On Some New Concepts of Weakly Neutrosophic Crisp Separation Axioms

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Abstract: In this paper, we shall study some new concepts of weakly neutrosophic crisp separation axioms, which are called “neutrosophic crisp $\alpha$-separation and neutrosophic crisp semi-$\alpha$-separation axioms” such as neutrosophic crisp $\alpha-T_i$ and neutrosophic crisp semi-$\alpha-T_i$, $\forall i = 0, 1, ..., 4$. Moreover, we shall study the relationship between usual neutrosophic crisp separation axioms and these kinds of weakly neutrosophic crisp separation axioms.

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1. Introduction

A. A. Salama et al. [1] give a concept of neutrosophic crisp topological space (briefly NCTS). A. A. Salama [2] provided some classes of neutrosophic crisp nearly open sets. A. H. M. Al-Obaidi et al. [3,4] give concepts of weakly neutrosophic crisp functions. Md. Hanif PAGE et al. [5] examined the view of neutrosophic generalized homeomorphism. Q. H. Imran et al. [6-8] established neutrosophic semi-$\alpha$-open sets, new types of weakly neutrosophic crisp continuity and new concepts of neutrosophic crisp open sets. R. Dhavaseelan et al. [9] examined the view of neutrosophic $\alpha^m$-continuity. R. K. Al-Hamido et al. [10] tendered the interpretation of neutrosophic crisp semi-$\alpha$-closed sets. A. B. Al-Nafee et al. [11] demonstrated the principle of separation axioms in neutrosophic crisp topological spaces. R. K. Al-Hamido et al. [12] provided neutrosophic crisp semi separation axioms. The objective of this paper is to study some new concepts of weakly neutrosophic crisp separation axioms, which are called “neutrosophic crisp $\alpha$-separation and neutrosophic crisp semi-$\alpha$-separation axioms” such as neutrosophic crisp $\alpha$-T$_i$ and neutrosophic crisp semi-$\alpha$-T$_i$, $\forall i = 0, 1, ..., 4$. Moreover, we shall study the relationship between usual neutrosophic crisp separation axioms and these kinds of weakly neutrosophic crisp separation axioms.

2. Preliminaries

Throughout this paper, $(\delta, \zeta)$ and $(\gamma, \eta)$ (or simply $\delta$ and $\gamma$) always mean NCTSs. The complement of a neutrosophic crisp open set (briefly NC-OS) is called a neutrosophic crisp closed...
set (briefly NC-CS) in \((S, \zeta)\). For a NCS \(\mathcal{B}\) in a NCTS \((S, \zeta)\), \(Nccl(\mathcal{B})\), \(NCint(\mathcal{B})\) and \(\mathcal{B}^c\) denote the NC-closure of \(\mathcal{B}\), the NC-interior of \(\mathcal{B}\) and the NC-complement of \(\mathcal{B}\) respectively.

**Definition 2.1** [1]:
For any nonempty under-consideration set \(S\), a neutrosophic crisp set (in short NCS) \(\mathcal{B}\) is an object holding the establish \(\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)\) where \(\mathcal{B}_1, \mathcal{B}_2\) and \(\mathcal{B}_3\) are mutually disjoint sets included in \(S\).

**Definition 2.2:**
A NC-subset \(\mathcal{B}\) of a NCTS \((S, \zeta)\) is said to be:
(i) neutrosophic crisp \(\alpha\)-open set (in short NC\(^{\alpha}\)-OS) [2] if \(\mathcal{B} \subseteq NCint(\mathcal{N}cl(\mathcal{N}cint(\mathcal{B})))\). The family of all NC\(^{\alpha}\)-OSs of \(S\) is denoted by \(NC^{\alpha}\text{O}(S)\). The complement of NC\(^{\alpha}\)-OS is called a neutrosophic crisp \(\alpha\)-closed set (in short NC\(^{\alpha}\)-CS). The family of all NC\(^{\alpha}\)-CSs of \(S\) is denoted by NC\(^{\alpha}\text{C}(S)\).

(ii) neutrosophic crisp semi-\(\alpha\)-open set (in short NC\(^{S\alpha}\)-OS) [10] if there exists a NC\(^{\alpha}\)-OS \(\mathcal{D}\) in \(S\) such that \(\mathcal{D} \subseteq \mathcal{B} \subseteq NCcl(\mathcal{D})\) or equivalently if \(\mathcal{B} \subseteq NCcl(\mathcal{N}ccl(\mathcal{N}cint(\mathcal{B})))\). The family of all NC\(^{S\alpha}\)-OSs of \(S\) is denoted by NC\(^{S\alpha}\text{O}(S)\). The complement of NC\(^{S\alpha}\)-OS is called a neutrosophic crisp semi-\(\alpha\)-closed set (in short NC\(^{S\alpha}\)-CS). The family of all NC\(^{S\alpha}\)-CSs of \(S\) is denoted by NC\(^{S\alpha}\text{C}(S)\).

**Example 2.3:**
Let \(S = \{k_1, k_2, k_3, k_4\}\). Then \(\zeta = \{\emptyset_N, \{k_1\}, \emptyset, \{k_2\}, \emptyset, \{k_2, k_4\}, \emptyset, \emptyset, \{k_1, k_2, k_3\}, \emptyset, \emptyset\}\) is a NCTS. The family of all NC\(^{\alpha}\)-OSs of \(S\) is : \(NC^{\alpha}\text{O}(S) = \zeta \cup \{(\{k_1, k_2, k_4\}, \emptyset, \emptyset)\}\).

The family of all NC\(^{S\alpha}\)-OSs of \(S\) is : \(NC^{S\alpha}\text{O}(S) = NC^{\alpha}\text{O}(S) \cup \{(\{k_1, k_3\}, \emptyset, \emptyset), (\{k_1, k_4\}, \emptyset, \emptyset), (\{k_1, k_2, k_4\}, \emptyset, \emptyset)\}\).

**Remark 2.4** [10,14]:
In a NCTS \((S, \zeta)\), then the following statements hold, and the opposite of each statement is not true:
(i) Every NC-OS (resp. NC-CS) is a NC\(^{\alpha}\)-OS (resp. NC\(^{\alpha}\)-CS) and NC\(^{S\alpha}\)-OS (resp. NC\(^{S\alpha}\)-CS).

(ii) Every NC\(^{\alpha}\)-OS (resp. NC\(^{\alpha}\)-CS) is a NC\(^{S\alpha}\)-OS (resp. NC\(^{S\alpha}\)-CS).

**Definition 2.5:**
(i) The NC\(^{\alpha}\)-interior of a NCS \(\mathcal{B}\) of a NCTS \((S, \zeta)\) is the union of all NC\(^{\alpha}\)-OSs contained in \(\mathcal{B}\) and is denoted by \(NC^{\alpha}\text{int}(\mathcal{B})\)[3].

(ii) The NC\(^{S\alpha}\)-interior of a NCS \(\mathcal{B}\) of a NCTS \((S, \zeta)\) is the union of all NC\(^{S\alpha}\)-OSs contained in \(\mathcal{B}\) and is denoted by \(NC^{S\alpha}\text{int}(\mathcal{B})\)[10].

**Definition 2.6:**
(i) The NC\(^{\alpha}\)-closure of a NCS \(\mathcal{B}\) of a NCTS \((S, \zeta)\) is the intersection of all NC\(^{\alpha}\)-CSs containing \(\mathcal{B}\) and is denoted by \(NC^{\alpha}\text{cl}(\mathcal{B})\)[3].

(ii) The NC\(^{S\alpha}\)-closure of a NCS \(\mathcal{B}\) of a NCTS \((S, \zeta)\) is the intersection of all NC\(^{S\alpha}\)-CSs containing \(\mathcal{B}\) and is denoted by \(NC^{S\alpha}\text{cl}(\mathcal{B})\)[10].

**Theorem 2.7:**
Let \((S, \zeta)\) and \((T, \eta)\) be two NCTSs. If \(\mathcal{B} \in NC^{\alpha}\text{O}(S)\) (resp. \(\mathcal{B} \in NC^{S\alpha}\text{O}(S)\)), \(\mathcal{D} \in NC^{\alpha}\text{O}(T)\) (resp. \(\mathcal{D} \in NC^{S\alpha}\text{O}(T)\)), then \(\mathcal{B} \times \mathcal{D} \in NC^{\alpha}\text{O}(S \times T)\) (resp. \(\mathcal{B} \times \mathcal{D} \in NC^{S\alpha}\text{O}(S \times T)\)).

**Proof:**
Since \(\mathcal{B} \subseteq NCint(\mathcal{N}ccl(\mathcal{N}cint(\mathcal{B})))\), \(\mathcal{D} \subseteq NCint(\mathcal{N}ccl(\mathcal{N}cint(\mathcal{D})))\).
Hence \(\mathcal{B} \times \mathcal{D} \subseteq NCint(\mathcal{N}ccl(\mathcal{N}cint(\mathcal{B}))) \times NCint(\mathcal{N}ccl(\mathcal{N}cint(\mathcal{D}))) = NCint(\mathcal{N}ccl(\mathcal{N}cint(\mathcal{B} \times \mathcal{D})))\).
Therefore $\mathcal{B} \times \mathcal{D} \subseteq NCint(NCcl(NCint(\mathcal{B} \times \mathcal{D}))) \Rightarrow \mathcal{B} \times \mathcal{D} \in NC^aO(\mathcal{S} \times \mathcal{J})$. The second case is similar.

**Corollary 2.8:**
Let $(\mathcal{S}, \mathcal{J})$ and $(\mathcal{J}, \eta)$ be two NCTSs. If $\mathcal{B} \in NC^aC(\mathcal{S})$ (resp. $\mathcal{B} \in NC^aS(\mathcal{S})$), $\mathcal{D} \in NC^aC(\mathcal{J})$ (resp. $\mathcal{D} \in NC^aS(\mathcal{J})$), then $\mathcal{B} \times \mathcal{D} \in NC^aC(\mathcal{S} \times \mathcal{J})$ (resp. $\mathcal{B} \times \mathcal{D} \in NC^aS(\mathcal{S} \times \mathcal{J})$).

**Proof:**
The proof of this is similar to that of theorem (2.6). 

**Proposition 2.9** [10]:
For any NC-subset $\mathcal{B}$ of a NCTS $(\mathcal{S}, \mathcal{J})$, then:

(i) $NCint(\mathcal{B}) \subseteq NC^aint(\mathcal{B}) \subseteq NC^{ca}int(\mathcal{B}) \subseteq NC^{ca}cl(\mathcal{B}) \subseteq NC^{ca}cl(\mathcal{S}) \subseteq NCcl(\mathcal{B})$.

(ii) $NCint(NC^{ca}cl(\mathcal{B})) = NC^{ca}int(NCint(\mathcal{B})) = NCint(\mathcal{B})$.

(iii) $NC^aint(NC^{ca}int(\mathcal{B})) = NC^{ca}int(NC^aint(\mathcal{B})) = NC^aint(\mathcal{B})$.

(iv) $NC^{ca}cl(NC^{ca}cl(\mathcal{B})) = NC^{ca}cl(NC^{ca}cl(\mathcal{S})) = NC^{ca}cl(\mathcal{B})$.

(v) $NC^{ca}cl(NC^{ca}cl(\mathcal{S})) = NC^{ca}cl(NC^{ca}cl(\mathcal{S})) = NC^{ca}cl(\mathcal{B})$.

(vi) $NC^{ca}cl(\mathcal{B}) = \mathcal{B} \cup NCint(NC^{ca}int(NC^{ca}cl(\mathcal{B})))$.

(vii) $NC^{ca}int(\mathcal{B}) = \mathcal{B} \setminus NCcl(NC^{ca}int(NC^{ca}cl(\mathcal{B})))$.

(viii) $NC^{ca}int(NC^{ca}cl(\mathcal{S})) \subseteq NC^{ca}int(NC^{ca}cl(\mathcal{S}))$.

**Definition 2.10** [1]:
Let $\rho: (\mathcal{S}, \mathcal{J}) \rightarrow (\mathcal{J}, \eta)$ be a function, then $\rho$ is said to be NC-continuous (in short NC-CF) iff $\forall \mathcal{B}$ NC-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC-OS in $\mathcal{S}$.

**Definition 2.11** [13]:
Let $\rho: (\mathcal{S}, \mathcal{J}) \rightarrow (\mathcal{J}, \eta)$ be a function, then $\rho$ is said to be NC$^a$-continuous (in short NC$^a$-CF) iff $\forall \mathcal{B}$ NC-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC$^a$-OS in $\mathcal{S}$.

**Definition 2.12** [10]:
Let $\rho: (\mathcal{S}, \mathcal{J}) \rightarrow (\mathcal{J}, \eta)$ be a function, then $\rho$ is said to be:

(i) NC$^{ca}$-continuous (in short NC$^{ca}$-CF) iff $\forall \mathcal{B}$ NC$^a$-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC$^a$-OS in $\mathcal{S}$.

(ii) NC$^{ca^*}$-continuous (in short NC$^{ca^*}$-CF) iff $\forall \mathcal{B}$ NC$^a$-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC-OS in $\mathcal{S}$.

**Definition 2.13** [10]:
Let $\rho: (\mathcal{S}, \mathcal{J}) \rightarrow (\mathcal{J}, \eta)$ be a function, then $\rho$ is said to be:

(i) NC$^{ca^*}$-continuous (in short NC$^{ca^*}$-CF) iff $\forall \mathcal{B}$ NC-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC$^{ca^*}$-OS in $\mathcal{S}$.

(ii) NC$^{ca}$-continuous (in short NC$^{ca}$-CF) iff $\forall \mathcal{B}$ NC$^{ca}$-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC$^{ca}$-OS in $\mathcal{S}$.

(iii) NC$^{ca^*}$-continuous (in short NC$^{ca^*}$-CF) iff $\forall \mathcal{B}$ NC$^{ca^*}$-OS in $\mathcal{J}$, then $\rho^{-1}(\mathcal{B})$ is a NC$^{ca^*}$-OS in $\mathcal{S}$.

3. Some New Concepts of Weakly Neutrosophic Crisp Separation Axioms

**Definition 3.1:**
(i) A NCTS $(\mathcal{S}, \mathcal{J})$ is said to be a NC$^a$-$T_0$-space if for each pair of distinct neutrosophic crisp points in $(\mathcal{S}, \mathcal{J})$ there exists NC$^a$-OS of $(\mathcal{S}, \mathcal{J})$ containing one neutrosophic crisp point but not the other.

(ii) A NCTS $(\mathcal{S}, \mathcal{J})$ is said to be a NC$^{ca}$-$T_0$-space if for each pair of distinct neutrosophic crisp points in $(\mathcal{S}, \mathcal{J})$ there exists NC$^{ca}$-OS of $(\mathcal{S}, \mathcal{J})$ containing one neutrosophic crisp point but not the other.
Theorem 3.3:
A NCTS $(S, \zeta)$ is NC$^a$-T$_0$-space (NC$^{Sa}$-T$_0$-space respectively) iff NC$^a$cl($\{u\}, \emptyset, \emptyset >) \neq NC^a$cl($\{v\}, \emptyset, \emptyset >) \neq NC^{Sa}$cl($\{v\}, \emptyset, \emptyset >)$ receptively for each $u \neq v$ in $S$.

Proof:
⇒ Let NC$^a$cl($\{u\}, \emptyset, \emptyset >) \neq NC^a$cl($\{v\}, \emptyset, \emptyset >) \neq NC^{Sa}$cl($\{v\}, \emptyset, \emptyset >) \neq NC^a$cl($\{v\}, \emptyset, \emptyset >)$. Hence NC$^a$cl($\{u\}, \emptyset, \emptyset >) \neq NC^a$cl($\{v\}, \emptyset, \emptyset >)$. Suppose that NC$^a$cl($\{u\}, \emptyset, \emptyset >) \neq NC^a$cl($\{v\}, \emptyset, \emptyset >) \Rightarrow u \in NC^a$cl($\{v\}, \emptyset, \emptyset >) \Rightarrow u \in (NC^a$cl($\{v\}, \emptyset, \emptyset >))^c$ but $(NC^a$cl($\{v\}, \emptyset, \emptyset >))^c$ is a NC$^a$-OS and $v \notin (NC^a$cl($\{v\}, \emptyset, \emptyset >))^c$. Therefore $S$ is a NC$^a$-T$_0$-space.

⇐ Let $S$ be a NC$^a$-T$_0$-space, $u \neq v \in S$. Hence there exists a NC$^a$-OS $B$ in $S$ such that $u \in B$, $v \notin B$ or $u \notin B$, $v \in B$. Then $B^c$ is a NC$^a$-CS and $u \in B^c$, $v \in B^c$. Therefore $u \notin NC^a$cl($\{v\}, \emptyset, \emptyset >)$. Hence $NC^a$cl($\{u\}, \emptyset, \emptyset >) \neq NC^a$cl($\{v\}, \emptyset, \emptyset >)$). The second case is similar.

Theorem 3.4:
If $(S, \zeta)$ is a NC$^a$-T$_0$-space (NC$^{Sa}$-T$_0$-space respectively), then NC$^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)) \cap NC^a$int(NC$^a$cl($\{v\}, \emptyset, \emptyset >)) = \emptyset_N$ (NC$^{Sa}$int(NC$^{Sa}$cl($\{u\}, \emptyset, \emptyset >)) \cap NC^{Sa}$int(NC$^{Sa}$cl($\{v\}, \emptyset, \emptyset >)) = \emptyset_N$ receptively, $u \neq v$ in $S$.

Proof:
Let $(S, \zeta)$ be a NC$^a$-T$_0$-space. Then there exists a NC$^a$-OS $B$ such that $u \in B$, $v \notin B$ or $u \notin B$, $v \in B$. If $u \notin B$, $v \notin B \Rightarrow u \notin B^c$, $v \in B^c$. Thus NC$^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)) \subseteq NC^a$cl($\{v\}, \emptyset, \emptyset >)$ (since $B^c$ is a NC$^a$-CS). Hence NC$^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)) \subseteq B^c \Rightarrow NC^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)$) = $B^c \Rightarrow NC^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)) \subseteq B$. Therefore, $u \in B \subseteq (NC^a$int(NC$^a$cl($\{v\}, \emptyset, \emptyset >)$))^c . Hence NC$^a$cl($\{u\}, \emptyset, \emptyset >) \subseteq (NC^a$int(NC$^a$cl($\{v\}, \emptyset, \emptyset >)$))^c \Rightarrow NC^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)) \subseteq NC^a$int(NC$^a$cl($\{v\}, \emptyset, \emptyset >)) \subseteq NC^a$int(NC$^a$cl($\{u\}, \emptyset, \emptyset >)) \subseteq \emptyset_N$. The second case is similar.

Remark 3.5:
(i) Every NC$^a$-T$_0$-space is a NC$^a$-T$_0$-space and NC$^{Sa}$-T$_0$-space.
(ii) Every NC$^a$-T$_0$-space is a NC$^{Sa}$-T$_0$-space.

Remark 3.6:
(i) NC$^a$-T$_0$ (NC$^{Sa}$-T$_0$ respectively) property is a NC$^{a^*}$ (NC$^{Sa^*}$ respectively) topological property.
(ii) NC$^a$-T$_0$ (NC$^{Sa}$-T$_0$ respectively) property is a NC$^{a^{**}}$ (NC$^{Sa^{**}}$ respectively) topological property.
(iii) NC$^a$-T$_0$ is a NC$^{a}$-hereditary property.

Proposition 3.7:
(i) Let $(S, \zeta)$ and $(J, \eta)$ be NC$^a$-T$_0$-spaces if and only if $S \times J$ is a NC$^a$-T$_0$-space.
(ii) If $(S, \zeta)$ and $(J, \eta)$ are NC$^{Sa}$-T$_0$-spaces, then $S \times J$ is a NC$^{Sa}$-T$_0$-space.

Proof:
(i) ⇒ Let $S$ and $J$ be NC$^a$-T$_0$-spaces. Let $(u_1, \sigma_1) \neq (u_2, \sigma_2)$ in $S \times J$. Then $u_1 \neq u_2$ in $S \Rightarrow$ there exists $B_1 \in NC^aO(S)$ such that $u_1 \in B_1, u_2 \notin B_1$ or $u_1 \notin B_1, u_2 \in B_1$.

Also $\sigma_1 \neq \sigma_2$ in $J \Rightarrow$ there exists $B_2 \in NC^aO(J)$ such that $\sigma_1 \in B_2, \sigma_2 \notin B_2$ or $\sigma_1 \notin B_2, \sigma_2 \in B_2$. Then $(u_1, \sigma_1) \in B_1 \times B_2, (u_2, \sigma_2) \in B_1 \times B_2$ or $(u_1, \sigma_1) \in B_1 \times B_2, (u_2, \sigma_2) \in B_1 \times B_2$.

But $B_1 \times B_2 \in NC^aO(S \times J)$ (since by theorem (2.6)). Hence $S \times J$ is a NC$^a$-T$_0$-space.
Definition 3.8:
(i) A NCTS \((S, \zeta)\) is said to be a \(NC^a-T_1\)-space if for each pair of distinct NC-points \(u\) and \(v\) of \(S\), there exist two NC\(^a\)-OSs \(B\) and \(D\) containing \(u\) and \(v\) respectively, such that \(u \in B, v \in D\).

(ii) A NCTS \((S, \zeta)\) is said to be a \(NC^a_{SO}-T_1\)-space if for each pair of distinct NC-points \(u\) and \(v\) of \(S\), there exist two NC\(^{a}_{S0}\)-OSs \(B\) and \(D\) containing \(u\) and \(v\) respectively, such that \(u \in B, v \in D\).

Proposition 3.9:
A NCTS \((S, \zeta)\) is \(NC^a-T_1\)-space (\(NC^{a}_{SO}-T_1\)-space respectively) if and only if \(<u>, 0, 0>\) is a NC\(^a\)-CS (NC\(^{a}_{SO}\)-CS respectively), \(\forall u \in S\).

Proof:
\(\Rightarrow\) Let \(S\) be a \(NC^a-T_1\)-space. Let \(w \in S\), to prove that \(<w>, 0, 0>\) is a NC\(^a\)-CS. Let \(u \in (<w>, 0, 0>)^c \neq u \neq \emptyset\) in \(S\). Hence there exists a NC\(^a\)-OS \(B\) such that \(u \in B, w \in B\) or \(u \notin B, w \in B\). If \(u \in B, w \notin B\) \(\Rightarrow u \in B \in (<w>, 0, 0>)^c \Rightarrow (<w>, 0, 0>)^c\) is a NC\(^a\)-OS \(\Rightarrow \emptyset\).

\(\Leftarrow\) Let \(<w>, 0, 0>\) be a NC\(^a\)-CS, \(\forall w \in S\), to prove that \(S\) is a \(NC^a-T_1\)-space. Let \(u \neq v\) in \(S\). Hence \(<u>, 0, 0>, <v>, 0, 0>\ are NC\(^a\)-CSs \(\Rightarrow (<u>, 0, 0>)^c, <v>, 0, 0>)^c\) are NC\(^a\)-OSs and \(u \notin (<u>, 0, 0>)^c, v \notin (<v>, 0, 0>)^c\). Therefore \(S\) is a \(NC^a-T_1\)-space. The second case is similar.

Remark 3.10:
(i) Every \(NC^a-T_2\)-space is a \(NC^a-T_1\)-space and \(NC^{a}_{SO}-T_1\)-space.
(ii) Every \(NC^a-T_1\)-space is a \(NC^{a}_{SO}-T_1\)-space.
(iii) Every \(NC^a-T_1\)-space is a \(NC^a_{SO}-T_0\)-space.
(iv) Every \(NC^{a}_{SO}-T_0\)-space is a \(NC^{a}_{SO}-T_0\)-space.

Remark 3.11:
(i) \(NC^a-T_1\) (\(NC^{a}_{SO}-T_1\) respectively) property is a \(NC^a_{SO}\) (\(NC^{a}_{SO}\) respectively) topological property.
(ii) \(NC^a-T_1\) (\(NC^{a}_{SO}-T_1\) respectively) property is a \(NC^a_{SO}\) (\(NC^{a}_{SO}\) respectively) topological property.
(iii) \(NC^a-T_1\) property is a \(NC^a\)-hereditary property.

Proposition 3.12:
(i) Let \(S\) and \(J\) be \(NC^a-T_1\)-spaces if and only if \(S \times J\) is a \(NC^a-T_1\)-space.
(ii) If \(S\) and \(J\) are \(NC^{a}_{SO}-T_1\)-spaces, then \(S \times J\) is a \(NC^{a}_{SO}-T_1\)-space.

Proof:
The proof of this is similar to that of proposition (3.7).
Proposition 3.14:
If $(\mathcal{S}, \mathcal{J})$ is a NC$^{a}$-$T_2$-space (NC$^{a}$-$T_2$-space respectively), then $\mathcal{B} = \{(u, v): u = v, u, v \in \mathcal{S}\}$ is a NC$^{a}$-CS (NC$^{a}$-$T_2$-CS respectively).

Proof:
Let $\mathcal{S}$ be a NC$^{a}$-$T_2$-space, to prove that $\mathcal{B}$ is a NC$^{a}$-CS. Let $(u, v) \in \mathcal{B}^c = \mathcal{S} \times \mathcal{S} - \mathcal{B}$. Hence $u \neq v$ in $\mathcal{S} \Rightarrow$ there exist $D_1, D_2 \in NC^{a}O(\mathcal{S})$ such that $u \in D_1, v \in D_2$ and $D_1 \cap D_2 = \emptyset_N$ (since $\mathcal{S}$ is a NC$^{a}$-$T_2$-space). Hence $D_1 \times D_2 \in NC^{a}O(\mathcal{S} \times \mathcal{S})$ by theorem (2.7) $(u, v) \in D_1 \times D_2 \subseteq \mathcal{B}^c$, hence $\mathcal{B}^c$ is a NC$^{a}$-OS. Therefore $\mathcal{B}$ is a NC$^{a}$-CS. The second case is similar.

Remark 3.15:
(i) Every NC-$T_2$-space is a NC$^{a}$-$T_2$-space and NC$^{a}$-$T_2$-space.
(ii) Every NC$^{a}$-$T_2$-space is a NC$^{a}$-$T_2$-space.
(iii) Every NC$^{a}$-$T_2$-space is a NC$^{a}$-$T_1$-space.
(iv) Every NC$^{a}$-$T_2$-space is a NC$^{a}$-$T_1$-space.

Remark 3.16:
(i) NC$^{a}$-$T_2$ (NC$^{a}$-$T_2$ respectively) property is a NC$^{a}$ (NC$^{a}$ respectively) topological property.
(ii) NC$^{a}$-$T_2$ (NC$^{a}$-$T_2$ respectively) property is a NC$^{a}$ (NC$^{a}$ respectively) topological property.
(iii) NC$^{a}$-$T_2$ property is a NC$^{a}$ hereditary property.

Proposition 3.17:
(i) Let $\mathcal{S}$ and $\mathcal{J}$ be NC$^{a}$-$T_2$-spaces if and only if $\mathcal{S} \times \mathcal{J}$ is a NC$^{a}$-$T_2$-space.
(ii) If $\mathcal{S}$ and $\mathcal{J}$ are NC$^{a}$-$T_2$-spaces, then $\mathcal{S} \times \mathcal{J}$ is a NC$^{a}$-$T_2$-space.

Proof:
The proof of this is similar to that of proposition (3.12).

Proposition 3.18:
(i) If $\rho, \mu: \mathcal{S} \rightarrow \mathcal{I}$ are NC$^{a}$-CF and $\mathcal{I}$ is a NC$^{a}$-$T_2$-space, then the NC-set $\mathcal{B} = \{u: u \in \mathcal{S}, \rho(u) = \mu(u)\}$ is a NC$^{a}$-CS.
(ii) If $\rho, \mu: \mathcal{S} \rightarrow \mathcal{I}$ are NC$^{a}$-CF and $\mathcal{I}$ is a NC-$T_2$-space, then the NC-set $\mathcal{B} = \{u: u \in \mathcal{S}, \rho(u) = \mu(u)\}$ is a NC$^{a}$-CS.

Proof:
(i) If $u \notin \mathcal{B} \Rightarrow u \in \mathcal{B}^c \Rightarrow \rho(u) \neq \mu(u)$ in $\mathcal{I}$. Hence there exist two NC$^{a}$-OSs $D_1$ and $D_2$ in $\mathcal{I}$ such that $\rho(u) \in D_1, \mu(u) \in D_2$ and $D_1 \cap D_2 = \emptyset_N$ (since $\mathcal{I}$ is a NC$^{a}$-$T_2$-space). But $\rho^{-1}(D_1), \mu^{-1}(D_2) \in NC^{a}O(\mathcal{I})$ (since $\rho, \mu$ are NC$^{a}$-CF). Therefore, $u \in \rho^{-1}(D_1)$ and $u \in \mu^{-1}(D_2)$. Hence $u \in \rho^{-1}(D_1) \cap \mu^{-1}(D_2)$.

Suppose that $\mathcal{U} \cap \mathcal{B} \neq \emptyset_N$ $\Rightarrow \exists \nu \in \mathcal{U} \cap \mathcal{B} \Rightarrow \nu \in \emptyset_N$ and $\nu \in \mathcal{B}$, i.e., $\nu \in \rho^{-1}(D_1)$ and $\nu \in \mu^{-1}(D_2)$ and $\nu \in \emptyset_N$. Hence $\rho(\nu) \in D_1, \mu(\nu) \in D_2$ and $\nu \in \mathcal{B}$. Therefore $\rho(\nu) = \mu(\nu)$ (since $\nu \in \mathcal{B}$). Hence $D_1 \cap D_2 \neq \emptyset_N$ which is a contradiction. Therefore $\mathcal{U} \subseteq \mathcal{B}^c \Rightarrow \mathcal{B} \subseteq NC^{a}O(\mathcal{I}) \Rightarrow \mathcal{B}$ is a NC$^{a}$-CS. The proof (ii) is evident for others.

4. Some New Concepts of Weakly Neutrosophic Crisp Regularity

Definition 4.1:
Let $(\mathcal{S}, \mathcal{J})$ be a NCTS, then $\mathcal{S}$ is said to be:
(i) NC$^{a}$-regular (NC$^{a}$-regular respectively) if every $u \in \mathcal{S}$ and every $\mathcal{Q}$ NC-CS such that $u \notin \mathcal{Q}$,
there exist two NC\(^a\)-OSs (NC\(^{Sa}\)-OSs respectively) \(\mathcal{B}\) and \(\mathcal{D}\) such that \(u \in \mathcal{B}, Q \in \mathcal{D}\) and \(\mathcal{B} \cap \mathcal{D} = \emptyset_N\).

(ii) NC\(^{a'}\)-regular (NC\(^{Sa'}\)-regular respectively) if every \(u \in \mathcal{S}\) and every \(Q\) NC\(^a\)-CS (NC\(^{Sa}\)-CS respectively) such that \(u \notin Q\), there exist two NC\(^a\)-OSs (NC\(^{Sa}\)-OSs respectively) \(\mathcal{B}\) and \(\mathcal{D}\) such that \(u \in \mathcal{B}, Q \in \mathcal{D}\) and \(\mathcal{B} \cap \mathcal{D} = \emptyset_N\).

(iii) NC\(^{a''}\)-regular (NC\(^{Sa''}\)-regular respectively) if every \(u \in \mathcal{S}\) and every \(Q\) NC\(^a\)-CS (NC\(^{Sa}\)-CS respectively) such that \(u \notin Q\), there exist two NC-OSs \(\mathcal{B}\) and \(\mathcal{D}\) such that \(u \in \mathcal{B}, Q \in \mathcal{D}\) and \(\mathcal{B} \cap \mathcal{D} = \emptyset_N\).

**Remark 4.2:**

The following diagram shows the relation between the different types of weakly NC-regular and weakly NC\(^a\)-regular (NC\(^{Sa}\)-regular respectively) spaces:

![Diagram](image-url)

**Fig. 4.1**

**Theorem 4.3:**

Let \((\mathcal{S}, \mathcal{Q})\) be a NCTS, then:

(i) \(\mathcal{S}\) is a NC\(^a\)-regular if and only if for each \(\mathcal{B}\) NC-OS containing \(u\), there exists \(\mathcal{D}\) NC\(^a\)-OS containing \(u\) such that \(u \in \mathcal{D} \subseteq NC^a cl(\mathcal{D}) \subseteq \mathcal{B}\).

(ii) \(\mathcal{S}\) is a NC\(^{a'}\)-regular if and only if for each \(\mathcal{B}\) NC-OS contains \(u\), there exists \(\mathcal{D}\) NC\(^{a'}\)-OS contains \(u\) such that \(u \in \mathcal{D} \subseteq NC^{a'} cl(\mathcal{D}) \subseteq \mathcal{B}\).

(iii) \(\mathcal{S}\) is a NC\(^{a''}\)-regular if and only if for each \(\mathcal{B}\) NC\(^a\)-OS contains \(u\), there exists \(\mathcal{D}\) NC-OS contains \(u\) such that \(u \in \mathcal{D} \subseteq NC cl(\mathcal{D}) \subseteq \mathcal{B}\).

**Proof:**

(i) \(\Rightarrow\) Let \(\mathcal{S}\) be a NC\(^a\)-regular space and let \(\mathcal{B}\) be a NC-OS containing \(u\). Hence \(\mathcal{B}^c\) is a NC-CS and \(u \notin \mathcal{B}^c\). Then there exist \(\mathcal{D}_1, \mathcal{D}_2\) NC\(^a\)-OSs in \(\mathcal{S}\) such that \(u \in \mathcal{D}_1, \mathcal{B}^c \subseteq \mathcal{D}_2\) and \(\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset_X\) (since \(\mathcal{S}\) is a NC\(^a\)-regular space). Hence \(u \in \mathcal{D}_1 \subseteq \mathcal{D}_1^c \subseteq \mathcal{B}\) (since \(\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset_X\) \(\Rightarrow \mathcal{D}_1 \subseteq \mathcal{D}_2^c\)). Therefore \(u \in \mathcal{D}_1 \subseteq NC^a cl(\mathcal{D}_1) \subseteq NC^a cl(\mathcal{D}_2^c) \subseteq NC^a cl(\mathcal{B})\). Therefore \(u \in \mathcal{D}_1 \subseteq NC^a cl(\mathcal{D}_1) \subseteq \mathcal{D}_2^c \subseteq \mathcal{B}\). The implies that \(u \in \mathcal{D}_1 \subseteq NC^a cl(\mathcal{D}_1) \subseteq \mathcal{B}\), where \(\mathcal{D}_1\) is a NC\(^a\)-OS.
\( \Leftarrow \) Let \( Q \) be a NC-CS such that \( u \notin Q \Rightarrow Q^c \) is a NC-OS contains \( u \). Hence there exists \( D \) NC\( a \)-OS contains \( u \) such that \( u \in D \subseteq NC^{a}(D) \subseteq Q^c \). We get \( Q \subseteq (NC^{a}(D))^c \), so it is \( (NC^{a}(D))^c \) is a NC\( a \)-OS and contains \( Q \). Now, to prove \( D \Cap (NC^{a}(D)) = \emptyset \). Since \( D \subseteq NC^{a}(D) \), but \( NC^{a}(D)(NC^{a}(D))^c = \emptyset \). Hence \( S \) is a NC\( a \)-regular space. The proofs (ii), (iii) are evident for others. 

**Theorem 4.4:**

Let \((S, \zeta)\) be a NCTS, then:

(i) \( S \) is a NC\( S^a \)-regular if and only if for each \( \mathcal{B} \) NC\( S^a \)-OS contains \( u \), there exists \( D \) NC\( S^a \)-OS contains \( u \) such that \( u \in D \subseteq NC^{S^a}(D) \subseteq \mathcal{B} \).

(ii) \( S \) is a NC\( S^\infty \)-regular if and only if for each \( \mathcal{B} \) NC\( S^a \)-OS contains \( u \), there exists \( D \) NC-OS contains \( u \) such that \( u \in D \subseteq NC(D) \subseteq \mathcal{B} \).

**Proof:**

The proof of this is similar to that of theorem (4.3). 

**Theorem 4.5:**

Let \((S, \zeta)\) be a NCTS, then:

(i) \( S \) is a NC\( a \)-regular if and only if \( u \notin Q \) where \( Q \) is a NC\( a \)-CS, there exist two NC\( a \)-OSs \( \mathcal{B} \) and \( D \) such that \( u \in \mathcal{B}, \mathcal{B} \subseteq D \) and NC\( a \)(\( \mathcal{B} \))\( NC^{a}(D) = \emptyset \).

(ii) \( S \) is a NC\( \infty \)-regular if and only if for each \( Q \) NC\( a \)-CS, such that \( u \notin Q \), there exist two NC-OSs \( \mathcal{B} \) and \( D \) such that \( u \in \mathcal{B}, \mathcal{B} \subseteq D \) and NC\( a \)(\( \mathcal{B} \))\( NC^{a}(D) = \emptyset \).

**Proof:**

(i) Let \( S \) be a NC\( a \)-regular space and let \( Q \) be a NC\( a \)-CS, such that \( u \notin Q \). Then there exist two NC\( a \)-OSs \( U \) and \( V \) such that \( u \in U \subseteq V \) and \( U \cap V = \emptyset \). Therefore \( U \) is a NC\( a \)-OS containing \( u \) in \( S \), where \( S \) is a NC\( a \)-regular space. Then there exists \( \mathcal{B} \) NC\( a \)-OS containing \( u \) such that \( u \in \mathcal{B} \subseteq NC^{a}(\mathcal{B}) \subseteq \mathcal{U} \) (since by theorem (4.3) (ii)). Hence NC\( a \)(\( \mathcal{B} \))\( NC^{a}(\mathcal{U}) = \emptyset \). Also, \( Q \subseteq V \subseteq NC^{a}(V) \), but NC\( a \)(\( V \))\( (NC^{a}(V))^c \) (since \( U \cap V = \emptyset \Rightarrow V \subseteq NC^{a}(U) \subseteq U \)). Hence \( Q \subseteq V \subseteq NC^{a}(V) \subseteq NC^{a}(U) \subseteq U \subseteq NC^{a}(U) \subseteq U \subseteq NC^{a}(U^c) \subseteq U^c \) (since \( U^c \) is a NC\( a \)-CS). Suppose that \( V = D \), hence \( Q \subseteq D \subseteq NC^{a}(D) \subseteq U \subseteq U^c \). Since \( U \Cap U^c = \emptyset \), hence NC\( a \)(\( \mathcal{B} \))\( NC^{a}(\mathcal{D}) = \emptyset \) (since NC\( a \)(\( \mathcal{B} \))\( U \subseteq \mathcal{U} \) and NC\( a \)(\( \mathcal{D} \))\( U^c \)). The other side is clear. The proof (ii) is evident for others. 

**Theorem 4.6:**

Let \((S, \zeta)\) be a NCTS, then:

(i) \( S \) is a NC\( S^a \)-regular if and only if \( u \notin Q \), where \( Q \) is a NC\( S^a \)-CS, there exist two NC\( S^a \)-OSs \( \mathcal{B} \) and \( D \) such that \( u \in \mathcal{B}, \mathcal{B} \subseteq D \) and NC\( S^a \)(\( \mathcal{B} \))\( NC^{S^a}(D) = \emptyset \).

(ii) \( S \) is a NC\( S^\infty \)-regular if and only if \( u \notin Q \), where \( Q \) is a NC\( S^a \)-CS, there exist two NC-OSs \( \mathcal{B} \) and \( D \) such that \( u \in \mathcal{B}, \mathcal{B} \subseteq D \) and NC\( S^a \)(\( \mathcal{B} \))\( NC^{S^a}(D) = \emptyset \).

**Proof:**

The proof of this is similar to that of theorem (4.5). 

**Remark 4.7:**

(i) NC\( a \)-regular property is a NC\( \infty \)-topological property.

(ii) NC\( a \)-regular property is a NC\( a \)-topological property.

(iii) NC\( \infty \)-regular property is a NC\( a \)-topological property.

(iv) NC\( S^a \)-regular property is a NC\( S^a \)-topological property.

(v) NC\( S^\infty \)-regular property is a NC\( S^a \)-topological property.
(vi) $NC^{sa''}$-regular property is a $NC^{sa'''}$-topological property.

**Proposition 4.8:**

(i) If $S \times J$ is a $NC^{a'''}$-regular, then both $S$ and $J$ are $NC^{a'''}$-regular spaces.

(ii) If $S \times J$ is a $NC^{sa''''}$-regular, then both $S$ and $J$ are $NC^{sa''''}$-regular spaces.

**Proof:**

(i) Suppose that $S \times J$ is a $NC^{a'''}$-regular, to prove that $S$ and $J$ are $NC^{a'''}$-regular spaces. Let $U$ and $V$ be two $NC^{a}$-OSs in $S$ and $J$ containing $u$ and $v$ respectively. Hence $(u, v) \in U \times V$ where $U \times V$ is a $NC^{a}$-OS in $S \times J$ (by theorem (2.7)). Hence there exists $NC$-OS $K$ in $S \times J$ such that $(u, v) \in K \subseteq NCcl(K) \subseteq U \times V$ (since $S \times J$ is a $NC^{a'''}$-regular). Then there exist two $NC$-OSs $B$ and $D$ in $S$ and $J$ such that $(u, v) \in B \times D \subseteq NCcl(B \times D) = NCcl(B) \times NCcl(D) \subseteq U \times V$. Hence $u \in B \subseteq NCcl(B) \subseteq U$ => $S$ is a $NC^{a'''}$-regular space. Also, $v \in D \subseteq NCcl(D) \subseteq V$ => $J$ is a $NC^{a'''}$-regular space. The proof (ii) is evident for others. •

**Theorem 4.9:**

If $(S, \zeta)$ is a $NC^{a'''}$-regular ($NC^{a''''}$-regular respectively), then $\zeta = NC^{a}O(S)$.

**Proof:**

It is clear that \( \zeta \subseteq NC^{a}O(S) \). Let $B$ be a $NC^{a}$-OS in $S$ containing $u$. Then there exists a $NC^{a}$-OS $D$ containing $u$ such that $u \in D \subseteq NC^{a}cl(D) \subseteq B$ (since $S$ is $NC^{a'''}$-regular). Therefore $NC^{a}int(D) \subseteq NC^{a}int(nc^{a}cl(D)) \subseteq B$. Thus $u \in D \subseteq NC^{a}cl(NC^{a}int(D)) \subseteq B$ (since by proposition (2.9)). Hence $B$ is a $NC$-OS => $NC^{a}O(S) \subseteq \zeta$. Therefore $\zeta = NC^{a}O(S)$. •

**Proposition 4.10:**

(i) If $\rho: S \rightarrow J$ is a $NC^{a}$-CF and $S$ is a $NC^{a'''}$-regular, then $\rho$ is a $NC$-CF.

(ii) If $\rho: S \rightarrow J$ is a $NC^{a}$-CF and $J$ is a $NC^{a'''}$-regular, then $\rho$ is a $NC^{a'''}$-CF.

(iii) If $\rho: S \rightarrow J$ is a $NC^{a}$-CF and $S$ is a $NC^{a'''}$-regular, then $\rho$ is a $NC^{a'''}$-CF.

**Proof:**

(i) Let $\rho: S \rightarrow J$ be a $NC^{a}$-CF, to prove that $\rho$ is a $NC$-CF. Let $B$ be a $NC$-OS in $J$, then $\rho^{-1}(B)$ is a $NC^{a}$-OS in $S$ (since $\rho$ is a $NC^{a}$-CF). But $S$ is a $NC^{a}$-regular space (by hypothesis). Hence $\rho^{-1}(B)$ is a $NC^{a}$-OS in $S$ (since by theorem (4.9)). Therefore $\rho$ is a $NC$-CF. The proofs (ii), (iii) are evident for others. •

**Definition 4.11:**

Let $(S, \zeta)$ be a NCTS, then $S$ is said to be:

(i) $NC^{a''}$-$T_2$-space if $S$ is a $NC^{a''}$-$T_2$-space and $NC^{a''}$-regular space.

(ii) $NC^{a''}$-$T_3$-space if $S$ is a $NC^{a''}$-$T_3$-space and $NC^{a''}$-regular space.

(iii) $NC^{a'''}$-$T_3$-space if $S$ is $NC^{a''}$-$T_3$-space and $NC^{a'''}$-regular space.

**Definition 4.12:**

Let $(S, \zeta)$ be a NCTS, then $S$ is said to be:

(i) $NC^{sa''}$-$T_3$-space if $S$ is a $NC^{sa''}$-$T_3$-space and $NC^{sa''}$-regular space.

(ii) $NC^{sa''}$-$T_3$-space if $S$ is $NC^{sa''}$-$T_3$-space and $NC^{sa''}$-regular space.

(iii) $NC^{sa'''}$-$T_3$-space if $S$ is $NC^{sa''}$-$T_3$-space and $NC^{sa'''}$-regular space.

**Remark 4.13:**

(i) $NC^{a''}$-$T_3$ ($NC^{sa''}$-$T_3$ respectively) property is a $NC^{a''}$ ($NC^{sa''}$ respectively) topological property.

(ii) $NC^{a'''}$-$T_3$ ($NC^{sa'''}$-$T_3$ respectively) property is a $NC^{a'''}$ ($NC^{sa'''}$ respectively) topological property.

**Remark 4.14:**

(i) Every $NC^\alpha T_3$-space is a $NC^{\alpha^*} T_3$-space and $NC^{Sa^*} T_3$-space.
(ii) Every $NC^\alpha T_3$-space is a $NC^{Sa} T_3$-space.
(iii) Every $NC^{\alpha^*} T_3$-space ($NC^{Sa^*} T_3$-space respectively) is a $NC^\alpha T_3$-space ($NC^{Sa} T_3$-space, respectively).
(iv) Every $NC^\alpha T_3$-space ($NC^{Sa} T_3$-space respectively) is a $NC^\alpha T_2$-space ($NC^{Sa} T_2$-space, respectively).

**Proposition 4.15:**

$S \times J$ is a $NC^{\alpha^*} T_3$-space if and only if both $S$ and $J$ are $NC^{\alpha^*} T_3$-spaces.

**Proof:**

Follow directly from proposition (3.12) part (i) and proposition (4.8) part (i). □

5. Some New Concepts of Weakly Neutrosophic Crisp Normality

**Definition 5.1:**

Let $(S, \zeta)$ be a NCTS, then $S$ is said to be:

(i) $NC^\alpha$-normal ($NC^{Sa}$-normal respectively) if for every two NC-CSs $Q_1$ and $Q_2$ such that $Q_1 \cap Q_2 = \emptyset_N$ there exist two $NC^\alpha$-OSs ($NC^{Sa}$-OSs respectively) $\mathcal{B}$ and $\mathcal{D}$ such that $Q_1 \subseteq \mathcal{B}$ and $Q_2 \subseteq \mathcal{D}$ and $\mathcal{B} \cap \mathcal{D} = \emptyset_N$.

(ii) $NC^{\alpha^*}$-normal ($NC^{Sa^*}$-normal respectively) if for every two NC-CSs ($NC^{Sa}$-CSs respectively) $Q_1$ and $Q_2$ such that $Q_1 \cap Q_2 = \emptyset_N$ there exist two $NC^{\alpha^*}$-OSs ($NC^{Sa^*}$-OSs respectively) $\mathcal{B}$ and $\mathcal{D}$ such that $Q_1 \subseteq \mathcal{B}$ and $Q_2 \subseteq \mathcal{D}$ and $\mathcal{B} \cap \mathcal{D} = \emptyset_N$.

(iii) $NC^{\alpha^{**}}$-normal ($NC^{Sa^{**}}$-normal respectively) if for every two NC-CSs ($NC^{Sa}$-CSs respectively) $Q_1$ and $Q_2$ such that $Q_1 \cap Q_2 = \emptyset_N$, there exist two NC-OSs $\mathcal{B}$ and $\mathcal{D}$ such that $Q_1 \subseteq \mathcal{B}$ and $Q_2 \subseteq \mathcal{D}$ and $\mathcal{B} \cap \mathcal{D} = \emptyset_N$.

**Remark 5.2:**

The following diagram shows the relation between the different types of weakly NC-normal and weakly $NC^\alpha$-normal ($NC^{Sa}$-normal respectively) spaces:

![Diagram showing the relation between different types of weakly NC-normal and weakly $NC^\alpha$-normal spaces](image)

**Theorem 5.3:**
Let \((\mathcal{S}, \xi)\) be a NCTS, then:

(i) \(\mathcal{S}\) is a NC\(^{a}\)-normal space if and only if for every NC-CS \(\mathcal{Q}\) and every NC-OS \(\mathcal{B}\) containing \(\mathcal{Q}\), there exists NC\(^{a}\)-OS say \(\mathcal{D}\), such that \(\mathcal{Q} \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\).

(ii) \(\mathcal{S}\) is a NC\(^{a}\)-normal space if and only if for every NC-CS \(\mathcal{Q}\) and every NC\(^{a}\)-OS \(\mathcal{B}\) containing \(\mathcal{Q}\), there exists NC\(^{a}\)-OS say \(\mathcal{D}\), such that \(\mathcal{Q} \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\).

(iii) \(\mathcal{S}\) is a NC\(^{a}\)-normal space if and only if for every NC-CS \(\mathcal{Q}\) and every NC\(^{a}\)-OS \(\mathcal{B}\) containing \(\mathcal{Q}\), there exists NC-OS say \(\mathcal{D}\), such that \(\mathcal{Q} \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\).

**Proof:**

(i) \(\Rightarrow\) Let \(\mathcal{S}\) be a NC\(^{a}\)-normal space. Let \(\mathcal{Q} \subseteq \mathcal{B}\), where \(\mathcal{Q}\) is a NC-CS and \(\mathcal{B}\) is a NC-OS \(\Rightarrow\ Q \cap \mathcal{B}^c = \emptyset\), where \(\mathcal{B}^c\) is a NC-CS. Hence there exist two NC\(^{a}\)-OSs \(\mathcal{D}_1, \mathcal{D}_2\) such that \(\mathcal{Q} \subseteq \mathcal{D}_1\) and \(\mathcal{B}^c \subseteq \mathcal{D}_2\) and \(\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset\), (since \(\mathcal{S}\) is a NC\(^{a}\)-normal space). Therefore \(\mathcal{Q} \subseteq \mathcal{D}_1 \subseteq NS\,cl(\mathcal{D}_1) \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\), where \(\mathcal{D}_1\) is a NC\(^{a}\)-OS in \(\mathcal{S}\).

\(\Leftarrow\) To prove \(\mathcal{S}\) is a NC\(^{a}\)-normal space. Let \(\mathcal{Q}_1\) and \(\mathcal{Q}_2\) be NC-CSs in \(\mathcal{S}\) such that \(\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset\). Hence \(\mathcal{Q}_1 \subseteq \mathcal{Q}_2^c\), where \(\mathcal{Q}_2^c\) is a NC-CS. Then there exists a NC-OS \(\mathcal{D}\) such that \(\mathcal{Q}_1 \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{Q}_2^c\) (by hypothesis). Hence \(\mathcal{Q}_1 \subseteq \mathcal{D}, \mathcal{Q}_2 \subseteq (NS\,cl(\mathcal{D}))^c\). On the other hand \(NS\,cl(\mathcal{D}) \cap (NS\,cl(\mathcal{D}))^c = \emptyset\). Hence \(\mathcal{D} \cap (NS\,cl(\mathcal{D}))^c = \emptyset\) (since \(\mathcal{D} \subseteq NS\,cl(\mathcal{D})\)). Therefore \(\mathcal{S}\) is a NC\(^{a}\)-normal space. The proofs (ii), (iii) are evident for others. □

**Theorem 5.4:**

Let \((\mathcal{S}, \xi)\) be a NCTS, then:

(i) \(\mathcal{S}\) is a NC\(^{a}\)-normal space if and only if for every NC-CS \(\mathcal{Q}\) and every NC-OS \(\mathcal{B}\) containing \(\mathcal{Q}\), there exists NC\(^{a}\)-OS say \(\mathcal{D}\), such that \(\mathcal{Q} \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\).

(ii) \(\mathcal{S}\) is a NC\(^{a}\)-normal space if and only if for every \(\mathcal{S}\) and every NC\(^{a}\)-OS \(\mathcal{B}\) containing \(\mathcal{Q}\), there exists NC\(^{a}\)-OS say \(\mathcal{D}\), such that \(\mathcal{Q} \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\).

(iii) \(\mathcal{S}\) is a NC\(^{a}\)-normal space if and only if for every NC\(^{a}\)-CS \(\mathcal{Q}\) and every NC\(^{a}\)-OS \(\mathcal{B}\) containing \(\mathcal{Q}\), there exists NC-OS say \(\mathcal{D}\), such that \(\mathcal{Q} \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\).

**Proof:**

(i) \(\Rightarrow\) Let \(\mathcal{S}\) be a NC\(^{a}\)-normal space. Let \(\mathcal{Q} \subseteq \mathcal{B}\), where \(\mathcal{Q}\) is a NC-CS and \(\mathcal{B}\) is a NC-OS \(\Rightarrow\ Q \cap \mathcal{B}^c = \emptyset\), where \(\mathcal{B}^c\) is a NC-CS. Hence there exist two NC\(^{a}\)-OSs \(\mathcal{D}_1, \mathcal{D}_2\) such that \(\mathcal{Q} \subseteq \mathcal{D}_1\) and \(\mathcal{B}^c \subseteq \mathcal{D}_2\) and \(\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset\), (since \(\mathcal{S}\) is a NC\(^{a}\)-normal space). Therefore \(\mathcal{Q} \subseteq \mathcal{D}_1 \subseteq NS\,cl(\mathcal{D}_1) \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{B}\), where \(\mathcal{D}_1\) is a NC\(^{a}\)-OS in \(\mathcal{S}\).

\(\Leftarrow\) To prove \(\mathcal{S}\) is a NC\(^{a}\)-normal space. Let \(\mathcal{Q}_1\) and \(\mathcal{Q}_2\) be NC-CSs in \(\mathcal{S}\) such that \(\mathcal{Q}_1 \cap \mathcal{Q}_2 = \emptyset\). Hence \(\mathcal{Q}_1 \subseteq \mathcal{Q}_2^c\), where \(\mathcal{Q}_2^c\) is a NC-CS. Then there exists a NC-OS \(\mathcal{D}\) such that \(\mathcal{Q}_1 \subseteq \mathcal{D} \subseteq NS\,cl(\mathcal{D}) \subseteq \mathcal{Q}_2^c\) (by hypothesis). Hence \(\mathcal{Q}_1 \subseteq \mathcal{D}, \mathcal{Q}_2 \subseteq (NS\,cl(\mathcal{D}))^c\). On the other hand \(NS\,cl(\mathcal{D}) \cap (NS\,cl(\mathcal{D}))^c = \emptyset\). Hence \(\mathcal{D} \cap (NS\,cl(\mathcal{D}))^c = \emptyset\) (since \(\mathcal{D} \subseteq NS\,cl(\mathcal{D})\)). Therefore \(\mathcal{S}\) is a NC\(^{a}\)-normal space. The proofs (ii), (iii) are evident for others. □

**Remark 5.5:**

(i) NC\(^{a}\)-normal property is a NC\(^{a}\)-topological property.

(ii) NC\(^{a}\)-normal property is a NC\(^{a}\)-topological property.

(iii) NC\(^{a}\)-normal property is a NC\(^{a}\)-topological property.

(iv) NC\(^{a}\)-normal property is a NC\(^{a}\)-topological property.

(v) NC\(^{a}\)-normal property is a NC\(^{a}\)-topological property.
(vi) $\text{NC}^{S_a^{**}}$-normal property is a $\text{NC}^{S_a^{**}}$-topological property.

**Proposition 5.6:**

(i) If $\mathcal{S} \times \mathcal{I}$ is a $\text{NC}^{a^{**}}$-normal space, then both $\mathcal{S}$ and $\mathcal{I}$ are $\text{NC}^{a^{**}}$-normal spaces.

(ii) If $\mathcal{S} \times \mathcal{I}$ is a $\text{NC}^{S_a^{**}}$-normal space, then both $\mathcal{S}$ and $\mathcal{I}$ are $\text{NC}^{S_a^{**}}$-normal spaces.

**Proof:**

(i) Suppose that $\mathcal{S} \times \mathcal{I}$ is a $\text{NC}^{a^{**}}$-normal space, to prove that $\mathcal{S}$ and $\mathcal{I}$ are $\text{NC}^{a^{**}}$-normal spaces. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two $\text{NC}^a$-OSs in $\mathcal{S}$ and $\mathcal{I}$ respectively, such that $Q_1 \subseteq \mathcal{B}_1$ and $Q_2 \subseteq \mathcal{B}_2$, where $Q_1$ and $Q_2$ are $\text{NC}^a$-CSs in $\mathcal{S}$ and $\mathcal{I}$ respectively. Hence $Q_1 \times Q_2 \subseteq \mathcal{B}_1 \times \mathcal{B}_2$ where $Q_1 \times Q_2$ is a $\text{NC}^a$-CS and $\mathcal{B}_1 \times \mathcal{B}_2$ is a $\text{NC}^a$-OS in $\mathcal{S} \times \mathcal{I}$ (by theorem (2.7) and corollary (2.8)). But $\mathcal{S} \times \mathcal{I}$ is a $\text{NC}^{a^{**}}$-normal space. Then there exists a $\text{NC}$-OS say $\mathcal{D}$ in $\mathcal{S} \times \mathcal{I}$ such that $Q_1 \times Q_2 \subseteq \mathcal{D} \subseteq \text{NCcl}(\mathcal{D}) \subseteq \mathcal{B}_1 \times \mathcal{B}_2$. Then there exist $\text{NC}$-OSs $\mathcal{U}_1$ and $\mathcal{U}_2$ in $\mathcal{S} \times \mathcal{I}$ such that $Q_1 \times Q_2 \subseteq \mathcal{U}_1 \times \mathcal{U}_2 \subseteq \text{NCcl}(\mathcal{U}_1 \times \mathcal{U}_2) = \text{NCcl}(\mathcal{U}_1) \times \text{NCcl}(\mathcal{U}_2) \subseteq \mathcal{B}_1 \times \mathcal{B}_2$. Hence $Q_1 \subseteq \mathcal{U}_1 \subseteq \text{NCcl}(\mathcal{U}_1) \subseteq \mathcal{B}_1 \implies \mathcal{S}$ is a $\text{NC}^{a^{**}}$-normal space. Also, $Q_2 \subseteq \mathcal{U}_2 \subseteq \text{NCcl}(\mathcal{U}_2) \subseteq \mathcal{B}_2 \implies \mathcal{I}$ is a $\text{NC}^{a^{**}}$-normal space. The proof (ii) is evident for others.

**Definition 5.7:**

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) $\text{NC}^a_2$-$T_4$-space if $\mathcal{S}$ is a $\text{NC}^a_2$-$T_1$-space and $\text{NC}^a$-normal space.

(ii) $\text{NC}^{a^{**}}_2$-$T_4$-space if $\mathcal{S}$ is a $\text{NC}^{a^{**}}_2$-$T_1$-space and $\text{NC}^{a^{**}}$-normal space.

(iii) $\text{NC}^{S_a^{**}}_2$-$T_4$-space if $\mathcal{S}$ is $\text{NC}^{S_a^{**}}_2$-$T_1$-space and $\text{NC}^{S_a^{**}}$-normal space.

**Definition 5.8:**

Let $(\mathcal{S}, \zeta)$ be a NCTS, then $\mathcal{S}$ is said to be:

(i) $\text{NC}^{S_a}_2$-$T_4$-space if $\mathcal{S}$ is a $\text{NC}^{S_a}_2$-$T_1$-space and $\text{NC}^{S_a}$-normal space.

(ii) $\text{NC}^{S_a^{**}}_2$-$T_4$-space if $\mathcal{S}$ is $\text{NC}^{S_a^{**}}_2$-$T_1$-space and $\text{NC}^{S_a^{**}}$-normal space.

(iii) $\text{NC}^{S_a^{**}}_2$-$T_4$-space if $\mathcal{S}$ is $\text{NC}^{S_a^{**}}_2$-$T_1$-space and $\text{NC}^{S_a^{**}}$-normal space.

**Remark 5.9:**

(i) $\text{NC}^{a}_2$-$T_4$ ($\text{NC}^{S_a}_2$-$T_4$ respectively) property is a $\text{NC}^{a^{**}}$ ($\text{NC}^{S_a^{**}}$ respectively) topological property.

(ii) $\text{NC}^{a^{**}}_2$-$T_4$ ($\text{NC}^{S_a^{**}}_2$-$T_4$ respectively) property is a $\text{NC}^{a}$ ($\text{NC}^{S_a}$ respectively) topological property.

(iii) $\text{NC}^{a^{**}}_2$-$T_4$ ($\text{NC}^{S_a^{**}}_2$-$T_4$ respectively) property is a $\text{NC}^{a^{**}}$ ($\text{NC}^{S_a^{**}}$ respectively) topological property.

**Remark 5.10:**

(i) Every $\text{NC}$-$T_4$-space is a $\text{NC}^{a}_2$-$T_1$-space and $\text{NC}^{S_a}_2$-$T_4$-space.

(ii) Every $\text{NC}^{a}_2$-$T_4$-space is a $\text{NC}^{S_a}_2$-$T_4$-space.

(iii) Every $\text{NC}^{a^{**}}_2$-$T_4$-space is a $\text{NC}^{a^{**}}_2$-$T_4$-space and $\text{NC}^{S_a}_2$-$T_4$-space.

(iv) Every $\text{NC}^{a}_2$-$T_4$-space ($\text{NC}^{S_a}_2$-$T_4$-space respectively) is a $\text{NC}^{a}_2$-$T_3$-space ($\text{NC}^{S_a}_2$-$T_3$-space, respectively).

(v) Every $\text{NC}^{a^{**}}_2$-$T_4$-space ($\text{NC}^{S_a^{**}}_2$-$T_4$-space respectively) is a $\text{NC}^{a^{**}}_2$-$T_3$-space ($\text{NC}^{S_a^{**}}_2$-$T_3$-space, respectively).

**Remark 5.11:**

The following diagram explains the relationships between usual NC-separation axioms, $\text{NC}^a$-separation axioms and $\text{NC}^{S_a}$-separation axioms:
Also, we have the following diagram:

6. Conclusions

We have provided some new concepts of weakly neutrosophic crisp separation axioms. Some characterizations have been provided to illustrate how far topological structures are conserved by the new neutrosophic crisp notion defined. Furthermore, some new concepts of weakly neutrosophic crisp regularity are also studied. The study demonstrated some new concepts of weakly neutrosophic crisp normality and proved some of their related attributes.
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