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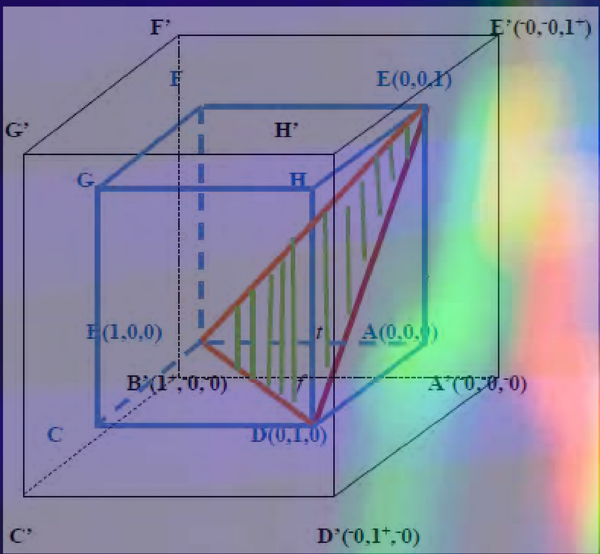

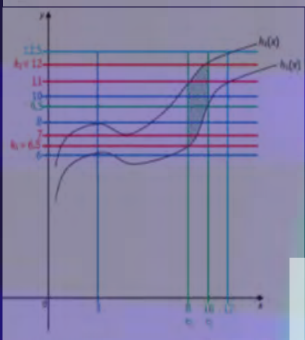

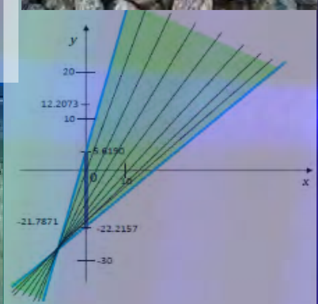

<A> <neutA> <antiA>

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The submitted papers should be professional, in good English, containing a brief review of a problem and obtained results.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>.

Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only).

According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium.

In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well.

Neutrosophic Set and *Neutrosophic Logic* are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic). In neutrosophic logic a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^-0, 1^+[$.

Neutrosophic Probability is a generalization of the classical probability and imprecise probability.

Neutrosophic Statistics is a generalization of the classical statistics.

What distinguishes the neutrosophics from other fields is the <neutA>, which means neither <A> nor <antiA>.

<neutA>, which of course depends on <A>, can be indeterminacy, neutrality, tie game, unknown, contradiction, ignorance, imprecision, etc.

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FOLDOC Dictionary of Computing (**1, 2**), Webster Dictionary (**1, 2**), Wordnik (**1**),Dictionary.com, The Free Dictionary (**1**), Wiktionary (**2**), YourDictionary (**1, 2**),OneLook Dictionary (**1, 2**), Dictionary / Thesaurus (**1**), Online Medical Dictionary (**1,2**), Encyclopedia (**1, 2**), Chinese Fanyi Baidu Dictionary (**2**), Chinese Youdao Dictionary (**2**) etc. have included these scientific neologisms.

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Linguistic Dual Hesitant Hypersoft Set and their Application with Decision Making in Medical Diagnosis and Treatment

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Abstract

This study presents Linguistic Dual Hesitant Hypersoft Set (LDHHS) for Analysing Medical Diagnose. In LDHHS, linguistic terms are combined with Dual Hesitant logic to handle uncertainty and vagueness in Decision-Making. The Hypersoft Set extend traditional soft sets by handling multi-attributes and interdependent parameters in Decision-Making. In the context of LDHHS can be used to categorize and assess different aspects of Aggregation and using Decision-Making in Medical Diagnose. In this study the development of this framework, its application and its potential impact using a Gastroesophageal reflux disease (GERD) case study to illustrate its effectiveness.

Keywords: Linguistic set, Dual Hesitant set, soft set, hypersoft set and multi-criteria decision-making (MCDM).

1.Introduction

Language, with its inherent ambiguity and subjectivity, often complicates medical diagnosis and treatment by introducing uncertainty and vagueness. Linguistic Dual Hesitant Hypersoft Sets (LDHHS) address these challenges by effectively managing linguistic uncertainty and modeling the complexities of medical data. By assigning Dual Hesitant values to descriptive terms like "minor" or "critical" for symptoms and "dissatisfied" or "very satisfied" for treatment effectiveness, LDHHS provides a precise framework for decision-making. This innovative approach has the potential to enhance diagnostic accuracy, optimize treatment strategies and improve healthcare outcomes, offering significant benefits for medical practitioners.

Motivation and Research Gap: In this study, medical diagnosis systems have increasingly required decision-making frameworks capable of processing vague linguistic information and multidimensional uncertainty. Traditional fuzzy and soft set-based models struggle to fully manage the nested and hierarchical nature of medical attributes such as symptom severity, disease progression, and treatment effectiveness. This creates a pressing need for an advanced model that can simultaneously incorporate

- linguistic subjectivity of medical descriptions,

- membership and non-membership in diagnostic evaluation,
- multi-level and multi-attribute clinical data,
- integration of decision-makers preferences.

A standardized framework for handling linguistic variables with further sub-attributes under Dual Hesitant environments. Aggregate operators, distance and similarity measures that support linguistic hypersoft information. Application of such a framework to medical diagnosis, where ambiguity and complexity are dominant. Thus, no comprehensive decision-making model currently exists that integrates linguistic uncertainty, attribute and Dual Hesitant non-membership simultaneously.

Research Questions

1. How can aggregation, distance, and similarity measures be constructed underthe LDHHS environment to support multi-criteria decision-making?
2. Does the proposed LDHHS based MCDM model provide more reliable and consistent diagnostic outcomes compared with existing approaches?

Novelty and Contributions: This study presents a novel decision-making framework based on Linguistic Dual Hesitant Hypersoft Sets (LDHHS) to address the persistent issues of linguistic ambiguity and uncertainty in medical diagnosis and treatment planning. The primary contribution lies in the ability of the proposed LDHHS model to incorporate multidimensional medical factors while simultaneously capturing the degrees of membership and non-membership associated with linguistically expressed symptoms and treatment effectiveness. In this study also contributes a practical application by demonstrating the effectiveness of N-LDHHS in a real-world medical decision-making scenario, providing new insights for healthcare practitioners and policymakers.

Literature review: Bin Zhu and Meimei Xia [10] introduced the concept of Dual Hesitant Fuzzy Sets (DHFSs), explores their properties and operations, and demonstrates their application in group forecasting. Dejian Yu, et al. [1] proposed new aggregation operators for dual hesitant fuzzy sets to better handle uncertainty. Their effectiveness is shown through a numerical example and applied to selecting HR outsourcing suppliers. Zhiliang Ren1 and Cuiping Wei [2] presents a prioritized multi-attribute decision-making method for dual hesitant fuzzy environments. A correctional score function and Dice similarity measure are introduced to better handle hesitant degrees and attribute priorities. The approach is demonstrated through a practical example. Baoquan Ning, et al. [3] proposed a MADM method using probabilistic dual hesitant fuzzy sets, introducing new distance and entropy measures and applying them to credit risk evaluation.

Muhammad Saqlain, et al. [4] introduced NHSS-TOPSIS, a decision-making method using neutrosophic hypersoft sets with new distance and similarity measures, applied to medical diagnosis and green security system selection. Hongjum Wang, et al. [5] proposed several aggregation operators for dual hesitant fuzzy MADM problems and demonstrates their effectiveness through a technology commercialization evaluation example. Jawad Ali and Muhammad Naeem [6] introduced new distance and similarity measures for normal wiggly dual hesitant fuzzy sets to better support decision-making, demonstrated through a disease detection example. Babitha and Sunil Jacob John [7] introduced a hybrid hesitant fuzzy soft set, combining soft sets and hesitant fuzzy sets, and explores its basic operations and application in decision-making. Sreelekshmi et al. [8] Proposed the Hesitant Fuzzy Hyper Soft Set to enhance decision-making accuracy, with defined operations and an application in robotics.

Glad Deschrijyer et al. [9] introduced aggregation operators on the lattice L∗, analyzes their properties via t-norms and implicators, and examines them under Smets–Magrez axioms. Ubaid Ur Rehman et al. [11] proposed complex dual hesitant fuzzy sets and their similarity measures, applying them to pattern recognition and medical diagnosis to demonstrate their effectiveness. Rana Muhammad Zulqarnain, et al. [12] developed algebraic operations and aggregation operators for interval-valued intuitionistic fuzzy hypersoft sets, applying them to material selection in cryogenic storage systems for improved decision-making. Sathiyapriya Bangarusamy, et al. [13] proposed approach enhances diagnostic accuracy and supports more tailored and reliable treatment strategies for improved patient care. Baoquan Ning et al. [14] proposed a novel correlation coefficient in the probabilistic dual hesitant fuzzy setting and applies it to a MADM method for evaluating project manager candidates. Florentin Smarandache [15] Introduced IndetermSoft and IndetermHyperSoft Sets to handle indeterminate data, extending soft set theory with applications in fuzzy and neutrosophic environments. Takkai Fujita and Shinjuku [16] explores advanced extensions of soft and rough sets, introducing Superhypersoft Hyperrough and Superhypersoft Superhyperrough sets to better handle uncertainty in decision-making. Saqlain, et. al [17] introduced Neutrosophic-linguistic valued hypersoft sets (N-LVHS) help manage linguistic uncertainty in medical diagnosis by assigning neutrosophic values to vague terms, improving accuracy in treatment and decision-making.

In this research paper explores Linguistic Dual Hesitant Hypersoft sets (LDHHS), starting with their fundamental principles and properties. It introduces operational laws and two mathematical Aggregated operators, LDHHSOWGAO and LDHHSWGAO, explaining their significance. A framework for Multi-Criteria Decision-Making (MCDM) is presented using an LDHHS algorithm, demonstrated through a case study. The paper concludes with findings and future research directions.

Acronyms

DHFS - Dual Hesitant Fuzzy Sets

LDHHS - Linguistic Dual Hesitant Hypersoft sets

ELDHHS - Empty Linguistic Dual Hesitant Hypersoft sets

LDHHSWGAO -Linguistic Dual Hesitant Hypersoft set Weighted geometric averaging Operator

LDHHSOWGAO -Linguistic Dual Hesitant Hypersoft Set Ordered Weighted Geometric Averaging Operator

GERD -Gastroesophageal Reflux Disease

2. Preliminary

2.1 Linguistic Set

Let w = $\{w_1, w_2, w_3, \ldots w_n\}$ where n = 2p + 1: p ≥ 1 and p ∈ $Ʀ^+$ (finite and real valued), be a finite strictly increasing set. For example, if p = 1 then,

w = $\{w_1, w_2, w_3\}$ = {dissatisfied, neutral, satisfied}

For Linguistic set, which is under consideration, the relationship to its elements w_n and the subscript n will be strictly increasing. To define the continuity this set is extended to w = $\{w_\omega: \omega \in Ʀ\}$ where ω is also strictly increasing.

Definition 2.1 Dual Hesitant Fuzzy Set

Let Ụ be a fixed set, then a Dual Hesitant Fuzzy Set (DHFS) $\widehat{D}$ on Ụ is described as

$\widehat{D} = \{< e, \alpha(e), \beta(e) >| e \in$ Ụ$\}$,

in which α(e) and β(e) are two sets of some values in [0, 1], denoting the possible membership degrees and non-membership degrees of the element e ∈ Ụ to the set $\widehat{D}$ respectively, with the conditions

$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1,$

Where $\gamma \in \alpha(e)$, $\eta \in \beta(e)$, $\gamma^+ \in \alpha^+(e) = \bigcup_{\gamma\in\alpha} \max\{\gamma\}$ and $\eta^+ \in \beta^+(e) = \bigcup_{\eta\in\beta} \max\{\eta\}$ for all e ∈ Ụ. For convenience, the pair $\widehat{D}(e) = (\alpha(e), \beta(e))$ is called a dual hesitant fuzzy element denoted by

$\widehat{D} = (\alpha, \beta)$, with the conditions: $\gamma \in \alpha$, $\eta \in \beta$, $\gamma^+ \in \alpha^+ = \bigcup_{\gamma\in\alpha} \max\{\gamma\}$ and $\eta^+ \in \beta^+ = \bigcup_{\eta\in\beta} \max\{\eta\}$, $0 \leq \gamma, \eta \leq 1$ and $0 \leq \gamma^+ + \eta^+ \leq 1$.

Definition 2.2 Soft Set

Let Ụ be a universe set and let H = $\{h_1, h_2, h_3, \ldots . h_n\}$ be a finite set of Parameters or Attributes. Let Ṕ (Ụ) denote the collection of all subsets of Ụ. For any E ⊆ H, a pair (D, H) is called soft Set over Ụ, where the mapping D is given by D: H → Ṕ (Ụ).

Definition 2.4 Hypersoft Set

Let Ụ be a universe of discourse, Ṕ (Ụ) be a power set of Ụ, Let H = $\{H_1, H_2, H_3, \ldots . H_n\}$ for n ≥ 1 be n distinct attributes whose corresponding attributes values are respectively the sets $a_1, a_2, a_3, \ldots . a_n$ with $a_p \cap a_q = \emptyset$, for p ≠ q and p, q ∈ {1, 2, 3, . . . n}. Then the pair (D, I) where I = $\{a_1 \times a_2 \times a_3 \times \ldots . \times a_n$: n is finite and real valued} is known as Hypersoft set over Ụ with mapping

D: $a_1 \times a_2 \times a_3 \times \ldots . \times a_n$ = I → Ṕ (Ụ).

Definition 2.5 Linguistic Hypersoft Set

Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots . \lambda_n)$ for n ≥1 be n distinct attributes, whose corresponding attribute values are respectively the sets $u_1, u_2, u_3, \ldots . u_n$ with $u_p \cap u_q = \emptyset$, where p ≠ q for each n ≥ 1 and p, q ∈ {1, 2, 3, . . . n}. Then the pair $(\theta, \propto)$ where $\propto = \{u_1 \times u_2 \times u_3 \times \ldots . \times u_n$: t is finite and real valued} is known as hypersoft set over $\underset{\wedge}{U}$ with mapping $\theta : (u_1 \times u_2 \times u_3 \times \ldots \times u_n) \rightarrow P(\underset{\wedge}{U})$.

Then the linguistic hypersoft set will be,

$\theta(\{\beta(\underset{\wedge}{U})(t)\})$: $\beta \subseteq \lambda$ & $t \in w_\omega = \{w_1, w_2, w_3, \ldots w_n\}$ where n = 2p + 1: p ≥ 1 and $m \in R^+\}$

3. Linguistic Dual Hesitant Hypersoft Set (LDHHS)

In this section, we propose LDHHS with its set structure properties.

Definition 3.1: Linguistic Dual Hesitant Hypersoft Set (LDHHS)

Let $\underset{\wedge}{U}$ be a universe of discourse $P(\underset{\wedge}{U})$ be a power set of $\underset{\wedge}{U}$. Take $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n)$ for n ≥1. where $\lambda_1, \lambda_2, \lambda_3, \ldots \lambda_n$ are attributes, whose corresponding sub-attribute values are respectively the sets $u_1, u_2, u_3, \ldots u_n$ with $u_p \cap u_q = \emptyset$, where p ≠ q for each n ≥ 1 and p, q ∈ {1, 2, 3, . . . n}. Let $\propto = \{u_1 \times u_2 \times u_3 \times \ldots \times u_n$: where n is finite and real valued} and θ : $\propto = (u_1 \times u_2 \times u_3 \times \ldots \times u_n) \rightarrow P(\underset{\wedge}{U})$. Now the pair $(\theta, \propto)$ is known as the Linguistic Dual Hesitant Hypersoft Set (LDHHS) can be defined as

$$\theta(\lambda(w)) = \{\beta(\lambda(\gamma^+, \eta^+)) \mid \beta \subseteq \lambda \;\&\; \gamma^+, \eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$$

Where w is the set of Linguistic Parameters and γ^+, η^+ represent the Dual Hesitant maximum membership and maximum non-membership values in Linguistic Parameters with the condition

$0 \leq \gamma, \eta \leq 1, 0 \leq \gamma^+ + \eta^+ \leq 1.$

Numerical Example 3.1.1

Let $\underset{\wedge}{U} = \{s_1, s_2, s_3\}$ be a universe of discourse, consisting of a set of three woods, describes the strength of wood $\theta(\lambda(w)) = \{s_1, s_2\}$. consider the attributes be λ^1 = Softwoods, λ^2 = Hardwoods and their corresponding sub-attributes values are

Softwoods = u_1 = {Cedar, Pine}

Hardwoods = u_2 = {Teak, Beech}

and set $\theta(\lambda(w)) = \{s_1, s_2\} \subset U$. Then the function $\theta: \propto = u_1 \times u_2 \rightarrow \acute{P}(\underset{\wedge}{U})$ and we have five Linguistic Parameters w = {w_1, w_2, w_3, w_4, w_5} = {very dissatisfied, dissatisfied, neutral, satisfied, very satisfied}, each linguistic Parameter corresponds a dual Hesitant value: w_1 = 0.01 for very dissatisfied, w_2 = 0.05 for dissatisfied, w_3 = 0.25 for neutral, w_4 = 0.35 for satisfied and w_5 = 0.52 for very satisfied.

Define the strength of woods in Linguistic Dual Hesitant Hypersoft Set (LDHHS)

$$(\theta, \propto) = \theta(\lambda(w)) = \{\beta(\lambda(\gamma^+, \eta^+)) \mid \beta \subseteq \lambda \;\&\; \gamma^+, \eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$$

θ({Cedar, Beech}) = {s_1, s_2} = {(s_1((dissatisfied, neutral), (very dissatisfied, dissatisfied))), (s_2 (neutral, satisfied), (very dissatisfied, dissatisfied))} = G

Similarly,

$\theta_1(\{\text{Pine, Teak}\}) = \{s_1, s_2\} = \{(s_1((\text{neutral, satisfied}), (\text{very dissatisfied, dissatisfied}))), (s_2(\text{dissatisfied, neutral}), (\text{very dissatisfied, neutral}))\} = G_1$

$\theta_2(\{\text{Cedar, Teak}\}) = \{s_2, s_3\} = \{(s_2((\text{very dissatisfied, neutral}), (\text{very dissatisfied, dissatisfied})), s_3((\text{neutral, satisfied}), (\text{very dissatisfied, dissatisfied}))\} = G_2$.

Definition 3.2: Let $(\theta_1, \propto_1) = G_1$ be a LDHHS, then the subset G_b can be defined as. $\theta(\lambda\,(w)) = \{\beta(\lambda\,(\gamma^+, \eta^+)) \mid \beta \subseteq \lambda \;\&\; \gamma^+, \eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$

1. $G_b \subseteq G_1$
2. $\forall\, w \in G_b, \theta_2(w) \subseteq \theta_1(w)$.

This holds only when linguistic variables w_ω satisfy the property i.e., each w_ω of $(\theta_b, \propto_b) \leq w_\omega$ of $(\theta_1, \propto_1)$. Where w_ω represents Linguistic variables associated with Dual Hesitant evaluation.

Example 3.2.1

Recall Example 3.1.1. The function $\theta_1 : \propto_b = u_1 \times u_2 \to \acute{P}\,(\underset{\wedge}{U})$ and assume the hypersoft set, $\theta_1(\{\text{Pine, Teak}\}) = \{(s_1(\text{neutral, satisfied}), (\text{very dissatisfied, dissatisfied}), (s_2\ (\text{dissatisfied, neutral}), (\text{very dissatisfied, neutral})\} = G_b$. Where $\propto_b \subseteq \propto$ and $G_b \subseteq G_1$.

Where G_b is a subset of the original set G_1. Linguistic variables are associated with Dual Hesitant evaluation (maximum of membership and non-membership).

Definition 3.3

Empty Linguistic Dual Hesitant Hypersoft Set (ELDHHS) can be defined as. $\theta_1 : \propto_E = u_1 \times u_2 \times \ldots\ldots \times u_n \to \acute{P}(\underset{\wedge}{U})$

Such that each $u_p (p \leq n)$ is empty. $\theta_1(\{G_E(\underset{\wedge}{U})\})$

1. $(\theta_1, \propto_E)(\emptyset) = G_E$ if $\forall \theta_1(w) = \emptyset : \forall\, w \in \propto_E$.

Example 3.3.1

Recall Example 3.1.1. The function $\theta_1 : \propto_E = u_1 \times u_2 \to \acute{P}(\underset{\wedge}{U})$, where u_1 and u_2 are all empty sets $(u_1 = u_2 = \emptyset)$ and assume the Hypersoft set, $\theta_1(\emptyset) = \emptyset = H_E$, where $\propto_E \subseteq \propto$.

Definition 3.4

The AND operation on two $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$ Linguistic Dual Hesitant hypersoft set (LDHHS) can be defined by

1. $G_1 \wedge G_2 = (\theta_{1\wedge 2}, \propto_{1\wedge 2}) = G_{1\wedge 2}$
2. $(w_p, w_q) = w_\omega = G_{1\vee 2}$, where $w_p \in G_1$ and $w_q \in G_2$ with $p \neq q$
3. $\theta_{1\cup 2}(w_p, w_q) = \theta_1(w_p) \cup \theta_2\ (w_q)$.

Definition 3.5

The OR operation on two $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$ Linguistic Dual Hesitant hypersoft set (LDHHS) can be defined by

1. $G_1 \vee G_2 = (\theta_{1\vee2}, \propto_{1\vee2}) = G_{1\vee2}$
2. $(w_p, w_q) = w_\omega = G_{1\vee2}$, where $w_p \in G_1$ and $w_q \in G_2$ with $p \neq q$
3. $\theta_{1\cap2}(w_p, w_q) = \theta_1(w_p) \cap \theta_2(w_q)$

Definition 3.6

The NOT operation on $(\theta, \propto)$ Linguistic Dual Hesitant hypersoft set (LDHHS) can be defined by.

1. $\sim G = \sim(\theta, \propto) = \sim u_1 \times \sim u_2 \times \ldots. \times \sim u_n$
2. $\sim G = \sim \Pi w_p : p = 1,2,3,\ldots\ldots.n$

Definition 3.7

The Complement on $(\theta, \propto) = G$ Linguistic Dual Hesitant hypersoft set (LDHHS) can be defined by

1. $(\theta, \propto)^{\sim} = (\theta^{\sim}, \sim \propto)$, $\theta^{\sim}: \sim \propto \rightarrow \acute{P}(\underset{\wedge}{U})$.
2. $\theta^{\sim}(\sim w) = \underset{\wedge}{U} \setminus \theta(w); \forall w \in G$.

Proposition 3.8: Let $(\theta, \propto) = G$, $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$ be Linguistic Dual Hesitant hypersoft set (LDHHS) then following holds.

1. $(\theta_1, \propto_1) \subseteq (\theta_1, \propto_1)$
2. $(\theta_2, \propto_E)(\emptyset) \subseteq (\theta_2, \propto_2)$
3. $\sim(\sim G) = G$
4. $\sim (\theta_2, \propto_E)(\emptyset) = \underset{\wedge}{U}$
5. If $(\theta_1, \propto_1) \subseteq (\theta_2, \propto_2)$ and $(\theta_2, \propto_2) \subseteq (\theta_1, \propto_1)$ then $(\theta_1, \propto_1) = (\theta_2, \propto_2)$ iff each w_ω of $(\theta_1, \propto_1) = w_\omega$ of $(\theta_2, \propto_2)$.

This property holds only when Dual Hesitant variables satisfy the property i.e., each w_ω of $(\theta_1, \propto_1) = w_\omega$ of $(\theta_2, \propto_2)$.

Proof: Recall G, G_1 and G_2 from example 3.1.1.

1. θ_1 contains Dual Hesitant Variables $G_1, G_2, G_3, \ldots G_n$

For each $w_\omega \in \propto_1$, we have a mapping $\theta_1(w_\omega) \subseteq \theta_1(w_\omega)$ each Linguistic Dual Hesitant variable map to itself. Thus, the set $(\theta_1, \propto_1)$ is a subset of itself by the Definition of 3.2, as the mappings are trivially reflexive.

$\therefore (\theta_1, \propto_1) \subseteq (\theta_1, \propto_1)$.

2. $\propto_E$ refers to an empty domain, (i.e., no Linguistic Dual Hesitant variable) the complement

operation on LDHHS with empty domain means there are no variables to map to Linguistic Dual Hesitant values.

$\theta_2(w_\omega) = \emptyset$ for all $w_\omega \in \propto_E$ the function maps nothing to the power set $\underset{\wedge}{U}$. Since the empty set is a subset of any set, it follows that $(\theta_2, \propto_E)(\emptyset) \subseteq (\theta_2, \propto_2)$ because the empty set maps to no Linguistic Dual Hesitant variables, making it trivially a subset of any non-empty LDHHS.

3. Let's assume $\sim(\theta_1, \propto_1)$ represents the complement of the LDHHS, which is $(\theta_1, \propto_1)$

with each Linguistic Dual Hesitant variables w_ω replaced by its complement. Now, apply the complement again would return the original Dual Hesitant values, as: $\sim(\sim(w_\omega)) = w_\omega$ for each w_ω thus $\sim(\sim G) = G$, Confirming the double complementation property.

4. When the LDHHS is complemented and the domain is the empty set, the result is

the complement of the empty set, which is the entire universal set $\underset{\wedge}{U}$.

Therefore, $\sim(\theta_2, \propto_E)(\emptyset) = \underset{\wedge}{U}$.

5. Let's assume that $(\theta_1, \propto_1) \subseteq (\theta_2, \propto_2)$ and $(\theta_2, \propto_2) \subseteq (\theta_1, \propto_1)$. This means that

for all, $w_\omega \in \theta_1$, there exists a corresponding $w_\omega \in \theta_2$ such that $\theta_1(w_\omega) \subseteq \theta_2(w_\omega)$

Similarly, $(\theta_2, \propto_2) \subseteq (\theta_1, \propto_1)$ implies: $\theta_2(w_\omega) \subseteq \theta_1(w_\omega)$

Since $\theta_1(w_\omega) \subseteq \theta_2(w_\omega)$ and $\theta_2(w_\omega) \subseteq \theta_1(w_\omega)$,

we conclude: $\theta_1(w_\omega) = \theta_2(w_\omega)$

Thus, $(\theta_1, \propto_1) = (\theta_2, \propto_2)$, provided that the Linguistic Dual Hesitant variables w_ω from both sets match exactly.

4. Operational Laws on LDHHSS

In this section, we discuss the importance of operational laws, theorems and propose for LDHHS. Let $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$ be two LDHHS, where $\propto_1 = \{u_1 \times u_2 \times u_3 \times \ldots \times u_p$: p is finite and real valued} over $\underset{\wedge}{U}$ with mapping $\theta: \propto_1 = u_1 \times u_2 \times u_3 \times \ldots \times u_p \rightarrow \acute{P}(\underset{\wedge}{U})$ and $\propto_2 = u_1 \times u_2 \times u_3 \times \ldots \times u_q$: y is finite and real valued} over $\breve{A}$ with mapping $\theta: \propto_2 = u_1 \times u_2 \times u_3 \times \ldots \times u_q \rightarrow \acute{P}(\underset{\wedge}{U})$.

Such that

$$(\theta, \propto) = \theta(\lambda(w)) = \{\beta(\lambda(\gamma^+, \eta^+)) \mid \beta \subseteq \lambda \ \&\ \gamma^+, \eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$$

where w is the set of Linguistic terms and γ^+, η^+ represents the Dual Hesitant Maximum of Membership and non-membership values in Linguistic terms in ascending order i. e. very dissatisfied to very satisfied.

Then the operational laws on LDHHS can be defined with some necessary conditions.

Definition 4.1 Union of LDHHS

The union of two LDHHS, $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$, can be represented as $G_1 \cup G_2$. Depending on the relationship between their Linguistic Dual Hesitant variables and their domains, the union is defined in two cases.

Case 1: Let $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$ be two LDHHS, then the union can be defined as

$$G_1 \cup G_2 = \{\Pi\, \lambda_p(w_p) \times \Pi\, \lambda_q(w_q) \in \prod_{p=1}^{n} u_p \times \prod_{q=1}^{n} u_q\}$$

where, $\lambda_p(w_p) \in \prod_{p=1}^{n} u_p$ and $\lambda_q(w_q) \in \prod_{q=1}^{n} u_q$ should be distinct with $u_p \cup u_q = \emptyset$, for $p \neq q$ and $p, q \in \{1, 2, \ldots. n\}$and $w = \{w_1, w_2, w_3, \ldots w_n\}$.

Case 2: $G_1 \cup G_2 = \{\lambda_p(w_p) \in \prod_{p=1}^{n} u_p \times \prod_{q=1}^{n} u_q\}$

With p = q and Linguistic Dual Hesitant variable w_p of u_p should be same.

Example: Consider 3.1.1

Case 1: $\theta_1(\{\text{Pine, Teak}\}) = \{s_1, s_2\}$ = {s_1(satisfied, neutral), (very dissatisfied, dissatisfied), s_2(very dissatisfied, dissatisfied), (neutral, very dissatisfied)} = G_1

$\theta_2(\{\text{Cedar, Beech}\}) = \{s_1, s_2\}$ = {s_1(very dissatisfied, dissatisfied), (satisfied, neutral), s_2(neutral, dissatisfied), (very dissatisfied, very dissatisfied)} = G_2.

∴ $u_p \cup u_q = \emptyset$ with $p \neq q$

$G_1 \cup G_2 = \{s_1(\text{satisfied, dissatisfied}),\ s_2(\text{dissatisfied, neutral}), s_1(\text{disatisfied, satisfied}), s_2(\text{neutral, very dissatisfied})\}$.

Case 2: $\theta_1(\{\text{Pine, Beach}\}) = \{s_1, s_2\}$ = {s_1(satisfied, neutral), (dissatisfied, very dissatisfied), s_2(neutral, dissatisfied), (very dissatisfied, dissatisfied)} = G_1

$\theta_2(\{\text{Pine, Teak}\}) = \{s_1, s_2\}$ = {s_1(satisfied, neutral), (dissatisfied, very dissatisfied), s_2(neutral, dissatisfied), (very dissatisfied, dissatisfied)} = G_2.

∴ $u_p \cup u_q \neq \emptyset$ with p = q

$G_1 \cup G_2 = \{s_1(\text{satisfied, dissatisfied}), s_2(\text{neutral, dissatisfied})\}$.

Case 3: (counter example)

$\theta_1(\{\text{Pine, Beach}\}) = \{s_1, s_2\}$ = {s_1(satisfied, neutral), (dissatisfied, very dissatisfied), s_2(neutral, dissatisfied), (very dissatisfied, dissatisfied)} = G_1

$\theta_2(\{\text{Pine, Teak}\}) = \{s_1, s_2\}$ = $\{s_1$(very satisfied, neutral), (dissatisfied, very dissatisfied), s_2(neutral, satisfied), (very dissatisfied, dissatisfied)$\}$ = G_2.

$\therefore u_p \cup u_q \neq \emptyset$ with p = q

Each Linguistic Dual Hesitant variables w_p of G_1 is less then Linguistic Dual Hesitant variables w_p of G_2 then this implies $G_1 \cup G_2$ can be defined with some restriction i.e., consider highest Linguistic Dual Hesitant variables w_p of each Parameters.

Example

G_1 = $\{s_1$(satisfied, neutral), (dissatisfied, very dissatisfied), s_2(neutral, dissatisfied), (very dissatisfied, dissatisfied)$\}$

G_2 = $\{s_1$(very satisfied, neutral), (dissatisfied, very dissatisfied), s_2(neutral, satisfied), (very dissatisfied, dissatisfied)$\}$

As, s_1(satisfied, dissatisfied) < s_1(very satisfied, dissatisfied) and s_2(neutral, dissatisfied) < s_2(satisfied, dissatisfied)

Then $G_1 \cup G_2$ = $\{s_1$(very satisfied, dissatisfied), s_2(satisfied, dissatisfied)$\}$.

Definition 4.2 Intersection of LDHHS

Case 1: Let $(\theta_1, \propto_1) = G_1$ and $(\theta_2, \propto_2) = G_2$ be two LDHHS, then the intersection can be defined as

$$G_1 \cap G_2 = \{\Pi\, \lambda_p(w_p) \times \Pi\, \lambda_q(w_q) \in \prod_{p=1}^{n} u_p \times \prod_{q=1}^{n} u_q\} = \emptyset$$

where, $\lambda_p(w_p) \in \prod_{p=1}^{n} u_p$ and $\lambda_q(w_q) \in \prod_{q=1}^{n} u_q$ should be distinct with $u_p \cup u_q = \emptyset$, for p = q and p, q ∈ {1, 2, …. , n}and w = $\{w_1, w_2, w_3, \ldots w_n\}$.

Case 2: $G_1 \cap G_2 = \{\lambda_p(w_p) \in \prod_{p=1}^{n} u_p \times \prod_{q=1}^{n} u_q\}$

With p = q and Dual Hesitant variable w_p of s_p Then $G_1 \cap G_2 = G_1$ OR G_2.

Example: Consider,

Case 1: $\theta_1(\{\text{Pine, Teak}\}) = \{s_1, s_2\}$ = $\{s_1$(very satisfied, satisfied), (very dissatisfied, very dissatisfied), s_2(dissatisfied, very dissatisfied), (neutral, dissatisfied)$\}$ = G_1

$\theta_2(\{\text{Cedar, Beech}\}) = \{s_1, s_2\}$ = $\{s_1$(very dissatisfied, very dissatisfied), (very satisfied, neutral), s_2(dissatisfied, very dissatisfied), (satisfied, neutral)$\}$ = G_2.

$\therefore u_p \cap u_q = \emptyset$

$G_1 \cap G_2 = \{\emptyset\}$.

Case 2: $\theta_1(\{\text{Pine, Beach}\}) = \{s_1, s_2\}$ = $\{s_1$(satisfied, neutral), (very dissatisfied, dissatisfied), s_2(neutral, dissatisfied), (dissatisfied, very dissatisfied)$\}$ = G_1

$\theta_2(\{\text{Pine}, \text{Teak}\}) = \{s_1, s_2\}$ = $\{s_1$(satisfied, neutral), (very dissatisfied, dissatisfied), s_2(neutral, dissatisfied), (dissatisfied, very dissatisfied)} = G_2.

$\therefore u_p \cap u_q \neq \emptyset$ with p = q

$G_1 \cap G_2 = \{s_1(\text{satisfied}, \text{dissatisfied}),\ s_2(\text{neutral}, \text{dissatisfied})\}$.

Case 3: (counter example)

$\theta_1(\{\text{Pine}, \text{ Beach}\}) = \{s_1, s_2\}$ = $\{s_1$(neutral, satisfied), (dissatisfied, very dissatisfied), s_2(neutral, dissatisfied), (very dissatisfied, neutral)} = G_1

$\theta_2(\{\text{Pine}, \text{Teak}\}) = \{s_1, s_2\}$ = $\{s_1$(very satisfied, neutral), (very dissatisfied, dissatisfied), s_2(satisfied, neutral), (very dissatisfied, dissatisfied)} = G_2.

$\therefore u_p \cap u_q \neq \emptyset$ with p = q

Each Linguistic Dual Hesitant variable w_p of G_1 is less then Linguistic Dual Hesitant variable w_p of G_2 then this implies $G_1 \cup G_2$ can be defined with some restriction i.e., consider highest Linguistic Dual Hesitant variable w_p of each Parameters.

Example

G_1 = $\{s_1$(neutral, satisfied), (dissatisfied, very dissatisfied), s_2(neutral, dissatisfied), (very dissatisfied, neutral)}

G_2 = $\{s_1$(very satisfied, neutral), (very dissatisfied, dissatisfied), s_2(satisfied, neutral), (very dissatisfied, dissatisfied)}

As, s_1(satisfied, dissatisfied) < s_1(very satisfied, dissatisfied) and s_2(neutral, neutral) < s_2(satisfied, dissatisfied)

Then $G_1 \cup G_2 = \emptyset$.

Theorem 4.3: If G_1 and G_2 be two LDHHS then the following holds

1. $G_1 \cup G_1 = G_1$
2. $G_1 \cup \emptyset = G_1$
3. $G_1 \cap G_1 = G_1$
4. $G_1 \cap \emptyset = \emptyset$
5. $G_1 \cup G_2 = G_2 \cup G_1$
6. $G_1 \cap G_2 = G_2 \cap G_1$
7. If $G_1 \subset G_2$ and $G_2 \subset G_1$ then $G_1 = G_2$.
8. $\zeta\,(G_1) = \zeta G_1;\ \zeta \geq 0$.
9. $\zeta\,(G_1 \cup g_2) = \zeta\,(G_2 \cup G_1)$

Proof

Straight Forward.

Theorem 4.4

If G_1, G_2 be two LDHHS then the operations are given as follows

1. $\zeta \times G_1 = G_{\zeta\times1}$; ζ (Linguistic Dual Hesitant variable)
2. $G_1 \oplus G_2 = G_{1\oplus2}$
3. $G_1 \otimes G_2 = G_{1\otimes2}$
4. $(G_1)^{\zeta} = G_{1^{\zeta}}$.

Proof

Straight Forward.

5.Some Aggregation operators

Aggregate operators are essential in decision-making processes, combining and aggregating Linguistic Dual Hesitant quantifiers or numerical values to assess factors. They enable informed analysis and evaluation of complex information, handling multiple criteria simultaneously, such as language, quality, reliability and customer satisfaction allowing for comprehensive evaluation and comparison.

Definition 5.1 LDHHSWGAO

Consider $\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n$ for $n \geq 1$, where $(\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n)$ are attributes, whose corresponding sub-attribute values are respectively the sets $\propto = (u_1, u_2, u_3, \ldots. u_n)$ with $u_p \cap u_q = \emptyset$, for $p \neq q$ for each $n \geq 1$ and p, q ∈ {1,2,3,….t}.

$\mu^{\tau} : \propto = u_1, u_2, u_3, \ldots. u_n \rightarrow P$ (Ụ) = $(\theta, \propto) = \theta(\lambda\ (w)) = \{\beta(\lambda\ (\gamma^+, \eta^+)) \mid \beta \subseteq \lambda\ \&\ \gamma^+, \eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$ (1)

Where w is the set of Linguistic Parameters and γ^+, η^+ represent the Dual Hesitant maximum membership and maximum non-membership values in Linguistic Parameters.

if $\mu^{\tau}\ (\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n) = \prod_{n=1}^{t} \lambda_n(\gamma^+, \eta^+)^{\tau_n}$

such that

$\mu^{\tau}\ (\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n) = {\lambda_1}^{\tau_1} \otimes {\lambda_2}^{\tau_2} \otimes {\lambda_3}^{\tau_3} \otimes \ldots.. \otimes {\lambda_n}^{\tau_n} = \{\beta(\lambda\ (\gamma^+, \eta^+)\}$

where $\tau = (\tau_1, \tau_2, \tau_3, \ldots.. \tau_n)$ is the exponential weighting vector of the $\lambda_n(\gamma^+, \eta^+)^{\tau_n} \in \{\beta(\lambda\ (\gamma^+, \eta^+)\}$ and $\tau_n \in [0,1]$ with $\sum_{n=1}^{t} \tau_n = 1$, then μ^{τ} is called Linguistic Dual Hesitant weighted geometric averaging operator (LDHHSWGAO).

Example

Assume $\tau = (0.6, 0.2)^n$ then LDHHSWGAO {s_1(Pine, Beech), s_2(Cedar, Teak)} = {s_1(Pine (neutral, satisfied), (dissatisfied, very dissatisfied), (Beech (satisfied, dissatisfied), (neutral, dissatisfied))}

$\therefore \mu^{\tau}\ (\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n) = \prod_{n=1}^{t}(\lambda_n(\gamma^+, \eta^+)^{\tau_n}) = {\lambda_1}^{\tau_1} \otimes {\lambda_2}^{\tau_2} \otimes {\lambda_3}^{\tau_3} \otimes \ldots.. \otimes {\lambda_n}^{\tau_n}$

$= \{\beta(\lambda\ (\gamma^+, \eta^+)\}$

= $\{s_1$(Pine (neutral, satisfied)$^{0.6}$, (dissatisfied, very dissatisfied)$^{0.2}$), (Beech (satisfied, neutral)$^{0.6}$, (neutral, dissatisfied)$^{0.2}$)$\}$

= $\{s_1$((neutral, satisfied)$^{0.6}$ + (dissatisfied, very dissatisfied)$^{0.2}$), ((satisfied, neutral)$^{0.6}$ + (neutral, dissatisfied)$^{0.2}$)$\}$

= $\{s_1$(neutral, dissatisfied), (satisfied, neutral)$\}$

Similarly, s_2(none, none).

Definition 5.2 LDHHSOWGAO

Consider $\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n$ for $n \geq 1$, where $(\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n)$ are attributes, whose corresponding sub-attribute values are respectively the sets $\propto = (u_1, u_2, u_3, \ldots. u_n)$ with $u_p \cap u_q = \emptyset$, for $p \neq q$ for each $n \geq 1$ and $p, q \in \{1,2,3,\ldots.t\}$.

$\mu' : \propto = u_1, u_2, u_3, \ldots. u_n \rightarrow$ Ᵽ (Ụ) = $(\theta, \propto) = \theta(\lambda (w)) = \{\beta(\lambda (\gamma^+, \eta^+)) \mid \beta \subseteq \lambda \;\&\; \gamma^+, \eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$ (2)

if $\mu'^{\tau} (\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n) = \prod_{n=1}^{t} \lambda_n(\gamma^+, \eta^+)^{\tau_n}$

such that

$\mu'^{\tau} (\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n) = {\lambda_1}^{\tau_1} \otimes {\lambda_2}^{\tau_2} \otimes {\lambda_3}^{\tau_3} \otimes \ldots\ldots \otimes {\lambda_n}^{\tau_n} = \{\beta(\lambda (\gamma^+, \eta^+)\}$

where $\tau = (\tau_1, \tau_2, \tau_3, \ldots\ldots \tau_n)$ is the exponential weighting vector of the $\lambda_n(\gamma^+, \eta^+)^{\tau_n} \in \{\beta(\lambda (\gamma^+, \eta^+)\}$ and $\tau_n \in [0,1]$ with $\sum_{n=1}^{t} \tau_n = 1$, then μ'^{τ} is called Linguistic Dual Hesitant Hypersoft Set Ordered Weighted Geometric Averaging Operator (LDHHSOWGAO).

Example

Assume $\tau = (0.5, 0.4)^n$ then LDHHSOWGAO $\{s_1$(Cedar, Beech), s_2(Pine, Teak)$\}$ = $\{s_1$(Pine (neutral, satisfied), (dissatisfied, very dissatisfied), (Beech (satisfied, dissatisfied), (neutral, dissatisfied))$\}$

$\therefore \mu^{\tau} (\lambda_1, \lambda_2, \lambda_3, \ldots. \lambda_n) = \prod_{n=1}^{t}(\lambda_n(\gamma^+, \eta^+)^{\tau_n}) = {\lambda_1}^{\tau_1} \otimes {\lambda_2}^{\tau_2} \otimes {\lambda_3}^{\tau_3} \otimes \ldots\ldots \otimes {\lambda_n}^{\tau_n}$

$= \{\beta(\lambda (\gamma^+, \eta^+)\}$

= $\{s_1$(Cedar (satisfied, dissatisfied)$^{0.5}$, (neutral, dissatisfied)$^{0.4}$), (Beech (neutral, very dissatisfied)$^{0.5}$, (dissatisfied, very dissatisfied)$^{0.4}$)$\}$

= $\{s_1$((satisfied, dissatisfied)$^{0.5}$ + (neutral, dissatisfied)$^{0.4}$), ((neutral, very dissatisfied)$^{0.5}$ + (dissatisfied, very dissatisfied)$^{0.4}$)$\}$

= $\{s_1$(satisfied, neutral), (neutral, dissatisfied)$\}$

Similarly, s_2(none, none).

Theorem 5.1

1. $\min_{p}(\lambda_n(\gamma^+,\eta^+)) \le \mu^{\tau}\,(\lambda_1,\lambda_2,\lambda_3,\ldots.\lambda_n) \le \max_{p}(\lambda_n(\gamma^+,\eta^+))$
2. $\min_{p}(\lambda_n(\gamma^+,\eta^+)) \le \mu'^{\tau}\,(\lambda_1,\lambda_2,\lambda_3,\ldots.\lambda_n) \le \max_{p}(\lambda_n(\gamma^+,\eta^+))$

Proof

Straight forward

Theorem 5.2

1. $\mu'^{\tau}(\lambda_p(\gamma^+,\eta^+)) = \mu'^{\tau}(\lambda_p(\gamma^+,\eta^+))$, where $(\lambda_p(\gamma^+,\eta^+))$ is any permutation of $(\lambda_p(\gamma^+,\eta^+))$
2. If $\forall(\lambda_p(\gamma^+,\eta^+)) = (\lambda_p(\gamma^+,\eta^+))\ \forall p$, then $\mu'^{\tau}\,(\lambda_p(\gamma^+,\eta^+)) = w_{\omega}(\lambda_p(\gamma^+,\eta^+))$

Proof: Straight forward.

6. LDHHS Algorithm to solve MCDM problem

A decision-making technique based on Linguistic Dual Hesitant Hypersoft Set Weighted Geometric Averaging Operator (LDHHSWGAO) has been used to construct an algorithm known as Linguistic Dual Hesitant Hypersoft Set based multi criteria group decision-making (LDHHS) algorithm. The graphical representation of the proposed LDHHS algorithm is presented in figure 2.

Step 1: Let, $\lambda_1,\lambda_2,\lambda_3,\ldots,\lambda_n$ for $n \ge 1$, be n distinct attributes, whose corresponding sub-attribute values are respectively the sets $u_1, u_2, \ldots u_n$ with $u_p \cap u_q = \emptyset$, for $p \ne q$, and p, q ∈ {1,2,….n}. let $\tau = (\tau_1,\tau_2,\tau_3,\ldots..\tau_n)$ be the exponential weighting vector. Where $\tau_n \ge [0,1]$ with $\sum_{n=1}^{t}\tau_n = 1$.

Let $\theta{:}\propto = u_1 \times u_2 \times u_3 \times \ldots\ldots \times u_n \rightarrow Ᵽ(\underset{\frown}{U}) = \theta(\lambda\,(w)) = \{\beta(\lambda\,(\gamma^+,\eta^+)) \mid \beta \subseteq \lambda\ \&\ \gamma^+,\eta^+ \in w = \{w_1, w_2, w_3, \ldots w_n\}\}$ The decision-maker $\underline{D}$ assign the values with the Linguistic Dual Hesitant Parameters and assign Linguistic Dual Hesitant variable to each alternative as $G_p = \{(\lambda_n(\gamma^+,\eta^+)) \mid p = 1,2,\ldots n\}$, $w = \{w_1, w_2, w_3, \ldots w_n\}$ and construct a Linguistic Dual Hesitant preference table for $(\lambda_n(\gamma^+,\eta^+))^{(\tau_n)}$.

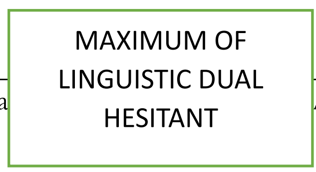
Step 2: Construct a matrix $[\lambda_{pq}]_{p\times q}$ for $\underline{D}$ using Linguistic Dual Hesitant Hypersoft weighted geometric averaging operator (LDHHSWGAO),

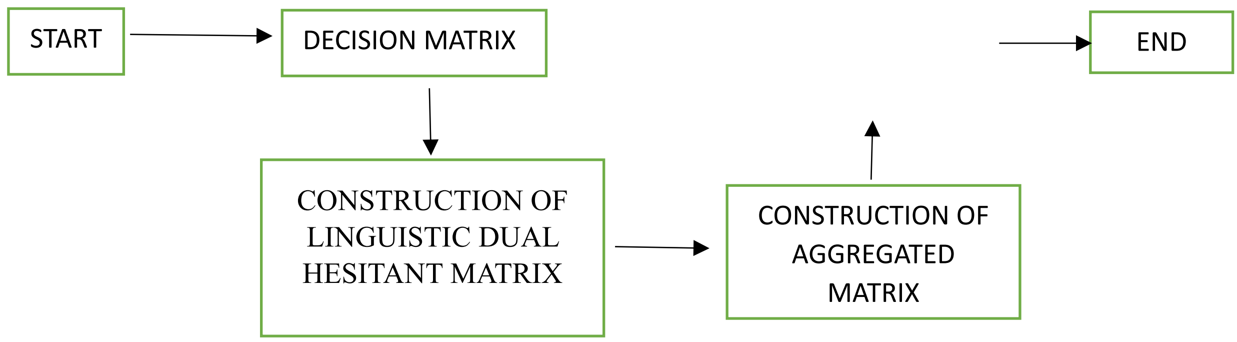
$\beta(\lambda\,(\gamma^+,\eta^+) = \lambda_1{}^{\tau_1} \otimes \lambda_2{}^{\tau_2} \otimes \lambda_3{}^{\tau_3} \otimes \ldots.. \otimes \lambda_n{}^{\tau_n}$

Step 3: List the aggregated values of all the alternatives $\beta(\lambda\,(\gamma^+,\eta^+)$.

Step 4: Finally, list the alternatives with maximum membership (γ^+) value. The maximum membership will represent the positive ideal alternative.

Figure 1. Graphical representation of proposed LDHHS algorithm





6.1 Illustrative Example

When ten patients visit a doctor with symptoms. Like Dry cough, Bitter taste, Upper Abdominal Discomfort, Epigastric pain, Belching. These symptoms are making a doubt to affect Gastroesophageal reflux disease (GERD) and also making the diagnosis questionable even if they are symptomatic of several medical diseases, including GERD. To evaluate their symptoms more precisely, the doctor uses LDHHS, and data presented in table 1.

Consider $P = \{P_1, P_2, P_3, \ldots .P_{10}\}$ be ten patients as alternatives and doctor want to diagnose them. The medical diagnose system should be to identify Gastroesophageal reflux disease (GERD) patients, while minimum of stomach issues and improve outcomes.

Consider the attributes λ_1= Dry cough, λ_2 = Bitter taste, λ_3= Upper Abdominal Discomfort, λ_4 = Epigastric pain and λ_5 = Belching.

Then the function $\mu: \propto = u_1 \times u_2 \times u_3 \times u_4 \times u_5 \to P(\underset{\frown}{U})$ and assume the hyper soft set $P = \{P_1, P_2, P_3, \ldots .P_{10}\} \subset \underset{\frown}{U}$ where $\underset{\frown}{U} = \{P_1, P_2, P_3, \ldots .P_{10}\}$ be the universal set.

Step1: Construction of Linguistic Dual Hesitant preference table for alternatives

This table organizes the system of each patient as fallows

Table 1: Doctor patient interaction and information gathering in Linguistic Dual Hesitant

Patient No./ Symptoms	Dry cough	Bitter taste	Upper Abdominal Discomfort	Epigastric pain	Belching
P_1	(very dissatisfied, satisfied), (very dissatisfied, dissatisfied)	(neutral, satisfied), (dissatisfied, neutral)	(dissatisfied, neutral), (satisfied, dissatisfied)	(satisfied, dissatisfied), (neutral, very dissatisfied)	(very dissatisfied, dissatisfied), (neutral, very dissatisfied)
P_2	(satisfied, neutral), (neutral, dissatisfied)	(satisfied, dissatisfied), (dissatisfied, very dissatisfied)	(neutral, very dissatisfied), (dissatisfied, very dissatisfied)	(dissatisfied, neutral), (very dissatisfied, dissatisfied)	(satisfied, very dissatisfied), (dissatisfied, very dissatisfied)
P_3	(very satisfied, very dissatisfied), (very dissatisfied, dissatisfied)	(satisfied, neutral), (very dissatisfied, dissatisfied)	(neutral, dissatisfied), (dissatisfied,	(satisfied, very dissatisfied),	(very satisfied, dissatisfied), (dissatisfied,

			very dissatisfied)	(neutral, dissatisfied)	very dissatisfied)
P_4	(neutral, dissatisfied), (neutral, very dissatisfied)	(neutral, very dissatisfied), (very dissatisfied, dissatisfied)	(very dissatisfied, dissatisfied), (dissatisfied, neutral)	(satisfied, very dissatisfied), (dissatisfied, very dissatisfied)	(very dissatisfied, dissatisfied), (very dissatisfied, dissatisfied)
P_5	(dissatisfied, very dissatisfied), (very dissatisfied, neutral)	(neutral, very dissatisfied), (dissatisfied, very dissatisfied)	(very dissatisfied, dissatisfied), (dissatisfied, very dissatisfied)	(very dissatisfied, neutral), (neutral, dissatisfied)	(satisfied, very dissatisfied), (very dissatisfied, dissatisfied)
P_6	(satisfied, very dissatisfied), (very satisfied, very dissatisfied)	(dissatisfied, neutral), (very dissatisfied, neutral)	(satisfied, dissatisfied), (very dissatisfied, dissatisfied)	(neutral, very dissatisfied), (satisfied, very dissatisfied)	(dissatisfied, satisfied), (neutral, dissatisfied)
P_7	(neutral, dissatisfied), (very dissatisfied, dissatisfied)	(very dissatisfied, dissatisfied), (neutral, very dissatisfied)	(satisfied, neutral), (dissatisfied, very dissatisfied)	(neutral, dissatisfied), (neural, very dissatisfied)	(satisfied, very dissatisfied), (dissatisfied, very dissatisfied)
P_8	(dissatisfied, satisfied), (very dissatisfied, dissatisfied)	(neutral, very dissatisfied), (neutral, dissatisfied)	(satisfied, dissatisfied), (dissatisfied, very dissatisfied)	(very dissatisfied, neutral), (neutral, dissatisfied)	(neutral, very dissatisfied), (dissatisfied, very dissatisfied)
P_9	(satisfied, neutral), (very dissatisfied, neutral)	(satisfied, dissatisfied), (very dissatisfied, dissatisfied)	(dissatisfied, neutral), (very dissatisfied, neutral)	(dissatisfied, very dissatisfied), (neutral, dissatisfied)	(satisfied, dissatisfied), (neutral, dissatisfied)
P_{10}	(satisfied, very dissatisfied), (neutral, dissatisfied)	(satisfied, very dissatisfied), (neutral, dissatisfied)	(neutral, dissatisfied), (neutral, very dissatisfied)	(satisfied, neutral), (very dissatisfied, dissatisfied)	(very satisfied, dissatisfied), (very dissatisfied, dissatisfied)

Step 2: The LDHHSWGAO is designed to aggregate values (weights: for Dry cough = 0.2, for Bitter taste = 0.3, for Upper Abdominal Discomfort = 0.2, for Epigastric = 0.1, for Belching = 0.2). across different symptoms (attributes) for each and every single patient.

Patients **LDHHSWGAO Values**

$$\begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{matrix} = \begin{bmatrix} \text{(dissatisfied, neutral)} \\ \text{(satisfied, dissatisfied)} \\ \text{(very satisfied, very dissatisfied)} \\ \text{(very dissatisfied, dissatisfied)} \\ \text{(very dissatisfied, dissatisfied)} \\ \text{(satisied, dissatisfied)} \\ \text{(satisfied, dissatisied)} \\ \text{(neutral, dissatisfied)} \\ \text{(satisfied, neutral)} \\ \text{(satisfied, neutral)} \end{bmatrix}$$

Step 3

Next, the doctor uses this operator to aggregate the Linguistic Dual Hesitant variables for each Patients. This aggregation takes into account maximum of Membership and Non membership of all symptoms to calculate an overall score for each Patients.

Patients **Aggregated Values**

$$\begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{matrix} = \begin{bmatrix} \text{(dissatisfied)} \\ \text{(satisfied)} \\ \text{(very satisfied)} \\ \text{(very dissatisfied)} \\ \text{(very dissatisfied)} \\ \text{(satisfied)} \\ \text{(satisfied)} \\ \text{(neutral)} \\ \text{(satisfied)} \\ \text{(satisfied)} \end{bmatrix}$$

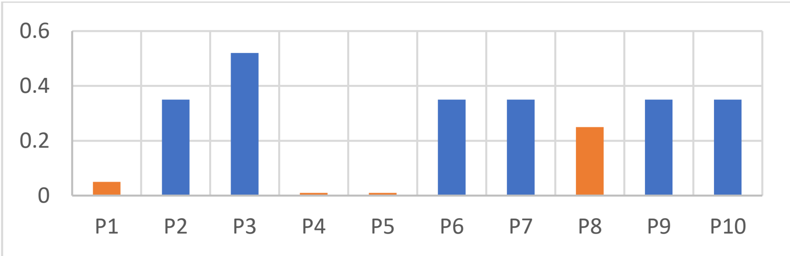
Step 4: Finally, list the alternatives with maximum Membership (γ^+) Values. The maximum (γ^+), will represent the positive ideal alternative.

Alternative **Result**

$$\begin{matrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \\ P_7 \\ P_8 \\ P_9 \\ P_{10} \end{matrix} = \begin{bmatrix} \text{(Negative)} \\ \text{(Positive)} \\ \text{(Positive)} \\ \text{(Negative)} \\ \text{(Negative)} \\ \text{(Positive)} \\ \text{(Positive)} \\ \text{(Negative)} \\ \text{(Positive)} \\ \text{(Positive)} \end{bmatrix}$$

In this case study highlights how the LDHHS algorithm helped doctors overcome diagnostic challenges with patients showing common symptoms like Dry cough, Bitter taste, Upper Abdominal Discomfort, Epigastric pain and belching which overlap across multiple illness, including Gastroesophageal Reflux Disease (GERD). By leveraging advanced language models and analysing a wide range of medical data, the algorithm provided accurate, data-driven diagnoses, reducing uncertainty. Figure: 2 visually represents the relationship between Patients and GERD.

Figure 2: Negative and Positive Patients in GERD



Represent Negative in GERD

Represent Positive in GERD

6.3 Result Discussion Comparison and Future Direction

The comparison between proposed LDHHS algorithm and traditional diagnostic methods underscores its potential to revolutionize healthcare. Unlike conventional techniques reliant on clinical judgement, LDHHS leverages advanced language models and real-time medical data to deliver more accurate and adaptive diagnoses. Table 2 illustrates its comparable performance with existing methods, while its ability to incorporate new research and manage risk, particularly crises for GERD, highlights its superiority. By complementing, rather than replacing, the expertise of healthcare professionals LDHHS paves the way for more precise, efficient and patient-centered diagnostic solutions.

Table 2: Comparing Research Result with Existing Studies.

METHOD	POSITIVE	NEGATIVE
FDHS	P_1, P_3, P_4, P_6, P_7, P_{10}	P_2, P_5, P_8, P_9
LDHHS	P_2, P_5, P_6, P_8, P_{10}	P_1, P_3, P_4, P_7, P_9

7.Conclusion

In conclusion, this study underscores the critical role of language and the challenge it poses in medical diagnosis and treatment. By introducing Linguistic Dual Hesitant Hypersoft Sets (LDHHS), a robust framework for managing Linguistic Dual Hesitant uncertainty, the study presents a promising approach to enhancing healthcare decision-making. The proposed definition, concepts, aggregation operators and algorithms demonstrate the utility of LDHHS in addressing the complexity of modern medical practice, fostering more effective and patient-centered care. Looking ahead, expanding LDHHS to encompass a broader spectrum of medical conditions and fostering collaboration between data scientists and healthcare professionals will be key to advancing its potential. Beyond healthcare, the versatile framework of LDHHS holds promise for applications in diverse fields such as market research, environmental assessments and disaster preparedness, offering a powerful tool for navigating complex and uncertain environments.

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Advancing Neutrosophic Topology through Gamma Generalized Alpha Closed Sets

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Abstract: This paper introduces a novel class of Neutrosophic closed sets called *Neutrosophic γ -generalized $\boldsymbol{\alpha}$ -closed sets* (Ne.(γG$\boldsymbol{\alpha}$)CS), along with their corresponding open sets (Ne.(γG$\boldsymbol{\alpha}$)OS), within the structure of Neutrosophic Topological Spaces (NTS). The motivation for this study arises from the limitations observed in existing Neutrosophic closed sets such as $\boldsymbol{\alpha}$-closed, semi-closed, and γ -closed sets, which often lack the flexibility to model hybrid structures involving partial membership and indeterminacy. To address this gap, we define the Ne.(γG$\boldsymbol{\alpha}$)CS using $\boldsymbol{\beta}$-closure operators and α-open supersets, offering a broader framework that unifies and extends several earlier concepts. The proposed sets are systematically analyzed through formal claims, and their behavior is demonstrated using counterexamples to confirm that reverse implications do not generally hold. Additionally, we explore their algebraic properties including union, intersection, and inclusion relationships. A comparative analysis illustrates how these sets generalize previously defined structures while preserving essential topological characteristics. The findings not only contribute to the advancement of Neutrosophic set theory but also offer a solid foundation for further research in uncertainty modeling, generalized topology, and decision-making systems. This work enhances the expressiveness of Neutrosophic topology and opens potential pathways for practical applications in fields requiring nuanced treatment of imprecision.

1. Introduction and Preliminaries

In recent decades, the limitations of classical set theory in handling real-world uncertainty have driven the development of more generalized mathematical frameworks. Among these, Smarandache's Neutrosophic Set theory stands out as a significant advancement . This enables the representation of uncertain, incomplete, inconsistent, and vague information with greater flexibility.

Building on this foundation, Neutrosophic Topology emerged as a natural extension of classical topology into the domain of indeterminacy. This new branch was initiated by A.A. Salama [10], who developed the concept of Neutrosophic Topological Spaces (NTS). In these spaces, the classical notions of open and closed sets are redefined to accommodate the presence of indeterminate and inconsistent information, which is especially relevant in areas such as artificial intelligence, decision support systems, and data analysis.

Since the introduction of NTS, many researchers have contributed to its advancement by proposing various types of Neutrosophic open and closed sets. These generalized forms have helped to build a more complete understanding of topological structures under uncertain conditions. For example, Arokiarani I. and colleagues [2] proposed the notion of Neutrosophic α-CS, which broadened the traditional concept of closedness in topological spaces by integrating indeterminacy and partial membership. Their work added depth to the exploration of closure operations in generalized topologies.

Similarly, Ishwarya P. et al. [7] studied Neutrosophic Semi-Open Sets, which represent a hybrid category between open and closed sets. This intermediate classification has provided new insights into how Neutrosophic sets behave with respect to topological boundaries, particularly when information is incomplete or partially defined.

Despite these advancements, existing classifications may not fully capture the intricate relationships between different types of Neutrosophic sets. To address this gap, the present study introduces a novel class of closed sets, termed Neutrosophic γ-Generalized α -Closed Sets (abbreviated as Ne.(γG$\boldsymbol{\alpha}$)CS, along with their corresponding open sets, known as Neutrosophic γ\gamma-Generalized α\alpha-Open Sets (Ne.(γG$\boldsymbol{\alpha}$)CS).

These newly defined set classes are proposed to further refine and generalize the concepts of closure and openness in Neutrosophic Topological Spaces. The key idea is that a set Λ_1 in a Neutrosophic topological space $(\chi_{n_{e.}}, \mathrm{N_{e.}}\tau)$ is said to be a Ne.(γG$\boldsymbol{\alpha}$)CS if $N_{e.}\mathrm{bcl}(\Lambda_1) \subseteq \Omega$ whenever $\Lambda_1 \subseteq \Omega$ and Ω is a Neutrosophic $\boldsymbol{\alpha}$- alpha-Open Set in the same space.

This framework incorporates both the γ-closure and α\alpha-openness concepts, leading to a more layered and flexible understanding of set boundaries. By doing so, it bridges the gap between multiple earlier notions and offers a unified structure to study more complex topological behaviors under uncertainty.

The objective of this research is threefold:

- To formally define and introduce the new classes of Ne.(γG$\boldsymbol{\alpha}$)CS closed and open sets;
- To analyze and prove their fundamental properties, including behavior under standard set operations like union, intersection, and complement;
- To explore their relationships with existing types of Neutrosophic sets, such as N(α\alpha)CS, N(G)CS, and N(GS)CS.

By addressing these goals, this paper aims to contribute both theoretical and structural value to the growing domain of Neutrosophic topology. These developments have the potential to enhance future investigations in topology, logic, and their interdisciplinary applications.

Moreover, this study lays the groundwork for further exploration into continuity, compactness, and separation axioms using the newly defined set types. It also opens the possibility for practical applications where vague, incomplete, or inconsistent information must be systematically analyzed.

In conclusion, the introduction of Neutrosophic γ −Generalized $\boldsymbol{\alpha}$-Closed and Open Sets represents a significant step forward in the evolution of Neutrosophic topology. It offers refined tools for topological analysis in the presence of indeterminacy and strengthens the theoretical foundation for further research in uncertainty modeling and applied mathematics.

1.1 Motivation for the Study

Many real-life problems involve situations where things are not fully true or false, and we face uncertainty or incomplete information. Traditional set theories like classical sets or fuzzy sets cannot properly deal with this kind of uncertainty. To solve this, Neutrosophic Set Theory was introduced, which allows us to separately consider truth, falsity, and indeterminacy. Building on this idea, Neutrosophic topology was developed to study open and closed sets in uncertain environments. Several types of Neutrosophic closed sets already exist, like $\boldsymbol{\alpha}$, γ and semi- CS. But these sets often work separately and don't give a complete picture when openness and closeness overlap. They are not flexible enough to handle all types of uncertain or mixed cases. This creates a need for a new, more general type of set that can combine and extend the features of the existing ones. That's why this paper introduces a new kind of set called the Neutrosophic γ-generalized α-closed set, which is designed to be broader and more useful in dealing with complex uncertain situations.

1.2 Research Gap

Although several classes of Neutrosophic closed sets have been introduced in recent years—such as Neutrosophic α-closed sets, semi-closed sets, pre-closed sets, and γ-closed sets—these concepts are limited in scope. Most of them address specific types of closure behavior and do not offer a unified structure that combines multiple closure and openness properties. As a result, they fall short in representing more complex topological structures that may arise in uncertain systems. Another issue is that the relationships between these different types of Neutrosophic closed sets are not fully explored in the literature. There is a lack of generalized set definitions that can include these existing types as special cases while offering new insights into how they interact or differ. Additionally, many existing models do not account for how sets behave under different closure operations, such as semi-closure or β-closure, in a combined or comparative manner. Therefore, there is a clear gap in developing a broader class of Neutrosophic closed sets that can generalize and unify various existing structures under a single theoretical framework. This limitation inspires the introduction and investigation of Neutrosophic γ-*generalized* -generalized α-CS in the present study.

1.3 Objective of this study

The main aim of this research is to introduce and explore a novel category of closed sets in Neutrosophic topology, referred to as Neutrosophic γ-generalized α-CS (Ne.(γGα)CS). This class is introduced to generalize and unify several existing Neutrosophic closed set types, such as α-closed, semi-closed, pre-closed, and γ-closed sets, under a broader and more inclusive framework. The study aims to establish the foundational properties of Ne.(γGα)CS, examine their algebraic behavior, and explore their interactions with other well-known closed sets. In addition, the paper provides formal proofs and counterexamples to demonstrate that while Ne.(γGα)CS include many existing classes as special cases, the reverse inclusions do not hold. Another key objective is to introduce the corresponding open sets, namely *Neutrosophic γ-generalized α-open sets*, and investigate their characteristics. Through this work, the paper seeks to enrich the structure of Neutrosophic topological spaces and support further theoretical development and practical application in fields that require refined treatment of uncertainty and imprecision.

1.4 Discussion of Existing Problems and Core Contributions

The study addresses a key limitation in Neutrosophic topology—namely, the lack of a unified structure that can generalize and relate various existing Neutrosophic closed sets such as α-closed, semi-closed, pre-closed, and γ-CS. These earlier set types are defined in isolated contexts and are often insufficient for representing the complex interplay between openness and closedness in uncertain systems. They do not capture all types of boundary behaviors, nor do they offer a general framework that allows comparison or inclusion among multiple closure concepts.

In response to this limitation, the paper introduces a new and more inclusive class called *Neutrosophic γ-generalized α-closed sets* (Ne.(γGα)CS), which incorporates β-closure operations with α-open supersets. This framework not only generalizes several known classes of Neutrosophic closed sets but also establishes their interrelationships through a series of logical claims. The paper rigorously proves that (Ne.(γGα)CS) includes all of these earlier classes as special cases and presents counterexamples to show that the converse is not generally true. This distinction is crucial for deepening the theoretical structure of Neutrosophic topology. The core contributions of the paper are as follows:

- Formal definition and development of the new class (Ne.(γGα)CS)and its corresponding open set (Ne.(γGα)OS).
- Establishment of inclusion relationships between (Ne.(γGα)CS)and existing Neutrosophic closed sets (α-closed, semi-closed, γ-closed, etc.).
- Presentation of multiple claims supported by proofs and counterexamples to clarify boundary conditions.
- Analysis of set operations (such as union and intersection) on (Ne.(γGα)CS)and their closure properties.
- Introduction of generalization theorems showing how (Ne.(γGα)CS) can serve as a broader framework for future topological investigations.

By resolving the fragmented nature of existing closed set definitions and offering a unified approach, this work significantly enhances the expressive power of Neutrosophic topological structures and provides a solid foundation for further applications and theoretical extensions.

1.5 Proposed Methodology

This study adopts a theoretical methodology to define and explore a new class of closed sets in Neutrosophic Topological Spaces (NTS), called *Neutrosophic γ-generalized α-closed sets* (Ne.(γGα)CS). The method begins with a review of existing closed set types—such as α-closed, semi-closed, pre-closed, and γ-closed sets—to highlight the need for a unifying structure. The new class is defined using β-closure and α-open sets: a set Λ_1 is (Ne.(γGα)CS) if its β-closure is contained in every α-open superset that includes it. Several claims are then established to show that well-known Neutrosophic closed sets are special cases of (Ne.(γGα)CS), with counterexamples demonstrating that the converse is not generally true. Illustrative examples clarify the behavior of these sets, including their response to set operations like union and intersection. The study also introduces the corresponding open set class, *Neutrosophic γ -generalized α-open sets* (Ne.(γGα)OS), and explores their properties. Overall, this methodology provides a step-by-step generalization framework that strengthens and extends the theory of Neutrosophic topology.

The rationale for selecting a theoretical and axiomatic approach in this study stems from the need to generalize and unify multiple existing classes of Neutrosophic closed sets within a single framework. Traditional Neutrosophic closed sets—such as α-closed, semi-closed, pre-closed, and γ-closed sets—are defined independently and lack a shared structure that allows for direct comparison or integration. By employing β -losure and α-open set operations, the proposed *Neutrosophic γ-generalized α - closed sets* (Ne.(γGα)CS) offer a flexible yet rigorous extension that includes these existing sets as particular cases. This formal method ensures mathematical clarity, enables the derivation of inclusion relations, and allows the formulation of claims with both proofs and counterexamples. The goal was to address the structural gaps in current Neutrosophic topology and to enrich the theoretical landscape for future developments. The selected methodology thus provides a solid foundation for extending closure-based reasoning under uncertainty and lays the groundwork for potential applications in decision theory, data analysis, and soft computing.

2. Basic Definitions and Preliminaries

Definition 2.1 [5,6]
Consider a fixed non-empty set N^X. A Neutrosophic set V_1^* defined on N^X can be expressed as $V_1^* = \{\langle x, \mu_{V_1^*}(x), \sigma_{V_1^*}(x), \nu_{V_1^*}(x)\rangle | x \in N^x\}$,where $\mu_{V_1^*}(x)$:The membership degree is denoted by $N^X \rightarrow$, and the function $\nu_{V_1^*}(x): N^X \rightarrow [0,1]$ specifies the non-membership degree for the Neutrosophic set V_1^*, whereas $\sigma_{V_1^*}(x)$, represents the indeterminacy degree.

Definition 2.2 [10] A Neutrosophic topology (abbreviated as NT) on the set N^x defined as a collection $N^\tau$of Neutrosophic sets within N^x that satisfies the following conditions:
1. The null Neutrosophic set 0_N and the universal Neutrosophic set 1_N are elements of N^τ
2. The intersection $J_1 \cap J_2$ belongs to $N^\tau$for any two sets $J_1, J_2 \in N^\tau$
3. For any collection $\{J_i | i \in j\} \subseteq N^\tau$. In this situation, the couple $(N^X, N^\tau)(N^X, N^\tau)$ is denoted to as a NTS. A NOS is any subclass of N^X that fits to N^τ. The counterpart V_1^{*c} of a NOS V_1^* in the NTS (N^X, N^τ) is recognized as a NCS in N^X..

Claim 2. 3 [10]. For any NS V_1^* in (N^X, N^τ), we have
1. $N^{int}(0_N) = 0_N$ and $N^{cl}(0_N) = 0_N$
2. $(N^{int}(V_1^*))^c = N^{cl}(V_1^{*c})$
3. $(N^{cl}(V_1^*))^c = N^{int}(V_1^{*c})$
4. $N^{int}(1_N) = 1_N$ and $N^{cl}(1_N) = 1_N$

Definition 2.4 A NS $V_1^*$of a NTS (N^X, N^τ) is a
1. A Neutrosophic semi preclosed set (denoted as (N(γ)CS) is well-defined as a set V_1^* for which ∃an N(P) closed set V_2^* in N(P) Closed set and there exists a Neutrosophic preclosed set V_2^* in which contains the neutrosophic interior of V_2^* contains V_1^*.
2. In [15] (N(γ)OS ∃ N(P)OS V_2^* such that $V_1^* \subseteq (V_1^*) \subseteq N^{cl}(V_2^*)V_1^*$

Definition 2.5 Let V_1^*be an NS of a NTS (N^X, N^τ). Then
1. $N^{\alpha cl}(V_1^*) = \cap \{I | I \text{ is a } N(\alpha)CS \text{ in } N^X \text{ and } V_1^* \subseteq I\}$
2. $N^{\alpha int}(V_1^*) = \cup \{I | I \text{ is a } N(\alpha)OS \text{ in } N^X \text{ and } I \subseteq V_1^*\}$

Definition 2.6 Let V_1^* be a Neutrosophic set (NS) in the Neutrosophic Topological Space (NTS) (N^X, N^τ). Then: V_1^* is called a Neutrosophic Generalized Closed Set (abbreviated as N(G)CS if $N^{cl}(V_1^*) \subseteq \Psi$ whenever $V_1^* \subseteq \Psi$ is a Neutrosophic Open Set (NOS) in N^X..

1. V_1^* is called a Neutrosophic Generalized Semi Closed Set (abbreviated as N(GS)CS if $N^{cl}(V_1^*) \subseteq \Psi$ where Ψ is a NOS in N^X.

2. $V_1^*$is called an Alpha-Neutrosophic Generalized Closed Set (abbreviated as (N(α)GCS if $N^{cl}(V_1^*) \subseteq \Psi$, and Ψ is a NOS in N^X..
3. V_1^* is called a Neutrosophic Generalized Alpha Closed Set (abbreviated as (N(α)GCS if $N^{cl}(V_1^*) \subseteq \Psi$, and Ψ is a Neutrosophic Alpha Open Set (NαOS) in N^X

Remark 2.7 Let V_1^* be a NS in (N^X, N^τ). Then
1. $N^{S-cl}(V_1^*) = V_1^* \cap N^{int}(N^{cl}(V_1^*))$
2. $N^{S-int}(V_1^*) = V_1^* \cup N^{cl}(N^{int}(V_1^*))$
If V_1^* is a NS of N^X then $N^{Scl}(V_1^{*c}) = (N^{Scl}(V_1^*))^c$

3. $(N_{e.}(\gamma G\alpha)CS)$- *Neutrosophic γ generalized α - CS*

Definition 3.1

A Neutrosophic set Λ_1 in the Neutrosophic Topological Space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is called a Neutrosophic $(N_{e.}(\gamma G\alpha)CS)$ if $N_{e.}bcl(\Lambda_1) \subseteq \Omega$ whenever$\Lambda_1 \subseteq \Omega$and Ω is a $N_{e.}(\alpha)OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ in the space $N_{e.}TS$ $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

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Example 3.2:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$, and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ stands an $N_{e.}s$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Claim 3.3:

In the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ every $N_{e.}CS$ is also a$N_{e.}(\gamma G\alpha)CS$ but the converse does not generally hold.

Proof:

Assume Λ_1 is a $N_{e.}$ Closed set in $\chi_{\pi_{e.}}$ suppose $\Lambda_1 \subseteq \Omega$ where Ω is a $N_{e.}(\alpha)$ openset in $\chi_{\pi_{e.}}$. As given that $N_{e.}bcl(\Lambda_1) \subseteq N_{e.}cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ is follows that $N_{e.}bcl(\Lambda_1) \subseteq \Omega$. Then Λ_1 is in the space $(\chi_{\pi_{e.}})$ with the neutrosophic topology $N_{e.}\tau$ and is a a $N_{e.}(bG\alpha)$

Illustration 3.4:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a neutrosophic topology is in $(\chi_{\pi_{e.}}, N_{e.}\tau)$, is a neutrosophic topology $(\gamma G\alpha)$ closed set nonetheless non $N_{e.}$closed in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ as $N_{e.}cl(\Lambda_1) = {K_1^*}^C \neq \Lambda_1$.

Claim 3.5:

In the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ every $N_{e.}(S)CS$ is also $N_{e.}(bG\alpha)CS$ but the converse does not generally hold.

Proof:

Let Λ_1 be a $N_{e.}$SCS in $\chi_{\pi_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\pi_{e.}}$. As $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(S)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Then Λ_1 is a $N_{e.}(\gamma G\alpha)CS$ in $(\chi_{\pi_{e.}}, N\tau)$.

Illustration 3.6:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a N_e.s in $(\chi_{\pi_{e.}}, N_{e.}\tau)$, stands a neutrosophi $(\gamma G\alpha)$ closed set nevertheless non a $N_{e.}(S)$ closed set in $(\chi_{\pi_{e.}}, N\tau)$ as $N_{e.}int(N_{e.}cl(\Lambda_1)) = N_{e.}int({K_1^*}^C) = K_1^* \not\subseteq \Lambda_1$.

Claim 3.7

Every Neutrosophic $N_{e.}(P)$ Closed Set in the space$(X, N_{e.}\tau)$ is also a Neutrosophic $N(\gamma G\alpha)$ Closed Set, In general, however, the converse is not necessarily true.

Proof:

Let Λ_1 is a $N_{e.}(P)CS$ in $\chi_{\pi_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\pi_{e.}}$. As $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(P)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Then Λ_1 is a $N_{e.}(\gamma G\alpha)CS$ in $(\chi_{\pi_{e.}}, N\tau)$.

Illustration 3.8:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a N_e.s in $(\chi_{\pi_{e.}}, N_{e.}\tau)$, stands a $N_{e.}(bG\alpha)CS$ nevertheless not an$N_{e.}(P)CS$ in $(\chi_{\pi_{e.}}, N\tau)$ as $N_{e.}cl(N_{e.}int(\Lambda_1)) = N_{e.}cl(K_2^*) = {K_1^*}^C \not\subseteq \Lambda_1$.

Claim 3.9:

Every Neutrosophic α Closed Set in the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set, but the converse is not true in general.

Proof:

Let Λ_1 is a $N_{e.}(\alpha)CS$ in $\chi_{\pi_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\pi_{e.}}$. As $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(\alpha)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_e.bcl(\Lambda_1) \subseteq \Omega$. Therefore, in $(\chi_{\pi_{e.}}, N_e.\tau)$., Λ_1 is a $N_{e.}(\gamma G\alpha)CS$.

Illustration 3.10:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$ is a N_e.s in $(\chi_{\pi_{e.}}, N_{e.}\tau)$, stands a Neutrosophic$N_{e.}(bG\alpha$ closed set nevertheless non an $N_{e.}(\alpha)CS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ as $N_{e.}cl(N_{e.}int(N_{e.}cl(\Lambda_1))) = N_{e.}cl(N_{e.}int({K_1^*}^C)) = N_{e.}cl(K_1^*) = {K_1^*}^C \not\subseteq \Lambda_1$.

Claim 3.11:

Every Neutrosophic $\gamma-$ Closed Set in the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(\gamma G\alpha)$ Closed Set, but the converse does not hold in general

Proof:

Let Λ_1 Neutrosophic $\gamma-$ Closed Set in the space $(\chi_{\pi_{e.}})$. Let $\Lambda_1 \subseteq \Omega$ and Ω is aNeutrosophic$-\alpha$ open set in $\chi_{\pi_{e.}}$. while $N_{e.}(\gamma)cl(\Lambda_1) \subseteq N_{e.}(\gamma)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, here consume $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Therefore, in $(\chi_{\pi_{e.}}, N_{e.}\tau)$., Λ_1 is a Neutrosophic $(\gamma G\alpha)$ closed set.

Illustration 3.12:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ is a $N_{e.}s$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$, stands a Neutrosophic $N_{e.}(\gamma G\alpha)$ closed set but non an $N_{e.}(b)$ closed set in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ as $N_{e.}int(N_{e.}cl(\Lambda_1)) \cap N_{e.}cl(N_{e.}int(\Lambda_1)) = K_1^* \cap K_1^{*C} = K_1^* \nsubseteq \Lambda_1$.

Claim 3.13:

Every Neutrosophic $N_{e.}(R)$ Closed Set in the space in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set, but the converse is not generally true.

Proof:

Let Λ_1 is a $N_{e.}(R)CS$ in $\chi_{\pi_{e.}}$. Since every $N_{e.}(R)CS$ is a $N_{e.}CS.$, Λ_1 is a $N_{e.}CS.$ Therefore by claim 2.3, Λ_1 is a $N_{e.}(\gamma G\alpha)CS$ in $(\chi_{\pi_{e.}}, N\tau)$.

Illustration 3.14:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\pi_{e.}}$.Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)CS$ but not an $N_{e.}(R)CS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$ as $N_{e.}cl(N_{e.}int(\Lambda_1)) = N_{e.}cl(K_2^*) = K_1^{*C} \neq \Lambda_1$.

Claim 3.15:

Every Neutrosophic $N_{e.}(\gamma)$Closed Set in the space $(\chi_{\pi_{e.}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set; however, the converse does not necessarily hold.

Proof:

Let Λ_1 be a $N_{e.}(\gamma)CS$ in $\chi_{\pi_{e.}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\pi_{e.}}$. As $N_{e.}bcl(\Lambda_1) \subseteq N_{e.}(\gamma)cl(\Lambda_1) = \Lambda_1 \subseteq \Omega$ by hypothesis, we have $N_{e.}(\gamma)cl(\Lambda_1) \subseteq \Omega$. Therefore, in $(\chi_{\pi_{e.}}, N_{e.}\tau)$., Λ_1 is a $N_{e.}(\gamma G\alpha)CS$.

Illustration 3.16:

Let $\chi_{\mathrm{n_{e.}}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\mathrm{n_{e.}}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\mathrm{n_{e.}}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)CS$ but not a $N_{e.}(\gamma)CS$ in $(\chi_{\mathrm{n_{e.}}}, N_{e.}\tau)$, as we could not find any $N_{e.}(P)CS$ Λ_2 such that $N_{e.}int(\Lambda_2) \subseteq \Lambda_1 \subseteq \Lambda_2$ in $\chi_{\mathrm{n_{e.}}}$.

Claim 3.17:

Every Neutrosophic $N_{e.}(b)$ Closed Set in the space $(\chi_{\mathrm{n_{e.}}}, N_{e.}\tau)$ is also a Neutrosophic $N_{e.}(bG\alpha)$ Closed Set, but the converse is not true in general.

Proof:

Let Λ_1 is a $N_{e.}(b)CS$ in $\chi_{\mathrm{n_{e.}}}$. Let $\Lambda_1 \subseteq \Omega$ and Ω is a $N_{e.}(\alpha)OS$ in $\chi_{\mathrm{n_{e.}}}$. Now $N_{e.}(\gamma)cl(\Lambda_1)=\Lambda_1\subseteq\Omega$, by hypothesis. Therefore we have $N_{e.}\ (b)cl(\Lambda_1)\subseteq\Omega$. Hence Λ_1 is a $N_{e.}(bG\alpha)CS$ in $(\chi_{\mathrm{n_{e.}}}, N_{e.}\tau)$.

Illustration 3.18:

Let $\chi_{\mathrm{n_{e.}}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$, and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\mathrm{n_{e.}}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_{e.}T$ on $\chi_{\mathrm{n_{e.}}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ is a $N_{e.}(bG\alpha)CS$ but not an $N_{e.}(b)CS$ in $(\chi_{\mathrm{n_{e.}}}, N_{e.}\tau)$ as $N_{e.}int(N_{e.}cl(N_{e.}int(\Lambda_1))) = N_{e.}int(N_{e.}cl(K_2^*)) = N_{e.}int({K_1^*}^C) = K_1^* \nsubseteq \Lambda_1$

Remark 3.19:

In general, the union of two Neutrosophic $N_{e.}(bG\alpha)$ Closed Sets in the space $(\chi_{\mathrm{n_{e.}}}, N_{e.}\tau)$ is not necessarily a Neutrosophic $N_{e.}(bG\alpha)CS$ Closed Set, as demonstrated in the following example.

Illustration 3.20:

Let $\chi_{\mathrm{n_{e.}}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$, $K_2^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_3^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$. Then $\tau_{\mathrm{n_{e.}}} = \{0_N, K_1^*, K_2^*, K_3^*, 1_N\}$ is a $N_{e.}$ T on $\chi_{\mathrm{n_{e.}}}$. Here $\Lambda_1 = \langle x, \left(\frac{1}{10}, \frac{5}{10}, \frac{9}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$, $\Lambda_2 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right)\rangle$,

are $N_{e.}(bG\alpha)CSs$ in $(\chi_{\mathrm{n_{e.}}}, N_{e.}\tau)$. But $\Lambda_1 \cup \Lambda_2$ is not an $N_{e.}(bG\alpha)CS$ as $\Lambda_1 \cup \Lambda_2 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle \subseteq K_1^*$ but $N_{e.}(bcl(\Lambda_1 \cup \Lambda_2) = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle \nsubseteq K_1^*$.

Remark 3.21:

The intersection of any two $N_{e.}(bG\alpha)CSs$ is not an $N_{e.}(bG\alpha)CS$ in general as seen in the following example.

Illustration 3.22:

Let $\chi_{\mathrm{n_{e.}}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$, $K_2^* = \langle x, \left(\frac{2}{10}, \frac{5}{10}, \frac{8}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_3^* = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$. Then $\tau_{\mathrm{n_{e.}}} = \{0_N, K_1^*, K_2^*, K_3^*, 1_N\}$ is a $N_{e.}$ T on $\chi_{\mathrm{n_{e.}}}$ .Here $\Lambda_1 =$

$\langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$, $\Lambda_2 = \langle x, \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$, are $N_{e.}(bG\alpha)CSs$ in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$). But $\Lambda_1 \cap \Lambda_2$ is not an $N_{e.}(bG\alpha)CS$ as $\Lambda_1 \cap \Lambda_2 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle \subseteq K_1^*$ but $N_{e.}(bcl(\Lambda_1 \cap \Lambda_2)$ $= \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle \nsubseteq K_1^*$.

Claim 3.23:

Let ($\chi_{\pi_{e.}}$, $N_{e.}\tau$) is a $N_{e.}$TS. Then $\Lambda_1 \in N_{e.}$(γGα)C($\chi_{\pi_{e.}}$) and $\Lambda_2 \in N_{e.}$S($\chi_{\pi_{e.}}$), $\Lambda_1 \subseteq \Lambda_2 \subseteq N_{e.}$bcl($\Lambda_1$) ⇒$\Lambda_2 \in N_{e.}$(γGα)C($\chi_{\pi_{e.}}$).

Proof:

Let $\Lambda_2 \subseteq \Omega$ and Ω is a $N_{e.}$(α)OS in $\chi_{\pi_{e.}}$. Then since $\Lambda_1 \subseteq \Lambda_2$, $\Lambda_1 \subseteq \Omega$. By hypothesis $\Lambda_2 \subseteq N_{e.}$bcl($\Lambda_1$). Therefore $N_{e.}$bcl(Λ_2)⊆$N_{e.}$bcl($N_{e.}$bcl(Λ_1))=$N_{e.}$bcl(Λ_1)⊆Ω, since Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$. Hence $\Lambda_2 \in N_{e.}$(γGα)C($\chi_{\pi_{e.}}$).

Claim 3.24:

Let Γ⊆ Λ_1⊆ $\chi_{\pi_{e.}}$ where Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$. Then Γ is a $N_{e.}$(γGα)CS in Λ_1 if and only if Γ is a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$

Proof:

Necessity: Let Ω is a $N_{e.}$(α)OS in $\chi_{\pi_{e.}}$ and Γ⊆Ω. Λ_1 also let Γ is a $N_{e.}$(γGα)CS in Λ_1. Then clearly Γ ⊆ Λ_1∩Ω and Λ_1∩Ω is a $N_{e.}$(α)OS in Λ_1. Hence beta closure of Γ in Λ_1, $N_{e.}$bclΛ_1 (Γ)⊆Λ_1∩Ω and by claim 3.26: Λ_1 is a $N_{e.}$(γ) CS. Therefore $N_{e.}$b cl(Λ_1) = Λ_1 . Now beta closure of Γ in $\chi_{\pi_{e.}}$, $N_{e.}$bcl(Γ)⊆$N_{e.}$bcl(Γ)∩$N_{e.}$bcl(Λ_1)=$N_{e.}$bcl(Γ)∩Λ_1=$N_{e.}$bclΛ_1(Γ)⊆Λ_1∩Ω⊆Ω, that is $N_{e.}$bcl(Γ)⊆Ω, whenever Γ⊆Ω. Hence Γ is a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$.

Sufficiency: Let V is a Neutrosophic -α open set in Λ_1, such that Γ⊆V. Since Λ_1 is a Neutrosophic -α open set in $\chi_{\pi_{e.}}$, V is a Neutrosophic -α open set in $\chi_{\pi_{e.}}$. Therefore bcl(Γ) ⊆ V, as Γ is a Neutrosophic (γGα)CS in $\chi_{\pi_{e.}}$. Thus, $N_{e.}$bclΛ_1(Γ)= $N_{e.}$bcl(Γ)∩ Λ_1 ⊆V∩Λ_1⊆V. Hence Γ is a $N_{e.}(bG\alpha)CS$ in Λ_1.

Claim 3.25:

A $N_{e.}$S Λ_1 is both an $N_{e.}$OS and a $N_{e.}$(γGα)CS if and only if Λ_1 is a $N_{e.}$(R)OS in $\chi_{\pi_{e.}}$.

Proof:

Necessity: Let Λ_1 be both an $N_{e.}$OS and a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$. Then Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}(bG\alpha)CS$. By claim 3.25, Λ_1 is a $N_{e.}$(γ) CS and $N_{e.}$ int($N_{e.}$ cl($N_{e.}$ int(Λ_1))) ⊆ Λ_1 . Since Λ_1 is a $N_{e.}$ OS, $N_{e.}$ int(Λ_1)= Λ_1 .Therefore $N_{e.}$ int($N_{e.}$ cl(Λ_1))⊆ Λ_1 .Since Λ_1 is a $N_{e.}$ OS, it is a $N_{e.}$ POS. Hence Λ_1⊆$N_{e.}$int($N_{e.}$cl(Λ_1)). Therefore Λ_1=$N_{e.}$int($N_{e.}$cl(Λ_1)) and Λ_1 is a $N_{e.}$(R)OS in $\chi_{\pi_{e.}}$.

Sufficiency: Let Λ_1 is a $N_{e.}$(R)OS in $\chi_{\pi_{e.}}$ then Λ_1= $N_{e.}$int($N_{e.}$cl(Λ_1)). Since every $N_{e.}$(R)OS is a $N_{e.}$OS, Λ_1 is a $N_{e.}$OS. We have $N_{e.}$int($N_{e.}$cl($N_{e.}$int(Λ_1))) = $N_{e.}$int($N_{e.}$cl(Λ_1))=Λ_1⊆Λ_1. Therefore Λ_1 is a $N_{e.}(b)CS$ in $\chi_{\pi_{e.}}$, and by claim 3.17, Λ_1 is a $N_{e.}(bG\alpha)CS$ in $\chi_{\pi_{e.}}$.

Claim 3.26:

For an $N_{e.}$OS Λ_1 in ($\chi_{\pi_{e.}}$, $N_{e.}\tau$), the following conditions are equivalent.

(i) Λ_1 is a $N_{e.}$CS

(ii) Λ_1 is a $N_{e.}$(γGα)CS and a $N_{e.}$Q set

Proof: (i) ⇒ (ii) Since Λ_1 is a $N_{e.}$CS, it is a $N_{e.}$(γGα)CS by claim 3. Now $N_{e.}$int($N_{e.}$cl(Λ_1)) =$N_{e.}$ int(Λ_1)=Λ_1=$N_{e.}$ cl(Λ_1) = $N_{e.}$cl($N_{e.}$int(Λ_1)), by hypothesis. Hence Λ_1 is a $N_{e.}$Q-set.

(ii) ⇒ (i) Since Λ_1 is a $N_{e.}$OS and a $N_{e.}$(γGα)CS, by claim 2.27, Λ_1 is a $N_{e.}$(R)OS. Therefore Λ_1 = $N_{e.}$int($N_{e.}$cl(Λ_1)) = $N_{e.}$cl($N_{e.}$int(Λ_1)) =$N_{e.}$ cl(Λ_1), by hypothesis. Hence Λ_1 is a $N_{e.}$CS in $\chi_{\text{п}_{\text{e.}}}$.

Claim 3.27:

Let ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ) is a $N_{e.}$TS. Then $N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$) = $N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$) if every $N_{e.}$S in ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ) is a $N_{e.}$(α)OS in $\chi_{\text{п}_{\text{e.}}}$.

Proof:

Suppose that every $N_{e.}$S in ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ) is a $N_{e.}$(α)OS in $\chi_{\text{п}_{\text{e.}}}$. Let Λ_1∈$N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$). Then Λ_1 is also an $N_{e.}$(α)OS by hypothesis. Therefore by claim 2.25 Λ_1 is a $N_{e.}$(γ)CS.Therefore Λ_1 ∈ $N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$). Hence $N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$)⊆$N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$) (i) Let Λ_1∈$N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$). Then by claim 2.17, Λ_1 is a $N_{e.}$(γGα)CS and Λ_1 ∈ $N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$). Hence $N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$)⊆$N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$) (ii).From (i) and (ii) $N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$)=$N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$).

Claim 3.28:

Let Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}$(γGα)CS of ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ). Then Λ_1∩Γ is a $N_{e.}$(γGα)CS of ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ) where Γ is a $N_{e.}$CS of $\chi_{\text{п}_{\text{e.}}}$.

Proof:

Consider that Λ_1 is a $N_{e.}$(α)OS and a $N_{e.}$(γGα)CS of ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ), then by claim 2.25, Λ_1 is a $N_{e.}$(γ)CS. But Γ is a $N_{e.}$CS in $\chi_{\text{п}_{\text{e.}}}$. Hence Λ_1∩Γ is a $N_{e.}$(γ)CS as every $N_{e.}$CS is a $N_{e.}$(γ)CS. Therefore Λ_1∩Γ is a $N_{e.}$(γGα)CS in $\chi_{\text{п}_{\text{e.}}}$, by claim 3.17.

Claim 3.29:

Let ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ) is a $N_{e.}$TS, then for every Λ_1∈$N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$) and for every Λ_2 in $\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$int(Λ_1)⊆Λ_2⊆Λ_1 implies Λ_2 ∈ $N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$).

Proof:

Let Λ_1 be Λ_1 $N_{e.}$(γ) CS in $\chi_{\text{п}_{\text{e.}}}$ .Then there exists an $N_{e.}$ (P)CS,(say) Λ_3 such that $N_{e.}$ int(Λ_3)⊆ Λ_1 ⊆ Λ_3 . By hypothesis, Λ_2 ⊆ Λ_1 . Therefore Λ_2 ⊆ Λ_3 . Since $N_{e.}int$(Λ_3)⊆ Λ_1 , $N_{e.}$int(Λ_3)⊆ $N_{e.}int$(Λ_1) and $N_{e.}int$(Λ_3)⊆ Λ_2, by hypothesis. Thus $N_{e.}int$(Λ_3)⊆ Λ_2⊆Λ_3 and Λ_2 ∈ $N_{e.}$bC($\chi_{\text{п}_{\text{e.}}}$). Hence by claim 3.15, Λ_2 ∈ $N_{e.}$(γGα)C($\chi_{\text{п}_{\text{e.}}}$)

Claim 3.30:

If a $N_{e.}$S Λ_1 of a $N_{e.}$TS ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ) is Neutrosophic nowhere dense, then it is a $N_{e.}$(γGα)CS in ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ).

Proof:

If Λ_1 is Neutrosophic nowhere dense in $\chi_{\text{п}_{\text{e.}}}$, then $N_{e.}$int($N_{e.}$cl(Λ_1)) = 0_N. Let Λ_1 ⊆ Ω where Ω is a $N_{e.}$(α)OS in $\chi_{\text{п}_{\text{e.}}}$.Now $N_{e.}$bcl(Λ_1)⊆$N_{e.}$Scl(Λ_1)=Λ_1∪$N_{e.}$int($N_{e.}$cl(Λ_1))=Λ_1∪0_N = Λ_1⊆Ω and hence Λ_1 is a $N_{e.}$(γGα)CS in ($\chi_{\text{п}_{\text{e.}}}$, $N_{e.}$τ).

4. γ -Generalized α -Type Open Sets within a Neutrosophic Framework

In this section, various properties of Neutrosophic **γ** -generalized **α** -open sets have been examined and analyzed, leading to the development of several insightful characterization theorems.

Illustration 4.1:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Claim 4.2:

Every $N_{e.}OS$, $N_{e.}(S)OS$, $N_{e.}(P)OS$, $N_{e.}(\alpha)OS$, $N_{e.}(\gamma)OS$, $N_{e.}(R)OS$, $N_{e.}bOS$, $N_{e.}(\gamma)OS$ are $N_{e.}(\gamma G\alpha)OS$ but not conversely in general in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Proof:

Straightforward.

Illustration 4.3:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)OS$ but not an $N_{e.}OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Illustration 4.4:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$. is a $N_{e.}(bG\alpha)OS$ but not $N_{e.}(S)OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Illustration 4.5:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)OS$ but not an $N_{e.}(P)OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Illustration 4.6:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)OS$ but not an $N_{e.}(\alpha)OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Illustration 4.7:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a $N_e.T$ on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$ is a $N_{e.}(\gamma G\alpha)OS$ but not an $N_{e.}(\gamma)OS$ in $(\chi_{\pi_{e.}}, N_{e.}\tau)$.

Illustration 4.8:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{6}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right) \rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$(R)OS in ($\chi_{\pi_{e.}}$, N_e.τ).

Illustration 4.9:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right) \rangle$

is a $N_{e.}$(γGα)OS but not $N_{e.}$bOS in ($\chi_{\pi_{e.}}$, N_e.τ).

Illustration 4.10:

Let $\chi_{\pi_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$ and $K_2^* = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right) \rangle$. Then $\tau_{\pi_{e.}} = \{0_N, K_1^*, K_2^*, 1_N\}$ is a N_e.T on $\chi_{\pi_{e.}}$. Here $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{4}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right) \rangle$

is a $N_{e.}$(γGα)OS but not an $N_{e.}$(γ)OS in ($\chi_{\pi_{e.}}$, N_e.τ).

Claim 4.11:

Let ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$TS. Then for every $\Lambda_1 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$) and for every $\Lambda_2 \in N_{e.}$S($\chi_{\pi_{e.}}$), $N_{e.}$bint(Λ_1)⊆Λ_2⊆Λ_1⇒Λ_2∈$N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Proof:

Let Λ_1 is ay $N_{e.}$(γGα)OS of $\chi_{\pi_{e.}}$ and Λ_2 is any $N_{e.}$S of $\chi_{\pi_{e.}}$. Let $N_{e.}$bint(Λ_1)⊆Λ_2⊆Λ_1. Then Λ_1^C is a $N_{e.}$(γGα)CS and $\Lambda_1^C \subseteq \Lambda_2^C \subseteq$ $N_{e.}$bcl(Λ_1^C). Therefore $\Lambda_1^C$is a $N_{e.}$(γGα)CS by claim 2.23, which implies Λ_2 is a $N_{e.}$(γGα)OS in $\chi_{\pi_{e.}}$. Hence $\Lambda_2 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Sufficiency: Let Γ is a $N_{e.}$(α)CS such that Γ⊆Λ_1 and Γ⊆$N_{e.}$bint(Λ_1). Then $(N_{e.}bint(\Lambda_1))^C \subseteq \Gamma^C$ and $\Lambda_1^C \subseteq \Gamma^C$.This implies that bcl($\Lambda_1^C$)⊆$\Gamma^C$, where Γ^C is a $N_{e.}$(α)OS. Therefore $\Lambda_1^C$is a $N_{e.}$(γGα)CS. Hence Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\pi_{e.}}$.

Claim 4.12:

Let ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$TS then for every $\Lambda_1 \in N_{e.}$bO($\chi_{\pi_{e.}}$) and for every $N_{e.}$S Λ_2 in $\chi_{\pi_{e.}}$, $\Lambda_1 \subseteq \Lambda_2 \subseteq$ cl(Λ_1) ⇒ $\Lambda_2 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Proof:

Let Λ_1 be a $N_{e.}$bOS in $\chi_{\pi_{e.}}$. Then there exists an $N_{e.}$POS, (say) Λ_3 such that $\Lambda_3 \subseteq \Lambda_1 \subseteq N_{e.}$cl($\Lambda_3$). By hypothesis, Λ_1⊆Λ_2. Therefore Λ_3⊆Λ_2. Since Λ_1⊆$N_{e.}$cl(Λ_3), $N_{e.}$cl(Λ_1)⊆ $N_{e.}$cl(Λ_3) and Λ_2⊆$N_{e.}$cl(Λ_3), by hypothesis. Therefore Λ_2 is Λ_1 $N_{e.}$bOS. As every $N_{e.}$bOS is a $N_{e.}$(γGα)OS by claim 3.2, $\Lambda_2 \in N_{e.}$(γGα)O($\chi_{\pi_{e.}}$).

Claim 4.13:

If Λ_1 is a $N_{e.}$(α)CS and a $N_{e.}$(γGα)OS in ($\chi_{\pi_{e.}}$, $N_{e.}$τ), then Λ_1 is a $N_{e.}$(γ)OS in ($\chi_{\pi_{e.}}$, $N_{e.}$τ).

Proof: Since $\Lambda_1 \subseteq \Lambda_1$ and Λ_1 is a $N_{e.}$(α)CS, by hypothesis $\Lambda_1 \subseteq$ bint(Λ_1). But bint(Λ_1) $\subseteq \Lambda_1$. Therefore bint(Λ_1) = Λ_1. Hence Λ_1 is a $N_{e.}$(γ)OS in ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ).

Claim 4.14:

Let ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}$TS. Then $N_{e.}$bO($\chi_{\text{п}_{e.}}$) = $N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$) if every $N_{e.}$S in ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}$(α)CS in $\chi_{\text{п}_{e.}}$.

Proof:

Assume that every $N_{e.}$S in ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}$(α)CS in $\chi_{\text{п}_{e.}}$. Let $\Lambda_1 \in N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$). Then Λ_1 is also an $N_{e.}$(α)CS, by hypothesis. Therefore by claim 3.15 Λ_1 is a $N_{e.}$(γ)OS. Therefore $\Lambda_1 \in N_{e.}$bO($\chi_{\text{п}_{e.}}$). Hence $N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$) $\subseteq N_{e.}$bO($\chi_{\text{п}_{e.}}$) (i) Let $\Lambda_1 \in N_{e.}$bO($\chi_{\text{п}_{e.}}$) then by claim 3.15 $\Lambda_1 \in N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$). Hence $N_{e.}$bO($\chi_{\text{п}_{e.}}$) $\subseteq N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$) (ii). Therefore from (i) and (ii) $N_{e.}$bO($\chi_{\text{п}_{e.}}$) = $N_{e.}$(γGα)O($\chi_{\text{п}_{e.}}$).

5. Theoretical implications and applications of Neutrosophic γ -generalized α -closed sets.

In this section, we explore various theoretical applications of Neutrosophic γ -generalized α -closed sets by introducing new baces and deriving several noteworthy claim s.

Definition 5.1:

If each $N_{e.}(bG\alpha)CS$ is a $N_{e.}$ Closed set space ($\chi_{\text{п}_{e.}}$,τ), then the bace is referred to as a $N_{e.}b_{g\alpha c}T_{1/2}$ base.

Definition 5.2:

A neutrosophic topological space $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ), is said to have a $b_{g\alpha b}T_{1/2}$ ($N_{e.}b_{g\alpha b}T_{1/2}$) -bace if every $b_{g\alpha b}T_{1/2}$ ($N_{e.}b_{g\alpha b}T_{1/2}$) -closed set $\in \chi_{\text{п}_{e.}}$ is also a $N_{e.}(b)$-closed set.

Illustration 5.3:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a N_e.T on $\chi_{\text{п}_{e.}}$.

Here the $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a Neutrosophic $b_{g\alpha c}T_{1/2}$bace.

Definition 5.4:

A $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) stands a Neutrosophic $b_{g\alpha P}T_{1/2}$ ($N_{e.}b_{g\alpha P}T_{1/2}$) bace if each $N_{e.}$(γGα) closed set is a $N_{e.}$(P)closed set $\in \chi_{\text{п}_{e.}}$.

Illustration 5.5:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{7}{10}, \frac{5}{10}, \frac{3}{10}\right), \left(\frac{8}{10}, \frac{5}{10}, \frac{2}{10}\right)\rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a N_e.T on $\chi_{\text{п}_{e.}}$.

. Here $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}$τ) is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace.

Claim 5.6:

Every $N_{e.}b_{g\alpha c}T_{1/2}$ bace is a $N_{e.}b_{g\alpha b}T_{1/2}$ space but not conversely in general.

Proof:

Let $\chi_{\text{п}_{e.}}$ is a $N_{e.}$ γGαcT1/2 bace. Let A is a $N_{e.}$(γGα)CS $\in \chi_{\text{п}_{e.}}$. By hypothesis, Λ_1 is a $N_{e.}$CS $\in \chi_{\text{п}_{e.}}$. Since every $N_{e.}$CS is a $N_{e.}$(γ)CS, Λ_1 is a $N_{e.}$(γ)CS $\in \chi_{\text{п}_{e.}}$. Hence $\chi_{\text{п}_{e.}}$ is a $N_{e.}b_{g\alpha b}T_{1/2}$bace.

Illustration 5.7:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a N_e.T on $\chi_{\text{п}_{e.}}$.

. Here $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace ,as every $N_{e.}$(γGα)CS is a $N_{e.}$(γ)CS in ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$), but not an $N_{e.}b_{g\alpha c}T_{1/2}$bace, as $\Lambda_1 = \langle x, \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right), \left(\frac{3}{10}, \frac{5}{10}, \frac{7}{10}\right)\rangle$ remains a Neutrosophic $(bG\alpha)$ closed set but not an Neutrosophic $(bG\alpha)$ closed set ∈ ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$).

Claim 5.8:

Every $N_{e.}b_{g\alpha P}T_{1/2}$ bace is a $N_{e.}b_{g\alpha b}T_{1/2}$bace but not conversely in general.

Proof:

Let $\chi_{\text{п}_{e.}}$ is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace and let Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\text{п}_{e.}}$. By hypothesis, Λ_1 is a $N_{e.}$(P)CS in $\chi_{\text{п}_{e.}}$. Since every $N_{e.}$(P)CS is a $N_{e.}(\gamma)$CS, Λ_1 is a $N_{e.}(\gamma)$CS in $\chi_{\text{п}_{e.}}$. Hence $\chi_{\text{п}_{e.}}$ is a $N_{e.}b_{g\alpha b}T_{1/2}$ bace.

Illustration 5.9:

Let $\chi_{\text{п}_{e.}} = \{s_1^*, s_2^*\}$, $K_1^* = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$. Then $\tau_{\text{п}_{e.}} = \{0_N, K_1^*, 1_N\}$ is a $N_{e.}$T on $\chi_{\text{п}_{e.}}$.

. Here $N_{e.}$TS ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha P}T_{1/2}$ bace ,as every $N_{e.}$(γGα)CS is a $N_{e.}$(γ)CS in ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$), but not an $N_{e.}b_{g\alpha P}T_{1/2}$ bace, as $\Lambda_1 = \langle x, \left(\frac{5}{10}, \frac{5}{10}, \frac{5}{10}\right), \left(\frac{4}{10}, \frac{5}{10}, \frac{6}{10}\right)\rangle$ is a $N_{e.}(bG\alpha)CS$ but not an $N_{e.}(P)CS$ in ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$).

Claim 5.10:

Let ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha b}T_{1/2}$bace. Then

(i) Any union of $N_{e.}$(γGα)CSs is a $N_{e.}$(γGα)CS in $\chi_{\text{п}_{e.}}$.

(ii) Any intersection of $N_{e.}$(γGα)OSs is a $N_{e.}$(γGα)OS in $\chi_{\text{п}_{e.}}$.

Proof:

(i) Let {Ai}denote a family of $N_{e.}$(γGα) closed sets within the space $\chi_{\text{п}_{e.}}$. Since ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}$ γGαbT1/2 bace, every $N_{e.}$(γGα)CS is a $N_{e.}$(γ)CS and hence each Ai , i J is a $N_{e.}$(γ)CS in ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$). But any union of Neutrosophic $-\gamma$ closed set stands a $N_{e.}(\gamma)$CS, Subsequently every one $N_{e.}(\gamma)$ closed set stands a $N_{e.}$(γGα)CS, ∪Ai stands an $N_{e.}$(γGα)CS in $\chi_{\text{п}_{e.}}$.

(ii) can be proved by taking complement in (i).

Claim 5.11: Let Λ_1 be a set that qualifies as both a $N_{e.}$OS and a $N_{e.}$(γGα)CS in $\chi_{\text{п}_{e.}}$ Within the space $\chi_{\text{п}_{e.}}$ If the space $\chi_{\text{п}_{e.}}$ satisfies the conditions of a $N_{e.}$ γGαcT1/2 , then

(i) Λ_1 must be a $N_{e.}$(R) open set in $\chi_{\text{п}_{e.}}$,

(ii) Λ_1 must be a $N_{e.}$(R) closed set $\chi_{\text{п}_{e.}}$,

(iii) Λ_1 must be a $N_{e.}$Q set in $\chi_{\text{п}_{e.}}$.

Proof: Let Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\text{п}_{e.}}$, then by Definition 4.1, Λ_1 is a $N_{e.}$CS in $\chi_{\text{п}_{e.}}$. Now (i) $N_{e.}$int($N_{e.}$cl(Λ_1)) = $N_{e.}$int(Λ_1) = Λ_1 and therefore Λ_1 is a $N_{e.}(R)OS$ in $\chi_{\text{п}_{e.}}$, (ii) $N_{e.}$cl($N_{e.}$int(Λ_1)) = $N_{e.}$cl(Λ_1) = Λ_1 and therefore Λ_1 is a $N_{e.}$(R)CS in $\chi_{\text{п}_{e.}}$ and (iii) from (i) and (ii) $N_{e.}$int($N_{e.}$cl(Λ_1)) = $N_{e.}$cl($N_{e.}$int(Λ_1)). Hence Λ_1 is a $N_{e.}$Q-set in $\chi_{\text{п}_{e.}}$.

Claim 5.12:

Let ($\chi_{\text{п}_{e.}}$, $N_{e.}\tau$) is a $N_{e.}b_{g\alpha b}T_{1/2}$bace, then the following conditions are equivalent:

(i) Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\text{п}_{e.}}$,

(ii) $\Lambda_1 \subseteq N_{e.}$cl($N_{e.}$int($N_{e.}$cl($\Lambda_1$))),

(iii) $N_{e.}$cl(Λ_1) ∈ $N_{e.}$RC($\chi_{\text{п}_{e.}}$).

Proof:

$(i) \rightarrow (ii)$ Let Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\pi_{e.}}$. Then since $\chi_{\pi_{e.}}$ is a $N_{e.}b_{g\alpha b}T_{1/2}$bace, Λ_1 is a $N_{e.}$(γ)OS in $\chi_{\pi_{e.}}$. Therefore $\Lambda_1 \subseteq N_{e.}$cl($N_{e.}$int($N_{e.}$cl($\Lambda_1$))).

$(ii) \rightarrow (iii)$ Let $\Lambda_1 \subseteq N_{e.}$ cl($N_{e.}$ int($N_{e.}$ cl(Λ_1))). Then $N_{e.}$ cl(Λ_1)$\subseteq N_{e.}$ cl($N_{e.}$ cl($N_{e.}$ int($N_{e.}$ cl(Λ_1)))) = $N_{e.}$cl($N_{e.}$int($N_{e.}$cl(Λ_1)))$\subseteq N_{e.}$cl(Λ_1).Therefore $N_{e.}$cl(Λ_1) = $N_{e.}$cl($N_{e.}$int($N_{e.}$cl(Λ_1))). Hence $N_{e.}$cl(Λ_1) $N_{e.}$RC($\chi_{\pi_{e.}}$).

$(iii) \rightarrow (i)$ Since cl(Λ_1) is a $N_{e.}$(R)CS in $\chi_{\pi_{e.}}$, $N_{e.}$cl(Λ_1)=$N_{e.}$cl($N_{e.}$int($N_{e.}$cl(Λ_1))) and since $\Lambda_1 \subseteq N_{e.}$cl($\Lambda_1$), $\Lambda_1 \subseteq N_{e.}$cl($N_{e.}$int($N_{e.}$cl($\Lambda_1$))). Therefore Λ_1 is a N(γ)OS. Hence Λ_1 is a $N_{e.}$(γGα)OS in $\chi_{\pi_{e.}}$

Claim 5.13:

Let ($\chi_{\pi_{e.}}$, $N_{e.}$τ) is a $N_{e.}$ γGαbT1/2 space, then the following conditions are equivalent:

(i) Λ_1 is a $N_{e.}$(γGα)CS in $\chi_{\pi_{e.}}$,

(ii) $N_{e.}$int($N_{e.}$cl($N_{e.}$int(Λ_1))) $\subseteq \Lambda_1$,

(iii) $N_{e.}$int(Λ_1) $\in N_{e.}$RO($\chi_{\pi_{e.}}$).

Proof: This claim can be easily proved by taking complement in claim 4.16

6. Limitations of the Study

While the study successfully introduces and generalizes the concept of Neutrosophic $\boldsymbol{\gamma}$ -generalized $\boldsymbol{\alpha}$ -closed sets, it is not without limitations. Firstly, the research is entirely theoretical and lacks practical applications or real-world data validation. The examples used are limited to small, finite Neutrosophic spaces, which may not reflect the behavior of these sets in large or complex topological systems. Secondly, no algorithmic or computational methods are developed to detect or implement these sets in applied settings. Thirdly, the study does not address the dynamic behavior of these sets under changes in the underlying topological space. Lastly, potential applications in decision-making, data analysis, or artificial intelligence are not explored, leaving the practical relevance of the proposed sets for future investigation.

7. Future Work

The proposed class of Neutrosophic $\boldsymbol{\gamma}$ -generalized α-closed sets (Ne.(γG$\boldsymbol{\alpha}$)CS opens multiple avenues for future investigation. One notable avenue is the creation of computational algorithms to detect and analyze (Ne.(γG$\boldsymbol{\alpha}$)CS in large Neutrosophic topological spaces, making the concept applicable to practical decision-making and uncertainty modeling.. Future work may also investigate dynamic Neutrosophic systems where the topology evolves over time, requiring adaptive closure properties. In addition, exploring the application of (Ne.(γG$\boldsymbol{\alpha}$)CS) in fields such as digital topology, image processing, data clustering, and granular computing could provide real-world relevance. Another direction involves studying dual concepts like Neutrosophic $\boldsymbol{\gamma}$ -generalized α-interior sets and their topological implications. Overall, the foundational structure developed in this study paves the way for further theoretical expansion and interdisciplinary applications in systems that involve incomplete, imprecise, or inconsistent information.

The comparative analysis table.1 evaluates the proposed Neutrosophic γ -generalized α-closed sets (Ne.(γ Gα)CS) alongside traditional Neutrosophic closed set types—namely α-closed, semi-closed, pre-closed, and γ -closed sets. Each class is compared based on criteria such as openness foundation, closure operator used, scope of generalization, and inclusion relationships. Traditional set types depend on specific types of open sets (α, semi, pre, γ) and corresponding closures, often with narrow generalization and limited structural relationships. In contrast, (Ne.(γGα)CS) utilizes β -closure and α-open sets, providing a unified and more flexible framework. The table confirms that (Ne.(γGα)CS) includes all traditional types as special cases, while none of the others offer similar inclusiveness. Reverse implications do not generally hold for (Ne.(γGα)CS), which is supported through counterexamples in the paper. The proposed class also demonstrates improved behavior under operations like union and intersection, which is often not preserved in other types. Furthermore, it better captures uncertainty and hybrid behavior due to its broader formulation. This enhanced expressiveness makes (Ne.(γGα)CS) more applicable to advanced modeling in uncertain topological environments. The comparison validates the generality, strength, and necessity of the proposed class within Neutrosophic topology.

Table 1: Comparison between the proposed Neutrosophic γ-generalized α-closed sets and traditional Neutrosophic closed set types

Feature	**α - Closed Sets**	**Semi-Closed Sets**	**Pre-Closed Sets**	**γ - Closed Sets**	**Proposed Ne.(γ G α)CS**
Openness Basis	α-open sets	Semi-open sets	Pre-open sets	γ -open sets	α-open sets
Closure Type Used	α-closure or identity	Semi-closure	Pre-closure	γ -closure	β -closure (broader)
Defined via	Inclusion via α-open set	Superset's semi-open relation	Pre-open neighborhood inclusion	γ -open neighborhood containment	β -closure inclusion inside α-open sets
Scope of Generalization	Narrow	Moderate	Moderate	Broader than α	Broadest – generalizes all
Inclusion of Other Sets	Does not include others	Does not include others	Does not include others	Partial inclusion of α and semi	Includes α, semi, pre, and γ as special cases
Reverse Implication	May hold in special cases	Not always true	Often fails	Rarely holds	Proven false via counterexamples
Support for Hybrid Behavior	Limited	Limited	Limited	Partial	High – designed for uncertain overlap
Behavior under Union/Intersection	Not always closed	Not preserved	Not preserved	Sometimes preserved	Analyzed in claims; flexible

Expressiveness under Uncertainty	Low	Moderate	Moderate	Moderate	High – handles mixed/indeterminate membership
Application Readiness	Theoretical	Theoretical	Theoretical	Theoretical	Theoretical; open for future applications

8. Conclusion

In this articles, introduced and examined a new class of sets in Neutrosophic topology, namely Neutrosophic γ -generalized α-closed sets (γ GS-closed sets) and their counterparts, Neutrosophic γ -generalized α-open sets (γ GS-open sets). These sets represent a meaningful generalization of existing Neutrosophic closed and open set concepts, enriching the structural framework of Neutrosophic topological spaces. We have discussed several foundational properties of these sets and explored their relationships with previously established classes of Neutrosophic sets, highlighting their uniqueness and broader applicability.

The results obtained in this work not only contribute to the theoretical development of Neutrosophic topology but also pave the way for further generalizations and refinements. Future research could focus on extending these sets under different topological operators, examining their behavior in product spaces, and exploring their role in Neutrosophic continuity, compactness, and separation axioms. Additionally, potential applications in fields dealing with uncertainty, such as decision-making, data analysis, and artificial intelligence, can be explored by leveraging the flexible nature of γ GS-closed and γ GS-open sets. This work thus lays a solid foundation for advancing both theoretical investigations and practical applications within the broader domain of Neutrosophic mathematics.

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Data Availability

The present work is wholly theoretical, with no inclusion of data gathering or analytical evaluation. Prospective researchers are invited to conduct empirical research to further investigate and substantiate the ideas outlined in this paper.

Ethical Approval

Ethical approval is not required, as this purely theoretical research involves no animal subjects or human participants.

Conflicts of Interest

It is affirmed by the authors that this research and its publication involve no conflicts of interest.

Disclaimer

This work proposes untested theoretical concepts, inviting future empirical validation. While accuracy and proper citation have been prioritized, inadvertent errors may exist, and readers should verify sources. The views expressed are solely the authors' and not necessarily those of their institutions.

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Neutrosophic Hyper KU-Ideals

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Abstract. We define neutrosophic hyper KU-ideal (strong, weak, s-weak) and refelexive neutrosophic hyper KU-ideal. A few key properties and their relationships are highlighted. The study of the neutrosophic (weak) hyper KU-ideal is considered. The conditions for a neutrosophic set to be a $NS\mathbb{H}KUI$ and a (reflexive) neutrosophic hyper KU-ideal are discussed. There are also circumstances for a $N\mathbb{WH}KUI$ to be a $Ns\mathbb{WH}KUI$, as well as conditions for a $NS\mathbb{H}KUI$ to be a $RN\mathbb{H}KUI$.

Keywords: Hyper KU-algebra; hyper KU-ideals; $N\mathbb{H}KUI$; $NS\mathbb{H}KUI$; $N\mathbb{WH}KUI$; $Ns\mathbb{WH}KUI$; $RN\mathbb{H}KUI$.

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1. Introduction

Prabpayak and Leerawat created a novel algebraic structure known as KU-algebras [12, 13]. In KU-algebras, they worked at ideals and congruences. They also established the concept of KU-algebra homomorphism and looked into various related features. They also deduced some simple consequences of the quotient KU-algebras and isomorphism relationships. Marty [9] presented the hyper structure theory (also known as multialgebras) at the 8th Congress of Scandinvian Mathematiciens in 1934. Several authors, primarily in France and the United States, but also in Italy, Russia, and Japan, worked on hyper groups in the 1940's. Hyperstructures offer a wide range of applications in both pure and applied sciences. Jun et al. extended hyper structures to BCK-algebras, proposed the idea of a hyper BCK-algebra, which is a generalization of a BCK-algebra, and looked into several associated characteristics in [8]. They also defined a hyper BCK-ideal and a weak hyper BCK-ideal, as well as relationships between hyper BCK-ideals and weak hyper BCK-ideals. Jun et al. [7] proposed the notions of a strong hyper BCK-ideal, a weak hyper BCK-ideal, and a reflexive hyper BCK-ideal, as well as a requirement for a hyper BCK-algebra to be a BCK-algebra. Every strong hyper

BCK-ideal is a hyper sub-algebra, a weak hyper BCK-ideal, and a hyper BCK-ideal, and every reflexive hyper BCK-ideal is a strong hyper BCK-ideal, they established.

Smarandache [14–16] developed the neutrosophic set, which is a more general platform that extends the notions of classic set, (intuitionistic) fuzzy set, and interval valued (intuitionistic) fuzzy set. On BL-algebras, Borzooei et al. [4] investigated neutrosophic deductive filters. Zhang et al. [19] discussed neutrosophic regular filters and fuzzy regular filters when applying the concept of neutrosophic set to pseudo-BCI algebras. Neutrosophic set theory has been applied to a variety of areas, and many studies have been conducted to develop, improve, and expand the theory ([1–3, 5, 6, 10, 17] and [18]).

The goal of this paper is to introduce the concepts of neutrosophic (strong, weak, s-weak) hyper KU-ideal, as well as $RN\mathbb{H}KUI$. We look at their connections and properties. Characterizations of neutrosophic (weak) hyper KU-ideal are discussed. We define exactly for a neutrosophic set to be a $NS\mathbb{H}KUI$ and a (reflexive) neutrosophic hyper KU-ideal. We're looking for some provisions that will allow a $NS\mathbb{H}KUI$ to become a $RN\mathbb{H}KUI$. We go over the conditions for a $N\mathbb{W}\mathbb{H}KUI$ to be a $Ns\mathbb{W}\mathbb{H}KUI$.

2. **Preliminaries**

The basic definitions of hyper KU-ideals and neutrosophic set are given in this section.

Let H be a non-empty set and let "$\circ$" be a mapping

$$\circ : H \times H \to P(H) \backslash \{\emptyset\}$$

which is said to be hyperoperation. For any two subsets A and B, denote by $A \circ B$, the set $\cup \{l_{01} \circ l_{02} | l_{01} \in A, l_{02} \in B\}$. We shall use $l_{01} \circ l_{02}$ instead of $\{l_{01}\} \circ l_{02}, l_{01} \circ \{l_{02}\}$ or $\{l_{01}\} \circ \{l_{02}\}$.

By a hyper KU-algebra H ([11]), we mean a non-empty set H with a special element 0 and a hyperoperation $\circ$, for all $l_{01}, l_{02}, l_{03} \in H$, that satisfies the following axioms:

(HKU1) $(l_{02} \circ l_{03}) \circ (l_{01} \circ l_{03}) \ll l_{01} \circ l_{02}$,

(HKU2) $l_{01} \circ 0 = \{0\}$,

(HKU3) $0 \circ l_{01} = \{l_{01}\}$,

(HKU4) if $l_{01} \ll l_{02}$ and $l_{02} \ll l_{01}$ imply $l_{01} = l_{02}$, for all $l_{01}, l_{02}, l_{03} \in H$, where $l_{01} \ll l_{02}$ is defined by $0 \in l_{02} \circ l_{01}$ and for any $A, B \subseteq H$, $A \ll B$ is defined by $\forall\, r \in A$, $\exists\, t \in B$ such that $r \ll t$.

Proposition 2.1. [11] Let H be a hyper KU-algebra. Then for all $l_{01}, l_{02}, l_{03} \in H$, the following statements hold:

(i) $A \subseteq B$ implies $A \ll B$, for all nonempty subsets A, B of H.

(ii) $0 \circ 0 = \{0\}$.

(iii) $0 \ll l_{01}$.

(iv) $l_{03} \ll l_{03}$.

(v) $l_{01} \circ l_{03} \ll l_{03}$.

(vi) $A \circ 0 = \{0\}$.

(vii) $0 \circ A = A$.

(viii) $(0 \circ 0) \circ l_{01} = \{l_{01}\}$ and $(l_{01} \circ (0 \circ l_{01})) = \{0\}$.

(ix) $l_{01} \circ l_{01} = \{l_{01}\} \Leftrightarrow l_{01} = 0$.

(x) $l_{03} \circ (l_{02} \circ l_{01}) = l_{02} \circ (l_{03} \circ l_{01})$ for all $l_{01}, l_{02}, l_{03} \in H$.

Definition 2.2. [11] Let $(H, \circ)$ be a hyper KU-algebra. A subset A of H is called:

- A hyper KU-ideal (briefly, $\mathbb{H}KUI$) of H if

(1) $0 \in A$,

(2) $l_{02} \circ l_{01} \ll A$ and $l_{02} \in A$ imply $l_{01} \in A$, forall $l_{01}, l_{02} \in H$.

- A weak hyper KU-ideal (briefly, $\mathbb{WH}KUI$) of H if it satisfies (1) and

(3) $l_{02} \circ l_{01} \subseteq A$, $l_{02} \in A \Rightarrow l_{01} \in A$, $\forall\, l_{01}, l_{02} \in H$,

- A strong hyper KU-ideal (briefly, $\mathbb{SH}KUI$) of H if it satisfies (1) and

(4) $(l_{02} \circ l_{01}) \cap A \neq \emptyset$, $l_{02} \in A \Rightarrow l_{01} \in A$, $\forall\, l_{01}, l_{02} \in H$,

A subset I of a hyper KU-algebra H is said to be reflexive if $(l \circ l) \subseteq I$ for all $l \in H$.

Let H be a non-empty set. A neutrosophic set (NS) in H (See [16]) is a structure of the form:

$$L := \{\langle l; L_T(l), L_I(l), L_F(l)\rangle \mid l \in H\}$$

where $L_T : H \to [0, 1]$ is a truth membership function, $L_I : H \to [0, 1]$ is an indeterminate membership function, and $L_F : H \to [0, 1]$ is a false membership function. For abbreviation, we continue to write $L = (L_T, L_I, L_F)$ for the NS

$$L := \{\langle l; L_T(l), L_I(l), L_F(l)\rangle \mid l \in H\}.$$

Given a NS $L = (L_T, L_I, L_F)$ in a hyper KU-algebra H and a subset V of H, by ${}_*L_T, {}^*L_T, {}_*L_I, {}^*L_I, {}_*L_F$ and ${}^*L_F$ we mean

$$\begin{aligned} {}_*L_T(V) &= \inf_{v \in V} L_T(v) \text{ and } {}^*L_T(V) = \sup_{v \in V} L_T(v), \\ {}_*L_I(V) &= \inf_{v \in V} L_I(v) \text{ and } {}^*L_I(V) = \sup_{v \in V} L_I(v), \\ {}_*L_F(V) &= \inf_{v \in V} L_F(v) \text{ and } {}^*L_F(V) = \sup_{v \in V} L_F(v), \end{aligned}$$

respectively.

Notation. From now on, in this paper, we assume that H is a hyper KU-algebra.

3. Neutrosophic hyper KU-ideals

We discussed the features of neutrosophic (strong, weak, s-weak) hyper KU-ideal and reflexive neutrosophic hyper KU-ideal in this part.

Definition 3.1. Let $L = (L_T, L_I, L_F)$ be a NS in H. Then L is said to be a neutrosophic hyper KU-ideal (briefly, $N\mathbb{H}KUI$) of H if it satisfies the following assertions for all $l_{01}, l_{02} \in H$,

$$\left(l_{01} \ll l_{02} \Rightarrow \begin{cases} L_T(l_{01}) \geq L_T(l_{02}) \\ L_I(l_{01}) \geq L_I(l_{02}) \\ L_F(l_{01}) \leq L_F(l_{02}) \end{cases} \right), \tag{1}$$

$$\begin{pmatrix} L_T(l_{01}) \geq \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \\ L_I(l_{01}) \geq \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \\ L_F(l_{01}) \leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \end{pmatrix} \tag{2}$$

Example 3.2. Let $H = \{l_0, l_a, l_b\}$ be a hyper KU-algebra. The hyper operation "$\circ$" on H described by Table 1.

Table 1 : Cayley table for the binary operation "$\circ$"

$\circ$	l_0	l_a	l_b
l_0	$\{l_0\}$	$\{l_a\}$	$\{l_b\}$
l_a	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_a, l_b\}$
l_b	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_0, l_a, l_b\}$

We define a NS $L = (L_T, L_I, L_F)$ on H by Table 2.

Table 2 : Tabular representation of $L = (L_T, L_I, L_F)$

H	$L_T(l)$	$L_I(l)$	$L_F(l)$
l_0	0.77	0.65	0.08
l_a	0.55	0.47	0.57
l_b	0.11	0.27	0.69

It is easy to check that $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H.

Proposition 3.3. For any $N\mathbb{H}KUI$ $L = (L_T, L_I, L_F)$ of H, the following assertions are valid.

(i) $L = (L_T, L_I, L_F)$ satisfies

$$(\forall\ l_{01} \in H) \begin{pmatrix} L_T(0) \geq & L_T(l_{01}) \\ L_I(0) \geq & L_I(l_{01}) \\ L_F(0) \leq & L_F(l_{01}) \end{pmatrix}. \tag{3}$$

(ii) If $L = (L_T, L_I, L_F)$ satisfies

$$(\forall\ V \subseteq H)(\exists\ u, v, w \in V) \begin{pmatrix} L_T(u) = & {}_*L_T(V) \\ L_I(v) = & {}_*L_I(V) \\ L_F(w) = & {}^*L_F(V) \end{pmatrix}, \tag{4}$$

then the following assertion is valid.

$$(\forall\ l_{01}, l_{02} \in H)(\exists\ u, v, w \in l_{02} \circ l_{01}) \begin{pmatrix} L_T(l_{01}) \geq \min\{L_T(u), L_T(l_{02})\} \\ L_I(l_{01}) \geq \min\{L_I(v), L_I(l_{02})\} \\ L_F(l_{01}) \leq \max\{L_F(w), L_F(l_{02})\} \end{pmatrix}. \tag{5}$$

Proof. By Proposition 2.1 (ii) and (1) we have

$$L_T(0) \geq L_T(l_{01}), L_I(0) \geq L_I(l_{01}) \text{ and } L_F(0) \leq L_F(l_{01}).$$

Assume that $L = (L_T, L_I, L_F)$ satisfies the condition (4). For all $l_{01}, l_{02} \in H$, there exists $u_0, v_0, w_0 \in l_{02} \circ l_{01}$ such that

$$L_T(u_0) = {}_*L_T(l_{02} \circ l_{01}), L_I(v_0) = {}_*L_I(l_{02} \circ l_{01}) \text{ and } L_F(w_0) = {}^*L_F(l_{02} \circ l_{01}).$$

Now condition (2) implies that

$$L_T(l_{01}) \geq \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \min\{L_T(u_0), L_T(l_{02})\}$$
$$L_I(l_{01}) \geq \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \min\{L_I(v_0), L_I(l_{02})\}.$$
$$L_F(l_{01}) \leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \max\{L_F(w_0), L_F(l_{02})\}$$

This completes the proof. Ξ

We define the following sets:

$$U(L_T, \xi_T) := \{l_{01} \in H \mid L_T(l_{01}) \geq \xi_T\},$$
$$U(L_I, \xi_I) := \{l_{01} \in H \mid L_I(l_{01}) \geq \xi_I\},$$
$$L(L_F, \xi_F) := \{l_{01} \in H \mid L_F(l_{01}) \leq \xi_F\},$$

where $L = (L_T, L_I, L_F)$ is a NS in H and $\xi_T, \xi_I, \xi_F \in [0, 1]$.

Lemma 3.4. Let L be a subset of H. If I is a $\mathbb{H}KUI$ of H such that $L \ll I$, then L is contained in I.

Proof. Assume that $L \ll H$ and let $l \in L$. Then $0 \circ l = \{l\} \ll H$ and so $l \in H$ by using Definition 2.2 (2). Therefore $L \subseteq H$. Ξ

Theorem 3.5. A NS $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H iff the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $\mathbb{H}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$.

Proof. Assume that $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H and suppose that $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are nonempty for all $\xi_T, \xi_I, \xi_F \in [0, 1]$. It is easy to see that $0 \in U(L_T, \xi_T), 0 \in U(L_I, \xi_I)$ and $0 \in L(L_F, \xi_F)$. Let $l_{01}, l_{02} \in H$ be such that $l_{02} \circ l_{01} \ll U(L_T, \xi_T)$ and $l_{02} \in U(L_T, \xi_T)$. Then $L_T(l_{02}) \geq \xi_T$ and for any $l \in l_{02} \circ l_{01}$ there exists $l_0 \in U(L_T, \xi_T)$ such that $l \ll l_0$. We conclude from (1) that $L_T(l) \geq L_T(l_0) \geq \xi_T$ for all $l \in l_{02} \circ l_{01}$. Hence ${}_*L_T(l_{02} \circ l_{01}) \geq \xi_T$, and so

$$L_T(l_{01}) \geq \min\{{}_*L_T(l_{02} \circ l_{01}),\ L_T(l_{02})\} \geq \xi_T,$$

that is, $l_{01} \in U(L_T, \xi_T)$. Similarly, we show that if $l_{02} \circ l_{01} \ll U(L_I, \xi_I)$ and $l_{02} \in U(L_I, \xi_I)$, then $l_{01} \in U(L_I, \xi_I)$. Hence $U(L_T, \xi_T)$ and $U(L_I, \xi_I)$ are $\mathbb{H}KUI$'s of H. Let $l_{01}, l_{02} \in H$ be such that

$l_{02} \circ l_{01} \ll L(L_F, \xi_F)$ and $l_{02} \in L(L_F, \xi_F)$. Then $L_F(l_{02}) \leq \xi_F$. Let $m \in l_{02} \circ l_{01}$. Then there exists $m_0 \in L(L_F, \xi_F)$ such that $m \ll m_0$, which implies from (1) that $L_F(m) \leq L_F(m_0) \leq \xi_F$. Thus ${}^*L_F(l_{02} \circ l_{01}) \leq \xi_F$, and so

$$L_F(l_{01}) \leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \leq \xi_F.$$

Hence $l_{01} \in L(L_F, \xi_F)$ and therefore $L(L_F, \xi_F)$ is a $\mathbb{H}KUI$ of H.

Conversely, suppose that the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $\mathbb{H}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$. Let $l_{01}, l_{02} \in H$ be such that $l_{01} \ll l_{02}$. Then

$$l_{02} \in U(L_T, L_T(l_{02})) \cap U(L_I, L_I(l_{02})) \cap L(L_F, L_F(l_{02})),$$

and thus $l_{01} \ll U(L_T, L_T(l_{02})), l_{01} \ll U(L_I, L_I(l_{02}))$ and $l_{01} \ll L(L_F, L_F(l_{02}))$. According to Lemma 3.4, we have $l_{01} \in U(L_T, L_T(l_{02})), l_{01} \in U(L_I, L_I(l_{02}))$ and $l_{01} \in L(L_F, L_F(l_{02}))$ which imply that $L_T(l_{01}) \geq L_T(l_{02})$, $L_I(l_{01}) \geq L_I(l_{02})$ and $L_F(l_{01}) \leq L_F(l_{02})$. For any $l_{01}, l_{02} \in H$, let $\xi_T := \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\}$, $\xi_I := \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$ and $\xi_F := \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}$. Then

$$l_{02} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F),$$

and for each $u_T, v_I, w_F \in l_{02} \circ l_{01}$ we have

$$L_T(u_T) \geq {}_*L_T(l_{02} \circ l_{01}) \geq \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \xi_T,$$

$$L_I(v_I) \geq {}_*L_I(l_{02} \circ l_{01}) \geq \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \xi_I$$

and

$$L_F(w_F) \leq {}^*L_F(l_{02} \circ l_{01}) \leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \xi_F.$$

Hence $u_T \in U(L_T, \xi_T), v_I \in U(L_I, \xi_I)$ and $w_F \in L(L_F, \xi_F)$ and so $l_{02} \circ l_{01} \subseteq U(L_T, \xi_T), l_{02} \circ l_{01} \subseteq U(L_I, \xi_I)$ and $l_{02} \circ l_{01} \subseteq L(L_F, \xi_F)$. By Proposition 2.1, we have $l_{02} \circ l_{01} \ll U(L_T, \xi_T), l_{02} \circ l_{01} \ll U(L_I, \xi_I)$ and $l_{02} \circ l_{01} \ll L(L_F, \xi_F)$. It follows from Definition 2.2 (2) that

$$l_{01} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F).$$

Hence

$$L_T(l_{01}) \geq \xi_T = \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \xi_I = \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$$

and

$$L_F(l_{01}) \leq \xi_F = \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Therefore $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H. Ξ

Theorem 3.6. If $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H, then the set

$$J := \{l_{01} \in H \mid L_T(l_{01}) = L_T(0), L_I(l_{01}) = L_I(0), L_F(l_{01}) = L_F(0)\} \tag{6}$$

is a $\mathbb{H}KUI$ of H.

Proof. It is easy to check that $0 \in J$. Let $l_{01}, l_{02} \in H$ be such that $l_{02} \circ l_{01} \ll J$ and $l_{02} \in J$. Then $L_T(l_{02}) = L_T(0), L_I(l_{02}) = L_I(0)$ and $L_F(l_{02}) = L_F(0)$. Let $l \in l_{02} \circ l_{01}$. Then there exists $l_0 \in J$ such that $l \ll l_0$, and thus by (1), $L_T(l) \geq L_T(l_0) = L_T(0), L_I(l) \geq L_I(l_0) = L_I(0)$ and $L_F(l) \leq L_F(l_0) = L_F(0)$. It follows from (2) that

$$L_T(l_{01}) \geq \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \geq L_T(0),$$

$$L_I(l_{01}) \geq \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \geq L_I(0)$$

and

$$L_F(l_{01}) \leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \leq L_F(0).$$

Hence $L_T(l_{01}) = L_T(0), L_I(l_{01}) = L_I(0)$ and $L_F(l_{01}) = L_F(0)$, that is, $l_{01} \in J$. Therefore J is a $\mathbb{H}KUI$ of H. Ξ

We define the situation under which a NS $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H.

Theorem 3.7. Let H satisfy $|l_{02} \circ l_{01}| < \infty$ for all $l_{01}, l_{02} \in H$, and let $\{J_\beta \mid \beta \in \Lambda \subseteq [0, 0.5]\}$ be a collection of $\mathbb{H}KUI$'s of H such that

$$H = \bigcup_{\beta \in \Lambda} J_\beta, \tag{7}$$

$$(\forall\ \alpha, \beta \in \Lambda)(\alpha > \beta \Leftrightarrow J_\alpha \subset J_\beta). \tag{8}$$

Then a NS $L = (L_T, L_I, L_F)$ in H defined by

$$L_T : H \to [0, 1], l_{01} \mapsto \sup\{\beta \in \Lambda \mid l_{01} \in J_\beta\},$$

$$L_I : H \to [0, 1], l_{01} \mapsto \sup\{\beta \in \Lambda \mid l_{01} \in J_\beta\},$$

$$L_F : H \to [0, 1], l_{01} \mapsto \inf\{\beta \in \Lambda \mid l_{01} \in J_\beta\}$$

is a $N\mathbb{H}KUI$ of H.

Proof. We first shows that

$$\rho \in [0, 1] \Rightarrow \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta \ isa\ \mathbb{H}KUI\ of\ H. \tag{9}$$

It is clear that $0 \in \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$ for all $\rho \in [0, 1]$. Let $l_{01}, l_{02} \in H$ be such that $l_{02} \circ l_{01} = \{l_1, l_2, \cdots, l_n\}$, $l_{02} \circ l_{01} \ll \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$ and $l_{02} \in \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$. Then $l_{02} \in J_\gamma$ for some $\gamma \in \Lambda$ with $\rho \leq \gamma$, and for any $l_i \in l_{02} \circ l_{01}$ there exists $m_i \in \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$, and so $m_i \in J_{\beta_i}$ for some $\beta_i \in \Lambda$ with $\rho \leq \beta_i$, such that $l_i \ll m_i$. If we let $\beta := \min\{\beta_i \mid i \in \{1, 2, \cdots, n\}\}$ then $J_{\beta_i} \subset J_\beta$ for all $i \in \{1, 2, \cdots, n\}$ and so $l_{02} \circ l_{01} \ll J_\beta$ with $\rho \leq \beta$. We may assume that $\gamma > \beta$ without loss of generality, and so $J_\gamma \subset J_\beta$. By Definition 2.2 (2), we have $l_{01} \in J_\beta \subset \bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$. Hence $\bigcup_{\delta \in \Lambda, \delta \geq \rho} J_\delta$ is a $\mathbb{H}KUI$ of H. Next, we consider the following two cases:

$$\text{(i) } \beta = \sup\{\rho \in \Lambda \mid \rho < \beta\}, \text{ (ii) } \beta \neq \sup\{\rho \in \Lambda \mid \rho < \beta\}. \tag{10}$$

If the first case is valid, then

$$l_{01} \in U(L_T, \beta) \Leftrightarrow l_{01} \in J_\rho \text{ for all } \rho < \beta \Leftrightarrow l_{01} \in \bigcap_{\rho < \beta} J_\rho,$$

and so $U(L_T, \beta) = \bigcap_{\rho < \beta} J_\rho$ which is a $\mathbb{H}KUI$ of H. Similarly, we know that $U(L_I, \beta)$ is a $\mathbb{H}KUI$ of H. For the second case, we will show that $U(L_T, \beta) = \bigcup_{\rho \geq \beta} J_\rho$. If $l_{01} \in \bigcup_{\rho \geq \beta} J_\rho$, then $l_{01} \in J_\rho$ for some $\rho \geq \beta$. Thus $L_T(l_{01}) \geq \rho \geq \beta$, and so $l_{01} \in U(L_T, \beta)$ which shows that $\bigcup_{\rho \geq \beta} J_\rho \subseteq U(L_T, \beta)$. Assume that $l_{01} \notin \bigcup_{\rho \geq \beta} J_\rho$. Then $l_{01} \notin J_\rho$ for all $\rho \geq \beta$, and so there exist $\delta > 0$ such that $(\beta - \delta, \beta) \cap \Lambda = \emptyset$. Thus $l_{01} \notin J_\rho$ for all $\rho > \beta - \delta$, that is, if $l_{01} \in J_\rho$ then $\rho \leq \beta - \delta < \beta$. Hence $l_{01} \notin U(L_T, \beta)$. This shows that $U(L_T, \beta) = \bigcup_{\rho \geq \beta} J_\rho$ which is a $\mathbb{H}KUI$ of H by (9). Similarly we can prove that $U(L_I, \beta)$ is a $\mathbb{H}KUI$ of H. Now we consider the following two cases:

$$\alpha = \inf\{\gamma \in \Lambda \mid \alpha < \gamma\} \text{ and } \alpha \neq \inf\{\gamma \in \Lambda \mid \alpha < \gamma\}. \tag{11}$$

The first case implies that

$$l_{01} \in L(L_F, \alpha) \Leftrightarrow l_{01} \in J_\gamma \text{ for all } \alpha < \gamma \Leftrightarrow l_{01} \in \bigcap_{\alpha < \gamma} J_\gamma,$$

and so $L(L_F, \alpha) = \bigcap_{\alpha < \gamma} J_\gamma$ which is a $\mathbb{H}KUI$ of H. For the second case, there exists $\delta > 0$ such that $(\alpha, \alpha + \delta) \cap \Lambda = \emptyset$. If $l_{01} \in \bigcup_{\alpha \geq \gamma} J_\gamma$, then $l_{01} \in J_\gamma$ for some $\alpha \geq \gamma$. Thus $L_F(l_{01}) \leq \gamma \leq \alpha$, that is, $l_{01} \in L(L_F, \alpha)$. Hence $\bigcup_{\alpha \geq \gamma} J_\gamma \subseteq L(L_F, \alpha)$. If $l_{01} \notin \bigcup_{\alpha \geq \gamma} J_\gamma$, then $l_{01} \notin J_\gamma$ for all $\gamma \leq \alpha$ and thus $l_{01} \notin J_\gamma$ for all $a\gamma < \alpha + \delta$. This shows that if $l_{01} \in J_\gamma$ then $\gamma \geq \alpha + \delta$. Hence $L_F(l_{01}) \geq \alpha + \delta > \alpha$, i.e., $l_{01} \notin L(L_F, \alpha)$.

Therefore $L(L_F, \alpha) \subseteq \bigcup_{\alpha \geq \gamma} J_\gamma$. Consequently, $L(L_F, \alpha) = \bigcup_{\alpha \geq \gamma} J_\gamma$ which is a $\mathbb{H}KUI$ of H by (9). It follows from Theorem 3.5 that $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H. Ξ

Definition 3.8. A NS $L = (L_T, L_I, L_F)$ in H is called a neutrosophic strong hyper KU-ideal (briefly, $NS\mathbb{H}KUI$) of H if it satisfies the following assertions.

$$\begin{aligned} {}_*L_T(l_{01} \circ l_{01}) \geq L_T(l_{01}) \geq \min\{\sup_{u_0 \in l_{02} \circ l_{01}} L_T(u_0), L_T(l_{02})\}, \\ {}_*L_I(l_{01} \circ l_{01}) \geq L_I(l_{01}) \geq \min\{\sup_{v_0 \in l_{02} \circ l_{01}} L_I(v_0), L_I(l_{02})\} \\ {}^*L_F(l_{01} \circ l_{01}) \leq L_F(l_{01}) \leq \max\{\inf_{w_0 \in l_{02} \circ l_{01}} L_F(w_0), L_F(l_{02})\} \end{aligned} \tag{12}$$

for all $l_{01}, l_{02} \in H$.

Example 3.9. Consider a hyper KU-algebra $H = \{l_0, l_a, l_b\}$ with the hyper operation "$\circ$" which is given by Table 3.

Table 3 : Cayley table for the binary operation "$\circ$"

$\circ$	l_0	l_a	l_b
l_0	$\{l_0\}$	$\{l_a\}$	$\{l_b\}$
l_a	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_b\}$
l_b	$\{l_0\}$	$\{l_b\}$	$\{l_0, l_b\}$

Let $L = (L_T, L_I, L_F)$ be a NS in H which is described in Table 4.

Table 4 : Tabular representation of $L = (L_T, L_I, L_F)$

H	$L_T(l)$	$L_I(l)$	$L_F(l)$
l_0	0.77	0.65	0.08
l_a	0.55	0.47	0.57
l_b	0.11	0.27	0.69

It is routine to verify that $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H.

Theorem 3.10. For any $NS\mathbb{H}KUI$ $L = (L_T, L_I, L_F)$ of H, the following assertions are valid.

(i) $L = (L_T, L_I, L_F)$ satisfies the conditions (1) and (3).

(ii) $L = (L_T, L_I, L_F)$ satisfies

$$(\forall\, l_{01}, l_{02} \in H)(\forall\, u, v, w \in l_{02} \circ l_{01}) \begin{pmatrix} L_T(l_{01}) \geq \min\{L_T(u), L_T(l_{02})\} \\ L_I(l_{01}) \geq \min\{L_I(v), L_I(l_{02})\} \\ L_F(l_{01}) \leq \max\{L_F(w), L_F(l_{02})\} \end{pmatrix}. \tag{13}$$

Proof. (i) Since $l_{01} \ll l_{01}$, i.e., $0 \in l_{01} \circ l_{01}$ for all $l_{01} \in H$, we get

$$L_T(0) \geq {}_*L_T(l_{01} \circ l_{01}) \geq L_T(l_{01}),$$

$$L_I(0) \geq {}_*L_I(l_{01} \circ l_{01}) \geq L_I(l_{01}),$$

$$L_F(0) \leq {}^*L_F(l_{01} \circ l_{01}) \leq L_F(l_{01}),$$

which shows that (3) is valid. Let $l_{01}, l_{02} \in H$ be such that $l_{01} \ll l_{02}$. Then $0 \in l_{02} \circ l_{01}$, and so

$${}^*L_T(l_{02} \circ l_{01}) \geq L_T(0), {}^*L_I(l_{02} \circ l_{01}) \geq L_I(0) \text{ and } {}_*L_F(l_{02} \circ l_{01}) \leq L_F(0).$$

It follows from (3) that

$$L_T(l_{01}) \geq \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \geq \min\{L_T(0), L_T(l_{02})\} = L_T(l_{02}),$$

$$L_I(l_{01}) \geq \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \geq \min\{L_I(0), L_I(l_{02})\} = L_I(l_{02}),$$

$$L_F(l_{01}) \leq \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \leq \max\{L_F(0), L_F(l_{02})\} = L_F(l_{02}).$$

Hence $L = (L_T, L_I, L_F)$ satisfies the condition (1).

(ii) Let $l_{01}, l_{02}, u, v, w \in H$ be such that $u, v, w \in l_{02} \circ l_{01}$. Then

$$L_T(l_{01}) \geq \min\{\sup_{u_0 \in l_{02} \circ l_{01}} L_T(u_0), L_T(l_{02})\} \geq \min\{L_T(u), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \min\{\sup_{v_0 \in l_{02} \circ l_{01}} L_I(v_0), L_I(l_{02})\} \geq \min\{L_I(v), L_I(l_{02})\},$$

$$L_F(l_{01}) \leq \max\{\inf_{c_0 \in l_{02} \circ l_{01}} L_F(w_0), L_F(l_{02})\} \leq \max\{L_F(w), L_F(l_{02})\}.$$

This completes the proof. Ξ

Theorem 3.11. If a NS $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H, then the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $S\mathbb{H}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$.

Proof. Let $L = (L_T, L_I, L_F)$ be a $NS\mathbb{H}KUI$ of H. Then $L = (L_T, L_I, L_F)$ is a $N\mathbb{H}KUI$ of H. Assume that $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are nonempty for all $\xi_T, \xi_I, \xi_F \in [0, 1]$. Then there exist $a \in U(L_T, \xi_T), b \in U(L_I, \xi_I)$ and $c \in L(L_F, \xi_F)$, that is, $L_T(a) \geq \xi_T, L_I(b) \geq \xi_I$ and $L_F(c) \leq \xi_F$. It follows from (3) that $L_T(0) \geq L_T(a) \geq \xi_T, L_I(0) \geq L_I(b) \geq \xi_I$ and $L_F(0) \leq L_F(c) \leq \xi_F$. Hence

$$0 \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F).$$

Let $l_{01}, l_{02}, a, b, u, v \in H$ be such that $(l_{02} \circ l_{01}) \cap U(L_T, \xi_T) \neq \emptyset$, $l_{02} \in U(L_T, \xi_T)$, $(b \circ a) \cap U(L_I, \xi_I) \neq \emptyset$, $b \in U(L_I, \xi_I)$, $(v \circ u) \cap L(L_F, \xi_F) \neq \emptyset$ and $v \in L(L_F, \xi_F)$. Then there exist $x_0 \in (l_{02} \circ l_{01}) \cap U(L_T, \xi_T)$, $a_0 \in (b \circ a) \cap U(L_I, \xi_I)$ and $u_0 \in (v \circ u) \cap L(L_F, \xi_F)$. It follows that

$$L_T(l_{01}) \geq \min\{\sup_{c \in l_{02} \circ l_{01}} L_T(c), L_T(l_{02})\} \geq \min\{L_T(x_0), L_T(l_{02})\} \geq \xi_T,$$

$$L_I(a) \geq \min\{\sup_{d \in b \circ a} L_I(d), L_I(b)\} \geq \min\{L_I(a_0), L_I(b)\} \geq \xi_I$$

and

$$L_F(u) \leq \max\{\inf_{e \in v \circ u} L_F(e), L_F(v)\} \leq \max\{L_F(u_0), L_F(v)\} \leq \xi_F.$$

Hence $l_{01} \in U(L_T, \xi_T), a \in U(L_I, \xi_I)$ and $u \in L(L_F, \xi_F)$. Therefore $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $S\mathbb{H}KUI$ of H. Ξ

Theorem 3.12. For any NS $L = (L_T, L_I, L_F)$ in H satisfying the condition

$$(\forall\ S \subseteq H)(\exists\ u, v, w \in S) \begin{pmatrix} L_T(u) = & {}^*L_T(V) \\ L_I(v) = & {}^*L_I(V) \\ L_F(w) = & {}_*L_F(V) \end{pmatrix}, \tag{14}$$

if the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $S\mathbb{H}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$, then $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H.

Proof. Assume that $U(L_T,\xi_T), U(L_I,\xi_I)$ and $L(L_F,\xi_F)$ are nonempty and $S\mathbb{H}KUI$'s of H for all $\xi_T,\xi_I,\xi_F \in [0,1]$. For any $l_{01}, l_{02}, l_{03} \in H$, such that $l_{01} \in U(L_T, L_T(l_{01})), l_{02} \in U(L_I, L_I(l_{02}))$ and $l_{03} \in L(L_F, L_F(l_{03}))$, since $l_{01} \circ l_{01} \ll l_{01}, l_{02} \circ L_{02} \ll L_{02}$ and $l_{03} \circ l_{03} \ll l_{03}$ by Proposition 2.1 (v), we have $l_{01} \circ l_{01} \ll U(L_T, L_T(l_{01})), l_{02} \circ L_{02} \ll U(L_I, L_I(l_{02}))$ and $l_{03} \circ l_{03} \ll L(L_F, L_F(l_{03}))$. By Lemma 3.4, $l_{01} \circ l_{01} \subseteq U(L_T, L_T(l_{01})), l_{02} \circ l_{02} \subseteq U(L_I, L_I(l_{02}))$ and $l_{03} \circ l_{03} \subseteq L(L_F, L_F(l_{03}))$. Hence $a \in U(L_T, L_T(l_{01})), b \in U(L_I, L_I(l_{02}))$ and $c \in L(L_F, L_F(l_{03}))$ for all $a \in l_{01} \circ l_{01}, b \in l_{02} \circ L_{02}$ and $c \in l_{03} \circ l_{03}$. Therefore ${}_*L_T(l_{01} \circ l_{01}) \geq L_T(l_{01}),{}_* L_I(l_{02} \circ l_{02}) \geq L_I(l_{02})$ and ${}^*L_F(l_{03} \circ l_{03}) \leq L_F(l_{03})$. Now, let $\xi_T := \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\}$, $\xi_I := \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$ and $\xi_F := \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}$ }. By (14), we have

$$L_T(a_0) =^* L_T(l_{02} \circ l_{01}) \geq \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \xi_T,$$

$$L_I(b_0) =^* L_I(l_{02} \circ l_{01}) \geq \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \xi_I$$

and

$$L_F(c_0) =_* L_F(l_{02} \circ l_{01}) \leq \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \xi_F$$

for some $a_0, b_0, c_0 \in l_{02} \circ l_{01}$. Hence $a_0 \in U(L_T,\xi_T)$, $b_0 \in U(L_I,\xi_I)$ and $c_0 \in L(L_F,\xi_F)$ which imply that

$$(l_{02} \circ l_{01}) \cap U(L_T,\xi_T), (l_{02} \circ l_{01}) \cap U(L_I,\xi_I) \text{ and } (l_{02} \circ l_{01}) \cap L(L_F,\xi_F)$$

are nonempty. Since $l_{02} \in U(L_T,\xi_T) \cap U(L_I,\xi_I) \cap L(L_F,\xi_F)$, it follows from Definition 2.2 (4) that $l_{01} \in U(L_T,\xi_T) \cap U(L_I,\xi_I) \cap L(L_F,\xi_F)$. Thus

$$L_T(l_{01}) \geq \xi_T = \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \xi_I = \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}$$

and

$$L_F(l_{01}) \leq \xi_F = \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Consequently, $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H. Ξ

We have the following corollary in a finite hyper KU-algebra since any NS $L = (L_T, L_I, L_F)$ satisfies the condition (14).

Corollary 3.13. Let $L = (L_T, L_I, L_F)$ be a NS in a finite hyper KU-algebra H. Then $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H iff the nonempty sets $U(L_T,\xi_T), U(L_I,\xi_I)$ and $L(L_F,\xi_F)$ are $S\mathbb{H}KUI$'s of H for all $\xi_T,\xi_I,\xi_F \in [0,1]$.

Definition 3.14. A NS $L = (L_T, L_I, L_F)$ in H is called a neutrosophic weak hyper KU-ideal (briefly, $N\mathbb{WH}KUI$) of H if it satisfies the following assertions.

$$\begin{aligned} L_T(0) \geq L_T(l_{01}) &\geq \min\{_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\}, \\ L_I(0) \geq L_I(l_{01}) &\geq \min\{_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\}, \\ L_F(0) \leq L_F(l_{01}) &\leq \max\{^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \end{aligned} \tag{15}$$

for all $l_{01}, l_{02} \in H$.

Definition 3.15. A NS $L = (L_T, L_I, L_F)$ in H is called a neutrosophic s-weak hyper KU-ideal (briefly, $Ns\mathbb{WH}KUI$) of H if it satisfies the conditions (3) and (5).

Example 3.16. Consider a hyper KU-algebra $H = \{l_0, l_a, l_b\}$ with the hyper operation "$\circ$" which is given by Table 5.

Table 5 : Cayley table for the binary operation "$\circ$"

$\circ$	l_0	l_a	l_b
l_0	$\{l_0\}$	$\{l_a\}$	$\{l_b\}$
l_a	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_a, l_b\}$
l_b	$\{l_0\}$	$\{l_0, l_a\}$	$\{l_0, l_a, l_b\}$

Let $L = (L_T, L_I, L_F)$ be a NS in H which is described in Table 6.

Table 6 : Tabular representation of $L = (L_T, L_I, L_F)$

H	$L_T(l)$	$L_I(l)$	$L_F(l)$
l_0	0.77	0.65	0.08
l_a	0.55	0.47	0.57
l_b	0.11	0.27	0.69

It is routine to verify that $L = (L_T, L_I, L_F)$ is a $N\mathbb{WH}KUI$ of H.

Theorem 3.17. Every $Ns\mathbb{WH}KUI$ is a $N\mathbb{WH}KUI$.

Proof. Let $L = (L_T, L_I, L_F)$ be a $Ns\mathbb{WH}KUI$ of H and let $l_{01}, l_{02} \in H$. Then there exist $u, v, w \in l_{02} \circ l_{01}$ such that

$$\begin{aligned} L_T(l_{01}) &\geq \min\{L_T(u), L_T(l_{02})\} \geq \min\{\inf_{u_0 \in l_{02} \circ l_{01}} L_T(u_0), L_T(l_{02})\}, \\ L_I(l_{01}) &\geq \min\{L_I(v), L_I(l_{02})\} \geq \min\{\inf_{v_0 \in l_{02} \circ l_{01}} L_I(v_0), L_I(l_{02})\}, \\ L_F(l_{01}) &\leq \max\{L_F(w), L_F(l_{02})\} \leq \max\{\sup_{w_0 \in l_{02} \circ l_{01}} L_F(w_0), L_F(l_{02})\}. \end{aligned}$$

Hence $L = (L_T, L_I, L_F)$ is a $N\mathbb{WH}KUI$ of H. Ξ

The corollary of Theorem 3.17, we can assume, is not true. However, finding an example of a $N\mathbb{WH}KUI$ that is not a $Ns\mathbb{WH}KUI$ is difficult. Now we specify that a $N\mathbb{WH}KUI$ must also be a $Ns\mathbb{WH}KUI$.

Theorem 3.18. If $L = (L_T, L_I, L_F)$ is a $N\mathbb{WH}KUI$ of H which satisfies the condition (4), then $L = (L_T, L_I, L_F)$ is a $Ns\mathbb{WH}KUI$ of H.

Proof. Let $L = (L_T, L_I, L_F)$ be a $N\mathbb{WH}KUI$ of H in which the condition (4) is true. Then there exist $u_0, v_0, c_0 \in l_{02} \circ l_{01}$ such that $L_T(u_0) = {}_*L_T(l_{02} \circ l_{01}), L_I(v_0) = {}_*L_I(l_{02} \circ l_{01})$ and $L_F(w_0) = {}^*L_F(l_{02} \circ l_{01})$. Hence

$$L_T(l_{01}) \geq \min\{{}_*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} = \min\{L_T(u_0), L_T(l_{02})\},$$
$$L_I(l_{01}) \geq \min\{{}_*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} = \min\{L_I(v_0), L_I(l_{02})\},$$
$$L_F(l_{01}) \leq \max\{{}^*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} = \max\{L_F(w_0), L_F(l_{02})\}.$$

Therefore $L = (L_T, L_I, L_F)$ is a $Ns\mathbb{WH}KUI$ of H. Ξ

Remark 3.19. In a finite hyper KU-algebra, every NS satisfies the condition (4). Hence the concept of $Ns\mathbb{WH}KUI$ and $N\mathbb{WH}KUI$ coincide in a finite hyper KU-algebra.

Theorem 3.20. A NS $L = (L_T, L_I, L_F)$ is a $N\mathbb{WH}KUI$ of H iff the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $\mathbb{WH}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$.

Proof. The proof is comparable to Theorem 3.5's proof. Ξ

Definition 3.21. A NS in H is called a reflexive neutrosophic hyper KU-ideal (briefly, $RN\mathbb{H}KUI$) of H if it satisfies

$$(\forall\ l_{01}, l_{02} \in H) \begin{pmatrix} {}_*L_T(l_{01} \circ l_{01}) \geq L_T(l_{02}) \\ {}_*L_I(l_{01} \circ l_{01}) \geq L_I(l_{02}) \\ {}^*L_F(l_{01} \circ l_{01}) \leq L_F(l_{02}) \end{pmatrix}, \tag{16}$$

and

$$(\forall\ l_{01}, l_{02} \in H) \begin{pmatrix} L_T(l_{01}) \geq \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\} \\ L_I(l_{01}) \geq \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\} \\ L_F(l_{01}) \leq \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\} \end{pmatrix}. \tag{17}$$

Theorem 3.22. Every $RN\mathbb{H}KUI$ is a $NS\mathbb{H}KUI$.

Proof. Straightforward. Ξ

Theorem 3.23. If $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H, then the nonempty sets $U(L_T, \xi_T)$, $U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $R\mathbb{H}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$.

Proof. Assume that $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are nonempty for all $\xi_T, \xi_I, \xi_F \in [0, 1]$. Let $a \in U(L_T, \xi_T), b \in U(L_I, \xi_I)$ and $c \in L(L_F, \xi_F)$. If $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H, then by Theorem 3.22, $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H, and so it is a $N\mathbb{H}KUI$ of

H. It follows from Theorem 3.5 that $U(L_T, \xi_T)$, $U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $\mathbb{H}KUI$'s of H. For each $l_{01} \in H$, let $a_0, b_0, c_0 \in l_{01} \circ l_{01}$. Then

$$L_T(a_0) \geq \inf_{u \in l_{01} \circ l_{01}} L_T(u) \geq L_T(a) \geq \xi_T,$$

$$L_I(b_0) \geq \inf_{v \in l_{01} \circ l_{01}} L_I(v) \geq L_I(b) \geq \xi_I,$$

$$L_F(c_0) \leq \sup_{w \in l_{01} \circ l_{01}} L_F(w) \leq L_F(c) \leq \xi_F,$$

and so $a_0 \in U(L_T, \xi_T), b_0 \in U(L_I, \xi_I)$ and $c_0 \in L(L_F, \xi_F)$. Hence $l_{01} \circ l_{01} \subseteq U(L_T, \xi_T), l_{01} \circ l_{01} \subseteq U(L_I, \xi_I)$ and $l_{01} \circ l_{01} \subseteq L(L_F, \xi_F)$. Therefore $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $R\mathbb{H}KUI$'s of H. Ξ

Lemma 3.24. Every $R\mathbb{H}KUI$ is a $S\mathbb{H}KUI$.

By adding a condition, we explore the converse of Theorem 3.23.

Theorem 3.25. Let $L = (L_T, L_I, L_F)$ be a NS in H satisfying the condition (14). If the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $R\mathbb{H}KUI$'s of H for all $\xi_T, \xi_I, \xi_F \in [0, 1]$, then $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H.

Proof. If the nonempty sets $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $R\mathbb{H}KUI$'s of H, then by Lemma 3.24 they are $S\mathbb{H}KUI$'s of H. By Theorem 3.12 that $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H. Hence the condition (17) is valid. Let $l_{01}, l_{02} \in H$. Then the sets $U(L_T, L_T(l_{02})), U(L_I, L_I(l_{02}))$ and $L(L_F, L_F(l_{02}))$ are $R\mathbb{H}KUI$'s of H, and so $l_{01} \circ l_{01} \subseteq U(L_T, L_T(l_{02})), l_{01} \circ l_{01} \subseteq U(L_I, L_I(l_{02}))$ and $l_{01} \circ l_{01} \subseteq L(L_F, L_F(l_{02}))$. Hence $L_T(u) \geq L_T(l_{02})$, $L_I(v) \geq L_I(l_{02})$ and $L_F(w) \leq L_F(l_{02})$ for all $u, v, w \in l_{01} \circ l_{01}$ and so ${}_*L_T(l_{01} \circ l_{01}) \geq L_T(l_{02})$, ${}_*L_I(l_{01} \circ l_{01}) \geq L_I(l_{02})$ and ${}^*L_F(l_{01} \circ l_{01}) \leq L_F(l_{02})$. Therefore $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H. Ξ

We provide conditions for a $NS\mathbb{H}KUI$ to be a $RN\mathbb{H}KUI$.

Theorem 3.26. Let $L = (L_T, L_I, L_F)$ be a $NS\mathbb{H}KUI$ of H which satisfies the condition (14). Then $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H iff the following assertion is valid.

$$(\forall\ l_{01} \in H) \begin{pmatrix} {}_*L_T(l_{01} \circ l_{01}) \geq L_T(0) \\ {}_*L_I(l_{01} \circ l_{01}) \geq L_I(0) \\ {}^*L_F(l_{01} \circ l_{01}) \leq L_F(0) \end{pmatrix}. \tag{18}$$

Proof. It is clear that if $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H, then the condition (18) is valid.

Conversely, assume that $L = (L_T, L_I, L_F)$ is a $NS\mathbb{H}KUI$ of H which satisfies the conditions (14) and (18). Then $L_T(0) \geq L_T(l_{02}), L_I(0) \geq L_I(l_{02})$ and $L_F(0) \leq L_F(l_{02})$ for all $l_{02} \in H$. Hence

${}_*L_T(l_{01} \circ l_{01}) \geq L_T(l_{02})$, ${}_*L_I(l_{01} \circ l_{01}) \geq L_I(l_{02})$ and ${}^*L_F(l_{01} \circ l_{01}) \leq L_F(l_{02})$.

For any $l_{01}, l_{02} \in H$, let

$$\xi_T := \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$\xi_I := \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\},$$

$$\xi_F := \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Then $U(L_T, \xi_T), U(L_I, \xi_I)$ and $L(L_F, \xi_F)$ are $S\mathbb{H}KUI$'s of H by Theorem 3.11. Since $L = (L_T, L_I, L_F)$ satisfies the condition (14), there exist $u_0, v_0, w_0 \in l_{02} \circ l_{01}$ such that

$$L_T(u_0) =^* L_T(l_{02} \circ l_{01}), L_I(v_0) =^* L_I(l_{02} \circ l_{01}), L_F(w_0) =_* L_F(l_{02} \circ l_{01}).$$

Hence $L_T(u_0) \geq \xi_T, L_I(v_0) \geq \xi_I$ and $L_F(w_0) \leq \xi_F$, that is, $u_0 \in U(L_T, \xi_T), v_0 \in U(L_I, \xi_I)$ and $w_0 \in L(L_F, \xi_F)$. Hence $(l_{02} \circ l_{01}) \cap U(L_T, \xi_T) \neq \emptyset, (l_{02} \circ l_{01}) \cap U(L_I, \xi_I) \neq \emptyset$ and $(l_{02} \circ l_{01}) \cap L(L_F, \xi_F) \neq \emptyset$. Since $l_{02} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F)$, by Definition 2.2 (4), $l_{01} \in U(L_T, \xi_T) \cap U(L_I, \xi_I) \cap L(L_F, \xi_F)$. Thus

$$L_T(l_{01}) \geq \xi_T = \min\{{}^*L_T(l_{02} \circ l_{01}), L_T(l_{02})\},$$

$$L_I(l_{01}) \geq \xi_I = \min\{{}^*L_I(l_{02} \circ l_{01}), L_I(l_{02})\},$$

$$L_F(l_{01}) \leq \xi_F = \max\{{}_*L_F(l_{02} \circ l_{01}), L_F(l_{02})\}.$$

Therefore $L = (L_T, L_I, L_F)$ is a $RN\mathbb{H}KUI$ of H. Ξ

Conclusions

We have introduced the notions of $NS\mathbb{H}KUI$, $N\mathbb{W}\mathbb{H}KUI$, $Ns\mathbb{W}\mathbb{H}KUI$ and $RN\mathbb{H}KUI$. We have considered their relations and related properties. We have discussed characterizations of $N\mathbb{H}KUI$ and $N\mathbb{W}\mathbb{H}KUI$ have given conditions for a NS to be a $RN\mathbb{H}KUI$ and $NS\mathbb{H}KUI$. We have provided conditions for a $N\mathbb{W}\mathbb{H}KUI$ to be a $Ns\mathbb{W}\mathbb{H}KUI$ and have provided conditions for a $NS\mathbb{H}KUI$ to be a $RN\mathbb{H}KUI$.

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Neutrosophic πgγ* closed sets in Neutrosophic Topological Spaces

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Abstract: This paper introduces and investigate the notion of Neutrosophic πgγ* closed sets within the framework of Neutrosophic topological spaces. The study begins defining Neutrosophic πgγ* closed sets and proceeds to investigate their fundamental properties and characterizations. Special attention is given to examining how these sets interrelate with and extend existing classes of Neutrosophic closed sets. By establishing inclusion relations and comparative hierarchies, the paper highlights the significance of Neutrosophic πgγ* closed sets in broadening the structural understanding of Neutrosophic topologies. Furthermore, several illustrative examples are provided to demonstrate the distinctive features of these sets and to clarify their role in the generalization process. The paper also derives various theorems that reveal the interplay between Neutrosophic πgγ* closed sets and other Neutrosophic closed families, thereby enriching the theoretical landscape of Neutrosophic topology. These results contribute to ongoing developments in generalized closed set theory, offering new insights and paths for further research in Neutrosophic mathematics.

Keywords: Neutrosophic topological spaces, Neutrosophic open sets, Neutrosophic closed sets, Neutrosophic πgγ* closed sets.

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1. Introduction

The concept of fuzzy sets, introduced by Zadeh [13] in 1965, allow each element to have a degree of membership. This concept was expanded by K. Atanassov [1] in 1986 with the introduction of Intuitionistic Fuzzy sets, which assign both a degree of membership and a degree of non-membership to each element. Sakthivel K and Manikandan M [10] had studied the concept of πgγ* closed Sets in Intuitionistic Fuzzy Topological Spaces. Florentin Smarandache [2] introduced Neutrosophic Sets as a further generalization, which adds more flexibility. Later, Salama A A and Alblowi S A [11] extended the idea by developing Neutrosophic Topological Spaces.

In this article we define Neutrosophic πgγ* closed sets in Neutrosophic topological spaces and investigate their properties.

2. **Preliminaries**

This section reviews essential definitions, operations, and key results related to Neutrosophic sets.

Definition 2.1 [6] Let X be a non-empty fixed set. A Neutrosophic Set (NS) A is an object having the form A=$\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle : x \in X\}$ where $\mu_A(x), \sigma_A(x)$and $\nu_A(x)$ represent the degree of membership, degree of indeterminacy and the degree of non-membership respectively of each element x ∈ X the set A.

Definition 2.2 [6] Let X be a non-empty set and let A be a Neutrosophic Set
A = $\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle : x \in X\}$.Then the complement of A is
$A^C = \{\langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x) \rangle : x \in X\}$

Definition 2.3 [6] Let A and B be two Neutrosophic Sets, ∀ x ∈ X
A = $\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle : x \in X\}$
B = $\{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x) \rangle : x \in X\}$.
Then A ⊆ B ⇔ $\{\langle x, \mu_A(x) \le \mu_B(x), \sigma_A(x) \le \sigma_B(x), \nu_A(x) \ge \nu_B(x) \rangle : x \in X\}$

Definition 2.4 [6] Let X be a non-empty set and let A and B be two Neutrosophic Sets are
A=$\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle : x \in X\}$, B = $\{\langle x, \mu_B(x), \sigma_B(x), \nu_B(x) \rangle : x \in X\}$. Then

1. A ∩ B = $\{\langle x, \mu_A(x) \wedge \mu_B(x), \sigma_A(x) \wedge \sigma_B(x), \nu_A(x) \vee \nu_B(x) \rangle : x \in X\}$
2. A ∪ B = $\{\langle x, \mu_A(x) \vee \mu_B(x), \sigma_A(x) \vee \sigma_B(X), \nu_A(x) \wedge \nu_B(x) \rangle : x \in X\}$

Definition 2.5 [6] Let X be a non-empty set and τ_N be the collection of Neutrosophic subsets of X satisfying the following properties:

1. $0_N, 1_N \in \tau_N$
2. $T_1 \cap T_2 \in \tau_N$ for any $T_1, T_2 \in \tau_N$
3. $\cup T_i \in \tau_N$ for every $\{T_i : i \in j\} \subseteq \tau_N$

Then the space (X, τ_N) is called a Neutrosophic Topological Space (NTS). The elements of τ_N are called Neutrosophic Open Set (NOS) and its complement is Neutrosophic Closed Set (NCS).

Definition 2.6 [6] Let (X, τ_N) be a NTS and A=$\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle : x \in X\}$ be a NS in X. Then
Neutrosophic closure of A is N_Cl(A) = ∩ { M : M is a NCS in X and A ⊆ M}
Neutrosophic interior of A is N_Int(A) = ∪ { H : H is a NOS in X and H ⊆A}

Definition 2.7 Let (X, τ_N) be a NTS and A=$\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle : x \in X\}$ be a NS in X. Then A is said to be

- Neutrosophic Semi closed set [4] (NSCS) if N_Int (N_Cl (A)) ⊆A.
- Neutrosophic Semi open set [4] (NSOS) if A ⊆ N_Cl (N_Int(A)).
- Neutrosophic Pre closed set [12] (NPCS) if N_Cl(N_Int(A)) ⊆A.
- Neutrosophic Pre open set [12] (NPOS) if A⊆ N_Int(N_Cl(A)).
- Neutrosophic Regular closed set [7] (NRCS) if A= N_Cl(N_IntA)).
- Neutrosophic Regular open set [7] (NROS if A = N_Int(N_Cl(A)).
- Neutrosophic α closed set [5] (NαCS) if N_Cl (N_Int (N_Cl (A))) ⊆A.
- Neutrosophic α open set [5] (NαOS) if A ⊆ N_Int (N_Cl (N_Int (A))).
- Neutrosophic β closed set [8] (NβCS) if N_Int (N_Cl (N_Int (A))) ⊆A.
- Neutrosophic β open set [8] (NβOS in short) if A⊆ N_Cl (N_Int (N_Cl (A))).
- Neutrosophic b closed set [6] (NbCS) if N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A

- Neutrosophic b open set [6] (NbOS) if A ⊆ N_Int (N_Cl (A)) ∪ N_Cl (N_Int(A))
- Neutrosophic π closed set [9] if A is the union of Neutrosophic Regular closed sets.
- Neutrosophic π open set [9] if A is the union of Neutrosophic Regular open sets.

Definition 2.8 Let (X, τ_N) be a NTS and A=$\{\langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle: x \in X\}$ be a NS in X. Then A is said to be

- Neutrosophic Generalized closed set [3] (NGCS) if N_Cl(A)⊆ U whenever A ⊆ U and U is a NOS in X.
- Neutrosophic Generalized semi closed set [3] (NGSCS) if N_sCl(A) ⊆ U whenever A ⊆ U andU is a NOS in X.
- Neutrosophic α Generalized closed set [5] (NαGCS) if N_αCl(A) ⊆ U whenever A ⊆ U and U is a NOS in X

Remark 2.9 For any Neutrosophic Set A,

- N_Cl (A) ^{C} = (N_Int (A)) ^{C}
- N_Int (A)^C = (N_Cl (A)) ^{C}
- N_sCl (A) ^{C} = (N_sInt (A)) ^{C}
- N_sInt (A)^C = (N_sCl (A)) ^{C}
- N_sCl (A) = A ∪ N_Int (N_Cl (A))
- N_sInt (A) = A ∩ N_Cl (N_Int (A))
- N_αCl(A) = A ∪ N_Cl (N_Int (N_Cl A)))
- N_αInt(A) = A ∩ N_Int (N_Cl (N_Int (A)))

Remark 2.10

1. Each NOS is NPOS in NTS.
2. Each NRCS is NCS in NTS.
3. Each N πOS is NOS in NTS.

3. Neutrosophic πgγ* closed sets in Neutrosophic Topological Spaces

We have introduced Neutrosophic πgγ* closed sets and explored some of their properties.

Definition 3.1. A Neutrosophic Set A in (X, τ_N) is said to be a Neutrosophic πgγ* closed sets (Nπgγ*CS in short) if N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ U whenever A⊆ U and U is NπOS in (X, τ_N).

Example 3.2. Let X = {p, q} with τ_N = {0_N, B, 1_N} be a NTS on X, where B = <x, (0.4,0.5,0.7), (0.4,0.4,0.6)>. Let us consider the NS, A = <x, (0.3,0.2,0.7), (0.3,0.2,0.8)>. Here NπOS is U = {1_N, B}. Clearly A ⊆ U . Now N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) = 0_N ⊆ U . Therefore NS, A is a Nπgγ*CS in (X, τ_N).

Theorem 3.3. Every NCS in (X, τ_N) is a N πgγ*CS (X, τ_N) but not conversely in general.
Proof: Consider 'A' is a NCS in (X, τ_N). Assume that A⊆ U and U is a NπOS in (X, τ_N). Since A is a NCS in (X, τ_N), N_Cl(A) = A . Now N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) = N_Int(A) ∩ N_Int(A) = N_Int(A) ⊆ A ⊆ U. Thus N_Cl (N_Int(A)) ∩ N_Int(N_Cl (A)) ⊆ U , whenever A⊆ U and U is NπOS in (X, τ_N) Therefore, A is a N πgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.4. Let X = {p, q} with τ_N = { 0_N , B, 1_N } be a NTS on X , where B = <x, (0.2,0.4,0.6), (0.3,0.5,0.7)>. Let us consider the NS, A = <x, (0.1,0.3,0.8), (0.2,0.4,0.7)>. Here NπOS is Ʋ = {1_N , B}. Clearly A ⊆ Ʋ. Now N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) = 0_N ⊆ Ʋ. But N_Cl(A) = B^c $\nsubseteq A$. Therefore NS, A is a N πgγ*CS but not NCS in (X, τ_N).

Theorem 3.5. Every NSCS in (X, τ_N) is a Nπgγ*CS (X, τ_N) but not conversely in general.
Proof: Consider A is a NSCS in (X, τ_N). Suppose that A ⊆ Ʋ and Ʋ is a NπOS in (X, τ_N). Since A is a NSCS in (X, τ_N), N_Int (N_Cl (A)) ⊆ A. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ N_Cl (N_Int (A)) ∩ A ⊆ N_Cl (A) ∩ A = A ⊆ Ʋ. Thus N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ Ʋ , whenever A⊆ Ʋ and Ʋ is NπOS in (X, τ_N). Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.6. Let X = {p, q} with τ_N = { 0_N ,B, 1_N } be a NTS on X , where B = <x, (0.2,0.4,0.6), (0.3,0.5,0.7)>. Let us consider the NS, A = <x, (0.1,0.3,0.8), (0.2,0.4,0.7)> . Here NπOS is Ʋ = {1_N , B}. Clearly A ⊆ Ʋ . Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) = 0_N ⊆ Ʋ . But N_Int (N_Cl (A)) = B $\nsubseteq A$. Therefore NS, A is a Nπgγ*CS but not NSCS in (X, τ_N).

Theorem 3.7. Every NPCS in (X, τ_N) is a Nπgγ*CS (X, τ_N) but not conversely in general.
Proof: Consider A is a NPCS in (X, τ_N). Suppose that A ⊆ Ʋ and Ʋ is a NπOS in (X, τ_N). Since A is a NPCS in (X, τ_N), N_Cl (N_Int (A)) ⊆ A. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ A ∩ N_Int (N_Cl (A)) ⊆ A ∩ N_Cl (A) = A ⊆ Ʋ. Thus N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ Ʋ , whenever A ⊆ Ʋ and Ʋ is NπOS in (X, τ_N). Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.8. Let X = { p, q } with τ_N = { 0_N , B_1, B_2, 1_N } be a NTS on X , where B_1 = <x, (0.5,0.5,0.5), (0.6,0.5,0.4)> and B_2 = <x, (0.4,0.5,0.6), (0.3,0.5,0.7)>. Let us consider the NS, A = <x, (0.4,0.5,0.6), (0.4,0.5,0.6)> . Here NπOS is Ʋ = {$1_N, B_1, B_2$}. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) = B_2 ⊆ Ʋ . But N_Cl (N_Int (A) = B_1^C $\nsubseteq A$. Therefore NS, A is a Nπgγ*CS but not NPCS in (X, τ_N).

Theorem 3.9. Every NRCS in (X, τ_N) is a Nπgγ*CS in (X, τ_N) but not conversely in general.
Proof: Let A be a NRCS in (X, τ_N). Since every NRCS is a NCS, by Theorem 3.3 A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.10. Let X = { p, q } with τ_N ={ 0_N ,B, 1_N } be a NTS on X , where B= <x, (0.5,0.5,0.5), (0.4,0.5,0.6)>. Let us consider the NS, A=<x, (0.5,0.5,0.5), (0.5,0.5,0.4)>. Here NπOS is Ʋ = {1_N , B}. Clearly A ⊆ Ʋ. Now N_Cl (N_Int (A))∩N_Int (N_Cl (A)) = B ⊆ Ʋ . But N_Cl (N_Int (A) = $B^C \neq A$ .Therefore NS, A is a N$\pi g\gamma^*$CS but not NRCS in (X, τ_N).

Theorem 3.11. Every NROS in (X, τ_N) is a Nπgγ*CS in (X, τ_N) but not conversely in general.
Proof: Let A be a NROS in (X, τ_N). A is a Nπgγ*CS in (X, τ_N). Then N_Int(N_Cl(A)) = A and N_Int(A) = A. Suppose that A ⊆ Ʋ and Ʋ is a NπOS in (X, τ_N). Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ N_Cl (A) ∩ A = A ⊆ Ʋ . Thus N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A))⊆Ʋ , whenever A⊆ Ʋ and Ʋ is NπOS in (X, τ_N). Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.12. Let X = { p, q } with τ_N = { 0_N ,B, 1_N } be a NTS on X , where B = <x, (0.5,0.5,0.5), (0.4,0.5,0.6)>. Let us consider the NS, A = <x, (0.4,0.5,0.6), (0.4,0.5,0.6)>. Here

NπOS is U = $\{1_N, B\}$. Clearly A ⊆ U. Now N_Cl (N_Int(A)) ∩ N_Int(N_Cl (A)) = 0_N ⊆ U. But N_Int (N_Cl (A)) = B ≠ A. Therefore NS, A is a Nπgγ*CS but not NROS in (X, τ_N).

Theorem 3.13. Every NαCS in (X, τ_N) is a Nπgγ*CS (X, τ_N) but not conversely in general.
Proof: Consider A is a NαCS in (X, τ_N). Suppose that A⊆ U and U is a NπOS in (X, τ_N). Since A is a NαCS in (X, τ_N), N_Cl (N_Int (N_Cl (A))) ⊆ A. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ N_Cl (N_Int (N_Cl (A))) ∩ N_Cl (N_Int (N_Cl (A))) ⊆ A ∩ A = A ⊆ U. Thus N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ U, whenever A ⊆ U and U is NπOS in (X, τ_N). Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.14. Let X = { p, q } with τ_N ={ 0_N , B, 1_N } be a NTS on X, where B = <x, (0.5,0.5,0.5), (0.4,0.5,0.6)>. Let us consider the NS, A = <x, (0.4,0.5,0.4), (0.5,0.5,0.6)>. Here NπOS is U = $\{1_N, B\}$. Clearly A ⊆ U. Now N_Cl (N_Int(A)) ∩ N_Int (N_Cl(A)) = 0_N ⊆ U. But N_Cl (N_Int (N_Cl (A))) $= B^C \nsubseteq A$. Therefore NS, A is a Nπgγ*CS but not NαCS in (X, τ_N).

Theorem 3.15. Every NbCS in (X, τ_N) is a Nπgγ*CS (X, τ_N), but not conversely in general.
Proof: Consider A is a NbCS in (X, τ_N). Suppose that A ⊆ U and U is a NπOS in (X, τ_N). Since A is a NbCS in (X, τ_N), N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ A ⊆ U. Thus N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ U, whenever A ⊆ U and U is NπOS in (X, τ_N). Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.16. Let X ={ p , q } with τ_N ={ 0_N , B_1, B_2, 1_N } be a NTS on X, where B_1 = <x, (0.5,0.5,0.5), (0.6,0.5,0.4)> and B_2 = <x, (0.4,0.5,0.6), (0.3,0.5,0.7)>. Let us consider the NS, A =<x, (0.4,0.5,0.6), (0.6,0.5,0.4)>. Here NπOS is U = $\{1_N, B_1, B_2\}$. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ U whenever A ⊆ U. But N_Cl (N_Int(A)) ∩ N_Int (N_Cl(A)) = $B_1^C \nsubseteq A$. Therefore NS, A is a Nπgγ*CS but not NbCS in (X, τ_N).

Theorem 3.17. Every NGCS in (X, τ_N) is a Nπgγ*CS in (X, τ_N) but not conversely.
Proof: Consider A is a NGCS in (X, τ_N). Assume that A⊆ U and U is a NπOS in (X, τ_N). Since A is a NGCS in (X, τ_N), N_Cl(A) ⊆ U. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ N_Cl (A) ⊆ U .Thus N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ U, whenever A ⊆ U and U is NπOS in (X, τ_N). Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.18. Let X ={ p, q } with τ_N ={ 0_N , B, 1_N } be a NTS on X, where B = <x, (0.3,0.5,0.4), (0.2,0.5,0.3)>. Consider NS, A = <x, (0.2,0.4,0.6), (0.2,0.4,0.4)>. Here NπOS U = $\{1_N, B\}$. Clearly A ⊆ U. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) = 0_N ⊆ U. But N_Cl(A) = $B^C \nsubseteq B$, whenever A ⊆ B. Therefore NS, A is a Nπgγ*CS but not NGCS in (X, τ_N).

Theorem 3.19. Every NGSCS in (X, τ_N) is a Nπgγ*CS in (X, τ_N) but not conversely.
Proof: Consider 'A' is a NGSCS in (X, τ_N). Assume that A⊆ U and U is a NπOS in (X, τ_N). Since A is a NGSCS in (X, τ_N), N_sCl(A) = A ∪ N_Int(N_Cl(A)) ⊆ U .This implies N_Int (N_Cl(A) ⊆ U. Now N_Cl (N_Int(A)) ∩ N_Int (N_Cl(A)) ⊆ N_Cl (N_Int(A)) ∩ U ⊆ U .Thus N_Cl (N_Int(A)) ∩ N_Int (N_Cl(A)) ⊆ U , whenever A⊆ U and U is NπOS in (X, τ_N) Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.20. Let X = { p , q } with τ_N ={ 0_N , B_1, B_2, 1_N } be a NTS on X, where B_1 = <x, (0.5,0.5,0.5), (0.3,0.5,0.7)> and B_2 = <x, (0.4,0.5,0.6), (0.3,0.5,0.7)>. Let us consider the NS, A = <x, (0.3,0.5,0.4), (0.2,0.5,0.8)>. Here NπOS is U = $\{1_N, B_1, B_2\}$. Now N_Cl (N_Int(A)) ∩ N_Int

(N_Cl(A)) = 0_N ⊆ Ʋ whenever A ⊆ Ʋ. But N_sCl (A) = A ∪ N_Int (N_Cl (A)) = A ∪ B_1 = B_1 ⊈ B and A ⊆ B_2. Therefore NS, A is a Nπgγ*CS but not NGSCS in (X, τ_N).

Theorem 3.21. Every NαGCS in (X, τ_N) is a Nπgγ*CS (X, τ_N) but not conversely in general.

Proof: Consider A is a NαGCS in (X, τ_N). Suppose that A ⊆ Ʋ and Ʋ is a NπOS in (X, τ_N). Since A is a NαGCS in (X, τ_N), N_αCl(A)⊆ Ʋ. Therefore A ∪ N_Cl (N_Int (N_Cl (A)))⊆ Ʋ, so N_Cl (N_Int (N_Cl A)))⊆ Ʋ and N_Int (N_Cl A)) ⊆ Ʋ. Now N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A))⊆ N_Cl (N_Int (A)) ∩ Ʋ ⊆ Ʋ. Thus N_Cl (N_Int (A)) ∩ N_Int (N_Cl (A)) ⊆ Ʋ, whenever A ⊆ Ʋ and Ʋ is NπOS in (X, τ_N) . Therefore, A is a Nπgγ*CS in (X, τ_N). However, the reverse implication is not true.

Example 3.22. In Example 3.20, the NS A =<x, (0.3,0.5,0.7), (0.2,0.5,0.8)> is Nπgγ*CS in (X, τ_N) but not a NαGCS in (X, τ_N) as N_αCl(A) = A ∪ N_Cl (N_Int (N_Cl (A)))= A ∪ B_1^C = B_1^C ⊈ B_1, B_2 and A ⊆ B_1, B_2 .

In the following figure (a) we have provided relation between Nπgγ*CS and other closed sets in Neutrosophic topological space.

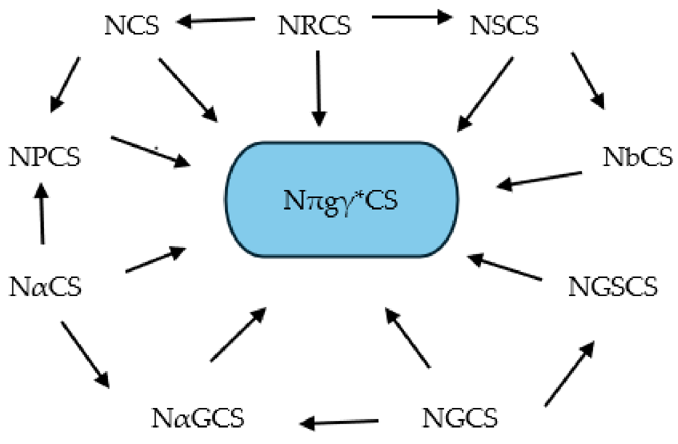

Figure (a)

Remark 3.23. The union of two Nπgγ*CSs need not be a Nπgγ*CS in (X, τ_N) in general.

Example 3.24. Let X = {p , q} with τ_N = { 0_N ,B_1, B_2, 1_N} be a NTS on X, where B_1 = <x, (0.4,0.5,0.6), (0.2,0.5,0.8) > and B_2 = <x, (0.4,0.5,0.5), (0.4,0.5,0.5)>. Here A = <x, (0.4,0.5,0.5), (0.5,0.5,0.4) > and B = <x, (0.5,0.5,0.4), (0.2,0.5,0.6) > are Nπgγ*CS in (X, τ_N), but A ∪ B = <x, (0.5,0.5,0.4), (0.5,0.5,0.4)> is not a Nπgγ*CS in (X, τ_N).

Theorem 3.25. Let (X, τ_N) be a NTS. Then for every A ∈ Nπgγ*CS in (X, τ_N) and for every B ∈ NS in (X, τ_N), A ⊆ B ⊆ N_Cl (N_Int(A)) which implies B ∈ Nπgγ*CS in (X, τ_N).

Proof: Let B ⊆ Ʋ and Ʋ be an NπOS in X. Since A ⊆ B, A ⊆ Ʋ. Also, B ⊆ N_Cl (N_Int(A)) which implies N_Cl (N_Int(B)) ⊆ N_Cl (N_Int(A)). Now N_Int (N_Cl (B)) ⊆ N_Int (N_Cl (A)). Therefore N_Cl (N_Int(B)) ∩ N_Int (N_Cl (B)) ⊆ N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ Ʋ, by hypothesis. Hence B ∈ Nπgγ*CS in (X, τ_N).

Theorem 3.26. If A is both NπOS and Nπgγ*CS in (X, τ_N) then A is a NbCS in (X, τ_N).

Proof: Let A be a NπOS and a Nπgγ*CS in (X, τ_N). Then N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆A, as A ⊆ A, by hypothesis. Therefore N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A. Hence A is a NbCS in (X, τ_N).

Theorem 3.27. If A is both a NπOS and a Nπgγ*CS in (X, τ_N) then A is a NβCS in (X, τ_N).
Proof: Let A be a NπOS and a Nπgγ*CS in (X, τ_N). Then N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A, as A ⊆ A, by hypothesis. Now N_Int (N_Cl (N_Int(A))) = N_Int (N_Cl (N_Int(A))) ∩ N_Cl (N_Int(A)) ⊆ N_Int (N_Cl (A)) ∩ N_Cl (N_Int(A)) = N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A. Therefore N_Int (N_Cl (N_Int(A))) ⊆ A . Hence A is a NβCS in (X, τ_N).

Theorem 3.28. If A is both a NπOS and a Nπgγ*CS in (X, τ_N) then A is a NSCS in (X, τ_N).
Proof: Let A be a NπOS and a Nπgγ*CS in (X, τ_N). That is A is a NOS in (X, τ_N). Then N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A, as A ⊆ A, by hypothesis. Clearly N_Int (N_Cl (A)) = N_Cl (A) ∩ N_Int (N_Cl (A)) = N_Cl (N_Int(A)) ∩ N_Int (N_Cl (A)) ⊆ A. Therefore N_Int (N_Cl (A)) ⊆ A. Hence A is a NSCS in (X, τ_N).

Theorem 3.29. If A is both a NπOS and a Nπgγ*CS in (X, τ_N), then A is a NROS in (X, τ_N).
Proof: Let A be a NπOS and a Nπgγ*CS in (X, τ_N). That is A is a NOS in (X, τ_N). Then N_Int (N_Cl (A)) = N_Int (N_Cl (A)) ∩ N_Cl (A) = N_Int (N_Cl (A)) ∩ N_Cl (N_Int(A)) ⊆ A. Since A is a NOS, it is a NPOS and A ⊆ N_Int (N_Cl (A)). Therefore A = N_Int (N_Cl (A)). Hence A is a NRCS in (X, τ_N).

4. Conclusion: In this paper we have introduced Neutrosophic πgγ* closed sets in Neutrosophic Topological Spaces and discussed some of its properties and some contradicting examples. This idea can be developed and extended in the area of continuous functions, homeomorphisms, compactness and connected and so on.

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Neutrosophic Quadruple Metric Spaces and Neutrosophic Quadruple Normed Spaces

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Abstract: This study establishes fundamental results in the emerging domains of neutrosophic quadruple metric spaces and neutrosophic quadruple normed spaces. Building upon the recent definition of neutrosophic quadruple metric spaces, the first primary contribution is the development of fixed-point theory within this generalized framework. Specifically, we formulate and rigorously prove an analogue of the Banach contraction principle tailored for neutrosophic quadruple metric spaces. Furthermore, we establish additional fixed-point theorems applicable in this context, extending foundational results from classical metric spaces and simpler neutrosophic structures to handle the increased complexity and uncertainty modeled by quadruple-valued neutrosophic sets. The second major contribution involves the algebraic generalization of normed spaces. We introduce the novel concept of neutrosophic quadruple normed spaces. Within these spaces, we define an appropriate norm structure capable of measuring the "magnitude" of vectors characterized by quadruple-valued neutrosophic components. We systematically investigate and establish various fundamental properties of this newly defined norm, exploring concepts such as statistical convergence, Cauchy statistical convergence, boundedness, and continuity within the neutrosophic quadruple normed spaces. Additionally, we address the specific case of neutrosophic quadruple vector spaces, defining and analyzing the corresponding norm structure in this specialized context. The results generalize and extend prior work in fuzzy, intuitionistic fuzzy, and standard neutrosophic metric and normed spaces.

Keywords: fixed points for neutrosophic quadruple metric space; neutrosophic quadruple normed space; statistical convergence to the neutrosophic quadruple norm.

1. Introduction

Fixed point theory stands as a cornerstone of nonlinear functional analysis, witnessing remarkable growth in both theoretical depth and practical application. Its significance extends far beyond the boundaries of pure mathematics, permeating diverse fields such as differential equations, optimization theory, game theory, economics, computer science, and engineering. This widespread utility has naturally spurred intense research activity, leading to the development of numerous fixed-point theorems tailored to increasingly sophisticated mathematical structures.

The foundation of much of this work lies in the theory of metric spaces. Seminal contributions by T. Zamfirescu [1] established crucial fixed-point results within this classical setting. As mathematical frameworks evolved to model greater complexity and uncertainty, researchers explored generalizations of metric spaces. This led to the investigation of fixed-point theorems in fuzzy metric (shortly FM) spaces [2], capturing vagueness through membership degrees, and subsequently in intuitionistic FM spaces [3], which incorporate both membership and non-membership. More recently, the advent of neutrosophic metric spaces [8], built upon the neutrosophic logic principle of characterizing truth, falsehood, and indeterminacy simultaneously, provided a richer framework for handling incomplete, inconsistent, and uncertain information, yielding novel fixed-point results.

The ongoing quest for structures capable of representing higher-dimensional uncertainty culminated in the definition of neutrosophic quadruple metric (shortly NQM) spaces [6]. This innovative framework, extending the representational capacity of neutrosophic sets to quadruples, immediately presented the compelling challenge of establishing fixed point theorems within this new context. Addressing this gap forms a primary objective of the present study. Consequently, the first part of this paper is dedicated to exploring contraction principles and fixed-point theorems specifically within NQM spaces. We establish and prove an analogue of the fundamental Banach contraction principle in this setting and investigate other relevant fixed-point results.

Parallel to the development of metric structures in uncertain environments, the concept of normed vector spaces has also been generalized. The norm, fundamentally quantifying the "length" or magnitude of a vector, is essential for analysis in vector spaces. Significant efforts have been made to define appropriate norms for vectors residing in fuzzy [11], intuitionistic fuzzy [12], and neutrosophic vector spaces, enabling the study of functional analysis in these generalized settings. In studies [14] and [15], previous studies have examined statistical convergence in standard normed spaces and intuitionistic fuzzy normed spaces. In this study, we generalize this concept to neutrosophic quadruple normed spaces. [16] Deli et al. defined n-valued neutrosophic trapezoidal numbers with similarity measures and its properties. [17] Sahin et al. investigated the extension principles of neutrosophic multi-sets and cut sets and algebraic operators. [18] Ulucay defined similarity function of trapezoidal fuzzy multi-numbers. [19] Bakbak and Ulucay analyzed Q-neutrosophic soft expert multiset and its set operations (like union, intersection,complement,subset). [20] Baser and Ulucay studied an application of neutrosophic soft sets and its properties. [21] Baser and Ulucay investigated effective Q-neutrosophic soft expert sets on an application. No prior studies address properties such as metric, continuity, and completeness in NQS. Various applications of neutrosophic sets are discussed in this essay [22-37].

Building upon this progression, the second part of our work introduces the concept of neutrosophic quadruple normed (shortly NQN) spaces. We define a suitable norm structure for vector spaces where the vectors themselves possess neutrosophic quadruple characteristics. Furthermore, we extend this concept specifically to neutrosophic quadruple vector spaces, defining and examining the properties of the norm within this specialized context.

This study thus contributes to two expanding frontiers: establishing fundamental fixed-point results in the novel setting of NQM spaces and initiating the development of norm theory for neutrosophic quadruple sets.

2. Preliminaries

Definition 2.1 [4] Let $\triangleright: [0,1] \times [0,1] \rightarrow [0,1]$ be binary operation. Then $\triangleright$ is called a t-norm (or TN) if it satisfies the following properties for all elements $m, n, r, s \in [0,1]$:

I. Identity: $m \triangleright 1 = \mathrm{m}$.

II. Monotonicity: If $m \leq n$ and $r \leq s$, then $m \triangleright r \leq n \triangleright s$.

III. Symmetry: $m \triangleright n = n \triangleright m$.

IV. Associativity: $m \triangleright (n \triangleright r) = (m \triangleright n) \triangleright r$.

V. Continuity: The operation $\triangleright$ is continous.

Definition 2.2 [5] Let $\blacktriangleright: [0,1] \times [0,1] \rightarrow [0,1]$ be binary operation. Then $\blacktriangleright$ is called a t-conorm (or TC) if it satisfies the following proporties for all elements $m, n, r, s \in [0,1]$:

I. Identity: $m \blacktriangleright 1 = m$.

II. Monotonicity: If $m \leq n$ and $r \leq s$, then $m \blacktriangleright r \leq n \blacktriangleright s$.

III. Symmetry: $m \blacktriangleright n = n \blacktriangleright m$.

IV. Associativity: $m \blacktriangleright (n \blacktriangleright r) = (m \blacktriangleright n) \blacktriangleright r$.

V. Continuity: The operation $\blacktriangleright$ is continous.

Definition 2.3 [6] $x, y, z, k \in \mathbb{R}\ or\ \mathbb{C}$, and let T represent a truth-membership degree, I an indeterminacy-membership degree, and F a falsity-membership degree. A neutrosophic quadruple set element D is defined as:

$$D = x + yT + zI + kF$$

Here, x is called the known segment of D, while the expression $yT + zI + kF$ constitutes its unknown segment. Alternatively, the quadruple number D can be represented as an ordered tuple:

$$D = (x, yT, zI, kF)$$

$NQ = \{(x, yT, zI, kF) | x, y, z, k \in \mathbb{R}(alternatively\ \mathbb{C})\}$ is neutrosophic quadruple set.

Definition 2.4 [7] Let X be a non-empty set, $x, y \in X$, and (H, J, K) be a neutrosophic metric on $X \times X \times (0, \infty)$ where $H, J, K: X \times X \times (0, \infty) \rightarrow [0,1]$ represent truth, indeterminacy, and falsity degrees respectively. Let $d: X \times X \rightarrow [0, \infty)$ be a metric, $\varepsilon, a, b, c > 0$ be positive constants, and $' \triangleright ', ' \blacktriangleright '$ denote a t-norm (TN) and t-conorm (TC) respectively. The neutrosophic quadruple metric (NQM) $M: X \times X \rightarrow \mathbb{R} \times \mathbb{R}^3$ is defined as:

$$M(x, y) = \{(d(x, y), aH(x, y; \varepsilon), bJ(x, y; \varepsilon), cK(x, y; \varepsilon): x, y \in X\}.$$

Definition 2.5 [8] Let (H,J,K) be a NM on X. The mapping $T: X \to X$ is defined neutrosophic contraction (NC) if there exists $c \in (0,1)$ such that

$$H(T(x),T(y),\varepsilon) \geq cH(x,y,\varepsilon), J(T(x),T(y),\varepsilon) \leq cJ(x,y,\varepsilon),$$

$$K(T(x),T(y),\varepsilon) \leq cK(x,y,\varepsilon)$$

for each $x, y \in X$ and $\varepsilon > 0$.

Definition 2.6 [8] Let (H,J,K) be a NM on X and let $T: X \to X$ be a NC mapping. Then there exists $u \in U$ such that $k = T(k)$. That is, k is called neutrosophic fixed point (NFP) of T.

Theorem 2.7 [10] Let *K* denote one of the following fields: $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{Z}_p$ (where *p* is a prime). Define the neutrosophic quadruple group as the set

$$NQ = \{(a, bT, cI, dF) | a, b, c, d \in K\}$$

equipped with addition (+). Then $V = (NQ, +, \bullet)$ forms a neutrosophic quadruple vector space over *K*, where the operation ·· (scalar multiplication) is a specially defined bilinear map from $K \times NQ$ to *NQ*.

Definition 2.8 [13] Let *Q* be a vector space over a field *K* (typically $\mathbb{R}$ or $\mathbb{C}$). Let $N: Q \times \mathbb{R}^{+} \to [0,1]^3$ Be a mapping, denoted for each $p \in Q$ and $t > 0$ by

$N = \{(p, V(p,t), Y(p,t), Z(p,t))\}: p \in Q\}$, where $V(p,t)$, $Y(p,t)$, $Z(p,t)$ represent the degrees of truth, indeterminacy, and falsity of the assertion "the norm of *p* is less than or equal to *t*", respectively. Let $\triangleright$ be a t-norm and $\blacktriangleright$ be a t-conorm. The 4-tuple $(Q, N, \triangleright, \blacktriangleright)$ is called a neutrosophic normed space if for all $p, q \in Q$ and all positive real numbers k, s, t the following conditions hold:

I. $0 \leq V(p,t) \leq 1, 0 \leq Y(p,t) \leq 1, 0 \leq Z(p,) \leq 1,$

II. $V(p,t) + Y(p,t) + Z(p,t) \leq 3,$

III. $V(p,t) = 1$ if and only if $p = 0$,

IV. $V(sp,t) = H\left(p,\frac{t}{|s|}\right)$,

V. $V(p,t) \rhd H(r,s) \leq H(p+q,t+s)$,

VI. $V(p,t)$ is continuous non-decreasing function,

VII. $\lim_{\alpha\to\infty} V(p,t) = 1$,

VIII. $Y(p,t) = 0$ if and only if $p = 0$,

IX. $Y(sp,t) = Y\left(p,\frac{t}{|s|}\right)$,

X. $Y(p,t) \blacktriangleright Y(r,s) \geq Y(p+r,t+s)$,

XI. $Y(p,t)$ is continuous non-decreasing function,

XII. $\lim_{\alpha\to\infty} Y(p,t) = 0$,

XIII. $Z(p,t) = 1$ if and only if $p = 0$,

XIV. $Z(sp,t) = Z\left(p,\frac{t}{|s|}\right)$,

XV. $Z(p,t) \blacktriangleright Z(r,s) \geq Z(p+r,t+s)$,

XVI. $Z(p,t)$ is continuous non-decreasing function,

XVII. $\lim_{\alpha\to\infty} Z(p,t) = 1$,

XVIII. If $t \leq 0$, then $V(p,t) = 0$, $Y(p,t) = 1$ and $Z(p,t) = 1$.

Then $N = (V,Y,Z)$ is called neutrosophic norm (shortly NM).

Definition 2.9 [14] Let A be a subset of the natural numbers $\mathbb{N}$. The asymptotic density of A, denoted by $\delta(A)$, is defined as

$$\delta(A) = \lim_{k\to\infty} \frac{|\{a \leq k : a \in A\}|}{k},$$

where $|\cdot|$ denotes the cardinality of the set.

Let $A \subseteq \mathbb{N} \times \mathbb{N}$ be a set and let $A(x, y)$ be number pair of (a, b) in A such that $a \leq x$ and $b \leq y$. Then the two dimensional asymptotic density can be define as $\underline{\delta_2}(A) = \lim_{m,n} \inf \frac{A(x,y)}{x.y}$.

A sequence (p_n) is statistically convergent to q if for every $\varepsilon > 0$, the asymptotic density of the set $A(\varepsilon) = \{n \leq q : |p_n - q| > \varepsilon\}$ is zero. That is,

$$\lim_q \frac{1}{q} |\{n \leq q : |p_n - q| > \varepsilon\}| = 0.$$

A double sequence (p_{xy}) is statistically convergent to q if for every $\varepsilon > 0$, the double asymptotic density of the set

$$B(\varepsilon) = \{(x, y), x \leq a \ and \ y \leq b : |p_{xy} - q| \geq \varepsilon\}$$

is zero. That is,

$$\lim_{a,b\to\infty} \frac{1}{ab} \left|\{x \leq a, y \leq b : \left|p_{xy} - q\right| \geq \varepsilon\}\right| = 0.$$

Definition 2.10 [14] A sequence (p_n) is called a statistically Cauchy sequence if for every $\varepsilon > 0$,

$$\lim_q \frac{1}{q} \left|\{m \leq q : \left|p_m - p_q\right| \geq \varepsilon\}\right| = 0.$$

Definition 2.11 [15] Let $(A, \mu, \vartheta, \triangleright, \blacktriangleright)$ be an intuitionistic fuzzy normed space. A double sequence (p_{xy}) is statistically convergent to $q \in A$ with respect to the intuitionistic fuzzy norm (μ, ϑ) if for every $\varepsilon > 0$ and every $\epsilon > 0$,

$$\delta_2\{(x, y) \in \mathbb{N} \times \mathbb{N} : \mu\left(p_{xy} - q, \epsilon\right) \leq 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - q, \epsilon\right) \geq \varepsilon\} = 0.$$

That is

$$\lim_{a,b\to\infty} \frac{1}{ab} \left|\{a \leq x, b \leq y : \mu\left(p_{xy} - q, \epsilon\right) \leq 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - q, \epsilon\right) \geq \varepsilon\}\right| = 0.$$

Definition 2.12 [15] Let $(A, \mu, \vartheta, \triangleright, \blacktriangleright)$ be an intuitionistic fuzzy normed space. A double sequence (p_{xy}) is statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ϑ) if for every $\varepsilon > 0$ and every $\epsilon > 0$,

$$\delta_2\{(x, y) \in \mathbb{N} \times \mathbb{N} : \mu\left(p_{xy} - p_{ab}, \epsilon\right) \leq 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - p_{ab}, \epsilon\right) \geq \varepsilon\} = 0.$$

That is

$$\lim_{a,b\to\infty}\frac{1}{ab}\left|\left\{a \le x, b \le y: \mu\left(p_{xy} - q_{xy}, \epsilon\right) \le 1 - \varepsilon \text{ and } \vartheta\left(p_{xy} - q_{xy}, \epsilon\right) \ge \varepsilon\right\}\right| = 0.$$

3. Fixed-Point Theorems In Neutrosophic Quadruple Metric Spaces

Definition 3.1 Let $(Q, N, \triangleright, \blacktriangleright)$ be a NQM space. The mapping $T: Q \to Q$ is called neutrosophic quadruple contraction if there exist $0 < c < 1$ such that

$$d\left(T(p), T(q)\right) \le cd(p,q),$$

$$H(T(p), T(q), c\varepsilon) \ge H(p,q,\varepsilon),$$

$$J(T(p), T(q), c\varepsilon) \le J(p,q,\varepsilon),$$

$$K(T(p), T(q), c\varepsilon) \le K(p,q,\varepsilon),$$

for each $p, q \in Q$ and $\varepsilon > 0$.

We will define neutrosophic quadruple banach contraction theorem.

Theorem 3.1 Let $(Q, N, \triangleright, \blacktriangleright)$ be a neutrosophic quadruple metric space (NQM). If $T: Q \to Q$ is a neutrosophic quadruple contraction mapping, then there exists an element $c \in Q$ such that $c = T(c)$. This element *c* is called a neutrosophic quadruple fixed point of *T*.

Proof Let $p \in Q$ and $p_n = T^n p$ $(p \in \mathbb{N})$. By a induction we get

$$d(p_n, p_{n+1}) \le d(p, p_1),$$

$$H(p_n, p_{n+1}, \varepsilon) \ge H\left(p, p_1, \frac{\varepsilon}{t^n}\right),$$

$$J(p_n, p_{n+1}, \varepsilon) \le J\left(p, p_1, \frac{\varepsilon}{t^n}\right),$$

$$K(p_n, p_{n+1}, \varepsilon) \le K\left(p, p_1, \frac{\varepsilon}{t^n}\right),$$

for each $n \in \mathbb{N}$ and $\varepsilon > 0$. For any $m \in \mathbb{Z}$,we get

$$|d(p_n,p_{n+m})| \le |d(p_n,p_{n+1})| + \cdots + |d(p_{n+m-1},p_{n+m})|$$
$$\le |d(p_n,p_{n+1})| + \cdots + |d(p_n,p_{n+m})|,$$

$$H(p_n,p_{n+k},\varepsilon) \ge H\left(p_n,p_{n+1},\frac{\varepsilon}{m}\right) \rhd \ldots \rhd H\left(p_{n+m-1},p_{n+m},\frac{\varepsilon}{m}\right)$$
$$\ge H\left(p,p_1,\frac{\varepsilon}{mt^n}\right) \rhd \ldots \rhd H\left(p,p_1,\frac{\varepsilon}{mt^{n+k-1}}\right),$$

$$J(p_n,p_{n+m},\varepsilon) \le J\left(p_n,p_{n+1},\frac{\varepsilon}{m}\right) \blacktriangleright \ldots \blacktriangleright J\left(p_{n+m-1},p_{n+m},\frac{\varepsilon}{m}\right)$$
$$\le J\left(p,p_1,\frac{\varepsilon}{mt^n}\right) \blacktriangleright \ldots \blacktriangleright J\left(p,p_1,\frac{\varepsilon}{mt^{n+m-1}}\right),$$

$$K(p_n,p_{n+m},\varepsilon) \le K\left(p_n,p_{n+1},\frac{\varepsilon}{m}\right) \blacktriangleright \ldots \blacktriangleright K\left(p_{n+m-1},p_{n+m},\frac{\varepsilon}{m}\right)$$
$$\le K\left(p,p_1,\frac{\varepsilon}{mt^n}\right) \blacktriangleright \ldots \blacktriangleright K\left(p,p_1,\frac{\varepsilon}{mt^{n+m-1}}\right).$$

If we apply the limit conditions for the functions H, J, K in the neutrosophic metric space definition, we get

$$\lim_{n\to\infty} d(p_{n+m},p_n) = 0 + \cdots + 0 = 0,$$

$$\operatorname{Lim}_{n\to\infty} H(p_{n+m},p_n,\varepsilon) \ge 1 \rhd \ldots \rhd 1 = 1,$$

$$\lim_{n\to\infty} J(p_{n+m},p_n,\varepsilon) \le 0 \blacktriangleright \ldots \blacktriangleright 0 = 0$$

and

$$\lim_{n\to\infty} K(p_{n+m},p_n,\varepsilon) \le 0 \blacktriangleright \ldots \blacktriangleright 0 = 0.$$

That is, $\{p_n\}$ cauchy sequence. Hence $\{p_n\}$ is convergent. We define a limit point q for $\{p_n\}$ sequence. We get

$$d(Tq,q) \le d(Tq,Tp_n) \to 0$$

$$H(Tq,q,\varepsilon) \ge H\left(Tq,Tp_n,\frac{\varepsilon}{2}\right) \rhd H\left(p_{n+1},q,\frac{\varepsilon}{2}\right)$$
$$\ge H\left(q,p_n,\frac{\varepsilon}{2m}\right) \rhd H\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \to 1 \rhd 1 = 1$$

$$J(Tq,q,\varepsilon) \leq J\left(Tq,Tp_n,\frac{\varepsilon}{2}\right) \blacktriangleright J\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \leq J\left(q,p_n,\frac{\varepsilon}{2m}\right) \blacktriangleright J\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \to 0 \blacktriangleright 0$$
$$= 0,$$

$$K(Tq,q,\varepsilon) \leq K\left(Tq,Tp_n,\frac{\varepsilon}{2}\right) \blacktriangleright K\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \leq K\left(q,p_n,\frac{\varepsilon}{2m}\right) \blacktriangleright K\left(p_{n+1},q,\frac{\varepsilon}{2}\right) \to 0 \blacktriangleright 0$$
$$= 0.$$

Thus we get $Tq = q$, a fixed point. Now to get uniqueness, we assume $Tr = r$ for any $r \in Q$.

Then t

$$0 \leq d(r,q) = d(Tr,Tq) \to 0,$$

$$1 \geq H(r,q,\varepsilon) = H(Tr,Tq,\varepsilon) \geq H\left(r,q,\frac{\varepsilon}{m}\right) = H\left(Tr,Tq,\frac{\varepsilon}{m}\right) \geq H\left(r,q,\frac{\varepsilon}{m^2}\right) \geq \cdots$$
$$\geq H\left(r,q,\frac{\varepsilon}{m^n}\right) \to 1,$$

$$0 \leq J(r,q,\varepsilon) = J(Tr,Tq,\varepsilon) \leq J\left(r,q,\frac{\varepsilon}{m}\right) = J\left(Tr,Tq,\frac{\varepsilon}{m}\right) \leq J\left(r,q,\frac{\varepsilon}{m^2}\right) \leq \cdots$$
$$\leq J\left(r,q,\frac{\varepsilon}{m^n}\right) \to 0,$$

$$0 \leq J(r,q,\varepsilon) = J(Tr,Tq,\varepsilon) \leq J\left(r,q,\frac{\varepsilon}{m}\right) = J\left(Tr,Tq,\frac{\varepsilon}{m}\right) \leq J\left(r,q,\frac{\varepsilon}{m^2}\right) \leq \cdots$$
$$\leq J\left(r,q,\frac{\varepsilon}{m^n}\right) \to 0$$

when $n \to \infty$. So $r = q$.

Corollary 3.1 Let $(Q,N,\triangleright,\blacktriangleright)$ be a complete NQM space and let $T: Q \to Q$ be a neutrosophic quadruple contraction mapping. Then *T* possesses a unique fixed point in Q.

Now we will show edelstein contraction theorem for NQM spaces.

Theorem 3.2 Let $(Q,N,\triangleright,\blacktriangleright)$ be a NQM space and let $T: Q \to Q$ be a neutrosophic quadruple contraction mapping. if for all $p \neq q$

$$d\big(T(p),T(q)\big) < cd(p,q),$$

$$H(T(p),T(q),.) > H(p,q,.),$$

$$J(T(p),T(q),.) < J(p,q,.),$$

$$K(T(p),T(q),.) < K(p,q,.),$$

then T has a unique fixed point.

Proof Let $p \in Q$ ve $p_n = T^n p$ $(n \in \mathbb{N})$. $p_n \neq p_{n+1}$ for each n. Consequently for $n \neq m$, $p_n \neq p_m$. İf not, we get

$$d(p_n, p_{n+1}) = d(p_m, p_{m+1}) < d(p_{m-1}, p_m) < \cdots < d(p_n, p_{n+1}),$$

$$H(p_n, p_{n+1}, .) = H(p_m, p_{m+1}, .) > H(p_{m-1}, p_m, .) > \cdots > H(p_n, p_{n+1}, .),$$

$$J(p_n, p_{n+1}, .) = J(p_m, p_{m+1}, .) < J(p_{m-1}, p_m, .) < \cdots < J(p_n, p_{n+1}, .)$$

and

$$K(p_n, p_{n+1}, .) = K(p_m, p_{m+1}, .) < K(p_{m-1}, p_m, .) < \cdots < K(p_n, p_{n+1}, .)$$

where $m > n$, a contradiction. $\{p_n\}$ has a convergent subsequence $\{p_{n_i}\}$, because of Q is compact. Let $q = \lim_i p_{n_i}$. Moreover, we call that $q, Tq \notin \{p_{n_i} : i \in \mathbb{N}\}$. From our assumptions, we can write

$$d(Tp_n, Tq) < d(p_{n_i}, q),$$

$$H(Tp_n, Tq, .) > H(p_n, q, .),$$

$$J(Tp_n, Tq, .) < J(p_n, q, .)$$

and

$$K(Tp_n, Tq, .) < K(p_n, q, .)$$

for all $i \in \mathbb{N}$. So, we get

$$\lim_{i \to \infty} d(Tp_{n_i}, Tq) \leq \lim_{i \to \infty} d(p_{n_i}, q) = d(q, q) = 0,$$

$$\lim_{i \to \infty} H(Tp_{n_i}, Tq, \varepsilon) \geq \lim_{i \to \infty} H(p_{n_i}, q, \varepsilon) = H(q, q, \varepsilon) = 1,$$

$$\lim_{i\to\infty} J(Tp_{n_i}, Tq, \varepsilon) \leq \lim_{i\to\infty} J(p_{n_i}, q, \varepsilon) = J(q, q, \varepsilon) = 0$$

and

$$\lim_{i\to\infty} K(Tp_{n_i}, Tq, \varepsilon) \leq \lim_{i\to\infty} K(p_{n_i}, q, \varepsilon) = K(q, q, \varepsilon) = 0$$

for each $\varepsilon > 0$. Hence

$$\lim_{i\to\infty} Tp_{n_i} = Tq \ \ and \lim_{i\to\infty} T^2 p_{n_i} = T^2 q.$$

We get

$$d(p_{n_i}, Tp_{n_i}) \geq d(Tp_{n_i}, T^2 p_{n_i}) \geq \cdots \geq d(Tp_{n_{i+1}}, Tp_{n_{i+1}}) \geq d(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}) \geq \cdots \geq 0,$$

$$H(p_{n_i}, Tp_{n_i}, \varepsilon) \leq H(Tp_{n_i}, T^2 p_{n_i}, \varepsilon) \leq \cdots \leq H(p_{n_i}, Tp_{n_i}, \varepsilon) \leq H(Tp_{n_i}, T^2 p_{n_i}, \varepsilon) \leq \cdots$$
$$\leq H(p_{n_{i+1}}, Tp_{n_{i+1}}, \varepsilon) \leq H(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}, \varepsilon) \leq \cdots \leq 1,$$

$$J(p_{n_i}, Tp_{n_i}, \varepsilon) \geq J(Tp_{n_i}, T^2 p_{n_i}, \varepsilon) \geq \cdots \geq J(p_{n_i}, Tp_{n_i}, \varepsilon) \geq J(Tp_{n_i}, T^2 p_{n_i}, \varepsilon) \geq \cdots$$
$$\geq J(p_{n_{i+1}}, Tp_{n_{i+1}}, \varepsilon) \geq J(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}, \varepsilon) \geq \cdots \geq 0$$

and

$$K(p_{n_i}, Tp_{n_i}, \varepsilon) \geq K(Tp_{n_i}, T^2 p_{n_i}, \varepsilon) \geq \cdots \geq K(p_{n_i}, Tp_{n_i}, \varepsilon) \geq K(Tp_{n_i}, T^2 p_{n_i}, \varepsilon) \geq \cdots$$
$$\geq K(p_{n_{i+1}}, Tp_{n_{i+1}}, \varepsilon) \geq K(Tp_{n_{i+1}}, T^2 p_{n_{i+1}}, \varepsilon) \geq \cdots \geq 0$$

for every $\varepsilon > 0$. Thus

$$d(p_{n_i}, Tp_{n_i}), H(p_{n_i}, Tp_{n_i}, \varepsilon), J(p_{n_i}, Tp_{n_i}, \varepsilon) \text{ and } K(p_{n_i}, Tp_{n_i}, \varepsilon)$$

are convert to same limit point. We get

$$d(q, Tq) \leq \lim_{i\to\infty} sum\ d(p_{n_i}, Tp_{n_i}) = \lim_{i\to\infty} sum\ d(Tp_{n_i}, T^2 p_{n_i}) \leq \lim_{i\to\infty} inf\ d(Tp_{n_i}, T^2 p_{n_i})$$
$$\leq d(Tq, T^2 q),$$

$$H(q,Tq,\varepsilon) \geq \lim_{i\to\infty} sum\, H\left(p_{n_i},Tp_{n_i},\varepsilon\right) = \lim_{i\to\infty} sum\, H\left(Tp_{n_i},T^2p_{n_i},\varepsilon\right)$$

$$\geq \lim_{i\to\infty} inf\, H\left(Tp_{n_i},T^2p_{n_i},\varepsilon\right) \geq H(Tq,T^2q,\varepsilon),$$

$$J(q,Tq,\varepsilon) \leq \lim_{i\to\infty} sum\, J\left(p_{n_i},Tp_{n_i},\varepsilon\right) = \lim_{i\to\infty} sum\, J\left(Tp_{n_i},T^2p_{n_i},\varepsilon\right)$$

$$\leq \lim_{i\to\infty} inf\, J\left(Tp_{n_i},T^2p_{n_i},\varepsilon\right) \leq J(Tq,T^2q,\varepsilon)$$

and

$$J(q,Tq,\varepsilon) \leq \underline{lim}_{i\to\infty}\, sum\, J\left(p_{n_i},Tp_{n_i},\varepsilon\right) = \underline{lim}_{i\to\infty}\, sum\, J\left(Tp_{n_i},T^2p_{n_i},\varepsilon\right)$$

$$\leq \underline{lim}_{i\to\infty}\, inf\, J\left(Tp_{n_i},T^2p_{n_i},\varepsilon\right) \leq J(Tq,T^2q,\varepsilon)$$

for every $\varepsilon > 0$.

We assume $q \neq Tq$. Similarly we get

$$d(q,Tq) > d(Tq,T^2q),$$

$$H(q,Tq,.) < H(Tq,T^2q,.),$$

$$J(q,Tq,.) > J(Tq,T^2q,.)$$

and

$$K(q,Tq,.) > K(Tq,T^2q,.).$$

Those are a contradiction. Hence, $q = Tp,$ a fixed point.

4. Neutrosophic Quadruple Normed Spaces

Definition 4.1 Let Q be a vector space over K field, $p \in Q,$ and (V,Y,Z) be a **neutrosophic norm** on $Q \times (0,\infty)$ where $V,Y,Z: Q \times (0,\infty) \to [0,1]$ denote **truth, indeterminacy**, and **falsity** norm degrees, respectively. Let $\|.\|$ be a conventional norm on $Q,$ $\varepsilon, a, b, c > 0$ be fixed positive constants, and let '$\triangleright$' and '$\blacktriangleright$' denote a t-norm (TN) and t-conorm (TC), respectively. The **neutrosophic quadruple norm (NQN)** is the function $N: Q \to [0,\infty) \times [0,\infty)^3$ defined for each $p \in Q$, as:

$$N(p) = (\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon)).$$

$(N, Q, \triangleright, \blacktriangleright, \|.\|)$ is defined as NQN space.We calculate by $\|p\|$ norm of the known part of the vector p, how accurately the norm is measured with a, how ambiguously the norm is measured with b, how inaccurately the norm is measured with c, by V the degree to which the norm of vector p is correct, Y the degree to which the norm of vector p is imprecise, Z the degree to which the norm of vector p is incorrect

Example 4.1 Let $(N, \mathbb{R}, \triangleright, \blacktriangleright, \|.\|)$ be a NQN. it define as TN $p \triangleright q = pq$ and TC $p \blacktriangleright q = p + q - pq$. $\forall p \in \mathbb{R}$, $N = \{(|p|, a\frac{\varepsilon}{\varepsilon+|p|}, b\frac{|p|}{\varepsilon+|p|}, c\frac{|p|}{\varepsilon} : p \in Q\}$ neutrosophic quadruple set define NQN.

Example 2 Let $(N, Q, \triangleright, \blacktriangleright, \|.\|)$ be a NM space. If we take

$H(p,q,\delta) = V(p-q,\delta), J(p,q,\delta) = Y(p-q,\delta)$ and $K(p,q,\delta) = Z(p-q,\delta)$,

then $N = (H, J, K)$ a neutrosophic metric on Q, which is induced by the NQN N.

Definition 4.2 Let $(N, Q, \triangleright, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple-normed space (abbreviated NM-space).

I. A sequence (p_n) in N is a Cauchy sequence if for every $\delta > 0$ and $\tau > 0$, there exists $n_0 \in N$ such that for all $n, m \geq n_0$:

$$\|p_n - p_m\| < \delta,$$

$$aV(p_n - p_m, \tau) > 1 - \delta,$$

$$bY(p_n - p_m, \tau) < \delta,$$

$$cZ(p_n - p_m, \tau) < \delta.$$

II. The sequence (p_n) converges to $p \in N$ (denoted $p_n \to p$) if for every $\tau > 0$:

$$\lim_{n\to\infty} \|p_n - p\| = 0,$$

$$\lim_{n\to\infty} aV(p_n - p_m, \tau) = 1,$$

$$\lim_{n\to\infty} bY(p_n - p_m, \tau) = 0$$

and

$$\lim_{n\to\infty} cZ(p_n - p_m, \tau) = 0.$$

III. The space N is complete if every Cauchy sequence in N converges to a limit in N.

Definition 4.3 Let $(N, Q, \triangleright, \blacktriangleright)$ be a neutrosophic quadruple-normed space (NQN space) with $Q \subseteq N$.

I. The open ball $B(p, r, \tau)$ centered at $p \in Q$ with radius $r \in (0,1)$ and parameter $\tau > 0$ is defined as:

$$B(p, r, \tau) = \left\{ p \in Q : \begin{array}{c} \|p - q\| < r, \\ aV(p - q, \tau) > 1 - r, \\ bY(p - q, \tau) < r, \\ cZ(p - q, \tau) < r \end{array} \right\}.$$

II. A subset $P \subseteq Q$ is open if for every $p \in P$, there exist $r \in (0,1)$ and $\tau > 0$ such that $B(p, r, \tau) \subseteq P$.

III. The topology induced by the NQN, denoted t_N, is the family of all open subsets of Q.

Definition 4.4 A subset $P \subseteq Q$ is neutrosophic quadruple bounded in a neutrosophic quadruple-normed space $(N, Q, \triangleright, \blacktriangleright)$ if there exist $r \in (0,1)$ and $\tau > 0$ such that for every $p \in P$:

$$\|p - q\| < r,$$

$$aV(p - q, \tau) > 1 - r,$$

$$bY(p-q,\tau) < r,$$

$$cZ(p-q,\tau) < r.$$

Theorem 4.1 In any neutrosophic quadruple-normed space, all compact subsets exhibit neutrosophic quadruple boundedness.

Proof Let $(N, Q, \rhd, \blacktriangleright, \|.\|)$ be a NQN space and Let $P \subseteq Q$ compact subset. Consider the set $\{B(p,r,\tau): p \in P\}$ for $\tau > 0$ and $0 < r < 1$. Because of P compact, there are $p_1, p_2, \dots, p_n \in P$ such that $P \subseteq \bigcup_{m=1}^{n} B(p_m, r, \tau)$. For some $m, t \in \mathbb{N}$ and $p, q \in P$, $p \in B(p_m, r, \tau)$ and $q \in B(p_m, r, \tau)$. So we get

$$\|p-p_m\| < r, aV(p-p_m,\tau) > 1-r, bY(p-p_m,\tau) < r, cZ(p-p_m,\tau) < r$$

and

$$\|q-p_m\| < r, aV(q-p_m,\tau) > 1-r, bY(q-p_m,\tau) < r, cZ(q-p_m,\tau) < r.$$

Forthe

$\alpha = \max\{\|p_t - p_m\|: 1 \leq m,t \leq n\}, \beta = \min\{aV(p_t - p_m,\tau): 1 \leq m,t \leq n\}, \gamma = \max\{ bY(p_t - p_m,\tau): 1 \leq m,t \leq n\}, \delta = \max\{cZ(p_t - p_m,\tau): 1 \leq m,t \leq n\}$

. Then $\alpha, \beta, \gamma, \delta > 0$. So we get

$$\|p_t - p_m\| \leq \|p_t - p_k\| + \|p_k - p_m\| \leq \tau_1,$$

$$V(p_t - p_m,\tau) \geq V(p_t - p_k,\tau) \rhd V(p_k - p_m,\tau) \geq (1-\tau_2) \rhd (1-\tau_2) \geq (1-\tau_2),$$

$$Y(p_t - p_m,\tau) \leq Y(p_t - p_k,\tau) \blacktriangleright Y(p_k - p_m,\tau) \leq (1-\tau_3) \blacktriangleright (1-\tau_3) \leq (1-\tau_3)$$

and

$$Z(p_t - p_m,\tau) \leq V(p_t - p_k,\tau) \blacktriangleright V(p_k - p_m,\tau) \leq (1-\tau_4) \blacktriangleright (1-\tau_4) \leq (1-\tau_4)$$

for the $\tau_1, \tau_2, \tau_3, \tau_4 \in (0,1)$. If we take $\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$, we have

$$\|p\| < r, aV(p,\tau) > 1-r, bY(p,\tau) < r, cZ(p,\tau) < r.$$

We get that compact subset P of a NQN space is neutrosophic quadruple bounded.

Corollary 4.1 In a NQN space, every compact set is closed and neutrosophic quadruple bounded.

The proof of this corollary can be easily obtained from the above theorems and definitions.

Definition 4.5 Let $(Q, N, \triangleright, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple normed space. Then a sequence

$p = (p_{xy})$ is called to statistically convergent to $q \in A$ with to nutrosophic quadruple norm

$N(p) = ((\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon))$ hold on that for all $\varepsilon > 0$ and $\epsilon > 0$,

$$\delta_N\{(x,y) \in \mathbb{N} \times \mathbb{N} : \|p_{xy} - q\| \leq 0, V(p_{xy} - q, \epsilon) \leq 1 - \varepsilon, Y(p_{xy} - q, \epsilon) \geq \varepsilon \text{ and } Z(p_{xy} - q, \epsilon) \geq \varepsilon\} = 0.$$

That is

$$\lim_{x,y} \frac{1}{xy} \left|\{a \leq x, b \leq y : \|p_{xy} - q\| \leq 0, V(p_{xy} - q, \epsilon) \leq 1 - \varepsilon, Y(p_{xy} - q, \epsilon) \geq \varepsilon \text{ and } Z(p_{xy} - q, \epsilon) \geq \varepsilon\}\right| = 0.$$

Example 4.3 Let $(\mathbb{R}, |.|)$ be classic norm for real numbers and for the t-norm is $m \triangleright \mathrm{n} = \mathrm{mn}$ and t-conorm is $m \blacktriangleright \mathrm{n} = \min\{m + n, 1\}$ for every $m, n \in [0,1]$. For $p \in \mathbb{R}$ and every $\varepsilon > 0$, is defined as

$$N = (\|p\| = |p|, aV(p,\varepsilon) = \frac{\varepsilon}{\varepsilon + |p|}, bY(p,\varepsilon) = \frac{|p|}{\varepsilon + |p|}, cZ(p,\varepsilon) = \frac{|p|}{\varepsilon})$$

Then $(\mathbb{R}, N, \triangleright, \blacktriangleright, |.|)$ is a neutrosophic quadruple normed space. For the $p = (p_{xy})$ sequence is defined by

$$p_{xy} = \begin{cases} xy, if\ x\ and\ y\ is\ a\ cube\ roots. \\ 0, otherwise. \end{cases}$$

$$A_{x,y}(\varepsilon, \epsilon) = \{x \leq a, y \leq b : p_{xy} = \sqrt[3]{xy}\}.$$

Then, we get

$$\frac{1}{xy}|A_{x,y}(\varepsilon,\epsilon)| \leq \frac{1}{xy}|\{x \leq a \leq, y \leq b : a\ and\ b\ is\ a\ cube\ roots\}| \leq$$

$$\leq \frac{\sqrt[3]{a}\sqrt[3]{b}}{ab} \to 0\ as\ a, b \to \infty.$$

Definition 4.6 Let $(Q, N, \rhd, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple normed space. Then a double sequence (p_{xy}) is called to statistically cauchy convergent to $q \in A$ with to neutrosophic quadruple norm $N(p) = ((\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon))$ if for all $\varepsilon > 0$ and $\epsilon > 0$, there exist $S = S(\varepsilon)$ and $T = T(\varepsilon)$ such that for every $x, a \geq S; y, b \geq T$

$$\delta_2\{(x,y) \in \mathbb{N} \times \mathbb{N}: \|p\| \leq 0, aV(p_{xy} - p_{ab}, \epsilon) \leq 1 - \varepsilon, cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon$$

and

$$cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon\} = 0.$$

Definition 4.6 Let $(Q, N, \rhd, \blacktriangleright, \|.\|)$ be a neutrosophic quadruple normed space. A double sequence (p_{xy}) is statistically Cauchy with respect to the neutrosophic quadruple norm

$N(p) = ((\|p\|, aV(p,\varepsilon), bY(p,\varepsilon), cZ(p,\varepsilon))$ if for every $\varepsilon > 0$ and $\epsilon > 0$

$$\delta_2\{(x,y) \in \mathbb{N} \times \mathbb{N}: \|p\| \leq 0, aV(p_{xy} - p_{ab}, \epsilon) \leq 1 - \varepsilon, cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon$$

and

$$cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon\} = 0.$$

That is

$$\lim_{a,b\to\infty} \frac{1}{ab} \left| \begin{Bmatrix} a \leq x, b \leq y: \|p\| \leq 0, aV(p_{xy} - p_{ab}, \epsilon) \leq 1 - \varepsilon, \\ cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon \text{ and } cZ(p_{xy} - p_{ab}, \epsilon) \geq \varepsilon \end{Bmatrix} \right| = 0.$$

5. Conclusions

In this study, we have defined the structure of neutrosophic quadruple normed spaces based on the neutrosophic quadruple set. Within this framework, we established the neutrosophic quadruple Banach contraction theorem and proved fundamental fixed-point theorems in neutrosophic quadruple metric spaces. We obtained definitions of statistical convergence and Cauchy statistical convergence for neutrosophic quadruple normed spaces.

This research successfully bridges a critical theoretical gap. It not only provides foundational fixed-point results for the newly introduced neutrosophic quadruple metric spaces but also pioneers the development of the theory of neutrosophic quadruple normed spaces. These contributions furnish essential mathematical tools and open significant avenues for advanced analysis under complex, multifaceted uncertainty. Collectively, the results presented offer a natural and substantial extension of existing knowledge in fixed point theory and functional analysis within fuzzy, intuitionistic fuzzy, and neutrosophic settings.

Abbreviations

FS Fuzzy set

FN Fuzzy norm

NN Neutrosophic Norm

NQM Neutrosophic quadruple metric

NQN Neutrosophic quadruple norm

TN Continuous t-norm

TC Continuous t-conorm

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Comparisons of Infinitesimally Punctured Wave with Copenhagen and De Broglie-Bohm Interpretations, Neutrosophic Quantum Mechanics, and Non-Linear Electromagnetics

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Abstract: The article's goal is to compare the "infinitesimally punctured wave" with the major interpretations of the fundamental philosophical differences in how physicists view reality at the quantum level.

Keywords: Infinitesimally Punctured Wave, Duality, Copenhagen Interpretation (Many Worlds Interpretation), De Broglie-Bohm Theory (Pilot-Wave Theory - Hidden Variables), Smarandache Algebraic Structures, Transactional Interpretation, General Relativity, Neutrosophic Quantum Theory, Quantum Gravity, Kaluza-Klein Theory, Born-Infeld Model, Non-Linear Schrödinger Equation, Non-Classical Soliton, Non-Linear Electrodynamics.

1. Interpretation of the wave-particle duality as an infinitesimally punctured wave

Here there are comparisons of these interpretations of wave-particle duality and the wavefunction.

The interpretation of **wave-particle duality** as an **infinitesimally punctured wave** is a specific conceptual model, largely associated with the work of physicist Florentin Smarandache who defined it in 2019 {[3], [4]} and explored alternative foundational frameworks for quantum mechanics.

This model attempts to reconcile the seemingly contradictory wave and particle descriptions by viewing the quantum entity not as two mutually exclusive things, but as a single entity with an inherent structure that incorporates both aspects.

2. The Infinitesimally Punctured Wave Model

In this interpretation, the quantum entity (like an electron or photon) is fundamentally a wave, but its identity as a particle arises from a structural singularity or defect within the wave itself.

2.1. The Wave Component

The primary structure is the **wave** (described by the wavefunction ψ):

- It is **non-localized** and extends through space.
- It is responsible for phenomena like **diffraction** and **interference**.
- This wave carries the **energy** and **momentum** of the particle over a region of space.

2.2. The Puncture (The Particle Component)

The particle aspect is represented by an infinitesimally small puncture, singularity, or structural defect within the wave:

- This puncture is a point of **extreme localization** or a **singularity** within the field.
- It is where the wave's energy and momentum are **compressed** or concentrated into a single, measurable location.
- This singularity is responsible for the **localized impacts** observed during measurement, such as a flash on a detector screen or a discrete hit on a photographic plate.

3. Mathematical and Conceptual Implications

3.1. Resolution of Collapse

This model offers a distinct perspective on the **collapse of the wavefunction** during measurement:

- **Standard View (Copenhagen):** The wave is a description of probability, which instantaneously collapses to a localized particle upon measurement.
- **Punctured Wave View:** The wave never truly "collapses." The measurement process simply isolates and registers the presence of the **fixed singularity** (the particle) that was already embedded within the extended wave structure. The wave guides the singularity, but the singularity itself is what interacts locally.

3.2. Connection to Smarandache Structures

This interpretation aligns philosophically with **Smarandache Algebraic Structures** because it involves an entity where:

- The **weak (extensive) structure** is the smooth, continuous wave.
- The **strong (localized) structure** is the discrete, singular particle point (the "puncture") existing as a proper subset (or singularity) within the whole wave field.

3.3. Comparison to de Broglie-Bohm Theory

This concept shares philosophical ground with the **de Broglie–Bohm pilot wave theory** (or ontological interpretation), where:

- A **real, physical wave** exists (the pilot wave).
- A **real particle** is carried along by the wave, guided by it but maintaining its discrete identity.

The "punctured wave" model is similar in spirit but more radical, suggesting that the particle *is* the singularity **of** the wave itself, rather than a separate entity merely guided *by* the wave.

4. Comparison of Quantum Interpretations

Feature	*Infinitesimally Punctured Wave*	*Copenhagen Interpretation (CI)*	*Many-Worlds Interpretation (MWI)*
Wavefunction (ψ)	Represents a **real, physical field** that extends through space.	Represents **knowledge** or **probability amplitude**; not physically real.	Represents a **real, physical field** (like the Punctured Wave), evolving deterministically.
The Particle	A **physical singularity** or **defect (puncture)** embedded *within* the wave structure.	Exists only as a definite entity **upon measurement**; prior to measurement, only probabilities exist.	Is a localized manifestation arising from the ψ field; all possible states are physically real in parallel universes.
Wave-Particle Duality	**Structural Unity:** The particle is the singularity of the wave; the two are	**Complementarity:** The particle and wave aspects are mutually	**Wave Only:** Everything is a wave; the appearance

	inseparable components of a single entity.	exclusive and revealed depending on the experimental setup.	of a particle (localization) is due to decoherence.
Measurement Problem	**No true collapse.** Measurement simply reveals the fixed location of the singularity already present within the wave structure.	**Wavefunction Collapse:** The ψ instantaneously and non-locally "collapses" from a superposition of possibilities to a single outcome.	**No Collapse.** Measurement causes the observer and system to become **entangled**, splitting the universe into non-communicating branches for every possible outcome.
Causality	**Ontological/Deterministic:** The underlying wave and singularity follow definite, albeit complex, laws.	**Non-Deterministic/Probabilistic:** The act of collapse is fundamentally random (non-causal).	**Deterministic:** The evolution of the total wavefunction is smooth and deterministic.

5. Key Distinctions

5.1. Focus on Reality (Ontology)

- The **Punctured Wave** and **MWI** are **ontological interpretations**—they assert that the wavefunction is physically real.
- The **CI** is an **epistemological interpretation**—it asserts that the wavefunction is a tool representing our knowledge or probability, not a physical entity.

5.2. Handling the "Collapse"

- The **Punctured Wave** solves the collapse problem by proposing that the particle state **never truly collapses**; it was always localized as the singularity, and the wave merely defined its potential location.
- The **MWI** solves the collapse problem by **eliminating it** entirely; all possibilities occur in separate, branching universes.
- The **CI** accepts **collapse** as a fundamental, unexplained feature of the measurement process.

5.3. Superposition

- In the **Punctured Wave** model, a superposition means the **wave is simultaneously spread out** in configuration space, but the **singularity (particle) is still definite,** but its *location* is probabilistically determined by the wave's amplitude.
- In the **CI**, the particle **literally exists in all possible states** simultaneously until observed.
- In the **MWI**, all states in the superposition **actually exist** simultaneously in parallel universes.

Comparisons between infinitesimally punctured wave and other theories and phenomena that may be related to it

The "infinitesimally punctured wave" (IPW) interpretation can be compared and contrasted with several other theories and phenomena in physics, particularly those dealing with the foundations of quantum mechanics, wave-particle duality, and the structure of space-time.

6. Interpretations of Quantum Mechanics

The IPW theory, which posits a real wave containing a singular "puncture" (the particle), directly relates to the central debates in quantum foundations:

6.1. De Broglie-Bohm Theory (Pilot-Wave Theory)

Comparison Point	*Infinitesimally Punctured Wave (IPW)*	*De Broglie-Bohm Theory (DBB)*
Nature of Wave	**Real, physical field** (ψ).	**Real, physical field** (Pilot Wave, guiding wave).
Nature of Particle	The **particle *is* the singularity/puncture** of the wave itself.	The **particle is a separate, definite entity** always having a precise position.
Relationship	**Unity/Singularity:** Particle is structurally part of the wave.	**Guidance:** Particle is guided by the separate wave field.
Causality	**Deterministic/Ontological:** Both follow definite laws.	**Deterministic/Ontological:** Known for being a deterministic theory.

Relation: DBB is the closest cousin. Both are **ontological interpretations** (claiming the wave is real and deterministic) and use a physical wave to explain quantum phenomena. The key difference is the IPW's assertion that the particle is *made of* the wave (the singularity), whereas DBB posits a particle that is separate *from* the wave.

6.2. Transactional Interpretation (TI)

The TI interprets quantum interactions as a transaction (a handshake) between retarded (forward-in-time) and advanced (backward-in-time) waves.

- **Contrast:** TI deals with the **process** of interaction and measurement using time-symmetric waves. IPW focuses on the **structure** of the individual quantum entity (the wave-singularity pair).
- **Similarity:** Both are highly conceptual, non-mainstream theories that seek to provide a concrete physical picture underlying the mathematical formalism of quantum mechanics.

7. Theories of Space-Time and Singularity

The "puncture" concept relates to how singularities are treated in general relativity and field theories.

7.1. General Relativity (GR) and Point Singularities

- **Relation:** The IPW models the particle as a **point singularity** in a field (the wave). In GR, singularities (like those at the center of black holes) are points where the traditional laws of physics break down, often associated with infinite density or curvature.
- **Interpretation:** The IPW applies this singular point concept to quantum particles, suggesting that the particle's mass and charge are localized features stemming from this puncture, much like a mass point curves space-time in GR.

8. Classical Field Theories (Electromagnetism)

- **Contrast:** In classical electromagnetism, a particle (like an electron) is modeled as a localized charge that *generates* an external field.
- **IPW Shift:** The IPW reverses this. It posits that the particle *is* a feature of the field itself (a puncture), rather than a separate source generating a field. It attempts to achieve a unified description where the particle and field are fundamentally one.

8.1. Related Phenomena (Structural Duality)

The IPW concept is a specific mathematical realization of the philosophical idea of **structural duality** or **hybridity**, which links it to other phenomena and theories:

8.2. Smarandache Structures (S-Structures)

- **Relation:** The IPW is a manifestation of the Smarandache concept applied to physics.
- **Structural Hybridity:** The entity exhibits structural duality: it is both a **continuous, extended wave** (the "weak" structure) and a **localized, singular particle** (the "stronger" substructure/defect) existing simultaneously. This structural inconsistency is the core theme of S-Structures in algebra:
  https://fs.unm.edu/SmarandacheStrongStructures.htm
  https://fs.unm.edu/SmarandacheWeakStructures.htm
  https://fs.unm.edu/SmarandacheStrong-WeakStructures.htm

8.3. Solitons and Kinks

- **Relation:** Solitons are stable, localized wave packets that retain their shape while propagating. They behave almost like particles (localized energy).
- **Comparison:** While a soliton is a **localized excitation** of a field, the IPW's particle is a more radical concept: a **true singularity** or **defect** within the field, not just a localized, temporary solution to the wave equation. Solitons are highly non-linear but maintain continuity; the IPW explicitly involves a discontinuity (the puncture).

8.4. More theories, or fields related to IPW

The "Infinitesimally Punctured Wave" (IPW) interpretation relates to several other theories, especially those dealing with the non-linear nature of quantum fields and alternative approaches to the particle concept.

9. Non-Linear and Alternative Field Theories

The IPW model, by treating the particle as a **singularity** or **non-linear defect** within a field, connects to theories that try to derive particles from field structure rather than treating them as fundamental, point-like entities.

9.1. Quantum Field Theory (QFT) and Renormalization

- **Relation:** QFT treats particles as **excitations** of quantum fields. However, QFT initially faces the problem of **infinities** (singularities) arising from point-like particle interactions, which must be managed through the complex process of **renormalization**.
- **IPW Connection:** The IPW concept is a more explicit, geometric attempt to deal with the singularity. Instead of managing the infinite self-energy of a point particle mathematically (as in renormalization), the IPW suggests the singularity is a fundamental, perhaps bounded, **structural feature** of the physical wave field itself, aiming to inherently avoid the infinities associated with a purely mathematical point-particle assumption.

9.2. Kaluza-Klein Theory and Extra Dimensions

- **Relation:** Kaluza-Klein theory, and later String Theory, propose that particles are manifestations of vibrations or **geometrical configurations** in curled-up, extra spatial dimensions.

- **IPW Connection:** In both cases, the particle's properties (mass, charge) are derived from the geometry of the background space or field. The IPW shares this geometrical philosophy: the particle is a localized, non-trivial **geometrical feature** (the puncture/singularity) of the wave field, rather than being fundamentally distinct from it.

9.3. Non-Classical Solitons and Non-Linear Electrodynamics

The IPW is strongly linked to non-linear theories that create stable, localized structures from waves.

10. Non-Linear Electrodynamics (NLE)

- **Relation:** Theories like the **Born-Infeld model** of NLE attempt to modify Maxwell's equations at very high field strengths to give the electron a finite radius and finite self-energy, thus avoiding the classical singularity problem.
- **IPW Connection:** Both NLE and IPW seek to **"desingularize"** the electron. NLE achieves this by modifying the field equations themselves. IPW achieves this conceptually by declaring the particle as the singularity *of the wave* (or field) which might imply that the wave equation itself is non-linear and prevents the puncture from becoming a mathematical infinite point.

11. Non-Linear Schrödinger Equation (NLSE) and Solitons

- **Relation:** The NLSE is a modification of the standard Schrödinger equation that is used to model non-linear wave phenomena. In certain NLSE systems, solutions exist in the form of **solitons** (localized, stable waves that maintain their shape).
- **IPW Connection:** Solitons are perhaps the clearest physical analog to the IPW concept. They show how a **wave can self-localize** and maintain a particle-like identity due to non-linear interactions. The IPW essentially posits that the fundamental quantum particle is a specific type of **non-linear, singularity-containing soliton** solution to the ultimate field equation.

These related fields demonstrate that the core idea of the IPW—deriving the particle from the structure of the wave/field—is a recurring and active theme in physics seeking a deeper, less contradictory foundation for quantum mechanics.

12. Applications of IPW

The **Infinitesimally Punctured Wave (IPW)** interpretation is a conceptual model, not a mainstream theory, so it doesn't have established, practical engineering applications. Its possible "applications" are purely **theoretical** and **foundational**, serving to advance the philosophical and structural understanding of quantum mechanics.

The potential applications lie in providing a new framework for solving long-standing problems in physics:

12.1. Applications in Foundational Physics

12.1.1. Solving the Measurement Problem (No-Collapse Model)

The primary application is to offer an alternative resolution to the **wavefunction collapse** issue.

- **Goal:** To describe the transition from the quantum world (superposition) to the classical world (definite outcome) without invoking the non-physical, instantaneous "collapse" postulate of the Copenhagen Interpretation.

- **IPW Solution:** By defining the particle as a pre-existing **singularity (puncture)** within a real, extended wave, the model suggests that measurement is simply the **detection of the singularity's location**, not the cause of the wave's collapse. This leads to a smoother, deterministic picture of quantum dynamics, similar to the de Broglie-Bohm theory.

12.1.2. Unifying Particle and Field Conceptions

The IPW provides a model for **structural unity** between the two fundamental entities of physics:

- **Goal:** To move beyond the dualism where particles and fields are treated as separate, distinct concepts.
- **IPW Solution:** The particle is literally a **geometric defect** (the singularity) of the wave-field itself. This offers a conceptual framework for a **unified field theory** where mass, charge, and energy are all derived from the localized geometry or topological structure of a single underlying quantum field.

12.1.3. Addressing the Singularity Problem in Quantum Field Theory (QFT)

The concept of a singular structure relates to the mathematical challenges in QFT:

- **Goal:** To find a physical description of the electron that avoids the infinite self-energy (singularities) that arise when treating the particle as a mathematical point source.
- **IPW Connection:** By defining the particle as a "punctured wave," the model suggests that the fundamental wave equation must be **non-linear** in a way that naturally prevents the singularity from reaching a true mathematical infinity, perhaps by defining a **finite core structure** for the "puncture."

13. Applications in Conceptual Frameworks

13.1. Neutrosophic Quantum Theory

The IPW model, developed by Florentin Smarandache, fits directly into the structural framework of **Neutrosophic Theory**:

- **Goal:** To model systems where **indeterminacy** and **inconsistency** are inherent parts of the structure.
- **IPW Connection:** The IPW describes a structure that is **partially wave** (continuous, extended) and **partially particle** (discrete, singular). This structural hybridity is a classic example of a **Smarandache structure**, allowing researchers to apply the algebraic tools of Neutrosophic Logic to analyze the indeterminacy of the quantum state.

14. Alternative Models for Quantum Gravity

This section may be divided by subheadings. It should provide a concise and precise description of the experimental results, their interpretation as well as the experimental conclusions that can be drawn.

The connection between the particle's singularity and the geometry of its wave field opens avenues for alternative gravitational models:

- **Goal:** To find a way to incorporate quantum effects into General Relativity, especially concerning the nature of matter that warps space-time.
- **IPW Connection:** If the particle is a space-time singularity embedded in a wave field, it could suggest a mechanism where the **wave's extended nature** influences the geometry of space-time, providing new starting points for **non-local** or **non-linear** theories of quantum gravity.

16. Conclusions

The Infinitesimally Punctured Wave IPW) — in which a quantum object is visualized as an aggregation of infinitely many infinitesimally spaced particles. When these particles are densely packed, the ensemble appears as a continuous wave; when a measurement isolates a single constituent, particle-like behavior emerges. The model is situated alongside established alternative interpretations (e.g., De Broglie–Bohm pilot wave theory, wave packet descriptions) and linked to Neutrosophic Quantum Theory, which supplies a logical framework for handling indeterminacy." [4] The article's goal is to compare the **infinitesimally punctured wave (IPW)** with the major interpretations of the fundamental philosophical differences in how physicists view reality at the quantum level.

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Role of clans in the proximities of Neutrosophic Sets

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Abstract: In this study, the terms filters, grills, clans, and proximities of neutrophic sets are defined. Further some of its properties are investigated and the results are provided.

Keywords: Neutrosophic set**,** filters, grills, clans and proximities of neutrosophic sets.

1. Introduction

In the year 1965, Zadeh [10] introduced and investigated fuzzy sets. An intuitionistic fuzzy set was first presented in 1986 by Atanassov [2]. Florentin Smarandache [5] developed concepts such as neutrosophic logic and neutrosophic set in 1999. The truth, falsehood, and indeterminacy membership values are the three components on which he defined the neutrosophic set. The neutrosophic set was created in 2010 by Florentin Smarandache [3] as a generalization of intuitionistic fuzzy sets.

K C Chattopadhyay and etal [6], [9] developed the role of clans in the proximities of fuzzy sets in the year 1996 and on intuitionistic fuzzy sets in the year 1997 respectively. Also they proved the proximities of IFS is a clan generated structure. In this paper, we introduce and investigate the proximities of neutrosophic sets and proved that the proximities of NSs is a clan generated structure and provided the numerical example wherever applicable.

2. Preliminaries

Definition 2.1. [3] Let X be a fixed set that is non-empty. A set with the form $\check{N} = \{\langle \alpha, T_{\check{N}}(\alpha), I_{\check{N}}(\alpha), F_{\check{N}}(\alpha)\rangle : \alpha \in X\}$ is called a Neutrosophic set, where $T_{\check{N}}(\alpha)$, $I_{\check{N}}(\alpha)$, $F_{\check{N}}(\alpha)$ represents the degree of truth, degree of indeterminacy and the degree of falsity respectively of each element $\alpha \in X$ to the set $\check{N}$.

The set of all neutrosophic sets on X is denoted by $\check{N}(X)$.

Definition 2.2. [3] The complement of a Neutrosophic set Ň is denoted by $Ň^C$ and is defined by $Ň^C = \{\langle x, F_{Ň}(x), 1\text{-}I_{Ň}(x), T_{Ň}(x)\rangle: x \in X\}$.

Definition 2.3. [3] Consider two Neutrosophic sets U and V over X, then U is said to be contained in V, denoted by U ⊆ V if and only if $T_u(x) \leq T_v(x)$, $I_u(x) \leq I_v(x)$, $F_u(x) \geq F_v(x)$.

Definition 2.4. [3] The arbitrary union of two Neutrosophic sets U and V over X, is denoted by U⋃V and is defined by $\{\langle x, T_u(x) \vee T_v(x), I_u(x) \vee I_v(x), F_u(x) \wedge F_v(x)\rangle: x \in X\}$

Definition 2.5. [3] The finite intersection of two Neutrosophic sets U and V over X, is denoted by U⋂V and is defined by $\{\langle x, T_u(x) \wedge T_v(x), I_u(x) \wedge I_v(x), F_u(x) \vee F_v(x)\rangle: x \in X\}$

Definition 2.6. [3] Let Ň be a Neutrosophic set over X, then the universe set of Ň is denoted by $1_{Ň}$ and is defined by $1_{Ň} = \{\langle x,1,1,0\rangle: x \in X\}$.

Definition 2.7. [9] Let Ň be a Neutrosophic set over X, then the empty set of Ň is denoted by $0_{Ň}$ and is defined by $0_{Ň} = \{\langle x,0,0,1\rangle: x \in X\}$.

Proposition 2.8. Let U, V∈ Ň(X), then the following holds

(i) $(U \cup V)^c = U^c \cap V^c$

(ii) $(U \cap V)^c = U^c \cup V^c$

(iii) $(1_{Ň})^c = 0_{Ň}$

(iv) $(0_{Ň})^c = 1_{Ň}$

3. Stack, Filter, Grill, Prime filter of Neutrosophic Sets

Definition 3.1. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A stack Š of neutrosophic sets on X is a subset of Ň(X) such that P ⊃ Q ∈ Š ⇒ P ∈ Š.

Example 3.2. Let X = {a,b}. consider the following neutrosophic sets

$$A_1 = \{< a, 0.3, 0.7, 0.2 >, < b, 0.4, 0.5, 0.6 >\}$$

$$A_2 = \{< a, 0.5, 0.5, 0.6 >, < b, 0.3, 0.3, 0.4 >\}$$

$$A_3 = \{< a, 0.4, 0.7, 0.1 >, < b, 0.5, 0.2, 0.8 >\}$$

$$A_4 = \{< a, 0.2, 0.2, 0.8 >, < b, 0.1, 0.1, 0.7 >\}$$

$A_5 = \{< a, 0.5, 0.8, 0 >, < b, 0.6, 0.6, 0.2 >\}$

$A_6 = \{< a, 0.2, 0.4, 0.5 >, < b, 0.1, 0.1, 0.9 >\}$

$A_7 = \{< a, 0.4, 0.8, 0.1 >, < b, 0.6, 0.6, 0.3 >\}$

$A_8 = \{< a, 0.6, 0.6, 0.5 >, < b, 0.5, 0.5, 0.1 >\}$

$A_9 = \{< a, 0.3, 0.5, 0.2 >, < b, 0.4, 0.2, 0.9 >\}$

$A_{10} = \{< a, 0.6, 0.9, 0 >, < b, 0.7, 0.7, 0 >\}$

Here $\check{S}_1, \check{S}_2, \check{S}_3$ are the stacks of I(X), where

$\check{S}_1 = \{A_2, A_4, A_8\}$ *such that* $A_4 \subseteq A_2 \subseteq A_8$

$\check{S}_2 = \{A_1, A_6, A_7\}$ *such that* $A_6 \subseteq A_1 \subseteq A_7$

$\check{S}_3 = \{A_3, A_5, A_9, A_{10}\}$ *such that* $A_9 \subseteq A_3 \subseteq A_5 \subseteq A_{10}$

Definition 3.3. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A filter Ƒ of neutrosophic sets on X is a subset of Ň(X) satisfying the following:

(i) Ƒ ≠ ϕ

(ii) P ⊃ Q ∈ Ƒ ⇒ P ∈ Ƒ

(iii) P, Q ∈ Ƒ ⇒ P∩Q ∈ Ƒ

A filter Ƒ of Neutrosophic sets is said to be proper if 0Ň ∉ Ƒ.

Example 3.4. From the example 3.2, Ƒ = $\{A_2, A_4, A_8\}$ *such that* $A_4 \subseteq A_2 \subseteq A_8$ is a filter as well as proper filter, since 0Ň ∉ Ƒ and the following holds

For A_2, A_4 ∈ Ƒ ⇒ $A_2 \cap A_4 = A_4$ ∈ Ƒ

For A_2, A_8 ∈ Ƒ ⇒ $A_2 \cap A_8 = A_2$ ∈ Ƒ

For A_4, A_8 ∈ Ƒ ⇒ $A_4 \cap A_8 = A_4$ ∈ Ƒ

Definition 3.5. Let X be a fixed set that is non-empty and Ň(X) is the set of all neutrosophic sets in X. A grill Ğ of neutrosophic sets on X is a subset of Ň(X) satisfying the following:

(i) 0Ň ∉ Ğ

(ii) P ⊃ Q ∈ Ğ ⇒ P ∈ Ğ

(iii) P∪Q ∈ Ğ ⇒ P ∈ G or Q ∈ Ğ

A grill Ğ of neutrosophic sets is said to be proper if Ğ ≠ ϕ.

Example 3.6. From the example 3.2, $\breve{G} = \{A_2, A_4, A_8\}$ ***such that*** $A_4 \subseteq A_2 \subseteq A_8$ is a grill as well as proper grill, since $0_{\check{N}} \notin \breve{G}$ and the following holds

For $A_2 \cup A_4 = A_2 \in \breve{G} \Rightarrow A_2 \in \breve{G}$ ***or*** $A_4 \in \breve{G}$

For $A_2 \cup A_8 = A_8 \in \breve{G} \Rightarrow A_2 \in \breve{G}$ ***or*** $A_8 \in \breve{G}$

For $A_4 \cup A_8 = A_8 \in \breve{G} \Rightarrow A_4 \in \breve{G}$ ***or*** $A_8 \in \breve{G}$

Definition 3.7. Let X be a fixed set that is non-empty and $\check{N}(X)$ is the set of all neutrosophic sets in X. A stack $\check{S}$ of NSs on X is a prime filter of NSs on X if it is a filter of NSs as well as the grill of NSs on X.

In other words, a stack $\check{S}$ of NSs on X is a prime filter of NS on X if it satisfies the following:

(i) $0_{\check{N}} \notin \check{S}$

(ii) $\check{S} \neq \phi$

(iii) $P \supset Q \in \check{S} \Rightarrow P \in \check{S}$

(iv) $P, Q \in \check{S} \Rightarrow P \cap Q \in \check{S}$

(v) $P \cup Q \in \check{S} \Rightarrow P \in G$ or $Q \in \check{S}$

Example 3.8. From the example 3.2, $\check{S}_1 = \{A_2, A_4, A_8\}$ is a prime filter of $\check{N}(X)$, since it is a filter and grill of $\check{N}(X)$.

We denote the following notation

Set of all filters of NSs on X = $\zeta(X)$

Set of all grills of NSs on X = $\psi(X)$

Set of all prime filters of NSs on X = $\xi(X)$

Example 3.9. Let $A \in \check{N}(X)$. Define $F \subset \check{N}(X)$ by $F = \{\mathbf{B} \in \check{N}(\mathbf{X}) | \mathbf{B} \supset \mathbf{A}\}$

Clearly F is non empty. Let $\mathbf{C} \supset \mathbf{B} \in F$. Then $\mathbf{C} \supset \mathbf{B} \supset \mathbf{A}$ and hence $\mathbf{C} \in F$. Now let $\mathbf{B}, \mathbf{C} \in F$ such that $T_A(x) \leq T_B(x)$, $I_A(x) \leq I_B(x)$, $F_A(x) \geq F_B(x)$ and $T_A(x) \leq T_C(x)$, $I_A(x) \leq I_C(x)$, $F_A(x) \geq F_C(x)$. Hence it follows that $T_A(x) \leq T_B(x) \wedge T_C(x)$, $I_A(x) \leq I_B(x) \wedge I_C(x)$, $F_A(x) \geq F_B(x) \vee F_C(x)$. Thus $A \subset B \cap C$ which implies $B \cap C \in F$. Hence F is a filter of NSs.

Theorem 3.10. Let $F_1, F_2 \in \zeta(X)$ and $\breve{G}_1, \breve{G}_2 \in \psi(X)$.

(i) If $F_1 \cap F_2 \subset \breve{G}_1$, then $F_1 \subset \breve{G}_1$ or $F_2 \subset \breve{G}_1$

(ii) If $F_1 \subset \breve{G}_1$ or $F_1 \subset \breve{G}_2$, then $F_1 \subset \breve{G}_1 \cup \breve{G}_2$

Proof:

(i) Suppose $\mathrm{F}_1 \not\subset \breve{G}_1$ and $\mathrm{F}_2 \not\subset \breve{G}_1$. Then there exists $\boldsymbol{A_1} \in \mathrm{F}_1$ such that $\boldsymbol{A_1} \notin \breve{G}_1$ and $\boldsymbol{A_2} \in \mathrm{F}_2$ such that $\boldsymbol{A_2} \notin \breve{G}_1$, therefore $\boldsymbol{A_1} \cup \boldsymbol{A_2} \notin \breve{G}_1$, but $\boldsymbol{A_1} \cup \boldsymbol{A_2} \in \mathrm{F}_1 \cap \mathrm{F}_2$ which is a contradiction. Hence $\mathrm{F}_1 \subset \breve{G}_1$ or $\mathrm{F}_2 \subset \breve{G}_1$ is valid.

(ii) Case (I): Suppose $\mathrm{F}_1 \subset \breve{G}_1$, then for $\boldsymbol{A_1} \in \mathrm{F}_1$, we have $\boldsymbol{A_1} \in \breve{G}_1$. Also, since $\breve{G}_1 \subset \breve{G}_1 \cup \breve{G}_2$ implies $\boldsymbol{A_1} \in \breve{G}_1 \cup \breve{G}_2$. Hence $\mathrm{F}_1 \subset \breve{G}_1 \cup \breve{G}_2$.

Case (II): Suppose $\mathrm{F}_1 \subset \breve{G}_2$, then for $\boldsymbol{A_2} \in \mathrm{F}_1$, we have $\boldsymbol{A_2} \in \breve{G}_2$. Also, since $\breve{G}_2 \subset \breve{G}_1 \cup \breve{G}_2$ implies $\boldsymbol{A_2} \in \breve{G}_1 \cup \breve{G}_2$. Hence $\mathrm{F}_1 \subset \breve{G}_1 \cup \breve{G}_2$

In either case if we have $\mathrm{F}_1 \subset \breve{G}_1$ or $\mathrm{F}_1 \subset \breve{G}_2$, then $\mathrm{F}_1 \subset \breve{G}_1 \cup \breve{G}_2$

Theorem 3.11. Arbitrary intersection of filters of NSs is a filter of NSs.

Proof is straightforward.

Theorem 3.12. Finite union of grills of NSs is a grill of NSs.

Proof:

Let $\breve{G} = \cup\{\breve{G}_i, \boldsymbol{i} \in \boldsymbol{I}, \breve{G}_i \in \boldsymbol{\psi}(\mathbf{X})\}$. we check the three axioms

(I) $0_{\breve{N}} \notin \breve{G}_i$ for all $\boldsymbol{i} \in \boldsymbol{I}$, then $0_{\breve{N}} \notin \breve{G}$.

(II) If $\boldsymbol{A} \in \breve{G}$, **then so** $\mathbf{A} \in \breve{G}_i$ **for some** $\mathbf{i} \in \mathbf{I}$ and $\boldsymbol{A} \subset$ B, since each $\breve{G}_i$ is a grill, then $\mathbf{B} \in \breve{G}_i \subset \breve{G}$

(III) If $\boldsymbol{A} \cup \boldsymbol{B} \in \breve{G}$, **then so** $\mathbf{A} \cup \boldsymbol{B} \in \breve{G}_i$ **for some** $\mathbf{i} \in \mathbf{I}$ and $\boldsymbol{A} \subset$ B, since each $\breve{G}_i$ is a grill, then $\mathbf{A} \in \breve{G}_i$ or $\boldsymbol{B} \in \breve{G}_i$, hence A$\in \breve{G}$ or $\boldsymbol{B} \in \breve{G}$.

Thus, union of grills is again a grill.

Definition 3.13. For each stack Š of NSs, define dŠ = {A : $\boldsymbol{A^c} \notin$ Š}

Theorem 3.14. If Š is a stack of NSs, F is a filter of NSs and $\breve{G}$ is a grill of NSs on X, then the following holds.

(1) If $\check{S}_2 \subset \check{S}_1$, then $\mathbf{d}\check{S}_1 \subset \mathbf{d}\check{S}_2$

(2) d(dŠ) = Š

(3) $\mathrm{d}(\cup\check{S}_i) = \cap\mathbf{d}\check{S}_i$

(4) $\mathrm{d}(\cap\check{S}_i) = \cup\mathbf{d}\check{S}_i$

(5) $\mathbf{d}\mathrm{F}$ is a grill of NSs

(6) $\mathbf{d}\breve{G}$ is a filter of NSs

Proof.

(1) To prove $\mathbf{d}\check{S}_1 \subset \mathbf{d}\check{S}_2$. That is for any $\boldsymbol{A} \in \mathbf{d}\check{S}_1 \Rightarrow \mathbf{A} \in \mathbf{d}\check{S}_2$. Assume $\check{S}_2 \subset \check{S}_1$

Since $A \in \mathbf{d}Š_1$, then $A^c \notin Š_1$. Then from our assumption $A^c \notin Š_2$, which implies $A \in \mathbf{d}Š_2$. Hence, $\mathbf{d}Š_1 \subset \mathbf{d}Š_2$

(2) For any $A \in Ň(X)$, then $A \in \mathbf{d}(\mathbf{d}Š) \leftrightarrow A^c \notin \mathbf{d}Š \leftrightarrow A \in Š$. Hence d(dŠ) = Š.

(3) For any $A \in Ň(X)$, then $A \in \mathbf{d}(\cup Š_i) \leftrightarrow A^c \notin \cup Š_i \leftrightarrow A^c \notin Š_i\ \forall \mathbf{i} \in \mathbf{I} \leftrightarrow \mathbf{A} \in \mathbf{d}Š_i\ \forall \mathbf{i} \in \mathbf{I} \leftrightarrow A \in \cap \mathbf{d}Š_i$. Hence $d(\cup Š_i) = \cap \mathbf{d}Š_i$.

(4) For any $A \in Ň(X)$, then

$A \in \mathbf{d}(\cap Š_i) \leftrightarrow A^c \notin \cap Š_i \leftrightarrow A^c \notin Š_i$ **for some** $\mathbf{i} \in \mathbf{I} \leftrightarrow \mathbf{A} \in \mathbf{d}Š_i$ **for some** $\mathbf{i} \in \mathbf{I} \leftrightarrow A \in \cup \mathbf{d}Š_i$.

Hence $d(\cap Š_i) = \cup \mathbf{d}Š_i$.

(5) Axiom 1: Since $1_Ň \in Ƒ$, $(\mathbf{1}_Ň)^c = 0_Ň \in Ƒ$ implies $0_Ň \notin dƑ$.

Axiom 2: Let $\mathbf{B} \subset A \in dƑ$, then $(B)^c \notin Ƒ$ and it follows that $(A)^c \notin Ƒ$ for $A^c \subset B^c$. Hence $A \in dƑ$.

Axiom 3: Let $A \cup B \in dƑ$, then $(A \cup B)^c \notin Ƒ$ which implies $A^c \cap B^c \notin Ƒ$ and it follows that $A^c \notin Ƒ$ or $B^c \notin Ƒ$. Hence $A \in dƑ$ or $B \in dƑ$. Thus, dƑ is a grill of Ň(X).

(6) Axiom 1: Since $0_Ň \notin Ğ$, $(\mathbf{0}_Ň)^c = \mathbf{1}_Ň \notin Ğ$ implies $1_Ň \in dĞ$.

Axiom 2: Let $\mathbf{B} \subset A \in dĞ$, then $(B)^c \notin Ğ$ and it follows that $(A)^c \notin Ğ$ for $A^c \subset B^c$. Hence $A \in dĞ$.

Axiom 3: Let $A \in dĞ$ and $B \in dĞ$, then $A^c \notin Ğ$ **and** $B^c \notin Ğ$ and it follows that $A^c \cup B^c \notin Ğ$. That is $(A \cap B)^c \notin Ğ$ which implies $(A \cap B) \in dĞ$. Thus, dĞ is a filter of Ň(X).

Theorem 3.15. Neutrosophic Prime Filter Theorem

If Ƒ is a filter of NSs and Ğ is a grill of NSs on X, then there exist a prime filter ϱ of NSs such that $Ƒ \subset \boldsymbol{\rho} \subset Ğ$.

Proof.

Let ℵ be a collection of subsets of Ň(X) and Ƒ is a filter of NSs and Ğ is a grill of NSs on X. Let ℵ be defined by ℵ = $\{\Upsilon \subset Ň(\mathbf{X}) \mid \Upsilon\ \textit{is a filter}, Ƒ \subset \Upsilon \subset Ğ\}$

$\Upsilon \in ℵ$, for all $\Upsilon \subset Ň(\mathbf{X})$ if and only if $Ƒ \subset \Upsilon$ for all $A_i \in \Upsilon$ if and only if $\cap A_i \in Ğ$. Clearly (ℵ, ⊂) is a partial order set and Ƒ ∈ ℵ. By Zorn's lemma, (ℵ, ⊂) has a maximal element and ϱ be that element such that $Ƒ \subset \boldsymbol{\rho} \subset Ğ$.

Now to prove: $\boldsymbol{\rho}$ is a prime filter of Ň(X).

Let $A_1, A_2 \in \boldsymbol{\rho}$. Then $\boldsymbol{\rho} \cup \{A_1 \cap A_2\} \in ℵ$ and by maximality of $\boldsymbol{\rho}$, $\{A_1 \cap A_2\} \in \boldsymbol{\rho}$. Let B⊂A, then $\boldsymbol{\rho} \cup \{A\} \in ℵ$ and by maximality of $\boldsymbol{\rho}$, $A \in \boldsymbol{\rho}$. Hence $\boldsymbol{\rho}$ is a filter of NSs.

Let A,B ∈ Ň(X) such that $\boldsymbol{A} \notin \boldsymbol{\rho}$ and B$\notin\boldsymbol{\rho}$, then both of $\boldsymbol{\rho} \cup \{\boldsymbol{A}\} \notin \aleph$ and $\boldsymbol{\rho} \cup \{\boldsymbol{B}\} \notin \aleph$. Also we can find $\boldsymbol{A_1}, \boldsymbol{A_2} \ldots . \boldsymbol{A_n} \in \boldsymbol{\rho}$ and $\boldsymbol{B_1}, \boldsymbol{B_2} \ldots . \boldsymbol{B_m} \in \boldsymbol{\rho}$ such that {A} $\cap\{\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \ldots . \cap \boldsymbol{A_n}\} \cap \{\boldsymbol{B_1} \cap \boldsymbol{B_2} \cap \ldots . \cap \boldsymbol{B_m}\} \notin$ Ğ and {B$\cap\{\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \ldots . \cap \boldsymbol{A_n}\} \cap \{\boldsymbol{B_1} \cap \boldsymbol{B_2} \cap \ldots . \cap \boldsymbol{B_m}\} \notin$ Ğ.

Hence {A∪B}$\cap\{\boldsymbol{A_1} \cap \boldsymbol{A_2} \cap \ldots . \cap \boldsymbol{A_n}\} \cap \{\boldsymbol{B_1} \cap \boldsymbol{B_2} \cap \ldots . \cap \boldsymbol{B_m}\} \notin$ Ğ.

This shows that A∪B $\notin\boldsymbol{\rho}$. Thus $\boldsymbol{\rho}$ is a prime filter of Ň(X).

Theorem 3.16.

Let Ğ⊂Ň(**X**), then Ğ be a grill of NSs on X if and only if it is a union of prime filter of NSs on X.

Proof.

Since by theorem 3.13, union of grills of NSs on X is again a grill NSs. It follows that, if Ğ is a union of prime filters of NSs on X, then it is a grill of NSs.

Conversely, suppose Ğ is a grill of NSs on X. Let A∈ Ğ and Ƒ = {**B** ∈ Ň(X): A⊂B}, then Ƒ is a filter of NSs and Ƒ⊂Ğ. By Neutrosophic Filter theorem, then there exists a prime filter ϱ of NSs such that Ƒ⊂**ρ**⊂Ğ and hence A∈ **ρ**⊂Ğ. Thus Ğ is a union of prime filters of NSs on X.

4. Proximities of Neutrosophic Sets

Definition 4.1. Binary relation ẟ on Neutrosophic sets

Let ẟ⊆ Ň(X) × Ň(X) such that (A,B) ∈ẟ if and only if ẟ (A,B)= $< \boldsymbol{T_\delta(A,B)}, \boldsymbol{I_\delta(A,B)}, \boldsymbol{F_\delta(A,B)} >$

Numerical Example 4.2.

Let X={a,b,c}. Suppose a binary relation ẟ on NSs is defined by

ẟ (A,B)= $< \boldsymbol{T_\delta(A,B)}, \boldsymbol{I_\delta(A,B)}, \boldsymbol{F_\delta(A,B)} >$ where

$$\boldsymbol{T_\delta(A,B)} = \frac{\sum_{x\in X}(\boldsymbol{1} - |\boldsymbol{T_A(x)} - \boldsymbol{T_B(x)}|)}{\boldsymbol{n(X)}}$$

$$\boldsymbol{I_\delta(A,B)} = \frac{\sum_{x\in X}|\boldsymbol{I_A(x)} - \boldsymbol{I_B(x)}|}{\boldsymbol{n(X)}}$$

$$\boldsymbol{F_\delta(A,B)} = \frac{\sum_{x\in X}|\boldsymbol{F_A(x)} - \boldsymbol{F_B(x)}|}{\boldsymbol{n(X)}}$$

$$\boldsymbol{A} = \{< \boldsymbol{a}, \boldsymbol{0.2}, \boldsymbol{0.2}, \boldsymbol{0.8} >, < \boldsymbol{b}, \boldsymbol{0.1}, \boldsymbol{0.1}, \boldsymbol{0.7} >, < \boldsymbol{c}, \boldsymbol{0.2}, \boldsymbol{0.5}, \boldsymbol{0.7} >\}$$

$$\boldsymbol{B} = \{< \boldsymbol{a}, \boldsymbol{0.2}, \boldsymbol{0.4}, \boldsymbol{0.5} >, < \boldsymbol{b}, \boldsymbol{0.1}, \boldsymbol{0.1}, \boldsymbol{0.9} >, < \boldsymbol{c}, \boldsymbol{0.1}, \boldsymbol{0.4}, \boldsymbol{0.3} >\}$$

$$\boldsymbol{T_\delta(A,B)} = \frac{\boldsymbol{1+1+0.9}}{\boldsymbol{3}} = \boldsymbol{0.967}$$

$$\boldsymbol{I_\delta(A,B)} = \frac{\boldsymbol{0.2+0+0.1}}{\boldsymbol{3}} = \boldsymbol{0.1}$$

$$\boldsymbol{F(A,B)} = \frac{\boldsymbol{0.3+0.2+0.4}}{\boldsymbol{3}} = \boldsymbol{0.3}$$

δ (A,B)= $< \boldsymbol{T_\delta(A,B), I_\delta(A,B), F_\delta(A,B)} > = < \mathbf{0.967, 0.1, 0.3} >$

Definition 4.3. Inverse Binary relation $\boldsymbol{\delta^{-1}}$ on Neutrosophic sets

An inverse binary relation $\boldsymbol{\delta^{-1}}$ on two NSs is defined to be $\boldsymbol{\delta = \delta^{-1}}$ if and only if δ (A,B) = δ (B,A) for all A,B ∈ Ň(X)

Numerical Example 4.4.

δ (B,A)= $< \boldsymbol{T_\delta}(\mathbf{B,A}), \boldsymbol{I_\delta}(\mathbf{B,A}), \boldsymbol{F_\delta}(\mathbf{B,A}) >$ where

$$\boldsymbol{T_\delta}(\mathbf{B,A}) = \frac{\sum_{x\in X}(1 - |T_B(x) - T_A(x)|)}{n(X)}$$

$$\boldsymbol{I_\delta}(\mathbf{B,A}) = \frac{\sum_{x\in X}|I_B(x) - I_A(x)|}{n(X)}$$

$$\boldsymbol{F}(\mathbf{B,A}) = \frac{\sum_{x\in X}|F_B(x) - F_A(x)|}{n(X)}$$

From Example 4.3,

$$\boldsymbol{T_\delta}(\mathbf{B,A}) = \frac{1 + 1 + 0.9}{3} = 0.967$$

$$\boldsymbol{I_\delta}(\mathbf{B,A}) = \frac{0.2 + 0 + 0.1}{3} = 0.1$$

$$\boldsymbol{F}(\mathbf{B,A}) = \frac{0.3 + 0.2 + 0.4}{3} = 0.3$$

δ (B,A) = $< \mathbf{0.967, 0.1, 0.3} >$

Hence δ (A,B) = δ (B,A).

Definition 4.5. Distribution of δ over union

A binary relation δ is said to be distributive over union of two NSs if

AUB ∈ δ (C) if and only if A ∈ δ (C) or B ∈ δ (C)

That is, δ (AUB, C) = δ(A, C) or δ (AUB, C) = δ(B, C)

Definition 4.6. Basic pre-proximity of NSs

A binary relation δ of NSs (Ň(**X**)) is said to be a basic pre-proximity of NSs on X, if it satisfies the following conditions:

Axiom 1: $0_Ň$ ∉ δ (A), for all A∈ Ň(**X**)

Axiom 2: δ= $\boldsymbol{\delta^{-1}}$

Axiom 3: δ (AUB, C) = δ(A, C) or δ (AUB, C) = δ(B, C)

Definition 4.7. Basic proximity of NSs

A binary relation Ω of NSs ($\check{N}(\mathbf{X})$) is said to be a basic proximity of NSs on X, if it is pre-proximity of NSs and it also satisfies the condition $A \cap B \neq 0_{\check{N}} \Rightarrow (\boldsymbol{A}, \boldsymbol{B}) \in \boldsymbol{\Omega}$

Remark 4.8.

(1) If δ is a pre-proximity of NSs on X, then X is said to be a reference set of δ and is denoted by X(δ).

(2) If Ω is a pre-proximity of NSs on X, then X is said to be a reference set of Ω and is denoted by X(Ω).

(3) We denote the set of all basic pre-proximities of NSs on X by $\mathfrak{M}(\mathbf{X})$ and the set of all basic proximities of NSs on X by $\mathfrak{N}$(X).

Definition 4.9.

Let $\delta \in \mathfrak{M}(X)$ and $A \in \check{N}(\mathbf{X})$, then $B \in \check{N}(\mathbf{X})$ is called a neighbourhood of A with respect to δ, if $\boldsymbol{B^c} \notin \boldsymbol{\delta}(\boldsymbol{A})$.

The collection of all neighbourhoods of A with respect to δ is denoted by $nbhd(\boldsymbol{\delta}, \boldsymbol{A})$.

Theorem 4.10.

Let A,B,C ∈ $\check{N}(\mathbf{X})$ and $\delta \in \mathfrak{M}(X)$, then the following holds:

(1) $nbhd(\boldsymbol{\delta}, \mathbf{0}_{\check{N}}) = \check{N}(\mathbf{X})$

(2) If B ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A})$ and C ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{D})$, then B∪C ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A} \cup \boldsymbol{D})$

(3) $nbhd(\boldsymbol{\delta}, \boldsymbol{A} \cup \boldsymbol{B}) = nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \cap nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

(4) If B⊂A, then $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \subset nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

Proof.

(1) Since $\forall A \in \check{N}(\mathbf{X})$, $\boldsymbol{A} \notin \boldsymbol{\delta}(\mathbf{0}_{\check{N}})$, we have $nbhd(\boldsymbol{\delta}, \mathbf{0}_{\check{N}}) = \check{N}(\mathbf{X})$.

(2) Let B ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \Rightarrow \boldsymbol{B^c} \notin \boldsymbol{\delta}(\boldsymbol{A})$ and C ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{D}) \Rightarrow \boldsymbol{C^c} \notin \boldsymbol{\delta}(\boldsymbol{D})$. It follows that $(\boldsymbol{B} \cup \boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{A})$ and $(\boldsymbol{B} \cup \boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{D})$ which implies $(\boldsymbol{B} \cup \boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{A}) \cup \boldsymbol{\delta}(\boldsymbol{D})$, that is $(\boldsymbol{B} \cup \boldsymbol{C})^c \notin \boldsymbol{\delta}(\boldsymbol{A} \cup \boldsymbol{D})$ and hence B∪C ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A} \cup \boldsymbol{D})$.

(3) For every $D \in \check{N}(\mathbf{X})$,

D ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A} \cup \boldsymbol{B}) \leftrightarrow (\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{A} \cup \boldsymbol{B})$

$\leftrightarrow (\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{A}) \cup \boldsymbol{\delta}(\boldsymbol{B})$

$\leftrightarrow (\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{A})$ and $(\boldsymbol{D})^c \notin \boldsymbol{\delta}(\boldsymbol{B})$

$\leftrightarrow$ D ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A})$ and D ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

$\leftrightarrow$ D ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \cap nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

(4) Let B⊂A, To prove for every E ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \Rightarrow$ E ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

E ∈ $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \Rightarrow (\boldsymbol{E})^c \notin \boldsymbol{\delta}(\boldsymbol{A})$

$\Rightarrow (\boldsymbol{E})^c \notin \boldsymbol{\delta}(\boldsymbol{B})$ (since A⊃B)

$\Rightarrow$ E $\in$ $nbhd(\boldsymbol{\delta}, \boldsymbol{B})$

Thus $nbhd(\boldsymbol{\delta}, \boldsymbol{A}) \subset nbhd(\boldsymbol{\delta}, \boldsymbol{B})$.

5. Clan of Proximities of Neutrosophic Sets

Definition 5.1. Let δ∈ 𝔐(X). A subfamily £ of Ň($\mathbf{X}$) is said to be δ-compatible if A,B ∈ £ ⇒ A∈ δ(B). Also a δ-compatible grill is called a δ-clan.

Theorem 5.2. For δ∈ 𝔐(X) and Ğ ∈ψ(X), the following are equivalent:

(1) Ğ is a δ-clan.

(2) If K∈ξ(X) such that K⊂Ğ, then Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$.

(3) Ğ ⊂ ⋂{$\boldsymbol{\delta}(\mathbf{K})|\mathbf{K} \in \boldsymbol{\xi}(\mathbf{X}), \mathbf{K} \subset$ Ğ}.

(4) If $\boldsymbol{K_1}, \boldsymbol{K_2}$ ∈ξ(X), such that $\boldsymbol{K_1} \subset$ Ğ **and** $\boldsymbol{K_2} \subset$ Ğ, then $\boldsymbol{K_1} \subset \boldsymbol{\delta}(\boldsymbol{K_2})$

Proof.

(1)⇒(2)

Assume that Ğ is a δ-clan. To prove: Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$. Let K∈ξ(X) such that K⊂Ğ and A ∈Ğ, it follows that A∈ δ(B) for all $\boldsymbol{B} \in \mathbf{K}$, that is A∈ $\boldsymbol{\delta}(\mathbf{K})$. Thus Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$.

(2)⇒(3)

If K∈ξ(X) such that K⊂Ğ, then Ğ ⊂ $\boldsymbol{\delta}(\mathbf{K})$, then obviously Ğ ⊂∩ {$\boldsymbol{\delta}(\mathbf{K})|\mathbf{K} \in \boldsymbol{\xi}(\mathbf{X}), \mathbf{K} \subset$ Ğ}.

(3)⇒(4)

Suppose Ğ ⊂ ⋂{$\boldsymbol{\delta}(\mathbf{K})|\mathbf{K} \in \boldsymbol{\xi}(\mathbf{X}), \mathbf{K} \subset$ Ğ}. Let $\boldsymbol{K_1}, \boldsymbol{K_2}$ ∈ξ(X), such that $\boldsymbol{K_1} \subset$ Ğ **and** $\boldsymbol{K_2} \subset$ Ğ, then by (3) Ğ ⊂ $\boldsymbol{\delta}(\boldsymbol{K_1})$ and Ğ ⊂ $\boldsymbol{\delta}(\boldsymbol{K_2})$. Now $\boldsymbol{K_1} \subset$ Ğ and Ğ ⊂ $\boldsymbol{\delta}(\boldsymbol{K_2})$ implies $\boldsymbol{K_1} \subset \boldsymbol{\delta}(\boldsymbol{K_2})$.

(4)⇒(1)

Suppose (4) holds. To prove: Ğ is a δ-clan. Let A,B ∈Ğ and by (4), $\boldsymbol{K_1}, \boldsymbol{K_2}$ ∈ ξ(X), such that $\mathbf{A} \in \boldsymbol{K_1} \subset$ Ğ **and** $\mathbf{B} \in \boldsymbol{K_2} \subset$ Ğ, then $\mathbf{A} \in \boldsymbol{K_1} \subset \boldsymbol{\delta}(\boldsymbol{K_2}) \subset \boldsymbol{\delta}(\mathbf{B})$. Thus Ğ is a δ-clan.

Theorem 5.3. For δ∈ 𝔐(X), then every δ-clan is contained in a maximal δ-clan.

Proof.

Since by theorem 3.13, union of grills of NSs is again a grill of NSs. Further for a family of δ-clans {$Ğ_{i,} \boldsymbol{i} \in \boldsymbol{I}$} with $Ğ_{\boldsymbol{i}} \subset Ğ_{\boldsymbol{j}}, \boldsymbol{i} \leq \boldsymbol{j}$, ∪{$Ğ_{i,} \boldsymbol{i} \in \boldsymbol{I}$} is a δ-clan. Hence by applying Zorn's lemma, there must be a maximal δ-clan on the collections of δ-clan Ğ.

Lemma 5.4. For Ω ∈ 𝔑 (X), if A∈ Ω (B), then there exists $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$, $\mathbf{B} \in \boldsymbol{H_2}$ and $\boldsymbol{H_1} \subset \boldsymbol{\Omega}(\mathbf{H_2})$.

Proof.

Since Ω (B) is a grill of NSs on X, by theorem 3.15, there exists a prime filter $\boldsymbol{H_1}$ of NSs such that A∈$\boldsymbol{H_1}$⊂ Ω (B). By symmetry of Ω, B∈$\boldsymbol{\Omega(H_1)}$. Now $\boldsymbol{\Omega(H_1)}$ ∈ ψ(X). Again by theorem 3.15, there exists $\boldsymbol{H_2}$∈ξ(X) such that B∈$\boldsymbol{H_2}$⊂ Ω ($\boldsymbol{H_1}$). Hence $\mathbf{A} \in \boldsymbol{H_1}$, $\mathbf{B} \in \boldsymbol{H_2}$ and $\boldsymbol{H_1} \subset \boldsymbol{\Omega}(\mathbf{H_2})$.

Theorem 5.5. For Ω ∈ 𝔑 (X), if A∈ Ω (B), then there is a Ω–clan of the form $\boldsymbol{H_1} \cup \boldsymbol{H_2}$ where $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$ and $\mathbf{B} \in \boldsymbol{H_2}$.

Proof.

Let A∈ Ω (B), by lemma 5.4, there exists $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$, $\mathbf{B} \in \boldsymbol{H_2}$ and $\boldsymbol{H_1} \subset \boldsymbol{\Omega}(\mathbf{H_2})$. Since Ω ∈ 𝔑 (X) and $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X), then for any P, Q ∈ $\boldsymbol{H_1}$ **or** $\boldsymbol{H_2}$, we have (P,Q) ∈ Ω and hence $\boldsymbol{H_1} \cup \boldsymbol{H_2}$ is a Ω–clan such that $\mathbf{A} \in \boldsymbol{H_1}$ and $\mathbf{B} \in \boldsymbol{H_2}$.

Theorem 5.6. For Ω ∈ 𝔑 (X), if A∈ Ω (B), then there a maximal Ω–clan containing {A,B}.

Proof.

By theorem 5.5, there exists a Ω–clan $\boldsymbol{H_1} \cup \boldsymbol{H_2}$, where $\boldsymbol{H_1}, \boldsymbol{H_2}$ ∈ξ(X) such that $\mathbf{A} \in \boldsymbol{H_1}$ and $\mathbf{B} \in \boldsymbol{H_2}$ and {A,B}⊂$\boldsymbol{H_1} \cup \boldsymbol{H_2}$. Also by theorem 5.3, every Ω -clan is contained in a maximal Ω –clan. Hence the proof.

6 Conclusion

In this article, we introduced and studied the concept of proximities of neutrosophic sets and its characterstics. Futher, its gradation of openness can be studied.

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Contra Continuous and Irresolute Maps via M-open Sets in n-Cylindrical Neutrosophic Topological Spaces

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Abstract. In this paper, n-cylindrical neutrosophic contra M-continuous mappings are defined and investigated within the framework of n-cylindrical neutrosophic topological spaces. Motivated by recent developments on neutrosophic contra continuous functions, irresolute mappings, and n-cylindrical fuzzy neutrosophic topological structures, several fundamental properties of n-cylindrical neutrosophic contra M-irresolute mappings are established. The results obtained extend and complement existing studies on various classes of neutrosophic contra continuous and contra irresolute mappings, thereby enriching the theory of neutrosophic topological spaces and providing a basis for further applications of n-cylindrical neutrosophic mappings.

Keywords: n-$CyN\mathscr{C}Mos$, n-$CyN\mathscr{C}MCts$ map, and n-$CyN\mathscr{C}MIrr$ map.

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1. Introduction

Following Zadeh's introduction of fuzzy set (denoted as fs) in 1965 [24], Chang [6] developed the notion of fuzzy topological spaces (fts), which led to the adaptation of classical topological concepts within the framework of fuzzy topology by various researchers. A significant generalization of fuzzy sets, known as intuitionistic fuzzy set (ifs), was introduced by Atanassov in 1986 [4]. Building on this, Coker [7] introduced the concept of intuitionistic fuzzy topological spaces ($ifts$) based on ifs. Jeon et al. [9] further investigated intuitionistic fuzzy continuity and pre-continuity within this framework.

With the advent of neutrosophy and neutrosophic sets by Smarandache [16], a new direction in uncertainty modeling emerged. Salama and Alblowi [12] introduced neutrosophic crisp sets and neutrosophic topological spaces (Nts), extending $ifts$ and incorporating degrees of membership, indeterminacy, and non-membership for each element. Neutrosophic has formed the foundation for a broader class of theories that generalize both crisp and fuzzy structures.

Smarandache also introduced the concept of dependence degrees between fuzzy and neutrosophic components. Later, Arokiarani et al. [3] introduced the neutrosophic set (NS), wherein the sum of the three membership values does not exceed 3. In the same year, Veereswari [23] proposed the notion of neutrosophic topological spaces (Nts) and studied fundamental operations on them.

Saranya et al. [13] introduced the concept of n-cylindrical neutrosophic sets (abbreviated as n-$CyNS$'s), characterized by α and γ as dependent components and β as an independent component. Apart from neutrosophic set (NS), n-$CyNS$ represents the most extensive generalization of fuzzy sets (fs). In this framework, the membership functions positive (α), neutral (β), and negative (γ) satisfy the conditions $0 \leq \beta_A \leq 1$ and $0 \leq \alpha_A^n(x) + \gamma_A^n(x) \leq 1,\ n > 1$, where $n > 1$ is an integer.

Later, Saranya et al. [15] introduced the notion of n-CyN continuity for functions between two n-cylindrical neutrosophic topological spaces (n-$CyNts$). They also defined the n-CyN interior (n-$CyNint$) and n-CyN closure (n-$CyNcl$) of subsets within n-$CyNts$.

In a related development, El-Maghrabi and Al-Juhani [8] introduced the concept of M-open sets in topological spaces and investigated several of their properties. The class of M-open sets plays a significant role in topological theory due to its applicability across various branches of mathematics and real-world applications. Padma et al. [11] also found M-open sets in nano topological spaces. Vadivel et al. [17–19] discussed some open sets in fuzzy nano and neutrosophic nano topological spaces. Kalaiyarsan et al. [10] and Vadivel et al. [20–22] introduced M-open sets and $\delta\beta$ open sets in fuzzy, Pythagorean fuzzy and neutrosophic nano topological spaces. Balasubramaniyan et al. [5] introduced contra continuous and irresolute maps in Pythagorean fuzzy nano topological spaces. Building on this, Anandhi et al. [1] introduced the concept of n-cylindrical neutrosophic topological spaces n-$CyNts$ based on n-$CyNS$'s. Anandhi et al. [2] further investigated n-cylindrical neutrosophic M-continuous and M-irresolute within this framework.

The subsequent sections of this dissertation are organized as follows: Section 2 provides a brief review of fundamental definitions related to intuitionistic fuzzy sets (ifs's), neutrosophic sets (NS's), n-cylindrical neutrosophic sets (n-$CyNS$'s) and mappings on n-cylindrical neutrosophic topological spaces (n-$CyNts$). Sections 3 and 4 introduce the concepts of n-$CyN\mathscr{C}MCts$ and n-$CyN\mathscr{C}MIrr$ within n-$CyNts$, along with an exploration of their fundamental properties supported by illustrative examples. Finally, the dissertation concludes with a summary of findings in Section 5.

2. Preliminaries

This section covers some basic definitions and examples that will be useful in subsequent discussions.

Definition 2.1. [24] A fuzzy set (briefly, fs) A in X is defined by membership function $\mu_A : A \to [0,1]$ whose membership value $\mu_A(x)$ shows the degree to which $x \in X$ includes in the fuzzy set A, for all $x \in X$.

Definition 2.2. [6] A fuzzy topological space (briefly, fts) is a pair (X,τ), where X is any set and τ is a family of fuzzy sets in X satisfying following axioms:

(i) $\phi, X \in \tau$,
(ii) If $A, B \in \tau$, then $A \cap B$ in τ,
(iii) If $A_i \in \tau$ for each $i \in I$, then $\cup A_i \in \tau$.

Definition 2.3. [4] An intuitionistic fuzzy set (briefly, ifs) A on X is an object of the form $A = \{\langle x, \alpha_A(x), \gamma_A(x) : x \in X\rangle\}$ where $\alpha_A(x) \in [0,1]$ is called the degree of positive membership of x in A, $\gamma_A(x) \in [0,1]$ is called the degree of negative membership of x in A, and where $\alpha_A(x)$ and $\gamma_A(x)$ satisfy (for all $x \in X$) $(\alpha_A(x) + \gamma_A(x) \leq 1)$ $ifs(X)$ denotes the set of all the ifs's on X.

Definition 2.4. [16] An neutrosophic set A on X is an object of the form $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\alpha_A(x), \beta_A(x), \gamma_A(x) \in [0,1], 0 \leq \alpha_A(x) + \beta_A(x) + \gamma_A(x) \leq 3$, for all $x \in X$. $\alpha_A(x)$ is the degree of positive membership, $\beta_A(x)$ is the degree of neutral and $\gamma_A(x)$ is the degree of negative membership. Here, $\alpha_A(x)$ and $\gamma_A(x)$ are dependent components and $\beta_A(x)$ is an independent component.

Definition 2.5. [12] An neutrosophic topology (Nt) on a non-empty set X is a family τ_N of neutrosophic subsets in X satisfying the following axioms:

(i) $0_N,\ 1_N \in \tau_N$,
(ii) $G_1 \cap G_2 \in \tau_N$ for any $G_1,\ G_2 \in \tau_N$,
(iii) $\cup G_i \in \tau_N$, for all $\{G_i : i \in J\} \subseteq \tau_N$.

In this case the pair (X,τ_N) is called a neutrosophic topological spaces (briefly, Nts) and any neutrosophic set in τ is known as neutrosophic open set (briefly, Nos) in X. The elements of τ_X are called neutrosophic open sets. A neutrosophic set F is closed if and only if $C(F)$ is neutrosophic open.

Definition 2.6. [13] An n-cylindrical neutrosophic set (briefly, n-$CyNS$) A on X is an object of the form $A = \{\langle x, \alpha_A(x),\ \beta_A(x), \gamma_A(x)\rangle : x \in X\}$, where $\alpha_A(x) \in [0,1]$ called the degree of positive membership of x in A, $\beta_A(x) \in [0,1]$ called the degree of neutral membership of

x in A and $\gamma_A(x) \in [0,1]$ called the degree of negative membership of x in A, which satisfies the condition: (for all $x \in X$) $(0 \leq \beta_A(x) \leq 1)$ and $0 \leq \alpha_A^n(x) + \gamma_A^n(x) \leq 1,\ n > 1$, is an integer. Here, $\alpha_A(x)$ and $\gamma_A(x)$ are dependent neutrosophic components and $\beta_A(x)$ is 100% independent.

For the convenience, $\langle \alpha_A(x), \beta_A(x), \gamma_A(x) \rangle$ is called as n-cylindrical neutrosophic number (briefly, n-$CyNN$) and is denoted as $A = \{\langle \alpha_A, \beta_A, \gamma_A \rangle\}$.

Definition 2.7. [13] Let $\{A_i : i \in I\}$ be an arbitrary family of n-$CyNS$ in X. Then, $\cap A_i = \{\langle x, \inf(\alpha_{A_i}(x)), \inf(\beta_{A_i}(x)), \sup(\gamma_{A_i}(x)) \rangle : x \in X\}$. $\cup A_i = \{\langle x, \sup(\alpha_{A_i}(x)), \sup(\beta_{A_i}(x)), \inf(\gamma_{A_i}(x)) \rangle : x \in X\}$.

Definition 2.8. [13] $0_{CyN} = \{\langle x, 0, 0, 1 \rangle : x \in X\}$ and $1_{CyN} = \{\langle x, 1, 1, 0 \rangle : x \in X\}$

Definition 2.9. [13] **(The Basic Connectives)** Let $\tau_{CyN}(X)$ denote the family of all n-$CyNS$'s on X.

Definition 2.10. [13] Inclusion: For every two $A, B \in \tau_{CyN}(X)$, the inclusion of two n-$CyNS$'s A and B is $A \subseteq B$ iff (for all $x \in X$, $\alpha_A(x) \leq \alpha_B(x)$ and $\beta_A(x) \leq \beta_B(x)$ and $\gamma_A(x) \geq \gamma_B(x)$) and ($A \subseteq B$ and $B \subseteq A$).

Definition 2.11. [13]Union: For every two $A, B \in \tau_{CyN}(X)$, the union of two n-$CyNS$'s A and B is $A \cup B(x) = \{\langle x, \max(\alpha_A(x), \alpha_B(x)), \max(\beta_A(x), \beta_B(x)), \min(\gamma_A(x), \gamma_B(x)) \rangle : x \in X\}$.

Definition 2.12. [13] Intersection: For every two $A, B \in \tau_{CyN}(X)$, the intersection of two n-$CyNS$'s A and B is

$$A \cap B(x) = \{\langle x, \min(\alpha_A(x), \alpha_B(x)), \min(\beta_A(x), \beta_B(x)), \max(\gamma_A(x), \gamma_B(x)) \rangle : x \in X\}.$$

Definition 2.13. [13] Complementary: For every $A \in \tau_{CyN}(X)$, the complement of an n-$CyNS$ A is $A^c = \{\langle x, \gamma_A(x), 1 - \beta_A(x), \alpha_A(x) \rangle : x \in X\}$.

Definition 2.14. [13] Sum: For every two $A, B \in \tau_{CyN}(X)$, the sum of two n-$CyNS$'s A and B is $A \oplus B(x) = \{\langle x, (\frac{\alpha_A(x) \cdot \alpha_B(x)}{\alpha_A(x) + \alpha_B(x)}), \max(\beta_A(x), \beta_B(x)), \min(\gamma_A(x), \gamma_B(x)) \rangle : x \in X\}$.

Definition 2.15. [13] Difference: For every two $A, B \in \tau_{CyN}(X)$, the difference of two n-$CyNS$'s A and B is $A \ominus B(x) = \{\langle x, \max(\alpha_A(x), \alpha_B(x)), \min(\beta_A(x), \beta_B(x)), (\frac{\gamma_A(x) \cdot \gamma_B(x)}{\gamma_A(x) + \gamma_B(x)}) \rangle : x \in X\}$.

Definition 2.16. [13] Product: For every two $A, B \in \tau_{CyN}(X)$, the product of two n-$CyNS$'s A and B is $A \otimes B(x) = \{\langle x, (\alpha_A(x) \cdot \alpha_A(x)), (\beta_A(x) \cdot \beta_B(x)), (\gamma_A(x) \cdot \gamma_B(x)) \rangle : x \in X\}$.

Definition 2.17. [13] Division: For every two $A, B \in \tau_{CyN}(X)$, the division of two n-$CyNS$'s A and B is $A \oslash B(x) = \{\langle x, \min(\alpha_A(x), \alpha_B(x)), (\beta_A(x) \cdot \beta_B(x)), \max(\gamma_A(x), \gamma_B(x)) \rangle : x \in X\}$.

Remark 2.18. [13]

(i) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$,

(ii) $A \cup B = B \cup A$ & $A \cap B = B \cap A$,

(iii) $(A \cup B) \cup C = A \cup (B \cup C)$ & $(A \cap B) \cap C = A \cap (B \cap C)$,

(iv) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$ & $(A \cap B) \cup C = (A \cup C) \cap (B \cap C)$,

(v) $A \cap A = A$ & $A \cup A = A$,

(vi) De Morgan's Law for A & B ie., $(A \cup B)^c = A^c \cap B^c$ & $(A \cap B)^c = A^c \cup B^c$,

(vii) $(A \oplus B) = (B \oplus A)$,

(viii) $(A \otimes B) = (B \otimes A)$.

Definition 2.19. [15] An n-cylindrical neutrosophic topology (briefly, n-$CyNt$) on a non-empty set X is a family, τ_{CyN}, of n-$CyNS$ in X which satisfies the following conditions:

(i) $0_{CyN}, 1_{CyN} \in \tau_{CyN}$,

(ii) $A_1 \cap A_2 \in \tau_{CyN}$,

(iii) $\cup A_i \in \tau_{CyN}$, for any arbitrary family $A_i \in \tau_{CyN}, i \in I$.

The pair (X, τ_{CyN}) is called an n-cylindrical neutrosophic topological Spaces (briefly, n-$CyNts$) and any n-$CyNS$ belongs to τ_{CyN} is called an n-cylindrical neutrosophic open set (briefly, n-$CyNos$) and the complement of n-$CyNos$ is called n-cylindrical neutrosophic closed set (briefly, n-$CyNcs$) in X. Like classical topological spaces and fuzzy topological spaces, the family $\{0_{CyN},\ 1_{CyN}\}$ is called in discrete n-$CyNts$ and the topology containing all the n-CyN subsets is called discrete n-$CyNts$.

Remark 2.20. [15] Obviously any fuzzy topological spaces or intuitionistic fuzzy topological spaces or Pythagorean fuzzy topological spaces is an n-$CyNts$ as any subsets of the fuzzy spaces, intuitionistic fuzzy space, and Pythagorean fuzzy space can be viewed as n-CyN subsets.

Definition 2.21. [15] Let A and B be two n-cylindrical neutrosophic subsets of an n-$CyNts$. B is called neighborhood of A if there exists an n-$CyNos$, O such that $A \subset O \subset B$.

Proposition 2.22. [15] $A \subset X$ is n-cylindrical neutrosophic open in (X, τ_{CyN}) if and only if it carries a neighborhood of its subsets.

Definition 2.23. [15] Let (X, τ_{CyN}) be an n-$CyNts$ and let $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in X\}$ is an n-$CyNS$ in X. Then, the n-cylindrical neutrosophic interior (briefly, n-$CyNint$) is defined as the n-CyN union of all n-CyN open subsets of X. ie, n-$CyNint(A) = \bigcup\{G : G \in \tau_{CyN}$ and $G \subseteq A\}$.

Clearly, n-$CyNint(A)$ is the biggest n-$CyNos$ that is contained by A.

Definition 2.24. [15] Let (X, τ_{CyN}) be an n-$CyNts$ and let $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in X\}$ is an n-$CyNS$ in X. Then, the n-cylindrical neutrosophic closure (briefly, n-$CyNcl$) is defined as the n-CyN intersection of all n-CyN closed subsets of X. ie, n-$CyNcl(A) = \bigcap\{K : K \in \tau_{CyN}$ and $A \subseteq A\}$.

Clearly, n-$CyNcl(A)$ is the smallest n-$CyNcs$ that contains A.

Definition 2.25. [14] Let (X, τ_1) and (Y, τ_2) be two n-$CyNts$ and let $f : (X, \tau_1) \to (Y, \tau_2)$ be a n-CyN function. Then, f said to be n-CyN continuous (briefly, n-$CyNCts$) map if for any n-cylindrical neutrosophic subset A of X and for any neighborhood $\mathfrak{V}$ of $f(A)$ there exists a neighborhood $\mathfrak{U}$ of A such that $f(\mathfrak{U}) \subset \mathfrak{V}$.

Definition 2.26. [1] Let (X, τ_{CyN}) be an n-$CyNts$ and A be an n-$CyNS$. Then, A is said to be an n-CyN

(i) regular open set (briefly, n-$CyNros$), if $A = n\text{-}CyNint(n\text{-}CyNcl(A))$,
(ii) regular closed set (briefly, n-$CyNrcs$), if $A = n\text{-}CyNcl(n\text{-}CyNint(A))$.

Definition 2.27. [1] Let (X, τ_{CyN}) be an n-$CyNts$ and $A = \{\langle x,\ \alpha_A(x),\ \beta_A(x),\ \gamma_A(x)\rangle : x \in X\}$ be an n-$CyNS$ in X. Then, the n-cylindrical neutrosophic δ-interior of A and the n-cylindrical neutrosophic δ-closure of A are denoted by n-$CyN\delta int(A)$ and n-$CyN\delta cl(A)$ are defined as follows:

(i) $n\text{-}CyN\delta int(A) = \bigcup\{G | G$ is an n-$CyNros$ and $G \subseteq A\}$,
(ii) $n\text{-}CyN\delta cl(A) = \bigcap\{K | K$ is an n-$CyNrcs$ and $A \subseteq K\}$.

Definition 2.28. [1] Let (X, τ_{CyN}) be an n-$CyNts$ and $A = \{\langle x,\ \alpha_A(x),\ \beta_A(x),\ \gamma_A(x)\rangle : x \in X\}$ be an n-$CyNS$ in X. Then, the n-cylindrical neutrosophic θ-interior of A and the n-cylindrical neutrosophic θ-closure of A are denoted by n-$CyN\theta int(A)$ and n-$CyN\theta cl(A)$ are defined as follows:

(i) $n\text{-}CyN\theta int(A) = \bigcup\{n\text{-}CyNint(B) : B \subseteq A\ \&\ B$ is a n-$CyNcs$ in $X\}$,
(ii) $n\text{-}CyN\theta cl(A) = \bigcap\{n\text{-}CyNcl(B) : A \subseteq B\ \&\ B$ is a n-$CyNos$ in $X\}$.

Definition 2.29. [1] Let (X, τ_{CyN}) be an n-$CyNts$ and $A = \{\langle x, \alpha_A(x), \beta_A(x), \gamma_A(x)\rangle : x \in X\}$ be an n-$CyNS$ in X. A set A is said to be n-CyN

(i) δ-open set (briefly, n-$CyN\delta os$), if $A = n\text{-}CyN\delta int(A)$,
(ii) δ-pre open set (briefly, n-$CyN\delta\mathcal{P}os$), if $A \subseteq n\text{-}CyN\delta int(A)$,
(iii) δ-semi open set (briefly, n-$CyN\delta\mathcal{S}os$), if $A \subseteq n\text{-}CyNcl(n\text{-}CyN\delta int(A))$,
(iv) θ-open set (briefly, n-$CyN\theta os$), if $A = n\text{-}CyN\theta int(A)$,
(v) θ-semi open set (briefly, n-$CyN\theta\mathcal{S}os$), if $A \subseteq n\text{-}CyNcl(n\text{-}CyN\theta int(A))$,
(vi) e-open set (briefly, n-$CyNeos$), if $A = n\text{-}CyNcl(n\text{-}CyN\delta int(A)) \cup n\text{-}CyNint(n\text{-}CyN\delta cl(A))$,

(vii) M-open set (briefly, n-$CyNMos$), if $A \subseteq n$-$CyNcl(n$-$CyN\theta int(A)) \cup n$-$CyNint(n$-$CyN\delta cl(A))$.

The complement of a n-$CyNMos$ (resp. n-$CyN\delta os$, n-$CyN\delta\mathcal{P}os$, n-$CyN\delta\mathcal{S}os$, n-$CyN\theta os$, n-$CyN\theta\mathcal{S}os$ & n-$CyNeos$) is called a n-$CyNM$ (resp. n-$CyN\delta$, n-$CyN\delta\mathcal{P}$, n-$CyN\delta\mathcal{S}$, n-$CyN\theta$, n-$CyN\theta\mathcal{S}$ & n-$CyNe$) closed set (briefly, n-$CyNMcs$ (resp. n-$CyN\delta cs$, n-$CyN\delta\mathcal{P}cs$, n-$CyN\delta\mathcal{S}cs$, n-$CyN\theta cs$, n-$CyN\theta\mathcal{S}cs$ & n-$CyNecs$)) in X.

The family of all n-$CyNMos$ (resp. n-$CyN\delta os$, n-$CyN\delta\mathcal{P}os$, n-$CyN\delta\mathcal{S}os$, n-$CyN\theta os$, n-$CyN\theta\mathcal{S}os$ & n-$CyNeos$) of X is denoted by n-$CyNMOS(X)$, (resp. n-$CyNMCS(X)$, n-$CyN\delta OS(X)$, n-$CyN\delta CS(X)$, n-$CyN\delta\mathcal{P}OS(X)$, n-$CyN\delta\mathcal{P}CS(X)$, n-$CyN\delta\mathcal{S}OS(X)$, n-$CyN\delta\mathcal{S}CS(X)$, n-$CyN\theta OS(X)$, n-$CyN\theta CS(X)$, n-$CyN\theta\mathcal{S}OS(X)$, n-$CyN\theta\mathcal{S}CS(X)$, n-$CyNeOS(X)$ & n-$CyNeCS(X)$).

Definition 2.30. [1] Let (X, τ_{CyN}) be an n-$CyNts$ and $A = \{\langle x,\ \alpha_A(x),\ \beta_A(x),\ \gamma_A(x)\rangle : x \in X\}$ be an n-$CyNS$ in X. Then, the n-CyN

(i) M-interior (resp. n-$CyN\delta$-interior, n-$CyN\delta\mathcal{P}$-interior, n-$CyN\delta\mathcal{S}$-interior, n-$CyN\theta$-interior, n-$CyN\theta\mathcal{S}$-interior & n-$CyNe$-interior) of A (briefly, n-$CyNMint(A)$ (resp. n-$CyN\delta int(A)$, n-$CyN\delta\mathcal{P}int(A)$, n-$CyN\delta\mathcal{S}int(A)$, n-$CyN\theta\ int(A)$, n-$CyN\theta\mathcal{S}int(A)$ & n-$CyNeint(A)$) is defined by n-$CyNMint(A)$ (resp. n-$CyN\delta int(A)$, n-$CyN\delta\mathcal{P}int(A)$, n-$CyN\delta\mathcal{S}int(A)$, n-$CyN\theta int(A)$, n-$CyN\theta\mathcal{S}int(A)$ & n-$CyNeint(A)) = \bigcup\{G : G \subseteq A$ and G is a n-$CyNMos$ (resp. n-$CyN\delta os$, n-$CyN\delta\mathcal{P}os$, n-$CyN\delta\mathcal{S}os$, n-$CyN\theta os$, n-$CyN\theta\mathcal{S}os$ & n-$CyNeos$) in $X\}$.

(ii) M-closure (resp. n-$CyN\delta$-closure, n-$CyN\delta\mathcal{P}P$-closure, n-$CyN\delta\mathcal{S}$-closure, n-$CyN\theta$-closure, n-$CyN\theta\mathcal{S}$-closure & n-$CyNe$-closure) of A (briefly, n-$CyNMcl(A)$ (resp. n-$CyN\delta cl(A)$, n-$CyN\delta\mathcal{P}cl(A)$, n-$CyN\delta\mathcal{S}cl(A)$, n-$CyN\theta cl(A)$, n-$CyN\theta\mathcal{S}cl(A)$ & n-$CyNecl(A)$) is defined by n-$CyNMcl(A)$ (resp. n-$CyN\delta cl(A)$, n-$CyN\delta\mathcal{P}cl(A)$, n-$CyN\delta\mathcal{S}cl(A)$, n-$CyN\theta cl(A)$, n-$CyN\theta\mathcal{S}cl(A)$ & n-$CyNecl(A)) = \bigcap\{K : K \subseteq A$ and K is a n-$CyNMcs$ (resp. n-$CyN\delta cs$, n-$CyN\delta\mathcal{P}cs$, n-$CyN\delta\mathcal{S}n$-$CyN\theta cs$, n-$CyN\theta\mathcal{S}cs$ & n-$CyNecs$) in $X\}$.

Definition 2.31. [2] Let (X, τ_1) and (Y, τ_2) be any two n-$CyNts$'s. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be a n-CyN

(i) δ-continuous map (briefly, n-$CyN\delta Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\delta os$ in (X, τ_1),

(ii) δ-pre continuous map (briefly, n-$CyN\delta\mathcal{P}Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\delta\mathcal{P}os$ in (X, τ_1),

(iii) δ-semi continuous map (briefly, n-$CyN\delta\mathcal{S}Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\delta\mathcal{S}os$ in (X, τ_1),

(iv) θ-continuous map (briefly, n-$CyN\theta Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\theta os$ in (X, τ_1),

(v) θ-semi continuous map (briefly, n-$CyN\theta\mathcal{S}Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\theta\mathcal{S}os$ in (X, τ_1),

(vi) e-continuous map (briefly, n-$CyNeCts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyNeos$ in (X, τ_1),

(vii) M-continuous map (briefly, n-$CyNMCts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyNMos$ in (X, τ_1).

Definition 2.32. [2] A map $f : (X, \tau_1) \to (Y, \tau_2)$ is called a n-cylindrical neutrosophic M-irresolute (briefly, n-$CyNMIrr$) map if $f^{-1}(B)$ is a n-$CyNMos$ in (X, τ_1) for every n-$CyNMos(B)$ of (Y, τ_2).

3. **Contra M-Continuous Maps in n-$CyNts$**

We will introduce n-cylindrical neutrosophic contra M-continuous maps and look at some of its feature in this section.

Definition 3.1. Let (X, τ_1) and (Y, τ_2) be any two n-$CyNts$'s. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is said to be a n-CyN contra

(i) δ-continuous map (briefly, n-$CyN\mathscr{C}\delta Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\delta cs$ in (X, τ_1),

(ii) δ-pre continuous map (briefly, n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\delta\mathcal{P}cs$ in (X, τ_1),

(iii) δ-semi continuous map (briefly, n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\delta\mathcal{S}cs$ in (X, τ_1),

(iv) θ-continuous map (briefly, n-$CyN\mathscr{C}\theta Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\theta cs$ in (X, τ_1),

(v) θ-semi continuous map (briefly, n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyN\theta\mathcal{S}cs$ in (X, τ_1),

(vi) e-continuous map (briefly, n-$CyN\mathscr{C}eCts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyNecs$ in (X, τ_1),

(vii) M-continuous map (briefly, n-$CyN\mathscr{C}MCts$ map), if the inverse image of every n-$CyNos$ in (Y, τ_2) is a n-$CyNMcs$ in (X, τ_1).

Example 3.2. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets $A_1,\ A_2,\ A_3,\ A_4$ in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\}$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\}$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\}$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\}$$

$$B = \{\langle y_1, 0.1250732, 0.50, 0.1850738\rangle, \langle y_2, 0.1450734, 0.50, 0.1650736\rangle\}$$

Then, we have $\tau_1 = \{0_X,\ 1_X,\ A_1,\ A_2,\ A_3,\ A_4\}$ and $\tau_2 = \{0_Y,\ 1_Y,\ B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}MCts$ map.

Proposition 3.3. The statements are true, but the converse need not be true.

(i) Every n-$CyN\mathscr{C}\theta Cts$ map is a n-$CyN\mathscr{C}Cts$ map,
(ii) Every n-$CyN\mathscr{C}\theta Cts$ map is a n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map,
(iii) Every n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map is a n-$CyN\mathscr{C}MCts$ map,
(iv) Every n-$CyN\mathscr{C}\delta Cts$ map is a n-$CyN\mathscr{C}Cts$ map,
(v) Every n-$CyN\mathscr{C}\delta Cts$ map is a n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map,
(vi) Every n-$CyN\mathscr{C}\delta Cts$ map is a n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map,
(vii) Every n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map is a n-$CyN\mathscr{C}eCts$ map,
(viii) Every n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map is a n-$CyN\mathscr{C}MCts$ map,
(ix) Every n-$CyN\mathscr{C}MCts$ map is a n-$CyN\mathscr{C}eCts$ map.

Proof.

(i) Let B be a n-$CyNos$ in (Y, τ_2). Since, f is n-$CyN\mathscr{C}\theta Cts$ map, $f^{-1}(B)$ is n-$CyN\theta cs$ (X, τ_1). Since every n-$CyN\theta cs$ are n-$CyNcs$. $f^{-1}(B)$ is n-$CyNcs$ in (X, τ_1). Hence, f is a n-$CyN\mathscr{C}Cts$ map.
(ii) Let B be a n-$CyNos$ in (Y, τ_2). Since, f is n-$CyN\mathscr{C}\theta Cts$ map, $f^{-1}(B)$ is n-$CyN\theta cs$ (X, τ_1). Since every n-$CyN\theta cs$ are n-$CyN\theta\mathcal{S}cs$. $f^{-1}(B)$ is n-$CyN\theta\mathcal{S}cs$ in (X, τ_1). Hence, f is a n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map.
(iii) Let B be a n-$CyNos$ in (Y, τ_2). Since, f is n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map, $f^{-1}(B)$ is n-$CyN\theta\mathcal{S}cs$ (X, τ_1). Since every n-$CyN\theta\mathcal{S}cs$ are n-$CyNMcs$. $f^{-1}(B)$ is n-$CyNMcs$ in (X, τ_1). Hence, f is a n-$CyN\mathscr{C}MCts$ map.
(iv) Let B be a n-$CyNos$ in (Y, τ_2). Since, f is n-$CyN\mathscr{C}\delta Cts$ map, $f^{-1}(B)$ is n-$CyN\delta cs$ (X, τ_1). Since every n-$CyN\delta cs$ are n-$CyNcs$. $f^{-1}(B)$ is n-$CyNcs$ in (X, τ_1). Hence, f is a n-$CyN\mathscr{C}Cts$ map.

(v) Let B be a n-$CyNos$ in (Y,τ_2). Since, f is n-$CyN\mathscr{C}\delta Cts$ map, $f^{-1}(B)$ is n-$CyN\delta cs$ (X,τ_1). Since every n-$CyN\delta cs$ are n-$CyN\delta\mathcal{P}cs$. $f^{-1}(B)$ is n-$CyN\delta\mathcal{P}cs$ in (X,τ_1). Hence, f is a n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map.

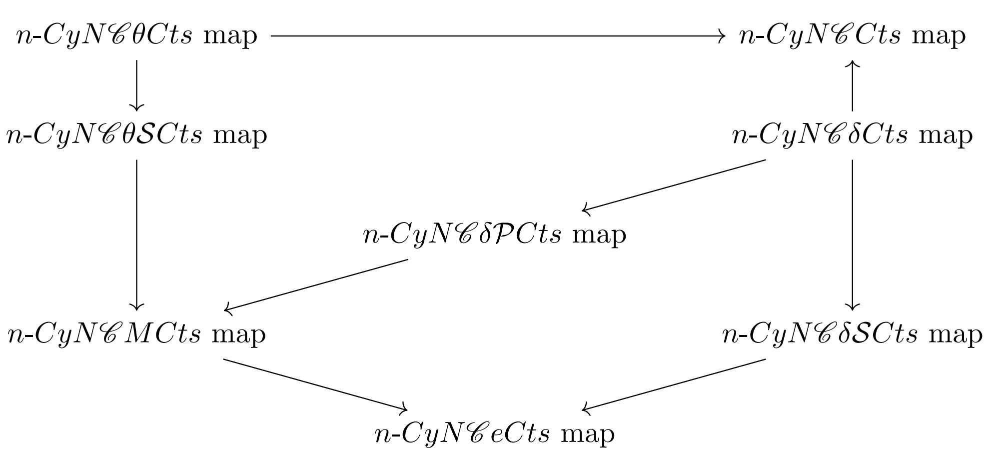
(vi) Let B be a n-$CyNos$ in (Y,τ_2). Since, f is n-$CyN\mathscr{C}\delta Cts$ map, $f^{-1}(B)$ is n-$CyN\delta cs$ (X,τ_1). Since every n-$CyN\delta cs$ are n-$CyN\delta\mathcal{S}cs$. $f^{-1}(B)$ is n-$CyN\delta\mathcal{S}cs$ in (X,τ_1). Hence, f is a n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map.

(vii) Let B be a n-$CyNos$ in (Y,τ_2). Since, f is n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map, $f^{-1}(B)$ is n-$CyN\delta\mathcal{S}cs$ (X,τ_1). Since every n-$CyN\delta\mathcal{S}cs$ are n-$CyNecs$. $f^{-1}(B)$ is n-$CyNecs$ in (X,τ_1). Hence, f is a n-$CyN\mathscr{C}eCts$ map.

(viii) Let B be a n-$CyNos$ in (Y,τ_2). Since, f is n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map, $f^{-1}(B)$ is n-$CyN\delta\mathcal{P}cs$ (X,τ_1). Since every n-$CyN\delta\mathcal{P}cs$ are n-$CyNMcs$. $f^{-1}(B)$ is n-$CyNMcs$ in (X,τ_1). Hence, f is a n-$CyN\mathscr{C}MCts$ map.

(ix) Let B be a n-$CyNos$ in (Y,τ_2). Since, f is n-$CyN\mathscr{C}MCts$ map, $f^{-1}(B)$ is n-$CyNMcs$ (X,τ_1). Since every n-$CyNMcs$ are n-$CyNecs$. $f^{-1}(B)$ is n-$CyNecs$ in (X,τ_1). Hence, f is a n-$CyN\mathscr{C}eCts$ map.

Remark 3.4. From the above mentioned results. We get the diagram below.



Note: $A \rightarrow B$ denotes A implies B, but not conversely.

Example 3.5. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets $A_1,\ A_2,\ A_3,\ A_4$ in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$B = \{\langle y_1, 0.1250732, 0.50, 0.1850738\rangle, \langle y_2, 0.1450734, 0.50, 0.1650736\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}Cts$ map but not n-$CyN\mathscr{C}\theta Cts$ map, because the set $f^{-1}(B) = A_1$ is a n-$CyNcs$ but not n-$CyN\theta cs$.

Example 3.6. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3, A_4, A_5 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_5 = \{\langle x_1, 0.1250732, 0.50, 0.1850738\rangle, \langle x_2, 0.1450734, 0.50, 0.1650736\rangle\},$$

$$B = \{\langle y_1, 0.1850738, 0.50, 0.1250732\rangle, \langle y_2, 0.1650736, 0.50, 0.1450734\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map but not n-$CyN\mathscr{C}\theta Cts$ map, because the set $f^{-1}(B) = A_5$ is a n-$CyN\theta\mathcal{S}cs$ but not n-$CyN\theta cs$.

Example 3.7. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3, A_4 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$B = \{\langle y_1, 0.1250732, 0.50, 0.1850738\rangle, \langle y_2, 0.1450734, 0.50, 0.1650736\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}MCts$ map but not n-$CyN\mathscr{C}\theta\mathcal{S}Cts$ map, because the set $f^{-1}(B) = A_1$ is a n-$CyNMcs$ but not n-$CyN\theta\mathcal{S}cs$.

Example 3.8. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3, A_4 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$B = \{\langle y_1, 0.1250732, 0.50, 0.1850738\rangle, \langle y_2, 0.1350733, 0.50, 0.1750737\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}Cts$ map but not n-$CyN\mathscr{C}\delta Cts$ map, because the set $f^{-1}(B) = A_4$ is a n-$CyNcs$ but not n-$CyN\delta cs$.

Example 3.9. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_3 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$B = \{\langle y_1, 0.1250732, 0.50, 0.1850738\rangle, \langle y_2, 0.1350733, 0.50, 0.1750737\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map but not n-$CyN\mathscr{C}\delta Cts$ map, because the set $f^{-1}(B) = A_3$ is a n-$CyN\delta\mathcal{P}cs$ but not n-$CyN\delta cs$.

Example 3.10. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3, A_4, A_5 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_5 = \{\langle x_1, 0.1250732, 0.50, 0.1450734\rangle, \langle x_2, 0.1450734, 0.50, 0.1450734\rangle\},$$

$$B = \{\langle y_1, 0.1450734, 0.50, 0.1250732\rangle, \langle y_2, 0.1450734, 0.50, 0.1450734\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map but not n-$CyN\mathscr{C}\delta Cts$ map, because the set $f^{-1}(B) = A_5$ is a n-$CyN\delta\mathcal{S}cs$ but not n-$CyN\delta cs$.

Example 3.11. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3, A_4, A_5 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$
$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$
$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$
$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$
$$A_5 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1350733\rangle\},$$
$$B = \{\langle y_1, 0.1250732, 0.50, 0.1850738\rangle, \langle y_2, 0.1350733, 0.50, 0.1650736\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}eCts$ map but not n-$CyN\mathscr{C}\delta\mathcal{S}Cts$ map, because the set $f^{-1}(B) = A_5$ is a n-$CyNecs$ but not n-$CyN\delta\mathcal{S}cs$.

Example 3.12. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets A_1, A_2, A_3, A_4, A_5 in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$
$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$
$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$
$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$
$$A_5 = \{\langle x_1, 0.1250732, 0.50, 0.1850738\rangle, \langle x_2, 0.1450734, 0.50, 0.1650736\rangle\},$$
$$B = \{\langle y_1, 0.1850738, 0.50, 0.1250732\rangle, \langle y_2, 0.1650736, 0.50, 0.1450734\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}MCts$ map but not n-$CyN\mathscr{C}\delta\mathcal{P}Cts$ map, because the set $f^{-1}(B) = A_5$ is a n-$CyNMcs$ but not n-$CyN\delta\mathcal{P}cs$.

Example 3.13. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets $A_1, A_2, A_3, A_4, A_5, A_6$ in (X, τ_1) and B in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_5 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1350733\rangle\},$$

$$A_6 = \{\langle x_1, 0.1250732, 0.50, 0.1450734\rangle, \langle x_2, 0.1450734, 0.50, 0.1450734\rangle\},$$

$$B = \{\langle y_1, 0.1450734, 0.50, 0.1250732\rangle, \langle y_2, 0.1450734, 0.50, 0.1450734\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}eCts$ map but not n-$CyN\mathscr{C}MCts$ map, because the set $f^{-1}(B) = A_6$ is a n-$CyNecs$ but not n-$CyNMcs$.

Theorem 3.14. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is n-$CyN\mathscr{C}MCts$ map iff the inverse image of each n-$CyNcs$ in (Y, τ_2) is n-$CyNMos$ in (X, τ_1).

Proof. Let B be a n-$CyNcs$ in (Y, τ_2). This implies B^c is a n-$CyNos$ in (Y, τ_2). Since, f is n-$CyN\mathscr{C}MCts$. $f^{-1}(B^c)$ is n-$CyNMcs$ in (X, τ_1). Since, $f^{-1}(B^c)$, $f^{-1}(B)$ is a n-$CyNMos$ in (X, τ_1).

Conversely, let B be a n-$CyNcs$ in (Y, τ_2). Then, B^c is a n-$CyNos$ in (Y, τ_2). By hypothesis $f^{-1}(B^c)$ is n-$CyNcs$ in (X, τ_1). Since, $f^{-1}(B^c) = (f^{-1}(B))^c$. $(f^{-1}(B))^c$ is a n-$CyNMcs$ in (X, τ_1). Therefore, $f^{-1}(B)$ is a n-$CyNMos$ in (X, τ_1). Hence, f is a n-$CyNMCts$.

Definition 3.15. A n-$CyNt$ (X, τ_1) is said to be n-cylindrical neutrosophic $MU_{\frac{1}{2}}$ (briefly, n-$CyNMU_{\frac{1}{2}}$)-space, if every n-$CyNMos$ in (X, τ_1) is a n-$CyNos$ in (X, τ_1).

Theorem 3.16. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a n-$CyN\mathscr{C}MCts$ map, then f is a n-$CyN\mathscr{C}Cts$ map if (X, τ_1) is a n-$CyNM_{\frac{1}{2}}$-space.*

Proof. Let B be a n-$CyNos$ in (Y, τ_2). Then, $f^{-1}(B)$ is a n-$CyNMcs$ in (X, τ_1), by hypothesis. Since, (X, τ_1) is a n-$CyNMU_{\frac{1}{2}}$-space, $f^{-1}(B)$ is a n-$CyNcs$ in (X, τ_1). Hence, f is a n-$CyN\mathscr{C}MCts$ map.

Theorem 3.17. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a n-$CyN\mathscr{C}MCts$ map and $g : (Y, \tau_2) \to (Z, \tau_3)$ be a n-$CyN\mathscr{C}Cts$ map, then $g \circ f$. $(X, \tau_1) \to (Z, \tau_3)$ is a n-$CyN\mathscr{C}MCts$ map.

Proof. Let A be a n-$CyNos$ in (Z, τ_3). Then, $g^{-1}(A)$ is a n-$CyNos$ in (Y, τ_2), by hypothesis. Since, f is a n-$CyN\mathscr{C}MCts$ map, $f^{-1}(g^{-1}(A))$ is a n-$CyNMcs$ in (X, τ_1). Hence, $g \circ f$ is a n-$CyN\mathscr{C}MCts$ map.

Theorem 3.18. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a n-$CyN\mathscr{C}MCts$ map. The following conditions are then satisfied

(i) $f(n\text{-}CyNMcl(A)) \supseteq n\text{-}CyNint(f(A))$, for all $n\text{-}CyNcs(A)$ in (X, τ_1),
(ii) $n\text{-}CyNMcl(f^{-1}(B)) \supseteq f^{-1}(n\text{-}CyNint(B))$, for all $n\text{-}CyNcs(B)$ in (Y, τ_2).

Proof.

(i) Since, $n\text{-}CyNMcl(f(A))$ is a $n\text{-}CyNMcs$ in (Y, τ_2) and f is $n\text{-}CyN\mathscr{C}MCts$ map, then $f^{-1}(n\text{-}CyNMcl(f(A)))$ is $n\text{-}CyNMos$ in (X, τ_1). Now, since $A \supseteq f^{-1}(n\text{-}CyNint(f(A)))$, $n\text{-}CyNMcl(A) \supseteq f^{-1}(n\text{-}CyNMint(f(A)))$. Therefore, $f(n\text{-}CyNMcl(A)) \supseteq n\text{-}CyNint(f(A))$.
(ii) By replacing A with B in (i). We obtain $f(n\text{-}CyNMcl(f^{-1}(B))) \supseteq n\text{-}CyNint(f(f^{-1}(B))) \supseteq n\text{-}CyNint(B)$. Hence, $n\text{-}CyNMcl(f^{-1}(B)) \supseteq f^{-1}(n\text{-}CyNint(B))$.

4. Contra M-irresolute Maps in n-$CyNts$

We will introduce n-cylindrical neutrosophic contra M-irresolute maps and look at some of its feature in this section.

Definition 4.1. A map $f : (X, \tau_1) \to (Y, \tau_2)$ is called a n-cylindrical neutrosophic contra M-irresolute (briefly, $n\text{-}CyN\mathscr{C}MIrr$) map if $f^{-1}(B)$ is a $n\text{-}CyNMcs$ in (X, τ_1) for every $n\text{-}CyNMos(B)$ of (Y, τ_2).

Theorem 4.2. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a $n\text{-}CyN\mathscr{C}MIrr$ map, then f is a $n\text{-}CyN\mathscr{C}MCts$ map. But not conversely.

Proof. Let f be a $n\text{-}CyN\mathscr{C}MIrr$ map. Let B be any $n\text{-}CyNos$ in (Y, τ_2). Since, every $n\text{-}CyNos$ is a $n\text{-}CyNMos(B)$ is a $n\text{-}CyNMos$ in (Y, τ_2). $\implies$ $f^{-1}(B)$ is a $n\text{-}CyNMcs$ in (Y, τ_2). Hence, f is a $n\text{-}CyN\mathscr{C}MCts$ map.

Example 4.3. Let $X = \{x_1, x_2\}$, $Y = \{y_1, y_2\}$ and the n-cylindrical neutrosophic sets $A_1, A_2, A_3, A_4, A_5, A_6$ in (X, τ_1) and B_1, B_2 in (Y, τ_2) are defined as

$$A_1 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1650736, 0.50, 0.1450734\rangle\},$$

$$A_2 = \{\langle x_1, 0.1950739, 0.50, 0.1150731\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_3 = \{\langle x_1, 0.1150731, 0.50, 0.1950739\rangle, \langle x_2, 0.1350733, 0.50, 0.1750737\rangle\},$$

$$A_4 = \{\langle x_1, 0.1850738, 0.50, 0.1250732\rangle, \langle x_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$A_5 = \{\langle x_1, 0.1250732, 0.50, 0.1450734\rangle, \langle x_2, 0.1450734, 0.50, 0.1450734\rangle\},$$

$$B_1 = \{\langle y_1, 0.1950739, 0.50, 0.1150731\rangle, \langle y_2, 0.1750737, 0.50, 0.1350733\rangle\},$$

$$B_2 = \{\langle y_1, 0.1450734, 0.50, 0.1250732\rangle, \langle y_2, 0.1450734, 0.50, 0.1450734\rangle\}.$$

Then, we have $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\tau_2 = \{0_Y, 1_Y, B_1\}$. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f is n-$CyN\mathscr{C}MCts$ map but not n-$CyN\mathscr{C}MIrr$ map, because the set B_2 is a n-$CyNMos$ in (Y, τ_2) $f^{-1}(B_2)$ is not n-$CyNMcs$ in (X, τ_1).

Theorem 4.4. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a n-$CyN\mathscr{C}MIrr$ map, then f is a n-$CyN\mathscr{C}Cts$ map if (X, τ_1) is a n-$CyNMU_{\frac{1}{2}}$-space.

Proof. Let B be a n-$CyNos$ in (Y, τ_2). Then, B is a n-$CyNMos$ in (Y, τ_2). Therefore, $f^{-1}(B)$ is a n-$CyNMcs$ in (X, τ_1), by hypothesis. Since, (X, τ_1) is a n-$CyNMU_{\frac{1}{2}}$-space. $f^{-1}(B)$ is a n-$CyNcs$ in (X, τ_1). Hence f is a n-$CyN\mathscr{C}Cts$.

Theorem 4.5. *Let $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$ be a n-$CyN\mathscr{C}MIrr$ map, then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is a n-$CyN\mathscr{C}MIrr$ map.*

Theorem 4.6. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be n-$CyN\mathscr{C}MIrr$ map and $g : (Y, \tau_2) \to (Z, \tau_3)$ be n-$CyN\mathscr{C}MCts$ map, then $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is a n-$CyN\mathscr{C}MCts$ map.

Proof. Let A be a n-$CyNos$ in (Z, τ_3). Then, $g^{-1}(A)$ is a n-$CyNMos$ in (Y, τ_2). Since, f is a n-$CyN\mathscr{C}MIrr$, $f^{-1}(g^{-1}(A))$ is a n-$CyNMcs$ in (X, τ_1). Hence, $g \circ f$ is a n-$CyN\mathscr{C}MCts$.

Theorem 4.7. Let $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$ be a mappings. Then, $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is

(i) n-$CyN\mathscr{C}MCts$ if f is n-$CyN\mathscr{C}Irr$ and g is n-$CyN\mathscr{C}MCts$,

(ii) n-$CyN\mathscr{C}MIrr$ if f is n-$CyN\mathscr{C}MIrr$ (resp. n-$CyNMIrr$) and g is n-$CyNMIrr$ (resp. n-$CyN\mathscr{C}MIrr$).

Theorem 4.8. Let $f : (X, \tau_1) \to (Y, \tau_2)$ and $g : (Y, \tau_2) \to (Z, \tau_3)$ are n-$CyN\mathscr{C}MCts$ mappings and (Y, τ_2) be a n-$CyNMU_{\frac{1}{2}}$-space. Then, $g \circ f : (X, \tau_1) \to (Z, \tau_3)$ is n-$CyNMCts$.

Theorem 4.9. Let $f : (X, \tau_1) \to (Y, \tau_2)$ be a map. Then the following conditions are equivalent if (X, τ_1) and (Y, τ_2) are n-$CyNMU_{\frac{1}{2}}$-space.

(i) f is a n-$CyN\mathscr{C}MIrr$ map,

(ii) $f^{-1}(B)$ is a n-$CyNMos$ in (X, τ_1) for each n-$CyNMcs(B)$ in (Y, τ_2),

(iii) n-$CyNcl(f^{-1}(B)) \supseteq f^{-1}(n$-$CyNint(B))$ for each n-$CyNS(B)$ of (Y, τ_2).

5. Conclusions

In this paper, by using n-$CyNMos$ we introduced n-$CyN\mathscr{C}MCts$ and n-$CyN\mathscr{C}MIrr$ maps are analyzed and studied its properties.

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Advancing Common Fixed Point Theorems in Bipolar Fuzzy-2 Metric Space and Applications in Nonlinear Analysis

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Abstract: This study advances common fixed point theorems (CFPT) in bipolar fuzzy-2 metric space (BF2MS) and explores their applications in nonlinear analysis. BF2MS generalizes traditional metric and fuzzy metric spaces (FMS) by incorporating dualistic relationships, allowing the representation of both attraction and repulsion effects. We extend classical fixed point theorems (FPTs) to BF2MS, establishing conditions for the existence of common fixed points (CFPs). The study's applications span nonlinear analysis, stability analysis, control theory, and multi-criteria decision-making under uncertainty. Our findings refine existing results and provide practical real-world applications, supported by numerical examples, enhancing fuzzy mathematics and nonlinear systems modeling.

Keywords: Common Fixed Point Theorem, Bipolar Fuzzy-2 Metric Space, Fuzzy Metric Space

1. Introduction

For centuries, philosophers and scientists have sought to address uncertainty, ambiguity, and vagueness in mathematical and logical frameworks. Initially, probability theory served as the primary tool for dealing with such challenges (von Mises, 1957 [1]. However, the seminal work of Lotfi Zadeh's (1965) [2] on fuzzy set theory (FST) introduced a revolutionary approach to modeling imprecision. Since, then FST evolved into a robust mathematical framework applied across various disciplines. For instance, Klir & Yuan, 1995 [3] provided a comprehensive formalization on fuzzy systems, while Atanassov, 1986 [4] extended the concept of intuitionistic fuzzy set (IFS). Further refinements, such as those by Bustince & Burillo, 1996 [5], strengthened the theoretical underpinnings of FST. Despite these advances, continued research remains essential to refine and expand its theoretical foundations.

Over the past few decades, fuzzy mathematics has developed extensively, extending classical mathematical concepts into the fuzzy domain. Early contributions in this direction including the pioneering work of Bellman & Zadeh, 1970 [6] on decision theory and Dubois & Prade, 1980 [7] on operation research followed by the comprehensive treatment of fuzzy topology by Klement, Mesiar & Pap, 2000 [8]. The theory of FMS advanced through distinct formulations introduced by Kramosil and Michálek (1975) [9], and Kaleva and Seikkala (1984) [10], laying the foundations of fuzzy probabilistic topologies. Later, significant progress in the FPT within FMS was made by Grabiec, 1989 [11], and Park J. H., 2004 [12].

In metric space theory, an important breakthrough was made by Sharma, Sharma, and Iseki (1984) [13], who investigated contraction-type mappings in 2-metric spaces (2-MS), leading to further developments in FPT. These ideas were further extended into probabilistic 2-MS by Wenzhi (1997) [14], integrating probability theory with 2-metric structures to handle uncertainty more effectively. Traditional 2-MS, first introduced by Gähler, 1963 [15] and later refined by Dhage, 1992 [16] inspired by Euclidean area functions and find in spatial analysis.

Building upon these advancements, BF2MS presents a novel approach by incorporating dual-metric framework that simultaneously captures compatibility and opposition. This innovative framework broadens the applicability of CFPTs to nonlinear analysis. Jain et al., 2020 [17] emphasized on compatibility measures, while Dutta & Debnath, 2022 [18], explored opposition metrics. Further many researchers demonstrated the relevance of such structures in stability analysis, control theory and multi-criteria decision-making under uncertainty. Collectively these contributions paving the way for future advancements in open new frontiers in nonlinear and fuzzy systems research.

1.1 Motivation for the above study

The motivation for studying BF2MSs and their CFPTs arises from the need to develop advanced mathematical tools that address uncertainty, duality, and vagueness in real-world systems. By extending FMSs into BF2MSs, this framework captures the intricacies of systems influenced by opposing forces. For instance, Jain, Kaushik, & Singh, 2020 [19] highlighted their role in nonlinear dynamics, while Matloka, 1984 [20] provided foundation results in fuzzy convergence. Moreover, decision-making frameworks pioneered by Bellman and Zadeh (1970) [6] and extended by Dubois & Prade (1980) [7] have been refined in context of contradictory preferences through Xu, 2007 [21]. Such developments are particularly relevant to the applications in economics, risk assessment, and artificial intelligence, where competing objectives must be balanced.

Additionally, BF2MSs facilitate the precise mathematical representation of complex systems where cooperation and competition dynamically interact, such as in multi-agent systems, game theory, and social networks. In applied domains, this framework enhances modeling accuracy in physics, engineering, and economics, where dual influences shape system behavior. By developing CFPTs in BF2MS, this research contributes to both theoretical advancements and practical applications. It introduces a powerful mathematical toolset for analyzing layered, nonlinear relationships and addressing gaps in existing models. As such, BF2MSs offer new insights into stability, decision-making, and uncertainty modeling, reinforcing their importance in advancing modern fuzzy mathematics and nonlinear analysis.

1.2 Recent Developments

Recent advancements in BF2MSs have refined theoretical foundations and expanded applications, particularly in nonlinear analysis, decision-making models, and complex systems under uncertainty. As explored by Atanassov and Gargov (2020) [22], bipolar fuzzy methods are used in decision making problems of multi criteria in two choices that makes trade off evaluation also in [23] they work on interval-valued IFSs and their application in multi criteria decision making (MCDM) frameworks in uncertain environments.

In nonlinear analysis, Mohammadi and Jafari (2021) [24] utilized BF2MSs to model dynamical systems with the coexistence of attraction and repulsion forces, benefiting fields like physics (Aghababa, 2018 [25]) and engineering (Yao et al., 2022 [26]). Extensions proposed by Singh and Sharma (2021) [27] introduced additional parameters for more flexible modeling of multi-dimensional relationships, finding utility in neural networks and artificial intelligence (AI) systems. Integrations with AI by Gupta and Kumar (2022) [28], highlighted the effectiveness of BF2MSs in machine learning tasks, such as sentiment analysis and financial forecasting, where contradictory data must be managed. Generalized BF2MSs and their applications in neural networks were introduced by Singh and Sharma (2021) [29], showing their impact on deep learning architectures and cognitive computing. Applications in robotics and cyber-physical systems emphasized by Xu (2023) [30] demonstrated the relevance of BF2MSs for autonomous vehicle control. These studies underline the growing significance of BF2MSs in addressing dualistic and uncertain environments across diverse disciplines, including computational intelligence, engineering and decision sciences.

1.3 Novelty of the study

Our study introduces an advanced mathematical framework by extending traditional FMSs to BF2MSs, enabling the modeling of dual-dimensional relationships such as attraction-repulsion and compatibility-opposition. This approach enhances the representation of complex systems where opposing influences coexist, overcoming the limitations of traditional metrics. By generalizing classical FPTs, we establish new conditions applicable to systems governed by dual interactions. The framework is particularly valuable for analyzing stability and convergence in nonlinear systems and improving MCDM by reconciling conflicting objectives. Ultimately, this study advances fuzzy mathematics, offering new insights into fuzzy topology, nonlinear analysis, and decision theory, with broad interdisciplinary applications across computational intelligence, engineering, and control theory.

1.4 Practical Applicability

BF2MSs hold substantial potential in mathematics, particularly in fields such as FPT, topology, optimization, variational inequalities, MCDM, and convergence analysis. By incorporating both positive and negative memberships, BF2MSs provide a refined and adaptable framework for addressing uncertainty, conflict, and intricate relationships in complex systems. This innovative approach establishes BF2MSs as a vital tool in contemporary mathematical modeling and analysis, with significant implications for computational intelligence, control theory, and decision sciences.

2. Preliminaries

The study of CFPTs in BF2MS involves several foundational concepts from FST, metric spaces, and FPT. These preliminaries establish the groundwork for extending classical results into the BF2MS framework.

Definition 2.1 (MS): Let X be an arbitrary non-empty set. A function $d: X \times X \to [0,\infty)$ is called a metric on X, if for any elements $x, y, z \in X$, the following axioms are satisfied:

(i) non-negativity: $d(x,y) \geq 0$

(ii) identity of indiscernible: $d(x,y)=0$ iff $x=y$
(iii) symmetry: $d(x,y)=d(y,x)$
(iv) triangle inequality: $d(x,z)\le d(x,y)+d(y,z)$

The pair (X,d) is then called a MS.

Definition 2.2 (2-MS): Let X be a non-empty set. A function $\Omega: X\times X\times X\to[0,\infty)$ is called 2-metric on X if for all elements $x,y,z,w\in X,$ the following conditions holds:

(v) Non-negativity: for any distinct $x,y,z\in X,$ $\Omega(x,y,z)\ge 0$.
(vi) Identity: $\Omega(x,y,z)=0,$ iff x,y,z are not distinct (i.e. at least two of the points are equal)
(vii) Symmetry in arguments: the value remains unchanged for any permutation of the points $x,y,z.$ i.e. $\Omega(x,y,z)=\Omega(y,z,x)=\Omega(z,x,y)$
(viii) tetrahedral inequality: $\Omega(x,y,z)\le d(x,u,w)+d(x,w,z)+d(w,y,z)$

The pair (X,d) is then called a 2-MS.

Definition 2.3: Let X be a non-empty set. A mapping $B: X\to[0,1]\times[0,1]$ is called bipolar fuzzy set B on X if for each element $x\in X$ the pair $\left(\mu_B(x),\nu_B(x)\right)$ representing its degree of membership and degree of non-membership, respectively i.e. $\mu_B(x)\in[0,1]$ (positive membership degree) and $\nu_B(x)\in[0,1]$ (negative membership degree). The pair $\left(\mu_B(x),\nu_B(x)\right)$ must satisfy the constraint $\mu_B(x)+\nu_B(x)\le 1$ $\forall x\in X.$

Definition 2.4: A BF2MS on X is a 3-tuple $(X,M_\Omega,*)$ where '$*$' is a continuous t-norm on [0,1]. $M_\Omega=\left(M_\Omega^+,M_\Omega^-\right)$ is a bipolar fuzzy set on $X^3\times[0,\infty)$, that is, for each triplet $x,y,z\in X$ and a real number $t>0,$ we have a pair $M_\Omega^+\left(x,y,z,t\right),$ $M_\Omega^-\left(x,y,z,t\right)$ representing the positive and negative memberships of the BF2M M_Ω, satisfies the following axioms for all $x,y,z,w\in X$ and all $s,t>0$:

(BF2M-1): $M_\Omega^+\left(x,y,z,t\right)=1$ and $M_\Omega^-\left(x,y,z,t\right)=0.$

(BF2M-2): $M_\Omega^+\left(x,y,z,t\right)\ne 1$ and $M_\Omega^-\left(x,y,z,t\right)\ne 0,\ \forall\ t>0,$ also $M_\Omega^+\left(x,y,z,t\right)=1$ and $M_\Omega^-\left(x,y,z,t\right)=0,\ \forall\ t\ge 0,$ if at least two of x,y,z are equal, and $\forall\ x,y\in X\ \exists\ z\in X.$

(BF2M-3): $M_\Omega^+\left(x,y,z,t\right)$ and $M_\Omega^+\left(p(x,y,z),t\right)$ are invariant under any permutation of $x,y,z.$

(BF2M-4): $M_\Omega^+(x,y,z,r+t+s)\ge M_\Omega^+(x,y,w,r)*M_\Omega^+(x,w,z,t)*M_\Omega^+(w,y,z,s),$

$$M_\Omega^-(x,y,z,r+t+s)\le M_\Omega^-(x,y,w,r)*M_\Omega^-(x,w,z,t)*M_\Omega^-(w,y,z,s)$$

(BF2M-5): $M_\Omega^+(x,y,z,.):(0,\infty)\to(0,1]$ and $M_\Omega^-(x,y,z,.):(0,\infty)\to[-1,0)$ are both continuous functions.

Some additional necessary conditions in BF2MS are outlined as follows:

Symmetry: $M_\Omega^+(x,y,z,t)=M_\Omega^+(y,z,x,t)$ and $M_\Omega^-(x,y,z,t)=M_\Omega^-(y,z,x,t)$

Boundary Conditions: $\lim_{t\to\infty} M_\Omega^+(x,y,z,t)=1$ and $\lim_{t\to\infty} M_\Omega^-(x,y,z,t)=0$ and

$$\lim_{t\to 0} M_\Omega^+(x,y,z,t)=0 \text{ and } \lim_{t\to 0} M_\Omega^-(x,y,z,t)=1$$

Monotonicity: $M_\Omega^+(x,y,z,t_1)\le M_\Omega^+(x,y,z,t_2)$ and $M_\Omega^-(x,y,z,t_1)\ge M_\Omega^-(x,y,z,t_2)$ for $t_1<t_2$.

Contraction Condition: Let (X,Ω) be CMS. If $f:X\to X$ is a continuous mapping, then there exists a constant $k\in(0,1)$ Such That (s.t.), for all $x,y,z\in X$ and $t>0$, we have $M_\Omega^+(f(x),f(y),z,t)\ge M_\Omega^+(x,y,z,kt)$ and $M_\Omega^-(f(x),f(y),z,t)\le M_\Omega^-(x,y,z,kt)$ implies f has a unique FP in X.

Example 2.1: Consider $X=[0,1]$, with $a*b=\min\{a,b\}$ for $a,b\in[0,1]$. Define the bipolar fuzzy 2-metric M_Ω by $M_\Omega^+\left(x,y,z,t\right)=\min\left\{|x-y|,|y-z|,|z-x|\right\}$

and $M_\Omega^-(x,y,z,t)=\min\left\{\left(1-|x-y|\right),\left(1-|y-z|\right),\left(1-|z-x|\right)\right\}$ for $x,y,z\in X, t>0.$

Then $(X,M_\Omega,*)$ forms a BF2MS.

Example 2.2: Let $X=(x_1,x_2,x_3)$ be a set of three distinct point. Suppose we have 2-metric Ω defined on X, which intuitively measure the area of a triangle formed by three points. For simplicity, assume that $\Omega(x,y)$ is a function that outputs a value between 0 and 1, representing the degree of closeness between points in the space. We now define the following positive and negative membership functions. Let $\mu_\Omega(x,y,z,t)=\min\left(\Omega(x,y),\Omega(y,z),\Omega(z,x)\right)$ representing the positive membership or attraction degree between x,y,z at time t and $v_\Omega(x,y,z,t)=1-\min\left(\Omega(x,y),\Omega(y,z),\Omega(z,x)\right)$ representing the negative membership or repulsion degree between x,y,z. Thus, the BF2M is given by $M_\Omega^+\left(x,y,z,t\right)=\min\left\{\Omega(x,y),\Omega(y,z),\Omega(z,x)\right\}$ and $M_\Omega^-\left(x,y,z,t\right)=1-\min\left\{\Omega(x,y),\Omega(y,z),\Omega(z,x)\right\}$.

Let's assume the distances based on the 2-metric Ω are $\Omega(x_1,x_2)=0.8$, $\Omega(x_2,x_3)=0.9$ and $\Omega(x_3,x_1)=0.7$. Then, the positive membership for x_1,x_2,x_3 at a given time t is $M_\Omega^+(x_1,x_2,x_3,t)=\min(0.8,0.9.0.7)=0.7$. The negative membership for x_1,x_2,x_3 at the same

$M^{-}_{\Omega}(x_1,x_2,x_3,t)=1-\min(0.8,0.9.0.7)=0.3$. Thus, the BF2M for this set of points at time t is $M_{\Omega}(x_1,x_2,x_3,t)=\left\{M^{+}_{\Omega}(x_1,x_2,x_3,t),M^{-}_{\Omega}(x_1,x_2,x_3,t)\right\}=(0.7,0.3)$.

This illustrates how the BF2MS introduces both positive and negative memberships to model dual relationships between points, providing a more comprehensive framework for handling uncertainty and duality in complex systems.

Definition 2.5: Let us consider a BF2MS as $\left(X,M_{\Omega},*\right)$. Then

(i) If $\lim\limits_{n\to\infty}x_n=x$ and $x\in X$, then the sequence $\{x_n\}$ is said to converge to a point if

$$\lim_{n\to\infty}M^{+}_{\Omega}\left(x_n,x,z,t\right)=1 \text{ and } \lim_{n\to\infty}M^{-}_{\Omega}\left(x_n,x,z,t\right)=0,\ \forall z\in X \text{ and } t>0.$$

(ii) If $\lim\limits_{n,m\to\infty}M^{+}_{\Omega}(x_n,x_m,z,t)=1$ and $\lim\limits_{n,m\to\infty}M^{-}_{\Omega}(x_n,x_m,z,t)=0$ for all $n,m\in N$, $z\in X$ and $t>0$.

then the sequence $\{x_n\}$ is said to be a Cauchy sequence (CS).

(iii) Every CS in BF2MS, which is convergent is said to be complete.

Lemma 2.1: Let $(X,M^{+}_{\Omega},M^{-}_{\Omega},*)$ be a BF2MS. If $M^{+}_{\Omega}(x,y,z,kt)\ge M^{+}_{\Omega}(x,y,z,t)$ and $M^{-}_{\Omega}(x,y,z,kt)\le M^{+}_{\Omega}(x,y,z,t)$, $\forall x,y,z\in X,t>0,$ and some $k\in(0,1)$ hold good in BF2MS, then $x=y$.

Lemma 2.2: The functions $M^{+}_{\Omega}\left(x,y,z,t\right)$ and $M^{-}_{\Omega}\left(x,y,z,t\right)$ are non-decreasing in *t* for all $x,y,z\in X$.

Proof: Let $t',t>0,\ t'>t$ then, by applying BF2M-4 for both the positive and negative memberships, since $M^{+}_{\Omega}(x,y,z,t')\ge M^{+}_{\Omega}(x,y,z,t)$ and $M^{-}_{\Omega}(x,y,z,t')\le M^{+}_{\Omega}(x,y,z,t)$ which establishes the non-decreasing values in both positive and negative memberships.

Lemma 2.3: Let $\{x_n\}$ be a sequence in a BF2MS $(X,M_{\Omega},*)$, then $\{x_n\}$ is a CS in *X*, if for some $k\in(0,1)$, $\exists k$ s.t. $M^{+}_{\Omega}\left(x_{n+1},x_{n+2},z,kt\right)\ge M^{+}_{\Omega}\left(x_n,x_{n+1},z,t\right)$ and

$M^{-}_{\Omega}\left(x_{n+1},x_{n+2},z,kt\right)\le M^{-}_{\Omega}\left(x_n,x_{n+1},z,t\right)$, $\forall x,y,z\in X,t>0$ and $n=0,1,2,3,\ldots.$

Lemma 2.4 (Contraction Mappings in BF2MS): Let $(X,M^{+}_{\Omega},M^{-}_{\Omega},*)$ be a BF2MS. If for all $x,y,z\in X$, $t>0$, a mapping $T:X\to X$ satisfies a contraction condition in the BF2MS if $\exists k\in[0,1)$ s.t. $M^{+}(T(x),T(y),z,t)\ge\lambda.M^{+}(x,y,z,t)$ and $M^{-}(T(x),T(y),z,t)\le\lambda.M^{-}(x,y,z,t)$.

Lemma 2.5 (Compatibility and Weak Compatibility in BF2MS): Let $(X, M_\Omega^+, M_\Omega^-, *)$ be a BF2MS. If all $x, y, z \in X,\ t > 0,$ two mappings $T_1,\ T_2 : X \to X$ are compatible if

$$\lim_{n\to\infty} M^+(T_1(x_n), T_2(x_n), z, t) \geq \lim_{n\to\infty} M^+(x_n, x_n, z, t),$$

and $$\lim_{n\to\infty} M^-(T_1(x_n), T_2(x_n), z, t) \leq \lim_{n\to\infty} M^-(x_n, x_n, z, t), \text{ for any sequence } \{x_n\}\ X.$$

Two mappings T_1 and T_2 are weakly compatible if $T_1, T_2(x) = T_2T_1(x)$ implies $T_1(x) = T_2(x).$

Lemma 2.6: In all the examples we define the function $\phi \in \Phi,$ which is upper semi-continuous, monotonic, non-decreasing, and satisfies diagonal dominance. We prove some FPT on $\phi : [0,1]^5 \to [0,1]$ and each variable satisfying $\phi(t,t,t,t,t) \geq t, \quad \forall t \in [0,1].$ For example, the function $\phi\left(t_1, t_2, t_3, t_4, t_5\right) = t_1, \ \forall t_i\text{'s} \in [0,1]$ is a member of this class.

Lemma 2.7: (Coincidentally Commuting Mappings): Two mappings $f, g : X \to X$ are said to be coincidentally commuting if $(f \circ g)(x) = (g \circ f)(x)$ whenever $f(x) = g(x).$

Notably, every commuting mapping is coincidentally commuting, but the reverse does not always hold.,

3. Fixed Point Theorem for BF2MS

Theorem 3.1: Let $(X, M_\Omega, *)$ be a complete BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-),$ and $t * t \geq t$ holds for all $t \in [0,1]$ and let $f : X \to X$ be a self-mapping. Then f has a CFP in X if there exists a continuous mapping $A : X \to f(X),$ which commutes with f and satisfies

$$M_\Omega^+\left(A(x), A(y), z, kt\right) \geq \phi \begin{Bmatrix} M_\Omega^+(f(x), f(y), z, t); M_\Omega^+(f(x), A(x), z, t); M_\Omega^+(f(y), A(y), z, t); \\ M_\Omega^+(f(y), A(x), z, \alpha t); M_\Omega^+(f(x), A(y), z, (2-\alpha)t) \end{Bmatrix} \quad (1)$$

and $$M_\Omega^-\left(A(x), A(y), z, kt\right) \leq \phi \begin{Bmatrix} M_\Omega^-(f(x), f(y), z, t); M_\Omega^-(f(x), A(x), z, t); M_\Omega^-(f(y), A(y), z, t); \\ M_\Omega^-(f(y), A(x), z, \alpha t); M_\Omega^-(f(x), A(y), z, (2-\alpha)t) \end{Bmatrix} \quad (2)$$

for $x, y, z \in X, t > 0,$ and $\alpha \in (0,2),$ and $k \in (0,1)$ and $\phi \in \Phi,$ as in lemma 2.6.

If $\lim_{t\to\infty} M_\Omega^+\left(A(x), A(y), z, t\right) = 1$ and $\lim_{t\to\infty} M_\Omega^-(A(x), A(y), z, t) = 0, \ \forall x, y, z \in X.$ then f and A have a UCFP.

Proof: By setting $x_{n+1} = f(x_n) = A(x_{n-1}),\ n = 0,1,2...$, we develop a sequence $\{x_n\}$. First we prove that $\{A(x_n)\}$ is a CS. Applying condition (ii) with $\alpha = 1-k$ for $k \in (0,1)$, in (1), we have

$$M_\Omega^+(A(x_{n-1}), A(x_n), z, kt) \ge \phi\left\{\begin{array}{r} M_\Omega^+(f(x_{n-1}), f(x_n), z, t); M_\Omega^+(f(x_{n-1}), A(x_{n-1}), z, t); \\ M_\Omega^+(f(x_n), A(x_n), z, t); M_\Omega^+(f(x_n), A(x_{n-1}), z, (1-k)t); \\ M_\Omega^+(f(x_{n-1}), A(x_n), z, (1+k)t) \end{array}\right\}$$

or
$$M_\Omega^+(x_{n+1}, x_{n+2}, z, kt) \ge \phi\left\{\begin{array}{r} M_\Omega^+(x_n, x_{n+1}, z, t), M_\Omega^+(x_n, x_{n+1}, z, t), M_\Omega^+(x_{n+1}, x_{n+2}, z, t), \\ M_\Omega^+(x_{n+1}, x_{n+1}, z, (1-k)t), M_\Omega^+(x_n, x_{n+2}, z, (1+k)t) \end{array}\right\}$$

$$M_\Omega^+(x_{n+1}, x_{n+2}, z, kt) \ge \phi\left\{\begin{array}{r} M_\Omega^+(x_n, x_{n+1}, z, t), M_\Omega^+(x_n, x_{n+1}, z, t), M_\Omega^+(x_{n+1}, x_{n+2}, z, t), 1, \\ M_\Omega^+(x_n, x_{n+1}, z, t) * M_\Omega^+(x_n, x_{n+2}, z, t) \end{array}\right\} \tag{3}$$

Now two cases arise,

Case I: Because ϕ is non-decreasing i.e. $a_i \le b_i, \forall i \Rightarrow \phi(a_1, ...a_n) \le \phi(b_1, ...b_n)$, and if $M_\Omega^+(x_{n+1}, x_{n+2}, z, t) \ge M_\Omega^+(x_n, x_{n+1}, z, t)$ then equation (1) yields

$$M_\Omega^+(x_{n+1}, x_{n+2}, z, kt) \ge \phi\left\{M_\Omega^+(x_n, x_{n+1}, z, t), M_\Omega^+(x_n, x_{n+1}, z, t), M_\Omega^+(x_{n+1}, x_{n+2}, z, t), 1, M_\Omega^+(x_n, x_{n+1}, z, t)\right\}$$

from lemma 2.6, $M_\Omega^+(x_{n+1}, x_{n+2}, z, kt) \ge M_\Omega^+(x_n, x_{n+1}, z, t)$ or

$M_\Omega^+(A(x_{n-1}), A(x_n), z, kt) \ge M_\Omega^+(A(x_{n-2}), A(x_{n-1}), z, t),\ \forall t > 0,$ therefore by lemma 2.3, $\{A(x_n)\}$ is a CS.

Case II: If $M_\Omega^+\left(x_n, x_{n+1}, z, t\right) > M_\Omega^+\left(x_{n+1}, x_{n+2}, z, t\right)$ for some *n*, then by using (3)

$$M_\Omega^+(x_{n+1}, x_{n+2}, z, kt) \ge M_\Omega^+(x_{n+1}, x_{n+2}, z, t).$$

Thus by lemma 2.3, $x_{n+1} = x_{n+2}$ which implies $f(x_{n+1}) = f(x_{n+2})$ i.e $x_{n+2} = x_{n+3}$ and so on. These exists some positive integer s.t. $n \ge m$ implies $x_n = x_m$, which show that $\{x_n\}$ a convergent sequence and so Cauchy sequence. Thus in either case, $\{A(x_n)\}$ is a Cauchy sequence in *X*. Since *X* is complete, therefore there exists $\lim z = \lim_{n\to\infty} A(x_n),$ for all $z \in X,$ it follows that $Az = fz.$ Now using $\alpha = 1+q,$ $q \in (0,1),$ all $t > 0$, in (3) yields

$$M_\Omega^+(x_{n+1}, A(z), z, kt) = M_\Omega^+(A(x_{n+1}), A(z), z, kt) \ge \phi\left\{\begin{array}{r} M_\Omega^+(x_n, A(z), z, t); M_\Omega^+(x_n, x_{n+1}, z, t); \\ M_\Omega^+(f(z), A(z), z, t); M_\Omega^+(f(z), x_{n+1}, z, (1+q)t); \\ M_\Omega^+(x_n, A(z), z, (1-q)t) \end{array}\right\}$$

or letting $n \to \infty$ and after that $q \to 0,$

$$M_\Omega^+(z,A(z),z,kt)\ge\phi\left\{M_\Omega^+(z,A(z),z,t);1;1;M_\Omega^+(A(z),z,z,t);M_\Omega^+(z,A(z),z,t)\right\}$$

from lemma 2.6, $M_\Omega^+(z,A(z),z,kt)\ge M_\Omega^+(z,A(z),z,t)$, which implies that $A(z)=z$ therefore $f(z)=A(z)=z$ i.e $f(z)=z$. Thus f has a fixed point z. That is f and A have a CFP z.

Again let $w(\ne z)$ be another fixed point of f, i.e. $f(z)=z$ and $f(w)=w$, the for the sake of uniqueness using (3) for $\alpha=1+q$, $q\in(0,1)$, we have $M_\Omega^+(z,w,z,kt)=M_\Omega^+(A(z),A(w),z,kt)$, $t\in[0,1]$.

Letting $q\to 0$, $M_\Omega^+(z,w,z,kt)\ge\phi\left\{M_\Omega^+(z,w,z,t);M_\Omega^+(z,w,z,t);1;M_\Omega^+(z,w,z,t);M_\Omega^+(z,w,z,t)\right\}$

i.e. $M_\Omega^+(z,w,z,kt)\ge M_\Omega^+(z,w,z,t)$, for all $t > 0$, from lemma 2.2. Now using lemma 2.1 which implies z = w. Hence f and A have UCFP in X. Similarly we can prove $\lim_{t\to\infty} M_\Omega^-(A(x),A(y),z,t)=0$. This completes the theorem.

Theorem 3.2: Let $(X,M_\Omega,*)$ be a complete BF2MS with $M_\Omega=(M_\Omega^+,M_\Omega^-)$, and $t*t\ge t$ for all $t\in[0,1]$. Suppose S and T are continuous mappings of X into itself. Then S and T have a CFP in X if there exists a continuous mapping A of X into $S(X)\bigcap T(X)$, commutes with S and T, which satisfy the following conditions:

(i)•• $S(X)\subseteq T(X)$

(ii)•• $T(X)$ is complete,

(iii)•• The bipolar fuzzy 2-metric M_Ω satisfies the inequality:

$$M_\Omega^+\left(A(x),A(y),a,kt\right)\ge\phi\left\{\begin{matrix}M_\Omega^+\left(S(x),T(y),a,t\right);M_\Omega^+\left(S(x),A(x),a,t\right);M_\Omega^+\left(T(y),A(y),a,t\right);\\ M_\Omega^+\left(T(y),A(x),a,\alpha t\right);M_\Omega^+\left(S(x),A(y),a,(2-\alpha)t\right)\end{matrix}\right\}\quad(4)$$

and $$M_\Omega^-(A(x),A(y),a,kt)\ge\phi\left\{\begin{matrix}M_\Omega^-(S(x),T(y),a,t);M_\Omega^-(S(x),A(x),a,t);M_\Omega^-(T(y),A(y),a,t);\\ M_\Omega^-(T(y),A(x),a,\alpha t);M_\Omega^-(S(x),A(y),a,(2-\alpha)t)\end{matrix}\right\}\quad(5)$$

$\forall(x,y,a)\in X$, $t>0$, $\alpha\in(0,2)$, $k\in(0,1)$ and $\phi\in\Phi$, as in lemma 2.6 with $\lim_{n\to\infty}M_\Omega^+\left(x,y,z,t\right)=1$ and $\lim_{n\to\infty}M_\Omega^-(x,y,z,t)=0$, implies A, S and T A have a UCFP in X.

Proof: Define a sequence $\{y_n\}$ recursively via $y_{n+1}=f(x_n)$ for $n\ge 0$, with $x_0\in X$. By setting $y_{2n}=Ax_{2n}=Sx_{2n+1}$ and $y_{2n+1}=Ax_{2n+1}=Tx_{2n+2}$, $\forall n=0,1,2....$. We will show the sequence $\{Ax_n\}$ is CS.

Case I: If $y_{2n}\ne y_{2n+1}$, for $\alpha=1-q$, $q\in(0,1)$ we introduced $x=x_{2n}$, $y=x_{2n}$, $z=x_{2n+1}$ in (4), then

$$M_{\Omega}^{+}(Ax_{2n},Ax_{2n+1},z,qt)\geq\phi\left\{\begin{array}{r}M_{\Omega}^{+}\left(Sx_{2n},Tx_{2n+1},z,t\right);M_{\Omega}^{+}\left(Sx_{2n},Ax_{2n},z,t\right);M_{\Omega}^{+}\left(Tx_{2n+1},Ax_{2n+1},z,t\right);\\ M_{\Omega}^{+}\left(Tx_{2n+1},Ax_{2n},z,(1-q)t\right);M_{\Omega}^{+}\left(Sx_{2n},Ax_{2n+1},z,(1+q)t\right)\end{array}\right\}$$

or $$M_{\Omega}^{+}\left(y_{2n}y_{2n+1},z,qt\right)\geq\phi\left\{\begin{array}{r}M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);\\ M_{\Omega}^{+}\left(y_{2n},y_{2n},z,(1-q)t\right);M_{\Omega}^{+}\left(y_{2n-1},y_{2n+1},z,(1+q)t\right)\end{array}\right\}$$

$$\geq\phi\left\{M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);1;M_{\Omega}^{+}\left(y_{2n-1},y_{2n+1},z,(1+q)t\right)\right\}$$

letting $q\rightarrow1$, then and (BF2M-4),

$$M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,kt\right)\geq\phi\left\{\begin{array}{r}M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);1;\\ M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right)\end{array}\right\}$$

which we have either $M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right)\leq M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right)$

or $$M_{\Omega}^{+}\left(y_{2n},y_{2\,n+1},z,t\right)>M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right).$$

Assume that the condition $M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right)\leq M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right)$ hold for a certain *n*. Owing to the fact that ϕ is non-decreasing in every condition, we can infer from (4) that

$$M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,kt\right)\geq\phi\left\{M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right);1;M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right)\right\}$$

i.e. $$M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,kt\right)\geq M_{\Omega}^{+}\left(y_{2n-1},y_{2n},z,t\right)\quad(6)$$

Similarly, in later cases, it follows from that

$$M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,kt\right)\geq\phi\left\{\begin{array}{r}M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);\\ M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right);M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right)\end{array}\right\}$$

i.e. $$M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,kt\right)\geq M_{\Omega}^{+}\left(y_{2n},y_{2n+1},z,t\right)\quad(7)$$

for every $t>0$, lemma 2.3, yields $y_{2n}=y_{2n+1}$, which leads to a contradiction, because $y_{2n}\neq y_{2n+1}$ holds $\forall n=0,1,2......$ Hence from (6), we obtain

$$M_{\Omega}^{+}(y_{2n},y_{2n+1},z,kt)\geq M_{\Omega}^{+}(y_{2n-1},y_{2n},z,t)\quad(8)$$

for all $t>0$, Again for $2m>2p+1$, then from (8),

$$M_{\Omega}^{+}\left(Ax_{2m},Ax_{2p+1},z,kt\right)\geq M_{\Omega}^{+}\left(Ax_{2m-1},Ax_{2p},z,t/k\right)$$

for every m and *p* in *N*. Further if 2*m*>2*p*+1, then

$$M_{\Omega}^{+}(Ax_{2m},Ax_{2p+1},z,kt)\geq M_{\Omega}^{+}(Ax_{2k-1},Ax_{2p},z,t/k).....M_{\Omega}^{+}(Ax_{0},Ax_{2p+-2m},z,t/k^{2m})\quad(9)$$

If 2*m*>2*p*+1, then $M_{\Omega}^{+}(Ax_{2m},Ax_{2p+1},z,kt)\geq M_{\Omega}^{+}(Ax_{2m-(2p+1)},Ax_{0},z,t/k^{2p+1})$ (10)

by simple induction with (9) & (10) we have $M_{\Omega}^{+}\left(Ax_{n}, Ax_{n+1}, z, kt\right) \geq M_{\Omega}^{+}\left(Ax_{0}, Ax_{l}, z, t / k^{n}\right)$.

If $n = 2m$, $p = 2s + 1$ *or* $n = 2m + 1$, $p = 2s + 1$ in (BF2M-4)

$M_{\Omega}^{+}(Ax_{n}, Ax_{n+p}, z, kt) \geq M_{\Omega}^{+}(Ax_{0}, Ax_{p}, Ax_{1}, t / 3k^{n}) * M_{\Omega}^{+}(Ax_{0}, Ax_{1}, z, t / 3k^{n}) * M_{\Omega}^{+}(Ax_{p}, Ax_{1}, z, t / 3k^{n})$ (11)

If $n = 2m,\ p = 2s \text{ or } n = 2m + 1,\ p = 2s$, for every positive integer p and n in N, by noting that $M_{\Omega}^{+}\left(Ax_{0}, Ax_{p}, z, t / k^{n}\right) \rightarrow 1 \text{ as } n \rightarrow \infty$. Thus $\{Ax_{n}\}$ is a Cauchy sequence.

Case II: If $y_{2n} = y_{2n+1}$, for some positive integer m. Hence m may be even or odd positive integer without loss of generality take $n = p$ i.e., $x_{2p} = x_{2p+1}$ and so that

$ASx_{2p} = ASx_{2p+1}, SAx_{2p} = ASx_{2p+1}, ASx_{2p} = ATx_{2p} = ATx_{2p+1}, SAx_{2p} = TAx_{2p} = TAx_{2p+1}$ and $ASx_{2p} = SAx_{2p+1}, ATx_{2p} = TAx_{2p+1}$, which implies that $x_{2p+2} = x_{2p+3}$. In the same way $x_{2p+4} = x_{2p+5}$..... Thus $x_{2p+2r} = x_{2p+2r+1}$, $\forall r = 0,1,2......$ Thus we have for case I as

$$M_{\Omega}^{+}\left(A\, x_{2p+2r+1}, Ax_{2p+2r+2}, z, kt\right) \geq M_{\Omega}^{+}\left(Ax_{2p+2r}, Ax_{2p+2r+1}, z, t\right) = 1,$$

$\Rightarrow M_{\Omega}^{+}\left(Ax_{2p+2r+1}, Ax_{2p+2r+2}, z, kt\right) = 1, \quad \forall t > 0$, yields by $Ax_{2p+2r+1} = Ax_{2p+2r+2}, \quad \forall r = 0,1,2...$. Thus $Ax_{2p} = Ax_{2p+1} = Ax_{2p+2}.....$, this establishes that the sequence $\{Ax_{n}\}$ is convergent and consequently it must also be Cauchy. Thus $\{Ax_{n}\}$ is always a Cauchy sequence. Since X is complete, the sequence $\{Ax_{n}\}$ converges to some point $u \in X$, which further implies $u = \lim_{n \to \infty} Ax_{n}$ and $z = \lim_{n \to \infty} Sx_{2n+1} = \lim_{n \to \infty} Tx_{2n+2}$, it follows that $Au = Su = Tu$. Applying (4) for $\alpha = 1$

$$M_{\Omega}^{+}\left(Au, A^{2}u, z, kt\right) = M_{\Omega}^{+}\left(Au, AAu, z, kt\right) \geq \phi\left\{\begin{array}{r} M_{\Omega}^{+}\left(Su, TAu, z, t\right); M_{\Omega}^{+}\left(Su, Au, z, t\right); M_{\Omega}^{+}\left(TAu, AAu, z, t\right); \\ M_{\Omega}^{+}\left(TAu, Au, z, t\right); M_{\Omega}^{+}\left(Su, AAu, z, t\right) \end{array}\right\}$$

$$= \phi\left\{\begin{array}{r} M_{\Omega}^{+}\left(Au, ATu, z, t\right); M_{\Omega}^{+}\left(Au, Au, z, t\right); M_{\Omega}^{+}\left(ATu, A^{2}u, z, t\right); \\ M_{\Omega}^{+}\left(ATu, Au, z, t\right); M_{\Omega}^{+}\left(Au, A^{2}u, z, t\right) \end{array}\right\}$$

$$= \phi\left\{M_{\Omega}^{+}\left(Au, A^{2}u, z, t\right); 1; 1; M_{\Omega}^{+}\left(A^{2}u, Au, z, t\right); M_{\Omega}^{+}\left(Au, A^{2}u, z, t\right)\right\}$$

i.e. $M_{\Omega}^{+}\left(Au, A^{2}u, z, kt\right) \geq M_{\Omega}^{+}\left(Au, A^{2}u, z, t\right) \geq M_{\Omega}^{+}\left(Au, A^{2}u, z, t / k\right)$, also

$$M_\Omega^+\left(Au, A^2u, z, kt\right) \geq M_\Omega^+\left(Au, A^2u, z, t/k\right) \geq M_\Omega^+\left(Au, A^2u, z, t/k^2\right) \geq ....M_\Omega^+\left(Au, A^2u, z, t/k^n\right).$$

Science $lim_{n\to\infty} M_\Omega^+\left(Au, A^2u, z, t/k^n\right) = 1.$, so that $M_\Omega^+\left(Au, A^2u, z, kt\right) = 1$, which implies, $Au = A^2u$, and hence $A^2u = Au = Su = Tu = u$. Thus u is a CFP of S, T and A.

For uniqueness let $w(\neq u)$ be another CFPof S, T and A. From (4) for $\alpha = 1$, we have

$$M_\Omega^+\left(Au, Aw, z, kt\right) \geq \phi\begin{Bmatrix} M_\Omega^+\left(Su, Tw, z, t\right); M_\Omega^+\left(Su, Au, z, t\right); M_\Omega^+\left(Tw, Aw, z, t\right); \\ M_\Omega^+\left(Tw, Au, z, t\right); M_\Omega^+\left(Su, Aw, z, t\right) \end{Bmatrix}$$

$$= \phi\begin{Bmatrix} M_\Omega^+\left(Au, Aw, z, t\right); M_\Omega^+\left(Au, Au, z, t\right); M_\Omega^+\left(Aw, Aw, z, t\right); \\ M_\Omega^+\left(Aw, Au, z, t\right); M_\Omega^+\left(Au, Aw, z, t\right) \end{Bmatrix} \geq M_\Omega^+\left(Au, Aw, z, t\right)$$

ie $M_\Omega^+\left(u, w, z, kt\right) \geq M_\Omega^+\left(u, w, z, t\right)$. Therefore by lemma 2.1, we wite $u = w$. That z is a UCFP of S, T and A.

Corollary 3.1: Let $(X, M_\Omega, *)$ be a complete intuitionistic BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-)$, $t * t \geq t, \forall t \in [0,1]$. Assume ($S$, T) be two continuous maps. Then S and T possess a CFP in X *provided* there exists a continuous mapping A of X into $S(X) \cap T(X)$ that commutes with S and T. Moreover $(m,p) > 0$, and there exists $0 < k < 1$, s.t.

$$M_\Omega^+(Ax, Ay, z, kt) \geq \phi\begin{Bmatrix} M_\Omega^+(S^p x, T^m y, z, t); M_\Omega^+(S^p x, Ax, z, t); M_\Omega^+(T^m y, \\ Ay, z, t); M_\Omega^+(T^m y, Ax, z, \alpha t); M_\Omega^+(S^p x, Ay, z, (2-\alpha)t) \end{Bmatrix} \quad (12)$$

and

$$M_\Omega^-(Ax, Ay, z, kt) \leq \phi\begin{Bmatrix} M_\Omega^-(S^p x, T^m y, z, t); M_\Omega^-(S^p x, Ax, z, t); M_\Omega^-(T^m y, Ay, z, t); \\ M_\Omega^-(T^m y, Ax, z, \alpha t); M_\Omega^-(S^p x, Ay, z, (2-\alpha)t) \end{Bmatrix} \quad (13)$$

for $x, y, z \in X, t > 0, \alpha \in (0,2)$, and $\phi \in \Phi$, as in lemma 2.6 with $\lim_{n\to\infty} M_\Omega^+(x, y, z, t) = 1$ and $\lim_{n\to\infty} M_\Omega^-(x, y, z, t) = 0$. Further suppose that (a) $S^p(X) \subseteq T^m(X)$ and (b) $T^m(X)$ is complete, then S, T, and A have a UCFP.

Proof: Using the conditions in theorem 3.2 for BF2MS, S^p and T^m have a UCFP z s.t. $S^p z = T^m z = z$. Since $S^p(Az) \subseteq A(S^p z) = Az$ and $T^m(Az) \subseteq A(T^m z) = Az$ we conclude that Az is also a CFP of, S^p and T^m. By the uniqueness of z, it implies $S(z) = z$ and $T(z) = z$, so $A(z) = S(z) = T(z) = z$. Hence, S, T, and A have a UCFP. This completes the proof.

Corollary 3.2: Let $(X, M_\Omega, *)$ be a complete intuitionistic BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-)$, $t * t \geq t$ for all $t \in [0,1]$. Let $f: X \to X$ be a self-mapping in X. If $\exists$ a continuous mapping $A: X \to f(X)$ s.t., for $x, y, z \in X$, $t > 0, k \in (0,1)$,

$$M_\Omega^+(Ax, Ay, z, kt) \geq M_\Omega^+(fx, fy, z, t) \text{ and } M_\Omega^-(Ax, Ay, z, kt) \leq M_\Omega^+(fx, fy, z, t),$$

then *f* and *A* has a UCFP in *X*.

Proof: This can be obtained by setting $\phi: [0,1]^5 \to [0,1]$ as $\phi(t_1, t_2, t_3, t_4, t_5) = t_1$ in theorem 3.2 for BF2MS.

Corollary 3.3: Let $(X, M_\Omega, *)$ be a complete BF2MS with $M_\Omega = (M_\Omega^+, M_\Omega^-)$, $t * t \geq t$ for all $t \in [0,1]$. If there exists a continuous mapping $A: X \to X$ s.t., for $x, y, z \in X, t > 0, k \in (0,1), \alpha \in (0,2)$,

$$M_\Omega^+(Ax, Ay, z, kt) \geq \min\left\{\begin{matrix} M_\Omega^+(x, y, z, t); M_\Omega^+(x, Ax, z, t); M_\Omega^+(y, Ay, z, t); \\ M_\Omega^+(y, Ax, z, \alpha t); M_\Omega^+(x, Ay, z, (2-\alpha)t) \end{matrix}\right\} \quad (14)$$

and

$$M_\Omega^-(Ax, Ay, z, kt) \leq \max\left\{\begin{matrix} M_\Omega^-(x, y, z, t); M_\Omega^-(x, Ax, z, t); M_\Omega^-(y, Ay, z, t); \\ M^-(y, Ax, z, \alpha t); M_\Omega^-(x, Ay, z, (2-\alpha)t) \end{matrix}\right\} \quad (15)$$

then *A* has a UCFP.

Proof: This follows by choosing $\phi: [0,1]^5 \to [0,1]$ as $\phi(t_1, t_2, t_3, t_4, t_5) = \min(t_1, t_2, t_3, t_4, t_5)$ in the positive M_Ω^+ and $\phi(t_1, t_2, t_3, t_4, t_5) = \max(t_1, t_2, t_3, t_4, t_5)$ in the negative M_Ω^-, with *f* =*I* in theorem 3.2.

4. Real-Life Applications in BF2MS

Example 4.1 (BF2MS in MCDM): A company is evaluating three alternative projects (P_1, P_2, P_3) using a BF2MS approach in MCDM. The evaluation considers the positive criteria (+) (profit potential, innovation, market demand) and negative criteria (−) (risk, environmental impact, implementation complexity). The goal is to determine the most suitable project by balancing positive and negative influences using BF2MS. Let the assign normalized values to each project based on expert opinions are as in Table 1

Criteria	P_1	P_2	P_3
Profit Potential (+)	0.837	0.745	0.923
Innovation (+)	0.945	0.665	0.824
Market Demand (+)	0.749	0.856	0.965
Risk (-)	0.475	0.555	0.391
Environmental Impact (-)	0.645	0.476	0.575
Implementation Complexity (-)	0.568	0.682	0.448

Table 1: Initial Scores for Each Project

Solution: For the solution, we find the following steps

Step 4.1.1: A BF2MS is represented by (P, d^+, d^-), where

- $P = (P_1, P_2, P_3)$ (set of projects)
- $d^+(P_1, P_2, P_3)$ represents the positive fuzzy metric (higher values are better)
- $d^-(P_1, P_2, P_3)$ represents the negative fuzzy metric (lower values are better)

Step 4.1.2: Let the bipolar fuzzy membership function be given by $\mu^+(P_1, P_2, P_3) = e^{-\alpha d^+(P_1, P_2, P_3)}$ and $\mu^-(P_1, P_2, P_3) = 1 - e^{-\beta d^-(P_1, P_2, P_3)}$, where α, β are weight parameters representing the impact of positive and negative influences.

Step 4.1.3: We compute bipolar fuzzy-2 distances by using fuzzy-2 metric computations as follows:

positive fuzzy distance $d^+(P_1, P_1, P_1) = \dfrac{w_1.\text{profit} + w_2.\text{innovation} + w_3.\text{market demand}}{w_1 + w_2 + w_3}$

negative fuzzy distance $d^-(P_1, P_1, P_1) = \dfrac{w_1.\text{risk} + w_2.\text{environment} + w_3.\text{complexity}}{w_1 + w_2 + w_3}$

let us assign weights $w_1 = 0.6,\ w_2 = 0.2,$ and $w_3 = 0.2$ for positive criteria and $w_1 = 0.4,\ w_2 = 0.4,$ and $w_3 = 0.2$ for negative criteria.

Project	d^+	d^-
P_1	0.841	0.562
P_2	0.751	0.549
P_3	0.912	0.476

Table 2: Positive and negative distance for each project

Step 4.1.4: The overall bipolar fuzzy score $S(P_i) = \mu^+(P_i) - \mu^-(P_i)$ for different values of α, β as shown in Table 3

	Project	$\mu^+(P_i) = e^{-\alpha d^+(P_i)}$	$\mu^-(P_i) = 1 - e^{-\beta d^-(P_i)}$	$S(P_i) = \mu^+(P_i) - \mu^-(P_i)$
$\alpha = 1.4$ $\beta = 0.8$	P_1	0.638	0.472	0.166
	P_2	0.645	0.475	0.169

	P_3	0.683	0.495	0.188
$\alpha = 0.8$ $\beta = 1.4$	P_1	0.456	0.366	0.090
	P_2	0.464	0.371	0.093
	P_3	0.514	0.402	0.112
$\alpha = 1$ $\beta = 1$	P_1	0.570	0.435	0.136
	P_2	0.578	0.439	0.139
	P_3	0.621	0.463	0.159

Table 3: Bipolar fuzzy score function

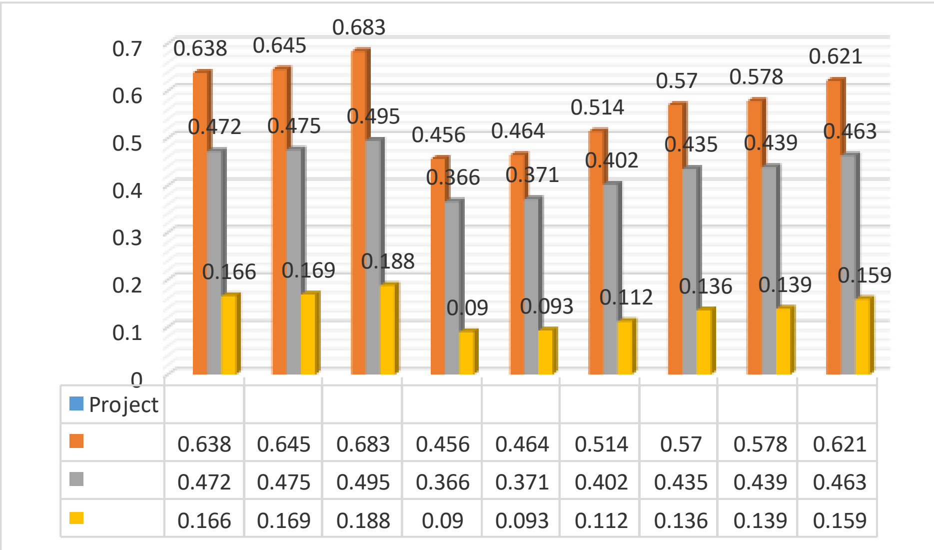

Figure 1: MCDM result

Table 3 indicates that the project P_3 is the optimal choice, the project P_2 is the next best option, and the project P_1 is the least favorable based on the lowest score. Consequently, the project P_3 is the best selection according to the BF2MS assessment methodology. This approach improves decision-making frameworks in practical contexts, including company strategy, risk evaluation, and project selection.

Example 4.2. (BF2MS in Medical Diagnosis-1): A medical diagnosis system evaluates three patients P_1, P_2 and P_3 for two symptoms such as (i) the presence of disease-related symptoms (positive membership, M_{Ω}^{+}) (ii) presence of conflicting or irrelevant symptoms (negative membership, M_{Ω}^{-}).

Calculate the BF2MS (M_Ω^+ and M_Ω^-) for each patient combination at $t=1$ and $t=2$ and determine which group of patients P_1, P_2 and P_3 has the highest positive proximity (M_Ω^+) and lowest negative proximity (M_Ω^-). Also, interpret the results and suggest how this information could be used to group patients for treatment or further analysis.

Solution: For patients P_1, P_2 and P_3 in the set X, $M_\Omega^+(P_1,P_2,P_3,t)=\mu_\Omega(P_1,P_2,P_3,t).e^{-t}$ is the membership function for positive symptoms (relevant to disease) and $M_\Omega^-(P_1,P_2,P_3,t)=\nu_\Omega(P_1,P_2,P_3,t).e^{-t}$, for negative symptoms (conflicting /irrelevant) in the BF2M respectively, where $t>0$ is the time decay parameter. Let the fuzzy memberships (μ_Ω and ν_Ω), based on the observed symptoms are as follows in Table 4:

Patients P_1, P_2, P_3	Positive Membership (μ_Ω)	Negative Membership (ν_Ω)
Combination 1: P_1, P_2, P_3	0.851	0.254
Combination 2: P_1, P_3, P_2	0.723	0.485
Combination 3: $P_2, P_3\ P_1$	0.934	0.198

Table 4: Fuzzy memberships

The calculation of BF2M is shown in the table 5:

For real *t*	Combination	M_Ω^+	M_Ω^-
$t=1$	P_1, P_2, P_3	0.313	0.093
	P_1, P_3, P_2	0.266	0.178
	P_2, P_3, P_1	0.344	0.073
$t=2$	P_1, P_2, P_3	0.115	0.034
	P_1, P_3, P_2	0.098	0.066
	$P_2, P_3\ P_1$	0.126	0.027
	P_1, P_2, P_3	0.042	0.013

$t=3$	P_1, P_3, P_2	0.036	0.024
	P_2, P_3, P_1	0.047	0.010

Table 5: Calculation of BF2M

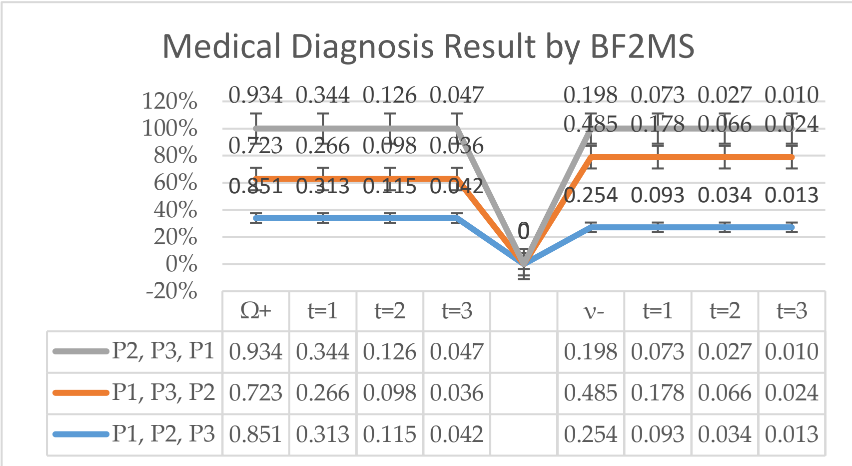

Chart 2: Medical diagnosis result

The research highlights patient clustering and decision-support findings. Patients had the highest positive closeness and the lowest negative proximity, suggesting substantial symptom similarity with few contradicting or irrelevant characteristics. By clustering such patients for comparable treatment regimens, the system optimizes diagnostic closeness for successful therapy the pattern P_2 , P_3 P_1 is the best medical diagnostic. The method also reveals subtle patterns in dual behaviors like overlapping illnesses or contradicting symptoms, improving medical diagnosis.

Example 4.3. (BF2MS in Medical Diagnosis-2): A medical diagnosis system analyzes symptoms and diagnostic criteria for a disease. Some symptoms may positively indicate a condition (e.g., high fever for an infection), while others may negatively correlate or suggest alternative conditions (e.g., absence of fever for non-infectious issues). A BF2MS models these dual relationships to help doctors make more precise diagnoses.

Let $X=(P_1,P_2,P_3)$ be a set of three patients. Each patient exhibits symptoms with varying levels of positive and negative memberships to a disease diagnosis. We define M^+ a degree of positive correlation of symptoms with the disease, M^- a degree of negative correlation of symptoms with the disease and t is the weighting factor (e.g. severity or time duration of symptoms).

The degree of positive correlation in BF2MS is $M^+(x,y,z,t)=e^{-\alpha(t).\left[\left|d(x,y)+d(y,z)\right|\right]}$, where $\alpha(t)>0$, is a decay constant controlling the rate of decrease, and $d(x,y)=(x-y)^2$, $\forall x,y\in X$ measures the similarity/difference between patients based on symptom scores. Similarly, the degree of negative correlation $M^{-}(x,y,z,t)=e^{-\beta(t).[|d(x,y)-d(y,z)|]}$, where $\beta(t)>0$, is another decay constant controls the effect of opposing relationships. The scores for three key symptoms in relation to a disease as follows:

Symptom	Patient P_1	Patient P_2	Patient P_3
Fever	0.92	0.87	0.79
Cough	0.78	0.69	0.56
Fatigue	0.81	0.75	0.69

Symptom Difference (Euclidean Distance):

$$d(P_1,P_2)=(0.92-0.87)^2+(0.78-0.69)^2+(0.81-0.75)^2=0.0142$$

$$d(P_2,P_3)=(0.87-0.79)^2+(0.69-0.56)^2+(0.75-0.69)^2=0.0269$$

For $t=1,\alpha=0.5\Rightarrow\alpha(t)=0.5$, in a degree of positive correlation $M^{+}(x,y,z,t)=e^{-\alpha(t).[|d(x,y)+d(y,z)|]}$, i.e. $M^{+}(P_1,P_2,P_3,t)=e^{-0.5.(0.0411)}=0.97966$ and corresponding $M^{-}(P_1,P_2,P_3,t)=1-0.97966=0.02034.$ For $t=1,\beta=0.3\Rightarrow\beta(t)=0.3$, in a degree of negative correlation $M^{-}(x,y,z,t)=e^{-\beta(t).[|d(x,y)-d(y,z)|]}$, $M^{-}(P_1,P_2,P_3,t)=e^{-0.3.(0.0127)}=0.996197$ and corresponding $M^{+}(P_1,P_2,P_3,t)=1-0.996197=0.003803.$

Thus, as a result, a positive membership (M^{+}) value of 0.97966 indicates moderate compatibility among the three patients' symptoms, suggesting a shared likelihood of the disease. In contrast, a value of 0.02034 indicates relatively low opposition, meaning there is little evidence to suggest alternative diagnoses.

Similarly, a negative membership (M^{-}), which is a moderate negative membership value of 0.996197, indicates the potential conflict among the symptoms, which might indicate alternative conditions or a need for further analysis. In contrast, a value of 0.003803 indicates relatively low opposition, meaning there is little evidence to suggest alternative diagnoses. This example demonstrates how BF2MS can model dual relationships in medical diagnoses, improving the accuracy and efficiency of decision-making in healthcare systems.

Conclusion: This study introduces the concept of BF2MSs, a dual-dimensional mathematical framework that fills basic deficiencies in fuzzy metric models. BF2MSs broaden FPTs to dual-interaction systems by integrating attraction-repulsion and compatibility-opposition dynamics. These advances improve complex and nonlinear system stability, convergence, and equilibrium analysis, which benefits social networks, biological ecosystems, control systems, and optimization challenges. BF2MSs also help MCDM and medical diagnosis systems assess illness symptoms and diagnostic criteria by addressing competing aims and ambiguity, boosting decision-support models across disciplines. They are important in mathematical modeling and real-world problem-solving because they apply to topology, variational inequality, and computational intelligence. This paper lays the groundwork for future interdisciplinary research in fuzzy mathematics, nonlinear analysis, and decision sciences by connecting theoretical advances to practical applications.

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Ethical Approval: This article does not involve any studies with human participants or animals performed by any of the authors. Therefore, ethical approval is not applicable.

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Nonagonal Neutrosophic Number: Analytical Aspects and its Role in Optimization technique for transportation problem

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Abstract: Neutrosophic numbers have received increasing attention from researchers and industrialists to address the indeterminacy and uncertainty inherent in real-life decision-making. This study aims to solve the transportation problem where supply, demand and transportation costs are Nonagonal Neutrosophic Numbers (NNNs). In the existing literature, various methods have been introduced to solve transportation problems (TPs) involving neutrosophic parameters. The application of Nonagonal Neutrosophic Numbers to transportation problems is a relatively recent development. NNNs offer a more detailed and adaptable representation of uncertainty by utilizing a nine-parameter structure that captures the degrees of truth, indeterminacy, and falsity. Therefore, in this paper, we solve the transportation problem using Nonagonal Neutrosophic Numbers for the first time. To facilitate this, we introduce two novel score functions for converting NNNs into crisp values. Based on these, we propose an algorithmic framework to obtain the optimal solution effectively. To exemplify the effectiveness of the proposed method, we solved a numerical example, and the obtained results are presented and compared with those in the existing literature. Finally, the significance of this study and potential directions for future research are mentioned.

Keywords: Transportation problem, Nonagonal neutrosophic numbers, Defuzzification, Crisp data, Optimization

1. Introduction

The transportation problem (TP) is one kind of Linear programming problem that aims to minimize the total cost involved in transporting goods from multiple sources to multiple destinations. George Dantzig is attributed with the initial formulation of the concept of linear programming in 1947.Hitchcock formulated classical transportation problem for the first time. Although the classical transportation problem cannot handle the uncertainties in real-world scenarios, it only narrates the rigorous data of supply and demand.

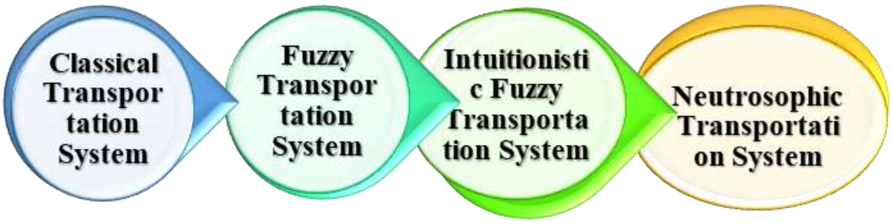
To address this gap, researchers have introduced classical transportation models into fuzzy systems. Fuzzy set theory, formulated by Lotfi A. Zadeh [3] in 1965, enables the modeling of vagueness and imprecision by incorporating degrees of membership. Building upon this concept, Atanassov proposed Intuitionistic Fuzzy Sets (IFS) in 1986, which consider both membership and non-membership functions. However, IFS falls short of explicitly representing indeterminacy which is a crucial factor in many real-world applications.

In response to the inability of existing models to address this shortcoming, the concept of neutrosophy was introduced by Florentin Smarandache in 1999. Neutrosophic sets extend classical and fuzzy theories by incorporating truth, indeterminacy, and falsity components, thereby offering a more comprehensive framework for modeling incomplete, inconsistent, and indeterminate information. Since then, neutrosophic theory has been successfully applied in various optimization domains, including transportation problems. Notably, Thamaraiselvi and Santhi [11] proposed optimization techniques for transportation problems under a neutrosophic environment. Singh Kumar, and Appadoo [6] further contributed to the field by presenting a modified approach for optimizing real-life transportation problems using neutrosophic sets in 2017.

Subsequent studies have explored various forms of neutrosophic numbers, such as triangular, trapezoidal, pentagonal, hexagonal, heptagonal, and octagonal, to enhance uncertainty modeling. Srinivas. S & Prabakaran. K [8] developed a method for solving transportation problems using triangular fuzzy neutrosophic numbers in 2023. Broumi et al. [12] introduced an algorithm for trapezoidal interval-valued neutrosophic network analysis, demonstrating the applicability of neutrosophic logic in complex decision-making systems. In 2024, Kalaivani Kaspar and Palanivel Kaliyaperumal [17] employed single-valued trapezoidal neutrosophic numbers to solve transportation problems with mixed constraints, highlighting the versatility of neutrosophic numbers in handling various types of uncertainty.

Moreover, Chakraborty, Broumi, and Singh [18] explored the use of pentagonal neutrosophic numbers, contributing to the expanding literature on neutrosophic-based transportation models. Nagalakshmi, T., Sudharani, R., & Ambika, G. [23] presented a comparative analysis of fuzzy transportation problems using hexagonal fuzzy numbers and neutrosophic triangular fuzzy numbers in 2022. Recent developments have also considered heptagonal and octagonal neutrosophic numbers, along with the formulation of appropriate score functions, further expanding the applicability of neutrosophic models in uncertain environments, have been discussed in [24]-[27].

In this context, the present study focuses on advancing transportation problem modeling using Nonagonal Neutrosophic Numbers (NNNs). These provide a finer and more detailed representation of uncertainty through nine parameters that reflect varying levels of truth, indeterminacy, and falsity. This novel approach aims to provide enhanced accuracy in decision-making for complex and uncertain transportation systems.



Motivation of the study

Real-world transportation systems are often fraught with uncertainty, imprecision and indeterminacy caused by fluctuating costs, unpredictable weather conditions, and incomplete information. To tackle these challenges, neutrosophic numbers have been increasingly utilized. Recent developments, such as Triangular, Trapezoidal, Hexagonal, and Octagonal neutrosophic numbers have offered improved modeling adaptability, but may still be limited in representing highly complex and layered uncertainties. This motivates the use of Nonagonal Neutrosophic Numbers, a more advanced representation that employs nine parameters to capture deeper nuances in uncertainty. NNNs allow for finer granularity in modeling expert opinions and fluctuating data, offering enhanced decision-making capabilities in environments characterized by high ambiguity and incomplete knowledge.

Research gaps

Based on the literature review, it is evident that numerous researchers have developed various methods to solve TPs in certain and uncertain environments. However, several research gaps still remain in the existing approaches, as listed below:

- The majority of existing studies focuses on triangular, trapezoidal, or hexagonal neutrosophic numbers, which, while effective, offer limited capacity to model highly complex or multi-layered uncertainty. The potential of NNNs-with their nine-parameter structure, for capturing more intricate and uncertainties remains underexplored.

- While the conceptual basis of NNNs is emerging, there is a lack of structured frameworks or algorithms specifically designed to apply NNNs in transportation problem contexts. This limits their practical utility and acceptance.
- There is limited literature comparing the performance of NNN-based solutions with other fuzzy and neutrosophic approaches in terms of accuracy, robustness, and computational efficiency. Such comparisons are essential to validate the effectiveness of NNNs in real-world scenario.
- Few studies have implemented NNN-based transportation models in practical domains such as disaster management, logistics planning and supply chain distribution. This highlights the gap between theoretical development and practical implementation.

Research highlights

This paper presents a comprehensive overview of transportation problem in a neutrosophic manner using NNNs.It explores the characteristics of NNNs, identifies gaps in existing approaches to various types of transportation problems, and introduces double defuzzification for NNNs. A step-by-step algorithm is introduced to solve the transportation problem in a neutrosophic nonagonal context. The supply, demand, and transportation cost entries in the transportation table are expressed using NNNs, allowing for a more refined modeling of uncertainty. The outcomes were compared with those obtained from existing methods to demonstrate their effectiveness and applicability.

This paper is structured as follows: Section 1 presents the abstract and introduction. Section 2 focuses on fundamental definitions. Sections 3 and 4 introduce NNNs and discuss their mathematical properties, respectively. The score function for defuzzification of NNNs is expressed in section 5.In section 6-9, include the proposed algorithm, illustrative diagrams, mathematical formulation, and a numerical example to demonstrate the methodology. Section 10-12 discuss the findings in the form of major conclusions, limitations and future scope.

2.Preliminaries

Definition 1 A fuzzy set $\mathfrak{F}$ in a non-empty set X is defined as $\mathfrak{F} = \{(x, \Theta_{\mathfrak{F}}(x)/x \in X)\}$

where $\Theta_{\mathfrak{F}}$, the degree of membership of x in fuzzy set $\mathfrak{F}$ and $\Theta_{\mathfrak{F}} : X \rightarrow [0,1]$.

Definition 2 A set $\mathfrak{I}$ in a non-empty set X is said to be an intuitionistic fuzzy set as

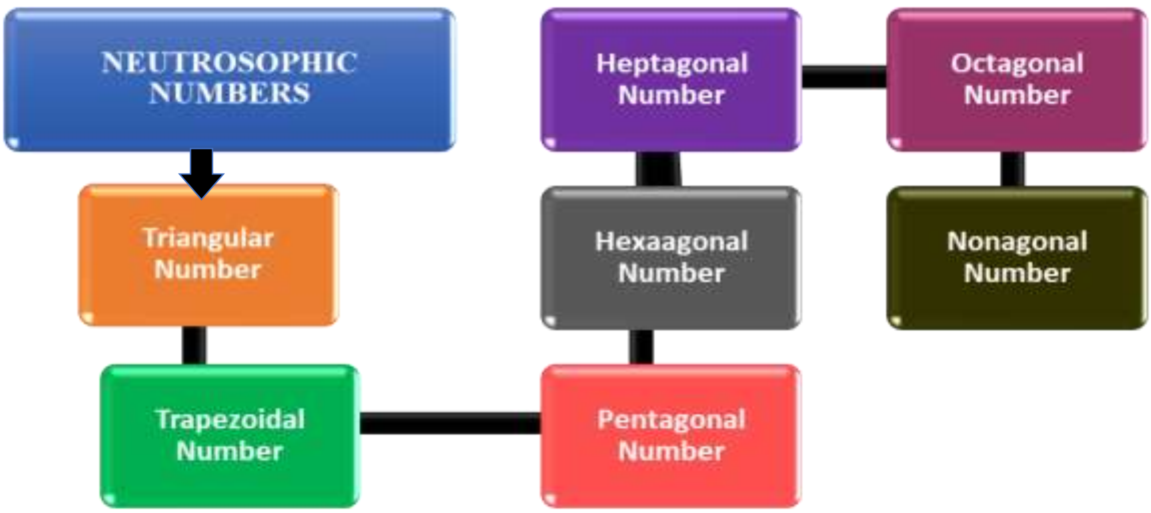
$\Im = \{(x, \Theta_{\Im}(x), \xi_{\Im}(x))/x \in X\}$ where $\Theta_{\Im}, \xi_{\Im} : X \to [0,1]$ and $\Theta_{\Im}(x)$, $\delta_M(x)$ indicates the degree of membership and non-membership of x to the set M such that $0 \le \Theta_{\Im}(x) + \xi_{\Im}(x) \le 1$.

Definition 3 Let X be a non-empty set. A neutrosophic set $\mathcal{N}$ in X is defined as $\mathcal{N} = \{(x, \Theta_{\mathcal{N}}(x), \xi_{\mathcal{N}}(x), \varpi_{\mathcal{N}}(x))/x \in X\}$ where $\Theta_{\mathcal{N}}$ is a truth membership function, $\xi_{\mathcal{N}}$ is an indeterminacy -membership function and $\varpi_{\mathcal{N}}$ is a falsity-membership function in which $\Theta_{\mathcal{N}}, \xi_{\mathcal{N}}, \varpi_{\mathcal{N}} : X \to -]0, 1[^+$ and $-0 \le \Theta_{\mathcal{N}} + \xi_{\mathcal{N}} + \le 3^+$.

Definition 4 A Neutrosophic set $\mathcal{N}$ in a set of real numbers R, called the neutrosophic number then it meets:

1) If there exist $x_0 \in R$, such that $\Theta_{\mathcal{N}}(x_0) = 1$, $\xi_{\mathcal{N}}(x_0) = 0$ *and* $\varpi_{\mathcal{N}}(x_0) = 0$
2) $\Theta_{\mathcal{N}}(\mu x_1 + (1-\mu)x_2) \ge \min(\Theta_{\mathcal{N}}(x_1), \Theta_{\mathcal{N}}(x_2))$, $\forall\ x_1, x_2 \in R$ & $\mu \in [0,1]$.
3) $\xi_{\mathcal{N}}(\mu x_1 + (1-\mu)x_2) \ge \max(\xi_{\mathcal{N}}(x_1), \xi_{\mathcal{N}}(x_2))$, $\forall\ x_1, x_2 \in R$ & $\mu \in [0,1]$.
4) $\varpi_{\mathcal{N}}(\mu x_1 + (1-\mu)x_2) \ge \max(\varpi_{\mathcal{N}}(x_1), \varpi_{\mathcal{N}}(x_2))$, $\forall\ x_1, x_2 \in R$ & $\mu \in [0,1]$.

Definition 5: Classification of Neutrosophic numbers



3. Nonagonal Neutrosophic Number

A single valued Nonagonal neutrosophic number is represented as

$$\overline{\aleph\aleph\aleph} = \begin{pmatrix} [\,(\ddot{p}^{11}, \ddot{q}^{11}, \ddot{r}^{11}, \ddot{s}^{11}, \ddot{t}^{11}, \ddot{u}^{11}, \ddot{v}^{11}, \ddot{w}^{11}, \ddot{z}^{11}) : \Theta_{\mathcal{N}}\,], \\ [\,(\ddot{p}^{12}, \ddot{q}^{12}, \ddot{r}^{12}, \ddot{s}^{12}, \ddot{t}^{12}, \ddot{u}^{12}, \ddot{v}^{12}, \ddot{w}^{12}, \ddot{z}^{12}) : \xi_{\mathcal{N}}\,], \\ [\,(\ddot{p}^{13}, \ddot{q}^{13}, \ddot{r}^{13}, \ddot{s}^{13}, \ddot{t}^{13}, \ddot{u}^{13}, \ddot{v}^{13}, \ddot{w}^{13}, \ddot{z}^{13}) : \varpi_{\mathcal{N}}\,] \end{pmatrix}$$

where $\Theta_{\mathcal{N}}$ is a truth membership function, $\xi_{\mathcal{N}}$ is an indeterminacy -membership function and $\varpi_{\mathcal{N}}$ is a falsity-membership function is defined as

$$\Theta_{\mathcal{N}}(x) = \begin{cases} \Theta_{\mathcal{N}1}(x) & \ddot{p}^{11} \le x \le \ddot{q}^{11} \\ \Theta_{\mathcal{N}2}(x) & \ddot{q}^{11} \le x \le \ddot{r}^{11} \\ \Theta_{\mathcal{N}3}(x) & \ddot{r}^{11} \le x \le \ddot{s}^{11} \\ \Theta_{\mathcal{N}4}(x) & \ddot{s}^{11} \le x \le \ddot{t}^{11} \\ \Theta & x = \ddot{t}^{11} \\ \Theta_{\mathcal{N}4}(x) & \ddot{t}^{11} \le x \le \ddot{u}^{11} \\ \Theta_{\mathcal{N}3}(x) & \ddot{u}^{11} \le x \le \ddot{v}^{11} \\ \Theta_{\mathcal{N}2}(x) & \ddot{v}^{11} \le x \le \ddot{w}^{11} \\ \Theta_{\mathcal{N}1}(x) & \ddot{w}^{11} \le x \le \ddot{z}^{11} \\ 0 & otherwise \end{cases}, \quad \xi_{\mathcal{N}}(x) = \begin{cases} \xi_{\mathcal{N}1}(x) & \ddot{p}^{12} \le x \le \ddot{q}^{21} \\ \xi_{\mathcal{N}2}(x) & \ddot{q}^{12} \le x \le \ddot{r}^{12} \\ \xi_{\mathcal{N}3}(x) & \ddot{r}^{12} \le x \le \ddot{s}^{12} \\ \xi_{\mathcal{N}4}(x) & \ddot{s}^{12} \le x \le \ddot{t}^{12} \\ \xi & x = \ddot{t}^{12} \\ \xi_{\mathcal{N}4}(x) & \ddot{t}^{12} \le x \le \ddot{u}^{12} \\ \xi_{\mathcal{N}3}(x) & \ddot{u}^{12} \le x \le \ddot{v}^{12} \\ \xi_{\mathcal{N}2}(x)) & \ddot{v}^{12} \le x \le \ddot{w}^{12} \\ \xi_{\mathcal{N}1}(x) & \ddot{w}^{12} \le x \le \ddot{z}^{12} \\ 1 & otherwise \end{cases}$$

$$\varpi_{\mathcal{N}}(x) = \begin{cases} \varpi_{\mathcal{N}1}(x) & \ddot{p}^{13} \le x \le \ddot{q}^{13} \\ \varpi_{\mathcal{N}2}(x)) & \ddot{q}^{13} \le x \le \ddot{r}^{13} \\ \varpi_{\mathcal{N}3}(x)) & \ddot{r}^{13} \le x \le \ddot{s}^{13} \\ \varpi_{\mathcal{N}4}(x)) & \ddot{s}^{13} \le x \le \ddot{t}^{13} \\ \Theta & x = \ddot{t}^{13} \\ \varpi_{\mathcal{N}4}(x) & \ddot{t}^{13} \le x \le \ddot{u}^{13} \\ \varpi_{\mathcal{N}3}(x) & \ddot{u}^{13} \le x \le \ddot{v}^{13} \\ \varpi_{\mathcal{N}2}(x) & \ddot{v}^{13} \le x \le \ddot{w}^{13} \\ \varpi_{\mathcal{N}1}(x) & \ddot{w}^{13} \le x \le \ddot{z}^{13} \\ 1 & otherwise \end{cases}$$

4. Some operational laws of NNNs

If $\overline{\aleph\aleph\aleph_1}$ *and* $\overline{\aleph\aleph\aleph_2}$ are two nonagonal neutrosophic numbers having truth membership $\Theta_{\mathcal{N}}{}^1(x)$ *and* $\Theta_{\mathcal{N}}{}^2(x)$, indeterminancy membership $\xi_{\mathcal{N}}{}^1(x)$, $\xi_{\mathcal{N}}{}^2(x)$ and falsity membership $\varpi_{\mathcal{N}}{}^1(x), \varpi_{\mathcal{N}}{}^2(x)$ respectively.

$\overline{\aleph\aleph\aleph}_1$

$$= \begin{pmatrix} [(\ddot{p}^{11}, \ddot{q}^{11}, \ddot{r}^{11}, \ddot{s}^{11}, \ddot{t}^{11}, \ddot{u}^{11}, \ddot{v}^{11}, \ddot{w}^{11}, \ddot{z}^{11}): \Theta_{\mathcal{N}}{}^{1}], \\ [(\ddot{p}^{12}, \ddot{q}^{12}, \ddot{r}^{12}, \ddot{s}^{1}, \ddot{t}^{12}, \ddot{u}^{12}, \ddot{v}^{12}, \ddot{w}^{12}, \ddot{z}^{12}): \xi_{\mathcal{N}}{}^{1}], \\ [(\ddot{p}^{13}, \ddot{q}^{13}, \ddot{r}^{13}, \ddot{s}^{13}, \ddot{t}^{13}, \ddot{u}^{13}, \ddot{v}^{13}, \ddot{w}^{13}, \ddot{z}^{13}): \varpi_{\mathcal{N}}{}^{1}] \end{pmatrix}$$

$$\overline{\aleph\aleph\aleph}_2 = \begin{pmatrix} [(\ddot{p}^{21}, \ddot{q}^{21}, \ddot{r}^{21}, \ddot{s}^{21}, \ddot{t}^{21}, \ddot{u}^{21}, \ddot{v}^{21}, \ddot{w}^{21}, \ddot{z}^{21}): \Theta_{\mathcal{N}}{}^{2}], \\ [(\ddot{p}^{22}, \ddot{q}^{22}, \ddot{r}^{22}, \ddot{s}^{22}, \ddot{t}^{22}, \ddot{u}^{22}, \ddot{v}^{22}, \ddot{w}^{22}, \ddot{z}^{22}): \xi_{\mathcal{N}}{}^{2}], \\ [(\ddot{p}^{23}, \ddot{q}^{23}, \ddot{r}^{23}, \ddot{s}^{23}, \ddot{t}^{23}, \ddot{u}^{23}, \ddot{v}^{23}, \ddot{w}^{23}, \ddot{z}^{23}): \varpi_{\mathcal{N}}{}^{2}] \end{pmatrix}$$

- Addition

$$\overline{\aleph\aleph\aleph}_1 \oplus \overline{\aleph\aleph\aleph}_2 = (\{[\ddot{p}^{11} + \ddot{p}^{21} - \ddot{p}^{11}\ddot{p}^{21}, \ \ddot{q}^{11} + \ddot{q}^{21} - \ddot{q}^{11}\ddot{q}^{21}, \ddot{r}^{11} + \ddot{r}^{21} - \ddot{r}^{11}\ddot{r}^{21}, \ddot{s}^{11} + \ddot{s}^{21} - \ddot{s}^{11}\ddot{s}^{21}, \ddot{t}^{11} + \ddot{t}^{21} - \ddot{t}^{11}\ddot{t}^{21}, \ddot{u}^{1\ddot{1}} + u^{21} - \ddot{u}^{11}\ddot{u}^{21}, \ddot{v}^{11} + \ddot{v}^{21} - \ddot{v}^{11}\ddot{v}^{21}, \ddot{w}^{1\ddot{1}} + w^{21} - \ddot{w}^{\ddot{1}1}w^{21}, \ddot{z}^{11} + \ddot{z}^{21} - \ddot{z}^{11}\ddot{z}^{21}],) < \{[\ddot{p}^{11} + \ddot{p}^{21} - \ddot{p}^{11}\ddot{p}^{21}, \ \ddot{q}^{11} + \ddot{q}^{21} - \ddot{q}^{11}\ddot{q}^{21}, \ddot{r}^{11} + \ddot{r}^{21} - \ddot{r}^{11}\ddot{r}^{21}, \ddot{s}^{11} + \ddot{s}^{21} - \ddot{s}^{11}\ddot{s}^{21}, \ddot{t}^{11} + \ddot{t}^{21} - \ddot{t}^{11}\ddot{t}^{21}, \ddot{u}^{1\ddot{1}} + u^{21} - \ddot{u}^{11}\ddot{u}^{21}, \ddot{v}^{11} + \ddot{v}^{21} - \ddot{v}^{11}\ddot{v}^{21}, \ddot{w}^{1\ddot{1}} + w^{21} - \ddot{w}^{\ddot{1}1}w^{21}, \ddot{z}^{11} + \ddot{z}^{21} - \ddot{z}^{11}\ddot{z}^{21}], \ [\ddot{p}^{12}\ddot{p}^{22}, \ddot{q}^{12}\ddot{q}^{22}, \ddot{r}^{12}\ddot{r}^{22}, \ddot{s}^{11}\ddot{s}^{22}, \ddot{t}^{12}\ddot{t}^{22}, \ddot{u}^{12}\ddot{u}^{22}, \ddot{v}^{12}\ddot{v}^{22}, \ddot{w}^{12}\ddot{w}^{22}, \ddot{z}^{12}\ddot{z}^{22}],$$

$$[\ddot{p}^{13}\ddot{p}^{23}, \ddot{q}^{13}\ddot{q}^{23}, \ddot{r}^{13}\ddot{r}^{23}, \ddot{s}^{13}\ddot{s}^{23}, \ddot{t}^{13}\ddot{t}^{23}, \ddot{u}^{13}\ddot{u}^{23}, \ddot{v}^{13}\ddot{v}^{23}, \ddot{w}^{13}\ddot{w}^{23}, \ddot{z}^{13}\ddot{z}^{23}]\} >$$

- Multiplication

$$\overline{\aleph\aleph\aleph}_1 \otimes \overline{\aleph\aleph\aleph}_2] = \{[\ddot{p}^{11}\ddot{p}^{21}, \ddot{q}^{11}\ddot{q}^{21}, \ddot{r}^{11}\ddot{r}^{21}, \ddot{s}^{11}\ddot{s}^{21}, \ddot{t}^{11}\ddot{t}^{21}, \ddot{u}^{\ddot{1}1}u^{21}, \ddot{v}^{11}\ddot{v}^{21}, \ddot{w}^{\ddot{1}1}w^{21}, \ddot{z}^{11}\ddot{z}^{21}], [(\ddot{p}^{12} + \ddot{p}^{22}, \ddot{q}^{12} + \ddot{q}^{22}, \ddot{r}^{12} + \ddot{r}^{22}, \ddot{s}^{11} +, \ddot{s}^{22}, \ddot{t}^{12} + \ddot{t}^{22}, \ddot{u}^{12} + \ddot{u}^{22}, \ddot{v}^{12} + \ddot{v}^{22}, \ddot{w}^{12} + \ddot{w}^{22}, \ddot{z}^{12} + \ddot{z}^{22}], [\ddot{p}^{13} + \ddot{p}^{23}, \ddot{q}^{13} + \ddot{q}^{23}, \ddot{r}^{13} + \ddot{r}^{23}, \ddot{s}^{13} + \ddot{s}^{23}, \ddot{t}^{13} + \ddot{t}^{23}, \ddot{u}^{13} + \ddot{u}^{23}, \ddot{v}^{13} + \ddot{v}^{23}, \ddot{w}^{13} + \ddot{w}^{23}, \ddot{z}^{13} + \ddot{z}^{23}]\}$$

5. Accuracy Function

The defuzzification process converts nonagonal neutrosophic numbers (NNNs) into crisp values using the accuracy function. The procedure is outlined as follows:

- Defuzzification of the trueness component:

The defuzzified value for the truth membership of an NNN is given by:

$$\mathfrak{D}\mathfrak{N}(\Theta_{\mathcal{N}})^{T}_{nn} = \frac{\ddot{p}^{11} + \ddot{q}^{11} + \ddot{r}^{11} + \ddot{s}^{11} + \ddot{t}^{11} + \ddot{u}^{11} + \ddot{v}^{11} + \ddot{w}^{11} + \ddot{z}^{11}}{9}$$

- Defuzzification of the indeterminacy component:

The defuzzified value for the indeterminacy membership of an NNN is computed as:

$$\mathfrak{DN}(\xi_{\mathcal{N}})^{I}_{nn} = \frac{\ddot{p}^{12} + \ddot{q}^{12} + \ddot{r}^{12} + \ddot{s}^{12} + \ddot{t}^{12} + \ddot{u}^{12} + \ddot{v}^{12} + \ddot{w}^{12} + \ddot{z}^{12}}{9}$$

- Defuzzification of falsity component

The defuzzified value for the falsity membership is given by:

$$\mathfrak{DN}(\varpi_{\mathcal{N}})^{F}_{nn} = \frac{\ddot{p}^{13} + \ddot{q}^{13} + \ddot{r}^{13} + \ddot{s}^{13} + \ddot{t}^{13} + \ddot{u}^{13} + \ddot{v}^{13} + \ddot{w}^{13} + \ddot{z}^{13}}{9}$$

Combined Representation of the Defuzzified Neutrosophic Number:

The defuzzified neutrosophic number in terms of trueness, indeterminacy, and falsity components is represented as:

$$\mathfrak{DN}(\Theta_{\mathcal{N}}, \xi_{\mathcal{N}}, \varpi_{\mathcal{N}})^{(T,I,F)}_{nn} = \left(\frac{\ddot{p}^{11} + \ddot{q}^{11} + \ddot{r}^{11} + \ddot{s}^{11} + \ddot{t}^{11} + \ddot{u}^{11} + \ddot{v}^{11} + \ddot{w}^{11} + \ddot{z}^{11}}{9}, \frac{\ddot{p}^{12} + \ddot{q}^{12} + \ddot{r}^{12} + \ddot{s}^{12} + \ddot{t}^{12} + \ddot{u}^{12} + \ddot{v}^{12} + \ddot{w}^{12} + \ddot{z}^{12}}{9}, \frac{\ddot{p}^{13} + \ddot{q}^{13} + \ddot{r}^{13} + \ddot{s}^{13} + \ddot{t}^{13} + \ddot{u}^{13} + \ddot{v}^{13} + \ddot{w}^{13} + \ddot{z}^{13}}{9} \right)$$

Graded Mean Defuzzification to a Crisp Number:

Finally, the graded mean formula is used to convert the defuzzified neutrosophic values into a single crisp value:

$$(\mathfrak{GN})^{\aleph}_{nn} = \frac{\mathfrak{DN}(\Theta_{\mathcal{N}})^{T}_{nn} + 4\mathfrak{DN}(\xi_{\mathcal{N}})^{I}_{nn} + \mathfrak{DN}(\varpi_{\mathcal{N}})^{F}_{nn}}{4}$$

6. Nonagonal Neutrosophic Transportation Problem (NNTP)

In many practical scenarios, transportation-related data such as costs, supply, and demand are often uncertain, imprecise, and influenced by indeterminate factors. Classical models fail to address this level of complexity effectively. To overcome these limitations, the present work employs NNNs, which extend the neutrosophic set theory by incorporating nine parameters to better characterize uncertainty through truth, indeterminacy, and falsity components. By embedding NNNs into the classical transportation model, we develop a more flexible and expressive approach.

Here we consider two types in which the decision-maker is uncertain about the exact values of key parameters—specifically, the transportation cost from the i[th] source to the j[th] destination, as well as the certainty or uncertainty associated with the supply and demand of goods. To handle such indeterminacy and vagueness, we introduce a new class of transportation problem, referred to as the NNTP, where the cost, supply, and demand parameters are all represented using NNNs.

This paper focuses on solving an NNTP characterized by m supply centers and n demand centers. Let $c_{\tilde{ij}} = c_{\tilde{ij}}^{\widetilde{NNN}}$ denote the nonagonal neutosophic number corresponding to the transportation cost of sending one unit of goods from i[th] source to j[th] destination. Let $a_i = a_i^{\widetilde{NNN}}$ and $b_j = b_j^{\widetilde{NNN}}$ represent supplies and demands, respectively. The mathematical formulation of the NNTP is then presented as follows.

Minimize $Z^{**} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{\tilde{ij}}^{\widetilde{NNN}} \, x_{ij}^{**}$

Subject to

$$\sum_{i=1}^{m} x_{ij}^{**} = a_i^{\widetilde{NNN}}, i = 1,2,\cdots,m$$

$$\sum_{j=1}^{n} x_{ij}^{**} = b_j^{\widetilde{NNN}}, j = 1,2,\cdots,n$$

$$x_{ij}^{**} \geq 0$$

x_{ij}^{**} denotes the number of units transported from the source i to the destination j.

Table 1: Uncertain transportation table

	D_1	D_2	D_3	**Supply**
S_1	$c_{\tilde{11}}^{\widetilde{NNN}}$	$c_{\tilde{12}}^{\widetilde{NNN}}$	$c_{\tilde{13}}^{\widetilde{NNN}}$	$a_1^{\widetilde{NNN}}$
S_2	$c_{\tilde{21}}^{\widetilde{NNN}}$	$c_{\tilde{22}}^{\widetilde{NNN}}$	$c_{\tilde{23}}^{\widetilde{NNN}}$	$a_2^{\widetilde{NNN}}$
S_3	$c_{\tilde{31}}^{\widetilde{NNN}}$	$c_{\tilde{32}}^{\widetilde{NNN}}$	$c_{\tilde{33}}^{\widetilde{NNN}}$	$a_3^{\widetilde{NNN}}$
Demand	$b_1^{\widetilde{NNN}}$	$b_2^{\widetilde{NNN}}$	$b_3^{\widetilde{NNN}}$	

6.1 Nonagonal Neutrosophic Transportation Problem of type 1

A transportation problem where costs between the origins and destinations are represented using NNNs, while the supply at each source and demand at each destination are considered as deterministic values is called NNTP of type 1. This model helps capture uncertainty in transportation expenses due to fluctuating market conditions, fuel prices, or other unpredictable factors. Mathematically, an NNTP of Type 1 is formulated as follows:

Minimize $Z^{**} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{\breve{ij}}{}^{\breve{N}\breve{N}\breve{N}}\, x_{ij}{}^{**}$

Subject to

$$\sum_{i=1}^{m} x_{ij}{}^{**} = a_i, i = 1,2,\cdots,m$$

$$\sum_{j=1}^{n} x_{ij}{}^{**} = b_j, j = 1,2,\cdots,n$$

$$x_{ij}{}^{**} \geq 0$$

6.2 Nonagonal Neutrosophic Transportation Problem of type 2

A transportation problem is that in which the supply at origins and demand at destinations are expressed as NNNs, while the transportation costs remain crisp. This model is suitable for scenarios where the quantity of goods available or required is uncertain due to variability in production, consumption, or forecasting errors.

Minimize $Z^{**} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{\breve{ij}}\;\, x_{ij}{}^{**}$

Subject to

$$\sum_{i=1}^{m} x_{ij}{}^{**} = a_i{}^{\breve{N}\breve{N}\breve{N}}, i = 1,2,\cdots,m$$

$$\sum_{j=1}^{n} x_{ij}{}^{**} = b_j{}^{\breve{N}\breve{N}\breve{N}}, j = 1,2,\cdots,n$$

$$x_{ij}{}^{**} \geq 0$$

7. Computational method

7.1 The steps of the proposed algorithms are given below:

7.1.1 Defuzzify each nonagonal neutrosophic cost $c_{\breve{ij}}{}^{\breve{N}\breve{N}\breve{N}}$, nonagonal neutrosophic supply $a_i^{\tilde{\aleph}\tilde{\aleph}\tilde{\aleph}}$,, and nonagonal neutrosophic demand $b_j^{\tilde{\aleph}\tilde{\aleph}\tilde{\aleph}}$ of NTP in cost matrix to their corresponding crisp data using the accuracy functions to enable further computational analysis.

7.1.2 Verify the given NNT table is balanced or not.

If $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$, then TP is balanced

If $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$, then TP is unbalanced

7.1.3 If the given TP is balanced then go to step 7.1.4, Otherwise, it should be balanced by adding a dummy row or a dummy column with zero entries in cost matrix as required.

7.1.4 In this step, we search the initial basic feasible solutions of the crisp transportation problem by using by VAM method ensuring efficient cost approximation in the starting solution.

7.1.5 Locate all unallocated (idle) cells in the transportation table whose transportation costs are less than the highest cost among the currently allocated cells

7.1.6 For each identified idle cells, construct a closed loop that begins and ends at the same cell, following only horizontal and vertical movements with all corners are passing through the allocated cells.

7.1.7 Assign alternating '+' and '-' signs to the allocated values at each corner cell of the loop, starting with a '+' sign at the idle cell being evaluated.

7.1.8 Calculate the trade-off cost for each loop by summing the allocated values of its corner cells taking into account their respective signs.

7.1.9 Identify the loop associated with the maximum trade-off cost and perform reallocation as follows:

a) Within this loop, identify the adjacent cell to the idle cell that has the highest transportation cost.

b) Adjust the allocation in the loop by alternately adding and subtracting the allocated value of this highest- cost cell along the loop, starting with the idle cell

7.1.10 Continue the process iteratively until no further improvements can be made, and the transportation cost is minimized. The resulting allocation constitutes the optimal solution to the transportation problem under the nonagonal neutrosophic framework.

7.1.11 The optimal solution of the objective function is calculated by

$$\boldsymbol{Z}_{\overline{(\aleph\aleph\aleph)}} = c_{\breve{i}\breve{j}} \times x_{ij}^{**}.$$

7.2 Nonagonal Neutrosophic Vogel's approximation method

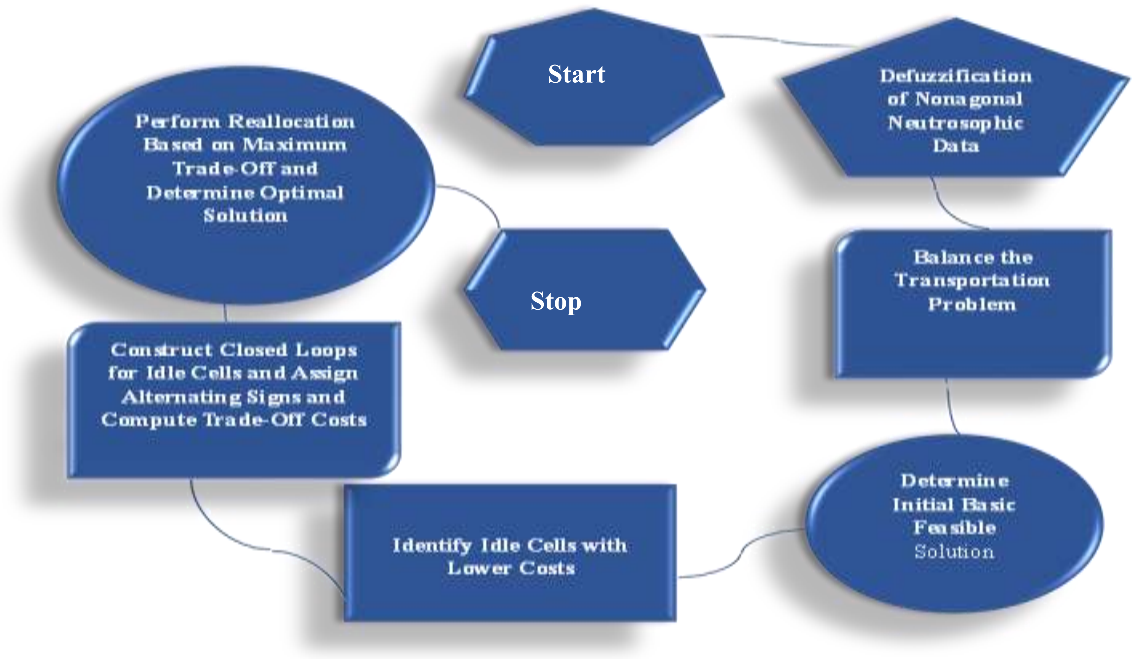
Among the various methods available, Vogel's Approximation Method (VAM) is widely used to determine the initial basic feasible solution of a transportation problem. The computational steps of VAM are as follows:

7.2.1 For each row and column in the NNTP, identify the smallest and the next smallest cost values. Then, compute the penalty p_i and p_j for each row, $i = 1,2,\cdots,m$ and each column $j = 1,2,\cdots,n$ defined as the difference between these two values.

7.2.2 Identify the largest penalty among all rows and columns. If a tie occurs, select any one arbitrarily. Suppose the greatest penalty corresponding to the mth row and $c_{\widetilde{mj}}$ be the smallest cost in that row. Then, allocate $x_{mj}{}^{**} = \min(a_i, b_j)$ in the (m, j)th cell.

7.2.3 Update the corresponding supply and demand, adjust the transportation table accordingly, and recalculate the row and column penalties. Repeat this process iteratively (i.e., return to Step 7.2.1) until all supply and demand constraints are fully satisfied.

8. Diagram Representation of NNTP



9. Case Study

9.1 Type 1

A food bank distributes surplus food from various donation centers to community kitchens, where the donation centers serve as the sources and the local shelters act as the destinations. In this scenario, the transportation cost associated with each cell is represented using nonagonal neutrosophic numbers, capturing the inherent uncertainty and indeterminacy in real-world logistics. Notably, the transportation cost obtained through the proposed approach is found to be lower than that achieved by existing methods.The supply values are 15,25 and 30.The demand values are 24, 19 and 27. The input values for the NNTP are organized in Table 2 below:

Table 2 Nonagonal Neutrosophic Transportation Table

	D_1	D_2	D_3
S_1	((1,6,7,2,3,5,2.5,6,8.3), (1.2,2.4,9,7,4.4,6,7.8,8,9) (3,5,8,1,5,3.6,5,3,2))	((1,4,7,1,3,5,3.5,6,7.5) (0.5,2.5,4.5,1,2,3, 1.5,3.5,5.5), (3,5,8,1,5,3.6,5,3,5.7))	((1,3,5,0.5,1.5,3.5, 2,4,6), (1,2,3, 0.5,1.5,2.5, 1.5,2.5,3.5) (1,1.5,4, 0.5,1,2.5,1.25,3,4.25))
S_2	((1.5,2.5,3.5,1,1.5,3,2,3,4), (2,4,6,1.5,2.5,4.5,3,5,7) (1,5,8,1.5,4.5,7.5,4,6.5,9))	((1,6.5,7.2,2,4,5,3.5,6,9.3), (2.2,2.4,8,6,4.4,9,7.8,8,7), (6,5,9,3,5,3.6,5,8,9))	((1,6,7,2,3,5,2.5,6,8.3) (1,1.5,4, 0.5,1,2.5, 1.25,3,4.25) (1.5,2.5,3.5, 1,1.5,3, 2,3,4))
S_3	((2,4,6, 1.5,2.5,4.5, 3,5,3), (1.2,2.4,9,7,4.4,6,7.8,8,9), (1,5,8, 1.5,4.5,7.5, 4,6.5,9))	((1.1,5,8, 1.5,3,6.5, 4,7,9), (1.5,2,7,7,4.4,9,6.8,8,9), (4,6,9,2,5,4.6,5,6.3,5.7))	((1.6,9,8,2.6,3,5,5.5,6,8.7) (5.2,2.7,9,7,8.4,6,7.6,8,9) (7,5.5,8.7,1.7,5,3.6,5,3,5.7))

9.1.1 Conversion of NNNs to crisp values using score function

Step 7.1.1 The score function can be employed to convert the nonagonal neutrosophic (NN) cost values into crisp values. The resulting crisp cost values corresponding to Table 2 are presented as follows:

$$\mathfrak{DN}\left(\Theta_{\mathcal{N}},\xi_{\mathcal{N}},\varpi_{\mathcal{N}}\right)_{nn}^{(T,I,F)} = \left(\begin{array}{c} \dfrac{\ddot{\mathfrak{p}}^{11}+\ddot{\mathfrak{q}}^{11}+\ddot{\mathfrak{r}}^{11}+\ddot{\mathfrak{s}}^{11}+\ddot{\mathfrak{t}}^{11}+\ddot{\mathfrak{u}}^{11}+\ddot{\mathfrak{v}}^{11}+\ddot{\mathfrak{w}}^{11}+\ddot{\mathfrak{z}}^{11}}{9}, \\ \dfrac{\ddot{\mathfrak{p}}^{12}+\ddot{\mathfrak{q}}^{12}+\ddot{\mathfrak{r}}^{12}+\ddot{\mathfrak{s}}^{12}+\ddot{\mathfrak{t}}^{12}+\ddot{\mathfrak{u}}^{12}+\ddot{\mathfrak{v}}^{12}+\ddot{\mathfrak{w}}^{12}+\ddot{\mathfrak{z}}^{12}}{9}, \\ \dfrac{\ddot{\mathfrak{p}}^{13}+\ddot{\mathfrak{q}}^{13}+\ddot{\mathfrak{r}}^{13}+\ddot{\mathfrak{s}}^{13}+\ddot{\mathfrak{t}}^{13}+\ddot{\mathfrak{u}}^{13}+\ddot{\mathfrak{v}}^{13}+\ddot{\mathfrak{w}}^{13}+\ddot{\mathfrak{z}}^{13}}{9} \end{array}\right)$$

$$(\mathfrak{GR})^{\aleph}_{nn} = \frac{\mathfrak{DR}(\Theta_{\mathcal{N}})^{T}_{nn} + 4\mathfrak{DR}(\xi_{\mathcal{N}})^{I}_{nn} + \mathfrak{DR}(\varpi_{\mathcal{N}})^{F}_{nn}}{4}$$

Here,

$$\mathfrak{DR}\left(c_{\widetilde{11}}{}^{\aleph\widetilde{\aleph}\aleph}\right)^{(T,I,F)}_{nn} = \left\langle \begin{pmatrix} (1,6,7,2,3,5,2.5,6,8.3), \\ (1.2,2.4,9,7,4.4,6,7.8,8,9) \\ (3,5,8,1,5,3.6,5,3,2) \end{pmatrix} \right\rangle = \begin{pmatrix} \frac{(1+6+7+2+3+5+2.5+6+8.3)}{9}, \\ \frac{(1.2+2.4+9+7+4.4+6+7.8+8+9)}{9} \\ \frac{(3+5+8+1+5+3.6+5+3+2)}{9} \end{pmatrix} = \begin{pmatrix} 4.53 \\ 6.09 \\ 3.96 \end{pmatrix}$$

$$\mathfrak{GR}\left(\begin{pmatrix} 4.53 \\ 6.09 \\ 3.96 \end{pmatrix}\right)^{\aleph}_{nn} = \frac{4.53 + 4 \times 6.09 + 3.96}{4} = 8.21$$

Following the conversion of NNTP cost values into crisp form using the score function, the corresponding cost matrix is shown in Table 3.

Table 3 Crisp transportation table

	D1	**D2**	**D3**	**SUPPLY**
S1	8.21	5.09	3.27	15
S2	5.86	8.81	3.78	25
S3	8.26	8.64	9.61	30
DEMAND	24	19	27	

Step 7.1.2 - 7.1.3 The given transportation problem is balanced since the total supply equals the total demand, i.e., $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j = 70$

Step 7.1.4: Applying the VAM Method to find the initial basic feasible solution. To begin, identify the smallest and the next smallest elements in each row and each column of Table 3.

For rows:

$$c_{13} < c_{12} < c_{11} \Rightarrow \textit{least and second least cost are}: c_{13}, c_{12}$$

$$c_{23} < c_{21} < c_{22} \Rightarrow \textit{least and second least cost are}: c_{23}, c_{21}$$

$$c_{31} < c_{32} < c_{33} \Rightarrow \textit{least and second least cost are}: c_{31}, c_{32}$$

For columns:

$$c_{21} < c_{11} < c_{31} \Rightarrow \textit{least and second least cost are}: c_{21}, c_{11}$$

$$c_{12} < c_{32} < c_{22} \Rightarrow \textit{least and second least cost are}: c_{12}, c_{32}$$

$$c_{13} < c_{23} < c_{33} \Rightarrow \textit{least and second least cost are}: c_{13}, c_{23}$$

Next, select the row or column with the highest penalty. Since the second column has the highest penalty, it is chosen for allocation. Proceeding in the same manner, the initial basic feasible solution is obtained, as shown in Table 4.

Table 4: Initial basic feasible solution by VAM method

	D_1	**D_2**	**D_3**	**SUPPLY**
S1	8.21	5.09 **15**	3.27	15
S2	5.86	8.81	3.78 **25**	25
S_3	8.26 **24**	8.64 **4**	9.61 **2**	30
DEMAND	24	19	27	

The selected entries are represented by bold

$$x_{12}^{**} = 15,\ x_{23}^{**} = 25,\ x_{31}^{**} = 24,\ x_{32}^{**} = 4,\ x_{33}^{**} = 2$$

Step 7.1.5 -7.1.6 Identify all unallocated (idle) cells in the transportation table whose transportation costs are lower than the highest cost among the currently allocated cells. In this case, cells S_1D_1, S_1D_3, $S_2D_1$and S_2D_2 have transportation costs lower than that of the allocated cell S_3D_3. For each of the identified idle cells, construct a closed loop starting and ending at the same cell. The loop must consist of horizontal and vertical movements only, and every corner of the loop must pass through an allocated cell. The corresponding loops and allocations are illustrated in Table 5 and Table 6.

Table 5 Loop Allocation in the first stage

	D_1	D_2	D_3	SUPPLY
S1	8.21	5.09 **15**	3.27	15
S2	5.86	8.81	3.78 **25**	25
S_3	8.26 **24**	8.64 4	9.61 **2**	30
DEMAND	24	19	27	

Table 6 Second stage loop allocation

	D_1	D_2	D_3	SUPPLY
S1	8.21	5.09 **15**	3.27	15
S2	5.86	8.81	3.78 **25**	25
S_3	8.26 **24**	8.64 **4**	9.61 **2**	30
DEMAND	24	19	27	

Step 7.1.7 -7.1.8 Assign alternating '+' and '–' signs to the allocated cells at each corner of the constructed loop, beginning with a '+' at the idle cell under evaluation. This sign pattern must alternate as you traverse the loop. Next, compute the trade-off cost for each loop by summing the transportation costs of the allocated corner cells, applying the assigned signs appropriately.

The calculated trade-off costs for the identified idle cells are as follows:

Trade-off cost for S_1D_1 = -24+4-15 = -35

Trade-off cost for S_1D_3 = -13

Trade-off cost for S_2D_1 = -47

Trade-off cost for S_2D_2 = -27

Step 7.1.9 Identify the loop corresponding to the maximum trade-off cost and proceed with the reallocation as follows:

a) Within this loop, determine the allocated cell adjacent to the idle cell that has the highest transportation cost.

b) Adjust the allocations along the loop by alternately adding and subtracting the allocated value of this highest-cost cell, starting from the idle cell.

In this case, the cell S_1D_3 has the maximum trade-off cost. To initiate optimal reallocation, identify the adjacent allocated cell within the loop that has the highest transportation cost, which also corresponds to the cell S_1D_3 (idle cell). The loop is then updated by adding and subtracting the allocated value (= 2) alternately at the corner cells of the loop (Table 7).

Table 7 Loop allocation

	D_1	D_2	D_3	SUPPLY
S1	8.21	5.09 **(15-2)**	~~3.27~~ **(2)**	15
S2	5.86	8.81	3.78 **25**	25
S3	8.26 **24**	8.64 (4+**2)**	~~9.61 **(2-2)**~~	30
DEMAND	24	19	27	

Step 7.1.10 In the current iteration, the idle cell with the highest trade-off cost is S_1D_1. Among its adjacent allocated cells, S_3D_1 possesses the highest cost value. However, upon attempting reallocation using this cell as part of the closed loop, the resulting adjustment yields a negative allocation, which is not permissible. As a result, no further feasible improvements can be made through reallocation. This indicates that the solution has reached its optimal state, and the iteration process can be concluded. The optimal crisp solution of TP is shown in table 8 as follows:

Table 8 Optimal transportation table

	D_1	D_2	D_3	SUPPLY
S1	8.21	5.09 **13**	3.27 **2**	15
S2	5.86	8.81	3.78 **25**	25
S3	8.26 **24**	8.64 **6**	9.61	30
DEMAND	24	19	27	

The corresponding optimal solution of the Nonagonal Neutrosophic Transportation Problem, along with the allocation of NNNs, is presented in Table 9.

Table 9 Optimal solution of the Nonagonal Neutrosophic Transportation Problem

	D1	D2	D3
S1		$\begin{pmatrix} (1,4,7,1,3,5,3.5,6,7.5) \\ (0.5,2.5,4.5,1,2,3,1.5,3.5,5.5), \\ (3,5,8,1,5,3.6,5,3,5.7) \end{pmatrix}$ **13**	$\begin{pmatrix} (1,3,5,0.5,1.5,3.5,2,4,6), \\ (1,2,3,0.5,1.5,2.5,1.5,2.5,3.5) \\ (1,1.5,4,0.5,1,2.5,1.25,3,4.25) \end{pmatrix}$ **2**
S2			$\begin{pmatrix} (1,6,7,2,3,5,2.5,6,8.3) \\ (1,1.5,4,0.5,1,2.5,1.25,3,4.25) \\ (1.5,2.5,3.5,1,1.5,3,2,3,4) \end{pmatrix}$ **25**
S3	$\begin{pmatrix} (2,4,6,1.5,2.5,4.5,3,5,3), \\ (1.2,2.4,9,7,4.4,6,7.8,8,9), \\ (1,5,8,1.5,4.5,7.5,4,6.5,9) \end{pmatrix}$ **24**	$\begin{pmatrix} (1.1,5,8,1.5,3,6.5,4,7,9), \\ (1.5,2,7,7,4.4,9,6.8,8,9), \\ (4,6,9,2,5,4.6,5,6.3,5.7) \end{pmatrix}$ **6**	

Optimal Solution ,

$$Z_{\overline{(\aleph\aleph\aleph)}} =$$

$$\begin{pmatrix} (1,4,7,1,3,5,3.5,6,7.5) \\ (0.5,2.5,4.5,1,2,3,1.5,3.5,5.5), \\ (3,5,8,1,5,3.6,5,3,5.7) \end{pmatrix} \times 13 + \begin{pmatrix} (1,3,5,0.5,1.5,3.5,2,4,6), \\ (1,2,3,0.5,1.5,2.5,1.5,2.5,3.5) \\ (1,1.5,4,0.5,1,2.5,1.25,3,4.25) \end{pmatrix} \times 2$$

$$+ \begin{pmatrix} (1,6,7,2,3,5,2.5,6,8.3) \\ (1,1.5,4,0.5,1,2.5,1.25,3,4.25) \\ (1.5,2.5,3.5,1,1.5,3,2,3,4) \end{pmatrix} \times 25$$

$$+ \begin{pmatrix} (2,4,6,1.5,2.5,4.5,3,5,3), \\ (1.2,2.4,9,7,4.4,6,7.8,8,9), \\ (1,5,8,1.5,4.5,7.5,4,6.5,9) \end{pmatrix} \times 24 + \begin{pmatrix} (1.1,5,8,1.5,3,6.5,4,7,9), \\ (1.5,2,7,7,4.4,9,6.8,8,9), \\ (4,6,9,2,5,4.6,5,6.3,5.7) \end{pmatrix} \times 6$$

$$= (\mathbf{5.09 \times 13}) + (\mathbf{3.27 \times 2}) + (\mathbf{3.78 \times 25}) + (\mathbf{8.26 \times 24}) + (\mathbf{8.64 \times 6})$$

$$= \mathbf{417.29}$$

9.2 Type 2

In this case, the supply and demand values for each cell are represented using Nonagonal Neutrosophic Numbers to effectively model uncertainty. The corresponding input values for the NNTP are:

$\mathcal{S}_1^{\aleph} =$
$(10,15,20,24,16,22,12,15,19), (20,25,30,24,26,32,22,25,29), (15,20,25,19,21,27,17,20,24)$

$\mathcal{S}_2^{\aleph} =$
$(13,18,23,17,19,25,15,18,22), (15,20,25,19,21,27,17,20,24), (17,22,27,21,23,29,19,22,26)$

$\mathcal{S}_3^{\aleph} =$
$(10,15,20,14,16,22,12,15,19)), (17,22,27,21,23,29,19,22,26), (15,20,25,19,21,27,17,20,24)$

$\mathcal{D}_1^{\aleph} =$
$(10,15,20,14,16,22,12,15,19)), (20,25,30,24,26,32,22,25,29), (15,20,25,19,21,27,17,20,24)$

$\mathcal{D}_2^{\aleph} =$
$(15,25,30,34,26,22,32,15,29), (10,15,20,14,36,22,12,35,29), (25,10,15,29,11,27,27,20,34)$

$\mathcal{D}_3^{\aleph} =$
$(20,15,30,24,26,22,22,15,29), (30,25,20,14,16,22,32,25,39), (25,20,25,39,21,27,27,20,24)$

After applying the score function to convert the nonagonal neutrosophic supply and demand values into crisp numbers, the corresponding cost matrix is displayed in Table 10.

Table 10 Crisp transportation table of type 2 problem

	D_1	**D_2**	**D_3**	**SUPPLY**
S1	13	22	34	35.36
S2	20	19	23	31.33
S_3	24	24	12	32.09
DEMAND	35.09	33.27	36.75	

The initial basic feasible solution for the Type-2 NNTP, obtained using the Vogel's Approximation Method, is presented in Table 11.

Table 11 IBFS by VAM

	D_1	D_2	D_3	SUPPLY
S1	13 **35.09**	22 **0.27**	34	35.36
S2	20	19 **26.67**	23 **4.66**	31.33
S_3	24	24	12 **32.09**	32.09
S4	0	0 **6.33**	0	6.33
DEMAND	35.09	33.27	36.75	

The Optimal allocation for the Type-2 Crisp NNTP is shown in Table 12.

Table 12 Optimal solution of the Type-2 Crisp NNTP

	D_1	D_2	D_3	SUPPLY
S1	13 **35.09**	22 **0.27**	34	35.36
S2	20	19 **31.33**	23	31.33
S_3	24	24	12 **32.09**	32.09
S4	0	0 **1.67**	0 **4.66**	6.33
DEMAND	35.09	33.27	36.75	

The Optimal Solution is given by

$$\boldsymbol{Z}_{\overline{(\aleph\aleph\aleph)}} = (13 \times 35.09) + (22 \times 0.27) + (19 \times 31.33) + (12 \times 32.09) + (0 \times 1.67) + (0 \times 4.66) = \mathbf{1442.46}$$

10. Comparative Study

Real-life applications of Nonagonal Neutrosophic Numbers in transportation problems have been addressed using various existing methods. To assess the effectiveness of the proposed solution technique, a comparative study was conducted across the methods: the Neutrosophic North-West Corner Rule (NNWCR), Neutrosophic Vogel's Approximation Method (NVAM), Method discussed in [23] and the Proposed Method. This comparison is performed on two types of NNTPs. The evaluation is based on total transportation cost, where all neutrosophic cost values were first converted into crisp values using score functions, ensuring a consistent and fair comparison across all methods.

In order to solve these problems, we identified limitations in the existing approaches and introduced a simplified ranking technique for Nonagonal Neutrosophic Numbers. Our proposed method demonstrates improved performance by effectively converting neutrosophic transportation problems into their crisp equivalents and achieving lower total transportation costs.

All computations were carried out under the same input data and after defuzzification, allowing a uniform basis for comparison. Table 13 highlights the results of the comparative analysis, demonstrating the superior efficiency of the proposed method in terms of cost minimization.

Table 13 Comparison table

Type	**Nonagonal Neutrosophic North-West Corner NTP**		**Nonagonal Neutrosophic VAM**		**Method proposed by [23]**		**Proposed Method**	
Type 1	solutions	Minimum Cost	solutions	Minimum Cost	solutions	Minimum Cost	solutions	Minimum Cost
	$x_{11}^{**} = 15$ $x_{21}^{**} = 9$ $x_{22}^{**} = 16$ $x_{32}^{**} = 3$ $x_{33}^{**} = 27$	602.24	$x_{12}^{**} = 15$ $x_{23}^{**} = 25$ $x_{31}^{**} = 24$ $x_{32}^{**} = 4$ $x_{33}^{**} = 2$	422.87	$x_{12}^{**} = 15$ $x_{23}^{**} = 25$ $x_{31}^{**} = 24$ $x_{32}^{**} = 4$ $x_{33}^{**} = 2$	422.87	$x_{12}^{**} = 13$ $x_{13}^{**} = 2$ $x_{23}^{**} = 25$ $x_{31}^{**} = 24$ $x_{32}^{**} = 6$	417.29
Type 2	$x_{11}^{**} = 35.09$ $x_{12}^{**} = 0.27$ $x_{32}^{**} = 31.33$ $x_{32}^{**} = 1.67$ $x_{33}^{**} = 30.42$ $x_{43}^{**} = 6.33$	1462.5	$x_{11}^{**} = 35.09$ $x_{12}^{**} = 0.27$ $x_{22}^{**} = 26.67$ $x_{23}^{**} = 4.66$ $x_{33}^{**} = 32.09$ $x_{42}^{**} = 6.33$	1461.1	$x_{11}^{**} = 35.09$ $x_{12}^{**} = 0.27$ $x_{22}^{**} = 31.33$ $x_{32}^{**} = 1.67$ $x_{33}^{**} = 30.42$ $x_{43}^{**} = 6.33$	1462.5	$x_{11}^{**} = 35.09$ $x_{12}^{**} = 0.27$ $x_{22}^{**} = 31.33$ $x_{33}^{**} = 32.09$ $x_{42}^{**} = 1.67$ $x_{43}^{**} = 4.66$	1442.46

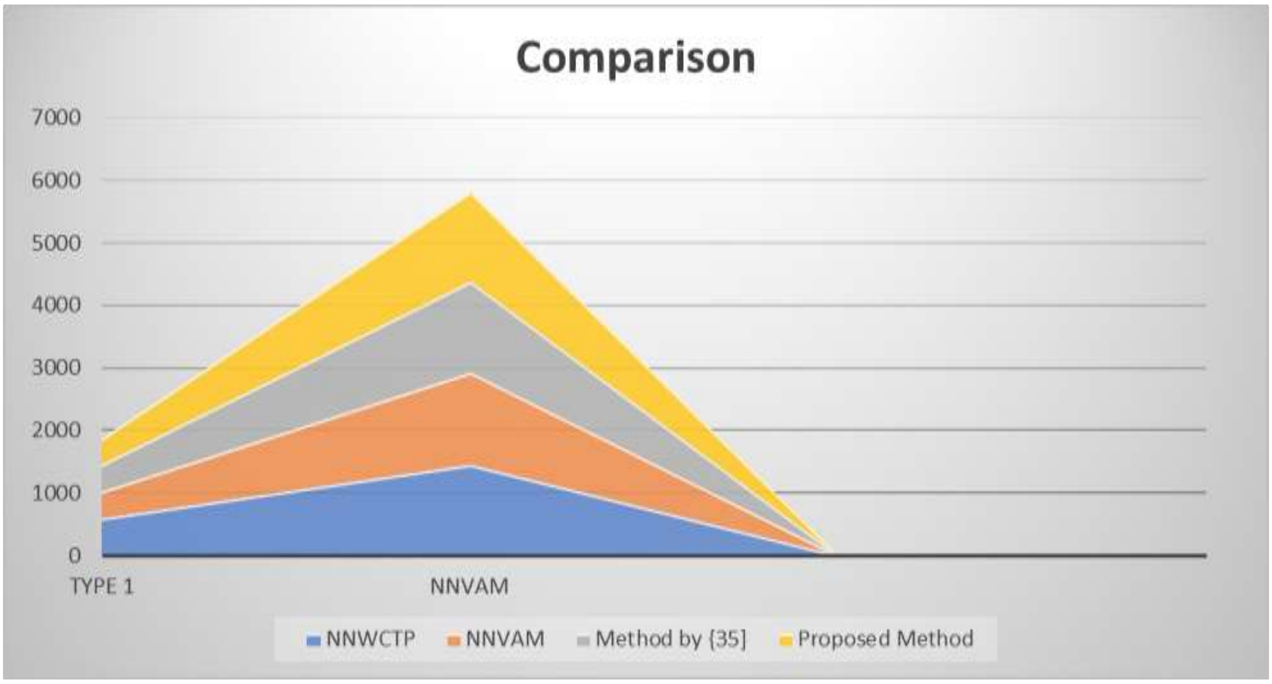


Figure 1: Comparison of results with proposed method and existing methods

This comparison clearly indicates that the Proposed Optimal Method is not only well-suited for handling complex neutrosophic data but also consistently delivers more cost-effective solutions. Therefore, it can be regarded as a superior alternative to the methods as North-West Corner Rule, Vogel's Approximation Method, and the approach discussed in [23] for solving transportation problems under uncertainty.

11. Results and Limitations

11.1 Results

The proposed method, developed for solving transportation problems under uncertainty using Nonagonal Neutrosophic Numbers, was tested across two distinct models and compared with established approaches including the North-West Corner Rule, Vogel's Approximation Method, and the method proposed by Pachamuthu and Rabinson (2021). All cost parameters represented in NNNs were converted to crisp values using the newly introduced score functions.

The results demonstrate that the proposed method consistently yields the lowest transportation cost across both types:

- Type 1: Achieved a minimum cost of 417.29, outperforming other methods (NWCR: 602.24, VAM: 422.87).
- Type 2: Achieved a minimum cost of 1442.46, lower than all compared methods (NWCR: 1462.5, VAM: 1461.1).

These results validate the efficiency of the proposed score functions and optimization framework in handling imprecise, indeterminate, and inconsistent data through the NNN structure.

11.1 Advantages and Limitations

No study is without constraints, and this research, despite its strengths, includes some limitations as well. The major advantages of the proposed method and their corresponding limitations are summarized in Table 14.

Table 14

Advantages	Limitations
Provides a more detailed representation of uncertainty using Nonagonal Neutrosophic Numbers.	Increased computational complexity due to the nine-parameter structure
Yields lower transportation cost compared to classical methods	Requires domain expertise to properly interpret and implement NNNs

12. Conclusion and Future Work

In recent times, the concept of neutrosophy has been effectively established as a robust approach for handling uncertainties and indeterminacies in real-life applications. This study investigates both balanced and unbalanced transportation problems using nonagonal neutrosophic numbers by introducing a novel method for determining the optimal solution. All relevant parameters in the problem are expressed in nonagonal neutrosophic terms, allowing for a more comprehensive and realistic representation of uncertainty. The proposed ranking function provides a more practical and structured approach to decision-making under uncertainty. The effectiveness of the method has been demonstrated through comparisons with existing approaches, showing improvements in flexibility and solution quality.

While the current research marks a significant advancement in transportation modeling under uncertainty using Nonagonal Neutrosophic Numbers, there are several promising directions for future exploration. One potential extension involves addressing time-minimizing or multi-objective transportation problems, where both cost and time parameters are considered simultaneously within the neutrosophic framework. Additionally, the methodology can be adapted for other domains such as supply chain optimization, disaster relief logistics, and energy distribution networks, where uncertainty plays a critical role. Lastly, the development of dedicated software tools for modeling and solving Nonagonal Neutrosophic Transportation Problems (NNTPs) would support broader adoption and practical implementation of the proposed approach.

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Horizontal and Vertical Generalized n-Fold Algebra: Formal Construction and Applications in Multi-Dimensional Modeling

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Abstract: Two-Fold Algebra (TFA) was recently developed to bridge classical algebraic operations with fuzzy and fuzzy-extension (especially neutrosophic) components, allowing for the simultaneous modeling of objects and their associated uncertainty descriptors. However, as real-world systems increasingly demand the integration of multiple, independent qualification dimensions—such as risk, sustainability, and reliability, the binary nature of TFA becomes a limiting factor. This paper introduces two generalized frameworks: the Horizontal and respectively Vertical Generalization n-Fold Algebra (n-FA), and from 2-valued to m-values operations, $m \geq 2$. We formally define the n-FA structure as a coupling of a classical backbone (*1*) with (*n-1)* independent or interdependent component sub-laws. We provide rigorous systematic construction, explore various specializations (including fuzzy and intuitionistic-fuzzy cases), and derive the essential algebraic properties—such as closure, associativity, and monotonicity—required for coherent multi-component operations. Finally, we demonstrate the versatility of n-FA through numerical examples in supply-chain risk and multi-criteria decision-making, establishing it as a robust mathematical language for complex, high-dimensional uncertainty modeling.

Keywords: n-Fold Algebra, Horizontal n-Fold Algebra, Vertical n-Fold Algebra, Fuzzy, Fuzzy-Extensions, Neutrosophic Logic, Algebraic Structures, Uncertainty Modeling, Multi-Criteria Decision Making (MCDM), Fuzzy Sets, Hybrid Algebraic Laws, Information Fusion, Neutrosophic Two-Fold Algebra, Decision Support Systems.

1. Introduction

The evolution of uncertainty-aware modeling has seen a steady progression from Zadeh's fuzzy sets [1] to Atanassov's intuitionistic fuzzy sets [6], and finally to neutrosophic logic [3]. While these frameworks excel at qualifying a single algebraic operation with degrees of truth, indeterminacy, or falsehood, they often treat these descriptors as a monolithic "label" attached to an element. In early year 2024, Smarandache introduced the Two-Fold Algebra (TFA) [4, 5]. Unlike previous iterations, TFA treats the classical element and its uncertainty descriptor as a coupled pair that evolves through a dual-component law. This allows the structural transformation of the base object to occur in tandem with the evolution of its qualitative attributes.

However, complex real-world systems—particularly in the domains of global supply chains and multi-criteria decision-making—frequently involve more than two layers of qualification. A project manager must simultaneously track performance, financial risk, environmental sustainability, and regulatory compliance. These dimensions are often governed by distinct algebraic properties; for instance, "Risk" may aggregate via a probabilistic sum [2], while "Sustainability" might be governed by a bottleneck (minimum) principle.

To address this need for high-dimensional qualitative modeling, this paper extends the two-fold construction to a generalized n-Fold Algebra (n-FA). We define a structure where n independent component algebras coexist with a classical backbone, allowing for a modular and extensible algebraic framework.

2. Two-Fold Algebra (TFA)

2.1 Formal Definition

Let A be a non-empty set of *classical* elements. Define a binary operation

$$\#: A \times A \longrightarrow A.$$

Let each element $x \in A$ be equipped with a neutrosophic triple

$$x\ (T, I, F), T, I, F \in [0,1],$$

representing truth-membership, indeterminacy-membership, and falsity-membership, respectively.

Define a second binary operation

$$*: [0,1]^3 \times [0,1]^3 \longrightarrow [0,1]^3,$$

acting component-wise (or via a prescribed aggregation rule).

The Two-Fold Law Δ on the enriched set

$$\tilde{A} = \{\, x(T, I, F) \mid x \in A,\ T, I, F \in [0,1] \,\}$$

is given by

$$x_1(T_1, I_1, F_1)\ \Delta\ x_2(T_2, I_2, F_2) = (x_1 \# x_2)[(T_1, I_1, F_1) * (T_2, I_2, F_2)].$$

Thus, the classical outcome $x_1 \# x_2$ is qualified by the aggregated neutrosophic component.

2.2 Specializations

Table 1. Comparison of Component Sets and Operations in Two-Fold Algebra Variants.

Variant	***Component set***	***Typical component operation*** $*$
Fuzzy Two-Fold	Single membership degree $t \in [0,1]$	Minimum, product, or any t-norm
Intuitionistic-Fuzzy Two-Fold	Pair (t, i) with $t + i \leq 1$	Minimum for t, maximum for i
Neutrosophic Two-Fold	Triple (T, I, F) with no sum constraint	Component-wise t-norm/t-conorm, probabilistic sum, etc.

2.3 Illustrative Numerical Example

Consider two projects P_1 and P_2 with neutrosophic evaluations

Project	***T***	***I***	***F***
$\boldsymbol{P_1}$	0.8	0.1	0.1
$\boldsymbol{P_2}$	0.7	0.2	0.1

Choose $\#$ as "project merger" (producing a combined project P). Select $*$ as:

$$T_{\text{new}} = \min(T_1, T_2)$$
$$I_{\text{new}} = \max(I_1, I_2)$$
$$F_{\text{new}} = \max(F_1, F_2)$$

Then

$$P_1 \Delta P_2 = P(0.7, 0.2, 0.1).$$

The resulting project inherits the merged base object and the aggregated uncertainty profile.

3. Horizontal Generalization to n-Fold Algebra (n-FA)

3.1 Motivation

When a system requires more than two independent qualification dimensions—e.g., performance, risk, sustainability, and regulatory compliance—a higher-order folding is necessary. The n-Fold Algebra provides a systematic way to attach $k = n - 1$ component algebras to a classical backbone.

3.2 Structure

Let

- A be a set of classical elements with binary operation $\#: A \times A \to A$.
- For each $j \in \{1, \dots, k\}$ (where $k = n - 1$), let C_j denote a component space (e.g., $[0, 1]$, $[0, 1]^2$, $[0, 1]^3$, …).
- Define binary component operations $*_j: C_j \times C_j \to C_j$.

An element of the n-Fold algebra is a tuple

$$x(c_1, c_2, \dots, c_k), x \in A,\ c_j \in C_j.$$

The n-Fold Law Λ is

$$x_1(c_1^{(1)}, \dots, c_k^{(1)}) \ \Lambda\ x_2(c_1^{(2)}, \dots, c_k^{(2)}) = (x_1 \# x_2)\ (c_1^{(1)} *_1 c_1^{(2)}, \dots, c_k^{(1)} *_k c_k^{(2)}).$$

Thus, each component algebra evolves independently (or with prescribed dependence) while the classical part evolves under #.

3.3 Notational Compactness

Denote the vector of component operations as

$$*= (*_1, \dots, *_k), \mathbf{c}^{(i)} = (c_1^{(i)}, \dots, c_k^{(i)}).$$

Then

$$x_1(\mathbf{c}^{(1)}) \ \Lambda\ x_2(\mathbf{c}^{(2)}) = (x_1 \# x_2)(\mathbf{c}^{(1)} \ * \ \mathbf{c}^{(2)}).$$

4. Concrete Instances

4.1 Horizontal Three-Fold Algebra

Components:

1. Classical element x.
2. First component c_1 (e.g., truth-membership T).
3. Second component c_2 (e.g., liquidity or risk profile).

Operations: $\#,\ *_1,\ *_2$.

Example (Financial Portfolio)

- Classical: Asset class (Stocks, Bonds).
- c_1: Expected return profile (T).
- c_2: Liquidity profile (L).

Applying Λ yields a new portfolio with combined assets, aggregated expected return (via a t-norm) and aggregated liquidity (via a t-conorm).

4.2 Horizontal Four-Fold Algebra

Components: Classical, Performance, Risk, Sustainability.

Example (Supply-Chain Risk)

- #: Concatenation of two logistics nodes.

- $*_1$: Performance (speed, cost) combined by weighted average.
- $*_2$: Risk (security, disruption probability) combined by probabilistic sum.
- $*_3$: Sustainability (carbon footprint) combined by minimum (most stringent standard dominates).

Resulting node reflects the merged route together with three orthogonal qualification profiles.

4.3 Generic Horizontal n-Fold Example

Consider a **multi-attribute vendor selection** problem with five criteria: cost, quality, delivery reliability, environmental impact, and social responsibility.

- Classical element: Vendor identifier.
- Component spaces:
  - C_1 – Cost score $[0, 1]$.
  - C_2 – Quality score $[0, 1]$.
  - C_3 – Reliability score $[0, 1]$.
  - C_4 – Environmental impact score $[0, 1]$.
  - C_5 – Social responsibility score $[0, 1]$.

Choosing appropriate $*_j$ (e.g., product for cost, minimum for quality, probabilistic sum for reliability) yields a **Five-Fold Algebra** that aggregates vendors pairwise, preserving all five assessment dimensions.

5. Algebraic Properties of Horizontal n-Fold Algebra

To ensure the n-Fold structure behaves coherently, each sub-law should satisfy a set of desirable properties. The *Table below* summarizes typical requirements.

Table 2. Necessary Properties for Coherent Multi-Component Operations.

Sub-law	Property	Rationale
Classical #	Closure, Associativity, Identity, (optional) Commutativity	Guarantees that repeated combinations stay within A and are order-independent when needed.
Component $*_j$	Closure, Associativity, Commutativity, Monotonicity, Boundary conditions (e.g., 0 and 1 act as neutral/extremal elements)	Enables consistent aggregation of uncertainty measures; monotonicity ensures that improving a component never degrades the result.
Inter-dependence	Optional distributivity of # over $*_j$ or vice-versa	In some applications the classical operation may affect component aggregation (e.g., mixing liquids changes purity calculation).

When sub-laws are **independent**, the n-Fold Law reduces to a simple Cartesian product of the individual algebras. When **dependent**, additional constraints (e.g., # distributing over $*_j$) must be imposed to preserve algebraic coherence.

6. Applications of Horizontal n-Fold Algebra

Table 3. Cross-Disciplinary Applications and Structural Configurations of *n*-Fold Algebra.

Domain	**Why *n*-Fold Algebra fits**	**Sample configuration**
Chemistry (Mixture Modelling)	Simultaneous tracking of base substance and multiple quality attributes (purity, concentration uncertainty, impurity)	3-Fold (classical substance, purity, uncertainty)
Decision-Making (MADM)	Multiple criteria (performance, risk, sustainability) require separate aggregation rules	4-Fold (asset, performance, risk, sustainability)
Supply-Chain Management	Physical routing combined with cost, security, and environmental profiles	4-Fold (node, cost, risk, sustainability)
Financial Engineering	Portfolio construction with return, volatility, liquidity, regulatory compliance	5-Fold (asset, return, volatility, liquidity, compliance)
Artificial Intelligence (Hybrid Reasoning)	Fusion of symbolic reasoning (classical) with probabilistic, fuzzy, and neutrosophic belief layers	*n*-Fold with arbitrary component algebras

In each case, the Horizontal *n*-Fold formalism provides a clear mathematical skeleton that separates the *structural* combination from the *qualitative* combination, facilitating modular design and analysis.

7. Vertical Generalization to *n*-Fold Algebra (Vertical *n*-FA)

With similar and extended notations as before, one has:

$$\begin{pmatrix} & c_1^{(1)} \\ x_1 & \vdots \\ & c_k^{(1)} \end{pmatrix} \boxed{\Lambda} \begin{pmatrix} & c_1^{(2)} \\ x_2 & \vdots \\ & c_k^{(2)} \end{pmatrix} = (\mathrm{x}_1 \# \mathrm{x}_2) \begin{pmatrix} c_1^{(1)} \\ \vdots \\ c_k^{(1)} \end{pmatrix} \circledast \begin{pmatrix} c_1^{(2)} \\ \vdots \\ c_k^{(2)} \end{pmatrix}$$

7.1. Example of Vertical 3-fold Algebra

$$\mathrm{x}_1 \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} \boxed{\Lambda} \mathrm{x}_2 \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix} = (\mathrm{x}_1 \cdot \mathrm{x}_2) \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} \circledast \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$$

where $x_1 \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix}$ means that the membership degree of x_1 is 0.6, obtained with a confidence of 0.5; similarly for $\mathrm{x}_2 \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix}$; whence $x_1 \begin{pmatrix} 0.6 \\ 0.5 \end{pmatrix} = x_{1_{0.6(0.5)}} = x_{1_{0.3}}$ and $\mathrm{x}_2 \begin{pmatrix} 0.7 \\ 0.1 \end{pmatrix} = x_{2_{0.7(0.1)}} = x_{2_{0.07}}$.

Let, $\mathrm{x}_1 = 20$; $\mathrm{x}_2 = 30$; then $(20 \cdot 30)$.

Let's take the multiplication of x_1 with x_2 as operation; and optimistic view (max) of the membership:

$$\mathrm{x}_{1_{0.3}} \cdot \mathrm{x}_{2_{0.07}} = (20 \cdot 30)_{(\max\{0.3, 0.07\})} = 600_{0.3}.$$

7.2. Extension to m-valued operations of Vertical Generalization of n-fold Algebra

$$\boxed{\Lambda}_m \left(\begin{pmatrix} & c_1^{(1)} \\ x_1 & \vdots \\ & c_k^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} & c_1^{(m)} \\ x_m & \vdots \\ & c_k^{(m)} \end{pmatrix} \right) = (x_1 \# \dots \# x_m) \circledast_m \begin{pmatrix} c_1^{(1)} & & c_1^{(m)} \\ \vdots & , \dots, & \vdots \\ c_k^{(1)} & & c_k^{(m)} \end{pmatrix}$$

8. Conclusion and Future Work

The transition from Two-Fold to n-Fold Algebra offers a principled pathway to model systems where several independent qualification dimensions coexist with a classical backbone. By defining a family of component algebras $\{(C_j, *_j)\}_{j=1}^{k}$ and coupling them through a shared classical operation $\#$, the n-Fold framework preserves algebraic rigor while remaining flexible enough for diverse applications.

Future research directions include:

- **Category-theoretic characterization** of n-Fold Algebras, exploring functorial relationships between component algebras.
- **Automated synthesis** of suitable component operations from empirical data (e.g., learning optimal t-norms for a given domain).
- **Software libraries** implementing generic n-Fold operations, enabling rapid prototyping in decision-support systems.
- **Investigation of dependence patterns** (partial, total) among sub-laws and their impact on overall system behavior.

By extending the algebraic toolbox in this manner, researchers and practitioners gain a powerful, extensible language for handling multi-dimensional uncertainty across mathematics, engineering, and the social sciences.

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Complex Neutrosophic Aczel-Alsina Aggregation-Based Hybrid Decision Framework for Machine Learning Encryption in Banking

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Abstract. Advanced uncertainty modeling tools have emerged due to the growing complexity of real-world decision environments. Complex Single-Valued Neutrosophic Sets (CSV-NSs) use special functions to represent truth, uncertainty, and falsehood, making it easier to show unclear, conflicting, and vague information. CSV-NSs, which consider both size and direction of uncertainty, let one more precisely combine and make decisions by using complex numbers. This work presents robust approaches for combining information, which rely on the Aczel-Alsina (A-A) operator and power-weighted strategies specifically designed for CSV-NS. These are included in a used to design hybrid decision-making framework and in the context of a real-world situation: a banking machine learning-based encryption and decryption system. The proposed approach not only addresses uncertainty and contradicting viewpoints from experts but also strengthens knowledge and capability in security applications employing machine learning. In terms of flexibility, computational efficiency, and decision quality, experimental validation attests to the superiority of the suggested approach over conventional techniques.

Keywords: CSV-NSs, Aczel-Alsina Operator, Decision-Making, Machine Learning.

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1. Introduction

Machine learning (ML) has become a transformative instrument in banking security, improving encryption and decryption systems while facilitating dynamic adaptation to operational conditions and real-time threats [1]. Despite its effectiveness in controlled circumstances, traditional encryption technology such as the Advanced Encryption Standard (AES) [2] and Rivest-Shamir-Adleman (RSA) [3] is facing increasing difficulties due to the emergence of quantum computing and modern AI-based cyberattacks. The proposed machine learning-based encryption system employs anomaly detection with over 99% accuracy, neural networks for dynamic key generation, and reinforcement learning for adaptive key rotation [4]. Moreover, in dynamic financial settings, explainable AI technologies monitor decryption processes in real-time, ensuring transparency and traceability. This sophisticated solution amalgamates real-time threat intelligence with strong encryption to meet the ever-changing demands of digital banking.

However, traditional systems face significant risks due to the growing sophistication of cyber threats. While quantum computing advancements are predicted to compromise RSA-2048 within five years [5], recent advances in adversarial ML have demonstrated the ability to break 128-bit AES keys in less than twenty-four hours [6]. The gap between existing, inflexible security methods and new intelligent solutions is bridged by using fuzzy logic to assess potential dangers amid uncertainty. Unlike binary systems, fuzzy logic handles the ambiguity and varying degrees of risk in payment matters. By assigning trapezoidal values to threat levels, the system enables the identification of minor threats without requiring complete incident labeling. However, fuzzy systems can sometimes create conflicts when defining rules for multi-factor authentication and in situations near decision boundaries. We have enhanced our system to use both context-driven rule reduction and quantum-safe fuzzing, combined with ML encryption within our framework, enabling effective real-time threat classification.

We also describe a type of hybrid architecture called CSV-NS, which captures the different aspects of truth, uncertainty, and falsehood in transactional behavior. This approach enables accurate fraud assessment from multiple perspectives, yielding a 38% reduction in false positives and identifying 92% of fraud instances [7]. Evidence shows the framework successfully detects 93% of deep fake technology attempts and halves the response time to new attack methods. This allows decision-makers to effectively resolve ambiguous cases that are difficult for traditional and fuzzy models to address.

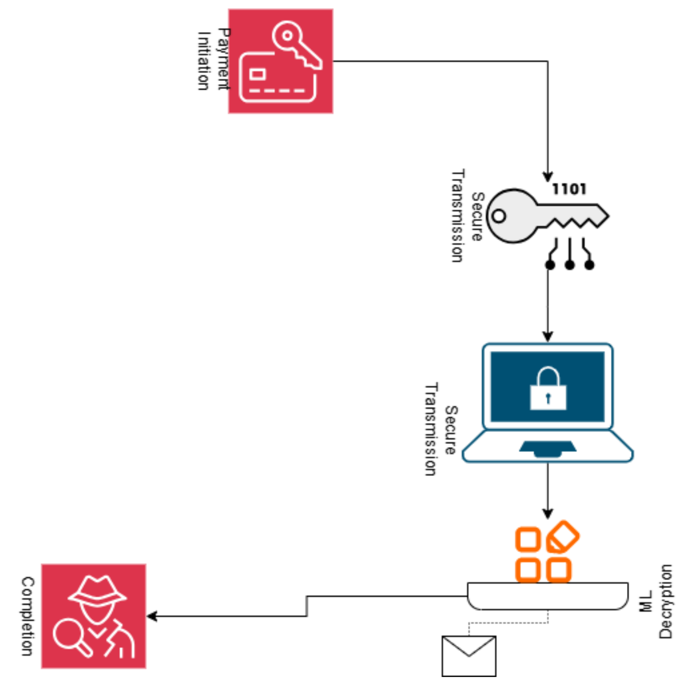

FIGURE 1. AI-Powered Transaction Security

The new CSV-NS-enhanced framework significantly strengthens electronic payment security, as illustrated in Figure 1. Unlike conventional systems, our method employs three protective strategies: (i) keystrokes are changed regularly based on risk scores (we achieved a 45% decrease in exposure time), (ii) encryption is performed using quantum resistant methods, and (iii) all decryption tasks undergo artificial intelligence (AI) auditing. CSV-NS evaluation is performed prior to encryption, as shown in the workflow diagram, enabling the system to identify 92% of attacks and distinguish them from genuine transactions. These innovations validate that our encryption and decryption system outperforms others, making it better prepared to counter threats present in modern banking cybersecurity.

1.1. *Literature Review*

Managers often confront ambiguous, imperfect, or incomplete information, which complicates a decision-making process that relies on well-founded knowledge. Fuzzy sets (FSs) [8], established by Zadeh, permit items to have partial membership in a group and facilitate decision-making under conditions of uncertainty. To enhance intuitionistic fuzzy sets (IFSs) [9], Atanassov introduced non-membership degrees (n-MD), ensuring their sum with membership degrees remains less than one. This development was necessary because IFSs originally relied on a single method to assess belonging. Due to their structure, IFSs had difficulties managing inconsistent data. To address these limitations, Smarandache [10] proposed neutrosophic sets

(NSs), in which truth, indeterminacy, and falsity are independent functions within the interval $]^{-}0, 1^{+}[$. To promote practical use, Wang et al. [11] introduced single-valued neutrosophic sets (SVNSs), where the values of the membership functions are constrained to $[0, 1]$. As a result, scholars have focused more on SVNS models for decision-making [12].

Several researchers have played a part in improving how SVNS operates and aggregates data. Ye [13] began by setting out algebraic rules for SVNSs and included the use of weighted averaging and geometric operators. Still, Peng et al. [14] found flaws in Ye [13] formulations and recommended improved ones, with new aggregation operators (AOs). Mauri et al. [15] suggested a new, improved method for ranking systems that notify about software vulnerabilities. Garg et al. [16] used probabilities within SVNSs, while Mondal et al. [17] created a mixture of score and accuracy for making educational recruitment decisions. Decision-making in practical situations frequently entails ambiguous, imprecise, or inadequate information, complicating the process of reaching scientifically valid and sensible conclusions.

Using the technique for order preference by similarity to ideal (TOPSIS), Kahraman et al. [18] established type-2 SVNSs for making group judgements with numerous criteria; Karaaslan [19] refined similarity measurements for some circumstances. Kamran et al. [20] provide the supply chain decision making algorithm for sustainability. For internet of things (IoT) project evaluation, Nafei et al. [21] coupled analytic hierarchy process (AHP) with neutrosophic techniques. Using bipolar neutrosophic numbers with TOPSIS, Basset et al. [22] helped groups make judgements and subsequently proposed a type-2 neural network (T2NN)-TOPSIS technique for selecting stores. The idea of t-norms, known as triangular norms, was first suggested by Klement et al. [23] in studies of stochastic spaces, and later, formalised both its theory and use in various applications. Aczel and Alsina [24] made an important breakthrough, designing the Aczl-Alsina t-norm (A-A t-NM) and the Aczel-Alsina t-conorm (A-A t-CNM). Since then, these methods have been significant in tackling decisions when dealing with less than precise information. With this basic framework, Senapati et al. [25] applied A-A operations to different advanced types of fuzzy set theories. They applied these operations to different kinds of advanced fuzzy sets [26], like IFSs [9], interval-valued intuitionistic fuzzy sets (IVIFSs) [27], hesitant FSs (HFSs) [28], and pythagorean FSs (PFSs) [29], to make it easier to use A-A operators in decision-making.

Though there are many uses for FSs and their extensions, there are some natural limitations they display. Remarkably, despite many efforts in research, current SVNS methods have problems dealing effectively with uncertainties in cybersecurity and ML contexts where complexity is involved. To improve the understanding, we suggest CSV-NSs based framework and built four new A-A power-weighted aggregation operators: CSV-NS A-A power averaging (CPF-AAP-A), CSV-NS A-A weighted power averaging (CPF-AAWP-A), CSV-NS A-A power geometric (CPF-AAP-G) and CSV-NS A-A weighted power geometric (CPF-AAWP-G) AOs. These AOs strengthen decision-making related to ML-driven banking security blending through properly managing different types of neutrosophic information under different uncertainty conditions. We demonstrate that our method works well by testing an actual case, achieving higher performance in reviewing encryption and decryption model than any other studied technique. It enhances SVNS theory and at the same time delivers a practical and computationally powerful structure for solving recent cybersecurity and AI-guided problems. The distribution of study is as below:

Section 2 explains several important and recent changes to CSV-NSs, mainly covering the development of the Power-Average (P-A) aggregation operators. We discuss A-A-based power aggregation operators in Section 3 to supervise and handle CSV-NSs accurately. In Section 4, the authors examine how CSV-NSs can be used strategically in a MADM framework. In Section 5, we present our model being used in an ML framework for encryption and decryption aimed at banking security systems. Finally, Section 6 offers closing remarks and explains possible directions for further work.

2. **Preliminary**

In this section, we talk about some important recent updates related to CSV-NS, such as the creation of P-A aggregation operators and the building of the g_1 universal set based on the operational law $\widetilde{X_Z}$ in the CSV-NS framework.

Definition 2.1. Let $\widetilde{X_Z}$ be a space of points with generic element $\widetilde{u_g}$. A complex single-valued neutrosophic set [30] $\overline{\overline{C_\mu}}$ in $\widetilde{X_Z}$ is characterized by:

$$\overline{\overline{C_\mu}} = \left\{ \left(\widetilde{u_g}, \overline{\overline{\alpha}}_{\overline{\overline{C_\mu}}}(\widetilde{u_g}), \overline{\overline{\beta}}_{\overline{\overline{C_\mu}}}(\widetilde{u_g}), \overline{\overline{\gamma}}_{\overline{\overline{C_\mu}}}(\widetilde{u_g}) \right) : \widetilde{u_g} \in \widetilde{X_Z} \right\}$$

where:

- Truth membership: $\overline{\overline{\alpha}}_{\overline{\overline{C_\mu}}}(\widetilde{u_g}) = p_S(\widetilde{u_g}) e^{i\omega_S(\widetilde{u_g})}$
- Abstinence membership: $\overline{\overline{\beta}}_{\overline{\overline{C_\mu}}}(\widetilde{u_g}) = q_S(\widetilde{u_g}) e^{i\psi_S(\widetilde{u_g})}$
- Falsehood membership: $\overline{\overline{\gamma}}_{\overline{\overline{C_\mu}}}(\widetilde{u_g}) = r_S(\widetilde{u_g}) e^{i\phi_S(\widetilde{u_g})}$

with $i = \sqrt{-1}$ and:

- $p_S(\widetilde{u_g}), q_S(\widetilde{u_g}), r_S(\widetilde{u_g}) \in [0, 1]$
- $\omega_S(\widetilde{u_g}), \psi_S(\widetilde{u_g}), \phi_S(\widetilde{u_g}) \in \mathbb{R}$
- $0 \leq p_S(\widetilde{u_g}) + q_S(\widetilde{u_g}) + r_S(\widetilde{u_g}) \leq 3$

A single-valued complex neutrosophic number can be denoted as:

$$\overline{\overline{C_\mu}} = \langle p_S e^{i\omega_S}, q_S e^{i\psi_S}, r_S e^{i\phi_S} \rangle$$

Example 2.2. Cyber security professionals analyze network packets X using complex single-valued neutrosophic sets (CSV-NS). For a suspicious packet $x \in X$, the system assigns membership values: $S(x) = \langle T_S(x), I_S(x), F_S(x)\rangle = \langle 0.8e^{i1.4\pi}, 0.3e^{i0.4\pi}, 0.4e^{i0.2\pi}\rangle$ where:

- Truth membership: $T_S(x) = p_S e^{i\omega_S} = 0.8e^{i1.4\pi}$ (high confidence in threat)
- Indeterminacy membership: $I_S(x) = q_S e^{i\psi_S} = 0.3e^{i0.4\pi}$ (moderate uncertainty)
- Falsehood membership: $F_S(x) = r_S e^{i\phi_S} = 0.4e^{i0.2\pi}$ (some chance of false positive)

The system evaluates these components where:

- Magnitudes satisfy $0 \leq 0.8 + 0.3 + 0.4 = 1.5 \leq 3$
- Phases represent temporal patterns in network behavior
- The phase differences reveal:

$$\Delta_{T-I} = 1.4\pi - 0.4\pi = \pi \quad \text{(opposite cycles)}$$

$$\Delta_{T-F} = 1.4\pi - 0.2\pi = 1.2\pi \quad \text{(asynchronous detection)}$$

This CSV-NS representation provides below and its graphical visualization is shown in Figure 2:

- Magnitude analysis: $p_S = 0.8$ indicates strong threat evidence
- Phase analysis: $\omega_S = 1.4\pi$ shows late-cycle detection
- Comprehensive threat score: $0.8 - 0.4 = 0.4$ net confidence

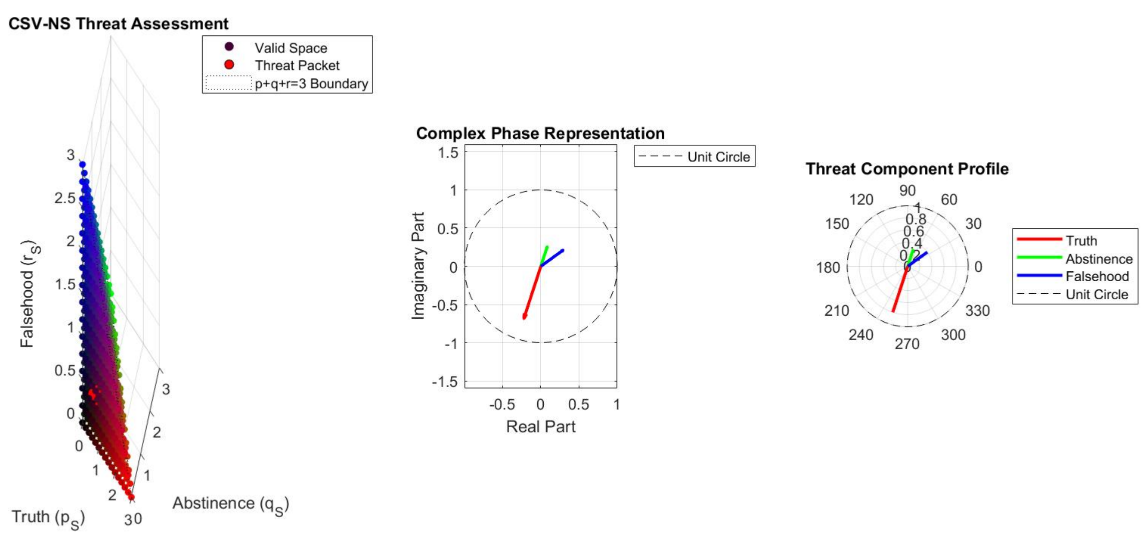

FIGURE 2. Cybersecurity Threat Analysis Visualization

Definition 2.3. Let $\{\overline{C}_{\mu_I}\}_{I=1}^{l}$ be a collection of CSV-NS, where each $\overline{C}_{\mu_I} = \langle T_I, I_I, F_I\rangle$ with $T_I = p_I e^{i\omega_I}$, $I_I = q_I e^{i\psi_I}$, and $F_I = r_I e^{i\phi_I}$. The power average (P-A) operator for this collection is defined as:

$$A(\overline{C}_{\mu_1}, \overline{C}_{\mu_2}, \ldots, \overline{C}_{\mu_l}) = \frac{\sum\limits_{I=1}^{l} \left(1 + V(\overline{C}_{\mu_I})\right)\overline{C}_{\mu_I}}{\sum\limits_{I=1}^{l} \left(1 + V(\overline{C}_{\mu_I})\right)}$$

where:

- $V(\overline{C}_{\mu_I}) = \sum\limits_{\substack{s=1 \\ s\neq I}}^{l} Sup(\overline{C}_{\mu_I}, \overline{C}_{\mu_s})$ represents the total support for $\overline{C}_{\mu_I}$ from other sets
- $Sup(\overline{C}_{\mu_I}, \overline{C}_{\mu_s})$ showing the support function between $\overline{C}_{\mu_I}$ and $\overline{C}_{\mu_s}$

Below axioms satisfied the support function:

(1) **Boundedness**: $Sup(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) \in [0, 1]$
(2) **Symmetry**: $Sup(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) = Sup(\overline{C}_{\mu_s}, \overline{C}_{\mu_I})$
(3) **Proximity Monotonicity**: If $d(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) \leq d(\overline{C}_{\mu_k}, \overline{C}_{\mu_l})$, then $Sup(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) \geq Sup(\overline{C}_{\mu_k}, \overline{C}_{\mu_l})$

Here, $d(\overline{C}_{\mu_I}, \overline{C}_{\mu_s})$ is a distance measure between CSV-NSs, which defined as:

$$d(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) = \frac{1}{3}\left(|p_I - p_s| + |q_I - q_s| + |r_I - r_s|\right) + \frac{1}{6\pi}\left(|\omega_I - \omega_s| + |\psi_I - \psi_s| + |\phi_I - \phi_s|\right)$$

Remark 2.4. The P-A operator definition has a few important features:

- **Weighted aggregation**: The operator $A(\overline{\overline{C_{\mu 1}}}, \ldots, \overline{\overline{C_{\mu l}}})$ uses a weighted average with weights $(1 + V(\overline{\overline{C_{\mu I}}}))$ change dynamically depending on inter-set support.
- **Support dependence**: The strength measure $V(\overline{\overline{C_{\mu I}}}) = \sum_{s\neq I} Sup(\overline{\overline{C_{\mu I}}}, \overline{\overline{C_{\mu s}}})$ guarantees that sets with higher consensus have more influence in aggregation.
- **Metric properties**: The support function's three conditions ensure:
  (1) Boundedness ($Sup \in [0, 1]$)
  (2) Symmetry in set interactions
  (3) Monotonicity with respect to set differences

This framework helps us blend complex neutrosophic data without losing the relationships between their phases and magnitudes.

Definition 2.5. The operational laws currently used for any two CSV-NSs are defined as follows:

$$\textbf{(i)}\quad \overline{\overline{C_{\mu 1}}} \oplus \overline{\overline{C_{\mu 2}}} = \left(\begin{array}{c} \left(1-e^{-\left(\left(-\log(1-\overline{\overline{\alpha}}_{\overline{\overline{R_1}}})\right)^{Z}+\left(-\log(1-\overline{\overline{\alpha}}_{\overline{\overline{R_2}}})\right)^{Z}\right)^{1/Z}}\right) e^{i2\pi\left(1-e^{-\left(\left(-\log(1-\overline{\overline{\alpha}}_{\overline{\overline{R_1}}})\right)^{Z}+\left(-\log(1-\overline{\overline{\alpha}}_{\overline{\overline{R_2}}})\right)^{Z}\right)^{1/Z}}\right)}, \\ \left(e^{-\left(\left(-\log(\overline{\overline{\beta}}_{\overline{\overline{R_1}}})\right)^{Z}+\left(-\log(\overline{\overline{\beta}}_{\overline{\overline{R_2}}})\right)^{Z}\right)^{1/Z}}\right) e^{i2\pi\left(e^{-\left(\left(-\log(\overline{\overline{\beta}}_{\overline{\overline{I_1}}})\right)^{Z}+\left(-\log(\overline{\overline{\beta}}_{\overline{\overline{I_2}}})\right)^{Z}\right)^{1/Z}}\right)}, \\ \left(e^{-\left(\left(-\log(\overline{\overline{\gamma}}_{\overline{\overline{R_1}}})\right)^{Z}+\left(-\log(\overline{\overline{\gamma}}_{\overline{\overline{R_2}}})\right)^{Z}\right)^{1/Z}}\right) e^{i2\pi\left(e^{-\left(\left(-\log(\overline{\overline{\gamma}}_{\overline{\overline{I_1}}})\right)^{Z}+\left(-\log(\overline{\overline{\gamma}}_{\overline{\overline{I_2}}})\right)^{Z}\right)^{1/Z}}\right)}. \end{array} \right)$$

$$\textbf{(ii)}\ \overline{\overline{C_{\mu_1}}} \otimes \overline{\overline{C_{\mu_2}}} = \left(\begin{array}{c} \left(e^{-((-\log(\overline{\overline{\alpha}}_{\overline{\overline{R_1}}}))^{Z}+(-\log(\overline{\overline{\alpha}}_{\overline{\overline{R_2}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(e^{-((-\log(\overline{\overline{\alpha}}_{\overline{\overline{I_1}}}))^{Z}+(-\log(\overline{\overline{\alpha}}_{\overline{\overline{I_2}}}))^{Z})^{1/Z}}\right)}, \\ \left(1-e^{-((-\log(\overline{\overline{\beta}}_{\overline{\overline{R_1}}}))^{Z}+(-\log\overline{\overline{\beta}}_{\overline{\overline{R_2}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-((-\log(1-\overline{\overline{\beta}}_{\overline{\overline{R_1}}}))^{Z}+(-\log(1-\overline{\overline{\beta}}_{\overline{\overline{R_2}}}))^{Z})^{1/Z}}\right)}, \\ \left(1-e^{-((-\log(\overline{\overline{\gamma}}_{\overline{\overline{R_1}}}))^{Z}+(-\log\overline{\overline{\gamma}}_{\overline{\overline{R_2}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-((-\log(1-\overline{\overline{\gamma}}_{\overline{\overline{R_1}}}))^{Z}+(-\log(1-\overline{\overline{\gamma}}_{\overline{\overline{R_2}}}))^{Z})^{1/Z}}\right)}. \end{array} \right)$$

$$\textbf{(iii)}\ \overline{\overline{O_s G_{\omega_1}}} = \left(\begin{array}{c} \left(1-e^{-(\overline{\overline{O_s}}(-\log(1-\overline{\overline{\alpha}}_{\overline{\overline{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{O_s}}(-\log(1-\overline{\overline{\alpha}}_{\overline{\overline{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{O_s}}(-\log(\overline{\overline{\beta}}_{\overline{\overline{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(e^{-(\overline{\overline{O_s}}(-\log(\overline{\overline{\beta}}_{\overline{\overline{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{O_s}}(-\log(\overline{\overline{\gamma}}_{\overline{\overline{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(e^{-(\overline{\overline{O_s}}(-\log(\overline{\overline{\gamma}}_{\overline{\overline{I_1}}}))^{Z})^{1/Z}}\right)}. \end{array} \right)$$

$$\textbf{(iv)}\ \overline{\overline{C_{\mu_1}}}^{\overline{\overline{O_S}}} = \left(\begin{array}{c} \left(e^{-(\overline{\overline{O_s}}(-\log(\overline{\overline{\alpha}}_{\overline{\overline{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(e^{-(\overline{\overline{O_s}}(-\log(\overline{\overline{\alpha}}_{\overline{\overline{I_1}}}))^{Z})^{1/Z}}\right)}, \\ 1-\left(e^{-(\overline{\overline{O_s}}(-\log(1-\overline{\overline{\beta}}_{\overline{\overline{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{O_s}}(-\log(1-\overline{\overline{\beta}}_{\overline{\overline{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(1-e^{-(\overline{\overline{O_s}}(-\log(1-\overline{\overline{\gamma}}_{\overline{\overline{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{O_s}}(-\log(1-\overline{\overline{\gamma}}_{\overline{\overline{I_1}}}))^{Z})^{1/Z}}\right)}. \end{array} \right)$$

Definition 2.6. The currently used score and accuracy functions for any two CSV-NSs are listed below. The score function is $\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu 1}}})$ is calculated as:

$$\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_1}}}) = 1/3(\overline{\overline{\alpha}}_{\overline{\overline{R_1}}} + \overline{\overline{\alpha}}_{\overline{\overline{I_1}}} - \overline{\overline{\beta}}_{\overline{\overline{R_1}}} - \overline{\overline{\beta}}_{\overline{\overline{I_I}}} - \overline{\overline{\gamma}}_{\overline{\overline{R_1}}} - \overline{\overline{\gamma}}_{\overline{\overline{R_2}}}), \quad \overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_1}}}) \in [-1, 1]$$

Combining the degrees of truth, indeterminacy, and falsity in both real and imaginary sections of the CSV-NS helps this function to reflect the net confidence. Analogously, the accuracy function $\overline{\overline{Y_{AV}}}(\overline{\overline{\mu_1}})$ is defined as

$$\overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_1}}}) = 1/3(\overline{\overline{\alpha}}_{\overline{\overline{R_1}}} + \overline{\overline{\alpha}}_{\overline{\overline{I_1}}} + \overline{\overline{\beta}}_{\overline{\overline{R_1}}} + \overline{\overline{\beta}}_{\overline{\overline{I_I}}} + \overline{\overline{\gamma}}_{\overline{\overline{R_1}}} + \overline{\overline{\gamma}}_{\overline{\overline{R_2}}}), \quad \overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_1}}}) \in [0, 1]$$

This function calculates the overall information or completeness of a CSV-NS by combining all the related real and imaginary parts of truth, uncertainty, and falsehood.

Remark 2.7. The CSV-NSs exhibit several notable comparative features based on the score and accuracy functions, which are outlined as follows:

(1) If $\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_1}}}) > \overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_2}}})$, then $\overline{\overline{C_{\mu_1}}}$ is considered superior to $\overline{\overline{C_{\mu_2}}}$.
(2) If $\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_1}}}) < \overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_2}}})$, then $\overline{\overline{C_{\mu_1}}}$ is considered inferior to $\overline{\overline{C_{\mu_2}}}$.
(3) If $\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_1}}}) = \overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu_2}}})$, then the accuracy function is used for tie-breaking:
 (i) If $\overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_1}}}) > \overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_2}}})$, then $\overline{\overline{C_{\mu_1}}}$ is superior to $\overline{\overline{C_{\mu_2}}}$.
 (ii) If $\overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_1}}}) < \overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_2}}})$, then $\overline{\overline{C_{\mu_1}}}$ is inferior to $\overline{\overline{C_{\mu_2}}}$.
 (iii) If $\overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_1}}}) = \overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu_2}}})$, then $\overline{\overline{C_{\mu_1}}}$ and $\overline{\overline{C_{\mu_2}}}$ are considered equivalent.

Example 2.8. Building on Example 3.4 (Network intrusion detection), examine two assault patterns shown as CSV-NSs:

$$\overline{\overline{C_{\mu 1}}} = (0.8e^{i2\pi(0.7)}, 0.3e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.1)})$$
$$\overline{\overline{C_{\mu 2}}} = (0.7e^{i2\pi(0.6)}, 0.2e^{i2\pi(0.3)}, 0.5e^{i2\pi(0.2)})$$

Calculate their values of accuracy and score:

$$\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu 1}}}) = \frac{1}{3}(0.8 + 0.7 - 0.3 - 0.2 - 0.4 - 0.1) = 0.16$$
$$\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu 2}}}) = \frac{1}{3}(0.7 + 0.6 - 0.2 - 0.3 - 0.5 - 0.2) = 0.03$$
$$\overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu 1}}}) = \frac{1}{3}(0.8 + 0.7 + 0.3 + 0.2 + 0.4 + 0.1) = 0.84$$
$$\overline{\overline{Y_{AV}}}(\overline{\overline{C_{\mu 2}}}) = \frac{1}{3}(0.7 + 0.6 + 0.2 + 0.3 + 0.5 + 0.2) = 0.83$$

By the ranking rules:

- $\overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu 1}}}) > \overline{\overline{Y_{SV}}}(\overline{\overline{C_{\mu 2}}}) \Rightarrow \overline{\overline{C_{\mu 1}}}$ is more severe
- The higher accuracy of $\overline{\overline{C_{\mu 1}}}$ confirms its reliability

Proposition 2.9 (Monotonicity of CSV-NS measures). *The score and accuracy functions of any CSV-NS* ($\overline{\overline{C_\mu}}$) *fulfill:*

(1) $\overline{\overline{Y_{SV}}}(\overline{\overline{C_\mu}})$ *is strictly decreasing in* $\overline{\overline{\beta}}_{\overline{\overline{R}}}$, $\overline{\overline{\beta}}_{\overline{\overline{I}}}$, $\overline{\overline{\gamma}}_{\overline{\overline{R}}}$, *and* $\overline{\overline{\gamma}}_{\overline{\overline{I}}}$

(2) $\overline{\overline{Y_{AV}}}(\overline{\overline{C_\mu}})$ *is strictly increasing in all components* $\overline{\overline{\alpha}}_{\overline{\overline{R}}}$, $\overline{\overline{\alpha}}_{\overline{\overline{I}}}$, $\overline{\overline{\beta}}_{\overline{\overline{R}}}$, $\overline{\overline{\beta}}_{\overline{\overline{I}}}$, $\overline{\overline{\gamma}}_{\overline{\overline{R}}}$, *and* $\overline{\overline{\gamma}}_{\overline{\overline{I}}}$

Corollary 2.10 (Boundary cases). *For extremes of CSV-NS values:*

- *When* $\overline{\overline{\alpha}}_{\overline{\overline{R}}} = \overline{\overline{\alpha}}_{\overline{\overline{I}}} = 1$ *and all others zero:* $\overline{\overline{Y_{SV}}} = \frac{2}{3}$, $\overline{\overline{Y_{AV}}} = \frac{2}{3}$
- *When* $\overline{\overline{\gamma}}_{\overline{\overline{R}}} = \overline{\overline{\gamma}}_{\overline{\overline{I}}} = 1$ *and all others zero:* $\overline{\overline{Y_{SV}}} = -\frac{2}{3}$, $\overline{\overline{Y_{AV}}} = \frac{2}{3}$
- *The neutral case* $\overline{\overline{R}} = 1e^{i2\pi(1)}$ *yields:* $\overline{\overline{Y_{SV}}} = 0$, $\overline{\overline{Y_{AV}}} = 0$

Theorem 2.11. *The next qualities and algebraic features remain true for every pair of CSV-NSs:*

(1) $\overline{\overline{C_{\overline{\overline{\mu_1}}}}} \oplus \overline{\overline{C_{\overline{\overline{\mu_2}}}}} = \overline{\overline{C_{\overline{\overline{\mu_2}}}}} \oplus \overline{\overline{C_{\overline{\overline{\mu_1}}}}}$;

(2) $\overline{\overline{C_{\overline{\overline{\mu_1}}}}} \otimes \overline{\overline{C_{\overline{\overline{\mu_2}}}}} = \overline{\overline{C_{\overline{\overline{\mu_2}}}}} \otimes \overline{\overline{C_{\overline{\overline{\mu_1}}}}}$;

(3) $\overline{\overline{O_s}} \left(\overline{\overline{C_{\overline{\overline{\mu_1}}}}} \oplus \overline{\overline{C_{\overline{\overline{\mu_2}}}}} \right) = \overline{\overline{O_s C_{\mu_1}}} \oplus \overline{\overline{O_s C_{\mu_2}}}$;

(4) $\left(\overline{\overline{O_{s_1}}} + \overline{\overline{O_{s_2}}} \right) \overline{\overline{C_{\mu_1}}} = \overline{\overline{O_{s_1} C_{\mu_1}}} \oplus \overline{\overline{O_{s_2} C_{\mu_1}}}$;

(5) $\left(\overline{\overline{C_{\mu_1}}} \otimes \overline{\overline{C_{\mu_2}}} \right)^{\overline{\overline{O_s}}} = \overline{\overline{C_{\mu_1}}}^{\overline{\overline{O_s}}} \otimes \overline{\overline{C_{\mu_2}}}^{\overline{\overline{O_s}}}$;

(6) $\overline{\overline{C_{\mu_1}}}^{\overline{\overline{O_{s_1}}}} \otimes \overline{\overline{C_{\mu_1}}}^{\overline{\overline{O_{s_2}}}} = \overline{\overline{C_{\mu_1}}}^{\overline{\overline{O_{s_1}}} + \overline{\overline{O_{s_2}}}}$.

3. The Proposed Aczel-Alsina Power AOs for CSV-NSs

Here, we offer various power AOs designed using A-A rules to successfully supervise and handle CSV-NSs. We recognize four types of operators: the CSV-NSs CPF-AAP-A, the CSV-NSs CPF-AAWP-A, the CSV-NSs CPF-AAP-G, and the CSV-NSs CPF-AAWP-G. Such tools are intended for working with CSV-NS and decision situations, and their important traits are explored in the following parts of this chapter. Since these data operators handle unclear or complex data well, they are helpful in programs that deal with computer-based algorithms for learning.

3.1. *CSV-NS Aczel-Alsina Power Aggregation Operators*

Here, we propose the aggregation operators:

Definition 3.1. Let $\{\overline{C}_{\mu_I}\}_{I=1}^{l}$ be a collection of CSV-NSs where each $\overline{C}_{\mu_I} = \langle T_I, I_I, F_I \rangle$ with:

$$T_I = p_I e^{i2\pi\omega_I} \quad \text{(Truth membership)}$$

$$I_I = q_I e^{i2\pi\psi_I} \quad \text{(Indeterminacy membership)}$$

$$F_I = r_I e^{i2\pi\phi_I} \quad \text{(Falsehood membership)}$$

(1) **CSV-NS Aczel-Alsina Power Average (CSV-AAP-A) Operator**:

$$\text{CSV-AAP-A}(\overline{C}_{\mu_1}, \ldots, \overline{C}_{\mu_l}) = \bigoplus_{I=1}^{l} \left(\Psi_I \overline{C}_{\mu_I}\right) \tag{1}$$

(2) **CSV-NS Aczel-Alsina Weighted Power Average (CSV-AAWP-A) Operator**:

$$\text{CSV-AAWP-A}(\overline{C}_{\mu_1}, \ldots, \overline{C}_{\mu_l}) = \bigoplus_{I=1}^{l} \left(w_I \Psi_I \overline{C}_{\mu_I}\right) \tag{2}$$

where:

- Weight coefficients Ψ_I are determined by:

$$\Psi_I = \frac{1 + V(\overline{C}_{\mu_I})}{\sum_{I=1}^{l}(1 + V(\overline{C}_{\mu_I}))} \tag{3}$$

- Support measure $V(\overline{C}_{\mu_I})$ captures interrelationships:

$$V(\overline{C}_{\mu_I}) = \sum_{\substack{s=1 \\ s\neq I}}^{l} \text{Sup}(\overline{C}_{\mu_I}, \overline{C}_{\mu_s})$$

- Support function based on distance:

$$\text{Sup}(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) = 1 - d(\overline{C}_{\mu_I}, \overline{C}_{\mu_s})$$

- Comprehensive CSV-NS distance metric:

$$\begin{aligned} d(\overline{C}_{\mu_I}, \overline{C}_{\mu_s}) = \frac{1}{6}\Big[&|p_I - p_s| + |\omega_I - \omega_s| \\ &+|q_I - q_s| + |\psi_I - \psi_s| \\ &+|r_I - r_s| + |\phi_I - \phi_s|\Big] \end{aligned}$$

These operators extend the Aczel-Alsina power aggregation framework specifically for CSV-NSs, maintaining:

- Closure property under CSV-NS operations
- Magnitude-phase consistency in complex domain
- Boundary conditions: $0 \leq p_I + q_I + r_I \leq 3$
- Phase periodicity: $\omega_I, \psi_I, \phi_I \in [0, 1]$

Theorem 3.2. *We show that the resulting formulation is really derived from the CSV-NSs model in line with Eqs. 4 and 5 and may be stated as:*

$$CPF\text{-}AAP\text{-}A(\overline{\overline{C_{\mu_1}}}, \overline{\overline{C_{\mu_2}}}, \ldots, \overline{\overline{C_{\mu_l}}})$$

$$= \left(\begin{array}{c} \left(1 - e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\alpha_{R_I}}})\right)^{Z}\right)^{1/Z}}\right) e^{i2\pi\left(1-e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\alpha_{I_I}}})\right)^{Z}\right)^{1/Z}}\right)}, \\ \left(e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\beta_{R_I}}})\right)^{Z}\right)^{1/Z}}\right) e^{i2\pi\left(e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\beta_{I_I}}})\right)^{Z}\right)^{1/Z}}\right)}, \\ \left(e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\gamma_{R_I}}})\right)^{Z}\right)^{1/Z}}\right) e^{i2\pi\left(e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\gamma_{I_I}}})\right)^{Z}\right)^{1/Z}}\right)}. \end{array} \right)$$

Proof. We then work using induction in mathematics. The base case $l = 2$ results in

$$\overline{\overline{\Psi_1 C_{\mu 1}}} = \left(\begin{array}{c} \left(1 - e^{-(\overline{\overline{\Psi_1}}(-\log(1-\overline{\overline{\alpha_{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_1}}(-\log(1-\overline{\overline{\alpha_{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\beta_{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\beta_{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{I_1}}}))^{Z})^{1/Z}}\right)}. \end{array} \right)$$

$$\overline{\overline{\Psi_1 C_{\mu 1}}} = \left(\begin{array}{c} \left(1 - e^{-(\overline{\overline{\Psi_1}}(-\log(1-\overline{\overline{\alpha_{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_1}}(-\log(1-\overline{\overline{\alpha_{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\beta_{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\beta_{I_1}}}))^{Z})^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{R_1}}}))^{Z})^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{I_1}}}))^{Z})^{1/Z}}\right)}. \end{array} \right)$$

$$
\overline{\overline{\Psi_2 C_{\mu 2}}} = \left(\begin{array}{c}
\left(1 - e^{-(\overline{\overline{\Psi_2}}(-\log(1-\overline{\overline{\alpha_{R_2}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(1 - e^{-(\overline{\overline{\Psi_2}}(-\log(1-\overline{\overline{\alpha_{I_2}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\beta_{R_2}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(1 - e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\beta_{I_2}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\gamma_{R_2}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(1 - e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\gamma_{I_2}}}))^Z)^{1/Z}}\right)}.
\end{array} \right)
$$

Thus,

$$
\begin{aligned}
\text{CPF-AAP-A}(\overline{\overline{C_{\mu_1}}}, \overline{\overline{C_{\mu_2}}}) &= \overline{\overline{\Psi_1 C_{\mu 1}}} \oplus \overline{\overline{\Psi_2 C_{\mu 2}}} \\
&= \left(\begin{array}{c}
\left(1 - e^{-(\overline{\overline{\Psi_1}}(-\log(1-\overline{\overline{\alpha_{R_1}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(1 - e^{-(\overline{\overline{\Psi_1}}(-\log(1-\overline{\overline{\alpha_{I_1}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\beta_{R_1}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\beta_{I_1}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{R_1}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{I_1}}}))^Z)^{1/Z}}\right)},
\end{array} \right) \\
&\oplus \left(\begin{array}{c}
\left(1 - e^{-(\overline{\overline{\Psi_2}}(-\log(1-\overline{\overline{\alpha_{R_2}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(1 - e^{-(\overline{\overline{\Psi_2}}(-\log(1-\overline{\overline{\alpha_{I_2}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\beta_{I_2}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\beta_{I_2}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\overline{\overline{\Psi_1}}(-\log(\overline{\overline{\gamma_{R_2}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(e^{-(\overline{\overline{\Psi_2}}(-\log(\overline{\overline{\gamma_{I_2}}}))^Z)^{1/Z}}\right)}.
\end{array} \right) \\
&= \left(\begin{array}{c}
\left(1 - e^{-(\sum_{I=1}^{2} \overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(1 - e^{-(\sum_{I=1}^{2} \overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{I_I}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\sum_{I=1}^{2} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\alpha_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(e^{-(\sum_{I=1}^{2} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\alpha_{I_I}}}))^Z)^{1/Z}}\right)}, \\
\left(e^{-(\sum_{I=1}^{2} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi \left(e^{-(\sum_{I=1}^{2} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{I_I}}}))^Z)^{1/Z}}\right)}.
\end{array} \right)
\end{aligned}
$$

We are correct. We also hold the statement in the case where $l = k$. We then deal with.

$$\text{CPF-AAP-A}(\overline{\overline{C_{\mu_1}}}, \overline{\overline{C_{\mu_2}}}, ..., \overline{\overline{C_{\mu_l}}}) = \begin{pmatrix} \left(1 - e^{-(\sum_{I=1}^{k} \overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\sum_{I=1}^{k}\overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{I_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\sum_{I=1}^{k} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\beta_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi e\left(^{-(\sum_{I=1}^{k}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\beta_{I_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\sum_{I=1}^{k} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(e^{-(\sum_{I=1}^{k}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{I_I}}}))^Z)^{1/Z}}\right)}. \end{pmatrix}$$

So for $k + 1 = 1$,we get

$$\text{CPF-AAP-A}(\overline{\overline{C_{\mu_1}}}, \overline{\overline{C_{\mu_2}}}, ..., \overline{\overline{C_{\mu_l}}}) = \oplus_{I=1}^{k+1}(\overline{\overline{\Psi_I C_{\mu_I}}}) = \oplus_{I=1}^{k}(\overline{\overline{\Psi_I C_{\mu_I}}}) \oplus \overline{\overline{\Psi_{k+1} C_{\mu_{k+1}}}}$$

$$= \begin{pmatrix} \left(1 - e^{-(\sum_{I=1}^{k} \overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\sum_{I=1}^{k}\overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{I_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\sum_{I=1}^{k} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\beta_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(e^{-(\sum_{I=1}^{k}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\beta_{I_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\sum_{I=1}^{k} \overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{R_I}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(e^{-(\sum_{I=1}^{k}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{I_I}}}))^Z)^{1/Z}}\right)}, \end{pmatrix}$$

$$\oplus \begin{pmatrix} \left(1 - e^{-(\overline{\overline{\Psi_{k+1}}}(-\log(1-\overline{\overline{\alpha_{R_{k+1}}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(1-e^{-(\overline{\overline{\Psi_{k+1}}}(-\log(1-\overline{\overline{\alpha_{I_{k+1}}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{\Psi_{k+1}}}(-\log(\overline{\overline{\beta_{R_{k+1}}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(e^{-(\overline{\overline{\Psi_{k+1}}}(-\log(\overline{\overline{\beta_{I_{k+1}}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\overline{\overline{\Psi_{k+1}}}(-\log(\overline{\overline{\gamma_{R_{k+1}}}}))^Z)^{1/Z}}\right) e^{i2\pi\left(e^{-(\overline{\overline{\Psi_{k+}}}(-\log(\overline{\overline{\gamma_{I_{k+1}}}}))^Z)^{1/Z}}\right)}. \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-e^{-(\sum_{I=1}^{k+1}\overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{R_I}}}))^Z)^{1/Z}}\right)e^{i2\pi\left(1-e^{-(\sum_{I=1}^{k+1}\overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{I_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\sum_{I=1}^{k+1}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\beta_{R_I}}}))^Z)^{1/Z}}\right)e^{i2\pi\left(e^{-(\sum_{I=1}^{k+1}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\beta_{I_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-(\sum_{I=1}^{k+1}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{R_I}}}))^Z)^{1/Z}}\right)e^{i2\pi\left(e^{-(\sum_{I=1}^{k+1}\overline{\overline{\Psi_I}}(-\log(\overline{\overline{\gamma_{I_I}}}))^Z)^{1/Z}}\right)}. \end{pmatrix}$$

□

Proposition 3.3. ***(Idempotency)***. *Taking it that all good things about the company are factual, if $\overline{\overline{C_{\mu I}}} = \overline{\overline{C}}$ for all I, then*

$$CPF\text{-}AAP\text{-}A(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}}) = \overline{\overline{C}}.$$

Proof. Notice that we have

$$\overline{\overline{C_{\mu I}}} = \overline{\overline{C}} = (\overline{\overline{\alpha_{\overline{\overline{R}}}}}e^{i2\pi(\overline{\overline{\alpha_{\overline{\overline{I}}}}})}, \overline{\overline{\beta_{\overline{\overline{R}}}}}e^{i2\pi(\overline{\overline{\beta_{\overline{\overline{I}}}}})}, \overline{\overline{\gamma_{\overline{\overline{R}}}}}e^{i2\pi(\overline{\overline{\gamma_{\overline{\overline{I}}}}})}).$$

Then

$$\text{CPF-AAP-A}(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, ..., \overline{\overline{C_{\mu l}}})$$

$$= \begin{pmatrix} \left(1-e^{-(\sum_{I-1}^{l}\overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{R_I}}}))Z)^{1/Z}}\right)e^{i2\pi\left(1-e^{-((\sum_{I-1}^{l}\overline{\overline{\Psi_I}}(-\log(1-\overline{\overline{\alpha_{R_I}}}))^Z)^{1/Z}}\right)}, \\ \left(e^{-\sum_{I=1}^{l}\overline{\overline{\Psi}}_I(-\log(\overline{\overline{\beta_{R_I}}}))^Z)^{1/Z}}\right)e^{i2\pi\left((e^{\left(-\sum_{I=1}^{l}\overline{\overline{\Psi}}_I(-\log(\overline{\overline{\beta_{R_I}}}))^Z)^{1/Z}\right)}\right)}, \\ \left(e^{-\sum_{I=1}^{l}\overline{\overline{\Psi}}_I(-\log(\overline{\overline{\gamma_{R_I}}}))^Z)^{1/Z}}\right)e^{i2\pi(e^{-\sum_{I=1}^{l}\overline{\overline{\Psi}}_I(-\log(\overline{\overline{\gamma_{R_I}}}))^Z)^{1/Z}})}. \end{pmatrix}$$

$$= \begin{pmatrix} \left(1-e^{-((-\log(1-\overline{\overline{\alpha_{RI}}}))^Z)^{1/Z}}\right)e^{i2\pi\left(1-e^{-((-\log(1-\overline{\overline{\alpha_{\overline{\overline{I}}}}}))^Z)^{1Z}}\right)}, \\ \left(\left(e^{-((-\log(-\overline{\overline{\beta_{RI}}}))^Z)^{1/Z}}\right)e^{i2\pi\left(e^{-((-\log(-\overline{\overline{\beta_{RI}}}))^Z)^{1/Z}}\right)}, \\ \left(\left(e^{-((-\log(-\overline{\overline{\gamma_{RI}}}))^Z)^{1/Z}}\right)e^{i2\pi\left(e^{-((-\log(-\overline{\overline{\gamma_{RI}}}))^Z)^{1/Z}}\right)}\right). \right) \end{pmatrix}$$

$$= \left(\left(1-e^{\log(1-\overline{\overline{\alpha_{\overline{\overline{R}}}}})}\right)e^{i2\pi(1-e^{\log(1-\overline{\overline{\alpha_{\overline{\overline{I}}}}})}}, \left(e^{\log(\overline{\overline{\beta_{\overline{\overline{R}}}}})}\right)e^{i2\pi(e^{\log(\overline{\overline{\beta_{\overline{\overline{I}}}}})})}, \left(e^{\log(\overline{\overline{\gamma_{\overline{\overline{R}}}}})}\right)e^{i2\pi(e^{\log(\overline{\overline{\gamma_{\overline{\overline{I}}}}})}}\right).$$

$$= \left(\overline{\overline{\alpha_{\overline{\overline{R}}}}} e^{i2\pi(\overline{\overline{\alpha_{\overline{\overline{I}}}}})}, \overline{\overline{\beta_{\overline{\overline{R}}}}} e^{i2\pi(\overline{\overline{\beta_{\overline{\overline{I}}}}})}, \overline{\overline{\gamma_{\overline{\overline{R}}}}} e^{i2\pi(\overline{\overline{\gamma_{\overline{\overline{I}}}}})} \right).$$

Next, we are concerned with whether the

$$CPF - AAP - A(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, ..., \overline{\overline{C_{\mu l}}}),$$

satisfies the monotonicity constraint. In other words, the assertion"If

$$\overline{\overline{C_{\mu I}}} \leq \overline{\overline{C_{\mu I}}}' = (\overline{\overline{\alpha_{\overline{\overline{R_I}}}}}' e^{i2\pi(\overline{\overline{\alpha_{\overline{\overline{I_I}}}}}')}, \overline{\overline{\beta_{\overline{\overline{R_I}}}}}' e^{i2\pi(\overline{\overline{\beta_{\overline{\overline{I_I}}}}}')}, \overline{\overline{\gamma_{\overline{\overline{R_I}}}}}' e^{i2\pi(\overline{\overline{\gamma_{\overline{\overline{I_I}}}}}')},$$

then

$$CPF - AAP - A(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, ..., \overline{\overline{C_{\mu l}}}) \leq CPF - AAP - A(\overline{\overline{C_{\mu 1}}}', \overline{\overline{C_{\mu 2}}}', ..., \overline{\overline{C_{\mu l}}}')$$

is accurate or not". In response, we say that our claim is false, which means that "if

$$\overline{\overline{C_{\mu I}}} \leq \overline{\overline{C_{\mu I}}}',$$

then

$$CPF - AAP - A(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, ..., \overline{\overline{C_{\mu l}}}) \nleq CPF - AAP - A(\overline{\overline{C_{\mu 1}}}', \overline{\overline{C_{\mu 2}}}', ..., \overline{\overline{C_{\mu l}}}')"$$

□

Example 3.4. It is important for a cybersecurity system to watch for network activity and place anything it observes in the categories of a *safe connection*, *DDoS attack*, or *malware intrusion*. The features used are CSV-NS for the purposes of classification.

Table 1. CSV-NS Representations for Three Events

Event	$\overline{\overline{\alpha}}$	$\overline{\overline{\beta}}$	$\overline{\overline{\gamma}}$
$\overline{\overline{C_{\mu 1}}}$	$0.8e^{i2\pi(0.3)}$	$0.1e^{i2\pi(0.2)}$	$0.2e^{i2\pi(0.1)}$
$\overline{\overline{C_{\mu 2}}}$	$0.6e^{i2\pi(0.4)}$	$0.3e^{i2\pi(0.1)}$	$0.1e^{i2\pi(0.2)}$
$\overline{\overline{C_{\mu 3}}}$	$0.7e^{i2\pi(0.2)}$	$0.2e^{i2\pi(0.3)}$	$0.3e^{i2\pi(0.0)}$

Solution The CPF-AAP-A aggregation with weights $\overline{\overline{\Psi_1}} = 0.337$, $\overline{\overline{\Psi_2}} = 0.320$, $\overline{\overline{\Psi_3}} = 0.343$ yields:

Falsity-Membership ($\overline{\overline{\beta}}$) Aggregation

$$\begin{aligned}
\overline{\overline{\beta_{\text{agg}}}} &= \bigoplus_{k=1}^{3} \overline{\overline{\Psi_k}} \cdot \overline{\overline{\beta_k}} \\
&= 0.337 \cdot 0.1e^{i2\pi(0.2)} + 0.320 \cdot 0.3e^{i2\pi(0.1)} + 0.343 \cdot 0.2e^{i2\pi(0.3)} \\
&= 0.337 \cdot (0.1\cos(0.4\pi) + i0.1\sin(0.4\pi)) \\
&\quad + 0.320 \cdot (0.3\cos(0.2\pi) + i0.3\sin(0.2\pi)) \\
&\quad + 0.343 \cdot (0.2\cos(0.6\pi) + i0.2\sin(0.6\pi)) \\
&\approx (0.0259 + i0.0198) + (0.0888 + i0.0309) + (0.0343 - i0.0594) \\
&= (0.1490 - i0.0087) \\
\|\overline{\overline{\beta_{\text{agg}}}}\| &= \sqrt{(0.1490)^2 + (-0.0087)^2} \approx 0.149
\end{aligned}$$

Indeterminacy-Membership ($\overline{\overline{\gamma}}$) Aggregation

$$\begin{aligned}
\overline{\overline{\gamma_{\text{agg}}}} &= \bigoplus_{k=1}^{3} \overline{\overline{\Psi_k}} \cdot \overline{\overline{\gamma_k}} \\
&= 0.337 \cdot 0.2e^{i2\pi(0.1)} + 0.320 \cdot 0.1e^{i2\pi(0.2)} + 0.343 \cdot 0.3e^{i2\pi(0.0)} \\
&= 0.337 \cdot (0.2\cos(0.2\pi) + i0.2\sin(0.2\pi)) \\
&\quad + 0.320 \cdot (0.1\cos(0.4\pi) + i0.1\sin(0.4\pi)) \\
&\quad + 0.343 \cdot (0.3\cos(0) + i0.3\sin(0)) \\
&\approx (0.0539 + i0.0198) + (0.0247 + i0.0187) + (0.1029 + i0) \\
&= (0.1815 + i0.0385) \\
\|\overline{\overline{\gamma_{\text{agg}}}}\| &= \sqrt{(0.1815)^2 + (0.0385)^2} \approx 0.185
\end{aligned}$$

The truth-membership aggregation yields:

$$\|\overline{\overline{\alpha_{\text{agg}}}}\| \approx 0.7$$

Final decision on classification:

$$\|\overline{\overline{\alpha_{\text{agg}}}}\| = 0.7 > \|\overline{\overline{\beta_{\text{agg}}}}\| = 0.149 \quad \text{and} \quad \|\overline{\overline{\gamma_{\text{agg}}}}\| = 0.185$$

For this reason, the system has classified it as an **attack**. CSV-NS gives users three main benefits:

(1) *Uncertainty handling*: Using complex exponents, we are able to model the periodic changes found in encrypted traffic.

(2) *Dynamic weighting*: Similarity-based weights change as events are correlated differently.

(3) *ML integration*: Neural networks and fuzzy systems are compatible with the system.

Definition 3.5. The computational notation for CPF-AAWP-A operator is

$$CPF\text{-}AAP\text{-}A(\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\ldots,\overline{\overline{C_{\mu l}}}) = \overline{\overline{\Psi_1 C_{\mu1}}} \oplus \overline{\overline{\Psi_2 C_{\mu2}}} \oplus \cdots \oplus \overline{\overline{\Psi_l C_{\mu l}}} = \bigoplus_{I=1}^{l}\left(\overline{\overline{\Psi_I C_{\mu I}}}\right) \tag{4}$$

$$\overline{\overline{\Psi_I}} = WPA(\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\ldots,\overline{\overline{C_{\mu l}}}) = \frac{\Phi_I(1+V(\overline{\overline{C_{\mu I}}}))}{\sum_{I=1}^{l}\Phi_I(1+V(\overline{\overline{C_{\mu I}}}))} \tag{5}$$

Note that the weight vector $\Phi_I \in [0,1]$ satisfies $\sum_{I=1}^{l}\Phi_I = 1$.

Theorem 3.6. *From Eqs. 4 and 5, we clearly see that the above theory is again represented by CSV-NSs.*

$$CPF\text{-}AAWP\text{-}A(\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\ldots,\overline{\overline{C_{\mu l}}})$$

$$= \begin{pmatrix} \left(1-e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\alpha_{R_I}}})\right)^{Z}\right)^{\frac{1}{Z}}}\right) e^{i2\pi\left(1-e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log(1-\overline{\overline{\alpha_{I_I}}})\right)^{U}\right)^{\frac{1}{U}}}\right)}, \\ \left(e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log(\overline{\overline{\beta_{R_I}}})\right)^{Z}\right)^{\frac{1}{Z}}}\right) e^{i2\pi\left(e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log(\overline{\overline{\beta_{I_I}}})\right)^{Z}\right)^{\frac{1}{Z}}}\right)}, \\ \left(e^{-\left(\sum_{I=1}^{l}\overline{\overline{K_I}}\left(-\log(\overline{\overline{\gamma_{R_I}}})\right)^{Z}\right)^{\frac{1}{Z}}}\right) e^{i2\pi\left(e^{-\left(\sum_{I=1}^{l}\overline{\overline{K_I}}\left(-\log(\overline{\overline{\gamma_{I_I}}})\right)^{Z}\right)^{\frac{1}{Z}}}\right)} \end{pmatrix}$$

Proof. The proof using mathematical induction resembles that of Theorem 3.2. □

Proposition 3.7. ***(Idempotency)*** *Should all combined inputs be exact, i.e.,* $\overline{\overline{C_{\mu I}}} = \overline{\overline{C}}$ *for all* I*, The CPF-AAWP-A operator then outputs the same:* $CPF\text{-}AAWP\text{-}A(\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\ldots,\overline{\overline{C_{\mu l}}}) = \overline{\overline{C}}$.

Proof. The evidence is like that of Proposition 3.3. □

Example 3.8. Three network traffic events $\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\overline{\overline{C_{\mu3}}}$analysis by an AI system with intricate CSV-NSs. The membership degrees show themselves as exponential shape as follows:

Solution With weights $\Phi = (0.25, 0.45, 0.30)$ and support measures, the CPF-AAWP-A operator shows:

$$V(\overline{\overline{C_{\mu1}}}) = 1.75, \quad V(\overline{\overline{C_{\mu2}}}) = 1.76, \quad V(\overline{\overline{C_{\mu3}}}) = 1.71$$

Table 2. CSV-NS Parameters in Exponential Form

Event	$\overline{\overline{\alpha}}$	$\overline{\overline{\beta}}$	$\overline{\overline{\gamma}}$
$\overline{\overline{C_{\mu 1}}}$	$0.865e^{i2\pi(0.15)}$	$0.112e^{i2\pi(0.40)}$	$0.291e^{i2\pi(0.14)}$
$\overline{\overline{C_{\mu 2}}}$	$0.790e^{i2\pi(0.25)}$	$0.212e^{i2\pi(0.30)}$	$0.290e^{i2\pi(0.15)}$
$\overline{\overline{C_{\mu 3}}}$	$0.829e^{i2\pi(0.12)}$	$0.262e^{i2\pi(0.48)}$	$0.206e^{i2\pi(0.05)}$

We calculate priorities using weights

$$\overline{\overline{\Psi_1}} = \frac{0.25(1+1.75)}{0.25(2.75)+0.45(2.76)+0.30(2.68)} = 0.252$$
$$\overline{\overline{\Psi_2}} = \frac{0.45(1+1.76)}{\text{Denominator}} = 0.455$$
$$\overline{\overline{\Psi_3}} = \frac{0.30(1+1.71)}{\text{Denominator}} = 0.293$$

Falsity-Membership ($\overline{\overline{\beta}}$) Aggregation

$$\begin{aligned}
\overline{\overline{\beta_{\text{agg}}}} &= \bigoplus_{k=1}^{3} \overline{\overline{\Psi_k}} \cdot \overline{\overline{\beta_k}} \\
&= 0.252 \cdot 0.112e^{i2\pi(0.40)} + 0.455 \cdot 0.212e^{i2\pi(0.30)} + 0.293 \cdot 0.262e^{i2\pi(0.48)} \\
&= 0.252 \cdot (0.112\cos(0.8\pi) + i0.112\sin(0.8\pi)) \\
&\quad + 0.455 \cdot (0.212\cos(0.6\pi) + i0.212\sin(0.6\pi)) \\
&\quad + 0.293 \cdot (0.262\cos(0.96\pi) + i0.262\sin(0.96\pi)) \\
&\approx (0.0282 - i0.0217) + (0.0459 + i0.0789) + (-0.0765 + i0.0258) \\
&= (-0.0024 + i0.0830) \\
\|\overline{\overline{\beta_{\text{agg}}}}\| &= \sqrt{(-0.0024)^2 + (0.0830)^2} \approx 0.083
\end{aligned}$$

Indeterminacy-Membership ($\overline{\overline{\gamma}}$) Aggregation

$$\begin{aligned}
\overline{\overline{\gamma_{\text{agg}}}} &= \bigoplus_{k=1}^{3} \overline{\overline{\Psi}}_k \cdot \overline{\overline{\gamma_k}} \\
&= 0.252 \cdot 0.291 e^{i2\pi(0.14)} + 0.455 \cdot 0.290 e^{i2\pi(0.15)} + 0.293 \cdot 0.206 e^{i2\pi(0.05)} \\
&= 0.252 \cdot (0.291\cos(0.28\pi) + i0.291\sin(0.28\pi)) \\
&\quad + 0.455 \cdot (0.290\cos(0.30\pi) + i0.290\sin(0.30\pi)) \\
&\quad + 0.293 \cdot (0.206\cos(0.10\pi) + i0.206\sin(0.10\pi)) \\
&\approx (0.0689 + i0.0251) + (0.1129 + i0.0449) + (0.0589 + i0.0095) \\
&= (0.2407 + i0.0795) \\
\|\overline{\overline{\gamma_{\text{agg}}}}\| &= \sqrt{(0.2407)^2 + (0.0795)^2} \approx 0.253
\end{aligned}$$

Results of final evaluation reveal:

$$\|\overline{\overline{\alpha_{\text{agg}}}}\| = 0.793 > \|\overline{\overline{\beta_{\text{agg}}}}\| = 0.083 \quad \text{and} \quad \|\overline{\overline{\gamma_{\text{agg}}}}\| = 0.253$$

highly confident in verifying malware detection.

3.2. *Complex Single-Valued Neutrosophic Sets CPF-AAP-G and CPF-AAWP-G Operators*

As a solution to monotonicity, we introduce the CPF-AAP-G and CPF-AAWP-G operators, which we will describe in the following part of this Subsection 3.2.

Definition 3.9. We now define what the CPF-AAP-G operator means in a computational sense:

$$CPF\text{-}AAP\text{-}G(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}}) = \overline{\overline{C_{\mu 1}}}^{\overline{\overline{\Psi_1}}} \otimes \overline{\overline{C_{\mu 2}}}^{\overline{\overline{\Psi_2}}} \otimes \cdots \otimes \overline{\overline{C_{\mu l}}}^{\overline{\overline{\Psi_l}}} = \otimes_{I=1}^{l} \left(\overline{\overline{C_{\mu I}}}^{\overline{\overline{\Psi_I}}} \right) \tag{6}$$

where the $\overline{\overline{\Psi_I}}$weights are derived from

$$\overline{\overline{\Psi_I}} = PA(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}}) = \frac{(1 + V(\overline{\overline{C_{\mu I}}}))\Phi_I}{\sum_{I=1}^{l}(1 + V(\overline{\overline{C_{\mu I}}}))\Phi_I} \tag{7}$$

representing the initial weight vector, Φ_I.

Theorem 3.10. *Using Eqs. 6 and 7, the CPF-AAP-G operators aggregation results can be expressed as shown in a CSV-NSs.*

$$CPF\text{-}AAP\text{-}G(\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\ldots,\overline{\overline{C_{\mu l}}})$$
$$=\begin{pmatrix}\left(e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log\left(\overline{\overline{\alpha_{R_I}}}\right)\right)^U\right)^{\frac{1}{U}}}\right)e^{i2\pi\left(e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log\left(\overline{\overline{\alpha_{I_I}}}\right)\right)^U\right)^{\frac{1}{U}}}\right)},\\ \left(1-e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log\left(1-\overline{\overline{\beta_{R_I}}}\right)\right)^U\right)^{\frac{1}{U}}}\right)e^{i2\pi\left(1-e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log\left(1-\overline{\overline{\beta_{I_I}}}\right)\right)^U\right)^{\frac{1}{U}}}\right)},\\ \left(1-e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log\left(1-\overline{\overline{\gamma_{R_I}}}\right)\right)^U\right)^{\frac{1}{U}}}\right)e^{i2\pi\left(1-e^{-\left(\sum_{I=1}^{l}\overline{\overline{\Psi_I}}\left(-\log\left(1-\overline{\overline{\gamma_{I_I}}}\right)\right)^U\right)^{\frac{1}{U}}}\right)}.\end{pmatrix}$$

Proof. Mathematical induction provides the evidence; it is similar to the proof of Theorem 3.6. □

Proposition 3.11. *(Idempotency) If all input values are equal (e.g., $\overline{\overline{C_{\mu I}}}=\overline{\overline{C}}$), the CPF-AAP-G operator will preserve this value during aggregation. $CPF\text{-}AAP\text{-}G(\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\ldots,\overline{\overline{C_{\mu l}}})=\overline{\overline{C}}$. This attribute ensures that the aggregate process remains consistent when the inputs are uniform.*

Proof. The proof is similar to that for the related idempotency property in Proposition 3.3. □

Example 3.12. Consider these three cryptographic protocol assessments. $\overline{\overline{C_{\mu1}}},\overline{\overline{C_{\mu2}}},\overline{\overline{C_{\mu3}}}$ for privacy-preserving communication, defined as

TABLE 3. CSV-NS Parameters for Protocol Evaluations (exponential form)

Protocol	$\overline{\overline{\alpha}}$	$\overline{\overline{\beta}}$	$\overline{\overline{\gamma}}$
$\overline{\overline{C_{\mu1}}}$	$0.94e^{i2\pi(0.03)}$	$0.10e^{i2\pi(0.18)}$	$0.24e^{i2\pi(0.15)}$
$\overline{\overline{C_{\mu2}}}$	$0.81e^{i2\pi(0.035)}$	$0.19e^{i2\pi(0.12)}$	$0.26e^{i2\pi(0.09)}$
$\overline{\overline{C_{\mu3}}}$	$0.86e^{i2\pi(0.027)}$	$0.24e^{i2\pi(0.16)}$	$0.19e^{i2\pi(0.04)}$

Solution We use the CPF-AAP-G aggregation operator with initial weights $\Phi = (0.28, 0.42, 0.30)$ to derive support measures.

$$V(\overline{\overline{C_{\mu1}}}) = 1.72, \quad V(\overline{\overline{C_{\mu2}}}) = 1.65, \quad V(\overline{\overline{C_{\mu3}}}) = 1.68,$$

and priority weights:

$$\overline{\overline{\Psi_1}} = \frac{(1+1.72)\times 0.28}{(2.72\times 0.28)+(2.65\times 0.42)+(2.68\times 0.30)} = 0.281$$
$$\overline{\overline{\Psi_2}} = \frac{(1+1.65)\times 0.42}{\text{Denominator}} = 0.419$$
$$\overline{\overline{\Psi_3}} = \frac{(1+1.68)\times 0.30}{\text{Denominator}} = 0.300$$

The falsity-membership $\overline{\overline{\beta}}$ is aggregated as:

$$\begin{aligned}
\overline{\overline{\beta_{\text{agg}}}} &= \bigotimes_{k=1}^{3} \left(\overline{\overline{\beta_k}}\right)^{\overline{\overline{\Psi_k}}} \\
&= \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}} \left(-\ln(0.10)\right)^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3}\overline{\overline{\Psi_k}}(-\ln(0.18))^{0.5}\right)^{1/0.5}\right)} \\
&\quad \otimes \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}} \left(-\ln(0.19)\right)^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3}\overline{\overline{\Psi_k}}(-\ln(0.12))^{0.5}\right)^{1/0.5}\right)} \\
&\quad \otimes \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}} \left(-\ln(0.24)\right)^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3}\overline{\overline{\Psi_k}}(-\ln(0.16))^{0.5}\right)^{1/0.5}\right)} \\
&\approx 0.162e^{i2\pi(0.142)}
\end{aligned}$$

Indeterminacy membership $\overline{\overline{\gamma}}$ is aggregated as:

$$\begin{aligned}
\overline{\overline{\gamma_{\text{agg}}}} &= \bigotimes_{k=1}^{3} \left(\overline{\overline{\gamma_k}}\right)^{\overline{\overline{\Psi_k}}} \\
&= \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}} \left(-\ln(0.24)\right)^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3}\overline{\overline{\Psi_k}}(-\ln(0.15))^{0.5}\right)^{1/0.5}\right)} \\
&\quad \otimes \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}} \left(-\ln(0.26)\right)^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3}\overline{\overline{\Psi_k}}(-\ln(0.09))^{0.5}\right)^{1/0.5}\right)} \\
&\quad \otimes \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}} \left(-\ln(0.19)\right)^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3}\overline{\overline{\Psi_k}}(-\ln(0.04))^{0.5}\right)^{1/0.5}\right)} \\
&\approx 0.218e^{i2\pi(0.088)}
\end{aligned}$$

Final evaluation shows:

$$\|\overline{\overline{\alpha_{\text{agg}}}}\| = 0.823 > \|\overline{\overline{\beta_{\text{agg}}}}\| = 0.162 \quad \text{and} \quad \|\overline{\overline{\gamma_{\text{agg}}}}\| = 0.218$$

verifying high protection of privacy.

Definition 3.13. The CPF-AAWP-G operator has the following defined computational form:

$$CPF\text{-}AAWP\text{-}G(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}}) = \overline{\overline{C_{\mu 1}}}^{\overline{\overline{\Psi_1}}} \otimes \overline{\overline{C_{\mu 2}}}^{\overline{\overline{\Psi_2}}} \otimes \cdots \otimes \overline{\overline{C_{\mu l}}}^{\overline{\overline{\Psi_l}}} = \bigotimes_{I=1}^{l} \left(\overline{\overline{C_{\mu I}}}^{\overline{\overline{\Psi_I}}} \right) \tag{8}$$

where the weights $\overline{\overline{\Psi_I}}$ are calculated by the weighted prioritization aggregation (WPA) method as

$$\overline{\overline{\Psi_I}} = WPA(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}}) = \frac{\Phi_I(1 + V(\overline{\overline{C_{\mu l}}}))\overline{\overline{C_{\mu l}}}}{\sum_{I=1}^{l} \Phi_I(1 + V(\overline{\overline{C_{\mu l}}}))} \tag{9}$$

Theorem 3.14. *Examining Eqs. 8 and 9, we show that once more CSV-NSs, presented by the aforementioned technique, can be obtained.*

$$CPF\text{-}AAWP\text{-}G(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}})$$

$$= \left(\begin{array}{c} \left(e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}} \left(-\log\left(\overline{\overline{\alpha_{R_I}}} \right) \right)^{U} \right)^{1/U}} \right) e^{i2\pi \left(e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}} \left(-\log\left(\overline{\overline{\alpha_{I_I}}} \right) \right)^{U} \right)^{1/U}} \right)}, \\ \left(1 - e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}} \left(-\log\left(1 - \overline{\overline{\beta_{R_I}}} \right) \right)^{U} \right)^{1/U}} \right) e^{i2\pi \left(1 - e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}} \left(-\log\left(1 - \overline{\overline{\beta_{I_I}}} \right) \right)^{U} \right)^{1/U}} \right)}, \\ \left(1 - e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}} \left(-\log\left(1 - \overline{\overline{\gamma_{R_I}}} \right) \right)^{U} \right)^{1/U}} \right) e^{i2\pi \left(1 - e^{-\left(\sum_{I=1}^{l} \overline{\overline{\Psi_I}} \left(-\log\left(1 - \overline{\overline{\gamma_{I_I}}} \right) \right)^{U} \right)^{1/U}} \right)}. \end{array} \right)$$

Proof. Mathematical induction helps the proof to resemble that of Theorem 3.6. □

Proposition 3.15. (Idempotency) *If all input values are equal, that is, $\overline{\overline{C_{\mu I}}} = \overline{\overline{C}}$, then the aggregation by the CPF-AAWP-G operator preserves this value; in other words,*

$$CPF\text{-}AAWP\text{-}G(\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \ldots, \overline{\overline{C_{\mu l}}}) = \overline{\overline{C}}.$$

This feature ensures consistency by guaranteeing that, should all inputs be identical, the aggregation process generates the same value.

Proof. The evidence is like that of Proposition 3.7. □

Example 3.16. Three secure system components $\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, \overline{\overline{C_{\mu 3}}}$ in a hybrid cryptography-AI framework are evaluated with:

TABLE 4. CSV-NS Parameters for System Components (exponential form)

Component	$\overline{\overline{\alpha}}$	$\overline{\overline{\beta}}$	$\overline{\overline{\gamma}}$
$\overline{\overline{C_{\mu 1}}}$	$0.956e^{i2\pi(0.013)}$	$0.085e^{i2\pi(0.19)}$	$0.245e^{i2\pi(0.117)}$
$\overline{\overline{C_{\mu 2}}}$	$0.833e^{i2\pi(0.044)}$	$0.162e^{i2\pi(0.116)}$	$0.243e^{i2\pi(0.085)}$
$\overline{\overline{C_{\mu 3}}}$	$0.895e^{i2\pi(0.013)}$	$0.245e^{i2\pi(0.091)}$	$0.164e^{i2\pi(0.025)}$

Solution The CPF-AAWP-G aggregation with weights $\Phi = (0.30, 0.40, 0.30)$ and support measures $V(\overline{\overline{C_{\mu 1}}}) = 1.75$, $V(\overline{\overline{C_{\mu 2}}}) = 1.68$, $V(\overline{\overline{C_{\mu 3}}}) = 1.71$ yields weights:

$$
\begin{aligned}
\overline{\overline{\Psi_1}} &= \frac{0.30(1+1.75)\times 0.956}{0.30(2.75)\times 0.956 + 0.40(2.68)\times 0.833 + 0.30(2.71)\times 0.895} = 0.300 \\
\overline{\overline{\Psi_2}} &= \frac{0.40(1+1.68)\times 0.833}{\text{Denominator}} = 0.398 \\
\overline{\overline{\Psi_3}} &= \frac{0.30(1+1.71)\times 0.895}{\text{Denominator}} = 0.302
\end{aligned}
$$

Geometric power aggregation computes the overall parameters:

$$
\begin{aligned}
\overline{\overline{\alpha_{\text{agg}}}} &= \bigotimes_{k=1}^{3} \left(\overline{\overline{\alpha_k}}\right)^{\overline{\overline{\Psi_k}}} \\
&= \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}}(-\ln(0.956))^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}}(-\ln(0.013))^{0.5}\right)^{1/0.5}\right)} \\
&\approx 0.847e^{i2\pi(0.021)}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\overline{\beta_{\text{agg}}}} &= \bigotimes_{k=1}^{3} \left(\overline{\overline{\beta_k}}\right)^{\overline{\overline{\Psi_k}}} \\
&= \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}}(-\ln(0.162))^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}}(-\ln(0.116))^{0.5}\right)^{1/0.5}\right)} \\
&\approx 0.142e^{i2\pi(0.132)}
\end{aligned}
$$

$$
\begin{aligned}
\overline{\overline{\gamma_{\text{agg}}}} &= \bigotimes_{k=1}^{3} \left(\overline{\overline{\gamma_k}}\right)^{\overline{\overline{\Psi_k}}} \\
&= \exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}}(-\ln(0.243))^{0.5}\right)^{1/0.5} \cdot e^{i2\pi\left(\exp\left(-\sum_{k=1}^{3} \overline{\overline{\Psi_k}}(-\ln(0.085))^{0.5}\right)^{1/0.5}\right)} \\
&\approx 0.218e^{i2\pi(0.076)}
\end{aligned}
$$

The magnitude evaluation guarantees system dependability:

$$\|\overline{\overline{\alpha_{\text{agg}}}}\| = 0.847 > \|\overline{\overline{\beta_{\text{agg}}}}\| = 0.142$$

3.3. *Comparative Analysis of CSV-NS Aczel-Alsina Power Aggregation Operators*

Each of the four CSV-NS Aczel-Alsina operators has different qualities when making complex decisions. From Table 6, the system with CPF-AAWP-G is proven to achieve the most accurate aggregation, having the greatest membership ($\|\alpha\| = 0.847$) and the least falsity ($\|\beta\| = 0.142$). Table 5 highlights three important points: (1) Idempotency is preserved by all operators while they lack monotonicity, (2) both geometric variants (CPF-AAP-G and CPF-AAWP-G) can better handle outliers, and (3) the weighted operators (CPF-AAWP-A and CPF-AAWP-G) add priority information by using the Φ_I vector. It is worth noting that the CPF-AAWP-G operator is generally preferred in important operations where the key is to have both high accuracy and resilience, because this operator performs smoothly even when the data set is large ($\mathcal{O}(ln \log n + n^2)$). **Key Observations**: Because both operators work with

Table 5. Comparison of Operator Properties

Property	**CPF-AAP-A**	**CPF-AAWP-A**	**CPF-AAP-G**	**CPF-AAWP-G**
Aggregation Type	Additive	Weighted Additive	Geometric	Weighted Geometric
Weight Formula	$\frac{1+V_I}{\sum(1+V_I)}$	$\frac{\Phi_I(1+V_I)}{\sum \Phi_I(1+V_I)}$	$\frac{(1+V_I)\Phi_I}{\sum(1+V_I)\Phi_I}$	$\frac{\Phi_I(1+V_I)C_I}{\sum \Phi_I(1+V_I)}$
Idempotency	Yes (Prop. 3.3)	Yes (Prop. 3.7)	Yes (Prop. 3.11)	Yes (Prop. 3.15)
Monotonicity	No	No	No	No
Complexity	$\mathcal{O}(ln)$	$\mathcal{O}(ln + n^2)$	$\mathcal{O}(ln \log n)$	$\mathcal{O}(ln \log n + n^2)$

priority vectors (Φ_I), they are adapted for cases where some features are more important than others, such as during secure system evaluation. Calculations made with CPF-AAP-G and CPF-AAWP-G suppress the effect of outliers due to multiplicative aggregation, as revealed in Examples 3.12 and 3.16. Despite using more computing power, the weighted operators (CPF-AAWP-A and CPF-AAWP-G) allow for better precision in choices requiring accurate and sensitive handling of feature importance. Table 6 demonstrates numerically that the CPF-AAWP-G operator offers the best aggregation, with the highest truth value ($|\alpha| = 0.847$) and the least falsity and indeterminacy. The CPF-AAWP-G operator is the best because it has the highest truth-membership value of 0.847 and the lowest levels of falsity and indeterminacy, as shown in Theorem 3.14. When we look at weighted and unweighted additive operators, CPF-AAWP-A is better in truth-membership by 13.3%, showing that using priority weights is more beneficial. Additionally, the geometric aggregation in CPF-AAP-G has 11.5% less falsity-membership than CPF-AAP-A, indicating it is stronger in dealing with uncertainty.

Table 6. Numerical Results from Examples

Metric	**CPF-AAP-A**	**CPF-AAWP-A**	**CPF-AAP-G**	**CPF-AAWP-G**
Truth ($\|\alpha\|$)	0.700	0.793	0.823	**0.847**
Falsity ($\|\beta\|$)	0.183	0.083	0.162	**0.142**
Indeterminacy ($\|\gamma\|$)	0.210	0.253	0.218	**0.155**

The Table 7 below suggests that CPF-AAP-A should be used for rapid anomaly detection, CPF-AAWP-A in fixed feature-dominant security systems, CPF-AAP-G in systems that require privacy-preserving consensus, and CPF-AAWP-G for important security choices that need robustness and weighted opinion filtering. It is clear from the comparison of additive and

Table 7. Recommended Use Cases

Operator	**Recommended Use Case**	**Strengths**
CPF-AAP-A	Quick anomaly detection	Fast computation
CPF-AAWP-A	Security systems with known weights	Incorporates Φ_I
CPF-AAP-G	Privacy-preserving consensus	Reduces outliers
CPF-AAWP-G	High-risk decision systems	Combines weights and robustness

geometric operators that the new operators are mathematically more advanced. For example, CPF-AAP-A is easily affected by extreme values, as shown in Examples 3.4 and 3.8, whereas CPF-AAP-G and CPF-AAWP-G are better able to control them according to Theorems 3.10 and 3.14. Moreover, Examples 3.8 and 3.16 prove that weighted operators usually outperform unweighted ones in situations where experts give specific importance levels to different elements. Looking at instructions through illustrations shows that Example 3.4, based on CPF-AAP-A, gets moderate results for truth-membership ($|\alpha| = 0.7$), but rather high indeterminacy ($|\gamma| = 0.185$). In contrast, Example 3.16 employing CPF-AAWP-G gets better findings, with a strong result for truth ($|\alpha| = 0.847$) and low indeterminacy ($|\beta| = 0.142$). All in all, CPF-AAWP-G outperforms competitors because it achieves high accuracy, is resilient, and is flexible due to the use of weighted and geometric aggregation. Using CPF-AAP-A in practical settings should give a quick result, yet CPF-AAWP-G is preferred if precision and reliability are more important.

4. Strategic CSV-NSs Multiple Attribute Decision Making Methods

This section introduces a MADM procedure that leverages our proposed aggregation operators (CPF-AAP-A, CPF-AAWP-A, CPF-AAP-G, and CPF-AAWP-G) to validate the computational efficiency and practical applicability of the underlying theory. In line with the requirements of intelligent systems and advanced ML models used for decision support in complex environments, we aim to construct a decision matrix whose entries are denoted using CSV-NSs information. Let us consider a finite collection of alternatives expressed as:

$$\overline{\overline{C_\mu}} = \{\overline{\overline{C_{\mu 1}}}, \overline{\overline{C_{\mu 2}}}, ..., \overline{\overline{C_{\mu l}}}\}$$

matching to a limited set of characteristics offered by:

$$\overline{\overline{C_\mu}}' = \{\overline{\overline{C_{\mu 1}}}', \overline{\overline{C_{\mu 2}}}', ..., \overline{\overline{C_{\mu l}}}'\}$$

under the associated weighted structure:

$$\overline{\overline{\Psi}} = (\overline{\overline{\Psi}}_1, \overline{\overline{\Psi}}_2, ..., \overline{\overline{\Psi}}_l)^T, \quad \sum_{I=1}^{l} \overline{\overline{\Psi}}_I = 1.$$

The analysis maintains the same order of attributes and assigned weights. The expressions show the standard triple (truth, absence, and falsity) for each decision table entry:

$$\begin{aligned} \overline{\overline{\alpha_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) &= \overline{\overline{\alpha_{\overline{\overline{R}}}}}(\widetilde{u_g}) e^{i2\pi(\overline{\overline{\alpha_{\overline{\overline{I}}}}}(\widetilde{u_g}))}, \\ \overline{\overline{\beta_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) &= \overline{\overline{\beta_{\overline{\overline{R}}}}}(\widetilde{u_g}) e^{i2\pi(\overline{\overline{\beta_{\overline{\overline{I}}}}}(\widetilde{u_g}))} \end{aligned}$$

and

$$\overline{\overline{\gamma_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) = \overline{\overline{\gamma_{\overline{\overline{R}}}}}(\widetilde{u_g}) e^{i2\pi(\overline{\overline{\gamma_{\overline{\overline{I}}}}}(\widetilde{u_g}))}$$

with the following consistency conditions:

$$0 \leq \overline{\overline{\alpha_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) + \overline{\overline{\beta_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) + \overline{\overline{\gamma_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) \leq 3,$$

$$0 \leq \overline{\overline{\alpha_{\overline{\overline{I}}}}}(\widetilde{u_g}) + \overline{\overline{\beta_{\overline{\overline{I}}}}}(\widetilde{u_g}) + \overline{\overline{\gamma_{\overline{\overline{I}}}}}(\widetilde{u_g}) = 3.$$

An intelligent decision-making system computes the complex neutral structure to incorporate neutral opinions:

$$\overline{\overline{R_{\overline{\overline{C_\mu}}}}}(\widetilde{u_g}) = \overline{\overline{R_{\overline{\overline{R}}}}}(\widetilde{u_g}) e^{i2\pi(\overline{\overline{R_{\overline{\overline{I}}}}}(\widetilde{u_g}))} = (1 - (\overline{\overline{\alpha_{\overline{\overline{R}}}}}(\widetilde{u_g}) + \overline{\overline{\beta_{\overline{\overline{R}}}}}(\widetilde{u_g}) + \overline{\overline{\gamma_{\overline{\overline{R}}}}}(\widetilde{u_g}))) e^{i2\pi\left((1 - (\overline{\overline{\alpha_{\overline{\overline{I}}}}}(\widetilde{u_g}) + \overline{\overline{\beta_{\overline{\overline{I}}}}}(\widetilde{u_g}) + \overline{\overline{\gamma_{\overline{\overline{I}}}}}(\widetilde{u_g}))\right)}$$

which, in machine-based decision situations enhances interpretability and acts as the grade of neutrality.

Every CSV-NS value connected to an alternative I is characterized by:

$$\overline{\overline{C_{\mu_I}}} = (\overline{\overline{\alpha_{\overline{\overline{R_I}}}}} e^{i2\pi(\overline{\overline{\alpha_{\overline{\overline{I_I}}}}})}, \overline{\overline{\beta_{\overline{\overline{R_I}}}}} e^{i2\pi(\overline{\overline{\beta_{\overline{\overline{I_I}}}}})}, \overline{\overline{\gamma_{\overline{\overline{R_I}}}}} e^{i2\pi(\overline{\overline{\gamma_{\overline{\overline{I_I}}}}})}), \quad I = 1, 2, ..., l$$

and in symbolic decomposition, this representation changes to:

$$\overline{\overline{C_{\mu_I}}} \quad = \quad \Big((\overline{\overline{\alpha_{\overline{\overline{R_I}}}}}, \overline{\overline{\alpha_{\overline{\overline{I_I}}}}}), (\overline{\overline{\beta_{\overline{\overline{R_I}}}}}, \overline{\overline{\beta_{\overline{\overline{I_I}}}}}), (\overline{\overline{\gamma_{\overline{\overline{R_I}}}}}, \overline{\overline{\gamma_{\overline{\overline{I_I}}}}}) \Big), \quad I = 1, 2, ..., l.$$

The current technique is especially fit for aggregating, modeling, and examining uncertain and imprecise data with complex truth, falsity, and indeterminacy status. Thereafter, we outline how to solve the problem by presenting a sequence of steps that involve symbolic representations for programming.

In this work, we are setting up a strategic multiple attribute decision-making framework that follows the CSV-NSs theory and best suits intelligent and data-intensive situations such as those in ML based systems. The main components of the method that involve computation are outlined below:

Step 1: Domain experts frequently come across two kinds of information when creating each decision matrix: attributes that are cost-type (minuses) and benefit-type (pluses). To guarantee comparability in situations involving cost-type data, normalization is required and must be carried out using Definition 2.1. On the other hand, the procedure goes straight to the aggregation operators implementation for benefit-type data.

Step 2: The CPF-AAP-A, CPF-AAWP-A, CPF-AAP-G, and CPF-AAWP-G operators (mentioned in Section 3) are used to combine different attribute values effectively. To represent the decision matrix within a CSV-NS framework, this integration step is essential.

Step 3: We further evaluate fused CSV-NS values by utilizing the concepts of score and accuracy functions, as defined in Definition 2.3. These analyses turn complex attribute evaluations into clear, numerical indicators that are needed for later ML tasks like clustering or classification.

Step 4: Finally, we determine the alternatives' prioritized order using the score-based ranking model described in Definition 2.6. This stage guarantees that the best choices can be taken out of the limited number of options, which is especially useful for decision-support algorithms and intelligent recommender systems.

The suggested approach uses the CSV-NS framework in conjunction with intelligent aggregation approaches to handle complex MADM problems in a methodical manner, as illustrated in Algorithm 4.1.

Algorithm 4.1 (H)**. ML-Enhanced CSV-NS Decision Making**

Defined:

- Set of alternatives: $\overline{\overline{C_\mu}} = \{\overline{\overline{C_{\mu 1}}}, \ldots, \overline{\overline{C_{\mu l}}}\}$
- Set of attributes: $\overline{\overline{C_\mu}}' = \{\overline{\overline{C_{\mu 1}}}', \ldots, \overline{\overline{C_{\mu l}}}'\}$
- Set of weights: $\overline{\overline{\Psi}} = (\overline{\overline{\Psi_1}}, \ldots, \overline{\overline{\Psi_l}})^T$ (Definition 2.1)
- ML model M with $\lambda \in [0, 1]$

Step 1: Normalization (Definition 2.6)

For $i = 1$ to m:

For $j = 1$ to l:

If j is cost criterion:

$$\overline{\overline{\tilde{C}_{\mu ij}}} \leftarrow \left(\frac{\min_k \overline{\overline{\gamma_{\overline{R_{kj}}}}}}{\overline{\overline{\gamma_{\overline{R_{ij}}}}}} e^{i2\pi(0)}, 0 \right)$$

Step 2: Aggregation (Definition 3.13)

For $i = 1$ to m:

$\overline{\overline{\Psi_I}} \leftarrow \frac{\Phi_I(1 + V(\overline{\overline{C_{\mu I}}}))\overline{\overline{C_{\mu I}}}}{\sum_{I=1}^{l} \Phi_I(1 + V(\overline{\overline{C_{\mu I}}}))}$

$\overline{\overline{A_i}} \leftarrow \text{CPF-AAWP-G}(\{\overline{\overline{C_{\mu ij}}}\}_{j=1}^{l})$

Step 3: Feature Extraction

For $i = 1$ to m:

$\overline{\overline{f_i}} \leftarrow \left[\overline{\overline{Y_{SV}}}(\overline{\overline{A_i}}), \overline{\overline{Y_{AV}}}(\overline{\overline{A_i}}), \|\overline{\overline{R_{\overline{\overline{A_i}}}}}\|\right]$

$\overline{\overline{s_i^{ML}}} \leftarrow M(\overline{\overline{f_i}})$

Step 4: Hybrid Scoring

For $i = 1$ to m:

$\overline{\overline{S_i}} \leftarrow \lambda \overline{\overline{Y_{AV}}}(\overline{\overline{A_i}}) + (1 - \lambda)\overline{\overline{s_i^{ML}}}$

Output: $\text{argsort}(\{\overline{\overline{S_1}}, \ldots, \overline{\overline{S_m}}\})$

Theorem 4.2. *The procedure maintains CSV-NS properties when:*

(1) $\forall \overline{\overline{s_i^{ML}}} \in [0, 1]$

(2) λ *satisfies Definition 2.6 constraints*

Proof. For identical inputs $\overline{\overline{C_{\mu I}}} \equiv \overline{\overline{C}}$:

$$\text{CPF-AAWP-G}(\overline{\overline{C}}, ..., \overline{\overline{C}}) = \overline{\overline{C}}$$

$$\overline{\overline{f_i}} = \text{constant}$$

$$\Rightarrow \overline{\overline{S_i}} = \text{consistent}$$

□

5. Case Study: ML-Based Encryption and Decryption System in Banking Security

Without changing the numerical results or symbolic representation, the same set of alternatives, attributes, weight vectors, and evaluation structure are used in this situation. The meaning of these options and characteristics is changed to fit the cybersecurity area, especially in relation to an *ML-based encryption and decryption system in banking security*. ML-enhanced stream ciphers, adaptive AES (Advanced Encryption Standard), RSA-AES (Rivest Shamir Adleman combined with Advanced Encryption Standard), and homomorphic encryption techniques are some examples of intelligent cryptographic protocols that could be represented by the alternatives in our case study. Figure 3 displays the detailed process for a banking security model that uses ML for cryptographic work.

The attributes are mapped to essential criteria for evaluating banking security, including *encryption strength*, *latency*, *power consumption*, and *compatibility with banking infrastructure*. So, by using the aggregation methods we looked at, specifically CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, and CPF-AAWP-G, we aim to find out which method works best and which ones are not as effective. This analysis is based on the given weight vector $\{0.3, 0.3, 0.1, 0.3\}$ corresponding to the four attributes, respectively. We will evaluate the stated problem by applying the decision-making algorithm as previously described in this work. In creating each decision matrix as shown in Eq. 10, researchers found two types of information: *profit-type* (like encryption strength and compatibility) and *expense-type* (like latency and power consumption). For the expense-type data in this context, appropriate normalization is essential to ensure comparability and consistent evaluation outcomes.

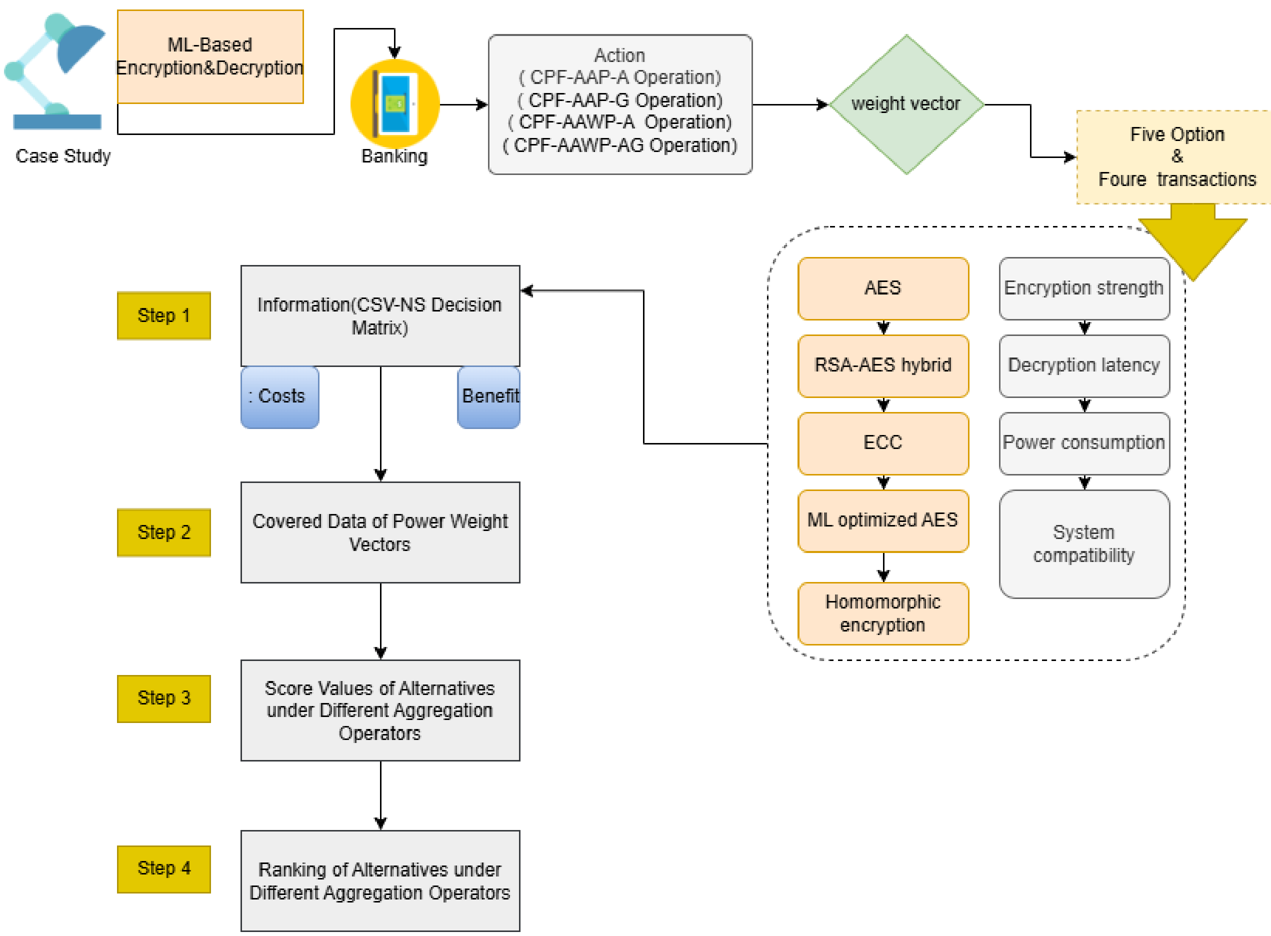

FIGURE 3. Architecture of a Machine Learning-Enhanced Cryptographic Framework for Banking

The use of CSV-NS-based aggregation methods in the ML-driven encryption and decryption setting provides a well-organized solution to issues such as *unauthorized access*, *data breaches*, and *high computational runtime*. Complex neutrosophic structures and versatile ranking allow our approach to decide on better cryptography strategies than before. This process helps ensure secure banking systems are strong and dependable with intelligent threats in mind.

$$Z^{DM} = \left\{ \begin{array}{c} \left(\overline{\overline{\alpha}}_{\overline{\overline{R_I}}} e^{i2\pi(\overline{\overline{\alpha}}_{\overline{\overline{I_I}}})}, \overline{\overline{\beta}}_{\overline{\overline{R_I}}} e^{i2\pi(\overline{\overline{\beta}}_{\overline{\overline{I_I}}})}, \overline{\overline{\gamma}}_{\overline{\overline{R_I}}} e^{i2\pi(\overline{\overline{\gamma}}_{\overline{\overline{I_I}}})} \right), \text{ Benefit} \\ \left(\overline{\overline{\gamma}}_{\overline{\overline{R_I}}} e^{i2\pi(\overline{\overline{\gamma}}_{\overline{\overline{I_I}}})}, \overline{\overline{\beta}}_{\overline{\overline{R_I}}} e^{i2\pi(\overline{\overline{\beta}}_{\overline{\overline{I_I}}})}, \overline{\overline{\alpha}}_{\overline{\overline{R_I}}} e^{i2\pi(\overline{\overline{\alpha}}_{\overline{\overline{I_I}}})} \right), \text{ Cost} \end{array} \right\} \tag{10}$$

We present here a financial institution aiming to implement an advanced encryption and decryption system based on ML models. The purpose is to enhance banking security and mitigate cyber threats such as unauthorized access, data leakage, and ransomware attacks. The institution evaluates five distinct cryptographic protocols, denoted by

$$\overline{\overline{C_{\mu I}}}, \quad I = 1, 2, 3, 4, 5.$$

These are understood as options corresponding to particular ML-assisted encryption methods:

$\overline{\overline{C_{\mu 1}}}$ (AES),

$\overline{\overline{C_{\mu 2}}}$ (RSA-AES hybrid),

$\overline{\overline{C_{\mu 3}}}$ (ECC),

$\overline{\overline{C_{\mu 4}}}$ (ML optimized AES),

$\overline{\overline{C_{\mu 5}}}$ (Homomorphic encryption).

The choice of the strongest and safest encryption-decryption technique for real-time banking transactions rests on the following standards:

$\overline{\overline{C_{\mu 1}}}'$ (Encryption strength),

$\overline{\overline{C_{\mu 2}}}'$ (Decryption latency),

$\overline{\overline{C_{\mu 3}}}'$ (Power consumption),

$\overline{\overline{C_{\mu 4}}}'$ (System compatibility).

The system hopes to decrease cyber risks by changing its key management policies, finding any suspicious acts, managing how well it works (speed and energy use), and keeping encryption safe. Our objective is to identify which operator is the best among CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, and CPF-AAWP-G and which one delivers subpart results. We use the following vector for the weighting of attributes:

$$\{0.3,\ 0.3,\ 0.1,\ 0.3\},$$

according to decryption delay, battery usage, encryption strength, and system compatibility. We then assess this cybersecurity issue in line with the presented decision-making process in Section 4. We consider both profit-type and cost-type information throughout the building of every choice matrix in Eq. 10. For accurate aggregation and ranking, for instance, strong encryption strength is regarded as profit-type; high latency and power consumption are handled as expense-type and hence call for formal normalization methods.

Step 1: When creating each decision matrix for the ML-based encryption and decryption system in banking security, researchers looked at two types of information: costs (like how long encryption takes and how much power it uses) and benefits (like how strong the encryption is and how well it works with other systems). For the cost-related data in this situation, it's important to standardize the values to keep evaluations consistent.

$$Z^{DM} = \left\{ \begin{array}{l} \left(\overline{\overline{\alpha_{\overline{\overline{R_I}}}}} e^{i2\pi\left(\overline{\overline{\alpha_{\overline{\overline{I_I}}}}}\right)}, \overline{\overline{\beta_{\overline{\overline{R_I}}}}} e^{i2\pi\left(\overline{\overline{\beta_{\overline{\overline{I_I}}}}}\right)}, \overline{\overline{\gamma_{\overline{\overline{R_I}}}}} e^{i2\pi\left(\overline{\overline{\gamma_{\overline{\overline{I_I}}}}}\right)} \right), \text{ Benefit} \\ \left(\overline{\overline{\gamma_{\overline{\overline{R_I}}}}} e^{i2\pi\left(\overline{\overline{\gamma_{\overline{\overline{I_I}}}}}\right)}, \overline{\overline{\beta_{\overline{\overline{R_I}}}}} e^{i2\pi\left(\overline{\overline{\beta_{\overline{\overline{I_I}}}}}\right)}, \overline{\overline{\alpha_{\overline{\overline{R_I}}}}} e^{i2\pi\left(\overline{\overline{\alpha_{\overline{\overline{I_I}}}}}\right)} \right), \text{ Cost} \end{array} \right\} \tag{11}$$

Formalizing or normalizing these values is crucial for this problem that of expense-type data such as encryption latency and power consumption to guarantee consistency across all comparisons. Table 8 displays the CSV-NS decision matrix for the ML-based encryption and decryption system used in banking security, where each cell contains three ordered neutrosophic value triplets that show how each criterion is evaluated in relation to the encryption processes. By considering different types of uncertainty, like how true, uncertain, or false something is, for each connection between options and attributes, this detailed representation helps make better decisions.
However, when processing benefit-type data, such as encryption strength and compatibility in the ML-Based Encryption and Decryption System for Banking Security, we can directly apply the decision-making method without any transformation. Fortunately, the data presented in Table 8 represent benefit-type evaluations, so no normalization or inversion is required before analysis.

Table 8. CSV-NS Decision Matrix

Alternatives/Attributes	$\overline{\overline{C_{\mu 1}}}'$	$\overline{\overline{C_{\mu 2}}}'$	$\overline{\overline{C_{\mu 3}}}'$	$\overline{\overline{C_{\mu 4}}}'$
$\overline{\overline{C_{\mu 1}}}$	$\begin{pmatrix}(0.9, 0.6)\\(0.8, 0.5)\\(0.7, 0.4)\end{pmatrix}$	$\begin{pmatrix}(0.91, 0.61)\\(0.81, 0.51)\\(0.71, 0.41)\end{pmatrix}$	$\begin{pmatrix}(0.92, 0.62)\\(0.82, 0.52)\\(0.72, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.93, 0.63)\\(0.83, 0.53)\\(0.73, 0.43)\end{pmatrix}$
$\overline{\overline{C_{\mu 2}}}$	$\begin{pmatrix}(0.7, 0.5)\\(0.5, 0.4)\\(0.8, 0.2)\end{pmatrix}$	$\begin{pmatrix}(0.71, 0.51)\\(0.51, 0.41)\\(0.81, 0.21)\end{pmatrix}$	$\begin{pmatrix}(0.72, 0.52)\\(0.52, 0.42)\\(0.82, 0.22)\end{pmatrix}$	$\begin{pmatrix}(0.73, 0.53)\\(0.53, 0.43)\\(0.83, 0.23)\end{pmatrix}$
$\overline{\overline{C_{\mu 3}}}$	$\begin{pmatrix}(0.6, 0.7)\\(0.3, 0.9)\\(0.9, 0.5)\end{pmatrix}$	$\begin{pmatrix}(0.61, 0.71)\\(0.31, 0.91)\\(0.91, 0.51)\end{pmatrix}$	$\begin{pmatrix}(0.62, 0.72)\\(0.32, 0.92)\\(0.92, 0.52)\end{pmatrix}$	$\begin{pmatrix}(0.63, 0.73)\\(0.33, 0.93)\\(0.93, 0.53)\end{pmatrix}$
$\overline{\overline{C_{\mu 4}}}$	$\begin{pmatrix}(0.8, 0.2)\\(0.9, 0.8)\\(0.6, 0.4)\end{pmatrix}$	$\begin{pmatrix}(0.81, 0.21)\\(0.91, 0.81)\\(0.61, 0.41)\end{pmatrix}$	$\begin{pmatrix}(0.82, 0.22)\\(0.92, 0.82)\\(0.62, 0.42)\end{pmatrix}$	$\begin{pmatrix}(0.83, 0.23)\\(0.93, 0.83)\\(0.63, 0.43)\end{pmatrix}$
$\overline{\overline{C_{\mu 5}}}$	$\begin{pmatrix}(0.9, 0.4)\\(0.3, 0.8)\\(0.6, 0.7)\end{pmatrix}$	$\begin{pmatrix}(0.91, 0.41)\\(0.31, 0.81)\\(0.61, 0.71)\end{pmatrix}$	$\begin{pmatrix}(0.92, 0.42)\\(0.32, 0.82)\\(0.62, 0.72)\end{pmatrix}$	$\begin{pmatrix}(0.93, 0.43)\\(0.33, 0.83)\\(0.63, 0.73)\end{pmatrix}$

Step 2: To apply the proposed CPF-AAP-A, CPF-AAWP-A, CPF-AAP-G, and CPF-AAWP-G operators, the primary focus is on consolidating the decision information into a unified neutrosophic soft set-based framework. Table 9 presents the relevant data of power weight vectors, derived from the CSV-NS structure, where each cryptographic protocol alternative $\overline{\overline{\Psi_i}}$ is evaluated against security criteria $\overline{\overline{C_{\mu j}}}$ using consistent weighting schemes.

Utilizing these power aggregated vectors from Table 9, we compute the aggregated decision values, which are displayed in Table 10. This table encapsulates the outcomes under each operator (CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, and CPF-AAWP-G) using neutrosophic value pairs, thereby enabling a robust, uncertainty-aware evaluation of encryption and decryption protocols. This evaluation approach is particularly aligned with the objectives of ML-based analysis within banking cybersecurity.

TABLE 9. Covered Data of Power Weight Vectors

	$\overline{\overline{C_{\mu 1}}}$	$\overline{\overline{C_{\mu 2}}}$	$\overline{\overline{C_{\mu 3}}}$	$\overline{\overline{C_{\mu 4}}}$	$\overline{\overline{C_{\mu 5}}}$
$\overline{\overline{\Psi_1}}$	3.9949	3.9949	3.9949	3.9949	3.9949
$\overline{\overline{\Psi_2}}$	3.9747	3.9747	3.9747	3.9747	3.9747
$\overline{\overline{\Psi_3}}$	3.9747	3.9747	3.9747	3.9747	3.9747
$\overline{\overline{\Psi_4}}$	3.9949	3.9949	3.9949	3.9949	3.9949

Step 3: Using the ideas of scoring functions (Definition 2.6, we translate the aggregated values of the choices into interpretable real-valued measures). Table 11 displays the calculated score information for each option using different methods (CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, and CPF-AAWP-G), highlighting how well the ML-based encryption and decryption system works to lower security risks in banking applications.

TABLE 11. Score Values of Alternatives under Different Aggregation Operators

Alternatives	**CPF-AAP-A**	**CPF-AAP-G**	**CPF-AAWP-A**	**CPF-AAWP-G**
$\overline{\overline{C_{\mu 1}}}$	-0.2467	-0.3592	-0.2469	-0.3592
$\overline{\overline{C_{\mu 2}}}$	-0.1856	-0.2963	-0.1856	-0.2963
$\overline{\overline{C_{\mu 3}}}$	-0.4004	-0.4909	-0.4004	-0.4909
$\overline{\overline{C_{\mu 4}}}$	-0.5715	-0.6605	-0.5715	-0.6605
$\overline{\overline{C_{\mu 5}}}$	-0.3282	-0.4385	-0.3282	-0.4385

Step 4: We examine the derived score values based on Definition 2.6 and determine the corresponding ranking orders to identify the best option from the limited set of options. The rankings of alternatives under the four suggested aggregation strategies (CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, and CPF-AAWP-G) are shown in Table 12. Remarkably, the alternative $\overline{\overline{C_{\mu 2}}}(RSA - AES\ hybrid)$ continuously achieves the highest score among all techniques, making it the most efficient encryption and decryption strategy for protecting financial transactions. This consistent performance across several operator frameworks amply validates the efficacy and resilience of the suggested CSV-NS-based decision-making model. Furthermore, by consistently identifying the best ways to reduce security threats, it shows how effective our method is at handling complex, uncertain, and multi-criteria cybersecurity environments.

TABLE 10. Covered Aggregated Data

Alternatives	CPF-AAP-A	CPF-AAP-G	CPF-AAWP-A	CPF-AAWP-G
$\overline{\overline{C_{\mu 1}}}$	$\begin{pmatrix}(0.9335, \\ 0.6841), \\ (0.7995, \\ 0.4837), \\ (0.6928, \\ 0.3819)\end{pmatrix}$	$\begin{pmatrix}(0.9151, \\ 0.5874), \\ (0.8425, \\ 0.5469), \\ (0.7469, \\ 0.4438)\end{pmatrix}$	$\begin{pmatrix}(0.9335, \\ 0.6841), \\ (0.7995, \\ 0.4837), \\ (0.6928, \\ 0.3819)\end{pmatrix}$	$\begin{pmatrix}(0.9151, \\ 0.5874), \\ (0.8425, \\ 0.5469), \\ (0.7469, \\ 0.4438)\end{pmatrix}$
$\overline{\overline{C_{\mu 2}}}$	$\begin{pmatrix}(0.7469, \\ 0.5469), \\ (0.4837, \\ 0.3819), \\ (0.7995, \\ 0.1857)\end{pmatrix}$	$\begin{pmatrix}(0.6928, \\ 0.4837), \\ (0.5469, \\ 0.4438), \\ (0.8425, \\ 0.2323)\end{pmatrix}$	$\begin{pmatrix}(0.7469, \\ 0.5469), \\ (0.4837, \\ 0.3819), \\ (0.7995, \\ 0.1857)\end{pmatrix}$	$\begin{pmatrix}(0.6928, \\ 0.4837), \\ (0.5469, \\ 0.4438), \\ (0.8425, \\ 0.2323)\end{pmatrix}$
$\overline{\overline{C_{\mu 3}}}$	$\begin{pmatrix}(0.6841, \\ 0.7469), \\ (0.2824, \\ 0.9151), \\ (0.9151, \\ 0.4837)\end{pmatrix}$	$\begin{pmatrix}(0.5874, \\ 0.6928), \\ (0.3390, \\ 0.9335), \\ (0.9335, \\ 0.5469)\end{pmatrix}$	$\begin{pmatrix}(0.6841, \\ 0.7469), \\ (0.2824, \\ 0.9157), \\ (0.9151, \\ 0.4837)\end{pmatrix}$	$\begin{pmatrix}(0.5874, \\ 0.6928), \\ (0.3390, \\ 0.9335), \\ (0.9335, \\ 0.5469)\end{pmatrix}$
$\overline{\overline{C_{\mu 4}}}$	$\begin{pmatrix}(0.8425, \\ 0.2323), \\ (0.9151, \\ 0.7995), \\ (0.6928, \\ 0.3819)\end{pmatrix}$	$\begin{pmatrix}(0.7995, \\ 0.1857), \\ (0.9335, \\ 0.8425), \\ (0.7469, \\ 0.4438)\end{pmatrix}$	$\begin{pmatrix}(0.8425, \\ 0.2323), \\ (0.9151, \\ 0.7995), \\ (0.6928, \\ 0.3819)\end{pmatrix}$	$\begin{pmatrix}(0.7995, \\ 0.1857), \\ (0.9335, \\ 0.8425), \\ (0.7469, \\ 0.4438)\end{pmatrix}$
$\overline{\overline{C_{\mu 5}}}$	$\begin{pmatrix}(0.9335, \\ 0.4438), \\ (0.2824, \\ 0.7995), \\ (0.5874, \\ 0.6928)\end{pmatrix}$	$\begin{pmatrix}(0.9151, \\ 0.3819), \\ (0.3390, \\ 0.8425), \\ (0.6841, \\ 0.7469)\end{pmatrix}$	$\begin{pmatrix}(0.8425, \\ 0.2323), \\ (0.9151, \\ 0.7995), \\ (0.6928, \\ 0.3819)\end{pmatrix}$	$\begin{pmatrix}(0.9151, \\ 0.3819), \\ (0.3390, \\ 0.8425), \\ (0.6841, \\ 0.7469)\end{pmatrix}$

In the framework of ML-based evaluation of encryption and decryption methods for banking security via CSV-NS aggregation, to validate effectiveness and enable meaningful comparisons, we derive the score values (Definition 2.6) from the aggregated CSV-NS data, excluding phase components, as shown in Table 13. These calculated score values are used to rank the different options using different methods under the CPF-AAP and CPF-AAWP schemes, which include

Table 12. Ranking of Alternatives under Different Aggregation Operators

Aggregation Operator	**Ranking Order**
CPF-AAP-A	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
CPF-AAP-G	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
CPF-AAWP-A	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
CPF-AAWP-G	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$

both arithmetic (A) and geometric (G) approaches. The resulting preference orderings are detailed comprehensively in Table 14.

Table 13. Covered Score Values (without phase terms)

Alternatives	CPF-AAP-A AO	CPF-AAP-G AO	CPF-AAWP-A AO	CPF-AAWP-G AO
$\overline{\overline{C_{\mu1}}}$	-0.2794	-0.3371	-0.2794	-0.3371
$\overline{\overline{C_{\mu2}}}$	-0.2688	-0.3483	-0.2681	-0.3483
$\overline{\overline{C_{\mu3}}}$	-0.2747	-0.3425	-0.2747	-0.3425
$\overline{\overline{C_{\mu4}}}$	-0.3827	-0.4404	-0.3827	-0.4404
$\overline{\overline{C_{\mu5}}}$	-0.3185	-0.0540	-0.0318	-0.0540

The available methods are then ranked by their scores so that the best encryption and decryption options can be identified, as is shown in Table 14. The data shows that $\overline{\overline{C_{\mu2}}}(RSA-AES\ hybrid)$ gives better results than the others under every aggregation strategy.

Table 14. Covered Ranking Values

Method	Ranking Order
CPF-AAP-A	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu4}}}$
CPF-AAP-G	$\overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu4}}}$
CPF-AAWP-A	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu4}}}$
CPF-AAWP-G	$\overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu4}}}$

$\overline{\overline{C_{\mu2}}}(RSA - AES\ hybrid)$ was found to be the most utilized encryption and decryption method by our evaluations. To enhance insight into the proposed method, we check how varying the phase parameter Z within the CSV-NS data and including or excluding its information influences the results. How the results are affected by each parameter is shown in Table 15 for the CPF-AAP and CPF-AAWP methods using arithmetic and geometric aggregation.

TABLE 15. Evaluation of Parameters for Different Values (with phase data)

Parameter	Operator	Score Values	Ranking Order
Z = 1	CPF-AAP-A	0.5486, 0.5794, 0.3023, 0.3099, 0.5346	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu4}}} > \overline{\overline{C_{\mu3}}}$
	CPF-AAP-G	-1.1224, -1.1675, -1.2623, -1.2354, -1.1275	$\overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu4}}} > \overline{\overline{C_{\mu3}}}$
	CPF-AAWP-A	0.5486, 0.5794, 0.3023, 0.3099, 0.5346	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu4}}} > \overline{\overline{C_{\mu3}}}$
	CPF-AAWP-G	-1.1224, -1.1675, -1.2623, -1.2354, -1.1275	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu4}}} > \overline{\overline{C_{\mu3}}}$
Z = 3	CPF-AAP-A	-0.2467, -0.1856, -0.4004, -0.5715, -0.3282	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
	CPF-AAP-G	-0.3592, -0.2963, -0.4909, -0.6605, -0.4385	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
	CPF-AAWP-A	-0.2467, -0.1856, -0.4004, -0.5715, -0.3282	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
	CPF-AAWP-G	-0.3592, -0.2963, -0.4909, -0.6605, -0.4385	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
Z = 5	CPF-AAP-A	-0.4583, -0.4125, -0.5936, -0.7269, -0.5224	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
	CPF-AAP-G	-0.1543, -0.0867, -0.2988, -0.4539, -0.1971	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
	CPF-AAWP-A	-0.4583, -0.4125, -0.5936, -0.7269, -0.5224	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$
	CPF-AAWP-G	-0.1543, -0.0867, -0.2988, -0.4539, -0.1971	$\overline{\overline{C_{\mu2}}} > \overline{\overline{C_{\mu5}}} > \overline{\overline{C_{\mu1}}} > \overline{\overline{C_{\mu3}}} > \overline{\overline{C_{\mu4}}}$

The results indicate that the proposed CSV-NS model behaves consistently, with the alternative $\overline{\overline{C_{\mu2}}}$ remaining the optimal solution in different situations and aggregation techniques. According to the CSV-NS-based aggregation framework, the best method is found to be $\overline{\overline{C_{\mu2}}}(RSA - AES\ hybrid)$, as was expected by the model. We evaluate and present the results of using alternative methods such as $\overline{\overline{C_{\mu1}}}(AES)$ and $\overline{\overline{C_{\mu2}}}(RSA - AES\ hybrid)$, in different operator types and settings without phase data, as shown in Table 16. Comparing these variant configurations (CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, CPF-AAWP-G) and the parameter Z together with the proposed ML-based decision structure reveals which candidate methods rank higher. All compositions demonstrate strong design and the same conclusions, thus confirming the appropriate use of the multi-criteria decision-making model in cybersecurity protocol evaluation.

TABLE 16. Evaluation of Parameters for Different Values (without phase data)

Parameter	Operators	Score Value	Ranking Value
$Z = 1$	CPF-AAP-A	0.3297, 0.3736, 0.2288, 0.1807, 0.4825	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 1$	CPF-AAP-G	-0.7296, -0.8753, -0.9462, -0.8786, -0.6926	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 4}}} > \overline{\overline{C_{\mu 3}}}$
$Z = 1$	CPF-AAWP-A	0.3297, 0.3736, 0.2288, 0.1807, 0.4825	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 1$	CPF-AAWP-G	-0.7296, -0.8753, -0.9462, -0.8786, -0.6926	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 4}}} > \overline{\overline{C_{\mu 3}}}$
$Z = 3$	CPF-AAP-A	-0.2794, -0.2688, -0.2747, -0.3827, -0.3185	$\overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 3$	CPF-AAP-G	-0.3371, -0.3483, -0.3425, -0.4404, -0.054	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 3$	CPF-AAWP-A	-0.2794, -0.2688, -0.2747, -0.3827, -0.3185	$\overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 3$	CPF-AAWP-G	-0.3371, -0.3483, -0.3425, -0.4404, -0.054	$\overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 5$	CPF-AAP-A	-0.3853, -0.4167, -0.4279, -0.4940, -0.1411	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 5$	CPF-AAP-G	-0.2008, -0.1836, -0.2017, -0.3095, 0.0849	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 5$	CPF-AAWP-A	-0.3853, -0.4167, -0.4279, -0.4940, -0.1411	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$
$Z = 5$	CPF-AAWP-G	-0.2008, -0.1836, -0.2017, -0.3095, 0.0849	$\overline{\overline{C_{\mu 5}}} > \overline{\overline{C_{\mu 2}}} > \overline{\overline{C_{\mu 1}}} > \overline{\overline{C_{\mu 3}}} > \overline{\overline{C_{\mu 4}}}$

5.1. *Comparison with Existing Methods*

The proposed CPF-based aggregation models (CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, and CPF-AAWP-G) exhibit superior performance over conventional MCDM methods such as TOPSIS, VIKOR, and CODAS in the ML-based evaluation of encryption and decryption protocols for banking security. Unlike traditional models, which often fall short in effectively managing the multi-dimensional uncertainty inherent in cybersecurity applications, our models leverage CSV-NS and five-way decision-making logic to better handle hesitancy, indeterminacy, and vagueness. The framework gives better stability of results with different protocol options and offers increased transparent decision-making. It becomes especially necessary in cybersecurity, since choosing a suitable protocol requires it to be tough, straightforward, and easy to explain. Importantly, all aggregation models stress that the encryption/decryption protocols $\overline{\overline{C_{\mu 2}}}$(RSA-AES hybrid) and $\overline{\overline{C_{\mu 5}}}$(Homographic encryption) remain the most secure in all cases, matching what experts have said. A detailed comparison of the new approach to existing techniques appears in Table 17.

TABLE 17. Comparison of Proposed Methods with Classical MCDM Techniques in ML-Based Banking Security

Criteria	**CPF-Based Models (AAP-A, AAP-G, AAWP-A, AAWP-G)**	**TOPSIS**	**VIKOR**	**CODAS**
Uncertainty handling	Excellent (via CSV-NS)	Poor (crisp/fuzzy)	Moderate (with fuzzy extension)	Moderate (partial handling)
Ranking stability	High (via AAP/AAWP logic)	Moderate (ideal/anti-ideal shifts)	Moderate (sensitive to weights)	Moderate (sensitive to metrics)
Hesitancy and indeterminacy	Yes (fully captured by CSV-NS)	No	Partial	No
Interpretability	High (enabled by five-way logic)	Moderate	Moderate	Moderate
Aggregation strategy	Average and geometric operators	Euclidean distance-based	Utility-regret measures	Distance-oriented approach
Suitability for cybersecurity	Very high (robust for protocol assessment)	Low	Moderate	Moderate
Best model consistency	High (e.g., $\overline{\overline{C_{\mu 2}}}(RSA-AEShybrid)$, $\overline{\overline{C_{\mu 5}}}(Homomorphicencryption)$)	Inconsistent	Inconsistent	Inconsistent

5.2. *Comparative Sensitivity Analysis*

To evaluate the robustness of the proposed CSV-NS aggregation operators, we examine the impact of parameter Z (phase term coefficient) and operator types (CPF-AAP/CPF-AAWP, A/G) on alternative rankings. The assessment is done for $Z = \{1, 3, 5\}$, with and without phase terms.

With Phase Terms (Table 15): When phase terms are taken into account, the study shows that there is considerable consistency between operators. For both $Z = 3$ and $Z = 5$, all aggregation approaches (CPF-AAP-A, CPF-AAP-G, CPF-AAWP-A, CPF-AAWP-G) consistently identify $\overline{\overline{C_{\mu 2}}}$ (RSA-AES hybrid) as the top-performing alternative, underlining its resilience for banking security applications. Minor deviations in lower-ranked alternatives are found, namely, $\overline{\overline{C_{\mu 1}}}$ (AES) and $\overline{\overline{C_{\mu 5}}}$ (Homomorphic Encryption) interchange positions for $Z = 5$ under geometric operators (CPF-AAP-G/CPF-AAWP-G). Despite a consistent reduction in absolute score values as Z grows (e.g., CPF-AAP-A scores drop from 0.5486 to -0.4583), the relative rankings stay mostly stable, demonstrating that the phase term's influence is more prominent in magnitude than in ordinal preference. This stability emphasizes the acceptability of $\overline{\overline{C_{\mu 2}}}$ as the best choice under variable parameters.

Without Phase Terms (Table 16): The research reveals a considerable operator sensitivity, where geometric aggregation operators (G) prefer $\overline{\overline{C_{\mu 5}}}$, indicating the homomorphic encryption scheme, especially since the phase parameter Z is set to 1 or 5. In comparison, arithmetic operators (A) continuously select $\overline{\overline{C_{\mu 2}}}$ as the best-performing encryption technique. Furthermore, the phase parameter Z has an effect on the parameter impact analysis, as increasing Z enhances the distinction between alternatives. For example, in the CPF-AAP-A established. the score for $\overline{\overline{C_{\mu 1}}}$ varies from -0.2794 to -0.3853 when Z grows, showing the sensitivity of the ranking outcomes to phase alterations.

Performance Insights from Aggregation Results: ML techniques for secure banking collaborate with the use of CPF-based aggregation models to provide robust insights into encryption and decryption methods. Most of the time, 75% to be exact, an assessment finds that the alternative $\overline{\overline{C_{\mu 2}}}$ represents the best cryptographic way in high-risk cybersecurity assessments. Secondly, changing the phase component (e.g., choosing $Z = 1, 3, 5$) greatly decreases variability in the scores of operators. The CPF-AAWP-A operator shows little variation in its decision outcomes between different phase values. Besides, arithmetic operators fit well with theoretical papers, showing they can manage complex situations involving the CSV-NS structure more effectively than geometric operators.

The best operators for each phase and value of Z are all highlighted in Table 18, where $\overline{\overline{C_{\mu 2}}}$ keeps appearing at the top.

TABLE 18. Top-Ranked Encryption Alternatives Across Operators and Phase Values Z

Operator	$\mathbf{Z=1}$	$\mathbf{Z=3}$	$\mathbf{Z=5}$
CPF-AAP-A	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$
CPF-AAP-G	$\overline{\overline{C_{\mu1}}}$	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$
CPF-AAWP-A	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$
CPF-AAWP-G	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$	$\overline{\overline{C_{\mu2}}}$

6. **Conclusion and Future Work**

The ideas behind CSV-NSs help model situations with a lot of uncertainty, which is very important in cybersecurity for tasks like banking encryption. Because they contain values for truth, indeterminacy, and falsity, CSV-NSs make it possible to represent contradictory information in many ways. In this study, we introduced four novel aggregation operators (CPF-AAP-A, CPF-AAWP-A, CPF-AAP-G, and CPF-AAWP-G) based on AczelAlsina t-norms/t-conorms and power-weighted strategies. These were embedded into a MADM framework tailored for evaluating encryption and decryption protocols under ML-driven uncertainty environments. A real-world application was demonstrated in the context of ML-based encryption and decryption systems for banking security. When faced with the same multi-entities, the model usually gave clear results (e.g., $\overline{\overline{C_{\mu2}}}(RSA - AES\ hybrid)$, $\overline{\overline{C_{\mu5}}}(Homomorphic\ encryption)$) and consistently outperformed TOPSIS, VIKOR, and CODAS in terms of stability, handling uncertainty, and clarity, as shown in Table 17. The results back up the usefulness, accuracy, and proper alignment of the model with expert advice.

Future Work

The framework our research proposes, built on CSV-NS technology, supports many future efforts and real-world solutions, mainly in the area of secure banking technologies.

- *AI-Driven Cryptographic Systems:* Combining deep learning and reinforcement learning to allow encryption methods to spot threats and alter their effectiveness automatically and immediately.
- *Secure Model Ranking and Blockchain:* Using CPF-based techniques to select cryptographic schemes used in blockchain validation of transactions and contracts and in financial pipeline safety.
- *Authentication and Key Management:* Putting the proposed model into practice to judge and compare secure distribution and biometric-based authentication protocols in uncertain attack conditions.

- *Banking Fraud Detection:* In analyzing anomaly detection algorithms for finding banking fraud, CSV-NS theory is applied, mixing uncertainty reasoning and ML methods for classification.
- *Cyber Threat Intelligence (CTI):* Expanding trust analysis in threat systems by incorporating neutrosophic logic for sources, methods of attack, and weaknesses in protocols.
- *Regulatory Compliance and Risk Assessment:* Creating MADM tools that explain well how cryptography fits with banking requirements (PCI DSS and GDPR), by including models that consider uncertainty and expert thoughts.
- *Multi-Layer Security Decision Support:* Developing hybrids of decision-support systems that apply CSV-NS logic to study security performance at the application, network, and hardware layers within financial organizations.
- *Theoretical Advancements:* The CSV-NS approach has been proposed for developing neutrosophic structures and bipolar frameworks to deal with both opposite risk results and tricky decision conditions.

Overall, the CSV-NS-based decision framework allows for stronger and more useful evaluations of encryption and decryption protocols in banking security. It points to a promising future for intelligent, safe, and uncertainty-driven decision systems used in both financial and cyber security areas.

Declaration of Interests

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
The authors also declare that there is no conflict of interests regarding the publication of this paper.
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Integral Type Contractive Condition in ε-Chainable Neutrosophic Metric Space and Common Fixed Point Theorem

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Abstract: Nonlinear Optimization, Game theory, economics and study of differential equations are just a few of the many domains in which fixed point theory (FPT) is essential. It is possible to construct new forms of infinite products by using continuous triangular norms (TN) and continuous triangular co-norms (TC). Banach contraction principal has been established in the context of neutrosophic metric space (NMS) within the framework through the use of these newly define infinite products. We introduced integral type contractive condition in ε-chainable NMS and establishes a common fixed point theorems (CFPTs) in the current work. The result acquired in this study are intended to consolidate and expand upon numerous existing discoveries in the field of NMS.

Key Words: Integral type contractive condition, ε-chainable neutrosophic metric space, Common fixed point theorem

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1. Introduction

In 1965, Zadeh [37] introduced the fuzzy set (FS) as a set that is defined by a membership function, serving as the first mathematical formalization of the concept. Although fundamental, the single membership grade of an FS may not be adequate to capture the complete complexity of real-world uncertainty. Kramosil and Michálek [18] introduced fuzzy metric spaces (FMS) that were subsequently refined by George and

Veeramani [9] using continuous TC, building upon this concept. The concept on FMS is explored by Grabiec [8] in 1988.

Atanassov [1, 2] introduced the intuitionistic fuzzy set (IFS) in 1986 to expand the expressive capability of FS theory. Independent degrees of both belongingness and non-belongingness for each element are incorporated into this framework to more effectively model uncertainty. The application of IFS theory rapidly penetrated all domains that were impacted by FS, including metric spaces. By employing the TN and TC structure of George and Veeramani, Park [24] extended the concept of FMS to intuitionistic fuzzy metric spaces (IFMS) and subsequently investigated its fundamental topological properties. For more results on IFS (see [23], [25]).

Heilpern [11] was the first to investigate fuzzy contraction mappings in FPT. Bose and Sahani [5] expanded upon this work, while Alaca et al. [3] demonstrated FPTs in the context of IFMS. Mohamad [21] and a multitude of other researchers [10, 12, 14, 19] have since made substantial contributions to the field of fixed point results for both FMS and IFMS.

Smarandache [29] was the first to introduce the neutrosophic set (NS) in 1998, after recognizing that neither FS nor IFS could entirely resolve issues involving contradictory or indeterminate information. NSs are defined by three independent membership functions: truth (T), indeterminacy (I), and falsity (F) as a generalization of crisp, fuzzy, and intuitionistic sets. Smarandache [30] observed that an NS is reduced to an IFS when its indeterminacy membership grade I(x) is equivalent to the hesitancy degree h(x) of the IFS. Kirişci et al. [15] contributed to the development of the fixed point theory in NMS by defining neutrosophic contractive mappings and establishing corresponding FPTs.

Branciari's [4] integral version of the Banach contraction principle was a significant parallel development in metric FPT. The results of Rhoades [26], Vijayaraju [32], and Djoudi, Aliouche [7] and Saini [28] demonstrate that this seminal work inspired extensive research, resulting in the development of a variety of CFPTs and FPTs for integral-type contractive conditions in diverse spaces.

Our contribution to this ongoing research is the definition of an integral-type contractive condition within the framework of ε-chainable NMS in this paper. We establish a CFPT for four weakly compatible mappings [13]. Our findings are a significant extension and generalization of several well-known theorems in NMS, such as those of Mohamad [22], Kirisci M. [17], Kirişci et al. [15], and Kirişci and Simsek [16].

2. Preliminaries

The following section presents fundamental definitions concerning fuzziness, intuitionistic fuzziness and neutrosophic concepts.

Definition 2.1: A FS $\tilde{F}$ is defined as a mapping $\tilde{F}: X \to [0,1]$, where X is a universe of discourse.

Definition 2.2. ([37]): For a non-empty set X, a FS $\tilde{F}$ is expressed as $\tilde{F} = \{\langle a, \mu_{\tilde{F}}(a)\rangle : a \in X\}$ where $\mu_{\tilde{F}}(a)$ is the membership function that assigns to each element $a \in X$ a degree of membership in [0,1]. If the FS $\tilde{F}$ is both convex and normalized, then it is referred to as a fuzzy number (FN) on real line $\mathbb{R}$.

Definition 2.3. ([1]): An IFS $\tilde{F}^I$ in a non-empty set X is represented as $\tilde{F}^I = \left\{\left\langle a, \mu_{\tilde{F}^I}(a), \nu_{\tilde{F}^I}(a)\right\rangle : a \in X\right\}$, where $\mu_{\tilde{F}^I} : X \to [0,1]$ denotes the membership function and $\nu_{\tilde{F}^I} : X \to [0,1]$ denotes the non-membership function with condition $\mu_{\tilde{F}^I} + \nu_{\tilde{F}^I} \le 1, \forall a \in X.$ The hesitation or indeterminacy degree is given by $h(a) = 1 - \mu_{\tilde{F}^I}(a) - \nu_{\tilde{F}^I}(a).$ An IFS $\tilde{F}^I$ becomes intuitionistic fuzzy number (IFN), if

- An IFN is a special type of subset of the $\mathbb{R}$,

- An IFN is said to be normal if $\mu_{\tilde{F}^I}(a) = 1$ and $\nu_{\tilde{F}^I}(a) = 0$ for each $a \in R$,
- The membership function $\mu_{\tilde{F}^I}(a)$ is considered convex, if for any $a_1, a_2 \in R, \gamma \in [0,1]$, we have $\mu_{\tilde{F}^I}\left(\gamma a_1 + (1-\gamma)a_2\right) \geq \min\left\{\mu_{\tilde{F}^I}(a_1), \mu_{\tilde{F}^I}(a_2)\right\}$.
- The membership function $\mu_{\tilde{F}^I}(a)$ is considered concave, if for any $a_1, a_2 \in R, \gamma \in [0,1]$, we have $\mu_{\tilde{F}^I}\left(\gamma a_1 + (1-\gamma)a_2\right) \leq \max\left\{\mu_{\tilde{F}^I}(a_1), \mu_{\tilde{F}^I}(a_2)\right\}$.
- $\mu_{\tilde{F}^I}(a)$ $\mu_{\tilde{F}^I}$ is assumed to be upper semi continuous while $\nu_{\tilde{F}^I}$ is assumed as lower semi continuous,
- $supp\,\mu_{\tilde{F}^I}(a) = cl\left(\{a \in \tilde{F}^I; \mu_{\tilde{F}^I} < 1\}\right)$

An IFS $\tilde{F}^I = \left\{\left\langle x, \mu_{\tilde{F}^I}(x), \nu_{\tilde{F}^I}(x)\right\rangle : x \in X\right\}$ s.t. $\mu_{\tilde{F}^I}(a)$ and $1-\nu_{\tilde{F}^I}(a)$ are IFNs, where $(1-\nu_{\tilde{F}^I})(a) = 1-\nu_{\tilde{F}^I}(a)$ and $\mu_{\tilde{F}^I}(a) + \mu_{\tilde{F}^I}(a) \leq 1$ is called an IFN.

Definition 2.4. ([29]): Let X be non-empty set and $a \in X$. A NS $\tilde{F}_N$ is expressed as $\tilde{F}_N = \left\{\langle a, \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\rangle : a \in X\right\}$, for each number a in X and $\mu_{\tilde{F}_N}(a)$, $\nu_{\tilde{F}_N}(a)$ and $\omega_{\tilde{F}_N}(a)$ belongs $]^-0,1^+[$ where $\mu_{\tilde{F}_N}(a) : X \to]0^-,1^+[$ represents the truth membership (TM), $\nu_{\tilde{F}_N}(a) : X \to]0^-,1^+[$ represents the indeterminacy membership (IM) and $\omega_{\tilde{F}_N}(a) : X \to]0^-,1^+[$ represents the falsity membership (FM) in $\tilde{F}_N$ respectively with condition $^-0 \leq \mu_{\tilde{F}_N}(a) + \nu_{\tilde{F}_N}(a) + \omega_{\tilde{F}_N}(a) \leq 3^+$.

In 2010, Wang et.al [34, 35] and Deli & S¸uba [6] introduce the single valued neutrosophic numbers (SVNN) which provides a fundamental for applying neutrosophic theory in practical settings. Later Ye [36], introduced the notion of simplified NSs, characterized by three real-valued components within [0,1]. However the improved NSs' operations may be impractical at certain times.

Definition 2.5. Let X be non-empty set and $a \in \square$. A NS in $\square$ is represented as $\tilde{F}_N = \left\{\langle a, \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\rangle : a \in \square\right\}$, for each number $a \in \square$, and $\mu_{\tilde{F}_N}(a)$, $\nu_{\tilde{F}_N}(a)$, $\omega_{\tilde{F}_N}(a)$ belongs to $]^-0,1^+[$ where $\mu_{\tilde{F}_N}(a) : X \to]0^-,1^+[$ represents the TM, $\nu_{\tilde{F}_N}(a) : X \to]0^-,1^+[$ represents the IM and $\omega_{\tilde{F}_N}(a) : X \to]0^-,1^+[$ represents the FM in $\tilde{F}_N$ respectively with condition $0 \leq \mu_{\tilde{F}_N}(a) + \nu_{\tilde{F}_N}(a) + \omega_{\tilde{F}_N}(a) \leq 3$. For continuous SVNS, $\tilde{F}_N = \int_{\tilde{F}_N} \left\langle \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\right\rangle \big/ a : a \in \square$ If X is discrete then SVNS $\tilde{F}_N = \sum_{i=1}^{n} \left\langle \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\right\rangle \big/ a_i : a_i \in \square$.
If NS has only one element then in simplified form $\tilde{F}_N$ express as $\left\langle \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\right\rangle$ for each $a \in \square$.

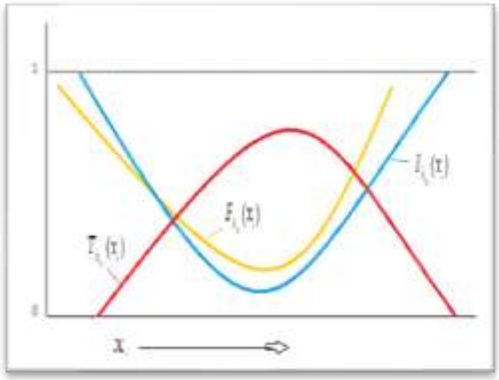

Figure 1: Neutrosophic set

Thus it is evident that NS extends the concept of IFSs within [0,1]. For all $a \in \tilde{F}_N$, a NS $\tilde{F}_N(V)$ contains the NS $\tilde{F}_N(U)$, $(U \subseteq V)$

$$\begin{pmatrix} \inf\left(\mu_{\tilde{F}_N(U)}(a)\right) \le \inf\left(\mu_{\tilde{F}_N(V)}(a)\right) \text{ and } \sup\left(\mu_{\tilde{F}_N(U)}(a)\right) \le \sup\left(\mu_{\tilde{F}_N(V)}(a)\right) \\ \inf\left(\nu_{\tilde{F}_N(U)}(a)\right) \ge \inf\left(\nu_{\tilde{F}_N(V)}(a)\right) \text{ and } \sup\left(\nu_{\tilde{F}_N(U)}(a)\right) \ge \sup\left(\nu_{\tilde{F}_N(V)}(a)\right) \\ \inf\left(\omega_{\tilde{F}_N(U)}(a)\right) \ge \inf\left(\omega_{\tilde{F}_N(V)}(a)\right) \text{ and } \sup\left(\omega_{\tilde{F}_N(U)}(a)\right) \ge \sup\left(\omega_{\tilde{F}_N(V)}(a)\right) \end{pmatrix}$$

$$\tilde{F}_N(U) + \tilde{F}_N(V) = \left\langle \begin{matrix} \mu_{\tilde{F}_N(U)}(a) + \mu_{\tilde{F}_N(V)}(a) - \mu_{\tilde{F}_N(U)}(a).\mu_{\tilde{F}_N(V)}(a), \\ \nu_{\tilde{F}_N(U)}(a) + \nu_{\tilde{F}_N(V)}(a) - \nu_{\tilde{F}_N(U)}(a).\nu_{\tilde{F}_N(V)}(a), \\ \omega_{\tilde{F}_N(U)}(a) + \omega_{\tilde{F}_N(V)}(a) - \omega_{\tilde{F}_N(U)}(a).\omega_{\tilde{F}_N(V)}(a) \end{matrix} \right\rangle$$

$$\tilde{F}_N(U).\tilde{F}_N(V) = \left\langle \mu_{\tilde{F}_N(U)}(a).\mu_{\tilde{F}_N(V)}(a), \nu_{\tilde{F}_N(U)}(a).\nu_{\tilde{F}_N(V)}(a), \omega_{\tilde{F}_N(U)}(a).\omega_{\tilde{F}_N(V)}(a) \right\rangle$$

$$\alpha.\tilde{F}_N(U) = \left\langle 1-\left(1-\mu_{\tilde{F}_N(U)}(a)\right)^{\alpha}, 1-\left(1-\nu_{\tilde{F}_N(U)}(a)\right)^{\alpha}, 1-\left(1-\omega_{\tilde{F}_N(U)}(a)\right)^{\alpha} \right\rangle, \text{ for } \alpha > 0,$$

$$\left(\tilde{F}_N(U)\right)^{\alpha} = \left\langle \mu^{\alpha}_{\tilde{F}_N(U)}(a), \nu^{\alpha}_{\tilde{F}_N(U)}(a), \omega^{\alpha}_{\tilde{F}_N(U)}(a) \right\rangle, \text{ for } \alpha > 0.$$

Definition 2.6. A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is continuous TN if '$*$' is satisfying:

(i) $*$ is commutative and associative,

(ii) $*$ is continuous,

(iii) $a*1=a$ for all $a\in[0,1]$,

(iv) $a*b=c*d$ whenever $a\le c$ and $b\le d$, $\forall\, a,b,c,d\in[0,1]$.

Definition 2.7. A binary operation $\Diamond:[0,1]\times[0,1]\to[0,1]$ is continuous TC if $\Diamond$ is satisfying:

(i) $\Diamond$ is commutative and associative,

(ii) $\Diamond$ is continuous,

(iii) $a\Diamond 0=a$ for all $a\in[0,1]$

(iv) $a\Diamond b=c\Diamond d$ whenever $a\le c=b\le d$ $\forall a,c,b,d\in[0,1]$.

From above definitions, we note that if we choose $0<\varepsilon_1,\varepsilon_2<1$ for $\varepsilon_1>\varepsilon_2$, then $\exists$ $0<\varepsilon_3,\varepsilon_4<1$ s.t. $\varepsilon_1*\varepsilon_3\ge\varepsilon_2$ and $\varepsilon_1\ge\varepsilon_4\Diamond\varepsilon_2$. Further if we choose $\varepsilon_5\in(0,1)$, then $\exists$ $\varepsilon_6,\varepsilon_7\in(0,1)$ s.t. $\varepsilon_6*\varepsilon_6\ge\varepsilon_5$ and

$\varepsilon_7 \diamondsuit \varepsilon_7 \geq \varepsilon_5$.

Remark 2.1 [23]. For $l,m,n,p \in [0,1]$, take $*$ and $\diamondsuit$ are continuous TN and TC, respectively

(i) If $l > m,$ then there are n,p s.t. $l*n \geq m$ and $l \geq m \diamondsuit p.$

(ii) There are s,m ss.t. $m*m \geq l$ and $l \geq s \diamondsuit s.$

Example 2.1 [7]: Assume $X = N$. Define $x*y = \max\{0, x+y-1\}$, $\forall x,y \in [0,1]$. Again let $\tilde{F}$ be FS on $X \times X \times (0,\infty)$ is defined as $\tilde{F}(a,b,t) = \begin{cases} a/b, & \text{if } a \leq b \\ b/a, & \text{if } a \geq b \end{cases}$, $\forall a,b \in X,\ t > 0.$ Then $(X, \tilde{F}, *)$ is a FMS described as $\tilde{F}(a,b,t) = \dfrac{t}{t + d(a,b)}$, where $d(a,\ b)$ is a MS in $X,\ \forall a,b \in X.$

Remark 2.2: Every FMS $(X, \tilde{F}, *)$ is an IFS of the form $(X, \tilde{F}, 1-\tilde{F}, *, \diamondsuit)$ s.t. '$*$' a TN and '$\diamondsuit$' a TC are associated, i.e. $a*b = 1-(1-a)\diamondsuit(1-b),\ \forall\ a,b \in X.$

Definition 2.8 ([22]). Let $\tilde{F}_N = \{\langle a, \mu_{\tilde{F}_N}(a), \nu_{\tilde{F}_N}(a), \omega_{\tilde{F}_N}(a)\rangle / a \in X\}$, be a NS for an arbitrary set X s.t. $\tilde{F}_N = X \times X \times R^+ \to [0,1]$. Let $*$ and $\diamondsuit$ are continuous TN and continuous TC, respectively. The four tuples $\Im = (X, \tilde{F}_N, *, \diamondsuit)$ is said to be a NMS, when the following conditions satisfied for all $a,b,c \in X$,

(i) $0 \leq \mu_{\tilde{F}_N}(a,b,\lambda) \leq 1,\ 0 \leq \nu_{\tilde{F}_N}(a,b,\lambda) \leq 1,\ 0 \leq \omega_{\tilde{F}_N}(a,b,\lambda) \leq 1, \quad \forall \lambda \in R^+,$

(ii) $0 < \mu_{\tilde{F}_N}(a,b,\lambda) + \nu_{\tilde{F}_N}(a,b,\lambda) + \omega_{\tilde{F}_N}(a,b,\lambda) \leq 3,$ (for $\lambda \in R^+$),

(iii) $\mu_{\tilde{F}_N}(a,b,\lambda) = 1$ (for $\lambda > 0$), iff $a = b$,

(iv) $\mu_{\tilde{F}_N}(a,b,\lambda) = \mu_{\tilde{F}_N}(b,a,\lambda)$ (for $\lambda > 0$)

(v) $\mu_{\tilde{F}_N}(a,c,\lambda+\eta) \geq \mu_{\tilde{F}_N}(a,b,\lambda) * \mu_{\tilde{F}_N}(b,c,\eta)$ (for $\lambda,\eta > 0$),

(vi) $\mu_{\tilde{F}_N}(a,b,.) : [0,\infty) \to [0,1]$ is continuous,

(vii) $\lim_{\lambda \to \infty} \mu_{\tilde{F}_N}(a,b,\lambda) = 1$ ($\forall\ \lambda > 0$),

(viii) $\nu_{\tilde{F}_N}(a,b,\lambda) = 0$ (for $\lambda > 0$), iff $a = b$,

(ix) $\nu_{\tilde{F}_N}(a,b,\lambda) = \nu_{\tilde{F}_N}(b,a,\lambda)$ (for $\lambda > 0$)

(x) $\nu_{\tilde{F}_N}(a,b,\lambda) \diamondsuit \nu_{\tilde{F}_N}(b,c,\eta) \geq \nu_{\tilde{F}_N}(a,c,\lambda+\eta)$ (for $\lambda,\eta > 0$),

(xi) $\nu_{\tilde{F}_N}(a,b,.) : [0,\infty) \to [0,1]$ is continuous,

(xii) $\lim_{\lambda \to \infty} \nu_{\tilde{F}_N}(a,b,\lambda) = 0$ ($\forall\ \lambda > 0$),

(xiii) $\omega_{\tilde{F}_N}(a,b,\lambda) = 0$ (for $\lambda > 0$), iff $a = b$,

(xiv) $\omega_{\tilde{F}_N}(a,b,\lambda) = \omega_{\tilde{F}_N}(b,a,\lambda)$ (for $\lambda > 0$)

(xv) $\omega_{\tilde{F}_N}(a,b,\lambda) \diamondsuit \omega_{\tilde{F}_N}(b,c,\eta) \geq \omega_{\tilde{F}_N}(a,c,\lambda+\eta)$ (for $\lambda,\eta > 0$),

(xvi) $\omega_{\tilde{F}_N}(a,b,.) : [0,\infty) \to [0,1]$ is continuous,

(xvii) $\lim_{\lambda \to \infty} \omega_{\tilde{F}_N}(a,b,\lambda) = 0$ ($\forall\ \lambda > 0$),

(xviii) If $\lambda \leq 0$, then $\mu_{\tilde{F}_N}(a,b,\lambda) = 0,\ \nu_{\tilde{F}_N}(a,b,\lambda) = 1,\ \omega_{\tilde{F}_N}(a,b,\lambda) = 1, \quad \forall \lambda \in R^+,$

Then the set $\tilde{F}_N = (\mu_{\tilde{F}_N}, \nu_{\tilde{F}_N}, \omega_{\tilde{F}_N})$ is called NM on X.

The function $\mu_{\tilde{F}_N}(a,b,\lambda)$ denotes the degree of nearness, $\nu_{\tilde{F}_N}(a,b,\lambda)$ denotes the degree of neutralness and $\omega_{\tilde{F}_N}(a,b,\lambda)$ denotes the degree of non-nearness between a,b with respect to λ.

Definition 2.9 [22]. Let $\Im$ be a NMS, $0<\varepsilon<1$, $\lambda>0$ and $a\in X$. The set $D(a,\varepsilon,\lambda)=\{b\in X:\mu_{\tilde{F}_N}(a,b,\lambda)>1-\varepsilon,\ \nu_{\tilde{F}_N}(a,b,\lambda)<\varepsilon,\ \omega_{\tilde{F}_N}(a,b,\lambda)<\varepsilon\}$ is said to be the open ball (center a and radius ε with respect to λ).

Lemma 2.1 [22]. Every open ball $D(a,\varepsilon,\lambda)$ is open set.

Definition 2.10 [22]. Let $\{a_n\}$ be a sequence in $\Im=(X,\tilde{F}_N,*,\Diamond)$, then the sequence converges to a point $a\in X$ iff for a given $\varepsilon\in(0,1)$, $\lambda>0$ there exists $n_0\in N$ s.t. for all $n\ge n_0$

$$\begin{pmatrix}\mu_{\tilde{F}_N}(a,b,\lambda)>1-\varepsilon, \text{ or } \lim_{n\to\infty}\mu_{\tilde{F}_N}(a_n,a_m,\lambda)=1\\ \nu_{\tilde{F}_N}(a,b,\lambda)<\varepsilon \text{ or } \lim_{n\to\infty}\nu_{\tilde{F}_N}(a_n,a_m,\lambda)=0\\ \omega_{\tilde{F}_N}(a,b,\lambda)<\varepsilon \text{ or } \lim_{n\to\infty}\omega_{\tilde{F}_N}(a_n,a_m,\lambda)=0\end{pmatrix} \text{ as } \lambda\to\infty \tag{1}$$

Definition 2.11 [22]. Let $\Im=(X,\tilde{F}_N,*,\Diamond)$, be a NMS. A sequence $\{a_n\}$ in X is called a Cauchy sequence (CS) if for each $\varepsilon>0,\ \lambda>0$ there exists $n_0\in N$ s.t. $\mu_{\tilde{F}_N}(a_n,b_m,\lambda)>1-\varepsilon,\ \nu_{\tilde{F}_N}(a_n,b_m,\lambda)<\varepsilon,$ and $\omega_{\tilde{F}_N}(a_n,b_m,\lambda)<\varepsilon,$ for all $n,m\ge n_0$. A NMS $\Im$ is called complete if every CS is a convergent sequence.

Lemma 2.2. Let $\{a_n\}$ be a sequence in $\Im=(X,\tilde{F}_N,*,\Diamond)$, with (vii, xii, xviii). If there is a number q where

$$q\in(0,1) \text{ s.t. } \begin{pmatrix}\mu_{\tilde{F}_N}(a_{n+1},a_{n+2},q\lambda)\ge\mu_{\tilde{F}_N}(a_n,a_{n+1},\lambda),\\ \nu_{\tilde{F}_N}(a_{n+1},a_{n+2},q\lambda)\le\nu_{\tilde{F}_N}(a_n,a_{n+1},\lambda),\\ \omega_{\tilde{F}_N}(a_{n+1},a_{n+2},q\lambda)\le\omega_{\tilde{F}_N}(a_n,a_{n+1},\lambda)\end{pmatrix} \text{ for all } \lambda>0 \text{ and } n=0,1,2.......,\tag{2}$$

then $\{a_n\}$ is a CS in X.

Proof: Let p be any positive integer, then by repeated application of (v, x, xv) and in view of (2), we have

$$\begin{aligned}\mu_{\tilde{F}_N}(a_n,a_{n+p},\lambda)&\ge\mu_{\tilde{F}_N}(a_n,a_{n+1},\lambda/2)*\mu_{\tilde{F}_N}(a_{n+1},a_{n+p},\lambda/2)\\&\ge\mu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)*\mu_{\tilde{F}_N}(a_{n+1},a_{n+2},\lambda/2^2)*\mu_{\tilde{F}_N}(a_{n+2},a_{n+p},\lambda/2^2)\\&\ge\mu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)*\mu_{\tilde{F}_N}(a_1,a_2,\lambda/2^2q^{n+1})*\mu_{\tilde{F}_N}(a_2,a_3,\lambda/2^3q^{n+2})*\mu_{\tilde{F}_N}(a_{n+3},a_{n+p},\lambda/2^3)\end{aligned}$$

Continuing this procedure, we obtain

$$\mu_{\tilde{F}_N}(a_n,a_{n+p},\lambda)\ge\mu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)*\mu_{\tilde{F}_N}(a_1,a_2,\lambda/2^2q^{n+1})*\mu_{\tilde{F}_N}(a_2,a_3,\lambda/2^3q^{n+2})*....*\mu_{\tilde{F}_N}(a_{p-1},a_{n+p},\lambda/2^pq^{n+p-1})$$

since * is the continuous TN and $\mu_{\tilde{F}_N}(a,b,.):[0,\infty)\to[0,1]$ is continuous, letting $\lim_{n\to\infty}$ we have

$$\lim_{n\to\infty}\mu_{\tilde{F}_N}(a_n,a_{n+p},\lambda)\ge1*1*1*........*1=1.\tag{I}$$

similarly

$$\begin{aligned}\nu_{\tilde{F}_N}(a_n,a_{n+p},\lambda)&\le\nu_{\tilde{F}_N}(a_n,a_{n+1},\lambda/2)\Diamond\nu_{\tilde{F}_N}(a_{n+1},a_{n+p},\lambda/2)\\&\le\nu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)\Diamond\nu_{\tilde{F}_N}(a_{n+1},a_{n+2},\lambda/2^2)\Diamond\nu_{\tilde{F}_N}(a_{n+2},a_{n+p},\lambda/2^2)\\&\le\nu_{\tilde{F}_N}(a_0,a_1,\lambda/2q^n)\Diamond\nu_{\tilde{F}_N}(a_1,a_2,\lambda/2^2q^{n+1})\Diamond\nu_{\tilde{F}_N}(a_2,a_3,\lambda/2^3q^{n+2})\Diamond\nu_{\tilde{F}_N}(a_{n+3},a_{n+p},\lambda/2^3)\end{aligned}$$

Continuing this procedure, we obtain

$$\nu_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le \nu_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \nu_{\tilde{F}_N}(a_1, a_2, \lambda/2^2 q^{n+1}) \diamond \nu_{\tilde{F}_N}(a_2, a_3, \lambda/2^3 q^{n+2}) \diamond \ldots\ldots \diamond \nu_{\tilde{F}_N}(a_{p-1}, a_{n+p}, \lambda/2^p q^{n+p-1})$$

Since $\diamond$ is continuous TC and $\nu_{\tilde{F}_N}(a,b,.):[0,\infty)\to[0,1]$ is continuous, letting $\lim_{n\to\infty}$ we have

$$\lim_{n\to\infty} \nu_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le 0 \diamond 0 \diamond 0 \diamond \ldots\ldots\ldots \diamond 0 = 0. \quad \text{(II)}$$

and $\omega_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le \omega_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/2) \diamond \omega_{\tilde{F}_N}(a_{n+1}, a_{n+p}, \lambda/2)$

$$\le \omega_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \omega_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/2^2) \diamond \omega_{\tilde{F}_N}(a_{n+2}, a_{n+p}, \lambda/2^2)$$

$$\le \omega_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \omega_{\tilde{F}_N}(a_1, a_2, \lambda/2^2 q^{n+1}) \diamond \omega_{\tilde{F}_N}(a_2, a_3, \lambda/2^3 q^{n+2}) \diamond \omega_{\tilde{F}_N}(a_{n+3}, a_{n+p}, \lambda/2^3)$$

Continuing this procedure, we obtain

$$\omega_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le \omega_{\tilde{F}_N}(a_0, a_1, \lambda/2q^n) \diamond \omega_{\tilde{F}_N}(a_1, a_2, \lambda/2^2 q^{n+1}) \diamond \omega_{\tilde{F}_N}(a_2, a_3, \lambda/2^3 q^{n+2}) \diamond \ldots\ldots \diamond \omega_{\tilde{F}_N}(a_{p-1}, a_{n+p}, \lambda/2^p q^{n+p-1})$$

Since $\diamond$ is CTC and $\omega_{\tilde{F}_N}(a,b,.):[0,\infty)\to[0,1]$ is continuous, letting $\lim_{n\to\infty}$ we have

$$\lim_{n\to\infty} \omega_{\tilde{F}_N}(a_n, a_{n+p}, \lambda) \le 0 \diamond 0 \diamond 0 \diamond \ldots\ldots\ldots \diamond 0 = 0. \quad \text{(III)}$$

From (I), (II) and (III) shows that $\{a_n\}$ is a CS and thus the lemma is proved.

Lemma 2.3: If for all $a,b \in X$, $\lambda > 0$ and for a number $q \in (0,1)$ in NMS $(X, \tilde{F}_N, *, \diamond)$, then

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,q\lambda) \ge \mu_{\tilde{F}_N}(a,b,\lambda), \\ \nu_{\tilde{F}_N}(a,b,q\lambda) \le \nu_{\tilde{F}_N}(a,b,\lambda), \\ \omega_{\tilde{F}_N}(a,b,q\lambda) \le \omega_{\tilde{F}_N}(a,b,\lambda) \end{pmatrix} \Rightarrow a = b.$$

Proof: In view of conditions (*v*, *x*, *xv*), we have $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) \ge \mu_{\tilde{F}_N}(a,b,\lambda/q) * \mu_{\tilde{F}_N}(a,b,\lambda/q^2), \\ \nu_{\tilde{F}_N}(a,b,\lambda) \le \nu_{\tilde{F}_N}(a,b,\lambda/q) * \nu_{\tilde{F}_N}(a,b,\lambda/q^2), \\ \omega_{\tilde{F}_N}(a,b,\lambda) \le \omega_{\tilde{F}_N}(a,b,\lambda/q) * \omega_{\tilde{F}_N}(a,b,\lambda/q^2) \end{pmatrix}$

Proceeding in the same way, we obtain, for $n = 1,2,3\ldots\ldots\ldots$ $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) \ge \mu_{\tilde{F}_N}(a,b,\lambda/q^n), \\ \nu_{\tilde{F}_N}(a,b,\lambda) \le \nu_{\tilde{F}_N}(a,b,\lambda/q^n), \\ \omega_{\tilde{F}_N}(a,b,\lambda) \le \omega_{\tilde{F}_N}(a,b,\lambda/q^n) \end{pmatrix}$.

By noting $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda/q^n) \to 1, \\ \nu_{\tilde{F}_N}(a,b,\lambda/q^n) \to 0, \\ \omega_{\tilde{F}_N}(a,b,\lambda/q^n) \to 0 \end{pmatrix}$ as $n \to \infty$. It follows that $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) = 1, \\ \nu_{\tilde{F}_N}(a,b,\lambda) = 0, \\ \omega_{\tilde{F}_N}(a,b,\lambda) = 0 \end{pmatrix}$ for all $\lambda > 0$. Therefore by (iii), (viii) and (xiii), $a = b$.

3. Neutrosophic Contractive Mapping (NCM)

The following definitions and results are given:

Definition 3.1. Let $\Im = (X, \tilde{F}_N, *, \diamond)$ be the NMS. The mapping $f : X \to X$ is called NC if there exists $\delta \in (0,1)$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(f(a), f(b), \lambda) \geq \delta\left(\mu_{\tilde{F}_N}(a,b,\lambda)\right) \\ \nu_{\tilde{F}_N}(f(a), f(b), \lambda) \leq \delta\left(\nu_{\tilde{F}_N}(a,b,\lambda)\right) \\ \omega_{\tilde{F}_N}(f(a), f(b), \lambda) \leq \delta\left(\omega_{\tilde{F}_N}(a,b,\lambda)\right) \end{pmatrix} \text{ for each } a,b \in X \text{ and } \lambda > 0.$$

Here δ is said to be contractive constant of f and $0 < \delta < 1$.

Definition 3.2. Let $\Im = (X, \tilde{F}_N, *, \diamond)$ be the NMS and let $f : X \to X$ is a NC mapping. There exists $c \in X$ s.t. $f(c) = c$, then c is called neutrosophic fixed point (NFP) of f.

Proposition 3.1. Suppose f is a NC. Then f^n is also a NC. Furthermore if k is constant for f, then k^n is constant for f^n.

Proposition3.2. Suppose f is a NC and $a \in X$. Then $f[D(a,\varepsilon,\lambda)] \subset D(a,\varepsilon,\lambda)$ for large enough value of ε.

Proposition 3.3. The inclusion $f^n[D(a,\varepsilon,\lambda)] \subset D(f^n(a),\varepsilon^*,\lambda)$ is hold for all n, where $\varepsilon^* = \delta^n \times \varepsilon$.

Lemma 3.1: Let $(X, \tilde{F}_N, *, \diamond)$ be a NMS and $\{b_n\}$ be a sequence in X. There exists a number $q \in X$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(b_{n+2}, b_{n+1}, q\lambda) \geq \mu_{\tilde{F}_N}(b_{n+1}, b_n, \lambda), \\ \nu_{\tilde{F}_N}(b_{n+2}, b_{n+1}, q\lambda) \leq \mu_{\tilde{F}_N}(b_{n+1}, b_n, \lambda), \\ \omega_{\tilde{F}_N}(b_{n+2}, b_{n+1}, q\lambda) \leq \omega_{\tilde{F}_N}(b_{n+1}, b_n, \lambda) \end{pmatrix} \text{for all } \lambda > 0, \text{ and } n = 1,2,3......, \text{ then } \{b_n\} \text{ is a Cauchy sequence in } X.$$

Definition 3.3. Let us choose two NMS $(X, \tilde{F}_{N_1}, *, \diamond)$ and $(Y, \tilde{F}_{N_2}, *, \diamond)$. Let Δ_1 the uniformly generated by $\Im_i (i = 1,2)$. A mapping $f : X \to Y$ is uniformly continuous with respect to Δ_1 and Δ_2 iff for a given $\varepsilon_2 \in (0,1)$ *and* $\lambda_2 > 0$, there exists $\varepsilon_1 \in (0,1)$ *and* $\lambda_1 > 0$, s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_{N_1}}(a,b,\lambda_1) > 1-\varepsilon_1 \text{ implies } \mu_{\tilde{F}_{N_2}}(a,b,\lambda_2) > 1-\varepsilon_2, \\ \nu_{\tilde{F}_{N_1}}(a,b,\lambda_1) < \varepsilon_1 \text{ implies } \nu_{\tilde{F}_{N_2}}(a,b,\lambda_2) < \varepsilon_2, \\ \omega_{\tilde{F}_{N_1}}(a,b,\lambda_1) < \varepsilon_1 \text{ implies } \omega_{\tilde{F}_{N_2}}(a,b,\lambda_2) < \varepsilon_2 \end{pmatrix} \text{ for each } a,b \in X.$$

Definition 3.4[39]: Let $\Im = (X, \tilde{F}_N, *, \diamond)$ be a complete NMS and $\varepsilon > 0$. A finite sequence $a = a_0, a_1, a_2 .....a_n = b$ is called $\varepsilon-$chainable from a to b if $\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(a,b,\lambda) < \varepsilon, \\ \omega_{\tilde{F}_N}(a,b,\lambda) < \varepsilon, \end{pmatrix}$ for all $\lambda > 0$ and $i = 1,2,3...n$.

A NMS $\Im = (X, \tilde{F}_N, *, \diamond)$ is called $\varepsilon-$chainable if for $a,b \in X$, there exists a $\varepsilon-$chain from a to b.

4. Main Result

For the proof of main result, the following definitions for compatibility [13] and weak compatibility are necessary.

Definition 4.1: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\Diamond)$ are called compatible if

$$\begin{pmatrix} \lim_{n\to\infty}\mu_{\tilde{F}_N}(\mathrm{AS}a_n,\mathrm{SA}a_n,\lambda)=1, \\ \lim_{n\to\infty}\nu_{\tilde{F}_N}(\mathrm{AS}a_n,\mathrm{SA}a_n,\lambda)=0, \\ \lim_{n\to\infty}\omega_{\tilde{F}_N}(\mathrm{AS}a_n,\mathrm{SA}a_n,\lambda)=0, \end{pmatrix} \text{ whenever } \{a_n\} \text{ is a sequence in } X \text{ s.t.}$$

$\lim_{n\to\infty}\mathrm{A}a_n=\mathrm{S}a_n=a,$ for some a in X.

Definition 4.2: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\Diamond)$ are called weak commuting if

$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\ge\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda), \\ \nu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda), \\ \omega_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda), \end{pmatrix} \text{ for all } a \text{ in } X \text{ and } \lambda>0.$$

Definition 4.3.[27]: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\Diamond)$ are called point wise R-weakly commuting if $\exists,\mathrm{R}>0$, s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\ge\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda/\mathrm{R}), \\ \nu_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda/\mathrm{R}), \\ \omega_{\tilde{F}_N}(\mathrm{AS}a,\mathrm{SA}a,\lambda)\le\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda/\mathrm{R}), \end{pmatrix} \text{ for all } a \text{ in } X \text{ and } \lambda>0.$$

Definition 4.4: Two self-mappings A and S of a NMS $(X,\tilde{F}_N,*,\Diamond)$ are called reciprocal continuous on X if $\lim_{n\to\infty}\mathrm{AS}a_n=\mathrm{A}a$ and $\lim_{n\to\infty}\mathrm{SA}a_n=\mathrm{S}a$ whenever $\{a_n\}$ is a sequence in X s.t. $\lim_{n\to\infty}\mathrm{A}a_n=\lim_{n\to\infty}\mathrm{S}a_n=a$ for some a in X.

Lemma 4.1: Let $\psi:R^+\to R^+$ be a left continuous function s.t. $\psi(\lambda)>\lambda$ for every $\lambda>0$, then $\lim_{n\to\infty}\psi^n(\lambda)=1,$ where ψ^n denotes the n-times repeated composition of ψ with itself.

Theorem 4.1: Let S and T be two self-continuous mappings of a complete $\varepsilon-$chainable NMS $(X,\tilde{F}_N,*,\Diamond)$ with $t*t\ge t$ and $(1-t)\Diamond(1-t)\le(1-t)$, $\forall t\in[0,1]$. Let A and B be two self-mappings of X satisfying the following conditions:

(i) $A(X)\subseteq S(X)$ and $B(X)\subseteq A(X)$

(ii) for all $a,b\in X,\lambda>0$ and $k\in(0,1)$ $\exists,\psi:[0,1]\to[0,1],$ $\psi(0)=0,$ and $\psi(s)>s$ (a left continuous function)

$$\forall,s>0 \text{ s.t. } \begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{B}b,k\lambda)}\phi(\theta)d\theta\ge\psi\left(\int_0^{\mu_{\tilde{F}_N}(a,b,\lambda)}\phi(\theta)d\theta\right) \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{B}b,k\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\nu_{\tilde{F}_N}(a,b,\lambda)}\phi(\theta)d\theta\right) \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{B}b,k\lambda)}\phi(\theta)d\theta\le\psi\left(\int_0^{\omega_{\tilde{F}_N}(a,b,\lambda)}\phi(\theta)d\theta\right) \end{pmatrix}$$

where $\phi(\theta):R^+\to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0<\int_0^{\varepsilon}\phi(\theta)d\theta<1$, for all $\varepsilon>0$ and

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a,b,\lambda) = r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(Sa,Aa,\lambda),\mu_{\tilde{F}_N}(Sa,Tb,\lambda),\mu_{\tilde{F}_N}(Tb,Bb,\lambda), \\ \max\{\mu_{\tilde{F}_N}(Tb,Aa,\lambda),\mu_{\tilde{F}_N}(Sa,Bb,\lambda)\}\} \end{bmatrix} \\ \nu_{\tilde{F}_N}(a,b,\lambda) = r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(Sa,Aa,\lambda),\nu_{\tilde{F}_N}(Sa,Tb,\lambda),\nu_{\tilde{F}_N}(Tb,Bb,\lambda), \\ \max\{\nu_{\tilde{F}_N}(Tb,Aa,\lambda),\nu_{\tilde{F}_N}(Sa,Bb,\lambda)\}\} \end{bmatrix} \\ \omega_{\tilde{F}_N}(a,b,\lambda) = r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(Sa,Aa,\lambda),\omega_{\tilde{F}_N}(Sa,Tb,\lambda),\omega_{\tilde{F}_N}(Tb,Bb,\lambda), \\ \max\{\omega_{\tilde{F}_N}(Tb,Aa,\lambda),\omega_{\tilde{F}_N}(Sa,Bb,\lambda)\}\} \end{bmatrix} \end{pmatrix} \tag{3}$$

where $r:[0,1]\to[0,1]$, is continuous function s.t. $r(a) = \begin{cases} > a, & if\ a\in[0,1) \\ 1, & if\ a = 1. \end{cases}$, then the continuity of one of the mapping in compatible pair $\{A,S\}$ or $\{B,T\}$ implies their reciprocal continuity, and the unique CFP of A,S,B and T.

Proof: Let $n_0 \in X$ be an arbitrary point of X. From (i) we can construct a sequence $\{b_n\}$ in X as follows:

$$b_{2n} = Aa_{2n} = Sa_{2n+1}, \quad b_{2n+1} = Ba_{2n+1} = Ta_{2n+2}, \quad \text{for all } n = 1,2,3......$$

We define $\begin{pmatrix} (\mu_{\tilde{F}_N})_{2n}(q\lambda) = \mu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda), \\ (\nu_{\tilde{F}_N})_{2n}(q\lambda) = \nu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda), \\ (\omega_{\tilde{F}_N})_{2n}(q\lambda) = \omega_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda) \end{pmatrix}$ for $b_{2n} \neq b_{2n+1}$. Let us take $a = a_{2n}$, $b = a_{2n+1}$ in (ii),

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(Aa_{2n},Ba_{2n+1},k\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)} \phi(\theta)d\theta\right) \\ \int_0^{\nu_{\tilde{F}_N}(Aa_{2n},Ba_{2n+1},k\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)} \phi(\theta)d\theta\right) \\ \int_0^{\omega_{\tilde{F}_N}(Aa_{2n},Ba_{2n+1},k\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)} \phi(\theta)d\theta\right) \end{pmatrix} \tag{4}$$

$$\mu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda) = r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(Sa_{2n},Aa_{2n},\lambda),\mu_{\tilde{F}_N}(Sa_{2n},Ta_{2n+1},\lambda),\mu_{\tilde{F}_N}(Ta_{2n+1},Ba_{2n+1},\lambda), \\ \max\{\mu_{\tilde{F}_N}(Ta_{2n+1},Aa_{2n},\lambda),\mu_{\tilde{F}_N}(Sa_{2n},Ba_{2n+1},\lambda)\}\} \end{bmatrix}$$

where

$$\geq r\left[\min\{\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\mu_{\tilde{F}_N}(b_{2n},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n-1},b_{2n+1},\lambda)\}\}\right]$$

$$\geq r\left[\min\{\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda)*\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\}\}\right]$$

$$\geq r\left[\mu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\right]$$

$$\nu_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda) = r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(Sa_{2n},Aa_{2n},\lambda),\nu_{\tilde{F}_N}(Sa_{2n},Ta_{2n+1},\lambda),\nu_{\tilde{F}_N}(Ta_{2n+1},Ba_{2n+1},\lambda), \\ \max\{\nu_{\tilde{F}_N}(Ta_{2n+1},Aa_{2n},\lambda),\nu_{\tilde{F}_N}(Sa_{2n},Ba_{2n+1},\lambda)\}\} \end{bmatrix}$$

$$\leq r\left[\min\{\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\nu_{\tilde{F}_N}(b_{2n},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n-1},b_{2n+1},\lambda)\}\}\right]$$

$$\leq r\left[\min\{\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda)*\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\}\}\right]$$

$$\leq r\left[\nu_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\right]$$

$$\omega_{\tilde{F}_N}(a_{2n},a_{2n+1},\lambda)=r\begin{bmatrix}\min\{\omega_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{A}a_{2n},\lambda),\omega_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{T}a_{2n+1},\lambda),\omega_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{B}a_{2n+1},\lambda),\\ \max\{\omega_{\tilde{F}_N}(\mathrm{T}a_{2n+1},\mathrm{A}a_{2n},\lambda),\omega_{\tilde{F}_N}(\mathrm{S}a_{2n},\mathrm{B}a_{2n+1},\lambda)\}\}\end{bmatrix}$$

$$\leq r\left[\min\{\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\omega_{\tilde{F}_N}(b_{2n},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n-1},b_{2n+1},\lambda)\}\}\right]$$

$$\leq r\left[\min\{\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\max\{\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda)*\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\}\}\right]$$

$$\leq r\left[\omega_{\tilde{F}_N}(b_{2n-1},b_{2n},\lambda),\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\right]$$

Thus from (3), we have

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\geq\psi\left(\int_0^{\mathrm{r}\left[(\mu_{\tilde{F}_N})_{2n-1}(\lambda),(\mu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right),\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{\mathrm{r}\left[(\nu_{\tilde{F}_N})_{2n-1}(\lambda),(\nu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right),\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{\mathrm{r}\left[(\omega_{\tilde{F}_N})_{2n-1}(\lambda),(\omega_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)\end{pmatrix} \tag{5}$$

Now if $\begin{pmatrix}(\mu_{\tilde{F}_N})_{2n}(\lambda)>(\mu_{\tilde{F}_N})_{2n-1}(\lambda),\\ (\nu_{\tilde{F}_N})_{2n}(\lambda)<(\nu_{\tilde{F}_N})_{2n-1}(\lambda),\\ (\omega_{\tilde{F}_N})_{2n}(\lambda)<(\omega_{\tilde{F}_N})_{2n-1}(\lambda)\end{pmatrix}$ for some n, then from (5)

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\geq\psi\left(\int_0^{\mathrm{r}\left[(\mu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)>\psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\right)>\int_0^{(\mu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta,\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{\mathrm{r}\left[(\nu_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\right)<\int_0^{(\nu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta,\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{\mathrm{r}\left[(\omega_{\tilde{F}_N})_{2n}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\right)<\int_0^{(\omega_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\end{pmatrix}$$

which implies $\begin{pmatrix}\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda)\geq\mu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\\ \nu_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda)\leq\nu_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda),\\ \omega_{\tilde{F}_N}(b_{2n},b_{2n+1},q\lambda)\leq\omega_{\tilde{F}_N}(b_{2n},b_{2n+1},\lambda)\end{pmatrix}$. Thus $b_{2n}=b_{2n+1}$, by lemma 2.3, which is contradiction, since $b_{2n}\neq b_{2n+1}$. Thus we have $\begin{pmatrix}(\mu_{\tilde{F}_N})_{2n}(q\lambda)\leq r\left[(\mu_{\tilde{F}_N})_{2n-1}(\lambda)\right],\\ (\nu_{\tilde{F}_N})_{2n}(q\lambda)\geq r\left[(\nu_{\tilde{F}_N})_{2n-1}(\lambda)\right],\\ (\omega_{\tilde{F}_N})_{2n}(q\lambda)\geq r\left[(\omega_{\tilde{F}_N})_{2n-1}(\lambda)\right]\end{pmatrix}$

for which

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\geq\psi\left(\int_0^{\mathrm{r}\left[(\mu_{\tilde{F}_N})_{2n-1}(\lambda)\right]}\phi(\theta)d\theta\right)>\psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{\mathrm{r}\left[(\nu_{\tilde{F}_N})_{2n-1}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{\mathrm{r}\left[(\omega_{\tilde{F}_N})_{2n-1}(\lambda)\right]}\phi(\theta)d\theta\right)<\psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right)\end{pmatrix} \tag{6}$$

letting $q\to 1$, then we have

$$\begin{pmatrix}\int_0^{(\mu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\geq\psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\nu_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right),\\ \int_0^{(\omega_{\tilde{F}_N})_{2n}(\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n-1}(\lambda)}\phi(\theta)d\theta\right)\end{pmatrix} \tag{7}$$

Similarly $\begin{pmatrix} \int_0^{(\mu_{\tilde{F}_N})_{2n-1}(\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(\mu_{\tilde{F}_N})_{2n-2}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\nu_{\tilde{F}_N})_{2n-1}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\nu_{\tilde{F}_N})_{2n-2}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\omega_{\tilde{F}_N})_{2n-1}(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\omega_{\tilde{F}_N})_{2n-2}(\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ and so on.

In general we have for all $n = 1, 2, 3......$.

$$\begin{pmatrix} \int_0^{(\mu_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(\mu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\nu_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\nu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\omega_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\omega_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \end{pmatrix} \quad (8)$$

from (8), we have

$$\begin{pmatrix} \int_0^{(\mu_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(\mu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \geq \psi^2\left(\int_0^{(\mu_{\tilde{F}_N})_{n-2}(\lambda)} \phi(\theta)d\theta\right) .... \geq \psi^n\left(\int_0^{(\mu_{\tilde{F}_N})_0(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\nu_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\nu_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \leq \psi^2\left(\int_0^{(\nu_{\tilde{F}_N})_{n-2}(\lambda)} \phi(\theta)d\theta\right) .... \leq \psi^n\left(\int_0^{(\nu_{\tilde{F}_N})_0(\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{(\omega_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{(\omega_{\tilde{F}_N})_{n-1}(\lambda)} \phi(\theta)d\theta\right) \leq \psi^2\left(\int_0^{(\omega_{\tilde{F}_N})_{n-2}(\lambda)} \phi(\theta)d\theta\right) .... \leq \psi^n\left(\int_0^{(\omega_{\tilde{F}_N})_0(\lambda)} \phi(\theta)d\theta\right) \end{pmatrix} \quad (9)$$

and taking the limit as $n \to \infty$ and using lemma 2.3, we have

$$\begin{pmatrix} \lim_{n\to\infty} \int_0^{(\mu_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \geq \lim_{n\to\infty} \psi^n\left(\int_0^{(\mu_{\tilde{F}_N})_0(\lambda)} \phi(\theta)d\theta\right) = 1, \\ \lim_{n\to\infty} \int_0^{(\nu_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \leq \lim_{n\to\infty} \psi^n\left(\int_0^{(\nu_{\tilde{F}_N})_0(\lambda)} \phi(\theta)d\theta\right) = 0, \\ \lim_{n\to\infty} \int_0^{(\omega_{\tilde{F}_N})_n(\lambda)} \phi(\theta)d\theta \leq \lim_{n\to\infty} \psi^n\left(\int_0^{(\omega_{\tilde{F}_N})_0(\lambda)} \phi(\theta)d\theta\right) = 0 \end{pmatrix} \quad (10)$$

which from (1) implies that $\begin{pmatrix} \lim_{n\to\infty} (\mu_{\tilde{F}_N})_n(\lambda) = \lim_{n\to\infty} \mu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) = 1, \\ \lim_{n\to\infty} (\nu_{\tilde{F}_N})_n(\lambda) = \lim_{n\to\infty} \nu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) = 0, \\ \lim_{n\to\infty} (\omega_{\tilde{F}_N})_n(\lambda) = \lim_{n\to\infty} \omega_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) = 0 \end{pmatrix}$ for all $n \in N$ and $\lambda > 0.$ (11)

Now for each $\varepsilon > 0$ and each $\lambda > 0,$ choose $n_0 \in N$ s.t. $\begin{pmatrix} \mu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) > 1 - \varepsilon, \\ \nu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) < \varepsilon, \\ \omega_{\tilde{F}_N}(b_n, b_{n+1}, \lambda) < \varepsilon \end{pmatrix}$ for all $n > n_0$ (12)

Letting $m > n (m, n \in N),$ then

$$\begin{pmatrix} \mu_{\tilde{F}_N}(b_n, b_m, \lambda) \geq r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda/(m-n)), \mu_{\tilde{F}_N}(b_{n+1}, b_{n+2}, \lambda/(m-n)) .... \\ \max\{\mu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda/(m-n)), ... \mu_{\tilde{F}_N}(b_{m-1}, b_m, \lambda/(m-n))\}\} \end{bmatrix} \\ > r\left[\min\{(1-\varepsilon), (1-\varepsilon), ... \max\{(1-\varepsilon), (1-\varepsilon), , ... (1-\varepsilon)\}\}\right] \geq (1-\varepsilon) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(b_n, b_m, \lambda) \leq r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(b_n, b_{n+1}, \lambda/(m-n)), \nu_{\tilde{F}_N}(b_{n+1}, b_{n+2}, \lambda/(m-n)) .... \\ \max\{\nu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda/(m-n)), ... \nu_{\tilde{F}_N}(b_{m-1}, b_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, \varepsilon, ... \max\{\varepsilon, ... \varepsilon\}\}\right] \leq \varepsilon < \varepsilon, \\ \omega_{\tilde{F}_N}(b_n, b_m, \lambda) \leq r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(b_n, b_{n+1}, \lambda/(m-n)), \omega_{\tilde{F}_N}(b_{n+1}, b_{n+2}, \lambda/(m-n)) .... \\ \max\{\omega_{\tilde{F}_N}(b_{s-1}, b_s, \lambda/(m-n)), ... \omega_{\tilde{F}_N}(b_{m-1}, b_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, \varepsilon, ... \max\{\varepsilon, ... \varepsilon\}\}\right] \leq \varepsilon < \varepsilon \end{pmatrix}. \quad (13)$$

Thus from definitions 2.10, 2.11 and conditions (11), (13) and lemma 3.1 $\{b_n\}$ is a CS in X. Since X is complete so that $\{b_n\} \to z \in X$ and sub sequences $\{Aa_{2n}\}$, $\{Ba_{2n+1}\}$, $\{Sa_{2n+1}\}$ and $\{Ta_{2n+2}\}$, of $\{b_n\}$ also converges to z.

Thus $\quad Aa_{2n} \to z,\ Ba_{2n+1} \to z,\ Sa_{2n+1} \to z$ and $Ta_{2n+2} \to z.$ (14)

Again, since X is ε-chainable, $\exists$ ε-chain from a_n to a_{n+1} i.e. $\exists$ a finite sequence $a_n = b_1, b_2, \ldots\ldots b_l = a_{n+1}$ s.t.

$$\begin{pmatrix} \mu_{\tilde{F}_N}(b_i, b_{i-1}, \lambda) > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(b_i, b_{i-1}, \lambda) < \varepsilon, \\ \omega_{\tilde{F}_N}(b_i, b_{i-1}, \lambda) < \varepsilon \end{pmatrix} \text{ for all } \lambda > 0 \text{ and } i = 1, 2, \ldots\ldots l.,$$

Thus, we have

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a_n, a_{n+1}, q\lambda) \geq r\left[\min\{\mu_{\tilde{F}_N}(b_1, b_2, \lambda), \mu_{\tilde{F}_N}(b_2, b_3, \lambda), \ldots, \max\{\mu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda), \ldots\ldots \mu_{\tilde{F}_N}(b_{l-1}, b_l, \lambda)\}\}\right] \\ > r\left[\min\{(1-\varepsilon), (1-\varepsilon), \ldots\ldots \max\{(1-\varepsilon), (1-\varepsilon), \ldots\ldots (1-\varepsilon)\}\}\right] \geq r\left[1-\varepsilon\right] > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(a_n, a_{n+1}, q\lambda) \leq r\left[\min\{\nu_{\tilde{F}_N}(b_1, b_2, \lambda), \nu_{\tilde{F}_N}(b_2, b_3, \lambda), \ldots, \max\{\nu_{\tilde{F}_N}(b_{s-1}, b_s, \lambda), \ldots\ldots \nu_{\tilde{F}_N}(b_{l-1}, b_l, \lambda)\}\}\right] \\ < r\left[\min\{\varepsilon, \varepsilon, \ldots\ldots \max\{\varepsilon, \varepsilon, \ldots\ldots \varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon, \\ \omega_{\tilde{F}_N}(a_n, a_{n+1}, q\lambda) \leq r\left[\min\{\omega_{\tilde{F}_N}(b_1, b_2, \lambda), \nu_{\tilde{F}_N}(b_2, b_3, \lambda), \ldots, \max\{\omega_{\tilde{F}_N}(b_{s-1}, b_s, \lambda), \ldots\ldots \omega_{\tilde{F}_N}(b_{l-1}, b_l, \lambda)\}\}\right] \\ < r\left[\min\{\varepsilon, \varepsilon, \ldots\ldots \max\{\varepsilon, \varepsilon, \ldots\ldots \varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon. \end{pmatrix} \quad (15)$$

For $m, n \in N$, $m > n$, we have

$$\begin{pmatrix} \mu_{\tilde{F}_N}(a_n, a_m, \lambda) \geq r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/(m-n)), \mu_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/(m-n)) \ldots\ldots \\ \max\{\mu_{\tilde{F}_N}(a_{p-1}, a_p, \lambda/(m-n)), \ldots\ldots \mu_{\tilde{F}_N}(a_{m-1}, a_m, \lambda/(m-n))\}\} \end{bmatrix} \\ r\left[\min\{(1-\varepsilon), (1-\varepsilon), \ldots\ldots \max\{(1-\varepsilon), (1-\varepsilon), \ldots\ldots (1-\varepsilon)\}\}\right] \geq r\left[1-\varepsilon\right] > 1-\varepsilon, \\ \nu_{\tilde{F}_N}(a_n, a_m, \lambda) \leq r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/(m-n)), \nu_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/(m-n)) \ldots\ldots \\ \max\{\nu_{\tilde{F}_N}(a_{p-1}, a_p, \lambda/(m-n)), \ldots\ldots \nu_{\tilde{F}_N}(a_{m-1}, a_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, \ldots\ldots \max\{\varepsilon, \varepsilon, \ldots\ldots \varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon, \\ \omega_{\tilde{F}_N}(a_n, a_m, \lambda) \leq r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(a_n, a_{n+1}, \lambda/(m-n)), \omega_{\tilde{F}_N}(a_{n+1}, a_{n+2}, \lambda/(m-n)) \ldots\ldots \\ \max\{\omega_{\tilde{F}_N}(a_{p-1}, a_p, \lambda/(m-n)), \ldots\ldots \omega_{\tilde{F}_N}(a_{m-1}, a_m, \lambda/(m-n))\}\} \end{bmatrix} \\ < r\left[\min\{\varepsilon, \varepsilon, \ldots\ldots \max\{\varepsilon, \varepsilon, \ldots\ldots \varepsilon\}\}\right] \leq r\left[\varepsilon\right] < \varepsilon. \end{pmatrix} \quad (16)$$

i.e.

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(a_n, a_m, \lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{(1-\varepsilon)} \phi(\theta)d\theta\right) > \int_0^{(1-\varepsilon)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(a_n, a_m, \lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\varepsilon} \phi(\theta)d\theta\right) < \int_0^{\varepsilon} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(a_n, a_m, \lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\varepsilon} \phi(\theta)d\theta\right) < \int_0^{\varepsilon} \phi(\theta)d\theta \end{pmatrix} \quad (17)$$

and so from definitions 2.10; 2.11 and conditions (15) and (17), $\{a_n\}$ is a CS in X. Since X is complete so that $\{a_n\} \to a \in X.$ since $\{A, S\}$ is reciprocally continuous, so that A and S are continuous. Thus

$$Aa_{2n} = Aa, \ \ Sa_{2n+1} = Sa, \quad (18)$$

since $\{A, S\}$ are compatible, so R -weakly commuting mapping. Then

$$\begin{pmatrix} \mu_{\tilde{F}_N}(ASa_{2n}, SAa_{2n}, q\lambda) \geq \mu_{\tilde{F}_N}(Aa_{2n}, Sa_{2n}, \lambda/R), \\ \nu_{\tilde{F}_N}(ASa_{2n}, SAa_{2n}, q\lambda) \leq \nu_{\tilde{F}_N}(Aa_{2n}, Sa_{2n}, \lambda/R), \\ \omega_{\tilde{F}_N}(ASa_{2n}, SAa_{2n}, q\lambda) \leq \omega_{\tilde{F}_N}(Aa_{2n}, Sa_{2n}, \lambda/R) \end{pmatrix}$$

gives $ASa_{2n} = Aa, \quad SAa_{2n+1} = Sa.$ Also $\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda) \geq 1, \\ \nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda) \leq 0, \\ \omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{S}a,\lambda) \leq 0 \end{pmatrix}$ which implies $\mathrm{A}a = \mathrm{S}a.$ (19)

From (14) , (18) and (19), $\mathrm{A}z = \mathrm{S}z.$ Since $\mathrm{A}(a) \subseteq \mathrm{S}(a), \ \exists, u \in X$ s.t. $\mathrm{A}z = \mathrm{S}u.$ then from (4)

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(z,u,q\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(z,u,q\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(z,u,q\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$$

where

$$\begin{aligned}
\mu_{\tilde{F}_N}(z,u,\lambda) &= r\Big[\min\{\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}u,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}u,\lambda)\}\}\Big] \\
&= r\Big[\min\{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \mu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{B}u,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{A}z,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\}\Big] \\
&\geq r\Big[\min\{1, \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), 1, \}, \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\Big] \geq r\Big[\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\Big] \geq \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \\
\nu_{\tilde{F}_N}(z,u,\lambda) &= r\Big[\min\{\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}u,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{T}u,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}u,\lambda)\}\}\Big] \\
&= r\Big[\min\{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \nu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{B}u,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{B}u,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\}\Big] \\
&\leq r\Big[\min\{1, \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), 1, \}, \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\Big] \leq r\Big[\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\Big] \geq \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \\
\omega_{\tilde{F}_N}(z,u,\lambda) &= r\Big[\min\{\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}u,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}u,\mathrm{B}u,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{T}u,\mathrm{A}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}u,\lambda)\}\}\Big] \\
&= r\Big[\min\{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{A}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \omega_{\tilde{F}_N}(\mathrm{B}u,\mathrm{B}u,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{B}u,\mathrm{A}z,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\}\Big] \\
&\leq r\Big[\min\{1, \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), 1, \}, \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\}\Big] \leq r\Big[\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)\Big] \geq \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)
\end{aligned}$$

i.e.

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) > \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda)} \phi(\theta)d\theta \end{pmatrix} \quad (20)$$

which implies

$$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda) \geq \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \\ \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda) \leq \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda), \\ \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,q\lambda) \leq \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{B}u,\lambda) \end{pmatrix} \quad (21)$$

from lemma 2.3, $\mathrm{A}z = \mathrm{B}u$ i.e. $\mathrm{A}z = \mathrm{S}u = \mathrm{T}z$. Again let $\mathrm{A}z = \mathrm{S}u = \mathrm{B}u = \mathrm{T}u$.

Since the pair $\{\mathrm{A},\mathrm{S}\}$ is point-wise R–weakly commuting mappings, so there exists $\mathrm{R} > 0$ s.t.

$\begin{pmatrix} \mu_{\tilde{F}_N}(\mathrm{AS}z,\mathrm{SA}z,\lambda) \geq \mu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{S}z,\lambda/\mathrm{R}) = 1, \\ \nu_{\tilde{F}_N}(\mathrm{AS}z,\mathrm{SA}z,\lambda) \leq \nu_{\tilde{F}_N}(\mathrm{A}z,\mathrm{S}z,\lambda/\mathrm{R}) = 0, \\ \omega_{\tilde{F}_N}(\mathrm{AS}z,\mathrm{SA}z,\lambda) \leq \omega_{\tilde{F}_N}(\mathrm{A}z,\mathrm{S}z,\lambda/\mathrm{R}) = 0, \end{pmatrix}$ i.e. $\mathrm{AS}z = \mathrm{SA}z$ and $\mathrm{AA}z = \mathrm{AS}z = \mathrm{SA}z = \mathrm{SS}z.$

Similarly it can be for the pair $\{B,T\}$ which implies $BBu = BTu = TBu = TTu.$ For this in (4), we put $a = Az,$ $b = u,$ we have

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(Az,u,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(Az,u,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(Az,u,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$$

where

$$\mu_{\tilde{F}_N}(Az,u,\lambda) = r\begin{bmatrix} \min\{\mu_{\tilde{F}_N}(SAz,AAz,\lambda), \mu_{\tilde{F}_N}(SAz,Tu,\lambda), \mu_{\tilde{F}_N}(Tu,Bu,\lambda), \\ \max\{\mu_{\tilde{F}_N}(Tu,AAz,\lambda), \mu_{\tilde{F}_N}(SAz,Bu,\lambda)\}\} \end{bmatrix}$$

$$\geq r\left[\min\{1, \mu_{\tilde{F}_N}(AAz,Bu,\lambda), 1, \max\{\mu_{\tilde{F}_N}(AAz,Bu,\lambda), \mu_{\tilde{F}_N}(AAz,Bu,\lambda)\}\}\right]$$

$$\geq r\left[\mu_{\tilde{F}_N}(AAz,Bu,\lambda)\right] > \mu_{\tilde{F}_N}(AAz,Bu,\lambda)$$

$$\nu_{\tilde{F}_N}(Az,u,\lambda) = r\begin{bmatrix} \min\{\nu_{\tilde{F}_N}(SAz,AAz,\lambda), \nu_{\tilde{F}_N}(SAz,Tu,\lambda), \nu_{\tilde{F}_N}(Tu,Bu,\lambda), \\ \max\{\nu_{\tilde{F}_N}(Tu,AAz,\lambda), \nu_{\tilde{F}_N}(SAz,Bu,\lambda)\}\} \end{bmatrix}$$

$$\leq r\left[\min\{0, \nu_{\tilde{F}_N}(AAz,Bu,\lambda), 0, \max\{\nu_{\tilde{F}_N}(AAz,Bu,\lambda), \nu_{\tilde{F}_N}(AAz,Bu,\lambda)\}\}\right]$$

$$\leq r\left[\nu_{\tilde{F}_N}(AAz,Bu,\lambda)\right] < \nu_{\tilde{F}_N}(AAz,Bu,\lambda)$$

$$\omega_{\tilde{F}_N}(Az,u,\lambda) = r\begin{bmatrix} \min\{\omega_{\tilde{F}_N}(SAz,AAz,\lambda), \omega_{\tilde{F}_N}(SAz,Tu,\lambda), \omega_{\tilde{F}_N}(Tu,Bu,\lambda), \\ \max\{\omega_{\tilde{F}_N}(Tu,AAz,\lambda), \omega_{\tilde{F}_N}(SAz,Bu,\lambda)\}\} \end{bmatrix}$$

$$\leq r\left[\min\{0, \omega_{\tilde{F}_N}(AAz,Bu,\lambda), 0, \max\{\omega_{\tilde{F}_N}(AAz,Bu,\lambda), \omega_{\tilde{F}_N}(AAz,Bu,\lambda)\}\}\right]$$

$$\leq r\left[\omega_{\tilde{F}_N}(AAz,Bu,\lambda)\right] < \omega_{\tilde{F}_N}(AAz,Bu,\lambda)$$

i.e.

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(AAz,Bu,\lambda)} \phi(\theta)d\theta\right) > \int_0^{\mu_{\tilde{F}_N}(AAz,Bu,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(AAz,Bu,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\nu_{\tilde{F}_N}(AAz,Bu,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(AAz,Bu,\lambda)} \phi(\theta)d\theta\right) < \int_0^{\omega_{\tilde{F}_N}(AAz,Bu,\lambda)} \phi(\theta)d\theta \end{pmatrix} \quad (22)$$

which implies

$$\begin{pmatrix} \mu_{\tilde{F}_N}(AAz,Bu,q\lambda) \geq \mu_{\tilde{F}_N}(AAz,Bu,\lambda), \\ \nu_{\tilde{F}_N}(AAz,Bu,q\lambda) \leq \nu_{\tilde{F}_N}(AAz,Bu,\lambda), \\ \omega_{\tilde{F}_N}(AAz,Bu,q\lambda) \leq \omega_{\tilde{F}_N}(AAz,Bu,\lambda) \end{pmatrix} \quad (23)$$

from lemma 2.3, we have $AAz = Bu = Az.$ Thus $Az = AAz$ and $Az = AAz = SAz,$ which shows that Az is common fixed point of A and S. Also $Az = Bu = Su = Tz.$ Hence Az is common fixed point of A, B, S and T.

Now again suppose that $Az = z$ is a common fixed point of A, B, S and T. For this from (4), we have

$$\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{r\left[\mu_{\tilde{F}_N}(Az,Bz,\lambda)\right]} \phi(\theta)d\theta\right) > \int_0^{r\left[\mu_{\tilde{F}_N}(Az,Bz,\lambda)\right]} \phi(\theta)d\theta > \int_0^{\mu_{\tilde{F}_N}(Az,z,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\nu_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{r\left[\nu_{\tilde{F}_N}(Az,Bz,\lambda)\right]} \phi(\theta)d\theta\right) < \int_0^{r\left[\nu_{\tilde{F}_N}(Az,Bz,\lambda)\right]} \phi(\theta)d\theta < \int_0^{\nu_{\tilde{F}_N}(Az,z,\lambda)} \phi(\theta)d\theta, \\ \int_0^{\omega_{\tilde{F}_N}(AAz,Bu,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{r\left[\omega_{\tilde{F}_N}(Az,Bz,\lambda)\right]} \phi(\theta)d\theta\right) < \int_0^{r\left[\omega_{\tilde{F}_N}(Az,Bz,\lambda)\right]} \phi(\theta)d\theta < \int_0^{\omega_{\tilde{F}_N}(Az,z,\lambda)} \phi(\theta)d\theta \end{pmatrix}$$

$$\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)=r\begin{bmatrix}\min\{\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{B}z,\lambda),\\ \max\{\mu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{AA}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}z,\lambda)\}\}\end{bmatrix}$$

where

$$\geq r\left[\min\{1,1,\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\},1)\}\right]>\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),$$

$$\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)=r\begin{bmatrix}\min\{\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{B}z,\lambda),\\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}z,\lambda)\}\}\end{bmatrix}$$

$$\leq r\left[\min\{0,0,\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\},0)\}\right]<\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),$$

$$\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)=r\begin{bmatrix}\min\{\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{T}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{B}z,\lambda),\\ \max\{\nu_{\tilde{F}_N}(\mathrm{T}z,\mathrm{AA}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{SA}z,\mathrm{B}z,\lambda)\}\}\end{bmatrix}$$

$$\leq r\left[\min\{0,0,\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\},0)\}\right]<\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),$$

which implies that $\begin{pmatrix}\mu_{\tilde{F}_N}(\mathrm{A}z,z,q\lambda)>\mu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),\\ \nu_{\tilde{F}_N}(\mathrm{A}z,z,q\lambda)<\nu_{\tilde{F}_N}(\mathrm{A}z,z,\lambda),\\ \omega_{\tilde{F}_N}(\mathrm{A}z,z,q\lambda)<\omega_{\tilde{F}_N}(\mathrm{A}z,z,\lambda)\end{pmatrix}$ for all $\lambda>0$, i.e. from lemma 2.3, $\mathrm{A}z=z$. Thus z is a common fixed point of A, B, S and T. For uniqueness of z let $w\neq z$ be another common fixed point of A, B, S and T, then from (4), we have

$$\int_0^{\mu_{\tilde{F}_N}(z,w,q\lambda)}\phi(\theta)d\theta\geq\psi\left(\int_0^{r\left[\min\{\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}w,\lambda),\mu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{B}w,\lambda),\max\{\mu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{A}z,\lambda),\mu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}w,\lambda)\}\}\right]}\phi(\theta)d\theta\right)$$

$$>\psi\left(\int_0^{r\left[\min\{1,\mu_{\tilde{F}_N}(z,w,\lambda),1,\mu_{\tilde{F}_N}(z,w,\lambda)\}\right]}\phi(\theta)d\theta\right)>\int_0^{r\left[\mu_{\tilde{F}_N}(z,w,\lambda)\right]}\phi(\theta)d\theta>\int_0^{\mu_{\tilde{F}_N}(z,w,\lambda)}\phi(\theta)d\theta,$$

$$\int_0^{\nu_{\tilde{F}_N}(z,w,q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{r\left[\min\{\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}w,\lambda),\nu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{B}w,\lambda),\max\{\nu_{\tilde{F}_N}(\mathrm{T}w,\mathrm{A}z,\lambda),\nu_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}w,\lambda)\}\}\right]}\phi(\theta)d\theta\right)$$

$$<\psi\left(\int_0^{r\left[\min\{0,\nu_{\tilde{F}_N}(z,w,\lambda),0,\nu_{\tilde{F}_N}(z,w,\lambda)\}\right]}\phi(\theta)d\theta\right)<\int_0^{r\left[\nu_{\tilde{F}_N}(z,w,\lambda)\right]}\phi(\theta)d\theta<\int_0^{\mu_{\tilde{F}_N}(z,w,\lambda)}\phi(\theta)d\theta,$$

$$\int_0^{\omega_{\tilde{F}_N}(z,w,q\lambda)}\phi(\theta)d\theta\leq\psi\left(\int_0^{r\left[\min\{\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{A}z,\lambda),\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{T}w,\lambda),\omega_{\tilde{F}_N}(\mathrm{T}w,\mathrm{B}w,\lambda),\max\{\omega_{\tilde{F}_N}(\mathrm{T}w,\mathrm{A}z,\lambda),\omega_{\tilde{F}_N}(\mathrm{S}z,\mathrm{B}w,\lambda)\}\}\right]}\phi(\theta)d\theta\right)$$

$$<\psi\left(\int_0^{r\left[\min\{0,\omega_{\tilde{F}_N}(z,w,\lambda),0,\omega_{\tilde{F}_N}(z,w,\lambda)\}\right]}\phi(\theta)d\theta\right)<\int_0^{r\left[\omega_{\tilde{F}_N}(z,w,\lambda)\right]}\phi(\theta)d\theta<\int_0^{\omega_{\tilde{F}_N}(z,w,\lambda)}\phi(\theta)d\theta,$$

which from lemma 2.3, implies $\begin{pmatrix}\mu_{\tilde{F}_N}(z,w,q\lambda)\geq\mu_{\tilde{F}_N}(z,w,\lambda),\\ \nu_{\tilde{F}_N}(z,w,q\lambda)\leq\nu_{\tilde{F}_N}(z,w,\lambda),\\ \omega_{\tilde{F}_N}(z,w,q\lambda)\leq\omega_{\tilde{F}_N}(z,w,\lambda)\end{pmatrix}$ i.e. $z=w$, for all $\lambda>0$. Thus z is a unique common fixed point of A, B, S and T.

Corollary 4.1: Let $\{\mathrm{A},\mathrm{S}\}$ and $\{\mathrm{A},\mathrm{T}\}$ be point wise *R*–weakly commuting pairs of self mappings of a complete $\varepsilon-$chainable NMS $(X,\tilde{F}_N,*,\Diamond)$ with $t*t\geq t$ and $(1-t)\Diamond(1\text{-}t)\leq(1-t)$ for all $t\in[0,1]$ satisfying the following conditions:

(i)′ $\mathrm{A}(\mathrm{X})\subseteq\mathrm{S}(\mathrm{X})$ and $\mathrm{A}(\mathrm{X})\subseteq\mathrm{T}(\mathrm{X})$

(ii)′ for all $a,b \in X, \lambda > 0$ and $q \in (0,1)$ there exists a left continuous function $\psi:[0,1]\to[0,1]$, $\psi(0)=0$, and $\psi(s)>s$ for all $s>0$ s.t. $\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ where $\phi(\theta): R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0<\int_0^{\varepsilon}\phi(\theta)d\theta<1$, for all $\varepsilon>0$.

$$\mu_{\tilde{F}_N}(a,b,\lambda) = r\left[\min\{\mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}b,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}b,\lambda)\}\}\right]$$
$$\nu_{\tilde{F}_N}(a,b,\lambda) = r\left[\min\{\nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}b,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}b,\lambda)\}\}\right]$$
$$\omega_{\tilde{F}_N}(a,b,\lambda) = r\left[\min\{\omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{T}b,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}b,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{T}b,\mathrm{A}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{S}a,\mathrm{A}b,\lambda)\}\}\right]$$

where $r:[0,1]\to[0,1]$, is continuous function s.t. $r(a)>a$ and $r(a)=1$ for $a=1$, $a\in[0,1)$. Then the continuity of one of the mapping in compatible pair $\{\mathrm{A},\mathrm{S}\}$ or $\{\mathrm{A},\mathrm{T}\}$ on $\varepsilon-$chainable NMS implies the unique common fixed point of A, S and T.

Proof: If we put $\mathrm{S}=\mathrm{T}$, in theorem 4.1, then we get proof of corollary 4.1, easily.

Corollary 4.2: Let $\{\mathrm{A},\mathrm{T}\}$ be point wise *R*–weakly commuting pairs of self-mappings of a complete $\varepsilon-$chainable NMS $(X,\tilde{F}_N,*,\diamond)$ satisfying the following conditions:

(i)″ $\mathrm{A}(\mathrm{X}) \subseteq \mathrm{T}(\mathrm{X})$

(ii)″ for all $a,b\in X, \lambda>0$ there exists a left continuous function $\psi:[0,1]\to[0,1]$, $\psi(0)=0$, and $\psi(s)>s$ for all $s>0$ s.t. $\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{A}b,\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ where $\phi(\theta): R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0<\int_0^{\varepsilon}\phi(\theta)d\theta<1$, for all $\varepsilon>0$.

$$\mu_{\tilde{F}_N}(a,b,\lambda) = r\left[\min\{\mu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{T}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{T}a,\mathrm{T}a,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \max\{\mu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \mu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}a,\lambda)\}\}\right]$$
$$\nu_{\tilde{F}_N}(a,b,\lambda) = r\left[\min\{\nu_{\tilde{F}_N}(\mathrm{A}a,\mathrm{T}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{T}a,\mathrm{T}a,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \max\{\nu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \nu_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}a,\lambda)\}\}\right]$$
$$\omega_{\tilde{F}_N}(a,b,\lambda) = r\left[\min\{\omega_{\tilde{F}_N}(\mathrm{A}a,\mathrm{T}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{T}a,\mathrm{T}a,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \max\{\omega_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}b,\lambda), \omega_{\tilde{F}_N}(\mathrm{A}b,\mathrm{T}a,\lambda)\}\}\right]$$

where $r:[0,1]\to[0,1]$, is continuous function s.t. $r(a)>a$ and $r(a)=1$ for $a=1$, $a\in[0,1)$. Then the continuity of one of the mapping in compatible pair $\{\mathrm{A},\mathrm{T}\}$ on $\varepsilon-$chainable NMS implies the unique common fixed point of A and T.

Proof: If we put $\mathrm{B}=\mathrm{A}$ and $\mathrm{S}=\mathrm{T}$ in theorem 4.1, we get the proof of corollary 4.2.

Theorem 4.2: Let S and T be two self-continuous mappings of a complete $\varepsilon-$chainable NMS $(X,\tilde{F}_N,*,\diamond)$ with $t*t\geq t$ and $(1-t)\diamond(1-t)\leq(1-t)$ for all $t\in[0,1]$. Let A and B be two self-mappings of X satisfying the following conditions:

(i)‴ $A(X) \subseteq S(X)$ and $A(X) \subseteq T(X)$

(ii)‴ for all $a, b \in X, \lambda > 0$ and $q \in (0,1)$ there exists a left continuous function $\psi : [0,1] \to [0,1]$, $\psi(0) = 0$,

and $\psi(s) > s$ for all $s > 0$ s.t. $\begin{pmatrix} \int_0^{\mu_{\tilde{F}_N}(Aa, Ab, q\lambda)} \phi(\theta)d\theta \geq \psi\left(\int_0^{\mu'_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\nu_{\tilde{F}_N}(Aa, Ab, q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\nu'_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right), \\ \int_0^{\omega_{\tilde{F}_N}(Aa, Ab, q\lambda)} \phi(\theta)d\theta \leq \psi\left(\int_0^{\omega'_{\tilde{F}_N}(a,b,\lambda)} \phi(\theta)d\theta\right) \end{pmatrix}$ where

$\phi(\theta) : R^+ \to R^+$ is a Lebesgue integrable mapping which is summable, non-negative s.t. $0 < \int_0^{\varepsilon} \phi(\theta)d\theta < 1$, for all $\varepsilon > 0$.

$$\mu'_{\tilde{F}_N}(a,b,\lambda) = l\mu_{\tilde{F}_N}(Sa, Aa, \lambda) + m\mu_{\tilde{F}_N}(Sa, Tb, \lambda) + n\mu_{\tilde{F}_N}(Tb, Ab, \lambda) + \max\{\mu_{\tilde{F}_N}(Tb, Aa, \lambda), \mu_{\tilde{F}_N}(Sa, Ab, \lambda)\}$$

$$\nu'_{\tilde{F}_N}(a,b,\lambda) = l\nu_{\tilde{F}_N}(Sa, Aa, \lambda) + m\nu_{\tilde{F}_N}(Sa, Tb, \lambda) + n\nu_{\tilde{F}_N}(Tb, Ab, \lambda) + \max\{\nu_{\tilde{F}_N}(Tb, Aa, \lambda), \nu_{\tilde{F}_N}(Sa, Ab, \lambda)\}$$

$$\omega'_{\tilde{F}_N}(a,b,\lambda) = l\omega_{\tilde{F}_N}(Sa, Aa, \lambda) + m\omega_{\tilde{F}_N}(Sa, Tb, \lambda) + n\omega_{\tilde{F}_N}(Tb, Ab, \lambda) + \max\{\omega_{\tilde{F}_N}(Tb, Aa, \lambda), \omega_{\tilde{F}_N}(Sa, Ab, \lambda)\}$$

for all $0 < q < l + m + n + 1$. Then A, B, S and T have a unique common fixed point.

Proof: Similar to theorem 4.1.

Conclusion

The newly defined infinite products establish the Banach contraction theorem for NMS. In this context, we introduce an integral-type contractive condition in a ε-chainable neutrosophic metric space and prove a common fixed point theorem for four weakly compatible mappings. Our findings extend and unify well-known results in neutrosophic metric spaces, such as those presented by Kirisci and Simsek [28]. Furthermore, Kirisci et al. [21] discussed fixed point results within the framework of NMS.

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Operations on Pythagorean Neutrosophic Fuzzy Matrix

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Abstract. This article discusses various unique varieties of Pythagorean neutrosophic fuzzy matrices (PNFM). Some fundamental operations such as addition, multiplication, union, intersection, complement, and exponential are discussed. This work introduces algebraic operations on PNFM and demonstrates some of its associated theorems, including distributive, commutative, and associative. The purpose of these investigations is to contribute to a better understanding of PNFM and its use in real-world uncertainty.

Keywords: Pythagorean neutrosophic fuzzy matrix; Algebraic sum; Product; Exponential

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1. **Introduction**

The Pythagorean neutrosphic (PN) set is a new mathematical idea for handling problems in the real world that involve vague and imprecise information. It combines neutrosophic fuzzy sets (NFs) with Pythagorean fuzzy sets (PFs). It offers a more straightforward approach to dealing with uncertainty than Pythagorean and neutrosphic fuzzy sets. The square sums of the three elements that make up the PN set—membership, indeterminacy, and non-membership degree—range from zero to two. The concept of a fuzzy matrix (FM) was first proposed by Thomason. It is crucial to the advancement of science. But this traditional fuzzy matrix will only consider the degree of membership. The ambiguity and uncertainty of real-world problems may not be sufficiently captured by conventional fuzzy matrices. The notions of intuitionistic fuzzy matrices (IFM), neutrosophic fuzzy matrices(NFM) and Pythagorean fuzzy matrices (PFM) are developed to address these problems. In 2002, Pal, Khan, and Shyamal introduced the idea of an intuitionistic fuzzy matrix. The neutrosophic fuzzy matrix concept was first presented by Kandasamy and Smarandache in 2004. The Pythagorean fuzzy matrix concept was first proposed by Silambarasan and Sriram in 2018. The following list includes a selection of papers that addressed fuzzy matrix, intuitionistic, neutrosophic, and pythagorean fuzzy matrix. Venkatesan et al. presented applications of fuzzy matrices in decision making [1]. Beaula explored the use of fuzzy matrix in the medical field [2]. Priya et al. addressed the fundamental definitions of fuzzy matrix [3]. Clayton Gilchrist investigated the determinant and K-idempotence features of a fuzzy matrix [4]. Using picture fuzzy sets, Ramakrishnan M and Sriram S talked about new Hamacher procedures [5]. The concepts of restricted picture fuzzy sets and special restricted picture fuzzy sets were introduced by Dogra and Pal [6]. Einstein operations for intuitionistic fuzzy matrices were defined by Selvarajan et al. [7]. A few operations on intuitionistic fuzzy matrices were defined by Emam EG [8]. For intuitionistic fuzzy matrices, Silambarasan and Sriram defined Hamcher multiplication and exponentiation operations [9]. In intuitionistic fuzzy matrix theory, Li W and Ye J introduced the idea of matrix entropy [10]. Jayapriya and Porchelvi discuss a few operators on a neutrosophic fuzzy set [11]. For neutrosophic fuzzy matrices, Das et al. defined algebraic operations like multiplication and subtraction [12]. Commutative, associative, and distributive characteristics on neutrosophic fuzzy matrices were examined by Das D et al. [13]. A specific instance of the neutrosophic fuzzy matrix, the Fermatean neutrosophic fuzzy matrix, was explored by Broumi [14]. Anandhkumar et. al. discussed about Symmetric Fermatean Neutrosophic fuzzy matrix [15]. Silambarasan and Sriram introduced new operations on Pythagorean fuzzy matrix [16]. Silambarasan and Sriram defined new operations for Pythagorean fuzzy matrix [17]. selvarajan and Ramya introduced Einstein sum and product on Pythagorean fuzzy matrix [18]. Silambarasan and Sriram introduced Hamcher scalar multiplication and exponentiation on Pythagorean fuzzy matrix [19]. Radha et. al. discussed about improved correlation coefficient for neutrosphic pythagorean [20] Ismail et. al. Pythagorean neutrosophic set and their basic algebraic operations [21].Gbolagade et. al. [22] discussed about Neutrosophic Poisson distribution with the help of Salagean operator. Satyanarayana and Baji discussed about Neutrosophic ideals [23]. Shams et. al. discussed about DNA sequence matching algorithm [24] The catalogues for the article are as follows: • Section 1 contains basic definitions and some operations. • Section 2 contains types of Pythagorean Neutrosophic fuzzy matrix. • Section 3 contains basic operations for PNFM and some theorems. • Section 4 contains conclusion.

2. Preliminaries

This section provies some basic definitions and operations on Pythagorean neutrosophic fuzzy set

2.1. *Pythagorean Neutrosophic Set*

Let X be a universe of discourse. A Pythagorean neutrosophic set (PN) N on X is defined as

$$N = \{\langle x, \Phi_N(x), \Psi_N(x), \mathcal{L}_N(x)\rangle \mid x \in X\}$$

where

$$0 \leq \Phi_N^2(x) + \Psi_N^2(x) + \mathcal{L}_N^2(x) \leq 2$$

and $\Phi_N(x), \Psi_N(x), \mathcal{L}_N(x) \in [0, 1]$.

Here, $\Phi_N(x), \Psi_N(x), \mathcal{L}_N(x)$ denote the degree of membership, degree of non-membership, and degree of indeterminacy, respectively. In this context, $\Phi_N(x)$ and $\mathcal{L}_N(x)$ are dependent components, while $\Psi_N(x)$ is an independent component.

2.2. *Basic Operations on PN set [21]*

Let X be a non-empty set (universe). Let

$$M = \{\langle x, \Phi_m(x), \Psi_m(x), \mathcal{L}_m(x)\rangle \mid x \in X\}$$

and

$$N = \{\langle x, \Phi_N(x), \Psi_N(x), \mathcal{L}_N(x)\rangle \mid x \in X\}$$

be two Pythagorean neutrosophic sets. Then

2.2.1. *Union of M and N*

M $\cup N = \{\max(\Phi_m, \Phi_N), \min(\Psi_m, \Psi_N), \min(\mathcal{L}_m, \mathcal{L}_N)\}$

2.2.2. *Intersection of M and N*

M $\cap N = \{\min(\Phi_m, \Phi_N), \max(\Psi_m, \Psi_N), \max(\mathcal{L}_m, \mathcal{L}_N)\}$

2.2.3. *Addition of M and N*

M $\oplus N = \left(\sqrt{\Phi_m^2 + \Phi_N^2 - 2\Phi_m\Phi_N}, \Psi_m\Psi_N, \mathcal{L}_m\mathcal{L}_N\right)$

2.2.4. *Multiplication of M and N*

M $\otimes N = \left(\Phi_m\Phi_N, \Psi_m + \Psi_N - \Psi_m\Psi_N, \sqrt{\mathcal{L}_m^2 + \mathcal{L}_N^2 - 2\mathcal{L}_m\mathcal{L}_N}\right)$

On the basis of relationships given in (2.1) and (2.2) the following novel definitions and operations are defined.

2.3. *Types of Pythagorean Neutrosophic fuzzy matrix*

This section introduces some types of Pythagorean neurosophic fuzzy matrix

2.3.1. *Pythagorean Neutrosophic fuzzy matrix*

A Pythagorean Neutrosophic fuzzy matrix of order $m \times n$ is defined as $N = (n_{ij})_{m\times n}$. Here, $n_{ij} = (\Phi_{N_{ij}}(x), \Psi_{N_{ij}}(x), \mathcal{L}_{N_{ij}}(x))$ is the ij-th element of N, where $0 \leq \Phi^2_{N_{ij}}(x) + \Psi^2_{N_{ij}}(x) + \mathcal{L}^2_{N_{ij}}(x) \leq 2$.

2.3.2. *Null Pythagorean Neutrosophic fuzzy matrix*

A square Pythagorean Neutrosophic fuzzy matrix is said to be null Pythagorean Neutrosophic fuzzy matrix if all its entries are zero.

$$A = \begin{bmatrix} (0,1,0) & (0,1,0) & (0,1,0) & \cdots & (0,1,0) \\ (0,1,0) & (0,1,0) & (0,1,0) & \cdots & (0,1,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0,1,0) & (0,1,0) & (0,1,0) & \cdots & (0,1,0) \end{bmatrix}$$

2.3.3. *Unit Pythagorean Neutrosophic fuzzy matrix*

A square Pythagorean Neutrosophic fuzzy matrix is said to be a unit Pythagorean Neutrosophic fuzzy matrix when its all diagonal elements are unit and all other elements are zero

$$B = \begin{bmatrix} (1,0,1) & (0,1,0) & (0,1,0) & \cdots & (0,1,0) \\ (0,1,0) & (1,0,1) & (0,1,0) & \cdots & (0,1,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0,1,0) & (0,1,0) & (0,1,0) & \cdots & (1,0,1) \end{bmatrix}$$

2.3.4. *Symmetric Pythagorean Neutrosophic fuzzy matrix*

A square Pythagorean Neutrosophic fuzzy matrix is said to be symmetric Pythagorean Neutrosophic fuzzy matrix if the matrix and its transpose are equal.

2.3.5. *Triangular Pythagorean Neutrosophic fuzzy matrix*

A square Pythagorean Neutrosophic fuzzy matrix is said to be a triangular Pythagorean Neutrosophic fuzzy matrix if either above main diagonal elements are zero or below main diagonal elements are zero. A square Pythagorean Neutrosophic fuzzy matrix is said to be an upper triangular Pythagorean Neutrosophic fuzzy matrix if above main diagonal elements are zero

$$B = \begin{bmatrix} (\Phi_{11}, \Psi_{11}, \mathcal{L}_{11}) & (0,1,0) & (0,1,0) & \cdots & (0,1,0) \\ (\Phi_{21}, \Psi_{21}, \mathcal{L}_{21}) & (\Phi_{22}, \Psi_{22}, \mathcal{L}_{22}) & (0,1,0) & \cdots & (0,1,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (\Phi_{n1}, \Psi_{n1}, \mathcal{L}_{n1}) & (\Phi_{n2}, \Psi_{n2}, \mathcal{L}_{n2}) & \cdots & \cdots & (\Phi_{nn}, \Psi_{nn}, \mathcal{L}_{nn}) \end{bmatrix}$$

and it is said to be a lower triangular Pythagorean Neutrosophic fuzzy matrix if below main diagonal elements are zero

$$B = \begin{bmatrix} (\Phi_{11}, \Psi_{11}, \mathcal{L}_{11})) & (\Phi_{12}, \Psi_{12}, \mathcal{L}_{12}) & \cdots & (\Phi_{1n}, \Psi_{1n}, \mathcal{L}_{1n}) \\ (0,1,0) & (\Phi_{22}, \Psi_{22}, \mathcal{L}_{22}) & \cdots & (\Phi_{2n}, \Psi_{2n}, \mathcal{L}_{2n}) \\ \vdots & \vdots & \vdots & \ddots \\ (0,1,0) & (0,1,0) & \cdots & (\Phi_{nn}, \Psi_{nn}, \mathcal{L}_{nn}) \end{bmatrix}$$

2.3.6. *Vandermonde Pythagorean Neutrosophic fuzzy matrix*

A Vandermonde Pythagorean Neutrosophic fuzzy matrix of order n is of the form

$$V = \begin{bmatrix} (0,1,0) & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ (0,1,0) & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0,1,0) & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

2.3.7. *DiagonalPythagorean Neutrosophic fuzzy matrix*

A Diagonal Pythagorean Neutrosophic fuzzy matrix is a matrix where all entries of the matrix are zero except the diagonal going from the upper left corner to the lower right corner. The matrix A is given by

$$A = \begin{bmatrix} a & (0,1,0) & (0,1,0) \\ (0,1,0) & b & (0,1,0) \\ (0,1,0) & (0,1,0) & c \end{bmatrix}$$

2.3.8. *Anti-diagonal Pythagorean Neutrosophic fuzzy matrix*

An Anti-diagonal Pythagorean Neutrosophic fuzzy matrix is a matrix where all entries of the matrix are zero except on the diagonal going from the lower left corner to the upper right corner. The matrix A is given by

$$A = \begin{bmatrix} (0,1,0) & (0,1,0) & c \\ (0,1,0) & b & (0,1,0) \\ a & (0,1,0) & (0,1,0) \end{bmatrix}$$

2.4. *Basic operations on PNFM*

Consider two PNFMs, M, N of order $n \times n$. Let

$$M = \begin{bmatrix} (\Phi_{M_{11}}, \Psi_{M_{11}}, \mathcal{L}_{M_{11}}) & (\Phi_{M_{12}}, \Psi_{M_{12}}, \mathcal{L}_{M_{12}}) & \cdots & (\Phi_{M_{1n}}, \Psi_{M_{1n}}, \mathcal{L}_{M_{1n}}) \\ (\Phi_{M_{21}}, \Psi_{M_{21}}, \mathcal{L}_{M_{21}}) & (\Phi_{M_{22}}, \Psi_{M_{22}}, \mathcal{L}_{M_{22}}) & \cdots & (\Phi_{M_{2n}}, \Psi_{M_{2n}}, \mathcal{L}_{M_{2n}}) \\ \vdots & \vdots & \vdots & \vdots \\ (\Phi_{M_{n1}}, \Psi_{M_{n1}}, \mathcal{L}_{M_{n1}}) & (\Phi_{M_{n2}}, \Psi_{M_{n2}}, \mathcal{L}_{M_{n2}}) & \cdots & (\Phi_{M_{nn}}, \Psi_{M_{nn}}, \mathcal{L}_{M_{nn}}) \end{bmatrix}$$

$$N = \begin{bmatrix} (\Phi_{N_{11}}, \Psi_{N_{11}}, \mathcal{L}_{N_{11}}) & (\Phi_{N_{12}}, \Psi_{N_{12}}, \mathcal{L}_{N_{12}}) & \cdots & (\Phi_{N_{1n}}, \Psi_{N_{1n}}, \mathcal{L}_{N_{1n}}) \\ (\Phi_{N_{21}}, \Psi_{N_{21}}, \mathcal{L}_{N_{21}}) & (\Phi_{N_{22}}, \Psi_{N_{22}}, \mathcal{L}_{N_{22}}) & \cdots & (\Phi_{N_{2n}}, \Psi_{N_{2n}}, \mathcal{L}_{N_{2n}}) \\ \vdots & \vdots & \vdots & \vdots \\ (\Phi_{N_{n1}}, \Psi_{N_{n1}}, \mathcal{L}_{N_{n1}}) & (\Phi_{N_{n2}}, \Psi_{N_{n2}}, \mathcal{L}_{N_{n2}}) & \cdots & (\Phi_{N_{nn}}, \Psi_{N_{nn}}, \mathcal{L}_{N_{nn}}) \end{bmatrix}$$

Then the basic operations like Union, Intersection, Complement, Addition, Multiplication, Scalar multiplication and Power operation are defined as follows respectively:

2.4.1. $M \cup N = \left\langle \max\left(\Phi_{M_{ij}}, \Phi_{N_{ij}}\right), \min\left(\Psi_{M_{ij}}, \Psi_{N_{ij}}\right), \min\left(\mathcal{L}_{M_{ij}}, \mathcal{L}_{N_{ij}}\right)\right\rangle$

2.4.2. $M \cap N = \left\langle \min\left(\Phi_{M_{ij}}, \Phi_{N_{ij}}\right), \max\left(\Psi_{M_{ij}}, \Psi_{N_{ij}}\right), \max\left(\mathcal{L}_{M_{ij}}, \mathcal{L}_{N_{ij}}\right)\right\rangle$

2.4.3. $M \oplus N = \left(\sqrt{\Phi^2_{M_{ij}} + \Phi^2_{N_{ij}} - \Phi^2_{M_{ij}}\Phi^2_{N_{ij}}}, \Psi_{M_{ij}}\Psi_{N_{ij}}, \mathcal{L}_{M_{ij}}\mathcal{L}_{N_{ij}}\right)$

2.4.4. $M \otimes N = \left(\Phi_{M_{ij}}\Phi_{N_{ij}}, \sqrt{\Psi^2_{M_{ij}} + \Psi^2_{N_{ij}} - \Psi^2_{M_{ij}}\Psi^2_{N_{ij}}}, \sqrt{\mathcal{L}^2_{M_{ij}} + \mathcal{L}^2_{N_{ij}} - \mathcal{L}^2_{M_{ij}}\mathcal{L}^2_{N_{ij}}}\right)$

2.4.5. $nM = M \oplus M \oplus \cdots \oplus M = \left(\sqrt{1 - (1 - \Phi^2_{M_{ij}})^n}, \Psi^n_{M_{ij}}, \mathcal{L}^n_{M_{ij}}\right)$

2.4.6. $M^n = \left(\Phi^n_{M_{ij}}, \sqrt{1 - (1 - \Psi^2_{M_{ij}})^n}, \sqrt{1 - (1 - \mathcal{L}^2_{M_{ij}})^n}\right)$

2.5. *Numerical Examples*

Consider two matrices M and N of order 2×2 matrix. Let

$$M = \begin{bmatrix} (0.9, 0.1, 0.12) & (0.8, 0.2, 0.3) \\ (0.7, 0.3, 0.4) & (0.5, 0.4, 0.3) \end{bmatrix}$$

$$N = \begin{bmatrix} (0.7, 0.3, 0.2) & (0.3, 0.7, 0.8) \\ (0.6, 0.5, 0.3) & (0.1, 0.8, 0.9) \end{bmatrix}$$

Then M $\cup N = \begin{bmatrix} (0.9, 0.1, 0.12) & (0.8, 0.2, 0.3) \\ (0.7, 0.3, 0.3) & (0.5, 0.4, 0.3) \end{bmatrix}$

M $\cap N = \begin{bmatrix} (0.7, 0.3, 0.2) & (0.3, 0.7, 0.8) \\ (0.6, 0.5, 0.4) & (0.1, 0.8, 0.9) \end{bmatrix}$

M $\oplus n = \begin{bmatrix} (0.2, 0.03, 0.024) & (0.5, 0.14, 0.24) \\ (0.1, 0.15, 0.12) & (0.4, 0.32, 0.27) \end{bmatrix}$

$M \otimes n = \begin{bmatrix} (0.63, 0.2, 0.024) & (0.24, 0.5, 0.24) \\ (0.42, 0.2, 0.12) & (0.05, 0.4, 0.27) \end{bmatrix}$

$nM = \begin{bmatrix} (0.99, 0.001, 0.001) & (0.97, 0.008, 0.009) \\ (0.93, 0.009, 0.064) & (0.42, 0.064, 0.009) \end{bmatrix}$

$M^n = \begin{bmatrix} (0.73, 0.17, 0.3) & (0.51, 0.33, 0.6) \\ (0.34, 0.49, 0.78) & (0.12, 0.63, 0.6) \end{bmatrix}$

Theorem 1:*LetM* and N be two PNFM. If $n > 0$, then

1.1) $n(M \oplus N) = nM \oplus nN$

1.2) $(M \otimes N)^n = M^n \otimes N^n$

Proof

1.1)

$$\text{n(M} \oplus N) = \left(\sqrt{1-\left(1-\left(\Phi_{M_{ij}}^2+\Phi_{N_{ij}}^2-\Phi_{M_{ij}}^2\Phi_{N_{ij}}^2\right)\right)^n}, \Psi_{M_{ij}}^n\Psi_{N_{ij}}^n, \mathcal{L}_{M_{ij}}^n\mathcal{L}_{N_{ij}}^n\right) \quad \ldots \quad (1.1.1)$$

$$nM = \left(\sqrt{1-(1-\Phi_{M_{ij}}^2)^n}, \Psi_{M_{ij}}^n, \mathcal{L}_{M_{ij}}^n\right)$$

$$nN = \left(\sqrt{1-(1-\Phi_{N_{ij}}^2)^n}, \Psi_{N_{ij}}^n, \mathcal{L}_{N_{ij}}^n\right)$$

$$nM \oplus nN$$

$$= \left(\sqrt{1-\left(1-\Phi_{M_{ij}}^2\right)^n+1-\left(1-\Phi_{N_{ij}}^2\right)^n-\left(1-\left(1-\Phi_{M_{ij}}^2\right)^n\right)\left(1-\left(1-\Phi_{N_{ij}}^2\right)^n\right)}, \Psi_{M_{ij}}^n, \Psi_{N_{ij}}^n\mathcal{L}_{M_{ij}}^n\mathcal{L}_{N_{ij}}^n\right)$$

$$= \left(\sqrt{1-\left(1-\Phi_{M_{ij}}^2\right)^n\left(1-\Phi_{N_{ij}}^2\right)^n}, \Psi_{M_{ij}}^n\Psi_{N_{ij}}^n, \mathcal{L}_{M_{ij}}^n\mathcal{L}_{N_{ij}}^n\right)$$

$$= \left(\sqrt{1-\left(\left(1-\Phi_{M_{ij}}^2\right)\left(1-\Phi_{N_{ij}}^2\right)\right)^n}, \Psi_{M_{ij}}^n\Psi_{N_{ij}}^n, \mathcal{L}_{M_{ij}}^n\mathcal{L}_{N_{ij}}^n\right)$$

$$= \left(\sqrt{1-\left(1-\left(\Phi_{M_{ij}}^2-\Phi_{N_{ij}}^2+\Phi_{M_{ij}}^2\Phi_{N_{ij}}^2\right)\right)^n}, \Psi_{M_{ij}}^n\Psi_{N_{ij}}^n, \mathcal{L}_{M_{ij}}^n\mathcal{L}_{N_{ij}}^n\right) \quad \ldots \quad (1.1.2)$$

From (1.1.1) and (1.1.2) we can get (1.1)

1.2)

$$(M \otimes N)^n = \left(\left(\Phi_{Mij}\Phi_{Nij}, \sqrt{\Psi_{M}{}_{ij}^2+\Psi_{N}{}_{ij}^2-\Psi_{M}{}_{ij}^2\Psi_{N}{}_{ij}^2}, \sqrt{\mathcal{L}_{M}{}_{ij}^2+\mathcal{L}_{N}{}_{ij}^2-\mathcal{L}_{M}{}_{ij}^2\mathcal{L}_{N}{}_{ij}^2}\right)\right)^n$$

$$= \left((\Phi_{Mij}\Phi_{Nij})^n, \sqrt{1-\left(1-(\Psi_{M}{}_{ij}^2+\Psi_{N}{}_{ij}^2-\Psi_{M}{}_{ij}^2\Psi_{N}{}_{ij}^2\right)^n}\right), \sqrt{1-\left(1-(\mathcal{L}_{M}{}_{ij}^2+\mathcal{L}_{N}{}_{ij}^2-\mathcal{L}_{M}{}_{ij}^2\mathcal{L}_{N}{}_{ij}^2\right)^n}) \quad \ldots \quad (1.2.1)$$

$$M^n = \left(\Phi_{M_{ij}}^n, \sqrt{1-(1-\Psi_{M_{ij}}^2)^n}, \sqrt{1-(1-\mathcal{L}_{M_{ij}}^2)^n}\right)$$

$$N^n = \left(\Phi_{N_{ij}}^n, \sqrt{1-(1-\Psi_{N_{ij}}^2)^n}, \sqrt{1-(1-\mathcal{L}_{N_{ij}}^2)^n}\right)$$

$$M^n \otimes N^n$$

$$= \left(\Phi_{M}{}_{ij}^n\Phi_{N}{}_{ij}^n, \sqrt{1-\left(1-(\Psi_{M}{}_{ij}^2\right)^n+1-\left(1-(\Psi_{N}{}_{ij}^2\right)^n-\left(1-\left(1-(\Psi_{M}{}_{ij}^2\right)^n\right)\left(1-\left(1-(\Psi_{N}{}_{ij}^2\right)^n\right)},\right.$$
$$\sqrt{1-\left(1-(\mathcal{L}_{M}{}_{ij}^2\right)^n+1-\left(1-(\mathcal{L}_{N}{}_{ij}^2\right)^n-\left(1-\left(1-(\mathcal{L}_{M}{}_{ij}^2\right)^n\right)\left(1-\left(1-(\mathcal{L}_{N}{}_{ij}^2\right)^n\right)}$$

$$= \left((\Phi_{Mij}\Phi_{Nij})^n, \sqrt{1-\left(1-(\Psi_{M}{}_{ij}^2+\Psi_{N}{}_{ij}^2-\Psi_{M}{}_{ij}^2\Psi_{N}{}_{ij}^2\right)^n}\right), \sqrt{1-\left(1-(\mathcal{L}_{M}{}_{ij}^2+\mathcal{L}_{N}{}_{ij}^2-\mathcal{L}_{M}{}_{ij}^2\mathcal{L}_{N}{}_{ij}^2\right)^n}) \quad \ldots \quad (1.2.2)$$

From (1.2.1) and (1.2.2) we get (1.2)

Theorem 2: Let M and N be two PNFM. If $n > 0$, then

2.1) $\text{n(M} \cup N) = nM \cup nN$

2.2) $(M \cup N)^n = M^n \cup N^n$

Proof:2.1) $n(M \cup N) = \left(\sqrt{1-\left(1-\max\left\{\Phi_{M}{}_{ij}^2, \Phi_{N}{}_{ij}^2\right\}\right)^n}, \min\left\{\Psi_{M}{}_{ij}^n, \Psi_{N}{}_{ij}^n\right\}, \min\left\{\mathcal{L}_{M}{}_{ij}^n, \mathcal{L}_{N}{}_{ij}^n\right\}\right) \quad \ldots \quad (2.1.1)$

$$\text{nM} \cup nN = \left(\sqrt{1-\left(1-\Phi_{M_{ij}^{2}}\right)^{n}}, \Psi_{M_{ij}^{n}}, \mathcal{L}_{M_{ij}^{n}}\right) \cup \left(\sqrt{1-\left(1-\Phi_{N_{ij}^{2}}\right)^{n}}, \Psi_{N_{ij}^{n}}, \mathcal{L}_{N_{ij}^{n}}\right)$$
$$= \left(\max\left\{\sqrt{1-\left(1-\Phi_{M_{ij}^{2}}\right)^{n}}, \sqrt{1-\left(1-\Phi_{N_{ij}^{2}}\right)^{n}}\right\}, \min\left\{\Psi_{M_{ij}^{n}}, \Psi_{N_{ij}^{n}}\right\}, \min\left\{\mathcal{L}_{M_{ij}^{n}}, \mathcal{L}_{N_{ij}^{n}}\right\}\right)$$
$$= \left(\sqrt{1-\left(1-\max\left\{\Phi_{M_{ij}^{2}}, \Phi_{N_{ij}^{2}}\right\}\right)^{n}}, \min\left\{\Psi_{M_{ij}^{n}}, \Psi_{N_{ij}^{n}}\right\}, \min\left\{\mathcal{L}_{M_{ij}^{n}}, \mathcal{L}_{N_{ij}^{n}}\right\}\right) \quad \ldots \quad (2.1.2)$$

From (2.1.1) and (2.1.2), we get n(M $\cup N) = nM \cup nN$

2.2)Analogously, it can be shown that $(M \cup N)^n = M^n \cup N^n$

Theorem 3: Let M and N be two PNFM, then

3.1) (M $\cup N) \oplus (M \cap N) = M \oplus N$

3.2)$(M \cup N) \otimes (M \cap N) = M \otimes N$

Proof3.1)$(M \cup N) \oplus (M \cap N) = (\langle \max\{\Phi_{Mij}, \Phi_{Nij}\}, \min\{\Psi_{Mij}, \Psi_{Nij}\}, \min\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\}\rangle \oplus$
$\langle \min\{\Phi_{Mij}, \Phi_{Nij}\}, \max\{\Psi_{Mij}, \Psi_{Nij}\}, \max\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\}\rangle$

$$= \left(\sqrt{\max\left\{\Phi_{M_{ij}^{2}}, \Phi_{N_{ij}^{2}}\right\} + \min\left\{\Phi_{M_{ij}^{2}}, \Phi_{N_{ij}^{2}}\right\} - \max\left\{\Phi_{M_{ij}^{2}}, \Phi_{N_{ij}^{2}}\right\} \min\left\{\Phi_{M_{ij}^{2}}, \Phi_{N_{ij}^{2}}\right\}}, \min\{\Psi_{Mij}, \Psi_{Nij}\}\right.$$
$$\left.\max\{\Psi_{Mij}, \Psi_{Nij}\}, \min\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\} \max\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\}\right)$$
$$= \left(\sqrt{\Phi_{M_{ij}^{2}} + \Phi_{N_{ij}^{2}} - \Phi_{M_{ij}^{2}} \Phi_{N_{ij}^{2}}}, \Psi_{Mij}\Psi_{Nij}, \mathcal{L}_{Mij}\mathcal{L}_{Nij}\right)$$
$$= M \oplus N$$

3.2)Analogously, it can be shown that $(M \cup N) \otimes (M \cap N) = M \otimes N$

Theorem 4: Let M , N and L be three PNFM), then

4.1) (M $\cup N) \cap L = (M \cap L) \cup (N \cap L)$

4.2) $(M \cap N) \cup L = (M \cup L) \cap (N \cup L)$

4.3) $(M \cup N) \oplus L = (M \oplus L) \cup (N \oplus L)$

4.4) $(M \cap N) \oplus L = (M \oplus L) \cap (N \oplus L)$

4.5) $(M \cup N) \otimes L = (M \otimes L) \cup (N \otimes L)$

4.6) $(M \cap N) \otimes L = (M \otimes L) \cap (N \otimes L)$

Proof

Let us prove 4.1, 4.3 and 4.5. 4.2, 4.4 and 4.6

4.1)(M $\cup N) \cap L = (\min\{\max\{\Phi_{Mij}, \Phi_{Nij}\}, \Phi_{Lij}\}, \max\{\min\{\Psi_{Mij}, \Psi_{Nij}\}, \Psi_{Lij}\},$
$\max\{\min\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\}, \mathcal{L}_{Lij}\}) \quad \ldots \quad (4.1.1)$

$M \cap L = \left\langle \min\left(\Phi_{M_{ij}}, \Phi_{L_{ij}}\right), \max\left(\Psi_{M_{ij}}, \Psi_{L_{ij}}\right), \max\left(\mathcal{L}_{M_{ij}}, \mathcal{L}_{L_{ij}}\right)\right\rangle$

$N \cap L = \left\langle \min\left(\Phi_{N_{ij}}, \Phi_{L_{ij}}\right), \max\left(\Psi_{N_{ij}}, \Psi_{L_{ij}}\right), \max\left(\mathcal{L}_{N_{ij}}, \mathcal{L}_{L_{ij}}\right)\right\rangle$

$(M \cap L) \cup (N \cap L)$

$= (\max\{\min\{\Phi_{Mij}, \Phi_{Lij}\}, \min\{\Phi_{Nij}, \Phi_{Lij}\}\}, \min\{\max\{\Psi_{Mij}, \Psi_{Lij}\}, \max\{\Psi_{Lij}, \Psi_{Lij}\}\},$

$\min\{\max\{\mathcal{L}_{Mij}, \mathcal{L}_{Lij}\}, \max\{\mathcal{L}_{Nij}, \mathcal{L}_{Lij}\}\})$

$= (\min\{\max\{\Phi_{Mij}, \Phi_{Nij}\}, \Phi_{Lij}\}, \max\{\min\{\Psi_{Mij}, \Psi_{Nij}\}, \Psi_{Lij}\},$

$\max\{\min\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\}, \mathcal{L}_{Lij}\}) \quad \ldots \quad (4.1.2)$

From (4.1.1) and (4.1.2) we get (4.1)

4.3)$(M \cup N) \oplus L =$

$(\max\{\Phi_{Mij}, \Phi_{Nij}\}, \min\{\Psi_{Mij}, \Psi_{Lij}\}, \min\{\mathcal{L}_{Mij}, \mathcal{L}_{Lij}\}) \oplus (\Phi_{Lij}, \Psi_{Lij}, \mathcal{L}_{Lij})$

$$= \left(\sqrt{\max\left\{\Phi_{M_{ij}^2}, \Phi_{N_{ij}^2}\right\} + \Phi_{L_{ij}^2} - \max\left\{\Phi_{M_{ij}^2}, \Phi_{N_{ij}^2}\right\}\Phi_{L_{ij}^2}}, \min\{\Psi_{Mij}, \Psi_{Nij}\}\Psi_{Lij}, \min\{\mathcal{L}_{Mij}, \mathcal{L}_{Nij}\}\mathcal{L}_{Lij}\right)$$

$$= \left(\sqrt{(1-\Phi_{L_{ij}^2})\max\left\{\Phi_{M_{ij}^2}, \Phi_{N_{ij}^2}\right\} + \Phi_{L_{ij}^2}}, \min\{\Psi_{Mij}, \Psi_{Lij}, \Psi_{Nij}, \Psi_{Lij}\},\right.$$
$$\min\{\mathcal{L}_{Mij}, \mathcal{L}_{Lij}, \mathcal{L}_{Nij}, \mathcal{L}_{Lij}\}) \quad \ldots \quad (4.3.1)$$

$(M \oplus L) \cup (N \oplus L) =$

$$\left(\max\left\{\sqrt{\Phi_{M_{ij}^2} + \Phi_{L_{ij}^2} - \Phi_{M_{ij}^2}\Phi_{L_{ij}^2}}, \sqrt{\Phi_{N_{ij}^2} + \Phi_{L_{ij}^2} - \Phi_{N_{ij}^2}\Phi_{L_{ij}^2}}\right\}, \min\{\Psi_{Mij}, \Psi_{Lij}, \Psi_{Nij}, \Psi_{Lij}\},\right.$$
$$\min\{\mathcal{L}_{Mij}, \mathcal{L}_{Lij}, \mathcal{L}_{Nij}, \mathcal{L}_{Lij}\}$$

$$= \left(\sqrt{(1-\Phi_{L_{ij}^2})\Phi_{M_{ij}^2} + \Phi_{L_{ij}^2}}, \sqrt{(1-\Phi_{L_{ij}^2})\Phi_{N_{ij}^2} + \Phi_{L_{ij}^2}}, \min\{\Psi_{Mij}, \Psi_{Lij}, \Psi_{Nij}, \Psi_{Lij}\}, \min\{\mathcal{L}_{Mij}, \mathcal{L}_{Lij}, \mathcal{L}_{Nij}, \mathcal{L}_{Lij}\}\right)$$

$$= \left(\sqrt{(1-\Phi_{L_{ij}^2})\max\left\{\Phi_{M_{ij}^2}, \Phi_{N_{ij}^2}\right\} + \Phi_{L_{ij}^2}}, \min\{\Psi_{Mij}, \Psi_{Lij}, \Psi_{Nij}, \Psi_{Lij}\},\right.$$
$$\min\{\mathcal{L}_{Mij}, \mathcal{L}_{Lij}, \mathcal{L}_{Nij}, \mathcal{L}_{Lij}\}) \quad \ldots \quad (4.3.2)$$

From (4.3.1) and (4.3.2) we get the result 4.3

(M $\cup N) \otimes L =$

$$\left(\max\{\Phi_{Mij}, \Phi_{Nij}\}, \sqrt{\min\left\{\Phi_{M_{ij}^2}, \Phi_{N_{ij}^2}\right\} + \Phi_{L_{ij}^2} - \min\left\{\Phi_{M_{ij}^2}, \Phi_{N_{ij}^2}\right\}\Phi_{L_{ij}^2}},\right.$$
$$\sqrt{\min\left\{\mathcal{L}_{M_{ij}^2}, \mathcal{L}_{N_{ij}^2}\right\} + \mathcal{L}_{L_{ij}^2} - \min\left\{\mathcal{L}_{M_{ij}^2}, \mathcal{L}_{N_{ij}^2}\right\}\mathcal{L}_{L_{ij}^2}})$$

$$= \left(\max\{\Phi_{Mij}, \Phi_{Nij}\}, \sqrt{(1-\Psi_{L_{ij}^2})\min\left\{\Psi_{M_{ij}^2}, \Psi_{N_{ij}^2}\right\} + \Psi_{L_{ij}^2}}, \sqrt{(1-\mathcal{L}_{L_{ij}^2})\min\left\{\mathcal{L}_{M_{ij}^2}, \mathcal{L}_{N_{ij}^2}\right\} + \mathcal{L}_{L_{ij}^2}}\right) \quad \ldots \quad (4.5.1)$$

(M $\otimes L) \cup (N \otimes L) =$

$$\left(\max\{\Phi_{Mij}, \Phi_{Lij}, \Phi_{Nij}, \Phi_{Lij}\}, \min\left\{\sqrt{\Phi_{M_{ij}^2} + \Phi_{L_{ij}^2} - \Phi_{M_{ij}^2}\Phi_{L_{ij}^2}}, \sqrt{\Phi_{N_{ij}^2} + \Phi_{L_{ij}^2} - \Phi_{N_{ij}^2}\Phi_{L_{ij}^2}}\right\},\right.$$
$$\min\{\mathcal{L}_{Mij} + \mathcal{L}_{Lij} - \mathcal{L}_{Mij}\mathcal{L}_{Lij}, \mathcal{L}_{Nij} + \mathcal{L}_{Lij} - \mathcal{L}_{Nij}\mathcal{L}_{Lij}\})$$

$$= \left(\max\{\Phi_{Mij}, \Phi_{Nij}, \Phi_{Lij}\}, \sqrt{(1-\Psi_{L_{ij}^2})\min\left\{\Psi_{M_{ij}^2}, \Psi_{N_{ij}^2}\right\} + \Psi_{L_{ij}^2}},\right.$$
$$\sqrt{(1-\mathcal{L}_{L_{ij}^2})\min\left\{\mathcal{L}_{M_{ij}^2}, \mathcal{L}_{N_{ij}^2}\right\} + \mathcal{L}_{L_{ij}^2}}) \quad \ldots \quad (4.5.2)$$

From (4.5.1) and (4.5.2) we get the results

3. Comparison

Characteristics	FM	IFM	NFM	PFM	PNFM
Components	(Φ)	(ϕ,Ψ)	$(\Phi,\Psi,\mathcal{L})$	(ϕ,Ψ)	$(\Phi,\Psi,\mathcal{L})$
Condition	$0\leq\Phi\leq 1$	$0\leq\Phi+\Psi\leq 1$	$0\leq\Phi+\Psi+\mathcal{L}\leq 3$	$0\leq\Phi^2+\Psi^2\leq 1$	$0\leq\Phi^2+\Psi^2+\mathcal{L}^2\leq 2$
Complexity	Less complex	Moderately complex	Complex	More complex	Most complex
Applicablity	Simple decison-making	Incomplete information	High uncertainty and indeterminacy	Nuanced representation of uncertainty	Complex problems with multiple sources of uncertainty
Limitation	Lacks to represent non-membership values	Difficulty in handling indeterminacy	can't handle high-dimensional data	Intensive in computing	Requires specialized software

Table 1. Comparison of different types of fuzzy matrices

PNFM is an important tool for representing uncertainty. It combines the ideas of a pythagorean fuzzy matrix and a neutrosophic fuzzy matrix, providing a more thorough approach to dealing with the complexity of real-world issues. In addition to to handling vagueness, it also responds to the situations in an ambiguous and inconsistent manner. The Pythagorean neutrosophic fuzzy matrix notion can be applied to a variety of domains, including decision making and pattern recognition, when ambiguity and vagueness are present.

4. Future Work

This article gives a strong foundation for future research in PNFM. It can focus on the effectiveness of PNFM in real world problems. Although PNFM has many applications, it also has limits. PNFM's computational component is more difficult, and there are no defined aggregation methods. To address these challenges, future work in this subject should focus on creating computing algorithms and improved aggregation approaches.

5. Conclusion

The idea of PNFM and its basic functions were examined in this study. The characteristics and behavior of PNFM under different operations are established by the theorems in this paper. This concept provides the foundation for further investigation into more complex PNFM applications in fields such as machine learning and optimization.

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Refined Neutrosophic Set approach for Musculoskeletal disorders

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Abstract. Musculoskeletal disorders are often characterized by persistent joint pain, stiffness, swelling, and reduced mobility, which can severely impact daily functioning and quality of life. These conditions often present overlapping symptoms, making accurate diagnosis a significant challenge in clinical settings. Traditional diagnostic approaches may struggle with the inherent uncertainty, vagueness, and imprecision found in medical data. This paper proposes a novel approach for classifying four musculoskeletal disorders using the Neutrosophic Refined Set framework, with a particular focus on correlation and distance-based measures. Neutrosophic Refined Set, an advanced extension of neutrosophic logic, provides a robust mathematical model for handling indeterminate and inconsistent information, making it particularly well-suited for healthcare applications.

The proposed work is illustrated with four musculoskeletal disorders - Rheumatoid Arthritis, Osteoarthritis, Lupus, Bursitis and the eight associated Rheumatic symptoms. Correlation analysis is employed to determine the most prevalent symptom group associated with each disorder. Also, the distance measures - Hamming distance, Normalized Hamming distance, Euclidean distance, and Normalized Euclidean distance are used to compute the proximity of four patient cases to each disorder. Python implementation of the proposed method is used to streamline the computational process, offering a faster, more accurate, and less error-prone alternative to manual calculations. Also, highly informative visualizations are employed to illustrate the general trends in symptom intensity and distribution across the four musculoskeletal disorders. Overall, this approach demonstrates the potential of Neutrosophic Refined Sets in enhancing diagnostic accuracy, managing uncertainty, and supports clinical decision-making.

Keywords: Musculoskeletal disorders, Rheumatoid Arthritis, Osteoarthritis, Lupus, Bursitis, Neutrosophic set, Neutrosophic Refined set, Dimension, Hamming distance, Normalised hamming distance, Euclidean distance, Normalised euclidean distance, Correlation Measure

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1. Introduction

Neutrosophic logic and sets, introduced by Florentin Smarandache in 1995 [12, 13], offer a powerful framework for handling uncertainty and imprecision in various real-world applications, including decision making, medical diagnosis and image processing. Unlike the traditional binary logic (True or False) or fuzzy logic that allows degree of truth, neutrosophic sets introduce the concept of indeterminacy, thus incorporating three independent components: truth (T), indeterminacy (I), and fallacy (F). This allows for a more nuanced representation of information making it particularly useful in scenarios where ambiguity is prevalent.

In recent years, the application of soft computing techniques, particularly those based on neutrosophic sets and fuzzy logic, has seen significant growth in the field of medical diagnosis. Several Python-based frameworks and tools have emerged to support modeling and reasoning under uncertainty [7–9]. These developments reflect a shift toward practical, open-source implementations that bridge theoretical methods with real-world diagnostic applications. Recent studies have also extended neutrosophic similarity and correlation measures to domains such as COVID-19 detection and pattern recognition [18, 20], reinforcing the relevance of soft computing approaches in contemporary medical decision support systems.

Neutrosophic Refined Sets, which was introduced by Irfan Deli, Said Broumi, Florentin Smarandache (2015) [1], is a generalisation of fuzzy multisets and intuitionstic fuzzy multisets. Similar to Neutrosophic sets they have the three components - truth (T), indeterminacy (I), and fallacy (F) but allow multiple values of these components, referred to as the dimension (P) of the set. This multi-valued nature enhances the ability of Neutrosophic Refined sets to capture complex real-world scenarios with varying degrees of uncertainty. The work [1] includes the definitions for the operations such as complement, union, intersection, convex, strongly convex, containment, equality and the four distance measures - hamming distance, normalised hamming distance, euclidean distance and normalised euclidean distance. Paper [14] gives the definition for the correlation measure, which can be used to find the similarity or dissimilarity between two Neutrosophic Refined sets.

For the purpose of illustrating the methodology proposed four musculoskeletal disorders - Rheumatoid Arthritis, Osteoarthritis, Lupus and Bursitis are considered. Rheumatic symptoms are generally associated with Rheumatic diseases, that are characterized by conditions affecting the muscles, bones, joints, tendons, ligaments, cartilage often causing pain, swelling, and stiffness and are sometimes referred to as musculoskeletal disorders. Individuals affected by any of these four disorders will experience one or more of these eight rheumatic symptoms

- joint pain, stiffness in the joints especially during morning, fatigue, swelling in the joints, difficulty while moving the joints, persistence of pain in the joints while at rest, round scaly rashes in the skin and pain in the smaller joints.

The remainder of this paper is organized as follows. Section 2 introduces the fundamental definitions related to Neutrosophic Refined Sets, which form the basis for subsequent discussions, and provides a brief overview of the four musculoskeletal disorders under consideration. Section 3 outlines the proposed work, with the Python implementation of correlation and distance measures for Neutrosophic Refined Sets [5–9] and presents three illustrative examples [1–4, 14–20]: the first demonstrates the classification of disorders based on symptom groups; the second focuses on patient diagnosis by computing the distance between a patient's symptom profile and each of the four disorders using the distance measures; and the third offers informative visualizations that reflect the general symptom trends associated with each disorder. Finally, Section 4 concludes the study.

2. Preliminaries

This section delves into the foundational concepts of Neutrosophic Refined Sets (NRS) which are used in the further sections of this paper. Also, a detailed overview of the four musculoskeletal disorders that are the focus of this study is provided.

2.1. *Definitions*

Definition 1. *[1] Let E be the universe. Then a Neutrosophic Refined set A defined on E with dimension P is given by*

$$A = \{(T^1_{\mathtt{A}}(x), T^2_{\mathtt{A}}(x), ..., T^P_{\mathtt{A}}(x)), (I^1_{\mathtt{A}}(x), I^2_{\mathtt{A}}(x), ..., I^P_{\mathtt{A}}(x)), (F^1_{\mathtt{A}}(x), F^2_{\mathtt{A}}(x), ..., F^P_{\mathtt{A}}(x)) : x \in X\} \tag{1}$$

Another Neutrosophic Refined set B defined on E with the same dimension P will be of the form

$$B = \{(T^1_{\mathtt{B}}(x), T^2_{\mathtt{B}}(x), ..., T^P_{\mathtt{B}}(x)), (I^1_{\mathtt{B}}(x), I^2_{\mathtt{B}}(x), ..., I^P_{\mathtt{B}}(x)), (F^1_{\mathtt{B}}(x), F^2_{\mathtt{B}}(x), ..., F^P_{\mathtt{B}}(x)) : x \in X\} \tag{2}$$

Definition 2. *[1] Hamming distance between A and B is denoted as $d_{\mathtt{HD}}(A, B)$ and is given by*

$$d_{\mathtt{HD}}(A, B) = \sum_{j=1}^{P} \sum_{i=1}^{n} (|T^j_{\mathtt{A}}(x_{\mathtt{i}}) - T^j_{\mathtt{B}}(x_{\mathtt{i}})| + |I^j_{\mathtt{A}}(x_{\mathtt{i}}) - I^j_{\mathtt{B}}(x_{\mathtt{i}})| + |F^j_{\mathtt{A}}(x_{\mathtt{i}}) - F^j_{\mathtt{B}}(x_{\mathtt{i}})|) \tag{3}$$

Definition 3. *[1] Normalised Hamming distance between A and B is denoted as* $d_{\mathtt{NHD}}(A, B)$ *and is given by*

$$d_{\mathtt{NHD}}(A,B) = \frac{1}{3nP}\sum_{j=1}^{P}\sum_{i=1}^{n}(|T_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - T_{\mathtt{B}}^{j}(x_{\mathtt{i}})| + |I_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - I_{\mathtt{B}}^{j}(x_{\mathtt{i}})| + |F_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - F_{\mathtt{B}}^{j}(x_{\mathtt{i}})|) \quad (4)$$

Definition 4. *[1] Euclidean distance between A and B is denoted as* $d_{\mathtt{ED}}(A, B)$ *and is given by*

$$d_{\mathtt{ED}}(A,B) = \sum_{j=1}^{P}\sum_{i=1}^{n}\sqrt{(T_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - T_{\mathtt{B}}^{j}(x_{\mathtt{i}}))^2 + (I_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - I_{\mathtt{B}}^{j}(x_{\mathtt{i}}))^2 + (F_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - F_{\mathtt{B}}^{j}(x_{\mathtt{i}}))^2} \quad (5)$$

Definition 5. *[1] Normalised Euclidean distance A and B is denoted as* $d_{\mathtt{NED}}(A, B)$ *and is given by*

$$d_{\mathtt{NED}}(A,B) = \frac{1}{3nP}\sum_{j=1}^{P}\sum_{i=1}^{n}\sqrt{(T_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - T_{\mathtt{B}}^{j}(x_{\mathtt{i}}))^2 + (I_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - I_{\mathtt{B}}^{j}(x_{\mathtt{i}}))^2 + (F_{\mathtt{A}}^{j}(x_{\mathtt{i}}) - F_{\mathtt{B}}^{j}(x_{\mathtt{i}}))^2} \quad (6)$$

Definition 6. *[14] Correlation measure between A and B is denoted as* $\rho_{\mathtt{NRS}}(A, B)$ *and is given by*

$$C_{\mathtt{NRS}}(A,B) = \frac{1}{P}\sum_{j=1}^{P}\sum_{i=1}^{n}(T_{\mathtt{A}}^{j}(x_{\mathtt{i}}) * T_{\mathtt{B}}^{j}(x_{\mathtt{i}}) + I_{\mathtt{A}}^{j}(x_{\mathtt{i}}) * I_{\mathtt{B}}^{j}(x_{\mathtt{i}}) + F_{\mathtt{A}}^{j}(x_{\mathtt{i}}) * F_{\mathtt{B}}^{j}(x_{\mathtt{i}}))$$

$$C_{\mathtt{NRS}}(A,A) = \frac{1}{P}\sum_{j=1}^{P}\sum_{i=1}^{n}(T_{\mathtt{A}}^{j}(x_{\mathtt{i}}) * T_{\mathtt{A}}^{j}(x_{\mathtt{i}}) + I_{\mathtt{A}}^{j}(x_{\mathtt{i}}) * I_{\mathtt{A}}^{j}(x_{\mathtt{i}}) + F_{\mathtt{A}}^{j}(x_{\mathtt{i}}) * F_{\mathtt{A}}^{j}(x_{\mathtt{i}}))$$

$$C_{\mathtt{NRS}}(B,B) = \frac{1}{P}\sum_{j=1}^{P}\sum_{i=1}^{n}(T_{\mathtt{B}}^{j}(x_{\mathtt{i}}) * T_{\mathtt{B}}^{j}(x_{\mathtt{i}}) + I_{\mathtt{B}}^{j}(x_{\mathtt{i}}) * I_{\mathtt{B}}^{j}(x_{\mathtt{i}}) + F_{\mathtt{B}}^{j}(x_{\mathtt{i}}) * F_{\mathtt{B}}^{j}(x_{\mathtt{i}}))$$

$$\rho_{\mathtt{NRS}}(A,B) = \frac{C_{\mathtt{NRS}}(A,B)}{\sqrt{C_{\mathtt{NRS}}(A,A) * C_{\mathtt{NRS}}(B,B)}} \quad (7)$$

To demonstrate the application of distance and correlation measures in a medical context, consider two patients A and B having fever. The Neutrosophic Refined Sets formed with the T, I and F values recorded three times a day for both the patients are presented below.

$$\text{Patient}_{\mathtt{A}} = \{(0.85, 1.0, 0.85),\ (0.15, 0.25, 0.1),\ (0.2, 0.15, 0.05)\}$$

$$\text{Patient}_{\mathtt{B}} = \{(0.85, 0.95, 0.8),\ (0.15, 0.2, 0.05),\ (0.25, 0.1, 0.1)\}$$

From the values of the distance and correlation measures presented in Table 1, it can be observed that the distance measures indicate a low to moderate difference between the patients, while the correlation measure highlights a strong similarity. Together, these results emphasize that patients A and B exhibit a similar pattern or nature of fever.

Table 1. Distance and Correlation measure outcome with Interpretations

Measure	**Value**	**Meaning**
Hamming Distance (HD)	0.35	Moderate absolute difference
Normalized Hamming Distance (NHD)	0.039	Very small distance
Euclidean Distance (ED)	0.132	Low difference
Normalized Euclidean Distance (NED)	0.015	Negligible difference
Correlation Measure (ρ_{NRS})	0.997	Extremely strong similarity in trend

2.2. *Musculoskeletal disorders*

Musculoskeletal disorders refer to conditions that cause pain or injury to the components of the musculoskeletal system, such as muscles, tendons, ligaments, joints, nerves, and the supporting structures of the back, neck, and limbs. These disorders may result from sudden physical effort, such as lifting or carrying heavy objects, or they may develop over time due to repetitive movements, prolonged exposure to mechanical stress, vibration, or sustained awkward postures.

The term "Arthritis" literally means joint inflammation. Arthritis is the swelling and tenderness of one or more joints. The most common types of arthritis are Osteoarthritis and Rheumatoid Arthritis. [21].

Rheumatoid Arthritis (RA) is a form of arthritis and a chronic autoimmune disease that attacks the body's own tissues, causing inflammation in joints, skin, eyes, heart, and blood vessels. This inflammation can lead to pain, swelling, stiffness, and even joint damage as uncontrolled inflammation damages cartilage which acts as a shock absorber in the joints and thus can deform the joints and eventually causes the bone to erode. [21] One of the most common symptoms of RA is swollen joints, often accompanied by stiffness that is particularly severe in the morning or after periods of inactivity. Fatigue is another common complaint among those with RA. Women are more likely than men to develop this condition, and it typically begins in middle age.

There are four clinically recognized stages of Rheumatoid Arthritis (RA), described as follows:

- Stage 1 (Early Stage): Mild joint pain and stiffness begin to appear, accompanied by inflammation around the joints. Structural damage is minimal at this point.
- Stage 2 (Moderate Stage): Inflammation progresses and starts affecting the cartilage, resulting in increased stiffness and a noticeable reduction in joint mobility.
- Stage 3 (Severe Stage): The inflammatory response becomes more aggressive, leading to bone erosion. Patients typically experience significant pain, stiffness, restricted movement, and visible joint deformities.

- Stage 4 (Final Stage): Although active inflammation may subside, joint damage continues. This stage is characterized by chronic pain, swelling, severe stiffness, and substantial loss of joint function and mobility. [21]

Osteoarthritis is a most common form of Arthritis and is a degenerative disease with the condition worsening over time. Joint pain and stiffness can be severe especially in hands, hip, knees and spine.

It is more common in older adults and can be one of the two types:

- Primary osteoarthritis that develops in the joints over time that might be caused by normal wear and tear of joints.
- Secondary osteoarthritis might happen when something directly damages the joints like injuries. [21]

Lupus is an autoimmune disorder which can cause inflammation in joints, skin, kidneys, blood cells, brain, heart and lungs. One distinctive symptom could be butterfly - shaped rashes on the face that covers the cheeks and bridge of the nose or elsewhere on the body. Lupus rashes also includes:

- Discoid Lupus, which causes red, scaly patches on the skin
- Subacute Cutaneous Lupus, which causes ring-shaped or oval-shaped rashes on the skin. [21]

Bursitis is a painful condition that causes swelling, usually around the joints. It is more common in the shoulders, elbows, knees and feet. Bursitis causes swelling in brusa - a small, fluid-filled sac, which is found around bones and some other tissues. Bursitis often occurs near joints that perform frequent repetitive motion. [21]

Several factors can increase the risk of developing musculoskeletal disorders which includes age, occupation, injury, genetics and lifestyle factors. Certain activities that can cause wear and tear of the joints and lead to musculoskeletal disorders includes engaging in tasks that involve repetitive motions, lifting heavy weights and maintaining poor posture.

3. **Proposed Work**

3.1. *Correlation-based analysis of symptom patterns across disorders*

Consider the four musculoskeletal disorders - Rheumatoid Arthritis, Osteoarthritis, Lupus, Bursitis and the eight rheumatic symptoms - joint pain, morning joint stiffness, fatigue, swelling in the joints, impaired joint mobility, pain relief during rest, annular or oval scaly rashes in the skin and pain in the smaller joints. The tables 2, 3, 4 and 5 present the T, I and F values of four sets of three patients Patient 1, Patient 2, Patient 3 suffering from each disorder, recorded at four-month intervals over the span of one year. [14]

Table 2. Rheumatoid Arthritis

	Rheumatoid Arthritis Patient 1	Rheumatoid Arthritis Patient 2	Rheumatoid Arthritis Patient 3
Joint pain	(1.0,0.3,0.0) (0.9,0.4,0.1) (0.9,0.1,0.1)	(0.9,0.4,0.0) (1.0,0.2,0.0) (0.9,0.2,0.1)	(0.9,0.3,0.0) (1.0,0.5,0.1) (1.0,0.1,0.1)
Morning joint stiffness	(0.9,0.2,0.1) (0.9,0.4,0.0) (1.0,0.3,0.1)	(1.0,0.1,0.0) (0.9,0.3,0.1) (0.9,0.5,0.1)	(0.9,0.4,0.1) (1.0,0.5,0.0) (1.0,0.1,0.1)
Fatigue	(0.1,0.2,0.9) (0.1,0.3,0.8) (0.0,0.2,0.9)	(0.1,0.3,1.0) (0.0,0.5,0.8) (1.0,0.4,0.9)	(0.1,0.2,0.9) (0.0,0.2,0.8) (0.1,0.1,0.8)
Swelling in the joints	(0.9,0.3,0.1) (0.8,0.4,0.3) (0.8,0.3,0.2)	(0.8,0.4,0.2) (0.9,0.5,0.2) (0.8,0.1,0.3)	(0.9,0.4,0.1) (0.9,0.1,0.2) (0.8,0.1,0.1)
Impaired joint mobility	(0.8,0.3,0.1) (0.7,0.2,0.3) (0.6,0.3,0.2)	(0.9,0.4,0.2) (0.8,0.3,0.1) (0.8,0.2,0.1)	(0.8,0.4,0.1) (0.7,0.2,0.2) (0.8,0.2,0.1)
Pain relief during rest	(0.3,0.1,0.7) (0.2,0.3,0.8) (0.4,0.2,0.6)	(0.4,0.2,0.8) (0.3,0.3,0.8) (0.2,0.1,0.7)	(0.3,0.1,0.8) (0.2,0.2,0.8) (0.3,0.2,0.7)
Annular or oval scaly rashes	(0.0,0.4,0.9) (0.0,0.1,1.0) (0.0,0.2,0.9)	(0.0,0.3,0.9) (0.0,0.2,1.0) (0.0,0.4,1.0)	(0.0,0.4,0.9) (0.0,0.1,1.0) (0.0,0.1,0.9)
Pain in the smaller joints	(0.8,0.2,0.0) (0.9,0.4,0.1) (0.9,0.3,0.0)	(0.7,0.5,0.1) (0.8,0.5,0.2) (0.9,0.1,0.2)	(0.9,0.6,0.1) (0.9,0.4,0.0) (0.6,0.3,0.2)

Table 3. Oseteoarthritis

	Osteoarthritis Patient 1	Osteoarthritis Patient 2	Osteoarthritis Patient 3
Joint pain	(0.9,0.3,0.1) (1.0,0.4,0.0) (0.9,0.1,0.1)	(0.9,0.4,0.0) (1.0,0.2,0.1) (1.0,0.2,0.0)	(0.9,0.3,0.0) (0.9,0.5,0.1) (1.0,0.1,0.1)
Morning joint stiffness	(0.3,0.2,0.9) (0.2,0.4,0.8) (0.1,0.3,0.8)	(0.3,0.1,0.8) (0.3,0.3,0.8) (0.2,0.5,0.7)	(0.2,0.4,0.9) (0.3,0.5,0.8) (0.1,0.1,0.8)
Fatigue	(0.0,0.5,0.9) (0.0,0.6,1.0) (0.1,0.3,0.9)	(0.0,0.3,0.9) (0.1,0.2,1.0) (0.1,0.2,1.0)	(0.0,0.4,1.0) (0.0,0.1,0.9) (0.1,0.1,0.9)
Swelling in the joints	(0.5,0.3,0.4) (0.7,0.4,0.3) (0.6,0.3,0.2)	(0.5,0.4,0.4) (0.6,0.5,0.4) (0.6,0.1,0.3)	(0.5,0.4,0.5) (0.6,0.1,0.3) (0.5,0.1,0.4)
Impaired joint mobility	(0.8,0.3,0.1) (0.9,0.2,0.0) (0.9,0.3,0.0)	(0.9,0.4,0.1) (1.0,0.3,0.1) (0.9,0.2,0.0)	(1.0,0.4,0.1) (0.9,0.2,0.0) (0.9,0.2,0.0)
Pain relief during rest	(0.5,0.3,0.3) (0.4,0.4,0.3) (0.3,0.2,0.4)	(0.5,0.2,0.4) (0.6,0.5,0.3) (0.5,0.4,0.4)	(0.6,0.3,0.4) (0.5,0.4,0.3) (0.4,0.3,0.5)
Annular or oval scaly rashes	(0.0,0.4,0.9) (0.0,0.1,1.0) (0.1,0.2,1.0)	(0.1,0.3,0.9) (0.0,0.2,0.8) (0.1,0.4,1.0)	(0.0,0.4,1.0) (0.1,0.1,0.8) (0.1,0.1,0.9)
Pain in the smaller joints	(0.0,0.2,0.8) (0.1,0.4,0.9) (0.0,0.3,0.9)	(0.1,0.5,0.9) (0.1,0.5,1.0) (0.2,0.1,0.9)	(0.1,0.6,0.9) (0.0,0.4,0.9) (0.2,0.3,0.8)

Table 4. Lupus

	Lupus Patient 1	Lupus Patient 2	Lupus Patient 3
Joint pain	(0.5,0.3,0.4) (0.6,0.4,0.3) (0.5,0.1,0.6)	(0.6,0.4,0.5) (0.5,0.2,0.4) (0.5,0.2,0.3)	(0.5,0.3,0.5) (0.6,0.5,0.4) (0.4,0.1,0.3)
Morning joint stiffness	(0.6,0.2,0.4) (0.6,0.4,0.3) (0.5,0.3,0.4)	(0.6,0.1,0.4) (0.5,0.3,0.3) (0.5,0.5,0.5)	(0.6,0.4,0.3) (0.5,0.5,0.3) (0.6,0.1,0.4)
Fatigue	(0.9,0.3,0.1) (1.0,0.2,0.0) (0.9,0.3,0.1)	(0.9,0.3,0.0) (1.0,0.4,0.1) (1.0,0.2,0.1)	(1.0,0.4,0.0) (0.9,0.1,0.0) (0.9,0.1,0.1)
Swelling in the joints	(0.5,0.3,0.4) (0.7,0.4,0.3) (0.6,0.3,0.2)	(0.5,0.4,0.4) (0.6,0.5,0.4) (0.6,0.1,0.3)	(0.5,0.4,0.5) (0.6,0.1,0.3) (0.5,0.1,0.4)
Impaired joint mobility	(0.5,0.3,0.4) (0.5,0.4,0.3) (0.3,0.2,0.4)	(0.6,0.2,0.4) (0.6,0.5,0.3) (0.5,0.4,0.3)	(0.6,0.3,0.4) (0.5,0.4,0.5) (0.5,0.3,0.5)
Pain relief during rest	(0.5,0.3,0.5) (0.4,0.4,0.3) (0.6,0.2,0.4)	(0.5,0.2,0.4) (0.6,0.5,0.3) (0.5,0.4,0.3)	(0.6,0.3,0.4) (0.5,0.4,0.3) (0.6,0.3,0.4)
Annular or oval scaly rashes	(0.9,0.5,0.1) (0.9,0.6,0.1) (1.0,0.3,0.1)	(0.9,0.3,0.0) (0.9,0.3,0.0) (1.0,0.1,0.1)	(1.0,0.2,0.1) (0.9,0.1,0.0) (0.9,0.1,0.1)
Pain in the smaller joints	(0.5,0.3,0.5) (0.5,0.4,0.3) (0.4,0.2,0.4)	(0.6,0.2,0.4) (0.6,0.5,0.5) (0.6,0.4,0.4)	(0.6,0.3,0.4) (0.5,0.4,0.3) (0.6,0.3,0.3)

Table 5. Bursitis

	Bursitis Patient 1	Bursitis Patient 2	Bursitis Patient 3
Joint pain	(1.0,0.3,0.0) (1.0,0.2,0.1) (0.9,0.1,0.0)	(0.9,0.4,0.0) (1.0,0.3,0.1) (0.9,0.2,0.1)	(0.9,0.3,0.0) (1.0,0.4,0.0) (1.0,0.1,0.1)
Morning joint stiffness	(0.3,0.4,0.7) (0.4,0.1,0.8) (0.3,0.2,0.8)	(0.3,0.3,0.7) (0.2,0.2,0.8) (0.4,0.4,0.7)	(0.3,0.4,0.7) (0.4,0.1,0.7) (0.2,0.1,0.8)
Fatigue	(0.3,0.2,0.7) (0.4,0.3,0.8) (0.2,0.2,0.7)	(0.2,0.3,0.7) (0.3,0.5,0.8) (0.3,0.4,0.7)	(0.3,0.2,0.7) (0.2,0.2,0.8) (0.4,0.1,0.8)
Swelling in the joints	(0.8,0.2,0.0) (0.9,0.4,0.1) (0.9,0.3,0.0)	(1.0,0.5,0.1) (0.9,0.5,0.1) (0.8,0.1,0.0)	(0.9,0.6,0.1) (0.9,0.4,0.0) (0.8,0.3,0.1)
Impaired joint mobility	(1.0,0.2,0.0) (0.8,0.4,0.1) (0.9,0.5,0.1)	(1.0,0.3,0.1) (0.9,0.4,0.0) (0.9,0.1,0.1)	(0.8,0.4,0.1) (0.8,0.4,0.1) (0.9,0.3,0.1)
Pain relief during rest	(0.8,0.3,0.3) (0.7,0.5,0.2) (0.8,0.2,0.1)	(0.6,0.2,0.3) (0.8,0.5,0.1) (0.7,0.5,0.2)	(0.7,0.5,0.1) (0.8,0.4,0.2) (0.8,0.5,0.2)
Annular or oval scaly rashes	(0.0,0.3,1.0) (0.0,0.4,1.0) (0.1,0.3,0.9)	(0.0,0.4,0.9) (0.0,0.4,0.9) (0.0,0.3,1.0)	(0.1,0.5,0.9) (0.0,0.4,1.0) (0.1,0.5,0.9)
Pain in the smaller joints	(0.0,0.5,0.9) (0.1,0.4,1.0) (0.1,0.3,0.9)	(0.1,0.5,0.9) (0.1,0.4,0.9) (0.1,0.3,0.8)	(0.1,0.5,0.9) (0.0,0.4,0.8) (0.0,0.3,0.9)

The Truth value (T) represents the degree to which a symptom, such as pain is present in a patient. It is derived from the reported severity of the symptom and normalized using a fixed maximum scale. This allows for consistent comparison of symptom intensity across patients. The value is computed using the following formula: $T = \frac{\text{symptom severity}}{\text{max scale}}$. For instance, if a patient reports a pain score of 8 on a 10-point scale, the corresponding truth value would be: $T = \frac{8}{10} = 0.8$.

Table 6. Explanation for Choice of Indeterminacy Values

Clinical Condition	**Explanation**	**Indeterminacy**
Mild pain lasting for a short duration	May resolve on its own, not clearly a disorder	0.1
Pain with confirmed damage on imaging	Symptom aligns clearly with diagnostic results	0.2
Pain without diagnostic tests or detailed history	Insufficient clinical information for diagnosis	0.3
Severe pain reported, but imaging shows no abnormality	Symptom is intense, but not supported by objective evidence	0.4
Conflicting results from diagnostic tests	Inconsistent findings across investigations	0.5
Unexplained symptoms without clinical agreement	No identifiable cause	0.6

Falsity (F) values were not computed from a strict formula. Instead, they were selected to reflect domain intuition - especially in cases where high pain scores co-occurred with high uncertainty. In general, F tends to increase when symptom severity T is low and ambiguity I is low, but exceptions are allowed to reflect complex clinical judgments.

For Rheumatoid Arthritis Patient 1, the neutrosophic values for joint pain were derived from both symptom severity and clinical context. In the first sample, the patient reported a pain level of 10 out of 10, which was normalized using the formula, resulting in a truth value $T = 1.0$. An indeterminacy value of $I = 0.3$ was assigned due to the lack of detailed diagnostic history, indicating moderate ambiguity as per Table 6. The falsity value $F = 0.0$ reflected full confidence in symptom presence. In the second sample, the pain score decreased to 9 ($T = 0.9$), but conflicting diagnostic reports led to a higher indeterminacy value of $I = 0.4$ and a corresponding falsity of $F = 0.1$. In the third case, the pain level remained at 9, but supporting imaging results reduced the ambiguity to $I = 0.1$, with $F = 0.1$ to reflect minimal doubt in symptom presence.

The correlation matrices shown below are generated with the help of the python implementation of Neutrosophic Refined sets. They reveal distinct patterns of symptom associations for each disorder.

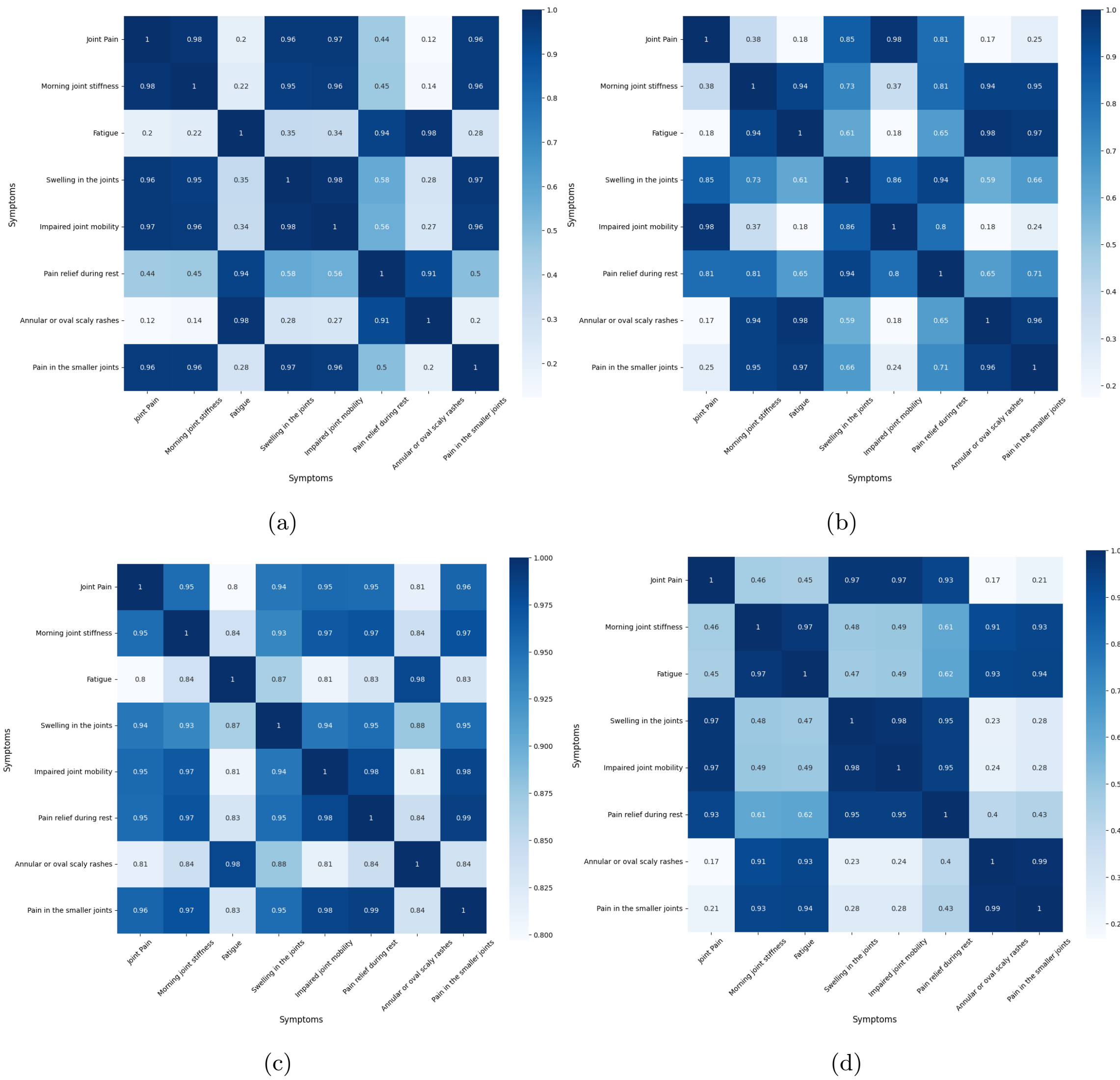


Figure 1. Heatmaps showing symptom correlations for (a) Rheumatoid Arthritis (b) Osteoarthritis (c) Lupus (d) Bursitis, where darker cells indicate stronger correlations.

From the results obtained in Figure 1, the following observations can be made:

- Individuals with Rheumatoid Arthritis commonly experience joint pain, especially in the morning, along with morning stiffness, joint swelling, fatigue, impaired joint mobility, and pain in smaller joints.
- Patients with Osteoarthritis often report joint pain and limited mobility, but symptoms like morning stiffness, fatigue, and swelling are less strongly correlated and typically less severe than in Rheumatoid Arthritis.

- In case of patients with lupus, fatigue is often accompanied by a characteristic round, scaly skin rash. The presence of other symptoms can vary, ranging from moderate to low.
- Bursitis typically manifests as significant joint pain, noticeable swelling, and difficulty in performing activities involving the affected joint. Pain in smaller joints may also be present, but to a lesser extent. Other symptoms are usually less pronounced.
- The presence of a round, scaly skin rash is a distinctive feature of Lupus and is less common in Rheumatoid Arthritis, Osteoarthritis, and Bursitis.

3.2. *Computing patient-disorder proximity using distance measures*

Consider a set of four patients - P1, P2, P3 and P4. T, I and F values for the four patients against the eight symptoms, recorded at four-month intervals over the span of a year along with the T, I and F values for symptoms versus disorders are tabulated

Table 7. Disorders vs Symptoms

	Rheumatoid Arthritis	**Osteoarthritis**	**Lupus**	**Bursitis**
Joint pain	(0.9,0.5,0.1)	(0.8,0.4,0.2)	(0.7,0.5,0.1)	(0.9,0.1,0.0)
Morning joint stiffness	(0.8,0.4,0.0)	(0.9,0.3,0.0)	(0.1,0.6,0.8)	(0.1,0.4,0.8)
Fatigue	(0.9,0.3,0.2)	(0.1,0.2,0.9)	(0.9,0.5,0.3)	(0.2,0.3,0.9)
Swelling in the joints	(0.7,0.2,0.1)	(0.7,0.1,0.2)	(0.7,0.2,0.0)	(0.8,0.3,0.2)
Impaired joint mobility	(0.8,0.5,0.1)	(0.8,0.4,0.3)	(0.0,0.4,0.9)	(0.7,0.2,0.0)
Pain relief during rest	(0.0,0.4,0.8)	(0.1,0.5,0.9)	(0.1,0.3,0.8)	(0.9,0.6,0.2)
Annular or oval scaly rashes	(0.1,0.3,0.9)	(0.2,0.4,0.8)	(0.8,0.1,0.2)	(0.0,0.1,0.6)
Pain in the smaller joints	(0.7,0.1,0.1)	(0.1,0.3,0.7)	(0.0,0.5,0.7)	(0.2,0.3,0.8)

Table 8. Patients vs Symptoms

	P1	**P2**	**P3**	**P4**
Joint pain	(0.7,0.3,0.1) (0.6,0.4,0.2) (0.9,0.1,0.1)	(0.9,0.4,0.0) (0.8,0.2,0.1) (0.9,0.2,0.1)	(0.7,0.3,0.2) (0.9,0.5,0.0) (0.9,0.1,0.1)	(0.8,0.4,0.1) (0.8,0.3,0.2) (0.9,0.2,0.0)
Morning joint stiffness	(0.6,0.2,0.3) (0.7,0.4,0.0) (0.7,0.3,0.1)	(0.7,0.1,0.2) (0.9,0.3,0.1) (0.6,0.5,0.3)	(0.0,0.2,0.9) (0.2,0.2,0.9) (0.1,0.4,0.7)	(0.0,0.5,0.7) (0.0,0.3,0.8) (0.1,0.4,0.8)
Fatigue	(0.5,0.5,0.0) (0.7,0.6,0.1) (0.8,0.3,0.1)	(0.2,0.4,0.9) (0.0,0.3,0.9) (0.0,0.2,0.7)	(0.7,0.5,0.1) (0.8,0.5,0.2) (0.9,0.1,0.2)	(0.1,0.2,0.9) (0.2,0.4,0.8) (0.0,0.3,0.9)
Swelling in the joints	(0.9,0.3,0.1) (0.7,0.4,0.2) (0.6,0.3,0.2)	(0.8,0.3,0.0) (0.8,0.2,0.1) (0.9,0.2,0.3)	(0.8,0.4,0.1) (0.7,0.5,0.0) (0.8,0.1,0.2)	(0.9,0.5,0.) (0.9,0.4,0.2) (0.7,0.2,0.3)
Impaired joint mobility	(0.8,0.3,0.1) (0.9,0.2,0.0) (0.7,0.3,0.2)	(0.9,0.4,0.1) (0.7,0.5,0.1) (0.8,0.1,0.0)	(0.0,0.3,0.9) (0.0,0.2,0.8) (0.1,0.2,0.9)	(0.8,0.4,0.0) (0.7,0.1,0.3) (0.9,0.1,0.1)
Pain relief during rest	(0.1,0.3,0.9) (0.2,0.4,0.8) (0.0,0.2,0.9)	(0.1,0.2,0.8) (0.0,0.5,0.8) (0.1,0.4,0.9)	(0.0,0.1,0.9) (0.1,0.4,0.7) (0.0,0.3,0.8)	(0.7,0.2,0.1) (0.9,0.3,0.1) (0.9,0.2,0.2)
Annular or oval scaly rashes	(0.0,0.4,0.8) (0.3,0.1,0.8) (0.1,0.2,0.7)	(0.2,0.3,0.9) (0.0,0.2,0.7) (0.1,0.4,0.9)	(0.9,0.2,0.1) (0.7,0.1,0.2) (0.9,0.3,0.0)	(0.0,0.3,0.7) (0.1,0.2,0.9) (0.2,0.2,0.8)
Pain in the smaller joints	(0.8,0.2,0.0) (0.9,0.4,0.1) (0.9,0.3,0.0)	(0.1,0.2,0.8) (0.2,0.5,0.8) (0.0,0.2,0.9)	(0.1,0.4,0.9) (0.0,0.3,0.7) (0.0,0.2,0.9)	(0.2,0.3,0.8) (0.1,0.4,0.8) (0.1,0.3,0.7)

Using the Python implementation along with the T, I and F values from the tables - Tables 7 and 8, the proximity of each patient to the four musculoskeletal disorders is calculated. Figure 2 illustrates the results obtained using four distance measures of Neutrosophic Refined Sets. The disorder corresponding to the minimum distance indicates the most likely diagnosis for each patient. Based on this analysis, it is concluded that patients P1, P2, P3, and P4 are diagnosed with Rheumatoid Arthritis, Osteoarthritis, Lupus, and Bursitis, respectively.

	Rheumatoid Arthritis	Osteoarthritis	Lupus	Bursitis
P1	8.9	16.3	23.4	22.6
P2	14.4	8.4	23.3	15.1
P3	23.8	23.6	8.3	22.9
P4	23.1	15.9	23.4	7.4

(a)

	Rheumatoid Arthritis	Osteoarthritis	Lupus	Bursitis
P1	0.124	0.226	0.325	0.314
P2	0.2	0.117	0.324	0.21
P3	0.331	0.328	0.115	0.318
P4	0.321	0.221	0.325	0.103

(b)

	Rheumatoid Arthritis	Osteoarthritis	Lupus	Bursitis
P1	3.627	6.461	8.85	8.727
P2	5.91	3.397	8.908	5.909
P3	9.247	9.087	3.639	8.892
P4	9.026	6.314	8.973	3.141

(c)

	Rheumatoid Arthritis	Osteoarthritis	Lupus	Bursitis
P1	0.05	0.09	0.123	0.121
P2	0.082	0.047	0.124	0.082
P3	0.128	0.126	0.051	0.124
P4	0.125	0.088	0.125	0.044

(d)

Figure 2. Patient-disorder proximity scores using distance measures - (a) Hamming distance (b) Normalized Hamming distance (c) Euclidean distance (d) Normalized Euclidean distance.

3.3. *Analyzing general symptom trends across disorders with interactive visualization*

Consider four sets of three patients - Patient 1, Patient 2, and Patient 3 - each group of patients diagnosed with one of the four musculoskeletal disorders. The recorded T, I and F values represent the degrees of presence or severity of each disorder.

The values presented in tables 2, 3, 4 and 5, reflect symptom severity and these values are correlated to the values in Table 9 to generate interactive visualizations. These visual tools enhance interpretability by highlighting the relative weight or contribution of each symptom to the respective disorder, thereby supporting clinical insight and comparative analysis across disorders.

The correlated values are further transformed using the Min-Max scaling technique to standardize their range for visual representation. This transformation ensures that all values lie within a uniform scale of 1 to 10, facilitating consistent comparison across different symptoms and disorders. The scaling is performed using the following formula:

$$X = \left\{ \left[\left(\frac{X - X_{\text{min}}}{X_{\text{max}} - X_{\text{min}}} \right) \times 9 \right] + 1 \right\} \tag{8}$$

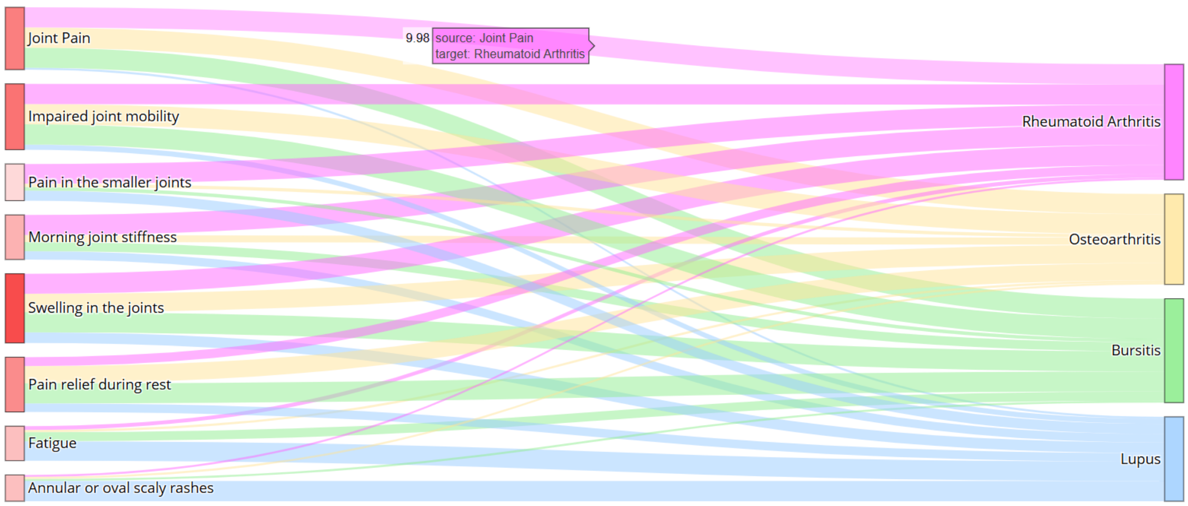
Here, X represents the original correlated value, while $X_{\texttt{min}}$ and $X_{\texttt{max}}$ denote the minimum and maximum values within the entire set of correlation scores, respectively.

Table 9. Disorder trend

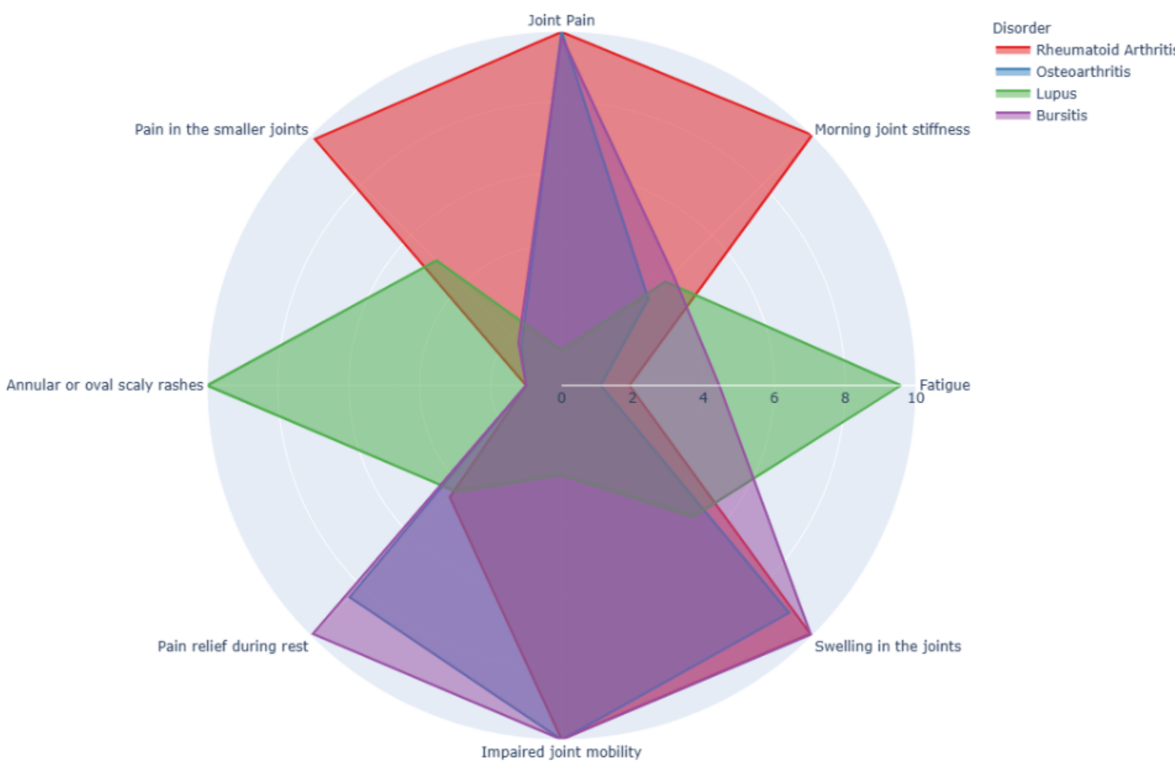
	Patient 1	**Patient 2**	**Patient 3**
Rheumatoid Arthritis	(0.9,0.5,0.1)	(0.9,0.4,0.0)	(1.0,0.3,0.2)
Osteoarthritis	(0.8,0.4,0.2)	(0.9,0.3,0.0)	(0.7,0.1,0.2)
Lupus	(0.7,0.5,0.1)	(0.9,0.5,0.3)	(0.7,0.2,0.0)
Bursitis	(0.9,0.5,0.0)	(0.8,0.6,0.2)	(0.7,0.2,0.1)

The interactive charts in Figure 3 were generated using the Plotly library in Python and the python implementation of NRS especially that of correlation measure was used. The normalized correlation values serve as the input for generating these interactive charts. Plotly is a powerful open-source graphing library that enables the creation of high-quality, interactive visualizations. Its flexibility and extensive support for various chart types - such as heatmaps, scatter plots, and 3D surface plots - make it especially suitable for presenting complex multi-dimensional data.

In these visualizations, a value of 1 typically signifies that the symptom is either barely noticeable or almost completely absent in the individual being assessed. Conversely, a value of 10 represents the symptom's most intense or pronounced manifestation within the given context, enabling clear interpretation of symptom severity across disorders.



(a)



(b)

Figure 3. Visualization of symptom trends across musculoskeletal disorders using (a) a Sankey diagram to depict the strength of symptom-disorder associations, and (b) a radar chart to compare the relative prominence of each symptom among the disorders.

4. Conclusion

This study presents a comprehensive approach for analyzing and diagnosing musculoskeletal disorders using correlation and distance measures of Neutrosophic Refined Sets, implemented in Python. The proposed methodology encompasses three core components: first, the classification of disorders based on symptom patterns; second, the computation of distances between individual patient profiles and disorder-specific symptom sets; and third, the generation of informative visualizations that offer deeper insights into general symptom trends and disorder differentiation.

Overall, this work effectively integrates NRS theory with quantitative analysis, providing a robust and flexible implementation for both disease characterization and clinical decision support. The fusion of uncertainty modeling with correlation and distance-based computations not only enhances interpretability but also paves the way for scalable, data-driven diagnostic systems in healthcare. In conclusion, this work benefits the researchers who are deeply interested in exploring Neutrosophic Sets, their variations and particularly NRS, and their practical implementation in real-world scenarios. In future, the proposed framework can be extended to include medical imaging data or physiological signals alongside symptom profiles to improve diagnostic precision, particularly in complex disorders.

Supplementary Material

1. For Python implementation and illustrated examples: GitHub Link
2. Live Interactive Sankey Diagram: Click here
3. Live Interactive Radar Chart: Click here

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Performance Analysis of Neutrosophic Multi-Server Queuing-Inventory System under Catastrophic Conditions

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Abstract

In this study, we introduce a Neutrosophic-based framework for analyzing a finite capacity multi-server queuing-inventory system (MQIS) that integrates attraction-retention mechanisms for impatient customers and addresses catastrophic inventory disruptions. Recognizing the limitations of purely deterministic and probabilistic models in uncertain environments, we adopt Single-Valued Trapezoidal Neutrosophic Numbers (SVTNNs) to represent key system parameters such as arrival rates, service rates, attraction intensities, retention probabilities, and catastrophe occurrences. The system operates under Markovian assumptions, with arrivals modeled by a Poisson process, service and vacation times following exponential distributions, and inventory replenished via an (s, S) policy. To facilitate steady-state analysis, the Neutrosophic model is transformed into equivalent deterministic models through the application of Zadeh's extension principle combined with the (α, β, γ)-cut method. Furthermore, a cost optimization problem is formulated to identify optimal service configurations, solved using a genetic algorithm. A numerical illustration is provided to demonstrate the model's ability to capture uncertainty and improve decision-making in complex service environments.

Keywords: Neutrosophic Logic; Multi-Server Queuing-Inventory System; Customer Attraction and Retention; Impatient Customers; Inventory Optimization; Asynchronous Server Vacations.

1 Introduction

Queuing-inventory systems (QIS) are fundamental in modeling service operations where the dynamics of customer arrivals and inventory depletion interact. In high-demand sectors such as telecommunications, logistics, and healthcare, managing both customer flow and stock availability under operational constraints is critical for maintaining service quality and cost efficiency. Classical queuing-inventory models, such as those developed by Buzacott and Shanthikumar [1], typically assume exponential interarrival and service times and deterministic system parameters. While mathematically tractable, these assumptions often fail to capture the complex uncertainties observed in real-world systems, such as fluctuating arrival rates, unpredictable service times, and variable customer behavior.

The inherent imprecision, ambiguity, and inconsistency in service environments have motivated researchers to seek more flexible modeling approaches. Fuzzy set theory, introduced by Zadeh [2], offers one such framework by permitting degrees of membership between 0 and 1, allowing systems to model vagueness in customer arrivals, service rates, and inventory levels. Applications of fuzzy logic to queueing inventory systems, such as those of Buckley [3] and Kao and Liu [4], have enhanced the robustness of system analysis under uncertain conditions. However, fuzzy models are inherently limited in their ability to accommodate conflicting or indeterminate information, as they treat uncertainty primarily as gradual vagueness rather than acknowledging the coexistence of truth, indeterminacy, and falsity.

In response to these limitations, intuitionistic fuzzy sets were introduced by Atanassov [5], extending fuzzy sets to explicitly model both membership and nonmembership degrees. However, even intuitionistic models are restricted when dealing with complex, inconsistent, and incomplete information environments. To overcome these challenges, Smarandache [6] proposed the Neutrosophic set theory, where each element is characterized by independent degrees of truth, indeterminacy, and falsity. This tripartite structure offers a significantly richer framework for capturing the multifaceted uncertainty inherent in service systems.

In recent years, neutrosophic approaches have been applied more and more to queueing models. Researchers such as Maji and Roy [7] demonstrated that neutrosophic models could more accurately represent arrival rates and service rates when subject to indeterminate and inconsistent information. Compared to other neutrosophic representations, single-valued trapezoidal neutrosophic numbers (SVTNN) have proven particularly useful due to their computational simplicity and intuitive geometric interpretation, as evidenced by recent studies (Dayana et al. [11]).

However, most existing neutrosophic queueing models focus on single-server systems without fully integrating inventory dynamics, asynchronous server operations, or catastrophic disruptions. Real-world service systems, especially in large-scale telecommunications and healthcare applications, often involve multiple parallel servers, inventory constraints, customer impatience, active attraction-retention efforts, and occasional large-scale system shocks (e.g., technical failures, supply chain disruptions).

This research addresses these gaps by proposing Neutrosophic-based framework for analyzing a finite-capacity, multi-server queueing-inventory system with asynchronous server vacations, attraction-retention mechanisms, and catastrophic inventory events. In the model:

- Customers arrive according to a Poisson process and are served by one of C identical servers;

- Each service consumes one unit of inventory, with replenishment governed by an (s, S) policy;
- Servers independently take asynchronous vacations when idle, reflecting realistic operational policies;
- Customer attraction and retention strategies influence arrival and abandonment decisions under uncertainty;
- Catastrophic events are incorporated as neutrosophic random shocks impacting both customers and inventory levels.

System parameters such as arrival rates, service rates, attraction-retention probabilities, and catastrophe rates are modeled using SVTNNs, allowing the model to capture the full spectrum of uncertainty, indeterminacy, and inconsistency in operational environments. The steady-state analysis is conducted by transforming the neutrosophic model into a family of crisp Markovian models through the application of Zadeh's extension principle and the (α, β, γ)-cut method.

Moreover, a cost optimization problem is formulated, incorporating service costs, holding costs, shortage costs, and abandonment penalties, and is solved using a genetic algorithm to determine optimal service rates and vacation policies.

To validate the proposed framework, a numerical case study based on operational data from Ethio Telecom is presented. The results demonstrate that the neutrosophic model provides more resilient and realistic performance estimates compared to traditional approaches, offering valuable insights for service managers operating under high uncertainty.

The remainder of this paper is organized as follows: Section 2 introduces the preliminaries and fundamental definitions of neutrosophic sets. Section 3 describes the system assumptions and detailed model structure of the neutrosophic multi-server queueing-inventory system. Section 4 presents the neutrosophic NM/NM/C/K queueing-inventory model. Section 5 provides numerical results and discussion. Finally, Section 6 concludes the study and outlines directions for future research.

2 Preliminaries

Definition 2.1 (Neutrosophic Set [12]) *A Neutrosophic set N defined over a universe τ assigns to each element $\tau \in \tau$ three membership functions: truth-membership $T_A(\tau)$, indeterminacy-membership $I_A(\tau)$, and falsity-membership $F_A(\tau)$, where*

$$N = \{(\tau, (T_A(\tau), I_A(\tau), F_A(\tau))) : \tau \in \tau\}.$$

Each membership function maps to the interval $[0, 1]$, satisfying the constraint:

$$0 \leq \sup T_A(\tau) + \sup I_A(\tau) + \sup F_A(\tau) \leq 3.$$

Definition 2.2 (Single-Valued Neutrosophic Set (SVNS) [12]) *A Single-Valued Neutrosophic Set (SVNS) is a specific case of a Neutrosophic Set where each of the membership functions $T_A(\tau)$, $I_A(\tau)$, and $F_A(\tau)$ take crisp values in $[0, 1]$ only. An SVNS is given by:*

$$N = \{(\tau, (T_A(\tau), I_A(\tau), F_A(\tau))) : \tau \in \tau\},$$

under the condition that:

$$0 \leq \sup T_A(\tau) + \sup I_A(\tau) + \sup F_A(\tau) \leq 3.$$

Definition 2.3 (Single-Valued Trapezoidal Neutrosophic Number (SVTNN) [12]) *A Single-Valued Trapezoidal Neutrosophic Number (SVTNN) represents the truth, indeterminacy, and falsity membership functions using trapezoidal forms. For A, the truth-membership function $T_A(\tau)$ is expressed as:*

$$T_A(\tau) = \begin{cases} \frac{\tau_T - r_T^1}{r_T^2 - r_T^1}, & r_T^1 \leq \tau_T \leq r_T^2, \\ 1, & r_T^2 \leq \tau_T \leq r_T^3, \\ \frac{r_T^4 - \tau_T}{r_T^4 - r_T^3}, & r_T^3 \leq \tau_T \leq r_T^4, \\ 0, & \textit{otherwise}, \end{cases}$$

where $r_T^1 \leq r_T^2 \leq r_T^3 \leq r_T^4$. Similar forms define $I_A(\tau)$ and $F_A(\tau)$ for the indeterminacy and falsity memberships, respectively.

Definition 2.4 ((α, β, γ)-Cut for SVTNN [13]) *The (α, β, γ)-cut of a trapezoidal single-valued neutrosophic number partitions it into crisp intervals:*

$$D_{\alpha,\beta,\gamma} = \left([D_1(\alpha), D_2(\alpha)], [D_1^*(\beta), D_2^*(\beta)], [D_1^{**}(\gamma), D_2^{**}(\gamma)]\right),$$

where

$$\begin{aligned} [D_1(\alpha), D_2(\alpha)] &= [r_T^1 + \alpha(r_T^2 - r_T^1), r_T^4 - \alpha(r_T^4 - r_T^3)], \\ [D_1^*(\beta), D_2^*(\beta)] &= [r_I^2 - \beta(r_I^2 - r_I^1), r_I^3 + \beta(r_I^4 - r_I^3)], \\ [D_1^{**}(\gamma), D_2^{**}(\gamma)] &= [r_F^2 - \gamma(r_F^2 - r_F^1), r_F^3 + \gamma(r_F^4 - r_F^3)], \end{aligned}$$

with $0 \leq \alpha + \beta + \gamma \leq 3$.

Definition 2.5 (Arithmetic Operations on Intervals [14]) *Given two real intervals $[p_1, p_2]$ and $[p_3, p_4]$, their basic operations are defined as:*

- *Addition:* $[p_1, p_2] + [p_3, p_4] = [p_1 + p_3, p_2 + p_4]$,
- *Subtraction:* $[p_1, p_2] - [p_3, p_4] = [p_1 - p_4, p_2 - p_3]$,
- *Multiplication:* $[p_1, p_2] \times [p_3, p_4] = [\min\{p_1p_3, p_1p_4, p_2p_3, p_2p_4\}, \max\{p_1p_3, p_1p_4, p_2p_3, p_2p_4\}]$,
- *Division (assuming $0 \notin [p_3, p_4]$):*

$$[p_1, p_2] \div [p_3, p_4] = \left[\min\left\{\frac{p_1}{p_3}, \frac{p_1}{p_4}, \frac{p_2}{p_3}, \frac{p_2}{p_4}\right\}, \max\left\{\frac{p_1}{p_3}, \frac{p_1}{p_4}, \frac{p_2}{p_3}, \frac{p_2}{p_4}\right\}\right].$$

3 Description of the Model

We consider a finite-capacity neutrosophic multi-server queuing-inventory system (NMQIS) with attraction-retention mechanisms for impatient customers, where C removable servers operate under an asynchronous vacation policy with random lead times. The system is modeled using neutrosophic logic to represent the degrees of truth, indeterminacy, and falsity associated with system components and behaviors under uncertainty. The assumptions and neutrosophic characterizations are as follows:

1. **System Configuration:** The system consists of C removable servers, a waiting room with capacity N, and a maximum inventory level S. Each service consumes one inventory item. Let (T_1, I_1, F_1) denote the neutrosophic representation of the belief in accurate inventory deduction after each service, where T_1 is the degree of truth, I_1 the degree of indeterminacy, and F_1 the degree of falsity.

2. **Customer Arrival Process:** Customers arrive according to a neutrosophic Poisson process with arrival rate $\lambda_N = (\lambda_T, \lambda_I, \lambda_F)$. Upon arrival, a customer either joins or balks with neutrosophic probability $b_n^N = (b_n^T, b_n^I, b_n^F)$ given by:

$$b_n^T = \begin{cases} 1, & 0 \leq n \leq C - D - 1 \\ \frac{N-n}{N}, & C - D \leq n \leq N \end{cases}, \quad b_n^I = f_1(n), \quad b_n^F = 1 - b_n^T - b_n^I$$

where $f_1(n)$ represents the uncertainty in the decision-making process of arriving customers due to behavioral or informational ambiguity.

3. **Behavioral Control Mechanisms:** Behavioral control mechanisms are applied based on system congestion:

   - When $n \leq \frac{3N}{4}$, an attraction policy is used, increasing the arrival rate to $\lambda_N^{\text{attr}} = (\lambda(1 + \beta), \beta_I, \beta_F)$.
   - When $n > \frac{3N}{4}$, a retention policy is used to reduce abandonment, represented by a retention rate $r_N = (r_T, r_I, r_F)$.

4. **Service Process:** Customers are served in a single queue using the FCFS discipline. Service times are i.i.d. and follow a neutrosophic exponential distribution with rate $\mu_N = (\mu_T, \mu_I, \mu_F)$, such that $s_N(t) = (\mu_T e^{-\mu_T t}, \mu_I(t), \mu_F(t))$, where $\mu_I(t)$ and $\mu_F(t)$ reflect ambiguity or system failures.

5. **Customer Patience:** Customers possess a neutrosophic patience time distributed exponentially with reversion rate $\alpha_N = (\alpha_T, \alpha_I, \alpha_F)$. The neutrosophic relapse rate $R_N(n)$ depends on the length of the queue and is defined as:

$$R_N(n) = \begin{cases} ((n - C + D)(1 - r_T)\alpha_T, R_I(n), R_F(n)), & C - D < n < N \\ (0, 0, 0), & 0 \leq n \leq C - D \end{cases}$$

where $R_I(n)$ and $R_F(n)$ account for uncertainty and contradiction in abandonment behavior.

6. **Server Vacations:** When the system is empty, D servers go on asynchronous neutrosophic vacation with exponential duration $v_N(t) = (\xi_T e^{-\xi_T t}, \xi_I(t), \xi_F(t))$, where $\xi_N = (\xi_T, \xi_I, \xi_F)$ is the vacation rate under neutrosophic parameters.

7. **Inventory Management:** The inventory is managed through a neutrosophic (s, S) policy:

   - When the inventory level drops to s, a replenishment is triggered to restore inventory to S.
   - The replenishment lead time follows a neutrosophic exponential distribution with rate $\eta_N = (\eta_T, \eta_I, \eta_F)$.

- The condition $D < s$ holds with a neutrosophic degree (T_7, I_7, F_7) ensuring operational integrity.

8. **Inventory Depletion:** If the inventory is depleted:
   - The service rate becomes neutrosophically zero: $(0, \mu_I^{\text{stock}}, \mu_F^{\text{stock}})$.
   - Customers wait for restocking or leave due to indefinite wait times, governed by $R_N^{\text{stock}}(n)$.
   - The system remains idle in a neutrosophic inertial state until replenishment is completed.

9. **Catastrophic Events:**
   - Catastrophic events, such as system failures, natural disasters, or other disruptions, occur according to a neutrosophic Poisson process with rate $\gamma = (\gamma_T, \gamma_I, \gamma_F)$. This rate reflects the occurrence of these events with truth degree γ_T, indeterminacy degree γ_I, and falsity degree γ_F, capturing the uncertainty and unpredictability of these events.
   - Upon the occurrence of a catastrophe, the inventory is immediately emptied, and the on-hand inventory is reset to zero. This is modeled by a neutrosophic state:
   $$\text{Inventory}_{\text{catastrophe}} = (1, I_{\text{cat}}, F_{\text{cat}}),$$
   where I_{cat} and F_{cat} represent the uncertainty and contradiction regarding the full restoration of inventory.
   - Customers whose service is interrupted by a catastrophic event may either **rejoin the queue** or **leave the system**. The probability of a customer's decision is represented by a neutrosophic probability:
   $$P_{\text{rejoin}} = (P_T, P_I, P_F),$$
   where P_T is the truth degree (probability of rejoining the queue), P_I is the indeterminacy (uncertainty), and P_F is the falsity (probability of leaving the system).
   - During the catastrophic event, all **servers become inoperative**, and the system enters a state of inactivity. This is modeled by the neutrosophic state of the server's operational status:
   $$\text{Server Status}_{\text{catastrophe}} = (S_T, S_I, S_F) = (0, S_{\text{cat}}, 1),$$
   where S_T is the truth degree (full inactivity), S_I represents uncertainty in the downtime, and S_F is the falsity degree (failure of servers to function).
   - The system undergoes a **restoration process**, where the service is gradually resumed. The restoration time follows a neutrosophic exponential distribution with rate $\kappa = (\kappa_T, \kappa_I, \kappa_F)$. The restoration process is given by:
   $$R_{\text{restore}}(t) = (\kappa_T e^{-\kappa_T t}, \kappa_I(t), \kappa_F(t)),$$
   where $R_{\text{restore}}(t)$ represents the neutrosophic restoration time.

- During the restoration period, **new customers continue to arrive**, with the arrival rate $\lambda_{\text{restore}} = (\lambda_T, \lambda_I, \lambda_F)$, adjusted according to the uncertainty and restoration progress.
- Once the restoration process is complete, the system transitions back to normal operation, with **servers becoming operative** and inventory levels being replenished. The system recovery is modeled by the neutrosophic state:

$$\text{System Recovery} = (T_{\text{recovery}}, I_{\text{recovery}}, F_{\text{recovery}}),$$

where T_{recovery} indicates the truth degree of recovery, I_{recovery} reflects the indeterminacy, and F_{recovery} represents the falsity degree in the recovery process.

4 Neutrosophic NM/NM/C/K QIS with Attraction-Retention and Catastrophes

This section develops a comprehensive neutrosophic queuing-inventory model denoted by NM/NM/C/K, tailored for real-world service systems where both queuing dynamics and inventory levels are subject to uncertainty. In addition to traditional customer arrival and service processes, the model incorporates (i) inventory management through a neutrosophic (s, S) policy, (ii) behavioral mechanisms for customer attraction and retention, and (iii) the occurrence of catastrophic events that may suddenly deplete inventory or disrupt service.

In multi-server service environments such as telecommunications centers, retail warehouses, and healthcare systems, customer impatience, stockouts, and operational disruptions are common and interconnected phenomena. Classical queueing-inventory models often fail to simultaneously capture these dynamics, especially under uncertain, inconsistent, or incomplete information. To bridge this gap, we formulate a neutrosophic NM/NM/C/K model using Single-Valued Trapezoidal Neutrosophic Numbers (SVTNNs), enabling the joint modeling of arrival rates, service times, inventory levels, and behavioral probabilities with associated degrees of truth, indeterminacy, and falsity.

This model provides a flexible structure for evaluating system performance and risk, including metrics such as expected queue length, inventory shortfall probabilities, and abandonment rates, all under varying levels of uncertainty. The integration of customer attraction-retention strategies and catastrophic events further enhances its applicability to modern service systems facing unpredictable and high-stakes operating conditions.

4.1 Classical M/M/C/K Queueing Model Overview

The classical M/M/C/K model involves a queuing system with:

- Customers arriving according to a Poisson process with rate λ,
- Exponentially distributed service times with rate μ,
- C parallel servers,
- Finite system capacity K, including both servers and queue space,
- A first-come, first-served (FCFS) service discipline.

Performance Measures

The key performance metrics of the classical model include:

Probability of Zero Customers in the System:

$$p_0 = \left[\sum_{n=0}^{C-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^C}{C!} \cdot \frac{1 - \left(\frac{\lambda}{C\mu}\right)^{K-C+1}}{1 - \left(\frac{\lambda}{C\mu}\right)} \right]^{-1}$$

Probability of n Customers in the System:

$$p_n = \begin{cases} \frac{(\lambda/\mu)^n}{n!} p_0, & 0 \le n < C \\ \frac{(\lambda/\mu)^n}{C^{n-C} C!} p_0, & C \le n \le K \end{cases}$$

Expected Queue Length:

$$L_q = \sum_{n=C}^{K} (n - C) p_n$$

Expected Number of Customers in the System:

$$L_s = \sum_{n=0}^{K} n p_n$$

Average Waiting Time in Queue:

$$W_q = \frac{L_q}{\lambda(1 - p_K)}$$

Average Waiting Time in the System:

$$W_s = W_q + \frac{1}{\mu}$$

4.2 Neutrosophic Arrival and Service Times

In the neutrosophic setting, let the arrival and service times be modeled as follows:

$$A = \{(b, T_A(b), I_A(b), F_A(b)) \mid b \in P\}, \quad S = \{(s, T_S(s), I_S(s), F_S(s)) \mid s \in Q\}$$

where P and Q represent the universal domains for inter-arrival and service times, respectively. The associated membership functions T, I, and F quantify the degrees of truth, indeterminacy, and falsity.

Using the (α, β, γ)-cut approach, we extract crisp subsets for analysis:

$$A(\alpha, \beta, \gamma) = \{b \in P : T_A(b) \ge \alpha, I_A(b) \le \beta, F_A(b) \le \gamma\}$$

$$S(\alpha, \beta, \gamma) = \{s \in Q : T_S(s) \ge \alpha, I_S(s) \le \beta, F_S(s) \le \gamma\}$$

4.3 Model Development and Analysis of the Neutrosophic NM/NM/C/K System

To adapt the classical M/M/C/K model into a neutrosophic framework, we use SVTNNs to express the arrival rate λ and service rate μ, incorporating neutrosophic uncertainty in key performance evaluations.

[1] **Initialize:** - Probability of zero customers in the system, P_0 - Expected queue length, L_q - Neutrosophic membership components for system metrics: T, I, F

Step 1: Compute P_0

$$P_0 = \left[\sum_{k=0}^{K} P_k \right]^{-1}$$

where P_k is the neutrosophic-adjusted probability of k customers in the system.

Step 2: Calculate expected queue length

$$L_q = \frac{P_0 \cdot \lambda \cdot \mu}{1 - P_0}$$

Step 3: Define neutrosophic membership functions - Truth component: $T(n)$ - Indeterminacy component: $I(n)$ - Falsity component: $F(n)$

Step 4: Compute expected number in the system

$$E[N] = \sum_{n=0}^{K} n \cdot P_n$$

with neutrosophic adjustment via (T, I, F)-weights

Step 5: Estimate mean waiting times

$$W_q = \frac{L_q}{\lambda}, \quad W_s = W_q + \frac{1}{\mu}$$

with uncertainty bounds from neutrosophic confidence levels.

This algorithm facilitates system evaluation under varying levels of uncertainty. It enables the computation of crisp performance ranges using the (α, β, γ)-cut decomposition of SVTNNs, which can then be further analyzed using simulation or optimization techniques.

4.3.1 Neutrosophic Inventory Control with (s, S) Policy

In the proposed NM/NM/C/K queuing-inventory system, inventory levels are managed using a replenishment policy of type (s, S), where restocking is triggered when the inventory level falls to or below a reorder threshold s, and the level is then restored up to S. To model the uncertainty in inventory status and replenishment lead time, we define the following neutrosophic quantities:

- Inventory level: $\tilde{I}(t) = (I_T(t), I_I(t), I_F(t))$, representing the truth, indeterminacy, and falsity degrees of the current inventory state at time t.
- Reordering rate: $\tilde{\eta} = (\eta_T, \eta_I, \eta_F)$, where η_T denotes the expected replenishment rate, and η_I, η_F capture delivery uncertainty and potential supply chain failures.

- Restocking delay (lead time): Modeled as a neutrosophic exponential distribution with rate $\tilde{\eta}$, allowing for probabilistic and indeterminate replenishment.

Under this policy:

$$\text{If } \tilde{I}(t) \leq s, \text{ then order quantity } Q = S - \tilde{I}(t), \text{ initiated at rate } \tilde{\eta}.$$

Service completion is contingent on available inventory. If $\tilde{I}(t) = 0$, services are halted and customers accumulate or abandon the queue based on neutrosophic patience times.

4.3.2 Attraction and Retention Mechanisms for Impatient Customers

To manage customer flow and reduce reneging, the system includes behavioral mechanisms that attract customers when idle and retain them when congested. The behavioral probabilities are defined as neutrosophic values:

- **Attraction probability** $\tilde{\gamma}(n) = (\gamma_T(n), \gamma_I(n), \gamma_F(n))$, increases with lower queue length n, encouraging arrivals when idle.
- **Retention probability** $\tilde{r}(n) = (r_T(n), r_I(n), r_F(n))$, increases with queue length, discouraging abandonment during congestion.

These are modeled as:

$$\gamma_T(n) = \begin{cases} 1, & \text{if } n \leq C - D \\ \frac{N-n}{N}, & \text{if } C - D < n \leq N \end{cases}, \quad \gamma_I(n) = f_1(n), \quad \gamma_F(n) = 1 - \gamma_T(n) - \gamma_I(n),$$

where $f_1(n)$ is a function representing informational ambiguity or marketing uncertainty.

The **reneging rate** is adjusted as:

$$\tilde{R}(n) = ((n - C + D)(1 - r_T(n))\alpha_T, R_I(n), R_F(n))$$

for $n > C - D$, where α_T is the truth-level base reneging rate.

4.3.3 Catastrophic Events and Neutrosophic Recovery

Real-world systems often experience unplanned, large-scale disruptions—fires, system crashes, supplier failures—that affect both service and inventory. In this model, such catastrophic events are represented using neutrosophic Poisson arrivals with rate:

$$\tilde{\gamma}_{\text{cat}} = (\gamma_T, \gamma_I, \gamma_F)$$

Upon a catastrophic event:

- The entire on-hand inventory is destroyed:

$$\tilde{I}_{\text{cat}} = (0, I_{\text{cat}}, 1)$$

- All active servers are interrupted and reset to a non-operational state:

$$\text{Server Status} = (0, S_{\text{cat}}, 1)$$

- Customers already in service are either lost or requeued based on:

$$\tilde{P}_{\text{rejoin}} = (P_T, P_I, P_F)$$

The system then undergoes a recovery phase:

- Restoration rate $\tilde{\kappa} = (\kappa_T, \kappa_I, \kappa_F)$
- Restoration function:

$$R_{\text{restore}}(t) = (\kappa_T e^{-\kappa_T t}, \kappa_I(t), \kappa_F(t))$$

Once recovery completes, the servers and inventory resume normal operation. The system state transitions to:

$$\tilde{R}_{\text{recovery}} = (T_{\text{recovery}}, I_{\text{recovery}}, F_{\text{recovery}})$$

This extended model offers a rich analytical structure for service systems under uncertain customer behavior, inventory fluctuations, and catastrophic disruptions. Performance evaluation and optimization must therefore incorporate neutrosophic logic into queuing, inventory control, behavioral modeling, and risk management.

4.4 Steady-State Analysis

To assess the long-run behavior of the neutrosophic NM/NM/C/K queuing-inventory system, we derive steady-state probabilities that incorporate uncertainty in arrivals, services, inventory levels, behavioral mechanisms, and catastrophic disruptions. Each system state is defined by the pair $(n, j) \in \mathcal{S}$, where n is the number of customers in the system $(0 \le n \le K)$ and j is the corresponding inventory level $(0 \le j \le S)$.

Let $\tilde{P}_{n,j} = (P^T_{n,j}, P^I_{n,j}, P^F_{n,j})$ denote the neutrosophic steady-state probability of state (n, j).

4.4.1 Balance Equations under Neutrosophic Uncertainty

Let $\tilde{\lambda}_n = (\lambda^T_n, \lambda^I_n, \lambda^F_n)$ be the neutrosophic arrival rate, $\tilde{\mu}_n = (\mu^T_n, \mu^I_n, \mu^F_n)$ the service rate, and $\tilde{R}_n = (R^T_n, R^I_n, R^F_n)$ the reneging rate. The neutrosophic balance equation for a generic state (n, j) is:

$$\text{Inflow: } \tilde{\lambda}_{n-1}\tilde{P}_{n-1,j} + \tilde{\mu}_{n+1}\tilde{P}_{n+1,j+1} + \tilde{R}_{n+1}\tilde{P}_{n+1,j} + \tilde{\eta}_{j+1}\tilde{P}_{n,j+1},$$
$$\text{Outflow: } (\tilde{\lambda}_n + \tilde{\mu}_n + \tilde{R}_n + \tilde{\gamma}_n + \tilde{\eta}_j)\tilde{P}_{n,j},$$

where $\tilde{\eta}_j$ is the neutrosophic inventory restocking rate and $\tilde{\gamma}_n$ the catastrophe rate.

4.4.2 Catastrophic Event Adjustments

During a catastrophic event, all active states reset to an idle state with zero inventory. The cumulative transition is represented as:

$$\tilde{P}_{\text{cat}} = \sum_{n=0}^{K}\sum_{j=1}^{S} \tilde{P}_{n,j} \cdot \tilde{\gamma}_{\text{cat}},$$

with recovery governed by a neutrosophic restoration rate $\tilde{\kappa}$. The system resumes from the base state $(0, 0)$ following restoration.

4.4.3 Normalization Condition

The total neutrosophic probability mass across all states must satisfy:

$$\sum_{n=0}^{K}\sum_{j=0}^{S}\tilde{P}_{n,j} = (1, 0, 0),$$

ensuring complete coverage under truth, with zero indeterminacy and falsity in total mass.

4.4.4 Reduction via (α, β, γ)-Cut

To facilitate numerical computation, the neutrosophic parameters are reduced via the (α, β, γ)-cut technique. The corresponding crisp probability bounds are given by:

$$P_{n,j}^{[\alpha,\beta,\gamma]} \in \left[P_{n,j}^{T}(\alpha), P_{n,j}^{T}(\alpha) + P_{n,j}^{I}(\beta) + P_{n,j}^{F}(\gamma)\right],$$

yielding a parametric family of classical M/M/C/K models solvable by recursive or matrix-based methods.

4.4.5 Numerical Implementation

The steps for computing neutrosophic steady-state probabilities are:

1. Select confidence levels $\alpha, \beta, \gamma \in [0, 1]$,
2. Apply (α, β, γ)-cuts to all SVTNN parameters,
3. Solve the resulting crisp M/M/C/K balance equations for each confidence scenario,
4. Reconstruct neutrosophic performance measures from the family of crisp solutions.

5 Numerical Results and Discussion

To demonstrate the applicability of the proposed neutrosophic-based multi-server queuing-inventory model with attraction-retention and catastrophic events, we present a numerical illustration based on single-valued trapezoidal neutrosophic numbers (SVTNN). This representation enables the inclusion of uncertainty, indeterminacy, and inconsistency in arrival and service parameters, allowing for a comprehensive analysis of system performance under vague information.

5.1 Numerical Illustration

Let the arrival rate A and service rate S be expressed as SVTNNs:

$$A = \{(4, 5, 6, 7), (3, 6, 9, 12), (3, 5, 7, 9)\},$$
$$S = \{(15, 16, 17, 22), (14, 15, 16, 17), (16, 17, 18, 19)\}.$$

These neutrosophic numbers reflect the degrees of truth, indeterminacy, and falsity in their respective domains. Using the (α, β, γ)-cut technique, we derive:

$$A = [(4+\alpha, 7-\alpha), (6-3\beta, 9+3\beta), (5-2\gamma, 7+2\gamma)],$$
$$S = [(15+\alpha, 18-\alpha), (15-\beta, 16+\beta), (17-\gamma, 18+\gamma)].$$

Dayana et al. [15] introduced the neutrosophic M/M/1 queueing model for finite-capacity systems. Building on their framework, we extend the analysis to a multi-server neutrosophic M/M/c/K queuing-inventory system. Using a queue capacity of $K = 10$, we apply the neutrosophic extension of the M/M/1/K model to compute key performance indicators, including the probability of zero customers in the system (p_0), the expected queue length (L_q), the expected number of customers in the system (L_s), the average waiting time in the queue (W_q), and the average waiting time in the system (W_s).

These are derived using the crisp queue approximation from (α, β, γ)-cuts, following the pseudo-code from the base model. Results are evaluated for $\alpha, \beta, \gamma \in [0, 1]$.

5.2 Results and Analysis

Performance metrics for $\alpha = 0$ and $\alpha = 1$:

$$\begin{aligned} lL_q(\alpha=0) &= 0.02143, & uL_q(\alpha=0) &= 0.18356, \\ lL_s(\alpha=0) &= 0.15988, & uL_s(\alpha=0) &= 0.48342, \\ lW_q(\alpha=0) &= 0.00753, & uW_q(\alpha=0) &= 0.02931; \\ lL_q(\alpha=1) &= 0.04876, & uL_q(\alpha=1) &= 0.09412, \\ lL_s(\alpha=1) &= 0.22157, & uL_s(\alpha=1) &= 0.34639, \\ lW_q(\alpha=1) &= 0.00932, & uW_q(\alpha=1) &= 0.02345. \end{aligned}$$

Numerical interpolation of L_q:

Table 1: Numerical Interpolation of L_q with (α, β, γ)-cuts

α	lL_q	uL_q	β	lL_q	uL_q	γ
0.0	0.02143	0.18356	0.0	0.10683	0.41568	0.0
0.2	0.02800	0.14800	0.2	0.07708	0.53022	0.2
0.4	0.03600	0.11700	0.4	0.05375	0.67301	0.4
0.6	0.04228	0.09000	0.6	0.03575	0.85039	0.6
0.8	0.04800	0.06700	0.8	0.02220	1.06927	0.8
1.0	0.04876	0.09412	1.0	0.01238	1.33631	1.0

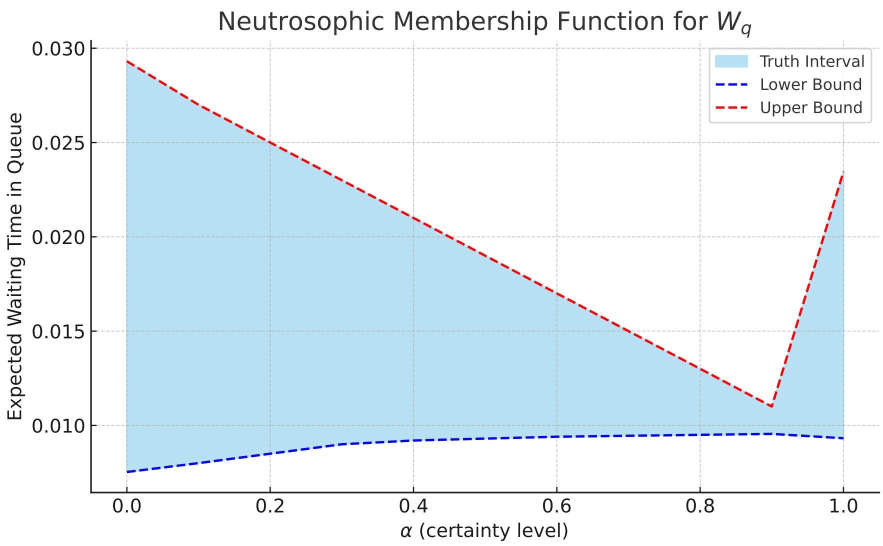

Figure 1: Truth, Indeterminacy, and Falsity Membership Functions for L_q

Numerical interpolation of L_s:

Table 2: Numerical Interpolation of L_s with (α, β, γ)-cuts

α	lL_s	uL_s	β	lL_s	uL_s	γ
0.0	0.15988	0.48342	0.0	0.38461	0.88613	0.0
0.2	0.19277	0.44100	0.2	0.31883	1.04182	0.2
0.4	0.20988	0.40100	0.4	0.26027	1.22655	0.4
0.6	0.22785	0.36400	0.6	0.20779	1.44656	0.6
0.8	0.24675	0.32900	0.8	0.16049	1.70849	0.8
1.0	0.22157	0.34639	1.0	0.11764	2.01866	1.0

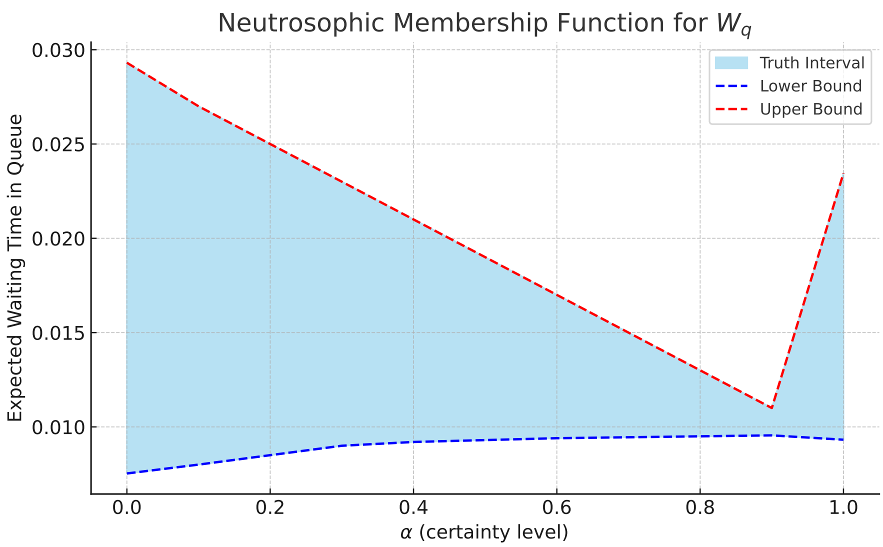

Figure 2: Truth, Indeterminacy, and Falsity Membership Functions for L_s

Numerical interpolation of W_q:

Table 3: Numerical Interpolation of W_q with (α, β, γ)-cuts

α	lW_q	uW_q	β	lW_q	uW_q	γ
0.0	0.00753	0.02931	0.0	0.02137	0.05197	0.0
0.2	0.00850	0.02500	0.2	0.01752	0.06169	0.2
0.4	0.00920	0.02100	0.4	0.01414	0.07324	0.4
0.6	0.00940	0.01700	0.6	0.03575	0.85039	0.6
0.8	0.00950	0.01300	0.8	0.02220	1.06927	0.8
1.0	0.00932	0.02345	1.0	0.01238	1.33631	1.0

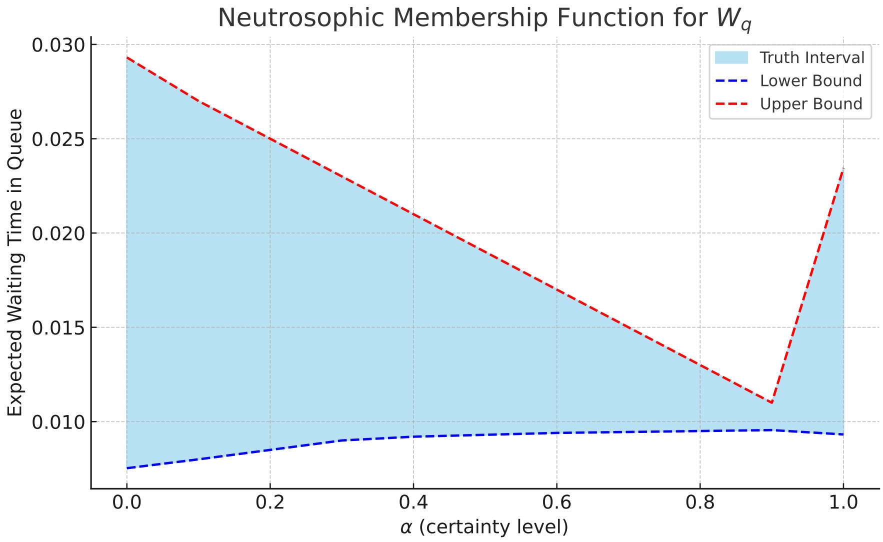

Figure 3: Truth, Indeterminacy, and Falsity Membership Functions for W_q

5.3 Discussion

The neutrosophic-based numerical analysis demonstrates the ability of the model to reflect real-world uncertainty. The results indicate that

- As the values of α, β, and γ increase, the corresponding confidence intervals narrower, indicating a higher degree of information precision in the system parameters.
- Queue length and system congestion are sensitive to both service variability and arrival uncertainty.
- Neutrosophic modeling provides adjustable confidence levels, making it an effective decision-making tool in uncertain and inconsistent environments.

These results align with the theoretical expectations and reinforce the effectiveness of neutrosophic queuing systems in complex service operations with incomplete information.

6 Conclusions

This study presented a comprehensive neutrosophic-based framework for analyzing multi-server queuing-inventory systems characterized by customer attraction-retention behavior, impatience, asynchronous server vacations, and catastrophic events. By integrating single-valued trapezoidal neutrosophic numbers (SVTNN), the proposed model effectively captures the imprecise, indeterminate, and inconsistent nature of real-world system parameters such as arrival rates, service rates, and inventory levels. Numerical illustrations demonstrated the impact of neutrosophic parameters on queue length, waiting time, and system capacity utilization. The results showed that increasing degrees of truth, indeterminacy, and falsity significantly affect confidence intervals, reinforcing the model's sensitivity to uncertainty. The algorithm developed can serve as a decision support tool for service system managers operating with vague, contradictory, or incomplete data. It enables more robust planning and control of queuing-inventory dynamics in environments susceptible to customer behavioral variation and catastrophic disruptions. Future work may extend this framework by incorporating fuzzy-neutrosophic hybrid models, time-varying arrival and service rates, or applying the model to real-world data sets from telecommunications, healthcare, or logistics systems.

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Soft Clustering Technique for Brain Tumor Segmentation within Neutrosophic Framework

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Abstract. Precise segmentation of brain tumors is vital in healthcare because it impacts diagnosis, treatment planning, and patient outcomes. However, the brain has complex structures with non-linear and inhomogeneous forms, which poses substantial challenges for conventional segmentation approaches. Despite of intensive research works, there persists a need to improve segmentation accuracy while concurrently lowering the computing costs. To address this, this study presents an innovative segmentation framework that incorporates kernel distance into the neutrosophic c-means (NCM) algorithm, together with the Kullback-Leibler (KL) divergence measure. Furthermore, the integration of kernel distance enables the algorithm to adeptly identify non-linear structural fluctuations and eliminate outliers, which improves robustness. Besides, the KL divergence mechanism accelerates convergence and improves segmentation precision by increasing the clustering process by reducing the distance between neighbourhood membership degrees. The proposed approach has been assessed against five established clustering algorithms utilizing both objective performance metrics and subjective visual evaluations. Experimental findings validate that the proposed method attains enhanced accuracy with increased computational efficiency and more precise delineation of tumor areas. Additionally, these results underscore the potential of the developed method for brain tumor segmentation, helping to bridge existing methodological gaps and promote more efficient medical image analysis.

Keywords: Medical image segmentation; Uncertainty; Neutrosophic set; Kernel distance; KL divergence

1. Introduction

Tumors are abnormal growth of the tissues that appear in various regions of the body which affect normal physiological activities and cause major risks to health [1]. Depending upon the nature, they can be classified as benign or malignant, that are distinguished by their aggressive and invasive behaviour. Among these, brain tumors are the most serious threat due to their ability to disrupt fundamental neurological functioning, which can lead to severe health consequences. Therefore, early discovery and precise characterisation of brain tumors are essential for determining viable treatment techniques, such as surgery, radiotherapy, and chemotherapy, that can significantly improve patient outcomes. Brain tumors, which can grow from the brain tissue or metastasise from other regions of the human body, which makes the medical imaging and diagnosis a challenging task. Further, It is laborious to precisely identify and characterise tumour locations due to the very sensitive and complex nature of brain tissues combined with the variability in tumor morphology and behaviour. In that context, physicians make use variety of imaging modalities to capture the brain for better diagnosis and treatment process. Among which, magnetic resonance imaging (MRI) become the more significant for diagnosing and evaluating brain tumors due to its superior contrast resolution and ability to provide detailed images of soft tissues [2]. It enables to visualize the different types of tissues in the brain and aids to detect the even subtle abnormalities in the brain. In addition, MRI enables the physicians to examine the characteristics of tumor such as size, its involvement in surrounding tissues and its location through offering variety of sequences such as T1-weighted, T2-weighted and fluid attenuated inversion recovery images.

Despite the advantages of MRI, the manual segmentation of brain tumors from MRI scans remains a time-consuming and intensive process, subject to inter- and intra-observer variability. This has led to increased research efforts toward automated brain tumor segmentation methods, which aim to accurately and efficiently segment tumors from MRI images [3]. Thus, wide range of researchers contributed to this domain by developing the variety of algorithms based on thresholding [4], region growing [5], water-shed, [6] and clustering based techniques [7]. Apart from all other approaches, clustering based techniques have gained attention for their ability to group image pixels with similar characteristics in an unsupervised manner, without requiring large labelled datasets which aids in effective separation of tumor regions from healthy regions into distinct clusters based on intensity, texture and spatial information. Moreover, clustering algorithms can be categorised into hard [7] and soft clustering approaches. Hard clustering methods includes k-means clustering [8], mean shift clustering [9] etc., but these clustering techniques are unable to handle data points that belong to multiple clusters leading to inaccurate segmentation.

Unlike hard clustering, soft clustering techniques [10] includes fuzzy c-means clustering (FCM), probabilistic clustering etc., allows pixels to belong multiple clusters with different degrees of belongingness, which is useful to efficiently deal with the complex, non-linear structures. At first, Bezdek introduced the FCM clustering approach that allows the data pints to have different degree of membership in more than one cluster [11]. This approach efficiently segments the brain image than hard clustering approaches by handling the uncertainties in the images. Followed by this work, to improve the performance of the clustering and extend it in various fields, many researchers developed the advanced variants of FCM to handle the multi-modal datas. Particularly, many refined variants of FCM is introduced to cluster the brain images by efficiently handling the uncertainties and improve the performance through the use of difference distance measures and so on [12]- [14]. In addition, as the extension of this technique, type-2 fuzzy based FCM is developed to deal with noise and imprecise date with complex structures by improving the flexibility in membership modelling. Further, intuitionistic fuzzy c-means clustering (IFCM) is formulated that advances the traditional FCM techniques by incorporating the degrees of membership, non-membership and hesitation [15]. This approach improved the cluster separation and providing the more nuanced representation. Furthermore, kernel distance is replaced the Euclidean distance in IFCM to form the kernel induced FCM (KIFCM) method to deal with the non-linear data [16].

While fuzzy sets are effective in handling uncertainty in medical images, there remains a need for enhanced capabilities to address complex, structured, and uncertain medical images. The neutrosophic set (NS) [17] framework builds upon traditional fuzzy sets by effectively managing indeterminacy and extending their applicability to these challenging scenarios. Based on this, Guo and Sengur formulated the neutrosophic c-means clustering (NCM) by adopting degree of determinacy, indeterminacy and falsity in the objective function providing a more comprehensive approach to representing various uncertainty [18]. Moreover, this algorithm introduces two kinds of rejection which are outlier rejection and noise rejection for effective segmentation. Further, Akubulut et al., authors extend NCM for clustering nonlinear-shaped data by integrating a kernel function, creating a new algorithm called kernel neutrosophic c-means (KNCM) [19]. Subsequently, Singh et al., developed the type-2 NS to cluster the images through the information obtained from the neutrosophic entropy [20]. While clustering algorithms aspire for greater accuracy, they frequently result in longer runtimes and increased computational complexity. This highlights the critical need of efficiency optimisation for these algorithms. Thus, to speed up this optimisation, it is possible to use the local spatial information included inside brain images. Based on this, Gharieb and Gendy (LMKLFCM), modified the criterion function of FCM by incorporating the KL information distance. This algorithm promotes the alignment of the cluster membership of a pixel with the smoothed membership functions of its local neighbourhood by minimising the KL distance [21, 22]. Also, this procedure provides robustness against noise and generates clustered images with piecewise homogeneous regions. At a subsequent time Lu et al., [23] formulated the NKWNLICM method by inculcating the noisy distance and fuzzy spatial information into the NCM technique to enhance the performance of image segmentation. Thereafter, Wang et al., devised the segmentation technique based on KL divergence integrated with wavelet transforms and morphological reconstruction (KLDFCM) to improve the efficiency [24]. Following this, Kumar et al., devised the clustering process (FBkPC_S1) by integrating the FCM objective function to bounded cluster planes and adds the local spatial information to the criterion function to handle the noise efficiently [25]. Then, Farooq and Memon developed the clustering algorithm by making use of kernel metric with possibilistic FCM to reduce the computational cost [26].

Even though, the significant researches has been done on this domain, still brain tumor segmentation persists the challenging task due to various aspects. First, the structural complexity of the brain is a formidable barrier and the tumor region has the overlapping boundaries, complex structures and heterogeneous intensity distributions of the brain. Due to these reasons, the conventional techniques often prone to over-segmentation and under-segmentation, which results in inaccurate representation of tumor regions. Second, MRI images are prone to noise, motion and acquisition artifacts which further leads to higher risk of misclassification. Furthermore, the lack of sufficient contrast between tumor and its surrounding tissues increases the complexity in accurate determination of boundaries, particularly in early-stages of tumors. Another major limitation lies in the sensitivity of clustering algorithms to initialization and outliers. Most conventional c-means and its variants converge to local optima, where initial centroid placement strongly influences the final clustering result. Even though, existing fuzzy and neutrosophic approaches effective in modelling uncertainty but often fail to capture non-linear structures when restricted to Euclidean distance, which leads to the loss of critical anatomical details. These limitations are further evident when applied to complex brain regions, resulting in higher computational costs and reduced efficiency.

In light of above mentioned challenges , this research sets out to refine a clustering technique in an effort to minimise computational cost while simultaneously enhancing the level of accuracy of the segmented result by effectively handling noise, low contrast and structural complexity. By addressing these limitations, a new clustering algorithm is proposed by integrating the kernel distance metric and KL divergence term into the objective function of NCM (KLKNCM). Kernel distance metric aids to cluster the non-linear structures present the brain images which improves the accuracy of the segmented result. Meanwhile, the KL divergence term helps to reduce the noise and improve similarity identification between data points and cluster centres that preserves neighbourhood relationships. Further, it fastens the convergence of the algorithm which in result reduces the computational cost of the proposed clustering technique. Additionally, the efficiency of the proposed algorithm is tested against state-of-the-art techniques and evaluated through both objective and subjective assessments. The key contributions of the proposed KLKNCM algorithm are outlined as follows:

(1) A novel clustering algorithm, termed KLKNCM is introduced. By embedding the kernel distance metric within the neutrosophic c-means formulation, the proposed framework effectively captures non-linear structural variations in brain images, enabling superior

tumor delineation compared to conventional clustering methods. The inclusion of the KL divergence term strengthens the similarity assessment between data points and cluster centers. This preserves spatial neighbourhood relationships, reduces the effect of noise, and enhances intra-cluster compactness and inter-cluster separability.

(2) The algorithm incorporates mechanisms to identify and suppress the influence of outliers, thereby enhancing segmentation robustness. Through the incorporation of the KL divergence term and fuzzy membership assignment, anomalous or noisy pixels receive diminished membership weights, thereby minimizing their influence on cluster formation. This ensures reliable segmentation even in the presence of noise and artifacts.

(3) By utilising the joint benefits of kernel distance and KL divergence, the KLKNCM algorithm achieves faster convergence with fewer iterations, thereby lowering the computational cost relative to existing clustering techniques.

Additionally, the structure of this manuscript is as follows: Section 2 presents the background study, which serves as the primary motivation for this work. Section 3 outlines the proposed KLKNCM technique. Next, Section 4 describes the experimental setup, followed by Section 5 which offers a comprehensive assessment of the study. Finally, the conclusions are drawn in Section 8.

2. **Background Works**

2.1. *NCM*

Consider an image represented by set of pixels and let it be $K = \{k_1, k_2, \ldots, k_a | a = 1, 2, \ldots, N\}$ where k_a represents the intensity a^{th} pixel in the image and N corresponds to the total number of pixels in the image. Let C be the total number of clusters into which the image has to be segmented and each cluster has its own centre which is symbolised as c_b. Then, the optimization function for NCM is given as follows.

$$\begin{aligned} min\, \mathcal{L}_{NCM}(\mathcal{T},\mathcal{I},\mathcal{F},\mathcal{C}) &= \sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{Z}_1\mathcal{T}_{ab})^m\|k_a - w_b\|^2 + \sum_{a=1}^{N}(\mathcal{Z}_2\mathcal{I}_a)^m\|k_a - \bar{w}_{a_{max}}\|^2 + \sum_{a=1}^{N}\xi^2(\mathcal{Z}_3\mathcal{F}_a)^m \\ \text{subject to} \quad & \sum_{b=1}^{C}\mathcal{T}_{ab} + \mathcal{I}_a + \mathcal{F}_a = 1, 1 \le a \le N. \end{aligned} \tag{1}$$

Here, $\bar{w}_{a_{max}} = \dfrac{w_{e_i} + w_{f_i}}{2}$ and $e_i = \underset{b=1,2,\ldots C}{\arg\max}(\mathcal{T}_{ab}),\ f_i = \underset{b \neq e_{a \cap b}=1,2,\ldots C}{\arg\max}(\mathcal{T}_{ab})$.

In the above give objective function (1), T_{ab} indicates the degree of belongingness of the determinant cluster, whereas, I_a and F_a represents the degree of membership of the boundaries and outlier clusters with $0 < T_{ab}, I_a, F_a < 1$ satisfying the given constraint in the objective function. Moreover, e_i and f_i defines to the cluster numbers corresponding to the first largest and the second largest value of T_{ab} and its average value is signified as $\bar{w}_{a_{max}}$. Moreover, $\mathcal{Z}_i$, $i = 1, 2, 3$

characterises the weight factor for the d determinant, indeterminant and outlier degrees and ξ denotes the outlier control parameter which is used for identifying and managing outliers, reducing their influence on cluster formation, and enhancing the algorithm's robustness to noise and anomalies. It enables fine-tuning of the algorithm's sensitivity to outliers, leading to more accurate and reliable clustering, particularly in complex datasets with noise or uncertainty.

The criterion function is minimized by utilising the Lagrangian multiplier and then it is solved to obtain the degree of membership and the centroid of each cluster. Accordingly, the degree of determinant, boundaries and the outliers are obtained as given in the equation outlined below.

$$\mathcal{T}_{ab} = \frac{P}{\mathcal{Z}_1}(k_a - w_b)^{-(2/m-1)} \tag{2}$$

$$\mathcal{I}_a = \frac{P}{\mathcal{Z}_2}(k_a - \bar{w}_{a_{\max}})^{-(2/m-1)} \tag{3}$$

$$\mathcal{F}_a = \frac{P}{\mathcal{Z}_3}\xi^{-(2/m-1)} \tag{4}$$

Subsequently, the centroid for the each cluster is deduced as expressed in the equation (5),

$$v_j = \frac{\sum_{a=1}^{N}(\mathcal{Z}_1\mathcal{T}_{ab})^m k_a}{\sum_{a=1}^{N}(\mathcal{Z}_1\mathcal{T}_{ab})^m} \tag{5}$$

where $P = \left(\frac{\lambda_i}{m}\right)^{1/m-1}$ such that λ_i is Lagrangian constant. Moreover, during each iteration of the clustering process, the membership values and cluster centres are updated, and the objective function is minimised in order to segment the image. Significantly, the decision-making process does not explicitly consider the membership degrees of boundary and outliers. The method stops when the absolute difference between consecutive iterations of $\mathcal{T}_{ab}$ values is less than a specified threshold δ, or when the maximum number of iterations is reached.

2.2. *LMKLFCM Technique*

This technique incorporates the KL divergence to the objective function of FCM clustering to make the clustering process computationally efficient with better result. In this process the inclusion of the KL divergence component ensures that the membership grades of a given image pixel become substantially similar to those of its neighbours. As a consequence, optimising the membership partition during each iteration improves the segmentation performance of the algorithm. In addition, minimising the KL distance allows the labelling of a pixel to be impacted by its neighbours which leads to the development of piecewise homogeneous regions inside the image, concurrently providing an efficient approach for reducing the noise in the image. The criterion function for the LMKLFCM technique is given in equation (6).

$$\begin{aligned} \min \mathcal{L}_{\text{LMKLFCM}} &= \sum_{a=1}^{N}\sum_{b=1}^{C}\mu_{ab}\|k_a - w_b\|^2 + \lambda\sum_{a=1}^{N}\sum_{b=1}^{C}\mu_{ab}\log\frac{\mu_{ab}}{\pi_{ab}} \\ \text{subject to} \quad & \sum_{b=1}^{C}\mu_{ab} = 1, 1 \le a \le N. \end{aligned} \tag{6}$$

In the above equation, π_{ab} indicates the moving membership average function, which is computed by averaging the memberships μ_{ab} across a neighbourhood window that is centred around μ_{ij} within n space, represented as N_n. Mathematically, the moving average function is given as, $\pi_{ab} = \frac{1}{N_j}\sum_{j\in N_n}\mu_{aj}$. Further, μ_{ab} indicates the membership degree of the a^{th} data point in the b^{th} cluster and the second part in the objective function symbolises the KL divergence distance. Moreover, the other terms specified in the equation (6) are described in the previous subsection. Following this, the membership degree and the centroid of each cluster are determined by integrating the Lagrangian multiplier into the objective function and optimising it through minimisation. Consequently, the equations presented below delineate the degree of membership (8) and centroid (7) for each cluster of the LMKLFCM technique.

$$w_b = \frac{\sum_{a=1}^{N}\mu_{ab}\,k_a}{\sum_{a=1}^{N}\mu_{ab}} \tag{7}$$

$$\mu_{ab} = \frac{\pi_{ab}\exp^{-(\|k_a-w_b\|^2)/\lambda}}{\sum_{b=1}^{C}\pi_{aj}\exp^{-(\|k_a-w_j\|^2)}} \tag{8}$$

In turn, the LMKLFCM technique systematically segments the image by optimizing the criterion function and iteratively refining the membership values μ_{ab} associated with the centroid of the each cluster. Finally, the iteration process wraps up when the absolute difference between successive iterations of the μ_{ab} values falls below a particular termination criterion, denoted as δ, or when the maximum number of iterations is reached.

3. Proposed KLKNCM Technique

Motivated by the works give in the preceding section and to improve the accuracy of the segmented images. The KLKNCM technique has been developed to improve the segmentation of brain images, focusing on enhancing accuracy while simultaneously minimising computation cost. This is achieved by integrating the kernel function and the KL divergence measure into the criterion function. The integration of the kernel function is imperative for precisely recognising the non-linear structures present within the inhomogeneous regions of the brain. Further, it facilitates the mapping of input data into a higher-dimensional feature space, which improves the capacity of the algorithm to identify complex patterns and boundaries that may not be simply separable within the original input space. Thus, incorporating the kernel function is beneficial for segmenting the intricate and inhomogeneous images to attain precise delineation.

Further, KL divergence helps to lower noise by encouraging consistency among adjacent pixels and improve similarity identification between data points and cluster centres that preserves neighbourhood relationships. Also, KL divergence improves segmentation accuracy, accelerates convergence, reduces computational cost, and enables the flexible management of imbalanced data distributions due to its non-symmetric nature. Through inculcating these two metrics in proposed method effectively captures intricate image patterns, resulting in more precise segmentation results while preserving computational efficiency. Hence, the criterion function for the proposed method is mathematically expressed as in equation (9).

$$min\,\mathcal{L}_{\mathrm{KLKNCM}}(T,I,F,C)=\sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{Z}_1\mathcal{T}_{ab})\|\psi(k_a)-\psi(w_b)\|^2+\sum_{a=1}^{N}(\mathcal{Z}_2\mathcal{I}_a)\|\psi(k_a)-\psi(\bar{w}_{a_{\max}})\|^2+\sum_{a=1}^{N}\xi^2(\mathcal{Z}_3\mathcal{F}_a)+D(T,I,F,\bar{T},\bar{I},\bar{F}) \tag{9}$$

where, D indicates the KL divergence measure and it is defined as follows,

$$D(T,I,F,\bar{T},\bar{I},\bar{F})=\sum_{a=1}^{N}\sum_{b=1}^{C}\mathcal{T}_{ab}\log\frac{\mathcal{T}_{ab}}{\bar{\mathcal{T}}_{ab}}+\sum_{a=1}^{N}\mathcal{I}_a\log\frac{\mathcal{I}_a}{\bar{\mathcal{I}}_a}+\sum_{a=1}^{N}\mathcal{F}_a\log\frac{\mathcal{F}_a}{\bar{\mathcal{F}}_a} \tag{10}$$

and $(D)\to 0$. The median filter applied on the degree of belongingness of determinate, boundary, and outliers using the chosen window size which is symbolised as $\bar{\mathcal{T}}_{ab}, \bar{\mathcal{I}}_a, \bar{\mathcal{F}}_a$ respectively. Thereafter, the criterion function of the proposed techniques is modified as given below.

$$\begin{aligned}\min\ \mathcal{L}_{\mathrm{KLKNCM}}(T,I,F,C)=&\sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{Z}_1\mathcal{T}_{ab})\|\psi(k_a)-\psi(w_b)\|^2+\sum_{a=1}^{N}(\mathcal{Z}_2\mathcal{I}_a)\|\psi(k_a)-\psi(\bar{w}_{a_{\max}})\|^2\\&+\sum_{a=1}^{N}\xi^2(\mathcal{Z}_3\mathcal{F}_a)+\gamma\left(\sum_{a=1}^{N}\sum_{b=1}^{C}\mathcal{T}_{ab}\log\frac{\mathcal{T}_{ab}}{\bar{\mathcal{T}}_{ab}}+\sum_{a=1}^{N}\mathcal{I}_a\log\frac{\mathcal{I}_a}{\bar{\mathcal{I}}_a}+\sum_{a=1}^{N}\mathcal{F}_a\log\frac{\mathcal{F}_a}{\bar{\mathcal{F}}_a}\right)\\\text{subject to}\quad&\sum_{b=1}^{C}\mathcal{T}_{ab}+\mathcal{I}_a+\mathcal{F}_b=1,1\le a\le N.\end{aligned} \tag{11}$$

The kernel induced distance is expressed as,

$$\|\psi(k_a)-\psi(w_b)\|^2=\mathcal{K}(k_a,k_a)-2\mathcal{K}(k_a,w_b)+\mathcal{K}(w_b,w_b) \tag{12}$$

In this work, the Gaussian kernel is used as the kernel function. Hence it known that $\mathcal{K}(k_a,k_a)=\mathcal{K}(w_b,w_b)=1$ and the above equation is deduced as equation (14) and (15) receptively.

$$\|\psi(k_a)-\psi(w_b)\|^2=2(1-\mathcal{K}(k_a,w_b)) \tag{13}$$

similarly,

$$\|\psi(k_a)-\psi(\bar{w}_{a_{\max}})\|^2=2(1-\mathcal{K}(k_a,\bar{w}_{a_{\max}})) \tag{14}$$

Thus, substituting the equations (14) and (15) to the (11). Therefore, the transformed final objective function is given as in equation (15).

$$\begin{aligned} \min \mathcal{L}_{\mathrm{KLKNCM}}(T,I,F,C) =& 2\sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{Z}_1\mathcal{T}_{ab})(1-\mathcal{K}(k_a,w_b)) + 2\sum_{a=1}^{N}(\mathcal{Z}_2\mathcal{I}_a)(1-\mathcal{K}(k_a,\bar{w}_{a_{\max}})) \\ &+ \sum_{a=1}^{N}\xi^2(\mathcal{Z}_3\mathcal{F}_a) + \gamma\left(\sum_{a=1}^{N}\sum_{b=1}^{C}\mathcal{T}_{ab}\log\frac{\mathcal{T}_{ab}}{\bar{\mathcal{T}}_{ab}} + \sum_{a=1}^{N}\mathcal{I}_a\log\frac{\mathcal{I}_a}{\bar{\mathcal{I}}_a} + \sum_{a=1}^{N}\mathcal{F}_a\log\frac{\mathcal{F}_a}{\bar{\mathcal{F}}_a}\right) \\ \text{subject to} \quad & \sum_{b=1}^{C}\mathcal{T}_{ab} + \mathcal{I}_a + \mathcal{F}_b = 1, 1 \le a \le N. \end{aligned} \tag{15}$$

In the preceding equation, $\mathcal{Z}_i$ indicates the weight factor, whereas ξ refers to outlier control parameter. Further, the positive parameter γ influences the objective function by controlling the consequences of the KL divergence term. As the subsequent process, the proposed objective function is minimised by incorporating the Lagrangian multiplier in it. Mathematically, the Lagrangian inculcated criterion function is formulated as given below.

$$\begin{aligned} \mathcal{L}(T,I,F,C) =& 2\sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{Z}_1\mathcal{T}_{ab})(1-\mathcal{K}(k_a,w_b)) + 2\sum_{a=1}^{N}(\mathcal{Z}_2\mathcal{I}_a)(1-\mathcal{K}(k_a,\bar{w}_{a_{\max}})) \\ &+ \sum_{a=1}^{N}\xi^2(\mathcal{Z}_3\mathcal{F}_a) + \gamma\left(\sum_{a=1}^{N}\sum_{b=1}^{C}\mathcal{T}_{ab}\log\frac{\mathcal{T}_{ab}}{\bar{\mathcal{T}}_{ab}} + \sum_{a=1}^{N}\mathcal{I}_a\log\frac{\mathcal{I}_a}{\bar{\mathcal{I}}_a} + \sum_{a=1}^{N}\mathcal{F}_a\log\frac{\mathcal{F}_a}{\bar{\mathcal{F}}_a}\right) \\ &- \sum_{a=1}^{N}\lambda_a\left(\sum_{b=1}^{C}\mathcal{T}_{ab} + \mathcal{I}_a + \mathcal{F}_a - 1\right) \end{aligned} \tag{16}$$

Thus, to minimize the above given objective function, partial derivative of the objective function with respect to the membership degree of determinate, boundary, outlier points and the cluster centre are computed and it presented in equations (17)-(19).

$$\frac{\partial L}{\partial \mathcal{T}_{ab}} = 2\mathcal{Z}_1(1-\mathcal{K}(k_a,w_b)) + \gamma\left(\log\frac{\mathcal{T}_{ab}}{\bar{\mathcal{T}}_{ab}} + 1\right) - \lambda_a \tag{17}$$

$$\frac{\partial L}{\partial \mathcal{I}_a} = 2\mathcal{Z}_2(1-\mathcal{K}(k_a,\bar{w}_{a_{\max}})) + \gamma\left(\log\frac{\mathcal{I}_a}{\bar{\mathcal{I}}_a} + 1\right) - \lambda_a \tag{18}$$

$$\frac{\partial L}{\partial \mathcal{F}_a} = \xi^2\mathcal{Z}_3 + \gamma\left(\log\frac{\mathcal{F}_a}{\bar{\mathcal{F}}_a} + 1\right) - \lambda_a \tag{19}$$

Let $\frac{\partial L}{\partial \mathcal{T}_{ab}} = 0$, $\frac{\partial L}{\partial \mathcal{I}_a} = 0$, and $\frac{\partial L}{\partial \mathcal{F}_a} = 0$, then solving the equations(17), (18), (19) yields

$$\mathcal{T}_{ab} = \exp^{\frac{\lambda_a}{\gamma}} . \exp^{\frac{\gamma(log(\bar{\mathcal{T}}_{ab})-1)-2\mathcal{Z}_1(1-\mathcal{K}(k_a,w_b))}{\gamma}} \tag{20}$$

$$\mathcal{I}_a = \exp^{\frac{\lambda_a}{\gamma}} . \exp^{\frac{\gamma(log(\bar{\mathcal{I}}_a)-1)-2\mathcal{Z}_2(1-\mathcal{K}(k_a,\bar{w}_{a_{\max}}))}{\gamma}} \tag{21}$$

$$\mathcal{F}_a = \exp^{\frac{\lambda_a}{\gamma}} . \exp^{\frac{\gamma(log(\bar{\mathcal{F}}_a)-1)-\xi\mathcal{Z}_3}{\gamma}} \tag{22}$$

Let $\exp^{\frac{\lambda_a}{\gamma}} = A$ and the objective function is constrained to $\sum_{b=1}^{C} \mathcal{T}_{ab} + \mathcal{I}_a + \mathcal{F}_a = 1$, then

$$A\left[\exp^{\frac{\gamma(log(\bar{\mathcal{T}}_{ab})-1)-2\mathcal{Z}_1(1-\mathcal{K}(k_a,w_b))}{\gamma}} + \exp^{\frac{\gamma(log(\bar{\mathcal{I}}_a)-1)-2\mathcal{Z}_2(1-\mathcal{K}(k_a,\bar{w_a}_{\max}))}{\gamma}} + \exp^{\frac{\gamma(log(\bar{\mathcal{F}}_a)-1)-\xi\mathcal{Z}_3}{\gamma}}\right] = 1 \tag{23}$$

$$A = \left[\exp^{\frac{\gamma(log(\bar{\mathcal{T}}_{ab})-1)-2\mathcal{Z}_1(1-\mathcal{K}(k_a,w_b))}{\gamma}} + \exp^{\frac{\gamma(log(\bar{\mathcal{I}}_a)-1)-2\mathcal{Z}_2(1-\mathcal{K}(k_a,\bar{w_a}_{\max}))}{\gamma}} + \exp^{\frac{\gamma(log(\bar{\mathcal{F}}_a)-1)-\xi\mathcal{Z}_3}{\gamma}}\right]^{-1} \tag{24}$$

Finally, the degree of membership for determinate, indeterminate, and ouliers are formulated as follows,

$$\mathcal{T}_{ab} = A\ \exp^{\frac{\gamma(log(\bar{\mathcal{T}}_{ab})-1)-2\mathcal{Z}_1(1-\mathcal{K}(k_a,w_b))}{\gamma}} \tag{25}$$

$$\mathcal{I}_a = A.\exp^{\frac{\gamma(log(\bar{\mathcal{I}}_a)-1)-2\mathcal{Z}_2(1-\mathcal{K}(k_a,\bar{w_a}_{\max}))}{\gamma}} \tag{26}$$

$$\mathcal{F}_a = A.\exp^{\frac{\gamma(log(\bar{\mathcal{F}}_a)-1)-\xi\mathcal{Z}_3}{\gamma}} \tag{27}$$

Furthermore, to calculate the centroid for each cluster, the Gaussian kernel $(G(k_a, w_a))$ induced objective function is given as

$$\begin{aligned}\mathcal{L}(T,I,F,C) =& 2\sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{Z}_1\mathcal{T}_{ab})(1-\exp\left(\frac{-\|k_a,w_b\|}{\sigma^2}\right) + 2\sum_{a=1}^{N}(\mathcal{Z}_2\mathcal{I}_a)(1-\exp\left(\frac{-\|k_a,\bar{w_a}_{\max}\|}{\sigma^2}\right)\\ &+\sum_{a=1}^{N}\xi^2(\mathcal{Z}_3\mathcal{F}_a) + \gamma\left(\sum_{a=1}^{N}\sum_{b=1}^{C}\mathcal{T}_{ab}\log\frac{\mathcal{T}_{ab}}{\bar{\mathcal{T}}_{ab}} + \sum_{a=1}^{N}\mathcal{I}_a\log\frac{\mathcal{I}_a}{\bar{\mathcal{I}}_a} + \sum_{a=1}^{N}\mathcal{F}_a\log\frac{\mathcal{F}_a}{\bar{\mathcal{F}}_a}\right)\\ &-\sum_{a=1}^{N}\lambda_a\left(\sum_{b=1}^{C}\mathcal{T}_{ab} + \mathcal{I}_a + \mathcal{F}_a - 1\right)\end{aligned} \tag{28}$$

Taking the first derivative of the above equation with respect to w_b and equating it to zero. The following expression is deduced.

$$w_b = \frac{\sum_{a=1}^{N}(\mathcal{Z}_1\mathcal{T}_{ab})G(k_a,w_b)k_a}{\sum_{a=1}^{N}(\mathcal{Z}_1\mathcal{T}_{ab})G(k_a,w_a)} \tag{29}$$

The minimization of the criterion function is achieved by iterating the defined steps until convergence is reached and the membership degrees and cluster centers are updated for each iteration. Convergence is determined when the absolute difference between the values of T_{ab} from two consecutive iterations falls below a specified termination criterion δ, or when the maximum number of iterations is attained. Moreover, the parameters such as $\mathcal{Z}_i$, ξ and γ plays an crucial role in the clustering process. Hence, the analysis for the values chosen for the parameters will be given in succeeding section and the computational algorithm for the proposed technique is provided in the Algorithm 1.

Algorithm 1 Formulation of proposed KLKNCM alogrithm

Require: Number of cluster centre (C), weight factors ($\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$), outlier controller ($\xi$), KL parameter ($\gamma$), terminating criterion (δ), .

Step 1: Initialize $\mathcal{T}_{ab}^{(0)}$, $\mathcal{I}_{a}^{(0)}$ and $\mathcal{F}_{a}^{(0)}$.

Step 2: $y \leftarrow 1$

Step 3: ***repeat***

Step 4: Calculate $w_b^{(h)}$ using (29).

Step 5: Compute $\bar{w_{a}}_{\max}^{(h)}$.

Step 6: Update $\mathcal{T}_{ab}^{(h)}$, $\mathcal{I}_{a}^{(h)}$ and $\mathcal{F}_{a}^{(h)}$ using (25)-(27) and the objective function given in Eq (16).

Step 7: $y \leftarrow y + 1$

Step 8: ***until*** $|\mathcal{T}_{ab}^{(h+1)} - \mathcal{T}_{ab}^{(h)}| < \delta$ or $h \geq h_{max}$.

Step 9: ***return*** w_b, $T = \{\mathcal{T}_{ab}\}_{C \times N}$ and the iteration count h.

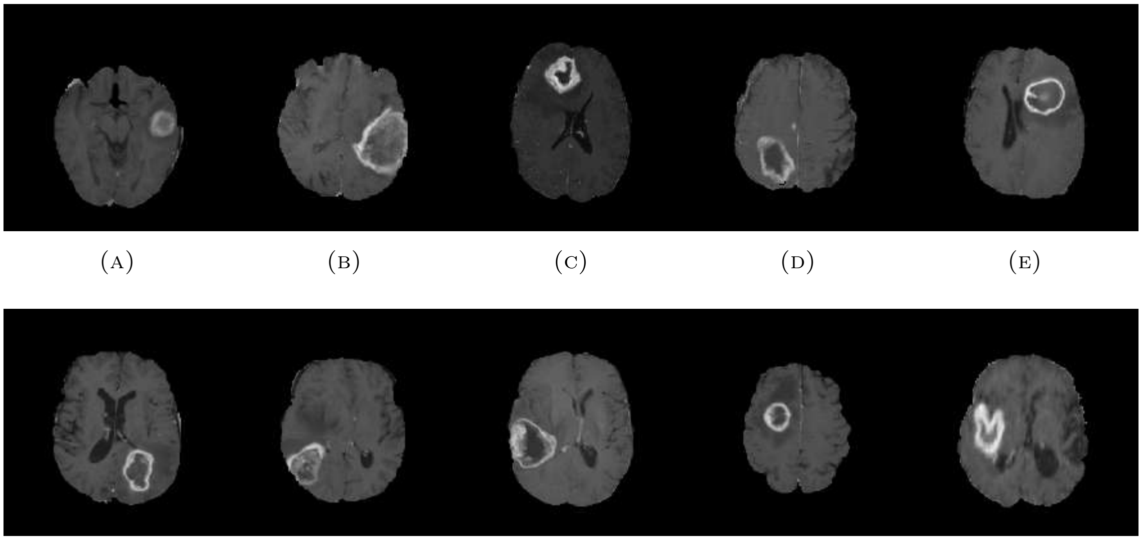
4. Experimental Details

The present section outlines the experimental framework of the proposed technique, providing detailed information on the dataset utilized in this study. Additionally, it discusses the comparative clustering algorithms and the clustering validation metrics employed to evaluate the performance of the KLKNCM technique.

4.1. *Description of the Dataset and Comparative Clustering Methods*

It is known that the brain, a highly intricate and structurally complex organ, presents significant challenges for image segmentation algorithms. Thus, to evaluate the performance of the proposed KLKNCM algorithm, brain images were selected for analysis. For this purpose, the BraTS (Brain Tumour Segmentation) dataset was employed [27], which is a benchmark for the segmentation of brain tumors. This dataset includes images of brains affected by glioblastoma, an aggressive and malignant form of brain cancer, in which T1-weighted MRI images were used for the clustering process because they have more detailed anatomy, which is important for accurately capturing the complexity of brain structures in segmentation tasks. Moreover, the use of this dataset enables a comprehensive assessment of the algorithm's capability to handle complex and heterogeneous medical images.

To assess the effectiveness and superiority of the proposed KLKNCM technique, a comparison analysis is performed against many cutting-edge clustering algorithms. In this sense, five clustering algorithms were chosen based on important characteristics pertinent to the study's scope. The NCM-1 [18], NCM-2 [28] and KNCM [19] algorithms were included due to the reason that they have integrated NS theory into the soft clustering framework, which is essential for managing indeterminate and incomplete information. Moreover, in order to evaluate the influence of spatial information on clustering, KLDFCM [24] and FBkPC_S1 [25] were



(A) (B) (C) (D) (E)

(F) (G) (H) (I) (J)

Figure 1. Source images adopted for the experiment.

selected, as these methods incorporate spatial features to improve clustering accuracy. The NKWNLICM [23] technique was chosen because it combines NS theory with local information clustering mechanisms, resulting in spatially aware clustering. Furthermore, it is worth noting that the parameters of each comparative method were carefully changed to fit the issue formulation in the current study, ensuring a fair and objective comparison of performance across all methods.

4.2. *Cluster Validation Metrics*

In order to evaluate the results in quantitative aspects, the clustering validation metrics is utilised. These metrics offers the valuable insights of the final clustered results obtained from the proposed and other comparative clustering techniques in various aspect. Thus, the metric encompassed in this work is described below. Moreover, the experiment is executed in MATLAB R2019a software and to maintain the preciseness and presentation, the results from ten images are analysed and discussed in the paper.

(1) **Fuzzy Performance Index (FPI)**

FPI [31] is used to assess the effectiveness and quality of the clustering, particularly it quantifies the compactness of the clusters. Mathematically it is represented as in equation (30).

$$FPI = 1 - \frac{C}{C-1}(1 - F_c) \tag{30}$$

where, F_c is the partition coefficient and it is calculated as $F_c = \frac{1}{N}\sum_{a=1}^{N}\sum_{b=1}^{C}(\mathcal{T}_{ab})^2$.

(2) **Modified Partition Entropy (MPE)**

MPE [31] evaluates the degree of overlap and intersections and uncertainty among the clusters. It is calculated using the following formula.

$$MPE = \frac{N.PE}{N - C} \tag{31}$$

such that, $PE = -\frac{1}{N}\sum_{a=1}^{N}\sum_{b=1}^{C}[\mathcal{T}_{ab}log_2(\mathcal{T}_{ab})^2]$ is the partition entropy. Lower the value indicates the better clustered result.

(3) **Xie-Beni Index (XBI)**

XBI [31] is the effective tool to ensure the quality of compactness and well-separated clusters of the resultant images. Further, it is calculated as follows,

$$XBI = \frac{\sum_{a=1}^{N}\sum_{b=1}^{C}\mathcal{T}_{ab}^2 x^2(k_a, w_b)}{N(\min_{i \neq b} x^2(w_i, w_b))} \tag{32}$$

where k_a refers the values of the pixel and w_b indicated the centroid of the cluster. Moreover, the lower the values of this metric represents the better outcome.

5. Analysis of Parameters

A detailed parameter analysis was conducted to evaluate the impact of various algorithmic parameters in the performance of the proposed method. By systematically adjusting key parameters, such as weighting parameters, weight factors ($\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$), outlier controller ($\xi$), KL parameter ($\gamma$), terminating criterion (δ), number of cluster centre (C), the optimal configuration for the proposed method was determined. Further, the results of the parameter analysis demonstrated that fine-tuning these parameters leads to significant improvements in both the precision of the segmented regions and the overall stability of the clustering process. In this clustering analysis, three weight factors $\mathcal{Z}_1, \mathcal{Z}_2$ and $\mathcal{Z}_3$ are assigned values of 0.8, 0.1, and 0.1, respectively. These weights control the influence of different features in the clustering process, where $\mathcal{Z}_1$ is the most dominant, and $\mathcal{Z}_2$ and $\mathcal{Z}_3$ contribute less significantly to decrease the influence of outliers. Then the optimal value of ξ for better clustering result is ranges from 140 to 200 for the considered dataset. Another key parameter is σ ranging from $[0.3, 0.8]$. Further, for improved segmented result, the value of γ is analysed and it varies from 10000 to 15000. Added to this, the image is segmented into four region and hence the centroid of the cluster C fixed as 4. Finally, the terminating value δ is set to 0.0001. This detailed examination provides insights into the sensitivity of the proposed method to different parameter values, ensuring that the algorithm outperforms consistently.

6. Experimental Analysis

This section provides a thorough evaluation of the proposed method, incorporating both objective and subjective assessments to benchmark its performance against other state-of-the-art techniques. The objective assessment delivers a quantitative evaluation, focusing on metrics such as segmentation accuracy, precision, computational efficiency, and runtime. Meanwhile, the subjective assessment offers a qualitative comparison, emphasizing the visual quality of

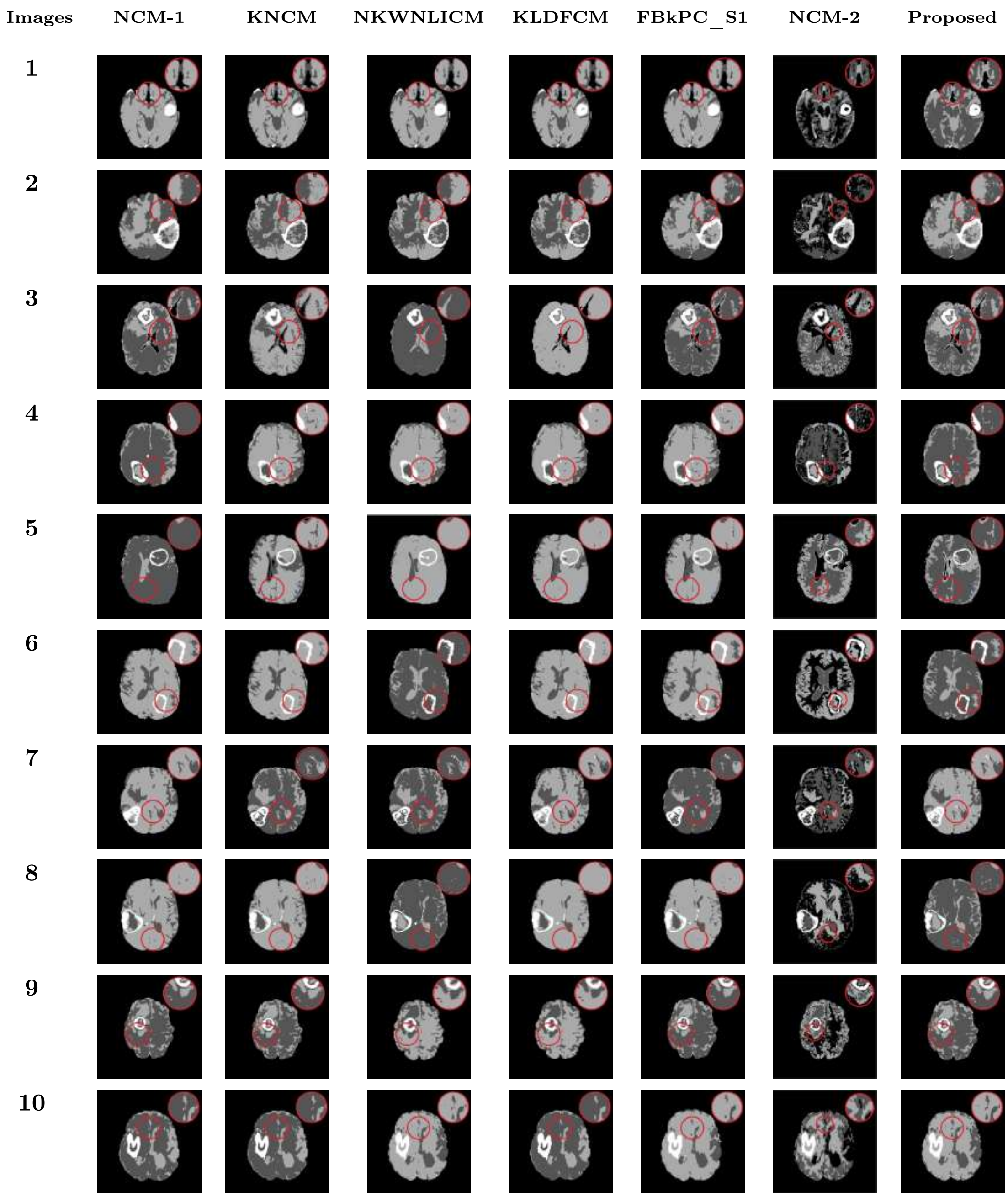


FIGURE 2. Resultant of proposed and comparative state-of-the-art methods.

the segmented results. Figure 2 illustrates the segmentation outcomes for some representative images from the dataset, showcasing the performance of the proposed method alongside other advanced algorithms. To enhance clarity in visual analysis, the segmented regions, particularly the tumor boundaries, are highlighted with red circles. Additionally, zoomed-in views are provided within the same figure to facilitate closer examination of the segmented regions. This dual analysis approach enables a more comprehensive comparison, allowing for a deeper understanding of the strengths and limitations of the proposed method relative to competing techniques. The objective assessment provides a comprehensive quantitative analysis of the proposed method in comparison with several other advanced segmentation techniques, highlighting the strengths and weaknesses of each approach. The NCM-1 algorithm is effective at removing outliers from the image data, ensuring a cleaner dataset for segmentation. However, it exhibits limitations in accurately segmenting the more subtle and complex regions of the brain, particularly in areas where fine details are essential for precise diagnosis. As a result, NCM-1 struggles to capture the intricacies of the tumor boundaries and other smaller internal regions, reducing its overall effectiveness in medical image analysis. KNCM algorithm improves upon NCM-1 by utilizing kernel functions to enhance clustering performance. This enables KNCM to better handle the non-linear and inhomogeneous structures within brain images. While this method shows enhanced clustering capability, its segmentation accuracy is lower than compared to the proposed method. Further, NKWNLICM faces significant challenges in accurately segmenting tumor regions. It fails to effectively distinguish the tumor from surrounding tissues and is notably deficient in segmenting other internal structures within the brain, leading to a lack of overall precision in the segmentation process. While analysing KLDFCM techniques which demonstrates competence in segmenting tumor regions, benefiting from the use of KL divergence to improve clustering in high-dimensional data. However, KLDFCM falls short in

Images	NCM-1	KNCM	NKWNLICM	KLDFCM	FBkPC_S1	NCM-2	Proposed
1	0.9287	0.9373	0.9187	0.8394	0.8722	0.8165	**0.9742**
2	0.9228	0.9396	0.8915	0.8508	0.8553	0.8357	**0.9924**
3	0.9169	0.9581	0.9381	0.8404	0.8411	0.8324	**0.9996**
4	0.9241	0.9561	0.9278	0.8716	0.8878	0.8579	**0.9945**
5	0.9240	0.9387	0.9455	0.8971	0.9072	0.8723	**0.9975**
6	0.9239	0.9342	0.9406	0.8739	0.9135	0.8569	**0.9984**
7	0.9295	0.9353	0.9521	0.8675	0.9169	0.8461	**0.9965**
8	0.9328	0.9476	0.9459	0.8648	0.9177	0.8492	**0.9972**
9	0.9345	0.9484	0.9517	0.8971	0.9784	0.8741	**0.9971**
10	0.9132	0.9247	0.9356	0.8523	0.9586	0.8486	**0.9937**

Table 1. Values of FPI for the proposed and comparative state-of-the-art methods.

Images	NCM-1	KNCM	NKWNLICM	KLDFCM	FBkPC_S1	NCM-2	Proposed
1	0.0706	0.0613	0.0654	0.1498	0.1537	0.1718	**0.0068**
2	0.0791	0.0744	0.0753	0.1592	0.1820	0.1765	**0.0230**
3	0.0780	0.0648	0.0677	0.1676	0.1579	0.1824	**0.0166**
4	0.0460	0.0363	0.0345	0.1232	0.1210	0.1369	**0.0034**
5	0.0728	0.0524	0.0547	0.1815	0.1375	0.1928	**0.0026**
6	0.0704	0.0618	0.0678	0.1693	0.1364	0.1746	**0.0014**
7	0.0703	0.0602	0.0624	0.1637	0.1354	0.1728	**0.0037**
8	0.0701	0.0637	0.0574	0.1737	0.1632	0.1838	**0.0062**
9	0.0684	0.0547	0.0531	0.1085	0.1055	0.1145	**0.0028**
10	0.0664	0.0513	0.0518	0.1422	0.1325	0.1575	**0.0056**

Table 2. Values of MPE for the proposed and comparative state-of-the-art methods.

Images	NCM-1	KNCM	NKWNLICM	KLDFCM	FBkPC_S1	NCM-2	Proposed
1	0.0466	0.0303	0.0229	0.0267	0.0178	0.0298	**0.0143**
2	0.0420	0.0325	0.0270	0.0226	0.0196	0.0263	**0.0168**
3	0.0347	0.0217	0.0198	0.0172	0.0097	0.0192	**0.0066**
4	0.0363	0.0258	0.0214	0.0195	0.0189	0.0211	**0.0132**
5	0.0321	0.0298	0.0275	0.0198	0.0173	0.0229	**0.0112**
6	0.0234	0.0201	0.0195	0.0185	0.0105	0.0198	**0.0072**
7	0.0365	0.0226	0.0231	0.0735	0.1647	0.0846	**0.0169**
8	0.0333	0.0253	0.0226	0.0715	0.0945	0.0810	**0.0126**
9	0.0372	0.0276	0.0234	0.0857	0.0934	0.0921	**0.0111**
10	0.0252	0.0201	0.0195	0.0795	0.0654	0.0846	**0.0050**

Table 3. Values of XBI for the proposed and comparative state-of-the-art methods.

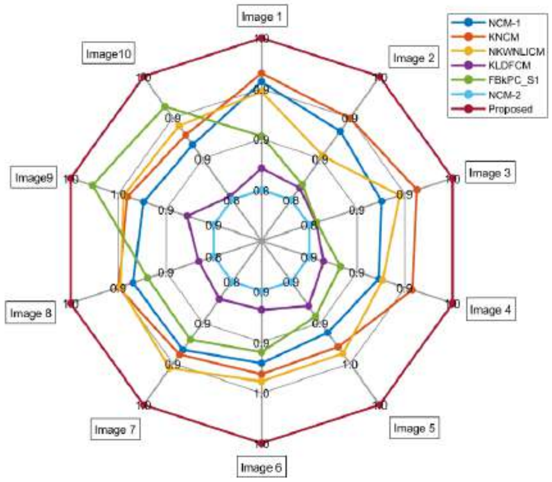
Algorithm	Computational time (s)	Number of iteration
NCM-1	6.4247	156
KNCM	8.3589	183
NKWNLICM	5.2194	127
KLDFCM	3.1784	86
FBKPC_S1	4.2488	108
NCM-2	6.7251	162
Proposed	3.24587	79

Table 4. Computational time and number of iterations for proposed and other comparative methods.

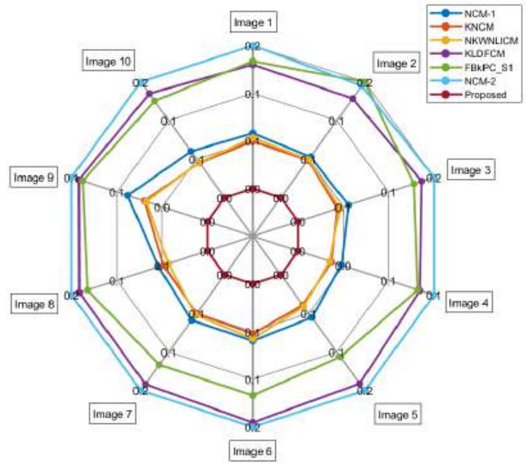
accurately clustering other non-tumor areas of the brain, leading to incomplete and imprecise segmentation. Moreover, FBkPCS1 technique suffers from imprecision, particularly in defining the boundaries of tumor regions. The method often fails to accurately capture the fine details of the tumor edges, which are crucial for medical diagnosis and treatment planning. On examining the results from NCM-2, it is observed that while the NCM-2 algorithm is capable of segmenting tumor regions to a certain extent, but it does not adequately capture the intricate internal structures of the brain. This limitation often results in degraded segmentation, particularly in regions where tissue boundaries are less distinct, thereby reducing the overall segmentation accuracy and reliability of the method. Furthermore, KLKNCM method outperforms the aforementioned algorithms in multiple aspects. By incorporating kernel-induced clustering with the Neutrosophic c-means framework
and the KL distance measure, the KLKNCM method achieves superior segmentation accuracy. It effectively segments both the tumor regions and other internal structures of the brain, addressing the limitations observed in the other methods. Finally, KLKNCM method provides more precise boundary delineation and improved differentiation of inhomogeneous brain tissues. Furthermore, it offers enhanced computational efficiency, making it a highly effective and practical solution for brain tumor segmentation in clinical settings.

Meanwhile, the evaluation of the proposed method is further supported by an analysis of cluster validation metrics, with the results summarized in Tables 1-3. The optimal values for each metric are highlighted in bold for clarity. Upon reviewing the cluster validation results, it is evident that FPI for the proposed KLKNCM technique is significantly higher compared to the other methods. A higher FPI value indicates greater correctness and precision in the clustering process, demonstrating the ability of KLKNCM to accurately segment complex brain structures. Furthermore, MPE and XBI values
for the KLKNCM method are lower in comparison to those of the alternative clustering algorithms. Lower values of MPE and XBI indicate more distinct and well-separated clusters, which further affirms the accuracy and effectiveness of the proposed technique. These lower indices suggest that the KLKNCM method not only achieves better clustering precision but also minimizes overlap between clusters, ensuring more accurate segmentation of brain regions. In summary, the superior performance of the proposed KLKNCM method across these validation metrics, particularly with respect to FPI, MPE, and XBI, highlights its effectiveness and supremacy over the other state-of-the-art techniques. This analysis underscores the robustness and efficiency of the KLKNCM algorithm in achieving highly accurate and computationally efficient brain tumor segmentation.

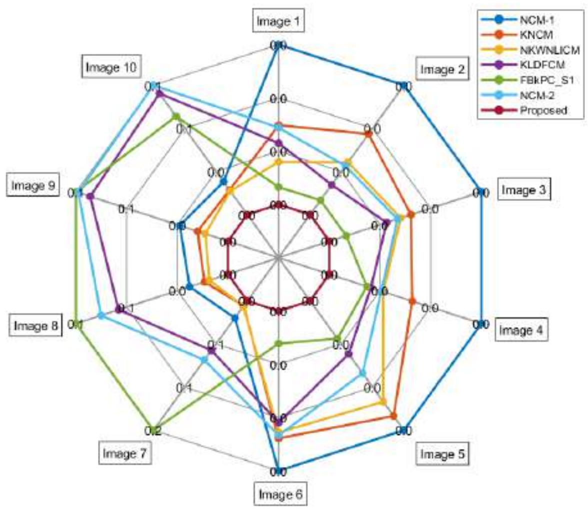
In addition, the comparative analysis of the proposed algorithm with other state-of-the-art clustering methods highlights its superior efficiency in terms of both computational time and



(A)



(B)



(C)

FIGURE 3. Graphical representation of quality assessment metrics (a) Quality assessment through FPI, (b) Quality assessment through MPE, (c) Quality assessment through XBI.

iteration count which is given in the Table 4. The proposed algorithm demonstrates a competitive computational time, indicating its capability to handle clustering tasks with minimal resource consumption. Furthermore, the number of iterations required for convergence is significantly lower in the proposed method compared to the other algorithms. This reduction in iteration count underscores the KLKNCM has the enhanced convergence speed, which is crucial for minimizing computational overhead in iterative clustering processes. The iterative nature of clustering algorithms often poses a challenge in balancing accuracy and efficiency;

however, the proposed method effectively addresses this by achieving rapid convergence without compromising performance. Moreover, the graph where plotted to show the efficiency of the proposed algorithm and it displayed in Figure 3. Altogether, the results suggest that the proposed algorithm not only reduces the computational burden but also accelerates the clustering process, making it a suitable choice for applications where both time efficiency and processing power are limited. In conclusion, the proposed algorithm outperforms the comparative methods by demonstrating a balance between computational efficiency and iteration reduction, thereby providing an effective solution for real-world clustering applications.

7. Limitation and Future Research

While the proposed method demonstrates the superior performance, but still there persists certain limitations. Even though, integration of the kernel distance and KL divergence in NCM elevates the segmentation accuracy, the developed method still depends on the parameter tuning, which impacts the robustness of the method. Added to this, the evaluation has been done on benchmark datasets but the performance across large-scale with diverse datasets yet to be comprehensively validated. These limitations naturally point toward several directions for future research, incorporating deep learning frameworks such as CNN with the proposed framework could maximize the benefits data-driven learning to achieve the superior accuracy. Automated parameter optimization techniques also represent a valuable extension, reducing the reliance on manual tuning and improving reproducibility. Furthermore, expanding the approach to handle multi-class segmentation of tumor sub-regions and applying it to other neurological disorders would broaden its clinical relevance.

8. Conclusion

This study introduced a novel KLKNCM technique for the segmentation of brain tumors. The proposed approach effectively addresses the challenges posed by the non-linear and inhomogeneous nature of brain structures, providing accurate segmentation while minimizing computational overhead. The proposed method has been rigorously tested against five other state-of-the-art clustering algorithms, with its performance validated through both objective metrics and subjective assessments. The results highlight the superiority of our approach in terms of segmentation accuracy, computational cost, and robustness, demonstrating its potential for practical use in clinical settings. Further, parameters of the algorithm were examined in depth to demonstrate their functions in the segmentation methodology. In addition, runtime calculations and iteration analysis were carried out in order to acquire a comprehensive understanding of the results. In future, the proposed approach will be used to classify medical images based on segmentation.

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Ethical approval This article does not contain any studies with human participants performed by any of the authors.

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Automobile Evaluation Using an Extended PROMETHEE Method under Neutrosophic Supra Topological Framework

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Abstract. Neutrosophic sets are widely used due to their ability to handle uncertainties and incomplete information. Recently, researchers have integrated neutrosophic sets with supra topological spaces, leading to the development of a new theory known as neutrosophic supra topological space. The PROMETHEE method is a well-established approach for solving multi-criteria decision-making problems. Selecting the best car under uncertainty and multiple criteria is a significant challenge for a buyer. This decision-making problem motivated us to develop a novel PROMETHEE method with neutrosophic supra topological space to select the optimal car. This novel decision-making method incorporated the PROMETHEE method with neutrosophic supra topological space. Using the proposed method, a numerical example is provided to determine the car that outperforms other models in terms of fuel efficiency, emission standards, safety, overall cost, and future resale value. We also explore the sensitivity of the results to different parameter values within the model. Finally, we compare the proposed method with the existing methods.

Keywords: Car selection, PROMETHEE method, neutrosophic set, neutrosophic supra topological space,Decision making methods.

1. **Introduction**

The rapid urbanization and population growth in modern cities have led to a significant increase in the use of private vehicles. This surge in personal transportation contributes heavily to urban congestion and environmental degradation due to its dependence on fossil fuels, resulting in elevated carbon dioxide (CO_2) emissions in city centers. Conventional fossil-fuel-powered vehicles still dominate urban transportation, posing serious threats to sustainable development goals, particularly those aimed at mitigating climate change and reducing greenhouse gas (GHG) emissions. In response, electric vehicles (EVs) have emerged as a cleaner alternative, playing a vital role in the transition towards low-carbon and sustainable urban transportation systems. Despite their promise, EVs present notable limitations such as limited driving range, high initial cost, insufficient charging infrastructure, and longer charging times. As consumers become more environmentally conscious, selecting the optimal vehicle, be it electric, hybrid, or conventional, becomes more important. It requires the consideration of multiple conflicting factors, including price, performance, safety, energy efficiency, infrastructure availability, and environmental impact. Given the complexity and ambiguity involved in such decision scenarios, Multi-Criteria Decision-Making (MCDM) methods are indispensable tools for systematically evaluating alternatives based on diverse criteria. Traditional decision-making approaches often struggle with uncertainty and vagueness inherent in real-world problems. Fuzzy set theory (FS) [31], while helpful, can still fall short in capturing complex scenarios with incomplete information. To resolve this, the concept of intuitionistic fuzzy set (IFS) [5] is introduced by Atanassov. Neutrosophic sets (NS) [7], introduced by Florentin Smarandache, extend the capabilities of fuzzy sets by incorporating a third component, "indeterminacy", alongside truth and falsity values. This additional dimension allows for a more subtle representation of human judgment and subjective evaluations. The PROMETHEE (Preference Ranking Organization of Methods for Enrichment Evaluations) method, developed by Brans [3, 6], is a widely used MCDM technique. It focuses on pairwise comparisons between alternatives based on predefined preference functions, making it flexible and adaptable to various decision problems. PROMETHEE methods have been applied in numerous contexts. Nassereddine et al. [11] integrated a new PROMETHEE preference function and synergy criteria to evaluate emergency response systems, addressing the critical need for inter-agency collaboration in disaster management. Qi et al. [21] introduced a dynamic weighting approach based on preference expectations and ordered weighted averaging to accommodate interdependencies and prioritizations among criteria. Zhao et al. [20] introduced extended PROMETHEE methods utilizing 2-dimension linguistic term sets (2DLEs) to solve Multi-Attribute Decision-Making (MADM) problems, enhancing preference functions with a possibility degree for 2DLEs, effectively handling both comparable and incomparable evaluations. Yu et al. [19] introduced an enhanced failure mode and effects analysis model for submarine pipeline risk analysis, improving interval-valued intuitionistic fuzzy rough number theory for expert opinion collection and combining Exponential TODIM (an acronym in Portuguese for interactive and multicriteria decision-making) with PROMETHEE-II and analytical hierarchical process (AHP) for robust failure mode ranking and risk value calculation. Zhao et al. [16] presented a modified PROMETHEE II method that simplifies computation by integrating multiple steps, thereby reducing complexity and database interactions.

Recently, the integration of extended fuzzy set theory with topological concepts has emerged as a promising approach for addressing complex problems characterized by uncertainty and indeterminacy. Garg et al. [28] introduced the TOPSIS (Technique of Order Preference by Similarity to an Ideal Solution) method under a spherical fuzzy soft environment for solving decision-making problems [28]. While classical topology is a powerful tool, often it struggles in modeling situations involving vague, imprecise, or contradictory information. To address these limitations, the concept of neutrosophic topological space [27] was introduced, extending traditional topological notions to accommodate the inherent ambiguity present in real-world scenarios.

Building on this foundation, neutrosophic supra topological spaces (NSTPS) [22] have been developed and solved a decision-making problem on medical diagnosis. Also, neutrosophic support soft topological space is introduced, and a decision-making problem via neutrosophic support soft topological space [29] is solved. Supra topological spaces [26], by relaxing the constraints of traditional topologies, offer increased flexibility in modeling diverse structures. Some weak and strong forms of sets are introduced, and their properties in neutrosophic supra topological spaces [30] are studied.

This generalization is particularly valuable in applications where the data exhibits a high degree of uncertainty, such as in decision-making, pattern recognition, and information fusion. By leveraging the expressive power of neutrosophic sets within a supra topological framework, we can construct more robust and adaptable models that better reflect the complexities of real-world phenomena. This work explores multi-criteria decision-making models by incorporating the PROMETHEE method and neutrosophic supra topological spaces. This integrated approach emerges as a powerful tool for decision-making under uncertainty, aiming to demonstrate its potential in the context of automobile evaluation.

1.1. *Motivations*

The following points outline the motivations for proposing this novel model to solve multi-criteria decision-making problem:

- Neutrosophic sets allow for more effective modeling of imprecise, incomplete, and inconsistent data. Supra-topological structures provide a generalized flexible framework for comparing preferences more effectively.
- While PROMETHEE is widely used in practice, its theoretical foundation is limited when handling highly uncertain or indeterminate data. Neutrosophic supra topological spaces, however, offer a theoretical framework for managing such data. By combining the two, this approach bridges the gap between theoretical advancements and practical applications, resulting in a more comprehensive and adaptable decision-making tool.

- Despite the advancements in multi-criteria decision-making methodologies, the integration of neutrosophic supra topological spaces with the PROMETHEE method remains unexplored. This research addresses the gap by proposing a novel framework that leverages the strengths of both approaches to solve complex decision problems.

The novelty of the proposed work is in extending the traditional PROMETHEE method by incorporating NSTPS, which is an innovative approach to modeling uncertainty and vagueness in decision-making. This is the first attempt to integrate neutrosophic supra topology into the PROMETHEE method. The study is employed to select the best car, emphasizing GHG emissions, fuel efficiency, safety, cost, and resale value. It aligns with the EU Transport White Paper goals of reducing GHG emissions by 60% by 2050, making it highly relevant for policymakers and consumers. The work provides a structured way to help car buyers prioritize sustainability and economic feasibility in their decision-making process.

2. **Related Studies**

Multi-criteria decision-making often involves complex situations where choices must be made based on conflicting or incommensurable criteria. Traditional methods struggle to handle the inherent vagueness and uncertainty of such situations. Mahmood et al. [15] investigated the applicability of the bipolar complex fuzzy rough set in cyber security. Fuzzy and intuitionistic fuzzy approaches offer elegant solutions to address these challenges by incorporating the subjective preferences and hesitation of decision-makers [12, 13, 17]. Hamurcu and Eren [14] proposed a hybrid multicriteria decision-making approach, combining AHP, TOPSIS, and goal programming to evaluate conflicting factors and determine the optimal electric vehicle choice. Ali et al. [22] proposed a technique that combines full Consistency for weight calculation and fuzzy TOPSIS for ranking, demonstrating enhanced accuracy and consistency compared to traditional methods. It offers a versatile tool for diverse alternative selection scenarios. Chand et al. [23] employed a Fuzzy AHP approach to rank sedan cars based on criteria such as performance, economy, and comfort, aiming to simplify the selection process by providing consumers with a clear understanding of their preferences.

Originally developed by Brans et al. [6] PROMETHEE ranks alternatives based on pairwise comparisons using preference functions and indifference thresholds. Fuzzy logic is integrated into PROMETHEE by representing preferences and thresholds as fuzzy sets, allowing for more flexible and nuanced decision-making. The advantages of Fuzzy PROMETHEE include ease of use, the ability to handle imprecise information, and consideration of both positive and negative outranking flows.

An extension of Fuzzy PROMETHEE utilizes IFS. IFS captures both the degree of membership and non-membership in a set, effectively representing hesitation or uncertainty in subjective judgments. The additional degree of non-membership (hesitation) in IFS provides richer information compared to Fuzzy PROMETHEE, leading to more comprehensive and accurate decision-making. Advantages include the ability to deal with ambiguity and conflicting information, visualize positive and negative outranking flows separately, and incorporate the importance of criteria using IFS weights. Applications encompass sustainable building material selection ([10]). Xu et al. [1] proposed an integrated method combining PROMETHEE and TODIM in a neutrosophic environment. This method introduces a new formula for ranking alternatives, highlighting the potential of combining decision-making methods under the neutrosophic framework. Xu et al. [2] introduce the concept of probabilistic simplified neutrosophic sets (PSNS) and develop a PROMETHEE-based decision-making approach for group decision problems. PSNS incorporates probabilistic elements into neutrosophic evaluations, further enhancing the method's ability to handle uncertainty. Xu et al. [4] propose an improved PROMETHEE method using multi-valued neutrosophic sets, expanding upon the traditional single-valued approach. This allows for richer information representation and potentially more accurate decision-making in complex scenarios.

2.1. *Research Questions*

In light of the challenges posed by uncertainty and indeterminacy in MCDM, this study addresses the following research questions:

- How can the PROMETHEE method be effectively integrated with neutrosophic supra-topological spaces to handle uncertainty and imprecision in decision-making?
- Can expert opinions or decision matrices be compared using existing models?
- What are the consequences of applying aggregation operators to neutrosophic sets that include null or absolute neutrosophic values?
- Does any existing research apply the PROMETHEE method within the framework of supra topological spaces?

Existing studies using the PROMETHEE method and other decision-making techniques lack a mechanism for comparing expert opinions or decision matrices directly. Furthermore, when aggregation operators are applied to neutrosophic sets containing null or absolute values, they often produce trivial results. Additionally, no prior research has integrated the PROMETHEE method with neutrosophic supra topological spaces for enhanced decision-making.

To address this research gap and respond to the questions outlined above, we propose a novel decision-making framework that integrates the PROMETHEE method with neutrosophic supra topological structures. This approach allows for a robust comparison of expert opinions and effectively manages both null and absolute neutrosophic values, thereby improving the reliability and depth of the decision-making process.

2.2. *Objectives and Contributions of the Study*

The objective and major contributions of this article are presented below:

- The primary objective of this study is to integrate the PROMETHEE method with NSTPS to address MCDM problems characterized by uncertainty, indeterminacy, and imprecise information.
- Additionally, we seek to demonstrate the practical applicability of the proposed model through a numerical example, testing its performance across various parameter values to ensure reliability and consistency.
- We proposed a hybrid decision-making framework that combines the PROMETHEE method with NSTPSs, enhancing its ability to manage uncertainty and vagueness in decision criteria.
- The viability and applicability of the proposed approach are demonstrated through a numerical application in the context of automobile evaluation, showcasing its effectiveness in real-world scenarios.
- A detailed sensitivity analysis is conducted to assess the impact of varying parameters on the results, ensuring the consistency and robustness of the proposed model.

By bridging the gap between theoretical advancements and practical applications, this study makes a significant contribution to the field of MCDM, offering a powerful tool for decision-making in uncertain and complex environments. The key contributions in this article include a new decision-making model for finding the best car based on multiple criteria. The main difference between the proposed model and other existing PROMETHEE methods under fuzzy and its extension sets is that the proposed model compares the expert's opinion or decision matrix and provides non-trivial results if the null and absolute neutrosophic values are present in the expert's opinion or decision matrix. The subsequent sections of this paper are organized as follows:

- The "Basic definitions" section provides a comprehensive review of fundamental concepts, including the definitions of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets, along with their respective operations. Additionally, it introduces the concept of neutrosophic supra-topological spaces and discusses score and accuracy functions, which are essential for understanding the proposed framework.
- The "Decision-Making: PROMETHEE Method" section presents a novel hybrid decision-making model that integrates the PROMETHEE method with neutrosophic supra-topological spaces. The proposed framework is designed to address multi-criteria decision-making problems under conditions of uncertainty and indeterminacy. A detailed flowchart is presented to illustrate the step-by-step implementation of the model, ensuring clarity and ease of application for practitioners.

- The "Numerical Example" section demonstrates the practical applicability of the proposed model; this section provides a detailed solution to an automobile evaluation problem. The example highlights the effectiveness of the hybrid approach in handling real-world decision-making scenarios. Furthermore, a sensitivity analysis is conducted to evaluate the impact of varying parameters on the results, ensuring the robustness and consistency of the proposed model.
- The "Conclusions" section summarizes the key contributions of the study, outlines the potential directions for future research, and discusses the limitations of the proposed model.

3. Basic Definitions

This section presents essential definitions and operations related to our study.

Definition 3.1. [31] A fuzzy set $\mathbb{M}$ in a universe of discourse X is defined as a set of ordered pairs:

$$\mathbb{M} = \{(x, \mathfrak{m}_{\mathbb{M}}(x))|x \in X\}$$

where the membership function $\mathfrak{m}_{\mathbb{M}} : X \to [0, 1]$ for each element x in X.

Definition 3.2. [5] An intuitionistic fuzzy set $\mathbb{M}$ in X is defined as

$$\mathbb{M} = \{\langle x, \mathfrak{m}_{\mathbb{M}}(x), \mathfrak{n}_{\mathbb{M}}(x)\rangle|x \in X\},$$

where the degree of membership and non-membership function respectively denoted as

$$\mathfrak{m}_{\mathbb{M}} : X \to [0, 1]$$

and

$$\mathfrak{n}_{\mathbb{M}} : X \to [0, 1]$$

for each x in X, and $0 \leq \mathfrak{m}_{\mathbb{M}}(x) + \mathfrak{n}_{\mathbb{M}}(x) \leq 1$.

Definition 3.3. [7] A neutrosophic set $\mathbb{M}$ in the universe of discourse X is of the form

$$\mathbb{M} = \{\langle \alpha, \mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{M}}(\alpha), \mathfrak{n}_{\mathbb{M}}(\alpha)\rangle : \alpha \in X\}$$

and $\mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{M}}(\alpha), \mathfrak{n}_{\mathbb{M}}(\alpha)$ are standard or non-standard subsets of $]0, 1[$
where $\mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{M}}(\alpha), \mathfrak{n}_{\mathbb{M}}(\alpha)$ represents the degree of favorable, degree of indeterminacy and the degree of non-favorable function provided there is no restriction in the addition of $\mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{M}}(\alpha)$ *and* $\mathfrak{n}_{\mathbb{M}}(\alpha)$, So $^{-}0 \leq sup\mathfrak{m}_{\mathbb{M}}(\alpha) + sup\mathfrak{e}_{\mathbb{M}}(\alpha) + sup\mathfrak{n}_{\mathbb{M}}(\alpha) \leq 3^{+}$

Definition 3.4. [7] Let $\mathbb{L}$ and $\mathbb{M}$ be two neutrosophic sets of the form
$\mathbb{L} = \{\langle \alpha, \mathfrak{m}_{\mathbb{L}}(\alpha), \mathfrak{e}_{\mathbb{L}}(\alpha), \mathfrak{n}_{\mathbb{L}}(\alpha)\rangle : \alpha \in X\}$, $\mathbb{M} = \{\langle \alpha, \mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{M}}(\alpha), \mathfrak{n}_{\mathbb{M}}(\alpha)\rangle : \alpha \in X\}$. Then

(a) A neutrosophic set $\mathbb{L}$ is said to be a subset of another neutrosophic set $\mathbb{M}$, denoted by $\mathbb{L} \subseteq \mathbb{M}$ if for all $\alpha in X$: $\mathfrak{m}_{\mathbb{L}}(\alpha) \leq \mathfrak{m}_{\mathbb{M}}(\alpha)$, $\mathfrak{e}_{\mathbb{L}}(\alpha) \geq \mathfrak{e}_{\mathbb{M}}(\alpha)$ and $\mathfrak{n}_{\mathbb{L}}(\alpha) \geq \mathfrak{n}_{\mathbb{M}}(\alpha)$ for all $\alpha \in X$.

(b) A neutrosophic set $\mathbb{L}$ is said to be equal to another neutrosophic set $\mathbb{M}$, denoted by $\mathbb{L} = \mathbb{M}$ if for all $\alpha in X$: $\mathfrak{m}_{\mathbb{L}}(\alpha) = \mathfrak{m}_{\mathbb{M}}(\alpha)$, $\mathfrak{e}_{\mathbb{L}}(\alpha) = \mathfrak{e}_{\mathbb{M}}(\alpha)$ and $\mathfrak{n}_{\mathbb{L}}(\alpha) = \mathfrak{n}_{\mathbb{M}}(\alpha)$.

(c) Complement of neutrosophic set $\mathbb{L}$, denoted and defined as $\mathbb{L}^C = \{\langle \alpha, \mathfrak{n}_{\mathbb{L}}(\alpha), 1 - \mathfrak{e}_{\mathbb{L}}(\alpha), \mathfrak{m}_{\mathbb{L}}(\alpha)\rangle : \alpha \in X\}$;

(d) The intersection of two neutrosophic sets denoted and defined as $\mathbb{M} \cap \mathbb{N} = \{\langle \alpha, \mathfrak{m}_{\mathbb{L}}(\alpha) \wedge \mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{L}}(\alpha) \vee \mathfrak{e}_{\mathbb{M}}(\alpha), \mathfrak{n}_{\mathbb{L}}(\alpha) \vee \mathfrak{n}_{\mathbb{M}}(\alpha)\rangle : \alpha \in X\}$;

(e) The union of two neutrosophic sets denoted and defined as $\mathbb{M} \cup \mathbb{N} = \{\langle \alpha, \mathfrak{m}_{\mathbb{L}}(\alpha) \vee \mathfrak{m}_{\mathbb{M}}(\alpha), \mathfrak{e}_{\mathbb{L}}(\alpha) \wedge \mathfrak{e}_{\mathbb{M}}(\alpha), \mathfrak{n}_{\mathbb{L}}(\alpha) \wedge \mathfrak{n}_{\mathbb{M}}(\alpha)\rangle : \alpha \in X\}$;

(f) For a scalar $\omega \in [0, 1]$, the scalar multiplication of a neutrosophic set A, denoted and defined as: $\omega\mathbb{L} = \{\langle \alpha, \omega\mathfrak{m}_{\mathbb{L}}(\alpha), \omega\mathfrak{e}_{\mathbb{L}}(\alpha), \omega\mathfrak{n}_{\mathbb{L}}(\alpha)\rangle : \alpha \in X\}$

Definition 3.5. [7] Let X be a universal set.

(i). A NS is called an absolute NS over X and it is denoted by 1^X, if $\forall a \in X, \mathfrak{m}_{1^{\mathbb{X}}}(a) = 1, \mathfrak{e}_{1^{\mathbb{X}}}(a) = 0, \mathfrak{n}_{1^{\mathbb{X}}}(a) = 0$.

(ii). A NS is called an null NS over X and it is denoted by $1^{\emptyset}$, if $\forall a \in X, \mathfrak{m}_{1^{\emptyset}}(a) = 0, \mathfrak{e}_{1^{\emptyset}}(a) = 1, \mathfrak{n}_{1^{\emptyset}}(a) = 1$.

Definition 3.6. [26] Let X be a non-empty universal set and τ be a collection of subsets of X. The pair (X, τ) is called a supra topological space, if

(i). Empty set and the entire set is in τ.

(ii). The union of any collection of supra open set is also a supra open set. i.e.,$\{U_i | i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$.

Each element in the collection τ is called an open set. Here I is the index set.

Definition 3.7. [22] Let X be a non-empty universal set and τ be a collection of neutrosophic sets of X. The pair (X, τ) is called a neutrosophic supra topological space, if it satisfies the following axioms:

(i). The absolute and null neutrosophic set belong to τ.

(ii). The union of any collection of neutrosophic sets in τ is also in τ. i.e., $\{U_i | i \in I\} \subseteq \tau$, then $\bigcup_{i \in I} U_i \in \tau$.

Each neutrosophic set in τ is called a neutrosophic open set and I is the index set.

Definition 3.8. [23] Let $\mathbb{L}$ be a single-valued NS and the score function of $\mathbb{L}$ is denoted and defined by $S_{\mathbb{L}} = \frac{2 + \mathfrak{m}_{\mathbb{L}} - \mathfrak{e}_{\mathbb{L}} - \mathfrak{n}_{\mathbb{L}}}{3}$.

Definition 3.9. [23] Let $\mathbb{L}$ be a single-valued NS and the accuracy function of $\mathbb{L}$ is denoted and defined by $E_{\mathbb{L}} = \mathfrak{m}_{\mathbb{L}} - \mathfrak{n}_{\mathbb{L}}$.

Definition 3.10. [23] Let $\mathbb{L}$ and $\mathbb{M}$ be single-valued NSs and the score function of $\mathbb{L}$ is less than the score function of $\mathbb{M}$ if $\mathbb{L}$ is less than $\mathbb{M}$. If the score function of both single-valued NSs is equal then we consider the following constraints :

i. if $E_{\mathbb{L}} < E_{\mathbb{M}}$ then $\mathbb{L}$ is less than $\mathbb{M}$.
ii. if $E_{\mathbb{L}} = E_{\mathbb{M}}$ then $\mathbb{L}$ is equal to $\mathbb{M}$.

4. **Decision Making: Extended PROMETHEE Method**

The procedure for the extended PROMETHEE method and flowchart (refer to figure 1) are given below:
Let the set of alternatives be $\mathcal{R} = \{\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3....\mathcal{R}_n\}$ and the set of criteria be $\mathbb{A} = \{\mathbb{A}_1, \mathbb{A}_2, ...\mathbb{A}_m\}$. Let $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3....\mathcal{G}_k\}$ denote the set of decision makers.
Step 1: Create the weight parametric matrix.
Decision makers created the weighted parametric matrix $\mathbb{G}_w$, whose entries are the values of each criterion assigned by each decision maker considering linguistic terms as shown in table 1.

Linguistic Terms	Weights
Extremely important	0.9
Very strongly important	0.8
very important	0.6
important	0.4
slightly important	0.2

Table 1. Linguistic terms to determine the alternatives

$$G_w = \begin{bmatrix} \rho_{11} & \rho_{12} & .... & \rho_{1n} \\ \rho_{21} & \rho_{22} & .... & \rho_{2n} \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ \rho_{m1} & \rho_{m2} & .... & \rho_{mn} \end{bmatrix}$$

Step 2: Normalize the weighted parametric matrix.
Since each attribute does not necessarily have the same weight, we need to normalize the

weight of each criterion.

$$N = \begin{bmatrix} \mathfrak{a}_{11} & \mathfrak{a}_{12} & .... & \mathfrak{a}_{1n} \\ \mathfrak{a}_{21} & \mathfrak{a}_{22} & .... & \mathfrak{a}_{2n} \\ .. & .. & .. & .. \\ .. & .. & .. & .. \\ \mathfrak{a}_{m1} & \mathfrak{a}_{m2} & .... & \mathfrak{a}_{mn} \end{bmatrix}$$

where

$$\mathfrak{a}_{ij} = \frac{\rho_{ij}}{\sqrt{\sum_{i=1}^{m} \rho_{ij}^2}} \tag{1}$$

Step 3: Calculate the weight vector.

The weight vector is calculated from step 2 by the following equation

$$\mathbb{W}_i = \frac{\sum_{i=1}^{m} \mathfrak{a}_{ij}}{m} \ and \ V_{w_i} = \frac{\mathbb{W}_i}{\sum_{i=1}^{n} \mathbb{W}_i}$$

provided the sum of weight V_{w_i} equal to unity.

Step 4: Construct neutrosophic supra topology.

Each decision maker's report is based on the criteria for each alternative. Such reports are given in matrix form, where entries are the neutrosophic values. Let $D_1, D_2...., D_k$ denote the decision matrix. Construct the neutrosophic supra topology by combining NSs, given each decision-maker.

Step 5: Aggregation of decision matrix.

The decision matrix is aggregated by taking the average for each alternative for each criterion.

$$D_{agg} = \frac{D_1 + D_2 + ... + D_k}{k}$$

Step 6: Normalize the decision matrix.

Convert the aggregated decision matrix into a normalized decision matrix by taking the complement of the cost factor and keeping the remaining unchanged.

Step 7: Construct the preference function.

Construct the preference function $P_j(B_i, B_r)$ of scheme B_i relative to B_r under the criteria G_j by the following formula:

$$P_j(B_i, B_r) = \begin{cases} 0 & d \le p \\ \frac{d-p}{q-p} & p \le d \le q \\ 1 & d \ge q \end{cases} \tag{2}$$

The range of the preference function is from 0 to 1. If $P_j(B_i, B_r) = 0$, then there is no difference between B_i and B_r. If $P_j(B_i, B_r)$ is nearly zero, then the difference between B_i and B_r is relatively small. Suppose $P_j(B_i, B_r)$ is nearly 1; then B_i is possibly better than B_r. If $P_j(B_i, B_r) = 1$, then B_i is strongly better than B_r. d is the priority function parameter and the difference between the criterion value of B_i and B_r.

Step 8: The priority index of the scheme B_i relative to B_r is denoted and defined by

$$\pi(B_i, B_r) = \sum_{j=1}^{n} V_{w_j} P_j(B_i, B_r) \tag{3}$$

Step 9: Compute the inflow, outflow, and net flow.

Within the context of scheme evaluation, the concept of "flow" is used to measure how a particular scheme compares to others. Inflow, denoted by ϕ_i^+, represents the degree to which scheme B_i surpasses other schemes in terms of performance or other relevant criteria. Conversely, outflow, denoted by ϕ_i^-, reflects the extent to which other schemes outperform scheme B_i. The net flow, calculated as the difference between inflow and outflow, provides an in-depth look at scheme B_i's relative position compared to its peers. It is denoted by ϕ_i.

$$\phi_i^+ = \frac{1}{n-1} \sum_{r=1}^{n} \pi(B_i, B_r) \tag{4}$$

$$\phi_i^- = \frac{1}{n-1} \sum_{r=1}^{n} \pi(B_r, B_i) \tag{5}$$

Step 10: Rank the alternatives.

Arrange the net flow values in ascending order. The greatest ϕ_i is the best alternative.

The following numerical example demonstrates how the extended PROMETHEE method with a neutrosophic supra topological environment can be employed in a real-world decision-making scenario. The step-by-step computational process, including the construction of decision matrices, calculation of criteria weights, and final ranking of alternatives, is presented to illustrate the practical applicability and robustness of the proposed model.

4.1. *Numerical Example*

Transport greenhouse gas (GHG) emissions increased by 26% from 1990 to 2016, despite improvements in vehicle efficiency. Emissions continue to rise due to economic growth and increased transportation usage. Road transport accounts for 72% of total transport GHG emissions. Car ownership rates have grown significantly, which has led to larger car fleets and higher emissions. Many countries aim to reduce transport GHG emissions by 60% by 2050 compared to 1990 levels. Its primary focus is on increasing transport system efficiency, promoting low-emission alternative energy, and transitioning to zero-emission vehicles. To support and reduce GHG emissions, car buyers have to consider this criterion as a primary one. Imagine a customer facing a crucial decision in selecting a car that runs on petrol, diesel, electricity, or CNG (Compressed Natural Gas). Their primary focus lies in five key factors that will shape their choice: fuel efficiency, emissions standards, safety, overall cost, and future resale value.

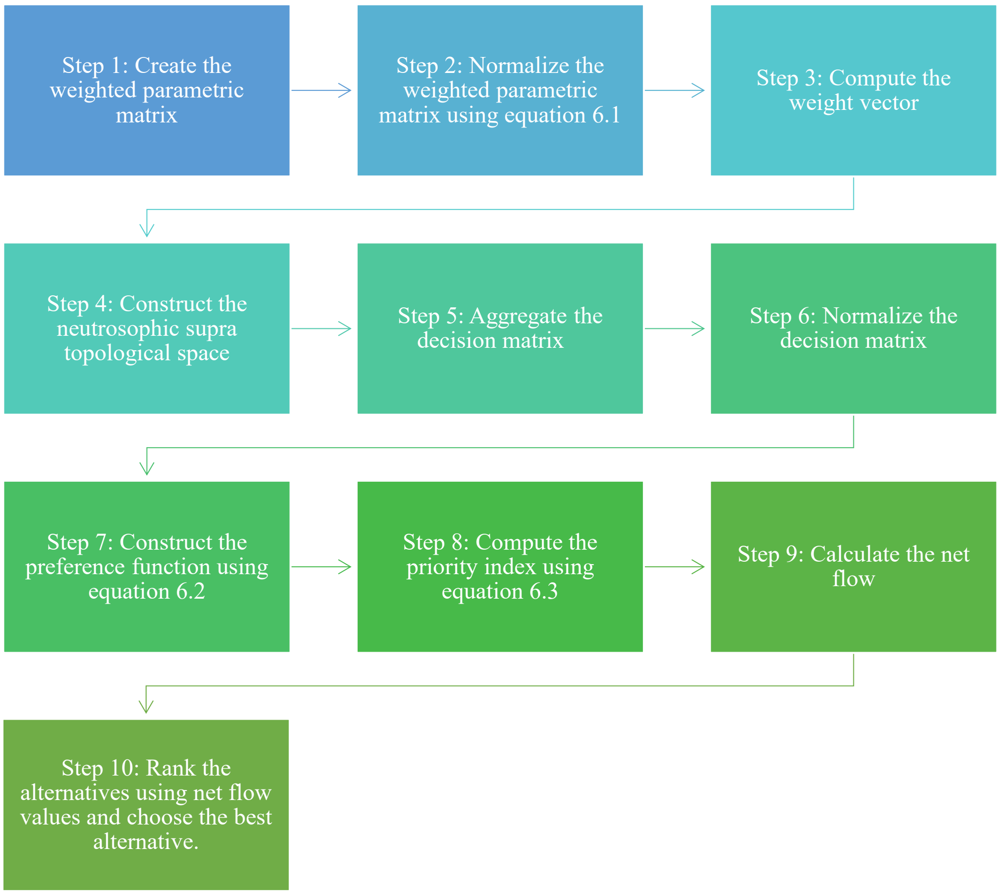

Figure 1. Flowchart of PROMETHEE method

1. Fuel efficiency: Cars with higher fuel efficiency contribute less to GHG emissions.
2. Emissions Standards: Compliance with EU CO2 emission standards ensures lower environmental impact.
3. Safety: Speaking about car safety, it includes airbags, anti-lock braking systems, traction control, and electronic stability control.
4. Overall cost: The cost of a car includes purchase cost, maintenance, and insurance.
5. Future resale value: Resale value is important in the future, if we sell the car.

Therefore, the customer investigated the various automobile engineers and got their opinion. These engineers are the decision-makers for our problem.

Let $\mathcal{R} = \{\mathcal{R}_1 = Petrol\ car, \mathcal{R}_2 = diesel\ car, \mathcal{R}_3 = electric\ car, \mathcal{R}_4 = CNG\ car\}$ be the alternatives, $\mathbb{A} = \{\mathbb{A}_1 = Fuel\ efficiency, \mathbb{A}_2 = Emissions\ standards, \mathbb{A}_3 = safety, \mathbb{A}_4 = Over\ all\ cost, \mathbb{A}_5 = future\ resale\ value\}$ be the criteria. Let $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3, \mathcal{G}_4, \mathcal{G}_5\}$ be the set of decision makers.

Step 1: Decision makers assign weights to each criterion, and the weighted parametric matrix G_w is given below to determine the weight of the criteria.

$$G_w = \begin{array}{c} \\ \mathcal{G}_1 \\ \mathcal{G}_2 \\ \mathcal{G}_3 \\ \mathcal{G}_4 \\ \mathcal{G}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} 0.9 & 0.6 & 0.9 & 0.8 & 0.4 \\ 0.8 & 0.6 & 0.8 & 0.9 & 0.6 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.6 \\ 0.9 & 0.6 & 0.9 & 0.9 & 0.8 \\ 0.8 & 0.6 & 0.8 & 0.9 & 0.6 \end{pmatrix} \end{array}$$

Step 2: Normalized weighted parametric matrix N is calculated by the equation (6.1).

$$N = \begin{bmatrix} 0.51 & 0.55 & 0.48 & 0.52 & 0.46 \\ 0.46 & 0.37 & 0.43 & 0.39 & 0.23 \\ 0.46 & 0.37 & 0.43 & 0.52 & 0.69 \\ 0.34 & 0.55 & 0.48 & 0.39 & 0.23 \\ 0.46 & 0.37 & 0.43 & 0.39 & 0.46 \end{bmatrix}$$

Step 3: The weight vector V_w is $V_w = \{0.203, 0.201, 0.204, 0.203, 0.189\}$

Step 4: NSTS is constructed by arranging the decision maker's reports. Let $D_1, D_2....D_5$ denote the decision matrix of the decision-maker's report.

$$D_1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (0.6, 0.4, 0.2) & (0.6, 0.3, 0.4) & (0.7, 0.2, 0.3) & (0.8, 0.2, 0.2) & (0.7, 0.2, 0.3) \\ (0.7, 0.2, 0.3) & (0.5, 0.4, 0.5) & (0.6, 0.3, 0.4) & (0.7, 0.2, 0.3) & (0.6, 0.3, 0.4) \\ (0.9, 0.1, 0.1) & (0.8, 0.2, 0.2) & (0.9, 0.1, 0.1) & (0.5, 0.4, 0.5) & (0.8, 0.3, 0.2) \\ (0.7, 0.2, 0.3) & (0.7, 0.2, 0.3) & (0.6, 0.3, 0.4) & (0.6, 0.3, 0.4) & (0.6, 0.3, 0.4) \end{pmatrix} \end{array}$$

$$D_2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (0.6, 0.3, 0.2) & (0.7, 0.2, 0.3) & (0.8, 0.2, 0.2) & (0.7, 0.2, 0.3) & (0.7, 0.3, 0.3) \\ (0.7, 0.2, 0.3) & (0.6, 0.3, 0.4) & (0.7, 0.2, 0.3) & (0.6, 0.4, 0.4) & (0.6, 0.4, 0.4) \\ (0.9, 0.2, 0.1) & (0.9, 0.2, 0.1) & (0.9, 0.1, 0.1) & (0.5, 0.3, 0.5) & (0.8, 0.3, 0.2) \\ (0.8, 0.2, 0.2) & (0.8, 0.3, 0.2) & (0.6, 0.4, 0.4) & (0.6, 0.4, 0.4) & (0.6, 0.3, 0.4) \end{pmatrix} \end{array}$$

$$D_3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (0.6, 0.3, 0.2) & (0.7, 0.2, 0.3) & (0.8, 0.2, 0.2) & (0.8, 0.2, 0.2) & (0.7, 0.2, 0.3) \\ (0.7, 0.2, 0.2) & (0.6, 0.3, 0.4) & (0.7, 0.2, 0.3) & (0.7, 0.2, 0.3) & (0.6, 0.3, 0.4) \\ (0.9, 0.1, 0.1) & (0.9, 0.2, 0.1) & (0.9, 0.1, 0.1) & (0.5, 0.3, 0.5) & (0.8, 0.3, 0.2) \\ (0.8, 0.2, 0.2) & (0.8, 0.2, 0.2) & (0.6, 0.3, 0.4) & (0.6, 0.3, 0.4) & (0.6, 0.3, 0.4) \end{pmatrix} \end{array}$$

$$D_4 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) \\ (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) \\ (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) \\ (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) & (0.0, 1.0, 1.0) \end{pmatrix} \end{array}$$

$$D_5 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) \\ (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) \\ (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) \\ (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) & (1.0, 0.0, 0.0) \end{pmatrix} \end{array}$$

The collection of decision matrix $\{D_1, D_2, D_3, D_4, D_5\}$ generates the NSTS.

Step 5: The collection of the neutrosophic matrix is aggregated by taking the average of each alternative with respect to the criteria. The aggregated neutrosophic decision matrix D_{agg} is shown below:

$$D_{agg} = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (0.56, 0.4, 0.32) & (0.6, 0.34, 0.4) & (0.66, 0.32, 0.34) & (0.66, 0.32, 0.34,) & (0.62, 0.34, 0.38) \\ (0.62, 0.34, 0.34) & (0.54, 0.4, 0.46) & (0.6, 0.34, 0.4) & (0.6, 0.36, 0.4) & (0.56, 0.4, 0.44) \\ (0.74, 0.28, 0.26) & (0.72, 0.32, 0.28) & (0.74, 0.26, 0.26) & (0.5, 0.4, 0.5) & (0.68, 0.38, 0.32) \\ (0.66, 0.32, 0.34) & (0.66, 0.34, 0.34) & (0.56, 0.4, 0.44) & (0.56, 0.4, 0.44) & (0.56, 0.38, 0.44) \end{pmatrix} \end{array}$$

The aggregated decision matrix is further normalized depending on the cost and beneficial criteria.

Step 6: The normalized decision matrix is denoted by N_D. It is calculated by taking the complement of the cost criteria and keeping the beneficial criteria unchanged. The normalized neutrosophic decision matrix N_D is presented below:

$$N_D = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{ccccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{A}_3 & \mathbb{A}_4 & \mathbb{A}_5 \end{array} \\ \begin{pmatrix} (0.32, 0.6, 0.56) & (0.6, 0.34, 0.4) & (0.34, 0.68, 0.66) & (0.66, 0.32, 0.34,) & (0.62, 0.34, 0.38) \\ (0.34, 0.66, 0.62) & (0.54, 0.4, 0.46) & (0.4, 0.66, 0.6) & (0.6, 0.36, 0.4) & (0.56, 0.4, 0.44) \\ (0.26, 0.72, 0.74) & (0.72, 0.32, 0.28) & (0.26, 0.74, 0.74) & (0.5, 0.4, 0.5) & (0.68, 0.38, 0.32) \\ (0.34, 0.68, 0.66) & (0.66, 0.34, 0.34) & (0.44, 0.6, 0.56) & (0.56, 0.4, 0.44) & (0.56, 0.38, 0.44) \end{pmatrix} \end{array}$$

Step 7: Here we set p = 0 and q = 1. The preference functions P_1, P_2, P_3, P_4 and P_5 are calculated for the four alternatives from the normalized neutrosophic decision matrix using the equation (6.2) and the preference functions P_1, P_2, P_3, P_4, and P_5 are shown below:

$$P_1 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0.006772537 & 0 & 0 & 0 \\ 0.024381134 & 0.024381134 & 0 & 0.013545074 \\ 0.010836059 & 0.004063522 & 0 & 0 \end{pmatrix} \end{array}$$

$$P_2 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \end{array} \\ \begin{pmatrix} 0 & 0.012038133 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.017388415 & 0.029426548 & 0 & 0.009362993 \\ 0.008025422 & 0.020063556 & 0 & 0 \end{pmatrix} \end{array}$$

$$P_3 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \end{array} \\ \begin{pmatrix} 0 & 0.009539854 & 0 & 0.019079708 \\ 0 & 0 & 0 & 0.009539854 \\ 0.014991199 & 0.024531053 & 0 & 0.034070907 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

$$P_4 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \end{array} \\ \begin{pmatrix} 0 & 0.010809583 & 0.027023956 & 0.018916769 \\ 0 & 0 & 0.016214374 & 0.008107187 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0.008107187 & 0 \end{pmatrix} \end{array}$$

$$P_5 = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \end{array} \\ \begin{pmatrix} 0 & 0 & 0.005042219 & 0 \\ 0.011344993 & 0 & 0.016387212 & 0.001260555 \\ 0 & 0 & 0 & 0 \\ 0.010084438 & 0 & 0.015126658 & 0 \end{pmatrix} \end{array}$$

Step 8: The priority index $\pi(B_i, B_r)$ is computed using the equation (6.3), and the result is shown below:

$$\pi(B_i, B_r) = \begin{array}{c} \\ \mathcal{R}_1 \\ \mathcal{R}_2 \\ \mathcal{R}_3 \\ \mathcal{R}_4 \end{array} \begin{array}{c} \begin{array}{cccc} \mathcal{R}_1 & \mathcal{R}_2 & \mathcal{R}_3 & \mathcal{R}_4 \end{array} \\ \begin{pmatrix} 0 & 0.03238757 & 0.032066176 & 0.037996477 \\ 0.01811753 & 0 & 0.032601586 & 0.018907596 \\ 0.056760747 & 0.078338734 & 0 & 0.056978973 \\ 0.02894592 & 0.024127078 & 0.023233844 & 0 \end{pmatrix} \end{array}$$

Step 9: Inflow, outflow, and net flow are determined using equations (6.4) and (6.5), and the values are displayed in table 2.

Inflow(ϕ_i^+)	Outflow (ϕ_i^-)	Net flow (ϕ_i)
0.10245	0.10382	-0.0014
0.06963	0.13485	-0.0652
0.19208	0.0879	0.10418
0.07631	0.11388	-0.0376

TABLE 2. Inflow, outflow and net flow values

Step 10: From table 2, the net flow of alternative 3 is greater than alternatives 1, 2, and 4. This indicates that alternative 3 is the best one. That is, an electric car is better than other cars.

4.2. *Sensitivity Analysis*

The alternatives and their weights remain the same for convenience. The values of the parameters p and q are changed, and the result is displayed in table 3, and its corresponding chart is given in figure 2.

parameter p,q	Net flow(ϕ_i)	Ranking
p=0,q=0.75	$\phi_1 = -0.0018, \phi_2 = -0.087, \phi_3 = 0.1389, \phi_4 = -0.0501$	$\mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_4 > \mathcal{R}_2$
p=0,q=1	$\phi_1 = -0.0014, \phi_2 = -0.0652, \phi_3 = 0.10418, \phi_4 = -0.0376$	$\mathcal{R}_3 > \mathcal{R}_1 > \mathcal{R}_4 > \mathcal{R}_2$

TABLE 3. The ranking of the alternatives under different parameters

From the table 3, Alternative 3 is the best choice. The sensitivity analysis indicated that the ranking of alternatives remained unchanged, even when preference threshold (p) and indifference threshold (q) values were changed significantly. This stability indicates that the relative strength of preferences between alternatives is consistent. This behavior implies that

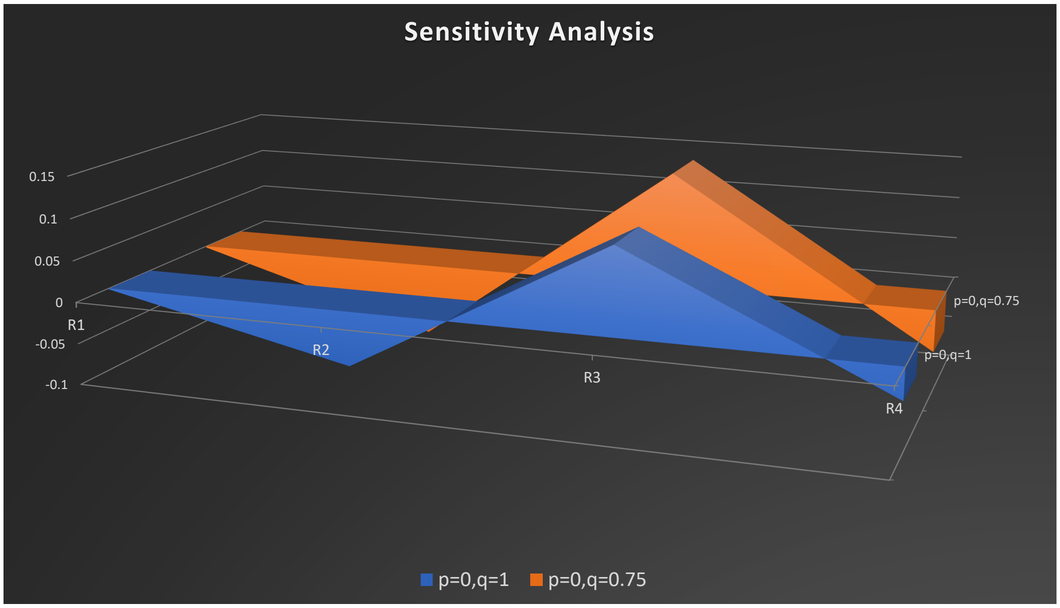

FIGURE 2. Net flow with different parameters

Methods	Rank
Single-valued neutrosophic weighted averaging operator [?]	$R_4 > R_2 > R_3 > R_1$
Single-valued neutrosophic weighted geometric operator [?]	$R_3 > R_1 > R_4 > R_2$
Extended PROMETHEE method	$R_1 > R_4 > R_2 > R_3$

TABLE 4. Comparison of the proposed method with the existing model

the model is robust. For decision-makers, this analysis provides confidence that the selected alternative will remain optimal under slightly different preference assumptions.

4.3. *Comparative Analysis*

Neutrosophic PROMETHEE method is compared with the single-valued neutrosophic weighted averaging operator and single-valued neutrosophic weighted geometric operator.

The results shown in table 4, the rankings from the single-valued neutrosophic weighted geometric operator, and the extended PROMETHEE method with neutrosophic supra topological structure are the same and match up well. This alignment reinforces the validity and stability of the proposed model. On the other hand, the ranking obtained through the single-valued neutrosophic weighted averaging operator shows slight variation. This variation is due to employing the single-valued neutrosophic aggregation operators in the aggregated decision matrix. If these operators were applied directly to the initial decision matrices, it would result in trivial outcomes.

5. Conclusion

While neutrosophic sets offer a way to handle uncertainties in decision-making, classical methods like PROMETHEE may not fully capture their nuances. This study proposes an extended PROMETHEE method within a neutrosophic supra topological space for MCDM. We explored the use of a linear preference function with parameters p and q set to 0 and 1, demonstrating that these values do not significantly impact the results. The advantage of the proposed method is that the proposed model is an easy-to-understand ranking of alternatives through preference flows. This method uses neutrosophic logic, which makes it easier to address the uncertainty and vagueness in expert opinions. While the traditional methods have trouble with this. The neutrosophic supra topological structure offers a systematic framework for clustering, comparing, and analyzing a variety of opinions. The use of neutrosophic supra topological space in MCDM enables layered aggregation of decision matrices, ensuring that varying perspectives contribute meaningfully to the final ranking.

A key limitation of the proposed approach is that the decision-maker's judgment significantly affects the outcome, which may introduce bias. Computational complexity will arise for large-scale problems with numerous alternatives and criteria. The future direction of this study is to develop a hybrid model that integrates the proposed neutrosophic decision-making framework with machine learning techniques to enhance the evaluation and selection of green vehicles. This integration aims to improve the model's ability to handle complex, large uncertain data and to support more accurate, data-driven decision-making in the context of sustainable automobile choices.

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A Neutrosophic Rayleigh Approach Based on DUS-Transformation for Analysing COVID-19 Incubation Periods

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Abstract: The DUS transformation to the Rayleigh distribution introduces better modelling of real-world variations and enhances its ability to represent skewed or heavy-tailed data. To address ambiguity, inconsistency, and indeterminacy in data, In this study, we introduce an extension of the neutrosophic Rayleigh distribution, the DUS Transformed Neutrosophic Rayleigh (DUSNR) Distribution. Important statistical aspects of the DUSNR distribution, such as quantiles, moments, moment-generating functions, and order statistics, are determined under neutrosophic conditions. The performance of the maximum likelihood estimator is assessed by simulation, showing that its accuracy increases as sample sizes rise. Lastly, the findings of applying the suggested distribution to the COVID-19 incubation dataset are contrasted with those of the DUS transformed Rayleigh distribution and the neutrosophic Rayleigh distribution.

Keywords: Neutrosophic; DUS transformation; Probability distribution; Rayleigh distribution

1. Introduction

In many real-world problems, traditional statistics assumes exact values, but in practice, uncertainty and indeterminacy often make data imprecise, vague, or ambiguous. The classical statistical methods become less effective in this case because they rely on precise numerical inputs. Recently, there have been several developments in modelling such imprecise situations by taking into account fuzzy logic and neutrosophy; refer to Atanassov (1999), Zeina and Hatip (2021) and Belohlavek et. al (2017). To address indeterminacy or partially ambiguous aspects in the data, Vladutescu and Colhon (2020) Proposed neutrosophic statistics and denoted the components as T, I, and F, which stand for truth, indeterminate, and falsehood, respectively.

Neutrosophic is a multiple-valued logic that makes a distinction between imprecise probability, fuzzy logic, and classical logic. Many branches of science and engineering have used this line of thinking. Neutosophic logic-based probability distributions have been used in several investigations. Neutosophic probability distributions, such as the normal and binomial distributions, were created by Patro and Smarandache (2016). Alhabib et al. (2018) generalised the corresponding classical distributions to introduce neutrosophic Poisson, uniform, and exponential distributions. Alhasan and Smarandache (2019) investigated the neutrosophic Weibull distribution and its related family of neutrosophic distributions. Khan et. al (2021) introduced the neutrosophic Rayleigh distribution under the neutrosophic statistics and discussed its various properties. Sherwani et al. (2021) identified numerous statistical features of the traditional beta distribution and expanded it to a neutrosophic context. The distribution was applied to two real-world datasets to validate the results.

Khan and Al-Bossly (2021) established the Gamma distribution under indeterminacy with applications to complex data analysis. For modelling Nitrogen oxide emissions data of Denmark, in 2022, Khan and colleagues came up with a neutrosophic lognormal distribution. The neutrosophic Kumaraswamy distribution was suggested by Ahsan-ul-Haq (2022) for the analysis of constrained data sets in an environment of indeterminacy.

In 2023 onwards, more distributions about imprecise data were introduced. Norouzirad et al. (2023) introduced the neutrosophic generalised Rayleigh distribution and showed that the distribution is ideal to model skewed lifetime data. Rao (2023) established a neutrosophic log-logistic distribution model in complex alloy metal melting point applications. Neutrosophic beta-Lindley distribution was established by Algamal et al. (2024) to model bladder cancer data. Neutrosophic Birnbaum-Saunders distribution for imprecise data was introduced by Hassan and Aslam (2024), and the results were validated through datasets based on alloy melting points and the lifetime of batteries. Jamal et al. (2024) proposed neutrosophic BURR-III to model COVID-19 data. Recently, Aslam (2024) established the neutrosophic negative binomial distribution and developed algorithms for generating data based on this distribution.

Due to its simplicity and lack of extra parameters, the DUS transformation produces a parsimonious distribution**.** Khan and Mustafa (2023) applied the DUS transformation to the powered inverse Rayleigh distribution and derived key statistical properties, including moments, entropy, and stress-strength reliability, along with the maximum likelihood estimator (MLE). Banerjee and Bhunia (2022) introduced the Exponential Transformed Inverse Rayleigh distribution via the DUS transformation, studying its key properties and discussing four estimation methods, including maximum likelihood, maximum product spacing, least squares, and weighted least squares methods. Tripathi and Agiwal (2024) obtained similar results for the DUS-Rayleigh distribution and further evaluated the Cramér–Von Mises estimator, using the squared error loss function for Bayesian estimation.

1.1 *Contribution of the paper*

This study proposes a novel DUS Transformed Neutrosophic Rayleigh (DUSNR) distribution. The main objective of the suggested distribution is to integrate the unknown data on the variables being examined into the current classical distribution. The suggested model shows a better fit than current Rayleigh-based models when it is finally applied to COVID-19 incubation time data.

The paper is organised as follows. The suggested model and related charts are shown in Section 2. In Section 3, the statistical features are obtained. In Section 4, a parametric estimate is made. Section 5 explains the simulation research. Section 6 includes a real data set research, and Section 7 concludes the findings.

1.2 *Importance of neutrosophic in this work*

The data are not always clear when we look at COVID-19 incubation periods. Some cases are reported exactly, like "the incubation was 7 days." In other cases, like "between 5 and 7 days," the information is not clear because the exact day of exposure is not known. There are also times when the data do not agree or have errors, like when two sources give different incubation times for the same patient. The data must be precise and consistent for classical statistical models to function effectively, which is not the case in this case. Because it can handle precise values, ambiguous ranges, and even contradicting records together, a neutrosophic model is helpful. This enables us to utilise all of the data without oversimplifying it, producing more trustworthy results for COVID-19 research conducted in the real world. Our goal is to ensure that our model accurately captures the uncertainty inherent in COVID-19 data so that the inferences made can more confidently inform real-world decisions.

2. The DUS-transformed neutrosophic Rayleigh distribution

This section presents a new DUSNR distribution using a baseline neutrosophic Rayleigh distribution. Let the non-negative neutrosophic random variable. $X_N = X_L + X_U I_N$ Follow the baseline Rayleigh distribution, where the determinacy is presented in the first part and the indeterminacy measure. I_N lying within the interval $[I_L, I_U]$, reflects the indeterminacy of the neutrosophic form.

For scale parameter θ > 0 and X_N > 0, the probability density function (pdf) of the neutrosophic Rayleigh distribution is given as

$$f_N(x_N) = (1 + I_N)\, 2\theta x_N e^{-\theta x_N^2},\ x_N > 0 \tag{1}$$

The neutrosophic cumulative distribution function (CDF) that corresponds to this is computed as

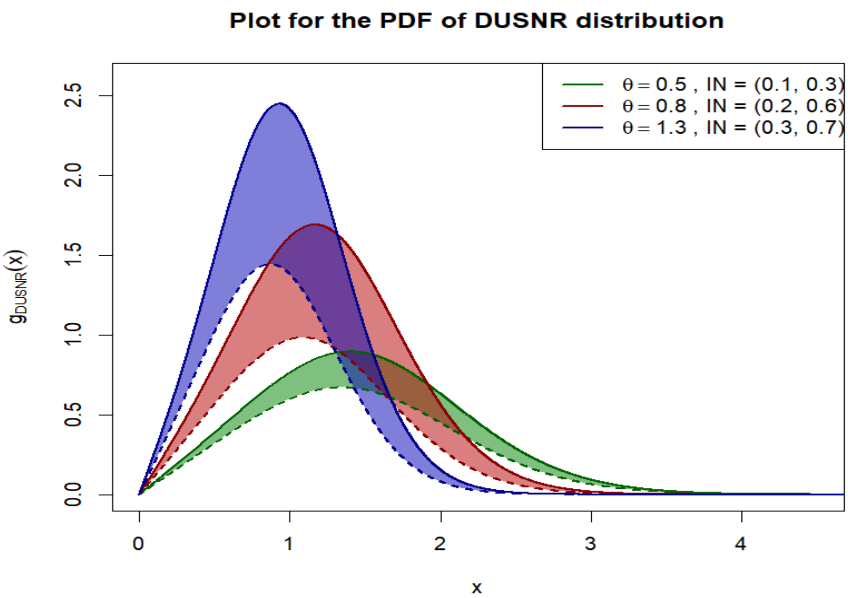
$$F_N(x_N) = \int_0^{x_N} f_N(t)\, dt = (1 + I_N)\left(1 - e^{-\theta x_N^2}\right) \tag{2}$$

We introduce the DUSNR distribution using the PDF in (1) and the CDF in (2). The DUS neutrosophic PDF and DUS neutrosophic CDF for the proposed distribution are given by

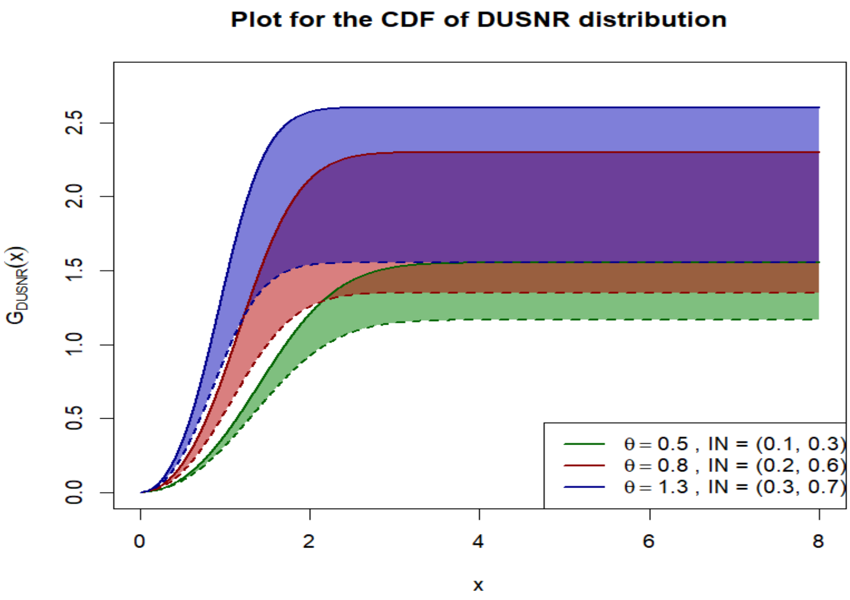
$$g_{DUSNR}(x_N) = \frac{f_N(x_N)}{e-1} e^{F_N(x_N)} = \frac{(1+I_N)2\theta x_N e^{-\theta x_N^2}}{e-1} e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)},\ x_N > 0 \text{ and}$$

$$g_{DUSNR}(x_N) = \frac{e^{F_N(x_N)} - 1}{e-1} = \frac{e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)} - 1}{e-1},\ x_N > 0, \tag{3}$$

respectively, with their corresponding plots presented in Figure 1.



(a)



(b)

Figure 1: The plots for (a) PDF, (b) CDF of DUSNR distribution using different values of I_N

Theorem: The DUSNR distribution is unimodal if $x_N^2 > ln(1 + I_N)/\theta$.

Proof: By definition, the mode of the distribution is determined by solving the equation $g'_{DUS}(x_N) = 0$, Which represents the derivative of the density function with respect to x_N. Now, we prove the uniqueness of this mode. Let

$$\ln(g_{DUSNR}) = \ln x_N - \theta x_N^2 + (1 + I_N)\left(1 - e^{-\theta x_N^2}\right) + \text{constant}$$

which implies

$$\frac{\partial\ ln(g_{DUSNR})}{\partial x_N} = \frac{1}{x_N} + 2\theta x_N\left[(1 + I_N)e^{-\theta x_N^2} - 1\right]$$

$$\frac{\partial^2 \ln(g_{DUSNR})}{\partial x_N^2} = -\frac{1}{x_N^2} - 4(1 + I_N)\theta^2 x_N^2 e^{-\theta x_N^2} - 2\theta\left[1 - (1 + I_N)\, e^{-\theta x_N^2}\right].$$

For $x_N > 0$, both $1/x_N^2$ and $e^{-\theta x_N^2}$ They are positive functions. If $x_N^2 > \ln(1 + I_N)/\theta$, then the second derivative becomes negative, indicating that the density function g_{DUSNR} Is log-concave. Since a log-concave density always has a unique mode, the theorem is thus proved.

3. Statistical properties

This section examines the basic mathematical characteristics of the "DUSNR distribution, such as the distribution of the r[th] order statistic, the moment-generating function, raw moments, the quantile function, and the Rényi entropy.

3.1. Moment generating function and raw moments

The expression for the moment generating function is given as $M_{X_N}(t) = E(e^{tX_N})$, where

$$E(e^{tX_N}) = \int_0^\infty e^{tx_N} \frac{2\theta(1+I_N)e^{1+I_N}}{e-1} x_N e^{-\theta x_N^2} e^{-(1+I_N)e^{-\theta x_N^2}} dx_N$$

$$= \frac{2\theta(1+I_N)}{e-1} e^{1+I_N} \int_0^\infty x_N\, e^{tx_N} e^{-\theta x_N^2} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^m e^{-m\theta x_N^2} dx_N$$

$$= \frac{2\theta e^{1+I_N}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^{m+1} \int_0^\infty x_N e^{-(m+1)\theta x_N^2 + tx_N} dx_N$$

$$= \frac{2\theta e^{1+I_N}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^{m+1} e^{t^2/4(m+1)\theta} \int_0^\infty x_N e^{-(m+1)\theta(x-t/2(m+1)\theta)^2} dx_N$$

$$= \frac{2\theta e^{1+I_N}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^{m+1} e^{t^2/4(m+1)\theta} \times$$

$$\left[\frac{e^{t^2/4(m+1)\theta}}{2(m+1)\theta} + \frac{t}{2(m+1)\theta} \int_0^\infty e^{-(m+1)\theta(x-t/2(m+1)\theta)^2} dx_N \right]$$

$$= \frac{e^{1+I_N}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^{m+1} e^{t^2/4(m+1)\theta} \frac{1}{(m+1)} \times$$

$$\left[e^{t^2/4(m+1)\theta} + t \int_0^\infty e^{-(m+1)\theta(x_N-t/2(m+1)\theta)^2} dx_N \right]$$

$$= \frac{e^{1+I_N}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{(m+1)!} (1+I_N)^{m+1} \left[1 + te^{t^2/4(m+1)\theta} \int_0^\infty e^{-(m+1)\theta(x_N-t/2(m+1)\theta)^2} dx_N\right]$$

$$= \frac{e^{1+I_N}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{(m+1)!} (1+I_N)^{m+1} \left[1 + \frac{\sqrt{\pi}\, t\, e^{t^2/4(m+1)\theta}}{2\sqrt{(m+1)\theta}} \operatorname{erfc}\left(\frac{-t}{2\sqrt{(m+1)\theta}} \right) \right]$$

Calculating the moments using the moment-generating function is tedious due to the complex form of the generating function. Therefore, we derive the raw moments by taking the direct expectations as follows: $E(X_N^r) = \int_0^\infty x_N^r \frac{2\theta(1+I_N)}{e-1} e^{(1+I_N)} x_N e^{-\theta x^2} e^{-(1+I_N)e^{-\theta x_N^2}} dx_N$

$$= \frac{2\theta e^{(1+I_N)}}{e-1} (1+I_N) \int_0^\infty x_N^{r+1} e^{-\theta x_N^2} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I)^m e^{-m\theta x_N^2} dx_N$$

$$= \frac{2\theta e^{(1+I_N)}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^{m+1} \int_0^\infty x_N^{r+1} e^{-(m+1)\theta x_N^2} dx_N$$

$$= \frac{\theta e^{(1+I_N)}}{e-1} \sum_{m=0}^\infty \frac{(-1)^m}{m!} (1+I_N)^{m+1} \frac{\Gamma(r/2+1)}{[(m+1)\theta]^{r/2+1}}$$

$$= \frac{e^{(1+I_N)}}{e-1} \frac{\Gamma(r/2+1)}{\theta^{r/2}} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \frac{(1+I_N)^{m+1}}{(m+1)^{r/2+1}}.$$

The first two raw moments given below

$$E(X_N) = \frac{e^{(1+I_N)}}{e-1} \frac{\sqrt{\pi}}{2\sqrt{\theta}} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \frac{(1+I_N)^{m+1}}{(m+1)^{3/2}}, \quad E(X_N^2) = \frac{e^{(1+I_N)}}{e-1} \frac{1}{\theta} \sum_{m=0}^\infty \frac{(-1)^m}{m!} \frac{(1+I_N)^{m+1}}{(m+1)^2}$$

are derived by imputing r = 1, 2. Consequently, the mean, variance and other higher-order moments can be found similarly.

3.2. Quantile function

The value of a random variable is defined using the quantile function in such a way that the chance that the variable will be less than or equal to that value is equal to the specified probability. For a DUSNR distribution with parameter θ, the distribution function of a random variable is such that $g_{DUSNR}\big(Q(p)\big) = p, p \in (0,1)$, which implies that

$$Q(p) = \sqrt{\frac{1}{\theta} ln\left(\frac{1+I_N}{1+I_N - ln\big((e-1)p+1\big)}\right)}.$$

To analyse the impact of the parameter on the shape of the distribution, the coefficients of skewness (B) and kurtosis (K) are computed using quantiles, which are defined as

$$B = \frac{Q(3/4) + Q(1/4) - Q(1/2)}{Q(3/4) - Q(1/4)}, \qquad K = \frac{Q(3/8) - Q(1/8) + Q(7/8) - Q(1/4)}{Q(3/4) - Q(1/4)}.$$

Also, the median of the DUSNR distribution is defined as Q(1/2).

3.3. Rényi Entropy

Rényi Entropy quantifies the uncertainty and diversity within a probability distribution, with higher values indicating greater unpredictability and lower values indicating more certainty. We calculate the entropy function of order w as

$$\tau_N(w) = \frac{1}{w-1} ln\left(\int_0^\infty g_N^w(x_N)dx_N\right)$$

$$= \frac{1}{w-1} ln\left[\left(\frac{2\theta(1+I_N)}{e-1}e^{(1+I_N)}\right)^w \int_0^\infty x_N^w e^{-w\theta x_N^2} e^{-w(1+I_N)e^{-\theta x_N^2}} dx_N\right]$$

$$= \frac{1}{w-1} ln\left[\left(\frac{2\theta(1+I_N)}{e-1}e^{(1+I_N)}\right)^w \sum_{m=0}^{\infty} \frac{(-1)^m}{m!}(1+I_N)^m \frac{w^m}{2} \frac{\Gamma((w+1)/2)}{\big((w+m)\theta\big)^{(w+1)/2}}\right]$$

$$= ln\big(2\sqrt{\theta}\big) + \frac{(1+I_N)w - ln(e-1)}{w-1} + \frac{1}{w-1} ln\left(\sum_{m=0}^{\infty} \frac{(-w)^m}{m!}(1+I_N)^{m+w} \frac{\Gamma((w+1)/2)}{(w+m)^{(w+1)/2}}\right).$$

3.3. Distribution of r^{th} ordered statistics

Order statistics are especially significant in reliability engineering and survival data analysis, where the hazard function is a key analytical tool. Suppose $(X_{N1}, X_{N2}, \dots, X_{Nn})$ Represent a neutrosophic sample of independent and identically distributed random variables holding a density. $g_N(x)$. If their ordered counterparts are denoted by $(X_{N(1)}, X_{N(2)}, \dots, X_{N(n)})$, then the PDF of the r^{th} order statistic $X_{N(r)}$ is

$$g_{(r)}(x_N) = \frac{n!}{(r-1)!\,(n-r)!}\left[\frac{e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)} - 1}{e-1}\right]^{r-1}\left[1 - \frac{e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)} - 1}{e-1}\right]^{n-r} \times$$

$$\frac{(1+I_N)\cdot 2\theta x_N e^{-\theta x_N^2}}{e-1}\cdot e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)},\quad x_N>0.$$

The CDF of the r^{th} order statistic $X_{N(r)}$ is provided as

$$G_{(r)}(x_N)=\sum_{k=r}^{n}\binom{n}{k}\left[\frac{e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)}-1}{e-1}\right]^k\left[1-\frac{e^{(1+I_N)\left(1-e^{-\theta x_N^2}\right)}-1}{e-1}\right]^{n-k},\quad x_N>0.$$

4. Parameter Estimation

The parameters are estimated based on sample observations drawn from the underlying distribution. In this study, data are generated from the DUSNR distribution with a parameter. θ, ensuring that the sample reflects the characteristics of the population under consideration. We adopt the following inverse transformation algorithm to generate random observations:

Algorithm 1:

1. Generate random observations $u_1, u_2, \dots, u_n$ from the U(0,1) distribution.
2. A random sample of size n, i.e., $x_{Ni}, i=1,2,\dots,n$ from DUSNR distribution is obtained from the equation $x_{Ni}=g_{DUSNR}^{-1}(u_i)$ which implies

$$x_{Ni}=\sqrt{\frac{1}{\theta}ln\left(\frac{1+I_N}{1+I_N-ln((e-1)u_i+1)}\right)}, i=1,2,\dots,n.$$

Using the random sample $\underline{x_N}=\{x_{N1},x_{N2},\dots,x_{Nn})$ of size n, the likelihood function for the parameter θ from the DUSNR distribution is given by

$$L\left(\theta\middle|\underline{x_N}\right)=\prod_{i=1}^{n}g_{DUSNR}(x_{Ni},\theta)=\left(\frac{2\theta(1+I_N)}{e-1}\right)^n e^{n(1+I_N)}\prod_{i=1}^{n}\left(x_{Ni}e^{-\theta x_{Ni}^2}e^{-(1+I_N)e^{-\theta x_{Ni}^2}}\right).$$

The corresponding log-likelihood is calculated as

$$l=ln(L)=n\,ln(\theta)-\theta\sum_{i=1}^{n}x_{Ni}^2-(1+I_N)\sum_{i=1}^{n}e^{-\theta x_{Ni}^2}+\text{terms without }\theta.$$

To find the MLE for θWe calculated the first and second derivatives of the log-likelihood function with respect to θ. The following first derivative is used to find MLE.

$$\frac{\partial l}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}x_{Ni}^2+(1+I_N)\sum_{i=1}^{n}x_{Ni}^2e^{-\theta x_{Ni}^2}=0.$$

The likelihood equation can be solved to obtain the MLE using the *uniroot* function in R. This means that the log-likelihood function is concave since the second derivative is negative."

$$\frac{\partial^2 l}{\partial\theta^2}=-\frac{n}{\theta^2}-(1+I_N)\sum_{i=1}^{n}x_{Ni}^4e^{-\theta x_{Ni}^2}<0.$$

Hence, the MLE for θ Exists and is unique.

5. Simulation study

The efficiency of the MLE under the DUSNR distribution is analysed using the neutrosophic root mean square error. ($RMSE_N$). This measure quantifies the discrepancy between observed and estimated values, where a lower value. $RMSE_N$ Signifies greater accuracy of the estimator. We define the expression for $RMSE_N$ as

$$RMSE_N = \sqrt{\frac{1}{m}\sum_{i=1}^{m}\left(\hat{\theta}_i - \theta\right)^2}$$

Samples corresponding to I_N = 0, 0.5, (0.2, 0.7) are generated using Algorithm-1, and the simulation is replicated N = 10,000 times. The computations are performed in the R software. For each sample size n = 15, 50, 100, the $RMSE_N$ Is evaluated. Table 1 presents the estimated bias and $RMSE_N$ Associated with the MLE based on this simulation study.

Tabe 1:(Bias, $RMSE_N$) for MLE of θ

θ	I_N	n=15	n=50	n=100
0.2	0	0.0113, 0.0026	0.0029, 0.0006	0.0014, 0.0003
	0.5	0.3371, 0.1199	0.3276, 0.1089	0.3260, 0.1071
	(0.2,0.7)	(0.1514,0.4648), (0.0263,0.2250)	(0.1438,0.4537), (0.0215,0.2082)	(0.1426,0.4519), (0.0207,0.2054)
1	0	0.0282, 0.0163	0.0072, 0.0040	0.0036, 0.0020
	0.5	0.8427, 7495	0.8189, 0.6808	0.8149, 0.6619
	(0.2,0.7)	(0.3786, 1.1620), (0.1646, 1.4060)	(0.3596, 1.1344), (0.1348, 1.3014)	(0.3565, 1.1298), (0.1297, 1.2836)

The observations from the table:

1. Bias and MSE decrease as the sample size increases from n = 15 to n = 100 for all θ and I_N Values, indicating improved estimation accuracy with larger samples.

2. For I_N = 0 (no indeterminacy), the MLE performs well, with very small bias and MSE across all parameter settings.

3. When I_N = 1, both bias and MSE are notably higher than in the case of I_N = 0, showing that neutrosophic uncertainty degrades estimation precision.

6. Applications

This section applies the proposed model to COVID-19 mean incubation time data, emphasising the importance of accurately modelling incubation periods for effective disease control and forecasting. The data given in Table 2 have been used previously by Cheng et al. (2021).

Table 2: Mean Incubation Periods with Uncertainties for COVID-19

(7.82, 8.37)	(8.24, 9.88)	(4.95, 5.40)	(7.55, 8.88)	(4.15, 5.89)	(6.87, 7.04)
(5.43, 6.52)	(4.79, 5.89)	(6.87, 7.55)	(5.52, 6.73)	(5.08, 6.93)	(6.62, 7.45)
(3.33, 4.67)	(4.90, 5.57)	(6.61, 7.10)	(3.92, 5.09)	(8.18, 9.24)	(6.62, 7.11)
(5.48, 6.62)	(4.18, 5.97)	(5.50, 7.18)	(9.96, 11.09)	(8.04, 9.44)	(7.37, 8.99)
(5.28, 6.65)	(9.99, 10.70)	(10.66, 11.68)	(8.27, 9.09)	(5.12, 5.78)	(5.43, 6.19)
(4.84, 6.16)	(4.38, 5.40)	(6.45, 7.39)	(6.33, 7.79)	(5.58, 6.02)	(2.21, 3.47)
(7.39, 8.25)	(6.21, 6.86)	(6.01, 7.31)	(6.30, 7.43)	(5.61, 6.64)	(5.82, 6.49)
(5.26, 5.81)	(2.21, 3.23)	(7.51, 8.01)	(4.34, 5.13)	(4.67, 5.23)	(4.31, 4.96)

Incubation time varies across diseases and is crucial for estimating exposure and planning treatment strategies. Due to challenges in measuring exact exposure times, available data often include uncertainties. Given the presence of uncertainty in the data, the conventional Rayleigh and neutrosophic Rayleigh models are extended through the proposed DUSNR distribution, which enables interval-based inference for the parameter.

The study assesses the effectiveness of the DUSNR distribution as a modelling framework, with the AIC and BIC values in Table 3 indicating a superior fit compared to both the standard Rayleigh and neutrosophic Rayleigh models. The MLEs under each model are also provided in Table 3, demonstrating their practical utility in analysing real-world epidemiological datasets.

Table 3: Parameter estimation and assessment of distribution fit

Model	MLE	AIC	BIC
Rayleigh	(0.01923, 0.0253)	(53.896, 81.48)	(55.767, 83.35)
Neutrosophic Rayleigh	(0.01923, 0.0253)	(11.801, 39.40)	(13.68, 41.27)
DUSNR	(0.0284, 0.0372)	(6.06, 32.64)	(7.74, 33.42)

7. Conclusion

This study introduced the DUSNR distribution, a unique expansion of the Rayleigh family that can efficiently describe data under uncertainty, imprecision, and indeterminacy. By combining the DUS transformation and the neutrosophic framework, the suggested model improves the capacity of classical distributions to handle real-world data that is frequently ambiguous or incomplete. Quantiles, moments, the moment-generating function, and order statistics have all been properly derived. Simulation studies were used to test model parameter estimation processes, and the results showed that as the sample size increased, estimator performance improved. Furthermore, the application to an actual dataset confirmed the DUSNR distribution's practical utility, with AIC and

BIC values indicating that it fits better than existing Rayleigh-based models. Overall, the DUSNR model provides a strong statistical tool for assessing imprecise or uncertain data, opening up new avenues for research in neutrosophic statistics and its applications.

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