Neutrosophic n-Valued Refined Sets and Topologies

Murad Arar *
Mathematics Department, Prince Sattam Bin Abdulaziz University-Al Kharj, Saudi Arabia; m.arar@psau.edu.sa , muradshhada@gmail.com
*Correspondence: m.arar@psau.edu.sa

Abstract. In n-Valued refined logic truth value $T$ can be split into many types of truths: $T_1, T_2, ..., T_p$ and $I$ into many types of indeterminacies: $I_1, I_2, ..., I_r$ and $F$ into many types of falsities: $F_1, F_2, ..., F_s$, where $p, r$ and $s$ are integers greater than 1, and $p + r + s = n$. Importance of n-valued refined logic and sets appeared in different applications specially in medical diagnosis. In this paper we post a condition on neutrosophic n-valued refined sets to make them functional to be applied in different mathematical branches. We define and study n-valued refined topological spaces. We defined neutrosophic n-valued refined $\alpha$-open, $\beta$-open, pre-open and semi-open sets and studied their properties. We constructed different counter examples to clarify the relations between these different types of neutrosophic n-valued refined generalized open sets.

Keywords: n-valued refined topology; refined logic; refined sets; n-valued refined $\alpha$-open; semi-open sets; n-valued refined generalized open sets.

1. INTRODUCTION

Neutrosophic sets are, first, introduced in 2005 by [26,27] as a generalization of intuitionistic fuzzy sets [13], where any element $x \in X$ we have three degrees; the degree of membership(T), indeterminacy(I), and non-membership(F). Neurosophic vague sets are introduced in 2015 by [30]. Neutrosophic vague topological spaces introduced in [21] we are many different notations are introduced and studied such as neurosophic vague continuity and compactness.

Neutrosophic topologies are defined and studied by Smarandache [27], Lupianez [19,20] and Salama [2]. Open and closed neutrosophic sets, interior, exterior, closure and boundary of neutrosophic sets can be found in [29].

Neutrosophic sets applied to generalize many notaions about soft topology and applications [18, 23, 16], generalized open and closed sets [31], fixed point theorems [18], graph theory...
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and rough topology and applications [22]. Neutrosophy has many applications especially in decision making, for more details about new trends of neutrosophic applications one can consult [1]-[7].

Generalized topology and continuity introduced in 2002 in [?] which is a generalization of topological spaces and has different properties than general topology, see for example [8], [11] and [12]. Neutrosophic generalized sets and topologies are introduced and studies by Murad M. Arar in 2020 see [9] and [10]. In n-valued refined logic truth value T can be split into many types of truths: \( T_1, T_2, ..., T_p \) and I into many types of indeterminacies: \( I_1, I_2, ..., I_r \) and \( F \) into many types of falsities: \( F_1, F_2, ..., F_s \), where \( p, r \) and \( s \) are integers greater than 1, and \( p + r + s = n \) see [28]. Importance of n-valued refined logic and sets appeared in different applications specially in medial diagnosis see [25] and [14], where a strong assumption is assumed to make them functional; that is \( p = r = s \).

**Definition 1.1.** [26]: We say that the set \( A \) is neutrosophic on \( X \) if

\[
A = \{ \langle x, \mu_A(x), \sigma_A(x), \nu_A(x) \rangle; x \in X \} ; \mu, \sigma, \nu : X \rightarrow \mathbb{R} \text{ and } -1 \leq \mu(x) + \sigma(x) + \nu(x) \leq 3.
\]

The class of all neutrosophic sets on the universe \( X \) will be denoted by \( \mathcal{N}(X) \). The basic neutrosophic operations (inclusion, union, and intersection) where first introduced by [24].

**Definition 1.2 (Neutrosophic sets operations).** Let \( A, A_\alpha, B \in \mathcal{N}(X) \) such that \( \alpha \in \Delta \). Then we define the neutrosophic:

1. (Inclusion) \( A \subseteq B \) if \( \mu_A(x) \leq \mu_B(x), \sigma_A(x) \geq \sigma_B(x) \) and \( \nu_A(x) \geq \nu_B(x) \).
2. (Equality) \( A = B \Leftrightarrow A \subseteq B \) and \( B \subseteq A \).
3. (Intersection) \( \bigcap_{\alpha \in \Delta} A_\alpha(x) = \{ \langle x, \wedge_{\alpha \in \Delta} \mu_{A_\alpha}(x), \vee_{\alpha \in \Delta} \sigma_{A_\alpha}(x), \vee_{\alpha \in \Delta} \nu_{A_\alpha}(x) \rangle; x \in X \} \).
4. (Union) \( \bigcup_{\alpha \in \Delta} A_\alpha(x) = \{ \langle x, \vee_{\alpha \in \Delta} \mu_{A_\alpha}(x), \wedge_{\alpha \in \Delta} \sigma_{A_\alpha}(x), \wedge_{\alpha \in \Delta} \nu_{A_\alpha}(x) \rangle; x \in X \} \).
5. (Complement) \( A^c = \{ \langle x, \nu_A(x), 1 - \sigma_A(x), \mu_A(x) \rangle; x \in X \} \)
6. (Universal set) \( 1_X = \{ \langle x, 1, 1, 1 \rangle; x \in X \} \); called the neutrosophic universal set.
7. (Empty set) \( 0_X = \{ \langle x, 0, 1, 1 \rangle; x \in X \} \); called the neutrosophic empty set.

**Proposition 1.3.** [24] For \( A, A_\alpha \in \mathcal{N}(X) \) for every \( \alpha \in \Delta \) we have:

1. \( A \cap ( \bigcup_{\alpha \in \Delta} A_\alpha) = \bigcup_{\alpha \in \Delta} (A \cap A_\alpha) \).
2. \( A \cup ( \bigcap_{\alpha \in \Delta} A_\alpha) = \bigcap_{\alpha \in \Delta} (A \cup A_\alpha) \).

**Definition 1.4.** [24] [Neutrosophic Topology] \( \tau \subseteq \mathcal{N}(X) \) is called a neutrosophic topology for \( X \) if

1. \( 0_X, 1_X \in \tau \).
2. If \( A_\alpha \in \tau \) for every \( \alpha \in \Delta \), then \( \bigcup_{\alpha \in \Delta} A_\alpha \in \tau \).
3. For every \( A, B \in \tau \), we have \( A \cap B \in \tau \).

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The ordered pair \((X, \tau)\) will be said a *neutrosophic space* over \(X\). The elements of \(\tau\) will be called *neutrosophic open sets*. For any \(A \in \mathcal{N}(X)\), If \(A^c \in \tau\), then we say \(A\) is *neutrosophic closed*.

2. **Neutrosophic n-valued refined sets and topology**

In neutrosophic \(n\)-valued refined logic (see [28]) the membership degree refined (split) into \(r\) values \(\mu_1, \mu_2, ..., \mu_r\), the indeterminacy refined into \(s\) values \(\sigma_1, \sigma_2, ..., \sigma_s\) and the nonmembership refined into \(t\) values \(\nu_1, \nu_2, ..., \nu_t\) such that \(n = r + s + t\) and

\[-0 \leq \sum_{i=1}^{r} \mu_i(x) + \sum_{i=1}^{s} \sigma_i(x) + \sum_{i=1}^{t} \nu_i(x) \leq n^+\]

Some authors assumes that \(r = s = t\) see for example [14]. Actually, there is no guarantee that the membership, indeterminacy and nonmembership degrees refined or split into the same number of values, and we will not get a functional system of Neutrosophic \(n\)-valued refined sets if no more restrictions are assumed on \(r, s\) and \(t\). This accurse when we define the basic set operations on the neutrosophic \(n\)-valued refined sets, especially when we try to define the neutrosophic \(n\)-valued refined complement of a given neutrosophic \(n\)-valued refined set; where \(r\) plays the role of \(t\) and vice versa. We will be back to this discussion after stating some definitions and theorems.

**Definition 2.1.** [26]: A is called a *neutrosophic \(n\)-valued refined set* on a universe \(X\) if \(A = \{x, \mu_A^1(x), \mu_A^2(x), ..., \mu_A^r(x); \sigma_A^1(x), \sigma_A^2(x), ..., \sigma_A^s(x); \nu_A^1(x), \nu_A^2(x), ..., \nu_A^t(x); x \in X\}\); \(\mu_A^i, \sigma_A^j, \nu_A^k: X \to -0, 1^+\) for every \(i = 1, ..., r, j = 1, ..., s, k = 1, ..., t\) such that \(r + s + t = n\) and

\[-0 \leq \sum_{i=1}^{r} \mu_A^i(x) + \sum_{j=1}^{s} \sigma_A^j(x) + \sum_{k=1}^{t} \nu_A^k(x) \leq n^+\]

The class of all neutrosophic \(n\)-valued refined sets on the universe \(X\) will be denoted by \(\mathcal{R}_n(X)\).

The following is the definition of the basic operations (inclusion, union, intersection and complement) on neutrosophic \(n\)-valued refined sets.

**Definition 2.2.** [*Neutrosophic \(n\)-valued refined sets operations*] Let \(A, A_\alpha, B \in \mathcal{R}_n(X)\) such that \(\alpha \in \Delta\). Then we define the neutrosophic \(n\)-valued refined:

1. *(Inclusion)*: \(A \subseteq R B\) If \(\mu_A^i(x) \leq \mu_B^i(x), \sigma_A^j(x) \geq \sigma_B^j(x)\) and \(\nu_A^k(x) \geq \nu_B^k(x)\) for every \(i = 1, ..., r, j = 1, ..., s, k = 1, ..., t\).

2. *(Equality)*: \(A = B \iff A \subseteq R B\) and \(B \subseteq R A\).

3. *(Intersection)*: \(\bigcap_{\alpha \in \Delta} A_\alpha(x) = \{x, \wedge_{\alpha \in \Delta} \mu_{A_\alpha}^1(x), ..., \wedge_{\alpha \in \Delta} \mu_{A_\alpha}^r(x); \vee_{\alpha \in \Delta} \sigma_{A_\alpha}^1(x), ..., \vee_{\alpha \in \Delta} \sigma_{A_\alpha}^s(x); \wedge_{\alpha \in \Delta} \nu_{A_\alpha}^1(x), ..., \wedge_{\alpha \in \Delta} \nu_{A_\alpha}^t(x); x \in X\}\).

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(4) \((\text{Union}) \bigcup_{\alpha \in \Delta} A_{\alpha}(x) = \{ (x, \bigvee_{\alpha \in \Delta} \mu_{A_{\alpha}}^{1}(x), ..., \bigvee_{\alpha \in \Delta} \mu_{A_{\alpha}}^{r}(x); \bigwedge_{\alpha \in \Delta} \sigma_{A_{\alpha}}^{1}(x), ..., \bigwedge_{\alpha \in \Delta} \sigma_{A_{\alpha}}^{s}(x); \bigwedge_{\alpha \in \Delta} \nu_{A_{\alpha}}^{1}(x), ..., \bigwedge_{\alpha \in \Delta} \nu_{A_{\alpha}}^{r}(x)); x \in X\}\). 

(5) \((\text{Complement}) A^{c} = \{ (x, \nu_{A}^{1}(x), ..., \nu_{A}^{r}(x); 1 - \sigma_{A}^{1}(x), ..., 1 - \sigma_{A}^{s}(x); \mu_{A}^{1}(x), ..., \mu_{A}^{r}(x)); x \in X\}\). 

(6) \((\text{Universal set}) 1_{X} = \{ (x, 1, ..., 1; 0, ..., 0; 0, ..., 0); x \in X\}; \text{ called the neutrosophic n-valued refined universal set.}\) 

(7) \((\text{Empty set}) 0_{X} = \{ (x, 0, ..., 0; 1, ..., 1; 1, ..., 1); x \in X\}; \text{ called the neutrosophic n-valued refined empty set.}\)

**Theorem 2.3.** Let \(A_{\alpha}, A, B \in \mathcal{R}_{n}(X)\) such that \(\alpha \in \Delta\). Then we have

1. If \(A \subseteq R B \subseteq R C\), then \(A \subseteq R C\).
2. If \(A \subseteq R B\), then \(B^{c} \subseteq R A^{c}\).
3. \((\bigcup_{\alpha \in \Delta} A_{\alpha}) \cap_{R} A = \bigcup_{\alpha \in \Delta} (A_{\alpha} \cap_{R} A)\)
4. \((\bigcap_{\alpha \in \Delta} A_{\alpha}) \cup_{R} A = \bigcap_{\alpha \in \Delta} (A_{\alpha} \cup_{R} A)\) \[\text{Demorgan’s Laws}\]
5. \((A \cup_{R} B)^{c} = A^{c} \cap_{R} B^{c}\)
6. \((A \cap_{R} B)^{c} = A^{c} \cup_{R} B^{c}\)

**Proof.** (1) and (2) are Straight forward! (3) and (4) can be proved using the following two propositions:

\[-(\bigwedge_{\alpha \in \Delta} a_{\alpha}) \land b = \bigvee_{\alpha \in \Delta} (a_{\alpha} \land b)\]
\[-(\bigvee_{\alpha \in \Delta} a_{\alpha}) \lor b = \bigwedge_{\alpha \in \Delta} (a_{\alpha} \lor b)\]

Now, we prove (3) and (4) can be proved by duality:

\[(A \cup_{R} B)^{c} = \{ \langle x, \mu_{A}^{1}(x) \lor \mu_{B}^{1}(x), ..., \mu_{A}^{r}(x) \lor \mu_{B}^{r}(x); \sigma_{A}^{1}(x) \land \sigma_{B}^{1}(x), ..., \sigma_{A}^{s}(x) \land \sigma_{B}^{s}(x); \nu_{A}^{1}(x) \land \nu_{B}^{1}(x), ..., \nu_{A}^{r}(x) \land \nu_{B}^{r}(x)); x \in X\}\}^{c}\]

\[= \{ \langle x, \nu_{A}^{1}(x) \land \nu_{B}^{1}(x), ..., \nu_{A}^{r}(x) \land \nu_{B}^{r}(x); 1 - (\sigma_{A}^{1}(x) \land \sigma_{B}^{1}(x)), ..., 1 - (\sigma_{A}^{s}(x) \land \sigma_{B}^{s}(x)); \mu_{A}^{1}(x) \lor \mu_{B}^{1}(x), ..., \mu_{A}^{r}(x) \lor \mu_{B}^{r}(x)); x \in X\}\}^{c}\]

\[= \{ \langle x, \nu_{A}^{1}(x) \land \nu_{B}^{1}(x), ..., \nu_{A}^{r}(x) \land \nu_{B}^{r}(x); (1 - \sigma_{A}^{1}(x)) \lor (1 - \sigma_{B}^{1}(x)), ..., (1 - \sigma_{A}^{s}(x)) \lor (1 - \sigma_{B}^{s}(x)); \mu_{A}^{1}(x) \lor \mu_{B}^{1}(x), ..., \mu_{A}^{r}(x) \lor \mu_{B}^{r}(x)); x \in X\}\}^{c}\]

\[= \{ \langle x, \nu_{A}^{1}(x), ..., \nu_{A}^{r}(x); 1 - \sigma_{A}^{1}(x), ..., 1 - \sigma_{A}^{s}(x); \mu_{A}^{1}(x), ..., \mu_{A}^{r}(x)); x \in X\}\}^{c}\]

\[\{ \langle x, \nu_{B}^{1}(x), ..., \nu_{B}^{r}(x); 1 - \sigma_{B}^{1}(x), ..., 1 - \sigma_{B}^{s}(x); \mu_{B}^{1}(x), ..., \mu_{B}^{r}(x)); x \in X\}\} = A^{c} \cap_{R} B^{c}\]

So, as the above theorem shows, the system defined in Definition 2.2 is rich to a certain extent, but it still needs to be stronger to deal with some situations: for example \(A \cap_{R} A^{c}\) is not well-defined if \(r \neq t\). The concept True (membership) and False (nonmembership) are related, it is reasonable to discuss them in any world simultaneously, so we can assume \(r = t\), and this is what F. Smarandache did in [28] when he discussed the relative (absolute)
truth and falsity simultaneously. The condition \( r = s = t \) mentioned in [14] is very strong and will not add any value to us, actually it implies that \( n \) is divisible by 3, since \( n = r + s + t \), so it does not include some worlds, for example a world of seven and five-valued logic which discussed in [28]. On the other hand if we, only, assume \( r = t \), then \( n \) can be any value since we have not assumed any condition on \( s \) and worlds of any \( n \)-valued logic will be included.

**Definition 2.4.** Let \( A \) be a *neutrosophic \( n \)-valued refined set* on a universe \( X \). If \( r = s \), then we call \( A \) a homogeneous neutrosophic \( n \)-valued refined set. \( n \) will be called the dimension of \( A \), and \( r, s \) will be called the sub-dimensions of \( A \). The class of all homogeneous neutrosophic \( n \)-valued refined sets on the universe \( X \) with sub-dimensions \( r, s \) will be denoted by \( \mathcal{R}_{(n,r,s)}(X) \).

The following is obvious:

**Proposition 2.5.** Let \( A, B \in \mathcal{R}_{(n,r,s)}(X) \). Then

1. \( A \cap_R B \in \mathcal{R}_{(n,r,s)}(X) \).
2. \( A \cup_R B \in \mathcal{R}_{(n,r,s)}(X) \).
3. \( A^c \in \mathcal{R}_{(n,r,s)}(X) \).

**Example 2.6.** Let \( X = \{a, b\} \), and let \( A, B \in \mathcal{R}_{(5,2,1)}(X) \) such that

\[
A = \{\langle a, 0.2, 0.1; 0.7; 0.1, 0.4 \rangle, \langle b, 0.5, 0.3; 0.2; 0.9, 0.5 \rangle\} \quad \text{and} \quad B = \{\langle a, 0.4, 0.01; 0.3; 0.4, 0.3 \rangle, \langle b, 0.4, 0.2; 0.1; 0.7, 0.7 \rangle\}.
\]

Then we have:

\[
A \cap_R B = \{\langle a, 0.2, 0.01; 0.7; 0.4, 0.4 \rangle, \langle b, 0.4, 0.2; 0.2; 0.9, 0.7 \rangle\} \in \mathcal{R}_{(5,2,1)}(X)
\]
\[
A \cup_R B = \{\langle a, 0.4, 0.1; 0.3; 0.1, 0.3 \rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5 \rangle\} \in \mathcal{R}_{(5,2,1)}(X)
\]
\[
A^c = \{\langle a, 0.1, 0.4; 0.3; 0.2, 0.1 \rangle, \langle b, 0.9, 0.5; 0.8; 0.5, 0.3 \rangle\} \in \mathcal{R}_{(5,2,1)}(X)
\]

**Definition 2.7 (Neutrosophic \( n \)-valued Refined Topology).** \( \tau \subset \mathcal{R}_{(n,r,s)}(X) \) is called a neutrosophic \( n \)-valued refined topology on \( X \) if

1. \( 0_X, 1_X \in \tau \).
2. For every \( A, B \in \tau \), we have \( A \cap_R B \in \tau \).
3. If \( A_\alpha \in \tau \) for every \( \alpha \in \Delta \), then \( \cup_{\alpha \in \Delta} A_\alpha \in \tau \).

Elements of \( \tau \) are called neutrosophic \( n \)-valued refined open sets. \( A \in \mathcal{R}_{(n,r,s)}(X) \) is said neutrosophic \( n \)-valued refined closed set if \( A^c \in \tau \).

The class of all neutrosophic \( n \)-valued refined topologies on \( X \) with sub-dimensions \( r, s \) will be denoted by \( \text{TOP}_{(n,r,s)}(X) \).

**Definition 2.8.** Let \( \tau \subset \mathcal{R}_{(n,r,s)}(X) \) be a neutrosophic \( n \)-valued refined topology on \( X \) and let \( A \in \mathcal{R}_{(n,r,s)}(X) \). Then:

1. The neutrosophic \( n \)-valued refined interior of \( A \) is defined to be

\[
\text{Int}_R(A) = \cup_R \{O \in \tau; O \subseteq_R A\}.
\]

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The neutrosophic n-valued refined closure of \( A \) is defined to be
\[
Cl_R(A) = \cap_R \{ C \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \text{ and } A \subseteq_R C \}
\]

Example 2.9. Let \( X = \{a, b\} \), and let \( \tau = \{0_X, 1_X, A, B, C, D\} \subset \mathcal{R}_{(5,2,1)}(X) \) where
\[
A = \{\langle a, 0.2, 0.1; 0.7; 0.1, 0.4\rangle, \langle b, 0.5, 0.3; 0.2; 0.9, 0.5\rangle\},
B = \{\langle a, 0.4, 0.01; 0.3; 0.4, 0.3\rangle, \langle b, 0.4, 0.2; 0.1; 0.7, 0.7\rangle\},
C = \{\langle a, 0.2, 0.01; 0.7; 0.4, 0.4\rangle, \langle b, 0.4, 0.2; 0.2; 0.9, 0.7\rangle\}
D = \{\langle a, 0.4, 0.1; 0.3; 0.1, 0.3\rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5\rangle\}
\]

Then \( \tau \) is a Neutrosophic 5-valued refined topology on \( X \). All closed set are:
\( 0_X, 1_X, A^c, B^c, C^c, D^c \)

\[
A^c = \{\langle a, 0.1, 0.4; 0.3; 0.2, 0.1\rangle, \langle b, 0.9, 0.5; 0.8; 0.5, 0.3\rangle\},
B^c = \{\langle a, 0.4, 0.3; 0.7; 0.4, 0.01\rangle, \langle b, 0.7, 0.7; 0.9; 0.4, 0.2\rangle\},
C^c = \{\langle a, 0.4, 0.4; 0.3; 0.2, 0.01\rangle, \langle b, 0.9, 0.7; 0.8; 0.4, 0.2\rangle\}
D^c = \{\langle a, 0.1, 0.3; 0.7; 0.4, 0.1\rangle, \langle b, 0.7, 0.5; 0.9; 0.5, 0.3\rangle\}
\]

Let \( K = \{\langle a, 0.43, 0.09; 0.2; 0.1, 0.2\rangle, \langle b, 0.5, 0.25; 0.1; 0.5, 0.6\rangle\}. \) Then the open sets in \( \tau \) contained in \( K \) are only \( 0_X, B, C \), so that \( Int_R(K) = 0_X \sqcup_R B \sqcup_R C = B \). Now, consider the set \( K^c = \{\langle a, 0.1, 0.2; 0.8; 0.43, 0.09\rangle, \langle b, 0.5, 0.6; 0.9; 0.5, 0.25\} \) and compute \( Cl_R(K^c) \); the only closed sets containing \( K^c \) are \( 1_X, B^c \) and \( C^c \), so that \( Cl_R(K^c) = 1_X \cap_R B^c \cap_R C^c = B^c \). Which means \( Cl_R(K^c) = B^c \) and so \( (Cl_R(K^c))^c = B = Int_R(K) \); that is \( Int_R(K) = (Cl_R(K^c))^c \) and this leads us to the following theorem:

**Theorem 2.10.** Let \( (X, \tau) \) be an n-valued refined topological space with sub-dimensions \( r, s \) and let \( A \in \mathcal{R}_{(n,r,s)}(X) \). Then we have:

1. \( Int_R(A) = (Cl_R(A^c))^c \)
2. \( Cl_R(K) = (Int_R(K^c))^c \)

**Proof.** Since \( \vee \) and \( \wedge \) has duality, we will, only, proof part (1).

Let \( A = \{(x, \mu_A^1(x), ..., \mu_A^r(x); \sigma_A^1(x), ..., \sigma_A^s(x); \nu_A^1(x), ..., \nu_A^s(x)); x \in X\} \). Then
\( A^c = \{(x, \nu_A^1(x), ..., \nu_A^s(x); 1 - \sigma_A^1(x), ..., 1 - \sigma_A^s(x); \mu_A^1(x), ..., \mu_A^s(x)); x \in X\} \), so
\( Cl_R(A^c) = \cap_R \{ C \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \text{ and } A^c \subseteq_R C \} \). We apply Demorgan’s Laws in Theorem 2.3 to get:
\( Cl_R(A^c)^c = \sqcup_R \{ C^c \in \mathcal{R}_{(n,r,s)}(X); C^c \in \tau \text{ and } A \subseteq_R C \} = \sqcup_R \{ O \in \mathcal{R}_{(n,r,s)}(X); O \in \tau \text{ and } O \subseteq_R A \} = Int_R(A) \).

\( \Box \)

**Theorem 2.11.** Let \( (X, \tau) \) be an n-valued refined topological space with sub-dimensions \( r, s \) and let \( A, B \in \mathcal{R}_{(n,r,s)}(X) \). Then we have:

1. \( Int_R(A) \subseteq_R A \).
2. If \( A \) is a neutrosophic n-valued refined open set, then \( Int_R(A) = A \).

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Proof. (1) Let $O \in \tau$ such that $O \subseteq_R A$. Then for every $x \in X$ we have $\mu_O^i(x) \leq \mu_A^i(x)$ for every $i = 1, \ldots, r$, $\sigma_O^i(x) \geq \sigma_A^i(x)$ for every $i = 1, \ldots, s$ and $\nu_O^i(x) \geq \nu_A^i(x)$ for every $i = 1, \ldots, r$, which implies that

$$\bigvee_{O \in \tau, O \subseteq_R A} \mu_O^i(x) \leq \mu_A^i(x) \text{ for every } i = 1, \ldots, r,$$

$$\bigwedge_{O \in \tau, O \subseteq_R A} \sigma_O^i(x) \geq \sigma_A^i(x) \text{ for every } i = 1, \ldots, s \quad \text{and} \quad \bigwedge_{O \in \tau, O \subseteq_R A} \nu_O^i(x) \geq \nu_A^i(x) \text{ for every } i = 1, \ldots, r;$$

that is $\text{Int}_R(A) \subseteq R A$.

(2) Since $A$ is open, then, from the definition of $\text{Int}_R(A)$, we have $A \subseteq_R \text{Int}_R(A)$, and from part (1) we have the converse, and we done.

(3) Since $\text{Int}_R(A)$ is a neutrosophic $n$-valued refined open set, we have (from part (2))

$$\text{Int}_R(\text{Int}_R(A)) = \text{Int}_R(A).$$

(4) Let $O$ be a neutrosophic $n$-valued refined open set such that $O \subseteq_R A$. Then since $A \subseteq_R B$, we have $O \subseteq_R B$, that is $\text{Int}_R(A) \subseteq_R \text{Int}_R(B)$.

(5) From part (4) we have $\text{Int}_R(A \cap_R B) \subseteq_R \text{Int}_R(A) \cap_R \text{Int}_R(B)$. On the other hand, $\text{Int}_R(A) \cap_R \text{Int}_R(B)$ is a neutrosophic $n$-valued refined open set contained in $A$ and $B$, so that $\text{Int}_R(A) \cap_R \text{Int}_R(B) \subseteq_R \text{Int}_R(A \cap_R B)$, and we done.

(6) Since $\text{Int}_R(A) \subseteq_R A$ and $\text{Int}_R(B) \subseteq_R B$, we have $\text{Int}_R(A) \cup_R \text{Int}_R(B)$ is a neutrosophic $n$-valued refined open set contained in $A \cup_R B$, which implies that $\text{Int}_R(A) \cup_R \text{Int}_R(B) \subseteq_R \text{Int}_R(A \cup_R B)$.

(7) Since $A_\alpha \subseteq_R \cup_R A_\alpha$ for every $\alpha \in \Delta$, $\text{Int}_R(A_\alpha) \subseteq_R \text{Int}_R(\cup_R A_\alpha)$ for every $\alpha \in \Delta$, that is $\cup_R \text{Int}_R(A_\alpha) \subseteq_R \text{Int}_R(\cup_R A_\alpha)$.

The remaining 5 parts can be proved by duality. □

Equality in parts (7) and (13) of Theorem 2.11 does not hold.

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Example 2.12. Consider the neutrosophic 5-valued refined topological space \((X, \tau)\) defined in Example 2.9 and let 
\[
\text{Example 2.12. Consider the neutrosophic 5-valued refined topological space } (X, \tau) \text{ defined in Example 2.9 and let } K = \{(a,0,1;0;1,1), (b,1,1;0,0,1)\}, \text{ and } L = \{(a,1,0;1;0,0), (b,0,0;1;1,0)\}. \text{ Then } K \uplus_R L = \{(a,1,0;0;0,0), (b,1,1;0,0,0)\} = 1_X. \text{ So we have } \text{Int}\,_R K \uplus_R L = 1_X, \text{ and since } K \text{ and } L \text{ contains no neutrosophic n-valued refined open set except } 0_X, \text{ we have } \text{Int}\,_R K = \text{Int}\,_R L = 0_X, \text{ which means } \text{Int}\,_R K \uplus_R \text{Int}\,_R L = 0_X, \text{ hence equality in parts (7) and (8) of Theorem 2.11 does not hold. For part (13) let } K = \{(a,0,1,0,4,0,6,0,5,0,1), (b,0,7,0,5,9,0,5,0,3)\}, \text{ and } L = \{(a,0,1,0,3,0,7,0,3,0,1), (b,0,7,0,5,9,0,5,0,3)\}. \text{ The only neutrosophic 5-valued Refined closed sets containing } K \text{ are: } 1_X, A^c \text{ and } C^c, \text{ so that we have } \text{Cl}\,_R K = 1_X \cap_R A^c \cap_R C^c = A^c. \text{ Again the only neutrosophic 5-valued Refined closed sets containing } L \text{ are: } 1_X, A^c \text{ and } C^c, \text{ so that we have } \text{Cl}\,_R L = 1_X \cap_R A^c \cap_R C^c = A^c, \text{ and } \text{Cl}\,_R K \cap_R \text{Cl}\,_R L = A^c \cap_R A^c = A^c, \text{ on the other hand the only neutrosophic 5-valued Refined closed sets containing } K \cap_R L \text{ are: } 1_X, A^c, B^c \text{ and } D^c, \text{ so that we have } \text{Cl}\,_R (K \cap_R L) = 1_X \cap_R A^c \cap_R B^c \cap_R D^c = D^c. \text{ Note that } D^c \text{ is a proper subset of } A^c, \text{ so equality in Theorem 2.11 part (13) does not hold.}

Question 2.13. Is there a neutrosophic n-valued refined topological space \((X, \tau)\) shows that equality in part (14) of Theorem 2.11 does not hold.

Definition 2.14 (Neutrosophic n-valued refined pre-open and pre-closed sets). Let \(\tau \in \text{TOP}_{(n,r,s)}(X)\) and \(A \in \mathcal{R}_{(n,r,s)}(X)\). Then \(A\) is said to be:

1. A neutrosophic n-valued refined semi-open set, if \(A \subseteq_R \text{Cl}\,_R (\text{Int}\,_R (A))\). The complement of a neutrosophic n-valued refined semi-open set is called a neutrosophic n-valued refined semi-closed set.
2. A neutrosophic n-valued refined pre-open set, if \(A \subseteq_R \text{Int}\,_R (\text{Cl}\,_R (A))\). The complement of a neutrosophic n-valued refined pre-open set is called a neutrosophic n-valued refined pre-closed set.
3. A neutrosophic n-valued refined \(\alpha\)-open set, if \(A \subseteq_R \text{Int}\,_R (\text{Cl}\,_R (\text{Int}\,_R (A)))\). The complement of a neutrosophic n-valued refined \(\alpha\)-open set is called a neutrosophic n-valued refined \(\alpha\)-closed set.
4. A neutrosophic n-valued refined \(\beta\)-open set, if \(A \subseteq_R \text{Cl}\,_R (\text{Int}\,_R (\text{Cl}\,_R (A)))\). The complement of a neutrosophic n-valued refined \(\beta\)-open set is called a neutrosophic n-valued refined \(\beta\)-closed set.

Theorem 2.15. Let \(\tau \in \text{TOP}_{(n,r,s)}(X)\) and \(A \in \mathcal{R}_{(n,r,s)}(X)\). Then:

1. Every Neutrosophic n-valued refined open (closed) set, is neutrosophic n-valued refined \(\alpha\)-open (closed) set.

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(2) Every Neutrosophic n-valued refined α-open (α-closed) set, is neutrosophic n-valued refined pre-open (pre-closed) set and neutrosophic n-valued refined semi-open (semi-closed) set.

(3) Every Neutrosophic n-valued refined pre-open (pre-closed) or semi-open (semi-closed) set, is a neutrosophic n-valued refined β-open (β-closed) set.

Proof. (1) Let \( A \) be a Neutrosophic n-valued refined open set. Then, from Theorem 2.11 part (2) and (8), we have \( \text{Int}_R(A) = A \) and \( A \subseteq_R \text{Cl}_R(A) \). So \( \text{Int}_R(\text{Cl}_R(\text{int}_R(A))) \supseteq_R \text{Int}_R(\text{Cl}_R(A)) \supseteq_R \text{Int}_R(A) = A \). That is \( A \) is a neutrosophic n-valued refined α-open set. Now, suppose that \( A \) is a Neutrosophic n-valued refined closed set. Then \( A^c \) is a Neutrosophic n-valued refined open set, which implies \( A^c \) is a neutrosophic n-valued refined α-open set, and so \( A \) is a neutrosophic n-valued refined α-closed set.

(2) Obvious! we only use Theorem 2.11 part (1).

(3) Obvious! we only use Theorem 2.11 part (8).

None of the above implications reverse. The following is an example of a neutrosophic 5-valued refined α-open set which is not open, and another example of a neutrosophic 5-valued refined pre-open (so it is β-open) set which is neither semi-open nor α-open.

Example 2.16. Consider \( \tau = \{0_X, 1_X, A, B, C, D\} \) in Example 2.9 and let
\[
H = \{\langle a, 0.5, 0.1; 0.3; 0.1, 0.3 \rangle, \langle b, 0.5, 0.3; 0.1; 0.7, 0.5 \rangle\}.
\]
Then the neutrosophic 5-valued refined open sets contained in \( H \) are \( 0_X, A, B, C, D \); so we have \( \text{Int}_R(H) = 0_X \sqcup_R A \sqcup_R B \sqcup_R C \sqcup_R D = D \), and since the only neutrosophic 5-valued refined close set containing \( D \) is \( 1_X \), we have \( \text{Cl}_R(\text{Int}_R(H)) = 1_X \), which implies \( \text{Int}_R(\text{Cl}_R(\text{int}_R(H))) = 1_X \), hence \( A \subseteq_R \text{Int}_R(\text{Cl}_R(\text{int}_R(A))) \) and \( H \) is a neutrosophic 5-valued refined α-open set but not a neutrosophic 5-valued refined open set.

Consider, again, the set \( K = \{\langle a, 0.1, 0.4; 0.6; 0.1, 0.3 \rangle, \langle b, 0.9, 0.2; 0.4; 0.1, 0.5 \rangle\} \). Since \( \mu^K_O(a) < \mu^K_O(a) \) for every \( O \in \tau - \{0_X\} \), we have the only Neutrosophic 5-valued refined open set contained in \( K \) is \( 0_X \) and \( \text{Int}_R(K) = 0_X \), which implies \( \text{Cl}_R(\text{Int}_R(K)) = 0_X \) and \( \text{Int}_R(\text{Cl}_R(\text{int}_R(K))) = 0_X \), so \( K \) is not a neutrosophic 5-valued refined semi-open nor α-open set; on the other hand, \( \mu^K_D(b) > \mu^K_D(b) \) for every neutrosophic 5-valued refined closed set \( D \) in \( \tau \) except for \( 1_X \), that means \( \text{Cl}_R(K) = 1_X \) and \( \text{int}_R(\text{Cl}_R(A)) = 1_X \), hence \( K \subseteq_R \text{Int}_R(\text{Cl}_R(A)) \) and \( K \) is a neutrosophic 5-valued refined pre-open set but not α-open. Since every neutrosophic 5-valued refined pre-open set is a neutrosophic 5-valued refined β-open set, \( K \) is, also, an example of a neutrosophic 5-valued refined β-open set which is not neutrosophic 5-valued refined semi-open.

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Here we give an example of a a neutrosophic 5-valued refined semi-open (so it is β-open) set which is neither pre-open nor α-open.

**Example 2.17.** Let \( X = \{a\} \), and let \( \tau = \{0_X, 1_X, A, B\} \subset \mathcal{R}_{(5,2,1)}(X) \) where
\[
A = \{(a, 0.2, 0.1; 0.7; 0.3, 0.4)\}, \quad B = \{(a, 0.3, 0.2; 0.5; 0.2, 0.3)\}.
\]
Since \( A \cap R B = A \) and \( A \cup R B = B \), \( \tau \) is a neutrosophic 5-valued refined topology on \( X \). The 5-valued refined closed sets in \( (X, \tau) \) are: \( 0_X, 1_X, A^c, B^c \) where
\[
A^c = \{(a, 0.3, 0.4; 0.3; 0.2, 0.1)\} \quad \text{and} \quad B^c = \{(a, 0.2, 0.3; 0.5; 0.3, 0.2)\}.
\]
Consider the neutrosophic 5-valued refined set \( L = \{(a, 0.2, 0.2; 0.5; 0.3, 0.3)\} \). Then the only neutrosophic 5-valued refined open sets contained in \( K \) are \( 0_X, A \), so that \( \text{Int}_R(L) = 0_X \cup_R A = A \). To find \( \text{Cl}_R(\text{Int}_R(L)) \) we note that the neutrosophic 5-valued refined closed sets containing \( \text{Int}_R(L) \) are \( 1_X, A^c, B^c \), so \( \text{Cl}_R(\text{Int}_R(L)) = 1_X \cap_R A^c \cap_R B^c = B^c \), and since \( L \subseteq B^c \), \( L \) is a neutrosophic 5-valued refined semi-open sets. Now, we will show that \( L \) is not α-open. First note that the neutrosophic 5-valued refined open sets contained in \( \text{Cl}_R(\text{Int}_R(K)) = B^c \) are \( 0_X \) and \( A \), so we have \( \text{Int}_R(\text{Cl}_R(\text{Int}_R(L))) = A \), and since \( L \) is not contained in \( A \), \( L \) is not a neutrosophic α-open set.

We will show \( L \) is not a neutrosophic 5-valued refined pre-open set. The only neutrosophic 5-valued refined closed sets containing \( L \) are \( 1_X, A^c \) and \( B^c \), so \( \text{Cl}_R(L) = 1_X \cap_R A^c \cap_R B^c = B^c \), and since the neutrosophic 5-valued refined open sets contained in \( B^c \) are \( 0_X \) and \( A \), we have \( \text{Int}_R(\text{Cl}_R(L)) = A \) which not containing \( L \), that is \( L \) is not a neutrosophic 5-valued refined pre-open set. So \( L \) is, also, an example of a a neutrosophic 5-valued refined semi-open set which is not pre-open. And since every neutrosophic 5-valued refined semi-open set is β-open set, \( K \) is an example of a a neutrosophic 5-valued refined β-open set which is not pre-open.

Finally we will give an example of a a neutrosophic 5-valued refined β-open set which is neither pre-open nor semi-open.

**Example 2.18.** Let \( (X, \tau) \) as in Example 2.17 and consider the neutrosophic 5-valued refined set \( M = \{(a, 0.2, 0.1; 0.9; 0.3, 0.5)\} \). Then the only neutrosophic 5-valued refined open sets in \( \tau \) contained in \( K \) is \( 0_X \), so \( \text{Int}_R(M) = 0_X \), which implies \( \text{Cl}_R(\text{Int}_R(M)) = 0_X \), and since \( M \) is not contained in \( 0_X \), we have \( M \) is not neutrosophic 5-valued refined semi-open set; on the other hand the neutrosophic 5-valued refined closed sets containing \( M \) are \( 1_X, A^c \) and \( B^c \), so that \( \text{Cl}_R(M) = B^c \), and since the only neutrosophic 5-valued refined open sets contained in \( B^c \) are \( 0_X \) and \( A \) we have \( \text{Int}_R(\text{Cl}_R(M)) = A \). Since \( \text{Int}_R(\text{Cl}_R(M)) = A \) and \( A \) does not contain \( M \), we have \( M \) is not a neutrosophic 5-valued refined pre-open set. Now, to find \( \text{Cl}_R(\text{Int}_R(\text{Cl}_R(M))) \) we note that the only neutrosophic 5-valued refined closed sets in \( \tau \).
Figure 1. Relations between different types of generalized neutrosophic n-valued refined open sets.

containing \( A \) are \( X, A^c \) and \( B^c \), so \( \text{Cl}_R(\text{Int}_R(\text{Cl}_R(M))) = B^c \) which contains \( M \), so \( M \) is a neutrosophic 5-valued refined \( \beta \)-open set but not semi-open nor pre-open.

The following diagram shows the relations between different types of generalized neutrosophic n-valued refined sets:

**Theorem 2.19.** Let \( \tau \in \text{TOP}_{(n,r,s)}(X) \) and \( K \in \mathcal{R}_{(n,r,s)}(X) \). Then

1. If there is a neutrosophic n-valued refined open set \( U \) such that \( K \subseteq_R U \subseteq_R \text{Cl}_R(K) \), then \( K \) is a neutrosophic n-valued refined pre-open set.
2. If there is a neutrosophic n-valued refined open set \( U \) such that \( U \subseteq_R K \subseteq_R \text{Cl}_R(U) \), then \( K \) is a neutrosophic n-valued refined semi-open set.

**Proof.**

1. \( K \subseteq_R U \subseteq_R \text{Int}_R(\text{Cl}_R(U)) \subseteq_R \text{Int}_R(\text{Cl}_R(\text{Cl}_R(K))) = \text{Int}_R(\text{Cl}_R(K)) \).
2. Since \( \text{Cl}_R(\text{Int}_R(U)) = \text{Cl}_R(U) \) we have
\[
\text{Cl}_R(\text{Int}_R(K)) \supseteq_R \text{Cl}_R(\text{Int}_R(U)) = \text{Cl}_R(U) \supseteq_R K.
\]

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Theorem 2.20. Let \( \tau \in \text{TOP}_{(n,r,s)}(X) \) and \( K \in \mathcal{R}_{(n,r,s)}(X) \). Then the union of any collection of neutrosophic \( n \)-valued refined \( \alpha \)-open, \( \beta \)-open, pre-open or semi-open sets is a neutrosophic \( n \)-valued refined \( \alpha \)-open, \( \beta \)-open, pre-open or semi-open set respectively.

Proof. We will prove it for neutrosophic \( n \)-valued refined \( \beta \)-open sets, and the remaining parts can be proved in the same manner. Let \( A_{\gamma} \) be a neutrosophic \( n \)-valued refined \( \beta \)-open set for every \( \gamma \in \Delta \). Then \( A_{\gamma} \subseteq \text{CL}_R(\text{int}_R(\text{CL}_R(A_{\gamma}))) \) for every \( \gamma \in \Delta \). Then from parts (7) and (14) of Theorem 2.11 we have:

\[
\text{CL}_R(\text{int}_R(\bigcup_{\gamma \in \Delta} A_{\gamma})) \supseteq \text{CL}_R(\bigcup_{\gamma \in \Delta} \text{int}_R(\text{CL}_R(A_{\gamma}))) \supseteq \text{CL}_R(\bigcup_{\gamma \in \Delta} \text{int}_R(\text{CL}_R(\text{int}_R(\text{CL}_R(A_{\gamma})))) \supseteq \text{CL}_R(\text{int}_R(\text{CL}_R(\bigcup_{\gamma \in \Delta} A_{\gamma}))) \subseteq \bigcup_{\gamma \in \Delta} A_{\gamma} \]

Funding: This study is supported via funding from Prince Sattam bin Abdulaziz University project number (PSAU/2023/R/1444).

Conflicts of Interest: The author declare that he has no conflict of interest.

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Received: August 10, 2022. Accepted: January 06, 2023