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Neutrosophic Hesitant Fuzzy Techniques and its Application to Structural Design

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Abstract:

Structural optimization in construction has attracted significant attention to sustainable development. In reality, structural model is associated with different imprecise parameters. Several factors influence the uncertain framework for optimization structural models. To tackle such structural difficulties, an effective design and optimization configuration is required. In this proposed work, we have created a solution procedure to solve multi objective problems under neutrosophic-hesitant fuzzy (NHF) environment in context of structural design. The suggested procedure is based on the NHF decision-making set that assigns a set of potential values for each objective function's membership, non-membership, and indeterminacy degrees in a NHF environment. The efficiency, applicability, and utility of the proposed technique are presented here by using a three-bar truss design model.

Keywords- Multi objective structural problem; Hesitant fuzzy optimization; Neotrosophic-hesitant fuzzy optimization; Pareto optimal solution; Indeterminacy hesitant membership function

1. Introduction

When it comes to tackling optimization challenges, optimization techniques have a big impact in real life. When dealing with real-life situations with various problems, various sorts of mathematical models exist. As a result, the mathematical models are formed with single or multi objective function/functions along with a branch of constraints. In multi-objective optimization problem (MOOP), objective functions are conflicting in nature. The objective functions of this mathematical models are maximization type or minimization type or mixed type. In this type of problems, it is very difficult to identify the suitable feasible solutions. That is why, decision maker (DM) prefers a compromise programming (CP) approach that currently meets each goal function is available. As a result, the idea of CP approach has a significant impact on the global optimality criterion. A large amount of research has been presented in the past era on the topic of MOOP. In MOOP, the difficult task as a DM is to discover an appropriate compromised solution set from a set of possible Pareto-optimal solutions.

Due to local and global optimal, multi-objective nonlinear programming problem (MONLPP) is a complex problem as compare to linear multi-objective programming problem. Professor Zadeh pioneered [2] the new idea of fuzzy set (FS) to address the uncertainty in 1965 and Professor Zimmermann [4] proposed a fuzzy programming technique (FPT) for several objective mathematical problem based on fuzzy set. The FPT was only concerned with the degree of acceptance, but it may be required to address the function of rejection in order to obtain more practical outcomes.

The FS (fuzzy set) theory was used in structural model as well. A new concept was implemented a sequence of optimal solution (OS) for structure with fuzzy constraints based on alpha-cut method by Wang et al. [3]. Rao [5] discussed a four–bar generating mechanism with a fuzzy goal function and fuzzy constraints. Yeh et al. [6] created structural optimization using imprecise parameters. Xu [7] solved a nonlinear structural model using fuzzy two-phase method. Shih et al. [8,9] suggested a novel approach to discover a unique solution using alpha-cut approaches of the 1 and 2 types to the structural model in fuzzy environment farther they had developed another alternative approach based on alpha-cut method to obtain the OS of a nonlinear structural problem. Dey et al. [10] addressed multi-objective structural design issues using generalized fuzzy programming. Also, a computational algorithm was developed by Dey et al. [11] for a structural model with three bar using basic triangular norm in fuzzy environment. The extension of ordinary fuzzy set (FS) or hesitant fuzzy set (HFS) was introduced by Torra et al. [12]. It provided an opportunity to allow more feasible values of an element to a set. The potential values of an element in HFS is a subinterval of [0,1]. Many research scholars have recently investigated HFSs and used them in different domains of research. In 2016, a computational programming technique based on HFS was developed by G.L. Xu, et al. [13] for hybrid MCGDM model. In the same domain another paper was published in 2017 by S.-P. Wan [14] based on hesitant fuzzy programming method. L. Dymova, [15] created a user-friendly computer application using a fuzzy MCDM technique. Farther, they [16] had applied this fuzzy MCDM technique in a rolled-steel heat treatment metallurgical plant in 2021. But in structural design optimization, hesitant fuzzy set is likewise not extensively utilized.

In 1986 [17], intuitionistic fuzzy set (IFS) was developed by Prof. Atanassov. IFS is an advanced version of FS. In FS, the membership degree is only consideration whereas in IFS, both the level of membership and non-membership are considered with the condition that the sum both membership values is not greater than one. P. P. Angelov [18] used the optimization for the first time in a widespread intuitionist fuzzy environment in 1997. B. Singh et al. [19] proposed an intuitionistic fuzzy optimization technique based on structural model. M. Sarkar et al. [20] proposed a new computational algorithm based on triangular-norm and triangular-conorm in intuitionistic fuzzy environment to solve a welded beam design issue. Kabiraj *et al.* [21] gave the utility of fuzzy logic has been used in linear programming in 2019. In 2019, S.F.Zhang, et al. [22] proposed GRA based IFMCGDM method for personnel selection. Kizilaslan et al. [23] proposed intuitionistic fuzzy function approaches utilizing ordinary least square estimation rather than ridge regression in 2019. Ahmadini and Ahmad [24] proposed intuitionistic fuzzy goal programming with preference relations to address a multi-objective problem in 2021. A. Ebrahimnejad, [25] introduced a novel approach to solve data envelopment analysis (DEA) models characterized by intuitionistic fuzzy data. Recently, many researchers have worked with intuitionistic hesitant fuzzy (IHF) sets and implemented them to many domains. S.K Bharati [26] in 2018 introduced hesitant fuzzy algorithm to solve multi objective linear optimization problem (MOLOP).K.B. Shailendra, [27] introduced IHF algorithm for MOOP in 2021. But in structural design optimization, IHF set is likewise not extensively utilized. The concept of neutrosophic theory was revealed to address the importance of indeterminacy in real life. In generalized FS and IFS were discussed about membership and non-membership function only but there is no information about the indeterminacy. New concept of neutrosophic theory was presented in front of researcher by Prof. Smarandache in 1995 [28], which is a dialectics extension. The neutrosophic set (NS) can manage both uncertain and partial information, whereas IFSs can only manage partial information. The word neutrosophic is derived from two words: neutron (neutral in French) and Sophia (skill or wisdom in Greek). The NS is described by using three functions namely belonging (truth) function, belonging to a certain point (indeterminacy) function, and not belonging (falsity) function. The Neutrosophical Programming Approach (NPA), based on NS, was implemented, and is now widely utilized in real-world applications. M. Sarkar et al. [35] applied neutrosophic fuzzy numbers in the area of structural design and application. Abdel-Basset et al. [1] offered a new technique for solving a completely neutral linear programming problem (LPP) that applies to production planning. In 2018, Ye et al. [29] suggested an effective technique for addressing the issue of non-linear programming of the neutrosophic number in neutrosophic numerical environments. An approach for solving MONLPPs in IFS was introduced by Rani et al. [39]. The develop method has been was compared with other existing methods that are already includes. Zhou and Xu [30] developed a novel portfolio selection and investment technique at risk in a widespread and faltering environment. All the sets mentioned above have their limits with respect to the presence of each component in the set. A new optimization technique based on a single-valued neutrosophic hesitant fuzzy set (SVNHFS) was proposed by Ahmad et al. [31]. This set includes the concept of truth hesitancy degrees, falsity hesitant degrees as well as indeterminacy hesitant degrees for various objective functions. The neutrosophic set of indeterminacy concepts examines potential future lines of research in the field of real-life application. Many researchers have contributed to the field of neutrosophic optimization techniques and real-world applications, including [36, 38]. In 2020, F. Ahmad, et al. [37] were developed a computational approach based on modified neutrosophic fuzzy set (NFS) to optimize a supply chain decision making problem. According to Giri et al. [40], TOPSIS for MADM has been extended through the use of single valued neutrosophic fuzzy sets (SVNFS). B. Tanuwijaya et al [41] developed fuzzy time series (FTS) model based on SVNFSs in 2020. In 2021, F. Ahmad [42] proposed interactive NPA based on Type-2 fuzzy in domain supplier selection problem. In order to tackle a PP issue, Khan et al. [43] studied the IVTN value and employed NS and IFS approach. S. Gupta et al [44] introduced Dash diet model and optimized the calorie consumption and minimized diet cost under neutrosophic goal programming (NGP). The multi objective NGP was used to solve the diet model, satisfy daily nutrient needs, and compared various approaches.

A wide range of methods have been used in the literature in order to solve the uncertainty in structural design problems, such as fuzzy, intuitionistic fuzzy and neutrosophic fuzzy optimization. But combination with HFS and NFS is very rare in literature survey in context of structural design.

This research is prompted by NHF emerging as a novel field of study with the capacity to attract the individuals responsible for making decisions. The subsequent are the impacts of the study:

- It serves as a supplementary addition to the existing literature on MOSOP.
- A case study is presented in which solution processes for MOSOP methodologies are documented.
- In this work, a novel technique based on NHF under various membership functions has been used.
- The method is contrasted with HFS and IHFS, and the findings indicate that the proposed study is effective.
- The proposed neutrosophic hesitant fuzzy programming approaches (NHFPAs) utilizing the neutrosophic fuzzy decision set is quite simple and easy.

The synopsis of rest of the manuscript is highlighted below: Section 2: we have highlighted the multi-objective structural optimization model (MOSOM). In section 3, we give some basic concepts about FS, IFS, SVNS, HFS, and SVNHFS. Section 4 proposes a computational algorithm to solve a

MOOP using neutrosophic hesitant fuzzy optimization technique (NHFOT). Section 5 outlines the approach for resolving the multi-objective structural model using NHFOT. An illustrative example is studies in section 6 which reflects the applicability and validity of the proposed method effectively. Finally, section 7 highlights the concluding remarks and finding based on the present work.

2. Mathematical Form of Multi-Objective Structural Optimization Problem (MOSOP)

In structural model, the basic parameters of a bar truss structure system (such as Young's modulus, material density, maximum permissible stress, and so on) are established, and the objective is to find the cross section area of the bar truss so that we can find the lightest weight of the structure and smallest node displacement under loading condition.

The MOSOP is formulated as follows:

$$\begin{aligned} &Minimize\ W(A) \\ &Minimize\ \rho(A) \\ &\text{s.t}\ \ \sigma(A) \le [\sigma] \\ &A \in [A_{\min}, A_{\max}] \end{aligned} \tag{1}$$

where n number design variables $A = [A_1, A_2, ..., A_n]^T$ are considered. The design parameters are the cross-sectional area of the truss bar, the total structural weight is $W(A) = \sum_{i=1}^{n} \delta_i A_i L_i$, the deflection of loaded joint is $\rho(A)$, length of bar= L_i, cross section area= A_i, and the i^{th} group bars density= δ_i, respectively. Under different conditions, the stress constraint=$\sigma(A)$ and maximum allowable stress of the group bars=$[\sigma]$, cross section area (minimum)= $A_{\min}$ and cross section area (maximum) = $A_{\max}$ respectively.

3. Preliminaries

Definition 1. [32] (Neutrosophic Set (NS)) Assume, U be the universe discourse such that $x \in U$. A NS $\tilde{A}$ in U is characterized by the membership functions as, truth $Tf_{\tilde{A}}(x)$, indeterminacy $If_{\tilde{A}}(x)$ and a falsity $Ff_{\tilde{A}}(x)$ and is denoted by the following form:

$$\tilde{A} = \{(x, Tf_{\tilde{A}}(x), If_{\tilde{A}}(x), Ff_{\tilde{A}}(x)) : x \in U\}$$

Where the subsets $Tf_{\tilde{A}}(x)$, $If_{\tilde{A}}(x)$ and $Ff_{\tilde{A}}(x)$ are truth, indeterminacy and falsity membership function lies in $E =]0^-, 1^+[$,also given as, $Tf_{\tilde{A}}(x): U \to E, If_{\tilde{A}}(x): U \to E,$ *and* $Ff_{\tilde{A}}(x): U \to E$ .There is no restriction on the sum of $Tf_{\tilde{A}}(x)$, $If_{\tilde{A}}(x)$ and $Ff_{\tilde{A}}(x)$,so we have,

$$0^- \le \sup Tf_{\tilde{A}}(x) + If_{\tilde{A}}(x) + \sup Ff_{\tilde{A}}(x) \le 3^+$$

Definition 2. [32] Let U be a universe set. A single valued neutrosophic set (SVNS) $\tilde{A}$ over U is given by $\tilde{A} = \{(x, Tf_{\tilde{A}}(x), If_{\tilde{A}}(x), Ff_{\tilde{A}}(x)) : x \in U\}$

Where $Tf_{\tilde{A}}(x)$, $If_{\tilde{A}}(x)$ and $Ff_{\tilde{A}}(x)$ lies in $[0,1]$ and $0 \le Tf_{\tilde{A}}(x) + If_{\tilde{A}}(x) + Ff_{\tilde{A}}(x) \le 3$ for every $x \in U$.

Definition 3. [33] (Hesitant Fuzzy Set (HFS) Torra et al. [12], created a new tool called HFSs and which allow the acceptance degree to the set of various possible values. The HFS is as follows:

Let U be a universe set, then a HFS on U is expressed as $\tilde{Y} = \{< x_j, h_{\tilde{Y}}(x_j) > | x_j \in U\}$, where $h_{\tilde{Y}}(x_j)$ is set of possible degree of acceptance of the element $x_j \in U$ in [0,1] .Also, we call $h_{\tilde{Y}}(x_j)$, a hesitant fuzzy element.

Definition 4. [34] (SVNHFS) Let's say there's a fixed set U ; an SVNHFS on U is represented as: $S_{\tilde{Y}} = \{(x, Tf_{\tilde{Y}}(x), If_{\tilde{Y}}(x), Ff_{\tilde{Y}}(x)) : x \in U\}$ where set of possible values of $Tf_{\tilde{Y}}(x)$, $If_{\tilde{Y}}(x)$ and $Ff_{\tilde{Y}}(x)$ are lies in $[0,1]$,indicating the possible truth, indeterminacy hesitant degree of acceptance and the falsehood hesitant degree of rejection of the element $x \in U$ to the set $S_{\tilde{Y}}$ accordingly with the conditions $0 \le \mu, \kappa, \gamma \le 1$ and $0 \le \mu^+, \kappa^+, \gamma^+ \le 3$, where $\mu \in Tf_{\tilde{Y}}(x)$, $\kappa \in If_{\tilde{Y}}(x)$, $\gamma \in Ff_{\tilde{Y}}(x)$ with $\mu^+ \in Tf_C{}^+(x) = U_{\mu \in Tf_{\tilde{Y}}(x)} \max\{\mu\}$, $\kappa^+ \in If_{\tilde{Y}}{}^+(x) = U_{\kappa \in If_{\tilde{Y}}(x)} \max\{\kappa\}$, $\gamma^+ \in Ff_{\tilde{Y}}{}^+(x) = U_{\gamma \in Ff_{\tilde{Y}}(x)} \max\{\gamma\}$ for all $x \in U$.

For ease, the three-tuple $S_{\tilde{Y}} = \{Tf_{\tilde{Y}}(x), If_{\tilde{Y}}(x), Ff_{\tilde{Y}}(x)\}$ is known as a single-valued neutrosophic hesitant fuzzy element (SVNHFE) or triple hesitant fuzzy element.

According to Definition 6, the SVNHFS has three types of membership functions: truth $Tf_{\tilde{A}}(x)$, indeterminacy $If_{\tilde{A}}(x)$ and a falsity $Ff_{\tilde{A}}(x)$ membership function, resulting in a more dependable structure and providing flexible options to allocate values for every element in the field, and may handle three types of uncertainty at the same time. As a result, FSs, IFSs, SVNFSs, and HFSs can be considered as specific instances of SVNHFSs.([33]

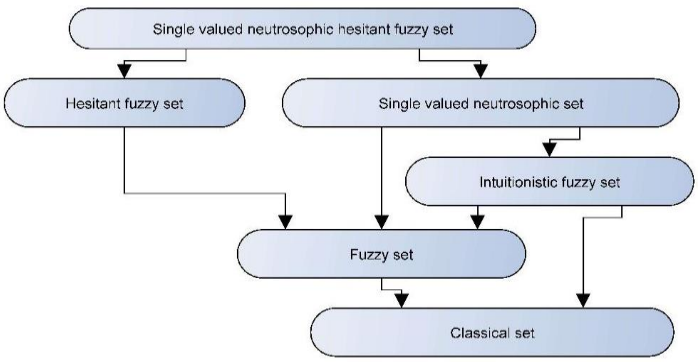

Figure 1: Dialogistic coverage of classical set to SVNHFS.

Definition 5. [34] Let there be two SVNHFSs, $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ in a universal set U . Then the union of $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ is described as:

$$S_{\tilde{Y}_1} \cup S_{\tilde{Y}_2} = \left\{ \begin{array}{l} Tf_{\tilde{Y}} \in \left(Tf_{\tilde{Y}_1} \cup Tf_{\tilde{Y}_2} \right) : Tf_{\tilde{Y}} \geq \max\left(\min\left\{ Tf_{\tilde{Y}_1} \cup Tf_{\tilde{Y}_2} \right\} \right), \\ If_{\tilde{Y}} \in \left(If_{\tilde{Y}_1} \cup If_{\tilde{Y}_2} \right) : If_{\tilde{Y}} \leq \min\left(\max\left\{ If_{\tilde{Y}_1} \cup If_{\tilde{Y}_2} \right\} \right), \\ Ff_{\tilde{Y}} \in \left(Ff_{\tilde{Y}_1} \cup Ff_{\tilde{Y}_2} \right) : Ff_{\tilde{Y}} \leq \min\left(\max\left\{ Ff_{\tilde{Y}_1} \cup Ff_{\tilde{Y}_2} \right\} \right) \end{array} \right\} \tag{2}$$

Definition 6. [34] Let there be two SVNHFSs, $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ in a universal set U . Then the intersection of $S_{\tilde{Y}_1}$ and $S_{\tilde{Y}_2}$ is described as:

$$S_{\tilde{Y}_1} \cap S_{\tilde{Y}_2} = \left\{ \begin{array}{l} Tf_{\tilde{Y}} \in \left(Tf_{\tilde{Y}_1} \cap Tf_{\tilde{Y}_2} \right) : Tf_{\tilde{Y}} \geq \min\left(\max\left\{ Tf_{\tilde{Y}_1} \cap Tf_{\tilde{Y}_2} \right\} \right), \\ If_{\tilde{Y}} \in \left(If_{\tilde{Y}_1} \cap If_{\tilde{Y}_2} \right) : If_{\tilde{Y}} \leq \max\left(\min\left\{ If_{\tilde{Y}_1} \cap If_{\tilde{Y}_2} \right\} \right), \\ Ff_{\tilde{Y}} \in \left(Ff_{\tilde{Y}_1} \cap Ff_{\tilde{Y}_2} \right) : Ff_{\tilde{Y}} \leq \max\left(\min\left\{ Ff_{\tilde{Y}_1} \cap Ff_{\tilde{Y}_2} \right\} \right) \end{array} \right\} \tag{3}$$

Definition 7. Assume that there is a set of feasible solution Λ of MOSOP (1). Then a point x^* taken into consideration to be a Pareto optimal solution of (1) iff there is no such point $x \in \Lambda$ such that $O_k(x^*) \geq O_k(x)\ \forall k$ as well as $O_k(x^*) > O_k(x)$ for as a minimum k.

Definition 8. A point $x^* \in \Lambda$ is called a weak Pareto OS of (1) iff there is not a point $x \in \Lambda$ such that $O_k(x^*) \geq O_k(x)\ \ \forall k$.

4. Proposed Algorithm

4.1 *To Solve MONLPPs using NHFPA*

One may take a MONLPP with k objectives.

$$Min.\ \ O(x) = \left[O_1(x), O_2(x), ...O_k(x) \right]^T \tag{4}$$

Subject to

$x \in U = \left\{ x \in \mathbb{R}^n \mid g_{ij}(x) \leq or = or \geq b_j\ , j = 1\ to\ m \in N \right\}$ and $L_i \leq x_i \leq U_i$ $\left(i = 1\ to\ n \in N, natural\ no. \right)$

Zimmermann [4] demonstrated that the MOOP can be resolved using fuzzy programming techniques.

The MONLPP is solved using the procedures listed below.

Step 1: The MONLPP (4) may be solved as a single objective nonlinear programming problem (SONLPP) by focusing on one objective at a time and overlooking the other objective goals which are called ideal solutions.

Step 2: The result achieved in step 1, the pay-off matrix may be created by identifying the corresponding listed values for every goal in the following manner:

$$\begin{array}{c} \\ x_1 \\ x_2 \\ \cdots \\ x_k \end{array} \begin{array}{c} \begin{array}{cccc} O_1(x) & O_2(x) & \dots & O_k(x) \end{array} \\ \begin{bmatrix} O_1^*(x_1) & O_2^*(x_1) & \dots & O_k^*(x_1) \\ O_1^*(x_2) & O_2^*(x_2) & \dots & O_k^*(x_2) \\ \dots & \dots & \dots & \dots \\ O_1^*(x_k) & O_2^*(x_k) & \dots & O_k^*(x_k) \end{bmatrix} \end{array}$$

In this case, the ideal solutions are $x_1, x_{2,}..........., x_k$ of the objective functions $O_1(x), O_2(x),\dots.., O_k(x)$ accordingly.

Step-3: In each column the highest possible value U_k denotes upper tolerance, or upper bound, for the k^{th} objective function $O_k(x)$, where $U_k = \max\{O_k(x_1), O_k(x_2),......, O_k(x_k)\}$ and the minimum value of each column L_k gives lower tolerance or lower limit for the k^{th} goal function $O_k(x)$, where $L_k = \min\{O_k(x_1), O_k(x_2),......, O_k(x_K)\}$ for $k = 1,2,...., K$.

$$U_k^T = U_k, L_k^T = L_k \quad \text{for truth membership}$$

$$L_k^I = U_k^T - s_k, U_k^I = U_k^T \quad \text{for indeterminacy membership}$$

$$U_k^F = U_k^T, L_k^F = L_k^T + t_k \quad \text{for falsity membership}$$

Where $0 \le s_k \le (U_k - L_k)$ and $0 \le t_k \le (U_k - L_k)$ are specific real numbers in $(0,1)$.

Step-4: Under a NHF environment, we can now define the various hesitant membership functions as linear, exponential, and hyperbolic. Each of them is specified for the membership functions truth, uncertainty, and falsehood, which appears to be more accurate.

4.1.1. Linear-type hesitant membership functions approach (LTHMFA)

The linear type truth membership $Tf_k^{L_i}(O_k(x))$, indeterminacy membership $If_k^{L_i}(O_k(x))$ and a falsehood membership $Ff_k^{L_i}(O_k(x))$ functions under NHF context can be described as below

For truth hesitant fuzzy membership functions:

$$Tf_k^{L_1}(O_k(x)) = \begin{cases} 1 & \text{if } O_k(x) \le L_k^T \\ \mu_1 \left[\dfrac{(U_k^T)^t - (O_k(x))^t}{(U_k^T)^t - (L_k^T)^t} \right] & \text{if } L_k^T \le O_k(x) \le U_k^T \\ 0 & \text{if } O_k(x) \ge U_k^T \end{cases}$$

$$Tf_k^{L_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\leq L_k^T\\ \mu_2\left[\dfrac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right] & \text{if } L_k^T\leq O_k\left(x\right)\leq U_k^T\\ 0 & \text{if } O_k\left(x\right)\geq U_k^T\end{cases}$$

....

$$Tf_k^{L_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\leq L_k^T\\ \mu_n\left[\dfrac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right] & \text{if } L_k^T\leq O_k\left(x\right)\leq U_k^T\\ 0 & \text{if } O_k\left(x\right)\geq U_k^T\end{cases}$$

For indeterminacy hesitant fuzzy membership functions:

$$If_k^{L_1}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\leq L_k^I\\ \kappa_1\left[\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right] & \text{if } L_k^I\leq O_k\left(x\right)\leq U_k^I\\ 0 & \text{if } O_k\left(x\right)\geq U_k^I\end{cases}$$

$$If_k^{L_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\leq L_k^I\\ \kappa_2\left[\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right] & \text{if } L_k^I\leq O_k\left(x\right)\leq U_k^I\\ 0 & \text{if } O_k\left(x\right)\geq U_k^I\end{cases}$$

...

$$If_k^{L_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\leq L_k^I\\ \kappa_n\left[\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right] & \text{if } L_k^I\leq O_k\left(x\right)\leq U_k^I\\ 0 & \text{if } O_k\left(x\right)\geq U_k^I\end{cases}$$

For Falsehood hesitant fuzzy membership functions

$$Ff_k^{L_1}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\leq L_k^F\\ \gamma_1\left[\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right] & \text{if } L_k^F\leq O_k(x)\leq U_k^F\\ 1 & \text{if } O_k(x)\geq U_k^F\end{cases}$$

$$Ff_k^{L_2}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_2\left[\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$\ldots.$$

$$Ff_k^{L_n}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_n\left[\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

The following is a mathematical explanation of objective functions.

$$\begin{aligned}&\text{Max}\ \min_{\mathrm{k}=1,2,..,\mathrm{K}} Tf_k^{L_i}(O_k(x))\\ &\text{Max}\ \min_{\mathrm{k}=1,2,..,\mathrm{K}} If_k^{L_i}(O_k(x))\\ &\text{Min}\ \max_{\mathrm{k}=1,2,..,\mathrm{K}} Ff_k^{L_i}(O_k(x))\end{aligned} \tag{5}$$

$(i=1,2,\ldots..,n)$, subject to all constraints of (4).

Assume that $Tf_k^{L_i}\left(O_k(x)\right)\ge\mu_i, If_k^{L_i}\left(O_k(x)\right)\ge\kappa_i$ and $Ff_k^{L_i}\left(O_k(x)\right)\le\gamma_i (i=1,2,\ldots..,n)$, for all k

Where the parameter $t>0$. Utilizing additional variables μ_i,κ_i and γ_i, the following problem (5) can be transformed to the problem (6)

LTNHMFA $\quad M\,\text{ax}\ \left(\sum_i \mu_i+\sum_i \kappa_i-\sum_i \gamma_i\right)$

Subject to

$$\alpha_1\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\ge\mu_1;\alpha_2\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\ge\mu_2,..,\alpha_n\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\ge\mu_n,$$

$$\beta_1\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\ge\kappa_1,\beta_2\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\ge\kappa_2,...,\beta_n\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\ge\kappa_n,$$

$$\lambda_1\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\le\gamma_1,\lambda_2\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\le\gamma_2,...,\lambda_n\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\le\gamma_n \qquad \ldots(6)$$

$\mu_i \geq \kappa_i, \mu_i \geq \gamma_i, 0 \leq \mu_i + \kappa_i + \gamma_i \leq 3, \mu_i, \kappa_i, \gamma_i \in (0,1)$ and $\alpha_i, \beta_i, \lambda_i \in (0,1)$, for all $(i = 1\, to\, n \in N)$

All the constraints of (4).

Theorem 1: There is only one OS $(x^*, \mu^*, \kappa^*, \gamma^*)$ of problem (6) (LTNHMFA) which is likewise an efficient solution of (4) where $\mu^* = (\mu_1^*, \mu_2^*, ..., \mu_n^*)$, $\kappa^* = (\kappa_1^*, \kappa_2^*, ..., \kappa_n^*)$ and $\gamma^* = (\gamma_1^*, \gamma_2^*, ..., \gamma_n^*)$.

Proof: Suppose that $(x^*, \mu^*, \kappa^*, \gamma^*)$ be the only OS of problem (6) which is an inefficient to solving the problem (4). Then there exist different feasible alternative $x'(x' \neq x^*)$ of the problem (4), so that $O_k(x^*) \leq O_k(x')\ \forall\, k$, $O_k(x^*) < O_k(x')$ for at least one k.

We have $\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t} \leq \frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}$ $\forall\, k$ and $\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t} < \frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}$ for at least one k.

Hence, $\underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t}\right) \leq \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}\right)$,

$\underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t}\right) < \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}\right)$ for at least one k

Similarly, $\underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x^*))^t}{(U_k^I)^t - (L_k^I)^t}\right) \leq \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x'))^t}{(U_k^I)^t - (L_k^I)^t}\right)$ and

$\underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x^*))^t}{(U_k^I)^t - (L_k^I)^t}\right) < \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x'))^t}{(U_k^I)^t - (L_k^I)^t}\right)$

Again,

$\underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x^*))^t}{(U_k^F)^t - (L_k^F)^t}\right) \geq \underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x'))^t}{(U_k^F)^t - (L_k^F)^t}\right)$, $\underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x^*))^t}{(U_k^F)^t - (L_k^F)^t}\right) > \underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x'))^t}{(U_k^F)^t - (L_k^F)^t}\right)$

for at least one k

Now, assume that $\mu^* = \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x^*))^t}{(U_k^T)^t - (L_k^T)^t}\right)$ and $\mu' = \underset{k}{Max}\left(\frac{(U_k^T)^t - (O_k(x'))^t}{(U_k^T)^t - (L_k^T)^t}\right)$,

$\kappa^* = \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x^*))^t}{(U_k^I)^t - (L_k^I)^t}\right)$, $\kappa' = \underset{k}{Max}\left(\frac{(U_k^I)^t - (O_k(x'))^t}{(U_k^I)^t - (L_k^I)^t}\right)$, $\gamma^* = \underset{k}{Min}\left(\frac{(U_k^F)^t - (O_k(x^*))^t}{(U_k^F)^t - (L_k^F)^t}\right)$ and

$\lambda' = \underset{k}{Min}\left(\frac{\left(U_k^F\right)^t - \left(O_k(x')\right)^t}{\left(U_k^F\right)^t - \left(L_k^F\right)^t}\right)$. Then $\mu^* \le (<)\mu'$, $\kappa^* \le (<)\kappa'$ and $\gamma^* \ge (>)\gamma'$ which gives $\left(\mu^* + \kappa^* - \gamma^*\right) < \left(\mu' + \kappa' - \gamma'\right)$ that implies the solution is not optimal which contradicts that $\left(x^*, \mu^*, \kappa^*, \gamma^*\right)$ is a unique OS of (6). As a result, it is a successful problem-solving strategy (6). Thus, the proof is finished.

4.1.2. Exponential-type hesitant membership functions approach (ETHMFA)

The truth membership function of exponential type $T_k^{E_i}(O_k(x))$, indeterminacy membership of exponential type $I_k^{E_i}(O_k(x))$ and a falsehood membership of exponential type $F_k^{E_i}(O_k(x))$ functions under NHF context can be can be described as follows

For truth hesitant fuzzy membership functions:

$$Tf_k^{E_1}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \le L_k^T \\ \mu_1\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t - \left(O_k(x)\right)^t}{\left(U_k^T\right)^t - \left(L_k^T\right)^t}\right)\right\}\right] & \text{if } L_k^T \le O_k(x) \le U_k^T \\ 0 & \text{if } O_k(x) \ge U_k^T \end{cases}$$

$$Tf_k^{E_2}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \le L_k^T \\ \mu_2\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t - \left(O_k(x)\right)^t}{\left(U_k^T\right)^t - \left(L_k^T\right)^t}\right)\right\}\right] & \text{if } L_k^T \le O_k(x) \le U_k^T \\ 0 & \text{if } O_k(x) \ge U_k^T \end{cases}$$

$$\cdots$$

$$Tf_k^{E_n}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \le L_k^T \\ \mu_n\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t - \left(O_k(x)\right)^t}{\left(U_k^T\right)^t - \left(L_k^T\right)^t}\right)\right\}\right] & \text{if } L_k^T \le O_k(x) \le U_k^T \\ 0 & \text{if } O_k(x) \ge U_k^T \end{cases}$$

For indeterminacy hesitant fuzzy membership functions:

$$If_k^{E_1}\left(O_k(x)\right) = \begin{cases} 1 & \text{if } O_k(x) \le L_k^I \\ \kappa_1\left[1 - \exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t - \left(O_k(x)\right)^t}{\left(U_k^I\right)^t - \left(L_k^I\right)^t}\right)\right\}\right] & \text{if } L_k^I \le O_k(x) \le U_k^I \\ 0 & \text{if } O_k(x) \ge U_k^I \end{cases}$$

$$If_k^{E_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_2\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

$$\ldots$$

$$If_k^{E_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_n\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

For Falsity hesitant fuzzy membership functions

$$Ff_k^{E_1}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_1\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$Ff_k^{E_2}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_2\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$\ldots$$

$$Ff_k^{E_n}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_n\left[1-\exp\left\{-\psi\left(\dfrac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

Where ψ is the measure of ambiguity degree or shape parameter which designated by the DM.

Assume that $T_k^{E_i}\left(O_k\left(x\right)\right)\ge\mu_i, I_k^{E_i}\left(O_k\left(x\right)\right)\ge\kappa_i$ and $F_k^{E_i}\left(O_k\left(x\right)\right)\le\gamma_i\left(\left(i=1\, to\, n\in N\right)\right)$, for all k

Where the parameter $t>0$. Utilizing additional variables μ_i,κ_i and γ_i, the given problem (5) can be converted to (7).

ETNHMFA $\quad Max \left(\sum_i \mu_i + \sum_i \kappa_i - \sum_i \gamma_i\right)$

Subject to

$$\alpha_1\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\geq \mu_1;\alpha_2\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\geq \mu_2$$

$$,...,\alpha_n\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\geq \mu_n,\beta_1\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^T\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\geq \kappa_1,$$

$$\beta_2\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\geq \kappa_2,...,\beta_n\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\geq \kappa_n$$

$$\lambda_1\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\leq \gamma_1,\lambda_2\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\leq \gamma_2, \tag{7}$$

$$...,\lambda_n\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\leq \gamma_n$$

$\mu_i \geq \kappa_i, \mu_i \geq \gamma_i, 0 \leq \mu_i + \kappa_i + \gamma_i \leq 3, \mu_i, \kappa_i, \gamma_i \in (0,1)$ and $\alpha_i, \beta_i, \lambda_i \in (0,1)$, for all $(i = 1\, to\, n \in N)$

All the constraints of (4).

Theorem 2: There is only one OS $(x^*, \mu^*, \kappa^*, \gamma^*)$ of problem (7) (ETNHMFA) which is likewise an efficient solution for (4) where $\mu^* = (\mu_1^*, \mu_2^*, ..., \mu_n^*)$, $\kappa^* = (\kappa_1^*, \kappa_2^*, ..., \kappa_n^*)$ and $\gamma^* = (\gamma_1^*, \gamma_2^*, ..., \gamma_n^*)$.

Proof: Suppose that $(x^*, \mu^*, \kappa^*, \gamma^*)$ be the only OS of (7) which is an inefficient to solving the problem (4). Then there exist different feasible alternative $x'(x' \neq x^*)$ of the problem (4), so that $O_k(x^*) \leq O_k(x')$ $\forall\ k$ and $O_k(x^*) < O_k(x')$ for at least one k.

Therefore, $1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\leq 1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}$ $\forall\ k$ and

$1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}< 1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}$ for at least one k.

Hence, $\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)\le \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$,

$\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)< \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$ for at least one k

Similarly, $\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)\le \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x')\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$ and

$\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)< \underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x')\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$ for at least one k

Again, $\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)\ge \underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x')\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$,

$\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)> \underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x')\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$ for at least one k

Now, assume that

$\mu^*=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$ and $\mu'=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^T\right)^t-\left(O_k(x')\right)^t}{\left(U_k^T\right)^t-\left(L_k^T\right)^t}\right)\right\}\right)$,

$\kappa^*=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$, $\kappa'=\underset{k}{Max}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^I\right)^t-\left(O_k(x')\right)^t}{\left(U_k^I\right)^t-\left(L_k^I\right)^t}\right)\right\}\right)$,

$\gamma^*=\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x^*)\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$ and $\lambda'=\underset{k}{Min}\left(1-\exp\left\{-\psi\left(\frac{\left(U_k^F\right)^t-\left(O_k(x')\right)^t}{\left(U_k^F\right)^t-\left(L_k^F\right)^t}\right)\right\}\right)$.

Then $\mu^*\le(<)\mu'$, $\kappa^*\le(<)\kappa'$ and $\gamma^*\ge(>)\gamma'$ which gives $\left(\mu^*+\kappa^*-\gamma^*\right)<\left(\mu'+\kappa'-\gamma'\right)$ that implies the solution is not optimal which contradicts that $\left(x^*,\mu^*,\kappa^*,\gamma^*\right)$ is a unique OS of (7). As a result, it is a successful problem-solving strategy (7). Thus, the proof is finished.

4.1.3. Hyperbolic-type hesitant membership functions approach (HTHMFA)

The truth membership function of hyperbolic type $T_k^{H_i}(O_k(x))$, indeterminacy membership of hyperbolic $I_k^{H_i}(O_k(x))$ and a falsity membership of hyperbolic $F_k^{H_i}(O_k(x))$ functions under NHF context can be can be described as follows

For truth hesitant fuzzy membership functions:

$$Tf_k^{H_1}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_1\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

$$Tf_k^{H_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_2\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

$$\cdots$$

$$Tf_k^{H_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^T\\ \mu_n\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^T\le O_k\left(x\right)\le U_k^T\\ 0 & \text{if } O_k\left(x\right)\ge U_k^T\end{cases}$$

For indeterminacy hesitant fuzzy membership functions:

$$If_k^{H_1}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_1\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

$$If_k^{H_2}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_2\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

$$\cdots .$$

$$If_k^{H_n}\left(O_k\left(x\right)\right)=\begin{cases}1 & \text{if } O_k\left(x\right)\le L_k^I\\ \kappa_n\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right] & \text{if } L_k^I\le O_k\left(x\right)\le U_k^I\\ 0 & \text{if } O_k\left(x\right)\ge U_k^I\end{cases}$$

For Falsity hesitant fuzzy membership functions

$$Ff_k^{H_1}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_1\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

$$Ff_k^{H_2}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_2\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

....

$$Ff_k^{H_n}\left(O_k(x)\right)=\begin{cases}0 & \text{if } O_k(x)\le L_k^F\\ \gamma_n\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right] & \text{if } L_k^F\le O_k(x)\le U_k^F\\ 1 & \text{if } O_k(x)\ge U_k^F\end{cases}$$

where $\tau_k=\dfrac{6}{U_k-L_k}$ is the measure of ambiguity degree or shape parameter which designated by the DM. Assume that $T_k^{H_i}\left(O_k\left(x\right)\right)\ge\mu_i, I_k^{H_i}\left(O_k\left(x\right)\right)\ge\kappa_i$ and $F_k^{H_i}\left(O_k\left(x\right)\right)\le\gamma_i((i=1\, to\, n\in N))$, for all k

Where the parameter $t>0$. Utilizing additional variables μ_i,κ_i and γ_i, the given problem (5) can be converted to the problem (8)

HTNHMFA $\quad Max\left(\sum_i\mu_i+\sum_i\kappa_i-\sum_i\gamma_i\right)$

Subject to

$$\alpha_1\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\mu_1;\alpha_2\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\mu_2$$

$$,..,\alpha_n\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\mu_n,\beta_1\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\kappa_1,$$

$$\beta_2\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\kappa_2,...,\beta_n\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x)\right)^t\right)\tau_k\right\}\right)\ge\kappa_n$$

$$\lambda_1\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left((O_k(x))^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\le\gamma_1,\lambda_2\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left((O_k(x))^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\le\gamma_2,$$

$$\ldots,\lambda_n\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left((O_k(x))^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right)\le\gamma_n \qquad \mu_i\ge\kappa_i,\mu_i\ge\gamma_i,0\le\mu_i+\kappa_i+\gamma_i\le 3\quad,$$

$\mu_i,\kappa_i,\gamma_i\in(0,1)$ and $\alpha_i,\beta_i,\lambda_i\in(0,1)$, for all $(i=1\,to\,n\in N)$ (8)

$\tau_k=\dfrac{6}{U_k-L_k}$. All the constraints of (4).

Theorem 3: There is only one OS $(x^*,\mu^*,\kappa^*,\gamma^*)$ of problem (8) (HTNHMFA) which is likewise an efficient solution for the issue (4) where $\mu^*=(\mu_1^*,\mu_2^*,.......,\mu_n^*)$, $\kappa^*=(\kappa_1^*,\kappa_2^*,.......,\kappa_n^*)$ and $\gamma^*=(\gamma_1^*,\gamma_2^*,.......,\gamma_n^*)$.

Proof: Suppose that $(x^*,\mu^*,\kappa^*,\gamma^*)$ be the only OS of (8) which is an inefficient to solving the problem (4). Then there exist different feasible alternative $x'(x'\ne x^*)$ of (4), so that $O_k(x^*)\le O_k(x')\ \ \forall\ \ k$ and $O_k(x^*)<O_k(x')$ for at least one k.

Therefore, $\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\le\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}$ $\forall$ k and $\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}<\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}$ for at least one k.

Hence,

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)\le\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right),$$

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)<\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right)$$ for at least one k

Similarly,

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right)\le\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right)$$ and

$$\underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right) < \underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right) \text{ for}$$

at least one k.

Again

$$\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x^*)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right) \geq \underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x')\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right),$$

$$\underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x^*)\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right) > \underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x')\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right) \text{ for}$$

at least one k

Now, assume that

$$\mu^* = \left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right) \text{and}$$

$$\mu' = \underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^T\right)^t+\left(L_k^T\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right),$$

$$\kappa^* = \underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x^*)\right)^t\right)\tau_k\right\}\right),$$

$$\kappa' = \underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right),$$

$$\gamma^* = \underset{k}{Max}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\frac{\left(U_k^I\right)^t+\left(L_k^I\right)^t}{2}-\left(O_k(x')\right)^t\right)\tau_k\right\}\right) \text{and}$$

$$\lambda' = \underset{k}{Min}\left(\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(\left(O_k(x')\right)^t-\frac{\left(U_k^F\right)^t+\left(L_k^F\right)^t}{2}\right)\tau_k\right\}\right).$$

Then $\mu^* \leq (<)\mu'$, $\kappa^* \leq (<)\kappa'$ and $\gamma^* \geq (>)\gamma'$ which gives $\left(\mu^*+\kappa^*-\gamma^*\right) < \left(\mu'+\kappa'-\gamma'\right)$ that implies the solution is not optimal which contradicts that $\left(x^*,\mu^*,\kappa^*,\gamma^*\right)$ is a unique OS of (8). As a result, it is a successful problem-solving strategy (8). Thus, the proof is finished.

A numerical example is given in Appendix A.

5.1. Solution procedure for MOSOP using NHFPA.

Step 1. The MOSOP (1) may be solved as a single objective by focusing on one objective at a time subject to the constraints given. Determine the values of the decision variables (DVs) and goal functions.

Step 2. Calculate the values of the remaining objectives based on the values of these DVs.

Step 3. For the remaining objective functions, repeat Steps 1 and 2.

Step 4: Then, according to step 3, the pay-off matrix may be shown as follows:

$$\begin{array}{c} \\ A^1 \\ A^2 \end{array}\begin{array}{c} \begin{array}{cc} W(A) & \rho(A) \end{array} \\ \begin{bmatrix} W(A^1) & \rho(A^1) \\ W(A^2) & \rho(A^2) \end{bmatrix} \end{array}$$

Step 5: The upper and lower bounds are $U_1 = \max\{W(A^1), W(A^2)\}$, $L_1 = \min\{W(A^1), W(A^2)\}$ for weight function $W(A)$,where $W(A) \in [L_1, U_1]$ and the upper and lower limits of objective are $U_2 = \max\{\rho(A^1), \rho(A^2)\}$, $L_2 = \min\{\rho(A^1), \rho(A^2)\}$ for deflection function $\rho(A)$,where $\rho(A) \in [L_2, U_2]$ are identified.

Step 6: Now the NHFPA for MOSOP with linear (or exponential or hyperbolic) accuracy, uncertainty, and falsehood neutrosophic membership functions yield equivalent MONLPP as.

$$\begin{aligned} &Max \min T_k^{\Omega_i}(W(A)); Max \min T_k^{\Omega_i}(\rho(A)); \\ &Max \min I_k^{\Omega_i}(W(A)), Max \min I_k^{\Omega_i}(\rho(A)); \\ &Min \max F_k^{\Omega_i}(W(A)); Min \max F_k^{\Omega_i}(\rho(A)) \end{aligned} \tag{9}$$

Subject to $\sigma(A) \le [\sigma_0]$ $A \in [A_{\min}, A_{\max}], \Omega_i = L_i, H_i, E_i; i = 1, 2, ..., n$ and $x_i \in [L_i, U_i]$ $(i = 1\, to\, n \in N)$.

Now, by utilizing the arithmetic aggregation operator, the equation (9) can be expressed in the subsequent manner:

$$Max\zeta = \frac{\mu_1 + \mu_2 + ... + \mu_n}{n} + \frac{\kappa_1 + \kappa_2 + ... + \kappa_n}{n} - \frac{\gamma_1 + \gamma_2 + ... + \gamma_n}{n}$$

Subject to

$$\begin{aligned} &T_k^{\Omega_1}(W(A)) \ge \mu_1, T_k^{\Omega_2}(W(A)) \ge \mu_2, ..., T_k^{\Omega_n}(W(A)) \ge \mu_n \\ &I_k^{\Omega_1}(W(A)) \ge \kappa_1, I_k^{\Omega_2}(W(A)) \ge \kappa_2, ..., I_k^{\Omega_n}(W(A)) \ge \kappa_n \\ &F_k^{\Omega_1}(W(A)) \le \gamma_1, F_k^{\Omega_2}(W(A)) \le \gamma_2, ..., F_k^{\Omega_n}(W(A)) \le \gamma_n \\ &T_k^{\Omega_1}(\rho(A)) \ge \mu_1, T_k^{\Omega_2}(\rho(A)) \ge \mu_2, ..., T_k^{\Omega_n}(\rho(A)) \ge \mu_n \\ &I_k^{\Omega_1}(\rho(A)) \ge \kappa_1, I_k^{\Omega_2}(\rho(A)) \ge \kappa_2, ..., I_k^{\Omega_n}(\rho(A)) \ge \kappa_n \\ &F_k^{\Omega_1}(\rho(A)) \le \gamma_1, F_k^{\Omega_2}(\rho(A)) \le \gamma_2, ..., F_k^{\Omega_n}(\rho(A)) \le \gamma_n \end{aligned} \tag{10}$$

Subject to $\sigma(A) \le [\sigma_0]$ $A \in [A_{\min}, A_{\max}], \Omega_i = L_i, H_i, E_i; i = 1, 2, ..., n$ and $x_i \in [L_i, U_i]$ $(i = 1\, to\, n \in N)$.

$A \ge 0,\ \mu_n, \kappa_n, \gamma_n \in (0,1); \mu_n + \kappa_n + \gamma_n \le 3, \mu_n \ge \kappa_n, \mu_n \ge \gamma_n, \forall n.$

Step 8: An appropriate mathematical programming algorithm can easily solve the above non-linear programming problem (10).

5.2. Numerical solution of a three-bar truss MOSOP

A well-known planar truss framework of three bars is depicted in Figure 2 in order to decrease the mass of the structure $W(A_1, A_2)$ and decrease the vertical bending at loading point $\rho(A_1, A_2)$ of a statically loaded three-bar planar truss under stress $\sigma_i(A_1, A_2)$ limitations on each of the truss elements $i = 1,2,3$.

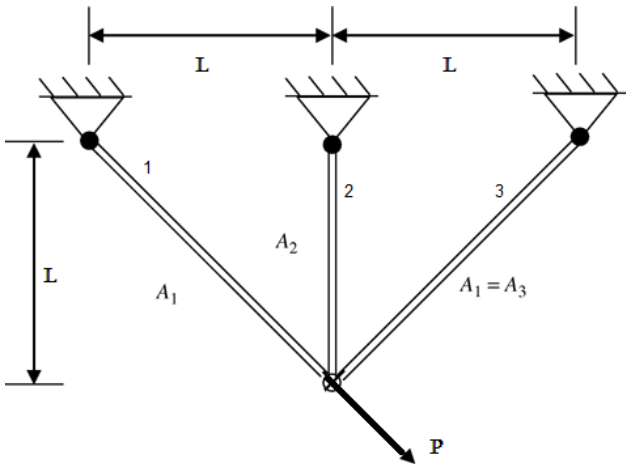

Figure 2. Design of the three-bar planar truss

In the following way, the MOSOP may be expressed:

$$\begin{aligned}
&\text{Minimize } W(A_1, A_2) = \delta L\left(2\sqrt{2}A_1 + A_2\right),\\
&\text{Minimize } \rho(A_1, A_2) = \frac{PL}{Y\left(A_1 + \sqrt{2}A_2\right)}\\
&\text{subject to } \sigma_1(A_1, A_2) \equiv \frac{P\left(\sqrt{2}A_1 + A_2\right)}{2A_1A_2 + \sqrt{2}A_1^2} \le \left[\sigma_1^T\right],\\
&\qquad \sigma_2(A_1, A_2) \equiv \frac{P}{A_1 + \sqrt{2}A_2} \le \left[\sigma_2^T\right],\\
&\qquad \sigma_3(A_1, A_2) \equiv \frac{PA_2}{2A_1A_2 + \sqrt{2}A_1^2} \le \left[\sigma_3^C\right],\\
&\qquad A_i \in \left[A_i^{\min}, A_i^{\max}\right], i = 1,2.
\end{aligned} \qquad ..(11)$$

where, applied load= P ; material density= δ , L= Length of each bar, $\left[\sigma_i^T\right]$=maximum tensile stress limit for $i = 1,2$. $\left[\sigma_3^C\right]$=maximum compressive stress limit, Y = Young's modulus, A_1 = cross sections of bar 1 and bar 3 and A_2 = cross section of bar 2.

The input information for MOSOP (11) are as follows:

P (applied force) = 20 KN, δ (material density)= 100 KN/m^3, L (bar length) = 1 m, $[\sigma_T]$ (maximum tensile stress limit for bars 1 and 3)= 20 KN/m^2, $[\sigma_C]$ (maximum limit of compressive stress for bar 2)= 15 KN/m^2, Y (Young's modulus)= 2×10^8 KN/m^2, range of bar cross section $0.1\times10^{-4}m^2 \le A_1, A_2 \le 5\times10^{-4}m^2$.

Solution: The tabulated values obtained in payoff matrix according to step 2 is as follows:

$$\begin{array}{c c} & \begin{array}{cc} W(A_1,A_2) & \rho(A_1,A_2) \end{array} \\ \begin{array}{c} A^1 \\ A^2 \end{array} & \begin{bmatrix} 2.638958 & 14.64102 \\ 19.14214 & 1.656854 \end{bmatrix} \end{array}$$

Here, $W_U^T = W_U = 19.12412$, $W_L^T = W_L = 2.638958$, $\rho_U^T = \rho_U = 14.64102$, $\rho_L^T = \rho_L = 1.656854$,

$W_L^I = 19.12412 - s_1, W_U^I = W_U^T = 19.12412$, $\rho_L^I = 14.64102 - s_2, \rho_U^I = \rho_U^T = 14.64102$,
$W_U^F = W_U^T = 19.12412, W_L^F = 2.638958 + t_1$, $\rho_U^\upsilon = \rho_U^\mu = 14.64102, \rho_L^\upsilon = 1.656854 + t_2$

where, $s_1, t_1 \in (19.12412 - 2.638958)$ and $s_2, t_2 \in (14.64102 - 1.656854)$.

Using the Linear type hesitant membership functions approach (LTHMFA) (6) the problem (11) equivalent to the following (12)

$$Maximize\ \xi = \frac{1}{3}\left(\sum_{i=1}^{3}\mu_i + \sum_{i=1}^{3}\kappa_i - \sum_{i=1}^{3}\gamma_i\right) \tag{12}$$

Subject to

$$\begin{aligned}
&\left(2\sqrt{2}A_1 + A_2\right)^t + \left((19.14214)^t - (2.63896)^t\right)\mu_1/0.98 \le (19.14214)^t \\
&\left(2\sqrt{2}A_1 + A_2\right)^t + \left((19.14214)^t - (2.63896)^t\right)\mu_2/0.99 \le (19.14214)^t \\
&\left(2\sqrt{2}A_1 + A_2\right)^t + \left((19.14214)^t - (2.63896)^t\right)\mu_3 \le (19.14214)^t
\end{aligned}$$

$$\begin{aligned}
&\left(2\sqrt{2}A_1 + A_2\right)^t + (s_1)^t\kappa_1/0.98 \le (19.14214)^t \\
&\left(2\sqrt{2}A_1 + A_2\right)^t + (s_1)^t\kappa_2/0.99 \le (19.14214)^t \\
&\left(2\sqrt{2}A_1 + A_2\right)^t + (s_1)^t\kappa_3 \le (19.14214)^t
\end{aligned}$$

$$\begin{aligned}
&\left(2\sqrt{2}A_1 + A_2\right)^t - (2.63896)^t - (t_1)^t \le \left((19.14214)^t - (2.63896)^t - (t_1)^t\right)\gamma_1/0.98 \\
&\left(2\sqrt{2}A_1 + A_2\right)^t - (2.63896)^t - (t_1)^t \le \left((19.14214)^t - (2.63896)^t - (t_1)^t\right)\gamma_2/0.99 \\
&\left(2\sqrt{2}A_1 + A_2\right)^t - (2.63896)^t - (t_1)^t \le \left((19.14214)^t - (2.63896)^t - (t_1)^t\right)\gamma_3
\end{aligned}$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left((14.64102)^t-(1.65685)^t\right)\mu_1/0.98\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left((14.64102)^t-(1.65685)^t\right)\mu_2/0.99\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left((14.64102)^t-(1.65685)^t\right)\mu_3\le(14.64102)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+(s_2)^t\kappa_1/0.98\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+(s_2)^t\kappa_2/0.99\le(14.64102)^t$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+(s_2)^t\kappa_3\le(14.64102)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-(1.65685)^t-(t_2)^t\le\left((14.64102)^t-(1.65685)^t-(t_2)^t\right)\gamma_1/0.98$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-(1.65685)^t-(t_2)^t\le\left((14.64102)^t-(1.65685)^t-(t_2)^t\right)\gamma_2/0.99$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-(1.65685)^t-(t_2)^t\le\left((14.64102)^t-(1.65685)^t-(t_2)^t\right)\gamma_3$$

$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le3,\mu_i,\kappa_i,\gamma_i\in(0,1)$ for $i=1,2,3$ and all the constraints of (11).

Using the Exponential type hesitant membership functions approach (ETHMFA) (7) the problem (11) equivalent to the following (13)

$$Maximize\ \ \xi=\frac{1}{3}\left(\sum_{i=1}^{3}\mu_i+\sum_{i=1}^{3}\kappa_i-\sum_{i=1}^{3}\gamma_i\right) \tag{13}$$

Subject to

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left((19.14214)^t-(2.63896)^t\right)\ln\left(1-\frac{\mu_1}{0.98}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t-\left((19.14214)^t-(2.63896)^t\right)\ln\left(1-\frac{\mu_2}{0.99}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t-\left((19.14214)^t-(2.63896)^t\right)\ln\left(1-\mu_3\right)/\psi\le(19.14214)^t$$

$$\left(2\sqrt{2}A_1+A_2\right)^t+(s_1)^t\ln\left(1-\frac{\kappa_1}{0.98}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t+(s_1)^t\ln\left(1-\frac{\kappa_2}{0.99}\right)/\psi\le(19.14214)^t$$
$$\left(2\sqrt{2}A_1+A_2\right)^t+(s_1)^t\ln\left(1-\kappa_3\right)/\psi\le(19.14214)^t$$

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\le\left(\left(19.14214\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\right)\left(-\ln\left(1-\frac{\gamma_1}{0.98}\right)/\psi\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\le\left(\left(19.14214\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\right)\left(-\ln\left(1-\frac{\gamma_2}{0.99}\right)/\psi\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\le\left(\left(19.14214\right)^t-\left(2.63896\right)^t-\left(t_1\right)^t\right)\left(-\ln\left(1-\gamma_3\right)/\psi\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(\left(14.64102\right)^t-\left(1.65685\right)^t\right)\ln\left(1-\frac{\mu_1}{0.98}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(\left(14.64102\right)^t-\left(1.65685\right)^t\right)\ln\left(1-\frac{\mu_2}{0.99}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(\left(14.64102\right)^t-\left(1.65685\right)^t\right)\ln\left(1-\mu_3\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left(s_1\right)^t\ln\left(1-\frac{\kappa_1}{0.98}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left(s_1\right)^t\ln\left(1-\frac{\kappa_2}{0.99}\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t+\left(s_1\right)^t\ln\left(1-\kappa_3\right)/\psi\le\left(14.64102\right)^t$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\le\left(\left(14.64102\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\right)\left(-\ln\left(1-\frac{\gamma_1}{0.98}\right)/\psi\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\le\left(\left(14.64102\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\right)\left(-\ln\left(1-\frac{\gamma_2}{0.98}\right)/\psi\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\le\left(\left(14.64102\right)^t-\left(1.65685\right)^t\text{-}\left(t_2\right)^t\right)\left(-\ln\left(1-\gamma_3\right)/\psi\right)$$

$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le3,\mu_i,\kappa_i,\gamma_i\in(0,1)$ for $(i=1\,to\,n\in N)$ and all the constraints of (11).

Using the Hyperbolic type hesitant membership functions approach (HTHMFA) (8) the problem (11) equivalent to the following (14)

$$Maximize\ \xi=\frac{1}{3}\left(\sum_{i=1}^{3}\mu_i+\sum_{i=1}^{3}\kappa_i-\sum_{i=1}^{3}\gamma_i\right)\tag{14}$$

Subject to

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\mu_1}{0.98}-1\right)\le\frac{\tau_{W(A)}}{2}\left(\left(19.14214\right)^t+\left(2.63896\right)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\mu_2}{0.99}-1\right)\le\frac{\tau_{W(A)}}{2}\left(\left(19.14214\right)^t+\left(2.63896\right)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(2\mu_3-1\right)\le\frac{\tau_{W(A)}}{2}\left(\left(19.14214\right)^t+\left(2.63896\right)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\kappa_1}{0.98}-1\right)\le\frac{\tau_{W(A)}}{2}\left(2\times(19.14214)^t-(s_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(\frac{2\kappa_2}{0.99}-1\right)\le\frac{\tau_{W(A)}}{2}\left(2\times(19.14214)^t-(s_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}+\tanh^{-1}\left(2\kappa_3-1\right)\le\frac{\tau_{W(A)}}{2}\left(2\times(19.14214)^t-(s_1)^t\right)$$

$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}-\tanh^{-1}\left(\frac{2\gamma_1}{0.98}-1\right)\le\frac{\tau_{W(A)}}{2}\left((19.14214)^t+(2.63896)^t+(t_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}-\tanh^{-1}\left(\frac{2\gamma_2}{0.99}-1\right)\le\frac{\tau_{W(A)}}{2}\left((19.14214)^t+(2.63896)^t+(t_1)^t\right)$$
$$\left(2\sqrt{2}A_1+A_2\right)^t\tau_{W(A)}-\tanh^{-1}\left(2\gamma_3-1\right)\le\frac{\tau_{W(A)}}{2}\left((19.14214)^t+(2.63896)^t+(t_1)^t\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\mu_1}{0.98}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\mu_2}{0.99}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(2\mu_3-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\kappa_1}{0.98}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left(2\times(14.64102)^t-(s_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(\frac{2\kappa_2}{0.99}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left(2\times(14.64102)^t-(s_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}+\tanh^{-1}\left(2\kappa_3-1\right)\le\frac{\tau_{\rho(A)}}{2}\left(2\times(14.64102)^t-(s_2)^t\right)$$

$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}-\tanh^{-1}\left(\frac{2\gamma_1}{0.98}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t+(t_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}-\tanh^{-1}\left(\frac{2\gamma_2}{0.99}-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t+(t_2)^t\right)$$
$$\left(20/\left(A_1+\sqrt{2}A_2\right)\right)^t\tau_{\rho(A)}-\tanh^{-1}\left(2\gamma_3-1\right)\le\frac{\tau_{\rho(A)}}{2}\left((14.64102)^t+(1.65685)^t+(t_2)^t\right)$$

Where $\tau_{W(A)}=\dfrac{6}{19.14214-2.638958}$ and $\tau_{\rho(A)}=\dfrac{6}{14.64102-1.656854}$

$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le3,\mu_i,\kappa_i,\gamma_i\in(0,1)$ for $(i=1\ to\ n\in N)$ and all the constraints of (11). On solving the neutrosophic hesitant optimization model (12), (13) and (14) the solution outcomes are outlined in Table 2 and Table 3.

Table 1. Input data for MOSOP (11)

P (KN) (Applied load)	$\delta(KN/m^3)$ (Material density)	L (Length)	$[\sigma_T]\,(KN/m^2)$ (maximum limit of tensile stress)	$[\sigma_C]\,(KN/m^2)$ (maximum limit of compressive stress)	Y (KN/m^2) (Young's modulus)	$A_i^{\min}\left(10^{-4}m^2\right)$ and $A_i^{\max}\left(10^{-4}m^2\right)$ (Cross section of bars)
20	100	1	20	15	2×10^8	$A_1^{\min}=0.1$ $A_1^{\max}=5$ $A_2^{\min}=0.1$ $A_2^{\max}=5$

A Comparative result of MOSOP (11) on basis of different membership function is given in table 2.

Table 2: A comparative optimal results on structural weight and deflection for t=2

Membership functions	Methods	$A_1\times10^{-4}m^2$	$A_2\times10^{-4}m^2$	$W(A_1,A_2)10^2KN$	$\rho(A_1,A_2)10^{-7}m$
Linear Type	FO [10]	0.5995887	3.789761	5.485654	3.356200
	IFO [45]	0.5766526	3.694181	5.325201	3.447673
	NFO	0.581611	3.462786	5.140011	3.628012
	Proposed NHFT	0.5932745	3.391146	5.069180	3.711209
Exponential Type	FO	0.5985788	3.779858	5.472895	3.364678
	IFO	0.5765578	3.678758	5.309510	3.460728
	NFO	0.5965065	3.437476	5.124651	3.664459
	Proposed NHFT	0.5934251	3.399846	5.078306	3.702652
Hyperbolic Type	FO	0.8535467	5.000000	7.414195	2.523782
	IFO	0.8354725	5.000000	7.363073	2.529551
	NFO	0.7994567	5.000000	7.261205	2.541127
	Proposed NHFT	0.7934604	5.000000	7.244245	2.543064

FO: Fuzzy Optimization; IFO: Intuitionistic Fuzzy Optimization; NFO: Neutrosophic fuzzy optimization

A comparative analysis for MOSOP based on several techniques using different membership functions as linear, exponential, hyperbolic types are shown in the Table 2. For all membership functions, it is obvious that the objective values are much superior to other current methods. Furthermore, the proposed NHFT performance measurements for different membership functions may be represented as Hyperbolic>Exponential>Linear. (9.787309 > 8.947535 > 8.780389(sum of weight and deflection)). However, the maximum acceptance degree of our suggested NHFT approach is better attained, demonstrating its superiority over other existing methods.

Table 3: Result comparison with minimizing the indeterminacy membership and maximizing the indeterminacy membership under proposed method at t=2

	Membership functions	$A_1 \times 10^{-4} m^2$	$A_2 \times 10^{-4} m^2$	$W(A_1, A_2)10^2 KN$	$\rho(A_1, A_2)10^{-7} m$
Maximize indeterminacy membership	Linear	0.5932745	3.391146	5.069180	3.711209
	Exponential	0.5934251	3.399846	5.078306	3.702652
	Hyperbolic	0.7934604	5.000000	7.244245	2.543064
Minimize indeterminacy membership	Linear	0.5925640	3.350599	5.026623	3.751623
	Exponential	0.5927716	3.362361	5.038972	3.739808
	Hyperbolic	0.7942572	5.000000	7.246499	2.542807

The comparison of proposed method under maximize and minimize indeterminacy membership function are displayed in the Table 3. From the above table, it is evident that the objective values under maximizing the indeterminacy membership are quite better than minimizing the indeterminacy membership under proposed method. However, our suggested approach's highest attainment of acceptance level is more effectively reached and demonstrates its superiority over reducing uncertainty membership level.

Table 4: Optimal results of different acceptance tolerance on Structural Weight and Deflection for t=2

Acceptance tolerance	Membership functions	$A_1 \times 10^{-4} m^2$	$A_2 \times 10^{-4} m^2$	$W(A_1, A_2)10^2 KN$	$\rho(A_1, A_2)10^{-7} m$
s_1=0.96, s_2=0.98, t_1=0.78, t_2=0.86	Linear	0.5932745	3.391145	5.069179	3.711208
	Exponential	0.5928106	3.364578	5.041298	3.737591
	Hyperbolic	0.7943072	5.000000	7.246640	2.542790
s_1=0.95, s_2=0.98, t_1=0.88, t_2=0.86	Linear	0.5929295	3.371356	5.048413	3.730824
	Exponential	0.5929295	3.371356	5.048414	3.730824
	Hyperbolic	0.7934604	5.000000	7.244245	2.543064

5.3 Sensitivity Analysis

A comparative study for MOSOP based on various acceptance tolerances was conducted using the suggested NHFP approach using linear, exponential, and hyperbolic membership functions. The compromise solution based on various membership functions is presented in Table 2. This result is showing sensitivity in Table 4 with different tolerances. It also shows that the neutrosophic optimization technique with exponential membership functions gives the lightest structural weight and the hyperbolic membership functions gives the least deflection at loading point.

6. Conclusion and Future Research Scope

We developed a MOSOM in a NHF fuzzy environment in this article. A computational algorithm for solving multi-objective structural models using neutrosophic hesitant fuzzy optimization has been developed. We have discussed a comparative study to identify the best optimal result using different membership functions. A three-bar truss numerical example, it shows that exponential membership function gives lightest structural weight whereas hyperbolic membership function gives least deflection in loading point. This method is simple and easy to use.

Our proposed approach might be used in the following fields of research:

- Our proposed method may be used in linear optimization problems with hesitantly and uncertainty.
- It may use in real life decision making of multi objective transportations and assignment problems with interval values.
- It can be expanded to handle issues involving multi objective fractional programming.
- For better decision making, it might be applied in game theory as well as goal programming problem with uncertainty and hesitation.
- It may be implemented in multi objective stochastic linear programming problem.

Our suggested computational technique can be further enhanced for the agricultural, industrial and health management as well, and it may be successfully applied in the variety of field like aircraft control system development, chemical engineering where in multiple objectives with multiple objectives, supply chain management, and industrial neural network architecture.

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Appendix A

Experimental Study

To demonstrate the effectiveness and validity of the suggested approach, we illustrate the numerical instance of formulating a MONLPP as presented below:

$$M_1: \quad Minimize \quad f_1(x) = x_1^{-1} x_2^{-2}$$

$$Minimize \quad f_2(x) = 2x_1^{-2} x_2^{-3}$$

$$\text{s.t} \quad x_1 + x_2 \le 1, x_1, x_2 \ge 0.$$

By solving each objective function separately as stated in (M_1), we obtain the subsequent optimal solution, lower and upper limit for each objective. $X^1 = (0.333, 0.667), X^2 = (0.4, 0.6)$ along with $L_1 = 6.75, U_1 = 6.94, L_2 = 57.87$ and $U_2 = 60.75$.

Linear type membership functions

For f_1: The membership functions of first objective as.

$$Tf_k^{L_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.98\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{L_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.99\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{L_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ \left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{L_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.98\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{L_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.99\left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{L_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ \left[\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{L_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.98\left[\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{L_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.99\left[\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{L_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ \left[\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

For f_2: The membership functions of second objective as. (Linear type)

$$Tf_k^{L_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ 0.98\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{L_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_{21}(x)\le 57.87\\ 0.99\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{L_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ \left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{L_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.98\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{L_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.99\left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{L_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ \left[\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{L_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.98\left[\dfrac{\left(2x_1^{-2}x_2^{-3}\right)^t-(57.87)^t-(t_2)^t}{(60.75)^t-(57.87)^t-(t_2)^t}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{L_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.99\left[\dfrac{\left(2x_1^{-2}x_2^{-3}\right)^t-(57.87)^t-(t_2)^t}{(60.75)^t-(57.87)^t-(t_2)^t}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{L_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ \left[\dfrac{\left(2x_1^{-2}x_2^{-3}\right)^t-(57.87)^t-(t_2)^t}{(60.75)^t-(57.87)^t-(t_2)^t}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

Which is transformed into an equivalent MONLPP with linear type as:

$$Max\ \zeta_1=\frac{\mu_1+\mu_2+\mu_3}{3}+\frac{\kappa_1+\kappa_2+\kappa_3}{3}-\frac{\gamma_1+\gamma_2+\gamma_3}{3}$$

$$Tf_k^{Li}(f_i)\ge\mu_i, If_k^{Li}(f_i)\ge\kappa_i, Ff_k^{Li}(f_i)\le\gamma_i$$

$$x_1+x_2\le 1, x_1,x_2\ge 0; 0\le\mu_i,\kappa_i,\gamma_i\le 1; 0\le t_1,t_2\le 1, 0\le s_1,s_2\le 1$$

$$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le 3\ \text{ for }\ i=1,2,3; k=1,2$$

Exponential type membership functions

For f_1: The membership functions of first objective as.

$$Tf_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right)\right\}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right)\right\}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.75\\ \left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(6.94)^t-(6.75)^t}\right)\right\}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right)\right\}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right)\right\}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}1 & \text{if } f_1(x)\le 6.94-s_1\\ \left[1-\exp\left\{-\psi\left(\dfrac{(6.94)^t-\left(x_1^{-1}x_2^{-2}\right)^t}{(s_1)^t}\right)\right\}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 0 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right)\right\}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right)\right\}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ \left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(6.75)^t-(t_1)^t}{(6.94)^t-(6.75)^t-(t_1)^t}\right)\right\}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

For f_2: The membership functions of second objective as. (Exponential type)

$$Tf_k^{E_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right)\right\}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{E_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right)\right\}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{E_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 57.87\\ \left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(60.75)^t-(57.87)^t}\right)\right\}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{E_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right)\right\}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{E_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right)\right\}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{E_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}1 & \text{if } f_2(x)\le 60.75-s_2\\ \left[1-\exp\left\{-\psi\left(\dfrac{(60.75)^t-\left(2x_1^{-2}x_2^{-3}\right)^t}{(s_2)^t}\right)\right\}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 0 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{E_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.98\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(57.87)^t-(t_2)^t}{(6.94)^t-(57.87)^t-(t_2)^t}\right)\right\}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{E_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.99\left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(57.87)^t-(t_2)^t}{(6.94)^t-(57.87)^t-(t_2)^t}\right)\right\}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{E_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ \left[1-\exp\left\{-\psi\left(\dfrac{\left(x_1^{-1}x_2^{-2}\right)^t-(57.87)^t-(t_2)^t}{(6.94)^t-(57.87)^t-(t_2)^t}\right)\right\}\right] & \text{if } 57.87+t_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

Which is transformed into an equivalent MONLPP with exponential type as:

$$Max\ \zeta_2=\frac{\mu_1+\mu_2+\mu_3}{3}+\frac{\kappa_1+\kappa_2+\kappa_3}{3}-\frac{\gamma_1+\gamma_2+\gamma_3}{3}$$

$$Tf_k^{Ei}(f_i)\ge\mu_i, If_k^{Ei}(f_i)\ge\kappa_i, Ff_k^{Ei}(f_i)\le\gamma_i$$

$$x_1+x_2\le 1, x_1,x_2\ge 0; 0\le\mu_i,\kappa_i,\gamma_i\le 1; 0\le t_1,t_2\le 1, 0\le s_1,s_2\le 1$$

$$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le 3,\psi=4 \text{ for } i=1,2,3;k=1,2$$

Hyperbolic type membership functions

For f_1: The membership functions of first objective as.

$$Tf_k^{H_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75\\ 0.98\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{(6.94)^t+(6.75)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{H_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75\\ 0.99\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{(6.94)^t+(6.75)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Tf_k^{H_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75\\ \left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{(6.94)^t+(6.75)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.75\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{H_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.94-s_1\\ 0.98\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{2(6.94)^t-(s_1)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{H_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.94-s_1\\ 0.99\left[\dfrac{1}{2}+\dfrac{1}{2}\tanh\left\{\dfrac{2(6.94)^t-(s_1)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$If_k^{H_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.94-s_1\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{2(6.94)^t-(s_1)^t}{2}-\left(x_1^{-1}x_2^{-2}\right)^t\right\}\tau_{f_1(x)}\right] & \text{if } 6.94-s_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_1}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(x_1^{-1}x_2^{-2}\right)^t-\frac{(6.94)^t+(6.75)^t+(t_1)^t}{2}\right\}\tau_{f_1(x)}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_2}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(x_1^{-1}x_2^{-2}\right)^t-\frac{(6.94)^t+(6.75)^t+(t_1)^t}{2}\right\}\tau_{f_1(x)}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_3}\left(x_1^{-1}x_2^{-2}\right)=\begin{cases}0 & \text{if } f_1(x)\le 6.75+t_1\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(x_1^{-1}x_2^{-2}\right)^t-\frac{(6.94)^t+(6.75)^t+(t_1)^t}{2}\right\}\tau_{f_1(x)}\right] & \text{if } 6.75+t_1\le f_1(x)\le 6.94\\ 1 & \text{if } f_1(x)\ge 6.94\end{cases}$$

For f_2: The membership functions of second objective as. (Hyperbolic type)

$$Tf_k^{H_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{(60.75)^t+(57.87)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{H_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{(60.75)^t+(57.87)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Tf_k^{H_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{(60.75)^t+(57.87)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 57.87\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{H_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 60.75-s_2\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{2(60.75)^t-(s_2)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{H_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 60.75-s_2\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{2(60.75)^t-(s_2)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$If_k^{H_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 60.75-s_2\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\frac{2(60.75)^t-(s_2)^t}{2}-\left(2x_1^{-2}x_2^{-3}\right)^t\right\}\tau_{f_2(x)}\right] & \text{if } 60.75-s_2\le f_2(x)\le 60.75\\ 1 & \text{if } f_2(x)\ge 60.75\end{cases}$$

$$Ff_k^{H_1}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.98\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(2x_1^{-2}x_2^{-3}\right)^t-\frac{(60.75)^t+(57.87)^t+(t_2)^t}{2}\right\}\tau_{f_2(x)}\right] & \text{if } 57.87+t_2\le f_2(x)\le 6.94\\ 1 & \text{if } f_2(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_2}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ 0.99\left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(2x_1^{-2}x_2^{-3}\right)^t-\frac{(60.75)^t+(57.87)^t+(t_2)^t}{2}\right\}\tau_{f_2(x)}\right] & \text{if } 57.87+t_2\le f_2(x)\le 6.94\\ 1 & \text{if } f_2(x)\ge 6.94\end{cases}$$

$$Ff_k^{H_3}\left(2x_1^{-2}x_2^{-3}\right)=\begin{cases}0 & \text{if } f_2(x)\le 57.87+t_2\\ \left[\frac{1}{2}+\frac{1}{2}\tanh\left\{\left(2x_1^{-2}x_2^{-3}\right)^t-\frac{(60.75)^t+(57.87)^t+(t_2)^t}{2}\right\}\tau_{f_2(x)}\right] & \text{if } 57.87+t_2\le f_2(x)\le 6.94\\ 1 & \text{if } f_2(x)\ge 6.94\end{cases}$$

Which is reduced to equivalent MONLPP with hyperbolic type as:

$$Max\ \zeta_3=\frac{\mu_1+\mu_2+\mu_3}{3}+\frac{\kappa_1+\kappa_2+\kappa_3}{3}-\frac{\gamma_1+\gamma_2+\gamma_3}{3}$$

$$Tf_k^{Hi}(f_i)\ge\mu_i, If_k^{Hi}(f_i)\ge\kappa_i, Ff_k^{Hi}(f_i)\le\gamma_i$$

$$x_1+x_2\le 1, x_1, x_2\ge 0; 0\le\mu_i,\kappa_i,\gamma_i\le 1; 0\le t_1,t_2\le 1, 0\le s_1,s_2\le 1$$

$$\mu_i\ge\kappa_i,\mu_i\ge\gamma_i,\mu_i+\kappa_i+\gamma_i\le 3,\ \tau_{f_k(x)}=\frac{6}{U_k-L_k}\quad\text{for}\ \ i=1,2,3; k=1,2$$

At t = 2, the OS of the MONLPP using the suggested NHFPA under various membership functions are as given below:

Method	Membership function type	x_1	x_2	$f_1(x)$	$f_2(x)$
Proposed NHFT	Linear	0.3659222	0.6340778	6.797139	58.59018
	Exponential	0.3656531	0.6343469	6.796371	58.60181
	Hyperbolic	0.3595183	0.6373039	6.848346	59.77892

References

1. M. Abdel-Basset, M. Gunasekaran, M. Mohamed, and F. Smarandache, "A novel method for solving the fully neutrosophic linear programming problems", *Neural Computing and Applications*, **2018,** Volume 6, 2018.
2. R. E. Bellman and L. A. Zadeh, Decision-Making in a Fuzzy Environment. Management Science, **1970,** Volume 17(4), pp. B141–B164.
3. G.Y. Wang & W.Q. Wang, "Fuzzy optimum design of structure". *Engineering Optimization*, **1985,** Volume 8, pp. 291- 300.
4. H. J. Zimmermann, "Fuzzy programming and linear programming with several objective functions", *Fuzzy Sets and Systems*, **1978,** Volume 1(1), pp.45–55.
5. S. S. Rao, "Description and optimum design of fuzzy mechanical systems", *Journal of Mechanisms, Transmissions, and Automation in Design*, **1987,** Volume 109(1), pp. 126- 132.
6. Y.C. Yeh, & D.S. Hsu, "Structural optimization with fuzzy parameters". *Computer and Structure*, **1990,** Volume 37(6), pp. 917–924.
7. X. Changwen, "Fuzzy optimization of structures by the two-phase method". *Computers & Structures*, **1989,** Volume 31(4), pp. 575-580.
8. C. J. Shih, C. C. Chi, & J. H. Hsiao, "Alternative α-level-cuts methods for optimum structural design with fuzzy resources". *Computers & structures*, **2003,** Volume 81(28), pp. 2579- 2587.
9. C.J. Shih, & H. W. Lee, "Level-cut approaches of first and second kind for unique solution design in fuzzy engineering optimization problems". *Tamkang Journal of Science and Engineering,* **2004,** Volume 7(3), pp. 189-198.
10. S. Dey, & T. K. Roy, "A Fuzzy programming Technique for Solving Multi-objective Structural Problem". *International Journal of Engineering and Manufacturing*, **2014,** Volume 4(5), pp. 24-42.
11. S. Dey, & T. K. Roy, "Multi-objective structural design problem optimization using parameterized t-norm based fuzzy optimization programming Technique". *Journal of Intelligent and Fuzzy Systems*, **2016,** Volume 30(2), pp. 971-982.
12. V. Torra and Y. Narukawa. On hesitant fuzzy sets and decision. *IEEE International Conference on Fuzzy Systems*, 2009, pp. 1378–1382.

13. S.-P. Wan, Y.-L. Qin and J.-Y. Dong, A hesitant fuzzy mathematical programming method for hybrid multi-criteria group decision making with hesitant fuzzy truth degrees. *Knowl.Based Syst*. **2017,** Volume 138, pp. 232–248.
14. Z.S. Xu, S.P. Wan, Zhang, X.L. A Hesitant Fuzzy Programming Method for Hybrid MADM with Incomplete Attribute Weight Information, *INFORMATICA*, 2016, Vol. 27, No. 4, 863–892
15. L. Dymova, K. Kaczmarek, P. Sevastjanov, J. A fuzzy multiple criteria decision making approach with a complete user-friendly computer implementation. *Entropy* **2021,** Volume 23(2), pp. 1-15.
16. L. Dymova, K. Kaczmarek, P. Sevastjanov, Multiple-Criteria Fuzzy Optimization of the Heat Treatment Processes for Two Steel Rolled Products. *Appl. Sci***. 2021**, Volume 11(5), pp. 1-15
17. K. T. Atanassov, Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, **1996,** Volume 20(1), pp. 87–96.
18. P. P. Angelov, Optimization in an intuitionistic fuzzy environment. *Fuzzy Sets and Systems*, **1997,** Volume 86(3), pp. 299–306.
19. B. Singh, M. Sarkar, T.K. Roy, "Intuitionistic fuzzy optimization of truss design: a comparative study" *International Journal of Computer Organization Trends*,**2016,** Volume 6(6), pp.25-33
20. M. Sarkar, T.K. Roy, multi-objective welded beam design optimization using t-norm and t-co-norm based on intuitionistic fuzzy optimization technique, *Advanced in Fuzzy mathematics*, **2017,** Volume 12(3), pp. 549-575.
21. B. Kizilaslan, E. Egrioglu , A.A. Evren , "Intuitionistic fuzzy ridge regression functions", *Communications in Statistics - Simulation and Computation*, 2019, volume 49(3), pp. 1-10.
22. S.F. Zhang, and S. Y. Liu,. A GRA-based intuitionistic fuzzy multi-criteria group decision making method for personnel selection. *Expert Systems with Applications*, **2019,** Volume 38, pp. 11401–11405.
23. A. Kabiraj, P. Nayak, and S. Raha, Solving Intuitionistic Fuzzy Linear Programming Problem—II. *International Journal of Intelligence Science*, **2019,** Volume 9, pp. 93-110.
24. A.A.H. Ahmadini and F. Ahmad, A novel intuitionistic fuzzy preference relations for multi-objective goal programming problems. *Journal of Intelligence & Fuzzy Systems,* **2021,** Volume 40, pp. 4761–4777.
25. F. J.,Santos Arteaga, A. Ebrahimnejad, , & A. Zabihi. A New Approach for Solving Intuitionistic Fuzzy Data Envelopment Analysis Problems. *Fuzzy Optimization and Modelling Journal*, **2021,** Volume 2(2), 2021, pp. 46-57.
26. I. Beg, T. Rashid, Hesitant intuitionistic fuzzy linguistic term sets, *Notes on Intuitionistic Fuzzy Sets*, **2014,** Volume 20(3), pp. 53–64.
27. K.B. Shailendra, introduced hesitant intuitionistic fuzzy algorithm for multi objective optimization problem, *Operational Research*, springer, **2021**, pp. 1-27.
28. F. Smarandache, Neutrosophy, neutrosophic probability, set and logic, Amer. Res. Press, Rehoboth, USA,**1998** Volume 105, pp. 1-105.
29. J. Ye, W. Cui, and Z. Lu. "Neutrosophic number nonlinear programming problems and their general solution methods under neutrosophic number environments", *Axioms*, **2018**, Volume 7(1), pp. 1-9.
30. W. Zhou and Z. Xu. "Portfolio selection and risk investment under the hesitant fuzzy environment", *Knowledge-Based Systems*, **2018,** Volume 144, pp. 21–31.

31. Ahmad F, Smarandache F "Single Valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multiobjective Nonlinear Optimization Problem", *Neutrosophic Sets and Systems*, **2018,** Volume 22, pp. 134-148.
32. F. Smarandache. A Unifying Field in Logics: *Neutrosophic Logic*. **1999**, pp. 134-148.
33. V. Torra and Y. Narukawa. "On hesitant fuzzy sets and decision", *IEEE International Conference on Fuzzy Systems*, **2009,** Volume 10 (7), pp. 1378–1382.
34. J. Ye., "Multiple-attribute decision-making method under a single-valued neutrosophic hesitant fuzzy environment", *Journal of Intelligent Systems*, **2015,** Volume 24(1), pp. 23–36.
35. M. Sarkar, S. Dey, & T. K. Roy, "Multi-objective Neutrosophic optimization Technique and its application to structural design", *International Journal of Computer Applications* **2016,** Volume 148(12), pp. 0975-8887.
36. F. Ahmad, A.Y. Adhami and F. Smarandache, "Single Valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multiobjective Nonlinear Optimization Problem", *Neutrosophic Sets and Systems*, **2018,** Volume 22, pp.76–86.
37. F. Ahmad, A.Y. Adhami and F. Smarandache, "Modified neutrosophic fuzzy optimization model for optimal closed-loop supply chain management under uncertainty", *Optimization theory based on neutrosophic and plithogenic sets*, Elsevier, 2020, pp. 343–403.
38. S. Ahmad, F. Ahmad and M. Sharaf, "Supplier selection problem with type-2 fuzzy parameters: A neutrosophic optimization approach", *International journal of Fuzzy Systems,* **2021,** Volume 23, 2021, pp. 755–775.
39. D. Rani, T. Gulati, H. Garg, "Multi-objective non-linear programming problem in intuitionistic fuzzy environment: Optimistic and pessimistic view point", *Expert systems with applications*, **2016** Volume 64, 2016, pp.228–238.
40. B.C. Giri, M. U. Molla, P. Biswas, "TOPSIS Method for Neutrosophic Hesitant Fuzzy Multi-Attribute Decision Making", *INFORMATICA*, **2020,** Volume 31(1), pp. 35–63.
41. B. Tanuwijaya, G. Selvachandran, L. H. Son, Md. Abdel-Basset, H. X. Huynh, Van-Huy Pham and Md. Ismail, "Novel SVNHFTS Model: Applications in Indonesian and Argentinian Stock Index Forecasting", Digital Object Identifier 10.1109/IEEE ACCESS.2020.2982825, **2020,** Volume 8, 2020, pp. 60126- 60141.
42. F. Ahmad, "Supplier selection problem with type-2 fuzzy parameters: A neutrosophic optimization approach", *International Journal of Fuzzy Systems*, **2021**, Volume 23(5), pp. 1-21.
43. M.F. Khan, A. Haq, A. Ahmed, & I. Ali. Multiobjective Multi-Product Production Planning Problem Using Intuitionistic and Neutrosophic Fuzzy Programming. IEEE Access, **2021,** Volume 9, pp. 37466-37486.
44. S. Gupta, A. Haq, I. Ali, Neutrosophic Goal Programming Approach for the Dash Diet Model. Neutrosophic Sets and Systems, 2022, Volume 50, pp. 189-207.
45. S. Dey, "Intuitionistic Fuzzy Multi–Objective Structural Optimization using Non-linear Membership Functions", International Journal of Computer & Organization Trends (IJCOT), **2017** Volume 41(1), pp. 14-20.

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