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An Introduction to NeutroHyperVector Spaces

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Abstract. The aim of this paper is to combine the notion of NeutroAlgebra, that includes the partiality and indeterminacy in the operations and axioms of algebraic structures, with algebraic hyperstructures. In this regard, a NeutroHyperVector space is introduced and various examples are given. Next, some types of linear transformations between NeutroHyperVector spaces are presented and some properties of mentioned concepts are studied. Finally, by giving a suitable field, the Cartesian product of NeutroHyperVector spaces over a common field is constructed, under certain conditions, and supported by interesting examples.

Keywords: Hypervector space; NeutroHyperoperation; NeutroAxiom; NeutroHyperVector space; SubNeutroHyperspace; NeutroTransformation

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1. Introduction

There are various mathematical tools for modeling the real facts. For example, the theory of fuzzy sets introduced by Zadeh [1] in 1934 expresses the vague and uncertain properties, and the theory of intuitionistic fuzzy sets introduced by Atanassov [2] in 1983 adds the non-membership information. Neutrosophic theory introduced by Smarandache in 1995 as a generalization of fuzzy sets and intuitionistic fuzzy set, is another way to this goal. In this idea, information related to the truth (T), falsity (F) and indeterminacy (I) of the problem is considered. In fact, the indeterminacy distinguishes neutrosophy from the other philosophy. Then some linked methods have been studied, such as neutrosophic rough set [3], complex neutrosophic set [4], neutrosophic soft set [5,6]. The theory of neutrosophic set applied in various branches such as in medical diagnosis [7], decision-making process [8], pattern recognition [9], economics [10] and operation research [11].

In 2019 Smarandache [12] introduced the notion of NeutroAlgebra, as a generalization of the classic algebraic structures, where it has operations and axioms that are partially well-defined, partially indeterminate, and partially false. By applying the new idea to another algebraic structures, many NeutroAlgebras have been studied, for example NeutroGroups [13], NeutroRings [14] and NeutroOrderedAlgebra [15].

On the other hand, the theory of algebraic hyperstructures was born in 1934 by Marty [16] as a generalization of algebraic structures, where the hyperoperation of two elements is a non-empty set. This theory has been extended in many branches of mathematics such as fields, lattices, rings, quasigroups, semigroups, ordered structures, combinatorics, topology, geometry, graphs, codes, etc.; for example, see the books [17–19]. The notion of hypervector space was introduced by Scafati-Tallini [20] in 1990 and has been investigated by herself, Ameri [21], Sedghi [22] and the author [23–27].

Recently, NeutroAlgebraic structures have been extended to NeutroHyperalgebraic structures; Ibrahim [28] defined NeutroHypergroups and Al-Tahan [29] and Rezaei [30] studied some properties of NeutroSemihypergroups. Now in this paper, we apply the theory of neutrosophy in hypervector spaces and introduce NeutroHyperVector spaces as an alternative structure and a type of generalization for hypervector spaces. In Section 3, we introduce the notion of NeutroHyperVector spaces, present some interesting examples and shortly study the concept of SubNeutroHyperspaces. In Section 4, we investigate the relation of two NeutroHyperVector spaces by using of different types of transformations, especially, the behavior of SubNeutroHyperspaces under transformations and their inverse, again supported by some examples. Finally, we make new NeutroHyperVector spaces by Cartesian product of NeutroHyperVector spaces and give some examples.

2. Preliminaries

In this section, we present some definitions and propositions that we shall use in later.

A hyperoperation over a non-empty set S is a mapping "$\circ : S \times S \to P_*(S)$", where $P_*(S)$ is the set of non-empty subsets of S. If $A, B \subseteq S$, then $A \circ B = \bigcup_{a\in A, b\in B} a \circ b$. Especially, $A \circ b = A \circ \{b\} = \bigcup_{a\in A} a \circ b$ and $a \circ B = \{a\} \circ B = \bigcup_{b\in B} a \circ b$.

Definition 2.1. [20] Let K be a field, $(V, +)$ be an Abelian group and $P_*(V)$ be the set of all non-empty subsets of V. We define a hypervector space over K to be the quadruplet $(V, +, \circ, K)$, where " $\circ$ " is an external hyperoperation

$$\circ : K \times V \longrightarrow P_*(V), \tag{1}$$

such that for all $a, b \in K$ and $x, y \in V$ the following conditions hold:

(HV_1) $a \circ (x + y) \subseteq a \circ x + a \circ y$, right distributive law,

(HV_2) $(a+b)\circ x \subseteq a\circ x + b\circ x$, left distributive law,

(HV_3) $a\circ(b\circ x) = (ab)\circ x$,

(HV_4) $a\circ(-x) = (-a)\circ x = -(a\circ x)$,

(HV_5) $x \in 1\circ x$,

where in (HV_1), $a\circ x + a\circ y = \{p+q : p\in a\circ x, q\in a\circ y\}$. Similarly, it is in ($HV_2$). Also, in ($HV_3$), $a\circ(b\circ x) = \bigcup\limits_{t\in b\circ x} a\circ t$.

V is called strongly right distributive, if we have equality in (HV_1). In a similar way we define the strongly left distributive hypervector spaces.

In the sequel of this paper, V denotes a hypervector space over the field K, unless otherwise is specified.

Example 2.2. Let $K = \mathbb{Z}_2 = \{0,1\}$ be the field of two numbers with the following operations:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

Also, let $V = \mathbb{Z}_3 = \{0,1,2\}$ be an Abelian group with the following operation:

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(\mathbb{Z}_3, +, \circ_1, \mathbb{Z}_2)$ and $(\mathbb{Z}_3, +, \circ_2, \mathbb{Z}_2)$ are hypervector spaces over the field $\mathbb{Z}_2$ with the following external hyperoperations, where they are not strongly left or right hypervector spaces:

$\circ_1$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0,1,2\}$	$\{0,1,2\}$

and

$\circ_2$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1,2\}$	$\{1,2\}$

(2)

3. NeutroHyperVector Spaces

Let U be the universe of discourse. Then a *neutrosophic set* is an object having the form

$$A = \{(x, \mu_A(x), \omega_A(x), \nu_A(x)), x \in U\}, \tag{3}$$

where the functions $\mu, \omega, \nu : U \to [0,1]$ define respectively the degree of membership, the degree of indeterminacy and the degree of non-membership of x to the set A.

For example, the following functions define a neutrosophic set in $U = \{1, 2, 3, 4\}$:

x	1	2	3	4
$\mu_A(x)$	0.4	0	0.7	0.7
$\omega_A(x)$	0.6	0.2	0	0.1
$\nu_A(x)$	0.5	0.8	0.3	0.2

A hyperoperation "$\circ : S \times S \to P_*(U)$", where U is the universe of discourse containing S, is called a *NeutroHyperoperation* on S, if $x \circ y \subseteq S$, for some $x, y \in S$ (the degree of truth "T") and some (or all) of the following conditions hold:

(1) $x \circ y \nsubseteq S$, for some $x, y \in S$ (the degree of falsity "F");

(2) $x \circ y$ is indeterminate in S, for some $x, y \in S$ (the degree of indeterminacy "I").

For example, if $U = \{1, 2, \ldots, 10\}$ and $S = \{2, 5, 8\}$, then the followings are NeutroHyperoperations on S:

$\circ$	2	5	8
2	$\{3\}$	$\{2,4\}$	$\{2,8\}$
5	$\{2,5\}$	$\{2,5,8\}$	$\{5\}$
8	$\{2\}$	$\{2\}$	$\{2\}$

and

$\circ$	2	5	8
2	$\{8\}$	$\{2,5\}$	
5	$\{2,5\}$	$\{2,5,8\}$	$\{5\}$
8	$\{2\}$		$\{2\}$

The hyperoperation " $\circ$ " is called an *AntiHyperoperation* on S, if $x \circ y \nsubseteq S$, for all $x, y \in S$. The following example is an AntiHyperoperation on $S = \{2, 5, 8\}$:

$\circ$	2	5	8
2	$\{1,2\}$	$\{1,5\}$	$\{4\}$
5	$\{10\}$	$\{2,4,8\}$	$\{3\}$
8	$\{1,2\}$	U	$\{2,3\}$

A hyperoperation "$\circ : S \times S \to P_*(U)$", where U is the universe of discourse containing S, is called *NeutroAssociative* on S, if $x \circ (y \circ z) = (x \circ y) \circ z$, for some $x, y, z \in S$ (the degree of truth "T") and some (or all) of the following conditions hold:

(1) $x \circ (y \circ z) \neq (x \circ y) \circ z$, for some $x, y, z \in S$ (the degree of falsity "F");

(2) for some $x, y, z \in S$, $x \circ (y \circ z)$ is indeterminate in S, or $(x \circ y) \circ z$ is indeterminate in S, or we can not find if $x \circ (y \circ z)$ and $(x \circ y) \circ z$ are equal (the degree of indeterminacy "I").

For example, if $U = \{1, 2, \ldots, 10\}$ and $S = \{2, 5, 8\}$, then the following hyperoperations are NeutroAssociative on S:

$\diamond_1$	2	5	8
2	$\{2,5\}$	$\{2,5\}$	$\{5\}$
5	$\{2,5\}$	$\{5,8\}$	$\{2\}$
8	$\{8\}$	$\{8\}$	$\{5\}$

$\diamond_2$	2	5	8
2	$\{2,5\}$	$\{2,5\}$	$\{5\}$
5	$\{2,5\}$	$\{5,8\}$	$\{2\}$
8	$\{8\}$		$\{5\}$

$\diamond_3$	2	5	8
2	$\{2,5\}$	$\{2,5\}$	$\{5\}$
5	$\{2,5\}$	$\{5,8\}$	$\{2\}$
8		$\{8\}$	$\{5\}$

More precisely, $\diamond_1$ is NeutroAssociative on S, since $2 \diamond_1 (2 \diamond_1 2) = (2 \diamond_1 2) \diamond_1 2$ and $5 \diamond_1 (5 \diamond_1 5) \neq (5 \diamond_1 5) \diamond_1 5$. $\diamond_2$ is NeutroAssociative on S, since $2 \diamond_2 (2 \diamond_2 2) = (2 \diamond_2 2) \diamond_2 2$ and $(5 \diamond_2 5) \diamond_2 5$ is indeterminate in S. $\diamond_3$ is NeutroAssociative on S, since $2 \diamond_3 (2 \diamond_3 2) = (2 \diamond_3 2) \diamond_3 2$, $5 \diamond_3 (5 \diamond_3 5) \neq (5 \diamond_3 5) \diamond_3 5$ and $(8 \diamond_3 2) \diamond_3 5$ is indeterminate in S.
The hyperoperation " $\circ$ " is called *AntiAssociative* on S, if $x \circ (y \circ z) \neq (x \circ y) \circ z$, for all $x, y, z \in S$. The following hyperoperation is AntiAssociative on S:

$\diamond$	2	5	8
2	$\{2,5\}$	$\{2,8\}$	$\{2\}$
5	$\{5\}$	$\{8\}$	$\{2\}$
8	$\{5\}$	$\{8\}$	$\{5,8\}$

Definition 3.1. Let K be a field and $(G, +)$ be a group. Then an external hyperoperation "$\circ : K \times G \to P_*(U)$", where U is the universe of discourse containing G, is called an *external NeutroHyperoperation* on G, if $a \circ x \subseteq G$, for some $a \in K,\ x \in G$ (the degree of truth "T") and at least one of the following conditions hold:

(1) $a \circ x \nsubseteq G$, for some $a \in K,\ x \in G$ (the degree of falsity "F");

(2) $a \circ x$ is indeterminate in G, for some $a \in K,\ x \in G$ (the degree of indeterminacy "I").

The external hyperoperation " $\circ$ " is called an *external AntiHyperoperation* on G, if $a \circ x \nsubseteq G$, for all $a \in K,\ x \in G$.

Example 3.2. Consider the field $K = \mathbb{Z}_2 = \{0, 1\}$ and the Abelian group $G = \mathbb{Z}_3 = \{0, 1, 2\}$ defined in Example 2.2. Then the followings are external NeutroHyperoperations on G, where $U = \{0, 1, 2, 3, 4, 5\}$:

$\circ_3$	0	1	2
0	$\{0,4\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{1,2\}$	$\{0,1,2\}$

$\circ_4$	0	1	2
0	$\{0,1\}$	$\{1,2\}$	$\{1,2\}$
1	$\{1\}$		$\{0,1,2\}$

$\circ_5$	0	1	2
0	$\{0,2\}$	$\{1,2\}$	$\{1,2\}$
1	$\{2,4\}$		$\{0,1,2\}$

Note that in the first table, $0 \circ_3 0 = \{0, 4\}$ is not a subset of $\mathbb{Z}_3$; in the second table, $1 \circ_4 1$ is indeterminate in $\mathbb{Z}_3$ and in the third table, $0 \circ_5 1 = \{2, 4\}$ is not a subset of $\mathbb{Z}_3$ and $1 \circ_5 1$ is indeterminate in $\mathbb{Z}_3$. Also, the following is an external AntiHyperoperation on G:

$\circ$	0	1	2
0	$\{0,4\}$	$\{0,3\}$	$\{5\}$
1	$\{1,4,5\}$	$\{1,3\}$	$\{0,5\}$

A NeutroHyperVector space is an alternative of a hypervector space $(V, +, \circ, K)$ such that "$+ : V \times V \to P_*(V)$" is a NeutroHyperoperation, or "$\circ : K \times V \to P_*(V)$" is an external NeutroHyperoperation, or at least it has one NeutroAxiom. Thus, there are several types of NeutroHyperVector spaces, based on the number of NeutroOperation, NeutroHyperoperation and NeutroAxioms.

In this paper, we consider the following definition for a NeutroHyperVector space:

Definition 3.3. Let $(V, +, \circ, K)$ be a hypervector space over the field K such that "$+ : V \times V \to V$" is an operation on V and "$\circ : K \times V \to P_*(U)$" is an external NeutroHyperoperation, where U is the universe of discourse containing V. Then V is called a *NeutroHyperVector space* over the field K, if at least one of the following NeutroAxioms hold:

NHV$_1$) " $\circ$ " is *right NeutroDistributive* on " $+$ ", i.e. $a \circ (x + y) \subseteq a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $a \circ (x + y) \nsubseteq a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$ (the degree of falsity "F");
- for some $a \in K,\ x, y \in V$, $a \circ (x + y)$ is indeterminate in V, or $a \circ x + a \circ y$ is indeterminate in V, or we can not find if $a \circ (x + y)$ is a subset of $a \circ x + a \circ y$ (the degree of indeterminacy "I").

NHV$_2$) " $\circ$ " is *left NeutroDistributive* on " $+$ ", i.e. $(a + b) \circ x \subseteq a \circ x + b \circ x$, for some $a, b \in K,\ x \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $(a + b) \circ x \nsubseteq a \circ x + b \circ x$, for some $a, b \in K,\ x \in V$ (the degree of falsity "F");
- for some $a, b \in K,\ x \in V$, $(a + b) \circ x$ is indeterminate in V, or $a \circ x + b \circ x$ is indeterminate in V, or we can not find if $(a + b) \circ x$ is a subset of $a \circ x + b \circ x$ (the degree of indeterminacy "I").

NHV$_3$) $a \circ (b \circ x) = (ab) \circ x$, for some $a, b \in K,\ x \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $a \circ (b \circ x) \neq (ab) \circ x$, for some $a, b \in K,\ x \in V$ (the degree of falsity "F");
- for some $a, b \in K,\ x \in V$, $a \circ (b \circ x)$ is indeterminate in V, or $(ab) \circ x$ is indeterminate in V, or we can not find if $a \circ (b \circ x)$ and $(ab) \circ x$ are equal (the degree of indeterminacy "I").

NHV$_4$) $a \circ (-x) = (-a) \circ x = -(a \circ x)$, for some $a \in K,\ x \in V$ (the degree of truth "T") and at least one of the following conditions hold:

- $a \circ (-x) \neq (-a) \circ x$ or $(-a) \circ x \neq -(a \circ x)$ or $a \circ (-x) \neq -(a \circ x)$, for some $a \in K,\ x \in V$ (the degree of falsity "F");
- for some $a \in K,\ x \in V$, $a \circ (-x)$ is indeterminate in V, or $(-a) \circ x$ is indeterminate in V, or $-(a \circ x)$ is indeterminate in V, or we can not find if $a \circ (-x)$, $(-a) \circ x$ and $-(a \circ x)$ are equal (the degree of indeterminacy "I").

NHV$_5$) $x \in 1 \circ x$ and ($y \notin 1 \circ y$ or $1 \circ z$ is indeterminate in V), for some $x, y, z \in V$.

We say that $(V, +, \circ, K)$ is strongly right distributive NeutroHyperVector space, if $a \circ (x + y) = a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$. In a similar way, the strongly left distributive

NeutroHyperVector space is defined. V is said to be strongly distributive, if it is both strongly left distributive and strongly right distributive.

Example 3.4. Consider the external NeutroHyperoperations $\circ_3, \circ_4, \circ_5 : \mathbb{Z}_2 \times \mathbb{Z}_3 \to P_*(U)$ defined in Example 3.2. Then $V_3 = (\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$, $V_4 = (\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$ and $V_5 = (\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ are NeutroHyperVector spaces over the field $\mathbb{Z}_2$, since they are satisfied in the NeutroAxioms (NHV_1)-(NHV_5) of Definition 3.3, as follows:

$V_3 = (\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space:

NHV_1) $a \circ_3 (x + y) = a \circ_3 x + a \circ_3 y$, for $a = 1,\ x = 0,\ y = 2$, $a \circ_3 (x + y) \nsubseteq a \circ_3 x + a \circ_3 y$, for $a = 1,\ x = 0,\ y = 1$, and $a \circ_3 x + a \circ_3 y$ is indeterminate in V, for $a = 0,\ x = 0,\ y = 0$.

NHV_2) $(a+b) \circ_3 x = a \circ_3 x + b \circ_3 x$, for $a = 0,\ b = 1,\ x = 2$ and $a \circ_3 x + b \circ_3 x$ is indeterminate in V, for $a = b = 0,\ x = 0$.

NHV_3) $a \circ_3 (b \circ_3 x) = (ab) \circ_3 x$, for $a = b = 1,\ x = 2$, $a \circ_3 (b \circ_3 x) \neq (ab) \circ_3 x$, for $a = b = 1,\ x = 0$, and $1 \circ_3 (0 \circ_3 0)$ and $(01) \circ_3 0$ are indeterminate in V.

NHV_4) $a \circ_3 (-x) = (-a) \circ_3 x = -(a \circ_3 x)$, for $a = 0,\ x = 1$, but $1 \circ_3 (-1) = 1 \circ_3 2 = \{0, 1, 2\} \neq (-1) \circ_3 1 = 1 \circ_3 1 = \{1, 2\} = -(1 \circ_3 1) = -(\{1, 2\})$ and $-(0 \circ_3 0)$ is indeterminate in V.

NHV_5) $1 \in 1 \circ_3 1$ and $2 \in 1 \circ_3 2$, but $0 \notin 1 \circ_3 0$.

Similarly, $V_4 = (\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space:

NHV_1) $1 \circ_4 (0 + 2) = 1 \circ_4 2 = \{0, 1, 2\} = 1 \circ_4 0 + 1 \circ_4 2 = \{1\} + \{0, 1, 2\}$, $1 \circ_4 (0 + 0) = 1 \circ_4 0 = \{1\} \nsubseteq 1 \circ_4 0 + 1 \circ_4 0 = \{1\} + \{1\} = \{2\}$, and $1 \circ_4 0 + 1 \circ_4 1$ is indeterminate in V.

NHV_2) $(0 + 1) \circ_4 2 = 1 \circ_4 2 = \{0, 1, 2\} = 0 \circ_4 2 + 1 \circ_4 2 = \{0, 2\} + \{0, 2\}$, $(1 + 1) \circ_4 0 = 0 \circ_4 0 = \{0, 1\} \nsubseteq 1 \circ_4 0 + 1 \circ_4 1 = \{1\} + \{1\} = \{2\}$, and $0 \circ_4 1 + 1 \circ_4 1$ is indeterminate in V.

NHV_3) $0 \circ_4 (0 \circ_4 1) = 0 \circ_4 (\{1, 2\}) = \{1, 2\} = (00) \circ_4 1$, $0 \circ_4 (0 \circ_4 0) = 0 \circ_4 (\{0, 1\}) = \{0, 1, 2\} \neq (00) \circ_4 0 = \{0, 1\}$, and $1 \circ_4 (1 \circ_4 1)$ is indeterminate in V.

NHV_4) $0 \circ_4 (-1) = 0 \circ_4 2 = \{1, 2\} = (-0) \circ_4 1 = -(0 \circ_4 1) = -(\{1, 2\})$, while $1 \circ_4 (-0) = 1 \circ_4 0 = \{1\} = (-1) \circ_4 0 = 1 \circ_4 0 \neq -(1 \circ_4 0) = -(\{1\}) = \{2\}$, and $-(1 \circ_4 1)$ and $1 \circ_4 (-2)$ are indeterminate in V.

NHV_5) $2 \in 1 \circ_4 2$, $0 \notin 1 \circ_4 0$, and we can not find $1 \in 1 \circ_4 1$, since $1 \circ_4 1$ is indeterminate in V.

The NeutroHyperVector space $V_5 = (\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ is strongly left distributive and it is not strongly right distributive:

NHV_1) $0 \circ_5 (2 + 2) = 0 \circ_5 1 = \{0, 2\} \subseteq 0 \circ_5 2 + 0 \circ_5 2 = \{1, 2\} + \{1, 2\} = \{0, 1, 2\}$, $1 \circ_5 (2 + 2)$ is indeterminate in V and we can not find $1 \circ_5 (2 + 2)$ and $1 \circ_5 2 + 1 \circ_5 2$ are equal. There don't exist $a \in K,\ x, y \in V$, such that $a \circ_5 (x + y) = a \circ_5 x + a \circ_5 y$.

NHV_2) $(0 + 1) \circ_5 2 = 1 \circ_5 2 = \{0, 1, 2\} = 0 \circ_5 2 + 1 \circ_5 2 = \{1, 2\} + \{0, 1, 2\}$, and $1 \circ_5 1 + 1 \circ_5 1$ is indeterminate in V, so we can not find $(1 + 1) \circ_5 1$ and $1 \circ_5 1 + 1 \circ_5 1$ are equal.

NHV$_3$) $0 \circ_5 (0 \circ_5 1) = 0 \circ_5 (\{1,2\}) = 0 \circ_5 1 \cup 0 \circ_5 2 = \{1,2\} = (00) \circ_5 1$, and $0 \circ_5 (0 \circ_5 0) = 0 \circ_5 (\{0,2\}) = 0 \circ_5 0 \cup 0 \circ_5 2 = \{0,1,2\} \neq (00) \circ_5 0 = \{0,2\}$.

NHV$_4$) $0 \circ_5 (-1) = 0 \circ_5 1 = \{1,2\} = (-0) \circ_5 1 = -(0 \circ_5 1)$ and $0 \circ_5 (-0) = 0 \circ_5 0 = \{0,2\} \neq -(0 \circ_5 0) = \{0,1\}$.

NHV$_5$) $2 \in 1 \circ_5 2$, $0 \notin 1 \circ_5 0$, and we can not find $1 \in 1 \circ_5 1$, since $1 \circ_5 1$ is indeterminate in V.

Example 3.5. Consider the field $K = \mathbb{Z}_2 = \{0,1\}$ defined in Example 2.2. Let $(\mathbb{Z},+)$ be the Abelian group of integer numbers and t be an arbitrary nonzero element of $\mathbb{Z}$. Define a mapping "$\circ_6 : \mathbb{Z}_2 \times \mathbb{Z} \to P_*(\mathbb{R})$" by:

$$\forall x \in \mathbb{Z},\ 0 \circ_6 x = \{0\}, \qquad \text{and} \qquad 1 \circ_6 x = \begin{cases} \{1, x\} & x \in \mathbb{Z}\backslash\{t\}, \\ \{t+1, \pi\} & x = t. \end{cases} \tag{4}$$

Then $V_6 = (\mathbb{Z}, +, \circ_6, \mathbb{Z}_2)$ is a NeutroHyperVector space over the field K. Note that "$\circ_6$" is an external NeutroHyperoperation, since $1 \circ_6 t \nsubseteq \mathbb{Z}$. Also, all axioms (HV$_1$)-(HV$_5$) are replaced by the NeutroAxioms (NHV$_1$)-(NHV$_5$); more details are listed below, choosing $t = 6$:

NHV$_1$) $1 \circ_6 (0+0) = 1 \circ_6 0 = \{0,1\} \subseteq 1 \circ_6 0 + 1 \circ_6 0 = \{0,1\} + \{0,1\} = \{0,1,2\}$ and $1 \circ_6 (2+3) = 1 \circ_6 5 = \{1,5\} \nsubseteq 1 \circ_6 2 + 1 \circ_6 3 = \{1,2\} + \{1,3\} = \{2,3,4,5\}$.

NHV$_2$) $(0+1) \circ_6 2 = 1 \circ_6 2 = \{1,2\} = 0 \circ_6 2 + 1 \circ_6 2 = \{0\} + \{1,2\}$, and $(1+1) \circ_6 2 = 0 \circ_6 2 = \{0\} \nsubseteq 1 \circ_6 2 + 1 \circ_6 2 = \{1,2\} + \{1,2\} = \{2,3,4\}$. Thus, V is strongly left distributive.

NHV$_3$) $0 \circ_6 (0 \circ_6 x) = 0 \circ_6 0 = \{0\} = (0) \circ_6 x$, for all $x \in V$, and $1 \circ_6 (0 \circ_6 2) = 1 \circ_6 0 = \{0,1\} \neq 0 \circ_6 2 = \{0\}$.

NHV$_4$) $0 \circ_6 (-x) = (-0) \circ_6 x = -(0 \circ_6 x) = \{0\}$, for all $x \in V$, but $1 \circ_6 (-2) = \{1,-2\} \neq (-1) \circ_6 2 = \{-1,2\}$, $(-1) \circ_6 2 = \{-1,2\} \neq -(1 \circ_6 2) = \{-1,-2\}$ and $1 \circ_6 (-2) = \{1,-2\} \neq -(1 \circ_6 2) = \{-1,-2\}$.

NHV$_5$) $x \in 1 \circ_6 x$, for all $x \in V\backslash\{t\}$ and $t \notin 1 \circ_6 t$.

Example 3.6. Consider the field $K = \mathbb{Z}_2 = \{0,1\}$ defined in Example 2.2. Define the operation "$+ : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{Z}^2$" by $(x,y) + (m,n) = (x+m, y+n)$. Then $(\mathbb{Z}^2,+)$ is an Abelian group. Choose $(x_0,y_0) \in \mathbb{Z}^2$ and define "$\circ_7 : \mathbb{Z}_2 \times \mathbb{Z}^2 \to P_*(\mathbb{R}^2)$" by

$$a \circ_7 (x,y) = \begin{cases} \{(0,0)\} & a = 0, \\ \{(1,1),(x,y)\} & a = 1, (x,y) \in \mathbb{Z}^2\backslash\{(x_0,y_0)\}, \\ \{(\pi,\pi),(x_0+1,y_0+1)\} & a = 1, (x,y) = (x_0,y_0). \end{cases} \tag{5}$$

Then $V_7 = (\mathbb{Z}^2, +, \circ_7, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space over the field $\mathbb{Z}_2$. In fact, "$\circ_7$" is an external NeutroHyperoperation, since $1 \circ_7 (x_0,y_0) \nsubseteq \mathbb{Z}^2$ and all NeutroAxioms (NHV$_1$)-(NHV$_5$) are satisfied:

NHV$_1$) $0 \circ_7 ((x,y)+(m,n)) = 0 \circ_7 (x+m,y+n) = \{(0,0)\} = 0 \circ_7 (x,y) + 0 \circ_7 (m,n) = \{(0,0)\}+\{(0,0)\}$, and $1\circ_7((x,y)+(m,n)) = 1\circ_7(x+m,y+n) = \{(1,1),(x+m,y+n)\} \nsubseteq$

$1 \circ_7 (x, y) + 1 \circ_7 (m, n) = \{(1, 1), (x, y)\} + \{(1, 1), (m, n)\} = \{(2, 2), (x + 1, y + 1), (m + 1, n + 1), (x + m, y + n)\}$.

NHV$_2$) $(1+0) \circ_7 (x, y) = 1 \circ_7 (x, y) = \{(1, 1), (x, y)\} = 1 \circ_7 (x, y) + 0 \circ_7 (x, y) = \{(1, 1), (x, y)\} + \{(0, 0)\}$, for all $(x, y) \in \mathbb{Z}^2$ and $(1+1) \circ_7 (x, y) = 0 \circ_7 (x, y) = \{(0, 0)\} \nsubseteq 1 \circ_7 (x, y) + 1 \circ_7 (x, y) = \{(1, 1), (x, y)\} + \{(1, 1), (x, y)\} = \{(2, 2), (x + 1, y + 1), (2x, 2y)\}$, for $(x, y) \in \mathbb{Z}^2 \backslash \{(0, 0), (-1, -1)\}$.

NHV$_3$) $1 \circ_7 (1 \circ_7 (x, y)) = 1 \circ_7 \{(1, 1), (x, y)\} = \{(1, 1), (x, y)\} = (1 \cdot 1) \circ_7 (x, y)$, and $1 \circ_7 (0 \circ_7 (x, y)) = 1 \circ_7 \{(0, 0)\} = \{(1, 1), (0, 0)\} \neq (1 \cdot 0) \circ_7 (x, y) = \{(0, 0)\}$.

NHV$_4$) $0 \circ_7 (-(x, y)) = (-0) \circ_7 (x, y) = -(0 \circ_7 (x, y)) = \{(0, 0)\}$, for all $(x, y) \in \mathbb{Z}^2$, but $1 \circ_7 (-(x, y)) = \{(1, 1), (-x, -y)\} \neq (-1) \circ_7 (x, y) = \{(1, 1), (x, y)\} \neq -(0 \circ_7 (x, y)) = \{(-1, -1), (-x, -y)\}$, for all $(x, y) \in \mathbb{Z}^2 \backslash \{(0, 0)\}$.

NHV$_5$) $(x, y) \in 1 \circ_7 (x, y)$, for all $(x, y) \in \mathbb{Z}^2 \backslash \{(x_0, y_0)\}$ and $(x_0, y_0) \notin 1 \circ_7 (x_0, y_0)$.

Definition 3.7. Let $(V, +, \circ, K)$ be a NeutroHyperVector space and H be a nonempty subset of V. Then H is called a *SubNeutroHyperspace* of V, if $(H, +, \circ, K)$ is itself a NeutroHyperVector space. In other words, H is a SubNeutroHyperspace of V if and only if the following conditions hold:

(1) $x - y \in H$, for all $x, y \in H$;

(2) $a \circ x \subseteq H$, for some $a \in K,\ x \in H$ and ($b \circ y \nsubseteq H$ or $c \circ z$ is indeterminate in H, for some $b, c \in K,\ y, z \in H$);

(3) at least one of the NeutroAxioms of the Definition 3.3, is satisfied for H.

It is clear that every NeutroHyperVector space is a SubNeutroHyperspace of itself. The NeutroHyperVector spaces $(\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$, $(\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$ and $(\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ defined in Example 3.4, do not have any proper SubNeutroHyperspace. If we define $0 \circ 0 = \{0\}$ or $0 \circ 0 = \mathbb{Z}_3$, then $\{0\}$ is the only proper SubNeutroHyperspace of $(\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$. In the following examples, some nontrivial SubNeutroHyperspaces are presented:

Example 3.8. Consider the field $K = \mathbb{Z}_2 = \{0, 1\}$ defined in Example 2.2, $V = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ and $U = \{0, 1, 2, 3, 4, 5\}$. Then $V_8 = (\mathbb{Z}_4, +, \circ_8, \mathbb{Z}_2)$ is a strongly distributive NeutroHyperVector space over the field $\mathbb{Z}_2$, where the operation "$+ : V \times V \to V$" and the external NeutroHyperoperation "$\circ_8 : K \times V \to P_*(U)$" are defined by

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\circ_8$	0	1	2	3
0	$\{0\}$	$\{0, 1\}$	$\{0, 2\}$	$\{3, 5\}$
1	$\{1, 2, 3\}$	$\{1\}$	$\{1, 2\}$	$\{2, 3\}$

One can see that, $H = \{0, 2\}$ is a SubNeutroHyperspace of V, since $x - y \in H$, for all $x, y \in H$, and $0 \circ_8 2 = \{0, 2\} \subseteq H$, and $1 \circ_8 0 = \{1, 2, 3\} \nsubseteq H$. Also, $(H, +, \circ_8, K)$ is a NeutroHyperVector space over the field K, where the operation "$+ : H \times H \to H$" and the external NeutroHyperoperation "$\circ_8 : K \times H \to P_*(U)$" are defined by

$+$	0	2
0	0	2
2	2	0

$\circ_8$	0	2
0	$\{0\}$	$\{0, 2\}$
1	$\{1, 2, 3\}$	$\{1, 2\}$

In fact, $0 \circ_8 (0+0) = 0 \circ_8 0 = \{0\} = 0 \circ_8 0 + 0 \circ_8 0 = \{0\} + \{0\}$, and $1 \circ_8 2 + 1 \circ_8 2$ is indeterminate in H. $(0+0) \circ_8 0 = 0 \circ_8 0 = \{0\} = 0 \circ_8 0 + 0 \circ_8 0 = \{0\} + \{0\}$, and $1 \circ_8 0 + 0 \circ_8 0$ is indeterminate in H. $0 \circ_8 (0 \circ_8 0) = 0 \circ_8 (\{0\}) = \{0\} = (00) \circ_8 0$, and $1 \circ_8 (0 \circ_8 2) = 1 \circ_8 (\{0, 2\}) = \{1, 2, 3\} \neq (1 \cdot 0) \circ_8 2 = \{0, 2\}$. $0 \circ_8 (-2) = 0 \circ_8 2 = \{0, 2\} = (-0) \circ_8 2 = -(0 \circ_8 2)$, and $-(1 \circ_8 2)$ is indeterminate in H. $2 \in 1 \circ_8 2$ and $0 \notin 1 \circ_8 0$.

Example 3.9. For every $m \in \mathbb{Z} \setminus \{\pm 1\}$, the set $m\mathbb{Z} = \{mn : n \in \mathbb{Z}\}$ is a proper SubNeutroHyperspace of $(\mathbb{Z}, +, \circ_6, \mathbb{Z}_2)$, defined in Example 3.5, such that $0 \circ_6 mn = \{0\} \subseteq m\mathbb{Z}$, $1 \circ_6 mn = \{1, mn\} \nsubseteq m\mathbb{Z}$, for all $mn \in m\mathbb{Z}$ and so the restriction "$\circ_6$" into $m\mathbb{Z}$ is a NeutroHyperoperaion. Also, all NeutroAxioms (NHV_1)-(NHV_5) are satisfied:

NHV_1) $0 \circ_6 (mn + m\acute{n}) = \{0\} = 0 \circ_6 mn + 0 \circ_6 m\acute{n}$ and $1 \circ_6 (mn + m\acute{n}) = \{1, m(n + \acute{n})\} \nsubseteq 1 \circ_6 mn + 1 \circ_6 m\acute{n} = \{1, mn\} + \{1, m\acute{n}\} = \{2, mn + 1, m\acute{n} + 1, m(n + \acute{n})\}$, for all $mn, m\acute{n} \in m\mathbb{Z}$. Then $m\mathbb{Z}$ is strongly right distributive.

NHV_2) $(0+0) \circ_6 mn = 0 \circ_6 mn = \{0\} = 0 \circ_6 mn + 0 \circ_6 mn$, and $(1+1) \circ_6 mn = 0 \circ_6 mn = \{0\} \nsubseteq 1 \circ_6 mn + 1 \circ_6 mn = \{1, mn\} + \{1, mn\} = \{2, mn + 1, 2mn\}$, for all $mn \in m\mathbb{Z}$. So $m\mathbb{Z}$ is strongly left distributive.

NHV_3) $0 \circ_6 (0 \circ_6 mn) = 0 \circ_6 0 = \{0\} = (0) \circ_6 mn$, and $1 \circ_6 (0 \circ_6 mn) = 1 \circ_6 0 = \{0, 1\} \neq 0 \circ_6 mn = \{0\}$, for all $mn \in m\mathbb{Z}$.

NHV_4) $0 \circ_6 (-mn) = (-0) \circ_6 mn = -(0 \circ_6 mn) = \{0\}$, but $1 \circ_6 (-mn) = \{1, -mn\} \neq (-1) \circ_6 mn = \{-1, mn\}$, $(-1) \circ_6 mn = \{-1, mn\} \neq -(1 \circ_6 mn) = \{-1, -mn\}$ and $1 \circ_6 (-mn) = \{1, -mn\} \neq -(1 \circ_6 mn) = \{-1, -mn\}$, for all $mn \in m\mathbb{Z}$.

NHV_5) $mn \in 1 \circ_6 mn$, for all $mn \in m\mathbb{Z} \setminus \{t\}$ and $t \notin 1 \circ_6 t$.

Example 3.10. The sets $H = \{(x, 0); x \in \mathbb{Z}\}$ and $L = \{(0, y); y \in \mathbb{Z}\}$ are proper SubNeutroHyperspaces of $(\mathbb{Z}^2, +, \circ_7, \mathbb{Z}_2)$, defined in Example 3.6. The restrictions of "$\circ_7$" into H and L, are NeutroHyperoperaions, since $0 \circ_7 (x, 0) = \{(0, 0)\} \subseteq H$ and $0 \circ_7 (0, y) = \{(0, 0)\} \subseteq L$, but $1 \circ_7 (x, 0) = \{(1, 1), (x, 0)\} \nsubseteq H$ and $1 \circ_7 (0, y) = \{(1, 1), (0, y)\} \nsubseteq L$. The NeutroAxioms ($NHV_1$)-($NHV_4$) and the axiom ($HV_5$) for $(H, +, \circ_7, \mathbb{Z}_2)$ are given in the following (details for $(L, +, \circ_7, \mathbb{Z}_2)$ are similar):

NHV$_1$) $0 \circ_7 ((x,0)+(m,0)) = 0 \circ_7 (x+m,0) = \{(0,0)\} = 0 \circ_7 (x,0) + 0 \circ_7 (m,0) = \{(0,0)\} + \{(0,0)\}$, and $1 \circ_7 ((x,0)+(m,0))$ is indeterminate in H.

NHV$_2$) $(0+0) \circ_7 (x,0) = \{(0,0)\} = 0 \circ_7 (x,0) + 0 \circ_7 (x,0)$ and $1 \circ_7 (x,0) + 1 \circ_7 (x,0)$ is indeterminate in H, for $x \in \mathbb{Z}\backslash\{-1\}$.

NHV$_3$) $1 \circ_7 (1 \circ_7 (x,0)) = 1 \circ_7 \{(1,1),(x,0)\} = \{(1,1),(x,0)\} = (1 \cdot 1) \circ_7 (x,y)$, and $1 \circ_7 (0 \circ_7 (x,0)) = 1 \circ_7 \{(0,0)\}$ is indeterminate in H.

NHV$_4$) $0 \circ_7 (-(x,0)) = (-0) \circ_7 (x,0) = -(0 \circ_7 (x,0)) = \{(0,0)\}$, for all $(x,y) \in \mathbb{Z}^2$, but $1 \circ_7 (-(x,0))$, $(-1) \circ_7 (x,0)$ and indeterminate in H.

HV$_5$) $(x,0) \in 1 \circ_7 (x,0)$, for all $(x,0) \in H$.

It is well-known that if H, L are subgroups of $(V,+)$, then $H \cap L$ is a subgroup of V, but $H \cup L$ is a subgroup of V if and only if $H \subseteq L$ or $L \subseteq H$. Now, if H, L are SubNeutroHyperspaces of V, then $H \cap L$ may be a SubNeutroHyperspace of V. For example, the intersection of two arbitrary SubNeutroHyperspaces $m\mathbb{Z}, n\mathbb{Z}$ of the NeutroHyperVector space $(\mathbb{Z},+,\circ_6,\mathbb{Z}_2)$, presented in Example 3.9, is the SubNeutroHyperspace $[m,n]\mathbb{Z}$ of $(\mathbb{Z},+,\circ_6,\mathbb{Z}_2)$, where $[m,n]$ is the smallest common multiplication of m,n. Also, the intersection of the SubNeutroHyperspaces $H = \{(x,0);\ x \in \mathbb{Z}\}$ and $L = \{(0,y);\ y \in \mathbb{Z}\}$ of $(\mathbb{Z}^2,+,\circ,\mathbb{Z}_2)$, defined in Example 3.10, is the SubNeutroHyperspace $\{(0,0)\}$. Moreover, similar to the groups, $H \cup L$ is a SubNeutroHyperspace of V if and only if $H \subseteq L$ or $L \subseteq H$.

4. NeutroLinearTransformations

In this section, some types of transformations between NeutroHyperVector spaces are introduced and some properties of mentioned concepts are studied, supported by some examples.

Definition 4.1. Let $(V,+,\circ,K)$ and $(W,\acute{+},\acute{\circ},K)$ be NeutroHyperVector spaces over the field K. Then a mapping $T : V \longrightarrow W$ is called

(1) *NeutroLinearTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$ and $T(a \circ x) \subseteq a\acute{\circ}T(x)$, for some $a \in K,\ x \in V$;

(2) *NeutroGoodTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$ and $T(a \circ x) = a\acute{\circ}T(x)$, for some $a \in K,\ x \in V$;

(3) *NeutroStrongLinearTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$, $T(a \circ x) \subseteq a\acute{\circ}T(x)$ when $a \circ x \subseteq V$, $a\acute{\circ}T(x) \nsubseteq W$ when $a \circ x \nsubseteq V$, and $a\acute{\circ}T(x)$ is indeterminate in W when $a \circ x$ is indeterminate in V;

(4) *NeutroStrongGoodLinearTransformation*, if $T(x+y) = T(x)\acute{+}T(y)$, for all $x,y \in V$, $T(a \circ x) = a\acute{\circ}T(x)$ when $a \circ x \subseteq V$, $a\acute{\circ}T(x) \nsubseteq W$ when $a \circ x \nsubseteq V$, and $a\acute{\circ}T(x)$ is indeterminate in W when $a \circ x$ is indeterminate in V;

(5) *NeutroStrongGoodIsomorphism*, if T is a bijective NeutroStrongGoodLinearTransformation. In this case, V and W are called NeutroIsomorphic and it is denoted by $V \underset{NS}{\cong} W$.

Proposition 4.2. *If $(V, +, \circ, K)$ is a NeutroHyperVector space over the field K, then the identity function $i_V : V \to V$ is a NeutroStrongGoodIsomorphism.*

Proof. It is clear that i_V is a bijection and $i_V(x + y) = i_V(x) + i_V(y)$, for all $x, y \in V$. Now, if $a \circ x \subseteq V$, then $T(a \circ x) = \{T(t); t \in a \circ x\} = a \circ x = a \circ T(x)$, and if $a \circ x \nsubseteq V$, then $a \circ T(x) \nsubseteq V$. Also, $a \circ T(x)$ is indeterminate in V, when $a \circ x$ is indeterminate in V. □

Example 4.3. Consider the NeutroHyperVector spaces $V_3 = (\{0, 1, 2\}, +, \circ_3, K)$, $V_4 = (\{0, 1, 2\}, +, \circ_4, K)$ defined in Example 3.4. Then the mappings $T_{34} : V_3 \to V_4$ and $S_{34} : V_3 \to V_4$ with $T_{34}(x) = x$ and $S_{34}(x) = 2x$ are NeutroGoodTransformations, since $T_{34}(x + y) = x + y = T_{34}(x) + T_{34}(y)$ and $S_{34}(x + y) = 2(x + y) = 2x + 2y = S_{34}(x) + S_{34}(y)$, for all $x, y \in V_3$. Also, $T_{34}(1 \circ_3 0) = 1 \circ_4 T_{34}(0)$ and $S_{34}(0 \circ_3 1) = 0 \circ_4 T_{34}(1)$. But both of them are not NeutroStrongLinearTransformations, because $0 \circ_3 0 \nsubseteq V_3$, while $0 \circ_4 T_{34}(0) \subseteq V_4$ and $0 \circ_4 S_{34}(0) \subseteq V_4$.

Example 4.4. Consider the field $K = \{0, 1\}$ and the Abelian group $V = \mathbb{Z}_3 = \{0, 1, 2\}$ defined in Example 2.2. Then similar to the Example 3.4, one can see that $V_9 = (\mathbb{Z}_3, +, \circ_9, K)$ and $V_{10} = (\mathbb{Z}_3, +, \circ_{10}, K)$ are NeutroHyperVector spaces over the field $\mathbb{Z}_2$, where the external NeutroHyperoperations $\circ_9, \circ_{10} : \mathbb{Z}_2 \times \mathbb{Z}_3 \to P_*(U)$ are defined by the following tables:

$\circ_9$	0	1	2
0	$\{0\}$	$\{1, 2\}$	$\{1, 2\}$
1	$\{1\}$		$\{0, 2\}$

$\circ_{10}$	0	1	2
0	$\{0\}$	$\{0, 1, 2\}$	$\{0, 1, 2\}$
1	$\{0, 1\}$		$\{0, 2\}$

Now, define the mapping $\acute{T} : V_9 \to V_{10}$ by $\acute{T}(x) = x$. Then $\acute{T}$ is a NeutroStrongLinearTransformation which is not a NeutroStrongGoodLinearTransformation, since $\acute{T}(x+y) = \acute{T}(x)+\acute{T}(y)$, for all $x, y \in V_9$. Also, $\acute{T}(0 \circ_9 0) = \{0\} = 0 \circ_{10} \acute{T}(0)$, $\acute{T}(0 \circ_9 1) = \acute{T}(\{1, 2\}) = \{1, 2\} \subseteq \{0, 1, 2\} = 0 \circ_{10} \acute{T}(1)$, $\acute{T}(0 \circ_9 2) = \acute{T}(\{1, 2\}) = \{1, 2\} \subseteq \{0, 1, 2\} = 0 \circ_{10} \acute{T}(2)$, $\acute{T}(1 \circ_9 0) = \acute{T}(\{1\}) = \{1\} \subseteq \{0, 1\} = 1 \circ_{10} \acute{T}(0)$ and $\acute{T}(1 \circ_9 2) = \acute{T}(\{0, 2\}) = \{0, 2\} = 1 \circ_{10} \acute{T}(2)$, Moreover, $1 \circ_9 1$ and $1 \circ_{10} \acute{T}(1)$ are indeterminate in V;

Theorem 4.5. *$\underset{NS}{\cong}$ is an equivalence relation on the set of NeutroHyperVector spaces.*

Proof. By Proposition 4.2, the identity function $i_V : V \to V$ is a NeutroStrongGoodIsomorphism, i.e. $V \underset{NS}{\cong} V$ and so $\underset{NS}{\cong}$ is reflexive. If $V \underset{NS}{\cong} W$, where $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$, then there exists a NeutroStrongGoodIsomorphism $T : V \to W$. We show that $T^{-1} : W \to V$ is a NeutroStrongGoodIsomorphism. It is clear that T^{-1} is bijective and

$T^{-1}(w_1 \acute{+} w_2) = T^{-1}(w_1) + T^{-1}(w_2)$, for all $w_1, w_2 \in W$. For any $w \in W$, there exist $v \in V$ such that $T(v) = w$, so $T^{-1}(a \acute{\circ} w) = T^{-1}(a \acute{\circ} T(v))$, for all $a \in K$. We must check the following three cases:

(1) $a \acute{\circ} w \subseteq W$: In this case, $a \circ v \subseteq V$ (if $a \circ v \nsubseteq V$, then $a \acute{\circ} w = a \acute{\circ} T(v) \nsubseteq W$) and so

$$T^{-1}(a \acute{\circ} w) = T^{-1}(a \acute{\circ} T(v)) = T^{-1}(T(a \circ v)) = a \circ v = a \circ T^{-1}(w).$$

(2) $a \acute{\circ} w \nsubseteq W$: Let $a \circ v = a \circ T^{-1}(w) \subseteq V$ or $a \circ v = a \circ T^{-1}(w)$ is indeterminate in V. If $a \circ v \subseteq V$, then $T(a \circ v) = a \acute{\circ} T(v) = a \acute{\circ} w \subseteq T(v) \subseteq W$, which is a contradiction. Also, if $a \circ v$ is indeterminate in V, then $a \acute{\circ} T(v) = a \acute{\circ} w$ is indeterminate in W, which is a contradiction, too. Thus in this case, $a \circ T^{-1}(w) \nsubseteq V$.

(3) $a \acute{\circ} w$ is indeterminate in W: If $a \circ v \subseteq V$, then $a \acute{\circ} w = a \acute{\circ} T(v) = T(a \circ v) \subseteq T(V) \subseteq W$, which is a contradiction. Also, if $a \circ v \nsubseteq V$, then $a \acute{\circ} T(v) \nsubseteq W$, which is a contradiction, too. Hence in this case, $a \circ T^{-1}(w)$ is indeterminate in V.

Consequently, T^{-1} is a NeutroStrongGoodIsomorphism and so $\underset{NS}{\cong}$ is symmetric.
Now let $V \underset{NS}{\cong} W$ and $W \underset{NS}{\cong} U$. Then there exist NeutroStrongGoodIsomorphisms $T : V \to W$ and $S : W \to U$. We shall prove that $S \circ T : V \to U$ is a NeutroStrongGoodIsomorphism. It is clear that, $S \circ T$ is bijective and $(S \circ T)(x + y) = S(T(x + y)) = S(T(x) + T(y)) = S(T(x)) + S(T(y))$, for all $x, y \in V$. Suppose $a \in K$ and $x \in V$. If $a \circ x \subseteq V$, then by using the hypothesis that T and S are NeutroStrongGoodIsomorphisms, $a \circ T(x) = T(a \circ x) \subseteq T(V) \subseteq W$ and $S(a \circ T(x)) = a \circ S(T(x)) = a \circ (S \circ T)(x)$. If $a \circ x \nsubseteq V$, then $a \circ T(x) \nsubseteq W$, and so $a \circ (S \circ T)(x) \nsubseteq U$. If $a \circ x$ is indeterminate in V, then $a \circ T(x)$ is indeterminate in W and so $a \circ (S \circ T)(x)$ is indeterminate in U. Hence, $V \underset{NS}{\cong} U$. Therefore, $\underset{NS}{\cong}$ is transitive.
Consequentely, $\underset{NS}{\cong}$ is an equivalence relation. □

Theorem 4.6. *Let $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ be NeutroHyperVector spaces over the field K and $T : V \to W$ be an injective NeutroStrongGoodTransformation. If H is a SubNeutroHyperspace of V, then $T(H)$ is a SubNeutroHyperspace of W.*

Proof. For all $T(x), T(y) \in T(H)$, $x - y \in H$ and so $T(x) \acute{-} T(y) = T(x - y) \in T(H)$. Also, there exist $a \in K$ and $x \in H$ such that $a \circ x \subseteq H$. Then $T(x) \in T(H)$ and $a \acute{\circ} T(x) = T(a \circ x) \subseteq T(H)$. Moreover, if $b \circ y \nsubseteq H$, for some $b \in K$, $y \in H$, then by injectivity of T, it follows that $b \circ T(y) \nsubseteq T(H)$. Also, if $c \circ z$ is indeterminate in V, for some $c \in K$, $z \in H$, then $c \acute{\circ} T(z)$ is indeterminate in W. Now we show that if H is satisfied in every NeutroAxiom of Definition 3.3, then $T(H)$ is satisfied in the same NeutroAxiom:

NHV$_1$) $a \circ (x+y) \subseteq a \circ x + a \circ y$, for some $a \in K,\ x, y \in V$. Thus

$$\begin{aligned} a\acute{\circ}(T(x)\acute{+}T(y)) &= a\acute{\circ}T(x+y) \\ &= T(a \circ (x+y)) \\ &\subseteq T(a \circ x + a \circ y) \\ &= T(a \circ x)\acute{+}T(a \circ y) \\ &= a\acute{\circ}T(x)\acute{+}a\acute{\circ}T(y). \end{aligned}$$

Also, if $\acute{a} \circ (\acute{x}+\acute{y}) \nsubseteq \acute{a} \circ \acute{x} + \acute{a} \circ \acute{y}$, for some $\acute{a} \in K$, $\acute{x}, \acute{y} \in V$, then by injectivity of T, it follows that $\acute{a}\acute{\circ}(T(\acute{x})\acute{+}T(\acute{y})) \nsubseteq \acute{a}\acute{\circ}T(\acute{x})\acute{+}\acute{a}\acute{\circ}T(\acute{y})$. Moreover, if for some $\hat{a} \in K,\ \hat{x}, \hat{y} \in V$, $\hat{a} \circ (\hat{x}+\hat{y})$ is indeterminate in H, or $\hat{a} \circ \hat{x} + \hat{a} \circ \hat{y}$ is indeterminate in H, or we can not find if $\hat{a} \circ (\hat{x}+\hat{y})$ is a subset of $\hat{a} \circ \hat{x} + \hat{a} \circ \hat{y}$, then $\hat{a}\acute{\circ}(T(\hat{x})\acute{+}T(\hat{y}))$ is indeterminate in $T(H)$, or $\hat{a}\acute{\circ}T(\hat{x})\acute{+}\hat{a}\acute{\circ}T(\hat{y})$ is indeterminate in $T(H)$, or we can not find if $\hat{a}\acute{\circ}(T(\hat{x})\acute{+}T(\hat{y}))$ is a subset of $\hat{a}\acute{\circ}T(\hat{x})\acute{+}\hat{a}\acute{\circ}T(\hat{y})$, respectively.

NHV$_2$) $(a+b) \circ x \subseteq a \circ x + b \circ x$, for some $a, b \in K,\ x \in H$. Then

$$\begin{aligned} (a+b)\acute{\circ}T(x) &= T((a+b) \circ x) \\ &\subseteq T(a \circ x + b \circ x) \\ &= T(a \circ x)\acute{+}T(b \circ x) \\ &= a\acute{\circ}T(x)\acute{+}b\acute{\circ}T(x). \end{aligned}$$

Also, if $(\acute{a}+\acute{b}) \circ \acute{x} \nsubseteq \acute{a} \circ \acute{x} + \acute{b} \circ \acute{x}$, for some $\acute{a}, \acute{b} \in K$, $\acute{x} \in H$, then $(\acute{a}+\acute{b})\acute{\circ}T(\acute{x}) \nsubseteq \acute{a}\acute{\circ}T(\acute{x})\acute{+}\acute{b}\acute{\circ}T(\acute{x})$. Moreover, if $(\hat{a}+\hat{b}) \circ \hat{x}$ is indeterminate in H, or $\hat{a} \circ \hat{x} + \hat{b} \circ \hat{x}$ is indeterminate in H, or we can not find if $(\hat{a}+\hat{b}) \circ \hat{x}$ is a subset of $\hat{a} \circ \hat{x} + \hat{b} \circ \hat{x}$, for some $\hat{a}, \hat{b} \in K,\ \hat{x} \in H$, then $(\hat{a}+\hat{b})\acute{\circ}T(\hat{x})$ is indeterminate in $T(H)$, or $\hat{a}\acute{\circ}T(\hat{x}) + \hat{b}\acute{\circ}T(\hat{x})$ is indeterminate in $T(H)$, or we can not find if $(\hat{a}+\hat{b})\acute{\circ}T(\hat{x})$ is a subset of $\hat{a}\acute{\circ}T(\hat{x})\acute{+}\hat{b}\acute{\circ}T(\hat{x})$, respectively.

NHV$_3$) $a \circ (b \circ x) = (ab) \circ x$, for some $a, b \in K,\ x \in H$. Thus

$$a\acute{\circ}(b\acute{\circ}T(x)) = a\acute{\circ}(T(b \circ x)) = T(a \circ (b \circ x)) = T((ab) \circ x) = (ab)\acute{\circ}T(x).$$

Also, if $a \circ (b \circ x) \neq (ab) \circ x$, for some $a, b \in K,\ x \in H$, then $a\acute{\circ}(b\acute{\circ}T(x)) \neq (ab)\acute{\circ}T(x)$. Moreover, if $a \circ (b \circ x)$ is indeterminate in H, or $(ab) \circ x$ is indeterminate in H, or we can not find if $a \circ (b \circ x) = (ab) \circ x$ or $a \circ (b \circ x) \neq (ab) \circ x$, for some $a, b \in K,\ x \in H$, then $a\acute{\circ}(b\acute{\circ}T(x))$ is indeterminate in $T(H)$, or $(ab)\acute{\circ}T(x)$ is indeterminate in $T(H)$, or we can not find if $a\acute{\circ}(b\acute{\circ}T(x)) = (ab)\acute{\circ}T(x)$ or $a\acute{\circ}(b\acute{\circ}T(x)) \neq (ab)\acute{\circ}T(x)$.

NHV$_4$) $a \circ (-x) = (-a) \circ x = -(a \circ x)$, for some $a \in K,\ x \in H$. Thus

$$\begin{aligned} a\acute{\circ}(-T(x)) &= a\acute{\circ}(T(-x)) = T(a \circ (-x)) = T((-a) \circ x) = (-a)\acute{\circ}T(-x) \\ &= T(-(a \circ x)) = -(T(a \circ x)) = -(a\acute{\circ}T(x)). \end{aligned}$$

Also, if $a \circ (-x) \neq (-a) \circ x$ or $(-a) \circ x \neq -(a \circ x)$ or $a \circ (-x) \neq -(a \circ x)$, for some $a \in K,\ x \in H$, then $a\acute{\circ}(-T(x)) \neq (-a)\acute{\circ}T(x)$ or $(-a)\acute{\circ}T(x) \neq -(a\acute{\circ}T(x))$ or $a\acute{\circ}(-T(x)) \neq -(a\acute{\circ}T(x))$, respectively. Moreover, if $a \circ (-x)$ or $(-a) \circ x$ or $-(a \circ x)$ is indeterminate in H, for some $a \in K,\ x \in H$, or we can not find if two of them are equal, then $a\acute{\circ}(-T(x))$ or $(-a)\acute{\circ}T(x)$ or $-(a\acute{\circ}T(x))$ is indeterminate in $T(H)$, or we can not find if they are equal, respectively.

NHV$_5$) $x \in 1 \circ x$, for some $x \in H$. Thus $T(x) \in T(1 \circ x) = 1\acute{\circ}T(x)$. Also, if $y \notin 1 \circ y$, for some $y \in H$, then $T(y) \notin 1\acute{\circ}T(y)$. Moreover, if $1 \circ z$ is indeterminate or we can not find if $z \in 1 \circ z$ or $z \notin 1 \circ z$, then $1\acute{\circ}T(z)$ is indeterminate or we can not find if $T(z) \in 1\acute{\circ}T(z)$ or $T(z) \notin 1\acute{\circ}T(z)$.

Therefore, by Definition 3.7, $T(H)$ is a SubNeutroHyperspace of W. □

Theorem 4.7. *Let $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ be NeutroHyperVector spaces over the field K and $T : V \to W$ be a NeutroStrongGoodIsomorphism. If L is a SubNeutroHyperspace of W, then $T^{-1}(L)$ is a SubNeutroHyperspace of V.*

Proof. For all $x, y \in T^{-1}(L)$, $T(x), T(y) \in L$ and so $T(x - y) = T(x) - T(y) \in L$. Then $x - y \in T^{-1}(L)$. Also, there exist $a \in K,\ y \in L$ such that $a \circ y \subseteq L$. Since T is surjective, $y = T(x)$, for some $x \in V$. Thus $T(a \circ x) = a \circ T(x) = a \circ y \subseteq L$ and so $a \circ x \subseteq T^{-1}(L)$, where $x \in T^{-1}(L)$. Moreover, if $a \circ y \nsubseteq L$, for some $a \in K,\ y \in L$, then $a \circ x \nsubseteq T^{-1}(L)$, for some $x \in V$ such that $T(x) = y$. Next, if $a \circ y$ is indeterminate, for some $a \in K,\ y \in L$, then $a \circ x$ is indeterminate, for some $x \in T^{-1}(y)$. One can similar to the proof of Theorem 4.6, show that if L is satisfied in each NeutroAxiom of Definition 3.3, then $T^{-1}(L)$ is satisfied in the same NeutroAxiom. Therefore, $T^{-1}(L)$ is a SubNeutroHyperspace of V. □

Corollary 4.8. *Let $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ be NeutroHyperVector spaces over the field K and $T : V \to W$ be a NeutroStrongGoodIsomorphism. Then H is a SubNeutroHyperspace of V, if and only if $T(H)$ is a SubNeutroHyperspace of W.*

Proof. It follows from Theorems 4.5, 4.6, and 4.7. □

Corollary 4.9. *If $V = (V, +, \circ, K)$ and $W = (W, \acute{+}, \acute{\circ}, K)$ are NeutroHyperVector spaces over the field K such that $V \underset{NS}{\cong} W$, then V has no proper SubNeutroHyperspace, if and only if W has no proper SubNeutroHyperspace.*

Proof. It follows from Corollary 4.8. □

5. Cartesian Product of NeutroHyperVector Spaces

In order to make the Cartesian product of NeutroHyperVector spaces over a common field K, we need a suitable field, that is the Cartesian product $K \times K$ with the following operations:

$$(a,b)+(c,d)=(a+c,b+d), \quad (a,b)(c,d)=(ac-bd,ad-bc). \tag{6}$$

The zero and the multiplicative identity of the field $K \times K$ are $(0,0)$ and $(1,0)$, respectively.

Theorem 5.1. *If $V_1 = (V_1, +_1, \circ_1, K)$ and $V_2 = (V_2, +_2, \circ_2, K)$ are NeutroHyperVector spaces over the field K such that are satisfied in the same NeutroAxioms, $k \circ_1 (0 \circ_1 v_1) = 0 \circ_1 v_1$, $1 \circ_2 (0 \circ_2 v_2) = 0 \circ_2 v_2$ and $0_{V_2} \in 0 \circ_2 \acute{v}_2$, for some $k \in K$, $v_1 \in V_1$ and $v_2, \acute{v}_2 \in V_2$, then $V_1 \times V_2 = (V_1 \times V_2, +, \circ, K \times K)$ is a NeutroHyperVector space over the field $K \times K$, where*

$$\begin{array}{c}(x_1,x_2)+(y_1,y_2)=(x_1 +_1 y_1, x_2 +_2 y_2),\\ (a_1,a_2)\circ(x_1,x_2)=\{(r,s);\ r \in a_1 \circ_1 x_1, s \in a_2 \circ_2 x_2\}.\end{array} \tag{7}$$

Proof. It is easy to see that $(V_1 \times V_2, +)$ is an Abelian group. Since "$\circ_1$" and "$\circ_2$" are external NeutroHyperoperations, there exist $a_1, a_2 \in K$, $x_1 \in V_1$, $x_2 \in V_2$, such that $a_1 \circ_1 x_1 \subseteq V_1$ and $a_2 \circ_2 x_2 \subseteq V_2$. Then

$$(a_1,a_2)\circ(x_1,x_2)=\{(r,s);\ r \in a_1 \circ_1 x_1, s \in a_2 \circ_2 x_2\} \subseteq V_1 \times V_2.$$

If $a_1 \circ_1 x_1 \nsubseteq V_1$, for some $a_1 \in K$, $x_1 \in V_1$, then $(a_1,a_2)\circ(x_1,x_2) \nsubseteq V_1 \times V_2$, for all $a_2 \in K$, $x_2 \in V_2$. If $a_1 \circ_1 x_1$ is indeterminate in V_1, for some $a_1 \in K$, $x_1 \in V_1$, then $(a_1,a_2)\circ(x_1,x_2)$ is indeterminate in $V_1 \times V_2$, for all $a_2 \in K$, $x_2 \in V_2$. Similarly, If $a_2 \circ_2 x_2 \nsubseteq V_2$, for some $a_2 \in K$, $x_2 \in V_2$, then $(a_1,a_2)\circ(x_1,x_2) \nsubseteq V_1 \times V_2$, for all $a_1 \in K$, $x_1 \in V_1$ and if $a_2 \circ_2 x_2$ is indeterminate in V_2, for some $a_2 \in K$, $x_2 \in V_2$, then $(a_1,a_2)\circ(x_1,x_2)$ is indeterminate in $V_1 \times V_2$, for all $a_1 \in K$, $x_1 \in V_1$. Hence "$\circ$" is an external NeutroHyperoperation on $V_1 \times V_2$. Now we show that if V_1, V_2 are satisfied in each NeutroAxioms of Definition 3.3, then $V_1 \times V_2$ is satisfied in the same NeutroAxiom.

NHV$_1$) If $a_1 \circ_1 (x_1 +_1 y_1) \subseteq a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$, and $a_2 \circ_2 (x_2 +_2 y_2) \subseteq a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then

$$\begin{aligned}
(a_1,a_2)\circ((x_1,x_2)+(y_1,y_2)) &= (a_1,a_2)\circ(x_1 +_1 y_1, x_2 +_2 y_2)\\
&= \{(r,s);\ r \in a_1 \circ_1 (x_1 +_1 y_1), s \in a_2 \circ_2 (x_2 +_2 y_2)\}\\
&\subseteq \{(r,s);\ r \in a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1, s \in a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2\}\\
&= \left\{\begin{array}{c}(r_1 +_1 \acute{r}_1, s_2 +_2 \acute{s}_2);\ r_1 \in a_1 \circ_1 x_1, \acute{r}_1 \in a_1 \circ_1 y_1,\\ s_2 \in a_2 \circ_2 x_2, \acute{s}_2 \in a_2 \circ_2 y_2\end{array}\right\}\\
&= \{(r_1,s_2);\ r_1 \in a_1 \circ_1 x_1, s_2 \in a_2 \circ_2 x_2\}\\
&\quad + \{(\acute{r}_1,\acute{s}_2);\ \acute{r}_1 \in a_1 \circ_1 y_1, \acute{s}_2 \in a_2 \circ_2 y_2\}\\
&= (a_1,a_2)\circ(x_1,x_2)+(a_1,a_2)\circ(y_1,y_2).
\end{aligned}$$

If $a_1 \circ_1 (x_1 +_1 y_1) \nsubseteq a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$, and $a_2 \circ_2 (x_2 +_2 y_2) \nsubseteq a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then

$$(a_1, a_2) \circ ((x_1, x_2) + (y_1, y_2)) \nsubseteq (a_1, a_2) \circ (x_1, x_2) + (a_1, a_2) \circ (y_1, y_2).$$

If $a_1 \circ_1 (x_1 +_1 y_1)$ and $a_2 \circ_2 (x_2 +_2 y_2)$ are indeterminate in V_1 and V_2, respectively, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then $(a_1, a_2) \circ ((x_1, x_2) + (y_1, y_2))$ is indeterminate in $V_1 \times V_2$.

If $a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$ and $a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$ are indeterminate in V_1 and V_2, respectively, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then $(a_1, a_2) \circ (x_1, x_2) + (a_1, a_2) \circ (y_1, y_2)$ is indeterminate in $V_1 \times V_2$.

If we can not find if $a_1 \circ_1 (x_1 +_1 y_1)$ is a subset of $a_1 \circ_1 x_1 +_1 a_1 \circ_1 y_1$ and we can not find if $a_2 \circ_2 (x_2 +_2 y_2)$ is a subset of $a_2 \circ_2 x_2 +_2 a_2 \circ_2 y_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then we can not find if $(a_1, a_2) \circ ((x_1, x_2) + (y_1, y_2))$ is a subset of $(a_1, a_2) \circ (x_1, x_2) + (a_1, a_2) \circ (y_1, y_2)$.

NHV$_2$) It is similar to the NeutroAxiom (NHV$_1$).

NHV$_3$) By the hypothesis, there exist $k \in K$, $v_1 \in V_1$ and $v_2, \acute{v}_2 \in V_2$, such that $c \circ_1 (0 \circ_1 v_1) = 0 \circ_1 v_1$ and $1 \circ_2 (0 \circ_2 v_2) = 0 \circ_2 v_2$. Thus

$$\begin{aligned}
(k, 1) \circ ((0, 0) \circ (v_1, v_2)) &= \bigcup_{(r,s) \in (0,0) \circ (v_1, v_2)} (k, 1) \circ (r, s) \\
&= \bigcup_{\substack{p \in k \circ_1 r, q \in 1 \circ_2 s, \\ r \in 0 \circ_1 v_1, s \in 0 \circ_2 v_2}} (p, q) \\
&= \bigcup_{p \in k \circ_1 (0 \circ_1 v_1), q \in 1 \circ_2 (0 \circ_2 v_2)} (p, q) \\
&= \bigcup_{p \in 0 \circ_1 v_1, q \in 0 \circ_2 v_2} (p, q) \\
&= (0, 0) \circ (v_1, v_2) \\
&= ((k, 1)(0, 0)) \circ (v_1, v_2).
\end{aligned}$$

If $a \circ_1 (b \circ_1 x_1) \neq (ab) \circ_1 x_1$ and $c \circ_2 (d \circ_2 x_2) \neq (cd) \circ_2 x_2$, for some $a, b, c, d \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a, c) \circ ((b, d) \circ (x_1, x_2)) \neq ((a, c)(b, d)) \circ (x_1, x_2)$.

If $a \circ_1 (b \circ_1 x_1)$ and $c \circ_2 (d \circ_2 x_2)$ are indeterminate in V_1 and V_2, respectively, for some $a, b, c, d \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a, c) \circ ((b, d) \circ (x_1, x_2))$ is indeterminate in $V_1 \times V_2$.

If $(ab) \circ_1 x_1$ and $(cd) \circ_2 x_2$ are indeterminate in V_1 and V_2, respectively, for some $a, b, c, d \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $((a, c)(b, d)) \circ (x_1, x_2)$ is indeterminate in $V_1 \times V_2$.

If we can not find if $a \circ_1 (b \circ_1 x_1) = (ab) \circ_1 x_1$ and $c \circ_2 (d \circ_2 x_2) = (cd) \circ_2 x_2$, for some $a_1, a_2 \in K$, $x_1, y_1 \in V_1$, $x_2, y_2 \in V_2$, then we can not find if $(a, c) \circ ((b, d) \circ (x_1, x_2)) = ((a, c)(b, d)) \circ (x_1, x_2)$.

NHV$_4$) If $a \circ_1 (-x_1) = (-a) \circ_1 x_1 = -(a \circ_1 x_1)$ and $b \circ_2 (-x_2) = (-b) \circ_2 x_2 = -(b \circ_2 x_2)$, for some $a, b \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then

$$\begin{aligned}
(a,b) \circ (-(x_1,x_2)) &= (a,b) \circ (-x_1,-x_2) \\
&= \{(r,s); r \in a \circ_1 (-x_1), s \in b \circ_2 (-x_2)\} \\
&= \{(r,s); r \in (-a) \circ_1 x_1, s \in (-b) \circ_2 x_2\} \\
&= (-a,-b) \circ (x_1,x_2) \\
&= (-(a,b)) \circ (x_1,x_2),
\end{aligned}$$

and

$$\begin{aligned}
(a,b) \circ (-(x_1,x_2)) &= \{(r,s); r \in a \circ_1 (-x_1), s \in b \circ_2 (-x_2)\} \\
&= \{(r,s); r \in -(a \circ_1 x_1), s \in -(b \circ_2 x_2)\} \\
&= \{(-\acute{r},-\acute{s}); \acute{r} \in a \circ_1 x_1, \acute{s} \in b \circ_2 x_2\} \\
&= \{-(\acute{r},\acute{s}); \acute{r} \in a \circ_1 x_1, \acute{s} \in b \circ_2 x_2\} \\
&= -((a,b) \circ (x_1,x_2)).
\end{aligned}$$

If $a \circ_1 (-x_1) \neq (-a) \circ_1 x_1$ and $b \circ_2 (-x_2) \neq (-b) \circ_2 x_2$, for some $a, b \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a,b) \circ (-(x_1,x_2)) \neq (-(a,b)) \circ (x_1,x_2)$. Similarly, if $a \circ_1 (-x_1) \neq -(a \circ_1 x_1)$ and $b \circ_2 (-x_2) \neq -(b \circ_2 x_2)$, then $(a,b) \circ (-(x_1,x_2)) \neq -((a,b) \circ (x_1,x_2))$. Also, if $(-a) \circ_1 x_1 \neq -(a \circ_1 x_1)$ and $(-b) \circ_2 x_2 \neq -(b \circ_2 x_2)$, then $(-(a,b)) \circ (x_1,x_2) \neq -((a,b) \circ (x_1,x_2))$.
If $a \circ_1 (-x_1)$ and $b \circ_2 (-x_2)$ are indeterminate in V_1 and V_2, for some $a, b \in K$, $x_1 \in V_1$, $x_2 \in V_2$, then $(a,b) \circ (-(x_1,x_2))$ is indeterminate in $V_1 \times V_2$. Similarly, if $(-a) \circ_1 x_1$ and $(-b) \circ_2 x_2$ are indeterminate in V_1 and V_2, then $(-(a,b)) \circ (x_1,x_2)$ is indeterminate in $V_1 \times V_2$. Also, if $-(a \circ_1 x_1)$ and $-(b \circ_2 x_2)$ are indeterminate in V_1 and V_2, then $-((a,b) \circ (x_1,x_2))$ is indeterminate in $V_1 \times V_2$.

NHV$_5$) If $x_1 \in 1 \circ_1 x_1$ and $x_2 \in 1 \circ_2 x_2$, for some $x_1 \in V_1$, $x_2 \in V_2$, then by the hypothesis it follows that, $(x_1, \acute{v}_2) \in (1,0) \circ (x_1, \acute{v}_2)$, where $(1,0)$ is the identity of $K \times K$.
If $x_1 \notin 1 \circ_1 x_1$ and $x_2 \notin 1 \circ_2 x_2$, for some $x_1 \in V_1$, $x_2 \in V_2$, then $(x_1,x_2) \notin (1,0) \circ (x_1,x_2)$.
If $1 \circ_1 x_1$ and $1 \circ_2 x_2$ are indeterminate in V_1 and V_2, respectively, for some $x_1 \in V_1$, $x_2 \in V_2$, then $(1,0) \circ (x_1,x_2)$ is indeterminate in $V_1 \times V_2$.
If we can not fine if $x_1 \in 1 \circ_1 x_1$ and $x_2 \in 1 \circ_2 x_2$, for some $x_1 \in V_1$, $x_2 \in V_2$, then we can not find $(x_1,x_2) \in (1,0) \circ (x_1,x_2)$.

Therefore, by Definition 3.3, $(V_1 \times V_2, +, \circ, K \times K$ is a NeutroHyperVector space over the field $K \times K$. □

It is easy to see that, the NeutroHyperVector space $V_3 = (\mathbb{Z}_3, +, \circ_3, \mathbb{Z}_2)$ was defined in Example 3.4, does not satisfy in the hypothesis of the Theorem 5.1, then we can not construct $V_3 \times W$ or $U \times V_3$ for any NeutroHyperVector spaces W and U. Also, the NeutroHyperVector spaces $V_4 = (\mathbb{Z}_3, +, \circ_4, \mathbb{Z}_2)$, $V_5 = (\mathbb{Z}_3, +, \circ_5, \mathbb{Z}_2)$ were defined in Example 3.4, $V_6 = (\mathbb{Z}, +, \circ_6, \mathbb{Z}_2)$ was defined in Example 3.5, $V_7 = (\mathbb{Z}^2, +, \circ_7, \mathbb{Z}_2)$ was defined in Example 3.6, $V_8 = (\mathbb{Z}_4, +, \circ_8, \mathbb{Z}_2)$ was defined in Example 3.8, and $V_9 = (\mathbb{Z}_3, +, \circ_9, \mathbb{Z}_2)$, $V_{10} = (\mathbb{Z}_3, +, \circ_{10}, \mathbb{Z}_2)$, were defined in Example 4.4, satisfy in the first hypothesis of the Theorem 5.1, (i.e. $k \circ_i (0 \circ_i v) = 0 \circ_i v$, for some $k \in K$, $v \in V_i$, $4 \leq i \leq 10$), thus we can construct the Cartesian product $V_i \times W$, for suitable NeutroHyperVector space W over the field $K = \{0, 1\}$, where $W = (W, +, \circ, K)$ satisfies in the necessary conditions of the Theorem 5.1, (i.e. $1 \circ (0 \circ w_1) = 0 \circ w_1$ and $0_W \in 0 \circ w_2$, for some $w_1, w_2 \in W$). In the following example, such a suitable NeutroHyperVector space is given:

Example 5.2. Consider the field $K = \{0, 1\}$ and the Abelian group $\mathbb{Z}_3 = \{0, 1, 2\}$ defined in Example 2.2. Define an external NeutroHyperoperation $\circ_{11} : \mathbb{Z}_2 \times \mathbb{Z}_3 \to P_*(U)$ by

$\circ_{11}$	0	1	2
0	$\{0, 4\}$	$\{1, 2\}$	$\{1, 2\}$
1	$\{1\}$	$\{1\}$	$\{2\}$

Then, similar to the Examples 3.4, and 4.4, it follows that $V_{11} = (\mathbb{Z}_3, +, \circ_{11}, K)$ is a strongly right distributive NeutroHyperVector space (it is not strongly left distributive) over the field $\mathbb{Z}_2$, such that $1 \circ_{11} (0 \circ_{11} 1) = 1 \circ_{11} (\{1, 2\}) = 1 \circ_{11} 1 \cup 1 \circ_{11} 2 = \{1, 2\} = 0 \circ_{11} 1$ and $0 \in 0 \circ_{11} 0$. Thus, by Theorem 5.1, $V_i \times V_{11}$, $4 \leq i \leq 11$, is a NeutroHyperVector space over the field $\mathbb{Z}_2 \times \mathbb{Z}_2$. Note that, $V_i = (V_i, +_i, \circ_i, K)$ and $V_{11} = (V_{11}, +_{11}, \circ_{11}, K)$ are satisfied in the same NeutroAxioms of Definition 3.3.

For instance, in the following, we check the NeutroAxioms of Definition 3.3, for the NeutroHyperVector space $V_4 \times V_{11} = (\mathbb{Z}_3 \times \mathbb{Z}_3, +, \circ, \mathbb{Z}_2 \times \mathbb{Z}_2)$, which is a strongly right distributive NeutroHyperVector space, but it is not strongly left distributive:

NHV$_1$) $1 \circ_4 (0 + 2) = 1 \circ_4 0 + 1 \circ_4 2$, and $1 \circ_{11} (1 + 1) = 1 \circ_{11} 1 + 1 \circ_{11} 1$, so $(1, 1) \circ ((0, 1) + (2, 1)) = (1, 1) \circ (0, 1) + (1, 1) \circ (2, 1)$, since

$$\begin{aligned} (1,1) \circ ((0,1) + (2,1)) &= (1,1) \circ (2,2) \\ &= \{(r, s); r \in 1 \circ_4 2, s \in 1 \circ_{11} 2\} \\ &= \{(0,2), (1,2), (2,2)\}, \end{aligned}$$

and

$$\begin{aligned}(1,1)\circ(0,1)+(1,1)\circ(2,1) &= \{(r,s); r\in 1\circ_4 0, s\in 1\circ_{11} 1\}\\ &\quad +\{(p,q); p\in 1\circ_4 2, q\in 1\circ_{11} 1\}\\ &= \{(1,1)\}+\{(0,1),(1,1),(2,1)\}\\ &= \{(0,2),(1,2),(2,2)\}.\end{aligned}$$

Also, $1\circ_4(0+0)\not\subseteq 1\circ_4 0+1\circ_4 0$ and $0\circ_{11}(1+2)\not\subseteq 1\circ_{11} 1+0\circ_{11} 2$, thus $(1,0)\circ((0,1)+(0,2))\not\subseteq (1,0)\circ(0,1)+(1,0)\circ(0,2)$, since $(1,0)\circ((0,1)+(0,2))=(1,0)\circ(0,0)=\{(1,0),(1,4)\}$ and

$$\begin{aligned}(1,0)\circ(0,1)+(1,0)\circ(0,2) &= \{(1,1),(1,2)\}+\{(1,1),(1,2)\}\\ &= \{(2,0),(2,1),(2,2)\}.\end{aligned}$$

NHV$_2$) $(0+1)\circ_4 2=1\circ_4 2=\{0,1,2\}=0\circ_4 2+1\circ_4 2=\{0,2\}+\{0,2\}$ and $(0+0)\circ_{11} 1\subseteq 0\circ_{11} 1+0\circ_{11} 1$, so $((0,0)+(1,0))\circ(2,1)\subseteq(0,0)\circ(2,1)+(1,0)\circ(2,1)$, since

$$\begin{aligned}((0,0)+(1,0))\circ(2,1) &= (1,0)\circ(2,1)\\ &= \{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)\},\end{aligned}$$

and

$$\begin{aligned}(0,0)\circ(2,1)+(1,0)\circ(2,1) &= \{(1,1),(1,2),(2,1),(2,2)\}\\ &\quad +\{(0,1),(0,2),(1,1),(1,2),(2,1),(2,2)\}\\ &= \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)\}.\end{aligned}$$

Also, $(1+1)\circ_4 0\not\subseteq 1\circ_4 0+1\circ_4 1$ and $(0+1)\circ_{11} 1\not\subseteq 0\circ_{11} 1+1\circ_{11} 1$, thus $((1,0)+(1,1))\circ(0,1)\not\subseteq (1,0)\circ(0,1)+(1,1)\circ(0,1)$, since $((1,0)+(1,1))\circ(0,1)=(0,1)\circ(2,1)=\{(0,1),(1,1)\}$ and

$$\begin{aligned}(1,0)\circ(0,1)+(1,1)\circ(0,1) &= \{(1,1),(1,2)\}+\{(1,1)\}\\ &= \{(2,0),(2,2)\}.\end{aligned}$$

NHV$_3$) $0\circ_4(0\circ_4 1)=(00)\circ_4 1$ and $0\circ_{11}(0\circ_{11} 1)=(00)\circ_{11} 1$, so $(0,1)\circ((0,0)\circ(1,1))=((0,1)(0,0))\circ(1,1)$, since

$$\begin{aligned}(0,1)\circ((0,0)\circ(1,1)) &= (0,1)\circ\{(1,1),(1,2),(2,1),(2,2)\}\\ &= ((0,1)\circ(1,1))\cup((0,1)\circ(1,2))\\ &\quad \cup((0,1)\circ(2,1))\cup((0,1)\circ(2,2))\\ &= \{(1,1),(2,1)\}\cup\{(1,2),(2,2)\}\\ &\quad \cup\{(1,1),(2,1)\}\cup\{(1,2),(2,2)\}\\ &= \{(1,1),(1,2),(2,1),(2,2)\},\end{aligned}$$

and $((0,1)(0,0)) \circ (1,1) = (0,0) \circ (1,1) = \{(1,1),(1,2),(2,1),(2,2)\}$.
Also, $(1,1) \circ ((1,0) \circ (1,0))$ is indeterminate in $V_4 \times V_{11}$.

NHV$_4$) $0 \circ_4 (-1) = (-0) \circ_4 1 = -(0 \circ_4 1)$ and $0 \circ_{11} (-1) = (-0) \circ_{11} 1 = -(0 \circ_{11} 1)$, so $(0,0) \circ (-(1,1)) = (-(0,0)) \circ (1,1) = -((0,0) \circ (1,1))$, since

$$(0,0) \circ (-(1,1)) = (0,0) \circ (2,2) = \{(1,1),(1,2),(2,1),(2,2)\},$$

$$(-(0,0)) \circ (1,1) = (0,0) \circ (1,1) = \{(1,1),(1,2),(2,1),(2,2)\},$$

and

$$-((0,0) \circ (1,1)) = -\{(1,1),(1,2),(2,1),(2,2)\} = \{(1,1),(1,2),(2,1),(2,2)\}.$$

Also, $(-1) \circ_4 0 \neq -(1 \circ_4 0)$ and $(-1) \circ_{11} 1 \neq -(1 \circ_{11} 1)$, thus $(-(1,1)) \circ (0,1) \neq -((1,1) \circ (0,1))$, since $(-(1,1)) \circ (0,1) = (1,1) \circ (0,1) = \{(1,1)\}$ and $-((1,1) \circ (0,1)) = -\{(1,1)\} = \{(2,2)\}$.

NHV$_5$) $2 \in 1 \circ_4 2$ and $1 \in 1 \circ_{11} 1$, so $(2,0) \in (1,0) \circ (2,0)$. Moreover, $(0,0) \notin (1,0) \circ (0,0)$.

Theorem 5.3. *Let $V_1 = (V_1, +_1, \circ_1, K)$ be a NeutroHyperVector space over the field K such that $k \circ_1 (0 \circ_1 v_1) = 0 \circ_1 v_1$, for some $k \in K$, $v_1 \in V_1$, and let $V_2 = (V_2, +_2, \circ_2, K)$ be a hypervector space over the field K such that $0_{V_2} \in 0 \circ_2 \acute{v}_2$, for some $\acute{v}_2 \in V_2$, then $(V_1 \times V_2, +, \circ, K \times K)$ with operation "$+$" and the external NeutroHyperoperation "$\circ$" defined in Theorem 5.1, is a NeutroHyperVector space over the field $K \times K$.*

Proof. It is similar to the proof of Theorem 5.1. □

Example 5.4. Let V be the hypervector spaces $(\mathbb{Z}_3, +, \circ_1, \mathbb{Z}_2)$ or $(\mathbb{Z}_3, +, \circ_2, \mathbb{Z}_2)$ over the field $K = \mathbb{Z}_2$, were defined in Example 2.2. Then for every $4 \leq i \leq 11$, the Cartesian product $V_i \times V$, is a NeutroHyperVector space over the field $K \times K$, where the NeutroHyperVector spaces V_i were defined in Examples 3.4, 3.5, 3.6, 3.8, 4.4, and 5.2.

6. **Conclusions**

A NeutroHyperVector space is an alternative of a hypervector space $(V, +, \circ, K)$ such that "$+ : V \times V \to P_*(V)$" is a NeutroHyperoperation, or "$\circ : K \times V \to P_*(V)$" is an external NeutroHyperoperation, or at least it has one NeutroAxiom. Thus, there are several types of NeutroHyperVector spaces, based on the number of NeutroOperation, NeutroHyperoperation and NeutroAxioms.

In this paper, we considered an specific and expected type of NeutroHyperVector spaces and studied some of their basic properties, such as SubNeutroHyperspace, NeutroLinearTransformation and Cartesian product of NeutroHyperVector spaces. Throughout the paper, a variety of examples are provided for each concept. For future research, one can investigate another basic properties of vector spaces, fuzzy vector spaces, hypervector spaces and fuzzy hypervector spaces in NeutroHyperVector spaces. It is also possible to change the basic definition of NeutroHyperVector space by changing the number of operations/hyperoperations/Neutrohyperoperations and axioms/NeutroAxioms and check the features of the new structure. Finding the applications of the defined structures can be the subject of valuable research works.

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