Neutrosophic Generalized Rayleigh Distribution with Application

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Abstract. In this paper, we consider the neutrosophic generalized Rayleigh distribution (NGRD). Various neutrosophic properties of NGRD are developed and discussed. The developed distribution is specifically more useful to model indeterminate data which are skewed lifetime data. Also, the neutrosophic parameters are estimated using the well-known method of maximum likelihood (ML) estimation based on a neutrosophic environment. A simulation study is carried out to establish the achievement of the estimated neutrosophic parameters. As a final point, the proposed NGRD applications in the real world have been discussed with the help of real data. A comparative studies also carried out with some recent proposed neutrosophic distributions.

Keywords: Neutrosophic Mean and variance; generating functions; parametric estimation; simulation study; neutrosophic generalized Rayleigh distribution; survival function.

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1. Introduction

To model the real data, [9] invented twelve forms of new cumulative distribution functions. More researchers pay attention to two distributions among the twelve new distributions: Burr-Type X and Burr-Type XII distributions. The two-parameter Burr Type X distribution is developed by [35] under the name of *generalized Rayleigh distribution (GRD)*. Due to the increasing and decreasing hazard function nature of GRD, it is more applicable in survival analysis. The various applications of GRD in statistical inference, reliability, statistical quality control, sampling plans were studied by [22], [1], [25], [7], [19], [12], [17], [16], [18], [10], [24].

This paper aims to develop a new neutrosophic generalized Rayleigh distribution (NGRD) in lifetime data application. Nevertheless, it is more sensible to consider that the GRD is the best-depicted distribution for the lifetime data based on an interval set of quantities for vague parameters. In this situation, neutrosophic statistics is the better environment to address lifetimes based on interval data. The concept of the neutrosophic theory is invented and studied extensively by [29] for indeterminacies in the data. The new school of thought on neutrosophic theory is an expansion of fuzzy logics or fuzzy sets; for more details, see [6], [26], [30–32,34], [36].

Furthermore, neutrosophic statistics is pioneered by [33] and is an expansion of classical statistics, which addresses uncertain or vague data and corresponding statistical probability distributions. The generalization of interval statistics is neutrosophic and also studies fuzzy interval sets. Neutrosophic statistics becomes classical statistics when data is known or deterministic. Whereas in real-world applications, most of the data sets are vague, nondeterministic or unclear, partially unknown or incomplete than determinate data; in these situations, neutrosophic statistical procedures are desirable; for more details, refer [4], [15], [23].

In recent years, few researchers have been attracted to work on neutrosophic probability distributions. [5] developed neutrosophic Weibull distribution. Neutrosophic exponential distribution applications for complex data analysis studied by [11]. [28] presented neutrosophic beta distribution with properties and applications. [2] explored neutrosophic Kumaraswamy distribution with engineering application. [27] discussed the neutrosophic extension of the Maxwell model. [14] attempted on statistical development of neutrosophic gamma distribution with applications to complex data analysis.

Developing a new model is to initiate neutrosophic adaptation of the GRD. This type of conservatory can handle real-world practical issues dealing with undetermined data in either univarite or multivariate situations, mainly when the data reported interval statistics. The cumulative distribution function (cdf) and probability density function (pdf) of GRD are respectively given below:

$$F(x; \upsilon, \sigma) = \left[1 - \exp\left\{ -\left(\frac{x}{\sigma}\right)^2 \right\} \right]^{\upsilon}; \qquad x > 0,\ \upsilon > 0,\ \sigma > 0. \tag{1}$$

and

$$f(x; \upsilon, \sigma) = \frac{2\upsilon}{\sigma^2} x \exp\{ -\left(\frac{x}{\sigma}\right)^2 \} \left[1 - \exp\left\{ -\left(\frac{x}{\sigma}\right)^2 \right\} \right]^{\upsilon-1}; \quad x > 0,\ \upsilon > 0,\ \sigma > 0. \tag{2}$$

Where υ and σ are shape and scale parameters, respectively.

The remaining paper is reported under a description of NGRD in Section 2. The various statistical properties of NGRD are presented in Section 3. In Section 4, the estimation of neutrosophic parameters is explained. The extensive simulation study is carried out in Section 5. An industrial application of the developed NGRD using the real-life data is given in Section 6, and Section 7 presents the concluding remarks and future study.

2. Neutrosophic Generalized Rayleigh Distribution

Neutrosophic statistics is the generalization of classical statistics. We administer with specific or crumple values in classical statistics, but in neutrosophic statistics, the sample values are chosen from a population with uncertainty environment. In neutrosophic statistics, the information can be vague, imprecise, ambiguous, uncertain, incomplete, or even unknown. Neutrosophic numbers have a standard form based on classical statistics, which is given below.

$$X_N = E + I$$

Data is broken down into two parts, E and I, where E is the exact or determined data, and I is the uncertain, inexact, or indeterminate part of the data. It is equivalent to $X_N \in [X_L, X_U]$. A subscript N is used to distinguish the neutrosophic random variable, for example, X_N. Let us assume that $X_{N_i} \in [X_L, X_U]$, $i = 1, 2, \ldots, n_N$ is neutrosophic random variable following the neutrosophic generalized Rayleigh distribution (NGRD) with neutrosophic shape parameter $v_N \in [v_L, v_U]$ and neutrosophic scale parameter $\sigma_N \in [\sigma_L, \sigma_U]$. The neutrosophic cumulative distribution fucntion (ncdf) and probability density function (npdf) of NGRD are respectively given as follows:

$$F(x_N; v_N, \sigma_N) = \left[1 - \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \right]^{v_N}; \quad x_N > 0, \ v_N > 0, \ \sigma_N > 0. \tag{3}$$

and

$$f(x_N; v_N, \sigma_N) = \frac{2v_N}{\sigma_N^2} x_N \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \left[1 - \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \right]^{v_N - 1};$$

$$x_N > 0, \ v_N > 0, \ \sigma_N > 0. \tag{4}$$

Where v_N and σ_N are neutrosophic shape and scale parameters, respectively.

The survival function and hazard function of NGRD are respectively given below:

$$s(x_N; v_N, \sigma_N) = 1 - \left[1 - \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \right]^{v_N} \tag{5}$$

and

$$h(x_N) = \frac{\frac{2v_N}{\sigma_N^2} x_N \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \left[1 - \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \right]^{v_N - 1}}{1 - \left(1 - \exp\left\{ -\left(\frac{x_N}{\sigma_N}\right)^2 \right\} \right)^{v_N}} \tag{6}$$

In Figure 1 presented various shapes of the NGRD for various scale and shape parameters. From Figure 1 it is noticed that the nature of NGRD is right-skewed, left-skewed and symmetrical shapes for the given shape parameters. In Figure 2 the various forms of CDF curves are displayed for various scale and shape parameters. The various natures of survival function and hazard function are plotted in Figures 3 and 4. From Figure 4 it is noticed that when shape

parameter v_N less than or equal to [0.75,0.75] the nature of hazard function is approximately bathtub type and when shape parameter v_N greater than [0.75,0.75] the nature of hazard function as increasing. Hence the proposed model is more applicable in industrial data where the failure rate is in increasing tendency.
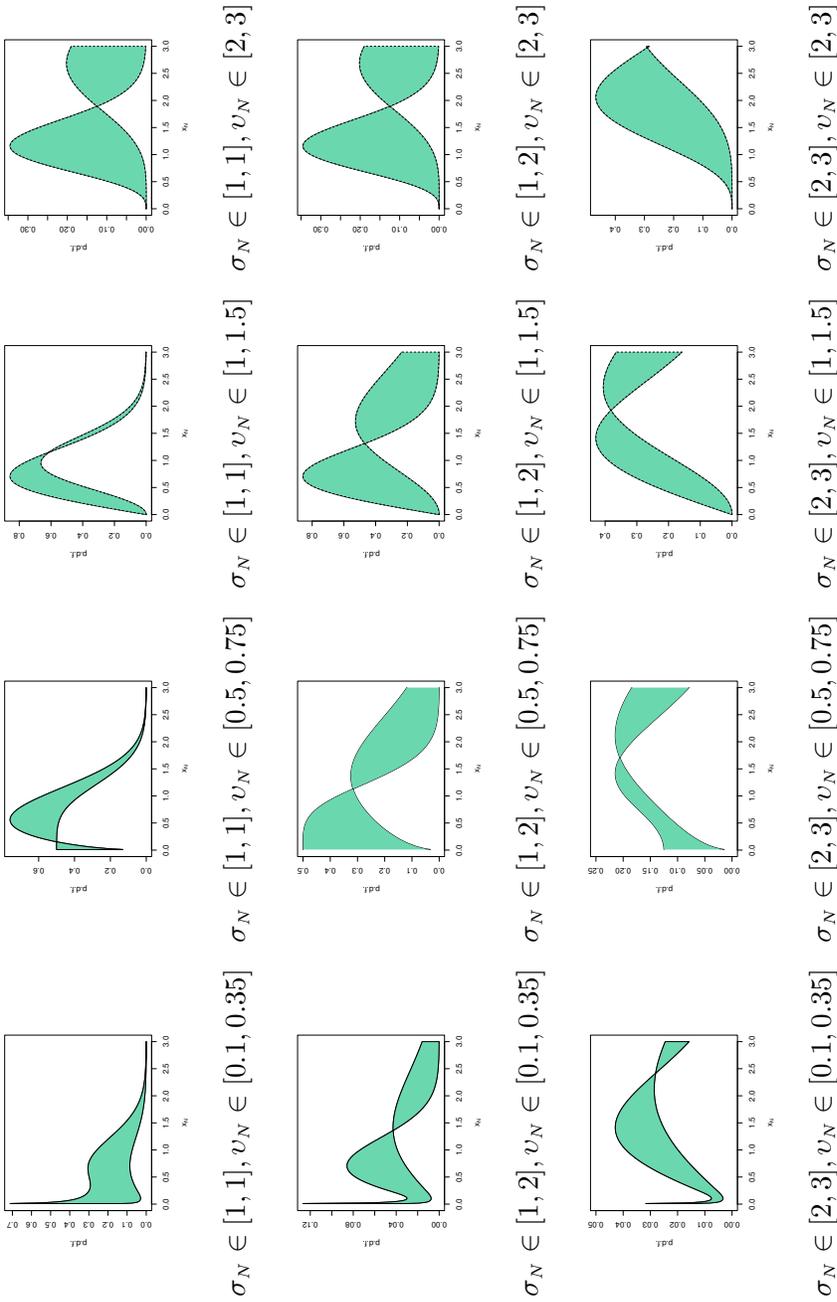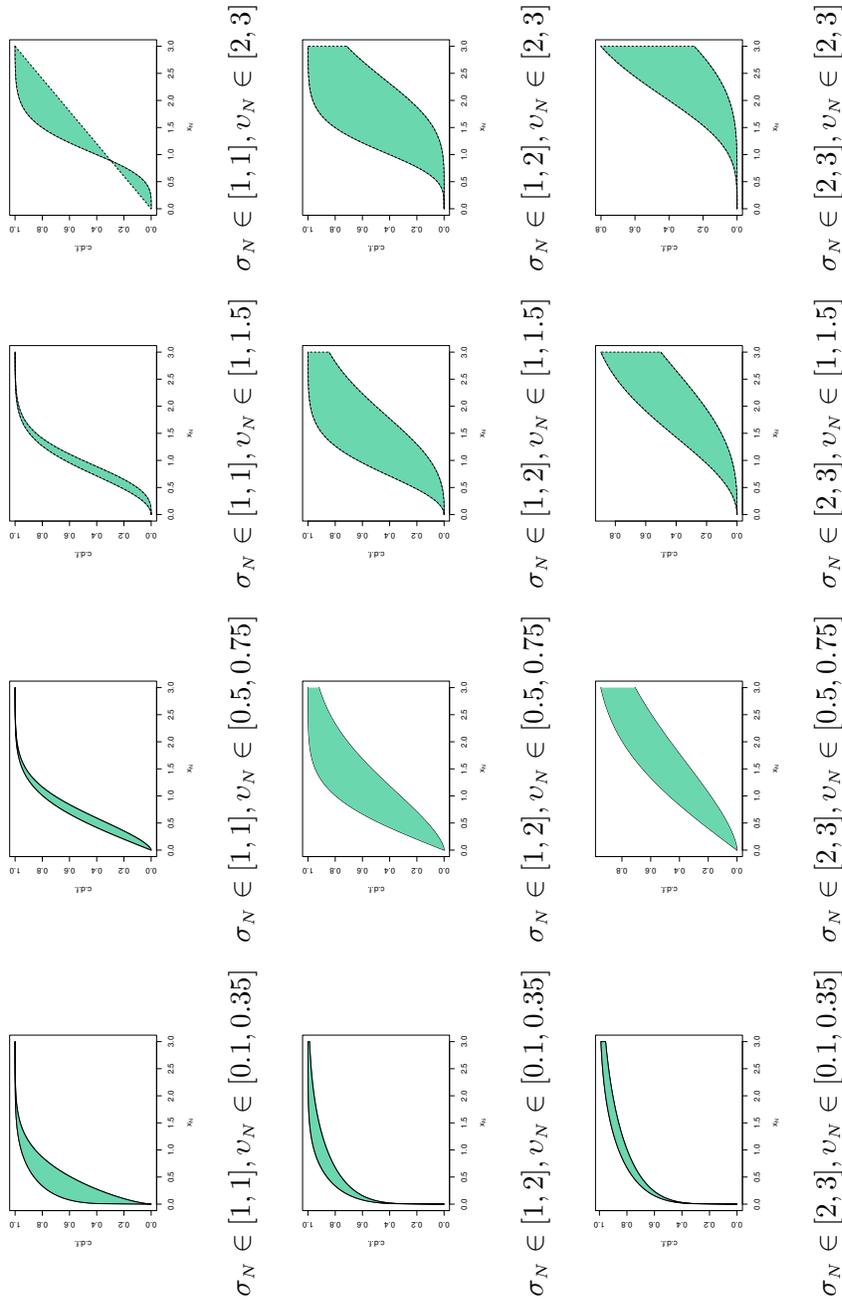
FIGURE 1. The p.d.f. plots of NGE distribution

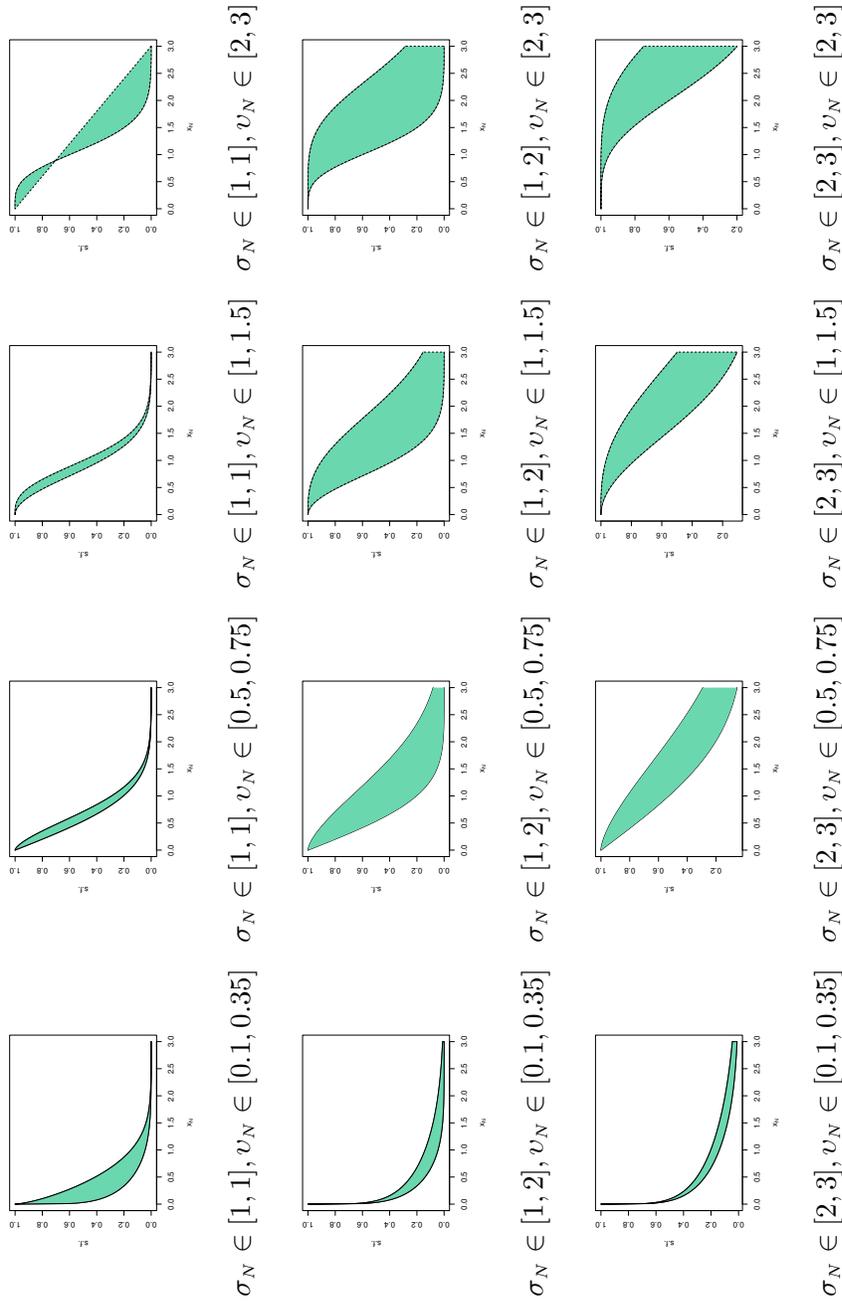FIGURE 2. The c.d.f. plots of NGE distribution

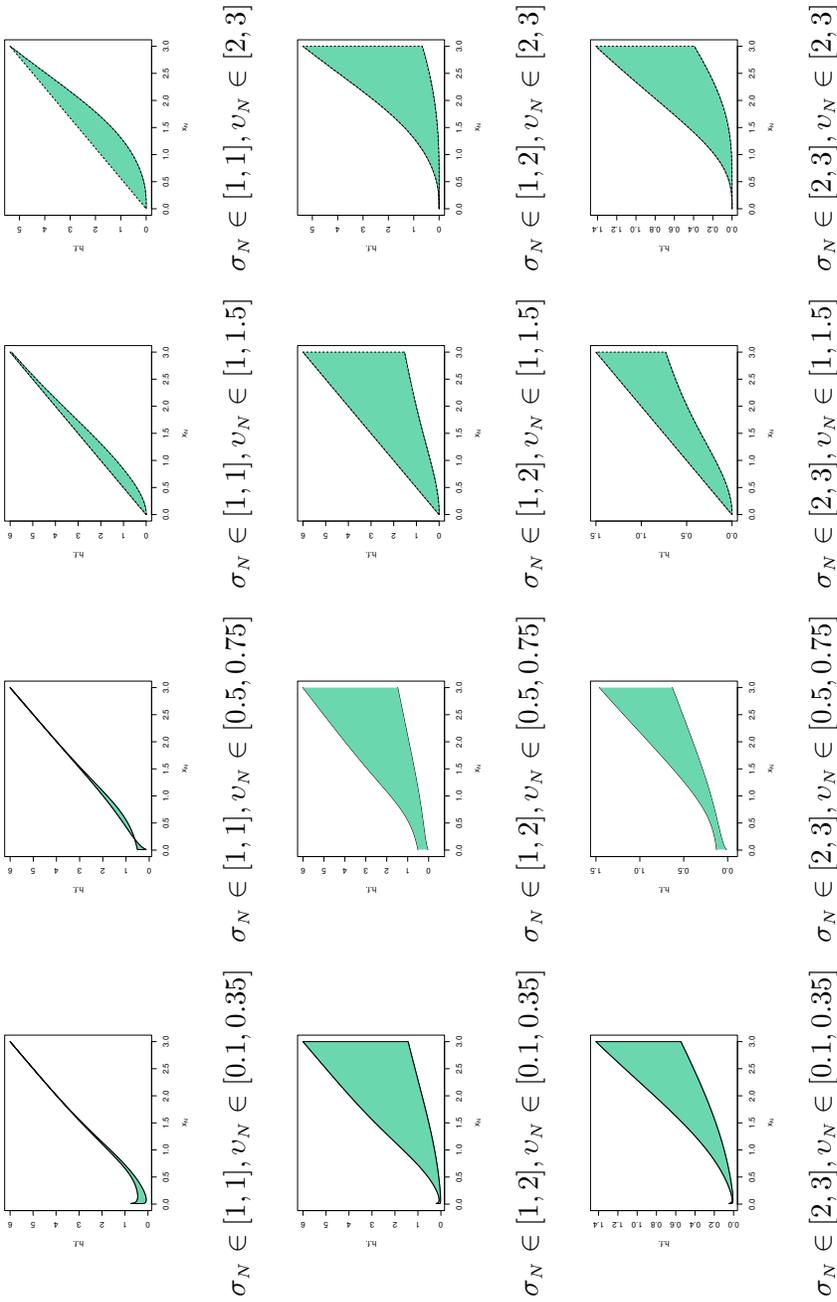FIGURE 3. The survival function plots of NGE distribution

FIGURE 4. The hazard function plots of NGE distribution

3. Properties of NGRD

In this section, we discuss the some statistical properties of the NGRD and the result are brought out as under:

Theorem 3.1. *The kth moment about origin of NGRD is*

$$\mu'_k = \upsilon_N \sigma_N^k \Gamma(\upsilon_N) \Gamma\left(\frac{k}{2}+1\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(\upsilon_N - j)\, j!(j+1)^{\frac{k}{2}+1}} \tag{7}$$

Proof. By definition, the kth raw moment is given as

$$\begin{aligned}
\mu'_k &= \mathrm{E}\left[X_N^k\right] = \int_0^{\infty} x_N^k f(x_N; \upsilon_N, \sigma_N)\, dx_N \\
&= \int_0^{\infty} x_N^k \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\frac{x_N^2}{\sigma_N^2}\right\}\left[1 - \exp\left\{-\frac{x_N^2}{\sigma_N^2}\right\}\right]^{\upsilon_N-1} dx_N \\
&= \int_0^{\infty} x_N^k \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\frac{x_N^2}{\sigma_N^2}\right\} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\upsilon_N) \exp\left\{-j\frac{x_N^2}{\sigma_N^2}\right\}}{\Gamma(\upsilon_N - j)\, j!} dx_N
\end{aligned}$$

where

$$(1-z)^{b-1} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(b)\, z^j}{\Gamma(b-j)\, j!}$$

Thus, we get

$$\begin{aligned}
\mu'_k &= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\upsilon_N)}{\Gamma(\upsilon_N - j)\, j!} \int_0^{\infty} x_N^k \frac{2\upsilon_N}{\sigma_N^2} x_N \exp\left\{-\left(\frac{x_N}{\sigma_N}\right)^2\right\} \exp\left\{-j\left(\frac{x_N}{\sigma_N}\right)^2\right\} dx_N \\
&= \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\upsilon_N)}{\Gamma(\upsilon_N - j)\, j!} \int_0^{\infty} \upsilon_N \sigma_N^k y_N^{\frac{k}{2}} \exp\left\{-(j+1)\, y_N\right\} dy_N
\end{aligned}$$

where $y = x_N^2/\sigma_N^2$. Therefore, we get

$$\mu'_k = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(\upsilon_N)\, \upsilon_N \sigma_N^k}{\Gamma(\upsilon_N - j)\, j!} \frac{\Gamma\left(\frac{k}{2}+1\right)}{(j+1)^{\frac{k}{2}+1}},$$

where $\upsilon_N \in [\upsilon_L, \upsilon_U]$ and $\sigma_N \in [\sigma_L, \sigma_U]$. $\square$

For $k=1$, in Eq. (7) we get the first raw moment (mean) of the NGRD is given by

$$\text{Mean} = \mu'_1 = \upsilon_N \sigma_N \Gamma(\upsilon_N) \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(\upsilon_N - j)\, j!(j+1)^{\frac{3}{2}}}. \tag{8}$$

For $k=2$, in Eq. 7 we get the second raw moment of the NGRD is given by

$$\mu'_2 = \upsilon_N \sigma_N^2 \Gamma(\upsilon_N) \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(\upsilon_N - j)\, j!(j+1)^2}.$$

Therefore, Neutrosophic variance (Nvar) is given by

$$\text{Nvar}(X_N) = \mu'_2 - (\mu'_1)^2$$

Similarly, we can obtain other raw and central moments, using first four central moments one can obtain skewness and kurtosis to study the nature of the NGRD.

3.1. *Quantile function*

A useful and the important statistical property of NGRD is a quantile function and it also useful to generate random sample from NGRD for simulation work. The quantile function of NGRD is given by

$$Q_N\left(q\right) = F_N^{-1}\left(q\right) = \sigma_N \left[-\ln\left(1 - q^{\frac{1}{v_N}}\right)\right]^{\frac{1}{2}},$$

where $v_N \in [v_\mathrm{L}, v_\mathrm{U}]$ and $\sigma_N \in [\sigma_\mathrm{L}, \sigma_\mathrm{U}]$. Thus,

$$\text{Median} = Q_N\left(0.5\right) = \sigma_N \left[-\ln\left(1 - 0.5^{\frac{1}{v_N}}\right)\right]^{\frac{1}{2}}.$$

3.1.1. *Measures of Skewness and Kurtosis based on Quantile Function*

The quantitative measure of skewness and Kurtosis based on quantile function is defined by [13] and [20], respectively. The following formula is used to determine the neutrosophic skewness and kurtosis of NGRD under neutrosophic environment.

$$\text{Skewness} = \frac{Q_N\left(\frac{6}{8}\right) - 2Q_N\left(\frac{4}{8}\right) + Q_N\left(\frac{2}{8}\right)}{Q_N\left(\frac{6}{8}\right) - Q_N\left(\frac{2}{8}\right)}$$

and

$$\text{Kurtosis} = \frac{Q_N\left(\frac{7}{8}\right) - Q_N\left(\frac{5}{8}\right) + Q_N\left(\frac{3}{8}\right) - Q_N\left(\frac{1}{8}\right)}{Q_N\left(\frac{6}{8}\right) - Q_N\left(\frac{2}{8}\right)}$$

The neutrosophic mean, neutrosophic variance, neutrosophic median, neutrosophic skewness, and neutrosophic kurtosis for various neutrosophic scale and shape parameters are displayed in Table 1. The results from Table 1 shows that for various statistic values are increases as neutrosophic share parametric values increases for fixed neutrosophic scale parameter.

4. **Neutrosophic Parametric Estimation**

To study the effectiveness of parametric estimation, a brief discussion is given in this section about neutrosophic maximum likelihood estimator(NMLE) of the parameters of NGRD. The asymptotic properties of NMLEs of v_N and σ_N also discussed. A simulation study is carried out to study the performance of classical MLEs of the parameters as well as other methods of parametric estimations have been considered widely in [16].

Let X_{N_1}, ..., X_{N_n} be a random sample from NGRD, then the log-likelihood function can be expressed as follows:

$$l\left(v_N, \sigma_N\right) \cong n\ln(2) + n\ln(v_N) + 2n\ln(\sigma_N) + \sum_{i=1}^{n} \ln(x_{N_i})$$

$$-\sum_{i=1}^{n}\left(\frac{x_{N_i}}{\sigma_N}\right)^2 + (v_N - 1)\sum_{i=1}^{n}\ln\left(1 - \exp\{-\left(\frac{x_N}{\sigma_N}\right)^2\}\right) \tag{9}$$

The NMLEs of v_N and σ_N can be obtained on maximize the Eq. (9) with respect to v_N and σ_N. Thus NMLEs v_N and σ_N would be the solution of the following two non-linear equations:

$$\frac{\partial l}{\partial v_N} = \frac{n}{v_N} + \sum_{i=1}^{n}\ln\left(1 - \exp\{-\left(\frac{x_{N_i}}{\sigma_N}\right)^2\}\right) = 0 \tag{10}$$

and

$$\frac{\partial l}{\partial \sigma_N} = \frac{-2n}{\sigma_N} + \frac{2}{\sigma_N}\sum_{i=1}^{n}\left(\frac{x_{N_i}}{\sigma_N}\right)^2 - \frac{2(v_N - 1)}{\sigma_N}\sum_{i=1}^{n}\frac{\left(\frac{x_{N_i}}{\sigma_N}\right)^2 \exp\left\{-\left(\frac{x_{N_i}}{\sigma_N}\right)^2\right\}}{\left[1 - \exp\{-\left(\frac{x_{N_i}}{\sigma_N}\right)^2\}\right]} = 0 \tag{11}$$

The NLME of v_N and σ_N, denoted by $\hat{v_N}$ and $\hat{\sigma_N}$ respectively, can be obtained by solving two non-linear Eqs. (10) and (11) simultaneously.

5. Simulation Study

To study the performance of the proposed NGRD distribution model, a simulation study is carried out. The accomplishment of NGRD estimated parameters and their performance are expressed as neutrosophic average estimates (AEs), neutrosophic average biased (Avg. Biases) and neutrosophic measure square error (MSEs) using simulation investigation. The simulation results of average Bias and MSE are summarized in Tables 2-4. It is noticed from tables that the average Bias and MSE are decrease when size of the sample increases, as

TABLE 1. The mean, variance, median, skewness and kurtosis for different neutrosophic parametric values

σ_N	v_N	Mean	Variance	Median	Skewness	Kurtosis
$[1, 1]$	$[0.1, 0.35]$	$[0.188, 0.502]$	$[0.117, 0.211]$	$[0.031, 0.385]$	$[0.207, 0.747]$	$[1.158, 1.936]$
$[1, 1]$	$[0.5, 0.75]$	$[0.626, 0.777]$	$[0.221, 0.222]$	$[0.536, 0.711]$	$[0.094, 0.138]$	$[1.167, 1.190]$
$[1, 1]$	$[1, 1.5]$	$[0.886, 1.039]$	$[0.200, 0.215]$	$[0.833, 0.997]$	$[0.063, 0.076]$	$[1.204, 1.219]$
$[1, 1]$	$[2, 3]$	$[0.779, 1.146]$	$[0.187, 0.894]$	$[1.108, 1.256]$	$[0.056, 0.058]$	$[1.227, 1.234]$
$[1, 2]$	$[0.1, 0.35]$	$[0.188, 1.003]$	$[0.118, 0.845]$	$[0.031, 0.771]$	$[0.207, 0.747]$	$[1.158, 1.936]$
$[1, 2]$	$[0.5, 0.75]$	$[0.626, 1.554]$	$[0.222, 0.884]$	$[0.536, 1.422]$	$[0.094, 0.138]$	$[1.167, 1.190]$
$[1, 2]$	$[1, 3]$	$[0.886, 2.659]$	$[0.215, 1.069]$	$[0.833, 1.994]$	$[0.063, 0.076]$	$[1.204, 1.219]$
$[1, 2]$	$[2, 3]$	$[1.146, 1.557]$	$[0.187, 3.574]$	$[1.108, 2.513]$	$[0.056, 0.058]$	$[1.227, 1.234]$
$[2, 3]$	$[0.1, 0.35]$	$[0.376, 1.505]$	$[0.470, 1.901]$	$[0.063, 1.156]$	$[0.207, 0.747]$	$[1.158, 1.936]$
$[2, 3]$	$[0.5, 0.75]$	$[1.252, 2.331]$	$[0.886, 1.989]$	$[1.073, 2.133]$	$[0.094, 0.138]$	$[1.167, 1.190]$
$[2, 3]$	$[1, 2]$	$[1.7723.988]$	$[0.858, 2.404]$	$[1.665, 2.991]$	$[0.063, 0.076]$	$[1.204, 1.219]$
$[2, 3]$	$[2, 3]$	$[2.292, 2.336]$	$[0.749, 8.043]$	$[2.216, 3.769]$	$[0.056, 0.058]$	$[1.227, 1.234]$

expected. According to Tables 2-4, Bias of shape parameters is negative and scale parameter is positive at different values of shape parametric and scale parametric values.

TABLE 2. $v_N = [1,1]$, $\sigma_N = [0.5, 0.75]$

	AEs		Avg. Biases		MSEs	
	$\hat{v}_N$	$\hat{\sigma}_N$	$\hat{v}_N$	$\hat{\sigma}_N$	$\hat{v}_N$	$\hat{\sigma}_N$
30	1.1005	[0.4902,0.7377]	0.1005	[-0.0098,-0.0123]	0.3204	[0.0604,0.0894]
50	1.0562	[0.4942,0.7425]	0.0562	[-0.0058,-0.0075]	0.2154	[0.0465,0.0686]
100	1.0274	[0.4974,0.7452]	0.0274	[-0.0026,-0.0048]	0.1415	[0.0326,0.0493]
200	1.0132	[0.4985,0.7474]	0.0132	[-0.0015,-0.0026]	0.0948	[0.0231,0.0347]
500	1.0051	[0.4994,0.7494]	0.0051	[-0.0006,-0.0005]	0.0598	[0.0146,0.0221]
1000	1.0025	[0.4998,0.7497]	0.0025	[-0.0002,-0.0003]	0.0419	[0.0103,0.0155]

TABLE 3. $v_N = [0.5, 0.75]$, $\sigma_N = [1,1]$

	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\sigma}_N$	$\hat{v}_N$	$\hat{\sigma}_N$	$\hat{v}_N$	$\hat{\sigma}_N$
30	[0.5388,0.8168]	0.9802	[0.0388,0.0668]	-0.0198	[0.1364,0.2132]	0.1383
50	[0.5225,0.7888]	0.9873	[0.0225,0.0388]	-0.0127	[0.0942,0.1524]	0.1070
100	[0.5108,0.7692]	0.9933	[0.0108,0.0192]	-0.0067	[0.0625,0.1008]	0.0760
200	[0.5052,0.7591]	0.9963	[0.0052,0.0091]	-0.0037	[0.0424,0.0676]	0.0536
500	[0.5018,0.7538]	0.9988	[0.0018,0.0038]	-0.0012	[0.0264,0.0434]	0.0341
1000	[0.5015,0.7525]	0.9992	[0.0015,0.0025]	-0.0008	[0.0226,0.0363]	0.029

TABLE 4. $v_N = [0.5, 0.75]$, $\sigma_N = [1, 3]$

	AEs		Avg. Biases		MSEs	
	$\hat{\alpha}_N$	$\hat{\sigma}_N$	$\hat{v}_N$	$\hat{\sigma}_N$	$\hat{v}_N$	$\hat{\sigma}_N$
30	[0.5397,0.8185]	[0.9757,2.9472]	[0.0397,0.0685]	[-0.0243,-0.0528]	[0.1363,0.2205]	[0.1484,0.3877]
50	[0.5225,0.7888]	[0.9854,2.9676]	[0.0225,0.0388]	[-0.0146,-0.0324]	[0.0942,0.1524]	[0.1149,0.2977]
100	[0.5108,0.7692]	[0.9936,2.9791]	[0.0108,0.0192]	[-0.0064,-0.0209]	[0.0625,0.1008]	[0.0807,0.2138]
200	[0.5052,0.7591]	[0.9963,2.9889]	[0.0052,0.0091]	[-0.0037,-0.0111]	[0.0424,0.0676]	[0.057,0.1508]
500	[0.5018,0.7538]	[0.9985,2.9974]	[0.0018,0.0038]	[-0.0015,-0.0026]	[0.0264,0.0434]	[0.0361,0.0962]
1000	[0.5015,0.7525]	[0.9988,2.9989]	[0.0015,0.0025]	[-0.0012,-0.0011]	[0.0226,0.0363]	[0.0307,0.0818]

6. Real Data Applications

A realistic attempt of NGE distribution model is studied with help a real data in this section. The Parameter estimates along with the values of AIC (Akaike's Information criteria),

BIC (Bayesian Information criteria) and KS (Kolmogorov–Smirnov) statistic are provided for comparison neutrosophic exponential distribution (NED), neutrosophic generalized exponential distribution (NGED), neutrosophic Weibull distribution (NWD), neutrosophic Rayleigh distribution (NRD) and neutrosophic generalized Rayleigh distribution (NGRD).

To demonstrate a real example here we considered an rough population compactness of few villages in rural USA. This data is taken from [3] and they studied for neutrosophic W/S test based on the data follows to neutrosophic normal distribution. This data consists of the population of 17 villages in USA and their neutrosophic data, which is reproduced in Table 5 for ready reference. The results in Table 6 also shows that NGED is more suitable to fit the data than the NED.

TABLE 5. Neutrosophic population density of some villages in the USA

Villages	Population density	Villages	Population density
Aranza	[4.13,4.14]	Charapan	[5.10,5.12]
Corupo	[4.53,4.55]	Comachuen	[5.25,5.27]
San Lorenzo	[4.69,4.70]	Pichataro	[5.36,5.38]
Cheranatzicurin	[4.76,4.78]	Quinceo	[5.94,5.96]
Nahuatzen	[4.77,4.79]	Nurio	[6.06,6.08]
Pomacuaran	[4.96,4.98]	Turicuaro	[6.19,6.21]
Servina	[4.97,4.99]	Urapicho	[6.30,6.32]
Arantepacua	[5.00,5.06]	Capacuaro	[7.73,7.98]
Cocucho	[5.04,5.06]		

TABLE 6. Estimates and Goodness-of-fit statistics for village data set

Model	Parameter	Estimates	LogLikelihood	AIC	BIC	KS
NED	υ	[0.1861,0.1873]	[-45.58152,-45.4788]	[94.9576,95.16304]	[102.2905,102.4959]	[0.5385,0.5372]
NGED	shape	[41.6889,44.5078]	[-20.86095,-20.2097]	[44.4194,45.72189]	[51.7523,53.0547]	[0.2707,0.3270]
	rate	[7.7599,8.3348]				
NWD	shape	[5.5773,5.8981]	[-23.94168,-23.04169]	[103.9359,107.1836]	[111.2687,114.5164]	[0.6552,0.6579]
	scale	[5.7621,5.7143]				
NRD	υ	[3.8224,3.8495]	[-34.4533,-34.3024]	[72.6048,72.9067]	[79.9377,80.2395]	[0.4438,0.4457]
	σ	-	-	-	-	-
NGRD	υ	[47.075,0.6187]	[-19.8662,-19.30757]	[42.6151,43.7323]	[49.9480,51.06525]	[0.1695,0.1728]
	σ	[2.5501,2.5915]				

7. Conclusions

In this article, a generalization Rayleigh distribution is developed under neutrosophic statistics environment. Very few researchers are studied probability distributions based on neutrosophic statistics. The mathematical properties of the developed neutrosophic generalization

Rayleigh distribution are studied. The nature of the distribution is studies through various neutrosophic parametric combinations. Using the maximum likelihood method the parameters are estimated. A simulation study is carried out under neutrosophic environment. The average Bias and MSE are decreases as sample size increases, as expected. Finally, the application of the proposed neutrosophic generalized Rayleigh distribution is presented through a real data set. A comparative study with other distribution is also done based real data set. Based on real data example, we conclude that the proposed distribution furnishes better performance over existing distributions.

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