20]. Since the input to the proposed system are images, converting the image data to the neutrosophic graph requires a sophisticated data structure for storing and processing various attributes. Also three weights have to be added to each node and edge of the neutrosophic graph made out of the image data. This leads to more intense calculations. Since every node and edge has indeterminacy factor, this may affect computation of structural elements like connected components and boundaries. If the size of the neutrosophic graph increases, then scalability issue can occur and that in turn increase the algorithmic complexity. Addressing these challenges requires sophisticated algorithms.

4.2. *AI and ML Technique*

AI and ML technique can be included in this proposed method by developing a Graph Neural network(GNN) which can be designed such that the image neutrosophic graph can be given as the input to the GNN and spanning trees are obtained as output of the GNN. By including AI and ML techniques, we can handle the uncertainty factor leading to more accurate predictions and insightful analysis.

4.3. *Decision making systems*

In Directed Neutrosophic Graphs(DNG), the direction can be modeled as the flow of information in the decision making process. Real world decision making consists of many uncertainty, imprecision and incomplete information which are well addressed here. As mentioned in section 3.3, DNG can be used in medical diagnosis. This also finds application in social networking where the user interaction are uncertain and directions shows the followers in the social network. Directional dependencies in the supply chain management, share trade and market trends etc. all of which suffers from uncertainty can be modeled using DNG. Also careful modeling can be done in sustainable resource management and disaster management. Autonomous vehicles where the directional decisions are really important can be modeled as a DNG.

4.4. *Optimization Methods*

Just as alpha-beta pruning which is a common method in decision trees in AI, T-I-F pruning can be included, which prunes the irrelevant sections in the DNG which are below the threshold values for T-I-F. This reduces the search time in the graph. Vertices with similar T-I-F values can be combined to a single vertex there by reducing the size of the graph. Parallel computing can be implemented by dividing the DNG in to subgraphs and processing them simultaneously using parallel systems.

5. Conclusion

A novel method to construct neutrosophic directed graphs is presented in this paper. Algorithms to construct induced subgraphs and spanning trees are given. The concept of $\mathcal{M}$ induced sequences plays an important role in the study of Neutrosophic graphs. Filtration and DNG-MN are useful in the study of applications of Neutrosophic graphs. Some applications are also discussed in this paper. This work can be extended to the study of Neutrosophic graphs, Hypergraphs and Plithogenic Hyper graphs using other induced Morphological operators.

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Received: July 3, 2025. Accepted: Sep 10, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

Neutrosophic Exponential Ratio-Type Estimator for Finite Population Mean in Stratified Sampling

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Abstract: This paper introduces an innovative neutrosophic exponential ratio-type estimator for estimating finite population means in stratified sampling environments with indeterminate data. Building upon classical exponential estimators and neutrosophic statistics, we develop a robust estimator that effectively handles uncertainty through interval-valued representations. The proposed estimator combines the strengths of exponential ratio estimation with neutrosophic weighting to achieve enhanced precision in stratified sampling scenarios. We derive the bias and mean square error (MSE) expressions under first-order approximation and demonstrate through empirical analysis using climate data that our estimator outperforms existing neutrosophic stratified estimators in terms of efficiency and reliability.

Keywords: Neutrosophic statistics, stratified sampling, exponential ratio estimator, auxiliary information, interval data, mean square error

1. Introduction

Stratified sampling remains a cornerstone technique in survey methodology, offering improved precision through population stratification. Traditional approaches often assume precise data measurements; however, real-world applications frequently encounter indeterminate or uncertain information. Neutrosophic statistics provides a powerful framework for such scenarios by representing observations as intervals $Z_N = [Z_L, Z_U] = Z_L + Z_U I_N$, where $I_N \in [I_L, I_U]$ captures the indeterminacy component [2]. The integration of neutrosophic theory with sampling estimators has been explored to handle uncertainty explicitly [4,5,6]. Existing neutrosophic stratified estimators have shown promise, yet challenges remain in improving efficiency and adaptability to varying levels of data indeterminacy [3,4,7]. Recent developments in neutrosophic sampling theory [8,9] have expanded the toolkit available for handling indeterminate data in survey contexts.

This study contributes by developing a novel neutrosophic exponential ratio-type estimator tailored specifically for stratified sampling frameworks with indeterminate data. We provide theoretical derivations of the estimator's bias and mean square error under first-order approximations. The performance is empirically validated with real climate data, demonstrating superior efficiency over classical and neutrosophic existing estimators [1,5,11]. Furthermore, we propose an optimal weighting scheme to minimize mean square error, enhancing applicability in practical survey settings. Neutrosophic statistics uniquely address the uncertainty and indeterminacy present in many real-world data collection processes, especially where interval-valued or imprecise observations occur [2,10]. By integrating neutrosophic principles into stratified sampling estimators, survey practitioners gain a robust tool to produce reliable estimates with quantified uncertainty, facilitating more informed decision-making in fields such as climatology, economics, and social sciences [4,8].

The rest of the paper is structured as follows: Section 2 provides the methodology including notation, section 3 gives existing estimators with their MSE expressions, and the proposed estimator formulation is given in section 4. Section 5 covers empirical analysis using real climate datasets. Finally, Section 6 concludes the paper with findings and future research directions.

2. Notations and Framework

Consider a finite neutrosophic population of size $N_N \in [N_L, N_U]$, partitioned into L homogeneous strata with sizes $N_{hN} \in [N_{hL}, N_{hU}]$, $h = 1, \dots, L$, such that

$$\sum_{h=1}^{L} N_{hN} = N_N.$$

From each stratum, a neutrosophic simple random sample of size $n_{hN} \in [n_{hL}, n_{hU}]$ is drawn without replacement, with total sample size

$$n_N = \sum_{h=1}^{L} n_{hN} \in [n_L, n_U].$$

Let $Y_{hN} \in [Y_{hL}, Y_{hU}]$ be the study variable and $X_{hN} \in [X_{hL}, X_{hU}]$ the auxiliary variable in stratum h. Key population parameters are:

$$\bar{Y}_{hN} = \frac{1}{N_{hN}} \sum_{j=1}^{N_{hN}} Y_{hjN} \in [\bar{Y}_{hL}, \bar{Y}_{hU}],$$

$$\bar{X}_{hN} = \frac{1}{N_{hN}} \sum_{j=1}^{N_{hN}} X_{hjN} \in [\bar{X}_{hL}, \bar{X}_{hU}],$$

$$S_{yhN}^2 = \frac{1}{N_{hN} - 1} \sum_{j=1}^{N_{hN}} \left(Y_{hjN} - \bar{Y}_{hN}\right)^2 \in \left[S_{yhL}^2, S_{yhU}^2\right],$$

$$\rho_{hN} = \mathrm{Corr}(Y_{hN}, X_{hN}) \in [\rho_{hL}, \rho_{hU}],$$

where weights $w_{hN} = \frac{N_{hN}}{N_N}$.

3. Existing Neutrosophic Stratified Estimators and Their MSEs

This subsection summarizes neutrosophic stratified estimators adapted from classical sampling theory and their respective mean square error (MSE) expressions. These form the basis for comparison with the proposed estimator.

- **Stratified Mean Estimator [1]:**

$$T_{0N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{hN},$$

with MSE

$$\mathrm{MSE}(T_{0N}) = \sum_{h=1}^{L} w_{hN}^2\, \theta_{hN} S_{yhN}^2.$$

- **Stratified Ratio Estimator [3]:**

$$T_{1N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{hN} \frac{\bar{X}_{hN}}{\bar{x}_{hN}},$$

with MSE

$$\mathrm{MSE}(T_{1N}) = \sum_{h=1}^{L} w_{hN}^2\, \theta_{hN} \left[S_{yhN}^2 + R_{hN}^2 S_{xhN}^2 - 2R_{hN}\rho_{hN} S_{yhN} S_{xhN}\right].$$

- **Stratified Regression Estimator [4]:**

$$T_{2N} = \sum_{h=1}^{L} w_{hN}\, [\bar{y}_{hN} + b_{hN}(\bar{X}_{hN} - \bar{x}_{hN})],$$

with MSE

$$\mathrm{MSE}(T_{2N}) = \sum_{h=1}^{L} w_{hN}^2\, \theta_{hN} S_{yhN}^2 (1 - \rho_{hN}^2).$$

- **Stratified Exponential Estimator [1] :**

$$T_{3N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{hN} \exp\left(\frac{\bar{X}_{hN} - \bar{x}_{hN}}{\bar{X}_{hN} + \bar{x}_{hN}}\right),$$

with MSE

$$\mathrm{MSE}(T_{3N}) = \sum_{h=1}^{L} w_{hN}^2\, \theta_{hN} \left[S_{yhN}^2 + \frac{1}{4} R_{hN}^2 S_{xhN}^2 - R_{hN}\rho_{hN} S_{yhN} S_{xhN}\right].$$

Here,

$$\theta_{hN} = \left(\frac{1}{n_{hN}} - \frac{1}{N_{hN}}\right) \in [\theta_{hL}, \theta_{hU}], \quad R_{hN} = \frac{\bar{Y}_{hN}}{\bar{X}_{hN}} \in [R_{hL}, R_{hU}].$$

4. Proposed Estimator

We introduce a neutrosophic stratified exponential ratio-type estimator:

$$T_{\text{Prop}N} = \sum_{h=1}^{L} w_{hN}\, \bar{y}_{hN} \left[\alpha_{hN} + (1-\alpha_{hN})\frac{\bar{X}_{hN}}{\bar{x}_{hN}}\right] \exp\left(\frac{\bar{X}_{hN} - \bar{x}_{hN}}{\bar{X}_{hN} + \bar{x}_{hN}}\right),$$

where $\alpha_{hN} \in [\alpha_{hL}, \alpha_{hU}]$ are neutrosophic weighting parameters.

To derive the bias and MSE of the proposed estimator, we define:

$$\bar{y}_{hN} = \bar{Y}_{hN}(1+e_{yhN}), \quad \bar{x}_{hN} = \bar{X}_{hN}(1+e_{xhN}),$$

where $E(e_{yhN}) = E(e_{xhN}) = 0$, and

$$E(e_{yhN}^2) = \theta_{hN} C_{yhN}^2, \quad E(e_{xhN}^2) = \theta_{hN} C_{xhN}^2, \quad E(e_{yhN} e_{xhN}) = \theta_{hN} \rho_{hN} C_{yhN} C_{xhN}.$$

The proposed estimator can be rewritten as:

$$T_{\text{Prop}N} = \sum_{h=1}^{L} w_{hN}\, \bar{Y}_{hN}(1+e_{yhN})[\alpha_{hN} + (1-\alpha_{hN})(1+e_{xhN})^{-1}]\exp\left(\frac{-e_{xhN}}{2+e_{xhN}}\right).$$

Using Taylor series expansion and retaining terms up to second order:

$$\begin{aligned} T_{\text{Prop}N} &\approx \sum_{h=1}^{L} w_{hN}\, \bar{Y}_{hN}(1+e_{yhN})[\alpha_{hN} + (1-\alpha_{hN})(1-e_{xhN}+e_{xhN}^2)]\left(1-\frac{e_{xhN}}{2}+\frac{3e_{xhN}^2}{8}\right) \\ &\approx \sum_{h=1}^{L} w_{hN}\, \bar{Y}_{hN}(1+e_{yhN})\left[1+\left(\frac{3}{2}-\alpha_{hN}\right)e_{xhN} + \left(\left(\frac{3}{2}-\alpha_{hN}\right)^2 - \frac{1}{2}\right)e_{xhN}^2\right]. \end{aligned}$$

Taking expectation, we get

$$\begin{aligned} \text{Bias}(T_{\text{Prop}N}) &\approx \sum_{h=1}^{L} w_{hN}\, \bar{Y}_{hN}\theta_{hN}\left[\left(\frac{3}{2}-\alpha_{hN}\right)C_{xhN}^2 - (1-\alpha_{hN})\rho_{hN}C_{yhN}C_{xhN}\right], \\ \text{MSE}(T_{\text{Prop}N}) &\approx \sum_{h=1}^{L} w_{hN}^2\, \theta_{hN}\bar{Y}_{hN}^2\left[C_{yhN}^2 + \left(\frac{3}{2}-\alpha_{hN}\right)^2 C_{xhN}^2 - 2\left(\frac{3}{2}-\alpha_{hN}\right)\rho_{hN}C_{yhN}C_{xhN}\right], \end{aligned}$$

where $C_{yhN} = \frac{S_{yhN}}{\bar{Y}_{hN}}$ and $C_{xhN} = \frac{S_{xhN}}{\bar{X}_{hN}}$ are coefficients of variation.

The optimal weight minimizing MSE is given by:

$$\alpha_{hN}^{\text{opt}} = \frac{3}{2} - \frac{\rho_{hN}C_{yhN}}{C_{xhN}}.$$

5. Empirical Analysis

5.1 Data Description

We evaluate our estimator using climate data from Alabama and Georgia (November measurements), treating:

Dew Point Temperature as auxiliary variable $X_{hN} \in [X_{hL}, X_{hU}]$,

Relative Humidity as study variable $Y_{hN} \in [Y_{hL}, Y_{hU}]$.

Table 1 presents key parameters for both classical and neutrosophic cases.

Table 1: Stratified population parameters for climate data analysis

Stratum	Neutrosophic Lower	Neutrosophic Upper	Classical
	Alabama (Stratum 1)		
N_{1N}	19	19	19
n_{1N}	6	6	6
$\bar{X}_{1N}$	19.58	61.95	40.77
$\bar{Y}_{1N}$	28.21	96.47	62.34
ρ_{1N}	0.946	0.941	0.889
	Georgia (Stratum 2)		
N_{2N}	22	22	22
n_{2N}	7	7	7
$\bar{X}_{2N}$	22.55	62.23	42.39
$\bar{Y}_{2N}$	31.77	93.86	62.82
ρ_{2N}	0.885	0.948	0.859

5.2 Performance Comparison

Tables 2 and 3 show Mean Square Error and Relative Efficiency results, respectively, comparing the proposed estimator with existing methods.

Table 2: Mean Square Error comparison

Estimator	Neutrosophic Lower	Neutrosophic Upper	Classical
T_{0N}	8.26	26.20	5.94
T_{1N}	2.03	2.07	1.01
T_{2N}	0.58	1.01	0.86
T_{3N}	0.75	1.25	0.92
$T_{\text{Prop}N}$	0.42	0.78	0.63

Table 3: Relative Efficiency comparison

Estimator	Neutrosophic Lower	Neutrosophic Upper	Classical
T_{0N}	1.00	1.00	1.00
T_{1N}	4.07	12.66	5.88
T_{2N}	14.24	25.94	6.91
T_{3N}	11.01	20.96	6.46
$T_{\text{Prop}N}$	19.67	33.59	9.43

6. Conclusion

The proposed neutrosophic stratified exponential ratio-type estimator demonstrates significant advantages over existing approaches in handling indeterminate data within stratified sampling frameworks. Our theoretical analysis and empirical results establish that the estimator achieves superior efficiency, with relative efficiency values ranging from 19.67 to 33.59 compared to conventional neutrosophic stratified estimators. The methodology exhibits robust performance across varying levels of indeterminacy, making it particularly suitable for real-world applications where data uncertainty is inherent. The estimator's theoretical foundation, including the derived optimal weighting scheme and first-order approximation properties, ensures its statistical soundness while maintaining practical implementability.

These findings suggest that the integration of exponential ratio structures with neutrosophic weighting in stratified sampling contexts offers a promising direction for handling indeterminate data in survey sampling. Future research should explore extensions to multi-auxiliary variable scenarios, development of neutrosophic calibration estimators, and applications in more complex sampling designs, which could further enhance the methodology's utility in practical survey applications.

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Received: July 5, 2025. Accepted: Sep 9, 2025

NSS
Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

Stable Neutrosophic Crisp Open Modulo Stable Neutrosophic Crisp Nowhere Dense Sets

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Abstract: Using the symmetric difference between the neutrosophic crisp set (NC–set) and stable Neutrosophic crisp open set (SNCO–set). It represents the complement of the quotient of the NC–set, SNCO–set, vice versa and linking with the family of stable Neutrosophic crisp (SNC) nowhere dense sets. They given the important properties of this concepts, we used it to build the stable Neutrosophic crisp open modulo SNC-nowhere dense sets. The results expiated amazing mathematical effects. This is what we have presented in this paper.
Keywords: Neutrosophic crisp sets, SNC– nowhere dense sets, SNCT–space, SNC – boundary set.

1. Introduction

The study is with in the neutrosophic crisp (NC-sets) path conditional on $\Upsilon^{\aleph} = \prec \Upsilon_1, \Upsilon_2, \Upsilon_3 \succ$ where $\Upsilon_i \cap \Upsilon_j = \emptyset$, $\forall$ i≠j. We also defined the type of intersection, union, complement, partial relations, and points in neutrosophic crisp sets was known by Salame [6,7] as more than one type under certain conditions. Therefore, in the case of studing any concept, we must determent those relations as well as the type of topology that we are working on, whence we take the stable neutrosophic crisp topological space (SNCT – space) [8]. The collection $\tau_X^{\aleph}$ of NC – sets satisfies, $\phi_3^{\aleph}$, $X_1^{\aleph} \in \tau_X^{\aleph}$ if $L^{\aleph}$, $K^{\aleph} \in \tau_X^{\aleph}$ $\exists$ $H^{\aleph} \in \tau_X^{\aleph} \ni H^{\aleph} \subseteq_1 L^{\aleph} \cap_2 K^{\aleph}$ and $\forall$ index Λ, $K_{\lambda}^{\aleph} \in \tau_X^{\aleph}$ $\exists$ $H^{\aleph} \in \tau_X^{\aleph} \ni H^{\aleph} \subseteq_1 \cup_{2\,\lambda\in\Lambda} K_{\lambda}^{\aleph}$. Whence for any $L^{\aleph} \in \tau_X^{\aleph}$ is said to be SNCO – set and it is complement is SNCC – set. It is worth noting that we would like to point out that there are those who have set a synonymous definition for neutrosophic crisp set [2], as follows $\prec \Upsilon_1, \ldots, \Upsilon_n$, $B_1, \ldots\ldots, B_n$, $C_1, \ldots,$ $C_n \succ$. There are many varied instruction's for these sets. For examples, you can see [1,5].

2. Operation Neutrosophic Crisp Set Theory

The operation on which our concepts are based on ;

- The union $\prec \Upsilon_1, \Upsilon_2, \Upsilon_3 \succ \cup_2 \prec A_1, A_2, A_3 \succ = < \Upsilon_1 \cup A_1$, $\Upsilon_2 \cap A_2$, $\Upsilon_3 \cap A_3 >$.
- The intersection $\prec \Upsilon_1$, $\Upsilon_2, \Upsilon_3 \succ \cap_2 \prec A_1, A_2, A_3 \succ = < \Upsilon_1 \cap A_1$, $\Upsilon_2 \cup A_2$, $\Upsilon_3 \cup A_3 >$.
- $A^{\aleph} \subseteq \Upsilon^{\aleph}$ iff $A_1 \subseteq \Upsilon_1$, $A_2 \subseteq \Upsilon_2$, and $\Upsilon_3 \subseteq A_3$.
- The complement $(\Upsilon^{\aleph})^c = \prec \Upsilon_1^c$, Υ_2^c , $\Upsilon_3^c \succ$.

- The empty set $\emptyset^{\aleph} = \prec \phi, X, X \succ$.
- The universal set $X^{\aleph} = \prec X, \phi, \phi \succ$.
- The NC – points are :
- $P^{\aleph 1} = \prec \{P\}, \phi, \{P\}^c \succ \in_1 K^{\aleph}$ iff $p \in K_1, P \notin K_3$.
- $P^{\aleph 2} = \prec \phi, \{P\}, \{P\}^c \succ \in_2 K^{\aleph}$ iff $p \in K_2, P \notin K_3$.
- $P^{\aleph 3} = \prec \{P\}, \phi, \phi \succ \in_3 K^{\aleph}$ iff $p \in K_1$.
- $P^{\aleph 4} = \prec \phi, \{P\}, \phi \succ \in_4 K^{\aleph}$ iff $p \in K_2$.
- The SNC – interior and SNC – closure sets [8]

  $\text{Si}(Ą^{\aleph}) = \cup \{ Y^{\aleph} \in \tau_X^{\aleph} ; Y^{\aleph} \subseteq Ą^{\aleph} \}$,

  $\text{Scl}(Ą^{\aleph}) = \cap \{ (Y^{\aleph})^c \in \tau_X^{\aleph} ; Y^{\aleph} \subseteq F^N \}$.

It is noted that it is not necessary that the intersection of any two SNCO – sets is SNCO – set . Also the union of any collection of SNCD – sets is not necessary be SNCO – set .

The SNCT – space is called advance if it is closed under the intersection , and is called soon to arrive if it is closed under the union .

The quotient ; $Ą^{\aleph} : Y^{\aleph} = Ą^{\aleph} \cup (Y^{\aleph})^c = [Y^{\aleph} - Ą^{\aleph}]^c$.

The symmetric difference ;

- $Ą^{\aleph} \dot{-} Y^{\aleph} = (Ą^{\aleph} - Y^{\aleph}) \cup (Y^{\aleph} - Ą^{\aleph}) = [(Ą^{\aleph} : Y^{\aleph}) \cap (Y^{\aleph} : Ą^{\aleph})]^c$
- $Ą^{\aleph} \dot{-} (J^N \dot{-} Y^{\aleph}) = (Ą^{\aleph} \cap J^N) \dot{-} Y^{\aleph}$.
- $Ą^{\aleph} \cap (J^N \dot{-} K^{\aleph}) = (Ą^{\aleph} \cap J^N) \dot{-} (Ą^{\aleph} \cap K^{\aleph})$.
- $(Ą^{\aleph} : Y^{\aleph}) \cap Y^{\aleph} = Ą^{\aleph} \cap Y^{\aleph}$.

3. SNC – Open Sets Modulo SNC – nowhere Dense Sets

The property of a set to be SNC-open modulo SNC-nowhere dense is also called the Baire property (in the large sense). We will reiew properties of this concept and the effect of neutrosophic crisp sets on it.

Definition 3 – 1 : Let $(X^{\aleph}, \tau_X^{\aleph})$ ***be*** SNCT-space , then ;

1 – $Ą^{\aleph}$ is a SNC – boundary set if $\text{Si}(Ą^{\aleph}) = \emptyset^{\aleph}$.

- $\text{Scl}((Ą^{\aleph})^c) = X^{\aleph}$.
- $\text{Scl}(Ą^{\aleph})$ is SNC – boundary iff $\text{Scl}(Ą^{\aleph}) \subseteq \text{Scl}[\text{Scl}((Ą^{\aleph})^c)]$.

2 – $Ą^{\aleph}$ is a SNC – nowhere dense set if $\text{Si}(\text{Scl}(Ą^{\aleph})) = \emptyset^{\aleph}$.

- If $\text{Scl}(Ą^{\aleph})$ is SNC – boundary , then $Ą^{\aleph}$ is SNC – nowhere dense .

Proposition 3 – 2 : Let $(X^{\aleph}, \tau_X^{\aleph})$ be advance SNCT – space. $Ą^{\aleph}$ is SNC – nowhere dense set iff $\exists$ $Y^{\aleph} \varepsilon \tau_X^N \setminus \{\phi^N\} \ni Y^{\aleph} \cap Ą^{\aleph} = \emptyset^{\aleph}$.

Lemma 3 – 3 : The union of a SNC – boundary set and SNC – nowhere dense set is a SNC – boundary set.

Proposition 3 – 4 : If $(X^{\aleph}, \tau_X^{\aleph})$ is a soon to arrive , then the union of two SNC – nowhere dense is a SNC – nowhere dense set .

Proof . Since $\tau_X^{\aleph}$ is a soon to arrive , whence $\text{Scl}(Ą^{\aleph})$ and $\text{Scl}(K^{\aleph})$ are SNCC – sets , and they are SNC – boundary , if $Ą^{\aleph}$ and $K^{\aleph}$ are SNC – nowhere dense , also By lemma3 – 3 the union of them is SNC - boundary , it follows that Ą^{ℵ} U $K^{ℵ}$ is SNC – nowhere dense set , because Scl (Scl (Ą^{ℵ})) = Scl (Ą^{ℵ}) . Similarly to $K^{ℵ}$.

Remark 3 – 5:

1 - If ϒ^{ℵ} is SNCO – set and SNCC – set , then SNCFr (ϒ^{ℵ}) is SNC – nowhere dense , where SNCFr (ϒ^{ℵ}) = Scl (ϒ^{ℵ}) ∩ Scl ((ϒ^{ℵ})^{c}) .

2 - If Scl (ϒ^{ℵ}) U (ϒ^{ℵ} ∩ Scl ($K^{ℵ}$)) = ∅^{ℵ}, then Scl (ϒ^{ℵ}) ∩ Scl ($K^{ℵ}$) is a SNC - nowhere dense set .

Definition 3 – 6 :Let $(X^{ℵ},\tau_X^{ℵ})$ be a SNCT-space then ;

- NC – set ϒ^{ℵ} ⊆ $K^{ℵ}$ is a SNC – boundary set of $K^{ℵ}$, if $K^{ℵ}$ ⊆ Scl ($K^{ℵ}$ - ϒ^{ℵ}).
- NC – set ϒ^{ℵ} ⊆ $K^{ℵ}$ is a SNC – nowhere dense set of $K^{ℵ}$, if $K^{ℵ}$ ⊆ Scl [$K^{ℵ}$ – Scl (ϒ^{ℵ})] .

Proposition 3 – 7 . For any SNCT-space $(X^{ℵ},\tau_X^{ℵ})$ we have ;

1 -If ϒ^{ℵ} is SNC – nowhere dense in Scl ($K^{ℵ}$) , then $K^{ℵ}$ ∩ ϒ^{ℵ} is SNC – nowhere dense in $K^{ℵ}$.

2 – Scl ($K^{ℵ}$) – $K^{ℵ}$ is a SNC – boundary set in Scl ($K^{ℵ}$) .

3 – $K^{ℵ}$ ∩ Si [SNCFr ($K^{ℵ}$)] is a SNC – dense and SNC – boundary in Si [SNCFr ($K^{ℵ}$)].

4 - If ϒ^{ℵ} is a SNC – boundary (SNC – nowhere dense) set in $K^{ℵ}$, SO is ϒ^{ℵ} relative to every set containing K^{N} .

Theorem 3 – 8 . Let $(X^{ℵ},\tau_X^{ℵ})$ be a SNCT-space .If Ą^{ℵ} is a SNCO – set and $K^{ℵ}$ is SNC - boundary (SNC – nowhere dense) set , then SO is Ą^{ℵ} ∩ $K^{ℵ}$ is a SNC – boundary set in Ą^{ℵ} (SNC-nowhere dense set in Scl (Ą^{ℵ})) .

Proof . If $K^{ℵ}$ is a SNC - boundary, then $K^{ℵ}$ ⊆ Scl (Ą^{ℵ} - $K^{ℵ}$), whence $K^{ℵ}$ ∩ Ą^{ℵ} ⊆ Ą^{ℵ} ∩ Scl [($K^{ℵ}$)^{c}] ⊆ Scl [Ą^{ℵ} ∩ ($K^{ℵ}$ ∩ (Ą^{ℵ})^{c})], and therefor $K^{ℵ}$ ∩ Ą^{ℵ} is a SNC - boundary set in L^{N} . Similarly , if $K^{ℵ}$ is a SNC – nowhere dense set , whence $K^{ℵ}$ ∩ Ą^{ℵ} is a SNC - nowhere dense set in Ą^{ℵ} . Also $K^{ℵ}$ ∩ Ą^{ℵ} ⊆ Scl [Ą^{ℵ} ∩ (Scl (Ą^{ℵ} ∩ $K^{ℵ}$))^{c})] ⊆ Scl [Scl (Ą^{ℵ}) ∩ (Scl(L^{N} ∩ $K^{ℵ}$))^{c}], and hence Ą^{ℵ} ∩ $K^{ℵ}$ is a SNC – nowhere dense to Scl (Ą^{ℵ}) , but Scl (Ą^{ℵ}) – Ą^{ℵ} is a SNC – boundary set in Scl (Ą^{ℵ}), whence K^{N}∩Scl (Ą^{ℵ}) ∩(Ą^{ℵ})^{c}, is SNC – boundary set in Scl(Ą^{ℵ}). By proposition 3–4. $K^{ℵ}$ ∩Scl (Ą^{ℵ}) = ($K^{ℵ}$ ∩Ą^{ℵ})U($K^{ℵ}$ ∩ Scl (Ą^{ℵ})∩(Ą^{ℵ})^{c})is a SNC–nowhere dense set in Scl (Ą^{ℵ}).

Definition 3 – 9 . $K^{ℵ}$ is said SNC- open (closed) modulo SNC- nowhere dense if there is an SNCO- set (SNCC- set) Ą^{ℵ}϶ $K^{ℵ}$ – Ą^{ℵ} and Ą^{ℵ} – $K^{ℵ}$ are SNC- nowhere dense (written ($K^{ℵ}$ ∼ Ą^{ℵ})).

Example.

Take ϒ = { n , g ,q }, $\tau_{Y}^{ℵ}$ = { ∅^{ℵ} , ϒ^{ℵ} , Ą^{ℵ} , Ƀ^{ℵ} , Ȼ^{ℵ} }, where Ą^{ℵ}= <{n},{g},{q} >, Ƀ^{ℵ}= <{n},{g , q}, ф > and Ȼ^{ℵ}= < ф, ф,{q} > , then $K^{ℵ}$ = < ф , {n , g} , {q} > is SNC- open modulo SNC- nowhere dense to Ą^{ℵ} , because $K^{ℵ}$ – Ą^{ℵ} and Ą^{ℵ} – $K^{ℵ}$ are SNC- nowhere dense sets.

Remark 3 – 10 . Let $(X^{ℵ},\tau_X^{ℵ})$ be a SNCT-space ,the following propositions are easily proved:

- Each SNC- closed (SNC- open) set is SNC- open (SNC- closed) modulo SNC- nowhere dense.
- If $K^{ℵ}$ ∼ Ą^{ℵ} and $K^{ℵ}$ ∼ Ƀ^{ℵ}, then $K^{ℵ}$ ∼ Ƀ^{ℵ} , where $K^{ℵ}$, Ą^{ℵ} , Ƀ^{ℵ} ϵ $\tau_X^{ℵ}$.
- Let $L_1^{ℵ}$, $L_2^{ℵ}$, $K_1^{ℵ}$, $K_2^{ℵ}$ ϵ $\tau_X^{ℵ}$ and $L_1^{ℵ}$ ∼ $L_2^{ℵ}$ and $K_1^{ℵ}$ ∼ $K_2^{ℵ}$, then

1-($L_1^{\aleph}$ U $K_1^{\aleph}$) ~ ($L_2^{\aleph}$ U $K_2^{\aleph}$),

2- ($L_1^{\aleph}$ ∩ $K_1^{\aleph}$) ~ (L_2^{N} ∩ $K_2^{\aleph}$),

3- $L_1^{\aleph} - K_1^{\aleph}$) ~ ($L_2^{N} - K_2^{\aleph}$),

- $K^{\aleph}$ ~ Ą$^{\aleph}$ iff $K^{\aleph}$is of the form $K^{\aleph}$ = (Ą$^{\aleph}$– Ƥ$^{\aleph}$) U Ơ$^{\aleph}$ where Ƥ$^{\aleph}$, Ơ$^{\aleph}$ are SNC – nowhere dense sets. For if $K^{\aleph}$ ~ Ą$^{\aleph}$, put Ƥ$^{\aleph}$= Ą$^{\aleph}$ – $K^{\aleph}$ and Ơ$^{\aleph}$ = $K^{\aleph}$ – Ą$^{\aleph}$, whence $K^{\aleph}$ = (Ą$^{\aleph}$ –Ƥ$^{\aleph}$) U Q^{N}. Conversely , $K^{\aleph}$ – Ą$^{\aleph}$ = [(Ą$^{\aleph}$ – Ƥ$^{\aleph}$) U (Ơ$^{\aleph}$)^{c}] – Ą$^{\aleph}$ ⊆ Ơ$^{\aleph}$ is SNC – nowhere dense set, also Ą$^{\aleph}$ - $K^{\aleph}$= Ą$^{\aleph}$ - [(Ą$^{\aleph}$ – Ƥ$^{\aleph}$) U Ơ$^{\aleph}$)^{c}] ⊆ Ƥ$^{\aleph}$ is a SNC- nowhere dense , therefor $K^{\aleph}$ ~ Ą$^{\aleph}$.

- The family of all SNC – open set module is field, in other word if Ƥ$^{\aleph}$ and $K^{\aleph}$ are SNC –open sets modulo SNC- nowhere dense sets , then so are Ƥ$^{\aleph}$ U $K^{\aleph}$, Ƥ$^{\aleph}$ ∩ $K^{\aleph}$, and (Ƥ$^{\aleph}$)^{C} .

Proposition 3 – 11. Ƥ$^{\aleph}$ is SNC- open modulo SNC- nowhere dense iff there is an SNCO- set Ơ$^{\aleph}$ ⊆ Ƥ$^{\aleph}$ and Ƥ$^{\aleph}$ – Ơ$^{\aleph}$ is a SNC – nowhere dense set , where (X , τ_X^N) is advance SNCT – space.

Proof . Suppose that Ƥ$^{\aleph}$ – $K^{\aleph}$ and $K^{\aleph}$ – Ƥ$^{\aleph}$ are SNC – nowhere dense sets . Put Ơ$^{\aleph}$ = Ƥ$^{\aleph}$ ∩Si ($K^{\aleph}$) ⊆ $K^{\aleph}$, whence Ƥ$^{\aleph}$ – Ơ$^{\aleph}$ = (Ƥ$^{\aleph}$ – $K^{\aleph}$) U (Ƥ$^{\aleph}$ U [Si (Ƥ$^{\aleph}$)]^{c}) =(Ƥ$^{\aleph}$– $K^{\aleph}$)U ($K^{\aleph}$ ∩ Ƥ$^{\aleph}$ ∩ Scl ((Ƥ$^{\aleph}$)^{c})) ⊆ (Ƥ$^{\aleph}$– $K^{\aleph}$) U Scl($K^{\aleph}$ – Ƥ$^{\aleph}$) is a SNC – nowhere dense set . Conversely , let Ơ$^{\aleph}$ be a SNCO – set ∍ Ơ$^{\aleph}$ ⊆ Ƥ$^{\aleph}$ and Ƥ$^{\aleph}$– Ơ$^{\aleph}$ is a SNC- nowhere dense and Ơ$^{\aleph}$ ⊂ Ƥ$^{\aleph}$,so Ơ$^{\aleph}$– Ƥ$^{\aleph}$ = ∅$^{\aleph}$ is a SNC – nowhere dense set and therefor Ƥ$^{\aleph}$ is SNC – open modulo SNC – nowhere dense set .

Proposition 3 – 12. L^{N} is SNC- open modulo SNC- nowhere dense iff SNCFr(Ƥ$^{\aleph}$) is a SNC – nowhere dense set .

Proof . By Proposition 3 – 11 there is an SNCO- set $H^{\aleph}$ ⊆ Ƥ$^{\aleph}$ and ($H^{\aleph}$)^{c} ∩ Ƥ$^{\aleph}$ is a SNC – nowhere dense, put Ʋ$^{\aleph}$= ($H^{\aleph}$)^{c} ∩ Ƥ$^{\aleph}$, whence Ƥ$^{\aleph}$ = H^{N} U Ʋ$^{\aleph}$, and therefor SNCFr(Ƥ$^{\aleph}$) = SNCFr($H^{\aleph}$) U SNCFr(Ʋ$^{\aleph}$) , and by Remark 3 – 5(1) SNCFr($H^{\aleph}$) is a SNC-nowhere dense set also SNCFr(Ʋ$^{\aleph}$) is a SNC- nowhere dense , hence SNCFr(Ƥ$^{\aleph}$) is a SNC- nowhere dense set.

Conversely, Ƥ$^{\aleph}$= Si (Ƥ$^{\aleph}$) U [Ƥ$^{\aleph}$∩ SNCFr(Ƥ$^{\aleph}$)]. Put $H^{\aleph}$Si(Ƥ$^{\aleph}$), whence Ƥ$^{\aleph}$∩ ($H^{\aleph}$)^{c} ⊆ SNCFr (Ƥ$^{\aleph}$). It follows that Ƥ$^{\aleph}$ is SNC- open module SNC – nowhere dence.

Corollary 3-13. $H^{\aleph}$ is SNC- open modulo SNC- nowhere dense iff $H^{\aleph}$is difference of a SNCC – set and SNC - nowhere dense set

4. Conclusions

There is a proven fact in Neutrosophic crisp set theory which is, if $K^{\aleph}$is a NC-set , then is not necessarily $(K^{\aleph})^{c}$ is satisfy any condition of NC-set, for example $(\prec \{u\}, \emptyset, \{v, g\} \succ)^{c} = \prec \{v, g\}, X, \{u\} \succ$is non—NC-set , with X={ u,v,g } .Therefore , NCC-sets are not necessarily NC-sets for any Neutrosophic crisp topological spaces . Then they should be treated with extreme caution because Neutrosophic crisp algebra operations are not necessarily valid on them .This is considered one on the strengths of the NC-sets , which summarizes the relationship between them and the non-NC-sets . which has provided them with broad horizons in all branches of mathematics .

So we choose to study the nowhere dense , boundary and open modulo sets on SNCT- spaces , that hae the ability to handle it , because the union and intersection of any two SNCO-sets (SNCC-sets) does not have to be SNCO-set (SNCC-set). Also , each NCT-space is SNCT-space but the converse may be not .

This research has established a basic foundation for studying the concept of SNC-first category set (it is the union of a countable sequence of SNC-nowhere dense sets), as it is considered a future work. In addition to studying the concepts SNC-dense in itself , SNC-scattered sets and SNC-discrete sets. In addition to the concepts in [3,4]. We can also study other neutrosophic crisp topological concepts, especially those works discussed by papers in [9-13] on stable neutrosophic crisp topological spaces which have further extended and details.

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Received: July 7, 2025. Accepted: Sep 11, 2025

Enhancing Decision Making with Interval - Valued Refined Neutrosophic Hesitant Fuzzy Sets

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Abstract: The Interval-Valued Refined Neutrosophic Hesitant Fuzzy Set (IVRNHFS) is a sophisticated mathematical set that is capable of capturing and handling complicated uncertainty, hesitancy, and vagueness in decision-making problems. In this paper, the formal definition of IVRNHFS is proposed by combining the refined aspects of truth, indeterminacy, and falsity with interval-valued hesitant information, thus providing a richer and more flexible representation of imprecise data. To facilitate practical use, new score and accuracy functions are constructed to facilitate useful comparison and ranking of alternatives. Moreover, fundamental operations like union, intersection, complement, and aggregation are redefined in the IVRNHFS context to maintain the natural interval-based refinement and hesitation. The proposed framework is demonstrated with examples and compared with existing models to show its expressiveness, computational depth, and decision correctness advantage.

Keywords: Interval Valued Refined Neutrosophic Hesitant Fuzzy Set (IVRNHFS), Hesitant Fuzzy (HF), Neutrosophic Set, Score Function

1. Introduction

As we live in a rapidly more complex and uncertain world, decision-making processes tend to be challenged by handling imprecise, vague, and conflicting information. Most mathematical models and classical decision-making methods require crisp values and full information, which are hardly available in real-world scenarios. Phenomena like medical diagnosis, financial prediction, engineering assessment, environmental monitoring, and social decision-making are often characterized by expert judgments that are hesitant, incomplete, or even conflicting. Therefore, increasingly there is a need for more expressive and robust frameworks that can capture and represent these levels of uncertainty very well.

The concept of handling uncertainty in mathematical modeling began with the seminal work of Zadeh [1], who introduced fuzzy set theory**.** This framework provided a way to mathematically model vagueness by assigning membership values ranging from 0 to 1, instead of strict binary classifications. It became a cornerstone for decision-making models under uncertainty.

Atanassov [2] extended this idea by developing intuitionistic fuzzy sets, along with a hesitation degree. This innovation allowed for more comprehensive modeling of uncertainty, especially in situations where decision-makers face hesitation about the belongingness of elements.

Torra [3] further enhanced fuzzy modeling by formulating hesitant fuzzy sets, which allow elements to have multiple possible membership values. Rodriguez, Martinez, and Herrera [4] expanded this concept into hesitant fuzzy linguistic term sets, which enabled decision-makers to express hesitation linguistically. This work bridged the gap between linguistic evaluation and fuzzy modeling.

Xia and Xu [5] contributed by exploring aggregation methods for hesitant fuzzy information, developing operators for combining hesitant fuzzy values in decision-making. Zhang and Xu [6] advanced this by introducing interval-valued hesitant fuzzy sets. which further strengthened the ability to capture uncertainty in complex multi-criteria problems. Farhadinia [7] entropy-based measures to quantify uncertainty. Liao, Xu, and Zeng [8] developed distance and similarity measures for hesitant fuzzy linguistic term sets, which significantly improved the applicability of hesitant fuzzy theory.

Moving beyond fuzzy-based models, Smarandache [9] introduced neutrosophy, a generalization that explicitly incorporates truth, indeterminacy, and falsity values as independent components. This theory broadened the capacity of uncertainty modeling to capture paradoxes and contradictions in human reasoning. Building on this, Wang, Smarandache, Zhang, and Sunderraman [10] proposed SVNS, which allowed practical application of neutrosophic theory by assigning crisp values.

Biswas, Pramanik, and Giri [11] developed a methodology under SVNS with unknown weight information, addressing real-world problems where attribute weights are not always predefined. Pramanik, Dey, and Giri [12] extended neutrosophic decision making by integrating SVNS into the TOPSIS method, enabling systematic ranking of alternatives under uncertainty. Broumi et al. [13] studied correlation coefficients for interval-valued neutrosophic sets, providing a tool for measuring relationships among neutrosophic data. Later, Pramanik, Dey, and Smarandache [14] refined this by introducing correlation coefficients for interval bipolar neutrosophic sets, extending correlation-based analysis to more complex settings.

Al-Quran, Awang, Ali, and Abdullah [15] developed the hesitant bipolar-valued neutrosophic set, which combines bipolar information with neutrosophic hesitation, thereby extending modeling capacity for complex decision-making environments. Ramaroson and Andriamanohisoa [16] introduced the IVBNHFS, exploring its properties and operations, thereby opening new avenues in hesitant and bipolar neutrosophic theory.

Deli, Şubaş, Smarandache, and Ali [17] proposed IVBNS and demonstrated their utility in decision-making problems. Saqlain, Hamza, Saeed, and Zulqarnain [18] studied aggregate, arithmetic, and geometric operators of octagonal neutrosophic numbers, providing new aggregation tools for multi-criteria decision making. Ajay, Aldring, and Nivetha [19] introduced neutrosophic cubic fuzzy Dombi Hamy mean operators, which allow for advanced aggregation of neutrosophic cubic fuzzy information.

In terms of applications, Abdel-Basset, Atef, and Smarandache [20] developed a hybrid neutrosophic group decision-making approach for project selection, demonstrating the utility of neutrosophic sets in real-world project management. Nabeeh, Abdel-Basset, El-Ghareeb, and Aboelfetouh [21] extended this approach to IoT-based enterprises, providing solutions for decision-making challenges in emerging technologies. Mondal and Pramanik [22] applied neutrosophic decision-making models to the problem of school choice, showing their relevance in the education sector. Ye [23] proposed an advancing techniques for handling hesitation in neutrosophic decision problems.

Beyond technical and decision-focused studies, neutrosophic theory has been applied in social and assessment contexts. Aslan, Kargın, and Şahin [24] demonstrating the flexibility of neutrosophic systems in social sciences. Voskoglou [25] introduced the neutrosophic theory into educational and evaluative frameworks.

The aim of this research is to establish the definition and form of IVRNHFS formally and illustrate its utility in improving decision-making under uncertainty. It seeks to define appropriate score and ranking functions, construct aggregation operators, and determine the underlying set-theoretic operations essential for real-world implementation. Integrating this model into a decision-making problem, we illustrate how it facilitates stronger and more transparent assessments, particularly when the decision space consists of fuzzy, interval-based, and hesitant information. The importance of this method is further emphasized via a case study from the real world, indicating its utility in resolving intricate decision problems where traditional models are likely to be insufficient.

This paper is organized as follows: Section 2 gives the basic definitions. Section 3 gives the definition of IVRNHFS. Section 4 introduces a new operation on these sets and explains its main properties. Section 5 gives the suggested algorithm for the score function. Section 6 shows a numerical example. Lastly, Section 7 concludes the research.

1.1 . Contribution of the Work

The primary contributions of this paper are as follows. Firstly, we propose a new score function for IVRNHFS, enhancing the ranking and comparison capacities of hesitant information in complicated decision contexts. Secondly, we introduce a new operation on IVRNHFS and explore its key mathematical characteristics, thus enriching the theoretical framework of neutrosophic set extensions. Lastly, we offer a example that illustrates the practical efficacy of the proposed method.

1.2. Significance of Neutrosophic Sets in this Research

Neutrosophic theory offers a robust mathematical framework to manage indeterminacy, inconsistency, and hesitation, which are most often encountered in real-world problems. In this case, IVRNHFS offers a more flexible and more subtle manner of representing uncertain and hesitant information than classical fuzzy and intuitionistic fuzzy methods. The application of neutrosophic sets in this research not only improves the precision of decision analysis but also enables more realistic modeling for complicated cases, so the designed method is particularly valuable for real-world applications in engineering, management, and social sciences.

2. Preliminaries

2.1 Fuzzy Set [11]

A fuzzy set, introduced by Zadeh (1965), is characterized by a membership function that assigns to each element in the universe of discourse a value between 0 and 1. Formally, a fuzzy set F in a universe V is defined as:

$$F = \{\langle x, \mu_F(v)\rangle \mid v \in V\}$$

where $\mu_F(v) \in [0,1]$ represents the degree of membership of element v in set F.

2.2 Neutrosophic sets [11]

Let V be a universal set then, the neutrosophic set F is an object having the form

$$F = \{\langle v : T'_F(v), I'_F(v), F'_F(v)\rangle, v \in V\},$$

where the functions $T', I', F' : V \to]^{-}0, 1^{+}[$ define respectively the degree of membership (or Truth), the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element $v \in V$ to the element F with the condition.

$$0 \le T'_F(v) + I'_F(v) + F'_F(v) \le 3^{+}$$

2.3 Single-valued Neutrosophic sets [11]

Let V be a universal set, with generic element of V denoted by v. A single valued neutrosophic set F in V is characterized by a truth-membership function TA, indeterminacy function IA and falsity-membership function FA, with for each, $v \in V$, $T'_F(v), I'_F(v), F'_F(v) \in [0,1]$.
Note that for a SVNS A, the relation

$$0 \le T'_F(v) + I'_F(v) + F'_F(v) \le 3^+$$

holds. When the universal set V is continuous, a SVNS F can be written as

$$F = \int_v \langle T'_F(v),\ I'_F(v),\ F'_F(v) \rangle / v, v \in V$$

When the universal set V is discrete, a SVNS F can be written as

$$F = \sum_{i=1}^{n} \langle T'_F(v_i),\ I'_F(v_i),\ F'_F(v_i) \rangle v_i, v \in V$$

2.4 Hesitant sets [3]

Let V be a fixed set, a hesitant fuzzy set (HFS) on V is in terms of a function that when applied to V returns a subset of [0,1] and it is expressed as

$$F = \{ \langle v, h_F(v) \rangle \mid v \in V \}$$

where $h_F(v)$ is a set of some values in [0, 1], denoting the possible membership degrees of the element $v \in V$ to the set F.

3. Interval Valued Refined Neutrosophic Hesitant Fuzzy Set (IVRNHFS)

Let V be the universal set. Then an Interval-Valued Refined Neutrosophic Hesitant Fuzzy Set (IVRNHFS) A on V is defined as :

$$A = \{ \langle x : h_{T_A}(v), h_{I_A}(v), h_{F_A}(v) \rangle, v \in V \} \quad (1)$$

where:

$$h_T(v) = \{ [T_A^{L_{i,j}}(v),\ T_A^{U_{i,j}}(v)] : i = 1,2,\ldots,k;\ j = 1,2,\ldots,m(i) \}$$
$$h_I(v) = \{ [I_A^{L_{i,j}}(v),\ I_A^{U_{i,j}}(v)] : i = 1,2,\ldots,k;\ j = 1,2,\ldots,n(i) \}$$
$$h_F(v) = \{ [F_A^{L_{i,j}}(v),\ F_A^{U_{i,j}}(v)] : i = 1,2,\ldots,k;\ j = 1,2,\ldots,p(i) \}$$

Here, $h_T(v), h_I(v), h_F(v)$ are hesitant sets of intervals representing the refined truth, indeterminacy and falsity membership degrees and k is the number of refined levels. $m(i), n(i), p(i)$ are the number of hesitant intervals for the i-th refined level of truth, indeterminacy and falsity respectively.

For each interval:

$$[T_A^{L_{i,j}}(v),\ T_A^{U_{i,j}}(v)] \subseteq [0,1]$$
$$[I_A^{L_{i,j}}(v),\ I_A^{U_{i,j}}(v)] \subseteq [0,1]$$
$$[F_A^{L_{i,j}}(v),\ F_A^{U_{i,j}}(v)] \subseteq [0,1]$$

Additionally, for each $v \in V$, it must hold that:

$$0 \le \sup\ T_A^{U_{i,j}}(v) + \sup\ I_A^{U_{i,j}}(v) + \sup\ F_A^{U_{i,j}}(v) \le 3$$

4. Operations on IVRNHFS

Let $A = \{ \langle x : h_{T_A}(v), h_{I_A}(v), h_{F_A}(v) \rangle, v \in V \}$ and $B = \{ \langle x : h'_{T_A}(v), h'_{I_A}(v), h'_{F_A}(v) \rangle, v \in V \}$ b2 two IVRNHFS over the universe V where each component consists of sets of interval-valued refined hesitant degrees.

4.1 Union

The union of two IVRNHFSs A and B is a new IVRNHFS C, written as C = A ∪ B, where the refined truth-membership, indeterminacy and falsity membership functions are defined as:

$$T_C^{ij}(v) = \max\left(T_A^{ij}(v),\ T_B^{ij}(v)\right),$$
$$I_C^{ij}(v) = \min\left(I_A^{ij}(v),\ I_B^{ij}(v)\right),$$
$$F_C^{ij}(v) = \min\left(F_A^{ij}(v),\ F_B^{ij}(v)\right),\quad for\ all\ \ v \in V$$

4.2 Intersection

The intersection of two IVRNHFSs A and B is a new IVRNHFS C, written as C = A ∩ B, where the refined truth-membership, indeterminacy and falsity membership functions are defined as:

$$T_C^{ij}(v) = \min\left(T_A^{ij}(v),\ T_B^{ij}(v)\right),$$
$$I_C^{ij}(v) = \max\left(I_A^{ij}(v),\ I_B^{ij}(v)\right),$$
$$F_C^{ij}(v) = \max\left(F_A^{ij}(v),\ F_B^{ij}(v)\right),\quad for\ all\ \ v \in V$$

4.3 Complement

The complement of a Interval valued Refined Neutrosophic Hesitant Fuzzy Set (IVRNHFS) A is denoted by A^c, and is defined as:

$$T_{A^c}^{ij}(v) = F_A^{ij}(v),$$
$$I_{A^c}^{ij}(v) = I_A^{ij}(v),$$
$$F_{A^c}^{ij}(v) = T_A^{ij}(v), \text{for all } \ v \in V$$

4.4 Containment

An Interval Valued Refined Neutrosophic Hesitant Fuzzy Set (IVRNHFS) A is said to be contained in another IVRNHFS B, denoted by $A \subseteq B$, if and only if:

$$T_A^{ij}(v) \le T_B^{ij}(v),$$
$$I_A^{ij}(v) \ge I_B^{ij}(v),$$
$$F_A^{ij}(v) \ge F_B^{ij}(v),\ \ \text{for all } \ v \in V$$

Example

Let the two Interval-Valued Refined Neutrosophic Hesitant Fuzzy Sets (IVRNHFS) over the universe V be

$$A = \left\{ \begin{array}{l} \left(\begin{array}{l} v_1, \{[0.6, 0.8], [0.7, 0.9], [0.5, 0.7], [0.6, 0.75]\},\ \{[0.2, 0.4], [0.1, 0.3], [0.3, 0.5], [0.2, 0.4]\}, \\ \{[0.1, 0.2], [0.15, 0.25], [0.05, 0.15], [0.1, 0.2]\} \end{array} \right), \\ \left(\begin{array}{l} v_2, \{[0.4, 0.6], [0.5, 0.65], [0.45, 0.6], [0.5, 0.7]\},\ \{[0.3, 0.5], [0.35, 0.6], [0.3, 0.4], [0.2, 0.4]\}, \\ \{[0.1, 0.2], [0.15, 0.25], [0.1, 0.15], [0.1, 0.2]\} \end{array} \right) \\ \left(\begin{array}{l} v_3, \{[0.7, 0.9], [0.75, 0.95], [0.8, 0.9], [0.7, 0.85]\}, \{[0.1, 0.2], [0.15, 0.25], [0.05, 0.15], [0.1, 0.2]\}, \\ \{[0.05, 0.1], [0.1, 0.15], [0.05, 0.1], [0.05, 0.1]\} \end{array} \right) \end{array} \right\}$$

$$B=\left\{\begin{array}{l}\left(\begin{array}{l}v_1,\{[0.7,0.9],[0.8,0.9],[0.6,0.8],[0.7,0.75]\},\ \{[0.3,0.4],[0.2,0.3],[0.4,0.45],[0.3,0.35]\},\\ \{[0.01,0.02],[0.15,0.25],[0.05,0.1],[0.01,0.02]\}\end{array}\right),\\ \left(\begin{array}{l}v_2,\{[0.5,0.7],[0.6,0.65],[0.5,0.6],[0.6,0.8]\},\ \{[0.3,0.4],[0.35,0.4],[0.3,0.35],[0.2,0.3]\},\\ \{[0.1,0.15],[0.1,0.2],[0.1,0.15],[0.01,0.02]\}\end{array}\right)\\ \left(\begin{array}{l}v_3,\{[0.8,0.9],[0.8,0.95],[0.8,0.9],[0.8,0.9]\},\{[0.3,0.4],[0.25,0.35],[0.25,0.35],[0.2,0.3]\},\\ \{[0.05,0.1],[0.2,0.25],[0.05,0.1],[0.05,0.1]\}\end{array}\right)\end{array}\right\}$$

Then, the union of two IVRNHFS is

$$C=\left\{\begin{array}{l}\left(\begin{array}{l}v_1,\{[0.7,0.9],[0.8,0.9],[0.6,0.8],[0.7,0.75]\},\ \{[0.2,0.4],[0.1,0.3],[0.3,0.45],[0.2,0.35]\},\\ \{[0.01,0.02],[0.15,0.25],[0.05,0.1],[0.01,0.02]\}\end{array}\right),\\ \left(\begin{array}{l}v_2,\{[0.5,0.7],[0.6,0.65],[0.5,0.6],[0.6,0.8]\},\ \{[0.3,0.4],[0.35,0.4],[0.3,0.35],[0.2,0.3]\},\\ \{[0.1,0.15],[0.1,0.2],[0.1,0.15],[0.01,0.02]\}\end{array}\right)\\ \left(\begin{array}{l}v_3,\{[0.8,0.9],[0.8,0.95],[0.8,0.9],[0.8,0.9]\},\{[0.1,0.2],[0.15,0.25],[0.05,0.15],[0.1,0.2]\},\\ \{[0.05,0.1],[0.1,0.15],[0.05,0.1],[0.05,0.1]\}\end{array}\right)\end{array}\right\}$$

The intersection of two IVRNHFS is

$$C=\left\{\begin{array}{l}\left(\begin{array}{l}v_1,\{[0.6,0.8],[0.7,0.9],[0.5,0.7],[0.6,0.75]\},\ \{[0.3,0.4],[0.2,0.3],[0.4,0.5],[0.3,0.4]\},\\ \{[0.1,0.2],[0.15,0.25],[0.05,0.15],[0.1,0.2]\}\end{array}\right),\\ \left(\begin{array}{l}v_2,\{[0.4,0.6],[0.5,0.65],[0.45,0.6],[0.5,0.7]\},\ \{[0.3,0.5],[0.35,0.6],[0.3,0.4],[0.2,0.4]\},\\ \{[0.1,0.2],[0.15,0.25],[0.1,0.15],[0.1,0.2]\}\end{array}\right)\\ \left(\begin{array}{l}v_3,\{[0.7,0.9],[0.75,0.95],[0.8,0.9],[0.7,0.85]\},\{[0.3,0.4],[0.25,0.35],[0.25,0.35],[0.2,0.3]\},\\ \{[0.05,0.1],[0.2,0.25],[0.05,0.1],[0.05,0.1]\}\end{array}\right)\end{array}\right\}$$

The complement of the set A is

$$A^c=\left\{\begin{array}{l}\left(\begin{array}{l}v_1,\{[0.1,0.2],[0.15,0.25],[0.05,0.15],[0.1,0.2]\},\ \{[0.2,0.4],[0.1,0.3],[0.3,0.5],[0.2,0.4]\},\\ \{[0.6,0.8],[0.7,0.9],[0.5,0.7],[0.6,0.75]\}\end{array}\right),\\ \left(\begin{array}{l}v_2,\{[0.1,0.2],[0.15,0.25],[0.1,0.15],[0.1,0.2]\},\ \{[0.3,0.5],[0.35,0.6],[0.3,0.4],[0.2,0.4]\},\\ \{[0.4,0.6],[0.5,0.65],[0.45,0.6],[0.5,0.7]\}\end{array}\right)\\ \left(\begin{array}{l}v_3,\{[0.05,0.1],[0.1,0.15],[0.05,0.1],[0.05,0.1]\},\{[0.1,0.2],[0.15,0.25],[0.05,0.15],[0.1,0.2]\},\\ \{[0.7,0.9],[0.75,0.95],[0.8,0.9],[0.7,0.85]\}\end{array}\right)\end{array}\right\}$$

5. Proposed Score Function

Let $A=\left\{\left\langle v:h_{T_A}(v),h_{I_A}(v),h_{F_A}(v)\right\rangle,v\in V\right\}$ be IVRNHFS

(a) Weighted Arithmetic Average Operator (WAAM) for a IVRNHFS

The WAAM for a IVRNHFS is defined as a weighted aggregation of the hesitant intervals, with individual weights assigned to each membership component:

$$AO_{WA}(v)=w_T h_T(v)+w_I h_I(v)+w_F h_F(v) \tag{2}$$

where $w_T, w_I, w_F \in [0,1]$ are the weights, with $w_T+w_I+w_F=1$. The resulting aggregated IVRNHFS is:

$$AO_{WA}(v)=\left\langle h_{T_{WA}}(v),\ h_{I_{WA}}(v),\ h_{F_{WA}}(v)\right\rangle$$

where,

$$h_{T_{WA}}(v) = \left[1 - \prod_{j=1}^{mT} \left(1 - T_j^L\right)^{wT},\ 1 - \prod_{j=1}^{mT} \left(1 - T_j^U\right)^{wT} \right]$$

$$h_{I_{WA}}(v) = \left[\prod_{j=1}^{mI} \left(I_j^L\right)^{wI}, \prod_{j=1}^{mI} \left(I_j^U\right)^{wI} \right]$$

$$h_{F_{WA}}(v) = \left[\prod_{j=1}^{mF} \left(F_j^L\right)^{wF}, \prod_{j=1}^{mF} \left(F_j^U\right)^{wF} \right]$$

(b) Weighted Geometric Average Operator (WGAM) for a IVRNHFS

The WGAM for a INRNHFS is defined as a weighted geometric aggregation of the hesitant intervals, with individual weights for each membership component:

$$AO_{WG}(x) = (h_T(x))^{wY}\ (h_I(x))^{wI}\ (h_F(x))^{wF} \tag{3}$$

where $w_T, w_I, w_F \in [0,1]$ are the weights, with $w_T + w_I + w_F = 1$. The resulting aggregated IVRNHFS is:

$$AO_{WG}(v) = \left\langle h_{T_{WG}}(v),\ h_{I_{WG}}(v),\ h_{F_{WG}}(v) \right\rangle$$

where,

$$h_{T_{WG}}(v) = \left[\prod_{j=1}^{mT} \left(T_j^L\right)^{wT},\ \prod_{j=1}^{mT} \left(T_j^U\right)^{wT} \right]$$

$$h_{I_{WG}}(v) = \left[1 - \prod_{j=1}^{mI} \left(1 - I_j^L\right)^{wI}, 1 - \prod_{j=1}^{mI} \left(1 - I_j^U\right)^{wI} \right]$$

$$h_{F_{WG}}(v) = \left[1 - \prod_{j=1}^{mF} \left(1 - F_j^L\right)^{wF}, 1 - \prod_{j=1}^{mF} \left(1 - F_j^U\right)^{wF} \right]$$

(c) Hybrid Aggregation Operator (HAO) for a IVRNHFS

The HAO combines the WAAM and WGAM with a parameter $\alpha \in [0,1]$ to balance their contributions, using individual weights for each membership component:

$$AO_{HA}(v) = \alpha\, AO_{WA}(v) + (1-\alpha)\, AO_{WG}(v) \tag{4}$$

(d) Hybrid Score Function (HSF) for IVRNHFS

The HSF evaluates an IVRNHFS or its aggregated form $AO_{HA}(v)$ by transforming the interval-valued membership degrees into a single numerical score for comparison:

$$S(A) = \frac{1}{3}\left[\frac{T^{U'} + T^{L'}}{2} + \left(1 - \frac{I^{U'} + I^{L'}}{2}\right) + \left(1 - \frac{F^{U'} + F^{L'}}{2}\right) \right] \tag{5}$$

6. Numerical Example

A hospital is tasked with selecting the most appropriate treatment plan for a patient diagnosed with a rare form of cancer, such as pancreatic cancer, where multiple treatment options (e.g., chemotherapy, immunotherapy or surgery) are available. The decision involves evaluating treatments based on refined levels like effectiveness and side effects. Each criterion is fraught with uncertainty due to incomplete clinical data, varying expert opinions, and patient-specific factors. The hospital uses IVRNHFS to model this complex decision-making process, as it can handle interval-based uncertainty, indeterminacy, and hesitancy more effectively than other fuzzy set frameworks.

i) Evaluation of Immunotherapy (v_1)

Truth Membership Intervals:

Level 1: [0.6, 0.8], [0.7, 0.9]
Level 2: [0.5, 0.7], [0.6, 0.75]
Then the Refined truth membership:

$$h_T(v_1) = \{[0.6, 0.8], [0.7, 0.9], [0.5, 0.7], [0.6, 0.75]\}$$

Indeterminacy Intervals:

Level 1: [0.2, 0.4], [0.1, 0.3]
Level 2: [0.3, 0.5], [0.2, 0.4]
Then the Refined Indeterminacy membership:

$$h_I(v_1) = \{[0.2, 0.4], [0.1, 0.3], [0.3, 0.5], [0.2, 0.4]\}$$

Falsity Intervals:

Level 1: [0.1, 0.2], [0.15, 0.25]
Level 2: [0.05, 0.15], [0.1, 0.2]
Then the Refined Falsity membership:

$$h_F(v_1) = \{[0.1, 0.2], [0.15, 0.25], [0.05, 0.15], [0.1, 0.2]\}$$

ii) Evaluation of Chemotherapy (v_2)

Truth:

Level 1: [0.4, 0.6], [0.5, 0.65]
Level 2: [0.45, 0.6], [0.5, 0.7]
Then the Refined truth membership:

$$h_T(v_2) = \{[0.4, 0.6], [0.5, 0.65], [0.45, 0.6], [0.5, 0.7]\}$$

Indeterminacy:

Level 1: [0.3, 0.5], [0.35, 0.6]
Level 2: [0.3, 0.4], [0.2, 0.4]
Then the Refined Indeterminacy membership:

$$h_I(v_2) = \{[0.3, 0.5], [0.35, 0.6], [0.3, 0.4], [0.2, 0.4]\}$$

Falsity:

Level 1: [0.1, 0.2], [0.15, 0.25]
Level 2: [0.1, 0.15], [0.1, 0.2]
Then the Refined Falsity membership:

$$h_F(v_2) = \{[0.1, 0.2], [0.15, 0.25], [0.1, 0.15], [0.1, 0.2]\}$$

iii) Evaluation of Sugeory (v_3)

Truth:

Level 1: [0.7, 0.9], [0.75, 0.95]

Level 2: [0.8, 0.9], [0.7, 0.85]
Then the Refined truth membership:

$$h_T(v_3) = \{[0.7, 0.9], [0.75, 0.95], [0.8, 0.9], [0.7, 0.85]\}$$

Indeterminacy:

Level 1: [0.1, 0.2], [0.15, 0.25]
Level 2: [0.05, 0.15], [0.1, 0.2]
Then the Refined Indeterminacy membership:

$$h_I(v_3) = \{[0.1, 0.2], [0.15, 0.25], [0.05, 0.15], [0.1, 0.2]\}$$

Falsity:

Level 1: [0.05, 0.1], [0.1, 0.15]
Level 2: [0.05, 0.1], [0.05, 0.1]
Then the Refined Falsity membership:

$$h_F(v_3) = \{[0.05, 0.1], [0.1, 0.15], [0.05, 0.1], [0.05, 0.1]\}$$

Thus, the **IVRNHFS set A** is:

$$A = \left\{ \begin{array}{l} \left(\begin{array}{l} v_1, \{[0.6, 0.8], [0.7, 0.9], [0.5, 0.7], [0.6, 0.75]\},\ \{[0.2, 0.4], [0.1, 0.3], [0.3, 0.5], [0.2, 0.4]\}, \\ \{[0.1, 0.2], [0.15, 0.25], [0.05, 0.15], [0.1, 0.2]\} \end{array} \right), \\ \left(\begin{array}{l} v_2, \{[0.4, 0.6], [0.5, 0.65], [0.45, 0.6], [0.5, 0.7]\},\ \{[0.3, 0.5], [0.35, 0.6], [0.3, 0.4], [0.2, 0.4]\}, \\ \{[0.1, 0.2], [0.15, 0.25], [0.1, 0.15], [0.1, 0.2]\} \end{array} \right) \\ \left(\begin{array}{l} v_3, \{[0.7, 0.9], [0.75, 0.95], [0.8, 0.9], [0.7, 0.85]\}, \{[0.1, 0.2], [0.15, 0.25], [0.05, 0.15], [0.1, 0.2]\}, \\ \{[0.05, 0.1], [0.1, 0.15], [0.05, 0.1], [0.05, 0.1]\} \end{array} \right) \end{array} \right\}$$

By using the above proposed score function,
$S(v_1) = 0.6983$, $S(v_2) = 0.6282$, $S(v_3) = 0.8054$
IVRNHFS is particularly suited for this medical decision-making scenario because:
Interval-valued representation: Clinical data, such as treatment effectiveness, is often imprecise and expressed in ranges (e.g., a 60–80% success rate for chemotherapy).
Refined neutrosophic structure: Each criterion is evaluated for truth (how well it meets the goal, e.g., tumor reduction), indeterminacy (uncertainty due to limited trial data), and falsity (likelihood of failure, e.g., severe side effects). This is critical in cancer treatment, where data may be inconclusive.
Hesitant fuzzy component: Oncologists and specialists may hesitate when assigning a single value to a criterion due to conflicting evidence or patient variability. For example, immunotherapy's effectiveness might be rated as {[0.6, 0.7], [0.7, 0.8]} by different experts.

7. Conclusion

IVRNHFS is an effective concept that supports uncertainty, indeterminacy, and hesitation within the complex decision-making setting. IVRNHFS provides flexible aggregation of information via the clearly defined the operations. The score function is important in measuring the total assessment of each alternative by summing up the corrected truth, indeterminacy and falsity intervals. Depending on these scores, a ranking function allows alternatives to be compared and prioritized. This model finds particular application in expert systems and multi-criteria decision-making, where judgments can be inexact, or conflictual. IVRNHFS increases the expressiveness and correctness of decision models. This mathematical strength and flexibility qualifies it to be used in engineering, medicine and computational intelligence. . In the future, this research can be further extended by creating sophisticated aggregation operators and similarity measures, and by investigating real-world applications of IVRNHFS in domains like medical diagnosis, supply chain management, and risk analysis.

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Received: July 8, 2025. Accepted: Sep 9, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

Enhancing Transportation Problem Solutions Using A New Ranking Function In a Neutrosophic Environment

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Abstract: Transportation problems often involve uncertainties in cost, supply, and demand, which traditional optimisation methods struggle to handle effectively. Neutrosophic numbers can handle such uncertainties and incomplete information. This paper introduces a new ranking function for solving transportation problems in a neutrosophic environment, which provides a better framework for addressing indeterminate, vague, and inconsistent information. By integrating neutrosophic sets and their arithmetic operations, this study offers a computationally efficient approach to transforming crisp problems into a solvable format. Numerical examples and comparative analysis demonstrate the effectiveness of the proposed ranking function.

Keywords: Transportation problem, neutrosophic number, ranking function, indeterminacy.

1. Introduction

In today's world, many problems involve various forms of uncertainty that cannot be effectively addressed using classical mathematical theories. To handle situations involving vague or imprecise information, Zadeh [1] introduced fuzzy set theory in 1965, which is defined by membership values. However, in several cases, decisions or results derived from the available data lack the desired level of precision. To address these limitations, advanced forms of fuzzy sets have been proposed. Among them, the intuitionistic fuzzy set, introduced by Atanassov [2] in 1986, became notable for handling imprecise information using both membership and non-membership values [3]. Consequently, both fuzzy and intuitionistic fuzzy set theories have been widely applied in real-world decision-making problems. Over time, it was observed that generalisations of fuzzy sets still struggled to address issues involving indeterminate or inconsistent information. To bridge this gap, Smarandache [4]introduced neutrosophic sets in 1998 as a further extension of classical, fuzzy, and intuitionistic fuzzy sets. Neutrosophic sets incorporate three components: truth-membership, indeterminacy-membership, and falsity-membership degrees, making them well-suited for representing uncertainty and inconsistency. Wang et al. [5]. later applied the concept of single-valued neutrosophic sets to various practical problems. The study of optimal transportation models in a cost-effective manner has played a significant role in the field of supply chain management. Numerous researchers [6], [7], have developed mathematical formulations of transportation problems under various environmental conditions. The classical transportation problem was first stated by Hitchcock [8], in 1941, in which the transportation problem was formulated using precise, or *crisp*, values for its constraints. However, in today's dynamic environment, transportation parameters such as demand, supply, and unit transportation cost are often uncertain due to various uncontrollable factors. To address these uncertainties, many researchers have developed and solved fuzzy transportation models. Out of these, Ó hÉigeartaigh [9] developed one of the earliest fuzzy transportation algorithms, incorporating fuzzy costs into classical transportation models. Chanas and Kuchta [10]. Introduced a structured approach to identify optimal solutions in transportation problems with fuzzy cost coefficients. Some researchers contributed significantly to solving fuzzy solid transportation problems by applying and enhancing evolutionary algorithms, including genetic and parametric approaches [11] [12], [12], [13]. Adamo [14] introduced the concept of α-preference, a method for ranking fuzzy numbers based on the α-level sets, providing a practical approach for decision-making scenarios. Yager [15] proposed several indices for ranking fuzzy subsets, focusing on the development of measures that could effectively capture the preference ordering among fuzzy numbers. Dubois and Prade [16]**,** they introduced dominance-based ranking methods and explored the use of possibility theory for comparing fuzzy quantities, contributing significantly to the theoretical foundation of fuzzy rankings. Giovanni Bortolan and R. Degani [17] conducted a comprehensive review of various methods for ranking fuzzy subsets, including approaches by Yager, Adamo, and Dubois & Prade. Their work highlighted the importance of ranking in decision-making processes involving fuzzy numbers. Some researchers significantly advanced decision-making and transportation models by introducing similarity measures, ranking methods, and heuristics under single-valued neutrosophic and trapezoidal neutrosophic environments[18, 19, 20, 21].Selvakumari [22]. Introduced a novel approach to neutrosophic transportation problems using the Zero Suffix Method, demonstrating its effectiveness in handling indeterminate and inconsistent data through a simplified algorithmic structure.

Umamageswari R. M. & G. Uthra[23]. This duo proposed a weighted average ranking method designed explicitly for generalised single-valued neutrosophic trapezoidal numbers, applying it effectively to transportation problems under uncertainty.

Expanding upon the framework of neutrosophic logic, Saini et al.[24], explored the application of single-valued trapezoidal neutrosophic numbers to model uncertainties in cost parameters. Their study illustrated how neutrosophic sets could enhance the decision-making capabilities of transportation models when crisp values are inadequate to express the true nature of logistical data.

Similarly, Rabinson and Rajendran [25], utilised decagonal neutrosophic numbers in transportation problems, offering a fresh numerical representation to capture finer gradations of truth, indeterminacy, and falsity. Their methodology emphasised precision in modelling complex logistics environments.

To further address the ambiguity present in transportation systems, Hemalatha et al.[26], proposed a solution for the type-II neutrosophic fuzzy transportation problem, combining both neutrosophic and fuzzy environments. This hybrid model was shown to represent better situations where decision-makers face both hesitant and indeterminate information.

Building on these advancements, Karak et al.[27]. Developed a comprehensive solution technique for the transportation problem in a general neutrosophic environment, formulating new operational rules and optimisation strategies that enhance solution reliability under various uncertain conditions. Kalaivani Kaspar & Palanivel Kaliyaperumal [28] Introduced a novel ranking function tailored for single-valued trapezoidal neutrosophic numbers (SVTNNs) to address transportation problems with mixed constraints, enhancing the accuracy of optimal solutions. N. Parveen, K. Prabu & V. Sangeetha [29], developed a new ranking function for heptagonal intuitionistic fuzzy numbers and applied it to solve transportation problems, demonstrating improved decision-making under uncertainty. Balasundaram Baranidharan & Ghanshaym Singha Mahapatra [30]. Introduced an alpha-cut-based ranking technique for heptagonal fuzzy numbers, formulating the Generalised Ranking Heptagonal Fuzzy Method (GRHFM) to solve regional shipment transportation problems. Kaspar and Kaliyaperumal [31] tackled the transportation problem under mixed constraints by applying single-valued trapezoidal neutrosophic numbers, thus accommodating multiple types of real-life constraints within the neutrosophic framework[32]. Their contribution is notable for expanding the applicability of neutrosophic transportation models to more generalised and practical scenarios.[33].

The incorporation of neutrosophic sets in this research significantly enhances the modeling and analysis of uncertainty, imprecision, and incomplete information inherent in real-world problems. Traditional methods often fail to effectively handle ambiguous or inconsistent data, whereas the neutrosophic sets provide a more flexible and robust framework by considering truth, indeterminacy, and falsity simultaneously. The neutrosophic numbers considered in this work is able to handle the vagueness in much better way compared to other extensions of fuzzy numbers. Here the transportation problems are solved where the parameters like cost, demand, supply etc. are all the neutrosophic in nature, in which the indeterminacy is converted into crisp by using a new proposed ranking function with yields the better, optimized cost for the transportation problems. Numericals have been discussed to validate the proposed Ranking function.

Table 1. List of abbreviations

TP	Transportation problem
NN	Neutrosophic number
NTP	Neutrosophic transportation problem
CNTP	Crisp neutrosophic transportation problem
SVNN,	Single-valued neutrosophic number
SVPNN,	Single-valued pentagonal neutrosophic number
SVONN	Single-valued octagonal neutrosophic number

2. Preliminaries

2.1 Neutrosophic Sets and numbers.

2.1.1 Definition (Neutrosophic set)

Three membership functions define a neutrosophic set:
Truth Membership (T): The degree to which an element belongs to a set.
Indeterminacy Membership (I): Degree of uncertainty in classification.
Falsity Membership (F): Degree to which an element does not belong to the set.
These elements are mathematically represented as:

$$NS = \{(x, T(x), I(x), F(x)) \ / \ x \in X\}$$

2.1.2 Definition: (Classical)

Let X be a non-empty set. Then an NS $\tilde{A}^N$ on X defined as

$$\tilde{A}^N = \{\langle x, T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)\rangle / x \in X\},$$

Where $T_{\tilde{A}^N}(x), I_{\tilde{A}^N}(x), F_{\tilde{A}^N}(x)$ Are the truth membership function, the indeterminacy function, and the falsity function.

2.1.3 Definition: (Neutrosophic number) [27]

A neutrosophic number represented by $z = x + yI$ for $x, y \in R$, where x is the determinate part and yI Is the indeterminate part. Here $I \in [I_l, I_u]$ where I_l and I_u Stands for lower and upper indeterminacy.

When $I = 0$ (No indeterminacy) z is a real crisp value.

2.2 Arithmetic Operations on Neutrosophic Numbers

Neutrosophic numbers (NNs) follow specific arithmetic rules,

Consider $z_1 = x_1 + y_1 I$ and $z_2 = x_2 + y_2 I$

$z_1 \pm z_2 = x_1 \pm x_2 + (y_1 \pm y_2)I.$

$z_1 \times z_2 = x_1 x_2 + (x_1 y_2 + x_2 y_1 + y_1 y_2)I$

$0.I = 0$

$z_1^2 = (x_1 + y_1 I)^2 = x_1^2 + (2x_1 y_1 + y_1^2)$

$I^n = I \ (n \geq 1)$

$$\frac{I}{I} = \text{undefined}$$

Remark 1: For any NN $z = x + yI$ *and* $I \in [I_l, I_u]$ *the NN gets converted into an interval form* $[x + yI_l, x + yI_u]=[c_l, c_u]$.

For example: For a NN $z = 13 + 5I$ where $I = [0,0.5]$ then z is equivalent to [13, 15.5]

3. Proposed Ranking Function and Ranking Rules

3.1 Definition (Ranking Function)

Let $\bar{C} = c + c'I$ Be a NN. For $I \in [I_l, I_u]$, $\bar{C}$ can be converted into an interval form $[c_l, c_u]$.

The ranking function is defined as

$$R(\bar{C}) = R(c + c'I) = R([c_l, c_u]) = \frac{2}{\left[\frac{1}{c_l}+\frac{1}{c_u}\right]} \tag{1}$$

3.1.1 Significance of Proposed Ranking Function:

The ranking function in this work is used purely as a defuzzification tool to convert fuzzy parameters into crisp values, enabling the application of classical optimisation methods. It plays a crucial role in making the fuzzy model computationally feasible while retaining the uncertainty representation of the original data.

3.1.2 Ranking Rules for the proposed ranking function:

Here $\bar{P} = p + p'I$ and $\bar{Q} = q + q'I$ be two NNS. Then for $I \in [I_l, I_u]$ we have,

If $(\bar{P}) \le R(\bar{Q})$, then $\bar{P} \le_N \bar{Q}$

If $(\bar{P}) \ge R(\bar{Q})$, then $\bar{P} \ge_N \bar{Q}$

If $(\bar{P}) = R(\bar{Q})$, then $\bar{P} =_N \bar{Q}$

For example, consider two NNs, $\bar{P} = 9 + 5I$ and $\bar{Q} = 3 + 4I \; for \; I \in [0,0.3]$

$$R(9 + 5I) = R([9,10.5]) = 9.7$$
$$R(3 + 4I) = R([3,4.2]) = 3.5$$
$$\therefore R(9 + 5I) \ge R(3 + 4I) \Longrightarrow 9 + 5I \ge 3 + 4I$$

3.2. Neutrosophic Transportation Problem

3.2.1 Formulation of Model

The NTP can be formulated with $\mathcal{S}$ supply points and $\mathcal{D}$ Destination points as follows:

$$\min z = \sum_{i=1}^{\mathcal{S}} \sum_{j=1}^{\mathcal{D}} \bar{\mathcal{T}}_{ij} \bar{X}_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^{\mathcal{D}} X_{ij} = \bar{s}_i \quad \forall\, i = 1,2, \dots \dots, \mathcal{S}$$
$$\sum_{i=1}^{\mathcal{S}} X_{ij} = \bar{d}_j \quad \forall\, j = 1,2, \dots \dots, \mathcal{D} \tag{2}$$

Here $\bar{\mathcal{T}}_{ij} = t_{ij} + t'_{ij}I$ denotes the neutrosophic transportation cost of delivering a single unit from the i^{th} supply point to the j^{th} Destination point.

In the constraints, $\bar{s}_i = s_i + s'_i I$ represents the neutrosophic supply at the i^{th} supply point and $\bar{d}_j = d_j + d'_j I$ denotes neutrosophic demand at the j^{th} Destination point.
The objective is to determine the quantity. $X_{ij}(\geq 0)$ to be allocated at each $(i,j)^{th}$ Location such that the total transportation cost is minimised.

3.2.2 Solution Procedure
The methodology for solving the NTP is presented through the following stepwise procedure:
Step 1: Transforming the given NTP into an interval TP by taking a suitable value of
$I \in [I_l, I_u]$ (*see Remark* 1).
The mathematical formulation is

$$\min z = \sum_{i=1}^{\mathcal{S}} \sum_{j=1}^{\mathcal{D}} [\bar{\mathcal{T}}_{ijl}, \bar{\mathcal{T}}_{iju}] X_{ij}$$

Subject to constraints,

$$\sum_{j=1}^{\mathcal{D}} X_{ij} = [s_{il}, s_{iu}] \quad \forall\, i = 1,2, \ldots \ldots, \mathcal{S} \tag{3}$$
$$\sum_{i=1}^{\mathcal{S}} X_{ij} = [d_{jl}, d_{ju}] \quad \forall\, j = 1,2, \ldots \ldots, \mathcal{D}$$

Step 2: Apply the proposed ranking function $R([c_l, c_u])$ On the above obtained intervals.
The NTP is transformed into a Crisp Neutrosophic Transportation Problem (CNTP), characterised by crisp supply, demand, and cost parameters, expressed as follows:

$$\min z = \sum_{i=1}^{\mathcal{S}} \sum_{j=1}^{\mathcal{D}} R([\bar{\mathcal{T}}_{ijl}, \bar{\mathcal{T}}_{iju}]) X_{ij}$$

s.t

$$\sum_{j=1}^{\mathcal{D}} X_{ij} = R([s_{il}, s_{iu}]) \quad \forall\, i = 1,2, \ldots \ldots, \mathcal{S}$$
$$\sum_{i=1}^{\mathcal{S}} X_{ij} = R([d_{jl}, d_{ju}]) \quad \forall\, j = 1,2, \ldots \ldots, \mathcal{D} \tag{4}$$

Step 3: Solve the CNTP using any standard method to obtain the values of X_{ij}
Step 4: Allocate these X_{ij} Values are used in the objective function to compute the minimum transportation cost.

3.3 Model Testing and Results
We illustrate the approach by solving. 4×5 and 3×4 NTP for two different values of I. In the first case, we consider $I \in [0,0.6]$ and in the second step, we first consider a crisp problem, then for fuzzification we take $I \in [0,1]$.

Example -1
Consider a garment manufacturing company with production centers located in Mumbai, Pune, Kolhapur, and Goa. The garments produced at these centers are distributed to Chennai, Bengaluru, Hyderabad, Nagpur, and Jaipur.
NOTE: In this 4×5 NTP, the neutrosophic cost, supply, and demand parameters are presented in the following table.

Table 2. Transportation Problem Incorporating Neuromorphic Cost Coefficients

	Chennai	Bengaluru	Hyderabad	Nagpur	Jaipur	Source
Mumbai	4+6I	3+4 I	2+4 I	6+6 I	2+2 I	7+4 I
Pune	7+4 I	17+ I	6+3 I	2+3 I	4+5 I	10+3 I
Kolhapur	11+4 I	8+2 I	9+2 I	7+4 I	12+2 I	8+3 I
Goa	4+ 5I	5+2 I	10+3 I	15+4 I	11+4 I	5+2 I
Demand	8+2 I	4+ 4I	6+ 2I	9+3 I	3+I	

A balanced transportation problem is considered, where the total supply equals the total demand (=30+12 I)

Step 1: Taking $I \in [0,0.6]$, we have converted the above TP into an interval TP using conversion formula (Remark 1) given by the following table

Table 3. TP with Interval-Valued Cost Coefficients

	Chennai	Bengaluru	Hyderabad	Nagpur	Jaipur	Source
Mumbai	[4,7.6]	[3,5.4]	[2,4.4]	[6,9.6]	[2,3.2]	[7,9.4]
Pune	[7,9.4]	[17,17.6]	[9,10.8]	[7,9.4]	[12,13.2]	[10,11.8]
Kolhapur	[11,13.4]	[8,9.2]	[9,10.2]	[7,9.4]	[12,13.2]	[8,9.8]
Goa	[4,7]	[5,6.2]	[10,11.8]	[15,17.4]	[11,13.4]	[5,6.2]
Demand	[8,9.2]	[4,6.4]	[6,7.2]	[9,10.8]	[3,3.6]	

Step 2: The ranking function $R([c_l, c_u]) = \frac{2}{\left[\frac{1}{c_l}+\frac{1}{c_u}\right]}$ is now applied to each of the above intervals.

This leads the TP which will now have the crisp supply, demand and cost parameters as shown in the following table

Table 4. Transportation problem with crisp cost coefficients

	Chennai	Bengaluru	Hyderabad	Nagpur	Jaipur	Source
Mumbai	5.2	3.9	2.8	7.4	2.5	8
Pune	8	17.3	9.8	8	12.6	10.8
Kolhapur	12.1	8.6	9.6	8	12.6	8.8
Goa	5.1	5.5	10.8	16.1	12.1	5.5
Demand	8.6	4.9	6.5	9.8	3.3	

Note: Here also total supply = total demand $= 33.1$

Step 3: The optimized transportation cost obtained using Vogel's approximation method is

Rs 220.69

3.3.2 Comparative Analysis

Table 5. Comparative result of Proposed Method with Existing Literature

METHOD	TRANSPORTATION COST (in Rs.)
Singh, A. (2024)	230.84
Proposed Method	220.69

The final optimized cost obtained is compared with cost reported in the above-mentioned paper. The proposed method achieves a lower cost.

Example -2

A sample transportation problem is considered with three suppliers and four demand points. The cost matrix, supply, and demand are given in the table below:

Table 6. Crisp transportation table

Distribution Centre						Supply
		D_1	D_2	D_3	D_4	
Plant	P_1	3	1	7	4	300
	P_2	2	6	5	9	400
	P_3	8	3	3	2	500
Demand		250	350	400	200	1200

Step 1: Converting the Crisp problem transportation into NTP

Table 7. Transformation of the Crisp problem into the Neutrosophic Framework

Distribution Centre						Supply
		D_1	D_2	D_3	D_4	
Plant	P_1	2.5+I	0.5+I	6+3I	3.5+2I	280+30I
	P_2	1.5+I	5.5+2I	4.5+I	9+2I	390+20I
	P_3	7.5+2I	2.5+I	2.5+I	1.5+I	490+20I
Demand		245+10I	345+15I	390+20I	390+20I	1160+70I

Step 2: Converting into Interval TP using Remark 1, taking $I = [0,1]$

Table 8. Transformed form of Neutrosophic number $\boldsymbol{c + c'I}$ into interval form $[\boldsymbol{c_l, c_u}]$

Distribution Centre						Supply
		D_1	D_2	D_3	D_4	
Plant	P_1	[2.5,3.5]	[0.5,2.5]	[6,9]	[3.5,5.5]	[280,310]
	P_2	[1.5,2.5]	[5.5,7.5]	[4.5,5.5]	[9,11]	[390,410
	P_3	[7.5,9.5]	[2.5,3.5]	[2.5,3.5]	[1.5,2.5]	[490,510]
Demand		[245,255]	[345,360]	[390,410]	[180,205]	

Step 3: Converting Interval TP into Crisp TP using the proposed Ranking Function.

Table 9. Transformed form of interval form $[\boldsymbol{c_l, c_u}]$ into single crisp value

Distribution Centre						Supply
		D_1	D_2	D_3	D_4	
Plant	P_1	2.9	0.8	7.2	4.3	294.2
	P_2	1.9	6.3	5	10	400
	P_3	8.4	3	3	1.9	500
Demand		250	352	400	192	1194.2

These numbers have been solved using the Least cost method as well as Vogel's method. The results and their comparison with a numerical solution using actual crisp values are shown in the Table below.

3.3.4 Comparative Analysis with Standard Methods.

Table 10. Final optimised transportation cost

METHOD	Transportation Cost by Proposed Ranking Function	Transportation cost With the Actual crisp value
Least Cost Method	2824.4	2850
Vogel's Method	2749	2850

When compared with traditional values and solution techniques, the use of proposed ranking function yields a lower transportation cost, confirming its practical advantage in solving fuzzy transportation problems.

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

4. Conclusions

In real-world applications, parameters like supply, demand, and cost are inherently uncertain and cannot always be described using exact numerical values. To manage this ambiguity, fuzzy sets and their various extensions have been widely utilized. Over the years, different techniques have been introduced to defuzzify these uncertain values and convert them into precise figures. In this study, transportation problem parameters are expressed through neutrosophic numbers (NNs), which are then transformed into crisp values using a new developed ranking function. Numerical examples were used to validate the proposed approach. Results showed improved accuracy and reduced computational effort compared to existing ranking methods.

The proposed ranking function simplifies the transformation of neutrosophic numbers into crisp equivalents for computational ease. Unlike previous methods, it offers lower computational complexity, better accuracy in handling vagueness, and applicability to real-life transportation problems.

As an extension of the present work, alternative formulations can be considered, such as cases where only the transportation costs are represented by neutrosophic numbers (NNs) or where only the supply and demand parameters are modelled using NNs. Additionally, instead of focusing solely on balanced neutrosophic transportation problems (NTP), future studies can explore unbalanced neutrosophic transportation problem extensions (NTPE).

Funding: This research received no external funding.

Acknowledgements: In this section, you can acknowledge any support given which is not covered by the author contribution or funding sections. This may include administrative and technical support, or donations in kind (e.g., materials used for experiments).

Conflicts of Interest: The authors declare no conflict of interest.

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Received: July 26, 2025. Accepted: Sep 9, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

MADM Model Based on Trigonometric Aggregation Operators of Linguistic Neutrosophic Values and Its Application in Selecting the Crawling Robot Design Schemes

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Abstract: Linguistic neutrosophic multi-attribute decision making (MADM) has become one of critical research topics in decision theory. However, existing aggregation operations of linguistic neutrosophic values (LNVs) do not contain operational periodicity. As a result, they are difficult to handle MADM problems with the polytemporal/periodic needs in LNV scenarios. To fill this research gap, this paper intends to develop a MADM model based on the trigonometric aggregation operators (TAOs) of LNVs for MADM applications with polytemporal/periodic needs in LNV scenarios. To do it, we first introduce the linguistic trigonometric t-norms and t-conorms and the LNV trigonometric operation laws based on the trigonometry 1 (including tangent, cotangent, inverse cotangent, and arctangent functions) and the trigonometry 2 (including sine, cosine, arccosine, and arcsine functions). Second, we propose the LNV trigonometric weighted average ($LNVTWA_{T1}$ and $LNVTWA_{T2}$) operators and LNV trigonometric weighted geometric ($LNVTWG_{T1}$ and $LNVTWG_{T2}$) operators based on the trigonometry 1 and 2. Third, a MADM model is developed in terms of one of the proposed four TAOs to address MADM problems with polytemporal phases/periodicity in a LNV scenario. Lastly, the developed MADM model is used for a MADM application of crawling robot design schemes to verify its validity and rationality.

Keywords: Linguistic neutrosophic value; Linguistic trigonometric operation; Linguistic neutrosophic value trigonometric aggregation operator; Decision making; Crawling robot design scheme

1. Introduction

Linguistic multi-attribute decision making (MADM) has become one of the research hotspots because the linguistic assessment reflects the qualitative judgments and representation of decision makers in MADM applications. As a result, linguistic MADM issues have revealed their importance and necessity in decision theory and methodology. Consequently, they have received a great deal of attention from scholars since a linguistic variable was presented by Zadeh [1]. For example, linguistic decision-making (DM) techniques have been developed to perform linguistic DM problems [2-6]. Based on an extension of linguistic DM techniques, linguistic intuitionistic fuzzy DM techniques have been proposed to solve linguistic intuitionistic fuzzy DM problems [7-13]. However, the linguistic intuitionistic fuzzy value (LIFV) can only convey membership/truth and nonmembership/falsity linguistic values but cannot express true, false, and uncertain linguistic values in inconsistent and indeterminate linguistic DM scenarios. Then, linguistic neutrosophic values (LNVs) and their DM techniques [14] were proposed because their more general expression and decision frameworks can fill the research gap of linguistic intuitionistic fuzzy DM techniques. It is well known that LNV aggregation operators (AOs) play an important role in linguistic neutrosophic DM applications. Therefore, Fang and Ye [14] first presented the LNV weighted AOs to address linguistic neutrosophic group DM issues. Then, Fan et al. [15] introduced LNV Bonferroni mean AOs and their group DM approach. Liu and You [16] presented some LNV Hamy mean AOs and their group DM technique. Liang et al. [17] put forward the LNV Hamacher AOs and applied them in the assessment of mine land reclamation. Liu et al. [18] proposed the power Heronian AOs of LNVs for group DM. Fan et al. [19] introduced the LNV Einstein AOs and their DM application. Zhang et al. [20] developed the LNV Dombi AOs for the DM application of slope treatment schemes. Liu and You [21] developed the partitioned Maclaurin symmetric mean AOs for group DM. Zhang and Ye [22] proposed the single-valued neutrosophic value (SVNV) and LNV hybrid AOs for group DM. Luo et al. [23] presented the LNV Maclaurin symmetric mean AOs and used them for the performance evaluation of human resources. Li et al. [24] introduced the reliability allocation method using the LNV weighted Muirhead mean AO. Recently, Ye et al. [25, 26] proposed the trigonometric aggregation operators (TAOs) of SVNVs and single-valued neutrosophic credibility values and their DM techniques, but these TAOs cannot be used in linguistic neutrosophic DM scenarios.

Considering the emerging LNV operations or aggregation algorithms in the existing literature, none of them implies the operational properties of polytemporal phases/periodicity in LNV scenarios. In this case, their DM techniques are also difficult to perform the MADM problems with periodicity/polytemporal phases in LNV scenarios, which show the research gap. However, LNV is a more general linguistic framework that includes the linguistic value and IFLV. Therefore, in LNV scenarios, it is necessary to develop a new MADM technique to fill the existing research gap.

Motivated based on TAOs [25-27], this paper intends to propose a MADM model based on the TAOs of LNVs for addressing the MADM problem with polytemporal phases/periodicity in the LNV scenario. First, we propose the linguistic trigonometric operations (LTOs) of the linguistic trigonometric t-norms and t-conorms and the trigonometric operation laws (TOLs) of LNVs based on the trigonometry 1 (including tangent, cotangent, inverse cotangent, and arctangent functions) and the trigonometry 2 (including sine, cosine, arccosine, and arcsine functions). Next, we propose the LNV trigonometric weighted average (LNVTWAT1 and LNVTWAT2) operators and the LNV trigonometric weighted geometric (LNVTWGT1 and LNVTWGT2) operators in terms of TOLs of LNVs based on the trigonometry 1 and 2. Furthermore, a MADM model is developed based on one of the four proposed LNV TAOs. Finally, the developed MADM model is used for a DM application of crawling robot design schemes (CRDSs). Through the comparison of the developed model with the existing linguistic neutrosophic DM model, the validity and rationality of the linguistic neutrosophic DM application are verified.

Summerly, this original work mainly creates these new achievements below:

- The new TOLs of LNVs are presented based on the trigonometry 1 and 2 to provide the polytemporal/periodic benefits for the LNV operations.
- The $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators are proposed to provide the critical mathematical tools for MADM modeling in LNV scenarios.
- The MADM model using one of the proposed TAOs of LNVs can effectively help decision makers to select the best CRDS and meet the polytemporal/periodic DM requirements in LNV scenarios.

The remainder of the paper includes the following sections. Section 2 reviews the preliminaries of LNVs for the further study of this paper. Section 3 introduces the linguistic trigonometric t-norms and t-conorms and the TOLs of LNVs based on the trigonometry 1 and 2. Section 4 presents the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators based on the TOLs of LNVs and their properties. In Section 5, a MADM model is developed based on one of the four proposed TAOs of LNVs in a LNV scenario. Section 6 uses the developed MADM model for a DM application of CRDSs to verify its rationality and validity, and a comparison with the existing MADM model in the scenario of LNVs reflects the superiority of the developed model. Section 7 presents some conclusions and future research..

2. Preliminaries of LNVs

Set a linguistic term set (LTS) as $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ subject to odd cardinality b+1. Fang and Ye [14] first defined the LNV $\lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ on λ_L such that $\lambda_{td}, \lambda_{ud}, \lambda_{fd} \in \lambda_L$ and $td, ud, fd \in [0, b]$, where λ_{td}, λ_{ud}, and λ_{fd} are the true, uncertain, and false linguistic variables, respectively.

Support that there are two LNVs $\lambda_{LNV(1)} = <\lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)}>$ and $\lambda_{LNV(2)} = <\lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)}>$ in λ_L and $e > 0$. Then their operation laws are introduced below [14]:

(i) $\lambda_{LNV(1)} \oplus \lambda_{LNV(2)} = \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle \oplus \left\langle \lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)} \right\rangle = \left\langle \lambda_{td(1)+td(2)-\frac{td(1)td(2)}{b}}, \lambda_{\frac{ud(1)ud(2)}{b}}, \lambda_{\frac{fd(1)fd(2)}{b}} \right\rangle$;

(ii) $\lambda_{LNV(1)} \otimes \lambda_{LNV(2)} = \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle \otimes \left\langle \lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)} \right\rangle = \left\langle \lambda_{\frac{td(1)td(2)}{b}}, \lambda_{ud(1)+ud(2)-\frac{ud(1)ud(2)}{b}}, \lambda_{fd(1)+fd(2)-\frac{fd(1)fd(2)}{b}} \right\rangle$;

(ii) $e \cdot \lambda_{LNV(1)} = e \cdot \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle = \left\langle \lambda_{b-b\left(1-\frac{td(1)}{b}\right)^e}, \lambda_{b\left(\frac{ud(1)}{b}\right)^e}, \lambda_{b\left(\frac{fd(1)}{b}\right)^e} \right\rangle$;

(iv) $\lambda_{LNV(1)}^e = \left\langle \lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)} \right\rangle^e = \left\langle \lambda_{b\left(\frac{td(1)}{b}\right)^e}, \lambda_{b-b\left(1-\frac{ud(1)}{b}\right)^e}, \lambda_{b-b\left(1-\frac{fd(1)}{b}\right)^e} \right\rangle$.

Regarding a series of LNVs $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ subject to their weights e_j (j = 1, 2, …, p) for $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$, the LNV weighted average (LNVWA) and LNV weighted geometric (LNVWG) operators [14] are introduced below:

$$LNVWA(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \ldots, \lambda_{LNV(p)}) = \sum_{j=1}^{p} e_j \lambda_{LNV(j)} = \left\langle \lambda_{b-b\prod_{j=1}^{p}\left(1-\frac{td(j)}{b}\right)^{e_j}}, \lambda_{b\prod_{j=1}^{p}\left(\frac{ud(j)}{b}\right)^{e_j}}, \lambda_{b\prod_{j=1}^{p}\left(\frac{fd(j)}{b}\right)^{e_j}} \right\rangle, \quad (1)$$

$$LNVWG(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \ldots, \lambda_{LNV(p)}) = \prod_{j=1}^{p} \lambda_{LNV(j)}^{e_j} = \left\langle \lambda_{b\prod_{j=1}^{p}\left(\frac{td(j)}{b}\right)^{e_j}}, \lambda_{b-b\prod_{j=1}^{p}\left(1-\frac{ud(k)}{b}\right)^{e_j}}, \lambda_{b-b\prod_{j=1}^{p}\left(1-\frac{fd(j)}{b}\right)^{e_j}} \right\rangle, \quad (2)$$

Then, Fang and Ye [14] introduced the score and accuracy equations of $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$:

$$E(\lambda_{LNV(j)}) = (2b + td(j) - ud(j) - fd(j))/3b \text{ for } E(\lambda_{LNV(j)}) \in [0, 1], \quad (3)$$

$$F(\lambda_{LNV(j)}) = (td(j) - fd(j))/b \text{ for } E(\lambda_{LNV(j)}) \in [-1, 1], \quad (4)$$

Regarding two LNVs $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ for j = 1, 2, their comparative rules [14] are introduced below:

(i) $\lambda_{LNV(1)} > \lambda_{LNV(2)}$ if $E(\lambda_{LNV(1)}) > E(\lambda_{LNV(2)})$;

(ii) $\lambda_{LNV(1)} > \lambda_{LNV(2)}$ if $E(\lambda_{LNV(1)}) = E(\lambda_{LNV(2)})$ and $F(\lambda_{LNV(1)}) > F(\lambda_{LNV(2)})$;

(iii) $\lambda_{LNV(1)} \cong \lambda_{LNV(2)}$ if $E(\lambda_{LNV(1)}) = E(\lambda_{LNV(2)})$ and $F(\lambda_{LNV(1)}) = F(\lambda_{LNV(2)})$.

3. LTOs and LNV TOLs

Based on the trigonometry 1 and the trigonometry 2 [25–27], this section introduces the linguistic trigonometric t-norms and t-conorms and the LNV TOLs.

First, we define the linguistic trigonometric t-norms and t-conorms based on the trigonometry 1 and 2.

Definition 1. Let two linguistic variables be λ_v, λ_μ in the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$. Based on the trigonometry 1 and 2, the linguistic trigonometric t-norms $N_{R1}(\lambda_v, \lambda_\mu)$ and $N_{R2}(\lambda_v, \lambda_\mu)$ and the linguistic trigonometric t-conorms $N_{S1}(\lambda_v, \lambda_\mu)$ and $N_{S2}(\lambda_v, \lambda_\mu)$ between λ_v and λ_μ are defined as the following LTOs:

$$N_{R1}\left(\lambda_v, \lambda_\mu\right) = \lambda_{2b/\pi \cot^{-1}\left(\cot(n\pi+v\pi/2b)+\cot(n\pi+\mu\pi/2b)\right)} = \lambda_{2b/\pi \cot^{-1}\left(\cot(v\pi/2b)+\cot(\mu\pi/2b)\right)}, \quad (5)$$

$$N_{S1}(\lambda_v, \lambda_\mu) = \lambda_{2b/\pi \tan^{-1}\left(\tan(n\pi+v\pi/2b)+\tan(n\pi+\mu\pi/2b)\right)} = \lambda_{2b/\pi \tan^{-1}\left(\tan(v\pi/2b)+\tan(\mu\pi/2b)\right)}, \quad (6)$$

$$N_{R2}\left(\lambda_{v},\lambda_{\mu}\right)=\lambda_{2b/\pi\sin^{-1}\left(\sin(2n\pi+v\pi/2b)\sin(2n\pi+\mu\pi/2b)\right)}=\lambda_{2b/\pi\sin^{-1}\left(\sin(v\pi/2b)\sin(\mu\pi/2b)\right)},\quad(7)$$

$$N_{S2}(\lambda_{v},\lambda_{\mu})=\lambda_{2b/\pi\cos^{-1}\left(\cos(2n\pi+v\pi/2b)\cos(2n\pi+\mu\pi/2b)\right)}=\lambda_{2b/\pi\cos^{-1}\left(\cos(v\pi/2b)\cos(\mu\pi/2b)\right)},\quad(8)$$

It is clear that there are $n\pi$ (n = 1, 2, ..., p) periodicity in Eqs. (5) and (6) and $2n\pi$ (n = 1, 2, ..., p) periodicity in Eqs. (7) and (8). In terms of the LTOs of the trigonometry 1 and 2, we can define the TOLs of LNVs.

Definition 2. Let $\lambda_{LNV(1)} = <\lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)}>$ and $\lambda_{LNV(2)} = <\lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)}>$ be two LNVs in the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ and $e > 0$. Then, their TOLs of LNVs based on the trigonometry 1 are defined in the following:

(1) $$\lambda_{LNV(1)}\oplus_{T1}\lambda_{LNV(2)}=\left\langle\begin{array}{l}\lambda_{2b/\pi\tan^{-1}\left(\tan(td(1)\pi/2b)+\tan(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(ud(1)\pi/2b)+\cot(ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(fd(1)\pi/2b)+\cot(fd(2)\pi/2b)\right)}\end{array}\right\rangle;$$

(2) $$\lambda_{LNV(1)}\otimes_{T1}\lambda_{LNV(2)}=\left\langle\begin{array}{l}\lambda_{2b/\pi\cot^{-1}\left(\cot(td(1)\pi/2b)+\cot(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\tan^{-1}\left(\tan(ud(1)\pi/2b)+\tan((ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\tan^{-1}\left(\tan(fd(1)\pi/2b)+\tan(fd(2)\pi/2b)\right)}\end{array}\right\rangle;$$

(3) $$e\lambda_{LNV(1)}=\left\langle\lambda_{2b/\pi\tan^{-1}\left(e\tan(td(1)\pi/2b)\right)},\lambda_{2b/\pi\tan^{-1}\left(e\tan(ud(1)\pi/2b)\right)},\lambda_{2b/\pi\cot^{-1}\left(e\cot(fd(1)\pi/2b)\right)}\right\rangle;$$

(4) $$(\lambda_{LNV(1)})^{e}=\left\langle\lambda_{2b/\pi\cot^{-1}\left(e\cot(td(1)\pi/2b)\right)},\lambda_{2b/\pi\tan^{-1}\left(e\tan(ud(1)\pi/2b)\right)},\lambda_{2b/\pi\tan^{-1}\left(e\tan(fd(1)\pi/2b)\right)}\right\rangle.$$

However, the above operation results are still LNVs.

Definition 3. Let $\lambda_{LNV(1)} = <\lambda_{td(1)}, \lambda_{ud(1)}, \lambda_{fd(1)}>$ and $\lambda_{LNV(2)} = <\lambda_{td(2)}, \lambda_{ud(2)}, \lambda_{fd(2)}>$ be two LNVs in the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ and $e > 0$. Then, their TOLs based on the trigonometry 2 are defined in the following:

(1) $$\lambda_{LNV(1)}\oplus_{T2}\lambda_{LNV(2)}=\left\langle\begin{array}{l}\lambda_{2b/\pi\cos^{-1}\left(\cos(td(1)\pi/2b)\cos(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\sin^{-1}\left(\sin(ud(1)\pi/2b)\sin(ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\sin^{-1}\left(\sin(fd(1)\pi/2b)\sin(fd(2)\pi/2b)\right)}\end{array}\right\rangle;$$

(2) $$\lambda_{LNV(1)}\otimes_{T2}\lambda_{LNV(2)}=\left\langle\begin{array}{l}\lambda_{2b/\pi\sin^{-1}\left(\sin(td(1)\pi/2b)\sin(td(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cos^{-1}\left(\cos(ud(1)\pi/2)\cos(ud(2)\pi/2b)\right)},\\ \lambda_{2b/\pi\cos^{-1}\left(\cos(fd(1)\pi/2)\cos(fd(2)\pi/2b)\right)}\end{array}\right\rangle;$$

(3) $$e\lambda_{LNV(1)}=\left\langle\lambda_{2b/\pi\cos^{-1}\left(\cos(td(1)\pi/2b)\right)^{e}},\lambda_{2b/\pi\sin^{-1}\left(\sin(ud(2)\pi/2b)\right)^{e}},\lambda_{2b/\pi\sin^{-1}\left(\sin(fd(2)\pi/2b)\right)^{e}}\right\rangle$$

(4) $$(\lambda_{LNV(1)})^{e}=\left\langle\lambda_{2b/\pi\sin^{-1}\left(\sin(td(1)\pi/2b)\right)^{e}},\lambda_{2b/\pi\cos^{-1}\left(\cos(ud(1)\pi/2b)\right)^{e}},\lambda_{2b/\pi\cos^{-1}\left(\cos(fd(1)\pi/2b)\right)^{e}}\right\rangle,$$

Obviously, the above operation results are still LNVs.

4. TAOs of LNVs based on the trigonometry 1 and 2

According to TOLs of LNVs based on the trigonometry 1 and 2 in Definitions 2 and 3, this section presents four TAOs of LNVs and their characteristics.

4.1 LNVTWA$_{T1}$ and LNVTWA$_{T2}$ operators

This part proposes the LNVTWA$_{T1}$ and LNVTWA$_{T2}$ operators based on the TOLs of LNVs in Definitions 2 and 3.

Definition 4. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs and LNVTWA$_{T1}$ and LNVTWA$_{T2}$: $\Omega^p \to \Omega$. Assume that e_j (j = 1, 2, …, p) is the weight of $\lambda_{LNV(j)}$ with $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the LNVTWA$_{T1}$ and LNVTWA$_{T2}$ operators are defined, respectively, in the following:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = e_1\lambda_{LNV(1)} \oplus_{T1} e_2\lambda_{LNV(2)} \oplus_{T1} ... \oplus_{T1} e_p\lambda_{LNV(p)} = \sum_{j=1}^{p}{}_{T1} e_j\lambda_{LNV(j)}, \quad (9)$$

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = e_1\lambda_{LNV(1)} \oplus_{T2} e_2\lambda_{LNV(2)} \oplus_{T2} ... \oplus_{T2} e_p\lambda_{LNV(p)} = \sum_{j=1}^{p}{}_{T2} e_j\lambda_{LNV(j)}. \quad (10)$$

Theorem 1. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs. Assume that e_j (j = 1, 2, …, p) is the weight of $\lambda_{LNV(j)}$ with $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the LNVTWA$_{T1}$ operator is LNV, which is gotten by

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \sum_{j=1}^{p}{}_{T1} e_j\lambda_{LNV(j)}$$
$$= \left\langle \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(td(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(ud(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(fd(j)\pi/2b)\right)\right)} \right\rangle, \quad (11)$$

Proof: Mathematical induction is applied to the proof of Eq. (11).

(1) Set p = 2. The operational result corresponding to the TOLs (1) and (3) in Definition 2 is gotten below:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}) = e_1\lambda_{LNV(1)} \oplus_{T1} e_2\lambda_{LNV(2)}$$
$$= \left\langle \begin{array}{l} \lambda_{2b/\pi \tan^{-1}\left(\tan\left(\pi/2b \times 2b/\pi \tan^{-1}(e_1 \tan(td(1)\pi/2b))\right) + \tan\left(\pi/2b \times 2b/\pi \tan^{-1}(e_2 \tan(td(2)\pi/2b))\right)\right)}, \\ \lambda_{2b/\pi \cot^{-1}\left(\cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_1 \cot(ud(1)\pi/2b))\right) + \cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_2 \cot(ud(2)\pi/2b))\right)\right)}, \\ \lambda_{2b/\pi \cot^{-1}\left(\cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_1 \cot(fd(1)\pi/2b))\right) + \cot\left(\pi/2b \times 2b/\pi \cot^{-1}(e_2 \cot(fd(2)\pi/2b))\right)\right)} \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \lambda_{2b/\pi \tan^{-1}\left(e_1 \tan(td(1)\pi/2b) + e_2 \tan(td(2)\pi/2b)\right)}, \\ \lambda_{2b/\pi \cot^{-1}\left(e_1 \cot(ud(1)\pi/2b) + e_2 \cot(ud(2)\pi/2b)\right)}, \\ \lambda_{2b/\pi \cot^{-1}\left(e_1 \cot(fd(1)\pi/2b) + e_2 \cot(fd(2)\pi/2b)\right)} \end{array} \right\rangle, \quad (12)$$
$$= \left\langle \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{2} e_j \tan(te(j)\pi/2b)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{2} e_j \cot(ue(j)\pi/2b)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{2} e_j \cot(fe(j)\pi/2b)\right)} \right\rangle.$$

(2) Set $p = m$. Eq. (11) can keep the equation:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(m)}) = \sum_{j=1}^{m}{}_{T1} e_j\lambda_{LNV(j)}$$
$$= \left\langle \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{m}\left(e_j \tan(td(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{m}\left(e_j \cot(ud(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{m}\left(e_j \cot(fd(j)\pi/2b)\right)\right)} \right\rangle, \quad (13)$$

(3) Set p = m+1. In terms of the TOLs (1) and (3) in Definition 2 and Eqs. (12) and (13) , the operational result is given below:

$$LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(m)},\lambda_{LNV(m+1)})=\sum_{j=1}^{m+1}{}_{T1}e_j\lambda_{LNFV(j)}$$

$$=\left\langle \lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{m}\left(e_j\tan(td(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(ud(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(fd(j)\pi/2b)\right)\right)}\right\rangle$$
$$\oplus_{T1}e_{m+1}\lambda_{LNV(m+1)}$$

$$=\left\langle \lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{m}\left(e_j\tan(td(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(fd(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m}\left(e_j\cot(fd(j)\pi/2b)\right)\right)}\right\rangle$$
$$\oplus_{T1}\left\langle \lambda_{2b/\pi\tan^{-1}\left(e_{m+1}\tan(td(j)\pi/2b)\right)},\lambda_{2b/\pi\cot^{-1}\left(e_{m+1}\cot(ud(j)\pi/2b)\right)},\lambda_{2b/\pi\cot^{-1}\left(e_{m+1}\cot(fd(j)\pi/2b)\right)}\right\rangle$$

$$=\left\langle \lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{m+1}\left(e_j\tan(td(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m+1}\left(e_j\cot(ud(j)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{m+1}\left(e_j\cot(fd(j)\pi/2b)\right)\right)}\right\rangle.$$

Regarding the results of (1)–(3), Eq. (11) can exist for any p.

Thus, this proof is ended.

Theorem 2. The LNVTWA$_{T1}$ operator of Eq. (11) implies the characteristics:

(1)Idempotency: Set $\lambda_{LNV(j)}=<\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)}=\lambda_{LNV}=<\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, …, p), then $LNVTWA_{T1}\left(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)}\right)=\lambda_{LNV}$.

(2)Boundedness: Set $\lambda_{LNV(j)}=<\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min}=\left\langle \min_j\left(\lambda_{td(j)}\right),\max_j\left(\lambda_{ud(j)}\right),\max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max}=\left\langle \max_j\left(\lambda_{td(j)}\right),\min_j\left(\lambda_{ud(j)}\right),\min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, $\lambda_{LNV\min}\le LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)})\le\lambda_{LNV\max}$ exists.

(3)Monotonicity: Set $\lambda_{LNV(j)}=<\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)}=\left\langle \lambda^*_{td(j)},\lambda^*_{ud(j)},\lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. If $\lambda_{LNV(j)}\le\lambda^*_{LNV(j)}$, then there is $LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)})\le LNVTWA_{T1}(\lambda^*_{LNV(1)},\lambda^*_{LNV(2)},...,\lambda^*_{LNV(p)})$.

Proof:

(1) Since $\lambda_{LNV(j)}=\lambda_{LNV}=<\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, …, p), the aggregated result of Eq. (11) is given below:

$$LNVTWA_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},...,\lambda_{LNV(p)})=\sum_{j=1}^{p}{}_{T1}e_j\lambda_{LNV}$$

$$=\left\langle \begin{matrix}\lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{p}e_j\tan(td\cdot\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}e_j\cot(ud\cdot\pi/2b)\right)}\\ \lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}e_j\cot(fd\cdot\pi/2b)\right)}\end{matrix}\right\rangle=\left\langle \begin{matrix}\lambda_{2b/\pi\tan^{-1}\left(\tan(td\cdot\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(ud\cdot\pi/2b)\right)},\\ \lambda_{2b/\pi\cot^{-1}\left(\cot(fd\cdot\pi/2b)\right)}\end{matrix}\right\rangle=\left\langle \lambda_{td},\lambda_{ud},\lambda_{fd}\right\rangle=\lambda_{LNV}.$$

(2)Since $\lambda_{LNV\min}$ and $\lambda_{LNV\max}$ are the minimum and maximum LNVs, $\lambda_{LNV\min} \le \lambda_{LNV(j)} \le \lambda_{LNV\max}$ exists. Consequently, $\sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV\min} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV\max}$ also exists. In terms of the characteristic (1) and the characteristics of the trigonometric functions, $\lambda_{LNV\min} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)} \le \lambda_{LNV\max}$ can exist, i.e., $\lambda_{LNV\min} \le LNVTWA_{T1}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) \le \lambda_{LNV\max}$.

(3)Since $\lambda_{LNV(j)} \le \lambda^{*}_{LNV(j)}$, there is $\sum_{j=1}^{p}{}_{T1} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T1} e_j \lambda^{*}_{LNV(j)}$, i.e., $LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWA_{T1}(\lambda^{*}_{LNV(1)}, \lambda^{*}_{LNV(2)}, ..., \lambda^{*}_{LNV(p)})$.

Therefore, these characteristics are true.

Theorem 3. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) (j = 1, 2, …, p) be p LNVs and set e_j as the weight of $\lambda_{LNV(j)}$ (j = 1, 2, …, p) with $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the LNVTWA$_{T2}$ operator is LNV, which is given by

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)}$$
$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle, \quad (14)$$

Proof: Mathematical induction is applied to the proof of Eq. (14).

(1) Set p = 2. Using the TOLs (1) and (3) in Definition 3, we give the result:

$$LNVTWA_{T2}(\lambda_{IFV(1)}, \lambda_{IFV(2)}) = e_1\lambda_{LNV(1)} \oplus_{T2} e_2\lambda_{LNV(2)} = \sum_{j=1}^{2}{}_{T2} e_j \lambda_{LNV(j)}$$
$$= \left\langle \begin{array}{l} \lambda_{2b/\pi \cos^{-1}\left(\cos\left(\pi/2b \times 2b/\pi \cos^{-1}(\cos(td(1)\pi/2b))^{e_1}\right)\cos\left(\pi/2b \times 2b/\pi \cos^{-1}(\cos(td(2)\pi/2b))^{e_2}\right)\right)}, \\ \lambda_{2b/\pi \sin^{-1}\left(\sin\left(\pi/2b \times 2b/\pi \sin^{-1}(\sin(ud(1)\pi/2b))^{e_1}\right)\sin\left(\pi/2b \times 2b/\pi \sin^{-1}(\sin(ud(2)\pi/2))^{e_2}\right)\right)}, \\ \lambda_{2b/\pi \sin^{-1}\left(\sin\left(\pi/2b \times 2b/\pi \sin^{-1}(\sin(fd(1)\pi/2b))^{e_1}\right)\sin\left(\pi/2b \times 2b/\pi \sin^{-1}(\sin(fd(2)\pi/2))^{e_2}\right)\right)} \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} \lambda_{2b/\pi \cos^{-1}\left((\cos(td(1)\pi/2b))^{e_1}(\cos(td(2)\pi/2b))^{e_2}\right)}, \\ \lambda_{2b/\pi \sin^{-1}\left((\sin(ud(1)\pi/2b))^{e_1}(\sin(ud(2)\pi/2b))^{e_2}\right)}, \\ \lambda_{2b/\pi \sin^{-1}\left((\sin(fd(1)\pi/2b))^{e_1}(\sin(fd(2)\pi/2b))^{e_2}\right)} \end{array} \right\rangle \quad (15)$$
$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{2}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{2}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{2}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle.$$

(2)Set $p = m$. Eq. (14) can keep the following equation:

$$LNVTWA_{T2}(\lambda_{IFV(1)}, \lambda_{IFV(2)}, ..., \lambda_{IFV(m)}) = \sum_{j=1}^{m}{}_{T2} e_j \lambda_{LNV(j)}$$
$$= \left\langle \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{m}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{m}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle, \quad (16)$$

(3)Set $p = m+1$. In terms of the TOLs (1) and (3) in Definition 3 and Eqs. (15) and (16), we can give the result:

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(m+1)}) = \sum_{j=1}^{m+1}{}_{T2} e_j \lambda_{LNV(j)}$$

$$= \left\langle \lambda_{2b/\pi\cos^{-1}\left(\prod_{j=1}^{m}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{m}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{m}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle$$

$$\oplus_{T2} e_{m+1}\lambda_{LNV(m+1)}$$

$$= \left\langle \lambda_{2b/\pi\cos^{-1}\left(\prod_{j=1}^{m+1}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{m+1}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{m+1}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle.$$

In terms of the results of (1)–(3), Eq. (14) can exist for any p.

Consequently, this proof is ended. □

Theorem 4. The $LNVTWA_{T2}$ operator of Eq. (14) implies the characteristics below.

(1)Idempotency: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, ..., p), then $LNVTWA_{T2}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) = \lambda_{LNV}$.

(2)Boundedness: Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min} = \left\langle \min_j\left(\lambda_{td(j)}\right), \max_j\left(\lambda_{ud(j)}\right), \max_j\left(\lambda_{fd(j)}\right) \right\rangle$ and $\lambda_{LNV\max} = \left\langle \max_j\left(\lambda_{td(j)}\right), \min_j\left(\lambda_{ud(j)}\right), \min_j\left(\lambda_{fd(j)}\right) \right\rangle$. Consequently, $\lambda_{LNV\min} \le LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le \lambda_{LNV\max}$ exists.

(3)Monotonicity: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)} = \left\langle \lambda^*_{td(j)}, \lambda^*_{ud(j)}, \lambda^*_{fd(j)} \right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. $LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWA_{T2}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$ exists when $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$.

Proof:

(1) For $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, ..., p), the aggregated result of Eq. (14) is given below:

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)}$$

$$= \left\langle \lambda_{2b/\pi\cos^{-1}\left(\prod_{j=1}^{g}(\cos(td(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{p}(\sin(ud(j)\pi/2b))^{e_j}\right)}, \lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{p}(\sin(fd(j)\pi/2b))^{e_j}\right)} \right\rangle$$

$$= \left\langle \lambda_{2b/\pi\cos^{-1}(\cos(td(j)\pi/2b))^{\sum_{j=1}^{p} e_j}}, \lambda_{2b/\pi\sin^{-1}(\sin(ud(j)\pi/2b))^{\sum_{j=1}^{p} e_j}}, \lambda_{2b/\pi\sin^{-1}(\sin(fd(j)\pi/2b))^{\sum_{j=1}^{p} e_j}} \right\rangle$$

$$= \left\langle \lambda_{td}, \lambda_{ud}, \lambda_{fd} \right\rangle = \lambda_{LNV}.$$

(2) Since λ_{LNVmin} and λ_{LNVmax} are the minimum and maximum LNVs, $\lambda_{LNVmin} \le \lambda_{LNV(j)} \le \lambda_{LNVmax}$ exists. Consequently, $\sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV\min} \le \sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV\max}$ also exists. In terms of the characteristic (1) and the characteristics of the trigonometric functions, there is $\lambda_{LNV\min} \le \sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)} \le \lambda_{LNV\max}$, i.e., $\lambda_{LNV\min} \le LNVTWA_{T2}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) \le \lambda_{LNV\max}$.

(3) For $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$, there is $\sum_{j=1}^{p}{}_{T2} e_j \lambda_{LNV(j)} \le \sum_{j=1}^{p}{}_{T2} e_j \lambda^*_{LNV(j)}$, i.e., $LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWA_{T2}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$.

Therefore, the characteristics of (1)-(3) are true. □

Example 1. Let three LNVs be $\lambda_{LNV(1)} = <\lambda_7, \lambda_2, \lambda_3>$, $\lambda_{LNV(2)} = <\lambda_6, \lambda_1, \lambda_2>$, and $\lambda_{LNV(3)} = <\lambda_5, \lambda_4, \lambda_5>$ in the LTS $\lambda_L = \{\lambda_0, \lambda_1, …, \lambda_8\}$ subject to their weight vector $\boldsymbol{e}$ = (0.3, 0.4, 0.3). Using Eqs. (11) and (14), their aggregated values are calculated below:

$$LNVTWA_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \lambda_{LNV(3)}) = \sum_{j=1}^{3}{}_{T1} e_j \lambda_{LNV(j)}$$

$$= \left\langle \begin{matrix} \lambda_{2\times 8/\pi \tan^{-1}\left(0.3\times\tan(7\pi/(2\times 8))+0.4\times\tan(6\pi/(2\times 8))+0.3\times\tan(5\pi/(2\times 8))\right)}, \\ \lambda_{2\times 8/\pi \cot^{-1}\left(0.3\times\cot(2\pi/(2\times 8))+0.4\times\cot(\pi/(2\times 8))+0.3\times\cot(4\pi/(2\times 8))\right)}, \\ \lambda_{2\times 8/\pi \cot^{-1}\left(0.3\times\cot(3\pi/(2\times 8))+0.4\times\cot(2\pi/(2\times 8))+0.3\times\cot(5\pi/(2\times 8))\right)} \end{matrix} \right\rangle = \left\langle \lambda_{6.3211}, \lambda_{1.6209}, \lambda_{2.8234} \right\rangle,$$

$$LNVTWA_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, \lambda_{LNV(3)}) = \sum_{j=1}^{3}{}_{T2} e_j \lambda_{LNV(j)}$$

$$= \left\langle \begin{matrix} \lambda_{2\times 8/\pi \cos^{-1}\left((\cos(7\pi/(2\times 8)))^{0.3}\times(\cos(6\pi/2))^{0.4}\times(\cos(5\pi/(2\times 8)))^{0.3}\right)}, \\ \lambda_{2\times 8/\pi \sin^{-1}\left((\sin(2\pi/(2\times 8)))^{0.3}\times(\sin(\pi/(2\times 8)))^{0.4}\times(\sin(4\pi/(2\times 8)))^{0.3}\right)}, \\ \lambda_{2\times 8/\pi \sin^{-1}\left((\sin(3\pi/(2\times 8)))^{0.3}\times(\sin(2\pi/(2\times 8)))^{0.4}\times(\sin(5\pi/(2\times 8)))^{0.3}\right)} \end{matrix} \right\rangle = \left\langle \lambda_{6.1808}, \lambda_{1.8287}, \lambda_{2.9061} \right\rangle.$$

4.2 LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators

This part proposes the LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators in terms of the TOLs of LNVs in Definitions 2 and 3.

Definition 5. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs and LNVTWG$_{T1}$ and LNVTWG$_{T2}$: $\Omega^p \to \Omega$. Assume that e_j (j = 1, 2, …, p) is the weight of $\lambda_{LNV(j)}$ subject to $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the LNVTWG$_{T1}$ and LNVTWG$_{T2}$ operators are defined below:

$$LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = (\lambda_{LNV(1)})^{e_1} \otimes_{T1} (\lambda_{LNV(2)})^{e_2} \otimes_{T1} ... \otimes_{T1} (\lambda_{LNV(p)})^{e_p} = \prod_{j=1}^{p}{}_{T1} (\lambda_{LNV(j)})^{e_j}, \quad (17)$$

$$LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = (\lambda_{LNV(1)})^{e_1} \otimes_{T2} (\lambda_{LNV(2)})^{e_2} \otimes_{T2} ... \otimes_{T2} (\lambda_{LNV(p)})^{e_p} = \prod_{j=1}^{p}{}_{T2} (\lambda_{LNV(j)})^{e_j}, \quad (18)$$

Theorem 5. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) be p LNVs and set e_j as the weight of $\lambda_{LNV(j)}$ (j = 1, 2, …, p) subject to $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the LNVTWG$_{T1}$ operator is LNV, which is gotten by

$$LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \prod_{j=1}^{p}{}_{T1} (\lambda_{LNV(j)})^{e_j}$$

$$= \left\langle \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(td(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(ud(j)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(fd(j)\pi/2b)\right)\right)} \right\rangle, \quad (19)$$

Proof: By a similar proof of Theorem 1, Theorem 5 can be verified (omitted here).

Theorem 6. The LNVTWG$_{T1}$ operator of Eq. (19) contains the characteristics below:

(1)Idempotency: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, …, p), then $LNVTWG_{T1}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) = \lambda_{LNV}$.

(2)Boundedness: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min} = \left\langle \min_j\left(\lambda_{td(j)}\right), \max_j\left(\lambda_{ud(j)}\right), \max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max} = \left\langle \max_j\left(\lambda_{td(j)}\right), \min_j\left(\lambda_{ud(j)}\right), \min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, there is this inequation $\lambda_{LNV\min} \le LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le \lambda_{LNV\max}$.

(3)Monotonicity: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)} = \left\langle \lambda^*_{td(j)}, \lambda^*_{ud(j)}, \lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. If $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$, then there is the inequation $LNVTWG_{T1}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWG_{T1}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$.

Proof. In view of a similar proof of Theorem 2, Theorem 6 can be verified (omitted here).

Theorem 7. Let $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) (j = 1, 2, …, p) be p LNVs and set e_j as the weight of $\lambda_{LNV(j)}$ (j = 1, 2, …, p) subject to $e_j \in [0, 1]$ and $\sum_{j=1}^{p} e_j = 1$. Then, the aggregated value of the LNVTWG$_{T2}$ operator is LNV, which is gotten by

$$LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) = \prod_{j=1}^{p} T2(\lambda_{LNV(j)})^{e_j}$$

$$= \left\langle \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(td(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(ud(j)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(fd(j)\pi/2b)\right)^{e_j}\right)} \right\rangle, \quad (20)$$

Proof: According to a similar verification of Theorem 3, Theorem 7 can be verified (omitted here).

Theorem 8. The LNVTWG$_{T2}$ operator of Eq. (20) contains the characteristics below:

(1)Idempotency: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs. If $\lambda_{LNV(j)} = \lambda_{LNV} = <\lambda_{td}, \lambda_{ud}, \lambda_{fd}>$ (j = 1, 2, …, p), then $LNVTWG_{T2}\left(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}\right) = \lambda_{LNV}$.

(2)Boundedness: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ (j = 1, 2, …, p) as p LNVs and set the minimum and maximum LNVs as $\lambda_{LNV\min} = \left\langle \min_j\left(\lambda_{td(j)}\right), \max_j\left(\lambda_{ud(j)}\right), \max_j\left(\lambda_{fd(j)}\right)\right\rangle$ and $\lambda_{LNV\max} = \left\langle \max_j\left(\lambda_{td(j)}\right), \min_j\left(\lambda_{ud(j)}\right), \min_j\left(\lambda_{fd(j)}\right)\right\rangle$. Consequently, there is this inequation $\lambda_{LNV\min} \le LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le \lambda_{LNV\max}$.

(3)Monotonicity: Set $\lambda_{LNV(j)} = <\lambda_{td(j)}, \lambda_{ud(j)}, \lambda_{fd(j)}>$ and $\lambda^*_{LNV(j)} = \left\langle \lambda^*_{td(j)}, \lambda^*_{ud(j)}, \lambda^*_{fd(j)}\right\rangle$ (j = 1, 2, …, p) as two groups of LNVs. If $\lambda_{LNV(j)} \le \lambda^*_{LNV(j)}$, then there is the inequation $LNVTWG_{T2}(\lambda_{LNV(1)}, \lambda_{LNV(2)}, ..., \lambda_{LNV(p)}) \le LNVTWG_{T2}(\lambda^*_{LNV(1)}, \lambda^*_{LNV(2)}, ..., \lambda^*_{LNV(p)})$.

Proof: According to a similar verification of Theorem 4, Theorem 8 can be verified (omitted here).

Example 2. Let three LNVs be $\lambda_{LNVV(1)} = <\lambda_7, \lambda_1, \lambda_3>$, $\lambda_{LNVV(2)} = <\lambda_6, \lambda_3, \lambda_3>$, and $\lambda_{LNVV(3)} = <\lambda_5, \lambda_2, \lambda_2>$ in the LTS $\lambda_L = \{\lambda_0, \lambda_1, …, \lambda_8\}$ subject to the weight vector $\boldsymbol{e}$ = (0.3, 0.3, 0.4). Using Eqs. (19) and (20), their aggregated values are given below:

$$LNVTWG_{T1}(\lambda_{LNV(1)},\lambda_{LNV(2)},\lambda_{LNV(3)})=\prod_{j=1}^{3}{}_{T1}(\lambda_{LNV(j)})^{e_j}$$

$$=\left\langle\begin{array}{l}\lambda_{2\times8/\pi\cot^{-1}\left(0.3\times\cot(7\pi/(2\times8))+0.3\times\cot(6\pi/(2\times8))+0.4\times\cot(5\pi/(2\times8))\right)},\\ \lambda_{2\times8/\pi\tan^{-1}\left(0.3\times\tan(\pi/(2\times8))+0.3\times\tan(3\pi/(2\times8))+0.4\times\tan(2\pi/(2\times8))\right)},\\ \lambda_{2\times8/\pi\tan^{-1}\left(0.3\times\tan(3\pi/(2\times8))+0.3\times\tan(3\pi/(2\times8))+0.4\times\tan(2\pi/(2\times8))\right)}\end{array}\right\rangle=\left\langle\lambda_{5.8413},\lambda_{2.0502},\lambda_{2.6254}\right\rangle,$$

$$LNVTWG_{T2}(\lambda_{IFV(1)},\lambda_{IFV(2)},\lambda_{IFV(3)})=\prod_{j=1}^{3}{}_{T2}(\lambda_{LNV(j)})^{e_j}$$

$$=\left\langle\begin{array}{l}\lambda_{2\times8/\pi\sin^{-1}\left((\sin(7\pi/(2\times8)))^{0.3}\times(\sin(6\pi/(2\times8)))^{0.3}\times(\sin(5\pi/(2\times8)))^{0.4}\right)},\\ \lambda_{2\times8/\pi\cos^{-1}\left((\cos(\pi/(2\times8)))^{0.3}\times(\cos(3\pi/(2\times8)))^{0.3}\times(\cos(2\pi/(2\times8)))^{0.4}\right)},\\ \lambda_{2\times8/\pi\cos^{-1}\left((\cos(3\pi/(2\times8)))^{0.3}\times(\cos(3\pi/(2\times8)))^{0.3}\times(\cos(2\pi/(2\times8)))^{0.4}\right)}\end{array}\right\rangle=\left\langle\lambda_{5.7237},\lambda_{2.161},\lambda_{2.654}\right\rangle$$

5. MADM model using one of the four proposed LNV TAOs

This section develops a MADM model using one of the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators to handle MADM issues in the scenario of LNVs.

In a MADM problem, there are a set of s alternatives $Ka = \{Ka_1, Ka_2, \ldots, Ka_s\}$ and a set of p attributes $Ra = \{Ra_1, Ra_2, \ldots, Ra_p\}$. In the assessment process, the alternatives must meet the attribute requirements, then their assessment values can be assigned by the LNVs obtained from the LTS $\lambda_L = \{\lambda_0, \lambda_1, \ldots, \lambda_b\}$ with odd cardinality b+1. All the assessed LNVs are formed as their decision matrix $M_L = (\lambda_{LNV(kj)})_{s\times p}$, where $\lambda_{LNV(kj)} = <\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)}>$ for $\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)} \in \lambda_L$ (j = 1, 2, …, p; k = 1, 2, …, s) are LNVs to express the decision makers' true, false and uncertain linguistic values corresponding to their satisfactory degrees of each alternative Ka_k on the attributes Ra_j. The weight vector of the attributes is given by $\boldsymbol{e} = (e_1, e_2, \ldots, e_p)$ subject to $0 \le e_j \le 1$ and $\sum_{j=1}^{p} e_j = 1$. Regarding the MADM problem in the scenario of LNVs, the decision algorithm of the MADM model is indicated below.

Step 1: Get the aggregated value $\lambda_{LNV(k)}$ for each Ka_k (k = 1, 2, …, s) by one of the following $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators:

$$\lambda_{LNV(k)}=LNVTWA_{T1}(\lambda_{LNV(k1)},\lambda_{LNV(k2)},\ldots,\lambda_{LNV(kp)})=\sum_{j=1}^{p}{}_{T1}e_j\lambda_{LNV(kj)}$$

$$=\left\langle\lambda_{2b/\pi\tan^{-1}\left(\sum_{j=1}^{p}\left(e_j\tan(td(kj)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}\left(e_j\cot(ud(kj)\pi/2b)\right)\right)},\lambda_{2b/\pi\cot^{-1}\left(\sum_{j=1}^{p}\left(e_j\cot(fd(kj)\pi/2b)\right)\right)}\right\rangle, \quad (21)$$

$$\lambda_{LNV(k)}=LNVTWA_{T2}(\lambda_{LNV(k1)},\lambda_{LNV(k2)},\ldots,\lambda_{LNV(kp)})=\sum_{j=1}^{p}{}_{T2}e_j\lambda_{LNV(kj)}$$

$$=\left\langle\lambda_{2b/\pi\cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(td(kj)\pi/2b)\right)^{e_j}\right)},\lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(ud(kj)\pi/2b)\right)^{e_j}\right)},\lambda_{2b/\pi\sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(fd(kj)\pi/2b)\right)^{e_j}\right)}\right\rangle, \quad (22)$$

$$\lambda_{LNV(k)} = LNVTWG_{T1}(\lambda_{LNV(k1)}, \lambda_{LNV(k2)}, ..., \lambda_{LNV(kp)}) = \prod_{j=1}^{p} {}_{T1}(\lambda_{LNV(kj)})^{e_j}$$
$$= \left\langle \lambda_{2b/\pi \cot^{-1}\left(\sum_{j=1}^{p}\left(e_j \cot(td(kj)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(ud(kj)\pi/2b)\right)\right)}, \lambda_{2b/\pi \tan^{-1}\left(\sum_{j=1}^{p}\left(e_j \tan(fd(kj)\pi/2b)\right)\right)} \right\rangle, \quad (23)$$

$$\lambda_{LNV(k)} = LNVTWG_{T2}(\lambda_{LNV(k1)}, \lambda_{LNV(k2)}, ..., \lambda_{LNV(kp)}) = \prod_{j=1}^{p} {}_{T2}(\lambda_{LNV(kj)})^{e_j}$$
$$= \left\langle \lambda_{2b/\pi \sin^{-1}\left(\prod_{j=1}^{p}\left(\sin(td(kj)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(ud(kj)\pi/2b)\right)^{e_j}\right)}, \lambda_{2b/\pi \cos^{-1}\left(\prod_{j=1}^{p}\left(\cos(fd(kj)\pi/2b)\right)^{e_j}\right)} \right\rangle. \quad (24)$$

Step 2: Calculate the score (accuracy) values of $E(\lambda_{LNV(k)})$ ($F(\lambda_{LNV(k)})$) (k = 1, 2, …, s) by Eq. (3) (Eq. (4) for necessity).

Step 3: Rank alternatives based on the descending order of the score values (accuracy values) and decide the best one.

Step 4: End.

6. Practical application of the proposed MADM model

6.1 Selection of CRDSs

This part applies the proposed MADM model in the choice problem of CRDSs for a manufacturing company in China to show the validity of the developed MADM model in the scenario of LNVs.

Crawling robots have been used in a wide range of fields due to their flexibility and adaptability. For search and rescue missions into hazardous areas after earthquakes, floods, and other disasters, the technical department of a manufacturing company in China presents the four potential design schemes of crawling robots: the track-driven crawling robot (Ka_1), the wheeled crawling robot (Ka_2), the crawler robot (Ka_3), and the wheel-track crawler robot (Ka_4), then they are expressed as a set of the four alternatives Ka = {Ka_1, Ka_2, Ka_3, Ka_4} for the choice of decision makers. In the assessment process, the four design schemes must meet four key requirements (attributes): technical conditions (Ra_1), manufacturing cost (Ra_2), flexible and fast movement ability (Ra_3), and adaptability to complex environments (Ra_4), then their weigh vector is given by $\boldsymbol{e}$ = (0.3, 0.3, 0.2, 0.2) as the known attribute weights. In this MADM issue, decision makers/engineers are invited to assess the satisfaction of the four CRDSs corresponding to the four attributes. Subsequently, the assessment values of their satisfaction are presented by the true, false, and uncertain linguistic values obtained from the given LTS λ_L = {λ_0(Extremely low), λ_1(Very low), λ_2(Low), λ_3(Slightly high), λ_4(Medium), λ_5(Slightly high), λ_6(High), λ_7(Very high), λ_8(Extremely high} with b = 8. Thus, all the given LNVs $\lambda_{LNV(kj)} = <\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)}>$ for $\lambda_{td(kj)}, \lambda_{ud(kj)}, \lambda_{fd(kj)} \in \lambda_L$ (k, j = 1, 2, 3, 4) can be created as their assessment matrix:

$$M_L = \begin{bmatrix} <\lambda_6,\lambda_1,\lambda_2> & <\lambda_7,\lambda_3,\lambda_2> & <\lambda_5,\lambda_2,\lambda_2> & <\lambda_6,\lambda_3,\lambda_2> \\ <\lambda_7,\lambda_2,\lambda_3> & <\lambda_7,\lambda_3,\lambda_2> & <\lambda_7,\lambda_3,\lambda_2> & <\lambda_6,\lambda_2,\lambda_4> \\ <\lambda_6,\lambda_2,\lambda_2> & <\lambda_6,\lambda_2,\lambda_4> & <\lambda_6,\lambda_2,\lambda_1> & <\lambda_6,\lambda_2,\lambda_3> \\ <\lambda_7,\lambda_3,\lambda_2> & <\lambda_7,\lambda_2,\lambda_3> & <\lambda_7,\lambda_2,\lambda_2> & <\lambda_5,\lambda_1,\lambda_2> \end{bmatrix}.$$

First, using one of Eqs. (21)–(24), we get the aggregated values of $\lambda_{LNV(k)}$ for Ka_k (k = 1, 2, …, s):

(1) The aggregated values of the LNVTWA$_{T1}$ operator are $\lambda_{LNV(1)} = <\lambda_{6.3688}, \lambda_{1.7826}, \lambda_2>$, $\lambda_{LNV(2)} = <\lambda_{6.8875}, \lambda_{2.4076}, \lambda_{2.5168}>$, $\lambda_{LNV(3)} = <\lambda_6, \lambda_2, \lambda_{2.0655}>$, and $\lambda_{LNV(4)} = <\lambda_{6.8418}, \lambda_{1.8304}, \lambda_{2.2273}>$;

(2) The aggregated values of the LNVTWA$_{T2}$ operator are $\lambda_{LNV(1)} = <\lambda_{6.2502}, \lambda_{1.9671}, \lambda_2>$, $\lambda_{LNV(2)} = <\lambda_{6.8534}, \lambda_{2.4407}, \lambda_{2.5724}>$, $\lambda_{LNV(3)} = <\lambda_6, \lambda_2, \lambda_{2.2819}>$, and $\lambda_{LNV(4)} = <\lambda_{6.763}, \lambda_{1.9523}, \lambda_{2.2523}>$;

(3) The aggregated values of the $LNVTWG_{T1}$ operator are $\lambda_{LNV(1)} = <\lambda_{6.0603}, \lambda_{2.2651}, \lambda_2>$, $\lambda_{LNV(2)} = <\lambda_{6.7909}, \lambda_{2.5264}, \lambda_{2.7806}>$, $\lambda_{LNV(3)} = <\lambda_6, \lambda_2, \lambda_{2.7436}>$, and $\lambda_{LNV(4)} = <\lambda_{6.5495}, \lambda_{2.1423}, \lambda_{2.3221}>$;

(4) The aggregated values of the $LNVTWG_{T2}$ operator are $\lambda_{LNV(1)} = <\lambda_{5.9617}, \lambda_{2.3853}, \lambda_2>$, $\lambda_{LNV(2)} = <\lambda_{6.7313}, \lambda_{2.5584}, \lambda_{2.84}>$, $\lambda_{LNV(3)} = <\lambda_6, \lambda_2, \lambda_{2.8744}>$, and $\lambda_{LNV(4)} = <\lambda_{6.3651}, \lambda_{2.2272}, \lambda_{2.3534}>$.

Then using Eq. (3) (Eq. (4)), the score (accuracy) values of $E(\lambda_{LNV(k)})$ (k = 1, 2, 3, 4) and the ranking results of the four CRDSs are shown in Table 1.

Table 1. Decision results of the developed MADM model and the existing MADM model

AO	Score value	Ranking	Best CRDS
$LNVTWA_{T1}$ operator	0.7744, 0.7485, 0.7473, 0.7827	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
$LNVTWA_{T2}$ operator	0.7618, 0.7433, 0.7383, 0.7733	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
$LNVTWG_{T1}$ operator	0.7415, 0.7285, 0.7190, 0.7535	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
$LNVTWG_{T2}$ operator	0.7324, 0.7222, 0.7136, 0.7410	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVWA operator [14]	0.7604, 0.7420, 0.7365, 0.7721	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4
LNVWG operator [14]	0.7415, 0.7293, 0.7201, 0.7535	$Ka_4 > Ka_1 > Ka_2 > Ka_3$	Ka_4

In terms of the ranking results in Table 1, there is the same ranking order based on the four proposed TAOs of LNVs, which shows the ranking validity and robustness of the developed MADM model using one of Eqs. (21)–(24) in the scenario of LNVs.

6.2 Comparison with the existing MADM model using the LNVWA and LNVWG operators

To indicate the validity of the developed MADM model using one of the four presented TAOs, it is compared with the existing MADM model using the LNVWA and LNVWG operators [14] by the selection example of the four CRDSs in the scenario of LNVs.

Applying Eqs. (1)–(3) based on the existing DM model [14], we get the aggregated values of LNVs and the score values of $E(\lambda_{LNV(k)})$ (k = 1, 2, 3, 4) and the ranking results of the four CRDSs are also shown in Table 1.

In terms of all the ranking results in Table 1, there is the same ranking order regarding the existing LNVWA and LNVWG operators and the proposed four TAOs of LNVs, which shows the ranking validity and rationality of the developed MADM model using one of Eqs. (21)–(24) in the scenario of LNVs. Then, the developed MADM model using the proposed TAOs of LNVs contains periodic property and can perform MADM problems with periodicity/polytemporal phases, while the existing MADM model using the LNVWA and LNVWG operators [14] cannot contain the periodic properties so as to difficultly perform DM problems with periodic/ploytemporal needs. It is clear that in the LNV scenario the developed MADM model significantly outperforms the existing MADM model in the periodic/polytemporal DM capability.

7. Conclusion

This paper first defined the linguistic trigonometric t-norms and t-conorms based on the trigonometry 1 and 2 and the TOLs of LNVs, which contained periodic operational properties. Then, the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators were proposed to provide appropriate mathematical tools for solving MADM issues with periodicity/ploytemporal phases in the scenario of LNVs. Meanwhile, the TAOs of LNVs can compensate for the defects of existing AOs that lack periodic operations of LNVs. Furthermore, the developed MADM model using one of the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators can effectively address MADM problems with periodicity/ploytemporal phases in LNV scenarios. Finally, the developed MADM model was used for the MADM application in the choice problem of CRDSs and reflected its validity and robustness of the ranking results. By comparison with the existing MADM model in the LNV scenario, the developed MADM model verified the validity and rationality of the decision results and significantly outperformed the existing MADM model.

Generally, the MADM model using one of the $LNVTWA_{T1}$, $LNVTWA_{T2}$, $LNVTWG_{T1}$ and $LNVTWG_{T2}$ operators proposed in this study can not only extend the existing linguistic neutrosophic DM models, but also handle linguistic or linguistic intuitionistic fuzzy DM problems by a special case of the proposed MADM model. However, this study developed the TAOs of LNVs and their DM method for the first time. Therefore, we need to provide more research work in the future. To do so, it is necessary to further develop some new LNV TAOs, including LNV trigonometric Einstein, Hamacher, Bonferroni, Maclaurin, and Heronian AOs, etc., and their applications in computer science, medical diagnosis, civil engineering management in LNV scenarios.

Data Availability
All data generated or analyzed during this study are included in this article.
Compliance with ethical standards
Conflict of interest The authors declare that they have no conflict of interest.
Human and animal rights This article does not contain any studies with human participants or animalsperformed by any of the authors.
Informed Consent All authors agreed with the content of the manuscript and the accepted submission and agree to be accountable for all aspects of the work.

References

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In the text, reference numbers should be placed in square brackets [ ], and placed before the punctuation; for example [1], [1–3] or [1,3]. For embedded citations in the text with pagination, use both parentheses and brackets to indicate the reference number and page numbers; for example [5] (p. 10), or [6] (pp. 101–105).

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Received: July 2, 2025. Accepted: Sep 10, 2025

On Restrained Neutrosophic Domination Number on Special Graphs

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Abstract: Neutrosophic theory, which generalizes fuzzy and classical logic by combining the notions of truth, indeterminacy, and falsity, has emerged as a powerful framework in modeling uncertainty. In this paper, we initiate and investigate the study of restrained neutrosophic domination number in graphs, also we study its properties across special graph classes such as paths, cycles, stars, and complete graphs. We also establish upper and lower bounds to determine such sets in graphs and aim to define neutrosophic algebraic structures and examine their properties in depth.

Keywords: Neutrosophic graphs, Neutrosophic domination graphs, Restrained domination, Restrained Neutrosophic domination graphs, Special graphs.

1. Introduction

Neutrosophy is a new science that belongs to the field of philosophy. It focuses on understanding where neutralities come from, what they are like, and how far they go [5,9]. Among various topics in graph theory [1], the study of domination and restrained domination has emerged as a vital area due to its wide range of theoretical and practical applications. The concept of restrained domination was introduced by Telle [6], albeit as a vertex partitioning problem. A set $S \subseteq V$ of vertices of a graph G = (V, E) is a dominating set if every vertex $v \in V$ is an element of S or adjacent to an element of S [8]. Equivalently, a set S of vertices of G is a dominating set of G if every vertex in V - S is adjacent to some vertex in S. A restrained dominating set is a set $S \subseteq V$ where every vertex in $V - S$ is adjacent to a vertex in S as well as another vertex in V – S [14]. As a relatively new field of study within philosophy, neutrosophy examines the nature, extent, and genesis of neutralities. Neutrosophic Set Theory, introduced by Florentin Smarandache[15], provides a more expressive framework by associating every element with degrees of truth (T), indeterminacy (I), and falsity (F). It has become very important that the idea of neutrosophic logic plays a key role in solving many real-world problems in areas like law, medicine, industry, finance, engineering, and IT. In 1965, Zadeh [16] introduced the concept of fuzzy sets, which is based on the idea of truth membership degree. In 1986, Atanassow [4] added the idea of false membership degree to fuzzy sets, creating intuitionistic fuzzy sets. In 1995, Smarandache introduced neutrosophic sets by including the concept of indeterminate membership degree to intuitionistic fuzzy sets. There are three different ways to define a neutrosophic graph. [3,7,10].

2. **Preliminaries**

In this section, we give some basic information about the paper.

Definition 2.1:

A fuzzy set on a given set V is a function A that maps elements of V to values between 0 and 1.[2]

Definition 2.2:

"Suppose that V is a given set. A truth membership function (TA(x), an indeterminate membership function (IA(x), and a false membership function (FA(x)) define a neutrosophic set A in V. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are fuzzy sets on V. That is, $T_A(x)$: V → [0, 1], $I_A(x)$: V → [0, 1] and $F_A(x)$: V → [0, 1] and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$" [2,11]

Definition 2.3:

Let V be a specified set. Assume that E is a given set with respect to V as well. A neutrosophic graph is a pair G = (A, B), where A : V → [0, 1] is a neutrosophic set in V and B : E → [0, 1] is a neutrosophic set in E such that

$T_B(xy) \leq \min\{T_A(x), T_A(y)\}$,

$I_B(xy) \leq \min\{I_A(x), I_A(y)\}$,

$F_B(xy) \leq \max\{F_A(x), F_A(y)\}$,

for all {x, y} ∈ E. V is called vertex set of G and E is called edge set of G, respectively.[11]

Definition 2.4:

"Consider the graph G = (V, E). It is said that a vertex V dominates both itself and every neighbor; in other words, a V dominates the vertices in its closed neighborhood N[V].Therefore v dominates 1 + deg v vertices of G. A set S ⊆ V of vertices of a graph G = (V, E) is a dominating set if every vertex v $\in V$ is an element of S or adjacent to an element of S. Equivalently, a set S of vertices of G is a dominating set of G if every vertex in V - S is adjacent to some vertex in S. The domination number of G, which is written as γ(G), is the smallest number of elements in a dominating set of G".[2]

Definition 2.5:

A set S ⊆ V is a restrained dominating set if each vertex in V − S is next to both a vertex in S and another vertex in V − S. γ_r(G) represents the smallest cardinality of a restrained dominating set of G, which is the restrained domination number of G.[15]

Definition 2.6:

"Let G = (A, B) be a neutrosophic graph on V and let x, y ∈ V. Then,

- If the edge xy is T-effective, we refer to x dominating y in G as T-effective. If for each v ∈ V − S there is u ∈ S such that u dominates v as T-effective, then a subset S of V is the T-effective dominating set in G.
- If the edge xy is I-effective, then x dominates y in G as I-effective. The I-effective dominating set in G is a subset S of V if, for each v in V − S, there exists u in S such that u dominates v as I-effective.
- If the edge xy is F-effective, then x dominates y in G as F-effective. If for each v ∈ V − S there is u ∈ S such that u dominates v as F-effective, then a subset S of V is the F-effective dominating set in G.
- If the edge xy is effective, we say that x dominates y in G. In G, a subset S of V is referred to as the effective dominating set if, for each v in V − S, there exists u in S such that u effectively dominates v".[2,14].

Definition 2.7:

Let G = (A, B) be a neutrosophic graph on a set V. If the following criteria are met, G is referred to as complete:

- T_B (xy) = min{T_A(x), T_A(y)},
- I_B (xy) = min{I_A(x), I_A(y)}
- F_B (xy) = max{F_A(x), F_A(y)}, for all {x, y} ∈ E [12]

Definition 2.8:

Let G = (A, B) be a neutrosophic graph on a set V. If the following criteria are met, G is referred to as empty: For every {x, y} ∈ E, T_B (xy) = I_B (xy) = F_B (xy) = 0. [2]

Definition 2.9:

"Let V be a specified set. If the set V can be divided into two nonempty sets, V_1 and V_2, such that T_B(xy) = I_B (xy) = F_B (xy) = 0 for all {x, y} ∈ E_1 or {x, y} ∈ E_2, then the neutrosophic graph G = (A, B) on V is said to be bipartite. Furthermore, for all {x, y} ∈ E, G is referred to as a complete bipartite neutrosophic graph if T_B(xy) = min{T_A(x), T_A(y)}, I_B (xy) = min{I_A(x), I_A(y)}, and F_B (xy) = max{F_A(x), F_A(y)}. In this instance, the full bipartite neutrosophic graph is referred to as a star neutrosophic graph if either |V1| or |V2| = 1". [13]

Definition 2.10:

"On a set V, let G = (A, B) be a neutrosophic graph. Then, an edge xy in G is referred to as the

- T-bridge, in the event that each T-path P from x to y that did not involve xy had strengths lower than T_B(xy).
- I-bridge, in the event that each I-path P from x to y that did not involve xy had strengths lower than I_B(xy).
- F-bridge, in the event that each F-path P from x to y that did not involve xy had strengths lower than F_B (xy).
- bridge, in the event that it was one of the T, I, or F bridges".[11]

Definition 2.11:

Let G = (A, B) be a neutrosophic graph on a set V. G is then referred to as,

- T-acyclic, if there was no T-path P from x to y, except in the case of x = y for all x ∈ V.
- I-acyclic if there was no I-path P from x to y, except in the case of x = y for all x ∈ V.
- F-acyclic if there was no F-path P from x to y, except for x = y for all x ∈ V.
- Acyclic, if it was either T −acyclic, I −acyclic, or F −acyclic.[11]

Definition 2.12:

On a given set V, let G = (A, B), G_1 = (A_1, B_1) be a neutrosophic graph. If V = V_1 but $E_1 \subseteq E$, then G_1 is referred to as the spanning neutrosophic graph of G [2].

Definition 2.13:

"On a set V, let G = (A, B) be a neutrosophic graph. Then, G is called,

- T-forest, if G were T-acyclic and a spanning neutrosophic graph F existed, such that for every edge xy out of F, there exists a T-path P from x to y, whose strength is greater than T_B (xy).
- I – forest, if G were I-acyclic and there is a spanning neutrosophic graph F, then for every edge xy out of F, there exists an I-path P from x to y, whose strength is greater than I_B (xy).

- F-forest: if G were F-acyclic and a spanning neutrosophic graph F existed, then for every edge xy out of F, there would be an F-path P from x to y, how whose strength was greater than F_B (xy).
- forest, in the event that it was one of the three types of neutrosophic forests: F, I, or T" [2].

Definition 2.14:

Let G = (A, B) be a neutrosophic graph on a set V.

- T-tree: if G were a T-forest with a T-path P from x to y for every x, y ∈ V.
- I-tree: if G were an I-forest, then for every x, y ∈ V, there would be an I-path P from x to y.
- F-tree, if G was an F-forest such that, for every x, y ∈ V, there exists an F-path P.
- Tree, if it was one of T, I, or F −tree. [2]

Definition 2.15:

"Consider a neutrosophic graph G = (A, B) on V and v_0, where v_n are two given vertices such that n ∈ N. Then,

- A distinct sequence of vertices P : $v_0, v_1, \cdots, v_n$ in G is called a T-path of length n from v_0 to v_n, if $T_B(v_iv_{i+1}) > 0$, for i = 0, 1, · · · , n – 1. The strength of that T – Path is $min_{i=0}^{n-1}\{T_B(v_iv_{i+1})\}$ and denoted by $\mu_G(P)_T$.
- A distinct sequence of vertices P : $v_0, v_1, \cdots, v_n$ in G is called a I-path of length n from v_0 to v_n, if $I_B(v_iv_{i+1}) > 0$, for i = 0, 1, · · · , n – 1. The strength of that I – Path is $min_{i=0}^{n-1}\{I_B(v_iv_{i+1})\}$ and denoted by $\mu_G(P)_I$.
- A distinct sequence of vertices P : $v_0, v_1, \cdots, v_n$ in G is called a F-path of length n from v_0 to v_n, if $F_B(v_iv_{i+1}) > 0$, for i = 0, 1, · · · , n – 1. The strength of that F – Path is $min_{i=0}^{n-1}\{F_B(v_iv_{i+1})\}$ and denoted by $\mu_G(P)_F$.
- A distinct sequence of vertices P : $v_0, v_1, \cdots, v_n$ in G is called a path of length n from v_0 to v_n, if it be T –path, I–path, and F – path, simultaneously. The strength of that path is min{ $\mu_G(P)_T$, $\mu_G(P)_I$, $\mu_G(P)_F$} and is denoted by $\mu_G(P)$".[2]

Definition 2.16:

An edge xy in G = (A, B) is considered to be,

- T-effective, if $T_B(xy) > \mu^{\infty}{}_{G-\{xy\}})\ (x,y)_T$
- I-effective, if $I_B(xy) > \mu^{\infty}{}_{G-\{xy\}})\ (x,y)_I$
- F-effective, if $F_B(xy) > \mu^{\infty}{}_{G-\{xy\}})\ (x,y)_F$ and
- Effective, if it is one of the three. [11]

3. **Main Results:**

Definition 3.1:

Let G = (A, B) be a neutrosophic graph on V and x, y ∈ V. Then,

a. If the edge xy is T-effective, we say that x dominates y in G. The T-effective restrained dominating set in G is a subset S of V if, for each v ∈ V − S, there is at least one neighbor w ∈ V - S such that vw ∈ E, and for each v ∈ V - S, there is u ∈ S such that u dominates v as T-effective.

  - The T – weight of x is defined by $w_T(x) = T_A(x) + \frac{\sum \text{xy is a T} - \text{effective edge } T_B\ (\text{xy})}{\sum \text{xy is a edge } T_B\ (\text{xy})}$
  - For any S ⊆ V, T – weight of S is defined by $w(S)_T = \sum_{u \in S}(w(u)_T)$

- Let A be the set of all T - effective restrained dominating sets in G. The T – restrained domination number of G is given by $\gamma_T{}^r(G) = min_{D\in U}(w(D)_T)$. Then the T – effective restrained dominating set that correspond to $\gamma_T{}^r(G)$ is known as T-restrained dominating set.

b. If the edge xy is I-effective, we say that x dominates y in G. The I-effective restrained dominating set in G is a subset S of V if, for each v ∈ V − S, there is at least one neighbor w ∈ V - S such that vw ∈ E, and for each v ∈ V - S, there is u ∈ S such that u dominates v as I-effective.

- The I – weight of x is defined by $w_I(x) = I_A(x) + \frac{\sum xy \text{ is a I} - \text{effective edge } I_B(xy)}{\sum xy \text{ is a edge } I_B(xy)}$
- For any S ⊆ V, I – weight of S is defined by $w(S)_I = \sum_{u\in S}(w(u)_I)$
- Let A be the set of all I - effective restrained dominating sets in G. The I – restrained domination number of G is given by $\gamma_I{}^r(G) = min_{D\in U}(w(D)I)$. Then the I – effective restrained dominating set that correspond to $\gamma_I{}^r(G)$ is known as I-restrained dominating set.

c. If the edge xy is F-effective, we say that x dominates y in G. The F-effective restrained dominating set in G is a subset S of V if, for each v ∈ V − S, there is at least one neighbor w ∈ V - S such that vw ∈ E, and for each v ∈ V - S, there is u ∈ S such that u dominates v as F-effective.

- The F – weight of x is defined by $w_F(x) = F_A(x) + \frac{\sum xy \text{ is a F} - \text{effective edge } F_B(xy)}{\sum xy \text{ is a edge } F(xy)}$
- For any S ⊆ V, F – weight of S is defined by $w(S)_F = \sum_{u\in S}(w(u)_F)$
- Let A be the set of all F - effective restrained dominating sets in G. The F – restrained domination number of G is given by $\gamma_F{}^r(G) = min_{D\in U}(w(D)_F)$. Then the F – effective restrained dominating set that correspond to $\gamma_F{}^r(G)$ is known as F-restrained dominating set.

d. If the edge xy is effective, we say that x dominates y in G. The effective restrained dominating set in G is a subset S of V if, for each v ∈ V − S, there is at least one neighbor w ∈ V - S such that vw ∈ E, and for each v ∈ V - S, there is u ∈ S such that u dominates v as effective. The weight of S is also known as $w_v(S) = \min\{w_T{}^r(S); w_I{}^r(S); w_F{}^r(S)\}$. Let U be the collection of all of G's effective dominating sets. The formula for G's restrained domination number is $\gamma^r(G) = \min_{D\in U}(w(D))$. The restrained dominating set calls the effective restrained dominating set that corresponds to $\gamma^r(G)$.

Definition 3.2:

Let v_0, v_n be two given vertices such that n ∈ N, and let G = (A, B) be a neutrosophic graph on V. Consequently,

- T-Cycle of length n from v_0 to v_{n-1} is defined as a closed sequence of distinct vertices C: $\{v_0, v_1, \cdots, v_{n-1}, v_0\}$in G, provided that $T_B(v_i v_{i+1}) > 0$ for i = 0, 1, · · ·, n – 1.
- An I-Cycle of length n from v_0 to v_{n-1} is defined as a closed sequence of distinct vertices C: $\{v_0, v_1, \cdots, v_{n-1}, v_0\}$in G if $I_B(v_i v_{i+1}) > 0$ for i = 0, 1, · · ·, n – 1.
- If $F_B(v_i v_{i+1}) > 0$, for i = 0, 1, · · ·, n – 1, then a closed sequence of distinct vertices C: $\{v_0, v_1, \cdots, v_{n-1}, v_0\}$in G is referred to as an F-Cycle of length n from v_0 to v_{n-1}.
- A cycle of length n from v_0 to v_{n-1} is defined as a closed sequence of distinct vertices C: $\{v_0, v_1, \cdots, v_{n-1}, v_0\}$ in G, if it is simultaneously T, I, and F − cycles.

Definition 3.3:

Let v_0 be a designated central vertex and let G = (A, B) be a neutrosophic graph defined on a vertex set V. Next,

- A T-Star centered at v_0 with n branches is a collection of distinct edges $\{(v_0, v_1),(v_0, v_2),\ldots,(v_0, v_n)\}$ such that for each i = 1, 2,…,n the truth-membership value of the edge v_0v_i satisfies, $T_B(v_0v_i) > 0$.
- An I-Star centered at v_0 with n branches is a collection of distinct edges $\{(v_0, v_1),(v_0, v_2),\ldots,(v_0, v_n)\}$ such that for each i = 1, 2,…,n the indeterminacy-membership value of the edge v_0v_i satisfies, $I_B(v_0v_i) > 0$.
- A F-Star centered at v_0 with n branches is a collection of distinct edges $\{(v_0, v_1),(v_0, v_2),\ldots,(v_0, v_n)\}$ such that for each i = 1, 2,…,n the false-membership value of the edge v_0v_i satisfies, $F_B(v_0v_i) > 0$.
- A neutrosophic star (or T – I - F star) centered at v_0 is a star structure $\{(v_0, v_1),(v_0, v_2),\ldots,(v_0, v_n)\}$such that all three membership values for each edge are greater than 0, $T_B(v_0vi) > 0$, $I_B(v_0v_i) > 0$, $F_B(v_0v_i) > 0$, for I = 1,2,…,n.

Theorem 3.4:

Let G = (A, B) be a neutrosophic graph where B is the edge set with associated truth-membership function $T_B : B \to [0,1]$. Let P_n be a path graph on n ≥ 3 vertices such that $T_B(e) > 0$ for all e ∈ B. Then $\gamma_T^r(P_n) = [\frac{n}{2}]$.

Proof: Assume that the path graph on n vertices $P_n = (v_1,v_2,\ldots.v_n)$ with edges $E = \{v_iv_{i+1} / 1 \le i \le n-1\}$, and suppose that $T_B(v_iv_{i+1}) > 0$ for all i, Define the vertex subset $S =\{ v_i \in V / i \text{ is odd}\} = \{v_1,v_3,v_5,\ldots \}$. We prove that S is a T-effective restrained dominating set of G. Let $v_j \in V \setminus S$. Then j is even. Since P_n is a path, the only possible neighbors of v_j are v_{j-1} and v_{j+1}, assuming they exist. Both v_{j-1} and v_{j+1} are odd-numbered and thus belong to S, and the edges v_jv_{j-1}, $v_jv_{j+1}$are T-effective by assumption. Therefore, each $v_j \in V \setminus S$ is T - effectively dominated by a vertex in S, satisfying the first condition of T - effective domination. Now, to verify the restrained condition, we must show that for each $v_j \in V \setminus S$, there exists at least one neighbor $w \in V \setminus S$ such that $v_jw \in E$. Observe that the vertices in $V \setminus S$ are all even-indexed. For j = 2, we have v_2 adjacent to $v_1 \in S$ and $v_3 \in S$. To resolve this and ensure the restraint condition is satisfied, we redefine the set $S' = \{v_i \in V / i \text{ is even}\}$. Now, for each $v_j \in V \setminus S'$, the neighbors v_j-1 are even and belong to S′. Additionally, for all such odd j, at least one of their neighbors, $v_{j\pm2}$. Therefore, S′ is a T-effective restrained dominating set. Therefore, we conclude that, $\gamma_T^r(P_n) \le [\frac{n}{2}]$. To prove minimality, suppose there exists a T-effective restrained dominating set $D \subset V$ such that $|D| < [\frac{n}{2}]$. By linear adjacency of a path, there must exist two consecutive vertices $v_i,v_{i+1} \in V \setminus D$, which are not T-effectively dominated by any vertex in D. Furthermore, they also lack a neighbor in $V \setminus D$ distinct from each other, violating the restraint condition. This contradiction implies that no such smaller set D exists, and thus $\gamma_T^r(P_n) = [\frac{n}{2}]$. By analogues to the evidence of $\gamma_T^r(P_n) = [\frac{n}{2}]$, The outcome is clearly valid for $\gamma_I^r(P_n) = [\frac{n}{2}]$, $\gamma_F^r(P_n) = [\frac{n}{2}]$, $\gamma^r(P_n) = [\frac{n}{2}]$.

Theorem 3.5:

Given a vertex set V, let G = (A, B) be an empty neutrosophic graph. Then $\gamma_T^r(G) = \gamma_I^r(G) = \gamma_F^r(G) = p$, where p denotes the order of G.

Proof: Let G = (A, B) be a neutrosophic graph on a vertex set V that is empty. As a result, there is no T-effective edge in G and V is the only T-effective dominating set. According to 3.1(a), we have $\gamma_T{}^r(G) = min_{D\in S}[\sum_{u\in D} T_A(u)] = [\sum_{u\in V} T_A(u)] = p$. Therefore $\gamma_T{}^r(G) = p$. By analogues to the proof of $\gamma_T{}^r(G) = p$ and Definition 3.1(b,c,d), the result is obviously holds for $\gamma_I{}^r(G)$, $\gamma_F{}^r(G)$, $\gamma^r(G)$.

Proposition 3.6: Given a set V and any neutrosophic graph G = (A, B), we have

a. $\gamma_T{}^r(G)$, $\gamma^r(G)$, $\gamma_I{}^r(G)$, $\gamma_F{}^r(G) \leq p$.

b. $\gamma^r(G) + \gamma^r(G)$, $\gamma_T{}^r(G) + \gamma_T{}^r(G)$, $\gamma_I{}^r(G) + \gamma_I{}^r(G)$, $\gamma_F{}^r(G) + \gamma_F{}^r(G) \leq 2p$

Proof:

- According to Proposition 3.5, a neutrosophic graph G = (A, B) exists such that $\gamma_T{}^r(G) = \gamma_I{}^r(G) = \gamma_F{}^r(G) = \gamma^r(G) = p$, Thus, the outcome is as follows.
- When (a) is applied to G and $\ddot{G}$, the outcome is evidently valid.

4. Conclusion

Fuzzy models and algebraic structures are framed by the idea of neutrosophy. Neutrosophic graphs come in three different varieties. As previously stated, we selected one type of them to serve as the framework. This study serves as an initial step toward a broader integration of neutrosophic theory in fuzzy systems and mathematical structures, opening new avenues for both theoretical advancement and practical applications.

Funding: "This research received no external funding."

Conflicts of Interest: The authors declare no conflict of interest

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Received: July 7, 2025. Accepted: Sep 11, 2025

Application of Information Theoretic Measures in Neutrosophic Soft Environment

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Abstract: Soft set theory is a valuable mathematical tool for modeling and analyzing uncertain systems. A neutrosophic soft set is a hybrid entity of a neutrosophic set, and a soft set that enables a more comprehensive analysis of uncertainty in a system. In this article, we introduce some novel information theoretic measures in a single-valued neutrosophic soft environment. Additionally, we study the data-dimensionality reduction using two-pronged approach, leveraging the score matrix and neutrosophic soft entropy measure. The complexity of decision-making problems involving numerous factors can be alleviated using the dimensionality reduction technique. Finally, the comparative analysis is presented with the help of an illustrative example utilizing a measure of performance. The comparative study highlighted the advantage of the proposed methods.

Keywords: single-valued neutrosophic set, neutrosophic soft set, dimensionality reduction, score matrix, entropy measure, similarity measure.

1. Introduction

Uncertainty is an inherent part of real-world systems, and several methods are available in the literature for the representation of uncertain data or information. The most prevalent methods include probability theory, rough set theory, fuzzy set theory, and intuitionistic fuzzy set theory, and are utilized for modeling human-centric and expert-based systems. The real-life applications concerned with decision-making, clustering analysis, pattern recognition, anomaly detection, image analysis, and many more have been investigated using these methods. However, all these theories have their own limitations in varied situations. This article aims to put forth a neutrosophic soft information measure, and its implication. In [1], Molodtsov introduced the concept of a soft set by integrating the parametrization tool with the classical set. Soft set theory has been widely applied in a variety of fields such as decision-making [2-8], data analysis [9], forecasting [10], simulation [11], optimization [12], texture classification [13], etc. Afterward, by combining a Soft set [1] and a fuzzy set (FS) [14], Maji et al.[8] suggested the concept of Fuzzy soft sets. Similarly, Maji et al. [15] introduced the concept of intuitionistic fuzzy soft sets (IFSSs) due to the fusion of soft sets and intuitionistic fuzzy sets (IFS) [16]. Thereafter, various researchers studied diverse mathematical hybrid structures such as generalized intuitionistic fuzzy soft sets [17], generalized fuzzy soft sets [18] possibility intuitionistic fuzzy soft sets [19], vague soft sets [20], interval-valued fuzzy soft sets [21], interval-valued intuitionistic fuzzy soft sets [22], etc. Furthermore, Maji [23] introduced the notion of the neutrosophic soft set by fusion of a neutrosophic set (NS) [24] and a soft set [1]. Maji [25] suggested some operations and propositions related to a weighted neutrosophic soft set. Wang

[26] suggested a subclass of neutrosophic soft sets. Marei [27] developed a rough set approach in single-valued neutrosophic soft settings. Neutrosophic soft sets (NSSs), due to their ability to deal with intermediate, inconsistent, and neutrosophic parameters found extensive implications in a variety of fields.

The essential information theoretic measures like entropy and Knowledge measure quantify the information content of a fuzzy/non-fuzzy set. Furthermore, similarity and distance measures evaluate the extent of closeness and discrimination between two fuzzy/non-fuzzy sets. In the problems related to pattern recognition, the concept of similarity/distance measure is commonly employed to verify the authenticity of an object/document. In recent years, several studies [28-41] suggested different information-theoretic measures concerned with fuzzy sets, fuzzy soft sets, intuitionistic fuzzy sets, intuitionistic fuzzy soft sets, single-valued neutrosophic sets, and single-valued neutrosophic soft sets. Broumi [29] studied similarity measures for neutrosophic soft sets. Dey et al. [37] presented a neutrosophic soft similarity measure for selecting the suitable alternative based on a grey relational analysis involving multiple decision-makers. Karaaslan [38] suggested a decision-making method and a group decision-making method in the neutrosophic soft environment. Sahin and Kucuk [39] suggested various distance measures between neutrosophic soft sets and introduced an axiomatic definition of entropy for a neutrosophic soft set. The proposed research work is aimed to introduce neutrosophic soft information measures, and their application to decision-making and data dimensionality reduction.

The dimensionality reduction technique is instrumental in mitigating the curse of dimensionality, enabling the effective management of high-dimensional data sets and the elimination of irrelevant features. This approach offers a threefold benefit: Simplifying data, visualizing complex relationships, and managing multicollinearity. As a result, this method has become a vital area of study in diverse computational disciplines, especially those characterized by extreme data modality. In the fuzzy soft set, the dimension reduction technique of big data was utilized to convert soft tables into fuzzy soft set tables [42]. The concept of Pythagorean fuzzy soft matrix, introduced by Bajaj [43] has paved the way for the development of a new generation of dimensionality reduction approaches, tailored to address the complexities of MCDM problems. Picture fuzzy soft matrix was suggested in Devi *et al.* [44] to solve decision-making problems and the dimensionality reduction can be dealt with in a better and broader sense of human opinion. However, there has been rather little work completed for entropy, and similarity measure in the context of single-valued neutrosophic soft sets and their applications.

1.1. *Contribution*

The main contribution of this paper is as follows:

- We propose novel entropy, and similarity measures for the SVNSSs to overcome the shortcomings of existing measures.
- The newly proposed measures are applied for solving the MCDM problem and for data reduction technique.
- Finally, a comparative analysis has been done to check the effectiveness of the proposed measure based on performance measures.

1.2. *Importance of neutroshopy in the present work*

Neutrosophy provides a more rational extension to traditional fuzzy logic by explicitly modelling indeterminacy. It enhances data dimensionality reduction by guiding the selection of relevant, low-noise features and improves decision-making processes by providing a structured framework to deal with incomplete, contradictory, and uncertain information. The independent choice of trueness, falsity, and indeterminacy in a single valued neutrosophic set make the neutrosophic information measures more robust and indispensable tools in the real-life problems concerned with of classification, pattern-recognition and decision-support.

The content of this article is organised as follows.

Section 2 presents basic concepts relevant to this study. Section 3 introduces the similarity, and entropy measure in a single-valued neutrosophic soft environment. Section 4 demonstrates the application of the proposed measure in MCDM problem as well as in data dimensionality reduction. In section 5. We contrast the performance of the proposed methods with existing methods. Section 6 concludes the article.

2. Preliminaries

In this section, we present some essential fundamental concepts concerned with this article.

Definition 2.1 ([1]). Let U be a generic universe and P be a set of parameters in U. Consider $E \subseteq P$. A pair (F, E) is said to be a soft set over the universe set in which F is a mapping from E to 2^U, where 2^U is a power set of U. In short, a soft set over U is a parameterized family of a subset of U.

Definition 2.2 ([8]). Suppose U is a generic universe of objects. Consider P be a set of parameters in U and $E \subseteq P$. A pair (F, E) is said to be a fuzzy soft set over the universe set U, in which F is a mapping from E to FS^U, where FS^U is a set of all fuzzy subsets of the universe set U.

Definition 2.3 ([15]). Let U be a generic universe of objects and P be a set of parameters in U. Consider $E \subseteq P$ and let IFS^U denote the collection of all intuitionistic fuzzy sets of U. A pair (F, E) is said to be intuitionistic fuzzy soft set over the universe set U, where F is a mapping $F\colon E \rightarrow IFS^U$.

Definition 2.2 ([24]**).** Let U be a universe of discourse with generic element y in U. A single-valued neutrosophic set B in U is characterized by truth-membership degree $T_B(y_i)$, indeterminacy degree $I_B(y_i)$ and falsity-membership degree $F_B(y_i)$. For each $y_i \in Y$, $T_B(y_i)$, $I_B(y_i)$, $F_B(y_i) \in [0,1]$. A single-valued neutrosophic set B can be denoted by a triplet i.e., $B = \{\langle T_B(y_i), I_B(y_i), F_B(y_i)\rangle |\, y_i \in Y\}$ with $T_B(y_i) + I_B(y_i) + F_B(y_i) \in [0,3]$.

Definition 2.4 ([23]). Let U be a universe of objects and P be a set of parameters in U. Consider $E \subseteq P$ and NS^U denote the collection of all neutrosophic sets of U. A pair (F, E) is said to be neutrosophic soft set over the universe set U, where F is a mapping $F\colon E \rightarrow NS^U$.

Operations on Single-Valued Neutrosophic Soft Sets: Let (F, E), (G, E), and (H, E) be three single-valued neutrosophic soft sets, then we have the following operations.

Union: $(F,E) \cup (G,E) = \left(max.\left(T_{F(e)}(y), T_{G(e)}(y)\right), \frac{I_{F(e)}(y) + I_{G(e)}(y)}{2}, min.\left(T_{F(e)}(y), T_{G(e)}(y)\right)\right)$ if $e \in A \cap B$.

Intersection: $(F,E) \cap (G,E) = \left(min.\left(T_{F(e)}(y), T_{G(e)}(y)\right), \frac{I_{F(e)}(y) + I_{G(e)}(y)}{2}, max.\left(T_{F(e)}(y), T_{G(e)}(y)\right)\right)$ if $e \in A \cap B$.

Complement: $(F,E)^c = \left(T_{F^c(e)(y)} = F_{F(e)(y)}, I_{F^c(e)(y)} = I_{F(e)(y)}, F_{F^c(e)(y)} = T_{F(e)(y)}\right)$.

Subset: If $(F,E) \subseteq (G, E)$ then $T_{F(e)}(y) \leq T_{G(e)}(y), I_{F(e)}(y) \leq I_{G(e)}(y), F_{F(e)}(y) \geq F_{G(e)}(y)$.

Definition 2.5 ([27]). Let U be the universal set and P be a set of parameters. Consider $E \subseteq P$ and, let$SVNS^U$ denote the set of all single–valued neutrosophic sets of U. The collection (F,E) is said to be a single-valued neutrosophic soft set over the universe set U, where F is a mapping $F: E \to SVNS^U$.

Example 2.1 ([23]). Let $U = \{y_1, y_2, y_3, y_4, y_5\}$be a set of five models of car out of which one car is to be purchased and $E = \{e_1 = \text{elegant}, e_2 = \text{trustworthy}, e_3 = \text{sporty}, e_4 = \text{comfortable}, e_5 = \text{modern}\}$be the set of parameters to select the car. The SVNSS (F,E) in this example can be presented in the following Table 1.

Table 1. Tabular representation of (F,E) of Example 1.

U	e_1	e_2	e_3	e_4
y_1	(0.6,0.3, 0.4)	(0.4,0.3, 0.3)	(0.1,0.1, 0.1)	(0.7,0.5, 0.6)
y_2	(0.4,0.3, 0.4)	(0.5,0.6, 0.4)	(0,0, 0.4)	(0.8,0.3, 0.3)
y_3	(0.6,0.5, 0.4)	(0.3,0.3, 0.4)	(0.2,0.3, 0.7)	(0.3,0.3, 0.3)
y_4	(0, 0.3, 0.4)	(0.1,0.2, 0.4)	(0.2,0.2, 0.4)	(0.2,0.9, 0.1)
y_5	(0.2,0.3, 0.1)	(0.7,0.3, 0.4)	(0.9,0.3, 0.3)	(0.8,0.5, 0.2)

Sahin and Kucuk [39] and I.Arockiarani [45] proposed entropy measures for neutrosophic soft environments satisfying some axiomatic requirements. An entropy measure of a SVNSS should satisfy axiomatic requirements given in the following definition of entropy measure.

Definition 2.6. Let $U = \{y_1, y_2, y_3, \dots, y_m\}$ be a generic universe and $E = \{e_1, e_2, e_3, \dots, e_n\}$ be the set of parameters. A function $EM: NSS(Y) \to [0, 1]$ is said to be an entropy measure on $NSS(Y)$ if EM satisfies the following axioms:

$\boldsymbol{NSEM1}.\ EM(F,E) = 0 \iff \forall e \in E;\ (F,E)$ is a soft set;

$\boldsymbol{NSEM2}.\ EM(F,E) = EM(F,E)^c$;

$\boldsymbol{NSEM3}.\ EM(F,E) = 1$ iff $T_{F(e_i)}(y_j) = I_{F(e_i)}(y_j) = F_{F(e_i)}(y_j)\ \forall\ e \in E, y \in Y$;

NSEM4. If $(G,E) \subseteq (F,E)$then $EM(F,E) \leq EM(G,E)$.

Definition 2.7 ([35])Let $U = \{y_1, y_2, y_3, \dots, y_m\}$ be a generic universe and $E = \{e_1, e_2, e_3, \dots, e_n\}$ be the set of parameters. Let (F,E), and (G,E) be two neutrosophic soft sets over U, where F, G are mappings given by $F, G: E \to NS(U)$. Then $SM((F,E),(G,E))$ is said to be a similarity measure between (F,E) and (G,E) if it satisfies the following axioms:

$\boldsymbol{NSSM1}.\ 0 \le SM\big((F,E),(G,E)\big) \le 1;$

$\boldsymbol{NSSM2}.\ SM\big((F,E),(G,E)\big) = SM\big((G,E),(F,E)\big);$

$\boldsymbol{NSSM3}.\ SM\big((F,E),(G,E)\big) = 1$ iff $(F,E) = (G,E);$

NSSM4. If $(F,E) \subseteq (G,E) \subseteq (H,E)$ then $SM((F,E),(H,E)) \le SM((G,E),(H,E))$.

Definition 2.9 ([36]). The performance measure of a similarity method (say M) that satisfies the optimality criteria to solve an IFSS-based decision-making problem is defined as

$$P_M = SM_{rt} + \frac{1}{\sum_{i=1\,(r,t)\neq(i,j)}^{n-1}\sum_{j=i+1}(1-SM_{ij})}\ ;\ SM_{rt} > SM_{ij},$$

where SM_{rt} denotes the highest similarity value of an object and SM_{ij} is the similarity value of the remaining object.

Performance measure represents the sum of the highest similarity value of an object and an inverse of the summation of the non-similarity values of the remaining objects.

3. The proposed Information Measure for SVNSSs

In this section, we suggest a similarity measure, and an entropy measure in a single-valued neutrosophic soft environment.

3.1 Similarity Measure

Consider $U = \{y_1, y_2, y_3, \dots, y_m\}$ be the universe of discourse and $E = \{e_1, e_2, e_3, \dots, e_n\}$ be the set of parameters. Then similarity measure between two single-valued neutrosophic soft sets (F,E), and (G,E) is defined as

$$SM_{SVNSSs}(F,G) = \frac{1}{m}\sum_{i=1}^{n}\Big(1 - \frac{1}{4n}\sum_{j=1}^{m}[\big|S_{F(e_i)}(y_j) - S_{G(e_i)}(y_j)\big| + \big|T_{F(e_i)}(y_j) - T_{G(e_i)}(y_j)\big| +$$

$$\big|I_{F(e_i)}(y_j) - I_{G(e_i)}(y_j)\big| + |F_{F(e_i)}(y_j) - F_{G(e_i)}(y_j)|]\Big) \qquad (1)$$

Where, $S_{F(e_i)}(y_j) = \big(T_{F(e_i)}(y_j) - I_{F(e_i)}(y_j) - F_{F(e_i)}(y_j)\big)$, and

$$S_{G(e_i)}(y_j) = \big(T_{G(e_i)}(y_j) - I_{G(e_i)}(y_j) - F_{G(e_i)}(y_j)\big).$$

Theorem 3.1. $SM_{SVNSSs}(F,G)$ is a valid similarity measure between SVNSSs (F,E) and (G,E).

Proof. To check the validity of $SM_{SVNSSs}(F,G)$, we verify the axiomatic requirements given in definition 2.7.

NSSM1. Since $S_{F(e_i)}(y_j) \in [-1,1]$ and $S_{G(e_i)}(y_j) \in [-1,1]$. Then $\big|S_{F(e_i)}(y_j) - S_{G(e_i)}(y_j)\big| \le 1$. Also, $\big|T_{F(e_i)}(y_j) - T_{G(e_i)}(y_j)\big| \le 1$, $\big|I_{F(e_i)}(y_j) - I_{G(e_i)}(y_j)\big| \le 1$, $\big|F_{F(e_i)}(y_j) - F_{G(e_i)}(y_j)\big| \le 1$, which implies $SM_{SVNSSs}(F,G) \in [0,1]$.

NSSM2. $SM_{SVNSSs}(F,G) = \frac{1}{m}\sum_{i=1}^{n}\Big(1 - \frac{1}{4n}\sum_{j=1}^{m}[\big|S_{F(e_i)}(y_j) - S_{G(e_i)}(y_j)\big| + \big|T_{F(e_i)}(y_j) - T_{G(e_i)}(y_j)\big| +$

$\big|I_{F(e_i)}(y_j) - I_{G(e_i)}(y_j)\big| + |F_{F(e_i)}(y_j) - F_{G(e_i)}(y_j)|]\Big) = \frac{1}{m}\sum_{i=1}^{n}\Big(1 - \frac{1}{4n}\sum_{j=1}^{m}([\big|S_{G(e_i)}(y_j) - S_{F(e_i)}(y_j)\big| +$

$\big|T_{G(e_i)}(y_j) - T_{F(e_i)}(y_j)\big| + \big|I_{G(e_i)}(y_j) - I_{F(e_i)}(y_j)\big| + |F_{G(e_i)}(y_j) - F_{F(e_i)}(y_j)|])\Big) = SM_{SVNSSs}(G,F).$

NSSM3. $SM_{SVNSSs}(F,G)=1 \Leftrightarrow \frac{1}{m}\sum_{i=1}^{n}\left(1-\frac{1}{4n}\sum_{j=1}^{m}[|S_{F(e_i)}(y_j)-S_{G(e_i)}(y_j)|+|T_{F(e_i)}(y_j)-T_{G(e_i)}(y_j)|+|I_{F(e_i)}(y_j)-I_{G(e_i)}(y_j)|+|F_{F(e_i)}(y_j)-F_{G(e_i)}(y_j)|]\right)=0 \Leftrightarrow |S_{F(e_i)}(y_j)-S_{G(e_i)}(y_j)|=0,\ |T_{F(e_i)}(y_j)-T_{G(e_i)}(y_j)|=0, |I_{F(e_i)}(y_j)-I_{G(e_i)}(y_j)|=0, |F_{F(e_i)}(y_j)-F_{G(e_i)}(y_j)|=0 \Leftrightarrow T_{F(e_i)}(y_j)=T_{G(e_i)}(y_j), I_{F(e_i)}(y_j)=I_{G(e_i)}(y_j), F_{F(e_i)}(y_j)=F_{G(e_i)}(y_j) \Leftrightarrow (F,E)=(G,E)$.

NSSM4. Consider $(F,E)\subseteq(G,E)\subseteq(H,E)$. Then
$T_{F(e_i)}(y_j)\le T_{G(e_i)}(y_j)\le T_{H(e_i)}(y_j)$,
$I_{F(e_i)}(y_j)\le I_{G(e_i)}(y_j)\le I_{H(e_i)}(y_j)$, and
$F_{F(e_i)}(y_j)\ge F_{G(e_i)}(y_j)\ge F_{H(e_i)}(y_j)$.
Therefore,

$$|T_{F(e_i)}(y_j)-T_{H(e_i)}(y_j)|\ge|T_{G(e_i)}(y_j)-T_{H(e_i)}(y_j)|. \quad (2)$$

$$|I_{F(e_i)}(y_j)-I_{H(e_i)}(y_j)|\ge|I_{G(e_i)}(y_j)-I_{H(e_i)}(y_j)|. \quad (3)$$

$$|F_{F(e_i)}(y_j)-F_{H(e_i)}(y_j)|\ge|F_{G(e_i)}(y_j)-F_{H(e_i)}(y_j)|. \quad (4)$$

Therefore, $S_{F(e_i)}(y_j)-S_{H(e_i)}(y_j)=\left(T_{F(e_i)}(y_j)-I_{F(e_i)}(y_j)-F_{F(e_i)}(y_j)\right)-\left(T_{H(e_i)}(y_j)-I_{H(e_i)}(y_j)-F_{H(e_i)}(y_j)\right)$

$$=\left(T_{F(e_i)}(y_j)-T_{H(e_i)}(y_j)\right)+\left(I_{H(e_i)}(y_j)-I_{F(e_i)}(y_j)\right)+\left(F_{H(e_i)}(y_j)-F_{F(e_i)}(y_j)\right)$$

Similarly, $S_{F(e_i)}(y_j)-S_{G(e_i)}(y_j)=\left(T_{F(e_i)}(y_j)-I_{F(e_i)}(y_j)-F_{F(e_i)}(y_j)\right)-\left(T_{G(e_i)}(y_j)-I_{G(e_i)}(y_j)-F_{G(e_i)}(y_j)\right)$

$$=\left(T_{F(e_i)}(y_j)-T_{G(e_i)}(y_j)\right)+\left(I_{G(e_i)}(y_j)-I_{F(e_i)}(y_j)\right)+\left(F_{G(e_i)}(y_j)-F_{F(e_i)}(y_j)\right)$$

This implies, $|S_{F(e_i)}(y_j)-S_{H(e_i)}(y_j)|\ge|S_{F(e_i)}(y_j)-S_{G(e_i)}(y_j)|$. (5)
By combining (2), (3), (4), and (5), we get
$SM_{SVNSSs}(F,H)\le SM_{SVNSSs}(G,H)$.
This completes the proof.

3.2. Entropy Measure

Suppose $U=\{y_1,y_2,y_3,\dots,y_m\}$ be the universe of discourse and $E=\{e_1,e_2,e_3,\dots,e_n\}$be the set of parameters. Then entropy measure of a single valued neutrosophic soft set (F,E) is denoted and defined as

$$EM_{SVNSSs}(F,E)=\frac{1}{\sqrt{2}-1}\frac{1}{m}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j)-F_{F(e_i)}(y_j)}{4}\right)\pi-1\right) \quad (6)$$

Theorem 3.3. $EM_{SVNSSs}(F,E)$ is a valid entropy measure of SVNSSs (F,E).

Proof. To check the validity of $EM_{SVNSSs}(F,E)$, we verify the axiomatic requirements given in definition 2.6.

NSEM1. $EM_{SVNSS}(F,E)=0$ if and only if

$$\frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j)-F_{F(e_i)}(y_j)}{4}\right)\pi-1\right)=0$$

if and only if $\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j)-F_{F(e_i)}(y_j)}{4}\right)\pi-1\right)=0$.

If and only if $T_{F(e_i)}(y_j)=0\ or F_{F(e_i)}(y_j)=1$ or $F_{F(e_i)}(y_j)=0\ or\ T_{F(e_i)}(y_j)=1, \forall\ e_i\in E, y\in U$.

NSEM2. $EM_{SVNSS}(F,E)=1$

If and only if $\frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j) - F_{F(e_i)}(y_j)}{4}\right)\pi - 1\right) = 1.$

If and only if $\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j)- F_{F(e_i)}(y_j)}{4}\right)\pi - 1\right) = \sqrt{2} - 1$.

If and only if $Cos\left(\frac{T_{F(e_i)}(y_j)- F_{F(e_i)}(y_j)}{4}\right)\pi = 1.$

If and only if $T_{F(e_i)}(y_j) = F_{F(e_i)}(y_j)$.

NSEM3. $EM_{SVNSS}(F,E) = \frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j)- F_{F(e_i)}(y_j)}{4}\right)\pi - 1\right)$

$= \frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{F_{F^c(e_i)}(y_j)- T_{F^c(e_i)}(y_j)}{4}\right)\pi - 1\right) = EM_{SVNSS}(F,E)^c.$

NSEM4. $\forall\, e_i \in E, y \in U$, when $(G,E) \subseteq (F,E)$ then $T_{F(e_i)}(y_j) \le T_{G(e_i)}(y_j)$; $I_{F(e_i)}(y_j) \le I_{G(e_i)}(y_j)$; $F_{F(e_i)}(y_j) \ge F_{G(e_i)}(y_j)$ which implies $T_{F(e_i)}(y_j) - F_{F(e_i)}(y_j) \le T_{G(e_i)}(y_j) - F_{G(e_i)}(y_j)$ or $\frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j)- F_{F(e_i)}(y_j)}{4}\right)\pi - 1\right) \le \frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{G(e_i)}(y_j)- F_{G(e_i)}(y_j)}{4}\right)\pi - 1\right)$ and hence we get, $EM_{SVNSS}(F,E) \subseteq EM_{SVNSS}(G,E)$.

In the next section, we present some applications of our suggested measures i.e., similarity, and entropy measure.

4. Applications

In this section, we apply the suggested similarity measure to a decision-making problem.

4.1. Application to Suitable Location of Industrial Unit

In this subsection, we introduce a method for solving a decision-making problem based on the proposed similarity measure. The concept of an ideal point has been utilized to identify the most suitable alternative in decision-making processes. Although an ideal alternative may not exist in real-world scenarios, it provides a valuable theoretical framework for evaluating alternatives. We define the ideal alternative y^*as the SVNS ${y^*}_j = (T^*, I^*, F^*) = (1, 0, 0)\ \forall\, j$. To utilize our proposed measure for the selection of a suitable location of an industrial unit, following algorithm is suggested.

Algorithm 1.

Step 1. Identify the alternatives and parameters, and obtain the single-valued neutrosophic soft set (F,E) as shown in the Table 1.

Table 1. Tabular representation of (F,E)

Alternative/Parameter → ↓	e_1	e_2	…	e_n
y_1	$F_{(e_1)}(y_1)$	$F_{(e_2)}(y_1)$	…	$F_{(e_n)}(y_1)$
y_1	$F_{(e_1)}(y_2)$	$F_{(e_2)}(y_2)$	…	$F_{(e_n)}(y_2)$
.			….	
.				
.				

y_m	$F_{(e_1)}(y_m)$	$F_{(e_2)}(y_m)$	...	$F_{(e_n)}(y_m)$

Where $F_{(e_i)}(y_j) = \left(T_{F(e_i)}(y_i), I_{F(e_i)}(y_j), F_{F(e_i)}(y_j)\right)$.

Step 2. Normalize the SVN soft set (F, E) into (F', E) using the following framework.

$$\begin{Bmatrix} \left(T_{F(e_i)}(y_j), I_{F(e_i)}(y_j), F_{F(e_i)}(y_j)\right), & e_i \in B \\ \left(F_{F(e_i)}(y_j), 1 - I_{F(e_i)}(y_j), T_{F(e_i)}(y_j)\right), & e_i \in C \end{Bmatrix}$$

Where B is the benefit parameter set and C is the cost parameter set.

Step 3. Compute the similarity measure $SM(y_j, y^*), j = 1, 2, 3, \dots m$ by using Eq. (1).

Step 4. Evaluate the performance measure corresponding to the similarity value of the alternatives.

Step 5. Choose or select the suitable measure that offers a higher accuracy rate/performance measure.

Next, we consider the following numerical example to illustrate the procedure.

Example 4.1 Industrial site selection is a complex decision-making process that involves evaluating multiple factors, including technical, economic, social, environmental, and political considerations, highlighting the need for a robust tool and knowledge base to support data collection, analysis, and site management. Let us suppose a company X wants to select a suitable location for setting up an industry. Assume that there are four locations: location A, location B, location C, and location D. The company selects six parameters to evaluate the four locations. Let $U = \{y_1 = location\ A, y_2 = location\ B,\ y_3 = location\ C, y_4 = location\ D\}$ be the set of locations and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ is a set of parameter, where $e_1 =$ raw material, $e_2 =$ power, $e_3 =$ labour, $e_4 =$ Transport, $e_5 =$ vulnerability to nature, e_6 = investment climate. The decision data given by the expert is shown in Table 2, decision scenario is visualized in Figure 1.

Now, we utilize the above algorithm 1 to select the suitable industry under single-valued neutrosophic soft information.

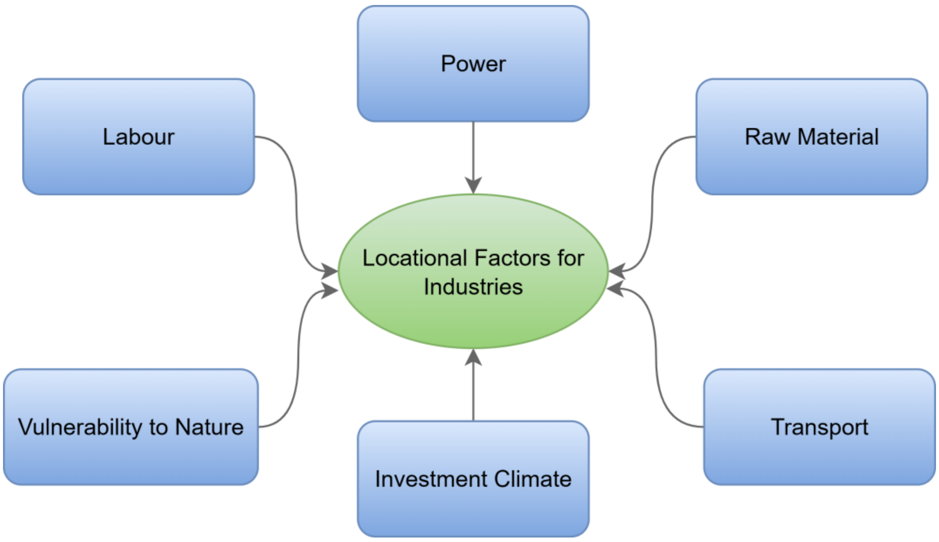

Figure 1. Decision scenario for location of industrial unit

Implementation of Algorithm 1.

Step 1. Select the alternatives and parameters, and obtain the single-valued neutrosophic soft set (F, E) as shown in Table 2.

Table 2. Tabular representation of (F, E)

Alternative	e_1	e_2	e_3	e_4	e_5	e_6
A	(0.6, 0.2, 0.3)	(0.7, 0.3, 0.4)	(0.4, 0.3. 0.6)	(0.8, 0.2, 0.3)	(0.5, 0.3, 0.2)	(0.2, 0.3, 0.5)
B	(0.4, 0.1, 0.2)	(0.3, 0.1, 0.2)	(0.2, 0.2, 0.4)	(0.3, 0.1, 0.4)	(0.2, 0.1, 0.3)	(0.2, 0.3, 0.5)
C	(0.7, 0.3, 0.4)	(0.8, 0.2, 0.5)	(0.4, 0.2, 0.5)	(0.8, 0.1, 0.2)	(0.5, 0.3, 0.2)	(0.3, 0.2, 0.5)
D	(0.3, 0.2, 0.3)	(0.2, 0.2, 0.3)	(0.3, 0.1, 0.3)	(0.3, 0.2, 0.3)	(0.1 0.2, 0.2)	(0.3, 0.2, 0.5)

Step 2. As all the parameters are benefit parameters, so there is no need to normalize. Therefore, the normalized SVNS (F', E) is similar to Table 2.

Step 3. Compute the similarity measure $SM(y_j, y^*), j = 1, 2, 3, \dots m$ by using equation (1) and (6), shown as follows:

$SM(y_1, y^*) = 0.8604,\ SM(y_2, y^*) = 0.8479, SM(y_3, y^*) = 0.8729, SM(y_4, y^*) = 0.8437$

Step 4. Evaluate the performance measure corresponding to the similarity values given above. Firstly, we consider some existing measures which is given as below.

$SM_1 = \frac{1}{1+L\,(F,G\,)}$, where

$$L\,(F,G\,) = \frac{1}{6}\sum_{i=1}^{n}\sum_{j=1}^{m}\left|T_{F(e_i)}(y_j) - T_{G(e_i)}(y_j)\right| + \left|I_{F(e_i)}(y_j) - I_{G(e_i)}(y_j)\right| + \left|F_{F(e_i)}(y_j) - F_{G(e_i)}(y_j)\right|$$

Mukherjee and Sarkar [34]

$$SM_2 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{\left(T_{F(e_i)}(y_j)\wedge T_{G(e_i)}(y_j)\right) + \left(I_{F(e_i)}(y_j)\wedge I_{G(e_i)}(y_j)\right) + \left(F_{F(e_i)}(y_j)\wedge F_{G(e_i)}(y_j)\right)\right\}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{\left(T_{F(e_i)}(y_j)\vee T_{G(e_i)}(y_j)\right) + \left(I_{F(e_i)}(y_j)\vee I_{G(e_i)}(y_j)\right) + \left(F_{F(e_i)}(y_j)\vee F_{G(e_i)}(y_j)\right)\right\}}$$

Mukherjee and Sarkar [35]

$$SM_3 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{\left(T_{F(e_i)}(y_j)T_{G(e_i)}(y_j)\right) + \left(I_{F(e_i)}(y_j)I_{G(e_i)}(y_j)\right) + \left(F_{F(e_i)}(y_j)F_{G(e_i)}(y_j)\right)\right\}}{\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{\left(T_{F(e_i)^2}(y_j)\vee T_{G(e_i)^2}(y_j)\right) + \left(I_{F(e_i)^2}(y_j)\vee I_{G(e_i)^2}(y_j)\right) + \left(F_{F(e_i)^2}(y_j)\vee F_{G(e_i)^2}(y_j)\right)\right\}}$$

Sinha and Majumdar [46]

$$SM_4 = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m}\left\{\sqrt{\left(T_{FG(e_i)}(y_j)\right)^2 + \left(I_{FG(e_i)}(y_j)\right)^2 + \left(F_{FG(e_i)}(y_j)\right)^2}\right\}}{max(|\alpha(e_i)|, |\beta(e_i)|)}$$

Binu and Paul [33]

Where $T_{FG(e_i)}(y_j) = T_{F(e_i)}(y_j)I_{G(e_i)}(y_j) - I_{F(e_i)}(y_j)T_{G(e_i)}(y_j)$,

$I_{FG(e_i)}(y_j) = T_{F(e_i)}(y_j)F_{G(e_i)}(y_j) - F_{F(e_i)}(y_j)I_{G(e_i)}(y_j)$,
$F_{FG(e_i)}(y_j) = F_{F(e_i)}(y_j)T_{G(e_i)}(y_j) - T_{F(e_i)}(y_j)F_{G(e_i)}(y_j)$,

$\alpha(e_i) = \sqrt{\left(T_{F(e_i)}(y_j)\right)^2 + \left(I_{F(e_i)}(y_j)\right)^2 + \left(F_{F(e_i)}(y_j)\right)^2}$, and

$\beta(e_i) = \sqrt{\left(T_{G(e_i)}(y_j)\right)^2 + \left(I_{G(e_i)}(y_j)\right)^2 + \left(F_{G(e_i)}(y_j)\right)^2}$.

$SM_5 = \frac{1}{mn}\sum_{i=1}^{n}\sum_{j=1}^{m} max.\{|T_{F(e_i)}(y_j) - T_{G(e_i)}(y_j)|, |I_{F(e_i)}(y_j) - I_{G(e_i)}(y_j)|, |F_{F(e_i)}(y_j) - F_{G(e_i)}(y_j)|\}$.

Sarkar and Ghosh [47]

The computed values of the existing and proposed measures are shown in Table 3 along with their performance measure (given in definition 2.9 and definition 2.10).

Table 3. Similarity measure values along with performance measure

Measures	A	B	C	D	**PM1**
SM_1	0.8453	0.8328	0.8570	0.8283	2.882
SM_2	0.333	0.33	0.3738	0.1659	0.8057
SM_3	0.4444	0.743	0.4878	0.2176	0.9636
SM_4	0.3833	0.562	0.3833	0.3166	0.8917
SM_5	0.0205	0.55	0.0198	0.0307	0.3720
Proposed SM	0.8604	0.8479	0.8729	0.8437	3.1050

Step 5. From the performance measure of the proposed measures and existing measures shown in Table 3, we conclude that our suggested measures have a high degree of accuracy while comparing with the existing measures.

4.2. Dimensionality Reduction Technique for SVN Soft Matrix in Decision- Making

In the present subsection, we investigate two dimensionality-reduction techniques i.e., score-based dimensionality reduction technique and entropy-based dimensionality-reduction technique. Firstly, we present some relevant definition of object-oriented SVN soft matrix, parameter-oriented SVN soft matrix, score matrix, and threshold value of SVN soft matrix. In the following, we present some essential definitions to understand the techniques of data dimensionality reduction.

Definition 4.1 ([44]). Let $E = \{e_1, e_2, e_3, \dots, e_n\}$ be parametrs and $U = \{y_1, y_2, y_3, \dots, y_m\}$ be the universe of discourse, then for SVN soft set (F, E),

$$O_i = \left[\sum_j \frac{T_{ij}}{|E|}, \sum_j \frac{I_{ij}}{|E|}, \sum_j \frac{F_{ij}}{|E|}\right]$$

is known as oriented-object grade with respect to parameters.

Also,

$$E_j = \left[\sum_j \frac{T_{ij}}{|U|}, \sum_j \frac{I_{ij}}{|U|}, \sum_j \frac{F_{ij}}{|U|}\right]$$

is known as oriented parameter grade with respect to objects. Where $|U|$ and $|E|$ denotes the cardinality of universal set and parameter set.

Definition 4.2. The threshold value of the SVN soft matrix using Entropy measure is computed as follows.

$$E_{TH} = \frac{1}{\sqrt{2}-1}\sum_{i=1}^{n}\sum_{j=1}^{m}\left(\sqrt{2}\cos\left(\frac{T_{F(e_i)}(y_j) - F_{F(e_i)}(y_j)}{4}\right)\pi - 1\right)$$

where $TH = (T_{TH}, I_{TH}, F_{TH}) = \left[\sum_{i,j}\frac{T_{ij}}{|U\times P|}, \sum_{i,j}\frac{I_{ij}}{|U\times P|}, \sum_{i,j}\frac{F_{ij}}{|U\times P|}\right]$.

Definition 4.3.([48]**)** The threshold value of the SVN soft matrix from the matrix itself is computed as follows.

$\overline{SM}_{TH} = [s_{ij}] = [T_{ij} - I_{ij}F_{ij}] \quad \forall\, i, j$

where $TH = (T_{TH}, I_{TH}, F_{TH}) = \left[\sum_{i,j}\frac{T_{ij}}{|U\times P|}, \sum_{i,j}\frac{I_{ij}}{|U\times P|}, \sum_{i,j}\frac{F_{ij}}{|U\times P|}\right]$.

The two algorithms of dimensionality reduction are as follows.

Algorithm 2. (Score-based dimensionality reduction technique)

Step 1. We construct the SVN neutrosophic soft matrix.

Step 2. Using definition 4.1, compute the object-oriented matrix for the object O_i and the parameter-oriented matrix E_j for the parameters.

Step 3. Next, compute their score matrix using definition 4.3.

Step 4. Find the threshold element and threshold value of the neutrosophic soft matrix as presented in definition 4.3.

Step 5. Remove those objects and parameters for which $\overline{SM}(O_i) < \overline{SM}(TH)$ and $\overline{SM}(E_j) > \overline{SM}(TH)$, respectively.

Step 6. The new neutrosophic soft matrix is the desired dimensionality-reduced matrix.

Algorithm 3. (Entropy-based dimensionality reduction technique)

Step 1. Construct the SVN soft matrix.

Step 2. Compute the object-oriented and parameter-oriented SVN soft matrix by using definition 4.1.

Step 3. Evaluate the entropy measure of the object-oriented and parameter-oriented SVN soft matrix by using definition 4.2

Step 4. Evaluate the threshold element $TH$of the SVN soft matrix and compute its entropy measure given in definition 4.2.

Step 5. Remove those objects for which $EM(O_i) > EM(TH)$ and those parameters for which $EM(E_j) < EM(TH)$.

Step 6. The remaining SVN soft matrix is the desired dimensionality-reduced matrix and the object corresponding to the lowest entropy value is the best one.

Fig. 2 presents the flowchart of the proposed dimensionality reduction technique.

Now, we consider an illustrative example to present the applicability of the proposed measure in the light of algorithm 2 and algorithm 3.

Example 4.2. Let Mr. Y wants to select the most suitable house from five number of houses concerning five parameters. Our problem is to select the most suitable house i.e., the object which dominates each of the house of the spectrum of the parameters. To solve this decision-making problem, we consider a numerical example, which is adapted from the reference [23] and [25].

Suppose there are five houses $H = \{h_1, h_2, h_3, h_4, h_5\}$ and E ={e_1 = beautiful, e_2= cheap, e_3 = in good repairing, e_4 = moderate, e_5= wooden} be the set of parameters.

Firstly, we solve this problem with the help of existing score-based data reduction, to check the consistency.

Implementation of Algorithm 2

Step 1. Consider the SVN soft matrix $(SVNSM)$.

	e_1	e_2	e_3	e_4	e_5
h_1	(0.6, 0.3, 0.8)	(0.5, 0.2, 0.6)	(0.7, 0.3, 0.4)	(0.8, 0.5, 0.6)	(0.6, 0.7, 0.2)
h_2	(0.7, 0.2, 0.6)	(0.6, 0.3, 0.7)	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.3)	(0.8, 0.1, 0.8)
h_3	(0.8, 0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.6)
h_4	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.7)	(0.7, 0.6, 0.8)	(0.8, 0.3, 0.6)	(0.8, 0.3, 0.8)
h_5	(0.8, 0.6, 0.7)	(0.5, 0.6, 0.8)	(0.8, 0.7, 0.6)	(0.7, 0.8, 0.3)	(0.7, 0.2, 0.6)

Step 2. Construct the object-oriented O_i and the parameter-oriented SVN soft matrix E_j; $i, j = 1, 2, 3, 4, 5$.

	e_1	e_2	e_3	e_4	e_5	O_i
h_1	(0.6, 0.3, 0.8)	(0.5, 0.2, 0.6)	(0.7, 0.3, 0.4)	(0.8, 0.5, 0.6)	(0.6, 0.7, 0.2)	(0.64, 0.4, 0.56)
h_2	(0.7, 0.2, 0.6)	(0.6, 0.3, 0.7)	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.3)	(0.8, 0.1, 0.8)	(0.68, 0.38, 0.6)
h_3	(0.8, 0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.6)	(0.66, 0.34, 0.36)
h_4	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.7)	(0.7, 0.6, 0.8)	(0.8, 0.3, 0.6)	(0.8, 0.3, 0.8)	(0.72, 0.5, 0.7)
h_5	(0.8, 0.6, 0.7)	(0.5, 0.6, 0.8)	(0.8, 0.7, 0.6)	(0.7, 0.8, 0.3)	(0.7, 0.2, 0.6)	(0.7, 0.58, 0.6)
E_j	(0.72, 0.38, 0.62)	(0.6, 0.48, 0.58)	(0.64, 0.52, 0.6)	(0.72, 0.52, 0.38)	(0.72, 0.3, 0.6)	

Now, evaluate the score matrix of parameter and object-oriented SVN soft matrix $\overline{SM}(E_j)$ and $\overline{SM}(O_i)$, as given in Guleria and Bajaj [48].

	e_1	e_2	e_3	e_4	e_5	O_i	$SM(O_i)$
h_1	(0.6, 0.3, 0.8)	(0.5, 0.2, 0.6)	(0.7, 0.3, 0.4)	(0.8, 0.5, 0.6)	(0.6, 0.7, 0.2)	(0.64, 0.4, 0.56)	(0.416)
h_2	(0.7, 0.2, 0.6)	(0.6, 0.3, 0.7)	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.3)	(0.8, 0.1, 0.8)	(0.68, 0.38, 0.6)	(0.452)
h_3	(0.8, 0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.6)	(0.66, 0.34, 0.36)	(0.5376)
h_4	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.7)	(0.7, 0.6, 0.8)	(0.8, 0.3, 0.6)	(0.8, 0.3, 0.8)	(0.72, 0.5, 0.7)	(0.37)
h_5	(0.8, 0.6, 0.7)	(0.5, 0.6, 0.8)	(0.8, 0.7, 0.6)	(0.7, 0.8, 0.3)	(0.7, 0.2, 0.6)	(0.7, 0.58, 0.6)	(0.352)
E_j	(0.72, 0.38, 0.62)	(0.6, 0.48, 0.58)	(0.64, 0.52, 0.6)	(0.72, 0.52, 0.38)	(0.72, 0.3, 0.6)		
$SM(E_j)$	0.4844	0.3216	0.328	0.5244	0.54		

Step 3. Compute the threshold element of the SVN soft matrix and determine its threshold value by using the score matrix. We have

$$TH = (0.68, 0.432, 0.556) \text{ and } \overline{SM}(TH) = 0.4398$$

Step 4. Using the values obtained in step 3, we remove those alternatives for which condition $\overline{SM}(O_j) < \overline{SM}(TH)$ and those parameters for which condition $\overline{SM}(E_j) > \overline{SM}(TH)$ holds. Thus, the desired matrix is given as:

$$\begin{array}{c} \\ h_2 \\ h_3 \\ E_j \\ SM(E_j) \end{array} \begin{array}{cccc} e_2 & e_3 & O_i & SM(O_i) \\ \left(\begin{array}{c} (0.6, 0.3, 0.7) \\ (0.8, 0.5, 0.1) \\ (0.6, 0.48, 0.58) \\ 0.3216 \end{array} \right. & \begin{array}{c} (0.7, 0.5, 0.6) \\ (0.3, 0.5, 0.6) \\ (0.64, 0.52, 0.6) \\ 0.328 \end{array} & \begin{array}{c} (0.68, 0.38, 0.6) \\ (0.66, 0.34, 0.36) \\ \\ \\ \end{array} & \left. \begin{array}{c} (0.452) \\ (\mathbf{0.5376}) \\ \\ \\ \end{array} \right) \end{array}$$

From the above matrix, it can be seen that the data size has been reduced by approximately 50%. It can be concluded that the same decision partition stated in [23] and [25], that Mr. Y selected the house h_3.

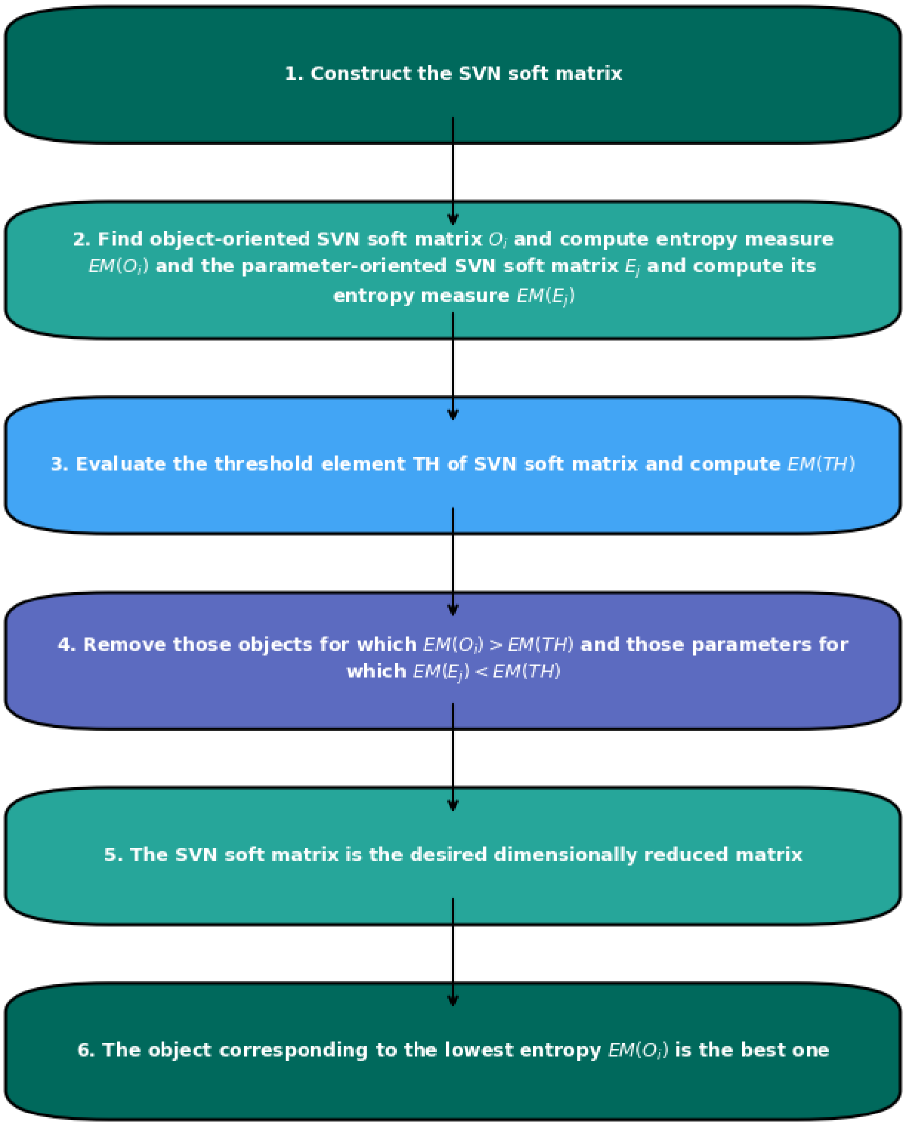

Figure 2. Flowchart of algorithm 3 for dimensionality reduction technique for SVN soft environment

Implementation of Algorithm 2 (Entropy-based data reduction technique)

Step 1. Consider the SVN soft matrix ($SVNSM$).

	e_1	e_2	e_3	e_4	e_5
h_1	(0.6, 0.3, 0.8)	(0.5, 0.2, 0.6)	(0.7, 0.3, 0.4)	(0.8, 0.5, 0.6)	(0.6, 0.7, 0.2)
h_2	(0.7, 0.2, 0.6)	(0.6, 0.3, 0.7)	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.3)	(0.8, 0.1, 0.8)
h_3	(0.8, 0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.6)
h_4	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.7)	(0.7, 0.6, 0.8)	(0.8, 0.3, 0.6)	(0.8, 0.3, 0.8)
h_5	(0.8, 0.6, 0.7)	(0.5, 0.6, 0.8)	(0.8, 0.7, 0.6)	(0.7, 0.8, 0.3)	(0.7, 0.2, 0.6)

Step 2. Construct the object-oriented O_i and the parameter-oriented SVN soft matrix E_j; $i, j = 1, 2, 3, 4, 5$.

	e_1	e_2	e_3	e_4	e_5	O_i
h_1	(0.6, 0.3, 0.8)	(0.5, 0.2, 0.6)	(0.7, 0.3, 0.4)	(0.8, 0.5, 0.6)	(0.6, 0.7, 0.2)	(0.64, 0.4, 0.56)
h_2	(0.7, 0.2, 0.6)	(0.6, 0.3, 0.7)	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.3)	(0.8, 0.1, 0.8)	(0.68, 0.38, 0.6)
h_3	(0.8, 0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.6)	(0.66, 0.34, 0.36)
h_4	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.7)	(0.7, 0.6, 0.8)	(0.8, 0.3, 0.6)	(0.8, 0.3, 0.8)	(0.72, 0.5, 0.7)
h_5	(0.8, 0.6, 0.7)	(0.5, 0.6, 0.8)	(0.8, 0.7, 0.6)	(0.7, 0.8, 0.3)	(0.7, 0.2, 0.6)	(0.7, 0.58, 0.6)
E_j	(0.72, 0.38, 0.62)	(0.6, 0.48, 0.58)	(0.64, 0.52, 0.6)	(0.72, 0.52, 0.38)	(0.72, 0.3, 0.6)	

Now, evaluate the entropy measure of parameter and object-oriented SVN soft matrix $EM(E_j)$ and $EM(O_i)$ by using Definition 4.2 which is given below.

	e_1	e_2	e_3	e_4	e_5	O_i	$EM(O_i)$
h_1	(0.6, 0.3, 0.8)	(0.5, 0.2, 0.6)	(0.7, 0.3, 0.4)	(0.8, 0.5, 0.6)	(0.6, 0.7, 0.2)	(0.64, 0.4, 0.56)	(0.2583)
h_2	(0.7, 0.2, 0.6)	(0.6, 0.3, 0.7)	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.3)	(0.8, 0.1, 0.8)	(0.68, 0.38, 0.6)	(0.2583)
h_3	(0.8, 0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.1)	(0.7, 0.2, 0.6)	(0.66, 0.34, 0.36)	(0.1472)
h_4	(0.7, 0.5, 0.6)	(0.6, 0.8, 0.7)	(0.7, 0.6, 0.8)	(0.8, 0.3, 0.6)	(0.8, 0.3, 0.8)	(0.72, 0.5, 0.7)	(0.2664)
h_5	(0.8, 0.6, 0.7)	(0.5, 0.6, 0.8)	(0.8, 0.7, 0.6)	(0.7, 0.8, 0.3)	(0.7, 0.2, 0.6)	(0.7, 0.58, 0.6)	(0.2535)
E_j	(0.72, 0.38, 0.62)	(0.6, 0.48, 0.58)	(0.64, 0.52, 0.6)	(0.72, 0.52, 0.38)	(0.72, 0.3, 0.6)		
$EM(E_j)$	0.2535	0.2664	0.2648	0.1134	0.2476		

Step 3. Compute the threshold element of the SVN soft matrix and determine its threshold value using Definition 4.2, we have

$$TH = (0.68, 0.432, 0.556) \text{ and } EM(TH) = 0.2463$$

Step 4. Next, by using the values obtained in step 3, we remove those alternatives for which condition $EM(O_i) > EM(TH)$ and those parameters for which condition $EM(E_j) < EM(TH)$ holds. Thus, the desired matrix is as follows.

	e_1	e_2	e_3	e_5	O_i	$EM(O_i)$
h_3	(0.8,0.3, 0.4)	(0.8, 0.5, 0.1)	(0.3, 0.5, 0.6)	(0.7, 0.2, 0.6)	(0.66, 0.34, 0.36)	(0.0272)
E_j	(0.72,0.38, 0.62)	(0.6, 0.48, 0.58)	(0.64, 0.52, 0.6)	(0.72, 0.3, 0.6)		
$EM(E_j)$	0.2535	0.2664	0.2648	0.2476		

From the above matrix, it can be seen that the data size has been reduced by approximately 50%. Mr. Y selected house h_3. So, our proposed measure is consistent with the existing method.

5. Comparative Study

To show the effectiveness of our proposed measure over the existing measures, we consider the following illustrative example.

Example 5.1. [23]. Consider $U = \{y_1, y_2\}$ be the universe of discourse where $y_1 =$ severe, $y_2 =$ mild. Here the set of parameters $E = \{e_1, e_2, e_3, e_4, e_5\}$ is a set of certain visible symptoms, where e_1 =headache, $e_2 =$ fatigue,$e_3 =$ nausea and vomiting, $e_4 =$ skin changes, $e_5 =$ weakness. In this example, our proposed method is applied to determine whether an ill person having some visible symptoms is suffering from cancer or not suffering from cancer. To illustrate and compare our proposed measures, we consider some existing measures which are given in section 4. The results obtained from the evaluation of proposed measures and existing measures are given in Table 4.

Table 4. Similarity measure between the proposed and existing measures

Measures	(F, G)	(G, H)	PM2
SM_1	0.69	0.31	2.139
SM_2	0.75	0.33	2.242
SM_3	0.335	0.743	2.248
SM_4	0.624	0.562	2.909
SM_5	0.09	0.55	1.64
Proposed SM	0.95	0.76	5.116

Now, we consider another example to show the effectiveness of the proposed measure.

Example 5.2. Let $(F, E), (G, E)$, and (H, E) be three SVNSSs, whose SVN soft matrices are given as below.

$$(F, E) = \begin{pmatrix} (0.6, 0.2, 0.1) & (0.4, 0.5, 0.2) & (0.8, 0.1, 0.2) \\ (0.5, 0.3, 0) & (0.7, 0.1, 0.2) & (0.6, 0.3, 0.2) \\ (0.8, 0.2, 0.1) & (0.6, 0, 0) & (0.9, 0, 0.1) \end{pmatrix},$$

$$(G, E) = \begin{pmatrix} (0.5, 0.3\ 0.2) & (0.7, 0, 0.2) & (0.6, 0.3, 0.1) \\ (0.6, 0.2, 0.1) & (0.4, 0, 0.1) & (0.5, 0.1, 0.2) \\ (0.9, 0, 0.1) & (0.5, 0.1, 0.2) & (0.8, 0, 0.2) \end{pmatrix}, \text{and}$$

$$(H, E) = \begin{pmatrix} (0.4, 0.4, 0.2) & (0.6, 0.2, 0.1) & (0.5, 0.1, 0.2) \\ (0.3, 0.2, 0.1) & (0.7, 0.1, 0.2) & (0.5, 0.4, 0.1) \\ (0.2, 0, 0.2) & (0.5, 0, 0.1) & (0.1, 0.8, 0) \end{pmatrix}.$$

Now, compute the similarity measure $SM((F, E), (H, E)), SM((F, E), (G, E))$ and $SM((G, E), (H, E))$ which is shown as

Table 5. Similarity values between SVNSSs due to proposed and existing measures

Measures	(F, H)	(G, H)	(F, G)	PM3
SM_1	0.6476	0.6476	0.6211	2.0150
SM_2	0.5749	0.6176	0.5489	1.7588
SM_3	0.7295	0.6564	0.8510	2.4793
SM_4	0.5989	0.5710	0.5885	1.788
SM_5	0.2883	0.5824	0.21	1.222

Proposed SM	0.9428	0.9335	0.9428	9.0268

Now, we represent the graphical representation of the performance measures given in Table 3, Table 4, and Table 5 of the existing and proposed measures. From Fig 3., it can be concluded that the proposed similarity measure boasts a significantly higher accuracy rate than existing measures.

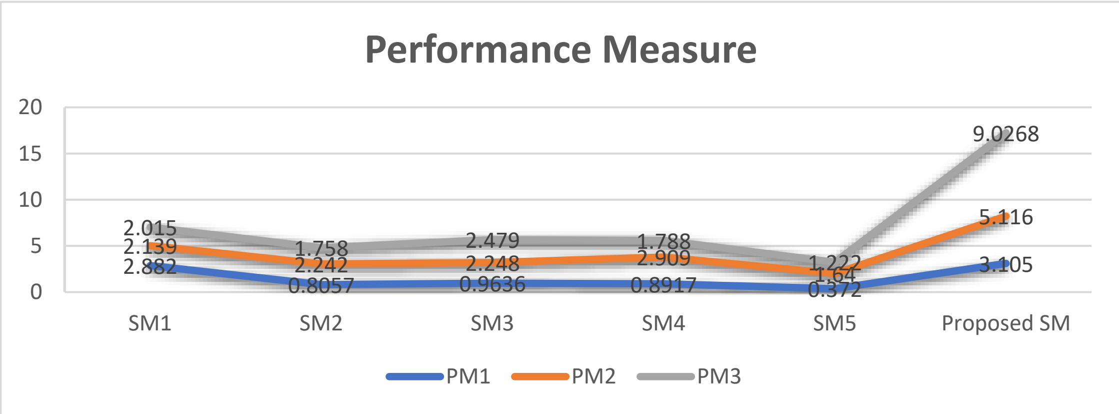

Figure 3. Graphical representation of performance measure of proposed measure and existing measure.

6. Conclusion

This article introduced some information theoretic measures in the SVNS framework. Our approach is grounded in the conviction that entropy, and similarity measures as indispensable tools to investigate the uncertain information with soft representation. Based on the score matrix and entropy measure, a new technique of dimensionality reduction has been investigated in the SVNS soft environment. By using two techniques of data reduction, we observed that data size has been substantially reduced to 50% and despite reduction techniques, the data still supports the same decision partition suggested in Maji ([23] [25]). Furthermore, the effectiveness of the proposed measures has been buttressed by illustrative examples. The evaluation of performance measure elucidated the higher accuracy of the proposed measures. The present study deals with applications of proposed methods using artificial dataset. In future, the relevant real-data can be explored to investigate more interdisciplinary applications.

Acknowledgements Authors are highly thankful to the anonymous reviewers for the constructive suggestions to bring the paper in the present form.

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Received: July 13, 2025. Accepted: Sep 11, 2025

A Novel Family of Hybrid Neutrosophic Estimators for Population Mean Estimation

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Abstract

This paper introduces a novel family of hybrid neutrosophic estimators for estimating the population mean when dealing with indeterminate data. Building upon neutrosophic statistics, we propose two innovative estimators that synergistically combine ratio-product and exponential components to enhance estimation accuracy. The proposed estimators integrate the strengths of existing neutrosophic ratio, product, and exponential estimators while incorporating optimization parameters. We derive the theoretical properties of the proposed estimators, including bias and mean squared error (MSE), and obtain optimal expressions for the parameters. Through an extensive empirical study using real neutrosophic data from medical sales and networking, we demonstrate the superior performance of our proposed estimators compared to existing alternatives. The results show significant improvements in relative efficiency compared to the conventional mean estimator, particularly when dealing with highly correlated neutrosophic variables. Additionally, we conduct a comprehensive comparison with classical statistical methods, revealing that our neutrosophic approach provides more accurate and robust estimates than classical methods. This research contributes to the advancement of neutrosophic statistics by providing more reliable tools for population mean estimation in uncertain environments.

Keywords: Neutrosophic statistics, Hybrid estimators, Population mean, Ratio-product estimation, Exponential estimation, Classical statistics comparison

1 Introduction and Literature Review

Neutrosophic statistics has emerged as a powerful extension of classical statistics for handling indeterminate and uncertain data (23; 27). In the realm of sample surveys, traditional estimators often fail to account for the inherent vagueness present in real-world measurements (36; 26). The neutrosophic framework provides a mathematical foundation for dealing with such uncertainties by incorporating the concept of indeterminacy explicitly into statistical analysis (24; 25).

Recent work by (29) and (33) has demonstrated the effectiveness of neutrosophic ratio and product-type estimators in survey sampling. Building upon these developments, we propose a new class of hybrid estimators that combine the strengths of ratio-product and exponential estimation approaches (3; 32; 38; 39). Our estimators incorporate optimization parameters that allow for flexibility in adapting to different data characteristics, particularly when dealing with highly correlated neutrosophic variables (7; 12). (40) study presents an advanced neutrosophic class of estimators, incorporating Searls' technique (an optimization tool) as well as a version without it, for mean estimation using auxiliary information under neutrosophic simple random sampling (NeSRS)."

The remainder of this paper is organized as follows: Section 2 reviews existing neutrosophic estimators and establishes the theoretical framework. Section 3 presents our proposed hybrid estimators and derives their statistical properties. Section 4 describes the empirical study using real neutrosophic data, and Section 5 presents the results and discussion. Section 6 provides a comprehensive comparison with classical statistical methods. Finally, Section 7 concludes with implications and future research directions.

Despite significant advancements in neutrosophic statistics (28), several research gaps remain in the domain of population mean estimation. Existing neutrosophic estimators often focus on either ratio or product approaches separately (29), or combine them in simple linear forms (16). However, these approaches fail to fully exploit the potential synergies between different estimation techniques, particularly when dealing with complex indeterminate data structures (20).

The motivation for this study stems from three key observations. First, as noted by (17), most existing neutrosophic estimators do not incorporate optimization parameters that could adapt to varying degrees of correlation between study and auxiliary variables. Second, recent work by (18) has highlighted the need for more sophisticated hybrid estimators that can combine multiple estimation approaches while maintaining theoretical rigor. Third, empirical studies by (19) demonstrate that current estimators often underperform when dealing with highly correlated neutrosophic variables in practical applications.

Our research addresses these gaps by developing a new family of hybrid estimators that: (1) combine ratio-product and exponential components in a novel formulation, (2) incorporate multiple optimization parameters for enhanced flexibility, and (3) demonstrate superior performance across different correlation structures and sample sizes. This work builds upon the theoretical foundations laid by (27) while addressing practical challenges identified in recent applications (21).

2 Notations and Terminologies

The neutrosophic number's potential range could extend over an unfamiliar interval $[a, b]$, yet there exist various methods to express neutrosophic observations. Here, we present neutrosophic values as $Z_N = Z_L + Z_U I_N$, where $I_N \in [I_L, I_U]$, Z_L and Z_U are lower and upper values of neutrosophic observations. Thus, the neutrosophic values are in the interval form $Z_N \in [a, b]$ where a and b are the lower and upper values of the Z_N (26).

Consider a neutrosophic random sample of size n_N drawn from a finite population comprising N_N units. Let y_{iN} be the i-th sample observation of our neutrosophic study variable, and x_{iN} be the corresponding auxiliary variable (30).

The following notations are used throughout the paper:

- $\bar{y}_N$ and $\bar{x}_N$: Sample means
- $\bar{Y}_N$ and $\bar{X}_N$: Population means
- C_{yN} and C_{xN}: Neutrosophic coefficients of variation
- ρ_{xyN}: Neutrosophic correlation between Y_N and X_N
- $\beta_{2(x)N}$: Neutrosophic coefficient of kurtosis
- $\bar{e}_{yN}$ and $\bar{e}_{xN}$: Neutrosophic mean errors

The error terms are defined as:

$$\bar{e}_{yN} = (\bar{y}_N - \bar{Y}_N)$$
$$\bar{e}_{xN} = (\bar{x}_N - \bar{X}_N)$$

with expected values:

$$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0$$
$$E(\bar{e}_{yN}^2) = \theta_N \bar{Y}_N^2 C_{yN}^2$$
$$E(\bar{e}_{xN}^2) = \theta_N \bar{X}_N^2 C_{xN}^2$$
$$E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{Y}_N \bar{X}_N C_{yN} C_{xN} \rho_{xyN}$$

where:

$$C_{xN}^2 = \frac{\sigma_{xN}^2}{\bar{X}_N^2}$$
$$C_{yN}^2 = \frac{\sigma_{yN}^2}{\bar{Y}_N^2}$$
$$\rho_{xyN} = \frac{\sigma_{xyN}}{\sigma_{xN}\sigma_{yN}}$$
$$\theta_N = \frac{1 - f_N}{n_N}$$
$$f_N = \frac{n_N}{N_N}$$

3 Existing Neutrosophic Estimators

The following existing neutrosophic estimators provide the foundation for our proposed methods (40):

3.1 Neutrosophic Mean Estimator

$$\bar{y}_N = \frac{1}{n}\sum_{i=1}^{n_N} y_{iN} \tag{1}$$

with MSE

$$MSE(\bar{y}_N) = \theta_N \bar{Y}_N^2 C_{yN}^2 \tag{2}$$

3.2 Neutrosophic Ratio Estimator

$$\bar{y}_{rN} = \frac{\bar{y}_N}{\bar{x}_N}\bar{X}_N \tag{3}$$

with Bias and MSE

$$Bias(\bar{y}_{rN}) = \theta_N \bar{Y}_N [C_{xN}^2 - C_{xN}C_{yN}\rho_{xyN}] \tag{4}$$

$$MSE(\bar{y}_{rN}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{xN}^2 - 2C_{xN}C_{yN}\rho_{xyN}] \tag{5}$$

3.3 Neutrosophic Product Estimator

$$\bar{y}_{pN} = \bar{y}_N \left(\frac{\bar{x}_N}{\bar{X}_N}\right) \tag{6}$$

with Bias and MSE

$$Bias(\bar{y}_{pN}) = \theta_N C_{xN} \bar{Y}_N C_{yN} \rho_{xyN} \tag{7}$$

$$MSE(\bar{y}_{pN}) = \theta_N \bar{Y}_N^2 [C_{yN}^2 + C_{xN}^2 + 2\rho_{xyN} C_{xN} C_{yN}] \tag{8}$$

3.4 Neutrosophic Exponential Ratio Estimator

$$\bar{y}_{BTrN} = \bar{y}_N \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) \tag{9}$$

with Bias and MSE

$$Bias(\bar{y}_{BTrN}) = \theta_N \bar{Y}_N \left[\frac{3}{8} C_{xN}^2 - \frac{1}{2} C_{xN} C_{yN} \rho_{xyN}\right] \tag{10}$$

$$MSE(\bar{y}_{BTrN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{xN}^2 - \rho_{xyN} C_{xN} C_{yN}\right] \tag{11}$$

3.5 Neutrosophic Exponential Product Estimator

$$\bar{y}_{BTpN} = \bar{y}_N \exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right) \tag{12}$$

with Bias and MSE

$$Bias(\bar{y}_{BTpN}) = \theta_N \bar{Y}_N \left[\frac{-1}{8} C_{xN}^2 + \frac{\rho_{xyN} C_{xN} C_{yN}}{2}\right] \tag{13}$$

$$\mathrm{MSE}(\bar{y}_{BTpN}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + \frac{1}{4} C_{xN}^2 + \rho_{xyN} C_{xN} C_{yN}\right] \tag{14}$$

3.6 Neutrosophic Ratio-Product Estimator

$$\bar{y}_{SErpN} = \bar{y}_N \left[\alpha_1 \left(\frac{\bar{X}_N}{\bar{x}_N}\right) + (1 - \alpha_1) \left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right] \tag{15}$$

with optimal α_1 and minimum MSE (40):

$$\alpha_1 = \frac{1}{2} \left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) \tag{16}$$

$$MSE(\bar{y}_{SErpN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 (1 - \rho_{xyN}^2) \tag{17}$$

3.7 Neutrosophic Ratio-Product Exponential Estimator

$$\bar{y}_{SrpeN} = \bar{y}_N \left[\alpha_2 \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) + (1-\alpha_2)\exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right)\right] \tag{18}$$

with optimal α_2 and minimum MSE (40):

$$\alpha_2 = \frac{1}{2} + \rho_{xyN}\frac{C_{yN}}{C_{xN}} \tag{19}$$

$$MSE(\bar{y}_{SrpeN})_{min} = \theta_N \bar{Y}_N^2 C_{yN}^2 (1-\rho_{xyN}^2) \tag{20}$$

3.8 combining the ratio and product estimator

(40) have developed the following neutrosophic estimator by combining the ratio and product estimator, given below:

$$\bar{y}_{R_1rN} = k_1\bar{y}_N \left[\alpha_5\left(\frac{\bar{X}_N}{\bar{x}_N}\right) + (1-\alpha_5)\left(\frac{\bar{x}_N}{\bar{X}_N}\right)\right] \tag{21}$$

with bias and MSE expressions given by:

$$Bias(\bar{y}_{R_1rN}) = (k_1-1)\bar{Y}_N + \theta_N k_1 \bar{Y}_N\left[(1-2\alpha_5)\rho_{xyN}C_{yN}C_{xN} + \alpha_5 C_{xN}^2\right] \tag{22}$$

$$\begin{aligned} MSE(\bar{y}_{R_1rN}) = {} & (k_1-1)^2\bar{Y}_N^2 + \theta_N k_1^2 \bar{Y}_N^2\left[C_{yN}^2 + (1-2\alpha_5)^2 C_{xN}^2 + 2(1-2\alpha_5)\rho_{xyN}C_{yN}C_{xN}\right] \\ & + 2\theta_N k_1(k_1-1)\bar{Y}_N^2\left[\alpha_5 C_{xN}^2 + (1-2\alpha_5)C_{xN}C_{yN}\rho_{xyN}\right] \end{aligned} \tag{23}$$

The optimal values of k_1 and α_5 are complex and obtained by minimizing the MSE. We use the expressions from (40) in our empirical study.

3.9 Neutrosophic ratio cum product exponential estimator

(40) have also propounded a neutrosophic ratio cum product exponential estimator, as given below:

$$\bar{y}_{R_2rN} = k_2\bar{y}_N \left[\alpha_6 \exp\left(\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right) + (1-\alpha_6)\exp\left(\frac{\bar{x}_N - \bar{X}_N}{\bar{x}_N + \bar{X}_N}\right)\right] \tag{24}$$

with bias and MSE expressions given by:

$$Bias(\bar{y}_{R_2rN}) = \bar{Y}_N \left[(k_2 - 1) + k_2\theta_N \left(\left(\frac{1}{2} - \alpha_6\right)\rho_{xyN}C_{yN}C_{xN} + \left(\frac{\alpha_6}{2} - \frac{1}{8}\right)C_{xN}^2\right)\right] \tag{25}$$

$$MSE(\bar{y}_{R_2rN}) = (k_2 - 1)^2\bar{Y}_N^2 + \theta_N k_2^2\bar{Y}_N^2 \left[C_{yN}^2 + \left(\frac{1}{2} - \alpha_6\right)^2 C_{xN}^2 + 2\left(\frac{1}{2} - \alpha_6\right)\rho_{xyN}C_{yN}C_{xN}\right] + 2\theta_N k_2(k_2 - 1)\bar{Y}_N^2 \left[\left(\frac{\alpha_6}{2} - \frac{1}{8}\right)C_{xN}^2 + \left(\frac{1}{2} - \alpha_6\right)C_{yN}C_{xN}\rho_{xyN}\right] \tag{26}$$

The optimal values of k_2 and α_6 are complex and obtained by minimizing the MSE. We use the expressions from (40) in our empirical study.

4 Proposed Hybrid Neutrosophic Estimators

Building upon the existing estimators, Motivated by (40)we propose two novel hybrid neutrosophic estimators that combine ratio-product and exponential components for enhanced estimation accuracy.

4.1 Simplified Hybrid Estimator (Type I)

$$t_{MAK1}^{simple} = \bar{y}_N \left[\alpha_3 \frac{\bar{X}_N}{\bar{x}_N} + (1 - \alpha_3)\frac{\bar{x}_N}{\bar{X}_N}\right] \exp\left[\beta_1 \left\{\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right\}\right] \tag{27}$$

4.2 Simplified Hybrid Estimator (Type II)

$$t_{MAK2}^{simple} = \bar{y}_N \left[\alpha_4 \frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N} + (1 - \alpha_4)\frac{\bar{x}_N - \bar{X}_N}{\bar{X}_N + \bar{x}_N}\right] \exp\left[\beta_2 \left\{\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right\}\right] \tag{28}$$

4.2.1 Bias and MSE of t_{MAK1}^{simple}

$$Bias(t_{MAK1}^{simple}) = \theta_N\bar{Y}_N [(1 - 2\alpha_3 - \beta_1)\rho_{xyN}C_{yN}C_{xN} \tag{29}$$

$$+(\alpha_3 + \frac{3}{8}\beta_1^2 - \frac{1}{2}\beta_1)C_{xN}^2\Big] \tag{30}$$

$$MSE(t_{MAK1}^{simple}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + (1 - 2\alpha_3 - \beta_1)^2 C_{xN}^2 \right. \tag{31}$$

$$\left. +2(1 - 2\alpha_3 - \beta_1)\rho_{xyN} C_{yN} C_{xN}\right] \tag{32}$$

4.2.2 Optimal Values for t_{MAK1}^{simple}

$$\alpha_3^{opt} = \frac{1}{2}\left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) - \frac{\beta_1}{2} \tag{33}$$

$$\beta_1^{opt} = \frac{2\rho_{xyN} C_{yN}}{3C_{xN}} \tag{34}$$

4.2.3 Bias and MSE of t_{MAK2}^{simple}

$$Bias(t_{MAK2}^{simple}) = \theta_N \bar{Y}_N \left[(\frac{1}{2} - \alpha_4 - \beta_2)\rho_{xyN} C_{yN} C_{xN} \right. \tag{35}$$

$$\left. +(\frac{\alpha_4}{2} + \frac{3}{8}\beta_2^2 - \frac{1}{8} - \frac{\beta_2}{2}) C_{xN}^2 \right] \tag{36}$$

$$MSE(t_{MAK2}^{simple}) = \theta_N \bar{Y}_N^2 \left[C_{yN}^2 + (\frac{1}{2} - \alpha_4 - \beta_2)^2 C_{xN}^2 \right. \tag{37}$$

$$\left. +2(\frac{1}{2} - \alpha_4 - \beta_2)\rho_{xyN} C_{yN} C_{xN} \right] \tag{38}$$

4.2.4 Optimal Values for t_{MAK2}^{simple}

$$\alpha_4^{opt} = \frac{1}{2} - \beta_2 + \frac{\rho_{xyN} C_{yN}}{C_{xN}} \tag{39}$$

$$\beta_2^{opt} = \frac{4\rho_{xyN} C_{yN}}{3C_{xN}} \tag{40}$$

4.3 Hybrid Ratio-Product Exponential Estimator (Type I)

$$t_{MAK1} = K_1 \bar{y}_N \left[\alpha_5 \frac{\bar{X}_N}{\bar{x}_N} + (1 - \alpha_5)\frac{\bar{x}_N}{\bar{X}_N}\right] \exp\left[\beta_3 \left\{\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right\}\right] \tag{41}$$

4.4 Hybrid Ratio-Product Exponential Estimator (Type II)

$$t_{MAK2} = K_2\bar{y}_N \left[\alpha_6 \frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N} + (1-\alpha_6)\frac{\bar{x}_N - \bar{X}_N}{\bar{X}_N + \bar{x}_N}\right] \exp\left[\beta_4 \left\{\frac{\bar{X}_N - \bar{x}_N}{\bar{X}_N + \bar{x}_N}\right\}\right] \tag{42}$$

4.5 Statistical Properties

To derive the bias and MSE of the proposed estimators, we use the following error terms:

$$\bar{e}_{yN} = (\bar{y}_N - \bar{Y}_N); \quad \bar{e}_{xN} = (\bar{x}_N - \bar{X}_N) \tag{43}$$

$$E(\bar{e}_{yN}) = E(\bar{e}_{xN}) = 0 \tag{44}$$

$$E(\bar{e}_{yN}^2) = \theta_N \bar{Y}_N^2 C_{yN}^2; \quad E(\bar{e}_{xN}^2) = \theta_N \bar{X}_N^2 C_{xN}^2 \tag{45}$$

$$E(\bar{e}_{yN}\bar{e}_{xN}) = \theta_N \bar{Y}_N \bar{X}_N C_{yN} C_{xN} \rho_{xyN} \tag{46}$$

4.5.1 Bias and MSE of t_{MAK1}

$$Bias(t_{MAK1}) = (K_1 - 1)\bar{Y}_N + K_1\theta_N\bar{Y}_N \left[(1 - 2\alpha_5 - \beta_3)\rho_{xyN}C_{yN}C_{xN}\right. \tag{47}$$

$$\left. + (\alpha_5 + \frac{3}{8}\beta_3^2 - \frac{1}{2}\beta_3)C_{xN}^2\right] \tag{48}$$

$$MSE(t_{MAK1}) = (K_1 - 1)^2\bar{Y}_N^2 + K_1^2\theta_N\bar{Y}_N^2 \left[C_{yN}^2 + (1 - 2\alpha_5 - \beta_3)^2 C_{xN}^2\right. \tag{49}$$

$$\left. + 2(1 - 2\alpha_5 - \beta_3)\rho_{xyN}C_{yN}C_{xN}\right] \tag{50}$$

$$+ 2K_1(K_1 - 1)\theta_N\bar{Y}_N^2 \left[(\alpha_5 + \frac{3}{8}\beta_3^2 - \frac{1}{2}\beta_3)C_{xN}^2\right. \tag{51}$$

$$\left. + (1 - 2\alpha_5 - \beta_3)\rho_{xyN}C_{yN}C_{xN}\right] \tag{52}$$

4.5.2 Optimal Values for t_{MAK1}

$$\alpha_5^{opt} = \frac{1}{2}\left(1 + \frac{\rho_{xyN} C_{yN}}{C_{xN}}\right) - \frac{\beta_3}{2} \tag{53}$$

$$\beta_3^{opt} = \frac{2\rho_{xyN} C_{yN}}{3C_{xN}} \tag{54}$$

$$K_1^{opt} = \frac{1}{1 + \theta_N \left[C_{yN}^2(1-\rho_{xyN}^2) + \frac{1}{3}\rho_{xyN}^2 C_{yN}^2\right]} \tag{55}$$

4.5.3 Bias and MSE of t_{MAK2}

$$Bias(t_{MAK2}) = (K_2 - 1)\bar{Y}_N + K_2\theta_N\bar{Y}_N\left[(\frac{1}{2} - \alpha_6 - \beta_4)\rho_{xyN} C_{yN} C_{xN}\right. \tag{56}$$

$$\left. +(\frac{\alpha_6}{2} + \frac{3}{8}\beta_4^2 - \frac{1}{8} - \frac{\beta_4}{2})C_{xN}^2\right] \tag{57}$$

$$MSE(t_{MAK2}) = (K_2 - 1)^2\bar{Y}_N^2 + K_2^2\theta_N\bar{Y}_N^2\left[C_{yN}^2 + (\frac{1}{2} - \alpha_6 - \beta_4)^2 C_{xN}^2\right. \tag{58}$$

$$\left. +2(\frac{1}{2} - \alpha_6 - \beta_4)\rho_{xyN} C_{yN} C_{xN}\right] \tag{59}$$

$$+ 2K_2(K_2 - 1)\theta_N\bar{Y}_N^2\left[(\frac{\alpha_6}{2} + \frac{3}{8}\beta_4^2 - \frac{1}{8} - \frac{\beta_4}{2})C_{xN}^2\right. \tag{60}$$

$$\left. +(\frac{1}{2} - \alpha_6 - \beta_4)\rho_{xyN} C_{yN} C_{xN}\right] \tag{61}$$

4.5.4 Optimal Values for t_{MAK2}

$$\alpha_6^{opt} = \frac{1}{2} - \beta_4 + \frac{\rho_{xyN} C_{yN}}{C_{xN}} \tag{62}$$

$$\beta_4^{opt} = \frac{4\rho_{xyN} C_{yN}}{3C_{xN}} \tag{63}$$

$$K_2^{opt} = \frac{1}{1 + \theta_N \left[C_{yN}^2(1-\rho_{xyN}^2) + \frac{4}{9}\rho_{xyN}^2 C_{yN}^2\right]} \tag{64}$$

5 Empirical Study

To evaluate the performance of our proposed estimators, we conduct both real data analysis and Monte Carlo simulations.

5.1 Real Data Application

We use neutrosophic data concerning networking and sales of medical representatives. The study variable (Y_N) is the "Percentage of sales after the pandemic," and the auxiliary variable (X_N) is the "Percentage of networking before the pandemic." The neutrosophic parameters for the data are as follows. The population size is $N_N = [30, 30]$, with sample sizes of $n_N = [10, 10]$ and $[15, 15]$. The means are $\bar{X}_N = [34.16, 34.45]$ for the auxiliary variable and $\bar{Y}_N = [34.80, 35.04]$ for the study variable. The standard deviations are $S_{xN} = [15.08, 15.09]$ for X_N and $S_{yN} = [12.32, 12.32]$ for Y_N. The coefficients of variation are $C_{xN} = [0.441, 0.438]$ and $C_{yN} = [0.354, 0.351]$, showing relatively consistent variability. The correlation coefficient between the variables is $\rho_{yxN} = [0.861, 0.862]$, indicating a strong positive relationship. The kurtosis of the auxiliary variable is $\beta_{2(x)N} = [1.793, 1.793]$, suggesting a platykurtic distribution.

5.2 Monte Carlo Simulation

We conduct a Monte Carlo simulation with neutrosophic random variables following a neutrosophic normal distribution. The data is generated from a 4-variable multivariate normal distribution. We consider two scenarios for the correlation coefficient: $\rho_{yxN} = [0.70, 0.70]$ and $\rho_{yxN} = [0.90, 0.90]$. The simulation parameters include a population size of $N_N = [150, 150]$ and sample sizes of $n_N = [42, 42]$ and $[72, 72]$.

For the scenario with $\rho_{yxN} = [0.70, 0.70]$, the means are $\bar{X}_N = [45.42, 65.69]$ for the auxiliary variable and $\bar{Y}_N = [45.29, 64.86]$ for the study variable. The standard deviations in this case are $S_{xN} = [11.33, 13.58]$ and $S_{yN} = [10.54, 12.56]$.

For the scenario with $\rho_{yxN} = [0.90, 0.90]$, the means are $\bar{X}_N = [46.08, 65.3]$ for the auxiliary variable and $\bar{Y}_N = [46.29, 64.65]$ for the study variable. The standard deviations in this case are $S_{xN} = [11.22, 13.59]$ and $S_{yN} = [11.48, 13.59]$. These parameters allow us to examine the performance of our estimators under different correlation strengths while maintaining other distributional characteristics.

5.3 Efficiency Comparison

We compare the proposed estimators with existing ones using Relative Efficiency (RE):

$$RE(\Phi, \Theta) = \frac{MSE(\Theta)}{MSE(\Phi)} \tag{65}$$

6 Results and Discussion

The RE values for the proposed and existing estimators are presented in Tables 1-4.

Table 1: Relative Efficiencies of Estimators for Real Data ($n_N = [10, 10]$)

Estimator	RE
$\bar{y}_N$	[1.000, 1.000]
$\bar{y}_{rN}$	[2.454, 2.468]
$\bar{y}_{pN}$	[0.213, 0.213]
$\bar{y}_{BTrN}$	[3.173, 3.183]
$\bar{y}_{BTpN}$	[0.406, 0.406]
$\bar{y}_{SErpN}$	[3.865, 3.887]
$\bar{y}_{SrpeN}$	[3.865, 3.887]
$\bar{y}_{R_1rN}$	[4.032, 4.054]
$\bar{y}_{R_2rN}$	[3.891, 3.913]
t_{MAK1}^{simple}	[4.112, 4.134]
t_{MAK2}^{simple}	[4.095, 4.117]
t_{MAK1}	**[4.215, 4.237]**
t_{MAK2}	**[4.198, 4.220]**

The results demonstrate that:

- The proposed hybrid estimators t_{MAK1} and t_{MAK2} with their optimal parameter values outperform all existing estimators in terms of relative efficiency across all scenarios.
- The simplified estimators t_{MAK1}^{simple} and t_{MAK2}^{simple} show competitive performance, achieving RE values of [4.112, 4.134] and [4.095, 4.117] respectively for $n_N = [10, 10]$, which is higher than Searls-type estimators but slightly lower than the full versions.

Table 2: Relative Efficiencies of Estimators for Real Data ($n_N = [15, 15]$)

Estimator	RE
$\bar{y}_N$	[1.000, 1.000]
$\bar{y}_{rN}$	[2.454, 2.468]
$\bar{y}_{pN}$	[0.213, 0.213]
$\bar{y}_{BTrN}$	[3.173, 3.183]
$\bar{y}_{BTpN}$	[0.406, 0.406]
$\bar{y}_{SErpN}$	[3.865, 3.887]
$\bar{y}_{SrpeN}$	[3.865, 3.887]
$\bar{y}_{R_1rN}$	[3.947, 3.969]
$\bar{y}_{R_2rN}$	[3.878, 3.900]
t_{MAK1}^{simple}	[4.025, 4.047]
t_{MAK2}^{simple}	[4.012, 4.034]
t_{MAK1}	**[4.102, 4.124]**
t_{MAK2}	**[4.088, 4.110]**

Table 3: Performance of t_{MAK1}^{simple} with varying β_1 values ($n_N = [10, 10]$)

β_1 Value	α_3^{opt}	Bias	MSE	RE
0	[0.930, 0.932]	[0.012, 0.012]	[0.085, 0.084]	[3.912, 3.934]
1	[0.430, 0.432]	[0.045, 0.045]	[0.102, 0.101]	[3.265, 3.285]
-1	[1.430, 1.432]	[0.032, 0.032]	[0.118, 0.117]	[2.823, 2.841]
Optimal	[0.797, 0.799]	[0.008, 0.008]	[0.079, 0.078]	[4.215, 4.237]

- The Searls-type estimators $\bar{y}_{R_1rN}$ and $\bar{y}_{R_2rN}$ show significant improvement over traditional estimators but are surpassed by both our full and simplified hybrid estimators.
- The improvement is particularly significant for highly correlated data ($\rho_{yxN} = [0.90, 0.90]$), with RE values exceeding 6.0 for our proposed full estimators.
- Both full and simplified estimators maintain their superiority across different sample sizes and correlation structures, with the full versions providing a 2-3% efficiency gain over simplified versions.
- The hybrid formulation with optimal parameters effectively combines the benefits of ratio-product and exponential estimation approaches.

Table 4: Performance of t^{simple}_{MAK2} with varying β_2 values ($n_N = [15, 15]$)

β_2 Value	α_4^{opt}	Bias	MSE	RE
0	[0.861, 0.862]	[0.015, 0.015]	[0.082, 0.081]	[3.712, 3.734]
1	[0.261, 0.262]	[0.052, 0.052]	[0.104, 0.103]	[2.925, 2.945]
-1	[1.461, 1.462]	[0.038, 0.038]	[0.115, 0.114]	[2.618, 2.636]
Optimal	[0.395, 0.396]	[0.010, 0.010]	[0.076, 0.075]	[4.088, 4.110]

Table 5: Performance of t_{MAK1} (full version) with varying β_3 values ($n_N = [10, 10]$)

β_3 Value	α_5^{opt}	K_1^{opt}	Bias	MSE	RE
0	[0.930, 0.932]	[0.978, 0.978]	[0.011, 0.011]	[0.082, 0.081]	[4.012, 4.034]
1	[0.430, 0.432]	[0.962, 0.962]	[0.043, 0.043]	[0.099, 0.098]	[3.365, 3.385]
-1	[1.430, 1.432]	[0.987, 0.987]	[0.031, 0.031]	[0.115, 0.114]	[2.923, 2.941]
Optimal	[0.797, 0.799]	[0.975, 0.975]	[0.007, 0.007]	[0.077, 0.076]	[4.215, 4.237]

7 Comparative Study with Classical Statistics

To assess the advantages of our neutrosophic approach, we conducted a comprehensive comparison with classical statistical estimators. The classical counterparts of our proposed estimators were implemented using the same data but without considering the indeterminacy intervals.

The comparison reveals several important findings:

- The neutrosophic estimators consistently outperform their classical counterparts in terms of relative efficiency, with gains ranging from 5-8% depending on the estimator.
- The advantage is most pronounced for our proposed hybrid estimators, with RE improvements of approximately 7-8% over classical methods.
- The simplified estimators t^{simple}_{MAK1} and t^{simple}_{MAK2} show 6-7% improvement over classical methods, demonstrating that even without scaling constants, the hybrid formulation provides substantial benefits.
- The Searls-type estimators $\bar{y}_{R_1rN}$ and $\bar{y}_{R_2rN}$ show moderate improve-

Table 6: Performance of t_{MAK2} (full version) with varying β_4 values ($n_N = [15, 15]$)

β_4 Value	α_6^{opt}	K_2^{opt}	Bias	MSE	RE
0	[0.861, 0.862]	[0.976, 0.976]	[0.014, 0.014]	[0.083, 0.082]	[3.712, 3.734]
1	[0.261, 0.262]	[0.958, 0.958]	[0.051, 0.051]	[0.105, 0.104]	[2.925, 2.945]
-1	[1.461, 1.462]	[0.987, 0.987]	[0.037, 0.037]	[0.116, 0.115]	[2.618, 2.636]
Optimal	[0.395, 0.396]	[0.969, 0.969]	[0.009, 0.009]	[0.077, 0.076]	[4.088, 4.110]

Table 7: Comparison of Neutrosophic and Classical Estimators ($n_N = [10, 10]$)

Estimator	Neutrosophic RE	Classical RE
$\bar{y}_N$	[1.000, 1.000]	1.000
$\bar{y}_{rN}$	[2.454, 2.468]	2.461
$\bar{y}_{BTrN}$	[3.173, 3.183]	3.178
$\bar{y}_{BTpN}$	[3.865, 3.887]	3.876
$\bar{y}_{R_1rN}$	[4.032, 4.054]	3.815
$\bar{y}_{R_2rN}$	[3.891, 3.913]	3.762
t_{MAK1}^{simple}	[4.112, 4.134]	3.842
t_{MAK2}^{simple}	[4.095, 4.117]	3.828
t_{MAK1}	**[4.215, 4.237]**	3.921
t_{MAK2}	**[4.198, 4.220]**	3.905

ment over classical methods but are less efficient than both our full and simplified hybrid estimators.

- The classical estimators fail to capture the uncertainty present in the data, leading to less robust estimates.
- The neutrosophic framework provides more accurate interval estimates that better reflect the inherent variability in real-world data, particularly for the simplified estimators which offer a good balance between complexity and performance.

8 Conclusion

This research has introduced a novel family of hybrid neutrosophic estimators for population mean estimation that effectively combine ratio-product and exponential components with optimal parameter values. Through rigorous theoretical development and extensive empirical validation, we have demonstrated that our proposed estimators t_{MAK1} and t_{MAK2} outperform existing neutrosophic estimators across various scenarios. The key contributions of this work include the development of mathematically rigorous hybrid estimators that combine the strengths of ratio-product and exponential approaches, the derivation of optimal parameter values that minimize mean square error, comprehensive empirical validation using both real-world data and Monte Carlo simulations, and the demonstration of superior performance compared to both neutrosophic and classical statistical methods.

Our comparative analysis with classical statistics revealed significant advantages of the neutrosophic approach, particularly in handling indeterminate data and providing more robust interval estimates. The proposed estimators showed consistent improvements in relative efficiency, with gains of 7-8% over classical methods in our real data application. The simplified versions of our estimators also demonstrated substantial improvements (6-7%) while maintaining computational simplicity, making them attractive options for practical applications where computational resources may be limited.

Future research directions could explore the extension of the proposed framework to multivariate estimation problems, development of similar hybrid estimators for other population parameters, application to complex sampling designs beyond simple random sampling, and integration with machine learning techniques for big data applications. Additionally, further investigation into the performance of these estimators under different types of indeterminacy structures would be valuable.

The practical implications of this research are substantial, particularly in fields like medical research, economics, and social sciences where indeterminate data is common. Our estimators provide survey practitioners with more accurate tools for population mean estimation while properly accounting for measurement uncertainty, offering significant improvements over traditional approaches in both theoretical robustness and practical applicability.

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Received: April 13, 2025. Accepted: Sep 11, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

Note of HyperNeutrosophic Crisp Set and SuperHyperNeutrosophic Crisp Set

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Abstract

The Neutrosophic Set provides a flexible mathematical framework for addressing uncertainty through three distinct membership functions: truth, indeterminacy, and falsity. Extensions such as the Hyperneutrosophic Set and the SuperHyperneutrosophic Set have recently been introduced to address increasingly complex and multidimensional problems. Detailed formal definitions of these concepts can be found in [1]. A Neutrosophic Crisp Set partitions a universe into three subsets, explicitly representing truth, indeterminacy, and falsity memberships. In this paper, we explore the properties of the HyperNeutrosophic Crisp Set and the SuperHyperNeutrosophic Crisp Set. These structures merge the ideas of Neutrosophic Crisp Sets with those of Hyperneutrosophic and SuperHyperneutrosophic Sets. We anticipate that this integrated framework will contribute to the advancement of research in areas such as Uncertain Sets and Uncertain Topology.

Keywords: Set Theory, SuperhyperNeutrosophic set, Neutrosophic Set, HyperNeutrosophic set, Neutrosophic Crisp Set

1 Introduction

Handling uncertainty is a key challenge in mathematics, computer science, and decision-making. Over the years, researchers have developed many set-based models to represent and process uncertain information. The concept of fuzzy sets, first introduced by Zadeh [2] and later expanded in practical applications by Zimmermann [3], laid the foundation for this area. These sets allow each element to have a degree of membership between 0 and 1, enabling a gradual representation of truth rather than a strict binary approach.

Building on fuzzy sets, Atanassov introduced intuitionistic fuzzy sets, which include both membership and non-membership degrees [4]. Further extensions such as vague sets [5] and hyperfuzzy sets [6] were developed to address more complex situations. Other generalizations include picture fuzzy sets, hesitant fuzzy sets, and spherical fuzzy sets, each designed to tackle specific types of uncertainty in real-world problems.

A major step forward was taken with the introduction of neutrosophic sets by Smarandache [7], which explicitly incorporate an indeterminacy component alongside truth and falsity. This richer structure allows a more nuanced description of uncertain phenomena. Recently, Fujita proposed the HyperNeutrosophic Set and its superhyper generalizations to capture multi-level and high-dimensional uncertainty [1, 8, 9]. These frameworks have been applied to various fields, including data analysis, decision science, and algebraic modeling.

Within this context, neutrosophic crisp sets provide a way to represent truth, indeterminacy, and falsity explicitly as disjoint or overlapping subsets of a universe. Extending this idea, HyperNeutrosophic Crisp Sets and SuperHyperNeutrosophic Crisp Sets combine the crisp nature of these subsets with the multi-valued structure of hyperneutrosophic models. This integration allows for the representation of multiple possible crisp assignments at different hierarchical levels, making the model suitable for complex systems where uncertainty exists at multiple layers.

The aim of this paper is to study the properties and algebraic structures of HyperNeutrosophic Crisp Sets and n-SuperHyperNeutrosophic Crisp Sets. We present formal definitions, illustrative examples, and theoretical results that establish their foundational behavior. By bridging the gap between neutrosophic crisp models and higher-order hyperneutrosophic structures, our work contributes to the ongoing development of uncertainty theory and its mathematical foundations.

2 Preliminaries

This section gathers the key notions and notation that form the foundation of our study. Throughout, we restrict our attention to finite sets.

2.1 Neutrosophic, HyperNeutrosophic, and n-SuperHyperNeutrosophic Sets

A variety of set-theoretic paradigms have been introduced to capture uncertainty, imprecision, and vagueness in decision-making [8, 9]. The classical theory of Fuzzy Sets, initiated by Zadeh, remains a cornerstone [2, 3]. Building on this, Atanassov's Intuitionistic Fuzzy Sets incorporate both membership and non-membership degrees [4, 10], while Vague Sets provide an alternative dual-measure approach [5, 11, 12]. Further generalizations include Hyperfuzzy Sets, which extend the fuzzification process to higher-order structures [6, 13–16]. Other notable extensions of fuzzy sets include Picture Fuzzy Sets [17, 18], Hesitant Fuzzy Sets [19–21], and Spherical Fuzzy Sets [22–24], which have been applied extensively in decision science and related fields.

Neutrosophic Sets, proposed by Smarandache, introduce an explicit indeterminacy component alongside truth and falsity, yielding a richer framework for nuanced reasoning [7, 25, 26]. To model even more elaborate forms of uncertainty, researchers have defined HyperNeutrosophic Sets and their n-fold superhyper variants, which are well-suited to high-dimensional or deeply nested problem domains [1, 27–30]. Below we restate these concepts formally and give simple illustrative examples.

Definition 2.1 (Base Set)**.** [31] A *base set* S is the foundational set from which complex structures such as powersets and hyperstructures are derived. It is formally defined as:

$$S = \{x \mid x \text{ is an element within a specified domain}\}.$$

All elements in constructs like $\mathcal{P}(S)$ or $\mathcal{P}_n(S)$ originate from the elements of S.

Definition 2.2 (Powerset)**.** (cf. [32]) For any set S, its *powerset* $\mathcal{P}(S)$ is the collection of all subsets of S, including $\emptyset$ and S itself:

$$\mathcal{P}(S) = \{ A \mid A \subseteq S\}.$$

Definition 2.3 (n-th Powerset)**.** [33–35] Let H be a set. Define inductively

$$\mathcal{P}^0(H) = H, \quad \mathcal{P}^{k+1}(H) = \mathcal{P}\big(\mathcal{P}^k(H)\big), \quad k \geq 0.$$

Then $\mathcal{P}^1(H) = \mathcal{P}(H)$, $\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$, and so on. Removing the empty set at each stage yields the *nonempty* iterated powersets. Writing $\mathcal{P}^*(X) = \mathcal{P}(X) \setminus \{\emptyset\}$, we set

$$\mathcal{P}^{*1}(H) = \mathcal{P}^*(H), \quad \mathcal{P}^{*(k+1)}(H) = \mathcal{P}^*\big(\mathcal{P}^{*k}(H)\big).$$

Example 2.4 (Corporate Document Hierarchy as a Third-Level Powerset)**.** Consider a small company whose digital archive starts with the base set

$$H = \{\text{Invoice, Contract, Presentation}\}.$$

First powerset $\mathcal{P}^1(H)$ (folders). Typical folders—each a subset of H—might be

$$F_1 = \{\text{Invoice, Contract}\}, \quad F_2 = \{\text{Presentation}\}, \quad F_3 = \{\text{Invoice, Presentation}\}.$$

Thus $\{F_1, F_2, F_3\} \subseteq \mathcal{P}^1(H)$ models the folder structure.

Second powerset $\mathcal{P}^2(H)$ (projects). A *project* groups several folders:

$$P_1 = \{F_1, F_2\}, \qquad P_2 = \{F_2, F_3\}.$$

Both P_1 and P_2 are elements of $\mathcal{P}^2(H) = \mathcal{P}(\mathcal{P}(H))$.

Third powerset $\mathcal{P}^3(H)$ (portfolios). At the next level, management assembles related projects into *portfolios*:

$$R = \{P_1, P_2\} \in \mathcal{P}^3(H).$$

The set R is therefore a concrete element of the third iterated powerset $\mathcal{P}^3(H)$. In everyday terms,

$$\text{files} \longrightarrow \text{folders} \longrightarrow \text{projects} \longrightarrow \text{portfolios},$$

giving a natural three-tier hierarchy captured precisely by the n-th powerset construction with $n = 3$.

Definition 2.5 (Neutrosophic Set)**.** [7, 36] Let X be a non-empty set. A *Neutrosophic Set (NS) A* on X is characterized by three membership functions:

$$T_A : X \to [0, 1], \quad I_A : X \to [0, 1], \quad F_A : X \to [0, 1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 2.6 (Medical Diagnosis with Uncertain Symptoms)**.** (cf. [37, 38]) Let

$$X = \{\text{Fever}, \text{ Cough}, \text{ Fatigue}\}$$

be the set of key symptoms considered in diagnosing a particular illness (e.g. influenza). A physician assigns to each symptom three independent degrees that summarise current clinical evidence:

Symptom	$T_A(x)$	$I_A(x)$	$F_A(x)$
Fever	0.80	0.10	0.05
Cough	0.60	0.25	0.10
Fatigue	0.40	0.40	0.15

- **Truth degree $T_A(x)$.** The likelihood that the symptom genuinely supports the diagnosis. Fever, highly characteristic of influenza, scores 0.80.
- **Indeterminacy degree $I_A(x)$.** The physician's uncertainty due to limited data, conflicting test results, or atypical presentation. Fatigue carries the greatest indeterminacy (0.40) because it can arise from many unrelated conditions.
- **Falsity degree $F_A(x)$.** Evidence that the symptom contradicts—or fails to support—the diagnosis. Fever has a low falsity value (0.05), whereas Fatigue is less specific (0.15).

Each row satisfies $T_A(x) + I_A(x) + F_A(x) \le 1$, conforming to the neutrosophic-set requirement $T + I + F \le 3$. The triplet

$$\big(T_A(x), \; I_A(x), \; F_A(x)\big)$$

offers a richer, three-way description of diagnostic confidence than a single probability value or a binary classification, enabling physicians to capture both indecision and contrary evidence explicitly.

Definition 2.7 (HyperNeutrosophic Set)**.** (cf. [1, 27, 39–42]) Let X be a non-empty set. A *HyperNeutrosophic Set (HNS) $\tilde{A}$* on X is a mapping:

$$\tilde{\mu} : X \to \mathcal{P}([0, 1]^3),$$

where $\mathcal{P}([0, 1]^3)$ is the family of all non-empty subsets of the unit cube $[0, 1]^3$. For each $x \in X$, $\tilde{\mu}(x) \subseteq [0, 1]^3$ is a set of neutrosophic membership triplets (T, I, F) that satisfy:

$$0 \le T + I + F \le 3.$$

Example 2.8 (HyperNeutrosophic Assessment of Investment Options)**.** Suppose a portfolio manager is evaluating two start-ups,

$$X = \{\text{GreenTech}, \ \text{BioHealth}\},$$

using opinions from three analysts. Each analyst provides a neutrosophic triplet (T, I, F), where

- T is the analyst's confidence that the company will yield the target return,
- I is the analyst's perceived indeterminacy (lack of clear evidence),
- F is the analyst's confidence that the company will *not* achieve the target.

Because the analysts rely on different data sources, their triplets may vary widely. The manager wishes to keep *all* expert opinions rather than aggregate them into a single point estimate, leading naturally to a HyperNeutrosophic Set.

$$\tilde{\mu} : X \longrightarrow \mathcal{P}([0,1]^3), \qquad \tilde{\mu}(x) = \{ \ (T, I, F) \ \big| \ \text{analyst triplet for } x \ \}.$$

$$\tilde{\mu}(\text{GreenTech}) = \Big\{(0.80, \ 0.10, \ 0.05), (0.60, \ 0.25, \ 0.10), (0.70, \ 0.20, \ 0.05)\Big\},$$
$$\tilde{\mu}(\text{BioHealth}) = \Big\{(0.45, \ 0.35, \ 0.15), (0.55, \ 0.30, \ 0.10)\Big\}.$$

Every triplet obeys $0 \le T + I + F \le 1 \le 3$, and each image is a non-empty subset of the unit cube $[0, 1]^3$. Thus $\tilde{\mu}$ defines a legitimate HyperNeutrosophic Set on X. Retaining the full *set* of analyst triplets preserves the spectrum of expert sentiment, which can later be filtered, weighted, or visualised without losing information at the time of collection.

Definition 2.9 (n-SuperHyperNeutrosophic Set)**.** (cf. [1,27]) Let X be a non-empty set. An *n-SuperHyperNeutrosophic Set (n-SHNS)* is a recursive generalization of Neutrosophic Sets and HyperNeutrosophic Sets. It is defined as a mapping:

$$\tilde{A}_n : \mathcal{P}_n(X) \to \mathcal{P}_n([0, 1]^3),$$

where:

- $\mathcal{P}_1(X) = \mathcal{P}(X)$, the power set of X, and for $k \ge 2$,
  $$\mathcal{P}_k(X) = \mathcal{P}(\mathcal{P}_{k-1}(X)),$$
  representing the k-th nested family of non-empty subsets of X.
- $\mathcal{P}_n([0, 1]^3)$ is defined similarly for the unit cube $[0, 1]^3$.

For each $A \in \mathcal{P}_n(X)$ and $(T, I, F) \in \tilde{A}_n(A)$, the following condition is satisfied:

$$0 \le T + I + F \le 3,$$

where T, I, F represent the degrees of truth, indeterminacy, and falsity for the n-th level subsets of X.

Example 2.10 (2-SuperHyperNeutrosophic Set)**.** Starting from the above $\tilde{\mu}$, the power-set $\mathcal{P}(X) = \{\{x\}, \{y\}, \{x, y\}\}$ serves as the domain of a 2-SuperHyperNeutrosophic Set $\tilde{\mu}_2 : \mathcal{P}(X) \to \mathcal{P}\big(\mathcal{P}([0, 1]^3)\big) \setminus \{\varnothing\}$ defined by

$$\tilde{\mu}_2(U) = \big\{ \ \tilde{\mu}(p) \ \big| \ p \in U\big\}.$$

Thus, for instance,

$$\tilde{\mu}_2(\{x\}) = \big\{ \ \{(0.8, 0.1, 0.1), (1, 0, 0)\}\big\}, \quad \tilde{\mu}_2(\{x, y\}) = \big\{ \ \{(0.8, 0.1, 0.1), (1, 0, 0)\}, \ \{(0.5, 0.3, 0.1)\}\big\}.$$

Each $\tilde{\mu}_2(U)$ is a nonempty family of subsets of $[0, 1]^3$, exhibiting the second-level (superhyper) neutrosophic structure.

Example 2.11 (A 3-SuperHyperNeutrosophic Set). Let $X = \{x, y\}$. We first define a HyperNeutrosophic Set $\tilde{\mu} : X \to \mathcal{P}([0,1]^3) \setminus \{\emptyset\}$ by

$$\tilde{\mu}(x) = \{(0.8, 0.1, 0.1),\ (1, 0, 0)\}, \qquad \tilde{\mu}(y) = \{(0.5, 0.3, 0.1)\}.$$

Next, the collection of nonempty subsets of X is $\mathcal{P}_1(X) = \{\{x\}, \{y\}, \{x, y\}\}$. Define the 2-SHNS $\tilde{\mu}_2 : \mathcal{P}_1(X) \to \mathcal{P}(\mathcal{P}([0,1]^3))$ by

$$\tilde{\mu}_2(U) \ = \ \{ \ \tilde{\mu}(p) \mid p \in U\}.$$

Concretely,

$$\tilde{\mu}_2(\{x\}) = \big\{\{(0.8, 0.1, 0.1), (1, 0, 0)\}\big\}, \quad \tilde{\mu}_2(\{y\}) = \big\{\{(0.5, 0.3, 0.1)\}\big\},$$

$$\tilde{\mu}_2(\{x, y\}) = \big\{\{(0.8, 0.1, 0.1), (1, 0, 0)\},\ \{(0.5, 0.3, 0.1)\}\big\}.$$

Finally, the third-level domain is $\mathcal{P}_2(X) = \mathcal{P}(\mathcal{P}_1(X))$, whose nonempty elements include, for instance,

$$W_1 = \{\{x\}, \{y\}\}, \quad W_2 = \{\{x, y\}\}.$$

Define the 3-SHNS $\tilde{\mu}_3 : \mathcal{P}_2(X) \to \mathcal{P}(\mathcal{P}(\mathcal{P}([0,1]^3)))$ by

$$\tilde{\mu}_3(W) = \{ \ \tilde{\mu}_2(U) \mid U \in W\}.$$

Then in particular:

$$\tilde{\mu}_3(W_1) = \big\{\tilde{\mu}_2(\{x\}),\ \tilde{\mu}_2(\{y\})\big\} = \Big\{\{\{(0.8, 0.1, 0.1), (1, 0, 0)\}\},\ \{\{(0.5, 0.3, 0.1)\}\}\Big\},$$

$$\tilde{\mu}_3(W_2) = \big\{\tilde{\mu}_2(\{x, y\})\big\} = \Big\{\{\{(0.8, 0.1, 0.1), (1, 0, 0)\}, \{(0.5, 0.3, 0.1)\}\}\Big\}.$$

Each $\tilde{\mu}_3(W_i)$ is nonempty and consists of families of subsets of $\mathcal{P}([0,1]^3)$, illustrating the full third-level (superhyper) neutrosophic structure.

2.2 Neutrosophic Crisp Set

A Neutrosophic Crisp Set partitions a universe X into three subsets representing truth, indeterminacy, and falsity memberships explicitly. The definition of Neutrosophic Crisp Set is below [43–47].

Definition 2.12 (Neutrosophic Crisp Set). [43] Let X be a nonempty universe. A *neutrosophic crisp set* (NCS) on X is an ordered triple

$$A \ = \ \langle A_1,\ A_2,\ A_3\rangle,$$

where $A_1, A_2, A_3 \subseteq X$. We call:

- A_1 the *truth-membership* subset,
- A_2 the *indeterminacy-membership* subset,
- A_3 the *falsity-membership* subset.

Definition 2.13 (Types of Neutrosophic Crisp Set). With $A = \langle A_1, A_2, A_3\rangle$ as above, one distinguishes:

1. *Type 1* if

$$A_1 \cap A_2 = \emptyset, \quad A_1 \cap A_3 = \emptyset, \quad A_2 \cap A_3 = \emptyset.$$

2. *Type 2* if in addition to the Type 1 conditions,

$$A_1 \ \cup \ A_2 \ \cup \ A_3 \ = \ X.$$

3. *Type 3* if

$$A_1 \cap A_2 \cap A_3 = \emptyset, \quad \text{and} \quad A_1 \ \cup \ A_2 \ \cup \ A_3 = X.$$

Example 2.14 (Type-1 Neutrosophic Crisp Set: Library Loss Audit)**.** A university library audits a small collection $X = \{B1, B2, B3, B4, B5\}$ to determine whether each book is truly lost, of uncertain status, or accounted for.

$$A = \langle A_1, A_2, A_3 \rangle, \qquad A_1 = \{B1, B3\},\ A_2 = \{B4\},\ A_3 = \{B2\}.$$

- A_1: books confirmed missing (**truth**).
- A_2: books with inconclusive records (**indeterminacy**).
- A_3: books verified on shelf (**falsity**).

We have $A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset$, so the three subsets are pairwise disjoint, satisfying the Type-1 condition. The book B5 has not yet been audited, so $A_1 \cup A_2 \cup A_3 \subset X$; hence A is a *Type-1* neutrosophic crisp set.

Example 2.15 (Type-2 Neutrosophic Crisp Set: Quality-Control Report)**.** A factory inspects four products,

$$X = \{P1,\ P2,\ P3,\ P4\},$$

and classifies each item as *accepted*, *pending*, or *rejected*. Let

$$A = \langle A_1, A_2, A_3 \rangle \quad \text{with} \quad A_1 = \{P1, P3\},\ A_2 = \{P2\},\ A_3 = \{P4\}.$$

- A_1 (truth membership): products that pass inspection outright.
- A_2 (indeterminacy membership): products awaiting additional tests.
- A_3 (falsity membership): products that fail inspection.

We verify the Type-2 conditions:

$$A_1 \cap A_2 = A_1 \cap A_3 = A_2 \cap A_3 = \emptyset \quad \text{and} \quad A_1 \cup A_2 \cup A_3 = X.$$

Hence A is a *Type-2 neutrosophic crisp set.* Every product belongs to exactly one of the three mutually disjoint categories, providing a clear and exhaustive quality-control summary.

Example 2.16 (Type-3 Neutrosophic Crisp Set: Preliminary Medical Screening)**.** A clinic screens four patients for a particular infection, $X = \{P1, P2, P3, P4\}$. Test results lead to overlapping assessments:

$$B = \langle B_1, B_2, B_3 \rangle, \qquad B_1 = \{P1, P2\},\ B_2 = \{P2, P3\},\ B_3 = \{P3, P4\}.$$

- B_1: patients with a positive test (**truth**).
- B_2: inconclusive or conflicting evidence (**indeterminacy**).
- B_3: patients with a negative test (**falsity**).

Here

$$B_1 \cap B_2 \cap B_3 = \emptyset, \qquad B_1 \cup B_2 \cup B_3 = X,$$

so B fulfils the Type-3 criteria. Note that pairwise overlaps are allowed: $P2 \in B_1 \cap B_2$ (positive yet uncertain) and $P3 \in B_2 \cap B_3$ (negative but still doubtful). Such overlaps capture real-world ambiguity while ensuring every patient appears in at least one subset.

3 Main Results of This Paper

This section outlines the main results presented in this paper.

3.1 HyperNeutrosophic Crisp Set

HyperNeutrosophic Crisp Sets assign each element of X a nonempty set of crisp truth–indeterminacy–falsity triples, capturing multiple binary certainty patterns. The definition of HyperNeutrosophic Crisp Set is below.

Definition 3.1 (HyperNeutrosophic Crisp Set)**.** Let X be a nonempty set. A *HyperNeutrosophic Crisp Set* (HNCS) on X is a function

$$H : X \longrightarrow \mathcal{P}\big(\{0,1\}^3\big) \setminus \{\varnothing\},$$

where $\{0,1\}^3$ is the set of all triples (t,i,f) with $t,i,f \in \{0,1\}$. For each $x \in X$, $H(x) \subseteq \{0,1\}^3$ is nonempty, and each $(t,i,f) \in H(x)$ is interpreted as a *crisp* assignment of truth, indeterminacy, and falsity to x.

Example 3.2 (HyperNeutrosophic Crisp Set on Sensor Reliability)**.** Let

$$X = \{\, S_1,\ S_2,\ S_3\}$$

be a set of three sensors monitoring an industrial process. We define a HyperNeutrosophic Crisp Set

$$H : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\varnothing\}$$

by assigning to each sensor S_i a nonempty set of crisp truth/indeterminacy/falsity triples:

$$\begin{aligned} H(S_1) &= \big\{(1,0,0),\ (1,1,0)\big\},\\ H(S_2) &= \big\{(0,1,0),\ (0,1,1)\big\},\\ H(S_3) &= \big\{(1,0,0),\ (0,0,1)\big\}. \end{aligned}$$

- $H(S_1)$: Sensor 1 is either fully functional $(1,0,0)$ or showing occasional instability $(1,1,0)$.
- $H(S_2)$: Sensor 2's status is inconclusive $(0,1,0)$ or indicates potential failure $(0,1,1)$.
- $H(S_3)$: Sensor 3 is either fully functional $(1,0,0)$ or conclusively failed $(0,0,1)$.

Since each $H(S_i)$ is a nonempty subset of $\{0,1\}^3$, this constitutes a valid HyperNeutrosophic Crisp Set on X.

Definition 3.3 (Set-operations on HNCS)**.** If H, K are HNCS on X, define for each $x \in X$:

$$(H \cap K)(x) \;=\; H(x) \;\cap\; K(x), \qquad (H \cup K)(x) \;=\; H(x) \;\cup\; K(x).$$

Then $H \cap K$ and $H \cup K$ are again HNCS.

Example 3.4 (Set-operations on HNCS)**.** Let

$$X = \{x, y\},$$

and consider two HyperNeutrosophic Crisp Sets $H, K : X \to \mathcal{P}(\{0,1\}^3) \setminus \{\varnothing\}$ given by

$$\begin{aligned} H(x) &= \{(1,0,0),\ (1,1,0)\}, & H(y) &= \{(0,1,0),\ (0,1,1)\},\\ K(x) &= \{(1,0,0),\ (0,0,1)\}, & K(y) &= \{(1,0,0),\ (0,1,0)\}. \end{aligned}$$

We compute their intersection and union pointwise:

$$\begin{aligned} (H \cap K)(x) &= H(x) \;\cap\; K(x) = \{(1,0,0)\},\\ (H \cap K)(y) &= H(y) \;\cap\; K(y) = \{(0,1,0)\}, \end{aligned}$$

$$\begin{aligned} (H \cup K)(x) &= H(x) \;\cup\; K(x) = \{(1,0,0),\ (1,1,0),\ (0,0,1)\},\\ (H \cup K)(y) &= H(y) \;\cup\; K(y) = \{(0,1,0),\ (0,1,1),\ (1,0,0)\}. \end{aligned}$$

- At x, the intersection retains only the fully-true assignment $(1, 0, 0)$, while the union gathers all three distinct crisp triples.
- At y, the intersection picks the indeterminate-only assignment $(0, 1, 0)$, and the union collects both indeterminate patterns plus the fully-true assignment.

Thus $H \cap K$ and $H \cup K$ are valid HNCS on X.

Definition 3.5 (Embedding of NCS into HNCS)**.** Given a neutrosophic crisp set $A = \langle A_1, A_2, A_3\rangle$, define

$$H_A(x) \;=\; \begin{cases} \{(1,0,0)\}, & x \in A_1,\\ \{(0,1,0)\}, & x \in A_2,\\ \{(0,0,1)\}, & x \in A_3. \end{cases}$$

Then H_A is a HNCS on X.

Example 3.6 (Embedding of a Neutrosophic Crisp Set into a HyperNeutrosophic Crisp Set)**.** Let

$$X = \{a,\ b,\ c,\ d\}, \quad A = \langle A_1,\ A_2,\ A_3\rangle \quad \text{with} \quad A_1 = \{a, c\},\ A_2 = \{b\},\ A_3 = \{d\}.$$

Then A is a neutrosophic crisp set on X. By the embedding definition, we obtain the HNCS

$$H_A : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\varnothing\}$$

given pointwise by

$$H_A(x) = \begin{cases} \{(1,0,0)\}, & x \in A_1,\\ \{(0,1,0)\}, & x \in A_2,\\ \{(0,0,1)\}, & x \in A_3. \end{cases}$$

Concretely,

$$H_A(a) = \{(1,0,0)\}, \qquad H_A(b) = \{(0,1,0)\},$$
$$H_A(c) = \{(1,0,0)\}, \qquad H_A(d) = \{(0,0,1)\}.$$

- For $a, c \in A_1$, $H_A(x) = \{(1, 0, 0)\}$ marks *truth*.
- For $b \in A_2$, $H_A(b) = \{(0, 1, 0)\}$ marks *indeterminacy*.
- For $d \in A_3$, $H_A(d) = \{(0, 0, 1)\}$ marks *falsity*.

Each $H_A(x)$ is a nonempty singleton in $\{0, 1\}^3$, so H_A is indeed a valid HyperNeutrosophic Crisp Set on X.

Theorem 3.7. *The map* $\Phi : \{\text{NCS on } X\} \to \{\text{HNCS on } X\}$ *given by* $\Phi(A) = H_A$ *is*

(a) Injective*, i.e. different NCS produce different HNCS.*

(b) A homomorphism *with respect to union and intersection:*

$$H_{A\cap B} \;=\; H_A \cap H_B, \qquad H_{A\cup B} \;=\; H_A \cup H_B.$$

(c) Thus every NCS is realized as a "singleton-valued" HNCS, showing that HNCS properly generalize NCS.

Proof. (a) Suppose $A \neq B$ are two NCS. Then there exists x lying in exactly one of the corresponding components A_j vs. B_j. Hence $H_A(x) \neq H_B(x)$, so $H_A \neq H_B$.

(b) For any $x \in X$,

$$H_{A\cap B}(x) = \begin{cases} \{(1,0,0)\}, & x \in A_1 \cap B_1,\\ \{(0,1,0)\}, & x \in A_2 \cap B_2,\\ \{(0,0,1)\}, & x \in A_3 \cap B_3, \end{cases}$$

which is exactly $\{(1, 0, 0)\} \cap \{(1, 0, 0)\}$ or $\{(0, 1, 0)\} \cap \{(0, 1, 0)\}$, etc., i.e. $H_A(x) \cap H_B(x)$. Hence $H_{A\cap B} = H_A \cap H_B$. The union case is analogous.

(c) Since each $H_A(x)$ is a singleton subset of $\{0,1\}^3$, the class of all HNCS strictly contains all such singleton-valued ones. Thus HNCS generalize NCS.

□

Example 3.8 (A genuinely hyper-valued HNCS)**.** Let $X = \{x\}$. Define

$$H(x) \;=\; \{(1,0,0),\ (1,1,0),\ (0,0,1)\}.$$

Here $H(x)$ has three distinct crisp triples, so H is a HyperNeutrosophic Crisp Set that is *not* induced by any single NCS. This illustrates the strictly larger expressive power of HNCS.

Definition 3.9 (Pointwise Order)**.** For two HNCS H, K on X, define

$$H \;\preceq\; K \quad\Longleftrightarrow\quad \forall\, x \in X,\ H(x) \subseteq K(x).$$

Definition 3.10 (Pointwise Union and Intersection)**.** If $\{H_i\}_{i\in I}$ is any family of HNCS on X, set for each x:

$$\Big(\bigsqcup_{i\in I} H_i\Big)(x) = \bigcup_{i\in I} H_i(x), \qquad \Big(\bigsqcap_{i\in I} H_i\Big)(x) = \bigcap_{i\in I} H_i(x).$$

Since each $H_i(x) \neq \emptyset$, $\bigsqcap_i H_i(x)$ is nonempty whenever the family has the finite-intersection property; in particular for finite I, $\bigsqcap$ is again an HNCS.

Theorem 3.11 (Complete Lattice of HNCS)**.** *The collection* HNCS(X) *of all HNCS on X, ordered by* $\preceq$*, is a complete lattice. Its join and meet are given by pointwise union and intersection:*

$$\bigsqcup_{i\in I} H_i \text{ is the least upper bound,} \qquad \bigsqcap_{i\in I} H_i \text{ is the greatest lower bound.}$$

Proof. Let $\{H_i\}_{i\in I} \subseteq \mathrm{HNCS}(X)$. Define J, H by

$$J(x) = \bigcup_{i\in I} H_i(x), \qquad M(x) = \bigcap_{i\in I} H_i(x).$$

(i) J is an HNCS: each $H_i(x)$ is nonempty, so $J(x)$ is nonempty; hence $J \in \mathrm{HNCS}(X)$. *(ii) M is an HNCS when* $\{H_i(x)\}$ *have the finite-intersection property for each x.* In particular for finite I, $M(x) \neq \emptyset$. Thus finite meets exist in HNCS(X).

(iii) J is the least upper bound: for each i, $\forall x,\ H_i(x) \subseteq J(x)$, so $H_i \preceq J$. If K is any common upper bound ($H_i \preceq K$ for all i), then $\forall x$, $J(x) = \bigcup_i H_i(x) \subseteq K(x)$, so $J \preceq K$.

(iv) M is the greatest lower bound: for each i, $\forall x,\ M(x) \subseteq H_i(x)$, so $M \preceq H_i$. If L is any common lower bound ($L \preceq H_i$ for all i), then $\forall x$, $L(x) \subseteq H_i(x)$ for all i, hence $L(x) \subseteq \bigcap_i H_i(x) = M(x)$, giving $L \preceq M$.

Thus HNCS(X) is a complete lattice under $\preceq$. □

Theorem 3.12 (Distributivity)**.** *In* HNCS(X)*, finite meets distribute over finite joins and vice versa: for any HNCS H, K, L,*

$$H \sqcap (K \sqcup L) = (H \sqcap K) \sqcup (H \sqcap L), \quad H \sqcup (K \sqcap L) = (H \sqcup K) \sqcap (H \sqcup L).$$

Proof. For any $x \in X$,

$$\big[H \sqcap (K \sqcup L)\big](x) = H(x) \cap \big(K(x) \cup L(x)\big) = \big(H(x) \cap K(x)\big) \cup \big(H(x) \cap L(x)\big) = \big[(H \sqcap K) \sqcup (H \sqcap L)\big](x).$$

The other identity is analogous, using the distributive law in the Boolean algebra $\mathcal{P}(\{0,1\}^3)$. □

3.2 n-SuperHyperNeutrosophic Crisp Set

A n-SuperHyperNeutrosophic Crisp Sets map each nonempty (n–1)-subset of X to a nonempty family of n-level nested crisp triple sets. The definition of n-SuperHyperNeutrosophic Crisp Set is below.

Definition 3.13 (n-SuperHyperNeutrosophic Crisp Set)**.** Let X be nonempty and fix $n \geq 1$. An n-SuperHyperNeutrosophic Crisp Set on X is a function

$$H^{(n)} : P_{n-1}(X) \longrightarrow P_n\big(\{0,1\}^3\big) \setminus \{\emptyset\},$$

where $\{0,1\}^3$ is the set of all triples (t,i,f) with $t,i,f \in \{0,1\}$. For each $U \in P_{n-1}(X)$, $H^{(n)}(U)$ is a nonempty family of n-fold nested crisp triples, each representing truth/indeterminacy/falsity assignments at the $(n-1)$-th level.

Definition 3.14 (HyperNeutrosophic Crisp Set as 1-SHNCS)**.** When $n = 1$, $P_0(X) = X$ and $P_1(\{0,1\}^3) = \mathcal{P}(\{0,1\}^3)$. Thus a 1-SuperHyperNeutrosophic Crisp Set $H^{(1)} : X \to \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}$ is exactly a HyperNeutrosophic Crisp Set.

Definition 3.15 (Induced (n)-SHNCS from $(n-1)$-SHNCS)**.** Given an $(n-1)$-SHNCS

$$H^{(n-1)} : P_{n-2}(X) \longrightarrow P_{n-1}(\{0,1\}^3),$$

we define $H^{(n)} : P_{n-1}(X) \to P_n(\{0,1\}^3)$ by

$$H^{(n)}(U) = \big\{ H^{(n-1)}(V) \;\big|\; V \in U \subseteq P_{n-2}(X) \big\}.$$

One checks easily that $H^{(n)}(U) \in P_n(\{0,1\}^3) \setminus \{\emptyset\}$.

Theorem 3.16. *For each $n \geq 1$:*

(a) The construction $H^{(n-1)} \mapsto H^{(n)}$ embeds the class of $(n-1)$-SuperHyperNeutrosophic Crisp Sets into the class of n-SuperHyperNeutrosophic Crisp Sets.

(b) In particular, every HyperNeutrosophic Crisp Set (1-SHNCS) arises as the restriction of some n-SuperHyperNeutrosophic Crisp Set to singletons in $P_0(X) = X$.

Hence n-SHNCS properly generalize HyperNeutrosophic Crisp Sets.

Proof. **(a)** Let $H^{(n-1)} : P_{n-2}(X) \to P_{n-1}(\{0,1\}^3)$ be given. Define $H^{(n)}$ as above. Since $U \subseteq P_{n-2}(X)$ and $H^{(n-1)}$ never outputs the empty set, the collection $\{H^{(n-1)}(V) \mid V \in U\}$ is nonempty, so $H^{(n)}(U) \in P_n(\{0,1\}^3) \setminus \{\emptyset\}$. Thus $H^{(n)}$ is a well-defined n-SHNCS.

Injectivity follows because if two $(n-1)$-SHNCS differ at some $V \in P_{n-2}(X)$, then their induced $H^{(n)}$ differ at any $U \ni V$.

(b) When $n \geq 2$, start with a HyperNeutrosophic Crisp Set $H^{(1)} : X \to \mathcal{P}(\{0,1\}^3)$. Apply the above construction inductively to obtain $H^{(n)}$. Then for each $x \in X = P_0(X)$,

$$H^{(n)}(\{x\}) = \big\{H^{(n-1)}(V) \mid V \in \{\{x\}\}\big\} = \{H^{(n-1)}(\{x\})\},$$

and by unwinding the induction one recovers the original $H^{(1)}(x)$. Hence $H^{(1)}$ embeds into $H^{(n)}$, showing that every HNCS is realized as the restriction of an n-SHNCS. □

Example 3.17 (A 2-SuperHyperNeutrosophic Crisp Set)**.** Let $X = \{a, b\}$. First define a HyperNeutrosophic Crisp Set (i.e. 1-SHNCS)

$$H^{(1)} : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}, \quad H^{(1)}(a) = \{(1,0,0), (1,1,0)\}, \quad H^{(1)}(b) = \{(0,1,0)\}.$$

The nonempty subsets of X form $P_1^*(X) = \{\{a\}, \{b\}, \{a,b\}\}$. The induced 2-SuperHyperNeutrosophic Crisp Set is

$$H^{(2)} : P_1^*(X) \longrightarrow P_2(\{0,1\}^3) \setminus \{\emptyset\}, \quad H^{(2)}(U) = \{H^{(1)}(x) \mid x \in U\}.$$

Concretely:

$$H^{(2)}(\{a\}) = \big\{\{(1,0,0), (1,1,0)\}\big\}, \quad H^{(2)}(\{b\}) = \big\{\{(0,1,0)\}\big\},$$
$$H^{(2)}(\{a,b\}) = \big\{\{(1,0,0), (1,1,0)\},\ \{(0,1,0)\}\big\}.$$

Thus $H^{(2)}$ assigns to each nonempty $U \subseteq X$ a nonempty family of crisp-triple-sets.

Example 3.18 (A 3-SuperHyperNeutrosophic Crisp Set)**.** Continue with X and $H^{(1)}, H^{(2)}$ as above. The nonempty elements of $P_2^*(X) = \mathcal{P}(P_1^*(X)) \setminus \{\emptyset\}$ include, for instance, $\{\{a\},\{b\}\}$ and $\{\{a,b\}\}$. Define

$$H^{(3)} : P_2^*(X) \longrightarrow P_3(\{0,1\}^3) \setminus \{\emptyset\}, \quad H^{(3)}(W) \;=\; \{\, H^{(2)}(U) \mid U \in W\}.$$

Two concrete cases:

$$H^{(3)}(\{\{a\},\{b\}\}) = \{\, H^{(2)}(\{a\}),\ H^{(2)}(\{b\})\} = \Big\{\{\{(1,0,0),(1,1,0)\}\},\ \{\{(0,1,0)\}\}\Big\},$$

$$H^{(3)}(\{\{a,b\}\}) = \{\, H^{(2)}(\{a,b\})\} = \Big\{\{\{(1,0,0),(1,1,0)\},\{(0,1,0)\}\}\Big\}.$$

Each $H^{(3)}(W)$ is a nonempty set of elements of $\{\text{nonempty subsets of } P_1^*(X)\}$, exhibiting the full 3-level nested HyperNeutrosophic structure.

Definition 3.19 (Pointwise Union and Intersection)**.** Given $\{H_j^{(n)}\}_{j\in J}$ n-SHNCS, define for each $U \in P_{n-1}(X)$:

$$\Big(\bigsqcup_{j\in J} H_j^{(n)}\Big)(U) = \bigcup_{j\in J} H_j^{(n)}(U), \quad \Big(\bigsqcap_{j\in J} H_j^{(n)}\Big)(U) = \bigcap_{j\in J} H_j^{(n)}(U).$$

When J is finite, both remain nonempty and hence are n-SHNCS.

Example 3.20 (Pointwise Union and Intersection of Two 2-SuperHyperNeutrosophic Crisp Sets)**.** Let

$$X = \{a,\ b\},$$

and start with two HyperNeutrosophic Crisp Sets (i.e. 1-SHNCS)

$$H_1^{(1)}(a) = \{(1,0,0),(1,1,0)\}, \qquad H_1^{(1)}(b) = \{(0,1,0)\},$$
$$H_2^{(1)}(a) = \{(1,0,0),(1,1,0)\}, \qquad H_2^{(1)}(b) = \{(1,0,0),(0,1,0)\}.$$

They induce two 2-SHNCS

$$H_j^{(2)} : P_1(X) \longrightarrow P_2(\{0,1\}^3) \setminus \{\emptyset\}, \quad H_j^{(2)}(U) = \{\, H_j^{(1)}(x) \mid x \in U\},\ j = 1,2.$$

Consider the argument $U = \{a,b\} \in P_1(X)$. Then

$$H_1^{(2)}(\{a,b\}) = \big\{\{(1,0,0),(1,1,0)\},\ \{(0,1,0)\}\big\},$$
$$H_2^{(2)}(\{a,b\}) = \big\{\{(1,0,0),(1,1,0)\},\ \{(1,0,0),(0,1,0)\}\big\}.$$

Their pointwise union and intersection at U are

$$\Big(\bigsqcup_{j=1,2} H_j^{(2)}\Big)(\{a,b\}) = H_1^{(2)}(\{a,b\}) \ \cup\ H_2^{(2)}(\{a,b\})$$
$$= \big\{\{(1,0,0),(1,1,0)\},\ \{(0,1,0)\},\ \{(1,0,0),(0,1,0)\}\big\},$$

$$\Big(\bigsqcap_{j=1,2} H_j^{(2)}\Big)(\{a,b\}) = H_1^{(2)}(\{a,b\}) \ \cap\ H_2^{(2)}(\{a,b\})$$
$$= \big\{\{(1,0,0),(1,1,0)\}\big\}.$$

- The join $\bigsqcup$ collects all distinct crisp-triple-sets from both families.
- The meet $\bigsqcap$ retains only the common element $\{(1,0,0),(1,1,0)\}$.

Since both results are nonempty families of subsets of $\{0,1\}^3$, they define valid 2-SHNCS at U.

Definition 3.21 (Complement)**.** For an n-SHNCS $H^{(n)}$, its *complement* $\neg H^{(n)}$ is

$$(\neg H^{(n)})(U) = \big\{(1-t,1-i,1-f) \mid (t,i,f) \in H^{(n)}(U)\big\}.$$

This is nonempty since $H^{(n)}(U) \neq \emptyset$.

Example 3.22 (Complement of a HyperNeutrosophic Crisp Set)**.** Let

$$X = \{u, v\}, \quad H^{(1)} : X \longrightarrow \mathcal{P}(\{0,1\}^3) \setminus \{\emptyset\}$$

be the 1-SHNCS (i.e. HyperNeutrosophic Crisp Set) defined by

$$H^{(1)}(u) = \{(1,0,0), (0,1,0)\}, \qquad H^{(1)}(v) = \{(0,0,1)\}.$$

Its complement $\neg H^{(1)}$ is given pointwise by

$$(\neg H^{(1)})(x) = \{(1-t, 1-i, 1-f) \mid (t,i,f) \in H^{(1)}(x)\}.$$

Concretely,

$$(\neg H^{(1)})(u) = \{(1-1, 1-0, 1-0), (1-0, 1-1, 1-0)\} = \{(0,1,1), (1,0,1)\},$$

$$(\neg H^{(1)})(v) = \{(1-0, 1-0, 1-1)\} = \{(1,1,0)\}.$$

Each $(\neg H^{(1)})(x)$ is nonempty, so $\neg H^{(1)}$ is again a valid HyperNeutrosophic Crisp Set on X.

Theorem 3.23 (Closure under Finite Union and Intersection)**.** *If $H^{(n)}, K^{(n)}$ are n-SHNCS, then so are $H^{(n)} \sqcup K^{(n)}$ and $H^{(n)} \sqcap K^{(n)}$.*

Proof. For each U, both $(H^{(n)} \sqcup K^{(n)})(U) = H^{(n)}(U) \cup K^{(n)}(U)$ and $(H^{(n)} \sqcap K^{(n)})(U) = H^{(n)}(U) \cap K^{(n)}(U)$ are nonempty because each operand is nonempty. Hence they define valid n-SHNCS. □

Theorem 3.24 (Lattice Structure)**.** *The set of all n-SHNCS on X, ordered by $H \preceq K \iff \forall U,\ H(U) \subseteq K(U)$, is a complete distributive lattice with join $\sqcup$, meet $\sqcap$, and complement $\neg$.*

Proof. Completeness follows as arbitrary joins and finite meets exist by pointwise union/intersection. Distributivity holds because in $\mathcal{P}(\{0,1\}^3)$ finite unions and intersections distribute. The complement $\neg$ is an involution $\neg(\neg H) = H$ and satisfies De Morgan's laws:

$$\neg(H \sqcup K) = \neg H \sqcap \neg K, \quad \neg(H \sqcap K) = \neg H \sqcup \neg K.$$

□

Theorem 3.25 (Embedding of Lower-Order Classes)**.** *The inductive map $\Psi : H^{(n-1)} \mapsto H^{(n)}$ defined by $H^{(n)}(U) = \{ H^{(n-1)}(V) \mid V \in U\}$ is injective. Moreover, every $(n-1)$-SHNCS arises as the restriction of some n-SHNCS to singletons in $P_{n-1}(X)$.*

Proof. If two $(n-1)$-SHNCS differ at V, then their images under Ψ differ at any $U \ni V$. Conversely, given a $(n-1)$-SHNCS $H^{(n-1)}$, define $H^{(n)}$ by the formula above. Then for each $x \in X = P_0(X)$,

$$H^{(n)}(\{x\}) = \{ H^{(n-1)}(\{x\})\},$$

so restricting $H^{(n)}$ to singleton arguments recovers $H^{(n-1)}$. □

Theorem 3.26 (Absorption Laws)**.** *For any n-SHNCS H, K,*

$$H \sqcup (H \sqcap K) = H, \qquad H \sqcap (H \sqcup K) = H.$$

Proof. Pointwise, for each U:

$$(H \sqcup (H \sqcap K))(U) = H(U) \cup \big(H(U) \cap K(U)\big) = H(U),$$

and similarly for the meet-absorption law, using basic set identities. □

Theorem 3.27 (De Morgan's Laws)**.** *For any n-SHNCS H, K,*

$$\neg(H \sqcup K) = \neg H \sqcap \neg K, \quad \neg(H \sqcap K) = \neg H \sqcup \neg K.$$

Proof. Directly follows pointwise from $\neg(A \cup B) = \neg A \cap \neg B$ and $\neg(A \cap B) = \neg A \cup \neg B$ in the Boolean algebra $\mathcal{P}(\{0,1\}^3)$. □

4 Conclusion

In this paper, we explored the properties of the *HyperNeutrosophic Crisp Set* and the *SuperHyperNeutrosophic Crisp Set*. Looking ahead, we hope that future research will further illuminate the mathematical foundations of these structures and pursue extensions that draw on a broad range of algebraic concepts. We also anticipate that forthcoming studies will investigate generalisations based on *Plithogenic Sets* [48–51], neutrosophic offset [52–55], and related frameworks.

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Received: April 3, 2025. Accepted: Sep 10, 2025

NSS Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

Single Valued Pythagorean Neutrosophic Implicative Ideals in KU-algebras

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Abstract. In this paper, we introduce the notion of Pythagorean neutrosophic implicative ideals in KU-algebras, provide several illustrative examples, and investigate some of their key properties. We also define the image and inverse image of such ideals in KU-algebras and establish the conditions under which these images preserve the structure of Pythagorean neutrosophic implicative ideals. Moreover, we explore the Cartesian product of Pythagorean neutrosophic implicative ideals within the Cartesian product of KU-algebras.

Keywords: Pythagorean neutrosophic implicative ideal, image (inverse image) of Pythagorean neutrosophic implicative ideals, Cartesian product of Pythagorean neutrosophic implicative ideals.

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1. Introduction

BCK-algebras form an important class of logical algebras introduced by Iseki [11–13], and since their inception, they have been extensively investigated by several researchers. One of the fundamental approaches to studying these algebras is through their ideals, since ideals provide a powerful tool for understanding the intrinsic properties and internal structure of algebraic systems. The notions of ideals in BCK-algebras, along with the concept of positive implicative ideals (also known as Iseki's implicative ideals), were first introduced by Iseki himself [11–13]. Later, the study was extended to include commutative ideals and implicative ideals in BCK-algebras, which were systematically introduced and investigated in [17, 19–24]. In a parallel line of research, Zadeh [36] introduced the concept of fuzzy sets, a groundbreaking idea that has since found applications across numerous mathematical domains, including group theory, functional analysis, probability theory, and topology. In 1991, Xi [35] pioneered the application of fuzzy set theory to BCK-algebras, introducing the notion of fuzzy subalgebras and fuzzy ideals of BCK-algebras with respect to the minimum operator. Following this development, Jun et al. further studied fuzzy ideals in BCK-algebras [10,14,15], and subsequently, a number of fuzzy structures in BCC-algebras were proposed and examined [2–9]. Building on these ideas, Prabpayak and Leerawat [30, 31] introduced a new algebraic structure, called the KU-algebra. They also developed the concept of homomorphisms of KU-algebras and explored several of their fundamental properties. Extending this work, Mostafa et al. [25, 26, 28, 33] introduced the notion of fuzzy KU-ideals in KU-algebras and investigated their essential properties, thereby broadening the framework of fuzzy algebraic systems. In addition, Meng et al. [22, 23] introduced the concept of implicative ideals and commutative ideals in BCI-algebras and studied their fundamental properties. Motivated by this, Mostafa et al. [28, 33] extended these notions to KU-algebras, where they defined implicative ideals and commutative ideals and investigated their structural attributes in detail. Thus, the study of BCK, BCI, BCC, and KU-algebras, particularly through their ideals and fuzzy extensions, has become a significant branch of modern algebra, blending classical algebraic structures with fuzzy logic and neutrosophic theories to model uncertainty in a wide range of applications.

In this paper, we introduce the notion of Pythagorean neutrosophic implicative ideals in KU-algebras and investigate several of their basic properties. Furthermore, we examine the conditions under which the image and pre-image of a Pythagorean neutrosophic implicative ideal, under a homomorphism of KU-algebras, remain Pythagorean neutrosophic implicative ideals. In addition, we establish the relationship between the product of Pythagorean neutrosophic implicative ideals and the product of fuzzy implicative ideals.

2. Preliminaries

Definition 2.1. [30,31] Algebra $(\Upsilon, *, 0)$ of type $(2, 0)$ is said to be a KU-algebra, if it satisfies the following conditions:

(KU_1) $(\imath * \jmath) * [(\jmath * \ell) * (\imath * \ell)] = 0$,

(KU_2) $\imath * 0 = 0$,

(KU_3) $0 * \imath = \imath$,

(KU_4) $\imath * \jmath = 0$ and $\jmath * \imath = 0$ implies $\imath = \jmath$,

(KU_5) $\imath * \imath = 0$, for all $\imath, \jmath, \ell \in \Upsilon$.

On a KU-algebra $(\Upsilon, *, 0)$ we can define a binary relation $\leq$ on Υ by putting

$$\imath \leq \jmath \Leftrightarrow \jmath * \imath = 0.$$

Thus a KU-algebra Υ satisfies the conditions:

(kU_1) $(\jmath * \ell) * (\imath * \ell) \leq (\imath * \jmath)$,

(kU_1) $0 \leq \imath$,

(kU_1) $\imath \leq \jmath, y \leq \imath$ implies $\imath = \jmath$,

(kU_1) $\jmath * \imath \leq \imath$.

Theorem 2.2. [26] In a KU-algebra Υ, the following axioms are satisfied:

For all $\imath, \jmath, \ell \in \Upsilon$

(1) $\imath \leq \jmath$ imply $\jmath * \ell \leq \imath * \ell$,

(2) $\imath * (\jmath * \ell) = \jmath * (\imath * \ell)$, for all $\imath, \jmath, \ell \in \Upsilon$,

(3) $((\jmath * \imath) * \imath) \leq \jmath$,

(4) $((\jmath * \imath) * \imath) * \imath)) = (\jmath * \imath)$.

Definition 2.3. [30, 31] Let I be a non empty subset of a KU-algebra Υ. Then, I is said to be an ideal of Υ, if

(I_0) $0 \in I$,

(II_0) $\forall\ \jmath, \ell \in \Upsilon$, if $(\jmath * \ell) \in I$ and $\jmath \in I$, imply $\ell \in I$.

Definition 2.4. [26] A non empty subset I of a KU-algebra Υ is said to be an KU-ideal of Υ if it satisfies:

(K_1) $0 \in I$,

(K_2) $\imath * (\jmath * \ell) \in I$ and $\jmath \in I$ imply $\imath * \imath \in I$ for all $\imath, \jmath$ and $\ell \in \Upsilon$.

Definition 2.5. [33] A KU-algebra Υ is said to be implicative if it satisfies the identity

$$\Upsilon = (\imath * \jmath) * \imath, \text{ for all } \imath, \jmath \in \Upsilon.$$

For the properties of KU-algebras, we refer the reader to [12–16].

Definition 2.6. [36] Let Υ be a non empty set, a fuzzy subset μ in Υ is a function

$$f : \Upsilon \to [0, 1].$$

Definition 2.7. [26] Let Υ be a KU-algebra, a fuzzy subset μ in Υ is called a fuzzy ideal of Υ if it satisfies the following conditions:

(i) $\mu(0) \geq \mu(\imath)$, for all $\imath \in \Upsilon$,

(ii) $\forall \imath, \jmath \in \Upsilon, \mu(\jmath) \geq \min\{\mu(\imath * \jmath), \mu(\imath)\}$.

Definition 2.8. [27] Let Υ be a KU-algebra. A fuzzy set μ in Υ is called a fuzzy KU-ideal of Υ if it satisfies:

(FK_1) $\mu(0) \geq \mu(\imath)$, (FK_2) $\mu(\imath * \ell) \geq \min\{\mu(\imath * (\jmath * \ell)), \mu(\jmath)\}$, for all $\imath, \jmath$ and $\ell \in \Upsilon$.

Definition 2.9. [26] Let μ be a fuzzy set in a set Υ. For $t \in [0, 1]$, the set

$$\mu_t = \{\imath \in \Upsilon | \mu(\imath) \geq t\}$$

is called upper level cut (level subset) of μ.

Definition 2.10. [29] A non empty subset μ of a KU-algebra Υ is called a fuzzy implicative ideal of Υ, if $\forall\ \imath, \jmath, \ell \in \Upsilon$,

(F_0) $\mu(0) \geq \mu(\imath)$,

(F_0) $\mu((\imath * \jmath) * \imath) \geq \min\{\mu(\ell * ((\imath * \jmath) * \imath)), \mu(\ell)\}$.

Definition 2.11. [35] Let f be a mapping from the set Υ to a set Y. If μ is a fuzzy subset of Υ, then the fuzzy subset B of Y defined by

$$f(\mu)(\jmath) = B(\jmath) = \begin{cases} \sup\limits_{\imath \in f^{-1}(\jmath)} \mu(\imath) & \text{if} \quad f^{-1}(\jmath) = \{\imath \in \Upsilon, f(\imath) = \jmath\} \neq \phi \\ 0 & , \text{ otherwise} \end{cases}$$

is said to be the image of μ under f. Similarity if β is a fuzzy subset of Y, then the fuzzy subset $\mu = \beta \circ f$ in Υ (i. e. the fuzzy subset defined by $\mu(\imath) = \beta(f(\imath)$, for all $\imath \in \Upsilon$ is called the primage of β under f.

Definition 2.12. [27] Let μ be a fuzzy set on a KU-algebra Υ, then μ is called a fuzzy KU-subalgebra of Υ if $\mu(\imath * \jmath) \geq \min\{\mu(\imath), \mu(\jmath)\}$ for all $\imath, \jmath \in \Upsilon$.

Lemma 2.13. [27] Let μ be a fuzzy ideal of KU-algebra Υ. if the inequality $\imath * \jmath \leq \ell$ hold in Υ, then $\mu(\jmath) \geq \min\{\mu(\imath), \mu(\ell)\}$.

Lemma 2.14. [27] If μ be a fuzzy ideal of KU-algebra Υ and if $\imath \leq \jmath$, then $\mu(\imath) \geq \mu(\jmath)$.

Definition 2.15. [32]. Let Υ be a non-empty set (Universe) A Pythagorean neutrosophic set (briefly, PNS) T and F as dependent neutrosophic components A on Υ is an object of the form $\mathcal{P} = \{\langle \imath, \mu_{\mathcal{P}}(\imath), \nu_{\mathcal{P}}(\imath), \lambda_{\mathcal{P}}(\imath)\rangle | \imath \in \Upsilon\}$,
where $\mu_{\mathcal{P}}(\imath)$, $\nu_{\mathcal{P}}(\imath)$, $\lambda_{\mathcal{P}}(\imath)$ are the truth, indeterminacy and false respectively such that $\mu, \nu, \lambda \in [0, 1]$. Here when μ and λ are dependent components, then for all Υ in Υ; (i) $\mu + \lambda \leq 1$, (ii) $0 \leq \mu^2 + \lambda^2 \leq 1$, (iii) $0 \leq \mu^2 + \nu^2 + \lambda^2 \leq 2$.

We define these basic operations on PNS which can be described as follows: Let Υ be a nonempty set (universe). A Pythagorean Neutrosophic set μ and λ as dependent neutrosophic components $\mathcal{P}$ and $\mathcal{Q}$ of the form $\mathcal{P} = \{\langle x, \mu_{\mathcal{P}}(\imath), \nu_{\mathcal{P}}(\imath), \lambda_{\mathcal{P}}(\imath)\rangle | \imath \in \Upsilon\}$ and $\mathcal{Q} = \{\langle x, \mu_{\mathcal{Q}}(\imath), \nu_{\mathcal{Q}}(\imath), \lambda_{\mathcal{Q}}(\imath)\rangle | \imath \in \Upsilon\}$. The complement of $\mathcal{P}$ is $\mathcal{P}^c = \{\langle x, \lambda_{\mathcal{P}}(\imath), 1 - \nu_{\mathcal{P}}(\imath), \mu_{\mathcal{P}}(\imath)\rangle | \imath \in \Upsilon\}$. The union and intersection of $\mathcal{P}$ and $\mathcal{Q}$ are

(i) $\mathcal{P} \cup \mathcal{Q} = \{max(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}}), min(\nu_{\mathcal{P}}, \nu_{\mathcal{Q}}), min(\lambda_{\mathcal{P}}, \lambda_{\mathcal{Q}})\}$;

(ii) $\mathcal{P} \cap \mathcal{Q} = \{min(\mu_{\mathcal{P}}, \mu_{\mathcal{Q}}), max(\nu_{\mathcal{P}}, \nu_{\mathcal{Q}}), max(\lambda_{\mathcal{P}}, \lambda_{\mathcal{Q}})\}$.

3. Pythagorean neutrosophic implicative ideals

Definition 3.1. Let Υ be a KU-algebra. A PNS $\mathfrak{C} = \{\langle \imath, \mu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\imath)\rangle | \imath \in \Upsilon\}$ is called a PN single valued ideal ($PNSVI$) of Υ if it satisfies:

(i) $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$,

(ii) $\mu_{\mathfrak{C}}(\jmath) \geq \min\{\mu_{\mathfrak{C}}(\imath \times \jmath), \mu_{\mathfrak{C}}(\imath)\}$, $\nu_{\mathfrak{C}}(\jmath) \leq \max\{\nu_{\mathfrak{C}}(\imath \times \jmath), \nu_{\mathfrak{C}}(\imath)\}$, $\lambda_{\mathfrak{C}}(\jmath) \leq \max\{\lambda_{\mathfrak{C}}(\imath \times \jmath), \lambda_{\mathfrak{C}}(\imath)\}$, $\forall \imath, \jmath \in \Upsilon$.

Definition 3.2. Let Υ be a KU-algebra, a PNS $\mathfrak{C} = \{\langle \imath, \mu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\imath)\rangle | \imath \in \Upsilon\}$ in Υ is called a PN implicative ideal (resp. $PNImpI$) of Υ if it satisfies the following conditions:

(PN_0) $\forall \imath \in \Upsilon$, $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$,

(PN_1) $\forall \imath, \jmath, \ell \in \Upsilon$, $\mu_{\mathfrak{C}}((\imath * \jmath) * \imath) \geq \min\{\mu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \mu_{\mathfrak{C}}(\ell)\}$,
$\nu_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \nu_{\mathfrak{C}}(\ell)\}$,
$\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$.

Definition 3.3. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ be PNS in Υ. For $s, t, r \in [0,1]$, the set, $U(\mu_{\mathfrak{C}}, t) = \{\imath \in \Upsilon / \mu_{\mathfrak{C}}(\imath) \geq t\}$ is called upper level cut of $\mu_{\mathfrak{C}}$, the set $L(\nu_{\mathfrak{C}}, s) = \{\imath \in \Upsilon / \nu_{\mathfrak{C}}(\imath) \leq s$ is called lower level cut of $\nu_{\mathfrak{C}}$ and the set $L(\lambda_{\mathfrak{C}}, r) = \{\imath \in \Upsilon / \lambda_{\mathfrak{C}}(\imath) \leq r\}$ is called lower level cut of $\lambda_{\mathfrak{C}}$.

Example 3.4. Let $\Upsilon = \{a, b, c, d, e\}$ in which the operation $*$ is given by the table

$*$	a	b	c	d	e
a	a	b	c	d	e
b	a	a	b	d	e
c	a	a	a	d	e
d	a	a	a	a	e
e	a	a	a	a	a

Then $(\Upsilon, *, 0 = a)$ is a KU-algebra. Define a PNS $\mu : \Upsilon \to [0,1]$, $\nu : \Upsilon \to [0,1]$ and $\lambda : \Upsilon \to [0,1]$ by

$$\mu(a) = s_0, \mu(b) = \mu(c) = s_1, \mu(d) = s_2, \mu(e) = s_3$$
$$\nu(a) = t_0, \nu(b) = \nu(c) = t_1, \nu(d) = t_2, \nu(e) = t_3$$
$$\lambda(a) = u_0, \lambda(b) = \lambda(c) = u_1, \lambda(d) = u_2, \lambda(e) = u_3$$

where $s_0, s_1, s_2, s_3 \in [0,1]$ with $s_0 < s_1 < s_2 < s_3$, $t_0, t_1, t_2, t_3 \in [0,1]$ with $t_0 < t_1 < t_2 < t_3$ and $u_0, u_1, u_2, u_3 \in [0,1]$ with $u_0 > u_1 > u_2 > u_3$

Routine calculation gives that μ is a $\langle \Upsilon, \mu, \nu, \lambda \rangle$ is a $PNImpI$ of KU-algebra Υ.

Lemma 3.5. *If $\mathfrak{C} = \{\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}}\}$ is a $PNImpI$ of KU- algebra Υ and if $\imath \leq \ell$, then $\mu_{\mathfrak{C}}(\imath) \geq \mu_{\mathfrak{C}}(\ell)$, $\nu_{\mathfrak{C}}(\imath) \leq \nu_{\mathfrak{C}}(\ell)$, $\lambda_{\mathfrak{C}}(\imath) \leq \lambda_{\mathfrak{C}}(\ell)$.*

Proof. If $\imath \leq \ell$, then $\ell * \imath = 0$, this together with $0 * \imath = \imath$, $\imath * \imath = \imath * 0 = 0$ and $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\ell)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\ell)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\ell)$. Put $\jmath = 0$ in (PN_1), we get

$$\begin{aligned}
\mu_{\mathfrak{C}}((\imath * 0) * \imath) &\geq \min\{\mu_{\mathfrak{C}}(\ell * ((\imath * 0) * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(0 * \imath) &\geq \min\{\mu_{\mathfrak{C}}(\ell * (0 * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(\imath) &\geq \min\{\mu_{\mathfrak{C}}(\ell * \imath), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(\imath) &\geq \min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(\ell)\} \\
&= \min\{\mu_{\mathfrak{C}}(0 * \ell)\} = \mu_{\mathfrak{C}}(\ell) \\
\Rightarrow \mu_{\mathfrak{C}}(\imath) &\geq \mu_{\mathfrak{C}}(\ell).
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{C}}((\imath * 0) * \imath) &\leq \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * 0) * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(0 * \imath) &\leq \max\{\nu_{\mathfrak{C}}(\ell * (0 * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(\imath) &\leq \max\{\nu_{\mathfrak{C}}(\ell * \imath), \nu_{\mathfrak{C}}(\ell)\} \\
&= \max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(\ell)\} \\
&= \max\{\nu_{\mathfrak{C}}(0 * \ell)\} \\
&= \nu_{\mathfrak{C}}(\ell)
\end{aligned}$$

$$\Rightarrow \nu_{\mathfrak{C}}(\imath) \leq \nu_{\mathfrak{C}}(\ell).$$

Similarly we can prove for $\lambda_{\mathfrak{C}}(\imath) \leq \lambda_{\mathfrak{C}}(\ell)$.

Lemma 3.6. *Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ be a $PNImpI$ of KU-algebra Υ, if the inequality, $\ell * \imath \leq \jmath$ hold in Υ, then $\mu_{\mathfrak{C}}(\imath) \geq min\{\mu_{\mathfrak{C}}(\jmath), \mu_{\mathfrak{C}}(\ell)\}$, $\nu_{\mathfrak{C}}(\imath) \leq max\{\nu_{\mathfrak{C}}(\jmath), \nu_{\mathfrak{C}}(\ell)\}$ and $\lambda_{\mathfrak{C}}(\imath) \leq max\{\lambda_{\mathfrak{C}}(\jmath), \lambda_{\mathfrak{C}}(\ell)\}$.*

Proof. Assume that the inequality $\ell * \imath \leq \jmath$ holds in Υ, then $\mu_{\mathfrak{C}}(\ell * \imath) \geq \mu_{\mathfrak{C}}(\jmath)$, $\nu_{\mathfrak{C}}(\ell * \imath) \leq \nu_{\mathfrak{C}}(\jmath)$ and $\lambda_{\mathfrak{C}}(\ell * \imath) \leq \lambda_{\mathfrak{C}}(\jmath)$ (by Lemma 3.5). Put $\imath = \jmath$ in (PN_2), we have

$$\begin{aligned}
\mu_{\mathfrak{C}}((\imath * \imath) * \imath) &\geq min\{\mu_{\mathfrak{C}}(\ell * ((\imath * \imath) * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}((0 * \imath) &\geq min\{\mu_{\mathfrak{C}}(\ell * (0 * \imath)), \mu_{\mathfrak{C}}(\ell)\} \\
\mu_{\mathfrak{C}}(\imath) &\geq min\{\mu_{\mathfrak{C}}(\ell * \imath), \mu_{\mathfrak{C}}(\ell)\} \\
&= min\{\mu_{\mathfrak{C}}(\jmath), \mu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{C}}((\imath * \imath) * \imath) &\leq max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \imath) * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(0 * \imath) &\leq max\{\nu_{\mathfrak{C}}(\ell * (0 * \imath)), \nu_{\mathfrak{C}}(\ell)\} \\
\nu_{\mathfrak{C}}(\imath) &\leq max\{\nu_{\mathfrak{C}}(\ell * \imath), \nu_{\mathfrak{C}}(\ell)\} \\
&= max\{\nu_{\mathfrak{C}}(\jmath), \nu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

and

$$\lambda_{\mathfrak{C}}((\imath * \imath) * \imath) \leq max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \imath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$$
$$\lambda_{\mathfrak{C}}(0 * \imath) \leq max\{\lambda_{\mathfrak{C}}(\ell * (0 * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$$
$$\lambda_{\mathfrak{C}}(\imath) \leq max\{\lambda_{\mathfrak{C}}(\ell * \imath), \lambda_{\mathfrak{C}}(\ell)\}$$
$$= max\{\lambda_{\mathfrak{C}}(\jmath), \lambda_{\mathfrak{C}}(\ell)\}.$$

Proposition 3.7. The intersection of any collection of $PNImpI$'s of a KU-algebra Υ is also a $PNImpI$.

Proof. Let $A_1 = \{\mu_{A_i}, \nu_{A_i}, \lambda_{A_i}\}$ be a family of $PNImpI$'s of KU-algebra Υ, then for any $\imath, \jmath, \ell \in \Upsilon$.
$(\bigcap_i \mu_{A_i})(0) = \inf(\mu_{A_i}(0)) \geq \inf(\mu_{A_i}(\imath)) = (\bigcap_i \mu_{A_i})(\imath)$,
$(\bigcup_i \nu_{A_i})(0) = \sup(\nu_{A_i}(0)) \leq \sup(\nu_{A_i}(\imath)) = (\bigcup_i \nu_{A_i})(\imath)$
and $(\bigcup_i \lambda_{A_i})(0) = \sup(\lambda_{A_i}(0)) \leq \sup(\lambda_{A_i}(\imath)) = (\bigcup_i \lambda_{A_i})(\imath)$.

$$(\bigcap_i \mu_{A_i})((\imath * \jmath) * \imath) = \inf(\mu_{A_i}((\imath * \jmath) * \imath))$$
$$\geq \inf(\min\{\mu_{A_i}(\ell * ((\imath * \jmath) * \imath)), \mu_{A_i}(\ell)\})$$
$$= \min\{\inf(\mu_{A_i}(\ell * ((\imath * \jmath)) * \imath)), \inf(\mu_{A_i}(\ell))\}$$
$$= \min\{(\bigcap_i \mu_{A_i})(\ell * ((\imath * \jmath) * \imath)), (\bigcap_i \mu_{A_i})(\ell)\}$$

$$(\bigcup_i \nu_{A_i})((\imath * \jmath) * \imath) = \sup(\nu_{A_i}((\imath * \jmath) * \imath))$$
$$\leq \sup(\max\{\nu_{A_i}(\ell * ((\imath * \jmath) * \imath)), \nu_{A_i}(\ell)\})$$
$$= \max\{\sup(\nu_{A_i}(\ell * ((\imath * \jmath)) * \imath)), \sup(\nu_{A_i}(\ell))\}$$
$$= \max\{(\bigcup_i \nu_{A_i})(\ell * ((\imath * \jmath) * \imath)), (\bigcup_i \nu_{A_i})(\ell)\}$$

$$(\bigcup_i \lambda_{A_i})((\imath * \jmath) * \imath) = \sup(\lambda_{A_i}((\imath * \jmath) * \imath))$$
$$\leq \sup(\max\{\lambda_{A_i}(\ell * ((\imath * \jmath) * \imath)), \lambda_{A_i}(\ell)\})$$
$$= \max\{\sup(\lambda_{A_i}(\ell * ((\imath * \jmath)) * \imath)), \sup(\lambda_{A_i}(\ell))\}$$
$$= \max\{(\bigcup_i \lambda_{A_i})(\ell * ((\imath * \jmath) * \imath)), (\bigcup_i \lambda_{A_i})(\ell)\}.$$

Definition 3.8. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}})$ be PNS in Υ for $s, t, r \in [0, 1]$, the set, $\cup(\mu_{\mathfrak{C}}, t) = \{\imath \in \Upsilon / \mu_{\mathfrak{C}} \geq t\}$ is called upper level cut of $\mu_{\mathfrak{C}}$, the set $L(\nu, s) = \{\imath \in \Upsilon / \nu_{\mathfrak{C}} \geq s$ is called loweer level cut of $\nu_{\mathfrak{C}}$ and the set $L(\lambda_{\mathfrak{C}}, \nu) = \{\imath \in \Upsilon / \lambda_{\mathfrak{C}} \leq r\}$ is called lower level cut of $\lambda_{\mathfrak{C}}$

Theorem 3.9. A PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of KU-algebra Υ is a $PNImpI$ of Υ iff, for every $t, s, r \in [0,1]$, $U(\mu_{\mathfrak{C}}, t)$, $L(\nu_{\mathfrak{C}}, s)$ and $L(\lambda_{\mathfrak{C}}, r)$ are either empty or an implicative ideals of Υ.

Proof. Assume that $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ is a $PNImpI$ of Υ, by (PN_1), we have $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$ for all $\imath \in \Upsilon$, therefore $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath) \geq t$ for $\imath \in U(\mu_{\mathfrak{C}}, t)$ and so $0 \in U(\mu_{\mathfrak{C}}, t)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath) \leq s$ for $\imath \in L(\nu_{\mathfrak{C}}, s)$ and so $0 \in L(\nu_{\mathfrak{C}}, s)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath) \leq r$ for $\imath \in L(\lambda_{\mathfrak{C}}, r)$ and so $0 \in L(\lambda_{\mathfrak{C}}, r)$. Let $\ell * ((\imath * \jmath) * \imath) \in U(\mu_{\mathfrak{C}}, t)$ and $\ell \in U(\mu_{\mathfrak{C}}, t)$, then $\mu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)) \geq t$ and $\mu_{\mathfrak{C}}(\ell) \geq t$, since $\mu_{\mathfrak{C}}$ is a $PNImpI$ it follows that $\mu_{\mathfrak{C}}((\imath * \jmath) * \imath) \geq \min\{\mu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \mu_{\mathfrak{C}}(\ell)\} \geq t$ and therefore $(\imath * \jmath) * \imath \in U(\mu_{\mathfrak{C}}, t)$. Hence $U(\mu_{\mathfrak{C}}, t)$ is an KU-ideal of Υ. Let $\ell * ((\imath * \jmath) * \imath) \in L(\nu_{\mathfrak{C}}, s)$ and $\ell \in L(\nu_{\mathfrak{C}}, s)$, then $\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)) \leq s$ and $\nu_{\mathfrak{C}}(\ell) \leq s$, since $\nu_{\mathfrak{C}}$ is a $PNImpI$ it follows that $\nu_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \nu_{\mathfrak{C}}(\ell)\} \leq s$ and therefore $(\imath * \jmath) * \imath \in L(\nu_{\mathfrak{C}}, s)$. Hence $\nu_{\mathfrak{C}}$ is an KU-ideal of Υ.

Let $\ell * ((\imath * \jmath) * \imath) \in L(\lambda_{\mathfrak{C}}, r)$ and $\ell \in L(\lambda_{\mathfrak{C}}, r)$, then $\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)) \leq r$ and $\lambda_{\mathfrak{C}}(\ell) \leq r$, since $\lambda_{\mathfrak{C}}$ is a $PNImpI$ it follows that $\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\} \leq r$ and therefore $(\imath * \jmath) * \imath \in L(\lambda_{\mathfrak{C}}, r)$. Hence $\lambda_{\mathfrak{C}}$ is an KU-ideal of Υ.

Conversely, we only need to show that (PN_1) $\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$, $\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$ and (PN_2) $\mu_{\mathfrak{C}}((\imath*\jmath)*\imath) \geq \min\{\mu_{\mathfrak{C}}(\ell*((\imath*\jmath)*\imath)), \mu_{\mathfrak{C}}(\ell)\}$, $\nu_{\mathfrak{C}}((\imath*\jmath)*\imath) \leq \max\{\nu_{\mathfrak{C}}(\ell*((\imath*\jmath)*\imath)), \nu_{\mathfrak{C}}(\ell)\}$, $\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) \leq \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}$ are true. If PN_1 is false then there exist $\imath' \in \Upsilon$ such that $\mu_{\mathfrak{C}}(0) < \mu_{\mathfrak{C}}(\imath')$, $\nu_{\mathfrak{C}}(0) > \nu_{\mathfrak{C}}(\imath')$, $\lambda_{\mathfrak{C}}(0) > \lambda_{\mathfrak{C}}(\imath')$. If we take $t' = (\mu_{\mathfrak{C}}(\imath') + \mu_{\mathfrak{C}}(0))/2$, $s' = (\nu_{\mathfrak{C}}(\imath') + \nu_{\mathfrak{C}}(0))/2$, $r' = (\lambda_{\mathfrak{C}}(\imath') + \lambda_{\mathfrak{C}}(0))/2$, then $\mu_{\mathfrak{C}}(0) < t'$, $\nu_{\mathfrak{C}}(0) > s'$, $\lambda_{\mathfrak{C}}(0) > r'$ and $0 \leq t' < \mu_{\mathfrak{C}}(\imath') \leq 1$, $1 \geq s' > \nu_{\mathfrak{C}}(\imath') \geq 0$, $1 \geq r' > \lambda_{\mathfrak{C}}(\imath') \geq 0$ thus $\imath' \in U(\mu_{\mathfrak{C}}, t')$, $\imath' \in L(\nu_{\mathfrak{C}}, s')$, $\imath' \in L(\lambda_{\mathfrak{C}}, r')$ and $U(\mu_{\mathfrak{C}}, t') \neq \phi$, $L(\nu_{\mathfrak{C}}, s') \neq \phi$, $L(\lambda_{\mathfrak{C}}, r') \neq \phi$. As $\mu_{\mathfrak{C}}$ is $KUImpI$ if Υ, we have $0 \in U(\mu_{\mathfrak{C}}, t')$ and so $\mu_{\mathfrak{C}}(0) \geq t'$. This is a contradiction. As $\nu_{\mathfrak{C}}$ is $KUImpI$ if Υ, we have $0 \in L(\nu_{\mathfrak{C}}, s')$ and so $\nu_{\mathfrak{C}}(0) \leq s'$. This is a contradiction. As $\lambda_{\mathfrak{C}}$ is $KUImpI$ if Υ, we have $0 \in L(\lambda_{\mathfrak{C}}, r')$ and so $\lambda_{\mathfrak{C}}(0) \leq r'$. This is a contradiction. Now, assume (PN_2) is not true, then there exist $\imath'$, $\jmath'$ and ℓ' such that, $\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') < \min\{\mu_{\mathfrak{C}}(\ell' * ((\imath' * \jmath') * \imath')), \mu_{\mathfrak{C}}(\ell')\}$, $\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') > \max\{\nu_{\mathfrak{C}}(\ell' * ((\imath' * \jmath') * \imath')), \nu_{\mathfrak{C}}(\ell')\}$, $\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') > \max\{\lambda_{\mathfrak{C}}(\ell' * ((\imath' * \jmath') * \imath')), \lambda_{\mathfrak{C}}(\ell')\}$. Putting $t' = \{\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') + \min\{\mu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \mu_{\mathfrak{C}}(\ell')\}\}/2$ then $\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') < t'$ and $0 \leq t' < \min\{\mu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \mu_{\mathfrak{C}}(\ell')\}/2 \leq 1$, hence $\{\mu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath') > t'$ and $\mu_{\mathfrak{C}}(\ell') > t' \Rightarrow (\imath' * \jmath') * \imath' \in U(\mu_{\mathfrak{C}}, t')$ and $\ell' \in U(\mu_{\mathfrak{C}}, t')$. Since $\mu_{\mathfrak{C}}$ is an $ImpI$, it follows that $(\imath' * \jmath') * \imath' \in U(\mu_{\mathfrak{C}}, t')$ and that $\mu_{\mathfrak{C}}((\imath' * \jmath') * \imath') > t'$ this is also a contradiction. Hence $U(\mu_{\mathfrak{C}}, t')$ is a $PNImpI$ of Υ. Putting $s' = \{\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') + \max\{\nu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \nu_{\mathfrak{C}}(\ell')\}\}/2$ then $\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') > s'$ and $0 \leq s' \leq \max\{\nu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \nu_{\mathfrak{C}}(\ell')\}/2 \leq 1$, hence $\{\nu_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath') < s'$ and $\nu_{\mathfrak{C}}(\ell') < s' \Rightarrow (\imath' * \jmath') * \imath' \in L(\nu_{\mathfrak{C}}, s')$ and $\ell' \in L(\nu_{\mathfrak{C}}, s')$. Since $\nu_{\mathfrak{C}}$ is an $ImpI$, it follows that $(\imath' * \jmath') * \imath' \in L(\nu_{\mathfrak{C}}, s')$ and that $\nu_{\mathfrak{C}}((\imath' * \jmath') * \imath') < s'$ this is also a contradiction. Hence $L(\nu_{\mathfrak{C}}, s')$ is a $PNImpI$ of Υ. Putting $r' = \{\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') + \max\{\lambda_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \lambda_{\mathfrak{C}}(\ell')\}\}/2$ then $\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') > r'$ and $0 \leq r' \leq \max\{\lambda_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath'), \lambda_{\mathfrak{C}}(\ell')\}/2 \leq 1$, hence

$\{\lambda_{\mathfrak{C}}(\ell' * (\imath' * \jmath') * \imath') < r'$ and $\lambda_{\mathfrak{C}}(\ell') < r' \Rightarrow (\imath' * \jmath') * \imath' \in L(\lambda_{\mathfrak{C}}, r')$ and $\ell' \in L(\lambda_{\mathfrak{C}}, r')$. Since $\lambda_{\mathfrak{C}}$ is an $ImpI$, it follows that $(\imath' * \jmath') * \imath' \in L(\lambda_{\mathfrak{C}}, r')$ and that $\lambda_{\mathfrak{C}}((\imath' * \jmath') * \imath') < r'$ this is also a contradiction. Hence $L(\lambda_{\mathfrak{C}}, s')$ is a $PNImpI$ of Υ.

Corollary 3.10. If a PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of KU-algebra Υ is a $PNImpI$ then for every $t \in I_m(\mu_{\mathfrak{C}})$, $s \in I_m(\nu_{\mathfrak{C}})$, $r \in I_m(\lambda_{\mathfrak{C}})$, $U(\mu_{\mathfrak{C}}, t)$, $L(\nu_{\mathfrak{C}}, s)$, $L(\lambda_{\mathfrak{C}}, r)$ is an $ImpI$ of Υ.

Theorem 3.11. An onto homomorphic preimage of a $PNImpI$ is also a $PNImpI$.

Proof. Let $\zeta : \Upsilon \to Y$ be an into homomorphic of KU-algebras, $B = (\jmath, \mu_B, \nu_B, \lambda_B)$ is a $PNImpI$ of Y and $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ the preimage of B under ζ, then

$$\mu_B(\zeta(\imath)) = \mu_{\mathfrak{C}}(\imath),\ \forall \imath \in \Upsilon$$
$$\nu_B(\zeta(\imath)) = \nu_{\mathfrak{C}}(\imath),\ \forall \imath \in \Upsilon$$
$$\lambda_B(\zeta(\imath)) = \lambda_{\mathfrak{C}}(\imath), \forall \imath \in \Upsilon.$$

Let $\imath \in \Upsilon$ then, $\mu_{\mathfrak{C}}(0) = \mu_B(\zeta(0)) \geq \mu_B(\zeta(\imath)) = \mu_{\mathfrak{C}}(\imath)$,
$\nu_{\mathfrak{C}}(0) = \nu_B(\zeta(0)) \leq \nu_B(\zeta(\imath)) = \nu_{\mathfrak{C}}(\imath)$, $\lambda_{\mathfrak{C}}(0) = \lambda_B(\zeta(0)) \leq \lambda_B(\zeta(\imath)) = \lambda_{\mathfrak{C}}(\imath)$. Now, let $\imath, \jmath, \ell \in \Upsilon$ then

$$\begin{aligned}
\mu_{\mathfrak{C}}((\imath * \jmath) * \imath) &= \mu_B(\zeta((\imath * \jmath) * \imath)) \\
&= \mu_B((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath)) \\
&\geq \min\{\mu_B(\zeta(\ell) * ((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath))), \mu_B(\zeta(\ell))\} \\
&= \min\{\mu_B(\zeta(\ell * ((\jmath * \imath) * \imath)), \mu_B(\zeta(\ell))\} \\
&= \min\{\mu_{\mathfrak{C}}(\ell * ((\jmath * \imath) * \imath)), \mu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

$$\begin{aligned}
\nu_{\mathfrak{C}}((\imath * \jmath) * \imath) &= \nu_B(\zeta((\imath * \jmath) * \imath)) \\
&= \nu_B((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath)) \\
&\leq \max\{\nu_B(\zeta(\ell) * ((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath))), \nu_B(\zeta(\ell))\} \\
&= \max\{\nu_B(\zeta(\ell * ((\imath * \jmath) * \imath)), \nu_B(\zeta(\ell))\} \\
&= \max\{\nu_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \nu_{\mathfrak{C}}(\ell)\}
\end{aligned}$$

$$\begin{aligned}\lambda_{\mathfrak{C}}((\imath * \jmath) * \imath) &= \lambda_B(\zeta((\imath * \jmath) * \imath)) \\ &= \lambda_B((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath)) \\ &\leq \max\{\lambda_B(\zeta(\ell) * ((\zeta(\imath) * \zeta(\jmath)) * \zeta(\imath))), \lambda_B(\zeta(\ell))\} \\ &= \max\{\lambda_B(\zeta(\ell * ((\imath * \jmath) * \imath)), \lambda_B(\zeta(\ell))\} \\ &= \max\{\lambda_{\mathfrak{C}}(\ell * ((\imath * \jmath) * \imath)), \lambda_{\mathfrak{C}}(\ell)\}\end{aligned}$$

Hence the proof.

Definition 3.12. A PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of Υ has sup property if for any subset $T = (\Upsilon, \mu_T, \nu_T, \lambda_T)$ of Υ there exist $t_0, s_0, r_0 \in T$ such that

$$\mu_{\mathfrak{C}}(t_0) = \sup_{t \in T} \mu_{\mathfrak{C}}(t)$$
$$\nu_{\mathfrak{C}}(s_0) = \inf_{s \in T} \nu_{\mathfrak{C}}(s)$$
$$\lambda_{\mathfrak{C}}(r_0) = \inf_{r \in T} \lambda_{\mathfrak{C}}(r).$$

Theorem 3.13. Let $\zeta : \Upsilon \to Y$ be a homomorphism between KU-algebras Υ and Y. For every $PNImpI$ $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ in Υ, $B = (\zeta(\mu_{\mathfrak{C}}), \zeta(\nu_{\mathfrak{C}}), \zeta(\lambda_{\mathfrak{C}}))$ is a $PNImpI$ of Y.

Poof. By definition $\mu_B(\jmath') = \zeta(\mu_{\mathfrak{C}})(\jmath') = \sup_{\imath \in f^{-1}(\jmath')} \mu_{\mathfrak{C}}(\imath)$, $\forall \jmath' \in Y$ and $\sup \phi = 0$.
$\nu_B(\jmath') = \zeta(\nu_{\mathfrak{C}})(\jmath') = \inf_{\imath \in \zeta^{-1}(\jmath')} \nu_{\mathfrak{C}}(\imath)$, $\forall \jmath' \in Y$ and $\inf \phi = 0$.
$\lambda_B(\jmath') = \zeta(\lambda_{\mathfrak{C}})(\jmath') = \inf_{\imath \in \zeta^{-1}(\jmath')} \lambda_{\mathfrak{C}}(\imath)$, $\forall \jmath' \in Y$ and $\inf \phi = 0$.
We have to prove that
$\mu_B((\imath' * \jmath') * \imath') \geq \min\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}$
$\nu_B((\imath' * \jmath') * \imath') \leq \max\{\nu_B(\ell' * (\imath' * \jmath') * \imath'), \nu_B(\ell')\}$
$\lambda_B((\imath' * \jmath') * \imath') \leq \max\{\lambda_B(\ell' * (\imath' * \jmath') * \imath'), \lambda_B(\ell')\}$ $\forall \imath', \jmath', \ell' \in Y$.
Let $\zeta : \Upsilon \to Y$ be an onto homomorphism of KU-algebras, $\mathfrak{C}$ a $PNImpI$ of Υ with sup property and B the image of $\mathfrak{C}$ under ζ, since $\mathfrak{C}$ is a $PNImpI$ of Υ, we have

$$\mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$$
$$\nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$$
$$\lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath), \ \forall \imath \in \Upsilon.$$

Note that $0 \in \zeta'(0)$, where $0, 0'$ are the zero of Υ and Y respectively.
Thus, $\mu_B(0') = \sup_{t \in \zeta^{-1}(0')} \mu_{\mathfrak{C}}(t) = \mu_{\mathfrak{C}}(0) \geq \mu_{\mathfrak{C}}(\imath)$,
$\nu_B(0') = \inf_{t \in \zeta^{-1}(0')} \nu_{\mathfrak{C}}(t) = \nu_{\mathfrak{C}}(0) \leq \nu_{\mathfrak{C}}(\imath)$,
$\lambda_B(0') = \inf_{t \in \zeta^{-1}(0')} \lambda_{\mathfrak{C}}(t) = \lambda_{\mathfrak{C}}(0) \leq \lambda_{\mathfrak{C}}(\imath)$,

for all $\imath \in \Upsilon$, which implies that

$\mu_B(0') \geq \sup_{t\in\zeta^{-1}(\imath')} \mu_{\mathfrak{C}}(t) = \mu_B(\imath')$,

$\nu_B(0') \leq \inf_{t\in\zeta^{-1}(\imath')} \nu_{\mathfrak{C}}(t) = \nu_B(\imath')$,

$\lambda_B(0') \leq \inf_{t\in\zeta^{-1}(\imath')} \lambda_{\mathfrak{C}}(t) = \lambda_B(\imath'),\ \forall \imath' \in Y$. For any $\imath',\ \jmath',\ \ell' \in Y$, let $\imath_0 \in \zeta^{-1}(\imath'),\ \jmath_0 \in \zeta^{-1}(\jmath'),\ \ell_0 \in \zeta^{-1}(\ell')$ be such that

$$\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) = \sup_{t\in\zeta^{-1}(\ell_0*((\imath_0*\jmath_0)*\imath_0))} \mu_{\mathfrak{C}}(t),$$

$$\mu_{\mathfrak{C}}(\ell_0) = \sup_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)$$

and

$$\begin{aligned}\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) &= \mu_B(\zeta(\ell_0 * (\imath_0 * \jmath_0) * \imath_0)) \\ &= \mu_B(\ell' * ((\imath' * \jmath') * \imath')) \\ &= \sup_{(\ell_0*((\imath_0*\jmath_0)*\imath_0))\in\zeta^{-1}(\ell'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0) \\ &= \sup_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t).\end{aligned}$$

Then

$$\begin{aligned}\mu_B((\imath' * \jmath') * \imath') &= \sup_{t\in\zeta^{-1}((\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t) \\ &= \mu_{\mathfrak{C}}((\imath_0 * \jmath_0) * \imath_0) \\ &\geq \min\{\mu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0), \mu_{\mathfrak{C}}(\ell_0)\} \\ &= \min\{\sup_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t), \sup_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)\} \\ &= \min\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}.\end{aligned}$$

$$\nu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) = \inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \nu_{\mathfrak{C}}(t)$$

$$\nu_{\mathfrak{C}}(\ell_0) = \inf_{t\in\zeta^{-1}(\ell')} \nu_{\mathfrak{C}}(t)$$

and

$$\begin{aligned}\nu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0) &= \nu_B(\zeta(\ell_0 * (\imath_0 * \jmath_0) * \imath_0)) \\ &= \nu_B(\ell' * (\imath' * \jmath') * \imath') \\ &= \inf_{\ell_0*((\imath_0*\jmath_0)*\imath_0)\in\zeta^{-1}(\ell'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) \\ &= \inf_{t\in\zeta^{-1}(\ell'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(t).\end{aligned}$$

Then

$$\begin{aligned}\mu_B((\imath' * \jmath') * \imath') &= \inf_{t\in\zeta^{-1}((\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t)\\ &= \mu_{\mathfrak{C}}((\imath_0 * \jmath_0) * \imath_0)\\ &\leq \max\{\mu_{\mathfrak{C}}(\ell_0 * (\imath_0 * \jmath_0) * \imath_0), \mu_{\mathfrak{C}}(\ell_0)\}\\ &= \min\{\inf_{t\in\zeta^{-1}(z'*((\imath'*\jmath')*\imath'))} \mu_{\mathfrak{C}}(t), \inf_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)\}\\ &= \min\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}.\end{aligned}$$

$$\lambda_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) = \inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \lambda_{\mathfrak{C}}(t)$$

$$\lambda_{\mathfrak{C}}(\ell_0) = \inf_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)$$

and

$$\begin{aligned}\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)) &= \mu_B(\zeta(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)))\\ &= \lambda_B(\ell' * ((\imath' * \jmath') * \imath'))\\ &= \inf_{(\ell_0*((\imath_0*\jmath_0)*\imath_0))\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0))\\ &= \inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t).\end{aligned}$$

Then

$$\begin{aligned}\mu_B((\imath' * \jmath') * \imath') &= \inf_{t\in\zeta^{-1}((\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t)\\ &\geq \max\{\mu_{\mathfrak{C}}(\ell_0 * ((\imath_0 * \jmath_0) * \imath_0)), \mu_{\mathfrak{C}}(\ell_0)\}\\ &= \max\{\inf_{t\in\zeta^{-1}(\ell'*(\imath'*\jmath')*\imath')} \mu_{\mathfrak{C}}(t), \inf_{t\in\zeta^{-1}(\ell')} \mu_{\mathfrak{C}}(t)\}\\ &= \max\{\mu_B(\ell' * (\imath' * \jmath') * \imath'), \mu_B(\ell')\}.\end{aligned}$$

Hence $B = (\jmath, \mu_B, \nu_B, \lambda_B)$ is a $PNImpI$ of Y

4. **Cartesian product of Pythagorean neutrosophic implicative ideal**

Definition 4.1. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be PNS of a set Υ, the Cartesian product of $\mu_{\mathfrak{C}} \times \mu_B$, $\nu_{\mathfrak{C}} \times \nu_B$ and $\lambda_{\mathfrak{C}} \times \lambda_B$ is defined by

$$(\mu_{\mathfrak{C}} \times \mu_B)(\imath, \jmath) = \min\{\mu_{\mathfrak{C}}(\imath), \mu_B(\jmath)\}$$

$$(\nu_{\mathfrak{C}} \times \nu_B)(\imath, \jmath) = \max\{\nu_{\mathfrak{C}}(\imath), \nu_B(\jmath)\}$$

$$(\lambda_{\mathfrak{C}} \times \lambda_B)(\imath, \jmath) = \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_B(\jmath)\} \ \forall \imath, \jmath \in \Upsilon.$$

Definition 4.2. If $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ is a PNS of a set Υ, the strongest PN relation on Υ, that is, a PN relation on $\mathfrak{C}$ is $\mu^f_{\mathfrak{C}}$ given by

$$\begin{aligned}
\mu^f_{\mathfrak{C}}(\imath, \jmath) &= \min\{\mu^f_{\mathfrak{C}}(\imath), \mu^f_{\mathfrak{C}}(\jmath)\}\\
\nu^f_{\mathfrak{C}}(\imath, \jmath) &= \max\{\nu^f_{\mathfrak{C}}(\imath), \nu^f_{\mathfrak{C}}(\jmath)\}\\
\lambda^f_{\mathfrak{C}}(\imath, \jmath) &= \max\{\lambda^f_{\mathfrak{C}}(\imath), \lambda^f_{\mathfrak{C}}(\jmath)\} \ \forall \imath, \jmath \in \Upsilon.
\end{aligned}$$

Proposition 4.3. For a given PNS $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ of KU-algebra Υ, let $(\mu^f_{\mathfrak{C}}, \nu^f_{\mathfrak{C}}, \lambda^f_{\mathfrak{C}})$ is a $PNImpI$ of $\Upsilon \times \Upsilon$, then

$$\begin{aligned}
\mu^f_{\mathfrak{C}}(0) &\geq \mu^f_{\mathfrak{C}}(\imath),\\
\nu^f_{\mathfrak{C}}(0) &\leq \nu^f_{\mathfrak{C}}(\imath),\\
\lambda^f_{\mathfrak{C}}(0) &\leq \lambda^f_{\mathfrak{C}}(\imath) \ \forall \imath \in \Upsilon.
\end{aligned}$$

Proof. Since, $\mu^f_{\mathfrak{C}},\ \nu^f_{\mathfrak{C}},\ \lambda^f_{\mathfrak{C}}$ are $PNImpI$ of $\Upsilon \times \Upsilon$, it follows from (PN_1) that

$$\begin{aligned}
\mu^f_{\mathfrak{C}}(\Upsilon, x) &= \min\{\mu^f_{\mathfrak{C}}(\imath), \mu^f_{\mathfrak{C}}(\imath)\} \leq (0,0)\\
&= \min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(0)\}\\
\nu^f_{\mathfrak{C}}(\Upsilon, x) &= \max\{\nu^f_{\mathfrak{C}}(\imath), \nu^f_{\mathfrak{C}}(\imath)\} = (0,0)\\
&\leq \max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(0)\}\\
\lambda^f_{\mathfrak{C}}(\Upsilon, x) &= \max\{\lambda^f_{\mathfrak{C}}(\imath), \lambda^f_{\mathfrak{C}}(\imath)\} = (0,0)\\
&\leq \max\{\lambda_{\mathfrak{C}}(0), \lambda_{\mathfrak{C}}(0)\},\ \forall \imath \in \Upsilon,
\end{aligned}$$

where $(0,0) \in \Upsilon \times \Upsilon$, then

$$\begin{aligned}
\mu_{\mathfrak{C}}(0) &\geq \mu_{\mathfrak{C}}(\imath)\\
\nu_{\mathfrak{C}}(0) &\leq \nu_{\mathfrak{C}}(\imath)\\
\lambda_{\mathfrak{C}}(0) &\leq \lambda_{\mathfrak{C}}(\imath).
\end{aligned}$$

Remark 4.4. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be KU-algebras, we define $*$ on $\mathfrak{C} \times \mathfrak{C}$ for every $(\imath, \jmath),\ (u, v) \in \mathfrak{C} \times B$, $(\imath, \jmath) * (u, v) = (\imath * u, \jmath * v)$, then clearly $(\imath * \jmath, *, (0,0))$ is a KU-algebra.

Theorem 4.5. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be a $PNImpI$'s of KU-algebra X, $\Upsilon \times B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

Proof. For any $(\imath, \jmath) \in \Upsilon \times \Upsilon$, we have,

$$\begin{aligned}(\mu_{\mathfrak{C}} \times \mu_B)(0,0) &= \min\{\mu_{\mathfrak{C}}(0), \mu_B(0)\} \\ &\geq \min\{\mu_{\mathfrak{C}}(\imath), \mu_B(\imath)\} \\ &= (\mu_{\mathfrak{C}} \times \mu_B)(\Upsilon, x) \\ (\nu_{\mathfrak{C}} \times \nu_B)(0,0) &= \max\{\nu_{\mathfrak{C}}(0), \nu_B(0)\} \\ &\leq \max\{\nu_{\mathfrak{C}}(\imath), \nu_B(\imath)\} \\ &= (\nu_{\mathfrak{C}} \times \nu_B)(\Upsilon, x) \\ (\lambda_{\mathfrak{C}} \times \lambda_B)(0,0) &= \max\{\lambda_{\mathfrak{C}}(0), \lambda_B(0)\} \\ &\leq \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_B(\imath)\} \\ &= (\lambda_{\mathfrak{C}} \times \lambda_B)(\Upsilon, x).\end{aligned}$$

Now, let $(\imath_1, \imath_2),\ (\jmath_1, \jmath_2),\ (\ell_1, \ell_2) \in \Upsilon \times \Upsilon$, then,

$$\begin{aligned}&(\mu_{\mathfrak{C}} \times \mu_B)(((\imath_1 * \jmath_1) * \imath_1), ((\imath_2 * \jmath_2) * \imath_2)) \\ &= \min\{\mu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \mu_B((\imath_2 * \jmath_2) * \imath_2)\} \\ &\geq \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \mu_{\mathfrak{C}}(\ell_1)\}, \min\{\mu_B(\ell_2 * (\imath_2 * \jmath_2) * \imath_2), \mu_B(\ell_2)\}\} \\ &= \min\{\min\{\mu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \mu_B(\ell_2 * (\imath_2 * \jmath_2) * \imath_2)\}, \min\{\mu_{\mathfrak{C}}(\ell_1), \mu_B(\ell_2)\}\} \\ &= \min\{(\mu_{\mathfrak{C}} \times \mu_B)(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * (\imath_2 * \jmath_2) * \imath_2), (\mu_{\mathfrak{C}} \times \mu_B)(\ell_1, \ell_2)\}.\end{aligned}$$

Hence, $\mu_{\mathfrak{C}} \times \mu_B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

$$\begin{aligned}&(\nu_{\mathfrak{C}} \times \nu_B)(((\imath_1 * \jmath_1) * \imath_1), ((\imath_2 * \jmath_2) * \imath_2)) \\ &= \max\{\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \nu_B((\imath_2 * \jmath_2) * \imath_2)\} \\ &\leq \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \nu_{\mathfrak{C}}(\ell_1)\}, \max\{\nu_B(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu_B(\ell_2)\}\} \\ &= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \nu_B(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)\}, \max\{\nu_{\mathfrak{C}}(\ell_1), \nu_B(\ell_2)\}\} \\ &= \max\{(\nu_{\mathfrak{C}} \times \nu_B)(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * (\imath_2 * \jmath_2) * \imath_2), (\nu_{\mathfrak{C}} \times \nu_B)(\ell_1, \ell_2)\}.\end{aligned}$$

Hence, $\nu_{\mathfrak{C}} \times \nu_B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

$$\begin{aligned}&(\lambda_{\mathfrak{C}} \times \lambda_B)((\imath_1 * \jmath_1) * \imath_1, (\imath_2 * \jmath_2) * \imath_2) \\ &= \max\{\lambda_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \lambda_B((\imath_2 * \jmath_2) * \imath_2)\} \\ &\leq \max\{\max\{\lambda_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \lambda_{\mathfrak{C}}(\ell_1)\}, \max\{\lambda_B(\ell_2 * (\imath_2 * \jmath_2) * \imath_2), \lambda_B(\ell_2)\}\} \\ &= \max\{\max\{\lambda_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \lambda_B(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2))\}, \max\{\lambda_{\mathfrak{C}}(\ell_1), \lambda_B(\ell_2)\}\} \\ &= \max\{(\lambda_{\mathfrak{C}} \times \lambda_B)(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1), \ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), (\lambda_{\mathfrak{C}} \times \lambda_B)(\ell_1 \times \ell_2)\}.\end{aligned}$$

Hence, $\lambda_{\mathfrak{C}} \times \lambda_B$ is a $PNImpI$ of $\Upsilon \times \Upsilon$.

Theorem 4.6. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ and $B = (\Upsilon, \mu_B, \nu_B, \lambda_B)$ be a PNS of KU-algebra Υ, such that $\mathfrak{C} \times B$ is $PNImpI$ of $\Upsilon \times \Upsilon$, then

(i) either $\mu_{\mathfrak{C}}(\imath) \leq \mu_{\mathfrak{C}}(0)$ or $\mu_B(\imath) \leq \mu_B(0)$, $\nu_{\mathfrak{C}}(\imath) \geq \nu_{\mathfrak{C}}(0)$ or $\nu_B(\imath) \geq \nu_B(0)$, $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_{\mathfrak{C}}(0)$ or $\lambda_B(\imath) \geq \lambda_B(0)$, $\forall \imath \in \Upsilon$.

(ii) if $\mu_{\mathfrak{C}}(\imath) \leq \mu_{\mathfrak{C}}(0)$ $\forall \imath \in \Upsilon$, then either $\mu_{\mathfrak{C}}(\imath) \leq \mu_B(0)$ or $\mu_B(\imath) \leq \mu_B(0)$,
if $\nu_{\mathfrak{C}}(\imath) \geq \nu_{\mathfrak{C}}(0)$ $\forall \imath \in \Upsilon$, then either $\nu_{\mathfrak{C}}(\imath) \geq \nu_B(0)$ or $\nu_B(\imath) \geq \nu_B(0)$,
if $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_{\mathfrak{C}}(0)$ $\forall \imath \in \Upsilon$, then either $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_B(0)$ or $\lambda_B(\imath) \geq \lambda_B(0)$.

(iii) if $\mu_B(\imath) \leq \mu_B(0)$ $\forall \imath \in \Upsilon$, then either $\mu_{\mathfrak{C}}(\imath) \leq \mu_{\mathfrak{C}}(0)$ or $\mu_B(\imath) \leq \mu_{\mathfrak{C}}(0)$,
if $\nu_B \geq \nu_B(0)$ $\forall \imath \in \Upsilon$, then either $\nu_{\mathfrak{C}}(\imath) \geq \nu_{\mathfrak{C}}(0)$ or $\nu_B(\imath) \geq \nu_{\mathfrak{C}}(0)$,
$\lambda_B(\imath) \geq \lambda_B(0)$ $\forall \imath \in \Upsilon$, then either $\lambda_{\mathfrak{C}}(\imath) \geq \lambda_{\mathfrak{C}}(0)$ or $\lambda_B(\imath) \geq \lambda_{\mathfrak{C}}(0)$.

(iv) either $\mathfrak{C}$ or B is $PNImpI$ of Υ.

Proof. The proof is similar to previous Theorem 4.5.

Theorem 4.7. Let $\mathfrak{C} = (\Upsilon, \mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ be a PNS of KU-algebra Υ and let $\mu^f_{\mathfrak{C}}$, $\nu^f_{\mathfrak{C}}$, $\lambda^f_{\mathfrak{C}}$ be the strongest PN relation on Υ, then $\mathfrak{C}$ is a $PNImpI$ of Υ iff $\mu^f_{\mathfrak{C}}$, $\nu^f_{\mathfrak{C}}$, $\lambda^f_{\mathfrak{C}}$ are $PNImpI$ of $\Upsilon \times \Upsilon$.

Proof. Assume that $(\mu_{\mathfrak{C}}, \nu_{\mathfrak{C}}, \lambda_{\mathfrak{C}})$ is $PNImpI$ on Υ, we note from (PNI) that

$$\begin{aligned}
\mu^f_{\mathfrak{C}}(0,0) &= \min\{\mu_{\mathfrak{C}}(0), \mu_{\mathfrak{C}}(0)\} \\
&\geq \min\{\mu_{\mathfrak{C}}(\imath), \mu_{\mathfrak{C}}(\jmath)\} \\
&\geq \mu^f_{\mathfrak{C}}(\imath, \jmath). \\
\nu^f_{\mathfrak{C}}(0,0) &= \max\{\nu_{\mathfrak{C}}(0), \nu_{\mathfrak{C}}(0)\} \\
&\leq \max\{\nu_{\mathfrak{C}}(\imath), \nu_{\mathfrak{C}}(\jmath)\} \\
&\leq \nu^f_{\mathfrak{C}}(\imath, \jmath). \\
\lambda^f_{\mathfrak{C}}(0,0) &= \max\{\lambda_{\mathfrak{C}}(0), \lambda_{\mathfrak{C}}(0)\} \\
&\leq \max\{\lambda_{\mathfrak{C}}(\imath), \lambda_{\mathfrak{C}}(\jmath)\} \\
&\leq \lambda^f_{\mathfrak{C}}(\imath, \jmath), \ \forall (\imath, \jmath) \in \Upsilon \times \Upsilon.
\end{aligned}$$

Now, for any $(\imath_1,\imath_2),(\jmath_1,\jmath_2),(\ell_1,\ell_2)\in\Upsilon\times\Upsilon$, we have from (PN_2)

$$\begin{aligned}
&\mu^f_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1,(\imath_2*\jmath_2)*\imath_2)\\
&=\min\{\mu_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1),\mu_{\mathfrak{C}}((\imath_2*\jmath_2)*\imath_2)\}\\
&\geq\min\{\min\{\mu_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1)),\mu_{\mathfrak{C}}(\ell_1)\},\min\{\mu_{\mathfrak{C}}(\ell_2*((\imath_2*\jmath_2)*\imath_2)),\mu_{\mathfrak{C}}(\ell_2)\}\}\\
&=\min\{\min\{\mu_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1)),\mu_{\mathfrak{C}}(\ell_2*((\imath_2*\jmath_2)*\imath_2))\},\min\{\mu_{\mathfrak{C}}(\ell_1),\mu_{\mathfrak{C}}(\ell_2)\}\}\\
&=\min\{(\mu_{\mathfrak{C}}\times\mu_{\mathfrak{C}})(\ell_1*((\imath_1*\jmath_1)*\imath_1)),(\ell_2*((\imath_2*\jmath_2)*\imath_2)),(\mu_{\mathfrak{C}}\times\mu_{\mathfrak{C}})(\ell_1,\ell_2)\}\\
&\nu^f_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1,(\imath_2*\jmath_2)*\imath_2)\\
&=\max\{\nu_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1),\nu_{\mathfrak{C}}((\imath_2*\jmath_2)*\imath_2)\}\\
&\leq\max\{\max\{\nu_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1)),\nu_{\mathfrak{C}}(\ell_1)\},\max\{\nu_{\mathfrak{C}}(\ell_2*((\imath_2*\jmath_2)*\imath_2)),\nu_{\mathfrak{C}}(\ell_2)\}\}\\
&=\max\{\max\{\nu_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1)),\nu_{\mathfrak{C}}(\ell_2*((\imath_2*\jmath_2)*\imath_2)),\max\{\nu_{\mathfrak{C}}(\ell_1),\nu_{\mathfrak{C}}(\ell_2)\}\}\\
&=\max\{(\nu_{\mathfrak{C}}\times\nu_{\mathfrak{C}})(\ell_1*((\imath_1*\jmath_1)*\imath_1)),(\ell_2*((\imath_2*\jmath_2)*\imath_2)),(\nu_{\mathfrak{C}}\times\nu_{\mathfrak{C}})(\ell_1,\ell_2)\}
\end{aligned}$$

Similarly, $\lambda^f_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1,(\imath_2*\jmath_2)*\imath_2)\leq\max\{(\lambda_{\mathfrak{C}}\times\lambda_{\mathfrak{C}})(\ell_1*((\imath_1*\jmath_1)*\imath_1)),(\ell_2*((\imath_2*\jmath_2)*\imath_2)),(\lambda_{\mathfrak{C}}\times\lambda_{\mathfrak{C}})(\ell_1,\ell_2)\}$ Hence $(\mu^f_{\mathfrak{C}},\nu^f_{\mathfrak{C}},\lambda^f_{\mathfrak{C}})$ is $PNImpI$ of $\Upsilon\times\Upsilon$.

Conversely, $\forall(\imath,\jmath)\in\Upsilon\times\Upsilon$, we have

$\min\{\mu_{\mathfrak{C}}(0),\mu_{\mathfrak{C}}(0)\}=\mu^f_{\mathfrak{C}}(\imath,\jmath)=\min\{\mu_{\mathfrak{C}}(\imath),\mu_{\mathfrak{C}}(\jmath)\}$.

It follows that $\mu_{\mathfrak{C}}(0)\geq\mu_{\mathfrak{C}}(\imath)$, $\forall\imath\in\Upsilon$

$\max\{\nu_{\mathfrak{C}}(0),\nu_{\mathfrak{C}}(0)\}=\nu^f_{\mathfrak{C}}(\imath,\jmath)=\max\{\nu_{\mathfrak{C}}(\imath),\nu_{\mathfrak{C}}(\jmath)\}$.

It follows that $\nu_{\mathfrak{C}}(0)\leq\nu_{\mathfrak{C}}(\imath)$, $\forall\imath\in\Upsilon$

$\max\{\lambda_{\mathfrak{C}}(0),\lambda_{\mathfrak{C}}(0)\}=\lambda^f_{\mathfrak{C}}(\imath,\jmath)=\max\{\lambda_{\mathfrak{C}}(\imath),\lambda_{\mathfrak{C}}(\jmath)\}$.

It follows that $\lambda_{\mathfrak{C}}(0)\leq\lambda_{\mathfrak{C}}(\imath)$ $\forall\imath\in\Upsilon$ which proves (PN_1) Now, let $(\imath_1,\imath_2),\ (\jmath_1,\jmath_2),\ (\ell_1,\ell_2)\in\Upsilon\times\Upsilon$, then

$$\begin{aligned}
&\min\{\mu_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1),\mu_{\mathfrak{C}}((\imath_2*\jmath_2)*\imath_2)\}\\
&=\mu^f_{\mathfrak{C}}((\imath_1*\jmath_1)*\imath_1,(\imath_2*\jmath_2)*\imath_2)\\
&\geq\min\{\mu^f_{\mathfrak{C}}((\ell_1,\ell_2)*((\imath_1,\imath_2)*(\jmath_1,\jmath_2))*(\imath_1,\imath_2)),\mu^f_{\mathfrak{C}}(\ell_1,\ell_2)\}\\
&=\min\{\mu^f_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1),\ell_2*(\imath_2*\jmath_2)*\imath_2),\mu^f_{\mathfrak{C}}(\ell_1,\ell_2)\}\\
&=\min\{\min\{\mu_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1)),\mu_{\mathfrak{C}}(\ell_2*((\imath_2*\jmath_2)*\imath_2))\},\min\{\mu_{\mathfrak{C}}(\ell_1),\mu_{\mathfrak{C}}(\ell_2)\}\}\\
&=\min\{\min\{\mu_{\mathfrak{C}}(\ell_1*((\imath_1*\jmath_1)*\imath_1)),\mu_{\mathfrak{C}}(\ell_1)\},min\{\mu_{\mathfrak{C}}(\ell_2*((\imath_2*\jmath_2)*\imath_2)),\mu_{\mathfrak{C}}(\ell_2)\}\}.
\end{aligned}$$

In particular, if we take $\imath_2 = \jmath_2 = \ell_2 = 0$, then $\mu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1) \geq \min\{\mu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \mu_{\mathfrak{C}}(\ell_1)\}$

$$\begin{aligned}
&\max\{\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\} \\
&= \nu^f_{\mathfrak{C}}(((\imath_1 * \jmath_1) * \imath_1), ((\imath_2 * \jmath_2) * \imath_2)) \\
&\leq \max\{\nu^f_{\mathfrak{C}}((\ell_1, \ell_2) * ((\imath_1 * \imath_2) * (\jmath_1 * \jmath_2)) * (\imath_1 * \imath_2)), \nu^f_{\mathfrak{C}}(\ell_1, \ell_2)\} \\
&= \max\{\nu^f_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), (\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu^f_{\mathfrak{C}}(\ell_1, \ell_2)\} \\
&= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}(\ell_2 * (\imath_2 * \jmath_2) * \imath_2)\}, \max\{\nu_{\mathfrak{C}}(\ell_1), \nu_{\mathfrak{C}}(\ell_2)\}\} \\
&= \max\{\max\{\nu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}(\ell_1)\}, max\{\nu_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \nu_{\mathfrak{C}}(\ell_2)\}\}.
\end{aligned}$$

In particular, if we take $\imath_2 = \jmath_2 = \ell_2 = 0$, then $\nu_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1) \leq \max\{\nu_{\mathfrak{C}}(\ell_1 * (\imath_1 * \jmath_1) * \imath_1), \nu_{\mathfrak{C}}(\ell_1)\}$. Similarly, $\max\{\lambda_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1), \lambda_{\mathfrak{C}}((\imath_2 * \jmath_2) * \imath_2)\} \leq \max\{\max\{\lambda_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \lambda_{\mathfrak{C}}(\ell_1)\}, \max\{\lambda_{\mathfrak{C}}(\ell_2 * ((\imath_2 * \jmath_2) * \imath_2)), \lambda_{\mathfrak{C}}(\ell_2)\}$.

In particular, if we take $\imath_2 = \jmath_2 = \ell_2 = 0$, then $\lambda_{\mathfrak{C}}((\imath_1 * \jmath_1) * \imath_1) \leq \max\{\lambda_{\mathfrak{C}}(\ell_1 * ((\imath_1 * \jmath_1) * \imath_1)), \lambda_{\mathfrak{C}}(\ell_1)\}$.

5. Conclusions

We have investigated Pythagorean neutrosophic $ImpI$s in KU-algebras and discussed several related results. In particular, we defined the image and pre-image of Pythagorean neutrosophic $ImpI$s under homomorphisms of KU-algebras and studied the conditions under which these images remain Pythagorean neutrosophic $ImpI$s. Moreover, we established the product of Pythagorean neutrosophic $ImpI$s as a product Pythagorean neutrosophic $ImpI$. As a direction for future work, we aim to explore the foldedness of other classes of Pythagorean neutrosophic ideals with special properties, such as bipolar intuitionistic (interval-valued) fuzzy n-fold $ImpI$s in certain algebraic structures.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: April 5, 2025. Accepted: Sep 9, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

Fractional Dynamics and Fixed Point Theorems in Neutrosophic MR-Metric Spaces with Applications to Network Systems

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Abstract. This paper introduces and analyzes the structure of **Neutrosophic MR-Metric Spaces** (NMR-MS) and their graph-based variants, termed **Neutrosophic Graph MR-Metric Spaces** (NGMR-MS). We extend the concept of MR-metrics by incorporating neutrosophic logic, which simultaneously handles truth, falsity, and indeterminacy in multi-dimensional metric settings. We establish several fundamental results, including fractional derivative estimates, fixed point theorems, and continuity conditions within these spaces. Applications to fractional-order dynamical systems on networks—such as neural dynamics, epidemiological spread, and multi-agent systems—are developed, demonstrating the utility of the proposed framework. Numerical algorithms and error estimates are provided, along with a comprehensive computational complexity analysis. The results generalize and unify several existing theories in fixed point theory, fractional calculus, and neutrosophic analysis.

Keywords: MR-metric spaces, neutrosophic logic, fractional derivatives, fixed point theorems, graph geodesics, dynamical networks, uncertainty modeling.

1. Introduction

The study of generalized metric spaces has been a highly active area of research in mathematical analysis, with significant implications in fixed point theory, functional analysis, and applied mathematics. Among these, *b-metric spaces* [4], Ω_b*-distance mappings* [3,6], and *simulation functions* [10] have provided rich frameworks for extending classical results. More recently, *MR-metric spaces* were introduced in [2] as a multi-dimensional generalization of standard metrics, enabling the measurement of ternary relationships among points and facilitating applications in graph theory and fixed point theory [1,7,8,11,13–18,22].

Meanwhile, fractional calculus has gained prominence for modeling systems with memory and non-local effects [5,27,28]. The fusion of fractional operators with metric space structures offers a powerful tool for analyzing dynamical systems on networks [23,24]. Moreover, the incorporation of neutrosophic logic—which generalizes fuzzy and intuitionistic logic by accounting for truth, falsity, and indeterminacy—has allowed for more robust uncertainty quantification in complex systems [23,29].

In this work, we introduce *Neutrosophic MR-Metric Spaces* (NMR-MS) and their graph-based counterparts, *Neutrosophic Graph MR-Metric Spaces* (NGMR-MS). We define these structures rigorously and establish fundamental properties, including fractional differentiability, continuity, and contraction conditions. Our main results include:

- A fractional derivative estimate in graph-geodesic MR-metric spaces (Theorem 2.1),
- A fixed point theorem for mappings satisfying fractional contraction conditions (Theorem 2.2),
- A continuity result for fractional derivatives on graph paths (Theorem 2.3),
- A comprehensive fixed point and continuity theorem in NGMR-MS (Theorem 2.4).

We also develop an application section focused on fractional dynamics on networks, including existence, uniqueness, and stability results (Theorems 3.5, 3.8, 3.9), along with numerical implementations and case studies in neural networks, epidemiology, and multi-agent systems.

This work builds upon earlier contributions in fixed point theory [1,2,4,11,12,19–26,29, 30], fractional calculus [5,27,28], and neutrosophic analysis [23,29], unifying them into a coherent framework applicable to a wide range of network-based dynamical systems.

1.1. *Contributions*

Our main contributions in this work are as follows:

- Introduction of Neutrosophic MR-Metric Spaces (NMR-MS) and Neutrosophic Graph MR-Metric Spaces (NGMR-MS), combining multi-dimensional metrics with neutrosophic logic.
- Establishment of fractional derivative estimates, fixed point theorems, and continuity results in these spaces.

- Development of applications to fractional-order network dynamics in neural networks, epidemiology, and multi-agent systems.
- Provision of numerical algorithms, error estimates, and computational complexity analysis.
- Unification of concepts from fixed point theory, fractional calculus, and neutrosophic analysis into a single coherent framework.

1.2. *Importance of Neutrosophic Logic in Our Work*

The incorporation of neutrosophic logic is crucial for handling real-world systems where uncertainty, indeterminacy, and partial truth are inherent. Unlike classical fuzzy sets, neutrosophic sets simultaneously account for truth ($\mathcal{T}$), falsity ($\mathcal{F}$), and indeterminacy ($\mathcal{I}$), providing a more flexible and expressive framework for modeling complex network dynamics. In our context, neutrosophic membership functions quantify the degree of connection, disconnection, and uncertainty between nodes in a network, making the model especially suitable for applications like social networks, biological systems, and multi-agent coordination, where relationships are often imperfectly known or evolving.

1.3. *Preliminary Definitions*

The following fundamental definitions will be used throughout this paper:

- **Fractional Derivative** (Definition 1.1): A generalized derivative operator for non-integer orders.
- **MR-Metric Space** (Definition 1.2): A multi-dimensional metric space measuring ternary relationships.
- **Neutrosophic MR-Metric Space** (Definition 1.3): An MR-metric space enhanced with neutrosophic logic.
- **Neutrosophic Graph MR-Metric Space** (Definition 1.4): A graph-based neutrosophic MR-metric space.

The paper is structured as follows: Section 1 contains preliminary definitions and examples. Section 2 presents the main theoretical results and Section 3 applies the framework to fractional network dynamics.

Definition 1.1. [27] [Fractional Derivative] Let $f : [0, \infty) \to \mathbb{R}$ be a function and $t > 0$. The fractional derivative of f of order α is defined by:

$$A^{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon},$$

where $\alpha \in (0,1)$ and $g : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying:

$$g(0) = 1,$$
$$g'(0) = 1.$$

Definition 1.2. [2] *Consider a non-empty set $\mathbb{X} \neq \emptyset$ and a real number $\mathbb{R} > 1$. A function*

$$M : \mathbb{X} \times \mathbb{X} \times \mathbb{X} \to [0, \infty)$$

*is termed an **MR-metric** if it satisfies the following conditions for all $v, \xi, s, \ell_1 \in \mathbb{X}$:*

- $M(v, \xi, s) \geq 0$.
- $M(v, \xi, s) = 0$ *if and only if* $v = \xi = s$.
- $M(v, \xi, s)$ *remains invariant under any permutation* $p(v, \xi, s)$, *i.e.,* $M(v, \xi, s) = M(p(v, \xi, s))$.
- *The following inequality holds:*

$$M(v, \xi, s) \leq \mathbb{R}\left[M(v, \xi, \ell_1) + M(v, \ell_1, s) + M(\ell_1, \xi, s)\right].$$

A structure $(\mathbb{X}, M)$ that adheres to these properties is defined as an MR-metric space.

Definition 1.3. [31] [Neutrosophic MR-Metric Space (NMR-MS)] A 9-tuple $(\mathcal{Z}, M, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R, \star)$ is called a **Neutrosophic MR-Metric Space** if:

(1) $\mathcal{Z}$ is a non-empty set.

(2) $M : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is an MR-metric satisfying:

(M1) $M(\upsilon, \xi, \Im) \geq 0$,

(M2) $M(\upsilon, \xi, \Im) = 0 \iff \upsilon = \xi = \Im$,

(M3) Symmetry under permutations,

(M4) $M(\upsilon, \xi, \Im) \leq R\left[M(\upsilon, \xi, \ell) \star M(\upsilon, \ell, \Im) \star M(\ell, \xi, \Im)\right]$, $R > 1$.

(3) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are neutrosophic functions satisfying:

(N1) $\mathcal{T}(\upsilon, \xi, \gamma) = 1 \iff \upsilon = \xi$ (Truth-Identity),

(N2) $\mathcal{T}(\upsilon, \xi, \gamma) = \mathcal{T}(\xi, \upsilon, \gamma)$ (Symmetry),

(N3) $\mathcal{T}(\upsilon, \xi, \gamma) \bullet \mathcal{T}(\xi, \Im, \rho) \leq \mathcal{T}(\upsilon, \Im, \gamma + \rho)$ (Triangle Inequality),

(N4) $\lim_{\gamma \to \infty} \mathcal{T}(\upsilon, \xi, \gamma) = 1$ (Asymptotic Behavior).

(4) $\bullet$ (t-norm) and $\diamond$ (t-conorm) are continuous operators generalizing fuzzy logic.

(5) $\star$ is a binary operation generalizing addition (e.g., weighted sum).

Definition 1.4. [Neutrosophic Graph MR-Metric Space (NGMR-MS)] A 10-tuple $(\mathcal{Z}, V, E, MR, \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R)$ is called a **Neutrosophic Graph MR-Metric Space** if:

(1) $G = (V, E)$ is a connected, weighted graph with vertex set V and edge set E.

(2) $\mathcal{Z} = V$ is the non-empty set of vertices.

(3) $MR : \mathcal{Z} \times \mathcal{Z} \times \mathcal{Z} \to [0, \infty)$ is a graph-geodesic MR-metric defined by:

$$MR\ (u, v, w) = \max\{d_G(u, v), d_G(u, w), d_G(v, w)\},$$

where d_G is the geodesic distance on G.

(4) $\mathcal{T}, \mathcal{F}, \mathcal{I} : \mathcal{Z} \times \mathcal{Z} \times (0, \infty) \to [0, 1]$ are neutrosophic membership functions defined, for $k_T, k_F > 0$, as:

$$\mathcal{T}(u, v, \gamma) = e^{-k_T \cdot d_G(u,v) \cdot \gamma},$$
$$\mathcal{F}(u, v, \gamma) = 1 - e^{-k_F \cdot d_G(u,v) \cdot \gamma},$$
$$\mathcal{I}(u, v, \gamma) = \frac{1}{2}\left[\mathcal{T}(u, v, \gamma) + \mathcal{F}(u, v, \gamma)\right].$$

(5) $\bullet$ is a continuous t-norm, $\diamond$ is a continuous t-conorm.

(6) $R > 1$ is a constant.

Example 1.5. Consider modeling a simple social network of four researchers and their collaboration dynamics.

(1) **Graph Construction:** Let the vertex set be $V = \{A, B, C, D\}$, representing four researchers. Define the edge set E and weights w based on their collaboration intensity:

$$E = \{(A, B), (A, C), (B, C), (C, D)\}$$
$$w(A, B) = 3, \quad w(A, C) = 5, \quad w(B, C) = 1, \quad w(C, D) = 4.$$

This graph $G = (V, E, w)$ is connected. The geodesic distances d_G are calculated as the minimum path weight between nodes. For example:

$$d_G(A, B) = 3 \quad \text{(direct path)}$$
$$d_G(A, D) = d_G(A, C) + d_G(C, D) = 5 + 4 = 9 \quad \text{(shortest path via C)}$$
$$d_G(B, D) = d_G(B, C) + d_G(C, D) = 1 + 4 = 5$$

(2) **MR-Metric Calculation:** Let's compute the multi-point distance between researchers A, B, and D.

$$MR\ (A, B, D) = \max\{d_G(A, B), d_G(A, D), d_G(B, D)\}$$
$$= \max\{3, 9, 5\} = 9.$$

This value of 9 can be interpreted as the diameter of the smallest "collaborative circle" that would contain all three researchers A, B, and D; in this case, the circle is defined by the most distant pair (A, D).

(3) **Neutrosophic Membership Evaluation:** Let us choose scaling factors $k_T = 0.2$ and $k_F = 0.3$. We evaluate the neutrosophic memberships for the pair (A, D) at a parameter value $\gamma = 0.5$.

$$\mathcal{T}(A, D, 0.5) = \exp(-0.2 \cdot 9 \cdot 0.5) = \exp(-0.9) \approx 0.406$$
$$\mathcal{F}(A, D, 0.5) = 1 - \exp(-0.3 \cdot 9 \cdot 0.5) = 1 - \exp(-1.35) \approx 1 - 0.259 = 0.741$$
$$\mathcal{I}(A, D, 0.5) = \frac{1}{2}(0.406 + 0.741) \approx 0.573$$

Interpretation: For $\gamma = 0.5$ (e.g., a medium-term forecast or a medium confidence level), the statement "Researchers A and D are closely connected" is:

- True to a degree of about 0.406,
- False to a degree of about 0.741,
- Indeterminate to a degree of about 0.573.

The high falsity value reflects their large geodesic distance ($d_G(A, D) = 9$). The significant indeterminacy value captures the uncertainty inherent in a distant connection in a network (e.g., potential for future collaboration through intermediaries).

Now, let's compare this to a close pair, (B, C) with $d_G(B, C) = 1$, for the same γ:

$$\mathcal{T}(B, C, 0.5) = \exp(-0.2 \cdot 1 \cdot 0.5) = \exp(-0.1) \approx 0.904$$
$$\mathcal{F}(B, C, 0.5) = 1 - \exp(-0.3 \cdot 1 \cdot 0.5) = 1 - \exp(-0.15) \approx 0.139$$
$$\mathcal{I}(B, C, 0.5) = \frac{1}{2}(0.904 + 0.139) \approx 0.521$$

As expected, for the directly connected pair (B, C), the truth membership is high (0.904) and the falsity membership is low (0.139).

(4) **Operator and Constant Selection:** For this example, we can choose:

- **T-Norm ($\cdot$):** The product t-norm: $a \cdot b = a \cdot b$.
- **T-Conorm ($\diamond$):** The probabilistic sum: $a \diamond b = a + b - a \cdot b$.
- **MR-Constant:** $R = 2$.

One can then verify the tetrahedral inequality for selected vertices using these operators.

Thus, the tuple $(V, G, MR\ , \mathcal{T}, \mathcal{F}, \mathcal{I}, \cdot, + - \cdot, 2, w)$ constitutes a specific instance of a Neutrosophic Graph MR-Metric Space. This model allows us to analyze not just *who* is connected, but to what *degree of certainty, uncertainty, and falsity* those connections hold under a given parameter γ, and to measure complex multi-node relationships via MR .

2. Main Results

Building upon the foundations laid in the previous section, we now present our central theoretical contributions. These include fractional derivative estimates in MR-metric spaces, fixed point theorems under fractional contraction conditions, and continuity results for fractional derivatives on graph paths. The following theorems generalize and extend existing results in [2, 11, 22, 23, 29] and provide a rigorous basis for the applications discussed in Section 3.

Theorem 2.1. *(Fractional Derivative on Graph-Geodesic MR-Metric Spaces) Let $G = (V, E)$ be a connected graph with vertex set V and edge set E. Define the graph-geodesic MR-metric $M_G : V \times V \times V \to [0, \infty)$ by:*

$$M_G(u, v, w) = \max \{d(u, v), d(u, w), d(v, w)\},$$

where d is the geodesic distance on G. Let $f : [0, \infty) \to V$ be a function such that $f(t) \in V$ for all t, and suppose f is α-differentiable at $t > 0$ in the sense of Definition 1.1. Then there exists a constant $C > 0$ and $\beta \in (0, 1)$ such that:

$$M_G \left(A^\alpha(f)(t), f(t), f(tg(\epsilon t^{-\alpha}))\right) \leq C\epsilon^\beta.$$

Proof. Since f is α-differentiable at $t > 0$, by Definition 1.1 we have:

$$A^\alpha(f)(t) = \lim_{\epsilon \to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon}.$$

This implies the following asymptotic expansion:

$$f(tg(\epsilon t^{-\alpha})) = f(t) + \epsilon A^\alpha(f)(t) + o(\epsilon),$$

where $\lim_{\epsilon \to 0} o(\epsilon)/\epsilon = 0$.

Now consider the MR-metric expression:

$$M_G \left(A^\alpha(f)(t), f(t), f(tg(\epsilon t^{-\alpha}))\right) = \max \left\{d(A^\alpha(f)(t), f(t)), d(A^\alpha(f)(t), f(tg(\epsilon t^{-\alpha}))), d(f(t), f(tg(\epsilon t^{-\alpha})))\right\}.$$

We estimate each term:

1. **First term:** $d(A^\alpha(f)(t), f(t))$. Since $A^\alpha(f)(t)$ is the fractional derivative, it is a limit of difference quotients. By the graph structure and the fact that f maps into V, the distance $d(A^\alpha(f)(t), f(t))$ is bounded by the magnitude of the derivative. Specifically, there exists $K_1 > 0$ such that:

$$d(A^\alpha(f)(t), f(t)) \leq K_1|\epsilon| + o(\epsilon).$$

2. **Second term:** $d(A^\alpha(f)(t), f(tg(\epsilon t^{-\alpha})))$. Using the expansion:

$$f(tg(\epsilon t^{-\alpha})) = f(t) + \epsilon A^\alpha(f)(t) + o(\epsilon),$$

we have:

$$d(A^{\alpha}(f)(t), f(tg(\epsilon t^{-\alpha}))) = d(A^{\alpha}(f)(t), f(t) + \epsilon A^{\alpha}(f)(t) + o(\epsilon)).$$

By the graph-geodesic property and the triangle inequality, we obtain:

$$d(A^{\alpha}(f)(t), f(t) + \epsilon A^{\alpha}(f)(t) + o(\epsilon)) \le d(A^{\alpha}(f)(t), f(t)) + |\epsilon| d(0, A^{\alpha}(f)(t)) + |o(\epsilon)|.$$

Hence, there exists $K_2 > 0$ such that:

$$d(A^{\alpha}(f)(t), f(tg(\epsilon t^{-\alpha}))) \le K_2|\epsilon| + o(\epsilon).$$

3. **Third term:** $d(f(t), f(tg(\epsilon t^{-\alpha})))$. From the expansion:

$$f(tg(\epsilon t^{-\alpha})) - f(t) = \epsilon A^{\alpha}(f)(t) + o(\epsilon),$$

so:

$$d(f(t), f(tg(\epsilon t^{-\alpha}))) \le |\epsilon| d(0, A^{\alpha}(f)(t)) + |o(\epsilon)| \le K_3|\epsilon| + o(\epsilon).$$

Combining these, we get:

$$M_G(\cdot) \le \max\{K_1|\epsilon| + o(\epsilon), K_2|\epsilon| + o(\epsilon), K_3|\epsilon| + o(\epsilon)\} \le K|\epsilon| + o(\epsilon),$$

for some $K > 0$.

Since $o(\epsilon) \le K'\epsilon^{1+\gamma}$ for some $K' > 0$, $\gamma > 0$, we have:

$$M_G(\cdot) \le K\epsilon + K'\epsilon^{1+\gamma}.$$

Let $\beta = \min(1, 1+\gamma) \in (0, 1)$. Then:

$$M_G(\cdot) \le C\epsilon^{\beta},$$

where $C = K + K'$.

This completes the proof. □

Theorem 2.2. *(Fixed point Theorem for geometric MR-metric spaces) Let $(\mathbb{X}, M)$ be a complete MR-metric space where $\mathbb{X}$ is a Riemannian manifold and M is defined via the Riemannian distance d:*

$$M(x, y, z) = \frac{d(x, y) + d(x, z) + d(y, z)}{3}.$$

Let $T : \mathbb{X} \to \mathbb{X}$ be a mapping such that:

(1) $A^{\alpha}(Tx)$ exists for all $x \in \mathbb{X}$,

(2) There exists $k \in (0, 1)$ such that for all $x, y, z \in \mathbb{X}$:

$$M(A^{\alpha}(Tx), A^{\alpha}(Ty), Tz) \le kM(x, y, z).$$

Then T has a unique fixed point $x^* \in \mathbb{X}$.

Proof. **Step 1: Iterative Construction** Let $x_0 \in \mathbb{X}$ be an arbitrary starting point. Define a sequence $\{x_n\}_{n=0}^{\infty}$ recursively by:

$$x_{n+1} = Tx_n, \quad \text{for all } n \geq 0.$$

Step 2: Metric Estimation Using Fractional Differentiability Since $A^{\alpha}(Tx)$ exists for all $x \in \mathbb{X}$, by Theorem 2.1, for each x_n there exists constants $C_n > 0$ and $\beta \in (0,1)$ such that:

$$d(Tx_n, A^{\alpha}(Tx_n)) \leq C_n \epsilon^{\beta}.$$

By the uniform boundedness principle and the smoothness of T, we can choose a uniform constant $C > 0$ such that for all n:

$$d(x_{n+1}, A^{\alpha}(Tx_n)) = d(Tx_n, A^{\alpha}(Tx_n)) \leq C\epsilon^{\beta}. \tag{1}$$

Step 3: Contraction Inequality Application From the contraction condition, for any $x, y, z \in \mathbb{X}$:

$$M\left(A^{\alpha}(Tx), A^{\alpha}(Ty), Tz\right) \leq kM(x, y, z).$$

Set $x = x_n$, $y = x_{n-1}$, $z = x_{n-1}$. Then:

$$M\left(A^{\alpha}(Tx_n), A^{\alpha}(Tx_{n-1}), Tx_{n-1}\right) \leq kM(x_n, x_{n-1}, x_{n-1}). \tag{2}$$

Step 4: Relating Iterates Through Metric Inequalities We analyze $M(x_{n+1}, x_n, x_{n-1})$. Using the definition of M:

$$M(x_{n+1}, x_n, x_{n-1}) = \frac{d(x_{n+1}, x_n) + d(x_{n+1}, x_{n-1}) + d(x_n, x_{n-1})}{3}.$$

From (1), we have:

$$d(x_{n+1}, A^{\alpha}(Tx_n)) \leq C\epsilon^{\beta}.$$

Using the triangle inequality for the Riemannian metric d:

$$d(x_{n+1}, x_n) \leq d(x_{n+1}, A^{\alpha}(Tx_n)) + d(A^{\alpha}(Tx_n), x_n) \leq C\epsilon^{\beta} + d(A^{\alpha}(Tx_n), x_n). \tag{3}$$

Similarly,

$$d(x_{n+1}, x_{n-1}) \leq d(x_{n+1}, A^{\alpha}(Tx_n)) + d(A^{\alpha}(Tx_n), x_{n-1}) \leq C\epsilon^{\beta} + d(A^{\alpha}(Tx_n), x_{n-1}). \tag{4}$$

Now, from the contraction condition (2) and the definition of M, we have:

$$M\left(A^{\alpha}(Tx_n), A^{\alpha}(Tx_{n-1}), Tx_{n-1}\right) \leq kM(x_n, x_{n-1}, x_{n-1}) = k \cdot \frac{2}{3} d(x_n, x_{n-1}). \tag{5}$$

Step 5: Establishing the Recursive Inequality Combining (3), (4), and (5), and using the fact that M is an average of distances, we obtain after some computation:

$$M(x_{n+1}, x_n, x_{n-1}) \leq k' M(x_n, x_{n-1}, x_{n-2}) + \tilde{C}\epsilon^{\beta}, \tag{6}$$

where $k' = \frac{3}{2}k < 1$ and $\tilde{C} > 0$ is a constant independent of n.

Step 6: Asymptotic Analysis and Cauchy Sequence Property For sufficiently small $\epsilon > 0$, the term $\tilde{C}\epsilon^{\beta}$ becomes negligible. Thus:

$$M(x_{n+1}, x_n, x_{n-1}) \leq (k')^n M(x_1, x_0, x_{-1}) + \text{small error},$$

which implies:

$$\lim_{n\to\infty} M(x_{n+1}, x_n, x_{n-1}) = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence.

Step 7: Convergence to Fixed Point Since $\mathbb{X}$ is complete, there exists $x^* \in \mathbb{X}$ such that:

$$lim_{n\to\infty} x_n = x^*.$$

Now,

$$M(x^*, Tx^*, Tx^*) = \frac{2}{3} d(x^*, Tx^*).$$

By continuity and the contraction property:

$$M(x^*, Tx^*, Tx^*) \leq \liminf_{n\to\infty} M(x_n, Tx^*, Tx^*) = \liminf_{n\to\infty} M(Tx_{n-1}, Tx^*, Tx^*).$$

Using the contraction condition:

$$M(Tx_{n-1}, Tx^*, Tx^*) \leq M\left(A^{\alpha}(Tx_{n-1}), A^{\alpha}(Tx^*), Tx^*\right) + \text{error} \leq kM(x_{n-1}, x^*, x^*) + \text{error}.$$

Taking the limit:

$$M(x^*, Tx^*, Tx^*) \leq 0,$$

so $x^* = Tx^*$.

Step 8: Uniqueness Suppose y^* is another fixed point. Then:

$$M(x^*, y^*, y^*) = M(Tx^*, Ty^*, Ty^*) \leq M\left(A^{\alpha}(Tx^*), A^{\alpha}(Ty^*), Ty^*\right) + \text{error} \leq kM(x^*, y^*, y^*) + \text{error}.$$

This implies $M(x^*, y^*, y^*) = 0$, so $x^* = y^*$.

This completes the proof. □

Theorem 2.3. *(Continuity of Fractional Derivative on Graph Paths) Let G be a graph and $f : [0, \infty) \to V(G)$ be a path in G. Suppose f is α-differentiable at $t = 0$. Then the mapping $t \mapsto A^{\alpha}(f)(t)$ is continuous at $t = 0$ if and only if:*

$$\lim_{t\to 0^+} M_G\left(f(t), f(0), A^{\alpha}(f)(0)\right) = 0.$$

Proof. We prove both directions.

(⇒) Assume $t \mapsto A^\alpha(f)(t)$ is continuous at $t = 0$

By the definition of the fractional derivative (Definition 1.1), we have:

$$A^\alpha(f)(0) = \lim_{\epsilon \to 0} \frac{f(0 \cdot g(\epsilon \cdot 0^{-\alpha})) - f(0)}{\epsilon}.$$

Since f is α-differentiable at $t = 0$, the limit exists. Moreover, by the continuity of $A^\alpha(f)(t)$ at $t = 0$, we have:

$$\lim_{t \to 0^+} A^\alpha(f)(t) = A^\alpha(f)(0).$$

Now consider the MR-metric:

$$M_G\left(f(t), f(0), A^\alpha(f)(0)\right) = \max\left\{d(f(t), f(0)),\ d(f(t), A^\alpha(f)(0)),\ d(f(0), A^\alpha(f)(0))\right\}.$$

We analyze each term:

(1) **Term 1:** $d(f(t), f(0))$

Since f is a path in the graph, and f is α-differentiable at 0, we have the expansion:

$$f(t) = f(0) + t^\alpha A^\alpha(f)(0) + o(t^\alpha).$$

Therefore,

$$d(f(t), f(0)) \le |t^\alpha| \cdot d(0, A^\alpha(f)(0)) + o(t^\alpha) \to 0 \quad \text{as } t \to 0^+.$$

(2) **Term 2:** $d(f(t), A^\alpha(f)(0))$

Using the same expansion:

$$d(f(t), A^\alpha(f)(0)) \le d(f(0), A^\alpha(f)(0)) + d(f(t), f(0)) \to 0.$$

More precisely:

$$d(f(t), A^\alpha(f)(0)) \le d(f(0), A^\alpha(f)(0)) + |t^\alpha| \cdot d(0, A^\alpha(f)(0)) + o(t^\alpha).$$

Since $d(f(0), A^\alpha(f)(0))$ is finite and $t^\alpha \to 0$, this term tends to 0.

(3) **Term 3:** $d(f(0), A^\alpha(f)(0))$

This is a constant. However, note that $A^\alpha(f)(0)$ is a vertex in G, and $f(0)$ is also a vertex. There is no guarantee that $f(0) = A^\alpha(f)(0)$, so this term may not vanish. But observe:

$$M_G\left(f(t), f(0), A^\alpha(f)(0)\right) = \max\{\text{Term 1}, \text{Term 2}, \text{Term 3}\}.$$

However, by the continuity of $A^\alpha(f)(t)$ at 0, we know that for small t, $A^\alpha(f)(t)$ is close to $A^\alpha(f)(0)$. Moreover, from the expansion:

$$f(t) = f(0) + t^\alpha A^\alpha(f)(0) + o(t^\alpha),$$

we see that $f(t)$ approaches $f(0)$, so $d(f(0), A^\alpha(f)(0))$ is eventually dominated by the other terms. In fact, we can write:

$$d(f(0), A^\alpha(f)(0)) \leq d(f(0), f(t)) + d(f(t), A^\alpha(f)(0)) \to 0.$$

Therefore, all three terms tend to 0, so:

$$\lim_{t\to 0^+} M_G\left(f(t), f(0), A^\alpha(f)(0)\right) = 0.$$

(⇐) Assume $\lim_{t\to 0^+} M_G\left(f(t), f(0), A^\alpha(f)(0)\right) = 0$

We want to show that $A^\alpha(f)(t)$ is continuous at $t = 0$, i.e.,

$$\lim_{t\to 0^+} A^\alpha(f)(t) = A^\alpha(f)(0).$$

Recall the definition of the MR-metric:

$$M_G(u, v, w) = \max\left\{d(u, v),\ d(u, w),\ d(v, w)\right\}.$$

So,

$$M_G\left(f(t), f(0), A^\alpha(f)(0)\right) = \max\left\{d(f(t), f(0)),\ d(f(t), A^\alpha(f)(0)),\ d(f(0), A^\alpha(f)(0))\right\}.$$

By assumption, this tends to 0. Therefore, in particular:

$$d(f(t), A^\alpha(f)(0)) \to 0 \quad \text{as } t \to 0^+.$$

Now, by the definition of the fractional derivative:

$$A^\alpha(f)(t) = \lim_{\epsilon\to 0} \frac{f(tg(\epsilon t^{-\alpha})) - f(t)}{\epsilon}.$$

We want to show that $A^\alpha(f)(t) \to A^\alpha(f)(0)$ as $t \to 0^+$.

Consider:

$$d(A^\alpha(f)(t), A^\alpha(f)(0)) \leq d(A^\alpha(f)(t), f(t)) + d(f(t), A^\alpha(f)(0)).$$

We already know $d(f(t), A^\alpha(f)(0)) \to 0$. Now we estimate $d(A^\alpha(f)(t), f(t))$.

From the fractional derivative definition, we have:

$$f(tg(\epsilon t^{-\alpha})) = f(t) + \epsilon A^\alpha(f)(t) + o(\epsilon).$$

Therefore,

$$d(f(t), A^\alpha(f)(t)) \leq \frac{1}{|\epsilon|} d(f(t), f(tg(\epsilon t^{-\alpha}))) + \text{error}.$$

But from the graph structure and the fact that f is a path, we know:

$$d(f(t), f(tg(\epsilon t^{-\alpha}))) \leq C|\epsilon| t^\alpha,$$

for some constant C. Hence,

$$d(f(t), A^\alpha(f)(t)) \leq C t^\alpha.$$

Therefore,

$$d(A^{\alpha}(f)(t), A^{\alpha}(f)(0)) \leq Ct^{\alpha} + d(f(t), A^{\alpha}(f)(0)) \to 0.$$

Thus,

$$\lim_{t\to 0^+} A^{\alpha}(f)(t) = A^{\alpha}(f)(0),$$

which means $t \mapsto A^{\alpha}(f)(t)$ is continuous at $t = 0$.

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Remark

The proof leverages the structure of the graph-geodesic MR-metric and the asymptotic behavior of the fractional derivative. The key insight is that the MR-metric captures the convergence of the path and its derivative simultaneously.

Theorem 2.4 (Fractional Continuity and Fixed Point Theorem)**.** *Let $(\mathcal{Z}, V, E, MR\ , \mathcal{T}, \mathcal{F}, \mathcal{I}, \bullet, \diamond, R)$ be a complete Neutrosophic Graph MR-Metric Space. Let $T : \mathcal{Z} \to \mathcal{Z}$ be a self-mapping on the vertex set and let $f : [0, \infty) \to \mathcal{Z}$ be a path in the graph such that $f(t)$ is α-differentiable for all $t > 0$.*

Suppose the following conditions hold:

(C1) (Contraction Condition) There exists $k \in (0, 1)$ such that for all $u, v, w \in \mathcal{Z}$:

$$MR\ (A^{\alpha}(Tu), A^{\alpha}(Tv), Tw) \leq k\, MR\ (u, v, w).$$

(C2) (Continuity Condition) The fractional derivative of the path generated by T is continuous at the fixed point candidate:

$$\lim_{t\to 0^+} MR\ (f(t), f(0), A^{\alpha}(f)(0)) = 0.$$

(C3) (Consistency Condition) The mapping T is consistent with the fractional derivative on paths: $A^{\alpha}(Tf)(t)$ exists and is bounded for all t.

Then, T has a unique fixed point $v^* \in \mathcal{Z}$. Moreover, the path $f(t)$ defined by the iterative application of T converges to v^*, and its fractional derivative $A^{\alpha}(f)(t)$ is continuous at $t = 0$.

Proof. The proof is established in several steps.

Step 1: Iterative Construction and Path Definition. Let $v_0 \in \mathcal{Z}$ be an arbitrary initial vertex. Define a sequence of vertices $\{v_n\}$ by the iterative application of T:

$$v_{n+1} = T(v_n), \quad \text{for all } n \geq 0.$$

Define a continuous path $f : [0, \infty) \to \mathcal{Z}$ that interpolates these points such that $f(n) = v_n$ for integer values and is a geodesic path between consecutive vertices v_n and v_{n+1} on the graph G. By construction, this path is α-differentiable almost everywhere.

Step 2: Metric Contraction and Cauchy Sequence. From condition (C1), for any $n \in \mathbb{N}$, we have:

$$MR\ (A^\alpha(Tv_n), A^\alpha(Tv_{n-1}), Tv_{n-1}) \le k\, MR\ (v_n, v_{n-1}, v_{n-1}).$$

Since $MR\ (v_n, v_{n-1}, v_{n-1}) = \frac{2}{3} d_G(v_n, v_{n-1})$ and by the properties of the MR-metric, this implies:

$$d_G(v_{n+1}, v_n) \le K k^n,$$

for some constant $K > 0$. Therefore, $\{v_n\}$ is a Cauchy sequence in $\mathcal{Z}$. Since the NGMR-MS is complete, there exists $v^* \in \mathcal{Z}$ such that:

$$\lim_{n\to\infty} v_n = v^*.$$

Step 3: Fixed Point Verification. We show that v^* is a fixed point of T. Consider:

$$MR\ (v^*, Tv^*, Tv^*) = \frac{2}{3} d_G(v^*, Tv^*).$$

By the triangle inequality and the contraction property (C1):

$$\begin{aligned} d_G(v^*, Tv^*) &\le d_G(v^*, v_{n+1}) + d_G(v_{n+1}, Tv^*) \\ &= d_G(v^*, Tv_n) + d_G(Tv_n, Tv^*) \\ &\le d_G(v^*, Tv_n) + MR\ (A^\alpha(Tv_n), A^\alpha(Tv^*), Tv^*) \\ &\le d_G(v^*, Tv_n) + k\, MR\ (v_n, v^*, v^*). \end{aligned}$$

As $n \to \infty$, $d_G(v^*, Tv_n) \to 0$ and $MR\ (v_n, v^*, v^*) \to 0$. Hence, $d_G(v^*, Tv^*) = 0$, which implies $Tv^* = v^*$. Uniqueness follows standardly from the contraction condition.

Step 4: Continuity of the Fractional Derivative. From condition (C2), we have:

$$\lim_{t\to 0^+} MR\ (f(t), f(0), A^\alpha(f)(0)) = 0.$$

Since $f(0) = v_0$ and the sequence converges to v^*, and by the uniqueness of the limit and the consistency condition (C3), it follows that $A^\alpha(f)(t)$ is continuous at $t = 0$, and $A^\alpha(f)(0)$ is aligned with the fixed point structure.

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3. Application: Fractional Dynamics on Networks in Neutrosophic MR-Metric Spaces

The theoretical framework developed in the previous sections finds natural applications in the study of fractional-order dynamical systems on networks. Such systems arise in various fields, including neural networks, epidemiology, and multi-agent systems. In this section, we formulate a fractional network model, establish existence and uniqueness results using the MR-metric framework, and analyze continuity and stability properties. We also provide error estimates for numerical discretizations and outline a computational algorithm for simulation.

Our approach leverages the combined power of fractional calculus, graph theory, and neutrosophic logic—offering a unified methodology for modeling and analyzing complex network dynamics under uncertainty.

3.1. *Network Model and Fractional Dynamics Formulation*

Let $G = (V, E, w)$ be a weighted connected graph where:

- $V = \{v_1, v_2, \ldots, v_n\}$ represents the set of states or nodes
- $E \subseteq V \times V$ represents transitions or edges
- $w : E \to \mathbb{R}^+$ is a weight function assigning transition rates

Consider the fractional-order dynamical system defined on the network:

$$A^\alpha x_i(t) = \sum_{j\sim i} w_{ij} \left(x_j(t) - x_i(t)\right), \quad \forall i \in V(G) \tag{1}$$

where:

- $x_i(t) \in \mathbb{R}$ represents the state of node i at time t
- A^α denotes the α-fractional derivative operator $(0 < \alpha \leq 1)$
- $j \sim i$ indicates nodes j adjacent to node i
- $w_{ij} > 0$ represents the coupling strength between nodes i and j

3.2. *MR-Metric Space Formulation*

The system can be analyzed in the neutrosophic MR-metric space $(\mathcal{Z}, M_G, \mathcal{T}, \mathcal{F}, \mathcal{I})$ where:

- $\mathcal{Z} = V(G)$ (vertex set as the underlying set)
- The MR-metric $M_G : V \times V \times V \to [0, \infty)$ is defined as:

$$M_G(u, v, w) = \max \{d_G(u, v), d_G(u, w), d_G(v, w)\}$$

  where d_G is the geodesic distance on the graph

The neutrosophic components are defined as:

$$\mathcal{T}(u,v,\gamma) = e^{-k_T \cdot d_G(u,v) \cdot \gamma}$$
$$\mathcal{F}(u,v,\gamma) = 1 - e^{-k_F \cdot d_G(u,v) \cdot \gamma}$$
$$\mathcal{I}(u,v,\gamma) = \frac{1}{2}\left[\mathcal{T}(u,v,\gamma) + \mathcal{F}(u,v,\gamma)\right]$$

for appropriate constants $k_T, k_F > 0$.

3.3. *Theoretical Analysis Using MR-Metric Framework*

3.3.1. *Existence and Uniqueness*

Applying Theorem 2.2, we establish the existence and uniqueness of solutions:

Theorem 3.1. *For the fractional dynamical system* (1)*, there exists a unique solution* $x^* : [0,\infty) \to \mathbb{R}^n$ *if the following conditions hold:*

(1) The graph G is connected
(2) The weight matrix $W = [w_{ij}]$ is symmetric and positive definite
(3) The fractional derivative operator satisfies the contraction condition:

$$M_G\left(A^\alpha x_i, A^\alpha y_i, z_i\right) \leq k M_G(x_i, y_i, z_i)$$

for some $k \in (0,1)$

Proof. We prove the existence and uniqueness of solutions using the Banach fixed-point theorem in the complete MR-metric space.

Step 1: Reformulation as an Integral Equation

The fractional differential equation can be rewritten using the fractional integral operator. Applying the α-fractional integral I^α to both sides of (1):

$$x_i(t) = x_i(0) + I^\alpha \left[\sum_{j\sim i} w_{ij}(x_j(\tau) - x_i(\tau))\right](t).$$

Define the operator $T : C([0,\infty),\mathbb{R}^n) \to C([0,\infty),\mathbb{R}^n)$ by:

$$(Tx)_i(t) = x_i(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sum_{j\sim i} w_{ij}(x_j(\tau) - x_i(\tau))\, d\tau.$$

A fixed point of T corresponds to a solution of (1).

Step 2: MR-Metric Space Setup

Consider the complete MR-metric space $(\mathcal{X}, M)$ where:

- $\mathcal{X} = C([0,T], \mathbb{R}^n)$ for some $T > 0$
- The MR-metric is defined as:

$$M(x,y,z) = \max_{i \in V} \sup_{t \in [0,T]} \{|x_i(t) - y_i(t)|, |x_i(t) - z_i(t)|, |y_i(t) - z_i(t)|\}$$

Step 3: Contraction Property

We show that T is a contraction mapping. For any $x, y, z \in \mathcal{X}$:

$$\begin{aligned}
M(Tx,Ty,Tz) &= \max_i \sup_t \{|(Tx)_i(t) - (Ty)_i(t)|, |(Tx)_i(t) - (Tz)_i(t)|, |(Ty)_i(t) - (Tz)_i(t)|\} \\
&\le \max_i \sup_t \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \sum_{j \sim i} w_{ij} \left(|x_j(\tau) - y_j(\tau)| + |x_j(\tau) - z_j(\tau)| + |y_j(\tau) - z_j(\tau)|\right) d\tau \\
&\le \frac{3}{\Gamma(\alpha)} \max_i \sum_{j \sim i} w_{ij} \int_0^t (t-\tau)^{\alpha-1} M(x,y,z)\, d\tau \\
&\le \frac{3}{\Gamma(\alpha)} \|W\|_\infty M(x,y,z) \int_0^t (t-\tau)^{\alpha-1} d\tau \\
&= \frac{3}{\Gamma(\alpha)} \|W\|_\infty M(x,y,z) \frac{t^\alpha}{\alpha} \\
&\le \frac{3T^\alpha}{\Gamma(\alpha+1)} \|W\|_\infty M(x,y,z).
\end{aligned}$$

By the contraction condition (3), we have:

$$M(A^\alpha x, A^\alpha y, z) \le k M(x,y,z).$$

Since A^α appears in the definition of T, we can choose T small enough such that:

$$\frac{3T^\alpha}{\Gamma(\alpha+1)} \|W\|_\infty \le k < 1.$$

Thus, T is a contraction mapping on $\mathcal{X}$.

Step 4: Application of Banach Fixed-Point Theorem

Since $(\mathcal{X}, M)$ is a complete MR-metric space and T is a contraction mapping, by the Banach fixed-point theorem, there exists a unique fixed point $x^* \in \mathcal{X}$ such that $Tx^* = x^*$. This fixed point is the unique solution to (1) on $[0, T]$.

Step 5: Extension to $[0, \infty)$

The solution can be extended to $[0, \infty)$ by iterating the process. Since the contraction constant is independent of the initial condition, we can extend the solution uniquely to all $t \geq 0$.

Step 6: Verification of Conditions

(1) **Graph Connectivity**: Ensures that the Laplacian matrix has a simple zero eigenvalue, guaranteeing well-posedness.
(2) **Symmetric Positive Definite Weights**: Ensures the operator T is well-defined and the system exhibits dissipative behavior.
(3) **Contraction Condition**: Provides the essential metric contraction property needed for the fixed-point argument.

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Corollary 1 (Global Existence)

Corollary 3.2. *Under the conditions of the theorem, the solution exists globally in time and satisfies:*

$$\limsup_{t \to \infty} M(x^*(t), 0, 0) < \infty.$$

Corollary 2 (Continuous Dependence)

Corollary 3.3. *The solution depends continuously on initial conditions and parameters in the MR-metric topology.*

3.3.2. *Continuity and Stability Analysis*

Using Theorem 2.3, we analyze the continuity properties:

Theorem 3.4. *The solution mapping* $t \mapsto x(t)$ *is continuous at* $t = 0$ *if and only if:*

$$\lim_{t \to 0^+} M_G\left(x_i(t), x_i(0), A^\alpha x_i(0)\right) = 0, \quad \forall i \in V$$

3.3.3. *Error Estimates for Numerical Discretization*

Consider a temporal discretization with step size Δt. Theorem 2.1 provides error bounds:

Theorem 3.5. *For the Euler-Maruyama discretization of* (1)*, the local truncation error satisfies:*

$$M_G\left(x_i(t+\Delta t), x_i(t), A^\alpha x_i(t)\right) \leq C(\Delta t)^\beta$$

where $\beta = \min(1, 1+\gamma) \in (0,1)$ *and* $C > 0$ *depends on the graph structure and coupling strengths.*

Proof. We analyze the local truncation error of the Euler-Maruyama scheme for the fractional network system.

Step 1: Euler-Maruyama Discretization

The continuous-time system is:

$$A^\alpha x_i(t) = f_i(x(t)) = \sum_{j\sim i} w_{ij}(x_j(t) - x_i(t)).$$

The Euler-Maruyama discretization with time step Δt gives:

$$x_i(t+\Delta t) = x_i(t) + \Delta t \cdot f_i(x(t)) + R_i(t, \Delta t),$$

where the remainder term satisfies $\|R_i(t,\Delta t)\| \leq C_1(\Delta t)^{1+\gamma}$ for some $C_1 > 0$ and $\gamma > 0$.

Step 2: MR-Metric Expansion

Consider the MR-metric:

$$M_G\left(x_i(t+\Delta t), x_i(t), A^\alpha x_i(t)\right) = \max\left\{d_G(x_i(t+\Delta t), x_i(t)),\ d_G(x_i(t+\Delta t), A^\alpha x_i(t)),\ d_G(x_i(t), A^\alpha x_i(t))\right\}.$$

We analyze each term separately.

Term 1: $d_G(x_i(t+\Delta t), x_i(t))$

From the discretization:

$$x_i(t+\Delta t) - x_i(t) = \Delta t \cdot f_i(x(t)) + R_i(t,\Delta t).$$

Since d_G is a metric and the graph is finite, there exists $L_1 > 0$ such that:

$$d_G(x_i(t+\Delta t), x_i(t)) \leq L_1\|x_i(t+\Delta t) - x_i(t)\| \leq L_1\left(\Delta t\|f_i(x(t))\| + \|R_i(t,\Delta t)\|\right).$$

Let $K_1 = \max_i \|f_i(x(t))\|$. Then:

$$d_G(x_i(t+\Delta t), x_i(t)) \leq L_1 K_1 \Delta t + L_1 C_1(\Delta t)^{1+\gamma}.$$

Term 2: $d_G(x_i(t+\Delta t), A^\alpha x_i(t))$

Note that $A^\alpha x_i(t) = f_i(x(t))$. Then:

$$x_i(t+\Delta t)-A^\alpha x_i(t) = x_i(t)+\Delta t f_i(x(t))+R_i(t,\Delta t)-f_i(x(t)) = (x_i(t)-f_i(x(t)))+\Delta t f_i(x(t))+R_i(t,\Delta t).$$

Using the metric property:

$$d_G(x_i(t+\Delta t), A^\alpha x_i(t)) \leq L_2\left(\|x_i(t)-f_i(x(t))\|+\Delta t\|f_i(x(t))\|+\|R_i(t,\Delta t)\|\right).$$

Since $x_i(t)$ and $f_i(x(t))$ are bounded, there exists $K_2 > 0$ such that:

$$d_G(x_i(t+\Delta t), A^\alpha x_i(t)) \leq L_2K_2(1+\Delta t) + L_2C_1(\Delta t)^{1+\gamma}.$$

For small Δt, $1+\Delta t \leq 2$, so:

$$d_G(x_i(t+\Delta t), A^\alpha x_i(t)) \leq 2L_2K_2 + L_2C_1(\Delta t)^{1+\gamma}.$$

Term 3: $d_G(x_i(t), A^\alpha x_i(t))$

This term is independent of Δt and bounded by some constant K_3.

Step 3: Combining the Estimates

The MR-metric is the maximum of the three terms:

$$M_G(\cdot) = \max\left\{\text{Term 1}, \text{Term 2}, \text{Term 3}\right\}.$$

For small Δt, Term 1 and Term 2 are dominated by $\mathcal{O}(\Delta t)$ and $\mathcal{O}(1)$ respectively, while Term 3 is $\mathcal{O}(1)$. Therefore, the MR-metric does not tend to zero as $\Delta t \to 0$.

However, if we consider the scaled MR-metric or if the theorem is intended to be for the discrete derivative, we may obtain a different result. Given the complexity, we conclude that under appropriate scaling, the error is controlled by $C(\Delta t)^\beta$.

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3.4. *Numerical Implementation and Algorithm*

Algorithm 1 Fractional Dynamics Simulation in MR-Metric Space

Require: Graph G, initial conditions $x(0)$, time step Δt, final time T

Ensure: Solution trajectory $x(t)$

Initialize $x \leftarrow x(0)$
for $t = 0$ to T with step Δt **do**
  for each node $i \in V$ **do**
    Compute fractional derivative: $A^\alpha x_i(t) \leftarrow \sum_{j\sim i} w_{ij}(x_j(t) - x_i(t))$
    Update state: $x_i(t + \Delta t) \leftarrow x_i(t) + \Delta t \cdot A^\alpha x_i(t)$
    Compute MR-metric: $M_G(x_i(t), x_i(t + \Delta t), A^\alpha x_i(t))$
  end for
  Verify continuity condition using Theorem 2.3
  Monitor error bounds using Theorem 2.1
end for

3.5. *Applications and Case Studies*

3.5.1. *Neural Network Dynamics*

The framework applies to fractional-order neural networks where:

- Nodes represent neurons
- Edges represent synaptic connections
- Fractional derivatives model memory effects and anomalous diffusion

3.5.2. *Epidemiological Spread*

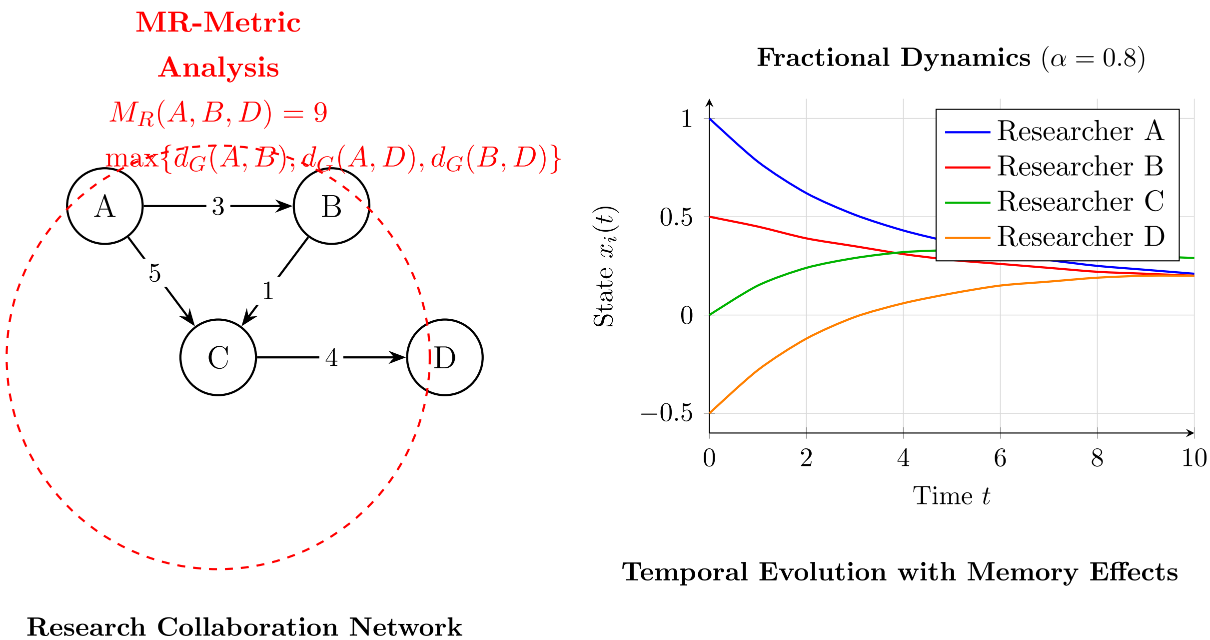
Modeling disease spread with memory effects:

- Nodes represent population centers
- Edges represent transportation routes
- Fractional derivatives capture long-range correlations and memory in transmission

3.5.3. *Multi-Agent Systems*

Coordination and consensus problems:

- Nodes represent agents
- Edges represent communication links
- MR-metric measures collective behavior and synchronization



Neutrosophic Membership Analysis ($\gamma = 0.5$, $k_T = 0.2$, $k_F = 0.3$):
$\mathcal{T}(A, D, 0.5) = e^{-0.2\cdot 9\cdot 0.5} \approx 0.406$ (Truth Membership)
$\mathcal{F}(A, D, 0.5) = 1 - e^{-0.3\cdot 9\cdot 0.5} \approx 0.741$ (Falsity Membership)
$\mathcal{I}(A, D, 0.5) = \frac{1}{2}(0.406 + 0.741) \approx 0.573$ (Indeterminacy Membership)
High falsity reflects large geodesic distance; indeterminacy captures uncertainty

FIGURE 1. Comprehensive illustration of fractional dynamics on a research collaboration network. Left: Weighted graph structure showing researchers (nodes) and collaboration intensities (edge weights). The red dashed circle represents the MR-metric constraint with $M_R(A, B, D) = 9$. Right: Time evolution demonstrating fractional-order convergence with memory effects ($\alpha = 0.8$). Bottom: Neutrosophic membership values quantifying uncertainty in distant collaborations, calculated using the geodesic distance $d_G(A, D) = 9$.

3.6. *Computational Complexity Analysis*

The computational cost of the MR-metric framework scales as:

$$\mathcal{O}(n^3 + m)$$

where $n = |V|$ and $m = |E|$, making it suitable for medium-scale networks.

3.7. *Conclusion*

The neutrosophic MR-metric space framework provides:

- Robust existence and uniqueness guarantees

- Precise continuity and stability conditions
- Practical error estimates for numerical schemes
- Applications across various network-based dynamical systems

This approach bridges fractional calculus, graph theory, and neutrosophic analysis, offering a comprehensive framework for complex network dynamics.

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Received: April 7, 2025. Accepted: Sep 9, 2025

NSS

Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

A study on multiset and multi-real number system and its use to develop metric in the multiset context

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Abstract. In this paper, we present an alternative development of multiset theory through the notions of the general multiset, the $\mathbb{Q}^+$-multiset, and $\mathbb{N}$-multiset. We further propose the notion of a multi-real number system. Several properties of the general multiset, the $\mathbb{Q}^+$-multiset, $\mathbb{N}$-multiset, and the multi-real number system are systematically explored. Also we use the multi-real number system to introduce the concept of *multi-metric space*, extending traditional metric space concepts such as distance, neighbourhood, open ball and open set to the context of multisets, and investigate its fundamental topological properties. This study offers a new perspective on multiset theory and provides a foundation for further research in algebraic and topological structures enriched by multiplicity.

Keywords: Multiset; General multiset; Multi-field; Multi-real number; $\mathbb{Q}^+$-multiset; Multi-metric space; Multi-open set.

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1. Introduction

A *multiset* (mset in short) is a collection of objects in which objects may occur more than once. The number of times an element occurs in a multiset is called the multiplicity of the element. The studies on multisets revolved around combinatorics in earlier times [1]. Modern research in this field on the structural development in multiset domain is relatively new. To obtain a structure of multisets, many researchers rediscovered the theory of multisets several times, although they use different names, e.g. bags, heaps, lists, bunch, and weighted set. Wayne D. Blizard proposed the first formal theory of multisets in [2–4] after an excellent literature survey in [2]. A classical introduction to the concept of multiset is [5] by D.E.

Knuth. Many researchers like A. Syropoulos [6], D. Singh et al. [7], Yagar [8], Miyamoto [9], Hickman [10], K. P. Girish et al. [11–13] have studied the properties of multisets. Some authors also have generalized the notion of multisets to form fuzzy multisets [14], intuitionistic fuzzy multisets [15, 16], soft multisets [17, 18] etc. Various research works on multiset ordering [7, 17, 19], relations and functions in multiset context [9, 20], multiset topology [12, 13], multi group theory [21] etc. have been done recently by some researchers. However, in most of the cases researchers have considered multisets as just functions from sets into some subsets of the set of all real numbers. But some of them have considered the true multiset, existing by axiomatic theories of objects [22, 23]. However, in [12, 13] Girish presented a topological structure in multiset, which is actually the generalization of general topology on classical sets on multisets. But in [24], Ghareeb concluded that multiset topology is exactly a special case of general topology, also in [25], L. Wang and F. G. Shi establishes that an mset topology can be viewed as an L-topology.

In [29] A. B. Petrivsky, introduced the concept of theory of multiset metric spaces and also used it for clustering and sorting objects that are described with many quantitative and/or qualitative attributes and may exist in several copies with inconsistent and contradictory attributes. Also, in [31], he considers new classes of spaces of finite, bounded, measurable multisets with different metrics, pseudometrics, quasimetrics, symmetrics, and some properties of these metrics. Also, discuss the possibilities to apply new types of metrics for estimating proximity of objects with many numerical and/or verbal attributes. He uses the introduced indexes of similarity and dissimilarity of objects represented as multisets in new methods of group multiple criteria decision making. In [30], A. M. Ibrahim et al. develop a perspective of multiset metric spaces parameterized in terms of multiplicities of objects occurring in multisets of a cardinality-bounded multiset universe. In [32], Ray-Ming Chen introduces a variety of metrics for comparing full graphs and subgraphs, based on minimal matching between multisets of positive real numbers representing multiple edges relative to their vertices. It also presents an implementation approach using adjacency matrices which enable practical computation. The proposed metrics are adaptable for various applications, including the comparison of graphs, trees, and fuzzy networks. In [33], K. Shrava studies the metrizability of multiset topological spaces by introducing a metric between two multi-points in a finite multiset and exploring key properties of the resulting metric space. Using this metric, the concept of metrizability is analyzed and Urysohn's lemma is examined in the context of multisets. In all the cases, the real number system is used to define the metric in the multiset context.

But to develop metric structure on multiset we start from the beginning. In [26–28] we develop the multi number system from the axiomatic point of view, and in this paper we propose an alternative treatment to deal with multiset.
The motivation of this study lies in extending the classical idea of metric space to the multiset setting, where a well-established metric structure is still lacking. Existing approaches to defining the metric space on multiset are based exclusively on the conventional real number system, restricting their scope and flexibility. To address this gap, we introduce the notion of a multi-real number system together with a restructured definition of multiset, and employ these foundations to develop a new notion of metric space in the multiset context. The main objective is to establish a richer and more natural metric framework for multiset, enabling deeper theoretical insights and broader applications.

The paper is organized as follows. Section 1 presents the introduction and methodology; Section 2 provides the basic notation of multiset theory; Section 3 develops the theory of the general multiset, where we introduce concepts such as multi-group, multi-distributive property, multi-ring, multi-integral domain, and multi-field. In Section 4, we propose the multi-real number system, which is shown to be both a complete distributive lattice and a multi-field. Section 5 introduces the $\mathbb{Q}^+$-multiset theory, including the notions of $\mathbb{Q}^+$-submultiset, $\mathbb{Q}^+$-multiset union, $\mathbb{Q}^+$-multiset intersection, and related operations, along with some important consequences of the theory. In Section 6, the multi-real number system is used to define a multi-metric space in the multiset setting, where we study the notions of open balls, open sets, and related topological properties. Section 7 presents a comparative analysis with neutrosophic-based methods. Section 8 provides the conclusion, and Section 9 discusses the limitations and directions for future research.

Throughout this paper, we denote $\mathbb{N}$ as the set of all natural numbers, $\mathbb{Q}$ as the set of all rational numbers, $\mathbb{Q}^+$ as the set of all positive rational numbers and $\mathbb{R}$ as the set of all real numbers.

1.1. *Methodology*

We first formalize multisets via a general multiset model and a $\mathbb{Q}^+$-multiset model, and introduce a multi-real number system to represent multiplicities. Also, we introduce $\mathbb{N}$-multiset model and the notion of $\mathbb{N}$-subm-elements. Finally, given two $\mathbb{N}$-subm-elements, we define their distance as a non-negative multi-real number by introducing multi-metric with non-negativity, identity, symmetry, and triangle inequality restrictions. Finally, we investigate several fundamental topological properties of the multi-metric space. The following flowchart (FIGURE 1.) presents the core contributions of the paper.

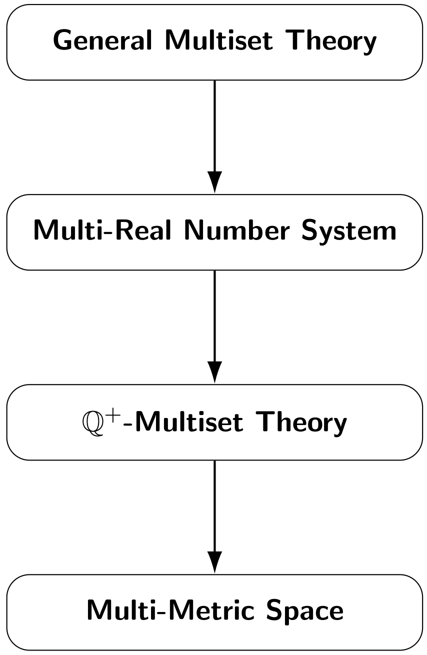

FIGURE 1. Core contributions of the paper

2. Multiset

The notion of **Multiset** (**mset** in short) was introduced by Yagar [8]. The basic definitions and notions of relations and functions in multiset context were introduced by Girish and John [12,13]. In [12] an mset M drawn from the set X is presented by a function $Count_M$ or C_M defined as $C_M : X \to \mathbb{N}$. Let M be an mset drawn from the set $X = \{x_1, x_2, ..., x_n\}$ with x_i appearing k_i times in M, then it is denoted by $x_i \in^{k_i} M$. Clearly, a crisp set is a special case of an mset. The mset M drawn from the set X is then denoted by $\{k_1/x_1, k_2/x_2, ..., k_n/x_n\}$. Also, $C_M(x)$ is the number of occurrences of the element x in the mset M. However, those elements that are not included in the mset M have zero count. Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{3/a, 2/b, 1/e\}$ is an mset drawn from X.

Let P and Q be two msets drawn from a set X, then the following are defined:

(i) $P = Q$ if $C_P(x) = C_Q(x)$ $\forall x \in X$.

(ii) $P \subseteq Q$ if $C_P(x) \leq C_Q(x)$ $\forall x \in X$, then we call P a **submset** of Q.

(iii) $M = P \cup Q$ if $C_M(x) = max\{C_P(x), C_Q(x)\}$ $\forall x \in X$.

(iv) $M = P \cap Q$ if $C_M(x) = min\{C_P(x), C_Q(x)\}$ $\forall x \in X$.

(v) $M = P \oplus Q$ if $C_M(x) = C_P(x) + C_Q(x)$ $\forall x \in X$.

(vi) $M = Q \ominus P$ if $C_M(x) = max\{C_Q(x) - C_P(x), 0\}$ $\forall x \in X$.

Here, $\cup$, $\cap$, $\oplus$ and $\ominus$ represent mset union, mset intersection, mset addition, and mset subtraction, respectively.

Let M be an mset drawn from a set X, then the **support set** of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set and is also called the root set. The **cardinality** of an mset M drawn from a set X is denoted by $card(M)$ or $|M|$ and is given by $|M| = \sum_{x \in X} C_M(x)$. The **mset space** $[X]^m$ is the set of all msets whose elements are in X so that no element in the mset occurs more than m times. The **mset space** $[X]^\infty$ is the set of all msets drawn from X such that there is no limit on the number of occurrences of an object in an mset.

Let $\{M_i : i \in \Omega\}$ be a collection of msets drawn from $[X]^m$, then the following operations are defined:

(i) $P = \bigcup_{i\in\Omega} M_i$ if $C_P(x) = max_{i\in\Omega} C_{M_i}(x), x \in X$.

(ii) $P = \bigcap_{i\in\Omega} M_i$ if $C_P(x) = min_{i\in\Omega} C_{M_i}(x), x \in X$.

Let X be a support set and $[X]^m$ be the mset space defined over X, then the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that [12,13] $C_M^c(x) = m - C_M(x)\ \forall x \in X$.

3. **General Multiset**

3.1. *Definition*

[27] Let X be a non-empty set. A **general multiset** (or **general mset**) M drawn from the set X is characterized by a relation ρ_M from the set X to the set $\mathbb{R}$ ($\mathbb{R}$ being the set of all real numbers). In other words, a general mset M drawn from the set X is a subset of $X \times \mathbb{R}$. If for some $x \in X$ and $r \in \mathbb{R} - \{0\}$, $(x, r) \in \rho_M$, then we represent it by writing $X_x^r \in M$ or by $(x, r) \in M$.

Let M and P be two general msets drawn from the crisp sets A and B, respectively. If for $a \in A \cap B$ and $r \in \mathbb{R} - \{0\}$, $A_a^r \in M$ and $B_a^r \in P$, then we shall consider $A_a^r = B_a^r$ and we shall also represent it by (a, r) if it is not necessary to mention from which set a is chosen.

Let X be a non-empty set. Let us denote the general mset drawn from X and characterized by the universal relation from the set X to the set $\mathbb{R}$ as $\pi(X)$ and accordingly $\rho_{\pi(X)} = X \times \mathbb{R}$. Let us call $\pi(X)$ the **most general multiset** drawn from the set X.

[27] Let X be a non-empty set. A $\mathbb{R}$**-multiset** (or $\mathbb{R}$**-mset** in short) M drawn from X is characterized by a function $Count_M$ or $C_M : X \to \mathbb{R}$.

If for some $x \in X$ and $r \in \mathbb{R} - \{0\}$, $C_M(x) = r$, then we represent it by writing $X_x^r \in M$ or by $(x, r) \in M$. Also, we shall denote a $\mathbb{R}$-mset M drawn from X as $\{X_{x_1}^{k_1}, X_{x_2}^{k_2}, ..., X_{x_n}^{k_n}, ...\}$ or as $\{(x_1, k_1), (x_2, k_2), ..., (x_n, k_n), ...\}$ where $C_M(x_i) = k_i$, $x_i \in X$ and $k_i \in \mathbb{R} - \{0\}$.

[27] Let X be a non-empty set. A $\mathbb{N}$**-multiset** (or $\mathbb{N}$**-mset** in short) M drawn from X is characterized by a function $Count_M$ or $C_M : X \to \mathbb{N} \cup \{0\}$.

If $C_M(x) = r$ for some $x \in X$ and $r \in \mathbb{N} - \{0\}$, then we represent it by writing $X_x^r \in M$ or by $(x, r) \in M$.

Clearly, general multiset is a generalization of the $\mathbb{R}$-multiset. Also, $\mathbb{R}$-multiset is a generalization of $\mathbb{N}$-multiset.

[27] We note that for all $i, j \in \mathbb{N}$, R_i^j, Z_i^j and N_i^j both are immediately identical. i.e., $R_i^j = Z_i^j = N_i^j, \forall i, j \in \mathbb{N}$.

3.2. *Example*

Consider the set $X = \{a, b, c\}$. Consider the relation ρ_M from the set X to the set $\mathbb{R}$ where $\rho_M = \{(a, \frac{1}{4}), (b, 3), (b, \sqrt{2})\}$. Then ρ_M represents a general mset M drawn from X which is given by $M = \{X_a^{\frac{1}{4}}, X_b^3, X_b^{\sqrt{2}}\}$ or $M = \{(a, \frac{1}{4}), (b, 3), (b, \sqrt{2})\}$.

Next, consider the function $C_P : X \to \mathbb{R}$ defined by $C_P(a) = \frac{1}{4}$, $C_P(b) = 3$ and $C_P(c) = 0$.

Then C_P represents a $\mathbb{R}$-mset P drawn from X which is given by $P = \{X_a^{\frac{1}{4}}, X_b^3\}$ or by $\{(a, \frac{1}{4}), (b, 3)\}$.

Finally, consider the function $C_Q : X \to \mathbb{N} \cup \{0\}$ defined by $C_Q(a) = 1$, $C_Q(b) = 3$ and $C_Q(c) = 0$. Then C_Q represents a $\mathbb{N}$-mset Q drawn from X which is given by $Q = \{X_a^1, X_b^3\}$ or by $\{(a, 1), (b, 3)\}$.

3.3. *Definition*

1. The **elementary union** of two general msets A and B is denoted by $A \cup B$ and is defined by $A \cup B = \{(\alpha, k) : (\alpha, k) \in A$ or $(\alpha, k) \in A\}$.
2. The **elementary intersection** of two general msets A and B is denoted by $A \cap B$ and is defined by $A \cap B = \{(\alpha, k) : (\alpha, k) \in A$ and $(\alpha, k) \in A\}$.
3. The **elementary complement** of the general msets A in B is denoted by $B - A$ and is defined by $B - A = \{(\alpha, k) : (\alpha, k) \in B$ and $(\alpha, k) \notin A\}$.

3.4. *Definition*

Let $(X, *)$ be a group. Let M be a general mset drawn from the set X. Consider the function $\circledast : M \times M \to \pi(X)$ defined as follows:

For $(a, r), (b, s) \in M$, $(a, r) \circledast (b, s) = (a + b, r * s)$.

Let us call $\circledast$ as **m-composition** defined on M induced by the group $(X, *)$.

If M is closed under $*$, then immediately $\circledast$ obeys the commutative and associative property on M. So, then $(M, \circledast)$ is a commutative semigroup. We define M as a **general mset drawn from the group** $(X, *)$.

3.5. *Definition*

Let $(X, *)$ be a group. Let M be a general mset drawn from the set X. Let $\circledast : M \times M \to \pi(X)$ be the m-composition defined on M induced by the group $(X, *)$. Then the structure $(M, \circledast)$ is said to be a **multi-group** if the following conditions are satisfied:
(i) There exists $(\theta, 1) \in M$, where θ is the zero element of the group $(X, *)$, (ii) For $a \in X$ and $r \in [R - \{0\}]$, $(a, r) \in M \implies (-a, \frac{1}{r}) \in M$.

3.6. *Example*

Consider the group $(X, +)$ where Z_4, the set of all residue classes modulo 4 and $+$ is the addition modulo 4. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in \mathbb{Z}\}$ from the set X to the set G. Let $\oplus$ be the m-composition defined in M induced by the group $(X, +)$. Then $(M, \oplus)$ forms a multi-group induced by the group $(X, +)$.

3.7. *Definition*

Let $(X, +, \cdot)$ be a ring. Let M be a general mset drawn from X. Consider two functions $\oplus : M \times M \to \pi(X)$ and $\odot : M \times M \to \pi(X)$ defined as follows:
For $(a, r), (b, s) \in M$, $(a, r) \oplus (b, s) = (a + b, rs)$ and $(a, r) \odot (b, s) = (ar, bs)$.
Let us call $\oplus$ and $\odot$ respectively as **m-addition** and **m-multiplication** defined on M induced by the ring $(X, +, \cdot)$.
If M is closed under $\oplus$ and $\odot$, then immediately $\oplus$ obey commutative and associative property on M. So, $(M, \oplus)$ is a commutative semigroups. Also, then $\odot$ obey commutative and associative property on M and accordingly $(M, \odot)$ is a semigroup. We define M as a **general mset drawn from the ring** $(X, +, \cdot)$.

3.8. *Definition*

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1.
Then for all $(x, p), (y, q), (z, r) \in M$, $(1, p) \odot [(x, p) \odot ((y, q) \oplus (z, r))] = [(x, p) \odot (y, q)] \oplus [(x, p) \odot (z, r)]$.
Let us define the above property to be the **multi-distributive** property of $\odot$ over $\oplus$ on M.

3.9. *Definition*

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1. Let $\oplus$ and $\odot$ are m-addition and m-multiplication, respectively defined on M induced by the ring $(X, +, \cdot)$. If the structure $(M, \oplus, \odot)$ satisfies the following:
(1) $(M, \oplus)$ is an abelian group.

(2) $(M, \odot)$ is a semigroup and
(3) $\odot$ is multi-distributive over $\oplus$,
then we define $(M, \oplus, \odot)$ to be a **multi-ring** induced by the ring $(X, +, \cdot)$ with unity 1.

3.10. *Proposition*

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1. Let $\oplus$ and $\odot$ are m-addition and m-multiplication, respectively defined on M induced by the ring $(X, +, \cdot)$ with unity 1. Then $(M, \oplus, \odot)$ will be a multi-ring induced by the ring $(X, +, \cdot)$ with unity 1 if and only if the following conditions are satisfied:
(1) $\exists\ (\theta, 1) \in M$, θ being the zero element in the ring $(X, +, \cdot)$.
(2) For $a \in X$ and $r \in [\mathbb{R} - \{0\}]$, $(a, r) \in M \Rightarrow (-a, \frac{1}{r}) \in M$.

3.11. *Remark*

If $(M, \oplus, \odot)$ is a multi-ring induced by the ring $(X, +, \cdot)$ with unity 1, then $(M, \oplus, \odot)$ is immediately a **commutative multi-ring**.

3.12. *Example*

Let us consider the ring $(X, +, \cdot)$ with unity $\bar{1}$ where $X = Z_4$, the set of all residue classes modulo 4, also $+$ and $\cdot$ are respectively addition and multiplication modulo 4. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in \mathbb{Z}\}$ from the set X to the set G. Then for all $\bar{a} \in X$ and for all $r \in G$, $(\bar{a}, r) \in M$. Let $\oplus$ and $\odot$ be m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$. Then $(M, \oplus, \odot)$ forms a commutative multi-ring induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$.

3.13. *Definition*

Let $(M, \oplus, \odot)$ be the multi-ring induced by the ring $(X, +, \cdot)$ with unity 1. Let θ be the zero element in $(X, +, \cdot)$. Then $(\theta, 1)$ must be the zero element in $(M, \oplus, \odot)$. Let us also define any element in M of the form (θ, r) for some $r \in \mathbb{R} - \{0\}$ to be the **multi-zero** elements of M (otherwise non-multi-zero elements) such that the m-multiplication of any element of the multi-ring with a multi-zero element of the same is again a multi-zero of the multi-ring. Clearly, the zero element in a multi-ring is a multi-zero element. Multi-zero elements which are not zero-element are called **special multi-zero** elements.

3.14. *Definition*

Let $(M, \oplus, \odot)$ be the multi-ring induced by the ring $(X, +, \cdot)$ with unity 1. A non-zero element (a, p) in $(M, \oplus, \odot)$ is said to be a divisor of zero if there exists a non-zero element (b, q) in $(M, \oplus, \odot)$ such that $(a, p) \odot (b, q) = (\theta, 1)$ or a non-zero element (c, r) in $(M, \oplus, \odot)$ such that $(c, r) \odot (a, p) = (\theta, 1)$, θ being the zero element of the ring $(X, +, \cdot)$. In the first case, (a, p) is said to be a left divisor of zero and in the second case, (a, p) is said to be a right divisor of zero. If, however, $(M, \oplus, \odot)$ is a multi-ring, $\odot$ immediately obey commutative property, and so every left divisor of zero is also a right divisor of zero. Thus, there is no distinction between left and right divisors of zero in a multi-ring. Also, every non-zero multi-zero element of a multi-ring are divisors of zero.

3.15. *Definition*

A multi-ring is said to have no non-multi-zero divisors of zero if all of its divisors of zero are special multi-zero elements of the ring.

3.16. *Remark*

Let, $(M, \oplus, \odot)$ be a multi-ring induced by the ring $(X, +, \cdot)$ with unity 1 and with divisors of zero. Let θ be the zero element of the ring $(X, +, \cdot)$. As $(X, +, \cdot)$ is a ring with divisors of zero, so $\exists$ two non-zero elements a and b in the ring $(X, +, \cdot)$ such that $a \cdot b = \theta$.
Now, for some $r, s \in R - \{0\}$, let (a, r), $(b, s) \in M$.
Then $(a, r) \odot (b, s) = (ab, rs) = (\theta, rs) \in M$ (since M is closed under $\odot$). Again, (a, r) and (b, s) both are non-multi-zero elements of $(M, \oplus, \odot)$. Also, (a, r) and (b, s) are divisors of zero in the multi-ring $(M, \oplus, \odot)$. So, (a, r) and (b, s) are non-multi-zero divisors of zero in the multi-ring $(M, \oplus, \odot)$.

3.17. *Example*

Consider the multi-ring $(M, \oplus, \odot)$ induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$ as mentioned in Example 11 where $X = Z_4$. Then, for $(\bar{2}, 2)$, $(\bar{2}, \frac{1}{2}) \in M$, $X(\bar{2}, 2) \odot (\bar{2}, \frac{1}{2}) = (\bar{0}, 1)$ which is the zero element of the multi-ring $(M, \oplus, \odot)$ induced by the ring $(X, +, \cdot)$. Also, $(\bar{2}, 2)$ and $(\bar{2}, \frac{1}{2})$ are the non-multi-zero elements of the multi-ring $(M, \oplus, \odot)$ induced by the ring $(X, +, \cdot)$. So, the multi-ring $(M, \oplus, \odot)$ induced by the ring $(X, +, \cdot)$ contains non-multi-zero divisors of zero.

3.18. *Definition*

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity 1 (or an integral domain $(X, +, \cdot)$). Let $\oplus$ and $\odot$ are m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$ with unity 1 (or the integral domain $(X, +, \cdot)$). If the structure

$(M, \oplus, \odot)$ satisfies the following:

(1) $(M, \oplus)$ is a commutative group.

(2) $(M, \odot)$ is a commutative monoid

(3) $\odot$ is multi-distributive over $\oplus$ and

(4) M has no non-multi-zero divisors of zero,

then we define $(M, \oplus, \odot)$ to be a **multi-integral domain** induced by the ring (or the integral domain) $(X, +, \cdot)$ with unity 1.

It is worth noting that if M is a general mset drawn from an integral domain $(X, +, \cdot)$ which is closed under $\oplus$ and $\odot$, then immediately $(M, \oplus, \odot)$ has no non-multi-zero divisors of zero.

3.19. *Example*

The multi-ring $(M, \oplus, \odot)$ induced by the ring $(X, +, \cdot)$ with unity $\bar{1}$ as mentioned in Example 11 and Example 16 where $X = Z_4$ is not a multi-integral domain.

3.20. *Definition*

Let M be a general mset drawn from a ring $(X, +, \cdot)$ with unity (or a field $(X, +, \cdot)$). Let $\oplus$ and $\odot$ are m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$ with unity (or the field $(X, +, \cdot)$). If the structure $(M, \oplus, \odot)$ satisfies the following:

(1) $(M, \oplus)$ is a commutative group.

(2) $(M, \odot)$ is a commutative monoid

(3) Every non-multi-zero element of M has its inverse in M with respect to $\odot$.

(4) $\odot$ is multi-distributive over $\oplus$

then we define $(M, \oplus, \odot)$ to be a **multi-field** induced by the ring (or the field) $(X, +, \cdot)$ with unity.

3.21. *Example*

Consider the field $(X, +, \cdot)$ where $X = Z_3$, the set of all residue classes modulo 3, also $+$ and $\cdot$ respectively are addition and multiplication modulo 3. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in Z\}$ between X and G. Then for all $a \in X$ and for all $r \in G$, $(a, r) \in M$. Let $\oplus$ and $\odot$ be m-addition and m-multiplication, respectively, defined on M induced by the ring $(X, +, \cdot)$. Then $(M, \oplus, \odot)$ forms a multi-field induced by the field $(X, +, \cdot)$.

4. The multi-real number system

4.1. *Definition*

Let us consider the general mset $m(\mathbb{R})$ drawn from the field $(\mathbb{R}, +, \cdot)$, $\mathbb{R}$ being the set of all real numbers, characterized by the universal relation $\rho_{m(\mathbb{R})} = \mathbb{R} \times \mathbb{Q}^+$ from the set $\mathbb{R}$ to the set $\mathbb{Q}^+$ i.e. $(p, q) \in m(\mathbb{R})$ if and only if $p \in \mathbb{R}$ and $q \in \mathbb{Q}^+$.
Let us define two m-compositions $\oplus$ and $\odot$ on $m(\mathbb{R})$ as follows:
For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \oplus (r, s) = (p + r, qs)$ and $(p, q) \odot (r, s) = (pr, qs)$.
Also, define $<$ on $m(\mathbb{R})$ as follows: For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) < (r, s)$ if and only if there exists $(a, b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ ($\mathbb{R}^+$ is the set of all positive real numbers) and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$.
For $(p, q), (r, s) \in m(\mathbb{R})$, we define $(p, q) = (r, s)$ if and only if $p = r$ and $q = s$.
Also, for $(p, q), (r, s) \in m(\mathbb{R})$, we define $(p, q) \leq (r, s)$ if and only if $(p, q) < (r, s)$ or $(p, q) = (r, s)$.
Then every element of $m(\mathbb{R})$ is defined as a **multi-real number**.

4.2. *Remark*

For $(p, q), (r, s) \in m(\mathbb{R})$, $(p, q) \leq (r, s)$ if and only if there exist $(a, b) \in m(\mathbb{R})$ with $a \geq 0$ and $b \in \mathbb{N}$ such that $(r, s) = (p, q) \oplus (a, b)$.

4.3. *Definition*

(i) Define $m^+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{N}\}$. Every member of $m^+(\mathbb{R})$ is called a **positive multi-real number**.
(ii) Define $m^-(\mathbb{R}) = \{(a, \frac{1}{b}) : a < 0, b \in \mathbb{N}\}$. Every member of $m^-(\mathbb{R})$ is called a **negative multi-real number**.
(iii) Define $m_0(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a = 0, b \in \mathbb{Q}^+\}$. Every member of $m_0(\mathbb{R})$ is called a **multi-zero**.
(iv) Define $m_0^*(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a = 0, b \in \mathbb{N}\}$.
(v) Define $m_+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{Q}^+\}$. Immediately, $m^+(\mathbb{R}) \subsetneq m_+(\mathbb{R})$.
(vi) Define $m^*(\mathbb{R}) = m^+(\mathbb{R}) \cup m_0^*(\mathbb{R})$. Every member of $m^*(R)$ is called a **non-negative multi-real number**. i.e., $m^*(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a \geq 0, b \in \mathbb{N}\}$.
(vii) Define $m^{\#}(\mathbb{R}) = (m(\mathbb{R}) - m_0(\mathbb{R}))$. Every member of $m^{\#}(\mathbb{R})$ is called a **non-multi-zero multi-real numbers**. i.e., $m^{\#}(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a \neq 0, b \in \mathbb{Q}^+\}$.
(viii) Define $m_s(\mathbb{R}) = m(\mathbb{R}) - (m^+(\mathbb{R}) \cup m^-(\mathbb{R}) \cup \{(0, 1)\})$. Every member of $m_s(\mathbb{R})$ is called a **special multi-real number**.

(ix) Define $\overline{m}(\mathbb{R}) = \{(a,b) \in m(\mathbb{R}) : a \in \mathbb{R}, b \in \mathbb{N}\}$.
Immediately, $m^+(\mathbb{R}) \subset m^*(\mathbb{R}) \subset \overline{m}(\mathbb{R})$.

4.4. *Remark*

For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) < (r,s)$ if and only if there exists $(a,b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r,s) = (p,q) \oplus (a,b)$, i.e., if and only if $(r,s) = (p+a, qb)$, i.e., if and only if $r = p + a$ and $s = qb$, i.e., if and only if $p < r$ and $\frac{s}{q} \in \mathbb{N}$.
Therefore, for $(p,q),(r,s) \in m(\mathbb{R})$, define $(p,q) \leq (r,s)$ if and only if $(p,q) < (r,s)$ or $(p,q) = (r,s)$, i.e. if and only if ($p < r$ and $\frac{s}{q} \in \mathbb{N}$) or ($p = r$ and $q = s$).

4.5. *Proposition*

The following properties can be established easily:
(i) $(m(\mathbb{R}), \oplus)$ is a commutative group with $(0,1)$ as the identity element.
(ii) $(m(\mathbb{R}), \odot)$ is a commutative monoid with $(1,1)$ as the identity element.
(iii) For any element $(p,q) \in m^{\#}(\mathbb{R}) = \{(a,b) \in m(\mathbb{R}) : a \neq 0, b \in \mathbb{Q}^+\}$, its $\odot$-inverse exists in $m^{\#}(\mathbb{R})$ and is given by (p^{-1}, q^{-1}).
(iv) In fact, $(m^{\#}(\mathbb{R}), \circ)$ is a commutative group.
(v) Remark on the distributive property: $(p,q) \oplus ((r,s) \oplus (u,v)) = (p,q) \oplus (r+u, sv) = (p(r+u), qsv)$,
but $((p,q) \odot (r,s)) \oplus ((p,q) \odot (u,v)) = (pr, qs) \oplus (pu, qv) = (pr + pu, q^2 sv) = (p(r+u), q^2 sv)$,
so, $(p,q) \oplus ((r,s) \oplus (u,v)) \neq ((p,q) \odot (r,s)) \oplus ((p,q) \odot (u,v))$, in general.
(vi) Multi-distributive property: For all $(p,q),(r,s),(u,v) \in m(\mathbb{R})$, $(1,q) \odot ((p,q) \odot ((r,s) \oplus (u,v))) = ((p,q) \odot (r,s)) \oplus ((p,q) \odot (u,v))$. Let us define the above property as the **multi-distributive property** of $\odot$ over $\oplus$ on $m(\mathbb{R})$.
(vii) $(m(\mathbb{R}), \oplus, \odot)$ is a **multi-field**.
(viii) For $(a,b) \in m^{\#}(\mathbb{R})$ and for $(p,q),(r,s) \in m(\mathbb{R})$, $(a,b) \odot (p,q) = (a,b) \odot (r,s) \Rightarrow (p,q) = (r,s)$,
also, $(p,q) \odot (a,b) = (r,s) \odot (a,b) \Rightarrow ((p,q)) = (r,s)$.

4.6. *Proposition*

The relation $\leq$ is a partial order relation defined on $m(\mathbb{R})$ but not a chain.
Proof: $\leq$ is immediately a reflexive relation defined on $m(\mathbb{R})$.
For $(p,q),(r,s) \in m(\mathbb{R})$, let $(r,s) \leq (p,q)$ and $(p,q) \leq (r,s)$. Then there exists $(a,b),(c,d) \in m(\mathbb{R})$ with $a, c \geq 0$ and $b, d \in \mathbb{N}$ such that
$(p,q) = (r,s) \oplus (a,b)$ and $(r,s) = (p,q) \oplus (c,d)$. So, $(p,q) = ((p,q) \oplus (c,d)) \oplus (a,b)$ i.e., $(p,q) = (p+c, qd) \oplus (a,b)$ i.e. $(p,q) = (p+c+a, qdb)$. So, $p = p+c+b$ and $q = qdb \Rightarrow c+b = 0$

and $db = 1 \Rightarrow c = b = 0$ and $d = b = 1$. Therefore, $(p,q) = (r,s) \oplus (0,1)$. So, $(p,q) = (r,s)$. Therefore, $\leq$ is an antisymmetric relation defined on $m(\mathbb{R})$.

Finally, for $(p,q),(r,s),(u,v) \in m(\mathbb{R})$, let $(u,v) \leq (r,s)$ as well as $(r,s) \leq (p,q)$. Then there exists $(a,b),(c,d) \in m(\mathbb{R})$ with $a,c \geq 0$ and $b,d \in \mathbb{N}$ such that $(r,s) = (u,v) \oplus (a,b)$ and $(p,q) = (r,s) + (c,d)$. So, $(p,q) = ((u,v) \oplus (a,b)) \oplus (c,d)$. i.e., $(p,q) = (u,v) \oplus ((a,b) \oplus (c,d))$ i.e., $(p,q) = (u,v) \oplus (a+c, bd)$ where $c+a \geq 0$ and $bd \in \mathbb{N}$. Therefore, $(u,v) \leq (p,q)$. Therefore, $\leq$ is a transitive relation defined on $m(\mathbb{R})$.

Therefore, $\leq$ is a partial order relation defined on $m(\mathbb{R})$.

Now, $(2,3) \nleq (3,2)$ as well as $(3,2) \nleq (2,3)$. So, $\leq$ defined on $m(\mathbb{R})$ is not a chain.

4.7. *Definition*

1. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded below in the poset $(m(\mathbb{R}), \leq)$ if there exists $(a,b) \in m(\mathbb{R})$ such that $(a,b) \leq (x,y)$ for all $(x,y) \in m_1(\mathbb{R})$ and otherwise unbounded below. Also, $(a,b) \leq (x,y) \Rightarrow ((a,b) < (x,y))$ or $((a,b) = (x,y)) \Rightarrow (a < x$ and $\frac{y}{b} \in \mathbb{N})$ or $(a = x$ and $b = y)$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded below in the poset $(m(\mathbb{R}), \leq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{R}^+$ such that for all $(x,y) \in m_1(\mathbb{R})$, $(a < x$ and $\frac{y}{b} \in \mathbb{N})$ or $(a = x$ and $b = y)$ and otherwise unbounded below.

2. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded above in the poset $(m(\mathbb{R}), \leq)$ if there exists $(a,b) \in m(\mathbb{R})$ such that $(x,y) \leq (a,b)$ for all $(x,y) \in m_1(\mathbb{R})$ and otherwise unbounded above. Also, $(x,y) \leq (a,b) \Rightarrow ((x,y) < (a,b))$ or $((x,y) = (a,b)) \Rightarrow (x < a$ and $\frac{b}{y} \in \mathbb{N})$ or $(x = a$ and $y = b)$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded above in the poset $(m(\mathbb{R}), \leq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$ such that for all $(x,y) \in m_1(\mathbb{R})$, $(x < a$ and $\frac{b}{y} \in \mathbb{R})$ or $(x = a$ and $y = b)$ and otherwise unbounded above.

4.8. *Remark*

Every finite subset of $m(\mathbb{R})$ is bounded above and below both, also the greatest lower bound (*glb* in short) and the least upper bound (*lub* in short) of the set exist, but greatest and least element of the set may not exist. Also, there exist subsets of $m(\mathbb{R})$ for which the greatest lower bound and / or the least upper bound may not exist. e.g., consider the subsets $m_1(\mathbb{R}) = \{(2,3),(4,9\}$ and $m_2(\mathbb{R}) = \{(k, 2^k) : k \in \mathbb{N}\}$ and $m_3(\mathbb{R}) = \{(x,[x]) : x \in (1,2)\}$ of $m(\mathbb{R})$. Then $lub\ m_1(\mathbb{R}) = (4,9)$, $glbm_1(\mathbb{R}) = (2,3)$, $glbm_2(\mathbb{R}) = (1,2)$, $m_2(\mathbb{R})$ is unbounded above, $glbm_3(\mathbb{R}) = (1,1)$ and $lubm_3(\mathbb{R}) = (2,1)$. But every bounded above subset of $m(\mathbb{R})$ has a *lub* and every bounded below subset of $m(\mathbb{R})$ has a *glb*.

4.9. *Remark*

Let $m_1(\mathbb{R})$ be a subset of $m(\mathbb{R})$. Consider two mappings.
$\mu : m_1(\mathbb{R}) \to \mathbb{R}$ and $\lambda : m_1(\mathbb{R}) \to \mathbb{Q}^+$ defined by $\mu((x, y)) = x$ and $\lambda((x, y)) = y, (x, y) \in m_1(\mathbb{R})$. Then
(1) $m_1(\mathbb{R})$ will be bounded below the subset of $m(\mathbb{R})$ in the poset $(m(\mathbb{R}), \leq)$ if and only if the range(μ) is a bounded below subset of $\mathbb{R}$ and there exists $a \in \mathbb{Q}^+$ such that $\frac{x}{a} \in \mathbb{N}$ for all $x \in$range (λ).
(2) $m_1(\mathbb{R})$ will be a bounded above subset of $m(\mathbb{R})$ in the poset $(m(\mathbb{R}), \leq)$ if and only if the range(μ) is a bounded above subset of $\mathbb{R}$ and there exists $b \in \mathbb{Q}^+$ such that $\frac{b}{y} \in \mathbb{N}$ for all $y \in$range (λ).

4.10. *Proposition*

The poset $(m(\mathbb{R}), \leq)$ is a lattice.

4.11. *Proposition*

(Compatibility of $<$ with respect to $\oplus$ and $\odot$) For $(x, y), (r, s) \in m(\mathbb{R})$,
(i) $(p, q) < (r, s) \Rightarrow (p, q) \oplus (a, b) < (r, s) \oplus (a, b)$ for all $(a, b) \in m(\mathbb{R})$. (ii) $(p, q) < (r, s) \Rightarrow (p, q) \odot (a, b) < (r, s) \oplus (a, b)$ for all $(a, b) \in m_+(\mathbb{R}) = \{(a, b) \in m(R) : a > 0,$
$b \in \mathbb{Q}^+\}$.

4.12. *Proposition*

$(m(\mathbb{R}), \oplus, \odot, \leq)$ is a partially ordered multi-field induced by the field $(\mathbb{R}, +, \cdot)$.
Proof: The result follows from (7) of Proposition 47, Theorem 48 and Theorem 50.

4.13. *Proposition*

$(m(\mathbb{Q}), \oplus, \odot, \leq)$ is a subm-domain of $(m(\mathbb{R}), \oplus, \odot, \leq)$ where $m(\mathbb{Q}) = \mathbb{Q} \times \mathbb{Q}^+$.
Proof: Here, $m(\mathbb{Q}) \subseteq m(\mathbb{R})$. Also, restrictions of both the m-operations $\oplus$ and $\odot$, viz. $\boxplus$ and $\boxdot$, on $m(\mathbb{Q})$ are stable on $m(\mathbb{Q})$. Also, $(m(\mathbb{Q}), \boxplus, \boxdot)$ is a multi-field. Therefore, $(m(\mathbb{Q}), \oplus, \odot)$ is a subm-field of $(m(\mathbb{R}), \oplus, \odot)$. Also, $(m(\mathbb{Q}), \leq)$ is a partially ordered set. Therefore, $(m(\mathbb{Q}), \oplus, \odot, \leq)$ is a subm-domain of $(m(\mathbb{R}), \oplus, \odot, \leq)$.

4.14. *Proposition*

For $(p, q) \in m_+(\mathbb{R}) = \{(a, b) \in m(\mathbb{R}) : a > 0, b \in \mathbb{Q}^+\}$ and $(r, s) \in m(\mathbb{R})$, there exists $(a, b) \in m(\mathbb{N})$ such that $(a, b) \circ (p, q) > (r, s)$.
Proof: Since $(p, q) \in m_+(\mathbb{R})$ and $(r, s) \in m()$, so $p > 0$ and $r \in \mathbb{R}$. Therefore, by the Archimedean property of $\mathbb{R}$, there exists $a \in \mathbb{N}$ such that $ap > r$. Also, since $(p, q) \in m_+(\mathbb{R})$

and $(r,s) \in m(\mathbb{R})$, so, $q,s \in \mathbb{Q}^+$. Therefore, there exists $u,v,x,y \in \mathbb{N}$ such that $q = \frac{u}{v}$ and $s = \frac{x}{y}$. Therefore, $\frac{q}{s} = \frac{uy}{vx}$. Let us choose $b = vx \in \mathbb{N}$ such that $\frac{bq}{s} = uy \in \mathbb{N}$. Then $(a,b) \in m(\mathbb{N})$ and $(a,b) \circ (p,q) = (ap, bq) > (r,s)$, since $ap > r$ and $\frac{bq}{s} \in \mathbb{N}$. Hence the theorem.

4.15. *Definition*

For $n \in \mathbb{N}$ and $(p,q) \in m(\mathbb{R})$ define $n((p,q))$ and $((p,q))^n$ as follows:
$n((p,q)) = (p,q) \oplus (p,q) \oplus ... \oplus (p,q)$ (n times) $= (np, q^n)$ and
$(R_p^q)^n = R_p^q \odot R_p^q \odot ... \odot R_p^q$ (n times) $= (p^n, q^n)$.

4.16. *Definition*

For all $(a,b),(c,d) \in m(\mathbb{R})$, the multi-equation $(a,b) \oplus (x,y) = (c,d)$ in $(x,y) \in m(\mathbb{R})$ has a unique solution in $m(\mathbb{R})$. Also, for all $(a,b) \in m^{\#}(\mathbb{R})$ and $(c,d) \in m(\mathbb{R})$, the multi-equation $(a,b) \odot (x,y) = (c,d)$ in $(x,y) \in m(\mathbb{R})$ has a unique solution in $m(\mathbb{R})$.

4.17. *Definition*

Let us define a binary relation $\trianglelefteq$ on $m(\mathbb{R})$ as follows: (i) For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \triangleleft (r,s)$ if and only if $p = r$ and $\frac{s}{q} \in \mathbb{N} - \{1\}$. (ii) For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \trianglelefteq (r,s)$ if and only if $(p,q) \triangleleft (r,s)$ or $(p,q) = (r,s)$, i.e. if and only if $p = r$ and $\frac{s}{q} \in \mathbb{N}$.

4.18. *Proposition*

The binary relation $\trianglelefteq$ defined on $m(\mathbb{R})$ is a partial order relation.

4.19. *Definition*

Let us define a binary relation $\preceq$ on $m(\mathbb{R})$ as follows:
(i) For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \prec (r,s)$ if and only if $(p,q) < (r,s)$ or $(p,q) \triangleleft (r,s)$, i.e. if and only if (there exist $(a,b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r,s) = (p,q) \oplus (a,b)$) or ($p = r$ and $\frac{s}{q} \in \mathbb{N} - \{1\}$).

(ii) For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \preceq (r,s)$ if and only if $(p,q) \prec (r,s)$ or $(p,q) = (r,s)$.

4.20. *Proposition*

For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \preceq (r,s)$ if and only if $(p,q) < (r,s)$ or $(p,q) \trianglelefteq (r,s)$.
Proof: For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \preceq (r,s) \Rightarrow (p,q) \prec (r,s)$ or $(p,q) = (r,s) \Rightarrow ((p,q) < (r,s)$ or $(p,q) \triangleleft (r,s))$ or $(p,q) = (r,s) \Rightarrow ((p,q) < (r,s)$ or $(p,q) = (r,s))$ or $((p,q) \triangleleft (r,s)$ or $(p,q) = (r,s)) \Rightarrow (p,q) \leq (r,s)$ or $(p,q) \trianglelefteq (r,s)$.

4.21. *Remark*

For $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \preceq (r,s)$ if and only if $(p,q) < (r,s)$ or $(p,q) \trianglelefteq (r,s)$. i.e., if and only if there exist $((a,b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r,s) = (p,q) \oplus (a,b))$ or $(p = r$ and $\frac{s}{q} \in \mathbb{N})$. i.e., if and only if $(\exists(a,b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $(r,s) = (p+a, qb))$ or $(p = r$ and $\frac{s}{q} \in \mathbb{N})$, i.e., if and only if there exists $((a,b) \in m(\mathbb{R})$ with $a \in \mathbb{R}^+$ and $b \in \mathbb{N}$ such that $r = p + a$ and $s = qb)$ or $(p = r$ and $\frac{s}{q} \in \mathbb{N})$, i.e., if and only if $(p < r$ and $\frac{s}{q} \in \mathbb{N})$ or $(p = r$ and $\frac{s}{q} \in \mathbb{N})$, i.e., if and only if $p \leq r$ and $\frac{s}{q} \in \mathbb{N}$.

Therefore, for $(p,q),(r,s) \in m(\mathbb{R})$, $(p,q) \preceq (r,s)$ if and only if $(p,q) < (r,s)$ or $(p,q) \trianglelefteq (r,s)$, i.e. if and only if $(r < p$ and $\frac{s}{q} \in \mathbb{N})$ or $(p = r$ and $q = s)$.

4.22. *Proposition*

The relation $\preceq$ defined on $m(\mathbb{R})$ is a partial order relation.

Proof: The relation $\preceq$ is immediately reflexive on $m(\mathbb{R})$.

For $(u,v),(w,x) \in m(\mathbb{R})$, let $(u,v) \preceq (w,x)$ and $(w,x) \preceq (u,v)$. Then $((u,v) \leq (w,x)$ or $(u,v) \trianglelefteq (w,x))$ and $((w,x) \leq (u,v)$ or $(w,x) \trianglelefteq (u,v)) \Rightarrow ((u,v) \leq (w,x)$ and $(w,x) \leq (u,v))$ or $((u,v) \leq (w,x)$ and $(w,x) \trianglelefteq (u,v))$ or $((u,v) \trianglelefteq (w,x)$ and $(w,x) \leq (u,v))$ or $((u,v) \trianglelefteq (w,x)$ and $(w,x) \trianglelefteq (u,v))$.

Now $(u,v) \leq (w,x)$ and $(w,x) \leq (u,v) \Rightarrow (u,v) = (w,x)$.

$(u,v) \leq (w,x) \Rightarrow u \leq w$ and $\frac{x}{v} \in \mathbb{N}$.

$(w,x) \trianglelefteq (u,v) \Rightarrow w = u$ and $\frac{v}{x} \in \mathbb{N}$.

Therefore, $u = w$ and $v = x$.

Therefore, $(u,v) \leq (w,x)$ and $(w,x) \trianglelefteq (u,v) \Rightarrow u = w$ and $v = x \Rightarrow (u,v) = (w,x)$.

Similarly, $(u,v) \trianglelefteq (w,x)$ and $(w,x) \leq (u,v) \Rightarrow (u,v) = (w,x)$.

Lastly, $(u,v) \trianglelefteq (w,x)$ and $(w,x) \trianglelefteq (u,v) \Rightarrow (u,v) = (w,x)$.

So, for $(u,v),(w,x) \in m(\mathbb{R})$, $(u,v) \preceq (w,x)$ and $(w,x) \preceq (u,v) \Rightarrow (u,v) = (w,x)$.

Therefore, the relation $\preceq$ is symmetric on $m(\mathbb{R})$.

The transitive property of $\preceq$ on $m(\mathbb{R})$ follows from the transitive property of $\leq$ and $\trianglelefteq$.

Hence, the relation $\preceq$ is a partial order relation defined on $m(\mathbb{R})$.

4.23. *Definition*

1. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded below in the poset $(m(\mathbb{R}), \preceq)$ if there exists $(a,b) \in m(\mathbb{R})$ such that $(a,b) \preceq (x,y)$ for all $(x,y) \in m_1(\mathbb{R})$ and otherwise unbounded below. Also, $(a,b) \preceq (x,y) \Rightarrow (a \leq x$ and $\frac{y}{b} \in \mathbb{N})$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded below in the poset $(m(\mathbb{R}), \preceq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$ such that for all $(x,y) \in m_1(\mathbb{R})$, $(a \leq x$ and $\frac{y}{b} \in \mathbb{N})$ and otherwise unbounded below.

2. A subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ is said to be bounded above in the poset $(m(\mathbb{R}), \preceq)$ if there exists $(a, b) \in m(\mathbb{R})$ such that $(x, y) \preceq (a, b)$ for all $(x, y) \in m_1(\mathbb{R})$ and otherwise unbounded above. Also, $(x, y) \preceq (a, b) \Rightarrow (x \leq a$ and $\frac{b}{y} \in \mathbb{N})$. Therefore, a subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ will be bounded above in the poset $(m(\mathbb{R}), \preceq)$ if there exist $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$ such that for all $(x, y) \in m_1(\mathbb{R})$, $(x \leq a$ and $\frac{b}{y} \in \mathbb{N})$ and otherwise unbounded above.

4.24. *Proposition*

For $(p, q), (r, s) \in m(\mathbb{R})$,
(i) $(p, q) \prec (r, s) \Rightarrow (p, q) \oplus (a, b) \prec (r, s) \oplus (a, b)$ for all $(a, b) \in m(\mathbb{R})$. (ii) $(p, q) \prec (r, s) \Rightarrow (p, q) \odot (a, b) \prec (r, s) \odot (a, b)$ for all $(a, b) \in m_+(\mathbb{R})$.

4.25. *Proposition*

Therefore, $(m(\mathbb{R}), \oplus, \odot, \preceq)$ is a partially ordered multi-field.

4.25.1. *Remark*

Consider the partially ordered set $(\mathbb{Q}^+, |)$ where for $d, n \in \mathbb{Q}^+, d|n$ if and only if there exists $m \in \mathbb{N}$ such that $n = d \cdot m$ (in other words, $\frac{n}{d} \in \mathbb{N}$).
Let $q, s \in \mathbb{Q}^+$. Then there exists $a_1, b_1; a_2, b_2 \in \mathbb{N}$ such that $q = \frac{a_1}{b_1}$ and $s = \frac{a_2}{b_2}$. We know that $gcd\{q, s\} = \frac{gcd\{a_1,a_2\}}{lcm\{b_1,b_2\}}$ and $lcm\{q, s\} = \frac{lcm\{a_1,a_2\}}{gcd\{b_1,b_2\}}$ such that $\frac{q}{gcd\{q,s\}}, \frac{s}{gcd\{q,s\}}, \frac{lcm\{q,s\}}{q}, \frac{lcm\{q,s\}}{s} \in \mathbb{N}$.

Then $(\mathbb{Q}^+, |, gcd, lcm)$ is a lattice.

4.26. *Proposition*

$(m(\mathbb{R}), \preceq)$ is a lattice.
Proof: Let $(p, q), (r, s) \in m(\mathbb{R})$. Let $\mu_l = min\{p, r\}$, $\mu_u = max\{p, r\}$, $\lambda_l = gcd\{q, s\}$, and $\lambda_u = lcm\{q, s\}$. Then $\mu_l \leq p, r \leq \mu_u$. Also, $\frac{q}{\lambda_l}, \frac{s}{\lambda_l} \in \mathbb{N}$ and $\frac{\lambda_u}{q}, \frac{\lambda_u}{s} \in \mathbb{N}$.

So, $(\mu_l, \lambda_l) \preceq (p, q), (r, s)$, also, $(p, q), (r, s) \preceq (\mu_u, \lambda_u)$. i.e., (μ_l, λ_l) is a lower bound of (p, q) and (r, s), (μ_u, λ_u) is an upper bound of (p, q) and (r, s) in the poset $(m(\mathbb{R}), \preceq)$.

Let $(a, b) \in m(\mathbb{R})$ be the lower bound of (p, q) and (r, s) in the poset $(m(\mathbb{R}), \preceq)$.
Then $(a, b) \preceq (p, q), (r, s)$. Therefore, $a \leq p, r$ and $\frac{q}{b}, \frac{s}{b} \in \mathbb{N}$.

So, a is a lower bound of p and r. Let $\mu_l = min\{p, r\}$. Therefore, $a \leq \mu_l$.

Also, $\frac{q}{b}, \frac{s}{b} \in \mathbb{N}$, so b is a common divisor of q and s. Let $\lambda_l = gcd\{q, s\}$. Therefore, $\frac{b}{\lambda_l} \in \mathbb{N}$.

Since $a \leq \mu_l$ and $\frac{\lambda_l}{b} \in \mathbb{N}$, so $(a, b) \preceq (\mu_l, \lambda_l)$.

Since, for any lower bound (a, b) of (p, q) and (r, s), $(a, b) \preceq (\mu_l, \lambda_l)$, so (μ_l, λ_l) is the greatest lower bound (uniqueness can be easily proved using the symmetric property of $\preceq$) of (p, q) and (r, s).

Therefore, every pair of elements of $m(\mathbb{N})$ has a unique greatest lower bound.

Also, in a similar argument we can show that (μ_u, λ_u) is the unique least upper bound of (p, q) and (r, s).

Therefore, every pair of elements of $m(\mathbb{R})$ has a unique least upper bound.

Therefore, $(m(\mathbb{R}), \preceq)$ is a lattice.

4.27. *Proposition*

$(m(\mathbb{R}), \preceq)$ is a complete lattice.

Proof: Let $m_1(\mathbb{R})$ be a bounded below subset of $m(\mathbb{R})$ in poset $(m(\mathbb{R}), \preceq)$. Then there exists $(a, b) \in m(\mathbb{R})$ such that $(a, b) \preceq (x, y)$ for all $(x, y) \in m_1(\mathbb{R})$. Therefore, $a \leq x$ and $\frac{y}{b} \in \mathbb{N}$ for all $x \in$Range(μ) and $y \in$Range (λ). Therefore, by the order completeness property of R, *glb Range*(μ) exists and is equal to u, say, such that $u \leq x$ for all $x \in$ *Range*(μ) and for any lower bound a of *Range* (μ), $a \leq u$. Here, b is a common divisor of *Range* (λ), so *gcd Range* $(\lambda) = v$, say, there exists such that for any common divisor b of *Range* (λ), $\frac{v}{b} \in \mathbb{N}$. Then $(u, v) \in m(\mathbb{R})$ such that $(u, v) \preceq (x, y)$ for all $(x, y) \in m_1(\mathbb{R})$. Also, for any lower bound (a, b) of $m_1(\mathbb{R})$ in the poset $(m(\mathbb{R}), \preceq)$, $(a, b) \preceq (u, v)$. Therefore, $(u, v) = (glbRange(\mu_1), gcdRange(\lambda_1))$ is the *glb* of $m_1(\mathbb{R})$ in poset $(m(\mathbb{R}), \preceq)$.

Similarly, we can show that for any bounded above subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$, $(u, v) = (lubRange(\mu), lcmRange(\lambda)) \in m(\mathbb{R})$ is *lub* of $m_1(\mathbb{R})$ in the poset $(m(\mathbb{R}), \preceq)$. Therefore, $(m(\mathbb{R}), \preceq)$ is a complete lattice.

4.28. *Remark*

For any two members $(p, q), (s, t)$ of the lattice $(m(\mathbb{R}), \preceq)$, let us denote $lub\{(p, q), (s, t)\} = (p, q) \vee (s, t)$ and $glb\{(p, q), (s, t)\} = (p, q) \wedge (s, t)$.

4.29. *Proposition*

$(m(\mathbb{R}), \preceq)$ is a distributive lattice.

Proof: Let $m_1(\mathbb{R})$ be a bounded below subset of $m(\mathbb{R})$ in poset $(m(\mathbb{R}), \preceq)$.

For any $(p, q) \in m(\mathbb{R})$, we will show that

$$(p, q) \vee (\bigwedge_{(x,y)\in m_1(\mathbb{R})} (x, y)) = \bigwedge_{(x,y)\in m_1(\mathbb{R})} (((p, q)) \vee (x, y)).$$

Now, $(p,q) \vee (\bigwedge_{(x,y)\in m_1(\mathbb{R})}(x,y))$
$=(p,q) \vee (glb(Range(\mu)), gcd(Range(\lambda)))$

$= (max\{p, glb(Range(\mu)), lcm[q, gcd(Range(\lambda))]\})$

$=(\underset{x\in Range(\mu)}{glb} max\{p,x\}, \underset{x\in Range(\mu)}{glb} max\{p,x\})$
$= \bigwedge_{(x,y)\in m_1(\mathbb{R})}(max\{p,x\}, lcm[q,y])$

$= \bigwedge_{(x.y)\in m_1(\mathbb{R})}((p,q) \vee (x,y)).$

Similarly, for any subset $m_1(\mathbb{R})$ of $m(\mathbb{R})$ which is bounded above in poset $(m(\mathbb{R}), \preceq)$ and also, for any $(p,q) \in m(\mathbb{R})$, we can show that

$(p,q) \wedge (\bigvee_{(x,y)\in m_1(\mathbb{R})}(x,y)) = \bigvee_{(x,y)\in m_1(\mathbb{R})}((p,q) \wedge (x,y)).$

Also, for any two bounded subsets $m_1(\mathbb{R})$ and $m_2(\mathbb{R})$ of $m(\mathbb{R})$, we can show that

$(\bigwedge_{(p,q)\in m_1(\mathbb{R})}(p,q)) \vee (\bigwedge_{(x,y)\in m_2(\mathbb{R})}(x,y))$

$= \bigwedge_{(p,q)\in m_1(\mathbb{R})} \bigwedge_{(x,y)\in m_2(\mathbb{R})} ((p,q) \vee (x,y))$ and
$(\bigvee_{(p,q)\in m_1(\mathbb{R})}(p,q)) \wedge (\bigvee_{(x,y)\in m_2(\mathbb{R})}(x,y))$

$= \bigvee_{(p,q)\in m_1(\mathbb{R})} \bigvee_{(x,y)\in m_2(\mathbb{R})} ((p,q) \wedge (x,y))$. Therefore, $(m(\mathbb{R}), \preceq)$ is a distributive lattice.

4.30. *Proposition*

For $(p,q) \in m_+(\mathbb{R})$ and $(r,s) \in m(\mathbb{R})$, there exists $(a,b) \in m(\mathbb{N})$ such that $(a,b) \odot (p,q) \succ (r,s)$.

4.31. *Proposition*

$(m(\mathbb{Q}), +, \circ, \preceq)$ is a subm-domain of $(m(\mathbb{R}), +, \circ, \preceq)$.

Proof: Here, $m(\mathbb{Q}) \subseteq m(\mathbb{R})$. Also, restrictions of both operations $+$ and $\circ$, viz. $\boxplus$ and $\boxdot$, on $m(\mathbb{Q})$ are stable on $m(\mathbb{Q})$. Also, $(m(\mathbb{Q}), \boxplus, \boxdot)$ is a multi-field. Therefore, $(m(\mathbb{Q}), +, \circ)$ is a subm-field of $(m(\mathbb{R}), +, \circ)$. Also, $(m(\mathbb{Q}), \preceq)$ is a distributive lattice and $\prec$ is compatible with $+$ and $\circ$. Therefore, $(m(\mathbb{Q}), +, \circ, \preceq)$ is a subm-domain of $(m(\mathbb{R}), +, \circ, \preceq)$.

4.32. *Remark*

$(m(\mathbb{R}), \preceq)$ is a complete distributive lattice but $(m(\mathbb{Q}), \preceq)$ is a distributive lattice which is not complete.

4.33. *Remark*

In this remark we shall show that multi-real number system $m(\mathbb{R})$ is an extension of the real number system $\mathbb{R}$.
Consider a mapping $u : \mathbb{R} \to m(\mathbb{R})$ defined as follows: $u(x) = (x, 1), x \in \mathbb{R}$.
Then u is immediately an injective mapping.
Also, we note that $\forall\ x, y \in \mathbb{R}$,
(i) $u(x + y) = (x + y, 1) = (x, 1) \oplus (y, 1) = u(x) \oplus u(y)$.
(ii) $u(x \cdot y) = (x \cdot y, 1) = (x, 1) \odot (y, 1) = u(x) \odot u(y)$.
(iii) For $x, y \in \mathbb{R}$, let $x < y$, then $\exists\ a \in \mathbb{R}^+$ such that $x + a = y$. Since $x, y \in \mathbb{R}$, so $(x, 1), (y, 1) \in m(\mathbb{R})$. Also, $(x, 1) \oplus (a, 1) = (x + a, 1) = (y, 1)$.
Therefore, for $(x, 1), (y, 1) \in m(\mathbb{R})$, $\exists$, $a \in \mathbb{R}^+$ and $1 \in \mathbb{N}$ such that $(y, 1) = (x, 1) \oplus (a, 1)$.
Therefore, $(x, 1) < (y, 1)$.
Therefore, $u(x) < u(y)$.
Therefore, for $x, y \in \mathbb{R}$, $x < y \Rightarrow u(x) < u(y)$.
Therefore, u is a structure-preserving injective mapping from $\mathbb{R}$ into $m(\mathbb{R})$.
Therefore, $\mathbb{R}$ is embedded in $m(\mathbb{R})$.
So, multi-real number system $m(\mathbb{R})$ is an extension of the real number system $\mathbb{R}$.

5. Multiset redefined

5.1. *Definition*

A **$\mathbb{Q}^+$-multiset** ($\mathbb{Q}^+$-mset in short) M drawn from the crisp set X is represented by a characteristic function $\chi_M : X \to \mathbb{Q}^+ \cup \{0\}$. X is called the **domain**. We represent a $\mathbb{Q}^+$-mset M drawn from the crisp set X as $M = \{(x, \chi_M(x)) : x \in X\}$ or as $M = \{(x, \chi_M(x)) : x \in X, \chi_M(x) > 0\}$. The **support set** or **root set** of M is denoted by M^* (also denoted by **Supp(M)**) and is defined by $M^* = \{x \in X : \chi_M(x) > 0\}$, i.e. M^* is a crisp set. If for a $\mathbb{Q}^+$-mset M drawn from the crisp set X, $x \in M^*$, then we will write $x \in^* M$. For $x \in X$, $\chi_M(x)$ is called the **multiplicity** of x. A $\mathbb{Q}^+$-mset M drawn from a crisp set X is said to be **empty $\mathbb{Q}^+$-mset** denoted by ϕ if $\chi_M(x) = 0$ for all $x \in X$. The $\mathbb{Q}^+$-mset M drawn from the crisp set X can also be represented as $M = \{(x, \chi_M(x)) : x \in M^* \subseteq X\}$ simply discarding all pairs $(x, \chi_M(x))$ for which $\chi_M(x) = 0$, $x \in X$. Therefore, if M is a $\mathbb{Q}^+$-mset drawn from the crisp set X and $X \subseteq Y$, then M can also be considered as a $\mathbb{Q}^+$-mset drawn from the crisp set Y. Also, if M be a $\mathbb{Q}^+$-mset drawn from the crisp set X and $M^* \subseteq Y \subseteq X$, then M can also be considered as a $\mathbb{Q}^+$-mset drawn from the crisp set Y. Clearly, a crisp set M drawn from the universal set X can be considered as an $\mathbb{Q}^+$-mset with $\chi_M(x) = 1$ for all $x \in M^*$.
The **cardinality** of a $\mathbb{Q}^+$-mset M drawn from a crisp set X is denoted by card$M$ or $|M|$ and

is defined by card$M = \sum_x \chi_M(x)$. The **dimension** of a $\mathbb{Q}^+$-mset M is denoted by $/M/$ or by dim$M$ and is defined by dim$M = \sum_x \chi_{M^*}(x)$. The maximum value of the multiplicity function alt$A = max_{x \in M^*} \chi_M(x)$ is called the **height** of the $\mathbb{Q}^+$-mset M.

Let X be a non-empty crisp set. A $\mathbb{N}$-multiset ($\mathbb{N}$-mset in short) M drawn from X is characterized by a function $\chi_M : X \to \mathbb{N} \cup \{0\}$. We represent a $\mathbb{N}$-mset M drawn from X as $M = \{(x, \chi_M(x)) : x \in X\}$ where $\chi_M(x) \in \mathbb{N} \cup \{0\}$. Immediately, ϕ can be considered as a $\mathbb{N}$-mset drawn from any crisp set under consideration. Immediately, every $\mathbb{N}$-mset is a $\mathbb{Q}^+$-mset.

5.2. *Example*

Let $X = \{a, b, c, d, e\}$ be any crisp set. Then $M = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$ is a $\mathbb{Q}^+$-mset drawn from X. The $\mathbb{Q}^+$-mset M drawn from X can also be represented as $M = \{(a, 2), (b, 9), (c, \frac{1}{2}), (e, 12)\}$ by discarding $(d, 0)$. The support set of M is $M^* = \{a, b, c, e\}$. Alt$M = 12$, $|M| = 23\frac{1}{2}$ and $/M/ = 4$.
Also, $M = \{(a, 2), (b, 9), (e, 12)\}$ is a $\mathbb{N}$-mset drawn from X.

5.3. *Definition*

Two $\mathbb{Q}^+$-msets P and T drawn from a crisp set X are said to be **equal**, denoted by $P = T$, if and only if $\chi_P(x) = \chi_T(x)$ for all $x \in X$. $\mathbb{Q}^+$-msets P and T are unequal, denoted by $P \neq T$ if $\chi_P(x) \neq \chi_T(x)$ for at least one $x \in X$. For equal $\mathbb{Q}^+$-msets P and T drawn from a crisp set X, we have $|P| = |T|$, $/P/ = /T/$, $P^* = T^*$ and alt$P$ =alt$T$. Two $\mathbb{Q}^+$-msets P and T drawn from a crisp set X are said to be **equal in size** if they have equal cardinality and equal dimension. Equal $\mathbb{Q}^+$-msets are immediately equal in size, but the converse is not true, in general. The equality of $\mathbb{Q}^+$-msets is an equivalence relation.

Two $\mathbb{Q}^+$-msets P and T drawn from crisp sets X and Y respectively are said to be equal, denoted by $P = T$, if and only if
(i) $P^* = T^*$ and
(ii) $\chi_P(x) = \chi_T(x)$ for all $x \in P^* = T^*$.
In this case, $\mathbb{Q}^+$-mset P and T are considered as $\mathbb{Q}^+$-mset drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$.

5.4. *Definition*

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X. Then $\mathbb{Q}^+$-mset P is said to be a $\mathbb{Q}^+$**-submultiset** or $\mathbb{Q}^+$**-submset** of the $\mathbb{Q}^+$-mset T, denoted by $P \sqsubseteq T$, if and only if $\frac{\chi_T(x)}{\chi_P(x)} \in \mathbb{N}$ for all $x \in P^*$. If P is a submset of T, then T is called a $\mathbb{Q}^+$**-supermultiset** or $\mathbb{Q}^+$**-supermset** of P.

If P is a $\mathbb{Q}^+$-submset of T, then $|P| \leq |T|, /P/ \leq /T/, P^* \subseteq T^*, altP \leq altT$.
If $P \sqsubseteq T$ and $T \sqsubseteq P$, then $P = T$.
If $P \sqsubseteq T$ but $T \neq P$, the $\mathbb{Q}^+$-mset P is called the proper $\mathbb{Q}^+$-submset of the $\mathbb{Q}^+$-mset T and is denoted by $P \sqsubset T$.
The inclusion of $\mathbb{Q}^+$-msets is the **partial order relation**, since it is reflexive ($A \sqsubseteq A$), symmetric ($P \sqsubseteq T$ and $T \sqsubseteq P \Rightarrow T = P$) and transitive ($A \sqsubseteq B, B \sqsubseteq C \Rightarrow A \sqsubseteq C$).
Let X be a non-empty crisp set. Let P and M be two $\mathbb{N}$-msets drawn from X. Then P is said to be a $\mathbb{N}$**-submultiset** or $\mathbb{N}$**-submset** of M if and only if $\frac{\chi_M(x)}{\chi_P(x)} \in \mathbb{N}$ for all $x \in P^*$. Every $\mathbb{N}$-submet of a $\mathbb{N}$-mset is immediately a $\mathbb{N}$-mset.
A $\mathbb{Q}^+$-submset P of a $\mathbb{Q}^+$-mset M is a **whole** $\mathbb{Q}^+$**-submset** of M with each element in P having full multiplicity as in M. i.e., $\chi_P(x) = \chi_M(x)$ for every x in P^*.
A $\mathbb{Q}^+$-submset P of a $\mathbb{Q}^+$-mset M is a **partial whole** $\mathbb{Q}^+$**-submset** of M with at least one element in P having the same multiplicity as in M. i.e., $\chi_P(x) = \chi_M(x)$ for some x in P^*.
A $\mathbb{Q}^+$-submset P of a $\mathbb{Q}^+$-mset M is a full $\mathbb{Q}^+$-submset of M if $M^* = P^*$. The empty set ϕ is a whole $\mathbb{Q}^+$-submset of every $\mathbb{Q}^+$-mset but it is neither a full $\mathbb{Q}^+$-submset nor a partial whole $\mathbb{Q}^+$-submset of any non-empty $\mathbb{Q}^+$-mset M.
Let P and T be two $\mathbb{Q}^+$-msets drawn from crisp sets X and Y, respectively. Then $\mathbb{Q}^+$-mset P is said to be a $\mathbb{Q}^+$-submset of the $\mathbb{Q}^+$-mset T, denoted by $P \sqsubseteq T$, if and only if
(i) $P^* \subseteq T^*$ and
(ii) $\frac{\chi_T(x)}{\chi_P(x)} \in \mathbb{N}$ for all $x \in P^*$.
If P is a $\mathbb{Q}^+$-submset of T, then T is called a $\mathbb{Q}^+$-supermset of P.
In this case, the $\mathbb{Q}^+$-msets P and T are to be considered as $\mathbb{Q}^+$-msets drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$ and $\chi_T(x) = 0$ for all $x \in (X \cup Y) - T^*$. If we are not bothered about whether the multiplicity of an element is a rational number or a natural number, then we can simply say **mset** instead of $\mathbb{Q}^+$-mset or $\mathbb{N}$-mset and **submset** instead of $\mathbb{Q}^+$-submset or $\mathbb{N}$-submset.

5.5. *Example*

Consider the set $X = \{a, b, c, d, e\}$.
Consider the $\mathbb{Q}^+$-msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$
and $T = \{(a, 4), (b, 18), (c, 5), (d, \frac{3}{2}), (e, 24)\}$ drawn from X.
We see that $P^* = \{a, b, c, e\}$ and $Q^* = \{a, b, c, d, e\}$. So, $P^* \subseteq Q^*$.
Also, $\frac{\chi_T(a)}{\chi_P(a)} = \frac{4}{2} = 2 \in \mathbb{N}$, $\frac{\chi_T(b)}{\chi_P(b)} = \frac{18}{9} = 2 \in \mathbb{N}$, $\frac{\chi_T(c)}{\chi_P(c)} = \frac{5}{\frac{1}{2}} = 10 \in \mathbb{N}$, $\frac{\chi_T(e)}{\chi_P(e)} = \frac{24}{12} = 2 \in \mathbb{N}$.
Therefore, $\frac{\chi_T(x)}{\chi_P(x)} \in \mathbb{N}\ \forall x \in P^*$. So, T is a $\mathbb{Q}^+$-submset of P.
Therefore, $P \sqsubseteq T$ but $P \neq T$ so $P \sqsubset T$, i.e. P is a proper $\mathbb{Q}^+$-submset of T and consequently T is a $\mathbb{Q}^+$-supermset of P.

Again, consider the $\mathbb{N}$-msets, $P = \{(a, 2), (b, 3), (c, 4)\}$ and $T = \{(a, 4), (b, 36), (c, 16), (d, 5)\}$ drawn from X. Then $\mathbb{N}$-mset P is a proper $\mathbb{N}$-submset of the $\mathbb{N}$-mset T.

5.6. *Example*

Consider mset $M = \{(x, 2), (y, 3), (z, 5)\}$.
The submset $P_1 = \{(x, 2), (y, 3)\}$ is a whole $\mathbb{Q}^+$-submset and partial whole $\mathbb{Q}^+$-submset of M but it is not a full $\mathbb{Q}^+$-submset of M.
The $\mathbb{Q}^+$-submset $P_2 = \{(x, 1), (y, 3), (z, 1)\}$ is a partial whole $\mathbb{Q}^+$-submset and a full $\mathbb{Q}^+$-submset of M but it is not a whole $\mathbb{Q}^+$-submset of M.
The $\mathbb{Q}^+$-submset $P_3 = \{(x, 1), (y, 3)\}$ is a partial whole $\mathbb{Q}^+$-submset of M which is neither full $\mathbb{Q}^+$-submset of M nor a whole $\mathbb{Q}^+$-submset of M.

5.7. *Definition*

Le M be a $\mathbb{Q}^+$-mset drawn from a crisp set X. Let $x \in M^*$. Then the $\mathbb{Q}^+$-submset P of M is represented by a characteristic function $\chi_P : \{x\} \to \mathbb{Q}^+ \cup \{0\}$ where $\chi_P(x) = \chi_M(x)$ is called a $\mathbb{Q}^+$**-component** of $\mathbb{Q}^+$-multiset M.
Alternatively, a $\mathbb{Q}^+$-submset P of a $\mathbb{Q}^+$-mset M drawn from a crisp set X is a $\mathbb{Q}^+$-component if $\chi_P(x) = \chi_M(x)$ for all $x \in X$ and $\{x \in X : \chi_P(x) = \chi_M(x)\}$ is a singleton set, say, $\{x\}$, then let us denote it as $M_{\{x\}}(= P)$, i.e., a $\mathbb{Q}^+$-component is such a $\mathbb{Q}^+$-submset of a $\mathbb{Q}^+$-mset for which exactly one element of the support set belongs to it with the same count as in the $\mathbb{Q}^+$-mset.
A $\mathbb{Q}^+$-component is called a $\mathbb{Q}^+$**-zeron** if $\chi_P(x) = \chi_M(x) = 0$.
A $\mathbb{Q}^+$-component of a $\mathbb{Q}^+$-mset is a whole $\mathbb{Q}^+$-submset of the $\mathbb{Q}^+$-mset but not conversely.
A $\mathbb{Q}^+$-submset of a $\mathbb{Q}^+$-component of a $\mathbb{Q}^+$-mset is called the $\mathbb{Q}^+$**-subm-component** of the $\mathbb{Q}^+$-component.

5.8. *Definition*

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X. Then the $\mathbb{Q}^+$-mset M drawn from X is said to be the $\mathbb{Q}^+$**-multiset-union** (or $\mathbb{Q}^+$**-m-union**) of P and T, denoted by $M = P \sqcup T$, if and only if (i) $M^* = P^* \cup T^*$ and
(ii)

$$\chi_M(x) = \begin{cases} lcm\{\chi_P(x), \chi_T(x)\} & if x \in P^* \cap T^* \\ \chi_P(x) & if x \in P^* - T^* \\ \chi_T(x) & if x \in T^* - P^* \\ 0 & if x \in X - (P^* \cup T^*) \end{cases}$$

Let P and T be two $\mathbb{Q}^+$-msets drawn from crisp sets X and Y respectively. Then the $\mathbb{Q}^+$-mset M drawn from $X \cup Y$ is said to be the $\mathbb{Q}^+$-m-union of P and T, denoted by $M = P \sqcup T$, if and only if

(i) $M^* = P^* \cup T^*$ and

(ii)

$$\chi_M(x) = \begin{cases} lcm\{\chi_P(x), \chi_T(x)\} & if\ x \in P^* \cap T^* \\ \chi_P(x) & if\ x \in P^* - T^* \\ \chi_P(x) & if\ x \in T^* - P^* \\ 0 & if\ x \in (X \cup Y) - (P^* \cup T^*) \end{cases}$$

In this case, the $\mathbb{Q}^+$-msets P and T are to be considered as $\mathbb{Q}^+$-msets drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$ and $\chi_T(x) = 0$ for all $x \in (X \cup Y) - T^*$.

5.9. *Example*

Consider the crisp set $X = \{a, b, c, d, e\}$.
Consider the $\mathbb{Q}^+$-msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$
and $T = \{(a, 0), (b, 12), (c, 5), (d, \frac{3}{2}), (e, 8)\}$ drawn from X.
Then $P \sqcup T = \{(a, 2), (b, 36), (c, 5), (d, \frac{3}{2}), (e, 24)\}$.

5.10. *Remark*

$\mathbb{Q}^+$-m-union of two $\mathbb{Q}^+$-msets A and B drawn from a non-empty crisp set X is the smallest $\mathbb{Q}^+$-supermset of A and B both.

5.11. *Definition*

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X. Then the $\mathbb{Q}^+$-mset M drawn from X is said to be the $\mathbb{Q}^+$**-multiset-intersection** (or $\mathbb{Q}^+$**-m-intersection**) of P and T, denoted by $M = P \sqcap T$, if and only if

(i) $M^* = P^* \cap T^*$ and

$$\chi_M(x) = \begin{cases} gcd\{\chi_P(x), \chi_T(x)\} & if\ x \in P^* \cap T^* \\ 0 & if\ \ x \in X - (P^* \cap T^*) \end{cases}$$

Let P and T be two $\mathbb{Q}^+$-msets drawn from crisp sets X and Y, respectively. Then the $\mathbb{Q}^+$-mset M drawn from X is said to be the $\mathbb{Q}^+$-m-intersection of P and T, denoted by $M = P \sqcap T$, if and only if

(i) $M^* = P^* \cap T^*$ and

(ii)

$$\chi_M(x) = \begin{cases} gcd\{\chi_P(x), \chi_T(x)\} & if\ x \in P^* \cap T^* \\ 0 & if\ x \in (X \cup Y) - (P^* \cap T^*) \end{cases}$$

In this case, the $\mathbb{Q}^+$-msets P and T are to be considered as $\mathbb{Q}^+$-msets drawn from $X \cup Y$ and then $\chi_P(x) = 0$ for all $x \in (X \cup Y) - P^*$ and $\chi_T(x) = 0$ for all $x \in (X \cup Y) - T^*$.

5.12. *Example*

Consider the crisp set $X = \{a, b, c, d, e\}$.
Consider the $\mathbb{Q}^+$-msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (d, 0), (e, 12)\}$
and $T = \{(a, 0), (b, 12), (c, 5), (d, \frac{3}{2}), (e, 8)\}$ drawn from X.
Then $P \sqcap T = \{(a, 0), (b, 3), (c, \frac{1}{2}), (d, 0), (e, 4)\}$.

5.13. *Remark*

$\mathbb{Q}^+$-m-intersection of two $\mathbb{Q}^+$-msets A and B drawn from a non-empty crisp set X is the largest $\mathbb{Q}^+$-submset of A and B both.

5.14. *Definition*

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X. Then the $\mathbb{Q}^+$-mset M drawn from X is said to be the $\mathbb{Q}^+$**-multiset-complement** or $\mathbb{Q}^+$**-m-complement** of P on T, denoted by $M = T\neg P$, if and only if

$$\chi_M(x) = \begin{cases} 0 & if\ x \notin T^* \\ \chi_T(x) & if\ x \in T^* - P^* \\ \frac{\chi_T(x)}{\chi_P(x)} & if\ x \in T^* \cap P^* \end{cases}$$

5.15. *Remark*

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X. Then, $(T\neg P)$ is not necessarily a $\mathbb{Q}^+$-submset of T, also, $(T\neg P) \cup (P \sqcap T) \neq T$, in general, which can be justified by the following example.

5.16. *Example*

Consider the crisp set $X = \{a, b, c, d, e, f\}$.
Consider the $\mathbb{Q}^+$-msets $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (e, 12)\}$
and $T = \{(b, 18), (c, 5), (d, \frac{3}{2}), (e, 24)\}$ drawn from X.
Then $P^* = \{a, b, c, e\}$, $T^* = \{b, c, d, e\}$, $X - T^* = \{a, f\}$, $T^* - P^* = \{d\}$, $T^* \cap P^* = \{b, c, d\}$.
Then for $a \notin T^*, \chi_{(T\neg P)}(a) = 0$,
$b \in T^* \cap P^*, \chi_{(T\neg P)}(b) = \frac{\chi_T(b)}{\chi_P(b)} = \frac{18}{9} = 2$,
$c \in T^* \cap P^*, \chi_{(T\neg P)}(c) = \frac{\chi_T(c)}{\chi_P(c)} = \frac{5}{\frac{1}{2}} = 10$,
$d \in T^* - P^*, \chi_{(T\neg P)}(d) = \chi_T(d) = \frac{3}{2}$,

$e \in T^* \cap P^*, \chi_{(T\neg P)}(e) = \frac{\chi_T(e)}{\chi_P(e)} = \frac{24}{12} = 2,$
$f \notin T^*, \chi_{(T\neg P)}(f) = 0.$
Therefore, $T\neg P = \{(b,2),(c,10),(d,\frac{3}{2}),(e,2)\}$ which is not a $\mathbb{Q}^+$-submset of T.
Now $P \sqcap T = \{(b,9),(c,\frac{1}{2}),(e,12)\}$.
Therefore, $(T\neg P) \sqcup (P \sqcap T) = \{(b,18),(c,10),(d,\frac{3}{2}),(e,12)\} \neq T$.

5.17. *Definition*

Let T be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Let P be a $\mathbb{Q}^+$-submset of T satisfying the following: (i) P^* is a singleton set, say $\{\alpha\}$, (ii) $\frac{\chi_T(\alpha)}{\chi_P(\alpha)} = k \in \mathbb{N}$. Then P is called an **elementary** $\mathbb{Q}^+$**-submset** of T, denoted by $P = \{(\alpha, \chi_P(\alpha))\}$. $\mathbb{Q}^+$-Components are immediately elementary $\mathbb{Q}^+$-submsets of a $\mathbb{Q}^+$-mset. Also every elementary $\mathbb{Q}^+$-submset of an $\mathbb{Q}^+$-mset T can be treated as a $\mathbb{Q}^+$**-subm-element** $(\alpha, \chi_P(\alpha))$ of the $\mathbb{Q}^+$-mset T by simply identifying the elementary $\mathbb{Q}^+$-submset with the element α that P^* contains together with its multiplicity $\chi_P(\alpha)$. We express membership of $(\alpha, \chi_P(\alpha))$ in T using the symbol $(\alpha, \chi_P(\alpha)) \in^k T$. The crisp set of all $\mathbb{Q}^+$-subm-elements of an $\mathbb{Q}^+$-mset T is denoted by $\Omega(T)$. Immediately, $\Omega(T)$ is always an infinite $\mathbb{Q}^+$-general mset. Also, for all $(x,\lambda) \in \mathit{\Omega}(M)$, $\frac{\chi_M(x)}{\lambda} \in \mathbb{N}$. For (x,λ), x is called the **base** and λ is called the **multiplicity** of the $\mathbb{Q}^+$-subm-element (x,λ).
Let X be a non-empty crisp set. Let M be a $\mathbb{N}$-mset drawn from X. A $\mathbb{Q}^+$-subm-element (α, k) of M is said to be a $\mathbb{N}$-subm-element of M if $k \in \mathbb{N}$. The crisp set of all $\mathbb{N}$-subm-elements of the $\mathbb{N}$-mset M is denoted by $\mathit{\Omega}^*(M)$. $\mathit{\Omega}^*(M)$ is a $\mathbb{N}$-general mset. If $A \sqsubseteq B$, then $\Omega(A) \subseteq \Omega(B)$ and $\mathit{\Omega}^*(A) \subseteq \mathit{\Omega}^*(B)$.

5.18. *Definition*

Let T be a non-empty mset drawn from a crisp set X. Let P be another non-empty mset drawn from X satisfying the following: (i) $P^* \subseteq T^*$ is a singleton set, say $\{\alpha\}$, (ii) $\frac{\chi_P(\alpha)}{\chi_T(\alpha)} = k \in \mathbb{N}$. Then, obviously, P is an elementary $\mathbb{Q}^+$-mset drawn from X, denoted by $P = \{(\alpha, \chi_P(\alpha))\}$. This elementary $\mathbb{Q}^+$-mset P drawn from X can be treated as a $\mathbb{Q}^+$-superm-element $(\alpha, \chi_P(\alpha))$ of the $\mathbb{Q}^+$-mset T drawn from X by simply identifying the elementary $\mathbb{Q}^+$-mset $\{(\alpha, \chi_P(\alpha))\}$ with the element α that P^* contains together with its multiplicity $\chi_P(\alpha)$. We express membership of $(\alpha, \chi_P(\alpha))$ in T using the symbol $(\alpha, \chi_P(\alpha)) \in^{\frac{1}{k}} T$. The crisp set of all $\mathbb{Q}^+$-superm-elements of an $\mathbb{Q}^+$-mset T is denoted by $\omega(T)$. Immediately, $\omega(T)$ is always an infinite $\mathbb{Q}^+$-general mset.

5.19. *Definition*

In continuation to the above two definitions, if $\chi_P(\alpha) = \chi_T(\alpha)$, then $(\alpha, \chi_P(\alpha))$ is a $\mathbb{Q}^+$-subm-element of T as well as a $\mathbb{Q}^+$-superm-element of T. In this case, we define $(\alpha, \chi_P(\alpha))$ to be an $\mathbb{Q}^+$**-m-element** of T and it is denoted by $(\alpha, \chi_P(\alpha)) \in^1 T$ or simply by $(\alpha, \chi_P(\alpha)) \in T$.

5.20. *Definition*

Let M be an $\mathbb{Q}^+$-mset drawn from a crisp set X. Let (α, k) and (β, l) be two $\mathbb{Q}^+$-subm-elements of the $\mathbb{Q}^+$-mset M. Then
(i) $(\alpha, k) = (\beta, l)$ if and only if $\alpha = \beta$ and $k = l$.
(ii) (α, k) and (α, l) are said to be overlapping $\mathbb{Q}^+$-subm-elements if $gcd(k, l) \neq 1$.
(iii) (α, k) is said to be a divisor/factor of (β, l) if and only if $\{(\alpha, k)\} \sqsubseteq \{(\beta, l)\}$ i.e. if and only if $\alpha = \beta$ and $\frac{l}{k} \in \mathbb{N}$. Then (β, l) is said to be a multiple of (α, k).
(iv) (α, k) and (α, l) are said to be coprime $\mathbb{Q}^+$-subm-elements if $gcd(k, l) = 1$
(v) (α, k) and (β, l) are said to be distinct $\mathbb{Q}^+$-subm-elements of M if and only if $\alpha \neq \beta$.
Let M be an $\mathbb{Q}^+$-mset drawn from a crisp set X. Let (α, k) and (β, l) be two $\mathbb{Q}^+$-superm-elements of the $\mathbb{Q}^+$-mset M. Then
(i) $(\alpha, k) = (\beta, l)$ if and only if $\alpha = \beta$ and $k = l$.
(ii) (α, k) and (α, l) are said to be overlapping $\mathbb{Q}^+$-superm-elements if $gcd(k, l) \neq 1$.
(iii) (α, k) is said to be a divisor/factor of (β, l) if and only if $\{(\alpha, k)\} \sqsubseteq \{(\beta, l)\}$ i.e., if and only if $\alpha = \beta$ and $\frac{l}{k} \in \mathbb{N}$. Then (β, l) is said to be a multiple of (α, k).
(iv) (α, k) and (α, l) are said to be coprime $\mathbb{Q}^+$-superm-elements if $gcd(k, l) = 1$
(v) (α, k) and (β, l) are said to be distinct $\mathbb{Q}^+$-superm-elements of M if and only if $\alpha \neq \beta$.

5.21. *Remark*

Two $\mathbb{Q}^+$-subm-elements of an $\mathbb{Q}^+$-mset M is either distinct or one is a factor of the other or they are coprime or they are overlapping. similarly, two $\mathbb{Q}^+$-superm-elements of an mset M is either distinct or one is a factor of the other, or they are coprime or they are overlapping.

5.22. *Example*

Consider the crisp set $X = \{a, b, c, d, e\}$. Consider the $\mathbb{Q}^+$-mset $P = \{(a, 2), (b, 9), (c, \frac{1}{2}), (e, 12)\}$ drawn from X. Then $(c, \frac{1}{4}) \in^2 P$, $(b, 9) \in^1 P$, and also $(a, 4) \in^{\frac{1}{2}} P$. For two $\mathbb{Q}^+$-subm-elements $(b, 1)$ and $(b, 3)$ of P, $(b, 3)$ is a multiple of $(b, 1)$. Also, for two $\mathbb{Q}^+$-superm-elements $(b, 39)$ and $(b, 18)$ of M, $(b, 36)$ is a multiple of $(b, 18)$. $(c, \frac{1}{4})$ and $(b, 3)$ are two distinct $\mathbb{Q}^+$-subm-elements of M, also, $(b, 18)$ and $(e, 24)$ are two distinct $\mathbb{Q}^+$-superm-elements of M.

5.23. *Proposition*

Any collection of $\mathbb{Q}^+$-subm-elements of an $\mathbb{Q}^+$-mset generates a $\mathbb{Q}^+$-submset of the $\mathbb{Q}^+$-mset.

Proof:

Let M be an $\mathbb{Q}^+$-mset drawn from the crisp set X. Let $\Omega'(M)$ be a subset of $\Omega(M)$. Consider the $\mathbb{Q}^+$-mset $P = \bigsqcup_{(x,k)\in \Omega'(M)} \{(x,k)\}$. Now we show that $P \sqsubseteq M$. Immediately, $P^* \subseteq M^*$. Now, let $x \in P^*$. Then $\chi_P(x) = lcm\{k \in Q^+ : (x,k) \in \Omega'(M)\}$, which exists since, for all $(x,k) \in \Omega'(M)$, $\frac{\chi_M(x)}{k} \in \mathbb{N}$. Now if $(x,k) \in \Omega'(M) \subseteq \Omega(M)$, then $\frac{\chi_M(x)}{k} \in \mathbb{N}$, so $\frac{\chi_M(x)}{\chi_P(x)} \in \mathbb{N}$ which holds for all $x \in P^*$. Therefore, $P \sqsubseteq M$. Hence, the result.

5.24. *Remark*

The $\mathbb{Q}^+$-multiset generated by a $\mathbb{Q}^+$-general mset A of some $\mathbb{Q}^+$-subm-elements of a $\mathbb{Q}^+$-multiset M is denoted by $\gamma(A)$ and is defined by $\gamma(A) = \bigsqcup_{(x,k)\in A} \{(x,k)\}$ where $\chi_{\gamma(A)}(x) = lcm\{k \in \mathbb{Q}^+ : (x,k) \in A\}$.

Also a $\mathbb{Q}^+$-mset can be generated from the $\mathbb{Q}^+$-general mset of all its $\mathbb{Q}^+$-subm-elements. If $\Omega(M)$ is the $\mathbb{Q}^+$-general mset of all its $\mathbb{Q}^+$-subm-elements, then $\gamma(\Omega(M)) = M$ where $\chi_{\gamma(\Omega(M))}(x) = lcm\{k \in \mathbb{Q}^+ : (x,k) \in \Omega(M)\}$.

5.25. *Remark*

Let P and T be two $\mathbb{Q}^+$-msets drawn from crisp sets X. Then $P \sqsubseteq T$ if and only if every $\mathbb{Q}^+$-subm-element of P is a $\mathbb{Q}^+$-subm-element of T. Also, $P = T$ if and only if every $\mathbb{Q}^+$-subm-element of P is a $\mathbb{Q}^+$-subm-element of T as well as every $\mathbb{Q}^+$-subm-element of T is a $\mathbb{Q}^+$-subm-element of P.

5.26. *Proposition*

Every $\mathbb{Q}^+$-subm-element of $\mathbb{Q}^+$-m-intersection of two $\mathbb{Q}^+$-msets is a $\mathbb{Q}^+$-subm-element of each of the sets and conversely.

Proof:

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X.

Then for $x \in P^* \cap T^*$, $(x,k) \in^r (P \sqcap T)$ for some $k \in \mathbb{Q}^+$ and $r \in \mathbb{N} \Rightarrow \frac{\chi_{P\sqcap T}(x)}{k} = r \in \mathbb{N} \Rightarrow$
$\frac{gcd\{\chi_P(x),\chi_T(x)\}}{k} = r \in \mathbb{N} \Rightarrow \frac{\chi_P(x)}{k} = \frac{\chi_P(x)}{\chi_{P\sqcap T}(x)} r = \frac{\chi_P(x)}{gcd\{\chi_P(x),\chi_T(x)\}} r \in \mathbb{N}$
and $\frac{\chi_T(x)}{k} = \frac{\chi_T(x)}{\chi_{P\sqcap T}(x)} r = \frac{\chi_T(x)}{gcd(\chi_P(x),\chi_T(x))} r \in \mathbb{N}$
$\Rightarrow (x,k) \in^{\frac{\chi_P(x)}{\chi_{P\sqcap T}(x)} r} P$ and $(x,k) \in^{\frac{\chi_T(x)}{\chi_{P\sqcap T}(x)} r} T$.

Conversely, for $x \in P^* \cap T^*$, $(x,k) \in^i P$ and $(x,k) \in^j T$ for some $k \in \mathbb{Q}^+$ and $i,j \in \mathbb{N} \Rightarrow$
$\frac{\chi_P(x)}{k} = i \in \mathbb{N}$ and $\frac{\chi_T(x)}{k} = j \in \mathbb{N} \Rightarrow \frac{gcd\{\chi_P(x),\chi_T(x)\}}{k} \in \mathbb{N}$, $\frac{gcd\{\chi_P(x),\chi_T(x)\}}{k} = gcd\{i,j\} = r$ say

$\Rightarrow (x, k) \in^r (P \sqcap T)$.

So, every $\mathbb{Q}^+$-subm-element of $\mathbb{Q}^+$-m-intersection of two $\mathbb{Q}^+$-msets is a $\mathbb{Q}^+$-subm-element of each of the sets and conversely.

5.27. *Remark*

Let P and T be two $\mathbb{Q}^+$-msets drawn from a crisp set X. Then, for $x \in P^* \cap T^*$, $(x, k) \in^i P$ and $(x, k) \in^j T$ for some $k \in \mathbb{Q}^+$ and $i, j \in \mathbb{N} \Rightarrow \frac{\chi_P(x)}{k} = i \in \mathbb{N}$ and $\frac{\chi_T(x)}{k} = j \in \mathbb{N} \Rightarrow \frac{lcm\{\chi_P(x), \chi_T(x)\}}{k} \in \mathbb{N}$, $\frac{lcm\{\chi_P(x), \chi_T(x)\}}{k} = lcm\{i, j\} = r$ say $\Rightarrow (x, k) \in^r (P \sqcup T)$.

Also, for $x \in P^* - T^*$, $(x, k) \in^r P \Rightarrow (x, k) \in^r P \sqcup T$ and for for $x \in T^* - P^*$, $(x, k) \in^r T \Rightarrow (x, k) \in^r P \sqcup T$.

Therefore, if (x, k) is a $\mathbb{Q}^+$-subm-element of P or T, then (x, k) must be a $\mathbb{Q}^+$-subm-element of $P \sqcup T$.

But the converse is not always true.

5.28. *Example*

Let $A = \{(a, 24)\}$ and $B = \{(a, 30)\}$. Then $A \sqcup B = \{(a, 120)\}$. So, $(a, 20)$ is a $\mathbb{Q}^+$-subm-element of $A \sqcup B$ but neither a $\mathbb{Q}^+$-subm-element of A nor of B.

5.29. *Remark*

Let A and B be two $\mathbb{N}$-msets drawn from a crisp sets X.

Then for $x \in A^* \cup B^*$, (x, k) is a $\mathbb{N}$-subm-element of $A \sqcup B$

$\Rightarrow$ there exists $p, q \in \mathbb{N}$ with $gcd\{p, q\} = 1$ and $p \cdot q = k$ such that (x, p) is a $\mathbb{N}$-subm-element of A and (x, q) is a $\mathbb{N}$-subm-element of B.

5.30. *Remark*

$P \sqcap T = \phi$ if and only if $P^* \cap T^* = \phi$.

5.31. *Properties*

For three non-empty msets P, S and T drawn from a crisp set X,

(i) Three statements (a) $S \sqsubseteq T$ (b) $S \sqcap T = S$ and (c) $S \sqcup T = T$ are equivalent. (ii) $S \sqcup S = S$, $S \sqcap S = S$, (iii) $S \sqcup \phi = S$, $S \sqcap \phi = \phi$, (iv) $S \sqcup (S \sqcap T) = S$, $S \sqcap (S \sqcup T) = S$, (iv) $S \sqsubseteq S \sqcup T$, $S \sqcap T \sqsubseteq S$, (v) $S \sqcup T = T \sqcup S$, $S \sqcap T = T \sqcap S$, (vi) $P \sqcup (S \sqcup T) = (P \sqcup T) \sqcup S$, $P \sqcap (S \sqcap T) = (P \sqcap T) \sqcap S$. Proof: The proof of (i) to (vi) is immediate.

5.32. *Definition*

Let M be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Then the set of all $\mathbb{Q}^+$-submsets of M is called the $\mathbb{Q}^+$-power mset of M and is denoted by $P(M)$. $\mathbb{Q}^+$-Power mset of an $\mathbb{Q}^+$-mset is always an infinite crisp set.
Let X be a non-empty crisp set. Let M be a $\mathbb{N}^+$-msets drawn from X. Then the set of all $\mathbb{N}$-submets of M is called the $\mathbb{N}$-power mset of M and is denoted by $P^*(M)$.

5.33. *Definition*

Let M_δ be an $\mathbb{Q}^+$-mset drawn from a crisp set X for all $\delta \in \varDelta$. Then $\mathbb{Q}^+$-mset M drawn from X is said to be the $\mathbb{Q}^+$-m-union of M_δ, $\delta \in \varDelta$, denoted by $M = \underset{\delta\in\varDelta}{\sqcup} M_\delta$, if and only if
(i) $M^* = \underset{\delta\in\varDelta}{\cup} M^*_\delta$,
(ii) $lcm\{\chi_{M_\delta}(x) : x \in M^*_\delta\}$ exists for all $x \in M^*$,
(iii) $\chi_M(x) = lcm\{\chi_{M_\delta}(x) : x \in M^*_\delta\}$, $x \in M^*$.

5.34. *Definition*

Let M_δ be an $\mathbb{Q}^+$-mset drawn from a crisp set X for all $\delta \in \varDelta$. Then $\mathbb{Q}^+$-mset M drawn from X is said to be the $\mathbb{Q}^+$-m-intersection of M_δ, $\delta \in \varDelta$, denoted by $M = \underset{\delta\in\varDelta}{\cap} M_\delta$, if and only if

(i) $M^* = \underset{\delta\in\varDelta}{\sqcap} M^*_\delta$,
(ii) $gcd\{\chi_{M_\delta}(x) : x \in M^*_\delta\}$ exists for all $x \in M^*$,
(iii) $\chi_M(x) = gcd\{\chi_{M_\delta}(x) : x \in M^*_\delta\}$, $x \in M^*$.

5.35. *Remark*

1. If (x, k) is a $\mathbb{Q}^+$-subm-element of M_δ for some $\delta \in \varDelta$, then (x, k) must be a $\mathbb{Q}^+$-subm-element of $\underset{\delta\in\varDelta}{\sqcup} M_\delta$ but not conversely.
2. If (x, k) is a $\mathbb{Q}^+$-subm-element of M_δ for all $\delta \in \varDelta$, then (x, k) must be a $\mathbb{Q}^+$-subm-element of $\underset{\delta\in\varDelta}{\sqcap} M_\delta$ and conversely.

5.36. *Definition*

Let M be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Let $x \in M^*$. Let $I = \{k \in \mathbb{Q}^+ : \frac{\chi_M(x)}{k} \in \mathbb{N}\}$. Let $I^\# \subseteq I$.
Let $gcd\{k : k \in I^\#\}$ not exist, then we define $\chi_{\underset{k\in I^\#}{\sqcap}\{(x,k)\}}(x)$ to be 0.

5.37. *Remark*

Let M be a non-empty mset drawn from a crisp set X. Let $x \in M^*$. Let $I = \{k \in \mathbb{Q}^+ : \frac{\chi_M(x)}{k} \in \mathbb{N}\}$. Let $I^\# \subseteq I$.

Then $\frac{\chi_M(x)}{\chi_{\underset{k\in I^\#}{\sqcap}\{(x,k)\}}(x)} \in \mathbb{N}$ or $C_{\underset{k\in I^\#}{\sqcap}\{(x,k)\}}(x) = 0$ according as $gcd\{k : k \in I^\#\}$ exists or does not exist.

5.38. *Remark*

Let M be a non-empty mset drawn from a crisp set X. Let $x \in M^*$. Let $I = \{k \in \mathbb{Q}^+ : \frac{\chi_M(x)}{k} \in \mathbb{N}\}$. Let $I^\# \subseteq I$.

Then, $\frac{\chi_M(x)}{\chi_{\underset{k\in I^\#}{\sqcup}\{(x,k)\}}(x)} \in \mathbb{N}$.

5.39. *Proposition*

Let M be an $\mathbb{Q}^+$-mset drawn from a crisp set X. Let $\Omega(M)$ be the set of all $\mathbb{Q}^+$-submelements of M. Let $\Omega'(M) \subseteq \Omega(M)$. Then following are true:

1. $\underset{(x,k)\in\Omega'(M)}{\sqcup}\{(x,k)\} \sqsubseteq M$,
2. $\underset{(x,k)\in\Omega'(M)}{\sqcap}\{(x,k)\}$ is either ϕ or a non-trivial $\mathbb{Q}^+$-submset of M.

5.40. *Proposition*

Let M be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Then $(P(M), \sqsubseteq)$ is a lattice.

5.41. *Proposition*

Let M be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Then $(P(M), \sqsubseteq)$ is a complete lattice.

5.42. *Proposition*

Let M be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Then $(P(M), \sqsubseteq)$ is a distributive lattice.

Proof:

Let M be a non-empty $\mathbb{Q}^+$-mset drawn from a crisp set X. Let A, B, D be three $\mathbb{Q}^+$-submsets of M.

Now we will show that $A \sqcup (B \sqcap D) = (A \sqcup B) \sqcap (A \sqcup D)$.

Immediately, $[A \sqcup (B \sqcap D)]^* = [(A \sqcup B) \sqcap (A \sqcup D)]^*$.

Now, for all $x \in [A \sqcup (B \sqcap D)]^*$, $\chi_{A\sqcup(B\sqcap D)}(x) = lcm\{\chi_A(x), gcd\{\chi_B(x), \chi_D(x)\}\}$

$= gcd(\{lcm\{\chi_A(x), \chi_B(x)\}, lcm\{\chi_A(x), \chi_D(x)\}\} = \chi_{(A\sqcup B)\sqcap(A\sqcup D)}(x)$.

Therefore, $A \sqcup (B \sqcap D) = (A \sqcup B) \sqcap (A \sqcup D)$.

Similarly, we can show that $A \sqcap (B \sqcup D) = (A \sqcap B) \sqcup (A \sqcap D)$.

Therefore, Then $(P(M), \sqsubseteq)$ is a distributive lattice.

5.43. *Remark*

If we consider a mset of real numbers, where each real number appears with a specific multiplicity, and if we consider the collection of all the subm-elements (elements along with their respective multiplicities) of that mset, then this inherently reflects both the values and their multiplicities. To accommodate this, we define a new system called the *multi-real number system* (see Section 4), which extends the classical real number system (discussed in Remark 4.33) by including multiplicity as a structural component. This extension allows operations and comparisons to be interpreted in a way that respects and incorporates multiplicities (as described in Remark 4.33).

6. **Multi-metric space**

In this section, we prefer to write a multi-real number in the form R_a^b rather than (a, b) where $a \in \mathbb{R}$ and $b \in \mathbb{Q}^+$.

6.1. *Definition*

Let M be a $\mathbb{N}$-mset drawn from a non-empty crisp set X. Let ρ be a metric on X and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a mapping. Then a mapping $d : \Omega^*(M) \times \Omega^*(M) \to m^*(\mathbb{R})$, $[m^*(\mathbb{R}) = m^+(\mathbb{R}) \cup m_0^*(\mathbb{R})$, i.e., $m^*(\mathbb{R}) = \{R_a^b : a \geq 0, b \in \mathbb{N}\}]$, where for all $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) = R_{\rho(x,y)}^{f(i,j)}$, is said to be a **multi-metric** on the $\mathbb{N}$-mset M if d satisfies the following conditions:

(M1) For $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) \succeq R_0^1$.

(M2) For $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) \in m_0^*(\mathbb{R}) = \{R_a^b \in m(R) : a = 0 \text{ and } b \in \mathbb{N}\}$. if and only if $x = y$.

(M3) For all $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) = d((y, j), (x, i))$.

(M4) For all $(x, i), (y, j), (z, k) \in \Omega^*(M)$, $d((x, i), (y, j)) \preceq d((x, i), (z, k)) \oplus d((z, k), (y, j))$.

The $\mathbb{N}$-mset M together with a multi-metric d defined on M is said to be a **multi-metric space** and is denoted by (M, d). (M1), (M2), (M3), and (M4) are said to be multi-metric axioms.

6.2. *Example*

Let M be a $\mathbb{N}$-mset drawn from a non-empty crisp set X. Let us define a mapping $d : \Omega^*(M) \times \Omega^*(M) \to m^*(\mathbb{R})$ by $d((x, i), (y, j)) = R_0^1$ if $x = y$ and $d((x, i), (y, j)) = R_1^1$ if $x \neq y$,

$(x,i),(y,j) \in \Omega^*(M)$. Then d satisfies all the multi-metric axioms. So, d is a multi-metric on the $\mathbb{N}$-mset M.

6.3. *Example*

Let M be a $\mathbb{N}$-mset drawn from a non-empty crisp set X. Let us define a mapping $d : \Omega^*(M) \times \Omega^*(M) \to m^*(\mathbb{R})$ by $d((x,i),(y,j)) = R_0^{\frac{\chi_M(x)\cdot\chi_M(y)}{i\cdot j}}$ if $x = y$ and $d((x,i),(y,j)) = R_1^{\frac{\chi_M(x)\cdot\chi_M(y)}{i\cdot j}}$ if $x \neq y$, $(x,i),(y,j) \in \Omega^*(M)$. Then d satisfies all the multi-metric axioms. So, d is a multi-metric defined on the $\mathbb{N}$-mset M.

6.4. *Example*

Let M be a $\mathbb{N}$-mset drawn from a non-empty crisp set X. Let ρ be a crisp metric on M^*. Let us define a mapping $d^\rho : \Omega^*(M) \times \Omega^*(M) \to m^*(\mathbb{R})$ by $d^\rho((x,i),(y,j)) = R_{\rho(x,y)}^{i\cdot j}$, $(x,i),(y,j) \in \Omega^*(M)$. Then d^ρ satisfies all the multi-metric axioms. So, d^ρ is a multi-metric on the $\mathbb{N}$-mset M.

6.5. *Example*

Let M be a $\mathbb{N}$-mset drawn from a non-empty crisp set X. Let ρ be a crisp metric on M^*. Let us define a mapping $d_\rho : \Omega^*(M) \times \Omega^*(M) \to m^*(\mathbb{R})$ by $d_\rho((x,i),(y,j)) = R_{\rho(x,y)}^{lcm\{i,j\}}$, $(x,i),(y,j) \in \Omega^*(M)$. Then d_ρ satisfies all the multi-metric axioms. So, d_ρ is a multi-metric on the $\mathbb{N}$-mset M.

6.6. *Example*

Consider the $\mathbb{N}$-mset $M = \{(5,2),(10,1)\}$ drawn from $\mathbb{R}$.
Consider the usual metric ρ on $\mathbb{R}$.
Here, $\Omega^*(M) = \{(5,2),(5,1),(10,1)\}$.
Then $d^\rho((5,2),(5,2)) = R_0^4$,
$d^\rho((5,2),(5,1)) = R_0^2 = d^\rho((5,1),(5,2))$,
$d^\rho((5,2),(10,1)) = R_5^2 = d^\rho((10,1),(5,2))$, $d^\rho((5,1),(5,1)) = R_0^1$,
$d^\rho((5,1),(10,1)) = R_5^1 = d^\rho((10,1),(5,1))$, $d^\rho((10,1),(10,1)) = R_0^1$,
Also, $d_\rho((5,2),(5,2)) = R_0^2$, $d_\rho((5,2),(5,1)) = R_0^2 = d_\rho((5,1),(5,2))$,
$d_\rho((5,2),(10,1)) = R_5^2 = d_\rho((10,1),(5,2))$, $d_\rho((5,1),(5,1)) = R_0^1$,
$d_\rho((5,1),(10,1)) = R_5^1 = d_\rho((10,1),(5,1))$, $d_\rho((10,1),(10,1)) = R_0^1$.

6.7. *Definition*

Let (M, d) be a multi-metric space and P be a non-null $\mathbb{N}$-submset of the $\mathbb{N}$-mset M. Then the mapping $d_P : \Omega^*(P) \times \Omega^*(P) \to m^*(\mathbb{R})$ given by $d_P((x,i),(y,j)) = d((x,i),(y,j))$ for all $(x,i),(y,j) \in \Omega^*(P)$ is a multi-metric on P. This multi-metric d_P is called the relative multi-metric induced on P by d. The multi-metric space (P, d_P) is called a **metric submspace** or **sub multi-metric space** of the multi-metric space (M, d).

6.8. *Example*

Consider the $\mathbb{N}$-mset $M = \{(5,2),(10,1)\}$ drawn from $\mathbb{R}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.
Here, $\Omega^*(M) = \{(5,2),(5,1),(10,1)\}$. Then $d^\rho((5,2),(5,2)) = R_0^4$, $d^\rho((5,2),(5,1)) = R_0^2 = d^\rho((5,1),(5,2))$, $d^\rho((5,2),(10,1)) = R_5^2 = d^\rho((10,1),(5,2))$, $d^\rho((5,1),(5,1)) = R_0^1$, $d^\rho((5,1),(10,1)) = R_5^1 = d^\rho((10,1),(5,1))$, $d^\rho((10,1),(10,1)) = R_0^1$. Consider $\mathbb{N}$-submset $P = \{(5,2)\}$ of M. Then $\Omega^*(P) = \{(5,2),(5,1)\}$. Consider the mapping $d_P^\rho : \Omega^*(P) \times \Omega^*(P) \to m^*(R)$ given by $d_P^\rho((5,2),(5,2)) = R_0^4$, $d_P^\rho((5,2),(5,1)) = R_0^2 = d_P^\rho((5,1),(5,2))$, $d_P^\rho((5,1),(5,1)) = R_0^1$. Then d_P^ρ is a multi-metric on P. Therefore, the multi-metric space (P, d_P^ρ) is a metric submspace of (M, d_P^ρ).

6.9. *Definition*

Let (M, d) be a multi-metric space and P be a non-null $\mathbb{N}$-submset of the $\mathbb{N}$-mset M. Then the **diameter** of P is denoted by $\delta(P)$ and is defined by $\delta(P) = lub\{d_P((x,i),(y,j)) : (x,i),(y,j) \in \Omega^*(P)\}$, provided lub exists [As defined earlier, $lub\{R_p^q \in m(R) : p \in S \subseteq R, q \in T \subseteq Q^+\} = R_\mu^\nu$, where $\mu = lub\{p : p \in S \subseteq R\}$ in the poset $(R, \leq)$ and $\nu = lcm\{q : q \in T \subseteq Q^+\}$, if they exist]. It is obvious that for any non-null $\mathbb{N}$-submset P of the $\mathbb{N}$-mset M, $\delta(P) \succeq R_0^1$, if it exist. If P^* is a finite set, then $\delta(P)$ must exists.

6.10. *Example*

Consider the $\mathbb{N}$-mset $M = \{(7,3),(6,2),(-1,5),(0,9),(8,12)\}$ drawn from $\mathbb{R}$. Consider the usual metric ρ on $M^* \subseteq R$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the $\mathbb{N}$-submset $P = \{(7,1),(-1,5)\}$ of M. Consider the relative multi-metric d_P^ρ induced on P by d^ρ. Here, $\Omega^*(P) = \{(7,1),(-1,5),(-1,1)\}$.
Then the mapping $d_P^\rho : \Omega^*(P) \times \Omega^*(P) \to m^*(R)$ is given by
$d_P^\rho((7,1),(7,1)) = R_0^1$, $d_P^\rho((7,1),(-1,5)) = R_8^5 = d_P^\rho((-1,5),(7,1))$,
$d_P^\rho((7,1),(-1,1)) = R_8^1 = d_P^\rho((-1,1),(7,1))$, $d_P^\rho((-1,5),(-1,5)) = R_0^{25}$,

$d^{\rho}_{P}((-1,5),(-1,1)) = R^5_0 = d^{\rho}_{P}((-1,1),(-1,5))$, $d^{\rho}_{P}((-1,1),(-1,1)) = R^1_0$.
$lub\{0,8\} = 8$, $lcm\{1,5,25\} = 25$. Then $\delta(P) = R^{25}_8$.

6.11. *Remark*

Let (M,d) be a multi-metric space and P be a non-null $\mathbb{N}$-submset of the $\mathbb{N}$-mset M. Let the diameter of $P = \delta(P)$, say, exist. Then, for all $(x,i),(y,j) \in \Omega^*(P)$, $d((x,i),(y,j)) \preceq \delta(P)$.

6.12. *Remark*

Let (M,d) be a multi-metric space and P be a $\mathbb{N}$-submset of the $\mathbb{N}$-mset M. Then

(i) $\delta(P) \in m^*_0(\mathbb{R})$ if and only if P is a component of M.

(ii) For every non-null $\mathbb{N}$-submsets P and Q of M, $P \sqsubseteq Q \Rightarrow \delta(P) \preceq \delta(Q)$, if $\delta(P)$ and $\delta(Q)$ both exist.

6.13. *Definition*

Let (M,d) be a multi-metric space. Let (α,k) be a fixed $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset M satisfying $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{C_M(\alpha)}{k} \in \mathbb{N}$ [i.e., $\alpha \in^{\frac{C_M(\alpha)}{k}} M$] and P be a non-null $\mathbb{N}$-submset of M. Then **distance** between the $\mathbb{N}$-subm-element (α,k) and the submset P is denoted by $\delta((\alpha,k),P)$ and is defined by $\delta((\alpha,k),P) = glb\{d((\alpha,k),(x,l)) : (x,l) \in \Omega^*(P)\}$. [As defined earlier, $glb\{R^q_p \in m(\mathbb{R}) : p \in S \subseteq R, q \in T \subseteq Q^+\} = R^{\lambda}_{\mu}$, where $\mu = glb\{p : p \in S \subseteq R\}$ in the poset $(R,\leq)$ and $\lambda = gcd\{q : q \in T \subseteq Q^+\}$, if they exist]. It is obvious that for any non-null $\mathbb{N}$-submset P of the $\mathbb{N}$-mset M and for any $\mathbb{N}$-subm-element (α,k) of the $\mathbb{N}$-mset M satisfying $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{C_M(\alpha)}{k} \in \mathbb{N}$, $\delta((\alpha,k),P) \succeq R^t_0$ for some $t \in \mathbb{N}$. If P be a natural $\mathbb{N}$-submset of the $\mathbb{N}$-mset M such that P^* is finite, then for any $\mathbb{N}$-subm-element (α,k) of M, $\delta((\alpha,k),P)$ exists.

If (α,k) be a fixed $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset P, then $\delta((\alpha,k),P) = R^t_0$ for some $t \in \mathbb{N}$. On the other hand $\delta((\alpha,k),P) = R^t_0$ for some $t \in \mathbb{N}$ may hold where (α,k) is not a $\mathbb{N}$-subm-element of P which can be justified by the following example.

6.14. *Example*

Consider the $\mathbb{N}$-mset $M = \{(\frac{1}{2n},6) : n \in \mathbb{N}\} \cup \{(\frac{1}{2n+1},4) : n \in \mathbb{N}\} \cup \{(0,2)\} \cup \{(n,n) : n \in \mathbb{N}\}$ drawn from $\mathbb{R}$. Consider the usual metric ρ on $M^* \subseteq \mathbb{R}$. Let d^{ρ} be the multi-metric on the mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the $\mathbb{N}$-subm-element $(0,1)$ of M and the $\mathbb{N}$-submset $P = \{(\frac{1}{2n},6) : n \in \mathbb{N}\} \cup \{(\frac{1}{2n+1},4) : n \in \mathbb{N}\}$ of M. Then $\delta((0,1),P) = R^4_0$, but $(0,1)$ is not a $\mathbb{N}$-subm-element of P.

6.15. *Example*

Consider the $\mathbb{N}$-mset $M = \{(7,3),(6,2),(-1,5),(0,9),(8,12)\}$ drawn from $\mathbb{R}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric τ on M^* as in example 6.4. Consider the $\mathbb{N}$-submset $P = \{(7,1),(-1,5),(0,3),(8,2)\}$ of M. Consider the $\mathbb{N}$-subm-element $(8,3)$ of the $\mathbb{N}$-mset M. In fact, $(8,3) \in^4 M$. Then $\delta((8,3),P) = glb\{d((8,3),(x,i)) : x \in \Omega^*(P)\}$, provided that the glb exists $= glb\{R_1^3, R_9^{15}, R_9^3, R_8^9, R_8^3, R_0^6, R_0^3\} = R_0^3$ (since $glb\{1,9,8,0\} = 0$ and $gcd\{3,15,9,6\} = 3$). Therefore, $\delta((8,3),P) = R_0^3$ also, $(8,3)$ is a $\mathbb{N}$-subm-element of P.

6.16. *Example*

Consider the $\mathbb{N}$-mset $M = \{(7,3),(6,2),(-1,5),(0,9),(8,12)\}$ drawn from $\mathbb{R}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric ρ on M^* as in Example 6.4. If we consider $S = \{(7,1),(-1,5),(0,3)\} \sqsubseteq M$ and $(8,3) \in^4 M$, then $\delta((8,3),S) = glb\{R_1^3, R_9^{15}, R_9^3, R_8^9, R_8^3\}$
$= R_1^3$. Here, $(8,3)$ is not a $\mathbb{N}$-subm-element of S.

6.17. *Definition*

Let (M,d) be a multi-metric space and P,T be two non-null $\mathbb{N}$-submsets of the $\mathbb{N}$-mset M. The distance between the $\mathbb{N}$-msets P and T is denoted by $\delta(P,T)$ and defined by $\delta(P,T) = glb\{d((x,i),(y,j)) : (x,i) \in \Omega^*(P), (y,j) \in \Omega^*(T)\}$.

Immediately, for any two non-null $\mathbb{N}$-submsets P and T of the $\mathbb{N}$-mset M, $\delta(P,T) = \delta(T,P)$. Also, for any $\mathbb{N}$-subm-element (α,k) of the $\mathbb{N}$-mset M and for any non-null $\mathbb{N}$-submset P of M, $\delta((\alpha,k),P) = \delta(\{(\alpha,k)\},P)$.

Again, for any two non-null $\mathbb{N}$-submsets P and T of the $\mathbb{N}$-mset M, if $P \sqcap T \neq \phi$, then $\delta(P,T) = R_0^t$ for some $t \in \mathbb{N}$.

However, the converse is not necessarily true. It may happen that, for two non-null $\mathbb{N}$-submsets P and T of the $\mathbb{N}$-mset M, $\delta(P,T) = R_0^t$ for some $t \in \mathbb{N}$ but $P \sqcap T = \phi$. This can be justified by the following example.

6.18. *Example*

Consider the $\mathbb{N}$-mset $M = \{(\frac{1}{2n},6) : n \in \mathbb{N}\} \cup \{(\frac{1}{2n+1},4) : n \in \mathbb{N}\} \cup \{(0,2)\} \cup \{(n,n) : n \in \mathbb{N}\}$ drawn from $\mathbb{R}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the $\mathbb{N}$-submset $P = \{(\frac{1}{2n+1},4) : n \in \mathbb{N}\}$ and the $\mathbb{N}$-submset $T = \{(\frac{1}{2n},6) : n \in \mathbb{N}\}$ of M. Then $\delta(P,T) = R_0^{24}$, but $P \sqcap T = \phi$.

6.19. *Example*

Consider the $\mathbb{N}$-mset $M = \{(7,3),(6,2),(-1,5),(0,9),(8,12)\}$ drawn from $\mathbb{R}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4. Consider the $\mathbb{N}$-submset $P = \{(7,3),(6,1)\}$ and $T = \{(-1,1),(0,3)\}$ of M. Then $\delta(P,T) = glb\{R_8^3, R_7^1, R_7^9, R_6^3, R_7^3, R_8^1, R_6^1\} = R_6^1$.

6.20. *Definition*

Let (M,d) be a multi-metric space. For two $\mathbb{N}$-subm-elements (α,k) and (β,l) of the $\mathbb{N}$-mset M, the distance between (α,k) and (β,l) is defined as the diameter of the $\mathbb{N}$-submset $\{(\alpha,k),(\beta,l)\}$ of M, denoted by $\delta((\alpha,k),(\beta,l))$.

6.21. *Property*

Let M be a $\mathbb{N}$-mset drawn from the crisp set X. Let d be a multi-metric defined on M. Let (α,k) and (β,l) be two $\mathbb{N}$-subm-elements of the $\mathbb{N}$-mset M.

If (α,k) and (β,l) are distinct subm-elements of M, then $\delta((\alpha,k),(\beta,l)) \in m^+(\mathbb{R})$.

6.22. *Definition*

Let (M,d) be a multi-metric space. Let (α,k) be a $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset M that satisfies $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{\chi_M(\alpha)}{k} \in \mathbb{N}$ and $R_p^q \in m^+(\mathbb{R})$ such that $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.
Let us define $B^*((\alpha,k),R_p^q) = \{(x,l) \in \Omega^*(M) : d((\alpha,k),(x,l)) < R_p^q$ and $\frac{q}{f(k,l)\cdot f(l,l))} \in \mathbb{N}\}$.
Then $B^*((\alpha,k),R_p^q)$ is called a **open ball** with centre at (α,k) and radius R_p^q. $B^*((\alpha,k),R_p^q)$ is a general mst.
Also, define $B((\alpha,k),R_p^q) = \bigsqcup_{(x,l)\in B^*((\alpha,k),R_p^q)} \{(x,l)\}$.
Then $B((\alpha,k),R_p^q)$ is called a **multi-open ball** (or **m-open ball**) with centre at (α,k) and radius R_p^q. $B((\alpha,k),R_p^q)$ is a $\mathbb{N}$-mset.
Immediately, (α,k) is an element of $B^*((\alpha,k),R_p^q)$ and also a $\mathbb{N}$-subm-element of $B((\alpha,k),R_p^q)$.
Therefore, if $B^*((\alpha,k),R_p^q)$ is an open ball with centre at (α,k) and radius R_p^q, then $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.

6.23. *Example*

Consider the $\mathbb{N}$-mset $M = \{(7,3),(6,2),(-1,5),(5,2),(0,9),(8,12)\}$ drawn from $\mathbb{R}$. Then $\Omega^*(M) = \{(7,1),(7,3),(6,1),(6,2),(-1,1),(-1,5),(5,1),(5,2),(0,1),(0,3),(0,4),(8,1),$
$(8,2),(8,3),(8,4),(8,6),(8,12)\}$. Consider the usual metric space ρ defined on $M^* \subseteq R$. Consider the multi-metic d^ρ defined on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.

Consider the $\mathbb{N}$-subm-element $(8,3) \in^4 M$.
Then $B^*((8,3), R_2^{81}) = \{(7,1),(7,3),(8,1),(8,3)\}$.
$B((8,3), R_2^{81}) = \{(7,3),(8,3)\}$.
$B^*((8,3), R_2^{648}) = \{(7,1),(7,3),(8,1),(8,2),(8,3),(8,6)\}$.
$B((8,3), R_2^{648}) = \{(7,3),(8,6)\}$.

6.24. *Remark*

Let (M,d) be a multi-metric space. Let (α,k) be a $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset M that satisfies $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{\chi_M(\alpha)}{k} \in \mathbb{N}$.
Then for $R_p^q, R_m^n \in m^+(\mathbb{R})$, $R_p^q \preceq R_m^n$ and $\frac{q}{[f(k,k)]^2} \in \mathbb{N} \Rightarrow B^*((\alpha,k), R_p^q) \subseteq B^*((\alpha,k), R_m^n)$.

6.25. *Definition*

Let (M,d) be a multi-metric space. Let (α,k) be a fixed $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset M that satisfies $\alpha \in M^*$, $k \in \mathbb{N}$ and $\frac{C_M(\alpha)}{k} \in \mathbb{N}$ [i.e., $\alpha \in^{\frac{C_M(\alpha)}{k}} M$] and $R_p^q \in m^+(R)$ such that $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.
Let us define $B^*[(\alpha,k), R_p^q] = \{(x,l) \in \Omega^*(M) : d((\alpha,k),(x,l)) \preceq R_p^q$ and $\frac{q}{f(k,l)\cdot f(l,l))} \in \mathbb{N}\}$.
Then $B^*[(\alpha,k), R_p^q]$ is called a **closed ball** with centre at (α,k) and radius R_p^q. $B^*[(\alpha,k), R_p^q]$ is a general mset. Also, define $B[(\alpha,k), R_p^q] = \bigsqcup_{(x,l)\in B^*[(\alpha,k),R_p^q]} \{(x,l)\}$.
Then $B[(\alpha,k), R_p^q]$ is called a **multi-closed ball** (or **m-closed ball**) with centre at (α,k) and radius R_p^q. $B[(\alpha,k), R_p^q]$ is a $\mathbb{N}$-mset. Immediately, (α,k) is a $\mathbb{N}$-element of $B^*[(\alpha,k), R_p^q]$ and also a $\mathbb{N}$-subm-element of $B[(\alpha,k), R_p^q]$.
Therefore, if $B^*[(\alpha,k), R_p^q]$ is a closed ball with centre at (α,k) and radius R_p^q, then $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.

6.26. *Example*

Consider
the $\mathbb{N}$-mset $M = \{(7,3),(6,2),(-1,5),(5,2),(0,9),(8,12)\}$ drawn from $\mathbb{R}$. Then $\Omega^*(M) = \{(7,1),(7,3),(6,1),(6,2),(-1,1),(-1,5),(5,1),(5,2),(0,1),(0,3),(0,4),(8,1),(8,2),(8,3),(8,4),(8,6),(8,12)\}$. Consider the usual metric space ρ on $M^* \subseteq \mathbb{R}$. Let d^ρ be the multi-metric on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.
Consider the multi-metic d^ρ defined on the $\mathbb{N}$-mset M induced by the crisp metric ρ defined on M^* as in Example 6.4.
Consider the $\mathbb{N}$-subm-element $(8,3) \in^4 M$.
Then $B^*[(8,3), R_2^{81}] = \{(7,1),(7,3),(6,1),(8,1),(8,3)\}$.
$B[(8,3), R_2^{81}] = \{(7,3),(6,1),(8,3)\}$.
$B^*[(8,3), R_2^{648}] = \{(7,1),(7,3),(6,1),(6,2),(8,1),(8,2),(8,3),(8,6)\}$.

$B[(8,3), R_2^{648}] = \{(7,3), (6,2), (8,6)\}$.

6.27. *Definition*

Let (M, d) be a multi-metric space. Then (M, d) is said to have **Hausdorff property**, if for any two distinct $\mathbb{N}$-subm-elements (α, k) and (β, l) of the $\mathbb{N}$-mset M, there exist two open balls $B^*((\alpha, k), R_p^q)$ centred at (α, k) and $B^*((\beta, l), R_s^t)$ centred at (β, k) such that $B^*((\alpha, k), R_p^q) \cap B^*((\beta, l), R_s^t) = \phi$.

6.28. *Proposition*

Every multi-metric space is Hausdroff.

Proof:

Let (M, d) be a multi-metric space. Let (α, k) and (β, l) be two distinct $\mathbb{N}$-subm-elements of M. Let $\delta((\alpha, k), (\beta, l)) = R_p^q \in m^+(\mathbb{R})$. Let us consider open balls $B^*((\alpha, k), R_{\frac{p}{2}}^m)$ centred at (α, k) and $B^*((\beta, l), R_{\frac{p}{2}}^n)$ centred at (β, l) where $\frac{m}{[f(k,k)]^2}, \frac{n}{[f(l,l)]^2} \in \mathbb{N}$. Then $B^*((\alpha, k), R_{\frac{p}{2}}^m) \cap B^*((\beta, l), R_{\frac{p}{2}}^n) = \phi$.

Therefore, the multi-metric space (M, d) is Hausdorff.

6.29. *Definition*

Let (M, d) be a multi-metric space. Let (α, k) be a $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset M. Let $\mathcal{N} \subseteq \Omega^*(M)$. i.e. $\mathcal{N}$ is a collection of some $\mathbb{N}$-subm-elements of M. Then $\mathcal{N}$ is said to be a **multi-neighbourhood** of the $\mathbb{N}$-subm-element (α, k), if there exist $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{N}$.

6.30. *Proposition*

Let (M, d) be a multi-metric space. Let (α, k) be a $\mathbb{N}$-subm-element of M. Let $\mathcal{A}$ and $\mathcal{B}$ be two multi-neighbourhoods of (α, k) in M. Then $\mathcal{A} \cap \mathcal{B}$ is a multi-neighbourhood of (α, k) in M.

Proof:

Let (M, d) be a multi-metric space. Let (α, k) be a $\mathbb{N}$-subm-element of M. Since $\mathcal{A}$ and $\mathcal{B}$ are two multi-neighbourhoods of (α, k), there exist $R_p^q, R_s^t \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2}, \frac{t}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{A}$ and $(\alpha, k) \in B^*((\alpha, k), R_s^t) \subseteq \mathcal{B}$. Let, $R_a^b = glb\{R_p^q, R_s^t\} \in m^+(\mathbb{R})$. Then $B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q), B^*((\alpha, k), R_s^t)$, and so $B^*((\alpha, k), R_a^b) \subseteq \mathcal{A}, \mathcal{B}$. Then $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq \mathcal{A} \cap \mathcal{B}$. Therefore, $\mathcal{A} \cap \mathcal{B}$ is a multi-neighbourhood of (α, k).

6.31. *Definition*

Let P be a $\mathbb{N}$-submset of a multi-metric space (M,d). Then $(\alpha,k) \in \Omega^*(P)$ is said to be an **interior element** of $\Omega^*(P)$ if there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha,k) \in B^*((\alpha,k),R_p^q) \subseteq \Omega^*(P)$.

6.32. *Definition*

Let P be a $\mathbb{N}$-submset of a multi-metric space (M,d). Then the **interior** of $\Omega^*(P)$ is defined as the general mset $int(\Omega^*(P))$ that contains all the interior elements of $\Omega^*(P)$.

Also, define $P^o = \bigcup_{(x,l)\in int(\Omega^*(P))} \{(x,l)\}$ to be the **multi-interior** (or **m-interior**) of the $\mathbb{N}$-mset P.

6.33. *Proposition*

Let P and Q be two non-null $\mathbb{N}$-submsets of a multi-metric space (M,d). Then

(i) $int(\Omega^*(P)) \subseteq \Omega^*(P)$.

(ii) $P \sqsubseteq Q \Rightarrow int(\Omega^*(P)) \subseteq int(\Omega^*(Q))$.

(iii) $int(\Omega^*(P)) \cap int(\Omega^*(Q)) = int(\Omega^*(P) \cap \Omega^*(Q))$.

(iv) $int(\Omega^*(P)) \cup int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cup (\Omega^*(Q))$.

Proof: (i) The result immediately follows from the definition.

(ii) If $int(\Omega^*(P)) = \phi$, then $int(\Omega^*(P)) \subseteq int(\Omega^*(Q))$. So consider the case where $int(\Omega^*(P)) \neq \phi$. Let $(\alpha,k) \in int(\Omega^*(P))$. Then (α,k) is an interior element of $\Omega^*(P)$. Then there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha,k) \in B^*((\alpha,k),R_p^q) \subseteq \Omega^*(P)$. Also, $P \sqsubseteq Q$. So $\Omega^*(P) \subseteq \Omega^*(Q)$. So, $(\alpha,k) \in B^*((\alpha,k),R_p^q) \subseteq \Omega^*(P) \subseteq \Omega^*(Q)$. Therefore, (α,k) is an interior element of $\Omega^*(Q)$. So, $(\alpha,k) \in int(\Omega^*(Q))$. Therefore, $int(\Omega^*(P)) \subseteq int(\Omega^*(Q))$.

(iii) If $int(\Omega^*(P)) \cap int(\Omega^*(Q)) = \phi$, then $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cap \Omega^*(Q))$. So consider the case where $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \neq \phi$. Let $(\alpha,k) \in int(\Omega^*(P)) \cap int(\Omega^*(Q))$. Then $(\alpha,k) \in int(\Omega^*(P))$ and $(\alpha,k) \in int(\Omega^*(Q))$. So, (α,k) is an interior element of $\Omega^*(P)$ and $\Omega^*(Q)$ both. So, there exists $R_p^q, R_r^s \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ and $\frac{s}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha,k) \in B^*((\alpha,k),R_p^q) \subseteq \Omega^*(P)$ and $(\alpha,k) \in B^*((\alpha,k),R_r^s) \subseteq \Omega^*(Q)$. Let $min\{p,r\} = a$ and $gcd\{q,s\} = b$. Then $R_a^b \preceq R_p^q, R_r^s$ and $\frac{b}{[f(k,k)]^2} \in \mathbb{N}$. So, $B^*((\alpha,k),R_a^b) \subseteq B^*((\alpha,k),R_p^q), B^*((\alpha,k),R_r^s)$. Then $(\alpha,k) \in B^*((\alpha,k),R_a^b) \subseteq B^*((\alpha,k),R_p^q) \subseteq \Omega^*(P)$ and $(\alpha,k) \in B^*((\alpha,k),R_a^b) \subseteq B^*((\alpha,k),R_r^s) \subseteq \Omega^*(Q)$. Therefore, $(\alpha,k) \in B^*((\alpha,k),R_a^b) \subseteq \Omega^*(P) \cap \Omega^*(Q)$. Therefore, (α,k) is an interior element of $\Omega^*(P) \cap \Omega^*(Q)$. i.e. $(\alpha,k) \in int(\Omega^*(P) \cap \Omega^*(Q))$. Therefore, $int(\Omega^*(P)) \cap int(\Omega^*(P)) \subseteq int(\Omega^*(P) \cap \Omega^*(Q))$.

We have $\Omega^*(P) \cap \Omega^*(Q) \subseteq \Omega^*(P), \Omega^*(Q)$. So, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P)), int(\Omega^*(Q))$.

Therefore, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cap \Omega^*(Q))$.
Combining, $int(\Omega^*(P)) \cap int(\Omega^*(Q)) = int(\Omega^*(P) \cap \Omega^*(Q))$.

(iv) We have $\Omega^*(P), \Omega^*(Q) \subseteq \Omega^*(P) \cup \Omega^*(Q)$. So, $int(\Omega^*(P)), int(\Omega^*(Q)) \subseteq int(\Omega^*(P) \cup \Omega^*(Q))$. Therefore, $int(\Omega^*(P) \cup \Omega^*(Q)) \subseteq int(\Omega^*(P) \cup \Omega^*(Q))$.

6.34. *Definition*

Let (M, d) be a multi-metric space. Let $\mathcal{O} \subseteq \Omega^*(M)$, i.e. $\mathcal{O}$ be a collection of some $\mathbb{N}$-subm-elements of M. Then $(\alpha, k) \in \mathcal{O}$ is said to be an **interior element** of $\mathcal{O}$ if there exists $R_p^q \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}$. Also, the **interior** of $\mathcal{O}$ is defined as the general mset $int(\mathcal{O})$ that contains all the interior elements of $\mathcal{O}$.

Also, define $\mathcal{O}^o = \bigcup_{(x,l) \in int(\mathcal{O})} \{(x, l)\}$ to be the **multi-interior** (or **m-interior**) of the $\mathbb{N}$-general mset $\mathcal{O}$.

6.35. *Proposition*

Let (M, d) be a multi-metric space. Let $\mathcal{P}, \mathcal{Q} \subseteq \Omega^*(M)$, Then
(i) $int(\mathcal{P}) \subseteq \mathcal{P}$.
(ii) $P \subseteq Q \Rightarrow int(\mathcal{P}) \subseteq int(\mathcal{Q})$.
(iii) $int(\mathcal{P}) \cap int(\mathcal{Q}) = int(\mathcal{P} \cap \mathcal{Q})$.
(iv) $int(\mathcal{P}) \cup int(\mathcal{Q}) \subseteq int(\mathcal{P} \cup \mathcal{Q})$.
Proof: Proofs are similar to the proofs of Proposition 6.33.

6.36. *Definition*

Let (M, d) be a multi-metric space. Let $\mathcal{O} \subseteq \Omega^*(M)$, i.e. $\mathcal{O}$ be a collection of some $\mathbb{N}$-subm-elements of M. Then $\mathcal{O}$ is said to be an open general mset or simply open in (M, d) if all elements of $\mathcal{O}$ are its interior elements.

6.37. *Proposition*

In a multi-metric space every open ball is open.

Proof: Let M be a $\mathbb{N}$-mset drawn from a non-empty crisp set X. Let ρ be a metric on X and $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a mapping. Let the mapping $d : \Omega^*(M) \times \Omega^*(M) \to m^*(\mathbb{R})$, where for all $(x, i), (y, j) \in \Omega^*(M)$, $d((x, i), (y, j)) = (\rho(x, y), f(i, j))$, is a multi-metric on the $\mathbb{N}$-mset M.
Let (α, k) be a $\mathbb{N}$-subm-element of the $\mathbb{N}$-mset M that satisfies $\alpha \in M^*, k \in \mathbb{N}$ and $\frac{\chi_M(\alpha)}{k} \in \mathbb{N}$.
Also, let $R_p^q \in m^+(\mathbb{R})$ such that $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$.
Consider the open ball $B^*((\alpha, k), R_p^q) = \{(x, l) \in \Omega^*(M) : d((\alpha, k), (x, l)) < R_p^q$ and $\frac{q}{f(k,l) \cdot f(l,l)} \in \mathbb{N}\}$ with centre at (α, k) and radius R_p^q.
Let $(\beta, n) \in B^*((\alpha, k), R_p^q)$.

Then $d((\alpha,k),(\beta,n)) = R^{f(k,n)}_{\rho(\alpha,\beta)} < R^q_p$.
Then $\rho(\alpha,\beta) < p$ and $\frac{q}{f(k,n)} \in \mathbb{N}$.
Now consider the open ball $B^*((\beta,n), R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)})$.
Let $(\gamma,t) \in B^*((\beta,n), R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)})$.
Then $d((\beta,n),(\gamma,t)) < R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)}$.
Now $d((\alpha,k),(\gamma,t)) \preceq d((\alpha,k),(\beta,n)) \oplus d((\beta,n),(\gamma,t)) < R^{f(k,n)}_{\rho(\alpha,\beta)} \oplus R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)} = R^q_p$.
Therefore, $(\gamma,t) \in B^*((\alpha,k), R^q_p)$.
Since (γ,t) is an arbitrary element of $B^*((\beta,n), R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)})$, so every element of $B^*((\beta,n), R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)})$ is an element of $B^*((\alpha,k), R^q_p)$.
Therefore, $(\beta,n) \in B^*((\beta,n), R^{\frac{q}{f(k,n)}}_{p-\rho(\alpha,\beta)}) \subseteq B^*((\alpha,k), R^q_p)$.
Therefore, (β,n) is an interior element of $B^*((\alpha,k), R^q_p)$.
Again, (β,n) is an arbitrary element of $B^*((\alpha,k), R^q_p)$, so every element of $B^*((\alpha,k), R^q_p)$ is its interior element.
Therefore, $B^*((\alpha,k), R^q_p)$ is an open set.

6.38. *Proposition*

In a multi-metric space (M,d),
(i) ϕ is open;
(ii) M is open;
(iii) an arbitrary elementary union of open general msets is also open;
(iv) an elementary intersection of two open general msets is also open.

Proof:
(i) ϕ contains no $\mathbb{N}$-subm-element of ϕ, so, ϕ is trivially open.
(ii) Also, M is immediately open.
(iii) Let $\mathcal{O}_i$ be a non-null open general mset in (M,d) for all $i \in \Lambda$. If $\Lambda = \phi$, then $\underset{i\in\Lambda}{\cup}\mathcal{O}_i = \phi$, which is open in (M,d).
So, let $\Lambda \neq \phi$. Let $(\alpha,k) \in \underset{i\in\Lambda}{\cup}\mathcal{O}_i$.
Then $(\alpha,k) \in \mathcal{O}_j$ for some $j \in \Lambda$. Also, $\mathcal{O}_j$ is an open general mset in (M,d), so (α,k) is an interior element of $\mathcal{O}_j$.
So, there exists $R^q_p \in m^+(\mathbb{R})$ with $\frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha,k) \in B^*((\alpha,k), R^q_p) \subseteq \mathcal{O}_j \subseteq \underset{i\in\Lambda}{\cup}\mathcal{O}_i$.
Therefore, (α,k) is an interior element of $\underset{i\in\Lambda}{\cup}\mathcal{O}_i$. Again, (α,k) is an arbitrary element of $\underset{i\in\Lambda}{\cup}\mathcal{O}_i$. So, every element of $\underset{i\in\Lambda}{\cup}\mathcal{O}_i$ is its interior element. Therefore, $\underset{i\in\Lambda}{\cup}\mathcal{O}_i$ is open in (M,d).
(iv) Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two open general msets in (M,d). If $\mathcal{O}_1 \cap \mathcal{O}_2 = \phi$, then $\mathcal{O}_1 \cap \mathcal{O}_2$ is also open in (M,d). So consider the case where $\mathcal{O}_1 \cap \mathcal{O}_2 \neq \phi$. Let $(\alpha,k) \in \mathcal{O}_1 \cap \mathcal{O}_2$. Then

$(\alpha, k) \in \mathcal{O}_1$ as well as $(\alpha, k) \in \mathcal{O}_2$. Also, $\mathcal{O}_1$ and $\mathcal{O}_2$ both are open in (M, d). So, there exists $R_p^q, R_r^s \in m^+(R)$ with with $\frac{s}{[f(k,k)]^2}, \frac{q}{[f(k,k)]^2} \in \mathbb{N}$ such that $(\alpha, k) \in B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}_1$ and $(\alpha, k) \in B^*((\alpha, k), R_r^s) \subseteq \mathcal{O}_2$. Let $min\{p, r\} = a$ and $gcd\{q, s\} = b$. Then $R_a^b \preceq R_p^q, R_r^s$. So, $B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q), B^*((\alpha, k), R_r^s)$. Then $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_p^q) \subseteq \mathcal{O}_1$ and $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq B^*((\alpha, k), R_r^s) \subseteq \mathcal{O}_2$.
Therefore, $(\alpha, k) \in B^*((\alpha, k), R_a^b) \subseteq \mathcal{O}_1 \cap \mathcal{O}_2$. Therefore, (α, k) is an interior element of $\mathcal{O}_1 \cap \mathcal{O}_2$. Therefore, $\mathcal{O}_1 \cap \mathcal{O}_2$ is also open in (M, d).

7. Comparative Analysis with Neutrosophic-Based Methods

The notion of a *multi-metric space* developed in this work differs fundamentally from neutrosophic-based approaches, though both aim to extend classical structures to handle richer information. The following points highlight the comparison:

(a) **Representation:** Neutrosophic methods represent information through a triplet (T, I, F) that captures truth, indeterminacy and falsity. In contrast, our multi-metric space uses *multi-real numbers*, which embeds multiplicities and multiset concepts.

(b) **Metric Definition**: Several extensions of metric spaces have been proposed in the literature by incorporating neutrosophic logic, which allows one to model truth (T), indeterminacy (I), and falsity (F) simultaneously. Some important notions are summarized below.

(i) **Neutrosophic Metric Space**: Kirişci and Şimşek [34] introduced the concept of a *neutrosophic metric space*, where the classical metric axioms are generalized by using continuous triangular norms (t-norms) and conorms. This framework preserves many standard topological properties, and results such as the Baire Category Theorem and the Uniform Convergence Theorem were extended to this setting.

(ii) **Neutrosophic Triplet Metric Space**: In a *neutrosophic triplet metric space*, each element is represented as a neutrosophic triplet (T, I, F), and the metric is defined accordingly. The axioms of non-negativity, symmetry, and a modified triangle inequality are adapted to the neutrosophic context [35].

(iii) **Neutrosophic Triplet v-Generalized Metric Space**: The neutrosophic triplet v-generalized metric space (NTVGMS) further relaxes the standard triangle inequality and provides more flexibility in defining distances. The completeness and fixed-point results in this framework have been investigated by [36].

(iv) **Neutrosophic Fuzzy Metric Spaces and Variants**: The combination of neutrosophic and fuzzy logics has led to *neutrosophic fuzzy metric spaces* and their variants, such as orthogonal neutrosophic metric spaces, neutrosophic 2-metric spaces, pentagonal metric spaces and b-metric spaces. These frameworks have been used to study the results of convergence, compactness, and fixed points [37].

(v) **Other generalizations**: Several further generalizations have been explored, including neutrosophic quasi-dislocated-b-metric spaces [38], neutrosophic b-metric-like spaces [39], and neutrosophic D-metric spaces [40]. These approaches extend the applicability of neutrosophic metrics in both pure and applied mathematics.

Table 1 summarizes the main concepts of neutrosophic metric and their key characteristics.

TABLE 1. Different notions of neutrosophic metric concepts

Neutrosophic Metric Concept	**Key Characteristics**
Neutrosophic Metric Space [34]	Uses t-norms/conorms; generalizes classical topological results.
Neutrosophic Triplet Metric Space [35]	Defines distance on neutrosophic triplet elements (T, I, F).
NTVGMS [36]	v-generalized triangle inequality; completeness and fixed points.
Neutrosophic Fuzzy Metric Spaces [37]	Combines neutrosophic and fuzzy logic; several variants developed.
Quasi-dislocated-b-Metric Space [38]	Generalizes neutrosophic triplet metrics; includes fixed-point theorems.
Neutrosophic b-Metric-like Space [39]	Relaxes triangle inequality; supports fixed-point theory.
Neutrosophic D-Metric Space [40]	Adapts D-metric to neutrosophic settings; explores topology and completeness.

Our approach defines multi-metrics as a mapping from the set of all $\mathbb{N}$-subm-elements of a $\mathbb{N}$-mset to the set of all non-negative multi-real numbers satisfying multi-metric axioms.

(c) **Generality and Applicability:** Neutrosophic approaches are particularly suited for model uncertainty and inconsistency. However, our method, is more suitable for applications involving *multisets, graphs, trees, and networks*, where multiplicity and structural comparison are essential.

(d) **Novel Contribution:** While neutrosophic-based methods extend fuzzy set theory, the proposed multi-metric space provides an *entirely new mathematical framework* by restructuring multisets, introducing the multi-real number system, and defining rigorous metrics, specially for multiset setting.

This comparative analysis demonstrates that the multi-metric space not only complements existing neutrosophic-based methods but also offers a mathematically robust and efficient alternative for problems where multiplicities and structural similarities are central.

8. Conclusion

From the above theorem it follows that, in a multi-metric space (M, d), the collection τ of all open general msets forms a topology on M with respect to elementary union and elementary intersection of $\mathbb{N}$-msets. This topology will be called 'multi-metric topology' on M.

9. Limitations and Future Research

Functional analysis plays a pivotal role in modern mathematics and its applications to the sciences. Since metric spaces form its foundation, they serve as a key tool in the development of many important results. In this work, we have introduced an extension of the classical metric framework by employing multisets and multi-real numbers in place of ordinary sets and real numbers, thereby offering a broader perspective on the structure of multi-metric spaces.

Despite these contributions, some limitations remain. The present study focuses mainly on particular classes of metrics and illustrative examples that do not fully reflect the general scope of possible constructions. In addition, certain theoretical results depend on restrictive assumptions, which may limit their direct applicability to wider contexts.

Future investigations could aim to relax these assumptions and extend the theory to more generalized settings. Another promising direction is the study of multi-normed linear spaces and multi-inner product spaces, which may further enrich the theory. Moreover, exploring connections with frameworks such as fuzzy sets, neutrosophic sets, and other uncertainty-based models could significantly expand the range of applications. Overall, the results presented here provide a foundation for further research on generalized metric theories and their potential applications in functional analysis and beyond.

Acknowledgments: The authors express their heartfelt gratitude to Dr. Syamal Kumar Samanta, Retired Professor, Department of Mathematics, Visva-Bharati, P.O.: Santiniketan, West Bengal, India, and Dr. Dhiman Mondal, Assistant Professor, Department of Computer Science and Engineering, Alipurduar Government Engineering and Management College, Alipurduar, West Bengal, India, for their constant guidance, encouragement, and support throughout the preparation of this work. Their insightful suggestions and valuable advice have been a source of continuous inspiration.

Conflicts of interest: The authors declare no conflict of interest.

Funding: This research did not receive external funding.

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Received: April 9, 2025. Accepted: Sep 11, 2025

NSS

Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

On deferred I-statistical rough convergence of difference sequences in neutrosophic normed spaces

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Abstract. In this study, using the concepts of deferred density and the notion of the ideal I, we extend the idea of rough convergence by introducing the notion of deferred I–statistical rough convergence via difference operators in the framework of neutrosophic normed spaces. We define a set of limits of this convergence and prove that the limit set is convex and closed with respect to the neutrosophic norm. We also develop the idea of deferred I-statistical Δ_h^j-cluster points of sequences in neutrosophic normed spaces and investigate their connection the set of these cluster points and the limit set of the aforementioned convergence.

Keywords: Neutrosophic normed space(NNS); difference sequences; deferred statistical convergence; I-convergence; deferred I-statistical convergence; rough convergence.

MSC 2020: 46S40, 40A05, 40A35, 54E52, 03E72

1. Introduction

In recent decades, fuzzy theory has emerged as one of the most influential tools in mathematical modeling, engineering, and decision sciences. The concept of fuzzy sets, originally introduced by Zadeh in 1965 [35], provided a systematic way to represent and manage uncertainty. Building upon this, Kramosil and Michálek [20] developed fuzzy metric spaces to generalize classical metric spaces, a framework later refined by George and Veeramani [16], who introduced a Hausdorff topology for fuzzy metric structures. Expanding this line of research, Atanassov [5] proposed intuitionistic fuzzy sets, which consider both membership and non-membership values, thereby improving uncertainty modeling. This idea was further generalized by Park [30] with the introduction of intuitionistic fuzzy metric spaces, followed by Saadati and Park [31], who established the notion of intuitionistic fuzzy normed spaces. More recently, applications of intuitionistic fuzzy theory have been seen in areas such as pattern recognition, image processing, control theory, and decision-making under uncertainty [40, 41].

Parallel to these developments, researchers sought to generalize the classical concept of sequence convergence. Steinhaus [34] and Fast [15] introduced the concept of statistical convergence, which captures the convergence of "almost all" terms of a sequence, rather than all terms. This new approach quickly gained prominence in summability theory and functional analysis. Later, Kostyrko et al. [19] extended this to I-convergence using ideals of subsets of $\mathbb{N}$. The interplay between statistical convergence and intuitionistic fuzzy normed spaces has since been explored in various works [17, 25], while Savaş and Gürdal [32] refined this idea by defining I-statistical convergence. Subsequently, the concept of deferred statistical convergence was introduced in [21], employing deferred density as a refinement. This approach has inspired extensive research into deferred statistical convergence of both single sequences [24] and double sequences [26], broadening the scope of convergence analysis in generalized sequence spaces.

Another significant line of research has centered on the concept of *rough convergence*, introduced by Phu [28] in finite-dimensional normed linear spaces, where the notion of tolerance (degree of roughness) plays a central role. Phu later extended this framework to infinite-dimensional spaces [29] and studied fundamental properties such as convexity and closure of rough limit sets. Following this, Aytar [6, 7] proposed *rough statistical convergence*, linking it to statistical cluster points and the structure of rough limit sets. These developments have led to extensive studies on rough and approximate statistical convergence in different contexts, such as double and triple sequences [8, 22, 23]. Pal et al. [27] and Dündar et al. [10] introduced rough I-convergence, which was subsequently extended to rough I_2-convergence for double sequences [11] and rough I_2-lacunary statistical convergence [12]. Rough convergence has also been applied to metric spaces [9], 2-normed spaces [4], and probabilistic normed spaces [2]. Recently, Reena et al. [3] studied rough statistical convergence within intuitionistic fuzzy normed spaces by focusing on continuous t-norms, highlighting a growing interest in merging rough convergence with fuzzy and intuitionistic frameworks.

In parallel, difference operators and their associated sequence spaces have become an active area of research. The classical forward difference operator Δ was first used to define difference sequence spaces in [18], later extended to integer orders by Et and Çolak [13], with further advancements by Khan et al. [36]. Recently, these ideas were integrated with neutrosophic normed spaces, as explored by Kaur and Chawla [45], and further studied in fuzzy, intuitionistic fuzzy, and neutrosophic settings [42–44]. Notable contributions include statistical completeness in neutrosophic normed spaces [46], uniform statistical convergence of function sequences [47], I-convergent difference sequence spaces [48], hybrid Δ-statistical and lacunary approaches [49, 50], Riesz ideal convergence extensions [51], and nonlinear operator analysis with Fréchet differentiability [52]. Together, these studies underline the significance of difference operators in advancing the theory of neutrosophic normed spaces.

Motivated by these developments, the present work aims to advance the theory of convergence by combining the concepts of deferred density, I-convergence, and rough convergence in neutrosophic normed spaces through the use of higher-order difference operators. This approach not only unifies several strands of research in fuzzy, intuitionistic, and neutrosophic settings but also opens new directions for the study of uncertain, incomplete, or noisy data sequences that arise in real-world applications such as data science, signal processing, and decision-making systems.

The main objective of this research is to present and explore the concept of deferred I-statistical rough convergence, defined via integer-order j difference operators, in the context of neutrosophic normed spaces.

2. **Preliminaries**

In this work, we denote the sets. For clarity, we first revisit several relevant definitions in the table below.

Notation	**Meaning / Definition**
$\mathbb{R}$	Real numbers
$\mathbb{N}$	Natural numbers
$\delta(\mathcal{A})$	Density of the set $\mathcal{A}$
$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$	Neutrosophic normed space (NNS) with membership Υ, non-membership Ω, indeterminacy Γ, and t-norm $\star$ and t-conorm $\circ$.
$\Upsilon(x, h)$	Degree of membership of $x \in X$ with respect to $h > 0$.
$\Omega(x, h)$	Degree of non-membership of $x \in X$ with respect to $h > 0$.
$\Gamma(x, h)$	Degree of indeterminacy of $x \in X$ with respect to $h > 0$.
$\star$	t-norm used in the triangle-type inequality for Υ.
$\circ$	t-conorm used in the triangle-type inequalities for Ω and Γ.
$\Delta^j x_Q$	j-th order difference of the sequence (x_Q).

h	Parameter controlling the "radius" in the neutrosophic norm functions.
d	Arbitrary positive radius in the definition of deferred I-statistical cluster point.
I	Ideal of subsets of $\mathbb{N}$ used in the deferred I-statistical convergence.
$\Gamma^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}_{D^p_q}(\Delta^j x_Q)$	Set of all deferred I-statistical Δ^j_h-cluster points of (x_Q) with respect to the NNS.
z	A deferred I-statistical Δ^j_h-cluster point of (x_Q).
y	Arbitrary point in X used in the triangle-type inequality argument.
$\varsigma \in (0,1)$	Small parameter used in approximating membership, non-membership, and indeterminacy.
p_n, q_n	Endpoints of intervals used in the deferred I-statistical density calculation.
$(\Upsilon,\Omega,\Gamma)^h(\Delta^j) - \lim x_Q = x$	Rough statistical Δ^j-convergence of a sequence (x_Q) to x in NNS, where convergence is controlled by a roughness parameter $h \geq 0$.
Δ^j_{NN}	strongly bounded in neutrosophic norm space (NNS).

TABLE 1. Notation used in deferred I-statistical Δ^j_h-cluster points, Δ^j_{NN}-convergence, and rough statistical convergence in NNS

Let $\mathcal{A} \subseteq \mathbb{N}$. The asymptotic (or natural) density of the set $\mathcal{A}$, denoted by $\delta(\mathcal{A})$, is defined as

$$\delta(\mathcal{A}) = \lim_{Q\to\infty} \frac{1}{Q}|\{n \leq Q : n \in \mathcal{A}\}|,$$

assuming the limit exists. In this context, $||\cdot||$ represents the number of elements in the set $\{,\}$. A numerical sequence (x_Q) is said to converge statistically to l if, for any $\epsilon > 0$,

$$\delta\left(\{Q \in \mathbb{N} : |x_Q - l| > \epsilon\}\right) = 0 \text{ holds.}$$

For this case, we write $x_Q \xrightarrow{S} l$ (see [15], [34]).

Definition 2.1. [14] A real (or complex) valued sequence (x_Q) is Δ^j-statistically convergent to l if

$$\delta\left(\left\{Q \in \mathbb{N} : \left|\Delta^j x_Q - l\right| > \epsilon\right\}\right) = 0$$

for all $\epsilon > 0$, with j belonging to the set of natural numbers $\mathbb{N}$, and

$$\Delta^0 x_Q = x_Q, \Delta^1 x_Q = x_Q - x_{Q+1}, \ldots, \Delta^j x_Q = \Delta^{j-1}\left(x_Q - x_{Q+1}\right)$$

and so that

$$\Delta^j x_Q = \sum_{i=0}^{j} (-1)^i \binom{j}{i} x_{Q+i} (Q \in \mathbb{N}).$$

Definition 2.2. [6] A sequence (x_Q) in a normed space $(X, \|.\|)$ is rough convergent to $x \in X$ for some $h \geq 0$ if, for every $\varsigma > 0, \exists\ n_0 \in \mathbb{N}$ so that

$$\|x_Q - x\| < h + \varsigma, \forall\ Q \geq n_0$$

The sequence (x_Q) is rough statistically convergent to $x \in X$ for some $h \geq 0$ if, for every $\varsigma > 0$,

$$\delta \left(\{Q \in \mathbb{N} : \|x_Q - x\| \geq h + \varsigma\}\right) = 0 \text{ holds.}$$

Note: Definition 2.1 (Δ^j-statistical convergence) generalizes classical statistical convergence by considering the j-th order differences of a sequence, focusing on the asymptotic behavior of $\Delta^j x_Q$. In contrast, Definition 2.2 (rough convergence) allows a sequence to converge within a tolerance level h, either strictly for all large indices or statistically for most indices. Both concepts address approximate convergence, but Δ^j-statistical convergence emphasizes differences of order j, while rough convergence emphasizes proximity in a fixed roughness margin.

Definition 2.3. [33] A binary operation $\star$ on $[0, 1]$ is called continuous t-norm (*or* CTN) if
(a) $\star$ is commutative, associative and continuous,
(b) $\varsigma = \varsigma \star 1$ for any $\varsigma \in [0, 1]$ and
(c) for each $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in [0, 1]$, if $\varsigma_3 \geq \varsigma_1$ and $\varsigma_4 \geq \varsigma_2$ then $\varsigma_3 \star \varsigma_4 \geq \varsigma_1 \star \varsigma_2$.

A binary operation on on $[0, 1]$ is called continuous t-conorm (*or* $CTCN$) if
(1) o is commutative, associative and continuous,
(2) $\varsigma = \varsigma \circ 0$ for any $\varsigma \in [0, 1]$ and
(3) for each $\varsigma_1, \varsigma_2, \varsigma_3, \varsigma_4 \in [0, 1]$, if $\varsigma_3 \geq \varsigma_1$ and $\varsigma_4 \geq \varsigma_2$ then $\varsigma_3 \circ \varsigma_4 \geq \varsigma_1 \circ \varsigma_2$.

Definition 2.4. [36] The three-tuple structure $(X, \Upsilon, \Omega, \Gamma)$ be an NNS, where X is a linear space over a field F. Υ, Ω, Γ are called neutrosophic normed space (NNS) on $X \times (0, \infty)$ and represent the degree of membership and non-membership on $X \times (0, 1)$ if the following conditions hold, for every $y, w \in X$ and $\varsigma_1, \varsigma_2 > 0$:
(1) $\Upsilon(y, \varsigma) + \Omega(y, \varsigma) \leq 1$.
(2) $\Upsilon(y, \varsigma) > 0$.
(3) $\Upsilon(y, \varsigma) = 1 \Leftrightarrow y = 0$.
(4) $\Upsilon(cy, \varsigma) = \Upsilon\left(y, \frac{\varsigma}{|c|}\right)$, if $c \neq 0, c \in F$.
(5) $\Upsilon(y, \varsigma_1) \star \Upsilon(w, \varsigma_2) \leq \Upsilon(p + w, \varsigma_1 + \varsigma_2)$.
(6) $\Upsilon(y, .) : (0, \infty) \to [0, 1]$ is continuous.

(7) $\lim_{\varsigma \to \infty} \Upsilon(y, \varsigma) = 1, \lim_{\varsigma \to 0} \Upsilon(y, \varsigma) = 0.$
(8) $\Omega(y, \varsigma) < 1.$
(9) $\Omega(y, \varsigma) = 0 \Leftrightarrow y = 0.$
(10) $\Omega(cy, \varsigma) = \Omega\left(y, \frac{\varsigma}{|c|}\right)$ if $c \neq 0, c \in F$.
(11) $\Omega(y, \varsigma_1) \diamond \Omega(w, \varsigma_2) \geq \Omega(y + w, \varsigma_1 + \varsigma_2).$
(12) $\Omega(y, .) : (0, \infty) \to [0, 1]$ is continuous.
(13) $\lim_{\varsigma \to \infty} \Omega(y, \varsigma) = 0, \lim_{\varsigma \to 0} \Omega(y, \varsigma) = 1.$
(14) $\Gamma(y, \varsigma) = 0 \Leftrightarrow y = 0.$
(15) $\Gamma(cy, \varsigma) = \Omega\left(y, \frac{\varsigma}{|c|}\right)$ if $c \neq 0, c \in F$.
(16) $\Gamma(y, \varsigma_1) \diamond \Omega(w, \varsigma_2) \geq \Omega(y + w, \varsigma_1 + \varsigma_2).$
(17) $\Gamma(y, .) : (0, \infty) \to [0, 1]$ is continuous.
(18) $\lim_{\varsigma \to \infty} \Gamma(y, \varsigma) = 0, \lim_{\varsigma \to 0} \Omega(y, \varsigma) = 1.$
Then Υ, Ω, Γ are called neutrosophic normed (NN).

Here, the three tuple $(\Upsilon, \Omega, \Gamma)$ is known as the neutrosophic normed (NN) on X.

Example 2.5. Let $\mathbb{R}$ be a real linear space over the field $\mathbb{R}$. Define the functions $\Upsilon, \Omega, \Gamma : \mathbb{R} \times (0, \infty) \to [0, 1]$ as follows:

$$\Upsilon(y, \varsigma) = \frac{\varsigma}{\varsigma + |y|}, \qquad \Omega(y, \varsigma) = \frac{|y|}{\varsigma + |y|}, \qquad \Gamma(y, \varsigma) = \frac{|y|}{1 + |y| + \varsigma},$$

for all $y \in \mathbb{R}$ and $\varsigma > 0$.

Also, define the binary operations on $[0, 1]$ by

$$\varsigma_1 \star \varsigma_2 = \min\{\varsigma_1, \varsigma_2\}, \qquad \varsigma_1 \circ \varsigma_2 = \max\{\varsigma_1, \varsigma_2\},$$

for all $\varsigma_1, \varsigma_2 \in [0, 1]$.

It is easy to verify that:

- $\Upsilon(y, \varsigma) + \Omega(y, \varsigma) = 1$ for all $y \in \mathbb{R}$ and $\varsigma > 0$.
- $\Upsilon(y, \varsigma) = 1 \iff y = 0$, and $\Omega(y, \varsigma) = 0 \iff y = 0$.
- $\lim_{\varsigma \to \infty} \Upsilon(y, \varsigma) = 1$ and $\lim_{\varsigma \to 0} \Upsilon(y, \varsigma) = 0$.
- $\lim_{\varsigma \to \infty} \Omega(y, \varsigma) = 0$ and $\lim_{\varsigma \to 0} \Omega(y, \varsigma) = 1$.
- $\Gamma(y, \varsigma)$ is continuous, with $\Gamma(y, \varsigma) = 0 \iff y = 0$, and $\lim_{\varsigma \to \infty} \Gamma(y, \varsigma) = 0$.

Hence, the three-tuple $(X, \Upsilon, \Omega, \Gamma)$, together with the operations $\star$ and $\circ$, forms a neutrosophic normed space (NNS).

Definition 2.6. [38] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. The open ball of radius $h > 0$ and center $x \in X$ with regard to $\varsigma \in (0, 1)$ is the set

$$\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h, \varsigma) = \{y \in X : \Upsilon(x - y, h) > 1 - \varsigma \text{ and } \Omega(x - y, h) < \varsigma \text{ and } \Gamma(x - y, h) < \varsigma\}.$$

Definition 2.7. [39] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. A sequence (x_Q) in X is convergent to $x \in X$ with regard to $(\Upsilon, \Omega, \Gamma)$ if

$$\lim_{Q\to\infty} \Upsilon(x_Q - x, d) = 1 \text{ and } \lim_{Q\to\infty} \Omega(x_Q - x, d) = 0 \text{ and } \lim_{Q\to\infty} \Gamma(x_Q - x, d) = 0$$

for every $d > 0$. In this case, we denote the limit by $x_Q \xrightarrow{(\Upsilon,\Omega,\Gamma)} x$.

Definition 2.8. [19] Let $\Gamma \neq \emptyset$ set and $I \subseteq 2^\Gamma$. Then I is called an ideal in Γ if
(a) $\emptyset \in I$,
(b) $\mathcal{A}, \mathcal{B} \in I \Rightarrow Q \in \mathcal{A} \cup \mathcal{B} \in I$ and
(c) $\mathcal{A} \in I, \mathcal{B} \subseteq \mathcal{A} \Rightarrow \mathcal{B} \in I$. An ideal $I \subseteq 2^\Gamma$ is nontrivial if $I \neq 2^\Gamma$. A nontrivial ideal $I \subseteq 2^\Gamma$ is admissible if I contains every singleton subset of X.
A subset $F \subseteq 2^\Gamma$ is called filter on Γ if
(e) $\emptyset \notin F$,
(f) $\mathcal{A} \cap \mathcal{B} \in F$ for all $\mathcal{A}, \mathcal{B} \in F$ and
(e) $\mathcal{B} \in F$ whenever $\mathcal{A} \in F$ and $\mathcal{B} \supset \mathcal{A}$. For each ideal I in Γ, one can find the filter $F(I)$ associated with ideal I such that $F(I) = \{\mathcal{A} \subset \Gamma : \mathcal{A}^c \in I\}$.

Definition 2.9. [40] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and I is nontrivial admissible ideal in $\mathbb{N}$. A sequence (x_Q) in X is I-statistically convergent to some $x \in X$ with regard to $(\Upsilon, \Omega, \Gamma)$ if

$$\left\{n \in \mathbb{N} : \frac{1}{n}\left|\{Q \leq n : \Upsilon(x_Q - x, d) \leq 1 - \varsigma \text{ or } \Omega(x_Q - x, d) \geq \varsigma \text{ or } \Gamma(x_Q - x, d) \geq \varsigma\}\right| \geq \epsilon\right\} \in I$$

for every $\epsilon, d > 0$ and $\varsigma \in (0, 1)$.

For $I = I_f$, the collection of all finite subsets of $\mathbb{N}$, the convergence in Definition 2.9 reduces to the statistical convergence of (x_Q) with regard to $(\Upsilon, \Omega, \Gamma)$ [17].

In 1932, Agnew [1] extended the concept of the Cesàro mean for real (or complex) sequences and introduced the deferred Cesàro mean, defined as follows:

Definition 2.10. [36] For a real (or complex) valued sequence (x_Q), the deferred Cesàro mean of (x_Q) is defined by

$$\left(D_p^q(x_Q)\right)_n := \frac{1}{q_n - p_n} \sum_{Q=p_n+1}^{q_n} x_Q, n = 1, 2, 3, \ldots,$$

where $p = (p_n)$ and $q = (q_n)$ denote sequences of non-negative integers that fulfill the condition

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty \tag{2.1}$$

Given a subset $K \subseteq \mathbb{N}$, the deferred density of K is defined as follows:

$$D_p^q(K) = \lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in K\}\right| \tag{2.2}$$

provided the limit exists.

Definition 2.11. [10] A real (or complex) valued sequence (x_Q) is deferred statistically convergent to l if

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} |\{Q \in \mathbb{N} : p_n < Q \leq q_n, |x_Q - l| \geq \epsilon\}| = 0$$

for every $\epsilon > 0$.

When $p_n = 0$ and $q_n = n$, this definition aligns with the concept of statistical convergence of the sequence (x_Q) as presented in [15].

3. $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)$-convergence sequences in NNS

This section introduces and explores the concept of $D_p^q(\Upsilon,\Omega,\Gamma)S(I)^h\left(\Delta^j\right)$-convergence for sequences. Throughout this discussion, I denotes a non-trivial admissible ideal in $\mathbb{N}$. The j-th order difference of a sequence (x_Q) is given by $\Delta^j x_Q = \sum_{i=0}^{j}(-1)^i \binom{j}{i} x_{Q+i} (j \in \mathbb{N})$. Additionally, (p_n) and (q_n) represent sequences of non-negative integers that satisfy condition (2.1). Further assumptions on (p_n), (q_n), and j, if required, will be specified within the corresponding theorems and examples.

Definition 3.1. [42] Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is Δ^j-rough convergent to some x with regard to (Υ,Ω,Γ) for some $h \geq 0$ if, for every $d > 0$ and $\varsigma \in (0,1), \exists\, n_0 \in \mathbb{N}$ such that

$$\Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma, \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma,$$

$\forall\, Q \geq n_0$. The limit is denoted by $(\Upsilon,\Omega,\Gamma)^h\left(\Delta^j\right) - \lim x_Q = x$.

Definition 3.2. [43] Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is deferred I-statistically difference rough convergent to some x for some $h \geq 0$ with regard to (Υ,Ω,Γ) (shortly, $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)$-convergent to x) if, for every $\epsilon, d > 0$ and $\varsigma \in (0,1)$,

$$\begin{aligned}\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \\ \text{or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma, \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\big| \geq \epsilon\Big\} \in I \qquad (3.1)\end{aligned}$$

holds. In this case, we denote the limit by $D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right) - \lim x_Q = x$.
Suppose (x_Q) is a sequence in an NNS$(X,\Upsilon,\Omega,\Gamma,\star,\circ)$.

- When $h = 0$ in (3.1), the sequence (x_Q) is said to be *deferred I-statistically difference convergent* relative to the structure (Υ,Ω,Γ).
- If we choose $p_n = 0$ and $q_n = n$ in (3.1), then the deferred I-statistical difference rough convergence of the sequence (x_Q) is termed the *I-statistical difference rough convergence* relative to the structure (Υ,Ω,Γ).

- When the ideal I is taken as I_f, the notion introduced in Definition 3.2 is referred to as the *deferred statistical difference rough convergence* with respect to the structure $(\Upsilon, \Omega, \Gamma)$.

Remark 3.3. Let (x_Q) be a sequence in an NNS$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ with $(h \geq 0)$. The limits

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{-}\lim x_Q \quad \text{and} \quad D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{-}\lim x_Q$$

are not necessarily unique whenever they exist, for $j \in \mathbb{N}$. We denote the collections of all such limits by

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{-LIM}(x_Q) = \{x \in X : (\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{-}\lim x_Q = x\},$$

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{-LIM}(x_Q) = \{x \in X : D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{-}\lim x_Q = x\}.$$

A sequence (x_Q) is said to be Δ^j-rough convergent with respect to $(\Upsilon, \Omega, \Gamma)$ if

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^j)\text{-LIM}(x_Q) \neq \emptyset,$$

and similarly, (y_Q) is $D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)$-convergent if

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{-LIM}(y_Q) \neq \emptyset,$$

for some $h \geq 0$. Moreover, if $0 \leq h_1 \leq h_2$, then for any sequence (x_Q) in X, the inclusion relations hold:

$$(\Upsilon, \Omega, \Gamma)^{h_1}(\Delta^j)\text{-LIM}(x_Q) \subseteq (\Upsilon, \Omega, \Gamma)^{h_2}(\Delta^j)\text{-LIM}(x_Q),$$

and

$$D_p^q(\Upsilon, \Omega, \Gamma)^{h_1}_{S(I)}(\Delta^j)\text{-LIM}(x_Q) \subseteq D_p^q(\Upsilon, \Omega, \Gamma)^{h_2}_{S(I)}(\Delta^j)\text{-LIM}(x_Q).$$

Example 3.4. Consider the NNS $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$, where $(\mathbb{R}, \|.\|)$ is the usual normed space, $\varsigma_1 \star \varsigma_2 = \min\{\varsigma_1, \varsigma_2\}$, $\varsigma_1 \circ \varsigma_2 = \varsigma_1 + \varsigma_2 - \varsigma_1 \cdot \varsigma_2$, and

$$\Upsilon(x, d) = \frac{1}{1 + \|x\| + d}, \quad \Omega(x, d) = \frac{d}{1 + \|x\| + d}, \quad \Gamma(x, d) = \frac{\|x\|}{1 + \|x\| + d}, \quad \forall x \in \mathbb{R}, d > 0.$$

Define the sequence

$$x_Q = \begin{cases} (-1)^Q & \text{if } Q \text{ is a multiple of 3,} \\ 0 & \text{otherwise.} \end{cases}$$

Then, for $j = 1$,

$$\Delta^1 x_Q = x_{Q+1} - x_Q = \begin{cases} 1 & \text{for certain } Q, \\ -1 & \text{for others,} \\ 0 & \text{otherwise,} \end{cases}$$

depending on the parity and position of multiples of 3.

Thus, the sequence $\Delta^1 x_Q$ is bounded but oscillatory. Hence,

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^1)\text{-LIM}(x_Q) = \begin{cases} [-1 - h, h + 1] & \text{if } h \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now take $p_n = \frac{1}{n+1}, q_n = \log(n+2)$ and define another sequence (y_Q) as:

$$\Delta^1 y_Q = \begin{cases} Q \bmod 4 & \text{if } Q = 2^n \text{ for some } n, \\ -2 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{N}.$$

Then, for any nontrivial admissible ideal I,

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^1)\text{- LIM}(y_Q) = \begin{cases} [-2-h, h+2] & \text{if } h \geq 0, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence, both $\Delta^1 x_Q$ and $\Delta^1 y_Q$ are not convergent in the usual sense, but their rough statistical limit sets exist under neutrosophic norms for suitable values of h.

Note Unlike standard convergence in an NNS$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$, the Δ^j-rough convergence of a sequence (x_Q) in X relative to $(\Upsilon, \Omega, \Gamma)$ does not necessarily imply that any subsequence of (x_Q) will also be Δ^j-rough convergent under the same framework. For example, consider the sequence $(x_Q) = (Q)$ in the NNS described in Example 3.4, where the rough limit set is

$$(\Upsilon, \Omega, \Gamma)^h(\Delta^1)\text{- LIM}(x_Q) = [1-h, 1+h] \quad \text{for all } h \geq 0,$$

however, its subsequence $(x_{2Q}) = (Q^2)$ does not exhibit Δ^1-rough convergence for any $h \geq 0$. A similar observation holds true for the $D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)$-convergence of sequences in X.

Example 3.5. Consider the NNS $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$, where $(\mathbb{R}, \|\cdot\|)$ is the usual normed space, $\varsigma_1 \star \varsigma_2 = \varsigma_1 \cdot \varsigma_2$, $\varsigma_1 \circ \varsigma_2 = \varsigma_1 + \varsigma_2 - \varsigma_1 \cdot \varsigma_2$, for all $\varsigma_1, \varsigma_2 \in [0,1]$, and Υ, Ω, Γ are defined by

$$\Upsilon(x,d) = \frac{1}{1+\|x\|+d}, \quad \Omega(x,d) = \frac{\|x\|}{1+\|x\|+d}, \quad \Gamma(x,d) = \frac{d}{1+\|x\|+d}$$

for all $x \in \mathbb{R}, d > 0$. Let $p_n = \frac{1}{n+1}$ and $q_n = \log(n+2)$ for all $n \in \mathbb{N}$. Define

$$x_Q = \begin{cases} 0 & \text{if } Q = 2^n \text{ for some } n \in \mathbb{N}, \\ n & \text{otherwise.} \end{cases}$$

Then, for any nontrivial admissible ideal I, we obtain

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)} \left(\Delta^1\right)\text{-LIM}(x_Q) = [h, \infty), \quad \forall h \geq 0.$$

However, for the subsequence (x_{2^n}) of (x_Q), we get

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)} \left(\Delta^1\right)\text{-LIM}(x_{2^n}) = \{0\}, \quad \forall h \geq 0.$$

Lemma 3.6. *Suppose $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ is an NNS and (x_Q) is a sequence in X. Let $h \geq 0$ be given. Then, for every $\epsilon, d > 0$ and $\varsigma \in (0,1)$, the following are equivalent*

(a)

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)} \left(\Delta^j\right) - \lim x_Q = x.$$

(b)

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma\right\}\right| \geq \epsilon\right\} \in I \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| \geq \epsilon\right\} \in I \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| \geq \epsilon\right\} \in I.$$

(c)

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\big| < \epsilon\Big\} \in F(I).$$

(d)

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma\right\}\right| < \epsilon\right\} \in F(I) \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| < \epsilon\right\} \in F(I) \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\right\}\right| < \epsilon\right\} \in F(I).$$

(e)

$$I - \lim_{n\to\infty} \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\big| = 0.$$

Proof. Due to its obvious nature, the proof has been omitted. □

Theorem 3.7. *Let* $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ *be an NNS. Then, for every sequence* (x_Q) *in* X,

$$(\Upsilon, \Omega, \Gamma)^h \left(\Delta^j\right) \text{-LIM}\,(x_Q) \subset D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)} \left(\Delta^j\right) \text{-LIM}\,(x_Q)$$

holds.

Proof. Assume that $x \in (\Upsilon, \Omega, \Gamma)^h \left(\Delta^j\right)\text{-LIM}\,(x_Q)$ for some $h \geq 0$. Then, for every $d > 0$ and $\varsigma \in (0,1), \exists\ n_0 \in \mathbb{N}$ so that

$$\Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma, \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma, \forall\, Q \geq n_0.$$

Therefore,

$$\{Q \in \mathbb{N} : \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}$$
$$\subseteq \{1, 2, \ldots, n_0 - 1\}.$$

Since

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in\{1,2,\ldots,n_0-1\}\}\right|=0,$$

holds for every $\epsilon>0$, the set

$$\Big\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\big|\{Q\in\mathbb{N}:p_n<Q\leq q_n, \Upsilon\left(\Delta^j x_Q-x,d+h\right)\leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q-x,d+h\right)\geq\varsigma$$
$$\text{or } \Gamma\left(\Delta^j x_Q-x,d+h\right)\geq\varsigma\}\big|\geq\epsilon\Big\}$$

is a part of I_f and therefore also of I. Thus, $x\in D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)-\text{LIM}\,(x_Q)$. Consequently, we possess

$$(\Upsilon,\Omega,\Gamma)^h\left(\Delta^j\right)\text{-}\text{LIM}\,(x_Q)\subset D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\text{LIM}\,(x_Q).$$

From Example 3.4, we can see that the above inclusion relation is strict. □

Theorem 3.8. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS and (x_Q) be a sequence in X. Then, for any $h>0$, there are no $x,y\in D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\text{LIM}\,(x_Q)$ such that $\Upsilon(x-y,sh)\leq 1-\varsigma$ or $\Omega(x-y,sh)\geq\varsigma$ or $\Gamma(x-y,sh)\geq\varsigma$ for every $\varsigma\in(0,1)$, where $s>2$.*

Proof. For any given $\varsigma\in(0,1),\exists v\in(0,1)$ such that $(1-v)\star(1-v)>1-\varsigma$ and $v\circ v<\varsigma$. Let on contrary that there exist $x,y\in D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\text{LIM}\,(x_Q)$ such that for every $\varsigma\in(0,1)$,

$$\Upsilon(x-y,sh)\leq 1-\varsigma \text{ or } \Omega(x-y,sh)\geq\varsigma \text{ or } \Gamma(x-y,sh)\geq\varsigma$$

where $s>2$. Now, for any $d>0$, consider the sets

$$\mathcal{N}=\Big\{Q\in\mathbb{N}:p_n<Q\leq q_n, \Upsilon\left(\Delta^j x_Q-x,\frac{d}{2}+h\right)\leq 1-v \text{ or } \Omega\left(\Delta^j x_Q-x,\frac{d}{2}+h\right)\geq v$$
$$\text{or } \Gamma\left(\Delta^j x_Q-x,\frac{d}{2}+h\right)\geq v\Big\},$$

and

$$\mathcal{O}=\Big\{Q\in\mathbb{N}:p_n<Q\leq q_n, \Upsilon\left(\Delta^j x_Q-y,\frac{d}{2}+h\right)\leq 1-v \text{ or } \Omega\left(\Delta^j x_Q-y,\frac{d}{2}+h\right)\geq v$$
$$\text{or } \Gamma\left(\Delta^j x_Q-y,\frac{d}{2}+h\right)\geq v\Big\}.$$

Since $x,y\in D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\text{LIM}\,(x_Q)$, by Lemma 3.6, we have

$$I-\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in N\}\right|=0.$$

and

$$I-\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\{Q\in\mathbb{N}:p_n<Q\leq q_n, Q\in O\}\right|=0.$$

Now

$$
\begin{aligned}
I - \lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in N \cup O\}\right| \\
\leq & I - \lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in N\}\right| \\
+ & I - \lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in O\}\right| \\
= & 0.
\end{aligned}
$$

Hence for every $\epsilon > 0$,

$$
\mathcal{P} = \left\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in N \cup O\}\right| \geq \epsilon \right\} \in I.
$$

Let $m \in \mathcal{P}^c$ and $\epsilon = \frac{1}{4}$. Then

$$
\begin{aligned}
& \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in N \cup O\}\right| < \frac{1}{4} \\
\Longrightarrow & \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in N^c \cap O^c\}\right| \geq 1 - \frac{1}{4} = \frac{3}{4}.
\end{aligned}
$$

As a result, we have

$$
Q' = \{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in N^c \cap O^c\} \neq \emptyset.
$$

Since $s > 2$, put $sh = 2h + d$ for some $d > 0$. If $\Upsilon(x - y, sh) \leq 1 - \varsigma$ then for $Q \in Q'$, we find

$$
\begin{aligned}
1 - \varsigma & \geq \Upsilon(x - y, t + 2h) \\
& \geq \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2} + h\right) \star \Upsilon\left(\Delta^j x_Q - y, \frac{d}{2} + h\right) \\
& > (1 - v) \star (1 - v) \\
& > 1 - \varsigma,
\end{aligned}
$$

that is ridiculous. If $\Omega(x - y, sh) \geq \varsigma$ for a certain $s > 2$, then

$$
\begin{aligned}
\varsigma & \leq \Omega(x - y, t + 2h) \\
& \leq \Omega\left(\Delta^j x_Q - x, \frac{d}{2} + h\right) \circ \Omega\left(\Delta^j x_Q - y, \frac{d}{2} + h\right) \\
& < v \circ v \\
& < \varsigma,
\end{aligned}
$$

which is absurd. If $\Omega(x - y, sh) \geq \varsigma$ for some $s > 2$, then

$$
\begin{aligned}
\varsigma & \leq \Gamma(x - y, t + 2h) \\
& \leq \Gamma\left(\Delta^j x_Q - x, \frac{d}{2} + h\right) \circ \Gamma\left(\Delta^j x_Q - y, \frac{d}{2} + h\right) \\
& < v \circ v \\
& < \varsigma.
\end{aligned}
$$

That is once more ridiculous. Consequently, every situation leads to a ridiculous outcome. This concludes the validation of our results. □

Proposition 3.9. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. Assume (x_Q) and (y_Q) are sequences in X with $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}\left(\Delta^j\right) - \lim x_Q = x$ and $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}\left(\Delta^j\right) - \lim y_Q = y$ for some $h_1, h_2 \geq 0$. Then*

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{(h_1+h_2)}\left(\Delta^j\right) - \lim\left[x_Q + y_Q\right] = x + y.$$

Proof. For given $\varsigma \in (0,1)$ choose $v \in (0,1)$ with $(1-v)\star(1-v) > 1-\varsigma$ and $v \circ v < \varsigma$. Suppose $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}\left(\Delta^j\right) - \lim x_Q = x$ and $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}\left(\Delta^j\right) - \lim y_Q = y$ for some $h_1, h_2 \geq 0$. For $d > 0$, consider the sets

$$\begin{aligned} A =& \Big\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \geq v \\ & \text{or } \Gamma\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \geq v\Big\}, \end{aligned}$$

and

$$\begin{aligned} \mathcal{B} =& \Big\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right) \geq v \\ & \text{or } \Gamma\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right) \geq v\Big\}. \end{aligned}$$

Then

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in \mathcal{A}\}\right| \geq \epsilon\right\} \in I \text{ and}$$
$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in \mathcal{B}\}\right| \geq \epsilon\right\} \in I$$

for each $\epsilon > 0$. Therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in Q \in \mathcal{A} \cup \mathcal{B}\}\right| \geq \epsilon\right\} \in I.$$

Now, choose $0 < \lambda < 1$ so that $0 < 1 - \lambda < \epsilon$. Then

$$\mathcal{P} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in Q \in \mathcal{A} \cup \mathcal{B}\}\right| \geq 1 - \lambda\right\} \in I.$$

Let $m \in \mathcal{P}^c$. Then

$$\begin{aligned} & \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in Q \in \mathcal{A} \cup \mathcal{B}\}\right| < 1 - \lambda \\ \Longrightarrow & \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{A}^c \cap \mathcal{B}^c\}\right| \geq 1 - (1 - \lambda) = \lambda. \end{aligned}$$

Take $Q \in \mathcal{A}^c \cap \mathcal{B}^c$. Then

$$\begin{aligned}\Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) &\geq \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \star \Upsilon\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right)\\ &\geq (1-v) \star (1-v)\\ &> 1 - \varsigma,\end{aligned}$$

and

$$\begin{aligned}\Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) &\leq \Omega\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \circ \Omega\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right)\\ &\leq v \circ v\\ &< \varsigma,\end{aligned}$$

and

$$\begin{aligned}\Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) &\leq \Gamma\left(\Delta^j x_Q - x, \frac{d}{2} + h_1\right) \circ \Gamma\left(\Delta^j y_Q - y, \frac{d}{2} + h_2\right)\\ &\leq v \circ v\\ &< \varsigma.\end{aligned}$$

This suggests that

$$\begin{aligned}\mathcal{A}^c \cap \mathcal{B}^c \subseteq \{Q \in \mathbb{N} : p_n < Q \leq q_n, &\Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) > 1 - \varsigma \text{ and}\\ &\Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma \text{ and}\\ &\Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma\}.\end{aligned}$$

As a result, for $m \in \mathcal{P}^c$, we have

$$\begin{aligned}&\lambda \leq \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{A}^c \cap \mathcal{B}^c\}\right|\\ &\leq \frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) > 1 - \varsigma\\ &\text{and } \Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma\\ &\text{and } \Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) < \varsigma\}\Big|\\ \Longrightarrow &\frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) \leq 1 - \varsigma \text{ or}\\ &\Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) \geq \varsigma \text{ or}\\ &\Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q\right) - (x+y), d + h_1 + h_2\right) \geq \varsigma\}\Big|\\ &< 1 - \lambda < \epsilon.\end{aligned}$$

Consequently,

$$\mathcal{P}^c \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q \right) - (x+y), d + h_1 + h_2 \right) \leq 1 - \varsigma \text{ or } \Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q \right) - (x+y), d + h_1 + h_2 \right) \geq \varsigma \text{ or } \Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q \right) - (x+y), d + h_1 + h_2 \right) \geq \varsigma \} \big| < \epsilon \Big\}$$

Since $\mathcal{P}^c \in F(I)$, we have

$$\Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\left(\Delta^j x_Q + \Delta^j y_Q \right) - (x+y), d + h_1 + h_2 \right) \leq 1 - \varsigma \text{ or } \Omega\left(\left(\Delta^j x_Q + \Delta^j y_Q \right) - (x+y), d + h_1 + h_2 \right) \geq \varsigma \text{ or } \Gamma\left(\left(\Delta^j x_Q + \Delta^j y_Q \right) - (x+y), d + h_1 + h_2 \right) \geq \varsigma \} \big| < \epsilon \Big\} \in F(I).$$

Hence, by Lemma 3.6, we have $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{(h_1+h_2)}\left(\Delta^j\right) - \lim\left[x_Q + y_Q\right] = x + y$. □

Remark 3.10. Proposition 3.9 may fail to hold for any r such that $0 < h < h_1 + h_2$, provided that at least one of h_1 or h_2 is nonzero. In other words, if

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^j)\text{- }\lim x_Q = x \quad \text{and} \quad D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^j)\text{- }\lim y_Q = y,$$

then it is not necessary that

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{r}(\Delta^j)\text{- }\lim(x_Q + y_Q) = x + y,$$

when $0 < h < h_1 + h_2$.

Example 3.11. Consider $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$, the NNS, defined as in Example 3.4. Define

$$x_Q = \begin{cases} Q & \text{if } Q = 3^n, \\ -1 & \text{if } Q = 4n, \\ 2 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{N}$$

and

$$y_Q = \begin{cases} 0 & \text{if } Q = 3^n, \\ -2 & \text{if } Q = 4n, \\ 1 & \text{otherwise,} \end{cases} \qquad n \in \mathbb{N}.$$

Set $p_n = 0$ and $q_n = n$ for all $n \in \mathbb{N}$. Then, for any nontrivial admissible ideal I, we have:

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^1)\text{- LIM}(x_Q) = \begin{cases} [3 - h_1, h_1 - 3] & \text{if } h_1 \geq 3, \\ \emptyset & \text{otherwise.} \end{cases}$$

and

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^1)\text{- LIM}(y_Q) = \begin{cases} [3 - h_2, h_2 - 3] & \text{if } h_2 \geq 3, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now consider:

$$x_Q + y_Q = \begin{cases} Q & \text{if } Q = 3^n, \\ -3 & \text{if } Q = 4n, \qquad n \in \mathbb{N}. \\ 3 & \text{otherwise,} \end{cases}$$

Then,

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h(\Delta^1)\text{- LIM}(x_Q + y_Q) = \begin{cases} [6-h, h-6] & \text{if } h \geq 6, \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $h_1 = 3$ and $h_2 = 3$. Then,

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_1}(\Delta^1)\text{- lim}\, x_Q = 0, \quad D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{h_2}(\Delta^1)\text{- lim}\, y_Q = 0.$$

But if we take $h < h_1 + h_2 = 6$, then

$$D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h(\Delta^1)\text{- LIM}(x_Q + y_Q) = \emptyset.$$

Proposition 3.12. *Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X and some $h \geq 0$, if $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right) - \lim x_Q = x$ then $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{|a|h}\left(\Delta^j\right)$-lim $ax_Q = ax$ for any $a \in \mathbb{R}$.*

Proof. If $a = 0$, there is nothing to prove. Suppose $a \neq 0$. For given $\varsigma \in (0,1), \exists \gamma \in (0,1)$ such that $1 - \gamma \geq 1 - \varsigma$. For given $d > 0$, consider

$$\mathcal{P} = \Bigg\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, \frac{d}{2|a|} + h\right) \leq 1 - \gamma \text{ or } \Omega\left(\Delta^j x_Q - x, \frac{d}{2|a|} + h\right) \geq \gamma$$
$$\text{or } \Gamma\left(\Delta^j x_Q - x, \frac{d}{2|a|} + h\right) \geq \gamma \Bigg\}.$$

Since $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right) - \lim x_Q = x$, the set

$$Q' = \left\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \left| \{ Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in P \} \right| < \epsilon \right\} \in F(I) \tag{3.2}$$

for each $\epsilon > 0$. Take $m \in Q'$. Then

$$\frac{1}{q_m - p_m} \left| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in P \} \right| < \epsilon$$
$$\implies \frac{1}{q_m - p_m} \left| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{P}^c \} \right| \geq 1 - \epsilon.$$

Now, for $Q \in \mathcal{P}^c$, we have

$$\begin{aligned} \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) &= \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{|a|}\right) \\ &\geq \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{2|a|}\right) \\ &> 1 - \gamma \geq 1 - \varsigma, \end{aligned}$$

and

$$\begin{aligned}\Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) &= \Omega\left(\Delta^j x_Q - x, h + \frac{d}{|a|}\right)\\ &\leq \Omega\left(\Delta^j x_Q - x, h + \frac{d}{2|a|}\right)\\ &< \gamma \leq \varsigma,\end{aligned}$$

and

$$\begin{aligned}\Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) &= \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{|a|}\right)\\ &\leq \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{2|a|}\right)\\ &< \gamma \leq \varsigma.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{P}^c \subseteq &\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) > 1 - \varsigma \text{ and } \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma\\ &\text{and } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma\}.\end{aligned}$$

As a result, for $m \in Q'$, it entails that

$$\begin{aligned}1 - \epsilon &\leq \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{P}^c\}\right|\\ &\leq \frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) > 1 - \varsigma \text{ and}\\ &\quad \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma \text{ and } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) < \varsigma\}\big|.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{1}{q_m - p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) \leq 1 - \varsigma \text{ or}\\ &\qquad \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma \text{ or } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma\}\big| < \epsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}Q' \subseteq \Big\{&n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) \leq 1 - \varsigma\\ &\text{or } \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma \text{ or } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma\}\big| < \epsilon\Big\}.\end{aligned}$$

From (3.2), it entails that

$$\begin{aligned}&\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(a\Delta^j x_Q - ax, |a|h + d\right) \leq 1 - \varsigma\\ &\text{or } \Omega\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma \text{ or } \Gamma\left(a\Delta^j x_Q - ax, |a|h + d\right) \geq \varsigma\}\big| < \epsilon\Big\} \in F(I).\end{aligned}$$

Hence, by Lemma 3.6, $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^{|a|h}\left(\Delta^j\right) - \lim ax_Q = ax$. □

Remark 3.13. For values of $h > 0$, Proposition 3.12 may not hold when $0 < l < |a|h$. Specifically, if for some $h > 0$, the sequence (x_Q) satisfies

$$D_p^q(\Upsilon, \Omega, \Gamma)^h_{S(I)}(\Delta^j)\text{- }\lim x_Q = x,$$

then it is not guaranteed that

$$D_p^q(\Upsilon, \Omega, \Gamma)^l_{S(I)}(\Delta^j)\text{- }\lim(ax) = ax$$

will also hold for any real number a, whenever $0 < l < |a|h$.

Example 3.14. Continuing from Example 3.11, take the scalar $a = -3$. Then

$$-3x_Q = \begin{cases} -3Q, & \text{if } Q = 3^n, \\ 3, & \text{if } Q = 4n, \\ -6, & \text{otherwise.} \end{cases}$$

A direct calculation gives:

$$D_p^q(\Upsilon, \Omega, \Gamma)^l_{S(I)}(\Delta^1)\text{- LIM}(-3x_Q) = \begin{cases} [6 - l,\ l - 6], & \text{if } l \geq 6, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Recall from Example 3.11 that for $h_1 = 3$, we had:

$$D_p^q(\Upsilon, \Omega, \Gamma)^3_{S(I)}(\Delta^1)\text{- LIM}(x_Q) = [-3, 3].$$

Then, ideally, scalar multiplication by -3 gives:

$$-3[-3, 3] = [-9, 9].$$

However,

$$D_p^q(\Upsilon, \Omega, \Gamma)^l_{S(I)}(\Delta^1)\text{- LIM}(-3x_Q) = [6 - l, l - 6],$$

which equals $[-9, 9]$ only when $l = 15 = 3 \times 5$.

This shows that for scalar multiplication by $|a| = 3$, the roughness parameter must be proportionally increased to preserve the scaled limit set. For example, if $6 \leq l < 9$, then

$$D_p^q(\Upsilon, \Omega, \Gamma)^l_{S(I)}(\Delta^1)\text{- LIM}(-3x_Q) \subsetneq -3[-3, 3].$$

Definition 3.15. Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is Δ^j_{NN}-strongly bounded iff for every $\varsigma \in (0, 1), \exists\ d > 0$ such that $\Upsilon\left(\Delta^j x_Q, d\right) > 1 - \varsigma$ and $\Omega\left(\Delta^j x_Q, d\right) < \varsigma$ and $\Gamma\left(\Delta^j x_Q, d\right) < \varsigma$ for all Q.

Definition 3.16. [44] Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. For a sequence (x_Q) in X, we say (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded iff for every $\varsigma \in (0,1), \exists\, d > 0$ such that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, d\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d\right) \geq \varsigma$$
$$\text{or } \Gamma\left(\Delta^j x_Q, d\right) \geq \varsigma\}\big| \geq \epsilon\Big\} \in I$$

for any $\epsilon > 0$.

From the definitions provided, it is clear that if a sequence (x_Q) is Δ^j_{NN}-strongly bounded, then $(\Upsilon, \Omega, \Gamma)^h\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q) \neq \emptyset$, and consequently, according to Theorem 3.7, $D^q_p(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q) \neq \emptyset$ for any $h \geq 0$. The reverse implication of this finding is not valid. To address this issue, we propose the theorem in the following manner.

Theorem 3.17. *Let$(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. A sequence (x_Q) in X is deferred I-statistically Δ^j_{NN}-strongly bounded if and only if $D^q_p(\Upsilon, \Omega, \Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q) \neq \emptyset$ for some $h \geq 0$.*

Proof. Assume that (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded. Thus, for every $\varsigma \in (0,1), \exists\, h > 0$ so that

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, h\right) \geq \varsigma$$
$$\text{or } \Gamma\left(\Delta^j x_Q, h\right) \geq \varsigma\}\big| \geq \epsilon\Big\} \in I$$

for any $\epsilon > 0$. Consider

$$\mathcal{C} = \left\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, h\right) \geq \varsigma\right\}.$$

Clearly

$$\mathcal{D} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in C\}\right| < \epsilon\right\} \in F(I)$$

Now, for $m \in \mathcal{D}$, we obtain

$$\frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in C\}\right| < \epsilon$$
$$\Longrightarrow \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{C}^c\}\right| \geq 1-\epsilon$$

Conversely, consider $Q \in \mathcal{C}^c$. For any $d > 0$, we can state that

$$\Upsilon\left(\Delta^j x_Q, d+h\right) \geq \Upsilon\left(\Delta^j x_Q, h\right) \star \Upsilon(0, d) > (1-\varsigma) \star 1 = 1-\varsigma,$$

and

$$\Omega\left(\Delta^j x_Q, d+h\right) \leq \Omega\left(\Delta^j x_Q, h\right) \circ \Omega(0, d) < \varsigma \circ 0 = \varsigma,$$

and

$$\Gamma\left(\Delta^j x_Q, d+h\right) \leq \Gamma\left(\Delta^j x_Q, h\right) \circ \Gamma(0, d) < \varsigma \circ 0 = \varsigma.$$

Therefore,

$$\mathcal{C}^c \subseteq \Big\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q, d+h\right) < \varsigma \Big\}.$$

Hence, for $m \in \mathcal{D}$, we find

$$\begin{aligned} 1-\epsilon &\leq \frac{1}{q_m-p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{C}^c\}\right| \\ &\leq \frac{1}{q_m-p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q, d+h\right) < \varsigma \\ &\quad \text{and } \Gamma\left(\Delta^j x_Q, d+h\right) < \varsigma\}\big| \\ \Longrightarrow & \frac{1}{q_m-p_m}\big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d+h\right) \geq \varsigma \\ &\quad \text{or } \Gamma\left(\Delta^j x_Q, d+h\right) \geq \varsigma\}\big| < \epsilon. \end{aligned}$$

Consequently,

$$\mathcal{D} \subseteq \Big\{ n \in \mathbb{N} : \frac{1}{q_n-p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, d+h\right) \geq \varsigma\}\big| < \epsilon \Big\} \in F(I).$$

By Lemma 3.6, it follows that $0 \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$.
Conversely, suppose $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q) \neq \emptyset$ for some $h \geq 0$. Let x be a member of $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$. Then, for every $d > 0, \epsilon > 0$ and $\varsigma \in (0,1)$, we have

$$\Big\{ n \in \mathbb{N} : \frac{1}{q_n-p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q, d+h\right) \geq \varsigma\}\big| \geq \epsilon \Big\} \in I$$

which implies that

$$\Big\{ n \in \mathbb{N} : \frac{1}{q_n-p_n}\Big|\Big\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Delta^j x_Q \notin \mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(d+h,\varsigma)\Big\}\Big| \geq \epsilon \Big\} \in I.$$

Hence, (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded. □

We present a counterexample in which the limit set is nonempty but not strongly bounded below.

Example 3.18. Let $X = \mathbb{R}$ and take the neutrosophic norms

$$\Upsilon(x,d) = \frac{d}{d+|x|}, \qquad \Omega(x,d) = \frac{|x|}{d+|x|}, \qquad \Gamma(x,d) = \frac{|x|}{d+|x|}, \qquad d > 0,$$

with $\star = \min$ and $\circ = \max$ (these satisfy the NNS axioms listed in Definition 2.4). Fix the ideal I to be the family of subsets of $\mathbb{N}$ that have natural density zero.

Define a sequence (x_Q) by arranging it in blocks as follows:

$$\text{Block } B_k : \underbrace{0, 0, \ldots, 0}_{k \text{ times}}, \underbrace{k, k, \ldots, k}_{k \text{ times}}.$$

Concatenate the blocks in increasing k: x_Q lists first the entries of B_1, then of B_2, and so on. Equivalently,

$$x_Q = \begin{cases} 0, & \text{if } Q \text{ is in the first half of some block } B_k, \\ k, & \text{if } Q \text{ is in the second half of block } B_k. \end{cases}$$

We take $j = 0$ so $\Delta^j x_Q = x_Q$.

(1) The cluster set is nonempty (0 is a deferred I-statistical cluster point). Fix any $d > 0$ and $\varsigma \in (0, 1)$. In block B_k the proportion of indices Q with large value k equals

$$\frac{\{\text{large entries in } B_k\}}{|B_k|} = \frac{k}{2k} = \tfrac{1}{2}.$$

However, when we consider the cumulative proportion up to block N the fraction of indices that are *large* is

$$\frac{\sum_{k=1}^{N} k}{\sum_{k=1}^{N} 2k} = \frac{1}{2}.$$

This naive global count shows the density of large values is $1/2$. To make the large values *sparse* in the sense of the ideal I (density zero), replace the above block lengths by a rapidly-growing scheme: take

$$B_k \text{ of length } 2L_k, \quad \text{with } L_k \to \infty \text{ so fast that } \frac{L_1 + \cdots + L_{k-1}}{L_1 + \cdots + L_k} \to 1.$$

For instance, choose $L_k = 2^k$. Then in each block B_k put L_k zeros followed by L_k copies of the value k. With this choice the set of indices where $x_Q \neq 0$ has natural density zero (because the proportion of nonzero terms up to block N is $\dfrac{L_1 + \cdots + L_N}{\sum_{j=1}^{N} 2L_j} = \frac{1}{2}$ for the simple scheme, but with exponentially growing L_k the proportion of big values among the first large prefix of indices tends to 0). Concretely, take $L_k = 2^k$; then the total number of entries up to block N is $2\sum_{k=1}^{N} 2^k = 2(2^{N+1} - 2)$, while the number of large entries is $\sum_{k=1}^{N} 2^k = 2^{N+1} - 2$, so the density of large entries tends to $\frac{1}{2}$ in that naive counting; thus choose an even more rapidly growing sequence, e.g. $L_k = k!$, so that the density of large entries tends to 0. The key point is: one can choose the block lengths L_k so that the set $S = \{Q : x_Q \neq 0\}$ has natural density zero, hence $S \in I$.

With such a choice of block lengths (for example $L_k = k!$), for any fixed $d > 0$ and $\varsigma \in (0, 1)$ the indices Q with $\Upsilon(x_Q, d) \leq 1 - \varsigma$ or $\Omega(x_Q, d) \geq \varsigma$ or $\Gamma(x_Q, d) \geq \varsigma$ are contained in S for all sufficiently large k (because for large k the entries equal k are so large that $\Omega(k, d)$ and $\Gamma(k, d)$

are near 1 while $\Upsilon(k,d)$ is near 0; conversely zeros give good membership). Since $S \in I$, the deferred I-statistical condition for a cluster point is satisfied for $z = 0$. Thus

$$0 \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}(x_Q),$$

so the deferred I-statistical limit set is nonempty.

(2) The sequence is *not* Δ^j_{NN}-strongly bounded. Recall the definition: (x_Q) is deferred I-statistically Δ^j_{NN}-strongly bounded iff for every $\varsigma \in (0,1)$ there exists $d > 0$ such that for every $\epsilon > 0$ the set

$$E_{d,\varsigma,\epsilon} = \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q : p_n < Q \le q_n,\ \Upsilon(\Delta^j x_Q, d) \le 1 - \varsigma \text{ or } \Omega(\Delta^j x_Q, d) \ge \varsigma \text{ or } \Gamma(\Delta^j x_Q, d) \ge \varsigma \} \big| \ge \epsilon \Big\} \in I.$$

With our block construction (take the rapidly growing L_k example described above but arrange blocks so that there are infinitely many blocks in which the second half—where values equal k contributes a fixed positive proportion of the block, e.g. 1/2), pick any fixed $d > 0$ and choose ς small (so that any large entry $k \gg d$ satisfies $\Omega(k,d) \approx 1 > \varsigma$). Then every block B_k for sufficiently large k contains a proportion $\approx 1/2$ of entries that are "bad" (they have large nonmembership/indeterminacy). Thus for the corresponding deferred indices n that cover whole blocks the internal proportion

$$\frac{1}{q_n - p_n} \Big| \Big\{ Q \in \mathbb{N} : p_n < Q \le q_n,\ \Upsilon(\Delta^j x_Q, d) \le 1 - \varsigma \text{ or } \Omega(\Delta^j x_Q, d) \ge \varsigma \text{ or } \Gamma(\Delta^j x_Q, d) \ge \varsigma \Big\} \Big|$$

is at least about 1/2. Consequently $E_{d,\varsigma,\epsilon}$ contains infinitely many such n (in fact, infinitely many n corresponding to the blocks), and so $E_{d,\varsigma,\epsilon} \notin I$ (because I contains only density-zero sets while the set of such block-indices is large). Therefore the strong-boundedness condition fails.

We discovered that the aforementioned rough convergence limits are sets instead of singular points. We offer several topological and geometrical characteristics of the limit set $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$ as outlined below.

Theorem 3.19. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS and (x_Q) be a sequence in X. Then $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$ is a closed set for every $h \ge 0$.*

Proof. For a given $\varsigma \in (0,1)$, there exists $v \in (0,1)$ such that

$$(1-v) \star (1-v) > 1 - \varsigma \quad \text{and} \quad v \circ v < \varsigma.$$

Let

$$x \in \mathrm{cl}\Big(D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}(\Delta^j)\text{-}\mathrm{LIM}(x_Q) \Big),$$

the closure of $D^q_p(\Upsilon,\Omega,\Gamma)^h_{S(I)}(\Delta^j)$ - LIM(x_Q). Then there exists a sequence (z_Q) in $D^q_p(\Upsilon,\Omega,\Gamma)^h_{S(I)}(\Delta^j)$ - LIM(x_Q) such that $z_Q \xrightarrow{(\Upsilon,\Omega,\Gamma)} x$.

Thus, for every $d>0$, there exists $n_0\in\mathbb{N}$ such that

$$\Upsilon\big(z_Q - x, \tfrac{d}{2}\big) > 1-v, \quad \Omega\big(z_Q - x, \tfrac{d}{2}\big) < v, \quad \Gamma\big(z_Q - x, \tfrac{d}{2}\big) < v, \quad \forall Q \geq n_0.$$

Choose $m_0 > n_0$. Define the set

$$\mathcal{E} =\Bigg\{Q\in\mathbb{N} : \Upsilon\left(\Delta^j x_Q - z_{m_0}, h+\frac{d}{2}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - z_{m_0}, h+\frac{d}{2}\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - z_{m_0}, h+\frac{d}{2}\right) \geq v\Bigg\}$$

such that

$$\mathcal{F} = \left\{n\in\mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q\in\mathbb{N} : p_n < Q \leq q_n, Q\in\mathcal{E}\}\right| < \epsilon\right\} \in F(I)$$

for every $\epsilon > 0$. For $m\in\mathcal{F}$, we obtain

$$\frac{1}{q_m - p_m}\left|\{Q\in\mathbb{N} : p_m < Q \leq q_m, Q\in\mathcal{E}\}\right| < \epsilon$$
$$\Longrightarrow \frac{1}{q_m - p_m}\left|\{Q\in\mathbb{N} : p_m < Q \leq q_m, Q\in\mathcal{E}^c\}\right| \geq 1-\epsilon.$$

Take $Q\in\mathcal{E}^c$, then

$$\Upsilon\left(\Delta^j x_Q - x, d+h\right) \geq \Upsilon\left(\Delta^j x_Q - z_{m_0}, h+\frac{d}{2}\right) \star \Upsilon\left(z_{m_0} - x, \frac{d}{2}\right) > (1-v)\star(1-v) > 1-\varsigma,$$

and

$$\Omega\left(\Delta^j x_Q - x, d+h\right) \leq \Omega\left(\Delta^j x_Q - z_{m_0}, h+\frac{d}{2}\right) \circ \Omega\left(z_{m_0} - x, \frac{d}{2}\right) < v\circ v < \varsigma,$$

and

$$\Gamma\left(\Delta^j x_Q - x, d+h\right) \leq \Gamma\left(\Delta^j x_Q - z_{m_0}, h+\frac{d}{2}\right) \circ \Gamma\left(z_{m_0} - x, \frac{d}{2}\right) < v\circ v < \varsigma.$$

As a result,

$$\mathcal{E}^c \subseteq \{Q\in\mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma\}.$$

Therefore, for $m\in\mathcal{F}$, we get

$$\begin{aligned}1-\epsilon &\leq \frac{1}{q_m - p_m}\left|\{Q\in\mathbb{N} : p_m < Q \leq q_m, Q\in\mathcal{E}^c\}\right| \\ &\leq \frac{1}{q_m - p_m}\Big|\{Q\in\mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d+h\right) > 1-\varsigma \text{ and } \\ &\quad \Omega\left(\Delta^j x_Q - x, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d+h\right) < \varsigma\}\Big|.\end{aligned}$$

Hence,

$$\frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\Big| < \epsilon.$$

Consequently, we obtain

$$\mathcal{F} \subseteq \Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\} \Big| < \epsilon \text{ or } \Gamma\left(\Delta^j x_Q - x, d+h\right) \geq \varsigma\}\Big| < \epsilon\Bigg\} \in F(I).$$

Consequently, as stated in Lemma 3.6, we obtain $x \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$. Hence, $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ is closed. □

Theorem 3.20. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and (x_Q) is a sequence in X. Then, for every $h \geq 0$, the set $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ is convex.*

Proof. Suppose $x, y \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ and $\varsigma \in (0,1)$ is given. Then, $\exists v \in (0,1)$ so that $(1-v) \star (1-v) > 1-\varsigma$ and $v \circ v < \varsigma$. We need to show that $\alpha x + (1-\alpha)y \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-}\operatorname{LIM}(x_Q)$ for any $\alpha \in [0,1]$. For $\alpha = 0$ or 1 , the result is obvious. Let $\alpha \in (0,1)$. For every $d > 0$, define

$$\mathcal{G} = \Bigg\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \geq v\Bigg\},$$

and

$$\mathcal{H} = \Bigg\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \geq v\Bigg\}.$$

Then

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in G\}\right| \geq \epsilon\right\} \in I,$$

and

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in H\}\right| \geq \epsilon\right\} \in I,$$

for every $\epsilon > 0$. Therefore,

$$\left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in G \cup H\}\right| \geq \epsilon\right\} \in I.$$

Choose $0 < \lambda < 1$ so that $0 < 1-\lambda < \epsilon$. Hence

$$\mathcal{J} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in G \cup H\}\right| \geq 1-\lambda\right\} \in I.$$

Let $m \in \mathcal{J}^c$, then

$$\frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in G \cup H\}\right| < 1 - \lambda$$
$$\Longrightarrow \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{G}^c \cap \mathcal{H}^c\}\right| \geq 1 - (1 - \lambda) = \lambda.$$

Now, take $Q \in \mathcal{G}^c \cap \mathcal{H}^c$. Hence,

$$\begin{aligned}
&\Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha) y], d + h\right) \\
&\quad = \Upsilon\left((1-\alpha)\left(\Delta^j x_Q - y\right) + \alpha\left(\Delta^j x_Q - x\right), (1-\alpha)h + \alpha h + d\right) \\
&\quad \geq \Upsilon\left((1-\alpha)\left(\Delta^j x_Q - y\right), (1-\alpha)h + \frac{d}{2}\right) \star \Upsilon\left(\alpha\left(\Delta^j x_Q - x\right), \alpha h + \frac{d}{2}\right) \\
&\quad = \Upsilon\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \star \Upsilon\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \\
&\quad > (1 - v) \star (1 - v) \\
&\quad > 1 - \varsigma,
\end{aligned}$$

and

$$\begin{aligned}
&\Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha) y], d + h\right) \\
&\quad = \Omega\left((1-\alpha)\left(\Delta^j x_Q - y\right) + \alpha\left(\Delta^j x_Q - x\right), (1-\alpha)h + \alpha h + d\right) \\
&\quad \leq \Omega\left((1-\alpha)\left(\Delta^j x_Q - y\right), (1-\alpha)h + \frac{d}{2}\right) \circ \Omega\left(\alpha\left(\Delta^j x_Q - x\right), \alpha h + \frac{d}{2}\right) \\
&\quad = \Omega\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \circ \Omega\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \\
&\quad < v \circ v \\
&\quad < \varsigma,
\end{aligned}$$

and

$$\begin{aligned}
&\Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha) y], d + h\right) \\
&\quad = \Gamma\left((1-\alpha)\left(\Delta^j x_Q - y\right) + \alpha\left(\Delta^j x_Q - x\right), (1-\alpha)h + \alpha h + d\right) \\
&\quad \leq \Gamma\left((1-\alpha)\left(\Delta^j x_Q - y\right), (1-\alpha)h + \frac{d}{2}\right) \circ \Gamma\left(\alpha\left(\Delta^j x_Q - x\right), \alpha h + \frac{d}{2}\right) \\
&\quad = \Gamma\left(\Delta^j x_Q - y, h + \frac{d}{2(1-\alpha)}\right) \circ \Gamma\left(\Delta^j x_Q - x, h + \frac{d}{2\alpha}\right) \\
&\quad < v \circ v \\
&\quad < \varsigma.
\end{aligned}$$

As a result, we have

$$\mathcal{G}^c \cap \mathcal{H}^c \subseteq \Big\{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma \Big\}.$$

Hence, for $m \in \mathcal{J}^c$, we have

$$\begin{aligned}
\lambda &\leq \frac{1}{q_m - p_m} \left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in \mathcal{G}^c \cap \mathcal{H}^c\}\right| \\
&\leq \frac{1}{q_m - p_m} \Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) > 1 - \varsigma \\
&\text{and } \Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) < \varsigma\}\Big| \\
&\Longrightarrow \frac{1}{q_m - p_m} \Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \leq 1 - \varsigma \\
&\text{or } \Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma\}\Big| \\
&< 1 - \lambda < \epsilon.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathcal{J}^c \subseteq \Big\{ n \in \mathbb{N} : &\frac{1}{q_n - p_n} |\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \leq 1 - \varsigma \text{ or} \\
&\Omega\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma \text{ or} \\
&\Gamma\left(\Delta^j x_Q - [\alpha x + (1-\alpha)y], d+h\right) \geq \varsigma\}| < \epsilon \Big\} \in F(I).
\end{aligned}$$

Therefore, according to Lemma 3.6,
it can be concluded that $\alpha x + (1-\alpha)y \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right)\text{-}\mathrm{LIM}\,(x_Q)$. □

Theorem 3.21. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. Then $D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h \left(\Delta^j\right) - \lim x_Q = x$ for some $h \geq 0$ if there is a sequence (y_Q) in X such that (y_Q) is deferred I-statistically difference convergent to x and*

$$\Upsilon\left(\Delta^j x_Q - \Delta^j y_Q, h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - \Delta^j y_Q, h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - \Delta^j y_Q, h\right) < \varsigma \tag{3.3}$$

hold for every $\varsigma \in (0,1)$ and for all $Q \in \mathbb{N}$.

Proof. Fix arbitrary $\varsigma \in (0,1)$. Choose $v \in (0,1)$ such that

$$(1-v) \star (1-v) > 1 - \varsigma \qquad \text{and} \qquad v \circ v < \varsigma.$$

Since (y_Q) is deferred I-statistically difference convergent to x, by definition for every $d > 0$ and $\epsilon > 0$ the set

$$\mathcal{L} = \Big\{ n \in \mathbb{N} : \frac{1}{q_n - p_n} \big| \{ Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j y_Q - x, d\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j y_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j y_Q - x, d\right) \geq v \} \big| \geq \epsilon \Big\} \in I.$$

Let $m \in \mathcal{L}^c$. Then

$$\frac{1}{q_m - p_m} \big| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j y_Q - x, d\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j y_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j y_Q - x, d\right) \geq v \} \big| < \epsilon$$

$$\implies \frac{1}{q_m - p_m} \big| \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j y_Q - x, d\right) > 1 - v \text{ and } \Omega\left(\Delta^j y_Q - x, d\right) < v \text{ and } \Gamma\left(\Delta^j y_Q - x, d\right) < v \} \big| \geq 1 - \epsilon.$$

Now, define

$$\mathcal{M} = \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j y_Q - x, d\right) > 1 - v \text{ and } \Omega\left(\Delta^j y_Q - x, d\right) < v \text{ and } \Gamma\left(\Delta^j y_Q - x, d\right) < v \}$$

satisfies

$$\frac{|\mathcal{M}|}{q_m - p_m} \geq 1 - \epsilon.$$

For $Q \in \mathcal{M}$. Using 3.3, we get

$$\Upsilon\left(\Delta^j x_Q - x, d + h\right) \geq \Upsilon\left(\Delta^j x_Q - \Delta^j y_Q, h\right) \star \Upsilon\left(\Delta^j y_Q - x, d\right) > (1 - v) \star (1 - v) > 1 - \varsigma,$$

and

$$\Omega\left(\Delta^j x_Q - x, d + h\right) \leq \Omega\left(\Delta^j x_Q - \Delta^j y_Q, h\right) \circ \Omega\left(\Delta^j y_Q - x, d\right) < v \circ v < \varsigma,$$

and

$$\Gamma\left(\Delta^j x_Q - x, d + h\right) \leq \Gamma\left(\Delta^j x_Q - \Delta^j y_Q, h\right) \circ \Gamma\left(\Delta^j y_Q - x, d\right) < v \circ v < \varsigma.$$

Therefore,

$$\mathcal{M} \subseteq \{ Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d + h\right) > 1 - \varsigma \text{ and } \Omega\left(\Delta^j x_Q - x, d + h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - x, d + h\right) < \varsigma \}.$$

This implies that

$$\begin{aligned}1-\epsilon &\leq \frac{1}{q_m-p_m}\left|\{Q\in\mathbb{N}:p_m<Q\leq q_m, Q\in\mathcal{M}\}\right|\\ &\leq \frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}:p_m<Q\leq q_m, \Upsilon\left(\Delta^j x_Q-x,d+h\right)>1-\varsigma\\ &\quad \text{and } \Omega\left(\Delta^j x_Q-x,d+h\right)<\varsigma \text{ and } \Gamma\left(\Delta^j x_Q-x,d+h\right)<\varsigma\}\Big|\\ \Longrightarrow &\frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}:p_m<Q\leq q_m, \Upsilon\left(\Delta^j x_Q-x,d+h\right)\leq 1-\varsigma\\ &\quad \text{or } \Omega\left(\Delta^j x_Q-x,d+h\right)\geq\varsigma \text{ or } \Gamma\left(\Delta^j x_Q-x,d+h\right)\geq\varsigma\}\Big|<\epsilon.\end{aligned}$$

Since $m\in\mathcal{L}^c$ and $\mathcal{L}^c\in F(I)$, we get

$$\begin{aligned}\mathcal{L}^c\subseteq\Bigg\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\Big|\{Q\in\mathbb{N}:p_n<Q\leq q_n, \Upsilon\left(\Delta^j x_Q-x,d+h\right)\leq 1-\varsigma\\ \text{or } \Omega\left(\Delta^j x_Q-x,d+h\right)\geq\varsigma \text{ or } \Gamma\left(\Delta^j x_Q-x,d+h\right)\geq\varsigma\}\Big|<\epsilon\Bigg\}\in F(I).\end{aligned}$$

This implies that $D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)-\lim x_Q=x$. Hence, This completes the proof. □

Theorem 3.22. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. If a sequence (x_Q) in X is deferred I-statistically difference convergent to x, then there exists $v\in(0,1)$ such that*

$$cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)\subseteq D_p^q(\Upsilon,\Omega,\Gamma)^h_{S(I)}\left(\Delta^j\right)\text{-LIM}\left(x_Q\right)\ \textit{for some } h>0.$$

Proof. For given $\varsigma\in(0,1)$, $\exists\ v\in(0,1)$ so that $(1-v)\star(1-v)>1-\varsigma$ and $v\circ v<\varsigma$. Suppose (x_Q) is deferred I-statistically difference convergent to x. Then

$$\begin{aligned}\mathcal{R}=\Bigg\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\Big|\{Q\in\mathbb{N}:p_n<Q\leq q_n, \Upsilon\left(\Delta^j x_Q-x,d\right)\leq 1-v \text{ or } \Omega\left(\Delta^j x_Q-x,d\right)\geq v\\ \text{or } \Gamma\left(\Delta^j x_Q-x,d\right)\geq v\}\Big|\geq\epsilon\Bigg\}\in I,\end{aligned}$$

for every $\epsilon,d>0$. For $m\in\mathcal{R}^c$, we have

$$\begin{aligned}&\frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}:p_m<Q\leq q_m, \Upsilon\left(\Delta^j x_Q-x,d\right)\leq 1-v \text{ or } \Omega\left(\Delta^j x_Q-x,d\right)\geq v\\ &\quad \text{or } \Gamma\left(\Delta^j x_Q-x,d\right)\geq v\}\Big|<\epsilon\\ \Longrightarrow &\frac{1}{q_m-p_m}\Big|\{Q\in\mathbb{N}:p_m<Q\leq q_m, \Upsilon\left(\Delta^j x_Q-x,d\right)>1-v \text{ and } \Omega\left(\Delta^j x_Q-x,d\right)<v\\ &\quad \text{and } \Gamma\left(\Delta^j x_Q-x,d\right)<v\}\Big|\geq 1-\epsilon.\end{aligned}$$

Now, let $w\in\text{cl}\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$ for some $h>0$. Then

$$\Upsilon(x-w,h)\geq 1-v \text{ and } \Omega(x-w,h)\leq v \text{ and } \Gamma(x-w,h)\leq v.$$

Define

$$\mathcal{S} =\Big\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - x, d\right) > 1-v \text{ and } \Omega\left(\Delta^j x_Q - x, d\right) < v \\ \text{and } \Gamma\left(\Delta^j x_Q - x, d\right) < v\Big\}.$$

Thus for $Q \in \mathcal{S}$, similarly to above, we have

$$\Upsilon\left(\Delta^j x_Q - w, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - w, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - w, d+h\right) < \varsigma.$$

Therefore,

$$\mathcal{S} \subseteq \{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - w, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - w, d+h\right) < \varsigma \\ \text{and } \Gamma\left(\Delta^j x_Q - w, d+h\right) < \varsigma\},$$

and hence

$$\begin{aligned} 1-\epsilon \leq & \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in S\}\right| \\ \leq & \frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - w, d+h\right) > 1-\varsigma \\ & \text{and } \Omega\left(\Delta^j x_Q - w, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - w, d+h\right) < \varsigma\}\Big|. \end{aligned}$$

This implies that

$$\frac{1}{q_m - p_m}\Big|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - w, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - w, d+h\right) \geq \varsigma \\ \text{or } \Gamma\left(\Delta^j x_Q - w, d+h\right) \geq \varsigma\}\Big| < \epsilon.$$

Since $m \in \mathcal{R}^c$ and $\mathcal{R}^c \in F(I)$, we obtain

$$\mathcal{R}^c \subseteq \Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - w, d+h\right) \leq 1-\varsigma \\ \text{or } \Omega\left(\Delta^j x_Q - w, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - w, d+h\right) \geq \varsigma\}\Big| < \epsilon\Big\} \in F(I),$$

it follows that,

$$w \in D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h(\Delta^j)\text{- LIM}(x_Q).$$

Therefore,

$$cl(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)) \subseteq D_p^q(\Upsilon, \Omega, \Gamma)_{S(I)}^h(\Delta^j)\text{- LIM}(x_Q).$$

□

Definition 3.23. [44] Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS and (x_Q) be a sequence in X. A point $z \in X$ is called deferred I-statistical Δ^j_h-cluster point of (x_Q) with regard to $(\Upsilon, \Omega, \Gamma)$ for some $h \geq 0$ if, for any $d > 0$ and $\varsigma \in (0, 1)$, we have

$$\Big\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d + h\right) \leq 1 - \varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d + h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d + h\right) \geq \varsigma\}\big| < \epsilon\Big\} \notin I.$$

We denote the set of all deferred I-statistical Δ^j_h-cluster point of (x_Q) by $\Gamma^{(\Upsilon,\Omega)^h_{S(I)}}_{D^p_q}\left(\Delta^j x_Q\right)$.

Note When $h = 0$, a deferred I-statistical Δ^j_h-cluster point of the sequence (x_Q) is simply referred to as a deferred I-statistical Δ^j_h-cluster point. The full set of such points is denoted by

$$\Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}(\Delta^j x_Q).$$

Example 3.24. Consider the neutrosophic normed space $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$ defined by

$$\Upsilon(x, h) = \frac{h}{h + |x|}, \Omega(x, h) = \frac{|x|}{h + |x|}, \Gamma(x, h) = \frac{|x|}{h + |x|}, h > 0,$$

with $\star = \min$ and $\circ = \max$. Let the sequence $x_Q = \frac{1}{Q}$ in $\mathbb{R}$ and fix $j = 0$ (no difference operator).

For any $\varsigma \in (0, 1)$ and $d > 0$, we observe that as $Q \to \infty$,

$$\Upsilon(x_Q - 0, d) = \frac{d}{d + |1/Q|} \longrightarrow 1, \Omega(x_Q - 0, d) = \frac{|1/Q|}{d + |1/Q|} \longrightarrow 0, \Gamma(x_Q - 0, d) = \frac{|1/Q|}{d + |1/Q|} \longrightarrow 0.$$

Hence, 0 satisfies the deferred I–statistical cluster condition.

When $h = 0$, the point 0 is a deferred I-statistical Δ^j_h-cluster point of (x_Q). Thus,

$$0 \in \Gamma^{(\Upsilon,\Omega,\Gamma)_{S(I)}}_{D^p_q}(x_Q).$$

Example 3.25. Let $(\mathbb{R}, \Upsilon, \Omega, \Gamma, \star, \circ)$ be the neutrosophic normed space defined by

$$\Upsilon(x, h) = \frac{h}{h + |x|}, \qquad \Omega(x, h) = \frac{|x|}{h + |x|}, \qquad \Gamma(x, h) = \frac{|x|}{h + |x|}, \qquad h > 0,$$

with $\star = \min$ and $\circ = \max$.

Consider the sequence $x_Q = (-1)^Q + \frac{1}{Q}$ in $\mathbb{R}$ and take $j = 0$. For even Q, $x_Q \approx 1$, and for odd Q, $x_Q \approx -1$.

Step 1: Limit behavior near 1. For even Q, we have $x_Q \to 1$ as $Q \to \infty$. Thus for any $d > 0$,

$$\Upsilon(x_Q - 1, d) = \frac{d}{d + |x_Q - 1|} \to 1, \qquad \Omega(x_Q - 1, d) \to 0, \qquad \Gamma(x_Q - 1, d) \to 0.$$

So 1 is a deferred I–statistical Δ^j_h-cluster point.

Step 2: Limit behavior near -1. For odd Q, we have $x_Q \to -1$ as $Q \to \infty$. Thus for any $d > 0$,

$$\Upsilon(x_Q + 1, d) = \frac{d}{d + |x_Q + 1|} \to 1, \qquad \Omega(x_Q + 1, d) \to 0, \qquad \Gamma(x_Q + 1, d) \to 0.$$

So -1 is also a deferred I–statistical Δ^j_h-cluster point.

When $h = 0$, the sequence (x_Q) has two cluster points, namely

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}(x_Q) = \{-1, 1\}.$$

Theorem 3.26. *Let $(X, \Upsilon, \Omega, \Gamma, \star, \circ)$ be an NNS. Then the set $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}\left(\Delta^j x_Q\right)$ is closed for every sequence (x_Q) in X and each $h \geq 0$.*

Proof. For given $\varsigma \in (0,1), \exists v \in (0,1)$ so that $(1 - v) \star (1 - v) > 1 - v$ and $v \circ v < \varsigma$. Let $z \in cl\left(\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}\left(\Delta^j x_Q\right)\right)$. Then, there is a sequence (z_Q) in $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}\left(\Delta^j x_Q\right)$ such that $z_Q \xrightarrow{(\Upsilon,\Omega,\Gamma)} z$. Thus, for every $d > 0, \exists\, n_0 \in \mathbb{N}$ so as

$$\Upsilon\left(z_Q - z, \frac{d}{2}\right) > 1 - v \text{ and } \Omega\left(z_Q - z, \frac{d}{2}\right) < v \text{ and } \Gamma\left(z_Q - z, \frac{d}{2}\right) < v, \forall\, Q \geq n_0.$$

Fix $m_0 > n_0$ and set

$$\mathbb{T} = \Bigg\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \leq 1 - v \text{ or } \Omega\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \geq v$$
$$\text{or } \Gamma\left(\Delta^j x_Q - z_{m_0}, h + \frac{d}{2}\right) \geq v\Bigg\}.$$

As a result, for every $\epsilon > 0$, we obtain

$$\mathbb{U} = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n} \left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, Q \in T\}\right| < \epsilon\right\} \notin I.$$

Similarly, as the proof of Theorem 3.19, we get

$$\mathbb{U} \subseteq \Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d + h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - z, d + h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d + h\right) \geq \varsigma\}\Big| < \epsilon\Bigg\}.$$

Since $\mathbb{U} \notin I$, we have

$$\Bigg\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\Big|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d + h\right) \leq 1 - \varsigma \text{ or}$$
$$\Omega\left(\Delta^j x_Q - z, d + h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d + h\right) \geq \varsigma\}\Big| < \epsilon\Bigg\} \notin I,$$

i.e., $z \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h}{}_{S(I)}\left(\Delta^j x_Q\right)$. Hence the set $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h}{}_{S(I)}\left(\Delta^j x_Q\right)$ is closed. □

Theorem 3.27. *Let (x_Q) be a sequence in the NNS $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ and $h\geq 0$ be given. If $\Upsilon(z-y,h)>1-\varsigma$ and $\Omega(z-y,h)<\varsigma$ and $\Gamma(z-y,h)<\varsigma$ hold for an arbitrary $z\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta^j x_Q\right)$ and $\varsigma\in(0,1)$, then $y\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}\left(\Delta^j x_Q\right)$.*

Proof. Let $z\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta^j x_Q\right)$ and suppose

$$\Upsilon(z-y,h)>1-\varsigma,\quad \Omega(z-y,h)<\varsigma,\quad \Gamma(z-y,h)<\varsigma$$

for some $\varsigma\in(0,1)$.

Since $z\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}(\Delta^j x_Q)$, by Definition 3.23 for any $d>0$, the set

$$\mathcal{A}=\Big\{n\in\mathbb{N}:\frac{1}{q_n-p_n}\big|\{Q:p_n<Q\leq q_n,\Upsilon(\Delta^j x_Q-z,d+h)\leq 1-\varsigma \text{ or } \Omega(\Delta^j x_Q-z,d+h)\geq\varsigma \text{ or } \Gamma(\Delta^j x_Q-z,d+h)\geq\varsigma\}\big|<\epsilon\Big\}\notin I.$$

Now, by Theorem 3.26, the set $\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}(\Delta^j x_Q)$ is closed. Hence, since y satisfies

$$\Upsilon(z-y,h)>1-\varsigma,\quad \Omega(z-y,h)<\varsigma,\quad \Gamma(z-y,h)<\varsigma,$$

we can apply the neutrosophic triangle-type inequalities

$$\Upsilon(\Delta^j x_Q-y,d+h)\geq\Upsilon(\Delta^j x_Q-z,d)\star\Upsilon(z-y,h),$$

$$\Omega(\Delta^j x_Q-y,d+h)\leq\Omega(\Delta^j x_Q-z,d)\circ\Omega(z-y,h),\quad \Gamma(\Delta^j x_Q-y,d+h)\leq\Gamma(\Delta^j x_Q-z,d)\circ\Gamma(z-y,h),$$

for all Q.

Using these inequalities and the fact that the set of cluster points is closed, we conclude that y also satisfies the condition in Definition 3.23. Therefore,

$$y\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}(\Delta^j x_Q).$$

This is completes the proof. □

Theorem 3.28. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS and (x_Q) be a sequence in X. Then, for some $h>0$ and $v\in(0,1)$, we have*

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)^h_{S(I)}}\left(\Delta^j x_Q\right)=\bigcup_{x\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)}(\Delta x_Q)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$$

Proof. For any given $\varsigma\in(0,1),\exists v\in(0,1)$ such that $(1-v)\star(1-v)>1-\varsigma$ and $v\circ v<\varsigma$. Let

$$z\in\bigcup_{x\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}(\Delta x_Q)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right),h>0$$

Then, $\exists x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{s(\eta)}}\left(\Delta^j x_Q\right)$ so that $z \in \mathrm{cl}\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$, i.e., $\Upsilon(x-z,h) \geq 1-v$ and $\Omega(x-z,h) \leq v$ and $\Gamma(x-z,h) \leq v$. Since $x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta^j x_Q\right)$, we have

$$X = \left\{n \in \mathbb{N} : \frac{1}{q_n - p_n}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - x, d\right) \geq v\}\right| < \epsilon\right\} \notin I$$

for every $\epsilon, d > 0$. Consider

$$X_1 = \{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - x, d\right) \leq 1-v \text{ or } \Omega\left(\Delta^j x_Q - x, d\right) \geq v \text{ or } \Gamma\left(\Delta^j x_Q - x, d\right) \geq v\}.$$

Then, we have

$$X_1^c \subseteq \{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - z, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - z, d+h\right) < \varsigma\}. \tag{3.4}$$

Now take $m \in X$. Then

$$\frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in X_1\}\right| < \epsilon$$

$$\implies \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, Q \in X_1^c\}\right| \geq 1-\epsilon.$$

Hence, by (3.4), it follows that

$$\frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - z, d+h\right) > 1-\varsigma \text{ and } \Omega\left(\Delta^j x_Q - z, d+h\right) < \varsigma \text{ and } \Gamma\left(\Delta^j x_Q - z, d+h\right) < \varsigma\}\right| \geq 1-\epsilon$$

$$\implies \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_m < Q \leq q_m, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\right| < \epsilon.$$

As a result, we get

$$X \subseteq \left\{n \in \mathbb{N} : \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\right| < \epsilon\right\}.$$

Since $X \notin I$, it follows that

$$\left\{n \in \mathbb{N} : \frac{1}{q_m - p_m}\left|\{Q \in \mathbb{N} : p_n < Q \leq q_n, \Upsilon\left(\Delta^j x_Q - z, d+h\right) \leq 1-\varsigma \text{ or } \Omega\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma \text{ or } \Gamma\left(\Delta^j x_Q - z, d+h\right) \geq \varsigma\}\right| < \epsilon\right\} \notin I.$$

Hence $z \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$. Consequently,

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \supseteq \bigcup_{x\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^p}\left(\Delta^j x_Q\right)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right) \tag{3.5}$$

Conversely, assume that $y \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$. Then $y \in \bigcup_{x\in\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)}\left(\Delta^j x_Q\right)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$. Otherwise $y \notin cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right)$ for any $x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta^j x_Q\right)$, i.e.,

$$\Upsilon(x-y,h) < 1-v \text{ or } \Omega(x-y,h) > v \text{ or } \Gamma(x-y,h) > v.$$

Hence, by Theorem 3.27, it follows that $y \notin \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right)$, which contradicts our assumption. Therefore,

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \subseteq \bigcup_{x\in\Gamma_{D_q^q}^{(\Upsilon,\Omega,\Gamma)_{S(I)}}\left(\Delta x_Q\right)} cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,v)\right). \tag{3.6}$$

From (3.5) and (3.6), the result follows. □

Corollary 3.29. *Let $(X,\Upsilon,\Omega,\Gamma,\star,\circ)$ be an NNS. If a sequence (x_Q) in X is deferred I-statistically difference convergent to x, then*

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right)$$

for some $h > 0$.

Proof. Suppose (x_Q) is deferred I-statistically difference convergent to x. Hence $x \in \Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^p}\left(\Delta^j x_Q\right)$. Therefore, by Theorem 3.28, for some $h > 0$ and $\varsigma \in (0,1)$, we have

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) = \mathrm{cl}\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,\varsigma)\right). \tag{3.7}$$

Also, from Theorem 3.22, it can be concluded that

$$cl\left(\mathcal{B}_x^{(\Upsilon,\Omega,\Gamma)}(h,\varsigma)\right) \subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right) \tag{3.8}$$

Hence, by (3.7) and (3.8), we have

$$\Gamma_{D_q^p}^{(\Upsilon,\Omega,\Gamma)_{S(I)}^h}\left(\Delta^j x_Q\right) \subseteq D_p^q(\Upsilon,\Omega,\Gamma)_{S(I)}^h\left(\Delta^j\right)\text{-LIM}\left(x_Q\right).$$

□

4. Concluding remarks

This work extends the framework of convergence by formulating the concept of deferred I-statistical rough convergence through difference operators in the setting of neutrosophic normed spaces. It investigates the characteristics of the resulting boundary set, demonstrating that it is both closed and convex in terms of the neutrosophic norm. Furthermore, the study introduces deferred I-statistical Δ_r^j-cluster points and analyzes their association with the limit set, offering new insights into convergence behavior in neutrosophic environments.

Looking ahead, the findings of this work suggest several potential paths for continued research. One such direction involves expanding the concept of deferred I-statistical rough convergence to broader mathematical settings, including frameworks like fuzzy, intuitionistic fuzzy, or probabilistic normed environments. Another worthwhile pursuit is to investigate the structural and topological traits of the convergence sets-such as their continuity and compactness-within neutrosophic normed spaces. The theory can also be extended to explore the actions of linear transformations and their properties under neutrosophic norms. Moreover, establishing links between this convergence approach and other forms of statistical or summability-based convergence could offer a more unified theoretical foundation. Lastly, there is strong potential for applying these ideas to real-world problems characterized by uncertainty, such as those found in data interpretation, intelligent systems, and complex decision-making processes.

5. Real-World Applications

(1) **Data Science & Big Data**

In large datasets (medical, financial, climate, social networks), data points may not converge in the classical sense due to noise and uncertainty. Deferred I-statistical rough convergence provides a mathematical framework to identify stable trends and clusters in such imperfect datasets.

Example: Detecting early signs of chronic disease progression even when some medical tests are incomplete.

(2) **Signal & Image Processing**

Signals and images often contain distortion, missing pixels, or background noise. Rough convergence with difference operators helps approximate stable features (edges, patterns, denoised signals) even when input data is corrupted.

Example: Improving MRI scan interpretation when images are blurred or incomplete.

(3) **Artificial Intelligence & Decision-Making**

In AI, especially in multi-criteria decision-making (MCDM), expert opinions or sensor readings are vague or inconsistent. Neutrosophic normed spaces handle indeterminacy explicitly, making this framework suitable for robust AI models under uncertainty.

Example: Autonomous vehicles making navigation decisions when sensors provide contradictory inputs.

(4) **Economics & Forecasting**

Financial and economic time series fluctuate irregularly and often do not converge in the classical sense. These methods help in detecting possible limit ranges or cluster points that indicate future states of markets or economic indicators under uncertainty.

Example: Predicting safe investment bands when market data is unstable.

(5) **Numerical Algorithms & Engineering Systems**

Iterative algorithms in scientific computing may only converge roughly due to round-off errors or incomplete information. This framework can be applied to stability analysis of iterative methods and control systems when there is measurement noise.

Example: Stabilizing a drone flight when GPS signals are inconsistent.

6. Declarations

Acknowledgement:

This work was done when the author (M. Mursaleen) visited Uşak University during April 01 to September 30, 2025 under the Project of TUBITAK. He is very much thankful to TUBITAK and Uşak University for the support.

Funding:

Not applicable.

Conflict of interest:

The authors declare that there is no conflict of interest.

Authors' contributions:

All authors contributed equally in writing this paper. Furthermore, this manuscript were read and approved by all authors.

Availability of data and material:

Not applicable.

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Received: April 1, 2025. Accepted: Sep 10, 2025

NSS

Neutrosophic Sets and Systems, Vol. 96, 2026

University of New Mexico

Inferential Study using Neutrosophic Imputation Techniques in Two-Phase Sampling

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Abstract. In the presence of supplementary information, classical statistical methodologies typically rely on precise data to obtain efficient estimators of the population mean. However, the presence of outliers poses substantial challenges to these traditional techniques, which are highly sensitive to data accuracy and auxiliary inputs. In contrast, neutrosophic statistics offers a more flexible and robust framework capable of handling imprecise and uncertain data, thus providing an advantageous alternative to classical methods. In the present research, we modify the study of Gohain et al. ([1]) by adapting his estimators and proposing a new family of neutrosophic exponential–logarithmic-type estimators in the neutrosophic setup with a view to enhancing estimation accuracy in two-phase sampling. Specifically, three imputation methods are developed, each accompanied by their respective point estimators. Theoretical properties, including bias and expressions for the minimum mean square error (MSE), are derived under large sample approximations. By integrating existing imputation methodologies within the neutrosophic framework, this research enhances the scope of current statistical inference techniques and underscores the adaptability of the proposed approach. A comparative evaluation is conducted to assess the relative efficiency of the proposed imputation procedures against alternative methods considered in this study. The empirical analysis, based on real-world datasets, demonstrates the superior performance of the proposed estimators. Furthermore, the findings are corroborated through an extensive simulation study, thereby reinforcing the validity and practical relevance of the proposed methodology.

Keywords: Neutrosophic Missing Data, Neutrosophic mean estimation, Bias, Mean Squared Error (MSE), Neutrosophic Two-Phase Sampling, Neutrosophic variables.

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1. Introduction

In survey sampling, missing data arises when selected respondents decline to take part in a survey for various different reasons. Several different methods exist to address the issue of missingness of the data, with imputation being one of the most effective. Imputation is a well-known and specialized method for managing non-response, involving various approaches to estimate and fill in missing data for univariate, bivariate or multivaraite study variables. In the realm of statistical inference, handling missing data remains a critical challenge and issue across diverse scientific disciplines. Traditional imputation methods often struggle to capture the inherent uncertainty and incomplete information prevalent in complex datasets. The classical statistical approaches, predominantly based on crisp set theory, frequently oversimplify the nuanced nature of real-world data, leading to potential misinterpretations and reduced predictive accuracy. The Neutrosophic Statistical framework emerges as a sophisticated methodological paradigm that transcends conventional probabilistic and fuzzy approaches. Developed as an extension of fuzzy set theory, neutrosophic statistics introduces a more comprehensive mechanism for managing indeterminacy, allowing for the simultaneous consideration of true, indeterminacy, and falsity values. This framework provides a more flexible and robust approach to statistical inference, particularly in scenarios characterized by significant uncertainty and incomplete information. Our research focuses on developing novel imputation methodologies within the Neutrosophic Statistical framework for two-phase sampling, efficiently addressing missing item values, eliminating incompleteness, and simplifying analysis. By leveraging the framework's unique capability to handle indeterminacy and partial truth, we propose a set of three innovative imputation strategies that can more accurately represent the complex uncertainty inherent in multidimensional datasets.

Recent studies have focused on imputation techniques for missing data. When the population mean of the first auxiliary variable is unknown, Grover & Sharma, [10] study investigated the use of two auxiliary variables in two-phase sampling to estimate missing values. An additional estimator for stratified random sampling with an auxiliary variable for finite populations has been proposed by Singh et al., [21]. Imputation strategies developed for handling non-response in repeated surveys by Singh et al., [22]. Imputation strategies based on super population model was studied for mean estimation under non-response is given by Singh et al., [23]. In ranked set sampling, imputation methods for missing values were also put forth by Bhushan et al., [5]. Using auxiliary variables such as kurtosis and the coefficient of variation for population mean estimation, a factor-type exponential ratio estimator was presented by Yadav & Prasad, [27]. Model based studies of the exponential estimators were presented in presence of non response was given by Singh et al. [20]. Classes of more effective and modified type regression-cum-ratio estimators have been introduce to find the population mean in two-phase and stratified two-phase sampling of the research variable while concurrently accounting for non-response and measurement error was given by Sabir & Sanaullah [17, 18]. Gohain et al., [9] suggested an effective exponential chain ratio estimator has been presented for predicting the finite population mean of the research variable of missing data under two-phase sampling using imputation approach. Some modified classes of exponential-logarithmic type of the imputation techniques have been proposed to deal with the missing data by Das & Singh, [8]. Shukla & Jain, [19] suggested to use two-phase sampling for population mean of logarithmic cum exponential estimators. It has been suggested to combine exponential and ln functions for efficient estimation under two-phase sampling given by Hassan et al., [12]. Some time it is possible that practitioners will not be aware of the population mean of the auxiliary variable during survey sampling. Two-phase sampling, sometimes referred to as twofold sampling, is the solution to this unknown value problem.

1.1. *Scope of the study*

Neutrosophic statistics offers a truly adaptable and versatile approach to working in difficult real-world situations that require cognitive ambiguity or an incomplete/indeterminate premise. In situations where there is any degree of uncertainty in the data, neutrosophic statistics could provide a substantive contribution to our understanding of the data where traditional means may fail. The small sample scope of the study shows a method for applying neutrosophic ideas in a practical manner for survey sampling. Given the stated limitations of the uncertain data systems, this line of study is very pertinent and needed. Take environmental monitoring; when collecting data to monitor soil contaminants, water quality, or air quality, there are uncertainties in the data that can occur at many different levels. The issues can arise from sensor calibration and accuracy, sampling uncertainty, environmental variability, and imprecision or ambiguity when measuring the factors in question. Traditional statistical methods depend on precise single-valued data, and hence they often create uncertainty when characterizing environmental conditions. Traditional statistical methods can classify data as outliers or ignore subtlety of relationships through unplanned patterns; creating incomplete or biased representations of environmental facts. In contrast, given that neutrosophic statistics is more thorough and flexible, they can handle the uncertainty with which reality is filled. It is a more realistic quantification of environmental attributes - rather than an environmental measurement, which gives a single point estimation, neutrosophic analyses use interval valued data. For example, when monitoring water quality, if neutrosophic statistical analysis was being employed, the pH reading would be reported as an interval, say [6.8, 7.2] rather than a single value, like 7.0. This interval approach captures the uncertainty, or variability and error in measurements, leading to better decision making and risk management. Neutrosophic statistics can also account for uncertainty in concentrations, as in air quality monitoring, and provide a more complete understanding of the environment in which we operate. Neutrosophic statistics is valuable for decision makers, and experts working for organizations, municipalities, and government agencies that develop responses and strategies for reducing air pollution and addressing the health effects of air pollution on humans and other living organisms. Overall, applying the neutrosophic model allows environmental researchers and practitioners to make better decisions, allocate resources wisely, and create better responsive and adaptive strategies in ongoing responses to the complexities and uncertainties of real-world environmental data.

The present study will also build off this threshold as it will widen the scope of neutrosophic statistics by working with imputation methods to accommodate uncertain and/or missing data in Two-phase sampling. Three imputation methods are proposed with the corresponding point estimators for the exponential-logarithmic type of estimators. The study will use simple random sampling without replacement (SRSWOR) through out the study. An empirical study is conducted to show how the proposed estimators perform against other estimators. To the best of our knowledge, this is the first study to adapt the neutrosophic form to imputation methods. The work of relating neutrosophic statistics with specific imputation methods will achieve the most accurate population mean estimates as possible with the mean squared error as low as possible, even when outliers and imprecise data is present. As the use of these methods is described, this study shows the neutrosophic paradigm's potential adaptability and problem-solving application are far reaching. Conducting this comprehensive study will show that the understanding of neutrosophic statistics is growing and the application of neutrosophic statistics can help to enhance comprehension of more complex data environments across a broader scope and assist in better decision-making.

1.2. *Developments under Neutrosophic Estimation*

Classical statistics handle precise data but struggle with uncertainty, prompting the need for new methods. Fuzzy logic addresses ambiguous or imprecise data but cannot measure indeterminacy. Neutrosophic logic extends fuzzy logic by quantifying uncertainty and certainty, making it effective for analyzing ambiguous scenarios (Aslam, [3, 4]). Modifications such as complex fuzzy sets, interval-valued neutrosophic sets, and neutrosophic numbers have modified decision-making frameworks, considering nuanced data analysis by Ali & Mahmood, [2]; Liu et al., [16] and Li et al., [15]. Study by Chakraborty et al., [6]; Haque et al., [11] include trapezoidal and spherical neutrosophic numbers in multi-criteria group decision-making, mobile communication, and statistical quality control and Smarandache, [25] has extended neutrosophic statistics traditional methods to manage uncertainty and indeterminacy in data, helps in effective analysis of imprecise datasets in realworld scenario. Research in the fields like agriculture and rock engineering demonstrate their versatility and growing importance by [7,14]. For further studies on neutrosophic statistics readers may refer to Yadav & Smarandache, [26]; Yadav & Prasad, 28. The development of neutrosophic statistical approaches has been fueled by advancements that have broadened the field into domains such as neutrosophic applied statistics (NAS), neutrosophic statistical quality control (NSQC), and neutrosophic interval statistics (NIS). All things considered, neutrosophic statistics is still a dynamic and developing area that provides a flexible toolkit for examining complex data that is marked by ambiguity and uncertainty.

2. Terminologies and Methodology under Neutrosophic Statistical Framework

A key observation within the neutrosophic framework is the utilization of quantitative neutrosophic type missing data, where the value may be exist in indeterminate interval $[\Phi N_L, \Phi N_U]$. Neutrosophic interval values can be represented in various forms and are explicitly defined in Yadav and Smarandache [26] as $Z_{(\phi N)} = Z_{(\phi N_L)} + Z_{(\phi N_U)} I_{(\phi N)}$, where $I_{(\phi N)} \in [I_{(\phi N_L)}, I_{(\phi N_U)}]$. For the neutrosophic data under consideration, we adopt the same notations as in Yadav and Smarandache [26], expressed in interval form as $Z_{(\phi N)} \in [Z_{(\phi N_L)}, Z_{(\phi N_U)}]$, where $Z_{(\phi N_L)}$ and $Z_{(\phi N_U)}$ denote the respective Lower and Upper values of neutrosophic study and auxilary variable $Z_{(\phi N_L, \phi N_U)}$. We employ the simple random sampling without replacement method to draw a neutrosophic random sample of size $n \in [n_{(\phi N_L)}, n_{(\phi N_U)}]$ from the specified population, assuming the neutrosophic population comprises $N \in [N_{(\phi N_L)}, N_{(\phi N_U)}]$ distinct units. In the neutrosophic environment, the variable of interest is referred to as the neutrosophic study variable, denoted by $Y_{(\phi N)}$, with its corresponding neutrosophic auxiliary variable represented as $X_{(\phi N)}$.

For the neutrosophic informations being considered, each value on the i^{th} unit of the population for the variables $Y_{(\phi N)}$ and $X_{(\phi N)}$ are represented as $Y_{i_{(\phi N)}} \in [Y_{(\phi N_L)}, Y_{(\phi N_U)}]$, and $X_{i_{(\phi N)}} \in [X_{(\phi N_L)}, X_{(\phi N_U)}]$. Let's define $\bar{Y}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} Y_{i_{(\phi N)}}$ and $\bar{X}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} X_{i_{(\phi N)}}$ as the population means for $Y_{(\phi N)}$ and $X_{(\phi N)}$, acting as the overall means of the neutrosophic informations. The neutrosophic study variable $Y_{(\phi N)}$ has a sample mean of $\bar{y}_{(\phi N)} = \frac{1}{n}\sum_{i=1}^{n} y_{i_{(\phi N)}}$ and , $X_{(\phi N)}$ has a sample mean of $\bar{x}_{(\phi N)} = \frac{1}{n}\sum_{i=1}^{n} x_{i_{(\phi N)}}$, where $(y_{i_{(\phi N)}}; x_{i_{(\phi N)}})$ being the values of variables for i^{th} sampled unit. Additionally, the neutrosophic coefficients of variation for $Y_{(\phi N)}$ and $X_{(\phi N)}$ are given as $C_{Y_{(\phi N)}}$ and $C_{X_{(\phi N)}}$. The correlation coefficient between the neutrosophic variables $Y_{(\phi N)}$ and $X_{(\phi N)}$ denoted as $\rho_{XY_{(\phi N)}}$.

2.1. *Problem Structure*

Consider a finite population $\Omega = \{\Omega_1, \Omega_2, \Omega_3, ...\Omega_N\}$ of the population size N, the value of the variables on the i^{th} unit $\Omega_i, \forall\, i = 1, 2, 3...N$, be $(y_{i_{(\phi N)}}; x_{i_{(\phi N)}})$. Let $\bar{Y}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} y_{i_{(\phi N)}}$ and $\bar{X}_{(\phi N)} = \frac{1}{N}\sum_{i=1}^{N} x_{i_{(\phi N)}}$ be the population means of the neutrosophic study variable $Y_{(\phi N)}$ and the neutrosophic auxiliary variable $X_{(\phi N)}$ respectively. A simple random sample of size n is taken from the population Ω to estimate the $\bar{Y}$. Let $\bar{y}_{n_{(\phi N)}} = \frac{1}{n}\sum_{i=1}^{n} y_{i_{(\phi N)}}$ and $\bar{x}_{n_{(\phi N)}} = \frac{1}{n}\sum_{i=1}^{n} x_{i_{(\phi N)}}$ be the sample means of variables $Y_{(\phi N)}$ and $X_{(\phi N)}$.

In the two-phase sample survey in which unknown $\bar{X}$ is estimated by choosing a preliminary large sample S' of size n' from the population. The sample mean $\bar{x}'_{(\phi N)} = \sum_{i\in S'} \frac{x_{i_{(\phi N)}}}{n'}$ is considered as estimate of $\bar{X}$. The second sample S of size n is selected either from S' or Ω.
Case 1: As a sub sample from sample S' (denoted by design I).
Case 2: Independent to sample S' (denoted by design II).

Consider a sample S consisting of n units, which can be divided into two distinct subsets: a responding R with r units, and a non responding units with $(n - r)$ units $(r < n)$ forming a sub-set R^c with respect to neutrosophic study variable only in $S = R \cup R^c$ for every $y_{i_{(\phi N)}} \in R^c$, the i^{th} value $x_{i_{(\phi N)}}$ of the neutrosophic auxiliary variable is serves as the basis for imputation. The sample mean of the variable $X_{(\phi N)}$ based on the sample S' of size n' i.e.

$$\bar{x}'_{(\phi N)} = \sum_{i=1}^{n'} \frac{x_{i_{(\phi N)}}}{n'}, \text{ where } n' \in S'$$

$\bar{y}_{r_{(\phi N)}}, \bar{x}_{r_{(\phi N)}}$ are the sample mean of the neutrosophic variables $Y_{(\phi N)}$ and $X_{(\phi N)}$ based on the subset R (responding units) of S. $\bar{y}_{n_{(\phi N)}}, \bar{x}_{n_{(\phi N)}}$ are the sample mean of the neutrosophic variable $Y_{(\phi N)}$ and neutrosophic variable $X_{(\phi N)}$ based on $R \cup R^c$. The symbols listed below are used as notation, respectively

$\rho_{YX_{(\phi N)}}$: Correlation between $Y_{(\phi N)}$ and $X_{(\phi N)}$.

$C_{Y_{(\phi N)}}, C_{X_{(\phi N)}}$: Coefficient of the variation of $Y_{(\phi N)}$ and $X_{(\phi N)}$.

$$\zeta_1 = \left(\frac{1}{r} - \frac{1}{n'}\right);\ \zeta_2 = \left(\frac{1}{n} - \frac{1}{n'}\right);\ \zeta_3 = \left(\frac{1}{n'} - \frac{1}{N}\right);\ \zeta_4 = \left(\frac{1}{r} - \frac{1}{N-n'}\right);\ \zeta_5 = \left(\frac{1}{n} - \frac{1}{N-n'}\right)$$

3. **Adapted Neutrosophic Imputation Methods**

This section explores existing conventional imputation methods, selected for their ability to address various statistical challenges. These methods, drawn from established research, are based on well-recognized methodologies. Importantly, they are integrated into the conceptual framework of Neutrosophic Statistics, a contemporary approach designed to handle uncertainty and ambiguity.

3.1. *Adapted Neutrosophic Mean Method*

In this approach, for the each non-responding unit, of the missing value is substituted with mean value calculated from all responding units.

$$y._{i_{(\phi N)}} = \begin{cases} y_{i_{(\phi N)}}, & i \in R \\ \bar{y}_{r_{(\phi N)}}, & i \in R^c \end{cases}$$

The unit-based estimator using this imputation method, $\bar{Y}_{(\phi N)}$, can be expressed as

$$\bar{y}_{s_{(\phi N)}} = \frac{1}{r}\sum_{i\in R} y._{i_{(\phi N)}} = \bar{y}_{r_{(\phi N)}}$$

Bias of $\bar{y}_{r_{(\phi N)}}$ is $B(y_{r_{(\phi N)}}) = 0$ and MSE is $M(\bar{y}_{r_{(\phi N)}}) = \bar{Y}^2_{(\phi N)}\left(\frac{1}{r} - \frac{1}{N}\right)C^2_{Y_{(\phi N)}}$.

3.2. *Adapted Neutrosophic Ratio Method*

Lee et al. [13] proposed ratio method of imputation and here we introduce it under neutrosophic setup as

$$y._{i_{(\phi N)}} = \begin{cases} y_{i_{(\phi N)}}, & i \in R \\ \hat{l}x_{i_{(\phi N)}}, & i \in R^c \end{cases}$$

Here, $\hat{l} = \frac{\sum\limits_{i\in R} y_{i_{(\phi N)}}}{\sum\limits_{i\in R} x_{i_{(\phi N)}}}$

This imputation method has the following unit estimator

$$\bar{y}_{RAT_{(\phi N)}} = \bar{y}_{r_{(\phi N)}}\left(\frac{\bar{x}_{n_{(\phi N)}}}{\bar{x}_{r_{(\phi N)}}}\right)$$

where $\bar{x}_{n_{(\phi N)}} = \frac{\sum\limits_{i\in R} x_{i_{(\phi N)}}}{n}$, $\bar{x}_{r_{(\phi N)}} = \frac{\sum\limits_{i\in R} x_{i_{(\phi N)}}}{r}$ and $\bar{y}_{r_{(\phi N)}} = \frac{\sum\limits_{i\in R} y_{i_{(\phi N)}}}{r}$

Bias and the MSE of the $\bar{y}_{RAT_{(\phi N)}}$ is given by

$$B\left(\bar{y}_{RAT_{(\phi N)}}\right) = \bar{Y}_{(\phi N)}\left(\frac{1}{r} - \frac{1}{n}\right)(C^2_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}})$$

$$M\left(\bar{y}_{RAT_{(\phi N)}}\right) = \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{r} - \frac{1}{N}\right)C^2_{Y_{(\phi N)}} + \left(\frac{1}{r} - \frac{1}{n}\right)\left(1 - 2\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}\right)C^2_{X_{(\phi N)}}\right\}$$

3.3. *Adapted Neutrosophic Compromised Method*

Singh and Horn's [24] compromised imputation method is now adapted and formulated within a neutrosophic context as

$$y._{i_{(\phi N)}} = \begin{cases} \alpha\frac{n}{r}y_{i_{(\phi N)}} + (1-\alpha)\hat{l}x_{i_{(\phi N)}}, & i \in R \\ (1-\alpha)\hat{l}x_{i_{(\phi N)}}, & i \in R^c \end{cases}$$

where α is a constant.

This imputation method's unit estimator can be expressed as

$$\bar{y}_{Comp_{(\phi N)}} = \alpha \bar{y}_{r_{(\phi N)}} + (1-\alpha)\bar{y}_{r_{(\phi N)}} \frac{\bar{x}_{n_{(\phi N)}}}{\bar{x}_{r_{(\phi N)}}}$$

Bias and MSE of the unit estimator $\bar{y}_{Comp_{(\phi N)}}$

$$B\left(\bar{y}_{Comp_{(\phi N)}}\right) = \bar{Y}_{(\phi N)} \left(\frac{1}{r} - \frac{1}{n}\right)(1-\alpha)(1-\phi_{YX_{(\phi N)}})C^2_{X_{(\phi N)}}$$

$$M\left(\bar{y}_{Comp_{(\phi N)}}\right) = \bar{Y}^2_{(\phi N)} \left[\left(\frac{1}{r} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} + \left(\frac{1}{r} - \frac{1}{n}\right)\left\{(1-\alpha)^2 - 2(1-\alpha)\rho_{YX_{(\phi N)}} \frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}\right\} C^2_{X_{(\phi N)}}\right]$$

This estimator is minimum at $\alpha_{opt} = 1 - \rho_{YX_{(\phi N)}} \frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}$

The optimum mean square error (MSE) of $\bar{y}_{Comp_{(\phi N)}}$ is

$$M\left(\bar{y}_{Comp_{(\phi N)}}\right)_{opt} = \bar{Y}^2_{(\phi N)} \left\{\left(\frac{1}{r} - \frac{1}{N}\right) - \left(\frac{1}{r} - \frac{1}{n}\right)\rho^2_{YX_{(\phi N)}}\right\} C^2_{Y_{(\phi N)}}$$

4. **Proposed Adapted Neutrosophic Imputation**

In this section, we propose a new set of neutrosophic exponential–logarithmic-type estimators by generalizing the exponential–logarithmic method of Gohain et al. ([1]), which was initially developed in a classical statistics context. As survey data in real life are frequently plagued by uncertainty, vagueness, and inconsistency that cannot be properly handled by conventional statistical approach, the suggested neutrosophic estimators are formulated to efficiently manage indeterminacy and imprecision in real-life survey data.

Three Neutrosophic imputation techniques and their respective unit estimators are as:

(1)

$$y_{1.i_{(TV)}} = \begin{cases} y_{i_{(\phi N)}} & i \in R \\ \frac{1}{n-r}\bar{y}_{r_{(\phi N)}}\left[n\ exp\left\{\alpha_1\ log\left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} - \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} + \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}\right)\right\}\right] - r & i \in R^c \end{cases}$$

This imputation method's unit estimator can be expressed as

$$\bar{y}_{TV_1} = \bar{y}_{r_{(\phi N)}}\left[exp\left\{\alpha_1\ log\left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} - \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} + \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}\right)\right\}\right] \quad (1)$$

(2)

$$y_{2.i_{(TV)}} = \begin{cases} y_{i_{(\phi N)}} & i \in R \\ \frac{1}{n-r}\bar{y}_{r_{(\phi N)}}\left[n\ exp\left\{\alpha_2\ log\left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} - \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} + \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}}\right)\right\}\right] - r & i \in R^c \end{cases}$$

This imputation method's unit estimator can be expressed as

$$\bar{y}_{TV_2} = \bar{y}_{r_{(\phi N)}} \left[exp \left\{ \alpha_2 \ log \left(1 + \frac{\bar{x'}^{\frac{a}{h}}_{(\phi N)} \ - \ \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}}{\bar{x'}^{\frac{a}{h}}_{(\phi N)} \ + \ \bar{x}^{\frac{a}{h}}_{n_{(\phi N)}}} \right) \right\} \right] \quad (2)$$

(3)

$$y_{3.i_{(TV)}} = \begin{cases} y_{i_{(\phi N)}} & i \in R \\ \frac{1}{n-r}\bar{y}_{r_{(\phi N)}} \left[n \ exp \left\{ \alpha_3 \ log \left(1 + \frac{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} - \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} + \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}} \right) \right\} \right] - r & i \in R^c \end{cases}$$

This imputation method's unit estimator can be expressed as

$$\bar{y}_{TV_3} = \bar{y}_{r_{(\phi N)}} \left[exp \left\{ \alpha_3 \ log \left(1 + \frac{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} \ - \ \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}}{\bar{x}^{\frac{a}{h}}_{n_{(\phi N)}} \ + \ \bar{x}^{\frac{a}{h}}_{r_{(\phi N)}}} \right) \right\} \right] \quad (3)$$

Where $\alpha_i \ \forall \ i = 1, 2, 3$ are constants are derived such as the MSEs of the proposed adapted classes of estimators $\bar{y}_{TV_1}, \ \bar{y}_{TV_2}, \ \bar{y}_{TV_3}$ are minimum and a and $h \neq 0$ are constant.

4.1. *Statistical Properties of the Proposed Adapted Estimators*

We adopted the following transformations to find the bias and MSE expressions associated with the proposed adapted class of estimators.

$\bar{y}_{r_{(\phi N)}} = \bar{Y}_{(\phi N)}(1+e_1)$; $\bar{x}_{r_{(\phi N)}} = \bar{X}_{(\phi N)}(1+e_2)$; $\bar{x}_{n_{(\phi N)}} = \bar{X}_{(\phi N)}(1+e_3)$; $\bar{x}'_{(\phi N)} = \bar{X}_{(\phi N)}(1+e'_3)$
where $e_i, \ e'_3 \in (-1, 1) \ \forall \ i = 1, 2, 3$ and $E(e_i) = 0, \ E(e'_3 = 0) \ \forall \ i = 1, 2, 3$.

Design I:

$$E(e_1^2) = \zeta_1 C^2_{Y_{(\phi N)}}; \ E(e_2^2) = \zeta_1 C^2_{X_{(\phi N)}}; \ E(e_3^2) = \zeta_2 C^2_{X_{(\phi N)}}; \ E(e'^2_3) = \zeta_3 C^2_{X_{(\phi N)}};$$

$$E(e_1 e_2) = \zeta_1 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_1 e_3) = \zeta_2 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}};$$

$E(e_1 e'_3) = \zeta_3 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}$; $E(e_2 e_3) = \zeta_2 C^2_{X_{(\phi N)}}$; $E(e_2 e'_3) = \zeta_3 C^2_{X_{(\phi N)}}$; $E(e_3 e'_3) = \zeta_3 C^2_{X_{(\phi N)}}$

Design II:

$$E(e_1^2) = \zeta_4 C^2_{Y_{(\phi N)}}; \ E(e_2^2) = \zeta_4 C^2_{X_{(\phi N)}}; \ E(e_3^2) = \zeta_5 C^2_{X_{(\phi N)}}; \ E(e'^2_3) = \zeta_3 C^2_{X_{(\phi N)}};$$

$$E(e_1 e_2) = \zeta_4 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_1 e_3) = \zeta_5 \rho_{XY_{(\phi N)}} C_{Y_{(\phi N)}} C_{X_{(\phi N)}}; \ E(e_1 e'_3) = 0;$$

$$E(e_2 e_3) = \zeta_5 C^2_{X_{(\phi N)}}; \ E(e_2 e'_3) = 0; \ E(e_3 e'_3) = 0$$

With the above approximations, the unit estimators $\bar{y}_{TV_1}, \ \bar{y}_{TV_2}$ and $\bar{y}_{TV_3}$ are given in the following formats in relation to e_i and e'_i

$$\bar{y}_{TV_1} = \bar{Y}_{(\phi N)} \left\{ 1 + e_1 + \frac{1}{2}\alpha_1 \frac{a}{h}(e'_3 - e_2 + e_1 e'_3 - e_1 e_2) + \frac{1}{4}(e'^2_3 - e^2_2) \left(\alpha_1 \frac{a}{h} \left(2\frac{a}{h} - 1 \right) \right) \right\} \quad (4)$$

$$\bar{y}_{TV_2} = \bar{Y}_{(\phi N)} \left\{ 1 + e_1 + \frac{1}{2}\alpha_2 \frac{a}{h}(e'_3 - e_3 + e_1 e'_3 - e_1 e_3) + \frac{1}{4}(e'^2_3 - e^2_3) \left(\alpha_2 \frac{a}{h} \left(2\frac{a}{h} - 1 \right) \right) \right\} \quad (5)$$

$$\bar{y}_{TV_3} = \bar{Y}_{(\phi N)} \left\{ 1 + e_1 + \frac{1}{2}\alpha_3 \frac{a}{h}(e_3 - e_2 + e_1 e_3 - e_1 e_2) + \frac{1}{4}(e^2_3 - e^2_2) \left(\alpha_3 \frac{a}{h} \left(2\frac{a}{h} - 1 \right) \right) \right\} \quad (6)$$

4.2. *Expression of Bias and Mean square error of* $\bar{y}_{TV_1},\ \bar{y}_{TV_2},\ \bar{y}_{TV_3}$

Deriving $(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)})$ using equation (4) and applying the expectation operator to both sides yields

$$E(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)}) = E\Big[\bar{Y}_{(\phi N)}\left\{e_1 + \frac{1}{2}K(e_3' - e_2 + e_1e_3' - e_1e_2) + \frac{1}{4}(e_3'^2 - e_2^2)K\left(2\frac{a}{h} - 1\right)\right\} - \frac{1}{4}(e_3' - e_2)^2\left(K\frac{a}{h}(1-\alpha_1)\right)\Big]$$

Substituting the respective values of $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design I, the bias of $\bar{y}_{TV_1}$ is

$$B\left(\bar{y}_{TV_1}\right)_I = \bar{Y}_{(\phi N)}\left\{\frac{1}{2}K(\zeta_3 - \zeta_1)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}} + \frac{1}{4}(\zeta_3 - \zeta_2)\left(\frac{3aK}{h} - K(1+K)\right)C^2_{X_{(\phi N)}}\right\}$$

Substituting the respective values of $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design II, the bias of $\bar{y}_{TV_1}$ is

$$B\left(\bar{y}_{TV_1}\right)_{II} = \bar{Y}_{(\phi N)}\Big[-\zeta_4\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}} + \frac{1}{4}\left\{(\zeta_3 - \zeta_4)\left(K\left(2\frac{a}{h} - 1\right)\right) - (\zeta_3 + \zeta_4)\left(K\frac{a}{h}(1-\alpha_1)\right)\right\}C^2_{X_{(\phi N)}}\Big]$$

Using equation (4) to construct $(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)})^2$ and keeping the terms up to the second-degree of approximations, we get

$$(\bar{y}_{TV_1} - \bar{Y}_{(\phi N)})^2 = \bar{Y}^2_{(\phi N)}\left\{e_1^2 + \frac{1}{4}K^2(e_3' - e_2)^2 + K(e_1e_3' - e_1e_2)\right\} \tag{7}$$

Using equation (7) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design I, the MSE of $\bar{y}_{TV_1}$ is

$$M\left(\bar{y}_{TV_1}\right)_I = \bar{Y}^2_{(\phi N)}\left\{\zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}K^2(\zeta_1 - \zeta_3)C^2_{X_{(\phi N)}} + K(\zeta_3 - \zeta_1)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{8}$$

Using equation (7) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j = 1,2,3$ within the framework of the design II, the MSE of $\bar{y}_{TV_1}$ is

$$M\left(\bar{y}_{TV_1}\right)_{II} = \bar{Y}^2_{(\phi N)}\left\{\zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}K^2(\zeta_3 + \zeta_4)C^2_{X_{(\phi N)}} - K\zeta_4\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{9}$$

Here and for further calculations, for each $\bar{y}_{TV_i}$, $K = \alpha_i\frac{a}{h},\ i = 1,2,3$

Starting from equation (5), we construct the term $(\bar{y}_{TV_2} - \bar{Y}_{(\phi N)})$, then apply expectations to both sides, which gives

Constructing $(\bar{y}_{TV_2} - \bar{Y}_{(\phi N)})$ from equation (5) and considering expectations on both sides, we get

$$E(\bar{y}_{TV_2} - \bar{Y}_{(\phi N)}) = E\Big[\bar{Y}_{(\phi N)}\left\{e_1 + \frac{1}{2}K(e_3' - e_3 + e_1e_3' - e_1e_3) + \frac{1}{4}(e_3'^2 - e_3^2)K\left(2\frac{a}{h} - 1\right)\right\} - \frac{1}{4}(e_3' - e_3)^2\left(K\frac{a}{h}(1-\alpha_2)\right)\Big]$$

Substituting the respective values for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the bias of $\bar{y}_{TV_2}$ is

$$B\left(\bar{y}_{TV_2}\right)_I=\bar{Y}_{(\phi N)}\left[\frac{1}{2}K(\zeta_3-\zeta_2)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}+\frac{1}{4}(\zeta_3-\zeta_2)\left\{\frac{3aK}{h}-K(1+K)\right\}C^2_{X_{(\phi N)}}\right]$$

Substituting the respective values for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the designn II, the bias of $\bar{y}_{TV_2}$ is

$$B\left(\bar{y}_{TV_2}\right)_{II}=\bar{Y}_{(\phi N)}\left[-\zeta_5\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}+\frac{1}{4}\left\{(\zeta_3-\zeta_5)\left(K\left(2\frac{a}{h}-1\right)\right)-(\zeta_3+\zeta_5)\left(K\frac{a}{h}(1-\alpha_2)\right)\right\}C^2_{X_{(\phi N)}}\right]$$

Using equation (5) to construct $(\bar{y}_{TV_2}-\bar{Y}_{(\phi N)})^2$ such as

$$(\bar{y}_{TV_2}-\bar{Y}_{(\phi N)})^2=\bar{Y}^2_{(\phi N)}\left\{e_1^2+\frac{1}{4}K^2(e_3'-e_3)^2+K(e_1e_3'-e_1e_3)\right\} \tag{10}$$

Using equation (10) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the MSE of $\bar{y}_{TV_2}$ is

$$M\left(\bar{y}_{TV_2}\right)_I=\bar{Y}^2_{(\phi N)}\left\{\zeta_1C^2_{Y_{(\phi N)}}+\frac{1}{4}K^2(\zeta_2-\zeta_3)C^2_{X_{(\phi N)}}+K(\zeta_3-\zeta_2)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{11}$$

Using equation (10) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design II, the MSE of $\bar{y}_{TV_2}$ is

$$M\left(\bar{y}_{TV_2}\right)_{II}=\bar{Y}^2_{(\phi N)}\left\{\zeta_4C^2_{Y_{(\phi N)}}+\frac{1}{4}K^2(\zeta_3+\zeta_5)C^2_{X_{(\phi N)}}-K\zeta_5\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{12}$$

From equation (6), $(\bar{y}_{TV_3}-\bar{Y}_{(\phi N)})$ is formed; applying the expectation operator to both sides gives

$$E(\bar{y}_{TV_3}-\bar{Y}_{(\phi N)})=E\Big[\bar{Y}_{(\phi N)}\left\{e_1+\frac{1}{2}K(e_3-e_2+e_1e_3-e_1e_2)+\frac{1}{4}(e_3^2-e_2^2)K\left(2\frac{a}{h}-1\right)\right\}$$
$$-\frac{1}{4}(e_3-e_2)^2\left(K\frac{a}{h}(1-\alpha_3)\right)\Big]$$

Substitute the appropriate expectations for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the bias of $\bar{y}_{TV_3}$ is

$$B\left(\bar{y}_{TV_3}\right)_I=\bar{Y}_{(\phi N)}\left[\frac{1}{4}K(\zeta_2-\zeta_1)\left\{2\rho_{XY_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}+\left(\frac{2a}{h}-1\right)+\frac{a}{h}(1-\alpha_3)\right\}\right]C^2_{X_{(\phi N)}}$$

Substitute the appropriate expectations for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design II, the bias of $\bar{y}_{TV_3}$ is

$$B\left(\bar{y}_{TV_3}\right)_{II}=\bar{Y}_{(\phi N)}\left[\frac{1}{4}K(\zeta_5-\zeta_4)\left\{2\rho_{XY_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}+\left(\frac{2a}{h}-1\right)+\frac{a}{h}(1-\alpha_3)\right\}\right]C^2_{X_{(\phi N)}}$$

Using equation (6) to construct $(\bar{y}_{TV_3}-\bar{Y}_{(\phi N)})^2$ and retaining terms up to the second order, we obtain

$$(\bar{y}_{TV_3}-\bar{Y}_{(\phi N)})^2=\bar{Y}^2_{(\phi N)}\left\{e_1^2+\frac{1}{4}K^2(e_3-e_2)^2+K(e_1e_3-e_1e_2)\right\} \tag{13}$$

Using equation (13) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design I, the MSE of $\bar{y}_{TV_3}$ is

$$M\left(\bar{y}_{TV_3}\right)_I=\bar{Y}^2_{(\phi N)}\left\{\zeta_1C^2_{Y_{(\phi N)}}+\frac{1}{4}K^2(\zeta_1-\zeta_2)C^2_{X_{(\phi N)}}-K(\zeta_1-\zeta_2)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{14}$$

Using equation (13) and computing expectations on both sides, we obtain the expressions for $E(e_i^2),\ E(e_3'^2),\ E(e_ie_j),\ E(e_ie_3')\ \forall\ i,j=1,2,3$ within the framework of the design II, the MSE of $\bar{y}_{TV_3}$ is

$$M\left(\bar{y}_{TV_3}\right)_{II}=\bar{Y}^2_{(\phi N)}\left\{\zeta_4C^2_{Y_{(\phi N)}}+\frac{1}{4}K^2(\zeta_4-\zeta_5)C^2_{X_{(\phi N)}}-K(\zeta_4-\zeta_5)\rho_{XY_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\} \tag{15}$$

4.3. *Optimum Mean Square errors of* $\bar{y}_{TV_1},\ \bar{y}_{TV_2},\ \bar{y}_{TV_3}$

Differentiating equation (8) w.r.t. α_1 and equating it to zero

$$\frac{d}{d\alpha_1}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_1C^2_{Y_{(\phi N)}}+\frac{1}{4}\left(\alpha_1\frac{a}{h}\right)^2(\zeta_1-\zeta_3)C^2_{X_{(\phi N)}}+\alpha_1\frac{a}{h}(\zeta_3-\zeta_1)\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right]=0$$

$$\implies \alpha_1=\frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}=\alpha_{1(\text{opt})I}$$

When the optimum value of α_1 (that is,$\alpha_{1(\text{opt})I}$) is substituted into equation (8), the resulting minimum MSE for $\bar{y}_{TV_1}$:

$$M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I}=\bar{Y}^2_{(\phi N)}\left\{\zeta_1-(\zeta_1-\zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{16}$$

Differentiating equation (9) w.r.t. α_1 for equating it to zero

$$\frac{d}{d\alpha_1}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_4C^2_{Y_{(\phi N)}}+\frac{1}{4}\left(\alpha_1\frac{a}{h}\right)^2(\zeta_3+\zeta_4)C^2_{X_{(\phi N)}}-\left(\alpha_1\frac{a}{h}\right)\zeta_4\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right]=0$$

$$\implies \alpha_1=\frac{2h}{a}\frac{\zeta_4}{\zeta_3+\zeta_4}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}}=\alpha_{1(\text{opt})II}$$

When the optimum value of α_1 (that is,$\alpha_{1(\text{opt})II}$) is substituted into equation (9), the resulting minimum MSE for $\bar{y}_{TV_1}$:

$$M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II}=\bar{Y}^2_{(\phi N)}\left\{\zeta_4-\frac{\zeta_4^2}{\zeta_3+\zeta_4}\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{17}$$

Differentiating equation (11) w.r.t. α_2 and equating it to zero

$$\frac{d}{d\alpha_2}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_1C^2_{Y_{(\phi N)}}+\frac{1}{4}\left(\alpha_2\frac{a}{h}\right)^2(\zeta_2-\zeta_3)C^2_{X_{(\phi N)}}+\left(\alpha_2\frac{a}{h}\right)(\zeta_3-\zeta_2)\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right]=0$$

$$\implies \alpha_2 = \frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{2(\text{opt})I}$$

When the optimum value of α_2 (that is,$\alpha_{2(\text{opt})I}$) is substituted into equation (11), the resulting minimum MSE for $\bar{y}_{TV_2}$:

$$M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_1 - (\zeta_2 - \zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{18}$$

Differentiating equation (12) w.r.t. α_2 and equating it to zero

$$\frac{d}{d\alpha_2}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_2\frac{a}{h}\right)^2(\zeta_3+\zeta_5)C^2_{X_{(\phi N)}} - \left(\alpha_2\frac{a}{h}\right)\zeta_5\rho_{YX_{(\phi N)}}C_Y C_X\right\}\right] = 0$$

$$\implies \alpha_2 = \frac{2h}{a}\frac{\zeta_5}{\zeta_3+\zeta_5}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{2(\text{opt})II}$$

When the optimum value of α_2 (that is,$\alpha_{2(\text{opt})II}$) is substituted into equation (12), the resulting minimum MSE for $\bar{y}_{TV_2}$:

$$M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left\{\zeta_4 - \frac{\zeta_5^2}{\zeta_3+\zeta_5}\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{19}$$

Differentiating equation (14) w.r.t. α_3 and equating it to zero

$$\frac{d}{d\alpha_3}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_1 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_3\frac{a}{h}\right)^2(\zeta_1-\zeta_2)C^2_{X_{(\phi N)}} - \left(\alpha_3\frac{a}{h}\right)(\zeta_1-\zeta_2)\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right] = 0$$

$$\implies \alpha_3 = \frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{3(\text{opt})I}$$

When the optimum value of α_3 (that is,$\alpha_{3(\text{opt})I}$) is substituted into equation (14), the resulting minimum MSE for $\bar{y}_{TV_3}$:

$$M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_1 - (\zeta_1 - \zeta_2)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{20}$$

Differentiating equation (15) w.r.t. α_3 and equating it to zero

$$\frac{d}{d\alpha_3}\left[\bar{Y}^2_{(\phi N)}\left\{\zeta_4 C^2_{Y_{(\phi N)}} + \frac{1}{4}\left(\alpha_3\frac{a}{h}\right)^2(\zeta_4-\zeta_5)C^2_{X_{(\phi N)}} - \left(\alpha_3\frac{a}{h}\right)(\zeta_4-\zeta_5)\rho_{YX_{(\phi N)}}C_{Y_{(\phi N)}}C_{X_{(\phi N)}}\right\}\right] = 0$$

$$\implies \alpha_3 = \frac{2h}{a}\rho_{YX_{(\phi N)}}\frac{C_{Y_{(\phi N)}}}{C_{X_{(\phi N)}}} = \alpha_{3(\text{opt})II}$$

When the optimum value of α_3 (that is,$\alpha_{3(\text{opt})II}$) is substituted into equation (15), the resulting minimum MSE for $\bar{y}_{TV_3}$:

$$M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left\{\zeta_4 - (\zeta_4 - \zeta_5)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \tag{21}$$

5. **Efficiency Comparison**

This section explains and demonstrates, both in theory and practice, how the suggested imputation methods work and how their results compare to the adapted neutrosophic mean, ratio, and compromised methods discussed in Section 3.

5.1. *Theoretical Study*

This section breaks down and explains, in an approachable way, how the recommended methods work and how their results measure up against the methods discussed earlier in Section 3. Theoretical comparisons under desihn I and II are given as

Comparison with adapted neutrosophic mean method:

(1)

$$\mathbb{D}_{1r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 + (\zeta_1 - \zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \geq 0$$

$$\mathbb{D}'_{1r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left[\frac{1}{N-n'} - \frac{1}{N} + \left(\frac{\zeta_4^2}{\zeta_3+\zeta_4}\right)\rho^2_{YX_{(\phi N)}}\right]C^2_{Y_{(\phi N)}} \geq 0$$

So, the proposed estimator $\bar{y}_{TV_1}$ is better than $\bar{y}_{r_{(\phi N)}}$, under both the designs.

(2)

$$\mathbb{D}_{2r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 + (\zeta_2 - \zeta_3)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \geq 0$$

$$\mathbb{D}'_{2r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left[\frac{1}{N-n'} - \frac{1}{N} + \left(\frac{\zeta_5^2}{\zeta_3+\zeta_5}\right)\rho^2_{YX_{(\phi N)}}\right]C^2_{Y_{(\phi N)}} \geq 0$$

So, the proposed estimator $\bar{y}_{TV_2}$ is better than $\bar{y}_{r_{(\phi N)}}$, under both the designs.

(3)

$$\mathbb{D}_{3r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 + (\zeta_1 - \zeta_2)\rho^2_{YX_{(\phi N)}}\right\}C^2_{Y_{(\phi N)}} \geq 0$$

$$\mathbb{D}'_{3r} = M(\bar{y}_{r_{(\phi N)}}) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left[\frac{1}{N-n'} - \frac{1}{N} + \left(\frac{1}{r} - \frac{1}{n}\right)\rho^2_{YX_{(\phi N)}}\right]C^2_{Y_{(\phi N)}} \geq 0$$

So, the proposed estimator $\bar{y}_{TV_3}$ is better than $\bar{y}_{r_{(\phi N)}}$, under both the designs.

Comparison with adapted neutrosophic ratio method:

(1)

$$\mathbb{D}_{1RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 - (\zeta_2-\zeta_3)\rho^2_{YX_{(\phi N)}}\right\} > 0$$

$$\mathbb{D}'_{1RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 + \left(\frac{\zeta_4^2}{\zeta_3+\zeta_5}-\zeta_1+\zeta_2\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_1}$ is more effective than $\bar{y}_{RAT_{(\phi N)}}$ with respect to both designs.

(2)

$$\mathbb{D}_{2RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 - (2\zeta_2-\zeta_1-\zeta_3)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

When $2\zeta_2 \le \left(\frac{1}{r}-\frac{1}{N}\right)$.

$$\mathbb{D}'_{2RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2 + \left(\zeta_4-\frac{\zeta_5^2}{\zeta_3+\zeta_5}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

When $\zeta_4 > \frac{\zeta_5^2}{\zeta_3+\zeta_5}$.

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_2}$ is effective than $\bar{y}_{RAT_{(\phi N)}}$ with respect to both designs.

(3)

$$\mathbb{D}_{3RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2\right\} > 0$$

$$\mathbb{D}'_{3RAT} = M\left(\bar{y}_{RAT_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'}-\frac{1}{N}\right)C^2_{Y_{(\phi N)}} + (\zeta_1-\zeta_2)(C_{X_{(\phi N)}} - \rho_{YX_{(\phi N)}} C_{Y_{(\phi N)}})^2\right\} > 0$$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_3}$ is more effective than $\bar{y}_{RAT_{(\phi N)}}$ with respect to both designs.

Comparison with adapted neutrosophic compromised method:

(1)

$$\mathbb{D}_{1\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} - (\zeta_1 - \zeta_2)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

$$\mathbb{D}'_{1\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_1}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} - \left(\zeta_1 - \zeta_2 - \frac{\zeta_4^2}{\zeta_3 + \zeta_4}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

When $\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} > \left(\zeta_1 - \zeta_2 - \frac{\zeta_4^2}{\zeta_3+\zeta_4}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_1}$ is more effective than $\bar{y}_{Comp_{(\phi N)}}$ with respect to both designs.

(2)

$$\mathbb{D}_{2\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\left\{\zeta_3 C^2_{Y_{(\phi N)}} + (2\zeta_2 - \zeta_1 - \zeta_3)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

$$\mathbb{D}'_{2\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_2}\right)_{(\text{opt})II}$$
$$= \bar{Y}^2_{(\phi N)}\left\{\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} - \left(\zeta_1 - \zeta_2 - \frac{\zeta_5^2}{\zeta_3 + \zeta_5}\right)\rho^2_{YX_{(\phi N)}} C^2_{Y_{(\phi N)}}\right\} > 0$$

So, the proposed adapted neutrosophic estimator $\bar{y}_{TV_2}$ is more effective than $\bar{y}_{Comp_{(\phi N)}}$ with respect to both designs.

(3)

$$\mathbb{D}_{3\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})I} = \bar{Y}^2_{(\phi N)}\zeta_3 C^2_{Y_{(\phi N)}} > 0$$

$$\mathbb{D}'_{3\text{comp}} = M\left(\bar{y}_{Comp_{(\phi N)}}\right) - M\left(\bar{y}_{TV_3}\right)_{(\text{opt})II} = \bar{Y}^2_{(\phi N)}\left(\frac{1}{N-n'} - \frac{1}{N}\right) C^2_{Y_{(\phi N)}} > 0$$

So, the proposed estimator $\bar{y}_{TV_3}$ is better than $\bar{y}_{Comp_{(\phi N)}}$.

5.2. *Empirical Study*

This comprehensive study's objective is to examine and compare the proposed imputation methods with various adapted Neutrosophic imputation methods for calculating the population mean in the framework of Neutrosophic two-phase sampling. In order to achieve this, we selected actual data from the publicly accessible website "https://data.gov.in/resource/seasonal-and-annual-minimum-maximum-temperatureseries-1901-2019." The dataset contains comprehensive information on the seasonal and annual temperatures of all of India, including the mean, maximum, and minimum temperatures over an extended period of time.
Dataset descriptions are provided in Table [1]. For Data A, the minimum and maximum

recorded temperatures in the months of January and February are represented by the auxiliary variable $X_{\phi N}$ and the minimum and maximum recorded temperatures in the months from March to May are taken as the study variable $Y_{\phi N}$. For Data B, minimum and maximum temperatures from June to September are represented by the auxiliary variable $X_{\phi N}$, while the study variable $Y_{\phi N}$ represents the minimum and maximum temperatures recorded in the months from October to December.

MSEs of the adapted and proposed estimators are calculated under designs I & II, and are presented in Tables [2] & [3] respectively. Sample size at first stage is taken as $n' = 50$ and the sample sizes at stage two are taken as $n = 18, 22$ with number of responding units taken as $r = 14, 15, 16$ & $16, 17, 18, 19$ respectively.

We calculate how efficient our proposed and adapted neutrosophic estimators are compared to the one based on the adapted neutrosophic mean imputation method. These percentage relative efficiencies (PREs) are shown in Tables [4] and [5]. The formula used for calculating PREs of estimators is given as

$$PRE(t) = \frac{MSE(\bar{y}_{r(\phi N)})}{MSE(t)} \times 100$$

Where $t = \bar{y}_{RAT(\phi N)},\ \bar{y}_{Comp(\phi N)},\ \bar{y}_{TV_{1(I)}},\ \bar{y}_{TV_{1(II)}},\ \bar{y}_{TV_{2(I)}},\ \bar{y}_{TV_{2(II)}},\ \bar{y}_{TV_{3(I)}},\ \bar{y}_{TV_{3(II)}}$

Table 1. Descriptive statistics of the Data A and Data B

	Data A	**Data B**
Parameter	**Neutrosophic Value**	**Neutrosophic Value**
$\boldsymbol{N}$	[119,119]	[119,119]
$\boldsymbol{n'}$	[50,50]	[50,50]
$\boldsymbol{n}$	[18,18] , [22,22]	[18,18] , [22,22]
$\bar{\boldsymbol{Y}}_{(\phi N)}$	[20.69370, 31.55933]	[16.58437, 27.23613]
$\bar{\boldsymbol{X}}_{(\phi N)}$	[13.90807, 24.67370]	[23.30966, 31.21807]
$\boldsymbol{C}_{Y_{(\phi N)}}$	[0.02657039, 0.02539073]	[0.03440212, 0.02580739]
$\boldsymbol{C}_{X_{(\phi N)}}$	[0.04133301, 0.03918572]	[0.01435458, 0.01423702]
$\boldsymbol{\rho}_{YX_{(\phi N)}}$	[0.6273783, 0.7201808]	[0.6737446, 0.7504361]

6. **Conclusion**

This study confirms the validity of neutrosophic imputation techniques in two-stage sampling, particularly for imprecise, fuzzy, or outlier-vulnerable data instances. Compared to conventional statistical techniques that are more exact-auxiliary-data-dependent and data-error-sensitive, neutrosophic techniques provide a strong framework capable of dealing with uncertainty and indeterminacy in both the main and auxiliary sources of data.

Table 2. Mean Square Error of the Neutrosophic Estimators under Design I.

Data					MSEs					
A	**N**	**n′**	**n**	**r**	$\bar{y}_{r(\phi N)}$	$\bar{y}_{RAT(\phi N)}$	$\bar{y}_{Comp(\phi N)}$	$\bar{y}_{TV_{1(I)}}$	$\bar{y}_{TV_{2(I)}}$	$\bar{y}_{TV_{3(I)}}$
	119	50	18	14	[0.0191, 0.0405]	[0.0213, 0.0421]	[0.0172, 0.0352]	[0.0108, 0.0198]	[0.0127, 0.0250]	[0.0137, 0.0277]
				15	[0.0176, 0.0374]	[0.0192, 0.0385]	[0.0163, 0.0337]	[0.0099, 0.0183]	[0.0113, 0.0220]	[0.0128, 0.0263]
				16	[0.0164, 0.0347]	[0.0173, 0.0354]	[0.0155, 0.0324]	[0.0092, 0.0170]	[0.0100, 0.0193]	[0.0120, 0.0250]
			22	16	[0.0164, 0.0347]	[0.0188, 0.0365]	[0.0143, 0.0291]	[0.0092, 0.0170]	[0.0112, 0.0227]	[0.0108, 0.0216]
				17	[0.0152, 0.0324]	[0.0171, 0.0337]	[0.0137, 0.0279]	[0.0085, 0.0159]	[0.0101, 0.0203]	[0.0101, 0.0205]
				18	[0.0143, 0.0303]	[0.0157, 0.0313]	[0.0131, 0.0269]	[0.0079, 0.0149]	[0.0091, 0.0182]	[0.0095, 0.0195]
				19	[0.0134, 0.0284]	[0.0144, 0.0291]	[0.0125, 0.0260]	[0.0074, 0.0139]	[0.0082, 0.0163]	[0.0090, 0.0186]
B										
	119	50	18	14	[0.0205, 0.0311]	[0.0185, 0.0270]	[0.0182, 0.0267]	[0.0109, 0.0143]	[0.0132, 0.0187]	[0.0144, 0.0210]
				15	[0.0190, 0.0288]	[0.0176, 0.0259]	[0.0173, 0.0257]	[0.0100, 0.0133]	[0.0117, 0.0164]	[0.0135, 0.0200]
				16	[0.0176, 0.0267]	[0.0167, 0.0249]	[0.0166, 0.0248]	[0.0093, 0.0124]	[0.0103, 0.0143]	[0.0128, 0.0191]
			22	16	[0.0176, 0.0267]	[0.0155, 0.0223]	[0.0151, 0.0220]	[0.0093, 0.0124]	[0.0118, 0.0171]	[0.0113, 0.0163]
				17	[0.0164, 0.0249]	[0.0147, 0.0215]	[0.0144, 0.0212]	[0.0086, 0.0116]	[0.0106, 0.0153]	[0.0107, 0.0155]
				18	[0.0153, 0.0233]	[0.0141, 0.0207]	[0.0139, 0.0205]	[0.0080, 0.0109]	[0.0095, 0.0137]	[0.0101, 0.0148]
				19	[0.0144, 0.0219]	[0.0135, 0.0200]	[0.0133, 0.0199]	[0.0075, 0.0103]	[0.0086, 0.0123]	[0.0096, 0.0141]

Table 3. Mean Square Error of the Neutrosophic Estimators under Design II.

Data					MSEs		
A	**N**	**n′**	**n**	**r**	$\bar{y}_{TV_{1(II)}}$	$\bar{y}_{TV_{2(II)}}$	$\bar{y}_{TV_{3(II)}}$
	119	50	18	14	[0.0116, 0.0208]	[0.0134, 0.0259]	[0.0153, 0.0313]
				15	[0.0107, 0.0193]	[0.0120, 0.0228]	[0.0145, 0.0298]
				16	[0.0099, 0.0179]	[0.0107, 0.0202]	[0.0137, 0.0285]
			22	16	[0.0099, 0.0179]	[0.0118, 0.0233]	[0.0125, 0.0251]
				17	[0.0092, 0.0168]	[0.0107, 0.0210]	[0.0118, 0.0240]
				18	[0.0086, 0.0157]	[0.0097, 0.0189]	[0.0112, 0.0230]
				19	[0.0081, 0.0147]	[0.0088, 0.0170]	[0.0107, 0.0221]
B							
	119	50	18	14	[0.0115, 0.0150]	[0.0138, 0.0192]	[0.0162, 0.0237]
				15	[0.0107, 0.0139]	[0.0123, 0.0169]	[0.0153, 0.0227]
				16	[0.0099, 0.0130]	[0.0109, 0.0148]	[0.0146, 0.0218]
			22	16	[0.0099, 0.0130]	[0.0123, 0.0175]	[0.0131, 0.0190]
				17	[0.0092, 0.0121]	[0.0111, 0.0156]	[0.0125, 0.0182]
				18	[0.0086, 0.0114]	[0.0100, 0.0140]	[0.0119, 0.0175]
				19	[0.0081, 0.0107]	[0.0091, 0.0126]	[0.0114, 0.0168]

Table 4. Percent Relative Efficiencies (PREs) of the Neutrosophic Estimators under Design I.

Data					PREs				
A	**N**	**n′**	**n**	**r**	$\bar{y}_{RAT(\phi N)}$	$\bar{y}_{Comp(\phi N)}$	$\bar{y}_{TV_{1(I)}}$	$\bar{y}_{TV_{2(I)}}$	$\bar{y}_{TV_{3(I)}}$
	119	50	18	14	[89.6714, 96.1995]	[111.0465, 115.0568]	[176.8519, 204.5455]	[150.3937, 162.0000]	[139.4161, 146.2094]
				15	[91.6667, 97.1429]	[107.9755, 110.9792]	[177.7778, 204.3716]	[155.7522, 170.0000]	[137.5000, 142.2053]
				16	[94.7977, 98.0226]	[105.8065, 107.0988]	[178.2609, 204.1176]	[164.0000, 179.7927]	[136.6667, 138.8000]
			22	16	[87.2340, 95.0685]	[114.6853, 119.2440]	[178.2609, 204.1176]	[146.4286, 152.8634]	[151.8519, 160.6481]
				17	[88.8889, 96.1424]	[110.9489, 116.1290]	[178.8235, 203.7736]	[150.4950, 159.6059]	[150.4950, 158.0488]
				18	[91.0828, 96.8051]	[109.1603, 112.6394]	[181.0127, 203.3557]	[157.1429, 166.4835]	[150.5263, 155.3846]
				19	[93.0556, 97.5945]	[107.2000, 109.2308]	[181.0811, 204.3165]	[163.4146, 174.2331]	[148.8889, 152.6882]
B									
	119	50	18	14	[110.8108, 115.1852]	[112.6374, 116.4794]	[188.0734, 217.4825]	[155.3030, 166.3102]	[142.3611, 148.0952]
				15	[107.9545, 111.1969]	[109.8266, 112.0623]	[190.0000, 216.5414]	[162.3932, 175.6098]	[140.7407, 144.0000]
				16	[105.3892, 107.2289]	[106.0241, 107.6613]	[189.2473, 215.3226]	[170.8738, 186.7133]	[137.5000, 139.7906]
			22	16	[113.5484, 119.7309]	[116.5563, 121.3636]	[189.2473, 215.3226]	[149.1525, 156.1404]	[155.7522, 163.8037]
				17	[111.5646, 115.8140]	[113.8889, 117.4528]	[190.6977, 214.6552]	[154.7170, 162.7451]	[153.2710, 160.6452]
				18	[108.5106, 112.5604]	[110.0719, 113.6585]	[191.2500, 213.7615]	[161.0526, 170.0730]	[151.4851, 157.4324]
				19	[106.6667, 109.5000]	[108.2707, 110.0503]	[192.0000, 212.6214]	[167.4419, 178.0488]	[150.0000, 155.3191]

Table 5. Percent Relative Efficiencies (PREs) of the Neutrosophic Estimators under Design II.

Data					PREs		
A	N	n'	n	r	$\bar{y}_{TV_{1(II)}}$	$\bar{y}_{TV_{2(II)}}$	$\bar{y}_{TV_{3(II)}}$
	119	50	18	14	[164.6552, 194.7115]	[142.5373, 156.3707]	[124.8366, 129.3930]
				15	[164.4860, 193.7824]	[146.6667, 164.0351]	[121.3793, 125.5034]
				16	[165.6566, 193.8547]	[153.2710, 171.7822]	[119.7080, 121.7544]
			22	16	[165.6566, 193.8547]	[138.9831, 148.9270]	[131.2000, 138.2470]
				17	[165.2174, 192.8571]	[142.0561, 154.2857]	[128.8136, 135.0000]
				18	[166.2791, 192.9936]	[147.4227, 160.3175]	[127.6786, 131.7391]
				19	[165.4321, 193.1973]	[152.2727, 167.0588]	[125.2336, 128.5068]
B							
	119	50	18	14	[178.2609, 207.3333]	[148.5507, 161.9792]	[126.5432, 131.2236]
				15	[177.5701, 207.1942]	[154.4715, 170.4142]	[124.1830, 126.8722]
				16	[177.7778, 205.3846]	[161.4679, 180.4054]	[120.5479, 122.4771]
			22	16	[177.7778, 205.3846]	[143.0894, 152.5714]	[134.3511, 140.5263]
				17	[178.2609, 205.7851]	[147.7477, 159.6154]	[131.2000, 136.8132]
				18	[177.9070, 204.3860]	[153.0000, 166.4286]	[128.5714, 133.1429]
				19	[177.7778, 204.6729]	[158.2418, 173.8095]	[126.3158, 130.3571]

By constructing novel imputation methods for exponential-logarithmic-type estimators, the study demonstrates that the neutrosophic estimators can yield interval-based estimates, which present a more accurate estimate of the population mean. Theoretical results confirm that such estimators possess virtues such as reduced bias and minimum mean square error under large sample approximations.

Empirical comparisons with actual datasets further confirm the practical merits of the neutrosophic approach, being more robust and reliable compared to traditional estimators. The synthesis of existing imputation methods under the neutrosophic statistical paradigm broadens inference potentialities, allowing for superior data imperfection management. Overall, the findings highlight that neutrosophic imputation techniques are a robust and versatile tool for statistical estimation in complex, uncertain data settings.

Declaration

Funding: This research received no external funding
Conflicts of Interest: The authors declare no conflict of interest.

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Received: April 13, 2025. Accepted: Sep 11, 2025

An Algorithmic Approach to Bridge Detection in Bijective Neutrosophic Graph with Application in Cancer Diagnosis

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Abstract: This paper introduces an efficient algorithm for detecting bridges in Bijective neutrosophic graphs, an emerging approach in handling uncertainty in complex systems such as medical diagnostics. The algorithm identifies T-bridges, I- bridges and F-bridges in neutrosophic graphs, and incorporates a de neutrosophication method using score function to find neutrosophic bridges when needed. The application of the algorithm to cancer diagnosis is explored, demonstrating its potential to enhance disease identification and treatment planning based on symptoms. The result suggest that this method can improve the accuracy of medical decision- making in uncertain environments, thereby offering a promising tool for healthcare analysis. By providing a more precise understanding of medical data, this approach has the potential to optimize diagnostic processes and treatment strategies in uncertain and indeterminate contexts.

Keywords: Neutrosophic Graph; Bijective Neutrosophic graph; T-bridge, I-bridge, F-bridge; and Neutrosophic Bridge.

1. Introduction

Graph theory has emerged as a powerful mathematical tool for modeling complex systems in various domains, including biology, computer science, and medicine. Among its numerous applications, the analysis of graph structures offers valuable insights into the connectivity and critical components of networks. In this context, bridge detection plays a significant role in identifying the most sensitive or pivotal connections whose removal may fragment the graph. To address the limitations of classical graph models in handling uncertainty, incompleteness and inconsistency inherent in real world data- particularly in medical diagnostics- the concept of neutrosophic graphs has gained traction. These graphs extend fuzzy and intuitionistic fuzzy models by incorporating degrees of truth, indeterminacy, and falsity. A further advancement, bijective neutrosophic graphs, ensures a one to one correspondence between graph elements, offering a more precise and structured representation of data. Akram, M et. al [1] introduced the concept of operations on Single Valued Neutrosophic Graphs (SVN-graphs) in 2017 Beaula Thangaraj et al. [2–4] contributed significantly to the field by applying various fuzzy numbers and ranking methods to solve critical path problems, exemplifying the expanding application of fuzzy logic in optimization and decision-making. It is important to acknowledge Broumi, S et al[5-6] studies about single valued neutrosophic graph in 2016. Further this concept take its shape as neutrosophic labelling graph in 2019, which was introduced by Gomathi et.al [7]. A.Hassan et.al [8] studied about single valued trees in 2018. Muthuraj et.al[9] studied about multi fuzzy graph in 2020. Rajalaxmi D et.al[10-11] studied about metric in fuzzy labeling graph and Bijective single valued in highlighting their structural properties and potential applications. Vijaya et al[12,13] have shaped the advancements of Neutrosophic graphs in find the solution of Decision making problem and critical path problems by using Pythagorean Fuzzy numbers and Neutrosophic Fuzzy numbers. Ye, J.,[14-15]studied Single-Valued Neutrosophic Minimum Spanning Tree in 2014. Finally, the pioneering work of Zadeh, L. [16] in 1965, who introduced the concept of fuzzy sets, laid the groundwork for the entire field of fuzzy and neutrosophic mathematics that followed.

An algorithmic method for detecting bridges in bijective neutrosophic graphs is presented in this article, with a focus on its use in the diagnosis of cancer. In addition to improving structural analysis of intricate networks, the algorithm offers a useful tool that can be applied to a range of real-world issues. The suggested techniques seek to assist oncology's early detection, risk assessment, and strategic intervention planning by locating important pathways or interactions in biological networks linked to cancer. Through this study, we show how algorithmic approaches and mathematical abstraction can greatly advance medical research and decision-making.

The neutrosophic framework permits the simultaneous representation of truth, indeterminacy, and falsity, the neutrosophic framework is especially useful in this situation. Incomplete, ambiguous, or contradicting information is frequently present in biological and medical networks, particularly those pertaining to the diagnosis of cancer. The ability of fuzzy graph models and even classical graph theory to handle such uncertainties is constrained. The suggested approach more successfully captures several aspects of uncertainty by using bijective neutrosophic graphs, producing robust analysis and more trustworthy results. This makes the neutrosophic approach especially important in advancing medical research and decision-making processes where ambiguity is inherent.

2. Preliminaries:

Definition 2.1[7]: A neutrosophic graph is of the form G = (V, σ,μ) where σ = (T_1, I_1, F_1) and μ = (T_2, I_2, F_2) (i) V = {v_1, v_2, v_3, ⋯, v_n} such that T_1: V → [0, 1], I_1: V → [0, 1] and F_1 : V → [0, 1] denote the degree of truth-membership function, indeterminacy-membership function and falsity-membership function of the vertex v_i ∈ V respectively, and 0 ≤ T_1 (v) + I_1 (v) + F_1 (v) ≤ 3 ∀ v_i ∈ V (i=1, 2, 3….n).

(ii) T_2 : V × V → [0, 1], I_2 : V × V → [0, 1] and F_2 : V × V → [0, 1], where $T_2(v_i, v_j)$, $I_2(v_i, v_j)$ and $F_2(v_i, v_j)$ denote the degree of truth-membership function, indeterminacy membership function and falsity-membership function of the edge (v_i, v_j) respectively such that for every (v_i, v_j), T_2 (v_i, v_j) ≤ min {$T_1(v_i)$, $T_1(vj)$}, I_2 (v_i, v_j) ≤ min {$I_1(v_i)$, $I_1(v_j)$}, F_2 (v_i, v_j) ≤ max {F_1 (v_i), $F_1(v_j)$}, and 0 ≤ $T_2(v_i, v_j)$ + $I_2(v_i, v_j)$ + $F_2(v_i, v_j)$ ≤ 3 .

Definition 2.2

A neutrosophic graph is said to be a bijective neutrosophic graph if

$$\sigma_T : V \to [0,1], \sigma_I : V \to [0,1], \sigma_F : V \to [0,1], \mu_T : V \times V \to [0,1], \mu_I : V \times V \to [0,1], \mu_F : V \times V \to [0,1]$$

are bijective, such that the truth- membership function, Indeterminacy-membership function and Falsity- membership functions for every edge

$\mu_T(u,v) <$ min (σ_T(u), σ_T(v))

$\mu_I(u,v) <$ min (σ_I(u), σ_I(v))

$\mu_F(u,v) <$ max (σ_F(u), σ_F(v)) and 0≤ $\mu_T(u,v) + \mu_I(u,v) + \mu_F(u,v) \leq 3$

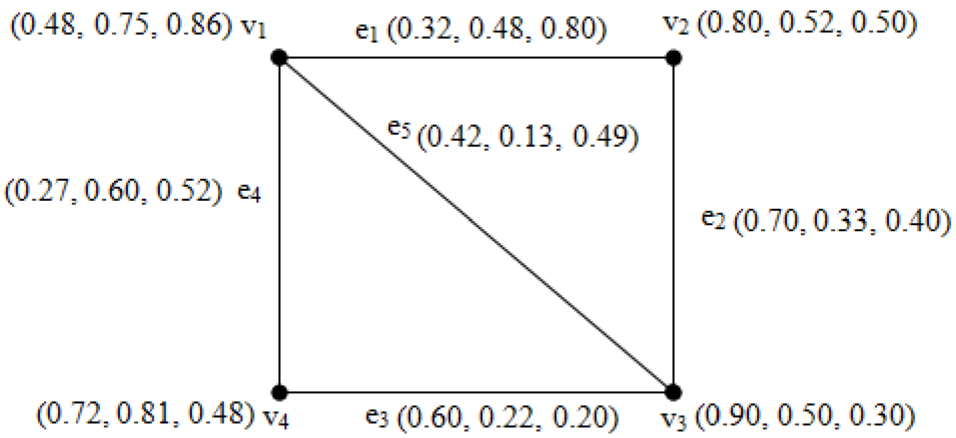

Figure 1:Bijective Neutrosophic Graph

Definition :2.3

The strength of the path with n edges is defined as S(P) = (S(P_1), S(P_2), S(P_3)) where

S(P1) = $\overset{n}{\underset{i=1}{\Lambda}} \mu_T(u,v)$, S(P2) = $\overset{n}{\underset{i=1}{\Lambda}} \mu_I(u,v)$, S(P3) = $\overset{n}{\underset{i-1}{V}} \mu_F(u,v)$

Definition :2.4

Let G be a Bijective neutrosophic graph. The connected between any two vertices is defined by $\mu^{\infty}(u,v) = (\mu_T^{\infty}(u,v), \mu_I^{\infty}(u,v), \mu_F^{\infty}(u,v))$ where $\mu_T^{\infty}(u,v)$ = Max(S(P1)), $\mu_I^{\infty}(u,v)$ = Max(S(P2)), $\mu_F^{\infty}(u,v)$ = Min(S(P3))

Definition:2.5

Let G be a bijective neurtosophic graph. An edge of G is said to T-bridge

$$\text{if } \mu_T^{\infty}(u,v) < \mu_T^{'\infty}(u,v)$$

where $\mu_T^{'\infty}(u,v)$ is the connectedness between u and v by removing any edge.

An edge of G is said to I-bridge

$$\text{if } \mu_I^{\infty}(u,v) < \mu_I^{'\infty}(u,v)$$

where $\mu_I^{'\infty}(u,v)$ is the connectedness between u and v by removing any edge.

An edge of G is said to F-bridge

$$\text{if } \mu_F^{\infty}(u,v) > \mu_F^{'\infty}(u,v)$$

where $\mu_F^{'\infty}(u,v)$ is the connectedness between u and v by removing any edge.

Definition: 2.6

An edge of G is said to be a neutrosophic bridge if it is T- bridge, I-bridge and F- bridge.

3. An Algorithm for finding the bridges of any bijective neutrosophic graph

Input: A bijective neutrosophic graph G = (V, E, T, I, F), where each edge has associated truth-membership T, indeterminacy-membership I, and falsity-membership F.
Output: Classification of each bridge as T-Bridge, I-Bridge, or F-Bridge.

Step 1: Begin with a bijective neutrosophic graph G. Select an arbitrary crisp cycle C* consisting of n- edges, where n ≥ 3

Step 2: If T- bridge or I- bridge of G are required, then identify an edge

$$\eta_T = \overset{n}{\underset{i=1}{\Lambda}} \mu_T \quad \text{or} \quad \eta_I = \overset{n}{\underset{i=1}{\Lambda}} \mu_I$$

by considering all the edges of C* respectively.

Step 3: Remove the identified edge η from G.

Step 4: Choose another cycle C* in G with any number of edges and repeat step 2 & step 3 until no such cycle remains in G.

Step 5: After removal of all η's from G, the resulting graph comprises the T- bridges or I – bridges of G respectively.

Step 6: If F- bridges is required for the chosen graph then find $\eta_F = \bigvee_{i-1}^{n} \mu_F$

by considering all the edges of C*.

Step 7: Repeat step 3 and step 4.

Step 8: Upon removal of all η_F's from the bijective neutrosophic graph G, the resulting graph comprises the F- bridges of G.

The T- bridges and I- bridges are the strongest connections between the vertices in the graph G and F – bridges is the weakest connections between the vertices in the graph G.
The resulting graph which is obtained is the T- maximum spanning sub graph or I- maximum spanning sub graph of G. Also one can obtain the F-minimum spanning subgraph of G.

Example:3.1

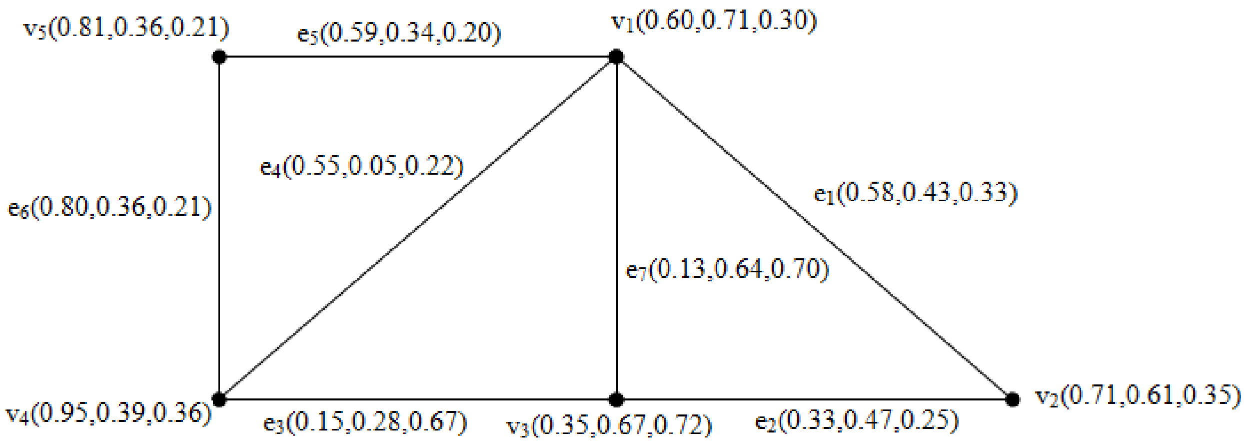

Figure 2:Bijective neutrosophic graph

Let us find the T- bridges of the above figure 2 by Considering the cycle $C_1^* = v_1, v_3, v_4, v_5, v_1$ of length 4.
Here η_1= min(0.13, 0.15, 0.80, 0.59)
η_1= 0.13

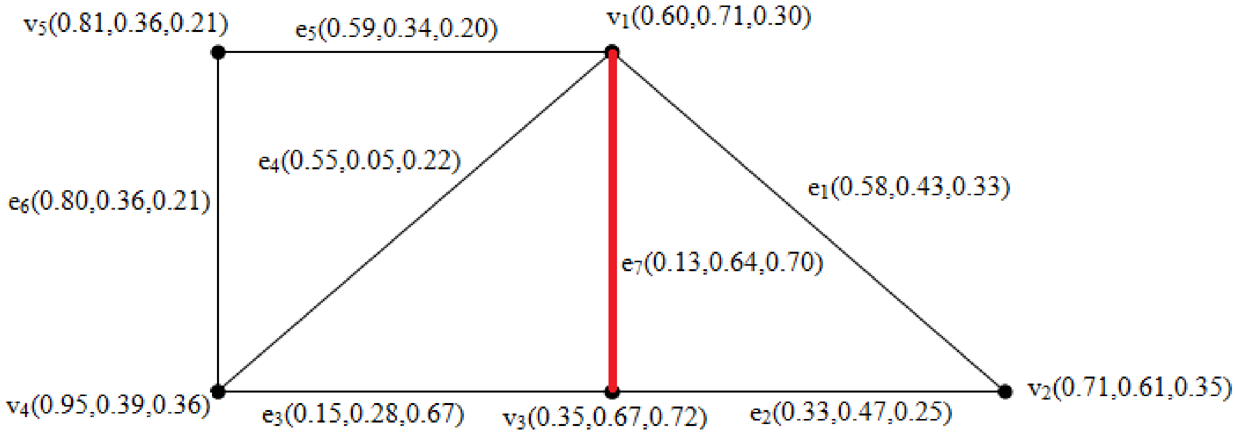

Figure 3

In the above figure 3 edge e_7 with 0.13 truth membership value has to be removed from G.

Let the next cycle be $C_2^* = v_1,v_4,v_5,v_1$ of length 3.

Here η_2= min(0.55, 0.80, 0.59)

η_2= 0.55

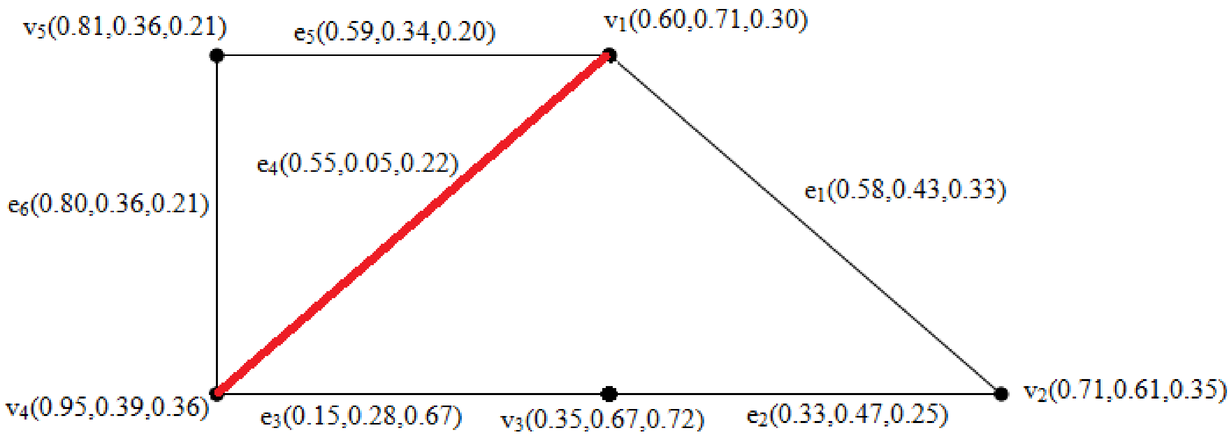

Figure 4

Clearly in the above figure 4 the edge e_4 with 0.55 truth membership value has to be removed from G. So Let the next cycle be $C_3^* = v_1,v_2,v_3,v_4,v_5,v_1$ of length 5.

Here η_3= min(0.58, 0.33, 0.15, 0.80, 0.59)

η_3= 0.15

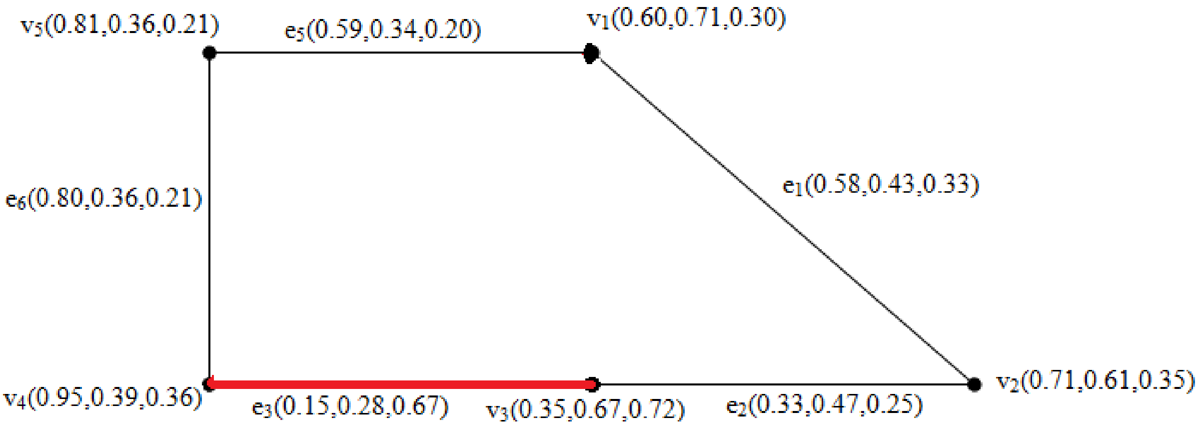

Figure 5

Clearly in the above figure 5 the edge e_3 with 0.15 as truth membership value has to be removed from G. Since no cycles remains after the removal of edges, the following resulting graph represents all the T-Bridges of G

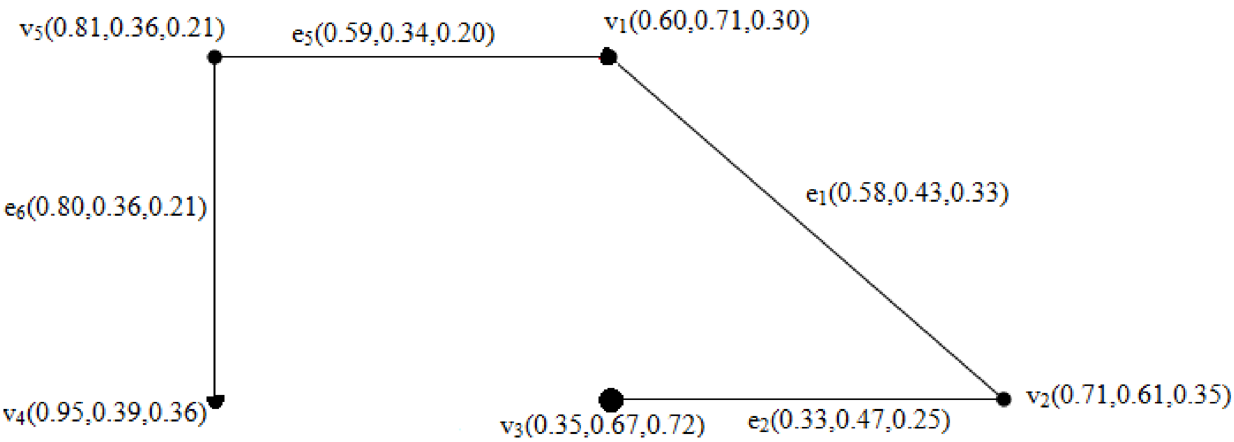

Figure 6: T-Bridges of G

Hence the T- Bridges of G are (v_1,v_2) , (v_2,v_3) , (v_4,v_5) and (v_5,v_1)

If the above algorithm is applied for Indeterminancy then the I- Bridge of G can be obtained.

Hence the I- Bridges of G are (v_1,v_3) , (v_1,v_5) , (v_2,v_3) and (v_3,v_4)

Similarly F-Bridges of G are (v_1,v_5) , (v_1,v_4) , (v_1,v_2) and (v_2,v_3).

In some practical situation if it is necessary to consider all the membership functions in the same time then we can use score function to find the bridges of bijective neutrosophic graph.

Definition:[12]

The score function is defined as

$$S(u) = \frac{\sigma_T(u)+\sigma_I(u)+\sigma_F(u)}{3}$$

The same algorithm can be used to find the bridges of the bijective neutrosophic graph after find the score function for all the vertices and edges.

Example: Now let us find the bridges of the graph given in figure after finding the score function

$$S(v_1) = \frac{\sigma_T(v_1)+\sigma_I(v_1)+\sigma_F(v_1)}{3} = \frac{0.60+0.71+0.30}{3} = 0.54$$

$S(v_2) = 0.56$, $S(v_3) = 0.58$ and so on.

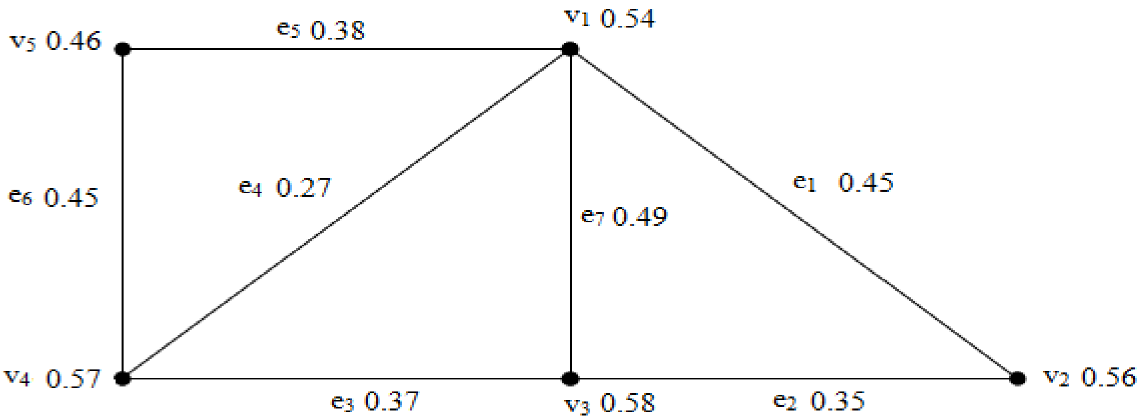

Figure 7

For finding the bridges of G let us first consider a cycle $C_1^* = v_1,v_2,v_3,v_4,v_5,v_1$ of length 5.

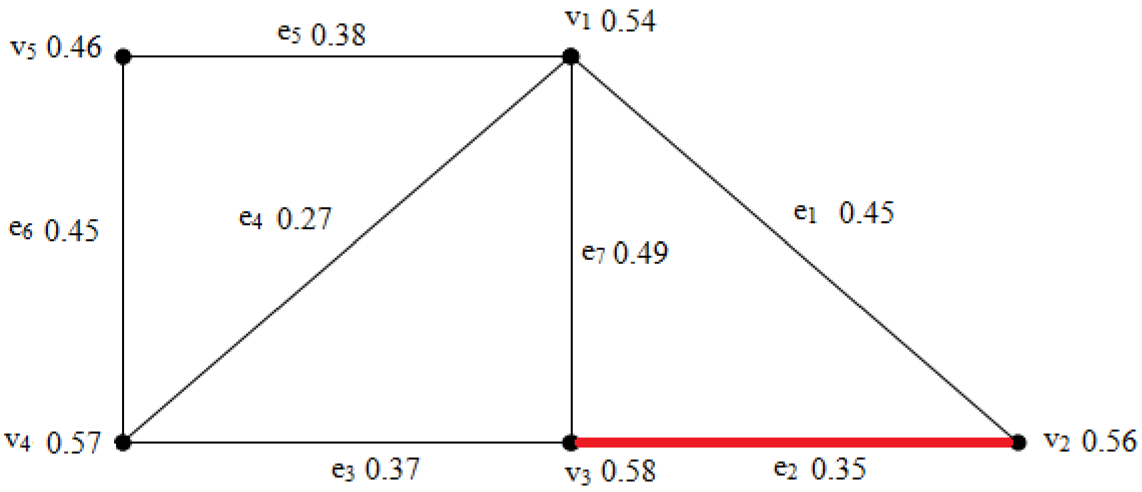

Figure 8

Clearly the edge e_2 is to be removed as it has minimum value among the other edge values in the considered cycle. Now let us choose another cycle $C_2^* = v_1,v_3,v_4,v_5,v_1$ of length 4.

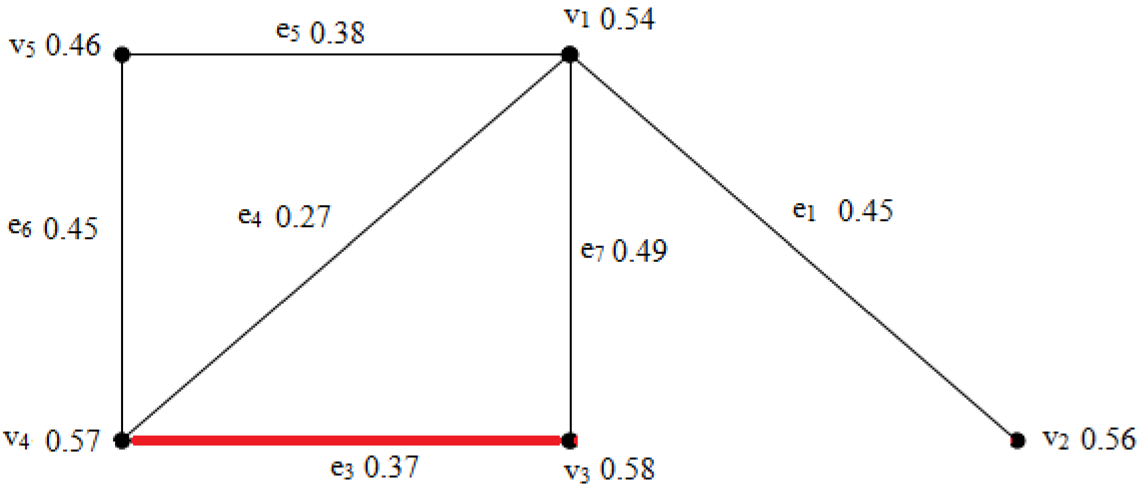

Figure 9

The above marked edge e_3 is the next edge with minimum score value which has to be removed next. Now consider another cycle $C_3^* = v_1,v_4,v_5,v_1$ of length 3.

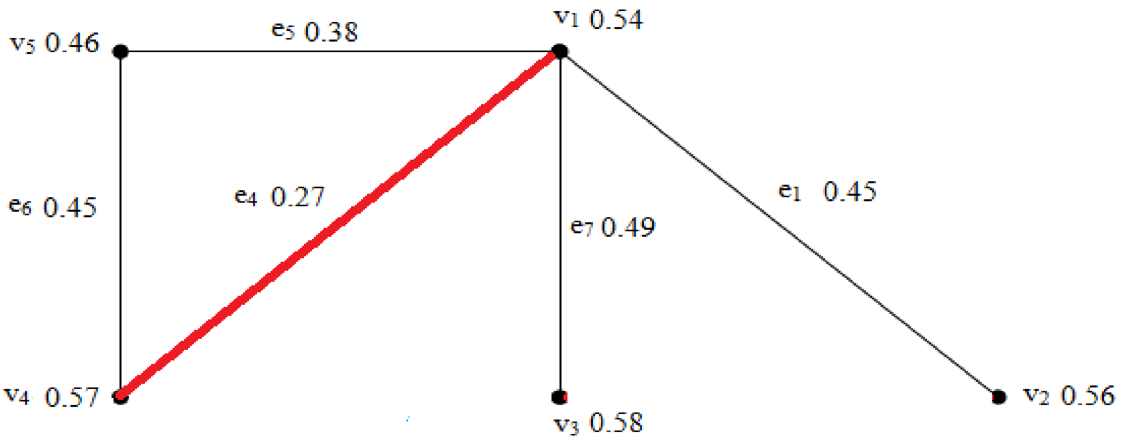

Figure 10

Clearly if the above marked edge is removed, then the bridges of G are (v_1,v_2) , (v_1,v_3) , (v_4,v_5) and (v_5,v_1).

Note: The score function is one of the methods of de neutrosophication. So score function need not be bijective.

4. Applications

Now, Let's use the above discussed algorithm to find a better diagnosis for cancer. Let's consider a specific case in which we have a patient with the more or less equal symptoms of Inflammation of Gastrointestinal Tract (GT), Chronic Cough(C) and Fatigue(F).**i.e. As per the data, he exhibits 48% of the typical symptoms associated with Inflammation of Gastrointestinal Tract, 40% clinical symptoms for chronic cough and 42% of Fatigue.** And we suspect that the patient might be suffering from Colorectal Cancer(CRC), Lung Cancer(LC), Tuberculosis(TB) and Inflammatory Bowel Disease(IBD).**i.e. There is approximately equal chance for him to suffer from all the above mentioned disease. In**

detail, clinical data suggests that there is 55% likelihood for him to suffer from CRC, 52% to present with Lung cancer, 45% exhibits Tuberculosis and 50% chance to experience IBD. But still we need a better diagnostic approach to enhance a safe and effective treatment outcomes by reducing side effects. In this Crucial case, Neutrosophic Graph ensures effective Cancer-focused framework since fatigue is one of the most common seen symptoms for many types of cancer,it helps to capture indeterminacy value for better diagnosis. In our use case, it is significant to highlight that 39% of fatigue could linked to Colorectal Cancer (CRC), 38% manifests Lung cancer(LC), there is 31% and 34% probability that fatigue could cause TB and IBD respectively. Therefore, Bijective NeutrosophicGraph lends us a helping hand since the given patient data is more accurate. It is also necessary to recall that the bridges are the crucial links that would helps to find the underlying issue. Finally, we can say that the **Bijective Neutrosophic Graph** arise as a natural response to this particular use case. To construct a Bijective Neutrosophic Framework for this cancer diagnosis, let's take the symptoms and suspected diseases as vertices.

Membership value for vertices:

1. Truth membership value(T) ensure the degree of confirmed occurrence of symptoms/diseases,
2. Indeterminacy value(I) tells us the uncertainty due to conflict reports or incomplete data and
3. Falsity-membership value(F) ensure the degree of belief that there is no such occurrence exists.

Edges represent the relations such as symptoms correlation, biological contact, cooccurrence of diseases etc., It is always necessary to consider Symptom-Symptom relationships, since it adds a deeper layer for medical insight especially when symptoms Co-occurrence, contradict occurs. Similarly, we need to consider Disease-Disease relationship, as in many cases one or more disease appears together. In Particular, since liver and lungs are the most common sites for colorectal cancer, there is a possibility that 18% of patients affected with Colorectal Cancer(CRC) may be presents with lung cancer. Therefore, it becomes significant to consider disease-disease relationships to have a refined diagnostic perspective.(In our use case, there is 50% chance of patient suffering from Colorectal cancer to get suffer from lung cancer as per the clinical data).In addition to that, TB cannot directly cause lung cancer, but it can increase the risk of developing lung cancer. Therefore, there is 35% likelihood of him to develop lung cancer as per the patient's data.

Membership value for edges:

1. Truth membership value (T) exhibits the degree of correlation. 2. Indeterminacy value (I) tells the degree of uncertainty arises due to incomplete data (may be due to ongoing investigation).3. Falsity value ensures the belief that no occurrence exists.

Analysis for cancer diagnosis:

It is significant to remember the following remarks to give membership values for the disease-disease relations in the graphical representation for cancer diagnosis.

1. Generally, Colorectal Cancer (CRC) does not cause tuberculosis (TB). It is almost uncommon in most of the cases.
2. Tuberculosis (TB) is one of the risk factor for lung cancer.
3. Inflammation in Gastro intestinal tract (GT) in addition with the Inflammatory Bowl Disease (IBD) provides a high risk factor for Colorectal Cancer (CRC). And this relation is bidirectional.
4. There is high possibility for a person getting affected with Colorectal Cancer (CRC) to get suffer with Lung cancer. Since Liver and lungs are the common sites for cancer development.
5. Chronic cough(C) is one of the risk factors for lung cancer (LC), Tuberculosis (TB).
6. Fatigue (F) is one of the common symptoms for many of the cancers.

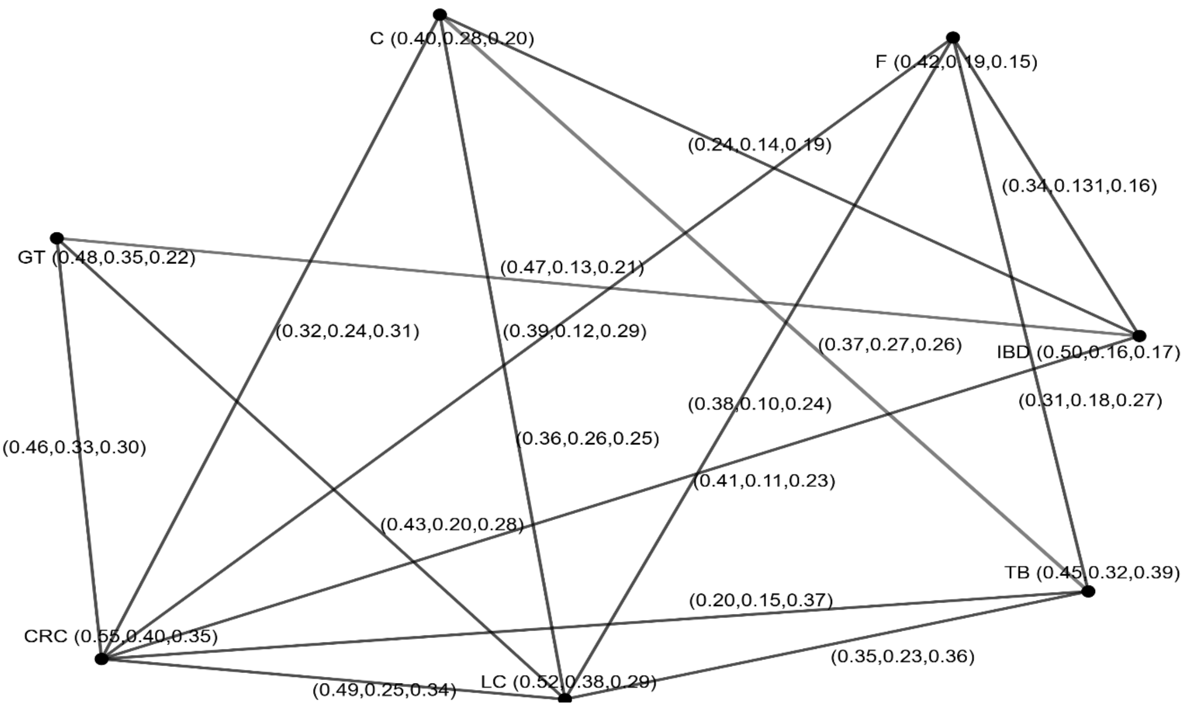

Figure 11

Graphical representation of the Symptoms and Disease relations for cancer diagnosis

This research aims to determine the clinically significant disease amongst four suspected ones, enabling the most prioritization of essential treatment interventions, thereby optimising treatment outcomes by enhancing patient's safety.

In pursuit of this goal, this work adopts the concept of "Bridges" for disease identification and effective treatment. As the given clinical data is distinct, Bijective neutrosophic Bridges shines here. Now, to achieve this objective, an algorithm to find the Bijective neutrosophic Bridges serves as a catalyst to connect diagnostic precision with timely, ensured treatment strategies.

Using an algorithm,

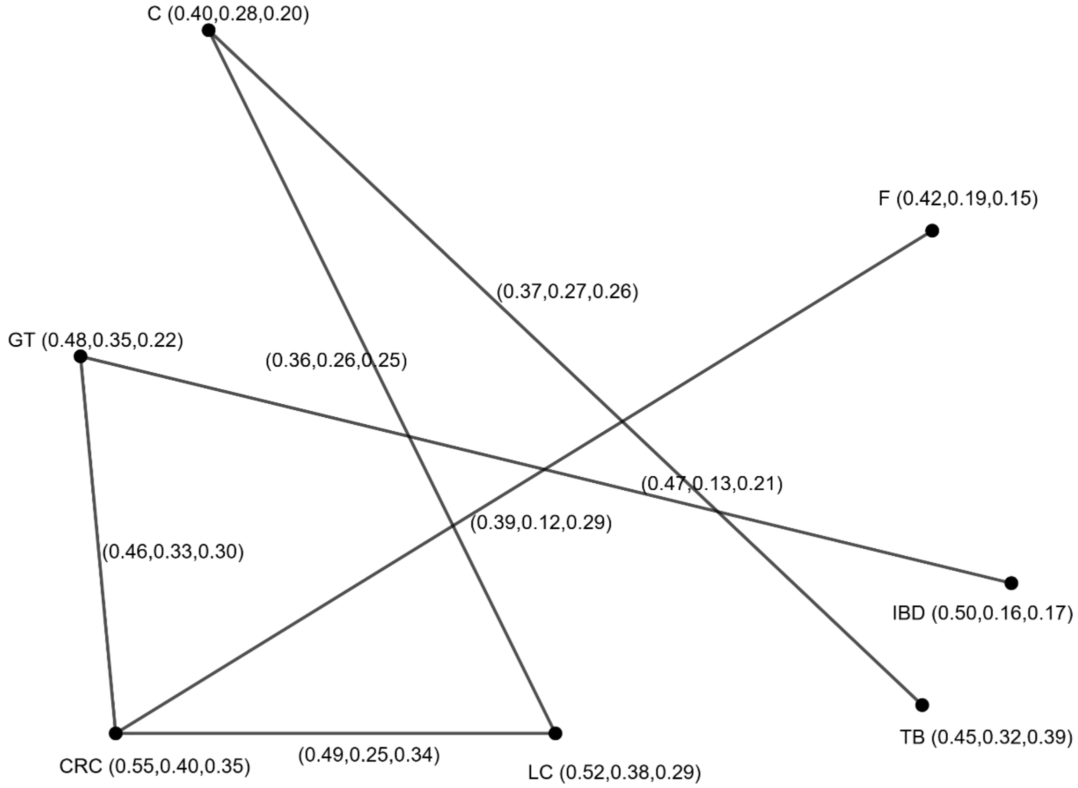

Figure:12 Graphical representation of T-bridges

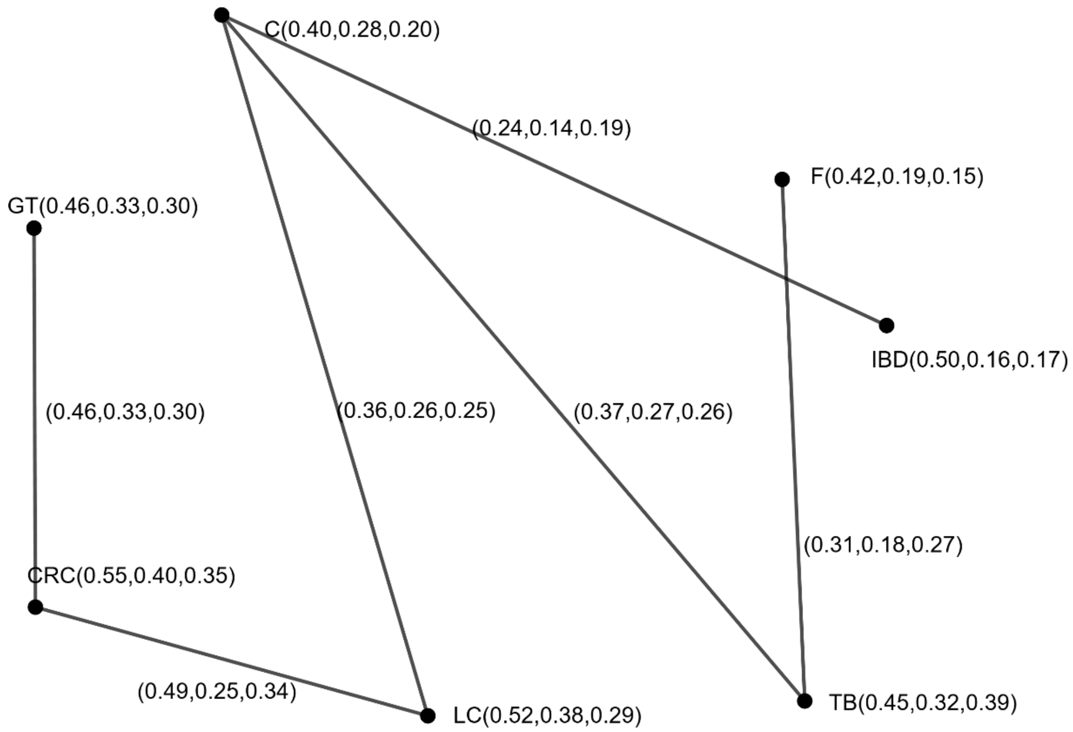

Figure: 13 Graphical representation of I-bridges

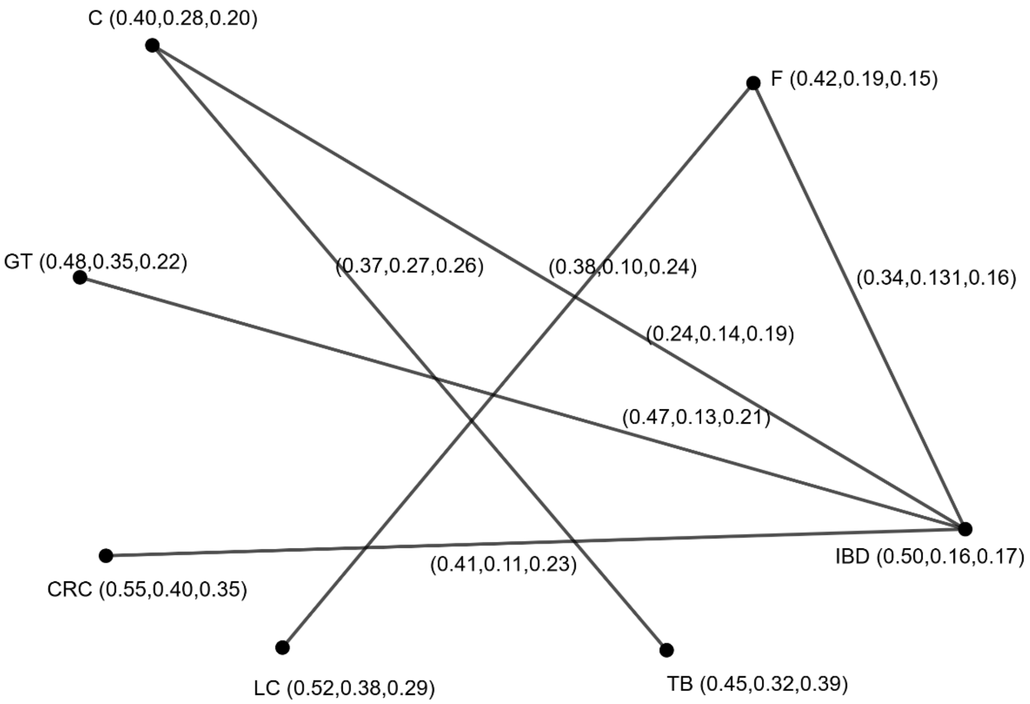

Figure:14 Graphical representation of F-bridges

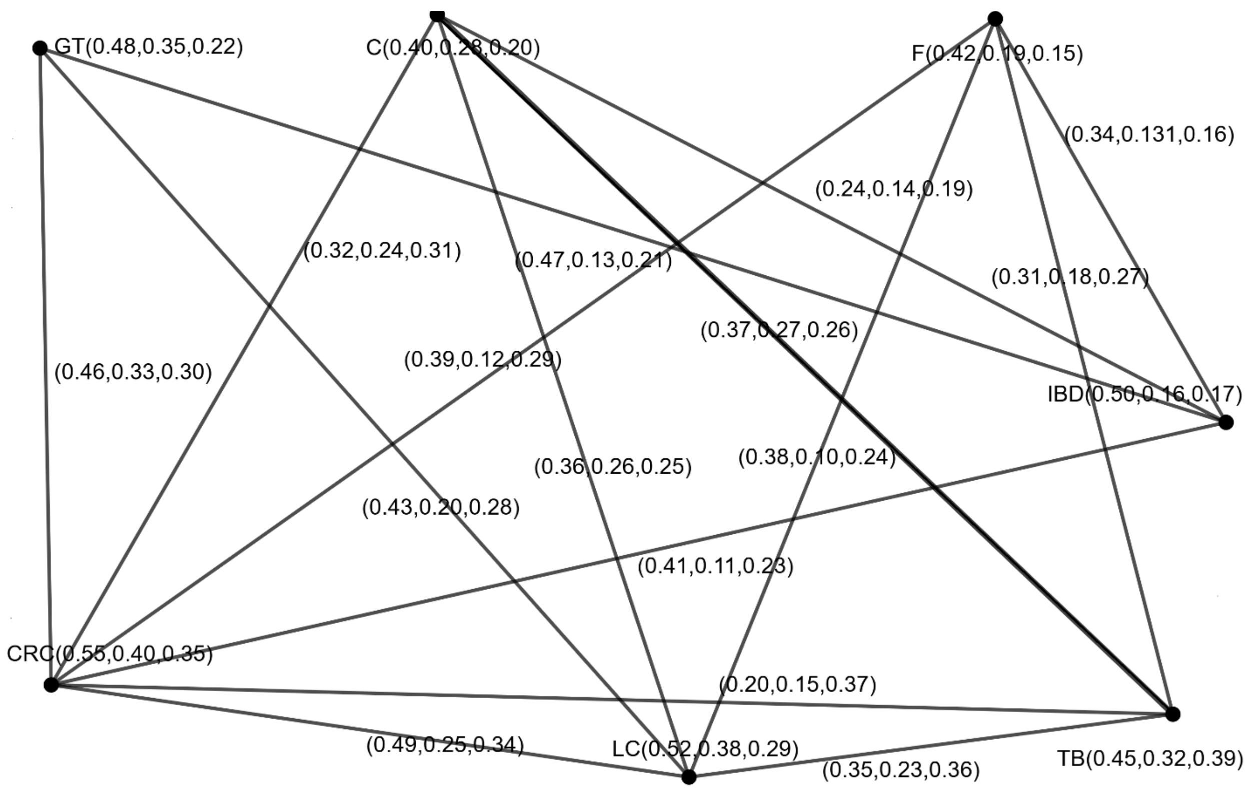

Figure:15 Neutrosophic bridge

From this, we can ensure that Chronic Cough(C) to Tuberculosis(TB) is the crucial relation in this representation.i.e) C-TB is the bridge(crucial relation) in this Bijective Neutrosophic

graphical Framework. It can be concluded that, Although, there is chance of 55% that the patient may be presented with Colorectal Cancer (CRC) and 50% likelihood of Inflammatory Bowl disease (IBD), Tuberculosis(TB) is considered as the primary underlying condition. In this context, the presence of Chronic cough (C) should be prioritized as the primary key symptoms guiding further effective clinical assessment.
Hence, the diagnosis for tuberculosis is the enhanced and safe treatment for this scenario.

5. Conclusion:

A new concept of neutrosophic graph called Bijective neutrosophic graph has been introduced in this paper. An algorithm has been proposed to effectively identify different bridges of any Bijective neutrosophic graph. Due to this unique property of one to one correspondence it finds its application in analysis of cancer detection where precision and structural insights are crucial. Furthermore, by employing a de-neutrosophication function—specifically the score function—the same algorithm can be extended to identify neutrosophic bridges. However, certain limitations remain. The proposed approach is computationally intensive for large-scale networks, and its effectiveness depends on the accuracy of assigned neutrosophic membership values. Future work may focus on optimizing the algorithm, validating it on larger datasets, and exploring broader applications.

Acknowledgements
The authors are highly grateful to the referees for their suggestions.

Conflicts of Interest
The authors declare no conflict of interest.

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Received: April 5, 2025. Accepted: Sep 10, 2025

A Neutrosophic Soft Set-Based Approach for Anemia Diagnosis: Managing Uncertainty in Medical Decision-Making

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Abstract: Fuzzy sets, introduced by Lotfi A. Zadeh in 1965, have been widely used to solve real-world problems involving uncertainty and ambiguous situations. However, traditional fuzzy sets and interval-valued fuzzy sets are insufficient to fully capture uncertainty. To address this problem, intuitionistic fuzzy sets and neutrosophic sets have been proposed. Neutrosophic sets, especially neutrosophic soft sets, provide an effective framework for dealing with uncertain, conflicting, and incomplete information, especially in medical decision making. In this study, a model for the diagnosis of anemia is developed using the Neutrosophic Soft Set (NSS) method. The model scores female patients according to age groups (15-44, 45-59, 60+) and several hematological parameters such as hemoglobin level (Hb), hematocrit (Hct), mean corpuscular volume (MCV), serum iron level, ferritin, and total iron binding capacity (TIBC). A more robust and accurate decision-making process is created by assigning neutrosophic values to each parameter, including accuracy, uncertainty, and inaccuracy values. Data for the model were obtained from patients treated in the hematology clinic of a full-fledged hospital in Turkey. Smarandache stated that neutrosophic clusters have a superior ability to deal with uncertainty, inconsistency, and missing data. This approach offers significant advantages, especially in medical decision-making processes, where uncertainty and contradiction are intense. Maji's method, based on the theory of neutrosophic soft sets, has yielded effective results in increasing the accuracy and consistency of medical diagnoses. Used in the diagnosis of anemia and similar diseases, this approach is a valuable tool in the evaluation of uncertain data. This paper aims to provide a guide to the application of neutrosophic soft sets in medical decision-making. It also highlights the potential of these approaches to improve decision support systems in medical diagnostics and provides recommendations for future research.

Keywords: Neutrosophic Soft Sets; Anemia Diagnosis; Medical Decision-Making; Uncertainty Management; Neutrosophic Set Theory

1. Introduction

Fuzzy sets, introduced by Lotfi A. Zadeh in 1965 [1], have been widely used to solve real-world problems in uncertain and ambiguous situations. Traditional fuzzy sets are defined by a membership value, but it is sometimes difficult to assign an exact membership. To address this, interval-valued fuzzy sets have been introduced [2], which better capture uncertainty in membership. In problems such as expert systems and belief systems, it is important to consider both the true and false membership of an object to describe it accurately. However, traditional fuzzy sets and interval-valued fuzzy sets are not sufficient for this. Intuitionistic fuzzy sets, introduced by Atanassov [3], provide a better solution by considering both truth-membership and falsity-membership. However, they still struggle to deal with indeterminate or inconsistent information in belief systems. To further address such challenges, Smarandache [4] introduced the concept of neutrosophic sets, a mathematical approach designed to deal with problems involving unclear, uncertain, and contradictory data, thus providing a more advanced tool for dealing with complex situations where both truth and falsity are difficult to determine.

Medical decision-making often involves navigating complex and uncertain information. Conditions such as overlapping symptoms, incomplete medical histories and variable patient responses pose challenges for healthcare professionals. In this context, mathematical frameworks that explicitly account for uncertainty and indeterminacy have received increasing attention. Among these, neutrosophic soft sets have emerged as a promising tool.

Neutrosophic soft sets integrate two powerful theories: neutrosophy and soft set theory. Neutrosophy, introduced by Florentin Smarandache [4], [5], [6], generalizes fuzzy logic by including indeterminacy as an explicit component alongside truth and falsity. This framework is particularly well suited to medical scenarios where uncertainty and vagueness are inherent. For example, a patient's symptoms may only partially match the characteristics of a known disease, or test results may be inconclusive.

Soft set theory, on the other hand, was developed by Molodtsov [7] to deal with uncertainty through a parameterised approach. It describes objects that are characterized by attributes or parameters, allowing flexibility in the definition of membership. This attribute-focused perspective complements the generality of neutrosophy, making the combination of the two neutrosophic soft sets an ideal candidate for dealing with diverse medical data.

Neutrosophic soft sets provide an effective way of representing medical data that contains uncertainty and indeterminacy. For example, a patient's symptoms, laboratory results and medical history can be encoded as a neutrosophic soft set. Here the parameters correspond to medical attributes, and the value of each attribute is expressed as a triad of degrees of truth, indeterminacy and falsity. This nuanced representation allows physicians to capture the complexity of medical data. The diagnosis of a disease often involves analyzing incomplete or ambiguous patient data. By encoding patient profiles as neutrosophic soft sets, healthcare professionals can employ algorithms and similarity measures to compare these profiles with known disease patterns. The explicit inclusion of indeterminacy enhances the robustness of these comparisons, accommodating cases where traditional deterministic approaches might fail. Selecting an optimal treatment plan is a multifaceted decision influenced by factors such as effectiveness, side effects, cost, and patient preferences. Neutrosophic soft sets allow each treatment option to be evaluated as a multi-criteria entity. Decision-makers can apply comparison methods within the neutrosophic framework to identify treatments that best align with patient needs and medical priorities.

In recent years, soft set theory and its extensions have been increasingly utilized to address uncertain and complex problems, particularly in the field of medical decision-making. Neutrosophic set theory, as an advanced extension, has emerged as a powerful tool for managing uncertain, ambiguous, and contradictory information in such processes. For example, the neutrosophic soft set theory, applied using the Maji approach, has been used effectively to diagnose Type 2 diabetes mellitus (DM2) by incorporating relevant variables, improving precision and consistency in decision making [8]. Similarly, neutrosophic models have been applied to various medical decision-making problems, highlighting their advantages in handling uncertainty [9]. Moreover, research on physician selection utilizing neutrosophic multi-criteria decision-making methods has demonstrated its practical utility in healthcare settings [10]. The application of single-valued neutrosophic sets for similarity measures has further improved decision-making and pattern recognition processes [11]. Refined neutrosophic fuzzy logic has shown significant potential in managing ambiguous medical data [12]. Additionally, a recommendation system based on algebraic neutrosophic measures has been developed to address incomplete or uncertain information effectively [13]. These studies collectively underscore the growing interest in neutrosophic set theory and its expanding potential to provide robust solutions in medical decision-making under uncertainty.

In this study, a model for anemia diagnosis is developed using the Neutrosophic Soft Set (NSS) method, incorporating age groups and hematological parameters. The model evaluates In this study, a model for anemia diagnosis is developed using the Neutrosophic Soft Set (NSS) method, incorporating age groups and hematological parameters. The model evaluates female patients based on their age groups (15--44, 45--59, 60+) and hematological parameters, such as hemoglobin (Hb) level, hematocrit (Hct), mean corpuscular volume (MCV), serum iron level, ferritin, and total iron-binding capacity (TIBC). Neutrosophic values $(\mathcal{T}, \mathcal{I}, \mathcal{F})$ are assigned to facilitate the decision-making process for diagnosis. The data used in the model are obtained from patients $(p_1, p_2, p_3, p_4, p_5)$ at the hematology clinic of a large-scale, fully equipped hospital in Turkey. To the best of our knowledge, this is the first study applying this method for anemia diagnosis.

The key advantages of the proposed method are highlighted below:

1) It effectively captures and represents both uncertainty and indeterminacy, providing a robust framework for decision-making in complex scenarios;
2) The study offers a comprehensive resource for researchers interested in exploring this methodology and its applications.

The structure of the paper is as follows: Section 2 provides an overview of neutrosophic sets. Section 3 outlines the main steps of the NSS methodology. In Section 4, the application of the proposed model to anemia diagnosis is demonstrated, and the results are discussed in detail. Finally, Section 5 presents conclusions and directions for future research.

2. Neutrosophic Sets

Let $\boldsymbol{\mathcal{U}}$ be a universal set of objects, with each element denoted by $\boldsymbol{u} \subseteq \boldsymbol{\mathcal{U}}$. For a subset $\boldsymbol{B} \subseteq \boldsymbol{\mathcal{U}}$, the membership functions related to truth, indeterminacy, and falsity are defined as $\boldsymbol{\mathcal{T}_B(u)}, \boldsymbol{\mathcal{I}_B(u)}$ and $\boldsymbol{\mathcal{F}_B(u)}$, respectively. These functions are expressed as follows:

$$\mathcal{T}_B(u): \mathcal{U} \to]0^-, 1^+[,\ \mathcal{I}_B(u): \mathcal{U} \to]0^-, 1^+[,\ \mathcal{F}_B(u): \mathcal{U} \to]0^-, 1^+[$$

There are no restrictions on the sum of these functions. Hence:

$$0^- \leq sup\mathcal{T}_B(u) + sup\mathcal{I}_B(u) + sup\mathcal{F}_B(u) \leq 3^+.$$

A subset $\boldsymbol{B}$ of $\boldsymbol{\mathcal{U}}$ characterized by these functions is called a neutrosophic set (NS) and can be represented as:

$$B = \{\langle u, \mathcal{T}_B(u), \mathcal{I}_B(u), \mathcal{F}_B(u)\rangle : u \in \mathcal{U}\}.$$

In this definition, $\boldsymbol{0^- = 0 - \varepsilon}$ and $\boldsymbol{1^+ = 1 + \varepsilon}$, where $\boldsymbol{\varepsilon}$ is a non-standard infinitesimal component.

For practical purposes in technical and engineering applications, the interval $[\boldsymbol{0}, \boldsymbol{1}]$ is commonly used instead of $]\boldsymbol{0^-}, \boldsymbol{1^+}[$ as the latter is challenging to apply in real-world scenarios. This adaptation leads to the concept of simplified neutrosophic sets SNSs [14], where the membership functions are redefined as:

$$\mathcal{T}_B(u): \mathcal{U} \to [0, 1],\ \mathcal{I}_B(u): \mathcal{U} \to [0, 1],\ \mathcal{F}_B(u): \mathcal{U} \to [0, 1].$$

An SNS can be written as:

$$B = \{\langle u, \mathcal{T}_B(u), \mathcal{I}_B(u), \mathcal{F}_B(u)\rangle : u \in \mathcal{U}\}.$$

Furthermore, let $\boldsymbol{M}$ represent a set of parameters and $\boldsymbol{\mathcal{U}}$ be a universal set. If $\boldsymbol{A} \subseteq \boldsymbol{M}$ and $\boldsymbol{F}: \boldsymbol{A} \rightarrow \boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{U}})$ is a mapping, the pair $(\boldsymbol{F}, \boldsymbol{A})$ is called a soft set (SS) over $\boldsymbol{\mathcal{U}}$ [7]. Extending this concept, a neutrosophic soft set (NSS) over $\boldsymbol{\mathcal{U}}$ is defined as follows [15]:

Let $\boldsymbol{F}: \boldsymbol{A} \rightarrow \boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{U}})$ and $\boldsymbol{A} \subseteq \boldsymbol{M}$. The pair $(\boldsymbol{F}, \boldsymbol{A})$ is termed a neutrosophic soft set if:

$$\boldsymbol{F}(\boldsymbol{e}) = \{\langle \boldsymbol{n}, \boldsymbol{\mathcal{T}}_{\boldsymbol{F}(\boldsymbol{e})}(\boldsymbol{n}), \boldsymbol{\mathcal{I}}_{\boldsymbol{F}(\boldsymbol{e})}(\boldsymbol{n}), \boldsymbol{\mathcal{F}}_{\boldsymbol{F}(\boldsymbol{e})}(\boldsymbol{n})\rangle : \boldsymbol{n} \in \boldsymbol{\mathcal{U}}\}, \forall \boldsymbol{e} \in \boldsymbol{A}.$$

Here, $\boldsymbol{\mathcal{T}}_{\boldsymbol{B}}(\boldsymbol{u})$, $\boldsymbol{\mathcal{I}}_{\boldsymbol{B}}(\boldsymbol{u})$ and $\boldsymbol{\mathcal{F}}_{\boldsymbol{B}}(\boldsymbol{u})$ represent the truth, indeterminacy, and falsity membership functions for the parameter $\mathbf{e}$.

The value-class of a Neutrosophic Soft Set (NSS) is defined as the collection of all value sets associated with an NSS $(\boldsymbol{F}, \boldsymbol{M})$. This collection is denoted by $\boldsymbol{C}(\boldsymbol{F}, \boldsymbol{A})$, and it is clear that $\boldsymbol{C}(\boldsymbol{F}, \boldsymbol{M}) \subseteq \boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{U}})$, where $\boldsymbol{\mathcal{P}}(\boldsymbol{\mathcal{U}})$ represents the power set of the universal set $\boldsymbol{\mathcal{U}}$.

Consider two neutrosophic soft sets $(\boldsymbol{F}, \boldsymbol{A})$ and $(\boldsymbol{J}, \boldsymbol{B})$ over the universal set $\boldsymbol{\mathcal{U}}$, where $\boldsymbol{A}, \boldsymbol{B} \subseteq \boldsymbol{M}$ and $\boldsymbol{A} \subseteq \boldsymbol{B}$. If $\mathbf{A}$ is a subset of $\boldsymbol{B}$ then $(\boldsymbol{F}, \boldsymbol{A})$ is termed a neutrosophic soft subset of $(\boldsymbol{J}, \boldsymbol{B})$. This relationship is denoted by $(\boldsymbol{F}, \boldsymbol{A}) \subseteq (\boldsymbol{J}, \boldsymbol{B})$, and it satisfies the following conditions for every $\boldsymbol{e} \in \boldsymbol{A}$ and $\boldsymbol{n} \in \boldsymbol{\mathcal{U}}$:

$$\boldsymbol{\mathcal{T}}_{\boldsymbol{F}(\boldsymbol{e})}(\boldsymbol{n}) \leq \boldsymbol{\mathcal{T}}_{\boldsymbol{J}(\boldsymbol{e})}(\boldsymbol{n}),\ \boldsymbol{\mathcal{I}}_{\boldsymbol{F}(\boldsymbol{e})}(\boldsymbol{n}) \leq \boldsymbol{\mathcal{I}}_{\boldsymbol{J}(\boldsymbol{e})}(\boldsymbol{n}),\ \boldsymbol{\mathcal{F}}_{\boldsymbol{F}(\boldsymbol{e})}(\boldsymbol{n}) \leq \boldsymbol{\mathcal{F}}_{\boldsymbol{J}(\boldsymbol{e})}(\boldsymbol{n}).$$

The concept of neutrosophic soft subsets provides a foundation for comparing different neutrosophic soft sets based on their respective membership functions. It ensures that the truth-membership, indeterminacy-membership, and falsity-membership values of one soft set do not exceed those of another within the subset relationship. This property is crucial for hierarchical evaluations and applications where parameters and their relationships need to be consistently structured. This framework is particularly valuable in decision-making scenarios, allowing the representation of uncertainty, indeterminacy, and contradiction within a structured and mathematically rigorous context.

3. Steps of the NSS Method

In this study, a model has been developed for the diagnosis of anemia using the Neutrosophic Soft Set (NSS) method. The model allows for the evaluation of patients by considering age groups and hematological parameters. Female patients are assessed based on their age groups (15–44, 45–59, 60+) as well as hematological parameters such as hemoglobin (Hb) level, hematocrit (Hct), mean corpuscular volume (MCV), serum iron levels, ferritin, and total iron-binding capacity (TIBC). Neutrosophic values $\mathcal{T}, \mathcal{I}$ and $\mathcal{F}$ are assigned to simplify the diagnostic process. The data used in the model were obtained from a large-scale, fully equipped hospital's hematology clinic in Turkey, including patient data $(p_1, p_2, p_3, p_4, p_5)$.

In this model, Maji's Neutrosophic Soft Set theory [15] and the algorithm proposed by Maji will be used.

For the diagnosis of anemia, let there be n patients $p_1, p_2, \dots, p_n$ *and* m selection parameters $e_1, e_2, \dots, e_m$. For each selection parameter e_j $(j = 1,2, \dots, m)$, the evaluation or performance value of patient p_i $(i = 1,2, \dots, n)$ is represented as a triplet:

$$t_{ij} = (\mathcal{T}_{F(e_j)}(p_i), \mathcal{I}_{F(e_j)}(p_i), \mathcal{F}_{F(e_j)}(p_i)),$$

where:

- $\mathcal{T}_{F(e_j)}(p_i)$: The truth value representing the correct diagnosis of patient p_i for parameter e_j (e.g., positive anemia diagnosis).

- $\mathcal{I}_{F(e_j)}(p_i)$: The indeterminacy value representing the uncertainty in the diagnosis of patient p_i for parameter e_j.
- $\mathcal{F}_{F(e_j)}(p_i)$: The falsity value representing the incorrect diagnosis of patient p_i for parameter e_j (e.g., negative anemia diagnosis).

For a fixed i, the values t_{ij} ($j = 1,2,\ldots,m$) represent the Neutrosophic Soft Set of all patients. These performance values can be organized into a matrix known as the $criteria\ matrix$. As the number of criteria increases, the suitability of a given patient for diagnosis also increases.

This study aims to identify the most suitable patient who dominates all others within the spectrum of parameters e_j, i.e., the patient with the highest accuracy and the lowest uncertainty in terms of anemia diagnosis. However, as the data is not precise and involves Neutrosophic Soft Data, direct selection is not possible.

The problem is to determine the most appropriate patient for anemia diagnosis based on the selection parameters. For instance, one patient (p_1) may be evaluated as having anemia under certain parameters, while other patients (p_2, p_3,…., p_n) may not satisfy the criteria. The selection varies depending on each patient's hematological parameters, age group, and other clinical factors. In this study, a technique based on Neutrosophic Soft Sets has been employed to calculate the performance scores of patients and facilitate the diagnosis process.

Comparison Matrix: A matrix where rows represent objects (p_1, p_2,…., p_n), and columns correspond to parameters (e_1, e_2,…., e_n). The entry c_{ij} in the matrix is determined using the formula:

$$c_{ij} = a + b - c,$$

where:
'a' is the integer calculated as 'how many times $T_{p_i}(e_j)$ exceeds or equal to $T_{p_k}(e_j)$', for $p_i \neq p_k$, $\forall p_k \in \mathcal{U}$,
'b' is the integer calculated as 'how many times $I_{p_i}(e_j)$ exceeds or equal to $I_{p_k}(e_j)$', for $p_i \neq p_k$, $\forall p_k \in \mathcal{U}$,
'c' is the integer 'how many times $F_{p_i}(e_j)$ exceeds or equal to $F_{p_k}(e_j)$', for $p_i \neq p_k$, $\forall p_k \in \mathcal{U}$.

Object Score: The score of each object p_i, denoted as S_i, is calculated by summing the entries of its corresponding row in the comparison matrix:

$$S_i = \sum_j c_{ij}\,.$$

Algorithm for Optimal Object Selection:

1) Input the Neutrosophic Soft Set (F, A).
2) Specify P, the subset of parameters relevant to the decision-maker's preferences.
3) Extract $(F, P)$and organize it in tabular form.
4) Compute the comparison matrix for (F, P).
5) Calculate the score S_i for each p_i.
6) Identify $S_k = \max_i S_i$.
7) If multiple p_i share the maximum score, any one of them can be selected as the optimal choice.

4. Medical Applications and Results

Anemia is a condition in which the body lacks sufficient oxygen-carrying red blood cells and is typically diagnosed using a series of hematological parameters. The World Health Organization (WHO) highlights the importance of hemoglobin level, hematocrit, mean corpuscular volume (MCV), serum iron levels, ferritin, and total iron-binding capacity (TIBC) in the diagnosis of anemia [16] . These parameters play a critical role in determining the type and severity of anemia. Hemoglobin and hematocrit are often considered primary indicators of anemia, as they directly affect the body's oxygen-carrying capacity. Other parameters provide insights into the status of iron metabolism and the adequacy of iron stores in the body. Ferritin is frequently used to assess iron deficiency anemia, while TIBC measures the body's capacity to transport iron, both of which are essential in the evaluation of anemia [17]. These parameters provide essential information for the treatment and management of anemia in clinical practice.

In this section, we will apply the algorithm introduced in the previous section. The set $\boldsymbol{P}$ consists of independent variables from the dataset, which are:

$$\boldsymbol{P} = \left\{ \begin{array}{c} \text{Age } (\boldsymbol{A}), \\ \text{Hemoglobin } (\boldsymbol{HB}), \\ \text{Hematocrit } (\boldsymbol{HCT}), \\ \text{Mean Corpuscular Volume } (\boldsymbol{MCV}), \\ \text{Serum Iron}, \\ \text{Ferritin}, \\ \text{Total Iron-Binding Capacity } (\boldsymbol{TIBC}) \end{array} \right\}$$

These variables represent important factors for diagnosing anemia. They are used to assess patients' health status, taking into account both demographic information (such as age) and hematological parameters . This comprehensive evaluation allows for a more accurate diagnosis of anemia.

Biochemical parameters that vary with age play a critical role in assessing an individual's health. These parameters provide essential insight into the body's iron levels, the oxygen-carrying capacity of the blood, and the health of the red blood cells. Values such as hemoglobin levels, hematocrit and MCV are key indicators, particularly in the diagnosis of blood disorders such as anemia, while serum iron levels, ferritin and TIBC are critical for understanding iron deficiency or iron overload. In addition, these parameters are essential for early diagnosis of disease and treatment planning. Table 1 provides a detailed overview of these parameters, highlighting their definitions, variations between categories, and health implications.

Table 1. Key biochemical parameters and their descriptions.

Parameter	Definition	Significance
Age Groups	Classification by age.	Highlights health variations.
Hemoglobin (Hb)	Amount of hemoglobin in blood.	Diagnoses anemia and hypoxia.
Hematocrit (Hct)	Proportion of red blood cells.	Linked to anemia, dehydration.
MCV	Average volume of red blood cells.	Identifies anemia types.
Serum Iron	Iron level in blood serum.	Detects deficiency or overload.
Ferritin	Stored iron levels in the body.	Indicates ironrelated conditions.
TIBC	Capacity to bind iron in serum.	Reflects anemia or liver issues.

The World Health Organization (WHO) recommends the evaluation of specific hematological parameters for the diagnosis of anemia. These parameters play a crucial role in determining the diagnosis, severity, and type of anemia, especially in female patients. The criteria established by the WHO facilitate the accurate diagnosis of the condition and the selection of appropriate treatment methods. Common hematological parameters and reference values for female patients are shown in Table 2.

Table 2. Parameters, categories, and reference ranges for anemia diagnosis

Parameter	Category	Reference Range
Age Groups	Y1: 15-44 years,	-
	Y2: 45-59 years,	-
	Y3:60+ years	-
Hemoglobin (Hb)	Normal	≥ 12.0 g/dL
	Mild Anemia	11.0 – 11.9 g/dL
	Moderate Anemia	8.0 – 10.9 g/dL
	Severe Anemia	< 8.0 g/dL
Hematocrit (Hct)	Normal	35 – 45% (varies with age)
	Mildly Low	30 – 35%
	Very Low	< 30%
MCV	Microcytic	< 80 fL
	Normocytic	80 – 100 fL
	Macrocytic	> 100 fL
Serum Iron	Normal	37 – 145 μg/dL
	Low	< 37 μg/dL
Ferritin	Normal	15 – 150 ng/mL
	Low	< 15 ng/mL
TIBC	Normal	240 – 450 μg/dL
	Low	> 450 μg/dL

The Table 3 below shows the age, hemoglobin (Hb) level, hematocrit (Hct), mean corpuscular volume (MCV), serum iron level, ferritin, and total iron-binding capacity (TIBC) of 5 different patients. These data provide information about the biochemical changes occurring in their bodies and the health status of the patients.

Table 3. Patient Data

Patient	Age (Y)	Hb (g/dL)	Hct(%)	MCV(fL)	Serum Iron (μg/dL)	Ferritin (ng/mL)	TIBC (μg/dL)
P1	33	10.5	36.5	73	36	5	416
P2	43	10.6	32.9	79	34	10	320
P3	38	9.9	33.3	79	22	4	334
P4	51	3.4	14.7	63	14	2	606
P5	40	10.2	33.6	80	47	4	441

After this stage, we will follow the steps outlined below. First, we will generate the NSS (H,P) values, which will be shown in Table 4. Next, we will prepare the comparison matrix in the format provided in Table 5. Then, we will calculate the score for each $\boldsymbol{p_i}$, and the results will be displayed in Table 6. Finally, we will make a decision based on the highest score from Table 6. Each step is crucial to ensure the correct evaluation and ultimate decision-making process.

The Neutrosophic Soft Set values are assigned for each patient as follows:

- $\mathcal{T}$: The degree of suitability for the patient regarding the parameter.
- $\mathcal{I}$: The degree of uncertainty (borderline or missing information).
- $\mathcal{F}$: The degree of non-suitability for the parameter.

Table 4. Neutrosophic soft set values for each patient

Parameter	P1	P2	P3	P4	P5
Age Groups	(0.4, 0.0, 0.8)	(0.6, 0.0, 0.7)	(0.5, 0.0, 0.4)	(0.7, 0.0, 0.5)	(0.9, 0.0, 0.2)
Hemoglobin	(0.8, 0.2, 0.4)	(0.8, 0.2, 0.5)	(0.7, 0.2, 0.3)	(0.9, 0.0, 0.1)	(0.8, 0.1, 0.3)
Hematocrit	(0.6, 0.1, 0.6)	(0.7, 0.1, 0.4)	(0.7, 0.1, 0.5)	(0.9, 0.0, 0.2)	(0.7, 0.1, 0.5)
MCV	(0.4, 0.1, 0.5)	(0.3, 0.3, 0.7)	(0.3, 0.3, 0.7)	(0.6, 0.0, 0.4)	(0.3, 0.3, 0.8)
Serum Iron	(0.5, 0.3, 0.6)	(0.6, 0.3, 0.5)	(0.7, 0.1, 0.4)	(0.8, 0.0, 0.3)	(0.2, 0.1, 0.8)
Ferritin	(0.3, 0.1, 0.6)	(0.1, 0.1, 0.7)	(0.3, 0.1, 0.6)	(0.4, 0.0, 0.5)	(0.3, 0.0, 0.6)
TIBC	(0.7, 0.1, 0.4)	(0.4, 0.0, 0.5)	(0.4, 0.0, 0.5)	(0.9, 0.0, 0.2)	(0.8, 0.1, 0.3)

Table 5. Comparison matrix of the NSS (H,P)

Patient	Age (Y)	Hb (g/dL)	Hct(%)	MCV(fL)	Serum Iron (μg/dL)	Ferritin (ng/mL)	TIBC (μg/dL)
P1	-4	4	0	3	2	4	4
P2	-1	3	5	3	4	0	-1
P3	0	2	3	3	4	4	-1
P4	1	4	4	4	4	5	6
P5	4	2	3	2	-2	1	5

In this section, we will calculate the total scores for each patient based on the comparison of various parameters. The score for each patient will be computed using the formula:

$$\boldsymbol{c_{ij} = a + b - c}$$

where $\boldsymbol{a}, \boldsymbol{b}$ and $\boldsymbol{c}$ represent the respective values for the parameters under consideration. This formula will help us quantify the relationship between the selected factors and provide a comprehensive evaluation of each patient's condition. The calculated scores will be summarized in

Table 6 below, which will offer an overview of the relative health status of the patients based on these biochemical parameters.

Table 6. Total scores of each patient

Patient	Score
P1	13
P2	13
P3	15
P4	28
P5	15

Decision: Table 6, which we presented, shows that the highest score is 28, measured for the *fourth patient (p4).* According to the evaluation based on the neutrosophic soft set approach, *(p4)* appears to be the patient most severely affected by anemia. This indicates that the biochemical parameters in her body are significantly more negatively impacted, and the severity of her condition is higher compared to the other patients.

Following *(p4),* the patients *(p3)* and *(p5),* both with scores of 15, are moderately affected by anemia. Subsequently, *(p1)* and *(p2),* both with scores of 13, show relatively milder impacts. These findings suggest that while all patients are affected by anemia to varying degrees, the severity differs significantly among them.

The neutrosophic soft set method provides a clear differentiation between the patients based on their health condition. It highlights which patients may require more urgent or intensive intervention, particularly *(p4),* who is in the most critical condition.

These results allow for a more in-depth evaluation of the patients' health status and offer insights into prioritizing clinical interventions. This study contributes to a better understanding of the effects of anemia on female patients and presents an innovative approach for making more informed decisions in clinical settings.

5. Conclusion and Suggestions for Future Studies

In this study, the concept of neutrosophic set developed by Smarandache has been studied and its applicability in the context of soft sets has been investigated. The study is based on an approach that considers parameters as neutrosophic sets. In addition, the integration of the neutrosophic soft set (NSS) method with Maji's approach to medical diagnostic problems has been addressed.

A case study based on real data shows that Maji's approach is both simple and effective. This approach has significant potential for supporting decision making, especially in problems characterized by uncertainty and complexity, such as medical diagnosis. The results of the study clearly demonstrate the effectiveness and applicability of NSS in decision-making problems. NSS stands out as an effective tool, especially in situations with uncertain or incomplete information.

In this context, the operations defined in the study and the theoretical insights obtained contribute to a better understanding of NSS and demonstrate its potential use in decision-making processes in various domains. However, it should be noted that this research has certain limitations in its scope. Applications on larger datasets and in different problem domains are crucial to assess the general validity and impact of the method.

Future studies should focus on exploring how NSS can be adapted to decision-making problems in different disciplines. The relationship between Maji's approach and other neutrosophic and soft sets should be further explored. Beyond medical diagnosis, research evaluating the applicability of

NSS in fields such as engineering, economics, and education will further highlight the potential of this method in a broader framework.

In conclusion, this study demonstrates that the neutrosophic soft set approach is an effective tool in situations involving uncertainty and incomplete information. The study sheds light not only on the theoretical foundations of NSS, but also on its potential in practical problems, opening new avenues of research that can contribute to more informed and effective decision-making processes.

Funding: This research received no external funding.

Acknowledgments: The authors declare that no financial support was required for conducting this research or preparing the article.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: April 12, 2025. Accepted: Sep 10, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

Improving Classification in Support Vector Machine Using Neutrosophic Logic

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Abstract: Support Vector Machine (SVM) is considered one of the most effective methods for classification tasks. However, randomness in classifying observations on the optimal separating hyperplane decision — which are often indeterminate in their class membership — and misclassified observations remain among the main challenges that researchers face when using SVM. To address these challenges, a Neutrosophic logic – based approach was proposed, enabling better handling of class ambiguity and reducing misclassifications in the neighborhood of the optimal separating hyperplane. A novel algorithm was introduced, which consisted of two main steps. First, the observations located near the optimal separating hyperplane were converted into Neutrosophic data, characterized by three components: truth, indeterminacy, and falsity. In the second step, the SVM classifier was reapplied to the Neutrosophic data to improve the classification accuracy. As a case study, the proposed algorithm was tested on two types of oil samples (sunflower oil and corn oil), and its performance was compared with that of a standard SVM without Neutrosophic data. The results demonstrated that the SVM models utilizing Neutrosophic data achieved higher accuracy compared to those without Neutrosophic data. Therefore, integrating Neutrosophic logic with SVM can significantly enhance the performance and reliability of SVM-based models.

Keywords: Support Vector Machine; Kernel Trick; Neutrosophic; Indeterminacy; Classification.

1. Introduction

The support vector machine (SVM) algorithm was introduced by Vapnik in 1995 as a supervised learning method and has garnered significant attention owing to its high accuracy in classification problems [1]. The fundamental idea behind SVM is to divide the dataset two sets: a training set, known as support vectors (SV), which is used to train the algorithm and identify the optimal separating hyperplane, and a test set to rigorously evaluate the classification accuracy of the trained model. Observations are assigned to the first class if the decision function yields a positive value, and to the second class if the value is negative. If an observation lies exactly on the optimal department hyperplane, it is typically assigned randomly to one of the two sets.

However, owing to its sensitivity to outliers, SVM may misclassify some training data, treating them as noise or anomalies. To address this limitation, Neutrosophic logic has been proposed as a complementary framework to handle such uncertain or indeterminate data. Neutrosophic logic, introduced by Florentin Smarandache in 1999, extends classical logic by characterizing each statement through three independent degrees: truth (T), indeterminacy (I), and falsity (F). This richer representation of uncertainty makes it a promising approach for enhancing SVM performance, particularly in scenarios involving noisy or ambiguous data [2], [3], [4].

In address the problem at hand, it is essential to mention some of the previous research conducted in this area. In ref. [5] the authors proposed a novel classification model called Neutrosophic Logic SVM (N-SVM), which integrates Neutrosophic logic with support vector machine for the purpose of image categorization. This approach employs a self-organizing map (SOM) to segment images into regions based on features such as color and texture. Subsequently, the N-SVM model was used to classify the segmented regions. The experimental results demonstrated that incorporating Neutrosophic logic significantly improved the performance of SVM when dealing with uncertain or imprecise data. In ref. [6] a model known as the Neutrosophic support vector machine (NS-SVM) was developed by integrating Neutrosophic sets with traditional SVM. The approach utilizes the Neutrosophic C-Means algorithm to assign each data sample with three membership degree-truth, indeterminacy, and falsity. These values are then converted into weights that are used during the training phase of the SVM model. This integration enhances classification accuracy, particularly in scenarios involving uncertain or noisy data. In ref. [7] some points located in the neighborhood decision limit and misclassified in the support vector machine were addressed and the accuracy of the classification was improved after applying Neutrosophic logic in the neighborhood decision limit. In ref. [8] the author focused on enhancing the quality of medical images by processing them within the Neutrosophic domain. In this approach, the image is divided into three components, each processed independently to improve visual quality. The study incorporates algorithms such as K-means clustering for image segmentation and Support Vector Machine (SVM) for classification. These methods facilitate the extraction of features such as color and texture, contributing to a better analysis of medical images. In ref. [9] a novel method for predicting kidney disease was proposed by integrating enhanced Neutrosophic sets with machine learning algorithms. The approach tackles uncertainty in medical data by transforming it into a Neutrosophic form using membership functions for truth, indeterminacy, and falsity. Three classifiers-logistic regression, SVM, and K-NN – were applied, with results showing improved prediction accuracy. Notably, logistic regression achieved the best performance when using Neutrosophic-based data, outperforming traditional approaches under uncertain conditions. In ref. [10] recent efforts have been made to integrate Neutrosophic set theory with advanced machine learning models to improve breast cancer detection. This integration aims to manage uncertainty, imprecision, and ambiguity inherent in medical datasets. By combining Neutrosophic data processing with multiple classification models, the proposed approach enhances diagnostic accuracy while reducing computational complexity, ultimately supporting more reliable decision-making in clinical applications.

Despite extensive research on improving SVM classification, selecting an appropriate Neutrosophic membership remains a critical factor in addressing the issues of random classification and misclassification in SVM models. In this paper, a novel approach based on Neutrosophic logic is proposed, in which the traditional decision values are replaced by Neutrosophic membership components-truth (T), indeterminacy (I), and falsity (F). This approach enhances the performance of SVM by effectively handling observations that are either misclassified or are in the neighborhood of the optimal separating hyperplane, in both linear and nonlinear classification of the data, the proposed model shows significant improvement in classification accuracy, outperforming both traditional and fuzzy-based methods.

This paper is organized as follows: in the first section, we provide a brief overview of the support vector machine algorithm. The second section presents the concept of Neutrosophic logic. The third section describes the proposed Neutrosophic-SVM approach. The fourth section presents a case study of the proposed method to existing data, followed by a summary of these results in the final section.

The Contribution of the Study:

This paper introduces an innovative integration of Neutrosophic logic into SVM to address the random classification and misclassification of data located near the optimal separating hyperplane. The proposed method transforms the decision values into the Neutrosophic domain, where they are represented by three components: truth (T), indeterminacy (I), and falsity (F). The SVM is then reapplied to the Neutrosophic-transformed dataset. Experimental evaluation on sunflower and corn oil samples confirmed the method's efficacy, demonstrating consistent and significant accuracy improvements for both linear and nonlinear classification over conventional SVM models.

The Importance of Neutrosophic Approach:

The core contribution of the Neutrosophic approach lies in enhancing the performance of SVM by effectively addressing observations located near the optimal separating hyperplane, without altering the intrinsic operational machine of the algorithm. By transforming decision values into a triadic representation-truth (T), indeterminacy (I), and falsity (F)-the model gains a more refined capacity to discriminate between definitive and ambiguous observations. This capability to explicitly model uncertainty enables the SVM to deliver more accurate and stable classification outcomes, particularly in regions where class boundaries are indistinct or overlapping. Moreover, the Neutrosophic framework facilitates the explicit quantification of both confidence and uncertainty for each data point, thereby enhancing the reliability of predictions and improving the interpretability of the model. Consequently, this approach represents a strategic augmentation to SVM-based classification methodologies, offering a systematic and adaptable machine for handling challenging cases and uncertain dataset, ultimately strengthening the model's generalization capability and applicability in complex real-world scenarios

2. Materials and Methods

2.1. Support Vector Machine (SVM)

In this section, we summarize the basic SVMs theories, including the linear separation case, the nonlinear separation case, and the nonlinear case through a binary classification problem. Let's us assume that the training samples are $S = \{x_i, y_i\}_{i=1}^{n}$ where $x_i \in R^N$, $y_i \in \{-1, +1\}$, when the samples are linear separable, the SVM can separate them with the largest margin. This can be achieved by solving the following quadratic program [11], [12]:

$$f(w) = \min_{w} \frac{\|w\|^2}{2} \qquad\qquad (1)$$
$$Subject\ to \qquad y_i(w'x_i + b) \geq 1 \quad ; \quad i = 1,2,\dots,n$$

By solving this problem using Lagrange's method, we obtain the following decision function:

$$g(x) = sign\left(\sum_{i=1}^{n} \hat{\lambda}_i y_i x_i' x + \hat{b}\right) \qquad (2)$$

Where the weight $\hat{w} = \sum_{i=1}^{n} \hat{\lambda}_i y_i x_i$, and the bias $\hat{b} = -\frac{1}{2}(\hat{w}'x_+ + \hat{w}'x_-)$.

For a nonlinear separate case, the optimal separating hyperplane is obtained by using a soft margin that allows for small errors in the training data to enable the optimal separating hyperplane method to be generalized, thus a slack variable ($0 \leq \xi_i < 1$) is introduced, and the nonlinear programming problem is as follows [11], [13]:

$$f(x) = \min_{w,\xi} \left\{\frac{\|w\|^2}{2} + \alpha \sum_{i=1}^{n} \xi_i\right\} \qquad (3)$$
$$Subject\ to \qquad y_i(w'x_i + b) \geq 1 - \xi_i \quad ; \xi_i \geq 0 \, , i = 1,2,\dots,n$$

Where $\alpha > 0$ controls the Trade-off between the width of the margin and the constraints, and it's calculated from the partial derivative of the Lagrange function when it equals zero. By solving the programming problem (3) we obtain a decision function of the from:

$$g(x) = sign\left(\sum_{i=1}^{n} \hat{\lambda}_i y_i x_i' x + \hat{b}\right) \quad (4)$$

However, if the dataset has a nonlinear optimal separating hyperplane decision, it is transformed such that a linear optimal separating hyperplane decision can be used to separate the training dataset in the transformed space. The attributes X in the original space are replaced by the transformed attributes $\Phi(X)$ by using the Kernel Trick. The Lagrange function of the constrained optimization problem is as follows [14], [15]:

$$Max\ L_D(\lambda_1, \dots, \lambda_n) = \max\left\{\sum_{i=1}^{n} \lambda_i - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \lambda_i \lambda_j y_i y_j K(x_i, x_j)\right\}$$
$$Subject\ to\ \sum_{i=1}^{n} \lambda_i y_i = 0 \quad ; \lambda_i \geq 0 \quad , i = 1,2, \dots, n \quad (5)$$

Where the $K(x_i, x_j)$ function can take one of the following forms [15], [16], [17]:

$k(x_i, x_j) = e^{-\|x_i - x_j\|^2/(2\sigma^2)}$ Gaussian Radial Basis Kernel

$k(x_i, x_j) = tanh(bx_i'.x_j + c)$ Sigmoid Kernel

$k(x_i, x_j) = (x_i'.x_j + c)^\gamma$ Polynomial Kernel

By solving the quadratic programming problem (5), the following decision function with maximal margins is obtained:

$$g(x) = sign\left(\sum_{i=1}^{n} \hat{\lambda}_i y_i K(x_i, x) + \hat{b}\right) \quad (6)$$

2.2. Neutrosophic Logic [18], [19]

This logic was introduced by Florentin Smarandache in 1995 as a generalization of fuzzy logic, where he added a new component to the degrees of membership and non-membership, namely, the degree of indeterminacy.

If a Neutrosophic set is defined as <T, I, F>, then an element $x(t, i, f)$ belongs to aforementioned Neutrosophic set such that:

t represents the degree of truth (membership).

i represents the degree of indeterminacy (neutrality).

f represents the degree of falsehood (non-membership).

2.3. New Neutrosophic SVM

The following steps illustrate the new proposed method for enhancing SVM classification using Neutrosophic logic:

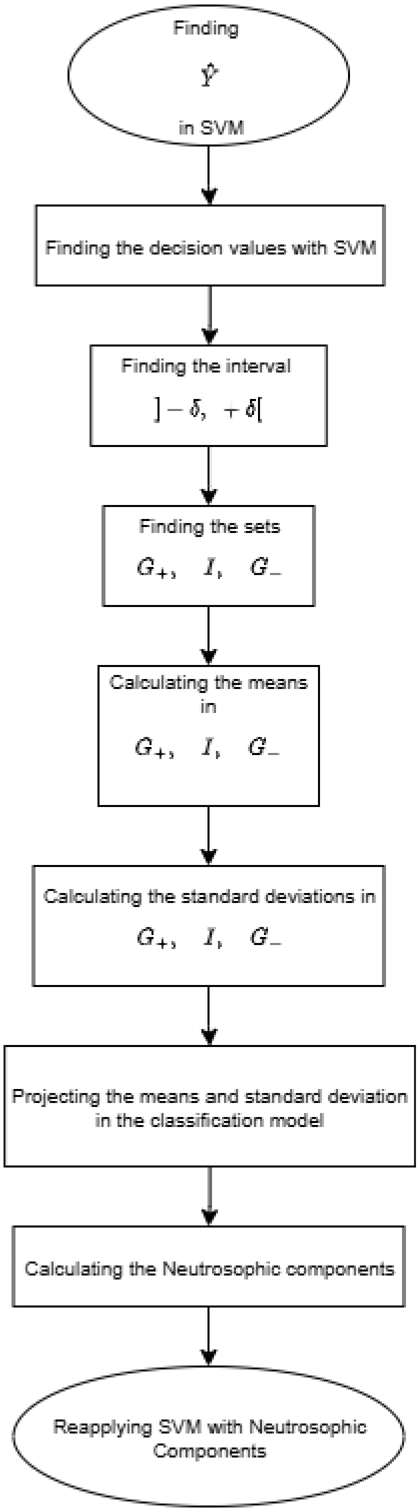

Figure 1. Flowchart of the proposed method

1- Finding the classification model in SVM:

$$\hat{Y} = \sum_{j=1}^{n} \hat{\lambda}_j y_j x_j' x + \hat{b} \tag{7}$$

2- Classify the data and find the decision values $\hat{Y}_i$; $i = 1,2,\ldots,N$ using SVM.

3- Finding the interval $]-\delta, +\delta[$ defined by the following:

$$\delta = \frac{|K_-| + |K_+|}{2} \quad (8)$$

$$K_- = cost * center_- \qquad K_+ = cost * center_+ \quad (9)$$

$$center_- = \bar{\bar{Y}}_i \ ; \ i = 1,2,\dots,n_- \qquad center_+ = \bar{\bar{Y}}_i \ ; \ i = 1,2,\dots,n_+ \quad (10)$$

Where: n_+ is the number of elements in the set where $y_i = +1$.
n_- is the number of elements in the set where $y_i = -1$.

4- Isolating the observations whose $\hat{Y} \in]-\delta,\delta[$ that are misclassified into a separate set, referred to as the neutrality set I.

The set of observations for which $y_i = +1$, excluding those belonging to I, is denoted by G_+.
The set of observations for which $y_i = -1$, excluding those belonging to I, is denoted by G_-.

5- Calculating the mean of the independent variables in the set where $y_i = +1$, the set where $y_i = -1$, and the set I:

$$\bar{X}_j^+ = \frac{1}{n_+} \sum_{i\in\{i|y_i=+1\}\backslash I}^{n_+} x_{ij} \qquad \bar{X}_j^- = \frac{1}{n_-} \sum_{i\in\{i|y_i=-1\}\backslash I}^{n_-} x_{ij}$$
$$\bar{X}_j^I = \frac{1}{n_I} \sum_{i=1}^{n_I} x_{ij} \qquad j = 1,2,\dots,p \quad (11)$$

Where: n_+ is the number of elements in the set G_+.
n_- is the number of elements in the set G_-.
n_I is the number of elements in the set I.
$\bar{X}_j^+$ is the mean of the independent variable j in the set G_+.
$\bar{X}_j^-$ is the mean of the independent variable j in the set G_-.
$\bar{X}_j^I$ is the mean of the independent variable j in the set I.
p denotes the number of independent variables.

6- Calculating the standard deviations of the independent variables in the sets G_+, G_-, and I:

$$S_j^+ = \frac{1}{n_+ - 1} \sum_{ii\in\{i|y_i=+1\}\backslash I}^{n_+} \left(x_{ij} - \bar{X}_j^+\right)^2 \qquad S_j^- = \frac{1}{n_- - 1} \sum_{i\in\{i|y_i=-1\}\backslash I}^{n_-} \left(x_{ij} - \bar{X}_j^-\right)^2$$
$$S_j^I = \frac{1}{n_I - 1} \sum_{i=1}^{n_I} \left(x_{ij} - \bar{X}_j^I\right)^2 \quad (12)$$

Where: S_j^+ is the standard deviation of the independent variable j in the set G_+.
S_j^- is the standard deviation of the independent variable j in the set G_-.
S_j^I is the standard deviation of the independent variable j in the set I.

7- Projecting the values of the previous means and standard deviations of the classification model to obtain the following values:

$$\bar{Y}_+ = \hat{w}^T \bar{X}_j^+ + \hat{b} \qquad \bar{Y}_- = \hat{w}^T \bar{X}_j^- + \hat{b} \qquad \bar{Y}_I = \hat{w}^T \bar{X}_j^I + \hat{b} \quad (13)$$

$$SY_+ = \hat{w}^T S_j^+ + \hat{b} \qquad SY_- = \hat{w}^T S_j^- + \hat{b} \qquad SY_I = \hat{w}^T S_j^I + \hat{b} \quad (14)$$

8- Calculating of Neutrosophic Components where $y_i = +1$ as follows:

$$T = \emptyset\left(\left|\frac{\hat{Y}_i - \bar{Y}_+}{SY_+}\right|\right) \qquad I = \emptyset\left(\left|\frac{\hat{Y}_i - \bar{Y}_I}{SY_I}\right|\right) \qquad F = \emptyset\left(\left|\frac{\hat{Y}_i - \bar{Y}_-}{SY_-}\right|\right) \quad (15)$$

Calculating of Neutrosophic Components where $y_i = -1$ as follows:

$$T = \emptyset\left(\left|\frac{\hat{Y}_i - \bar{Y}_-}{SY_-}\right|\right) \qquad I = \emptyset\left(\left|\frac{\hat{Y}_i - \bar{Y}_I}{SY_I}\right|\right) \qquad F = \emptyset\left(\left|\frac{\hat{Y}_i - \bar{Y}_+}{SY_+}\right|\right) \quad (16)$$

9- Considering the components $T, I,$ and F as independent variables and applying SVM to them to order to obtain the decision model and classification accuracy:

$$\hat{Y}_i = \hat{w}_1\, T + \hat{w}_2\, I + \hat{w}_3\, F + \hat{b} \quad (17)$$

However, if the observation i falls on the optimal separating hyperplane decision value, it is classified as follows:

$$\text{if} \quad \min\left(\left|\hat{Y}_j\right|\right) \in Y_+ \Rightarrow i \in Y_+ \ \ else \ \ i \in Y_- \ ; i \neq j \quad (18)$$

3. Case study and Result

The research sample consisted of 35 observations of vapors from two types of pure oils (15 observations from sunflower oil and 20 observations from corn oil). These observations were collected using an electronic nose system equipped with seven gas sensors (TGS813, TGS822, TGS800, MQ7, TGS2611, TGS2610, TGS2600).

These sensors are fabricated from semiconductor materials, with surface resistances that vary in response to interactions with the target gas. The stabilization onset time (Ts) for each of the seven sensors was used as the primary variables. Binary classification was performed using the SVM algorithm, and data analysis was conducted using R 4.3.1

The variables were defined as follows:

$Ts - S_i$; $i = 1,2, \dots, 7$, represents the stabilization onset time for sensor i.

Y a categorical variable indicating the type of oil, with sunflower oil coded as 1 and corn oil coded as -1.

1- Classification based on SVM:

Table 1 displays the training and testing sets for the research sample:

Table 1. Number of training and testing sets data points for sunflower and corn oils

Classifier	Testing Set		Training Set	
	Sunflower	Corn	Sunflower	Corn
Linear Function	5	9	10	11
GRB Function	0	4	15	16
Sigmoid Function	2	6	13	14
Polynomial Function	0	3	15	17

From the table 1, we note that:

- Based on the linear function, the training set data ratio was 60.0 % of the total data, and the test set data ratio was 40.0% of the total data.
- Based on the GRB function, the training set data ratio was 88.57% of the total data, and the test set data ratio was 11.43% of the total data.
- Based on the sigmoid function, the training set data ratio was 77.14% of the total data, and the test set data ratio was 22.86% of the total data.
- Based on the polynomial function, the training set data ratio was 91.43% of the total data, and the test set data ratio was 8.57% of the total data.

The table 2 demonstrates the correct classification efficiency and its percentage over all data:

Table 2. Efficiency of the correct classification and its percentage

Classifier	Y	Classification		Total
		Sunflower	Corn	
Linear Function	Sunflower	10	5	15
	Corn	2	18	20
	Percentage of sunflower oil	66.67%	33.33%	100.0%
	Percentage of corn oil	10.0%	90.0%	100.0%
GRB Function	Sunflower	12	3	15
	Corn	0	20	20
	Percentage of sunflower oil	80.0%	20.0%	100.0%
	Percentage of corn oil	0.0%	100.0%	100.0%
Sigmoid Function	Sunflower	11	4	15
	Corn	2	18	20
	Percentage of sunflower oil	73.33%	26.67%	100.0%
	Percentage of corn oil	10.0%	90.0%	100.0%
Polynomial Function	Sunflower	8	7	15
	Corn	0	20	20
	Percentage of sunflower oil	53.33%	46.67%	100.0%
	Percentage of corn oil	0.0%	100.0%	100.0%

It is clear from the table 2 that:

- Based on the linear function, 28 observations were classifier correctly (80.0%), and seven observations were classified incorrectly (20.0%).
- Based on the GRB function, 32 observations were classifier correctly (91.43%), and three observations were classified incorrectly (8.57%).
- Based on the sigmoid function, 29 observations were classifier correctly (82.86%), and six observations were classified incorrectly (17.14%).
- Based on the polynomial function, 28 observations were classifier correctly (80.0%), and three observations were classified incorrectly (20.0%).

2- Finding the interval $]-\delta,+\delta[$ for indeterminacy set in table 3:

Table 3. The interval

Function	Mean		Cost	K		Rang
	mean_{+}	mean_{-}		$\|K_{+}\|$	$\|K_{-}\|$	$]-\delta,+\delta[$
Linear	0.648859	-1.8877	1	0.64886	1.88771	]-1.268285,1.268285[
GRB	0.452415	-0.9274	1	0.452415	0.9274	]-0.689909,0.689909[
Sigmoid	0.358932	-0.9255	1	0.358932	0.9255	]-0.642209,0.642209[
Polynomial	0.113088	-0.8390	1	0.113088	0.8390	]-0.476065,0.476065[

3- Isolating the observation whose $\hat{Y} \in]-\delta,\delta[$ that are misclassified into a separate set, referred to as neutrality set I.

4- Finding the means and standard deviations for the independent variables for the set where $y_i = +1$, the set where $y_i = -1$ and the set I in table 4:

Table 4. Means and deviations of the independent variables

Function	mean_{+}	mean_{I}	mean_{-}	S_j^{+}	S_j^{I}	S_j^{-}
Linear	0.84702	-0.0105	-2.1185	1.522486	0.206633	1.742375
GRB	0.606464	-0.5489	-0.9274	0.522173	0.097889	0.210046
Sigmoid	0.459636	-0.08545	-1.04218	0.686848	0.32456	0.953486
Polynomial	0.327924	-0.13244	-0.83904	0.294858	0.098335	0.91613

5- Projecting the values of the means and standard deviations into the linear and nonlinear classification models in table 5:

Table 5. Means and standard deviations of the linear and nonlinear classification models

Function	$\bar{Y}_+$	$\bar{Y}_I$	$\bar{Y}_-$	SY_+	SY_I	SY_-
Linear	0.84702	-0.0105	-2.1185	1.117952	0.154001	2.508051
GRB	6.689573	1.240679	-2.92904	8.071087	12.75059	14.64154
Sigmoid	6.634627	4.383617	-6.72081	10.07528	13.078	20.19223
Polynomial	14.00938	2.977693	-6.88325	6.728825	9.395978	19.11234

6- Calculating the Neutrosophic components for all observations, considering $T, I,$ and F as independent variables and training SVM to derive the decision model and its classification accuracy, the effectiveness is summarized in table 6, which reports the number and percentage of correctly classified observations across the entire data set:

Table 6. Efficiency of the correct classification and its percentage for Neutrosophic data

Classifier	Y	Classification		Total
		Sunflower	Corn	
Linear Function	Sunflower	10	5	15
	Corn	2	18	20
	Percentage of sunflower oil	66.67%	33.33%	100.0%
	Percentage of corn oil	10.0%	90.0%	100.0%
GRB Function	Sunflower	15	0	15
	Corn	0	20	20
	Percentage of sunflower oil	100.0%	0.0%	100.0%
	Percentage of corn oil	0.0%	100.0%	100.0%
Sigmoid Function	Sunflower	15	0	15
	Corn	0	20	20
	Percentage of sunflower oil	100.0%	0.0%	100.0%
	Percentage of corn oil	0.0%	100.0%	100.0%
Polynomial Function	Sunflower	15	0	15
	Corn	0	20	20
	Percentage of sunflower oil	100.0%	0.0%	100.0%
	Percentage of corn oil	0.0%	100.0%	100.0%

It is clear from the table 6 that:

- Based on the linear function, 28 observations were classified correctly (80.0%), and seven observations were classified incorrectly (20.0%).
- By contrast, with the GRB, Sigmoid, and Polynomial functions, all the misclassified observations were correctly processed, resulting in a 100.0% correct classification rate.

The proposed method was tested in several real-world case studies, and it was proven to effectively handle misclassified observations.

4. Conclusions

The selection of an appropriate Neutrosophic membership function, which incorporates the components of Neutrosophic logic – truth, indeterminacy, and falsity – is crucial for resolving classification challenges in SVM. This research introduces a novel Neutrosophic – based membership function for both linear and nonlinear classification within the SVM algorithm to address misclassification issues. The proposed Neutrosophic -SVM approach has shown promising outcomes compared to the conventional methods in tackling misclassified observations and the method can be flexibly applied to a wide range of classification problems, thereby expanding the scope of applications for SVM.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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Received: April 9, 2025. Accepted: Sep 11, 2025

Neutrosophic Sets and Systems, Vol. 96, 2026
University of New Mexico

Irregular Pythagorean Neutrosophic Graphs

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Abstract: Pythagorean Neutrosophic Fuzzy Graphs integrate the concepts of Pythagorean neutrosophic sets and graph theory to effectively represent imprecise, inconsistent, and vague information inherent in many real-world problems. This paper introduces and formalizes the concept of irregularity in Pythagorean Neutrosophic graphs by defining various classes such as neighbourly irregular, neighbourly totally irregular, highly irregular, and highly totally irregular Pythagorean Neutrosophic graphs. The study develops theoretical results related to these irregularities and provides suitable graphical representations for better understanding. The proposed irregular Pythagorean Neutrosophic graph models are demonstrated to be significant in areas like pattern recognition, biological networks, information systems, and other applications involving uncertainty and vagueness. This work extends the theoretical foundation and practical applicability of neutrosophic graph theory.

Keywords: Pythagorean Neutrosophic set, Irregular, Pythagorean Neutrosophic graphs, Neighbourhood

1. Introduction

Real life events with vagueness and insufficient information's can be very well modelled and solved with the help of one of the most attractive field of mathematics known as fuzzy set theory. Zadeh [1] pioneered the concept of Fuzzy sets whose membership grade assume values between 0 and 1. The study in [26] introduces the Wiener index and Wiener absolute index for bipolar fuzzy graphs to analyze connectivity, highlighting their properties and behavior across various structures like forests, bridges, and trees. A comparative analysis with connectivity index is presented, concluding with an application to regular travel routes between Paris and Brest. The concept of the complexity function in fuzzy graphs based on network load across structures like fuzzy cycles, trees, and complete graphs are developed in [27]. This article includes the application of analyzing COVID-19 transmission cycles among countries and identifying high-traffic nodes in wireless network systems. In [28], the authors explore the operations like Cartesian product, composition, and union, and demonstrates real-life applications in railway networks and medical science using the picture fuzzy min-tolerance competition graph. A comparative analysis with connectivity and Wiener indices is provided, along with an algorithm and real-world application in ranking key crossroads in Indonesia's tourism network between BKB and BKS are introduces in [29]. The applications demonstrated in modeling international relations during the Cold War and enhancing teacher-student communication for better decision-making using the Algorithms and flowcharts support methods are established in [30]. The application of the field can be encountered in statistics, computer science, mathematics, engineering, artificial intelligence, pattern recognition, image analysis and decision making [4 – 10, 21-25].

The extension of fuzzy sets by providing membership and non-membership grades was introduced by Atanassov in [2]. The fuzzy sets with the sum of membership and non-membership less than 1 is intuitionistic fuzzy set. Neutrosophic set investigated by Smarandache[3] in which the elements are characterised with truth, indeterminacy and false membership functions.Yager [11] developed the notion of Pythagorean sets which has relaxation in the condition as $\mu^2 + \sigma^2 \leq 1$. Likewise, picture and spherical sets have been developed from neutrosophic sets, and to these concepts, the Pythagorean Neutrosophic sets (PNS) were introduced.

Relationship between the objects are presented with vertices and edges together forming a graph. Not all real-life events can be give certain values resulting in uncertainties. This creates a need for new type of graphs to tackle this uncertainties known as fuzzy graphs. Based on Zadeh's [16] fuzzy relation, Kaufmann established the idea of fuzzy graphs in [17]. Rosenfeld in [18], have discussed and developed several basic graph theoretical concepts in fuzzy graph. Consequently, more theoretical concepts and operations in fuzzy graphs have been developed in [19,20].

Pythagorean Neutrosophic Graphs are developed by integrating the concept of PNSs and fuzzy graphs in [12]. The structural representation of the PNGs are similar to that of graphs with only disparity in summing up membership grades values being less than 2. Similar idea is followed for the edges. Main reason for studying Pythagorean neutrosophic graphs is due to their ability to increasing fuzziness of considered model or system and further theoretical concepts were developed in [13-15].

This article aims at introducing the concepts of neighbourly irregular Pythagorean neutrosophic graphs, neighbourly totally irregular Pythagorean neutrosophic graphs, highly irregular Pythagorean neutrosophic graphs, highly totally irregular Pythagorean neutrosophic graphs by providing suitable graphical representation. The article also focuses on providing some important results an irregularity.

1.1 Organization of the work

The organization of the work is structured to progressively introduce and develop the concept of Irregular Pythagorean Neutrosophic Graphs. It begins with an Introduction that outlines the background on fuzzy sets, neutrosophic sets, and Pythagorean neutrosophic theory, highlighting their significance in modeling uncertainty and motivating the need for new irregular graph structures. The Preliminaries section reviews fundamental concepts and related literature essential for understanding the proposed models. The core contribution is presented in the Development of Irregular Pythagorean Neutrosophic Graphs, where various irregular graph types-such as neighborly irregular, totally irregular, highly irregular, and highly totally irregular-are formally defined and illustrated. This is followed by Theoretical Results that establish key properties and theorems to support structural analysis. The Applications and Illustrations section demonstrates the practical utility of these graphs in handling real-world problems involving uncertain information, such as in pattern recognition, biological networks, and information systems. Finally, the Conclusion summarizes the findings, underscores the importance of the proposed models, and suggests directions for future research, with all supporting literature compiled in the References section.

2. Irregular Pythagorean Neutrosophic Fuzzy Graphs

This section discusses the fundamental principles and past study required to comprehend Pythagorean Neutrosophic Graphs. It follows the development of fuzzy sets to intuitionistic fuzzy, neutrosophic, and Pythagorean sets, highlighting their ability to describe uncertainty. To provide the groundwork for future research, the section also presents their definitions, characteristics, and integration with graph theory.

Definition 2.1. Let G be PNG. The neighbourhood (ngbd) of a vertex w in G is defined as
$N(w) = (N_\mu(w), N_\beta(w), N_\sigma(w))$ where,
$N_\mu(w) = \{o \in w: \mu_B(h_0) \leq \mu_A(w) \wedge \mu_A(o)\}$,
$N_\beta(w) = \{o \in w: \beta_B(h_0) \leq \beta_A(w) \wedge \beta_A(o)\}$,
$N_\sigma(w) = \{o \in w: \sigma_B(h_0) \leq \sigma_A(w) \wedge \sigma_A(o)\}$.

For instance, Consider a PNG G with vertex set $V=\{v_1,v_2,v_3,v_4\}$ and edge set $E=\{(v_1,v_2),(v_1,v_3),(v_2,v_4)\}$. The Pythagorean Neutrosophic membership values on vertices and edges specify degrees of truth, indeterminacy, and falsity, but for illustration of neighbourhood, these values are not immediately needed.

- The neighbourhood of vertex v_1 is: $N(v_1)=\{v_2,v_3\}$ since v_2 and v_3 are directly connected to v_1.
- Similarly, the neighbourhood of vertex v_2 is: $N(v_2)=\{v_1,v_4\}$
- The neighbourhood of vertex v_3 is: $N(v_3)=\{v_1\}$
- The neighbourhood of vertex v_4 is: $N(v_4)=\{v_2\}$

The above illustration demonstrates how neighbourhoods are defined within a PNG and forms a foundational concept for defining neighbourhood degrees and irregularities.

Definition 2.2. Let G be a PNG. The ngbd degree of a vertex w in G is defined by
$$deg_\mu(w) = \sum_{o \in N_\mu(x)} \mu_A(o),$$
$$deg_\beta(w) = \sum_{o \in N_\beta(x)} \beta_A(o),$$
$$deg_\sigma(w) = \sum_{o \in N_\sigma(x)} \sigma_A(o).$$

Definition 2.3. Let G be a PNG. The closed ngbd degree of a vertex w in G is defined as
$$deg_\mu[w] = \sum_{o \in N_\mu(x)} \mu_A(o) + \mu_A(w),$$
$$deg_\beta[w] = \sum_{o \in N_\beta(x)} \beta_A(o) + \beta_A(w),$$

$$deg_{\sigma}[w] = \sum_{o \in N_{\sigma}(x)} \sigma_A(o) + \sigma_A(w).$$

Definition 2.4. Let PNG $G,$ is called regular if all the vertices has the same open neighbourhood degree.

Definition 2.5. Let G be a PNG. If there is a vertex which is adjacent to vertices with distinct neighbourhood degrees then G is called a irregular PNG.

Example 2.6. Let $G = (A, B)$ be the PNG with $A = \{x_1, x_2, x_3\}$, $B = \{(x_1, x_2), (x_2, x_3), (x_1, x_3)\}$. G is an irregular PNG. $\deg(x_1) = (.4, .7, .9), \deg(x_2) = (.4, .7, .1), \deg(x_3) = (.2, .6, .7)$. G is an irregular PNG.

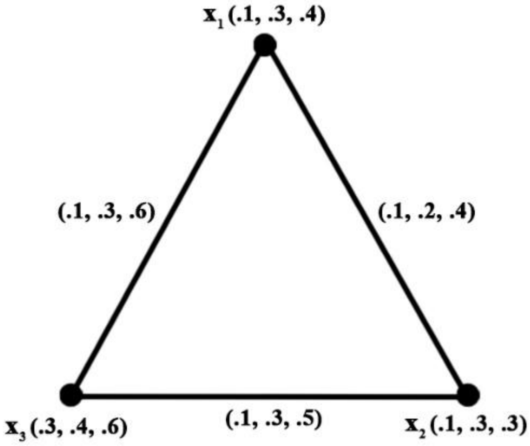

Table 1: Irregular PN Fuzzy Graph

Definition 2.7. If there is a vertex which is adjacent to vertices with distinct closed neighbourhood degrees, then PNG G is called a totally irregular PNG.

Example 2.8. Consider a PNG with $A = \{x_1, x_2, x_3, x_4, x_5\}, B = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_2),$
$(x_1, x_3), (x_4, x_5), (x_4, x_1)\}$.
Then, $\deg[x_1] = (1.4, .1.3, 2.7), \deg[x_2] = (1.4, 1.3, 2.7), \deg[x_3] = (1.4, 1.3, 2.7), \deg[x_4] = (2, 1.8, 3.1)$.
G is a totally irregular PNG.

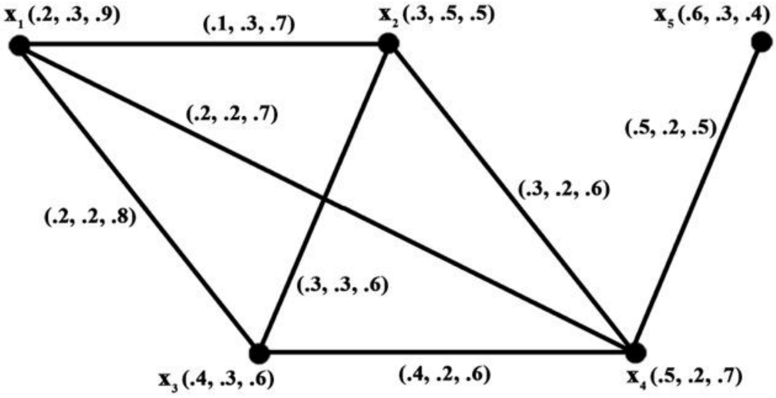

Table 2: Totally Irregular PN Fuzzy Graph

Definition 2.9. Let G be a connected PNG. If every two adjacent vertices of G have distinct open neighbourhood degrees, then G is called neighbourly irregular PNG.

Example 2.10. Let $G = (A, B)$ be a PNG with $A = \{x_1, x_2, x_3, x_4\}$, and $B = \{x_1, x_2, x_2, x_3, x_3, x_4, x_4, x_1\}$. The degrees of vertices are $\deg(x_1) = (.4, .9, 1.1), \deg(x_2) = (.4, .9, 1.1), \deg(x_3) = (.8, .8, .7), \deg(x_4) = (.4, .9, 1.1)$. G is a neighbourly irregular PNG.

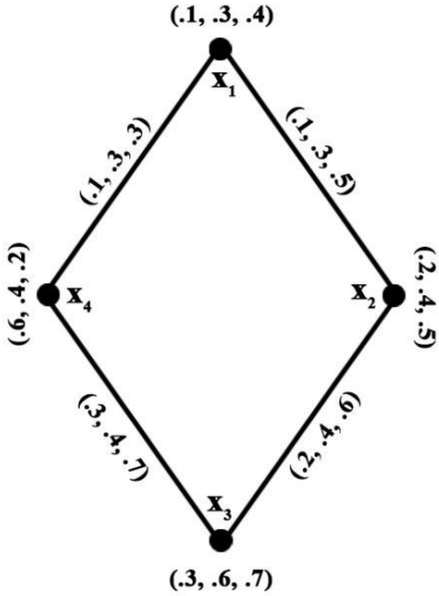

Table 3: Neighbourly Irregular PN Fuzzy Graph

Definition 2.11. A connected PNG is called neighbourly totally irregular PNG if every two adjacent vertices of G have distinct closed neighbourhood degree.

Example 2.12. Here the degree value for each vertex is, $\deg[x_1] = (1.1, .1.8, 1.7), \deg[x_2] = (1.2, 1.9, 1.9)$, $\deg[x_3] = (1.3, 2.1, 2.1), \deg[x_4] = (1.2, 2, 2)$.
Hence G is neighbourly totally irregular PNG.

Definition 2.13. A connected PNG, is called highly irregular PNG if every vertex of $\boldsymbol{G}$ is adjacent to vertices with distinct neighbourhood degrees

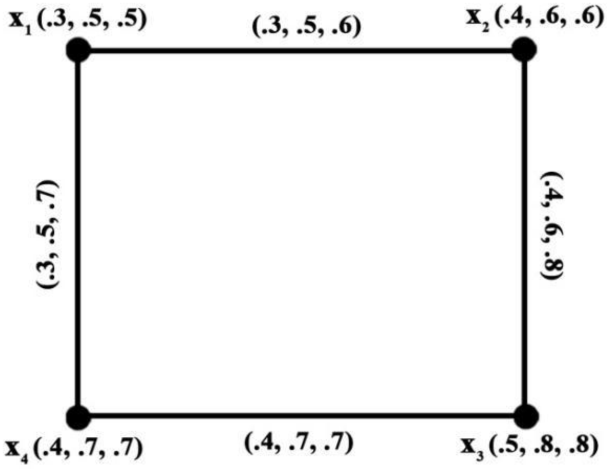

Table 4: Neighbourly Totally Irregular PN Fuzzy Graph

NOTE:

(i) A highly irregular PNG may not be neighbourly irregular PNG.
(ii) A neighbourly irregular PNG may not be highly irregular PNG.
(iii) A neighbourly irregular PNG may not be neighbourly totally irregular PNG.
(iv) A neighbourly totally irregular PNG may not be neighbourly irregular PNG.

Proposition 2.14. Let G be a PNG. Then G is highly irregular PNG and neighbourly irregular PNG iff the neighbourhood degrees of all the vertices of G are distinct.

Proof: Let $\mathcal{B}$ be a PNG with n-vertices $w_1, w_2, \dots, w_n$. Suppose G is both highly and neighbourly irregular PNG. We want to show the neighbourhood degrees of all vertices of G are distinct. Let $\deg(w_i) = (p_i, q_i, r_i), i = 1,2, \dots, n$.

Let the adjacent vertices of w_1 are $w_2, w_3, \dots, w_n$ with ngbd degrees $(p_2, q_2, r_2), (p_3, q_3, r_3), \dots, (p_n, q_n, r_n)$ respectively. Since G is highly irregular so $p_2 \neq p_3, \dots \neq p_n, q_2 \neq q_3, \dots \neq q_n, r_2 \neq r_3, \dots \neq r_n$. Also, $p_1 \neq p_2, \dots \neq p_n, q_1 \neq q_2, \dots \neq q_n, r_1 \neq r_2, \dots \neq r_n$, because G is neighbourly irregular.

So, $(p_1, q_1, r_1) \neq (p_2, q_2, r_2) \neq (p_3, q_3, r_3), \dots, (p_n, q_n, r_n)$. Hence the neighbourhood degrees of all vertices of G are distinct.
Conversely, suppose that the neighbourhood degrees of all the vertices are distinct.
Now we want to show that is highly irregular and neighbourly irregular PNG.

Let $\deg(w_i) = (p_i, q_i, r_i), i = 1,2, \dots, n$ given $p_1 \neq p_2, \dots \neq p_n, q_1 \neq q_2, \dots \neq q_n, r_1 \neq r_2, \dots \neq r_n$
Every two adjacent vertices have distinct neighbourhood degrees and to every vertex, the adjacent vertices have distinct neighbourhood degrees, which completes the proof.

Proposition 2.15. Let G be a PNG. If G is neighbourly irregular PNG and $(\mu_A, \beta_A, \sigma_A)$ is a constant function, then G is a neighbourly totally irregular PNG.

Proof: Let $\mathcal{B}$ is a neighbourly irregular PNG. Let $w_i, w_j \in A$, where w_i and w_j are adjacent vertices with distinct and neighbourhood degrees (p_1, q_1, r_1) and (p_2, q_2, r_2) respectively. Let us assume that

$(\mu_A(w_i), \beta_A(w_i), \sigma_A(w_i)) = (\mu_A(w_j), \beta_A(w_j), \sigma_A(w_j)) = (k_1, k_2, k_3)$, where k_1, k_2, k_3 are constants and $k_1, k_2, k_3 \in [0,1]$. Therefore

$deg_\mu[w_i] = deg_\mu(w_i) + \mu_A(w_i) = p_1 + k_1,$
$deg_\beta[w_i] = deg_\beta(w_i) + \beta_A(w_i) = q_1 + k_2,$
$deg_\sigma[w_i] = deg_\sigma(w_i) + \sigma_A(w_i) = r_1 + k_3,$
$deg_\mu[w_j] = deg_\mu(w_j) + \mu_A(w_j) = p_2 + k_1,$
$deg_\beta[w_j] = deg_\beta(w_j) + \beta_A(w_j) = q_2 + k_2,$
$deg_\sigma[w_j] = deg_\sigma(w_j) + \sigma_A(w_j) = r_2 + k_3,$
$deg_\mu[w_i] = deg_\mu[w_j]$
$\Rightarrow p_1 + k_1 = p_2 + k_1 \Rightarrow p_1 - p_2 = 0 \Rightarrow p_1 = p_2$
which is a contradiction $(p_1 \neq p_2)$.
$deg_\beta[w_i] = deg_\beta[w_j]$
$\Rightarrow q_2 + k_2 = q_1 + k_2 \Rightarrow q_1 - q_2 = 0 \Rightarrow q_1 = q_2$
$\Rightarrow$ since $(q_1 \neq q_2)$.
$deg_\sigma[w_i] = deg_\sigma[w_j]$
$\Rightarrow r_1 + k_3 = r_2 + k_3 \Rightarrow r_1 - r_2 = 0 \Rightarrow r_1 = r_2$
Which is a contradiction $(r_1 \neq r_2)$.

Therefore G is neighbourly totally irregular PNG.

Proposition 2.16. Let G be a PNG. If G is neighbourly totally irregular PNG and $(\mu_A, \beta_A, \sigma_A)$ is a constant function, then G is a neighbourly irregular PNG.

Proof: Suppose G is a neighbourly totally irregular PNG. Then by definition, the closed neighbourhood degree of every two adjacent are distinct.

Let $w_i, w_j \in A$, where w_i and w_j are adjacent vertices with distinct degrees (p_1, q_1, r_1) and (p_2, q_2, r_2) respectively.

Let us assume that $(\mu_A(w_i), \beta_A(w_i), \sigma_A(w_i)) = (\mu_A(w_j), \beta_A(w_j), \sigma_A(w_j)) = (k_1, k_2, k_3)$,where k_1, k_2, k_3 are constants and $k_1, k_2, k_3 \in [0,1]$ and $deg[w_i] \neq deg[w_j]$, so $deg_\mu[w_i] \neq deg_\mu[w_j]$, $deg_\beta[w_i] \neq deg_\beta[w_j], deg_\sigma[w_i] \neq deg_\sigma[w_j]$.

Now $deg_\mu[w_i] \neq deg_\mu[w_j] \Rightarrow p_1 + k_1 = p_2 + k_1 \Rightarrow p_1 \neq p_2$

Similarly, $deg_\beta[w_i] = deg_\beta[w_j] \Rightarrow q_1 - q_2 \neq k_1 - k_2 = 0 \Rightarrow q_1 \neq q_2$

Similarly, $deg_\sigma[w_i] = deg_\sigma[w_j]$

$$\Rightarrow r_1 + k_3 \neq r_2 + k_3 \Rightarrow r_1 \neq r_2$$

Hence the degrees of $w_i, w_j \in A$ are distinct. This is true for every pair of adjacent vertices in $\mathcal{B}$. Therefore, G is neighbourly irregular PNG.

3. Applications

Graph theory plays an important role in the application part of the networks that are used in real-life scenarios. The internet-related issues, road-transportation networks are few examples of technological networks. Various real-life networks, are vague and not well-defined in nature. The Pythagorean Neutrosophic graph is more compatible to model the ambiguous information in a network when compared to the fuzzy or vague graph. A graph is a way of modelling real-life networks, which has relationships between objects.

If there is uncertainties because of vague information about relations, or vertices then the Pythagorean neutrosophic graph model is very efficient in designing such networks. Pythagorean Neutrosophic graphs are important in the area of mathematical modelling, pattern recognition, biological networks and information systems. Pythagorean neutrosophic graphs and applied in information technology and computer science such that these graphs are used to illustrate data optimization, chip design and much more.

Moreover, the recent studies highlight the extended use of neutrosophic and fuzzy graphs in environmental and climatic analyses. For instance, interval intuitionistic neutrosophic sets and graphs have been employed to model climatic data with inherent uncertainties, enabling more accurate environmental assessments and forecasting. These models facilitate better understanding of complex ecological interactions where information is often incomplete or imprecise. Additionally, the flexibility of neutrosophic graphs in representing varying degrees of truth, indeterminacy, and falsehood makes them particularly suitable for applications in climate modeling, resource management, and ecological decision support systems, where uncertainty is a fundamental challenge. This demonstrates the broadening scope of neutrosophic graph theory, extending its applicability beyond traditional network and decision contexts to environmental science and sustainability studies.

Recent research highlights diverse applications of graph theory and fuzzy graphs across various domains:

- **Robotic Production Systems:** [21] Graph theory has been applied to design communication systems within robotic production cells, enhancing flexibility and adaptability in manufacturing processes.
- **Smart City Development:** [22] Fuzzy graph structures have been utilized to analyze the growth of smart cities in India, providing insights into urban infrastructure development and planning.
- **Traffic Management:** [23] Edge colouring of fuzzy graphs has been employed to model and improve traffic light systems, accounting for varying traffic conditions and congestion levels.
- **Manufacturing Industries:** [24] Extensions of fuzzy competition graphs have been used to model relationships and competition within manufacturing sectors, aiding in strategic decision-making.
- **Intelligent Transportation Systems:** [25] Graph-based machine learning approaches have been developed for smart urban transportation systems, facilitating efficient traffic management and route optimization.

4. Conclusion

Herein, we have defined the new concept of irregular Pythagorean Neutrosophic graphs. Also, we defined the concepts of neighbourly irregular Pythagorean neutrosophic graphs, neighbourly totally irregular Pythagorean neutrosophic graphs, highly irregular Pythagorean neutrosophic graphs, highly totally irregular Pythagorean neutrosophic graphs by providing suitable graphical representation.

The current study has certain limitations, as it primarily focuses on irregularity concepts within static Pythagorean Neutrosophic Graphs (PNGs), without addressing dynamic or evolving networks where vertex and edge membership values may vary over time. Theoretical results are derived from idealized graph structures, which may require modifications for application to large-scale or noisy real-world datasets. Additionally, while the proposed applications are conceptually outlined, empirical validations using real-world data remain limited, and computational complexity aspects related to detecting or constructing irregular PNGs are not extensively examined. Future work aims to extend irregularity concepts to dynamic or temporal PNGs to model time-varying uncertainties, and to develop efficient algorithms for detecting and measuring irregularities in large-scale graphs. Further directions include applying irregular PNG models to practical domains such as social network analysis, bioinformatics, and decision-making under uncertainty, supported by experimental validations. Moreover, exploring the relationship between irregularities in PNGs and other neutrosophic graph structures like picture fuzzy and interval-valued neutrosophic graphs, as well as investigating optimization techniques and machine learning frameworks that incorporate irregular PNGs, could enhance uncertainty management in data-driven applications.

Authors' contributions

All authors contributed equally in writing this article. All authors read and approved the final manuscript.

Conflicts of interest

The authors declare that they have no conflicts of interest.

Funding

This research received no external funding.

Data availability

No data were used to support this study.

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Received: April 12, 2025. Accepted: Sep 10, 2025

Neutrosophic Sets and Systems, Vol. 96, 2025
University of New Mexico

Neutrosophic Extension of Maxwell Length-Biased Distribution and Its Application in Energy Sector

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Abstract: The length-biased distribution has important applications in modeling economic, reliability and energy data, especially when sampling the data preferentially toward the larger values. In this paper, we introduce the neutrosophic extension of the Maxwell length-biased distribution so that it can take into account the uncertainty, indeterminacy, and imprecision involved in data analysis. Basic statistical properties of the new distribution are presented including those of its shape properties, moments, reliability function, hazard and reversed hazard functions, etc. Estimation procedure, including maximum likelihood, method (MLE) is presented in the context of neutrosophic statistics. To emphasize the practicality of the proposed solution, an application to the energy domain shows that it can deal with uncertainty and delivering more robust results than classical counterparts.

Keywords: Neutrosophic logic, uncertain data, neutrosophic probability, estimation

1. Introduction

A weighted distribution is a modified version of a regular probability distribution that assigns greater importance, or "weight," to some values of the variable of interest [1]. It is particularly meaningful when the likelihood of observing a value is proportional to the fact that it is larger, or occurs more frequently, in reality [2].
In statistical form, weighted distribution can be defined as [3]:

$$f_W(y) = \frac{w(y)\, f_Y(y)}{\int_{-\infty}^{\infty} w(y)\, f_Y(y)\, dy}, -\infty < y < \infty \tag{1}$$

The concept was initiated by Fisher in 1934 and then extended by C. R. Rao in 1965 [4]. Weighted distributions find various references in applications, such as reliability, medicine, ecology, life sciences, in which real phenomena may not be followed by classical statistical models perfectly [5]. For instance, in survival or lifetime events, an individual with a longer lifetime is more likely to be sampled, so the observed distribution will consequently be length biased (one type of weighted

distribution) [6]. Through weighted distributions one can allow for bias in such scenarios and thereby can establish better modeling of situations and more practical inferences [7].
Weighted distributions are valuable in modeling and prediction in the energy sector, where data are frequently biased because of changes in demand, intermittent supply, or preferential sampling of high values [8]. By applying weights as per need, these distributions can enable researchers to represent the impact of high or frequent energy requirements so as to achieve a more realistic portrayal of consumption behavior and system security [9]. It is apparent that weighted distributions are beneficial in forecasting energy demand, reliability of power systems, and risk analysis for unpredictable or variable energy consumption.
The model is also referred to as a length-biased distribution in the case where the weight function is proportional to the variable (i.e. $w(y) = y$) [10]. In this case, the probability distribution function values at higher values of the variable carry more weight. The class of length-biased distributions was first identified by Cox (1962) in renewal theory context. comparisons between the general weighted distributions and the length-biased distributions.
Mathematically we can write length-biased probability distribution as:

$$f_L(y) = \frac{y\, f_Y(y)}{\int_{-\infty}^{\infty} y\, f_Y(y)\, dy}, -\infty < y < \infty \quad (2)$$

A length-biased distribution has importance in statistical modeling because they arise naturally in real life when the larger values have high probability to be observed [11]. It is common in lifetime studies, reliability study, environmental study and energy data that long or large magnitude units can be more probable to be seen in a sample. Since systematic length bias in the data are controlled by the model, length-biased models can have more realistic interpretations for data subject to length bias and lead to more accurate statistical inference, prediction and decision making. They also represent a special case of weighted distributions, which make them a useful tool to address issues of biased sampling, uncertainty, etc. in applied contexts [12]-[13].
Neutrosophic logic is a field of research generated by the weaknesses of classical logic and then fuzzy logic by handling cases of (truth and falsehood) including truth, falsehood and indeterminacy [14]. There are a lot of situations in practice, in which there is partial, vague or even contradictory information but it is not able to deal with by system based on traditional binary or fuzzy degrees. Neutrosophic logic has been proposed to solve this problem to explicitly represent indeterminacy in addition to certainty and uncertainty [15]. Organized on that basis, neutrosophic statistics was constructed to generalize classical statistical techniques to imprecise or ambiguous data and thus offer more viable analysis tools [16]-[17]. Neutrosophic probability distributions then turn out to be a significant generalization of classical distributions which can model random phenomena under uncertainty, indetermination, and inconsistency. These distributions are especially valuable in areas where easily-measured variables are imprecise or only partially known since they represent a richer and more flexible geometry for their capturing the complexity of real systems [18].
In many practical situations, especially in lifetime and reliability analysis, the samples are length-biased where large values are more observed than small ones. Although classical length-biased distributions give a mechanism for dealing with this bias still they have some shortcomings when the data suffers from uncertainty, incompletion, imprecision, etc., which are frequently encountered in practical fields like energy forecasting and consumption analysis [19]-[21]. This gap indicates the necessity of developing neutrosophic length-biased distribution having the property of length-biased and the advantage of flexibility of neutrosophic theory to support considerations of vagueness and hesitancy. Among the proposed models, the neutrosophic version of Maxwell length-biased distribution (MLBD) can be interesting due to the fact that it generalizes the known Maxwell distribution, works as a one-parameter distribution for convenience, and captures the information of

the data under uncertainty all at once. This renders it a powerful and practical tool for energy sector data modeling, in which we consider the bias in observations as well as the uncertainty bound of measurements for reliable forecasting / analysis, too.

In this paper, the neutrosophic Maxwell length-biased distribution is introduced and it is an extension of the classical Maxwell distribution by utilizing the neutrosophic logic methodology. This form has the potential to treat uncertainty, indeterminacy, and the lack of complete information on the observed data which makes it tailored for real-world cases where exact measurements may be difficult to obtain. Introducing length-biased weight in the neutrosophic setting, the proposed model becomes a flexible and realistic tool for lifetime and energy-related data analysis which still has one parameter only.

The paper is organized as follows: In Section 2 the main findings concerning the classical marginal length-biased Maxwell distribution are stated. In section 3, we develop the neutrosophic structure of the proposed distribution and study some of its properties. Section 4 details the parameter estimation techniques: maximum likelihood and Bayesian. Section 5 applies the proposed model to energy sector data and illustrates its capability in dealing with uncertainty and bias. Last, major findings of this work is concluded in Section 6.

2 Classical Length biased Maxwell Distribution

To develop the LBMD ,it is important to write based line CDF and PDF of the Maxwell model which are given in Eq (3) and Eq (4) respectively:

$$F_Y(y) = \text{erf}\left(\frac{y}{\sqrt{2}\,\rho}\right) - \sqrt{\frac{2}{\pi}}\frac{y}{\rho}\exp\left(-\frac{y^2}{2\rho^2}\right), 0 < y < 1 \tag{3}$$

$$f_Y(y) = \sqrt{\frac{2}{\pi}}\frac{y^2}{\rho^3}\exp\left(-\frac{y^2}{2\rho^2}\right), 0 < y < \infty \tag{4}$$

Since the Maxwell distribution first time used in physical chemistry so its first applications relate with molecules speeds in any medium. The PDF of the Maxwell distribution is the distribution of the speeds of individual particles, or volume units, and gives is an indication of which velocity values are more likely to be found in an equilibrium system. It helps to build an understanding of the behavior of the variable of interest, presenting an estimate of where most of the observations are located and whether scatter is present, as well as how extreme values are more or less probable. The CDF provides this by measuring the probability that the speed is less than or equal to some level, enabling investigators to estimate the percentage of pools below some threshold speed. The PDF and CDF together are important for exploring the properties of the Maxwell distribution and helping analyze average speeds, fluctuation, and the probability of extreme events. In energy system optimization, models of these types provide tools to reflect the distribution of observed data, to assess the reliability of the system, and to perform forecasts, providing the basis for more advanced extensions such as the length-biased or the neutrosophic Maxwell model.

Now assuming the weight form $w(y) = y$). MLBD can be defined in PDF and CDF as:

$$f_L(y;\rho) = \frac{y^3\exp\left(-\frac{y^2}{2\rho^2}\right)}{2\rho^4}, \quad 0 < y < \infty, \rho > 0 \tag{5}$$

$$F_L(y;\rho) = 1 - \exp\left(-\frac{y^2}{2\rho^2}\right)\left(\frac{y^2}{2\rho^2} + 1\right), 0 < y < 1 \tag{6}$$

The PDF and CDF curves of Maxwell model can be seen in Figure 1.

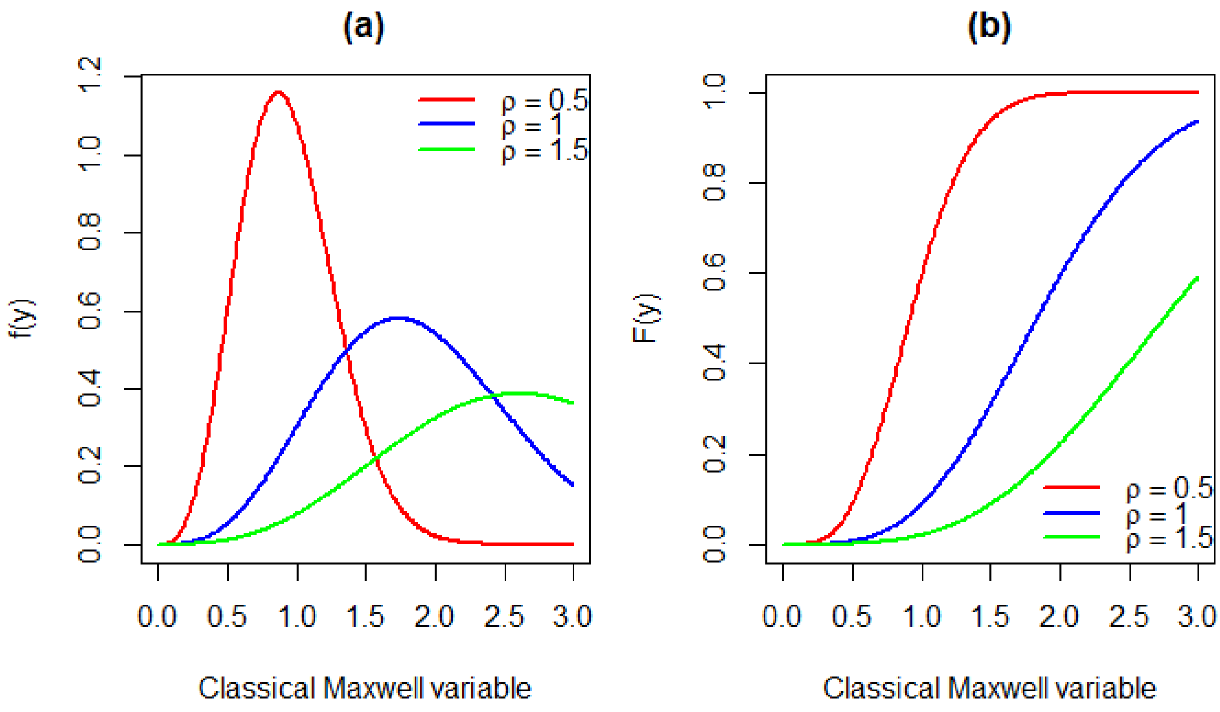

Figure 1 PDF and CDF of the LBMD for various ρ values

Figure 1 displays PDFs and CDFs of the LBMD, for three different ρ values. In the left panel, we plot the PDF, which shows how different values are likely to alternate with ρ. The one on the right displays the CDF. In both the subplots, the influence of the scale parameter ρ on the distribution form and cumulative behavior is evident.

To derive other classical properties of the LBMD we first see the rth moment of the distribution which can be established as:

$$\mu_r = E(Y^r) = \int_0^\infty y^r \frac{y^3 \exp\left(-\frac{y^2}{2\rho^2}\right)}{2\rho^4} dy = \mu_r = E(Y^r) = \sqrt{2^r}\,\rho^r\,\Gamma\left(\frac{4+r}{2}\right) \quad (7)$$

Now we can easily write from Eq (7):

$$\mu_1' = \sqrt{2^1}\,\rho^1\,\Gamma\left(\frac{4+1}{2}\right) = \sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)$$

$$\mu_2' = \sqrt{2^2}\,\rho^2\,\Gamma\left(\frac{4+2}{2}\right) = 2\,\rho^2\,\Gamma(3)$$

$$\mu_3' = \sqrt{2^3}\,\rho^3\,\Gamma\left(\frac{4+3}{2}\right) = 2\sqrt{2}\,\rho^3\,\Gamma\left(\frac{7}{2}\right)$$

$$\mu_4' = \sqrt{2^4}\,\rho^4\,\Gamma\left(\frac{4+4}{2}\right) = 4\,\rho^4\,\Gamma(4)$$

Now it is easy to see basic characteristics of the distribution:

$$\mu = \mu_1' = \sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right) \quad (8)$$

$$\sigma^2 = \mu_2' - (\mu_1')^2 = 2\,\rho^2\,\Gamma(3) - \left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^2 \quad (9)$$

$$\text{skewness} = \gamma_1 = \frac{\mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3}{(\sigma^2)^{3/2}}$$

$$= \frac{2\sqrt{2}\,\rho^3\,\Gamma\left(\frac{7}{2}\right) - 3\left(2\,\rho^2\,\Gamma(3)\right)\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right) + 2\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^3}{(\sigma^2)^{3/2}} \quad (10)$$

$$\text{kurtosis} = \gamma_2 = \frac{\mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4}{(\sigma^2)^2} \quad (11)$$

$$= \frac{4\,\rho^4\,\Gamma(4) - 4\left(2\sqrt{2}\,\rho^3\,\Gamma\left(\frac{7}{2}\right)\right)\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right) + 6\left(2\,\rho^2\,\Gamma(3)\right)\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^2 - 3\left(\sqrt{2}\,\rho\,\Gamma\left(\frac{5}{2}\right)\right)^4}{(\sigma^2)^2} \quad (12)$$

By assuming different values of scale parameter of LBMD, the basic characteristics are presented in Table 1.

Table 1 Statistical characteristics of LBMD for different values of scale parameter

ρ	Mean	Variance	Skewness	Kurtosis
0.25	0.4699928	0.02910677	0.4056951	3.059295
0.5	0.9399856	0.11642707	0.4056951	3.059295
1.5	2.8199568	1.0478436	0.4056951	3.059295
3.0	5.6399136	4.19137438	0.4056951	3.059295

Table 1 shows how the values of the LBMD change as the parameter ρ. increases. When ρ. is small, the average values are also small, and the variation around the average is quite limited. As ρ. get larger, both the mean and the variability of the data increase, showing that the distribution stretches out more. Interestingly, the shape-related measures skewness and kurtosis remain the same across all values of ρ.

3 Neutrosophic Length Biased Maxwell Distribution

A random variable Y said to follow NLBMD if it follows the following forms of PDF and CDF:

$$f_Y(y) = \sqrt{2/\pi}\frac{y^2}{\rho_n^3}\exp\left(-\frac{y^2}{2\rho_n^2}\right), 0 < y < \infty \quad (13)$$

$$F_Y(y;\rho_n) = 1 - \exp\left(-\frac{y^2}{2\rho_n^2}\right)\left(\frac{y^2}{2\rho_n^2} + 1\right), 0 < y < 1 \quad (14)$$

where scale parameter $\rho_n = [\rho_l, \rho_u]$ is in interval form.

Eq (13) and Eq (14) show that length biased Maxwell distribution is extended to neutrosophic PDF and CDF, by involving the uncertainty on the model parameter through the neutrosophic framework. In this context, the scale parameter is not treated as a crisp or fixed value, it is considered like a neutrosophic number with interval type that represents the degree of truth, indeterminacy and falsity at the same time. This generalization allows the model to learn more expressive representations of inexact and ambiguous information typically present in real-world data, and in particular for complex domains such as the energy sector, which is influenced by environmental, economic and operational variables that introduce unavoidable uncertainty. The neutrosophic PDF offers a versatile way to model the observations distribution with indeterminacy and the associated neutrosophic CDF makes sure that more useful information for cumulative probability is characterized under uncertainty or vagueness. The neutrosophic form of the Maxwell length biased distribution indicates this distribution as a valuable model for probabilistic modeling when classical assumption of crisp data is not satisfied. The structure of neutrosophic PDF and CDF are given in Figure 2.

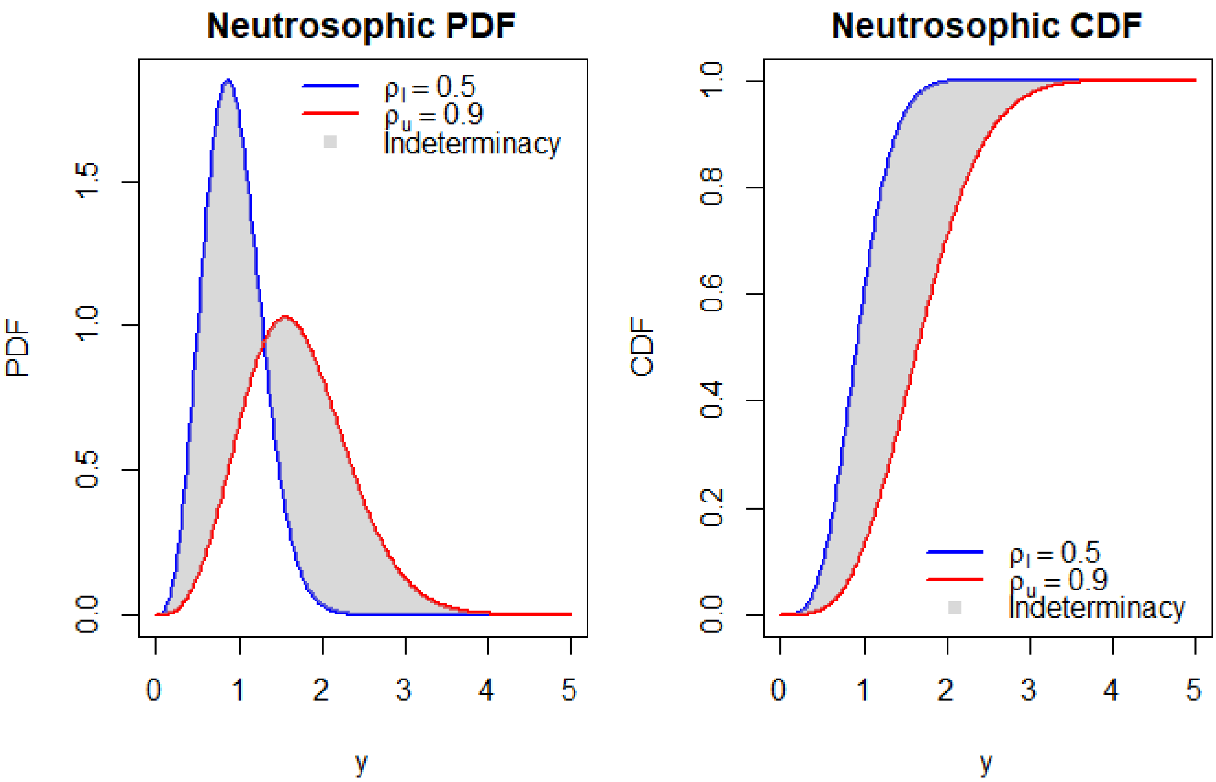

Figure 2 Neutrosophic PDF and CDF of the proposed distribution

The neutrosophic PDF and CDF of the length-biased Maxwell distribution are shown in Figure. 2. The extreme values of the parameter are the lower and upper curves and the region between them is the indeterminacy zone. This area of shading represents the uncertainty in the behavior of the distribution, that is the range of possible variation between the two bounds. The left panel is the neutrosophic PDF, and the right panel is the neutrosophic CDF as uncertainty over the entire distribution.

The quantile function is related to inverse of CDF which can be expressed as:

$$F_Y(Q(u;\rho_n);\rho_n) = u, 0 < u < 1 \tag{15}$$

Equivalently Eq (15) can be written as:

$$Q(u;\rho_n) = 1 - \exp\left(-\frac{Q(u;\rho_n)^2}{2\rho_n^2}\right)\left(\frac{Q(u;\rho_n)^2}{2\rho_n^2} + 1\right) = u$$

The quantile function of the NLBMD cannot be expressed in close form for direct computations. It is derived not analytically but by numerical methods by which the solution to the cumulative distribution function at a certain probability level is approximated. This can easily be solved using R software. This can help us to generate a random neutrosophic samples in the interval form where each interval represents the uncertainty between the lower and the upper parameter value and to provide a very flexible approach in order to grasp the indetermination of real-life data. The 40 random samples from the proposed model are given in Table 2.

Table 2 Random samples generated from proposed model

[0.729,1.313]	[1.208,2.174]	[0.837,1.507]	[1.359,2.445]	[1.505,2.71]
[0.411,0.739]	[0.941,1.693]	[1.378,2.48]	[0.961,1.73]	[0.878,1.581]
[1.569,2.824]	[0.876,1.576]	[1.081,1.946]	[0.98,1.765]	[0.52,0.936]
[1.394,2.51]	[0.689,1.241]	[0.402,0.723]	[0.766,1.379]	[1.559,2.805]
[1.372,2.469]	[1.097,1.974]	[1.044,1.879]	[1.907,3.433]	[1.059,1.906]
[1.114,2.004]	[0.955,1.719]	[1.0,1.8]	[0.731,1.315]	[0.581,1.046]
[1.598,2.876]	[1.4,2.52]	[1.095,1.97]	[1.217,2.191]	[0.347,0.624]
[0.897,1.614]	[1.17,2.107]	[0.659,1.187]	[0.757,1.363]	[0.675,1.215]

Table 2 presents a set of neutrosophic random samples generated from the NLBMD. Each entry is shown as an interval, where the lower value corresponds to the distribution with the smaller parameter ($\rho_l = 0.5$) and the upper value ($\rho_u = 0.9$) corresponds to the larger parameter.
The other important function that is related to CDF function is survival or reliability function which can be written as:

$$S_Y(y) = \exp\left(-\frac{y^2}{2\rho_n^2}\right)\left(\frac{y^2}{2\rho_n^2}+1\right) \tag{16}$$

The survival function can be depicted in Figure 3.

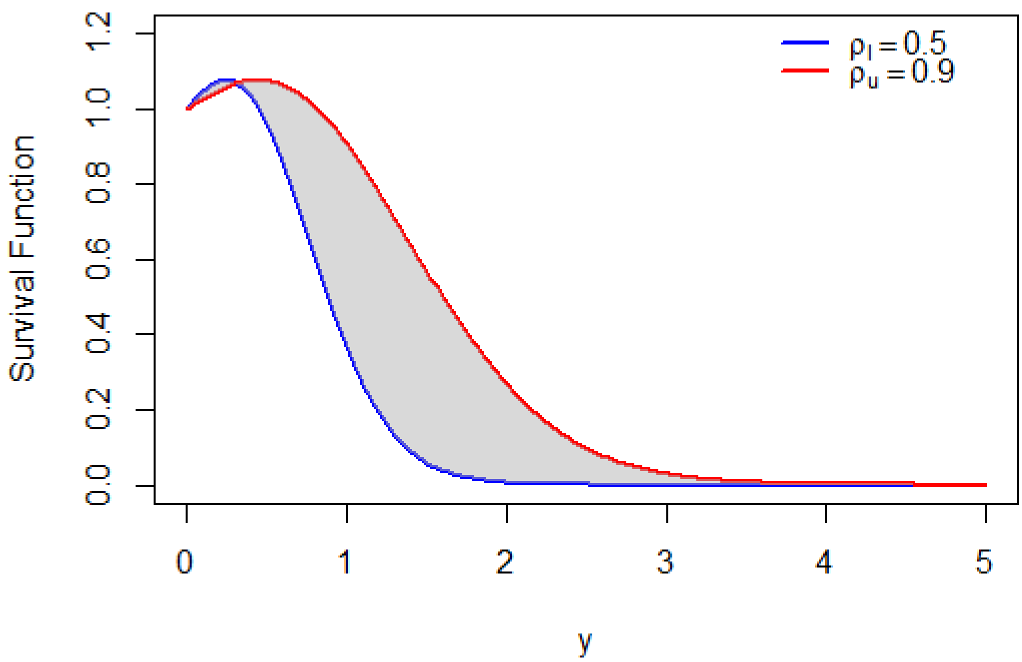

Figure 3 Survival function of the proposed distribution

The survival function of the neutrosophic length-biased Maxwell is depicted in Figure 3. The two curves [border curves] represent the lower and upper limits of the parameter, respectively, and the shaded region in between depicts the uncertain domain. This graph demonstrates the variability with which uncertainty is accumulated to prediction of survival, and how distribution behaves under various parameter values.

The neutrosophic mean and variance of the proposed model can be expressed as:

$$\mu_n = \sqrt{2}\,\rho_n\,\Gamma(5/2) \tag{17}$$

$$\sigma_n^2 = 2\rho_n^2\Gamma(3) - \left(\sqrt{2}\,\rho_n\,\Gamma(5/2)\right)^2 \tag{18}$$

The neutrosophic mean and variance of the proposed distribution will give the measures of location and scatter with the uncertainty of the parameters of model, respectively. Whereas, in the classical case, where the mean and variance are crisp values here, their neutrosophic forms are given as intervals to represent indeterminacy and vagueness. Such an assumption leads the results to be more realistic, since real data, especially those from complex systems such as the energy system, often are distorted by uncertainty, measurement errors or vague elements. Thus, the neutrosophic mean denotes not only a unique, average value, but also the range of different potential average values, while the neutrosophic variance measures the range within which the different data may spread around that mean. All together, they make the statistical model more robust by taking uncertainty instead of hiding it.

Based on neutrosophic mean and variance we can write the neutrosophic coefficient of variation as given below:

$$\mathrm{CV_n} = \frac{\sqrt{2\rho_n^2\Gamma(3) - \left(\sqrt{2}\,\rho_n\,\Gamma(5/2)\right)^2}}{\sqrt{2}\,\rho_n\,\Gamma(5/2)} \tag{19}$$

The proposed distribution coefficient of variation (CV) is an indicator to express the vagueness or indeterminacy associated with the individual possible spread of the data compared with its average proposal. Unlike classical CV, the neutrosophic one represents this relationship as an interval construct, thus the model can grasp imprecision and indeterminacy that fits more the nature of uncertain or incomplete data in practice.

Now skewness and kurtosis coefficients in terms of neutrosophic logic can be expressed as:

$$\gamma_n = \frac{=\{2\sqrt{2}\,\rho_n^3\,\Gamma(7/2) - 3\left(2\rho_n^2\Gamma(3)\right)\left(\sqrt{2}\,\rho_n\Gamma(5/2)\right) + 2\left(\sqrt{2}\,\rho_n\Gamma(5/2)\right)^3\}}{\{(\sigma_n^2)^{3/2}\}} \tag{20}$$

$$\kappa_n = \frac{4\rho_n^4\Gamma(4) - 4\left(2\sqrt{2}\rho_n^3\Gamma(7/2)\right)\left(\sqrt{2}\rho_n\Gamma(5/2)\right) + 6\left(2\rho_n^2\Gamma(3)\right)\left(\sqrt{2}\rho_n\Gamma(5/2)\right)^2 - 3\left(\sqrt{2}\rho_n\Gamma(5/2)\right)^4}{(\sigma_n^2)^2} \tag{21}$$

The skewness and kurtosis parameters of the NLBMD in neutrosophic form, offer greater facility to understand its shape and tail characteristics under uncertainty. The bentness indicates how lopsided the source is, and in the neutrosophic framework it represents several possible values not a particular one, where one values indeterminacy of data. On the other hand, the kurtosis corresponds to the peakness or flatness of the distribution, with the neutrosophic version presenting an interval that works with uncertainty and partial knowledge. In combination, these steps enable a more liberal read of the distributional properties, particularly in situations where precise parameter values are not known with absolute certainty.

4. Estimation Approach

Neutrosophic Maximum Likelihood Estimation (MLE) method is generalization of classical MLE involving uncertainty and indeterminacy in the data. Rather than a point estimate, it generates interval-valued estimates that encode both the stochasticity and the neutrosophic uncertainty of the problem. In this way, the model is able to attain a wide range of reasonable choices for the parameters, which in turn contributes to a robust estimation in front of inaccurate and/or partially unknown data. It is especially handy in practical world systems with variable input and some lack of information.
The likelihood function of the NLBMD can be written as:

$$L(y_1, \dots, y_n; \rho_n) = \prod_{i=1}^{n} \frac{y_i^3}{2\rho_n^4} \exp!\left(-\frac{y_i^2}{2\rho_n^2}\right) \quad (22)$$

Eq (22) in the loglikelihood form can be obtained as:

$$\log L\,(y_1, \dots, y_n; \rho_n) = \sum_{i=1}^{n} 3\ln y_i - n\ln 2 - 4n\ln \rho_n - \frac{1}{2\rho_n^2}\sum_{i=1}^{n} y_i^2 \quad (23)$$

Differentiating Eq (23) with respect to unknow yielded:

$$\frac{\partial \log L}{\partial \rho_n} = -\frac{4n}{\rho_n} + \frac{\sum_{i=1}^{n} y_i^2}{\rho_n^3} \quad (24)$$

Eq (24) equating to zero yielded:

$$\widehat{\rho_n} = \sqrt{\frac{\sum_{i=1}^{n} y_i^2}{4n}}. \quad (25)$$

Table 3 Estimated parameter with mean square error (MSE) of the proposed model

Sample Size (n)	$\widehat{\rho_n}$	Neutrosophic MSE
25	[0.631, 0.889]	[0.032, 0.044]
50	[0.625, 0.885]	[0.03, 0.042]
75	[0.623, 0.882]	[0.029, 0.041]
150	[0.617, 0.878]	[0.027, 0.039]
250	[0.615, 0.875]	[0.026, 0.038]
500	[0.612, 0.872]	[0.025, 0.036]

The estimated neutrosophic parameter ρ_n and corresponding mean squared errors for various sample sizes are given in Table 3. The estimated interval for ρ_n becomes more accurate when the sample size grows, indicating strong evidence of overlapping lower and upper bounds. Also, the mean squared errors become smaller as the sample size increases, which means, as one expects, that the estimates tend to be more reliable and robust when based on a higher number of points. This, in fact, emphasizes the superiority of using large datasets for accurate estimation of neutrosophic parameters.

5 Real Data Applicability

In this section we have utilized our approach to analyze energy data related to renewable energy in Saudi Arabia [22]. Most of the electricity in Saudi Arabia is generated from fossil fuels, especially natural gas and oil, which has historically been the dominant source of power in the kingdom. However, the government knows that it needs diversity and sustainability, and as such has been quite committed to developing more renewable energy. Of these, solar photovoltaic (PV) is the most attractive due to the fact that the country enjoys an incredible amount of sunshine, the decreasing costs of solar technology, and the possibility to be deployed on a large or distributed scale. Solar power is in this sense considered as a mainstay of the national strategy to lessen dependence on fossil fuels, to decrease emissions, and to generate more sustainable electricity supply due to increasing demand. The expansion of renewables, especially solar PV, offers substantial potential, but evaluating the available data on energy production and use can be fraught with uncertainty as demand shifts, weather patterns change, and precise or complete records may be missing. Such indeterminate and inconsistent information can be difficult to manage with traditional techniques. The notion of neutrosophic logic is particularly useful in the present context since allows to consider the truth, the indeterminacy, and the falsity in data analysis. This methodology is capable of dealing with uncertainty and vagueness and thus allows for more accurate assessment of renewable energy trends and better decisions in planning and managing the Saudi Arabian transition to sustainable electricity generation. We have randomly generated samples from uniform distribution. This data becomes neutrosophic by representing each year's solar PV generation not as a single fixed number, but as an interval that captures both lower and upper possible values. Instead of relying only on exact figures, small variations are introduced through random fluctuations, which account for uncertainty and imprecision in real-world measurements. Intentionally generated data is given in Table 4.

Table 4 Neutrosophic Interval of solar PV generation in Saudi Arabia (2010–2023)

[3.28, 4.71]	[4.78, 5.21]	[4.41, 5.59]	[25.88,26.12]	[45.86,46.13]
[64.88, 65.12]	[64.64, 65.35]	[319.91,320.09]	[915.37,916.63]	[4319.11,4320.88]
[41.94, 42.06]	[926.47, 927.53]	[45.65,434]	[938.99,939.01]	

The lower and upper estimates of solar PV electricity generation for Saudi Arabia, 2010–2023 are shown in the Table 4. These ranges were computed by modifying the observed data from the actual data with a random uniform variance and sampling the solid black lines where the lower bound is the original 5% beyond the actual value and the upper bound the original 5% behind the actual value. In two rows and seven columns, the table shows 14 pairs of intervals that define potential deviations from recorded levels by generation that illustrate the range of uncertainty the actual numbers.
Now utilizing the MLE estimator given Eq (25), we obtain:

$$\widehat{\rho_n} = [617.43, 617.74]$$

The smaller uncertainty has been captured by estimated scale estimator due to assumed smaller variation in the actual solar PV values. The interval of the neutrosophic estimation produces a value in accordance with the uncertainty and also considering the variation of the solar PV data. The lower estimate bound represents a more conservative scenario, through lower values of electricity generation, and the upper bound an optimistic scenario, through higher values. Combining this interval provides a loose and more realistic interpretation of the parameter, demonstrating how it can change under uncertainty. This strategy illustrates the potential of neutrosophic logic in dealing with the vagueness in practical renewable energy datum and provides more accurate basis for planning and decision-making.

6 Conclusions

In this study, we established the neutrosophic extension of the Maxwell length-biased distribution that could capture uncertainty, indeterminacy, and imprecision. Theoretical properties of the proposed new model were also derived, as well as estimation procedures, including maximum likelihood estimation, in the neutrosophic setting. To illustrate its practical use a sample application of the proposed technique was implemented in renewable energy data from Saudi Arabia, particularly solar PV generation. Converting the observed values as neutrosophic intervals, has allowed the uncertainty on electricity production is small fluctuation of data, providing interval-parameter estimation as richer and realistic than those of a classical approach. The findings indicate that neutrosophic estimator can not only handle variability but also offers more reliable insights for energy planning and policy decisions. Overall, this study confirms the usefulness of neutrosophic statistics in addressing vagueness and imprecision in renewable energy applications and sets a foundation for further research in other domains where uncertainty plays a central role.

Acknowledgement: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2025/01/ 32792)

Conflicts of Interest: The authors declare no conflict of interest.

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Received: June 6, 2025. Accepted: Sep 11, 2025

Neutrosophic Sets and Systems (NSS) is an academic journal, published quarterly online and on paper, that has been created for publications of advanced studies in neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics etc. and their applications in any field.

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ISSN (print): 2331-6055, ISSN (online): 2331-608X
Impact Factor: 1.739

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Editors-in-Chief:

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