BS-Neutrosophic Structures in BCK/BCI-Algebras

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Abstract: In this article we generalized a Smarandache's neutrosophic set as BS-neutrosophic set and it is applied to BCK/BCI-algebras. The concept of BS-neutrosophic subalgebra, BS-neutrosophic ideal and related properties are investigated.

Keywords: Neutrosophic set (NSS); BS-neutrosophic set (BS-NSS); BS-neutrosophic subalgebra (BS-NSSA); BS-neutrosophic ideal (BS-NSI).

1. Introduction

L.A.Zadeh [1], a professor of computer science at the University of California, introduced the concept of fuzzy set (FS) in 1965.Fuzzy sets analyzed the degree of membership of elements of set. In 1986 Atanassove [2] generalized a fuzzy set to an Intuitionistic Fuzzy Set (IFS) by including another function called a non-membership function. The neutrosophic Set (NS) concept was developed by Smarandache ([3],[4]) and is a more general framework that extends the concepts of Classical Set, fuzzy set, Intuitionistic fuzzy set, Interval valued fuzzy(Intuitionistic) set. Neutrosophic algebraic structures in BCK/BCI-algebras are described in articles [5],[6],[7],[8],[9],[10],[11],[12],[13],[14],[15] we know that Smarandache's NSS have many generalizations. The purpose of this paper is to consider a new generalization of the NS. A NS has true, false and indeterminate membership functions which are fuzzy sets. When we considering the generalization of a NS, an interval-valued fuzzy set is used as non-membership function, since the interval-valued fuzzy set is a generalization of the fuzzy set. We introduce the concept of BS-neutrosophic set, and we apply it to BCK/BCI-algebras. Also, the concept of BS-neutrosophic subalgebra, BS-neutrosophic ideal are introduced and the associated properties are investigated. We consider homomorphic inverse image of the BS-neutrosophic subalgebra and discuss the translation of the BS- neutrosophic Subalgebra. In a BCI-algebra, we provide conditions for a BS- neutrosophic ideal to be a BS-neutrosophic subalgebra.

2. Preliminaries

Definition: 2.1([16],[17],[18]) Let $\mathcal{K}$ be a non-empty set with a binary operation "$*$" and a constant "0" is called a BCI-algebra if it satisfies the following axioms for all $\wp_0, r_0, u_0 \in \mathcal{K}$

i. $((\wp_0 * r_0) * (\wp_0 * u_0)) * (u_0 * r_0) = 0$
ii. $(\wp_0 * (\wp_0 * r_0)) * r_0 = 0$
iii. $\wp_0 * \wp_0 = 0$
iv. $\wp_0 * r_0 = 0, r_0 * \wp_0 = 0 \Rightarrow \wp_0 = r_0$.
If a BCI-algebra $\mathcal{K}$ satisfies the following identity
v. $0 * \wp_0 = 0$ for all $\wp_0 \in \mathcal{K}$ then $\mathcal{K}$ is called a BCK-algebra
The following properties are hold in any BCK/BCI-algebra
i. $\wp_0 * 0 = 0$
ii. $\wp_0 \le r_0 \Rightarrow \wp_0 * u_0 \le r_0 * u_0, u_0 * r_0 \le u_0 * \wp_0$
iii. $(\wp_0 * r_0) * u_0 = (\wp_0 * u_0) * r_0$
iv. $(\wp_0 * u_0) * (r_0 * u_0) \le \wp_0 * r_0$ for all $\wp_0, r_0, u_0 \in \mathcal{K}$.
Where $\wp_0 \le r_0$ if and only if $\wp_0 * r_0 = 0$.
The following conditions are hold in any BCI-algebra $\mathcal{K}$ [16]
i. $\wp_0 * (\wp_0 * (\wp_0 * r_0)) = \wp_0 * r_0$
ii. $0 * (\wp_0 * r_0) = (0 * \wp_0) * (0 * r_0)$

Definition: 2.2[16] A BCI-algebra $\mathcal{K}$ is said to be p-semisimple if $0 * (0 * \wp_0) = \wp_0$ for all $\wp_0 \in \mathcal{K}$
In a p-semisimple BCI-algebra $\mathcal{K}$, the following holds for all $\wp_0, r_0 \in \mathcal{K}$
a. $0 * (\wp_0 * r_0) = r_0 * \wp_0$
b. $\wp_0 * (\wp_0 * r_0) = r_0$

Definition: 2.3[16] A BCI-algebra $\mathcal{K}$ is said to be associative if $(\wp_0 * r_0) * u_0 = (\wp_0 * u_0) * r_0$ for all $\wp_0, r_0, u_0 \in \mathcal{K}$

Definition: 2.4 [18] An (s)-BCK-algebra, we mean a BCK-algebra $\mathcal{K}$ such that, for any $\wp_0, r_0 \in \mathcal{K}$ the set $\{u_0 \in \mathcal{K} / u_0 * \wp_0 \le r_0\}$ has the greatest element, written by $\wp_0 \circ r_0$.

Definition: 2.5 A non-empty sub set $\mathcal{H}$ of a BCK/BCI-algebra $\mathcal{K}$ is called a sub algebra of $\mathcal{K}$ if $\wp_0 * r_0 \in \mathcal{H}$ for all $\wp_0, r_0 \in \mathcal{H}$.

Definition: 2.6 A non-empty sub set $\mathcal{H}$ of a BCK/BCI-algebra $\mathcal{K}$ is called an ideal of $\mathcal{K}$ if $0 \in \mathcal{H}, and \ r_0, \wp_0 * r_0 \in \mathcal{H} \Rightarrow \wp_0 \in \mathcal{H}$ for all $\wp_0, r_0 \in \mathcal{K}$.

Definition: 2.7 A non-empty sub set $\mathcal{H}$ of a BCI-algebra $\mathcal{K}$ is called a closed ideal of $\mathcal{K}$ if it is an ideal of $\mathcal{K}$ which satisfies $\wp_0 \in \mathcal{H} \Rightarrow 0 * \wp_0 \in \mathcal{H}$ for all $\wp_0 \in \mathcal{K}$

Definition: 2.8[1] Let $\mathcal{K}$ be non-empty set. A fuzzy set in $\mathcal{K}$ is a mapping $\mathcal{N}_T: \mathcal{K} \to [0,1]$

Definition: 2.9[1] The complement of fuzzy set $\mathcal{N}_T$ denoted by $(\mathcal{N}_T)^c$ is also a fuzzy set defined as $(\mathcal{N}_T)^c = 1 - \mathcal{N}_T$ for all $\wp_0 \in \mathcal{K}$. Also $((\mathcal{N}_T)^c)^c = \mathcal{N}_T$.

Definition: 2.10 A fuzzy set $\mathcal{N}_T: \mathcal{K} \to [0,1]$ is called fuzzy sub-algebra of $\mathcal{K}$, if $\mathcal{N}_T(\wp_0 * r_0) \ge min\{\mathcal{N}_T(\wp_0), \mathcal{N}_T(r_0)\}$.

By an interval number we mean a closed subinterval $\widehat{m} = [m^-, m^+]$ of $[I]$ where $0 \le m^- \le m^+ \le 1$. Denote by $[I]$ the set of all interval numbers. Let us define what is known as refined minimum (briefly, rmin) and refined maximum (briefly, rmax) of two elements in $[I]$. We also define the symbols " $\preccurlyeq$ " ," $\succcurlyeq$ "," = " in case of two elements in $[I]$. Consider two interval numbers $\widehat{m}_1 = [{m_1}^-, {m_1}^+]$ and $\widehat{m}_2 = [{m_2}^-, {m_2}^+]$. Then
$rmin\{\widehat{m}_1, \widehat{m}_2\} = [min\{{m_1}^-, {m_2}^-\}, \min\{{m_1}^+, {m_2}^+\}]$,
$rmax\{\widehat{m}_1, \widehat{m}_2\} = [max\{{m_1}^-, {m_2}^-\}, \max\{{m_1}^+, {m_2}^+\}]$,

$\hat{m}_1 \succcurlyeq \hat{m}_2 \Leftrightarrow m_1^- \geq m_2^-, m_1^+ \geq m_2^+$, and similarly, we may have $\hat{m}_1 \preccurlyeq \hat{m}_2$ and $\hat{m}_1 = \hat{m}_2$. To say $\hat{m}_1 \succ \hat{m}_2$ (resp. $\hat{m}_1 \prec \hat{m}_2$) we mean $\hat{m}_1 \succcurlyeq \hat{m}_2$ and $\hat{m}_1 \neq \hat{m}_2$ (resp. $\hat{m}_1 \preccurlyeq \hat{m}_2$ *and* $\hat{m}_1 \neq \hat{m}_2$). Let $\hat{m}_i \in [I]$ where $i \in \sqcap$. We define

$\underset{i \in \sqcap}{rinf}\ \hat{m}_i = [\underset{i \in \sqcap}{inf} m_i^-, \underset{i \in \sqcap}{inf} m_i^+]$ *and* $\underset{i \in \sqcap}{rsup}\ \hat{m}_i = [\underset{i \in \sqcap}{sup} m_i^-, \underset{i \in \sqcap}{sup} m_i^+]$.

Definition: 2.11[19] Let $\mathcal{K}$ be a non-empty set. A function $\widehat{\mathcal{N}}: \mathcal{K} \to [I]$ is called an interval-valued fuzzy set (briefly, an IVF set) in $\mathcal{K}$. Let $[I]^{\mathcal{K}}$ stand for the set of all IVF sets in $\mathcal{K}$. For every $\widehat{\mathcal{N}} \in [I]^{\mathcal{K}}$ and $p_0 \in \mathcal{K}$, $\widehat{\mathcal{N}}(p_0) = [\mathcal{N}^-(p_0), \mathcal{N}^+(p_0)]$ is called the degree of membership of an element $p_0 \in \widehat{\mathcal{N}}$, where $\mathcal{N}^-: \mathcal{K} \to [I]$ and $\mathcal{N}^+: \mathcal{K} \to [I]$ are fuzzy sets in $\mathcal{K}$ which are called a lower fuzzy set and an upper fuzzy set in $\mathcal{K}$, respectively. For simplicity, we denote $\widehat{\mathcal{N}} = [\mathcal{N}^-, \mathcal{N}^+]$.

Definition: 2.12[4] Let $\mathcal{K}$ be a non-empty set. A neutrosophic set (NS) in $\mathcal{K}$ is a structure of the form $\mathcal{N} = \{\langle p_0; \mathcal{N}_T(p_0), \mathcal{N}_I(p_0), \mathcal{N}_F(p_0)\rangle : p_0 \in \mathcal{K}\}$ where $\mathcal{N}_T: \mathcal{K} \to [0,1]$ is a degree of membership, $\mathcal{N}_I: \mathcal{K} \to [0,1]$ is a degree of indeterminacy, and $\mathcal{N}_F: \mathcal{K} \to [0,1]$ degree of non-membership. For the sake of simplicity, we shall use the symbol $\mathcal{N} = (\mathcal{N}_T, \mathcal{N}_I, \mathcal{N}_F)$ for the neutrosophic set $\mathcal{N} = \{\langle p_0; \mathcal{N}_T(p_0), \mathcal{N}_I(p_0), \mathcal{N}_F(p_0)\rangle : p_0 \in \mathcal{K}\}$.

3. BS-Neutrosophic Structures

Definition: 3.1 Let $\mathcal{K}$ be a non-empty set. BS-neutrosophic set in $\mathcal{K}$, is a structure of the form $\mathcal{N} = \{\langle p_0; \mathcal{N}_t(p_0), \mathcal{N}_i(p_0), \widehat{\mathcal{N}_f}(p_0)\rangle : p_0 \in \mathcal{K}\}$ where $\mathcal{N}_t$, $\mathcal{N}_i$ are fuzzy sets in $\mathcal{K}$, which are called a degree of indeterminacy and degree of non-membership, respectively, and $\widehat{\mathcal{N}_f}$ is an interval valued fuzzy set in $\mathcal{K}$ which is called an interval valued degree of non-membership

For the sake of simplicity, we shall use the symbol $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ for the BS-NSS $\mathcal{N} = \{\langle p_0; \mathcal{N}_t(p_0), \mathcal{N}_i(p_0), \widehat{\mathcal{N}_f}(p_0)\rangle : p_0 \in \mathcal{K}\}$.

In a BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ if we take $\widehat{\mathcal{N}_f}: \mathcal{K} \to [I]$, $p_0 \mapsto [\mathcal{N}_f^-(p_0), \mathcal{N}_f^+(p_0)]$ with $\mathcal{N}_f^-(p_0) = \mathcal{N}_f^+(p_0)$ then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a neutrosophic set in $\mathcal{K}$.

Definition: 3.2 Let $\mathcal{K}$ be a BCK/BCI algebra. A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is called a BS-neutrosophic subalgebra of $\mathcal{K}$ if it satisfies

(BS-NSSA 1) $\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

(BS-NSSA 2) $\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

(BS-NSSA 3) $\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Example: 3.3 Consider a set $\mathcal{K} = \{0, a, b, c\}$ with the binary operation $*$ which is given in table.1

*	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table.1 BCK-algebra

Then $(\mathcal{K}; *, 0)$ is a BCK-algebra. Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by table.2

$\mathcal{K}$	$\mathcal{N}_t(p_0)$	$\mathcal{N}_i(p_0)$	$\widehat{\mathcal{N}_f}(p_0)$
0	0.9	1	[0.1,0.4]
a	0.4	0.5	[0.3,0.5]

b	0.3	0.3	[0.2,0.6]
c	0	0.1	[0.4,1]

Table.2 BS-NSSA

It is routine to verify that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Proposition: 3.4 If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$ then $\mathcal{N}_t(0) \geq \mathcal{N}_t(\wp_0)$ $\mathcal{N}_i(0) \geq \mathcal{N}_i(\wp_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(\wp_0)$ for all $\wp_0 \in \mathcal{K}$

Proof: For any $\wp_0 \in \mathcal{K}$, we have

$\mathcal{N}_t(0) = \mathcal{N}_t(\wp_0 * \wp_0) \geq \ min\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(\wp_0)\} = \mathcal{N}_t(\wp_0)$

$\mathcal{N}_i(0) = \mathcal{N}_i(\wp_0 * \wp_0) \geq \ min\{\mathcal{N}_i(\wp_0)\ , \mathcal{N}_i(\wp_0)\ \} = \mathcal{N}_i(\wp_0)$

$\widehat{\mathcal{N}_f}(0) = \widehat{\mathcal{N}_f}(\wp_0 * \wp_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(\wp_0)\} = rmax\{[\mathcal{N}_f{}^-(\wp_0), \mathcal{N}_f{}^+(\wp_0)], [\mathcal{N}_f{}^-(\wp_0), \mathcal{N}_f{}^+(\wp_0)]\}$

$= [\mathcal{N}_f{}^-(\wp_0), \mathcal{N}_f{}^+(\wp_0)] = \widehat{\mathcal{N}_f}(\wp_0)$

Hence the proof is completed.

Proposition: 3.5 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$ if there exists a sequence $\{\wp_{0_n}\}$ in $\mathcal{K}$ such that $\lim_{n\to\infty} \mathcal{N}_t(\wp_{0_n}) = 1$, $\lim_{n\to\infty} \mathcal{N}_i(\wp_{0_n}) = 1$ and $\lim_{n\to\infty} \widehat{\mathcal{N}_f}(\wp_{0_n}) = [0,0]$, then $\mathcal{N}_t(0) = 1$, $\mathcal{N}_i(0) = 1$ and $\widehat{\mathcal{N}_f}(0) = [0,0]$.

Proof: Using the proposition 3.4, we know that $\mathcal{N}_t(0) \geq \mathcal{N}_t(\wp_{0_n})$ $\mathcal{N}_i(0) \geq \mathcal{N}_i(\wp_{0_n})$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(\wp_{0_n})$ for every positive integer n. Note that

$1 \geq \mathcal{N}_t(0) \geq \lim_{n\to\infty} \mathcal{N}_t(\wp_{0_n}) = 1,$

$1 \geq \mathcal{N}_i(0) \geq \lim_{n\to\infty} \mathcal{N}_i(\wp_{0_n}) = 1,$

$[0,0] \preccurlyeq \widehat{\mathcal{N}_f}(0) \preccurlyeq \lim_{n\to\infty} \widehat{\mathcal{N}_f}(\wp_{0_n}) = [0,0].$

Therefore $\mathcal{N}_t(0) = 1$, $\mathcal{N}_i(0) = 1$ and $\widehat{\mathcal{N}_f}(0) = [0,0]$.

Theorem: 3.6 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSS in $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA in $\mathcal{K}$ if and only if $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f{}^-)^c$ *and* $(\mathcal{N}_f{}^+)^c$ are fuzzy subalgebras of $\mathcal{K}$.

Proof: Suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA in $\mathcal{K}$ then for all $\wp_0, r_0 \in \mathcal{K}$ we have

$\mathcal{N}_t(\wp_0 * r_0) \geq rmin\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(\wp_0 * r_0) \geq min\{\mathcal{N}_i(\wp_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(\wp_0 * r_0) \preccurlyeq max\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(r_0)\}$

$[\mathcal{N}_f{}^-(\wp_0 * r_0)\ \ , \mathcal{N}_f{}^+(\wp_0 * r_0)] \preccurlyeq rmax\{[\mathcal{N}_f{}^-(\wp_0)\ \ , \mathcal{N}_f{}^+(\wp_0)], [\mathcal{N}_f{}^-(r_0)\ \ , \mathcal{N}_f{}^+(r_0)]\}$

$=[max\{\mathcal{N}_f{}^-(\wp_0), \mathcal{N}_f{}^-(r_0)\}, max\{\mathcal{N}_f{}^+(\wp_0), \mathcal{N}_f{}^+(r_0)\}]$

Therefore $\mathcal{N}_f{}^-(\wp_0 * r_0)\ \ \leq max\{\mathcal{N}_f{}^-(\wp_0), \mathcal{N}_f{}^-(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f{}^-(\wp_0 * r_0) \geq 1 - max\{\mathcal{N}_f{}^-(\wp_0), \mathcal{N}_f{}^-(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f{}^-(\wp_0 * r_0) \geq min\{1 - \mathcal{N}_f{}^-(\wp_0), 1 - \mathcal{N}_f{}^-(r_0)\}$

$\Rightarrow (\mathcal{N}_f{}^-)^c(\wp_0 * r_0) \geq min\{(\mathcal{N}_f{}^-)^c(\wp_0), (\mathcal{N}_f{}^-)^c(r_0)\}$ and

$\mathcal{N}_f{}^+(\wp_0 * r_0)\ \ \leq max\{\mathcal{N}_f{}^+(\wp_0), \mathcal{N}_f{}^+(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f{}^+(\wp_0 * r_0) \geq 1 - max\{\mathcal{N}_f{}^+(\wp_0), \mathcal{N}_f{}^+(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f{}^+(\wp_0 * r_0) \geq min\{1 - \mathcal{N}_f{}^+(\wp_0), 1 - \mathcal{N}_f{}^+(r_0)\}$

$\Rightarrow (\mathcal{N}_f{}^+)^c(\wp_0 * r_0) \geq min\{(\mathcal{N}_f{}^+)^c(\wp_0), (\mathcal{N}_f{}^+)^c(r_0)\}.$

Hence $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^{\ -})^c$ *and* $(\mathcal{N}_f^{\ +})^c$ are fuzzy subalgebras of $\mathcal{K}$.

Converse part is obvious.

Definition: 3.7 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSS in $\mathcal{K}$, we define the following sets

$\mathcal{U}(\mathcal{N}_t; l) = \{\wp_0 \in \mathcal{K} : \mathcal{N}_t(\wp_0) \geq l\}$

$\mathcal{U}(\mathcal{N}_i; m) = \{\wp_0 \in \mathcal{K} : \mathcal{N}_i(\wp_0) \geq m\}$

$\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2]) = \{\wp_0 \in \mathcal{K} : \widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq [n_1, n_2]\}$

Where $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

Theorem: 3.8 A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is a BS-NSSA of $\mathcal{K}$ if and only if the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are subalgebras of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

Proof: Suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Let $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$ be such that $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are non-empty. For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{K}$ if $a_1, a_2 \in \mathcal{U}(\mathcal{N}_t; l)$, $b_1, b_2 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1, c_2 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ then

$\mathcal{N}_t(a_1 * a_2) \geq rmin\{\mathcal{N}_t(a_1), \mathcal{N}_t(a_2)\} \geq rmin\{l, l\} = l$

$\mathcal{N}_i(b_1 * b_2) \geq min\{\mathcal{N}_i(b_1), \mathcal{N}_i(b_2)\} \geq min\{m, m\} = m$

$\widehat{\mathcal{N}_f}(c_1 * c_2) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(c_1), \widehat{\mathcal{N}_f}(c_2)\} \preccurlyeq rmax\{[n_1, n_2], [n_1, n_2]\} = [n_1, n_2]$

Therefore $a_1 * a_2 \in \mathcal{U}(\mathcal{N}_t; l)$, $b_1 * b_2 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1 * c_2 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$

Hence $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are subalgebras of $\mathcal{K}$.

Conversely, assume that the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are subalgebras of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

If $\mathcal{N}_t(a_0 * b_0) < min\{\mathcal{N}_t(a_0), \mathcal{N}_t(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$, then $a_0, b_0 \in \mathcal{U}(\mathcal{N}_t; l_0)$ but $a_0 * b_0 \notin \mathcal{U}(\mathcal{N}_t; l_0)$ for $l_0 = min\{\mathcal{N}_t(a_0), \mathcal{N}_t(b_0)\}$. This is a contradiction, and thus

$\mathcal{N}_t(\wp_0 * r_0) \geq min\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$. Similarly, we can show that $\mathcal{N}_i(\wp_0 * r_0) \geq min\{\mathcal{N}_i(\wp_0), \mathcal{N}_i(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$.

Suppose that $\widehat{\mathcal{N}_f}(a_0 * b_0) \succ rmax\{\widehat{\mathcal{N}_f}(a_0), \widehat{\mathcal{N}_f}(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$.

Let $\widehat{\mathcal{N}_f}(a_0) = [\delta_1, \delta_2]$, $\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4]$ and $\widehat{\mathcal{N}_f}(a_0 * b_0) = [n_1, n_2]$

Then $[n_1, n_2] \succ rmax\{[\delta_1, \delta_2], [\delta_3, \delta_4]\} = [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]$ and so $n_1 > max\{\delta_1, \delta_3\}$ and $n_2 > max\{\delta_2, \delta_4\}$

Taking $[\eta_1, \eta_2] = \frac{1}{2}\left[\widehat{\mathcal{N}_f}(a_0 * b_0) + rmax\{\widehat{\mathcal{N}_f}(a_0), \widehat{\mathcal{N}_f}(b_0)\}\right]$

$$= \frac{1}{2}\left[[n_1, n_2] + [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]\right]$$

$$= \left[\frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}), \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\})\right]$$

It follows that

$$n_1 > \eta_1 = \frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}) > max\{\delta_1, \delta_3\} \text{ and } n_2 > \eta_2 = \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\}) > max\{\delta_2, \delta_4\}$$

Hence $[max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2] \prec [n_1, n_2] = \widehat{\mathcal{N}_f}(a_0 * b_0)$

Therefore $a_0 * b_0 \notin \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$. On the other hand

$\widehat{\mathcal{N}_f}(a_0) = [\delta_1, \delta_2] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$

$\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$ that is $a_0, b_0 \in \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ .This is a contradiction and therefore $\widehat{\mathcal{N}_f}(\wp_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$.

Consequently $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Corollary: 3.9 If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$,then the sets $\mathcal{K}_{\mathcal{N}_t} = \{p_0 \in \mathcal{K}: \mathcal{N}_t(p_0) = \mathcal{N}_t(0)\}$, $\mathcal{K}_{\mathcal{N}_i} = \{p_0 \in \mathcal{K}: \mathcal{N}_i(p_0) = \mathcal{N}_i(0)\}$ and $\mathcal{K}_{\widehat{\mathcal{N}_f}} = \{p_0 \in \mathcal{K}: \widehat{\mathcal{N}_f}(p_0) = \widehat{\mathcal{N}_f}(0)\}$ are subalgebras of $\mathcal{K}$.

We say that the subalgebras as $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are BS-subalgebras of $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$

Theorem: 3.10 Every subalgebra of $\mathcal{K}$ can be realized as BS-subalgebra of a BS-NSSA of $\mathcal{K}$.

Proof: Let $\mathcal{J}$ be a subalgebra of $\mathcal{K}$ and let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by

$\mathcal{N}_t(p_0) = \begin{cases} l & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ $\mathcal{N}_i(p_0) = \begin{cases} m & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ and $\widehat{\mathcal{N}_f}(p_0) = \begin{cases} [\eta_1, \eta_2] & if \ \ p_0 \in \mathcal{J}, \\ [1,1] & otherwise, \end{cases}$

Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$. It is clear that $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}$, $\mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [\eta_1, \eta_2]) = \mathcal{J}$. Let $p_0, r_0 \in \mathcal{K}$. If $p_0, r_0 \in \mathcal{J}$ then $p_0 * r_0 \in \mathcal{J}$ and so

$\mathcal{N}_t(p_0 * r_0) = l = min\{l, l\} = min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) = m = min\{m, m\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) = [\eta_1, \eta_2] = rmax\{[\eta_1, \eta_2], [\eta_1, \eta_2]\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$.

If any one of p_0 *and* r_0 is contained in $\mathcal{J}$, say $p_0 \in \mathcal{J}$, then $\mathcal{N}_t(p_0) = l, \mathcal{N}_i(p_0) = m, \widehat{\mathcal{N}_f}(p_0) = [\eta_1, \eta_2], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and \ \widehat{\mathcal{N}_f}(r_0) = [1,1]$. Hence

$\mathcal{N}_t(p_0 * r_0) \geq 0 = min\{l, 0\} = min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq 0 = min\{m, 0\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq [1,1] = rmax\{[\eta_1, \eta_2], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$.

If $p_0, r_0 \notin \mathcal{J}$, then

$\mathcal{N}_t(p_0) = 0, \mathcal{N}_i(p_0) = 0, \widehat{\mathcal{N}_f}(p_0) = [1,1], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and \ \widehat{\mathcal{N}_f}(r_0) = [1,1]$ it follows that

$\mathcal{N}_t(p_0 * r_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq [1,1] = rmax\{[1,1], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Theorem:3.11 For any non-empty set $\mathcal{J}$ of $\mathcal{K}$, Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by

$\mathcal{N}_t(p_0) = \begin{cases} l & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ $\mathcal{N}_i(p_0) = \begin{cases} m & if \ \ p_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ and $\widehat{\mathcal{N}_f}(p_0) = \begin{cases} [\eta_1, \eta_2] & if \ \ p_0 \in \mathcal{J}, \\ [1,1] & otherwise, \end{cases}$ Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$. If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$,then $\mathcal{J}$ is subalgebra of $\mathcal{K}$.

Proof: let $p_0, r_0 \in \mathcal{J}$ then $\mathcal{N}_t(p_0) = l, \mathcal{N}_i(p_0) = m, \widehat{\mathcal{N}_f}(p_0) = [\eta_1, \eta_2]$, $\mathcal{N}_t(r_0) = l, \mathcal{N}_i(r_0) = m, and \ \widehat{\mathcal{N}_f}(r_0) = [\eta_1, \eta_2]$. Thus

$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\} = l$

$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\} = m$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\} = [\eta_1, \eta_2]$ and therefore $p_0 * r_0 \in \mathcal{J}$.

Hence $\mathcal{J}$ is a subalgebra of $\mathcal{K}$.

Theorem: 3.12 Given a BS-NSSA $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ of a BCI-algebra $\mathcal{K}$, Let $\mathcal{N}^* = ({\mathcal{N}_t}^*, {\mathcal{N}_i}^*, \widehat{\mathcal{N}_f}^*)$ is a BS-NSS defined by ${\mathcal{N}_t}^*(p_0) = \mathcal{N}_t(0 * p_0)$, ${\mathcal{N}_i}^*(p_0) = \mathcal{N}_i(0 * p_0)$ and $\widehat{\mathcal{N}_f}^*(p_0) = \widehat{\mathcal{N}_f}(0 * p_0)$ for all $p_0 \in \mathcal{K}$ then $\mathcal{N}^* = ({\mathcal{N}_t}^*, {\mathcal{N}_i}^*, \widehat{\mathcal{N}_f}^*)$ is a BS-NSSA of $\mathcal{K}$.

Proof: Note that $0 * (p_0 * r_0) = (0 * p_0) * (0 * r_0)$ for all $p_0, r_0 \in \mathcal{K}$. We have

${\mathcal{N}_t}^*(p_0 * r_0) = \mathcal{N}_t(0 * (p_0 * r_0)) = \mathcal{N}_t((0 * p_0) * (0 * r_0))) \geq min\{\mathcal{N}_t(0 * p_0), \mathcal{N}_t(0 * r_0)\}$
$= min\{{\mathcal{N}_t}^*(p_0), {\mathcal{N}_t}^*(r_0)\}$

${\mathcal{N}_i}^*(p_0 * r_0) = \mathcal{N}_i(0 * (p_0 * r_0)) = \mathcal{N}_i((0 * p_0) * (0 * r_0))) \geq min\{\mathcal{N}_i(0 * p_0), \mathcal{N}_i(0 * r_0)\}$
$= min\{{\mathcal{N}_i}^*(p_0), {\mathcal{N}_i}^*(r_0)\}$

$\widehat{\mathcal{N}_f}^*(p_0 * r_0) = \widehat{\mathcal{N}_f}(0 * (p_0 * r_0)) = \widehat{\mathcal{N}_f}((0 * p_0) * (0 * r_0))) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(0 * p_0), \widehat{\mathcal{N}_f}(0 * r_0)\} = rmax\{\widehat{\mathcal{N}_f}^*(p_0), \widehat{\mathcal{N}_f}^*(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$

Therefore $\mathcal{N}^* = ({\mathcal{N}_t}^*, {\mathcal{N}_i}^*, \widehat{\mathcal{N}_f}^*)$ is a BS-NSSA of $\mathcal{K}$.

Theorem: 3.13 Let $\Psi: \mathcal{K} \to \mathcal{Y}$ be a homomorphism of BCK/BCI-algebras. If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{Y}$, then $\Psi^{-1}(\mathcal{N}) = (\Psi^{-1}(\mathcal{N}_t), \Psi^{-1}(\mathcal{N}_i), \Psi^{-1}(\widehat{\mathcal{N}_f}))$ is a BS-NSSA of $\mathcal{K}$. Where $\Psi^{-1}(\mathcal{N}_t)(p_0) = \mathcal{N}_t(\Psi(p_0))$, $\Psi^{-1}(\mathcal{N}_i)(p_0) = \mathcal{N}_i(\Psi(p_0))$ and $\Psi^{-1}(\widehat{\mathcal{N}_f})(p_0) = \widehat{\mathcal{N}_f}(\Psi(p_0))$ for all $p_0 \in \mathcal{K}$.

Proof: Let $p_0, r_0 \in \mathcal{K}$. Then

$\Psi^{-1}(\mathcal{N}_t)(p_0 * r_0) = \mathcal{N}_t(\Psi(p_0 * r_0)) = \mathcal{N}_t(\Psi(p_0) * \Psi(r_0)) \geq min\{\mathcal{N}_t(\Psi(p_0)), \mathcal{N}_t(\Psi(r_0))\} = min\{\Psi^{-1}(\mathcal{N}_t)(p_0), \Psi^{-1}(\mathcal{N}_t)(r_0)\}$,

$\Psi^{-1}(\mathcal{N}_i)(p_0 * r_0) = \mathcal{N}_i(\Psi(p_0 * r_0)) = \mathcal{N}_i(\Psi(p_0) * \Psi(r_0)) \geq min\{\mathcal{N}_i(\Psi(p_0)), \mathcal{N}_i(\Psi(r_0))\} = min\{\Psi^{-1}(\mathcal{N}_i)(p_0), \Psi^{-1}(\mathcal{N}_i)(r_0)\}$,

And

$\Psi^{-1}(\widehat{\mathcal{N}_f})(p_0 * r_0) = \widehat{\mathcal{N}_f}(\Psi(p_0 * r_0)) = \widehat{\mathcal{N}_f}(\Psi(p_0) * \Psi(r_0)) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\Psi(p_0)), \widehat{\mathcal{N}_f}(\Psi(r_0))\} = rmax\{\Psi^{-1}(\widehat{\mathcal{N}_f})(p_0), \Psi^{-1}(\widehat{\mathcal{N}_f})(r_0)\}$.

Hence $\Psi^{-1}(\mathcal{N}) = (\Psi^{-1}(\mathcal{N}_t), \Psi^{-1}(\mathcal{N}_i), \Psi^{-1}(\widehat{\mathcal{N}_f}))$ is a BS-NSSA of $\mathcal{K}$

Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$. We denote

$\mathfrak{S} = 1 - sup\{\mathcal{N}_t(p_0): p_0 \in \mathcal{K}\}$

$\mathfrak{K} = 1 - sup\{\mathcal{N}_i(p_0): p_0 \in \mathcal{K}\}$

$\mathfrak{B} = r\inf\{\widehat{\mathcal{N}_f}(p_0): p_0 \in \mathcal{K}\}$.

For any $a \in [0, \mathfrak{S}]$, $b \in [0, \mathfrak{K}]$ and $\widehat{c} \in [[0,0], \mathfrak{B}]$ we define ${\mathcal{N}_t}^a(p_0) = \mathcal{N}_t(p_0) + a$, ${\mathcal{N}_i}^b(p_0) = \mathcal{N}_i(p_0) + b$ and $\widehat{\mathcal{N}_f}^{\widehat{c}} = \widehat{\mathcal{N}_f}(p_0) - \hat{c}$ then $\mathcal{N}^T = ({\mathcal{N}_t}^a, {\mathcal{N}_i}^b, \widehat{\mathcal{N}_f}^{\widehat{c}})$ is a BS-NSS in $\mathcal{K}$,which is called a $(a, b, \widehat{c}) -$ translative BS-NSS of $\mathcal{K}$.

Theorem: 3.14 If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ BS-NSSA of $\mathcal{K}$, then the $(a, b, \widehat{c}) -$ translative BS-NSS of $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is also a BS-NSSA of $\mathcal{K}$.

Proof: For any $p_0, r_0 \in \mathcal{K}$,we get

${\mathcal{N}_t}^a(p_0 * r_0) = \mathcal{N}_t(p_0 * r_0) + a \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\} + a = min\{\mathcal{N}_t(p_0) + a, \mathcal{N}_t(r_0) + a\} = min\{{\mathcal{N}_t}^a(p_0), {\mathcal{N}_t}^a(r_0)\}$,

${\mathcal{N}_i}^b(p_0 * r_0) = \mathcal{N}_i(p_0 * r_0) + b \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\} + b = min\{\mathcal{N}_i(p_0) + b, \mathcal{N}_i(r_0) + b\} = min\{{\mathcal{N}_i}^b(p_0), {\mathcal{N}_i}^b(r_0)\}$, and

$\widehat{\mathcal{N}_f}^{\widehat{c}}(p_0 * r_0) = \widehat{\mathcal{N}_f}(p_0 * r_0) - \widehat{c} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\} - \widehat{c} = rmax\{\widehat{\mathcal{N}_f}(p_0) - \widehat{c}, \widehat{\mathcal{N}_f}(r_0) - \widehat{c}\} = rmax\left\{\widehat{\mathcal{N}_f}^{\widehat{c}}(p_0), \widehat{\mathcal{N}_f}^{\widehat{c}}(r_0)\right\}$. Therefore $\mathcal{N}^T = (\mathcal{N}_t^{\hat{a}}, \mathcal{N}_i^{b}, \widehat{\mathcal{N}_f}^{c})$ is a BS-NSSA of $\mathcal{K}$.

Theorem: 3.15 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ such that its $(a, b, \widehat{c})-$ translative BS-NSS is a BS-NSSA of $\mathcal{K}$ for $a \in [0, \mathfrak{S}], b \in [0, \mathfrak{K}]$ and $\widehat{c} \in [[0,0], \mathfrak{B}]$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$is a BS-NSSA of $\mathcal{K}$.

Proof: Assume that $\mathcal{N}^T = (\mathcal{N}_t^{a}, \mathcal{N}_i^{b}, \widehat{\mathcal{N}_f}^{\widehat{c}})$ is a BS-NSSA of $\mathcal{K}$ for $a \in [0, \mathfrak{S}], b \in [0, \mathfrak{K}]$ and $\widehat{c} \in [[0,0], \mathfrak{B}]$. Let $p_0, r_0 \in \mathcal{K}$. Then

$\mathcal{N}_t(p_0 * r_0) + a = \mathcal{N}_t^{a}(p_0 * r_0) \geq min\{\mathcal{N}_t^{a}(p_0), \mathcal{N}_t^{a}(r_0)\} = min\{\mathcal{N}_t(p_0) + a, \mathcal{N}_t(r_0) + a\} = rmin\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\} + a,$

$\mathcal{N}_i(p_0 * r_0) + b = \mathcal{N}_i^{b}(p_0 * r_0) \geq min\{\mathcal{N}_i^{b}(p_0), \mathcal{N}_i^{b}(r_0)\} = min\{\mathcal{N}_i(p_0) + b, \mathcal{N}_i(r_0) + b\} = min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\} + b,$ and

$\widehat{\mathcal{N}_f}(p_0 * r_0) - \widehat{c} = \widehat{\mathcal{N}_f}^{\widehat{c}}(p_0 * r_0) \preccurlyeq rmax\left\{\widehat{\mathcal{N}_f}^{\widehat{c}}(p_0), \widehat{\mathcal{N}_f}^{\widehat{c}}(r_0)\right\} = rmax\{\widehat{\mathcal{N}_f}(p_0) - \widehat{c}, \widehat{\mathcal{N}_f}(r_0) - \widehat{c}\} = rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\} - \widehat{c}$. It follows that

$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Hence $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$is a BS-NSSA of $\mathcal{K}$.

4. BS-Neutrosophic Ideal (BS-NSI)

Definition:4.1 Let $\mathcal{K}$ be a BCK/BCI-algebra. A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is called a BS-NSI of $\mathcal{K}$ if it satisfies

(BS-NSI 1) $\mathcal{N}_t(0) \geq \mathcal{N}_t(p_0), \mathcal{N}_i(0) \geq \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $p_0 \in \mathcal{K}$

(BS-NSI 2) $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$

(BS-NSI 3) $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$

(BS-NSI 4) $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Example:4.2 Consider a set $\mathcal{K} = \{0,1,2,a\}$ with the binary operation '$*$' which is given in the table:3 Then $(\mathcal{K}\ ;\ *, 0)$ is a BCI-algebra.

*	0	a	b	1
0	0	0	0	1
a	a	0	0	1
b	b	b	0	1
1	1	1	1	0

Table.3 BCI-algebra

Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined in table:4

$\mathcal{K}$	$\mathcal{N}_t(p_0)$	$\mathcal{N}_i(p_0)$	$\widehat{\mathcal{N}_f}(p_0)$
0	0.9	0.8	[0.2,0.5]

a	0.7	0.6	[0.4,0.7]
b	0.4	0.3	[0.7,0.9]
1	0.2	0.1	[0.9,1]

Table.4 BS-NSI

It is routine to verify that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ is a BS-NSI of $\mathcal{K}$.

Proposition: 4.3 Let $\mathcal{K}$ be a BCK/BCI-algebra. Then every BS-NSI $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ of $\mathcal{K}$ satisfies the following assertion $p_0 * r_0 \le u_0 \Rightarrow \mathcal{N}_t(p_0) \ge min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \ge min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(r_0), \widehat{\mathcal{N}}_f(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$.

Proof: Let $p_0, r_0, u_0 \in \mathcal{K}$ be such that $p_0 * r_0 \le u_0$. Then

$\mathcal{N}_t(p_0 * r_0) \ge min\{\mathcal{N}_t((p_0 * r_0) * u_0), \mathcal{N}_t(u_0)\} = min\{\mathcal{N}_t(0), \mathcal{N}_t(u_0)\} = \mathcal{N}_t(u_0)$,

$\mathcal{N}_i(p_0 * r_0) \ge min\{\mathcal{N}_i((p_0 * r_0) * u_0), \mathcal{N}_i(u_0)\} = min\{\mathcal{N}_i(0), \mathcal{N}_i(u_0)\} = \mathcal{N}_i(u_0)$, and

$\widehat{\mathcal{N}}_f(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f((p_0 * r_0) * u_0), \widehat{\mathcal{N}}_f(u_0)\} = rmax\{\widehat{\mathcal{N}}_f(0), \widehat{\mathcal{N}}_f(u_0)\} = \widehat{\mathcal{N}}_f(u_0)$.

It follows that

$\mathcal{N}_t(p_0) \ge min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\} \ge min\{\mathcal{N}_t(u_0), \mathcal{N}_t(r_0)\}$,

$\mathcal{N}_i(p_0) \ge min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\} \ge min\{\mathcal{N}_i(u_0), \mathcal{N}_i(r_0)\}$,

$\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(p_0 * r_0), \widehat{\mathcal{N}}_f(r_0)\} \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(u_0), \widehat{\mathcal{N}}_f(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Hence the proof is completed.

Theorem: 4.4 Every BS-NSS in a BCK/BCI-algebra $\mathcal{K}$ satisfying (BS-NSI 1) and assertion $p_0 * r_0 \le u_0 \Rightarrow \mathcal{N}_t(p_0) \ge min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \ge min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(r_0), \widehat{\mathcal{N}}_f(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$ is a BS-NSI of $\mathcal{K}$.

Proof: Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ be a BS-NSS in $\mathcal{K}$ satisfying (BS-NSI 1) and assertion $p_0 * r_0 \le u_0 \Rightarrow \mathcal{N}_t(p_0) \ge min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \ge min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(r_0), \widehat{\mathcal{N}}_f(u_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Note that $p_0 * (p_0 * r_0) \le r_0$ for all $p_0, r_0 \in \mathcal{K}$. So, we have

$\mathcal{N}_t(p_0) \ge min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$,

$\mathcal{N}_i(p_0) \ge min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$,

$\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(p_0 * r_0), \widehat{\mathcal{N}}_f(r_0)\}$. There fore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ is a BS-NSI of $\mathcal{K}$.

Theorem: 4.5 Given a BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ in a BCK/BCI-algebra $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ is a BS-NSI if and only if $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^{\,-})^c, and\ (\mathcal{N}_f^{\,+})^c$ are fuzzy ideals of $\mathcal{K}$.

Proof: suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}}_f)$ be a BS-NSI in $\mathcal{K}$. Then we have $\mathcal{N}_t(0) \ge \mathcal{N}_t(p_0), \mathcal{N}_i(0) \ge \mathcal{N}_i(p_0)$ and $\widehat{\mathcal{N}}_f(0) \preccurlyeq \widehat{\mathcal{N}}_f(x)$ for all $p_0 \in \mathcal{K}$

$\mathcal{N}_t(p_0) \ge min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0) \ge min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(p_0 * r_0), \widehat{\mathcal{N}}_f(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$

Now $\widehat{\mathcal{N}}_f(0) \preccurlyeq \widehat{\mathcal{N}}_f(p_0) \Rightarrow [\mathcal{N}_f^{\,-}(0), \mathcal{N}_f^{\,+}(0)] \preccurlyeq [\mathcal{N}_f^{\,-}(p_0)\ , \mathcal{N}_f^{\,+}(p_0)]$

$\Rightarrow \mathcal{N}_f^{\,-}(0) \le \mathcal{N}_f^{\,-}(p_0)\ \ and\ \ \mathcal{N}_f^{\,+}(0) \le \mathcal{N}_f^{\,+}(p_0)$

$\Rightarrow (\mathcal{N}_f^{\,-})^c(0) \ge (\mathcal{N}_f^{\,-})^c(p_0)\ \ and\ \ (\mathcal{N}_f^{\,+})^c(0) \ge (\mathcal{N}_f^{\,+})^c(p_0)$

$\widehat{\mathcal{N}}_f(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}}_f(p_0 * r_0), \widehat{\mathcal{N}}_f(r_0)\}$

$$\Rightarrow [\mathcal{N}_f^{\,-}(p_0), \mathcal{N}_f^{\,+}(p_0)] \preccurlyeq rmax\{[\mathcal{N}_f^{\,-}(p_0 * r_0), \mathcal{N}_f^{\,+}(p_0 * r_0)], [\mathcal{N}_f^{\,-}(r_0), \mathcal{N}_f^{\,+}(r_0)]\}$$
$$= [max\{\mathcal{N}_f^{\,-}(p_0 * r_0), \mathcal{N}_f^{\,-}(r_0)\}, max\{\mathcal{N}_f^{\,+}(p_0 * r_0), \mathcal{N}_f^{\,+}(r_0)\}]$$

Therefore

$\mathcal{N}_f^{\,-}(\wp_0) \le max\{\mathcal{N}_f^{\,-}(\wp_0 * r_0), \mathcal{N}_f^{\,-}(r_0)\}$ and $\mathcal{N}_f^{\,+}(\wp_0) \le max\{\mathcal{N}_f^{\,+}(\wp_0 * r_0), \mathcal{N}_f^{\,+}(r_0)\}$

$\Rightarrow 1 - \mathcal{N}_f^{\,-}(\wp_0) \ge 1 - max\{\mathcal{N}_f^{\,-}(\wp_0 * r_0), \mathcal{N}_f^{\,-}(r_0)\}$

$\Rightarrow (\mathcal{N}_f^{\,-})^c(\wp_0) \ge min\{1 - \mathcal{N}_f^{\,-}(\wp_0 * r_0), 1 - \mathcal{N}_f^{\,-}(r_0)\}$

$\Rightarrow (\mathcal{N}_f^{\,-})^c(\wp_0) \ge min\{(\mathcal{N}_f^{\,-})^c(\wp_0 * r_0), (\mathcal{N}_f^{\,-})^c(r_0)\}$

Similarly

$(\mathcal{N}_f^{\,+})^c(\wp_0) \ge min\{(\mathcal{N}_f^{\,+})^c(\wp_0 * r_0), (\mathcal{N}_f^{\,+})^c(r_0)\}$

Therefore $\mathcal{N}_t, \mathcal{N}_i, (\mathcal{N}_f^{\,-})^c, and\ (\mathcal{N}_f^{\,+})^c$ are fuzzy ideals of $\mathcal{K}$.

Converse part is obvious.

Theorem: 4.6 A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ in $\mathcal{K}$ is a BS-NSI of $\mathcal{K}$ if and only if the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are ideals of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$

Proof: Suppose that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

Let $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$ be such that $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are non-empty.

Obviously $0 \in \mathcal{U}(\mathcal{N}_t; l)$, $0 \in \mathcal{U}(\mathcal{N}_i; m)$ and $0 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$

For any $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathcal{K}$ if $a_1 * a_2, a_2 \in (\mathcal{N}_t; l)$, $b_1 * b_2, b_2 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1 * c_2, c_2 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ then

$\mathcal{N}_t(a_1) \ge min\{\mathcal{N}_t(a_1 * a_2), \mathcal{N}_t(a_2)\} \ge min\{l, l\} = l$

$\mathcal{N}_i(b_1) \ge min\{\mathcal{N}_i(b_1 * b_2), \mathcal{N}_i(b_2)\} \ge min\{m, m\} = m$

$\widehat{\mathcal{N}_f}(c_1) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(c_1 * c_2), \widehat{\mathcal{N}_f}(c_2)\} \preccurlyeq rmax\{[n_1, n_2], [n_1, n_2]\} = [n_1, n_2]$

Therefore $a_1 \in \mathcal{U}(\mathcal{N}_t; l)$, $b_1 \in \mathcal{U}(\mathcal{N}_i; m)$ and $c_1 \in \mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$

Hence $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are ideals of $\mathcal{K}$.

Conversely, assume that the non-empty sets $\mathcal{U}(\mathcal{N}_t; l)$, $\mathcal{U}(\mathcal{N}_i; m)$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [n_1, n_2])$ are ideals of $\mathcal{K}$ for all $l, m \in [0,1]$ and $[n_1, n_2] \in [I]$.

Suppose that $\mathcal{N}_t(0) < \mathcal{N}_t(\wp_0)$, $\mathcal{N}_i(0) < \mathcal{N}_i(\wp_0)$ and $\widehat{\mathcal{N}_f}(0) \succ \widehat{\mathcal{N}_f}(\wp_0)$ for some $\wp_0 \in \mathcal{K}$.

Then $0 \notin \mathcal{U}(\mathcal{N}_t; \mathcal{N}_t(\wp_0)) \cap \mathcal{U}(\mathcal{N}_i; \mathcal{N}_i(\wp_0)) \cap \mathcal{L}(\widehat{\mathcal{N}_f}; \widehat{\mathcal{N}_f}(\wp_0))$.which is a contradiction.

Hence $\mathcal{N}_t(0) \ge \mathcal{N}_t(\wp_0), \mathcal{N}_i(0) \ge \mathcal{N}_i(\wp_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $\wp_0 \in \mathcal{K}$

If $\mathcal{N}_t(a_0) < min\{\mathcal{N}_t(a_0 * b_0), \mathcal{N}_t(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$, then $a_0 * b_0, b_0 \in \mathcal{U}(\mathcal{N}_t; l_0)$ but $a_0 \notin \mathcal{U}(\mathcal{N}_t; l_0)$ for $l_0 = min\{\mathcal{N}_t(a_0 * b_0), \mathcal{N}_t(b_0)\}$. This is a contradiction, and thus

$\mathcal{N}_t(a) \ge min\{\mathcal{N}_t(a * b), \mathcal{N}_t(b)\}$ for all $a, b \in \mathcal{K}$.

Similarly, we can show that $\mathcal{N}_i(a) \ge min\{\mathcal{N}_i(a * b), \mathcal{N}_i(b)\}$ for all $a, b \in \mathcal{K}$.

Suppose that $\widehat{\mathcal{N}_f}(a_0) \succ rmax\{\widehat{\mathcal{N}_f}(a_0 * b_0), \widehat{\mathcal{N}_f}(b_0)\}$ for some $a_0, b_0 \in \mathcal{K}$.

Let $\widehat{\mathcal{N}_f}(a_0 * b_0) = [\delta_1, \delta_2]$, $\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4]$ and $\widehat{\mathcal{N}_f}(a_0) = [n_1, n_2]$

Then $[n_1, n_2] \succ rmax\{[\delta_1, \delta_2], [\delta_3, \delta_4]\} = [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}]$ and so

$n_1 > max\{\delta_1, \delta_3\}$ and $n_2 > max\{\delta_2, \delta_4\}$

Taking $[\eta_1, \eta_2] = \frac{1}{2}\left[\widehat{\mathcal{N}_f}(a_0) + rmax\{\widehat{\mathcal{N}_f}(a_0 * b_0), \widehat{\mathcal{N}_f}(b_0)\}\right]$

$$= \frac{1}{2}\left[[n_1, n_2] + (max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\})\right]$$

$$= \left[\frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}), \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\})\right]$$

It follows that

$n_1 > \eta_1 = \frac{1}{2}(n_1 + max\{\delta_1, \delta_3\}) > max\{\delta_1, \delta_3\}$ and $n_2 > \eta_2 = \frac{1}{2}(n_2 + max\{\delta_2, \delta_4\}) > max\{\delta_2, \delta_4\}$

Hence $[max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2] \prec [n_1, n_2] = \widehat{\mathcal{N}_f}(a_0)$
Therefore $a_0 \notin \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$. On the other hand
$\widehat{\mathcal{N}_f}(a_0 * b_0) = [\delta_1, \delta_2] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$
$\widehat{\mathcal{N}_f}(b_0) = [\delta_3, \delta_4] \preccurlyeq [max\{\delta_1, \delta_3\}, max\{\delta_2, \delta_4\}] \prec [\eta_1, \eta_2]$ that is $a_0 * b_0, b_0 \in \mathcal{U}(\widehat{\mathcal{N}_f}; [n_1, n_2])$.This is a contradiction and therefore $\widehat{\mathcal{N}_f}(a_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(a_0 * b_0), \widehat{\mathcal{N}_f}(b_0)\}$ for all $a_0, b_0 \in \mathcal{K}$ .Consequently $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

Theorem: 4.7 Given an ideal $\mathcal{J}$ of a BCK/BCI-algebra $\mathcal{K}$, let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be an BS-NSS in $\mathcal{K}$ defined by $\mathcal{N}_t(\wp_0) = \begin{cases} l & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ $\mathcal{N}_i(\wp_0) = \begin{cases} m & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ and $\widehat{\mathcal{N}_f}(\wp_0) = \begin{cases} [\eta_1, \eta_2] & if\ \wp_0 \in \mathcal{J}, \\ [1,1] & otherwise. \end{cases}$ Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$ Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$ such that $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}$, $\mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [\eta_1, \eta_2]) = \mathcal{J}$.
Proof: Let $\wp_0, r_0 \in \mathcal{K}$
If $\wp_0 * r_0 \in \mathcal{J}$ and $r_0 \in \mathcal{J}$ then $\wp_0 \in \mathcal{J}$ and so
$\mathcal{N}_t(\wp_0) = l = min\{l, l\} = min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$
$\mathcal{N}_i(\wp_0) = m = min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$
$\widehat{\mathcal{N}_f}(\wp_0) = [\eta_1, \eta_2] = rmax\{[\eta_1, \eta_2], [\eta_1, \eta_2]\} = rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.
If any one of $\wp_0 * r_0$ *and* r_0 is contained in $\mathcal{J}$, say $\wp_0 * r_0 \in \mathcal{J}$, then $\mathcal{N}_t(\wp_0 * r_0) = l, \mathcal{N}_i(\wp_0 * r_0) = m, \widehat{\mathcal{N}_f}(\wp_0 * r_0) = [\eta_1, \eta_2], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and\ \widehat{\mathcal{N}_f}(r_0) = [1,1]$. Hence
$\mathcal{N}_t(\wp_0) \geq 0 = min\{l, 0\} = min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$
$\mathcal{N}_i(\wp_0) \geq 0 = min\{m, 0\} = min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$
$\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq [1,1] = rmax\{[\eta_1, \eta_2], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.
If $\wp_0 * r_0 \notin \mathcal{J}$ *and* $r_0 \notin \mathcal{J}$,then
$\mathcal{N}_t(\wp_0 * r_0) = 0, \mathcal{N}_i(\wp_0 * r_0) = 0, \widehat{\mathcal{N}_f}(\wp_0 * r_0) = [1,1], \mathcal{N}_t(r_0) = 0, \mathcal{N}_i(r_0) = 0, and\ \widehat{\mathcal{N}_f}(r_0) = [1,1]$ it follows that
$\mathcal{N}_t(\wp_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$,
$\mathcal{N}_i(\wp_0) \geq 0 = min\{0,0\} = min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$,
$\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq [1,1] = rmax\{[1,1], [1,1]\} = rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.
It is obvious that $\mathcal{N}_t(0) \geq \mathcal{N}_t(\wp_0), \mathcal{N}_i(0) \geq \mathcal{N}_i(\wp_0)$ and $\widehat{\mathcal{N}_f}(0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $\wp_0 \in \mathcal{K}$
Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.
Obviously, we have $\mathcal{U}(\mathcal{N}_t; l) = \mathcal{J}$, $\mathcal{U}(\mathcal{N}_i; m) = \mathcal{J}$ and $\mathcal{L}(\widehat{\mathcal{N}_f}; [\eta_1, \eta_2]) = \mathcal{J}$.
Theorem:4.8 For any non-empty set $\mathcal{J}$ of $\mathcal{K}$, Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by $\mathcal{N}_t(\wp_0) = \begin{cases} l & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ $\mathcal{N}_i(\wp_0) = \begin{cases} m & if\ \wp_0 \in \mathcal{J}, \\ 0 & otherwise, \end{cases}$ and $\widehat{\mathcal{N}_f}(\wp_0) = \begin{cases} [\eta_1, \eta_2] & if\ \wp_0 \in \mathcal{J}, \\ [1,1] & otherwise. \end{cases}$ Where $l, m \in (0,1]$ and $\eta_1, \eta_2 \in [0,1)$ with $\eta_1 < \eta_2$ .If $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$,then $\mathcal{J}$ is ideal of $\mathcal{K}$.
Proof: Obviously, $0 \in \mathcal{J}$. Let $\wp_0, r_0 \in \mathcal{K}$ be such that $\wp_0 * r_0, r_0 \in \mathcal{J}$ then $\mathcal{N}_t(\wp_0 * r_0) = l, \mathcal{N}_i(\wp_0 * r_0) = m, \widehat{\mathcal{N}_f}(\wp_0 * r_0) = [\eta_1, \eta_2], \mathcal{N}_t(r_0) = l, \mathcal{N}_i(r_0) = m, and\ \widehat{\mathcal{N}_f}(r_0) = [\eta_1, \eta_2]$. Thus

$\mathcal{N}_t(\wp_0) \geq min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\} = l$

$\mathcal{N}_i(\wp_0) \geq min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\} = m$

$\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\} = [\eta_1, \eta_2]$ and therefore $\wp_0 \in \mathcal{J}$. Hence $\mathcal{J}$ is an ideal of $\mathcal{K}$.

Theorem:4.9 In a BCK-algebra $\mathcal{K}$, every BS-NSI is a BS-NSSA of $\mathcal{K}$.

Proof: Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI of a BCK-algebra $\mathcal{K}$.

Since $(\wp_0 * r_0) * \wp_0 \leq r_0$ for $\wp_0, r_0 \in \mathcal{K}$, it follows from proposition 4.3 that

$\mathcal{N}_t(\wp_0 * r_0) \geq min\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(r_0)\}$, $\mathcal{N}_i(\wp_0 * r_0) \geq min\{\mathcal{N}_i(\wp_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(\wp_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$.

Hence $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSSA of a BCK-algebra $\mathcal{K}$.

The converse of the above theorem may not be true as seen in the following example.

Example: 4.10 Consider a BCK-algebra $\mathcal{K} = \{0, a, b, c\}$ with a binary operation '$*$' which is given in table.5 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by table.6 then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is BS-NSSA of $\mathcal{K}$, but it is not an BS-NSI of a BCK-algebra $\mathcal{K}$. Since $\mathcal{N}_t(a) \ngeq smin\{\mathcal{N}_t(a * b), \mathcal{N}_t(b)\}$

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Table.5 BCK-algebra

$\mathcal{K}$	$\mathcal{N}_t(\wp_0)$	$\mathcal{N}_i(\wp_0)$	$\widehat{\mathcal{N}_f}(\wp_0)$
0	1	0.8	[0.2,0.4]
a	0.3	0.5	[0.4,0.6]
b	0.3	0.8	[0.5,0.7]
c	0.5	0.5	[0.7,0.9]

Table.6 BS-NSSA

We give a condition for a BS-NSSA to be a BS-NSI in a BCK-algebra

Theorem:4.11 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSSA of a BCK-algebra $\mathcal{K}$ satisfying the conditions $\wp_0 * r_0 \leq u_0 \Rightarrow \mathcal{N}_t(\wp_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(\wp_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $\wp_0, r_0, u_0 \in \mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

Proof: For any $\wp_0 \in \mathcal{K}$, we get

$\mathcal{N}_t(0) = \mathcal{N}_t(\wp_0 * \wp_0) \geq min\{\mathcal{N}_t(\wp_0), \mathcal{N}_t(\wp_0)\} = \mathcal{N}_t(\wp_0)$

$\mathcal{N}_i(0) = \mathcal{N}_i(\wp_0 * \wp_0) \geq min\{\mathcal{N}_i(\wp_0), \mathcal{N}_i(\wp_0)\} = \mathcal{N}_i(\wp_0)$,

$\widehat{\mathcal{N}_f}(0) = \widehat{\mathcal{N}_f}(\wp_0 * \wp_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0), \widehat{\mathcal{N}_f}(\wp_0)\} \preccurlyeq rmax\{[\mathcal{N}_f^{-}(\wp_0), \mathcal{N}_f^{+}(\wp_0)], [\mathcal{N}_f^{-}(\wp_0), \mathcal{N}_f^{+}(\wp_0)]\}$

$= [\mathcal{N}_f^{-}(\wp_0), \mathcal{N}_f^{+}(\wp_0)] = \widehat{\mathcal{N}_f}(\wp_0)$.

Since $\wp_0 * (\wp_0 * r_0) \leq r_0$ for all $\wp_0, r_0 \in \mathcal{K}$. It follows that $\mathcal{N}_t(\wp_0) \geq min\{\mathcal{N}_t(\wp_0 * r_0), \mathcal{N}_t(r_0)\}$, $\mathcal{N}_i(\wp_0) \geq min\{\mathcal{N}_i(\wp_0 * r_0), \mathcal{N}_i(r_0)\}$, $\widehat{\mathcal{N}_f}(\wp_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(\wp_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $\wp_0, r_0 \in \mathcal{K}$. Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI in a BCK-algebra $\mathcal{K}$.

Definition:4.12 A BS- neutrosophic ideal of $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ of a BCI-algebra $\mathcal{K}$ is said to be closed if $\mathcal{N}_t(0 * \wp_0) \geq \mathcal{N}_t(\wp_0)$, $\mathcal{N}_i(0 * \wp_0) \geq \mathcal{N}_i(\wp_0)$ and $\widehat{\mathcal{N}_f}(0 * \wp_0) \preccurlyeq \widehat{\mathcal{N}_f}(x)$ for all $\wp_0 \in \mathcal{K}$.

Theorem:4.13 In a BCI-algebra $\mathcal{K}$, every closed BS-NSI is a BS-NSSA

Proof: Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a closed BS-NSI of a BCI-algebra $\mathcal{K}$

We have for all $p_0, r_0 \in \mathcal{K}$

$$\mathcal{N}_t(p_0 * r_0) \geq min\{\mathcal{N}_t((p_0 * r_0) * p_0), \mathcal{N}_t(p_0)\} \quad (\because \ \mathcal{N} \ is \ a \ \text{BS} - \text{neutrosophic ideal})$$
$$= min\{\mathcal{N}_t((p_0 * p_0) * r_0), \mathcal{N}_t(p_0)\} \quad (\because \ (p_0 * r_0) * u_0 = (p_0 * u_0) * r_0)$$
$$= min\{\mathcal{N}_t(0 * r_0), \mathcal{N}_t(p_0)\} \quad (\because \ p_0 * p_0 = 0)$$
$$\geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(p_0)\} \quad (\because \ \mathcal{N} \ is \ a \ closed \ \text{BS} - \text{neutrosophic ideal})$$

$$\mathcal{N}_i(p_0 * r_0) \geq min\{\mathcal{N}_i((p_0 * r_0) * p_0), \mathcal{N}_i(p_0)\} \quad (\because \ \mathcal{N} \ is \ a \ \text{BS} - \text{neutrosophic ideal})$$
$$= min\{\mathcal{N}_i((p_0 * p_0) * r_0), \mathcal{N}_i(p_0)\} \quad (\because \ (p_0 * r_0) * u_0 = (p_0 * u_0) * r_0)$$
$$= min\{\mathcal{N}_i(0 * r_0), \mathcal{N}_i(p_0)\} \quad (\because \ p_0 * p_0 = 0)$$
$$\geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(p_0)\} \quad (\because \ \mathcal{N} \ is \ a \ closed \ \text{BS} - \text{neutrosophic ideal})$$

And

$$\widehat{\mathcal{N}_f}(p_0 * r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}((p_0 * r_0) * p_0), \widehat{\mathcal{N}_f}(p_0)\} \quad (\because \ \mathcal{N} \ is \ a \ \text{BS} - \text{neutrosophic ideal})$$
$$= rmax\{\widehat{\mathcal{N}_f}((p_0 * p_0) * r_0), \widehat{\mathcal{N}_f}(p_0)\} \quad (\because \ (p_0 * r_0) * u_0 = (p_0 * u_0) * r_0)$$
$$= rmax\{\widehat{\mathcal{N}_f}(0 * r_0), \widehat{\mathcal{N}_f}(p_0)\} \quad (\because \ p_0 * p_0 = 0)$$
$$\preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(p_0)\} \quad (\because \ \mathcal{N} \ is \ a \ closed \ \text{BS} - \text{neutrosophic ideal})$$

Hence $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Theorem:4.14 In a weakly BCK-algebra $\mathcal{K}$, every BS-NSI is closed.

Proof: Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI of a weakly BCK-algebra $\mathcal{K}$. For any $p_0 \in \mathcal{K}$, we obtain

$\mathcal{N}_t(0 * p_0) \geq min\{\mathcal{N}_t((0 * p_0) * p_0), \mathcal{N}_t(p_0)\} = min\{\mathcal{N}_t(0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$,

$\mathcal{N}_i(0 * p_0) \geq min\{\mathcal{N}_i((0 * p_0) * p_0), \mathcal{N}_i(p_0)\} = min\{\mathcal{N}_i(0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,

$\widehat{\mathcal{N}_f}(0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}((0 * p_0) * p_0), \widehat{\mathcal{N}_f}(p_0)\} = rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

Corollary: 4.15 In a weakly BCK-algebra, every BS-NSI is a BS-NSSA of $\mathcal{K}$.

In a following example we show that any BS-NSSA is not an BS-NSI in a BCI-algebra.

Example: 4.16 Consider a BCI-algebra $\mathcal{K} = \{0,1,2,3,4,5\}$ with binary operation '$*$' in table.7

Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in $\mathcal{K}$ defined by table.8 It is routine to verify that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$. But it is not a BS-NSI of $\mathcal{K}$. Since $\mathcal{N}_t(4) \prec min\{\mathcal{N}_t(4 * 3), \mathcal{N}_t(3)\}$.

$*$	0	1	2	3	4	5
0	0	0	3	2	3	3
1	1	0	3	2	3	3
2	2	2	0	3	0	0
3	3	3	2	0	2	2
4	4	2	1	3	0	1
5	5	2	1	3	1	0

Table.7 BCI-algebra

$\mathcal{K}$	$\mathcal{N}_t(p_0)$	$\mathcal{N}_i(p_0)$	$\widehat{\mathcal{N}_f}(p_0)$
0	0.9	0.8	[0.2,0.6]
1	0.3	0.4	[0.5,0.9]

2	0.9	0.8	[0.2,0.6]
3	0.9	0.8	[0.2,0.6]
4	0.3	0.4	[0.5,0.9]
5	0.3	0.4	[0.5,0.9]

Table.8 BS-NSSA

Theorem: 4.17 In a p-semi simple BCI-algebra $\mathcal{K}$, the following are equivalent

(i). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

(ii). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Proof: $(i) \Longrightarrow (ii)$ see theorem 4.12

$(ii) \Longrightarrow (i)$ let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$. For any $p_0 \in \mathcal{K}$, we get

$\mathcal{N}_t(0) = \mathcal{N}_t(p_0 * p_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$,

$\mathcal{N}_i(0) = \mathcal{N}_i(p_0 * p_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,

$\widehat{\mathcal{N}_f}(0) = \widehat{\mathcal{N}_f}(p_0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$.

Hence $\mathcal{N}_t(0 * p_0) \geq min\{\mathcal{N}_t(0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$

$\mathcal{N}_i(0 * p_0) \geq min\{\mathcal{N}_i(0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$,$\widehat{\mathcal{N}_f}(0 * p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$ for all $p_0 \in \mathcal{K}$.

Let $p_0, r_0 \in \mathcal{K}$ then

$\mathcal{N}_t(p_0) = \mathcal{N}_t(r_0 * (r_0 * p_0)) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(r_0 * p_0)\}$

$= min\{\mathcal{N}_t(r_0), \mathcal{N}_t(0 * (p_0 * r_0))\} \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$,

$\mathcal{N}_i(p_0) = \mathcal{N}_i(r_0 * (r_0 * p_0)) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(r_0 * p_0)\}$

$= min\{\mathcal{N}_i(r_0), \mathcal{N}_i(0 * (p_0 * r_0))\} \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$, and

$\widehat{\mathcal{N}_f}(p_0) = \widehat{\mathcal{N}_f}(r_0 * (r_0 * p_0)) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(r_0 * p_0)\}$

$= rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(0 * (p_0 * r_0))\} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

Since every associative BCI-algebra is a p-semisimple, we have the following are corollary

Corollary:4.18 In a associative BCI-algebra $\mathcal{K}$, the following are equivalent

(i). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a closed BS-NSI of $\mathcal{K}$.

(ii). $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSSA of $\mathcal{K}$.

Definition: 4.19 Let $\mathcal{K}$ be an (s)-BCK-algebra. A BS-NSS $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is called a BS-neutrosophic $\circ$-subalgebra of $\mathcal{K}$ if the following assertions are valid

$\mathcal{N}_t(p_0 \circ r_0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0 \circ r_0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0 \circ r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$.

Lemma:4.20 Every BS-NSI of BCK/BCI-algebra $\mathcal{K}$ satisfies the following assertion

$p_0 \leq r_0 \Rightarrow \mathcal{N}_t(p_0) \geq \mathcal{N}_t(r_0), \mathcal{N}_i(p_0) \geq \mathcal{N}_i(r_0)$ and $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq \widehat{\mathcal{N}_f}(r_0)$ for all $p_0, r_0 \in \mathcal{K}$.

Proof: Assume that $p_0 \preccurlyeq r_0$ for all $p_0, r_0 \in \mathcal{K}$ then $p_0 * r_0 = 0$ and so

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\} = min\{\mathcal{N}_t(0), \mathcal{N}_t(r_0)\} = \mathcal{N}_t(r_0)$

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\} = min\{\mathcal{N}_i(0), \mathcal{N}_i(r_0)\} = \mathcal{N}_i(r_0)$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\} = rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(r_0)\} = \widehat{\mathcal{N}_f}(r_0)$ for all $p_0, r_0 \in \mathcal{K}$.

Hence the proof is completed.

Theorem:4.21 In a (s)-BCK-algebra, every BS-NSI is a BS-neutrosophic $\circ$-subalgebra.

Proof: Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSI of a (s)-BCK-algebra $\mathcal{K}$. Note that $(p_0 \circ r_0) * p_0 \leq r_0$ for $p_0, r_0 \in \mathcal{K}$. We have

$\mathcal{N}_t(p_0 \circ r_0) \geq min\{\mathcal{N}_t((p_0 \circ r_0) * p_0), \mathcal{N}_t(p_0)\} \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(p_0)\}$,

$\mathcal{N}_i(p_0 \circ r_0) \geq min\{\mathcal{N}_i((p_0 \circ r_0) * p_0), \mathcal{N}_i(p_0)\} \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(p_0)\}$, and

$\widehat{\mathcal{N}_f}(p_0 \circ r_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}((p_0 \circ r_0) * p_0), \widehat{\mathcal{N}_f}(p_0)\} \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(p_0)\}$.

Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-neutrosophic $\circ$-subalgebra of $\mathcal{K}$.

Theorem:4.22 Let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in a (s)-BCK-algebra $\mathcal{K}$. Then $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$ if and only if the following assertions are valid $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$ with $p_0 \leq r_0 \circ u_0$

Proof: Assume that $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$ and let $p_0, r_0, u_0 \in \mathcal{K}$ be such that $p_0 \leq r_0 \circ u_0$

Then we have

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * (r_0 \circ u_0)), \mathcal{N}_t(r_0 \circ u_0)\}$

$= min\{\mathcal{N}_t(0), \mathcal{N}_t(r_0 \circ u_0)\} = \mathcal{N}_t(r_0 \circ u_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$ (By theorem 4.20)

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * (r_0 \circ u_0)), \mathcal{N}_i(r_0 \circ u_0)\}$

$= min\{\mathcal{N}_i(0), \mathcal{N}_i(r_0 \circ u_0)\} = \mathcal{N}_i(r_0 \circ u_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$ (By theorem 4.20)

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * (r_0 \circ u_0)), \widehat{\mathcal{N}_f}(r_0 \circ u_0)\}$

$= rmax\{\widehat{\mathcal{N}_f}(0), \widehat{\mathcal{N}_f}(r_0 \circ u_0)\} = \widehat{\mathcal{N}_f}(r_0 \circ u_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ (By theorem 4.20)

Conversely, let $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ be a BS-NSS in a (s)-BCK-algebra $\mathcal{K}$ satisfying the conditions $\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(r_0), \mathcal{N}_t(u_0)\}$, $\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(r_0), \mathcal{N}_i(u_0)\}$, $\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(r_0), \widehat{\mathcal{N}_f}(u_0)\}$ for all $p_0, r_0, u_0 \in \mathcal{K}$ with $p_0 \leq r_0 \circ u_0$

Since $0 \leq p_0 \circ p_0$ for all $p_0 \in \mathcal{K}$, we have

$\mathcal{N}_t(0) \geq min\{\mathcal{N}_t(p_0), \mathcal{N}_t(p_0)\} = \mathcal{N}_t(p_0)$

$\mathcal{N}_i(0) \geq min\{\mathcal{N}_i(p_0), \mathcal{N}_i(p_0)\} = \mathcal{N}_i(p_0)$

$\widehat{\mathcal{N}_f}(0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0), \widehat{\mathcal{N}_f}(p_0)\} = \widehat{\mathcal{N}_f}(p_0)$.

Since $p_0 \leq (p_0 * r_0) \circ r_0$ for all $p_0, r_0 \in \mathcal{K}$, we have

$\mathcal{N}_t(p_0) \geq min\{\mathcal{N}_t(p_0 * r_0), \mathcal{N}_t(r_0)\}$

$\mathcal{N}_i(p_0) \geq min\{\mathcal{N}_i(p_0 * r_0), \mathcal{N}_i(r_0)\}$

$\widehat{\mathcal{N}_f}(p_0) \preccurlyeq rmax\{\widehat{\mathcal{N}_f}(p_0 * r_0), \widehat{\mathcal{N}_f}(r_0)\}$ for all $p_0, r_0 \in \mathcal{K}$. Therefore $\mathcal{N} = (\mathcal{N}_t, \mathcal{N}_i, \widehat{\mathcal{N}_f})$ is a BS-NSI of $\mathcal{K}$.

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