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INTRODUCTION OF NEUTROSOPHIC SOFT LIE ALGEBRAS

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Abstract. We introduce the concept of neutrosophic soft Lie subalgebras of a Lie algebra and investigate some of their properties. The Cartesian product of neutrosophic soft Lie subalgebras will be discussed. In particular, the homomorphisms of neutrosophic soft Lie algebras is introduced and investigated some of their properties.

Keywords: Lie algebra, subalgebra, neutrosophic soft set, neutrosophic soft Lie Algebras.

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1. **Introduction**

The contribution of mathematics to the present-day technology in reaching to a fast trend cannot be ignored. The theories presented differently from classical methods in studies such as fuzzy set [25], intuitionistic fuzzyset [9], soft set [18], neutrosophic set [17, 20], etc. The algebraic structure of set theories dealing with uncertainties has also been studied by some authors. After Molodtsov's work, some different applications of soft sets were studied in [18]. Maji et al. [16] presented the concept of fuzzy soft set. This kind of fuzzy sets have now gained a wide recognition as useful tool,in modeling of some uncertain phenomena, computer science, mathematics, medicine, chemistry, economics, astronomy etc. Smarandache [20,21] introduced the concept of neutrosophic set which is a mathematical tool for handling problems involving imprecise, indeterminacy and inconsistent data. Rosenfeld [19] proposed the concept of fuzzy groups in order to establish the algebraic structures of fuzzy sets. Definition of fuzzy module is given by some authors. Qiu- Mei Sun et al. defined soft modules and investigated their basic properties. Fuzzy soft modules and intuitionistic fuzzy soft modules was given and researched by C. Gunduz(Aras) and S. Bayramov [11, 12]. Neutrosophic soft modules are introduction in [23].

Lie algebras were first discovered by Sophus Lie (1842-1899) when he attempted to classify certain smoothsubgroups of general linear groups [13]. The groups he considered are now called Lie groups. By taking the tangent space at the identity element of such a group, he obtained the Lie algebra and hence the problems on groups can be reduced to problems on Lie algebras so that it becomes more tractable. There are many applications of Lie algebras in many branches of mathematics and physics. In [1–8, 10, 14, 15, 22, 24] there is an introduction the concept of fuzzy Lie subalgebras and investigation of some of their properties.

In this paper we have introduced the concept of neutrosophic soft Lie subalgebras of a Lie algebra and investigated some of their properties. The Cartesian product of neutrosophic soft Lie subalgebras will be discussed. In particular, the homomorphisms of neutrosophic soft Lie algebras is introduced and investigated some of their properties.

2. Preliminaries

In this section, we first review some elementary aspects that are necessary for this paper.

Definition 2.1. [24] An intuitionistic fuzzy set $A = (\mu_A, \lambda_A)$ on L is called an intuitionistic fuzzy Lie subalgebra if the following conditions are satisfied:

$$\mu_A(x+y) \geq min(\mu_A(x), \ \mu_A(y)) \text{ and } \lambda_A(x+y) \leq \max\ (\lambda_A(x), \lambda_A(y)), \tag{1}$$

$$\mu_A(\alpha x) \geq \mu_A(x) \text{ and } \lambda_A(\alpha x)) \leq \ \lambda_A(x) \tag{2}$$

$$\mu_A([x,y]) \geq min\{\mu_A(x), \mu_A(y)\} \text{ and } \lambda_A([x,y]) \leq max\ \{\lambda_A(x), \lambda_A(y)\} \tag{3}$$

for all $x, y \in L$ and $\alpha \in F$.

Definition 2.2. [22] A neutrosophic set A on the universe of X is defined as
$A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$, where $T, I, F : X \to]^{-}0, 1^{+}[$ and

$$^{-}0 < 0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^{+}$$

Definition 2.3. [17] Let X be an initial universe set and E be a set of parametres. Let $P(X)$ denote the set of all neutrosophic sets of X. Then, a neutrosophic soft set $\left(\tilde{F}\ , E\right)$ over X is a set defined by a set valued function $\tilde{F}$ representing a mapping $\tilde{F}\ : E \to P(X)$ where $\tilde{F}$ is called approximate function of the neutrosophic soft set $\left(\tilde{F}, E\right)$. In other words, the neutrosophic soft set is a parameterized family of some elements of the set $P(X)$ and therefore it can be written as a set of ordered pairs,

$$(\tilde{F}\ : E) = \left\{\left(e, \left\langle x,\ \ T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x)\right\rangle\ : x \epsilon X\right)\ : e \in E\right\}$$

where $T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x) \in [0,1]$ respectively called the truth-membership, indeterminacy-membership, falsity-membership function of $\tilde{F}(e)$. Since supremum of each $T,\ I,\ F$ is 1 so the inequality $0 \leq T_{\tilde{F}(e)}(x), I_{\tilde{F}(e)}(x), F_{\tilde{F}(e)}(x) \leq 3$ is obvious.

Definition 2.4. [17] Let $\left(\tilde{F}, E\right)$ be neutrosophic soft set over the common universe X. The complement of $\left(\tilde{F}, E\right)$ is denoted by $\left(\tilde{F}, E\right)^c$ and is defined by:

$$\left(\tilde{F}, E\right)^c = \left\{ \left(e, \left\langle x, F_{\tilde{F}(e)}(x), 1 - I_{\tilde{F}(e)}(x), T_{\tilde{F}(e)}(x), \right\rangle : x \in X\right) : e \in E\right\}.$$

Obvious that, $\left(\left(\tilde{F}, E\right)^c\right)^c = \left(\tilde{F}, E\right)$.

Definition 2.5. [17] Let $\left(\tilde{F}, E\right)$ and $\left(\tilde{G}, E\right)$ be two neutrosophic soft sets over the common universe X. $\left(\tilde{F}, E\right)$ is said to be neutrosophic soft subset of $\left(\tilde{G}, E\right)$ if $\mathrm{T}_{\tilde{F}(e)}(x) \leq \mathrm{T}_{\tilde{G}(e)}(x), I_{\tilde{F}(e)}(x) \leq I_{\tilde{G}(e)}(x), F_{\tilde{F}(e)}(x) \geq F_{\tilde{G}(e)}(x), \quad \forall e \in E,\ \forall x \in X$. It is denoted by $\left(\tilde{F}, E\right) \subseteq \left(\tilde{G}, E\right)$.

Definition 2.6. [17] Let $\left(\tilde{F}_1, E\right)$ and $\left(\tilde{F}_2, E\right)$ be two neutrosophic soft sets over the common universe X. Then their union is denoted by $\left(\tilde{F}_1, E\right) \bigcup \left(\tilde{F}_2, E\right) = \left(\tilde{F}_3, E\right)$ and is defined by:

$$\left(\tilde{F}_3, E\right) = \left\{ \left(e, \left\langle x, \mathrm{T}_{\tilde{F}_3(e)}(x), I_{\tilde{F}_3(e)}(x), F_{\tilde{F}_3(e)}(x), \right\rangle : x \in X\right) : e \in E\right\}$$

where

$$\begin{aligned} \mathrm{T}_{\tilde{F}_3(e)}(x) &= \max\left\{\mathrm{T}_{\tilde{F}_1(e)}(x), \mathrm{T}_{\tilde{F}_2(e)}(x)\right\}, \\ I_{\tilde{F}_3(e)}(x) &= \max\left\{I_{\tilde{F}_1(e)}(x), I_{\tilde{F}_2(e)}(x)\right\}, \\ F_{\tilde{F}_3(e)}(x) &= \min\left\{F_{\tilde{F}_1(e)}(x), F_{\tilde{F}_2(e)}(x)\right\}. \end{aligned}$$

Definition 2.7. [17] Let $\left(\tilde{F}_1, E\right)$ and $\left(\tilde{F}_2, E\right)$ be two neutrosophic soft sets over the common universe X. Then their intersection is denoted by $\left(\tilde{F}_1, E\right) \bigcap \left(\tilde{F}_2, E\right) = \left(\tilde{F}_3, E\right)$ and is defined by:

$$\left(\tilde{F}_3, E\right) = \left\{ \left(e, \left\langle x, \mathrm{T}_{\tilde{F}_3(e)}(x), I_{\tilde{F}_3(e)}(x), F_{\tilde{F}_3(e)}(x), \right\rangle : x \in X\right) : e \in E\right\}$$

where

$$\begin{aligned} \mathrm{T}_{\tilde{F}_3(e)}(x) &= \min\left\{\mathrm{T}_{\tilde{F}_1(e)}(x), \mathrm{T}_{\tilde{F}_2(e)}(x)\right\}, \\ I_{\tilde{F}_3(e)}(x) &= \min\left\{I_{\tilde{F}_1(e)}(x), I_{\tilde{F}_2(e)}(x)\right\}, \\ F_{\tilde{F}_3(e)}(x) &= \max\left\{F_{\tilde{F}_1(e)}(x), F_{\tilde{F}_2(e)}(x)\right\}. \end{aligned}$$

Definition 2.8. [17] Let $\left(\tilde{F}_1, E\right)$ and $\left(\tilde{F}_2, E\right)$ be two neutrosophic soft sets over the common universe X. Then " AND" operation on them is denoted by $\left(\tilde{F}_1, E\right) \wedge \left(\tilde{F}_2, E\right) = \left(\tilde{F}_3, E \times E\right)$ and is defined by:

$$\left(\tilde{F}_3, E \times E\right) =$$

$$\left\{\left((e_1,e_2),\left\langle x,\mathrm{T}_{\tilde{F}_{3(e_1,e_2)}}(x),I_{\tilde{F}_{3(e_1,e_2)}}(x),F_{\tilde{F}_{3(e_1,e_2)}}(x),\right\rangle : x\in X\right) : (e_1,e_2)\in E\times E\right\}$$

where

$$\mathrm{T}_{\tilde{F}_{3(e_1,e_2)}}(x)=\min\left\{\mathrm{T}_{\tilde{F}_{1(e_1)}}(x),\mathrm{T}_{\tilde{F}_{2(e_2)}}(x)\right\},$$
$$I_{\tilde{F}_{3(e_1,e_2)}}(x)=\min\left\{I_{\tilde{F}_{1(e_1)}}(x),I_{\tilde{F}_{2(e_2)}}(x)\right\},$$
$$F_{\tilde{F}_{3(e_1,e_2)}}(x)=\max\left\{F_{\tilde{F}_{1(e_1)}}(x),F_{\tilde{F}_{2(e_2)}}(x)\right\}.$$

3. **Introduction of neutrosophic Soft Lie algebras**

Definition 3.1. Let E be a set of all parameters, L be Lie algebra and $P(L)$ denotes all neutrosophic sets over L. Then a pair $\left(\tilde{F},E\right)$ is called a neutrosophic soft Lie algebra over L, where, $\tilde{F}$ is a mapping given by $\tilde{F}:E\to P(L)$, if for $\forall e\in E$, $\tilde{F}(e)=\left(T_{\tilde{F}}(e),I_{\tilde{F}}(e),F_{\tilde{F}}(e)\right)$ is a neutrosophic Lie algebra over L, i.e:

$$\begin{aligned}T_{\tilde{F}}(e)(x,y)&\geq\min\left(T_{\tilde{F}}(e)(x),T_{\tilde{F}}(e)(y)\right)\\ I_{\tilde{F}}(e)(x,y)&\geq\min\left(I_{\tilde{F}}(e)(x),I_{\tilde{F}}(e)(y)\right)\\ F_{\tilde{F}}(e)(x,y)&\leq\max\left(F_{\tilde{F}}(e)(x),F_{\tilde{F}}(e)(y)\right)\end{aligned}\tag{4}$$

$$\begin{aligned}T_{\tilde{F}}(e)(\alpha x)&\geq T_{\tilde{F}}(e)(x)\\ I_{\tilde{F}}(e)(\alpha x)&\geq I_{\tilde{F}}(e)(x)\\ F_{\tilde{F}}(e)(\alpha x)&\leq F_{\tilde{F}}(e)(x)\end{aligned}\tag{5}$$

$$\begin{aligned}T_{\tilde{F}}(e)([x+y])&\geq\min\left(T_{\tilde{F}}(e)(x),T_{\tilde{F}}(e)(y)\right)\\ I_{\tilde{F}}(e)([x+y])&\geq\min\left(I_{\tilde{F}}(e)(x),I_{\tilde{F}}(e)(y)\right)\\ F_{\tilde{F}}(e)([x+y])&\leq\max\left(F_{\tilde{F}}(e)(x),F_{\tilde{F}}(e)(y)\right)\end{aligned}$$

Definition 3.2. A neutrosophicsoft set $\left(\widetilde{F},E\right)$ on L is called neustrosophic soft Lie ideal if it satisfied the conditions (3.1), (3.2) and the following additional condition:
For each $e\in E$

$$T_{\widetilde{F}}(e)([x,y])\geq T_{\widetilde{F}}(e)\ (x)$$

$$I_{\widetilde{F}}(e)([x,y])\geq I_{\widetilde{F}}(e)\ (x)$$

$$F_{\widetilde{F}}(e)([x,y])\leq F_{\widetilde{F}}(e)\ (x)$$

For all $x,y\in L$

$$T_{\widetilde{F}}(e)(0)\geq T_{\widetilde{F}}(e)(x)\qquad T_{\widetilde{F}}(e)(-x)\geq T_{\widetilde{F}}(e)(x)$$

$$I_{\widetilde{F}}(e)(0)\geq I_{\widetilde{F}}(e)(x)\qquad I_{\widetilde{F}}(e)(-x)\geq I_{\widetilde{F}}(e)(x)$$

$$F_{\widetilde{F}}(e)(0)\leq F_{\widetilde{F}}(e)(x)\qquad F_{\widetilde{F}}(e)(-x)\leq F_{\widetilde{F}}(e)(x)$$

Proposition 3.1. Every neutrosophic soft Lie ideal is a neutrosophic soft Lie subalgebra.

Theorem 3.3. *Let $\left(\widetilde{F},E\right)$ be a neutrosophic soft Lie subalgebraover L.Then $\left(\widetilde{F},E\right)$ is a neutrosophic soft Lie subalgebra of L if and only if for each $e\in E$ the non-empty upper s-level cut.*

$$U_{T_{\widetilde{F}(e)}}(s)=\left\{x\in L\ /T_{\widetilde{F}}(e)\ (x)\geq s\right\}$$

$$U_{I_{\widetilde{F}(e)}}(s)=\left\{x\in L\ /I_{\widetilde{F}}(e)\ (x)\geq s\right\}$$

and the non-empty Lower s- lewel cut

$V_{F_{\widetilde{F}(e)}}(s)=\left\{x\in L/F_{\widetilde{F}}(e)(x)\leq s\right\}$ *are Lie subalgebras of L , for all $s\in[0,1]$*

Proof. Assume that $\left(\widetilde{F}\ ,E\right)$ is a neutrosophic soft Lie subalgebra of L and let $s\in[0,1]$ be such that $U_{T_{\widetilde{F}(e)}}(s)\neq\emptyset$. Let $x,y\in L$ be such that $x\in U_{T_{\widetilde{F}(e)}}(s)$ and $y\in U_{T_{\widetilde{F}(e)}}(s)$. It follows that

$$T_{\widetilde{F}}(e)\left(x+y\right)\geq\min\left(T_{\widetilde{F}}(e)\left(x\right),T_{\widetilde{F}}(e)\left(y\right)\right)\geq s,$$

$$T_{\widetilde{F}}(e)\left(\alpha x\ \right)\geq T_{\widetilde{F}}(e)\left(x\right)\geq s,$$

$$T_{\widetilde{F}}\left(e\right)\left([x,y]\right)\geq\min(T_{\widetilde{F}}\left(e\right)\left(x\right),T_{\widetilde{F}}(e)(y))\geq s$$

and hence, $x+y\in U_{T_{\widetilde{F}(e)}}(s)$, $\alpha x\in U_{T_{\widetilde{F}(e)}}(s)$, and $[x,y]\in U_{T_{\widetilde{F}(e)}}$. Thus, $U_{T_{\widetilde{F}(e)}}(s)$, forms a Lie subalgebra of L. For the case $U_{I_{\widetilde{F}(e)}}(s)$ and $V_{F_{\widetilde{F}(e)}}(s)$ the proof is analogously.

Conversely, suppose that $U_{T_{\widetilde{F}(e)}}(s)\neq\emptyset$, is a Lie subalgebra of L for every $s\in[0,1]$ and $e\in E$.

Assume that

$$T_{\widetilde{F}}\left(e\right)\left(x+y\right)<\min\left\{T_{\widetilde{F}}(e)\left(x\right),T_{\widetilde{F}}(e)\left(y\right)\right\}$$

For same $x,y\in L$ now, taking

$$S_O:=\frac{1}{2}\left\{T_{\widetilde{F}}(e)(x+y),\min\left\{T_{\widetilde{F}}(e)(x),T_{\widetilde{F}}(e)(y)\right\}\right\},$$

Then we have

$$T_{\widetilde{F}}\left(e\right)\left(x+y\right)<S_0\ <\min\left\{T_{\widetilde{F}}\left(e\right)\left(x\right),T_{\widetilde{F}}\left(e\right)\left(y\right)\right\}$$

And hence $x+y\in U_{T_{\widetilde{F}(e)}}\ (s)$ and $x\in U_{T_{\widetilde{F}(e)}}(s)$ and $y\in U_{T_{\widetilde{F}(e)}}(s)$.

However, this is clearly a contradiction.

Therefore,

$$T_{\widetilde{F}}\left(e\right)\left(x+y\right)\geq\min\left\{T_{\widetilde{F}}(e)\left(x\right),T_{\widetilde{F}}(e)\left(y\right)\right\}$$

For all $x,y\in L$ similarly we can show that:

$$T_{\widetilde{F}}\left(e\right)\left(\alpha x\right)\geq T_{\widetilde{F}}\left(e\right)\left(x\right),$$

$$T_{\widetilde{F}}\left(e\right)\left([x,y]\right)\geq\min\left\{T_{\widetilde{F}}\left(e\right)\left(x\right),T_{\widetilde{F}}\left(e\right)\left(y\right)\right\}$$

For the case $U_{I_{\widetilde{F}(e)}}(\ s)$ and $V_{F_{\widetilde{F}(e)}}(s)$ the proof is similar. □

Theorem 3.4. *If* $\left(\widetilde{F}^1, E_1\right)$ *and* $\left(\widetilde{F}^2, E_2\right)$ *be two neutrosophic soft Lie subalgebra over* L*, then intersection* $\left(\widetilde{F}^1, E_1\right) \cap \left(\widetilde{F}^2, E_2\right) = (\widetilde{F}^3, E_1 \cap E_2)$ *is a neutrosophic soft Lie subalgebra over* L*.*

Proof. For each $x, y \in L$, $e \in E_1 \cap E_2$,

$$T_{\widetilde{F}^3}(e)(x+y) = \min\left\{T_{\widetilde{F}^1}(e)(x+y), T_{\widetilde{F}^2}(e)(x+y)\right\} \geq$$
$$\geq \min\left\{\min\left\{T_{\widetilde{F}^1}(e)(x), T_{\widetilde{F}^1}(e)(y)\right\}, \min\left\{T_{\widetilde{F}^2}(e)(x), T_{\widetilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \min\{\min\{T_{\widetilde{F}^1}(e)(x), T_{\widetilde{F}^2}(e)(x)\}, \min\{T_{\widetilde{F}^1}(e)(y), T_{\widetilde{F}^2}(e)(y)\}\} =$$
$$= \min\{T_{\widetilde{F}^3}(e)(x), T_{\widetilde{F}^3}(e)(y)\}$$
$$I_{\widetilde{F}^3}(e)(x+y) = \min\left\{I_{\widetilde{F}^1}(e)(x+y), I_{\widetilde{F}^2}(e)(x+y)\right\} \geq$$
$$\geq \min\left\{\min\left\{I_{\widetilde{F}^1}(e)(x), I_{\widetilde{F}^1}(e)(y)\right\}, \min\left\{I_{\widetilde{F}^2}(e)(x), I_{\widetilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \min\left\{\min\left\{I_{\widetilde{F}^1}(e)(x), I_{\widetilde{F}^2}(e)(x)\right\}, \min\left\{I_{\widetilde{F}^1}(e)(y), I_{\widetilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \min\left\{I_{\widetilde{F}^3}(e)(x), I_{\widetilde{F}^3}(e)(y)\right\}$$
$$F_{\widetilde{F}^3}(e)(x+y) = \max\left\{F_{\widetilde{F}^1}(e)(x+y), F_{\widetilde{F}^2}(e)(x+y)\right\} \leq$$
$$\leq \max\{\max\{F_{\widetilde{F}^1}(e)(x), F_{\widetilde{F}^1}(e)(y)\}, \max\{F_{\widetilde{F}^2}(e)(x), F_{\widetilde{F}^2}(e)(y)\}\}$$
$$= \max\left\{\max\{F_{\widetilde{F}^1}(e)(x), F_{\widetilde{F}^2}(e)(x)\right\}, \max\{F_{\widetilde{F}^1}(e)(y), F_{\widetilde{F}^2}(e)(y)\}\} =$$
$$= \max\left\{F_{\widetilde{F}^3}(e)(x), F_{\widetilde{F}^3}(e)(y)\right\}$$
$$T_{\widetilde{F}^3}(e)(\alpha x) = \min\left\{T_{\widetilde{F}^1}(e)(\alpha x), T_{\widetilde{F}^2}(e)(\alpha x)\right\} \geq$$
$$\geq \min\left\{T_{\widetilde{F}^1}(e)(x), T_{\widetilde{F}^2}(e)(x)\right\} = T_{\widetilde{F}^3}(e)(x)$$
$$I_{\widetilde{F}^3}(e)(\alpha x) = \min\left\{I_{\widetilde{F}^1}(e)(\alpha x), I_{\widetilde{F}^2}(e)(\alpha x)\right\} \geq$$
$$\geq \min\left\{I_{\widetilde{F}^1}(e)(x), I_{\widetilde{F}^2}(e)(x)\right\} = I_{\widetilde{F}^3}(e)(x)$$
$$F_{\widetilde{F}^3}(e)(\alpha x) = \max\left\{F_{\widetilde{F}^1}(e)(\alpha x), F_{\widetilde{F}^2}(e)(\alpha x)\right\} \leq$$
$$\leq \max\left\{F_{\widetilde{F}^1}(e)(x), F_{\widetilde{F}^2}(e)(x)\right\} = F_{\widetilde{F}^3}(e)(x)$$

$$\begin{aligned}
&T_{\tilde{F}^3}(e)[x,y] = \min\left\{T_{\tilde{F}^1}(e)[x,y], T_{\tilde{F}^2}(e)[x,y]\right\} \geq \\
&\geq \min\left\{\min\left\{T_{\tilde{F}^1}(e)(x), T_{\tilde{F}^1}(e)(y)\right\}, \min\left\{T_{\tilde{F}^2}(e)(x), T_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{\min\left\{T_{\tilde{F}^1}(e)(x), T_{\tilde{F}^2}(e)(x)\right\}, \min\left\{T_{\tilde{F}^1}(e)(y), T_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{T_{\tilde{F}^3}(e)(x), T_{\tilde{F}^3}(e)(y)\right\}
\end{aligned}$$

$$\begin{aligned}
&I_{\tilde{F}^3}(e)[x,y] = \min\left\{I_{\tilde{F}^1}(e)[x,y], I_{\tilde{F}^2}(e)[x,y]\right\} \geq \\
&\geq \min\left\{\min\left\{I_{\tilde{F}^1}(e)(x), I_{\tilde{F}^1}(e)(y)\right\}, \min\left\{I_{\tilde{F}^2}(e)(x), I_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{\min\left\{I_{\tilde{F}^1}(e)(x), I_{\tilde{F}^2}(e)(x)\right\}, \min\left\{I_{\tilde{F}^1}(e)(y), I_{\tilde{F}^2}(e)(y)\right\}\right\} = \\
&= \min\left\{I_{\tilde{F}^3}(e)(x), I_{\tilde{F}^3}(e)(y)\right\}
\end{aligned}$$

$$F_{\tilde{F}^3}(e)[x,y] = \max\left\{F_{\tilde{F}^1}(e)[x,y], F_{\tilde{F}^2}(e)[x,y]\right\} \leq$$
$$\leq \max\left\{\max\left\{F_{\tilde{F}^1}(e)(x), F_{\tilde{F}^1}(e)(y)\right\}, \max\left\{F_{\tilde{F}^2}(e)(x), F_{\tilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \max\left\{\max\left\{F_{\tilde{F}^1}(e)(x), F_{\tilde{F}^2}(e)(x)\right\}, \max\left\{F_{\tilde{F}^1}(e)(y), F_{\tilde{F}^2}(e)(y)\right\}\right\} =$$
$$= \max\left\{F_{\tilde{F}^3}(e)(x), F_{\tilde{F}^3}(e)(y)\right\}$$

□

Theorem 3.5. *Let* $\left(\widetilde{F}^1, E_1\right)$ *and* $\left(\widetilde{F}^2, E_2\right)$ *be two neutrosophic soft Lie subalgebra over* L*. If* $E_1 \cap E_2 = \emptyset$*, then union* $\left(\widetilde{F}^1, E_1\right) \cup \left(\widetilde{F}^2, E_2\right) = (\widetilde{F}^3, E_1 \cup E_2)$ *is a neutrosophic soft Lie subalgebra over* L*.*

Proof. Since $E_1 \cap E_2 = \emptyset$, it follows that either $e \in E_1$ or $e \in E_2$ for all $e \in E_3$. If $e \in E_1$, then $(\widetilde{F}^3, E_1 \cup E_2) = (\widetilde{F}^1, E_1)$ is a neutrosophic soft Lie subalgebra of L, and if $e \in E_2$, then $(\widetilde{F}^3, E_1 \cup E_2) = (\widetilde{F}^2, E_2)$ is a neutrosophic soft Lie subalgebra of L. Hence $\left(\widetilde{F}^1, E_1\right) \cup \left(\widetilde{F}^2, E_2\right)$ is a neutrosophic soft Lie subalgebraover L. □

Theorem 3.6. $\left(\widetilde{F}, E\right)$ *be a neutrosophic soft Lie subalgebra over* L*, and let* $\left[\left(\widetilde{F}_i, E_i\right)\right]_{i\epsilon I}$ *be nonempty family of neutrosophic soft Lie subalgebra of* L*. Then*
1) $\prod_{i\epsilon I} \left(\widetilde{F}_i, E_i\right)$ *is a neutrosophic soft Lie subalgebra of* L*,*
2) If $E_i \cap E_j = \emptyset$*, for all* $i, j\epsilon I$*, then* $\bigcup_{i\in I}\left(\tilde{F}_i, E_i\right)$ *is a neutrosophic soft Lie subalgebra of* L*.*

Theorem 3.7. *Let* $\left(\widetilde{F}^1, E_1\right)$ *and* $\left(\widetilde{F}^2, E_2\right)$ *be two neutrosophic soft Lie algebras over* L_1 *and* L_2 *respectively. Then* $\left(\widetilde{F}^1, E_1\right) \wedge \left(\widetilde{F}^2, E_2\right) = \left(\widetilde{F}^3, E_1 \times E_2\right)$ *is a neutrosophic soft Lie algebra over* L*.*

Proof. For each $x, y \in L$, $e_1 \in E_1, e_2 \in E_2$, $\alpha \in K$,

$$T_{\tilde{F}^3}(e_1,e_2)(x+y) = T_{\tilde{F}^1}(e_1)(x+y) \wedge T_{\tilde{F}^2}(e_2)(x+y) \geq$$
$$\geq \left\{\min\left(T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(T_{\tilde{F}^2}(e_2)(x), T_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$= \min\left\{T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{T_{\tilde{F}^1}(e_1)(y), T_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$= T_{\tilde{F}^3}(e_1,e_2)(x) \wedge T_{\tilde{F}^3}(e_1,e_2)(y)$$

$$I_{\tilde{F}^3}(e_1,e_2)(x+y) = I_{\tilde{F}^1}(e_1)(x+y) \wedge I_{\tilde{F}^2}(e_2)(x+y) \geq$$
$$\geq \left\{\min\left(I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(I_{\tilde{F}^2}(e_2)(x), I_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$= \min\left\{I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{I_{\tilde{F}^1}(e_1)(y), I_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$= I_{\tilde{F}^3}(e_1,e_2)(x) \wedge I_{\tilde{F}^3}(e_1,e_2)(y)$$

$$F_{\tilde{F}^3}(e_1,e_2)(x+y) = F_{\tilde{F}^1}(e_1)(x+y) \vee F_{\tilde{F}^2}(e_2)(x+y) \leq$$
$$\leq \left\{\max\left(F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^1}(e_1)(y)\right)\right\} \vee \left\{\max\left(F_{\tilde{F}^2}(e_2)(x), F_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$= \max\left\{F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^2}(e_2)(x)\right\} \vee \max\left\{F_{\tilde{F}^1}(e_1)(y), F_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$= F_{\tilde{F}^3}(e_1,e_2)(x) \vee F_{\tilde{F}^3}(e_1,e_2)(y)$$

$$T_{\tilde{F}^3}(e_1,e_2)(\alpha(x)) = \left(T_{\tilde{F}^1}(e_1) \wedge T_{\tilde{F}^2}(e_2)\right)(\alpha(x)) =$$
$$\left(T_{\tilde{F}^1}(e_1) \wedge T_{\tilde{F}^2}(e_2)\right)(\alpha x) = \min\left(T_{\tilde{F}^1}(e_1)(\alpha x), T_{\tilde{F}^2}(e_2)(\alpha x)\right) \geq$$
$$\min\left(T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^2}(e_2)(x)\right) = \left(T_{\tilde{F}^1}(e_1) \wedge T_{\tilde{F}^2}(e_2)\right)(x) = T_{\tilde{F}^3}(e_1,e_2)(x),$$

$$I_{\tilde{F}^3}(e_1,e_2)(\alpha(x)) = \left(I_{\tilde{F}^1}(e_1) \wedge I_{\tilde{F}^2}(e_2)\right)(\alpha(x)) =$$
$$\left(I_{\tilde{F}^1}(e_1) \wedge I_{\tilde{F}^2}(e_2)\right)(\alpha x) = \min\left(I_{\tilde{F}^1}(e_1)(\alpha x), I_{\tilde{F}^2}(e_2)(\alpha x)\right) \geq$$
$$\min\left(I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^2}(e_2)(x)\right) = \left(I_{\tilde{F}^1}(e_1) \wedge I_{\tilde{F}^2}(e_2)\right)(x) = I_{\tilde{F}^3}(e_1,e_2)(x),$$

$$F_{\tilde{F}^3}(e_1,e_2)(\alpha(x)) = \left(F_{\tilde{F}^1}(e_1) \vee F_{\tilde{F}^2}(e_2)\right)(\alpha(x)) =$$
$$\left(F_{\tilde{F}^1}(e_1) \vee F_{\tilde{F}^2}(e_2)\right)(\alpha x) = \max\left(F_{\tilde{F}^1}(e_1)(\alpha x), F_{\tilde{F}^2}(e_2)(\alpha x)\right) \leq$$
$$\max\left(F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^2}(e_2)(x)\right) = \left(F_{\tilde{F}^1}(e_1) \vee F_{\tilde{F}^2}(e_2)\right)(x) = F_{\tilde{F}^3}(e_1,e_2)(x),$$

$$T_{\tilde{F}^3}(e_1,e_2)([x,y]) = T_{\tilde{F}^1}(e_1)([x,y]) \wedge T_{\tilde{F}^2}(e_2)([x,y]) \geq$$
$$\left\{\min\left(T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(T_{\tilde{F}^2}(e_2)(x), T_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$\min\left\{T_{\tilde{F}^1}(e_1)(x), T_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{T_{\tilde{F}^1}(e_1)(y), T_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$T_{\tilde{F}^3}(e_1,e_2)(x) \wedge T_{\tilde{F}^3}(e_1,e_2)(y),$$

$$I_{\tilde{F}^3}(e_1,e_2)([x,y]) = I_{\tilde{F}^1}(e_1)([x,y]) \wedge I_{\tilde{F}^2}(e_2)([x,y]) \geq$$
$$\left\{\min\left(I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^1}(e_1)(y)\right)\right\} \wedge \left\{\min\left(I_{\tilde{F}^2}(e_2)(x), I_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$\min\left\{I_{\tilde{F}^1}(e_1)(x), I_{\tilde{F}^2}(e_2)(x)\right\} \wedge \min\left\{I_{\tilde{F}^1}(e_1)(y), I_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$I_{\tilde{F}^3}(e_1,e_2)(x) \wedge I_{\tilde{F}^3}(e_1,e_2)(y),$$

$$F_{\tilde{F}^3}(e_1,e_2)([x,y]) = F_{\tilde{F}^1}(e_1)([x,y]) \vee F_{\tilde{F}^2}(e_2)([x,y]) \leq$$
$$\leq \left\{\max\left(F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^1}(e_1)(y)\right)\right\} \vee \left\{\max\left(F_{\tilde{F}^2}(e_2)(x), F_{\tilde{F}^2}(e_2)(y)\right)\right\} =$$
$$\max\left\{F_{\tilde{F}^1}(e_1)(x), F_{\tilde{F}^2}(e_2)(x)\right\} \vee \max\left\{F_{\tilde{F}^1}(e_1)(y), F_{\tilde{F}^2}(e_2)(y)\right\} =$$
$$F_{\tilde{F}^3}(e_1,e_2)(x) \vee F_{\tilde{F}^3}(e_1,e_2)(y).$$

□

Definition 3.8. Let $\left(\widetilde{F}^1, E_1\right)$ and $\left(\widetilde{F}^2, E_2\right)$ be two neutrosophic soft sets on a set L. Then the generalized Cartesian product $\left(\widetilde{F}^1, E\right) \times \left(\widetilde{F}^2, E\right) = (\widetilde{F}^1 \times \widetilde{F}^2, E_1 \times E_2)$ is defined as follow:

$$\widetilde{F}^1 \times \widetilde{F}^2 \ : E_1 \times E_2 \to NS\ (L)$$

$$\widetilde{F}^1 \times \widetilde{F}^2 (e_1,e_2) = (T_{\widetilde{F}^1}(e_1) \times (T_{\widetilde{F}^2}(e_2), (I_{\widetilde{F}^1}(e_1) \times (I_{\widetilde{F}^2}(e_2), (F_{\widetilde{F}^1}(e_1) \times (F_{\widetilde{F}^2}(e_2)),$$

where,

$$(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)(x,y) = \min(T_{\widetilde{F}^1}(e_1)(x),\ T_{\widetilde{F}^2}(e_2)(y)),$$

$$(I_{\widetilde{F}^1}(e_1) \times (I_{\widetilde{F}^2}(e_2))(x,y) = \min(I_{\widetilde{F}^1}(e_1)(x), I_{\widetilde{F}^2}(e_2)(y)),$$

$$(F_{\widetilde{F}^1}(e_1) \times (F_{\widetilde{F^2}}(e_2))(x,y) = \max(F_{\widetilde{F}^1}(e_1)(x), F_{\widetilde{F}^2}(e_2)(y)).$$

For each $(e_1, e_2) \in E_1 \times E_2$

Theorem 3.9. *Let* $\left(\widetilde{F}^1 , E_1\right)$ *and* $\left(\widetilde{F}^2 , E_2\right)$ *be two neutrosophic soft Lie subalgebras of* L*, then is* $\left(\widetilde{F}^1 , E_1\right) \times \left(\widetilde{F}^2 , E_2\right)$ *is neutrosophic soft Lie subalgebra of* $L \times L$

Proof. Let $x = (x_1x_2)$ and $y = (y_1y_2) \in L \times L$. Then for each $(e_1e_2) \in E_1 \times E_2$.

$$(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x+y) = (T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)((x_1,x_2)+(y_1,y_2)) =$$
$$= (T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))((x_1+y_1, x_2+y_2)) = \min(T_{\widetilde{F}^1}(e_1)(x_1+y_1), T_{\widetilde{F}^2}(e_2)(x_2+y_2))$$
$$\geq \min(\min(T_{\widetilde{F}^1}(e_1)(x_1), T_{\widetilde{F}^1}(e_1)(y_1)), \min(T_{\widetilde{F}^2}(e_2)(x_2), T_{\widetilde{F}^2}(e_2)(y_2))) =$$
$$= \min((T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x_1,x_2)), \left(\left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(y_1,y_2)\right)) =$$
$$= \min\left((T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x), \left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(y)\right)\right),$$
$$\left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)(x+y) = \left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)((x_1,x_2)+(y_1,y_2)) =$$
$$= (I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2))((x_1+y_1, x_2+y_2)) = \min(I_{\widetilde{F}^1}(e_1)(x_1+y_1), I_{\widetilde{F}^2}(e_2)(x_2+y_2))$$
$$\geq \min(\min(I_{\widetilde{F}^1}(e_1)(x_1), I_{\widetilde{F}^1}(e_1)(y_1)), \min(I_{\widetilde{F}^2}(e_2)(x_2), I_{\widetilde{F}^2}(e_2)(y_2))) =$$
$$= \min((I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2))(x_1x_2)), \left(\left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)(y_1,y_2)\right)) =$$
$$= \min\left((I_{\widetilde{F}^1}(e_1) \times I_{F^2}(e_2))(x), \left(I_{\widetilde{F}^1}(e_1) \times I_{\widetilde{F}^2}(e_2)\right)(y)\right),$$
$$\left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)(x+y) = \left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)((x_1,x_2)+(y_1,y_2))) =$$
$$= \left(F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2)\right)((x_1+y_1, x_2+y_2)) =$$
$$= \max(F_{\widetilde{F}^1}(e_1)(x_1+y_1), F_{\widetilde{F}^2}(e_2)(x_2+y_2)) \leq \max(\max(F_{\widetilde{F}^1}(e_1)(x_1), F_{\widetilde{F}^1}(e_1)(y_1)),$$
$$\max(F_{\widetilde{F}^2}(e_2)(x_2), F_{\widetilde{F}^2}(e_2)(y_2))) = \max((F_{\widetilde{F}^1}(e_1) \times F_{\widetilde{F}^2}(e_2))(x_1,x_2)),$$
$$\left(\left(\mathrm{F}_{\widetilde{\mathrm{F}}^1}(\mathrm{e}_1) \times \mathrm{F}_{\widetilde{\mathrm{F}}^2}(\mathrm{e}_2)\right)(\mathrm{y}_1,\mathrm{y}_2)\right)) = \max\left((\mathrm{F}_{\widetilde{\mathrm{F}}^1}(\mathrm{e}_1) \times \mathrm{F}_{\widetilde{\mathrm{F}}^2}(\mathrm{e}_2))(\mathrm{x}), \left(\mathrm{F}_{\widetilde{\mathrm{F}}^1}(\mathrm{e}_1)\mathrm{F}_{\widetilde{\mathrm{F}}^2}(\mathrm{e}_2)\right)(\mathrm{y})\right)\right).$$
$$\left(\mathrm{T}_{\widetilde{\mathrm{F}}^1}(\mathrm{e}_1) \times \mathrm{T}_{\widetilde{\mathrm{F}}^2}(e_2)\right)(\alpha\mathrm{x}) = \left(\mathrm{T}_{\widetilde{\mathrm{F}}^1}(\mathrm{e}_1) \times \mathrm{T}_{\widetilde{\mathrm{F}}^2}(\mathrm{e}_2)\right)(\alpha(x_1,x_2)) =$$
$$\left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)((\alpha x_1, \alpha x_2)) = \min\left(T_{\widetilde{F}^1}(e_1)(\alpha x_1), T_{\widetilde{F}^2}(e_2)(\alpha x_1)\right) \geq$$
$$\geq \min(T_{\widetilde{F}^1}(e_1)(x_1), T_{\widetilde{F}^2}(e_2)(x_2)) = (T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2))(x_1,x_2))$$
$$= \left(T_{\widetilde{F}^1}(e_1) \times T_{\widetilde{F}^2}(e_2)\right)(x),$$
$$(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(\alpha x) = (I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(\alpha(x_1,x_2)) = (I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))((\alpha x_1, \alpha x_2)) =$$
$$= \min(I_{\tilde{F}^1}(e_1)(\alpha x_1), I_{\tilde{F}^2}(e_2)(\alpha x_1)) \geq \min(I_{\tilde{F}^1}(e_1)(x_1), I_{\tilde{F}^2}(e_2)(x_2)) =$$
$$= \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)(x_1x_2) = \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)(x),$$
$$\left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)(\alpha x) = \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)(\alpha(x_1,x_2)) =$$

$$= \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)((\alpha x_1, \alpha x_2)) = \max\left(F_{\tilde{F}^1}\left(e_1\right)(\alpha x_1), F_{\tilde{F}^2}(e_2)\left(\alpha x_1\right)\right) \leq$$
$$\leq \max\left(F_{\tilde{F}^1}\left(e_1\right)(x_1), F_{\tilde{F}^2}(e_2)\left(x_2\right)\right) = \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)(x_1, x_2) =$$
$$= \left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}\ (e_2)\right)(x),$$
$$\left(T_{\tilde{F}^1}\ (e_1) \times T_{\tilde{F}^2}\ (e_2)\right)([x,y]) = \left(T_{\tilde{F}^1}\ (e_1) \times T_{\tilde{F}^2}\ (e_2)\right)([(x_1,x_2)+(y_1,y_2)]) \geq$$
$$\geq \min(\min(T_{\tilde{F}^1}(e_1)(x_1), T_{\tilde{F}^1}(e_2)(x_2)), \min(T_{\tilde{F}^1}(e_1)(y_1), T_{\tilde{F}^2}(e_2)(y_2))) =$$
$$\min\left(\left(T_{\tilde{F}^1}(e_1) \times T_{\tilde{F}^2}(e_2)\right)(x_1,x_2), \left(T_{\tilde{F}^1}\ (e_1) \times T_{\tilde{F}^2}\ (e_2)\right)(y_1,y_2)\right) =$$
$$= \min((T_{\tilde{F}^1}(e_1) \times T_{\tilde{F}^2}(e_2))(x), \left(T_{\tilde{F}^1}(e_1) \times T_{\tilde{F}^2}(e_2)\right)(y)),$$
$$\left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)([x,y]) = \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)([(x_1,x_2)+(y_1,y_2)]) \geq$$
$$\geq \min(\min(I_{\tilde{F}^1}(e_1)(x_1), I_{\tilde{F}^2}(e_2)(x_2)), \min(I_{\tilde{F}^1}(e_1)(y_1), I_{\tilde{F}^2}(e_2)(y_2)) =$$
$$= \min((I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(x_1,x_2), (I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2))(y_1,y_2)) =$$
$$= \min\left(\left(I_{\tilde{F}^1(e_1}) \times I_{\tilde{F}^2}\ (e_2)\right)(x), \left(I_{\tilde{F}^1}(e_1) \times I_{\tilde{F}^2}(e_2)\right)(y)\right),$$
$$\left(F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2)\right)([x,y]) = \left(F_{\tilde{F}^1}\ (e_1) \times F_{\tilde{F}^2}\ (e_2)\right)([(x_1),x_2)+(y_1,y_2)]) \leq$$
$$\leq \max(\max(F_{\tilde{F}^1}(e_1)(x_1), F_{\tilde{F}^2}(e_2)(x_2)), \max(F_{\tilde{F}^1}(e_1)(y_1), F_{\tilde{F}^2}\ (e_2)(y_2)) =$$
$$= \max((F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(x_1,x_2), (F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(y_1,y_2)) =$$
$$\max((F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(x)), (F_{\tilde{F}^1}(e_1) \times F_{\tilde{F}^2}(e_2))(y)).$$

□

This shows that $\left(\tilde{F}^1, E_1\right) \times \left(\tilde{F}^2, E_2\right)$ is a neutrosophic soft Lie subalgebra of $L \times L$.

Definition 3.10. Let $\left(\tilde{F}^1, E_1\right)$ and $\left(\tilde{F}^2, E_2\right)$be two neutrosophic soft Lie algebras over L_1 and L_2 respectively, and let $f : L_1 \to L_2$ be a homomorphism of Lie algebras, and let $g : E_1 \to E_2$ be a mapping of sets. Then we say that $(f,g) : \left(L_1, \left(\tilde{F}^1, E_1\right)\right) \to \left(L_2, \left(\tilde{F}^2, E_2\right)\right)$ is a neutrosophic soft Lie homomorphism of neutrosophic soft Lie algebras, if the following condition is satisfied:

$$f\left(T_{\tilde{F}^1}(e)\right) = \tilde{F}^2(g(e)) = T_{\tilde{F}^2}(g(e))$$
$$f\left(I_{\tilde{F}^1}(e)\right) = \tilde{F}^2(g(e)) = I_{\tilde{F}^2}(g(e))$$
$$f\left(F_{\tilde{F}^1}(e)\right) = \tilde{F}^2(g(e)) = F_{\tilde{F}^2}(g(e))$$

For the Lie algebras L_1 and L_2 it can be easily observed that if $f : L_1 \to L_2$ is a Lie homomorphism $g : E \to E'$ map of sets and $\left(\tilde{F}, E'\right)$ is neutrosophic soft Lie subalgebra of L_2, then the neutrosophic soft set $f^{-1}\left(\tilde{F}, E\right)$ of L_1 is also a neutrosophic soft Lie subalgebra, where,

$$f^{-1}\left(T_{\tilde{F}}(e)\right)(x) = T_{\tilde{F}}(e)(f(x))$$
$$f^{-1}\left(I_{\tilde{F}}(e)\right)(x) = I_{\tilde{F}}(e)(f(x))$$
$$f^{-1}\left(F_{\tilde{F}}(e)\right)(x) = F_{\tilde{F}}(e)(f(x))$$

Definition 3.11. Let L_1 and L_2 be two Lie algebras, $f : L_1 \to L_2$ is a Lie homomorphism,and let $\left(\tilde{F}, E\right)$ be neutrosophic soft set over L_1, $g : E \to E'$ map of sets then image of (f, g) is defined by:

$$f\left(T_{\tilde{F}}\right)(e)(y) = \sup\left\{T_{\tilde{F}}(e)(x) : x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\}$$
$$f\left(I_{\tilde{F}}\right)(e)(y) = \sup\left\{I_{\tilde{F}}(e)(x) : x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\}$$
$$f\left(F_{\tilde{F}}\right)(e)(y) = \inf\left\{F_{\tilde{F}}(e)(x) : x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\}$$

for $\forall e \in E$, $\forall\, y \in Y$.

Theorem 3.12. *Let $f : L_1 \to L_2$ ephimorfizm of Lie algebras and $\left(\tilde{F}, E\right)$ neutrosophic soft Lie subalgebra of L_1,then the homomorphic image of $\left(\tilde{F}, E\right)$ is neutrosophic soft Lie subalgebra of L_2.*

Proof. Let $y_1, y_2 \in L_2$. Then,

$$\left\{x \mid x \in f^{-1}(y_1+y_2)\right\} \supseteq \left\{x_1+x_2 \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2)\right\}.$$

Now, we have, for each $e \in E$

$$f\left(T_{\tilde{F}}(e^{'})\right)(y_1 + y_2) = \sup\left\{T_{\tilde{F}}(e)(x) \mid x \in f^{-1}(y_1 + y_2), e \in g^{-1}(e^{'})\right\}$$
$$\geq \sup\left\{T_{\tilde{F}}(e)(x_1 + x_2), \mid x_1 \in f^{-1}(y_1) \text{ and } \quad x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\right\} \geq$$
$$\geq \sup\left\{\min\{T_{\tilde{F}}(e)(x_1), T_{\tilde{F}}(e)(x_2)\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\right\} =$$
$$= \min\{\{\sup\left(T_{\tilde{F}}(e)\right)(x_1) \mid x_1 \in f^{-1}(y_1)\, e \in g^{-1}(e')\}, \{\sup T_{\tilde{F}}(e)(x_2) | x_2 \in f^{-1}(y_2), e \in g^{-1}(e^{'})\}\} =$$
$$= \min\left\{f(T_{\tilde{F}}(e'))(y_1), f(T_{\tilde{F}}(e'))(y_2)\right\}$$

For $y_2 \in L_2$ and $\alpha \in K$ we have

$$\left\{x \mid x \in f^{-1}(\alpha y)\right\} \supseteq \left\{\alpha x \mid x \in f^{-1}(y)\right\}.$$

$$f\left(T_{\tilde{F}}(e)\right)(\alpha y) = \sup\{T_{\tilde{F}}(e)(\alpha x) \mid x \in f^{-1}(y), e \in g^{-1}(e')\}$$
$$\geq \sup\left\{T_{\tilde{F}}(e)(x), |x \in f^{-1}(y), e \in g^{-1}(e^{'})\right\} = f\left(T_{\tilde{F}}(e^{'})\right)(y).$$

If $y_1, y_2 \in L_2$ then

$$\left\{x | x \in f^{-1}([y_1, y_2])\right\} \supseteq \left\{[x_1, x_2] | x_1 \in f^{-1}(y_1),\ x_2 \in f^{-1}(y_2)\right\}$$

Now

$$f\left(T_{\tilde{F}}(e')\right)([y_1, y_2]) = \sup\{\, T_{\tilde{F}}(e)(x) \ | x \in f^{-1}([y_1, y_2]), e \in g^{-1}(e')\}$$
$$\geq \sup\{T_{\tilde{F}}(e)[x_1, x_2], | x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\}$$
$$\geq \sup\left\{\min\left\{T_{\tilde{F}}(e)(x_1), T_{\tilde{F}}(e)(x_2)\right\} \mid x_1 \in f^{-1}(y_1) \text{ and } x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\right\}$$
$$= \min\{\{\sup T_{\widetilde{F}}(e)(x_1) | x_1 \in f^{-1}(y_1), e \in g^{-1}(e')\}, \{\sup T_{\widetilde{F}}(e)(x_2) | x_2 \in f^{-1}(y_2), e \in g^{-1}(e')\}\}$$
$$= \min\{f\left(T_{\tilde{F}}(e)\right)(y_1), f\left(T_{\tilde{F}}(e)\right)(y_2)\}.$$

Now , we can easily proof for $f\left(I_{\tilde{F}}(e')\right)(y_1{+}y_2) \geq \min\left\{f\left(I_{\tilde{F}}(e')\right)(y_1)\,,\, f\left(I_{\tilde{F}}(e')\right)(y_2)\right\}$

$$f(I_{\tilde{F}}(e'))(\alpha y) \geq f(I_{\tilde{F}}(e'))(y)$$

$$f\left(I_{\tilde{F}}(e')\right)([y_1,y_2]) \geq \min\{f\left(I_{\tilde{F}}(e')\right)(y_1)\,, f\left(I_{\tilde{F}}(e')\right)(y_2)\}$$

$$f\left(F_{\tilde{F}}(e')\right)(y_1{+}y_2) = \inf\{\ F_{\tilde{F}}(e)\,(x)\ \ |x{\in}f^{-1}(y_1{+}y_2), e\in g^{-1}(e')\}$$

$$\leq \inf\{F_{\tilde{F}}(e')(x_1{+}x_2),|x_1{\in}f^{-1}(y_1)\ \ ,x_2{\in}f^{-1}(y_2)\,, e\in g^{-1}(e')\}$$

$$\leq \inf\left\{\max\left\{F_{\tilde{F}}(e)\,(x_1)\,, F_{\tilde{F}}(e)\,(x_2)\right\}\,\middle|\ x_1{\in}f^{-1}(y_1)\ \ , x_2\in f^{-1}(y_2)\,, e\in g^{-1}(e')\right\}$$

$$= \max\left\{\left\{\inf F_{\tilde{F}}(e)\,(x_1)\,, |x_1\in f^{-1}(y_1), e\in g^{-1}(e')\right\}, \left\{\inf F_{\tilde{F}}(e)(x_2)|x_2\in f^{-1}(y_2)\,, e\in g^{-1}(e')\right\}\right\}$$

$$= \max\{f\left(F_{\tilde{F}}(e')\right)(y_1)\,, f\left(F_{\tilde{F}}(e')\right)(y_2)\}$$

For $y\in L_2$ and $\alpha\in K$ we have

$$f\left(F_{\tilde{F}}(e')\right)(\alpha y) = \inf\ \ \{\ F_{\tilde{F}}(e)\,(\alpha x)\ \ |x{\in}f^{-1}(y), e\in g^{-1}(e')\}$$

$$\leq \inf\left\{F_{\tilde{F}}(e)\,(x)\,,\middle|\ x{\in}f^{-1}(y)\,, e\in g^{-1}(e')\right\} = f\left(F_{\widetilde{F}}(e')\right)(y)$$

Now

$$f\left(F_{\widetilde{F}}\left(e'\right)\right)([y_1,\mathrm{y}_2]) = \inf\{\ \mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})\,(\mathrm{x})\ \ |\mathrm{x}{\in}\mathrm{f}^{-1}([\mathrm{y}_1,\mathrm{y}_2], e\in g^{-1}(e')\}$$

$$\leq \inf\left\{\mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})[\mathrm{x}_1,\mathrm{x}_2]\,,\middle|\ \mathrm{x}_1{\in}\mathrm{f}^{-1}(\mathrm{y}_1)\ \ , \mathrm{x}_2{\in}\mathrm{f}^{-1}(\mathrm{y}_2)\,, e\in g^{-1}(e')\right\}$$

$$\leq \inf\left\{\max\{\mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})\,(\mathrm{x}_1)\,, \mathrm{F}_{\widetilde{\mathrm{F}}}(\mathrm{e})\,(\mathrm{x}_2)\}\,\middle|\ \ \mathrm{x}_1{\in}\mathrm{f}^{-1}(\mathrm{y}_1)\,,\ \ \mathrm{x}_2{\in}\mathrm{f}^{-1}(\mathrm{y}_2)\,, e\in g^{-1}(e')\right\}$$

$$= \max\{\{\inf(F_{\widetilde{F}(e)}\,(x_1)\,|x_1\in f^{-1}(y_1), e\in g^{-1}(e')\}, \{\inf F_{\widetilde{F}(e)}\,(x_2))|x_2\in f^{-1}(y_2), e\in g^{-1}(e')\}$$

$$= \max\{f(F_{\widetilde{F}}(e)\,(y_1)\,, f(F_{\widetilde{F}}(e)\,(y_2)\}$$

Thus $f\left(\left(\widetilde{F},E\right)\right) = \left(f\left(T_{\widetilde{F}}(e')\right), f\left(I_{\widetilde{F}}(e')\right), f\left(F_{\widetilde{F}}(e')\right)\right)$ is a neutrosophic soft Lie algebra of L_2 . □

4. Conclusion

There we have introduced the concept of neutrosophic soft Lie subalgebras of a Lie algebra and investigated some of their properties.

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