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On S_λ-summability in neutrosophic soft normed linear spaces

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Abstract. In present paper, we aim to define λ-statistical convergence, λ-statistical Cauchy and λ-statistical completeness of sequences in neutrosophic soft normed linear spaces (briefly called $NSNLS$). We study certain properties of these notions and provide example to show that λ-statistical convergence is a more general method of summability in these spaces.

Keywords: λ-statistical convergence, λ-statistical Cauchy, soft sets, soft normed linear spaces.

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1. **Introduction**

For any non-decreasing sequence $\lambda = (\lambda_n)$ of positive reals with $\lambda_n \to \infty$, $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, the notion of the λ-statistical convergence was explored by Mursaleen [17] as a generalization of statistical convergence that was initially introduced by Fast [9] and Schoenberg [10] independently.

If we denote $I_n = [n - \lambda_n + 1, n]$, then the λ-density of any subset K of $\mathbb{N}$ is defined as follows.

"For $K \subseteq \mathbb{N}$, the λ-density of K is denoted by $\delta_\lambda(K)$ and is defined by

$$\delta_\lambda(K) = \lim_n \frac{1}{\lambda_n} |\{k \in I_n : k \in K\}|$$

provided the limit exists, where the vertical bars denote the cardinality of the enclosed set.

A sequence $x = (x_k)$ is said to be λ-statistical convergent to x_0 if for each $\epsilon > 0$,

$$\lim_n \frac{1}{\lambda_n} |\{k \in I_n : |x_k - x_0| \geq \epsilon\}| = 0,$$

i.e., $\delta_\lambda(K_\epsilon) = 0$, where $K_\epsilon = \{k \in I_n : |x_k - x_0| \geq \epsilon\}$. We write, in this case $S_\lambda - \lim_n x_k = x_0$."

Subsequently, statistical convergence and its generalizations have been developed by numerous authors including Hazarika et al.[5], Maddox[11], Fridy[12], Connor[13], Šalát[28], Kumar et al.[32] and many others.

On the other side, many problems of the real world are so complicated due to the uncertainty of data. Therefore, it is very difficult to model these problems mathematically via crisp set theory. So far we have many approaches including, the theory of probability, theory of rough sets[35], theory of fuzzy sets[16], theory of intuitionistic fuzzy sets[15], and theory of neutrosophic sets[7,8] to deal with such situations. In the present study, we are interested in the latter one i.e., the neutrosophic sets, which were initially introduced by Smarandache[7,8] as a generalization of fuzzy sets and intuitionistic fuzzy sets. He used the idea of indeterminacy function along with membership and non-membership functions to define a neutrosophic set. These sets have been further developed by numerous authors in [1], [14], [21], [22], [23], etc.

Kirişçi and Şimşek[18] used neutrosophic logic to define a new kind of norm, called neutrosophic norm and studied statistical convergence in neutrosophic normed linear spaces. Their pioneer work attracted many researchers to work in this direction and nowadays many interesting methods of summability theory have been extended in neutrosophic normed linear spaces. For a wide view in this direction, we refer to the reader [2], [3], [31].

Many approaches discussed above to minimize the uncertainty have their own drawbacks due to the inadequacy of the parametrization. In view of this, Molodtsov[6] proposed a new approach, called soft set theory to reduce the uncertainty during mathematical modelling. These sets turn out very useful tools in many areas of engineering and medical sciences. For instance: Maji et al.[20] applied the theory of soft sets in decision-making problems. Kong et al.[36] presented a heuristic algorithm of normal parameter reduction of soft sets. Zou and Xiao[33] presented a data analysis approach of soft sets under incomplete information. Recently, Yuksel et al.[24] applied soft set theory to diagnose the prostate cancer risk in human beings whereas Çelik and Yamak[34] applied fuzzy soft set theory for medical diagnosis using fuzzy arithmetic operations. Shabir and Naz[19] used soft sets to define soft topological spaces and studied some of their properties. However, Das et al.[25] defined soft normed linear spaces and investigated some of their properties. Recently, Bera and Mahapatra [29] united the concepts of softness and neutrosophic logic to define a generalized norm and called it as neutrosophic soft norm. They also studied some properties of $NSNLS$ and developed fundamental concepts of sequences in these spaces. In present study, we continue to define a more generalized convergence which we called S_λ-convergence in $NSNLS$. We also introduce the concepts of S_λ-Cauchy sequence, S_λ-completeness and develop some of their properties.

2. Preliminaries

This section starts with a brief information on soft sets, soft vector spaces and neutrosophic soft normed spaces. We begin with the following notations and definitions.

Throughout this work, $\mathbb{N}$ will denote the set of positive integers, $\mathbb{R}$ the set of reals and $\mathbb{R}^+$ the set of positive real numbers.

Definition 2.1 [4] A binary operation $\circ : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $\circ$ satisfies the following conditions:

(i) $x \circ y = y \circ x$ and $x \circ (y \circ z) = (x \circ y) \circ z$.

(ii) $\circ$ is continuous.

(iii) $x \circ 1 = 1 \circ x = x$ for all $x \in [0,1]$.

(iv) $w \circ x \leq y \circ z$ if $w \leq y, x \leq z$ with $w, x, y, z \in [0,1]$.

Definition 2.2 [4] A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is continuous t-conorm(s-norm) if $\diamond$ satisfies the following conditions:

(i) $x \diamond y = y \diamond x$ and $x \diamond (y \diamond z) = (x \diamond y) \diamond z$.

(ii) $\diamond$ is continuous.

(iii) $x \diamond 0 = 0 \diamond x = x$ for all $x \in [0,1]$.

(iv) $w \diamond x \leq y \diamond z$ if $w \leq y, x \leq z$ with $w, x, y, z \in [0,1]$.

For any universe set U and the set E of the parameters, the soft set is defined as follows:

Definition 2.3 [6] A pair (H, E) is called a soft set over U if and only if H is a mapping from E into the set of all subsets of the set U. i.e., the soft set is a parametrized family of subsets of the set U.

Moreover, every set $H(\epsilon), \epsilon \in E$, from this family may be considered as the set of ϵ-elements of the soft set (H, E), or as the set of ϵ-approximate elements of the set.

Definition 2.4 [6] A soft set (H, E) over U is said to be absolute soft set if for all $\epsilon \in E$, $H(\epsilon) = U$. We will denote it by $\widetilde{U}$.

Definition 2.5 [26] Let $\mathbb{R}$ be the set of real numbers, $B(\mathbb{R})$ be the collection of all non-empty bounded subsets of $\mathbb{R}$ and E taken as a set of parameters. Then a mapping $F : E \to B(\mathbb{R})$ is called a soft real set. If a soft real set is a singleton soft set, then it is called a soft real number and denoted by $\widetilde{r}, \widetilde{s}, \widetilde{t}$, etc. $\widetilde{0}, \widetilde{1}$ are the soft real numbers where $\widetilde{0}\ (e) = 0, \widetilde{1}\ (e) = 1$ for all $e \in E$ respectively.

Let $\mathbb{R}(E)$ and $\mathbb{R}^+(E)$ respectively denote the sets of all soft real numbers and all positive soft real numbers.

Definition 2.6 [27] Let (H, E) be a soft set over U. The set (H, E) is said to be a soft point, denoted by H_e^u if there is exactly one $e \in E$ s.t $H(e) = \{u\}$ for some $u \in U$ and $H(e') = \phi$ for all $e' \in E - \{e\}$.

Two soft points $H_e^u, H_{e'}^w$ are said to be equal if $e = e'$ and $u = w$. Let $\Delta_{\widetilde{U}}$ denotes the set of all soft points on $\widetilde{U}$.

In case U is a vector space over $\mathbb{R}$ and the parameter set $E = \mathbb{R}$, the soft point is called a soft vector.

Soft vector spaces are used to define soft norm as follows:

Definition 2.7 [30] Let $\widetilde{U}$ be a absolute soft vector space. Then a mapping $\| \cdot \| : \widetilde{U} \to \mathbb{R}^+(E)$ is said to be a soft norm on $\widetilde{U}$, if $\| \cdot \|$ satisfies the following conditions:

(i) $\|u_e\| \geq \widetilde{0}$ for all $u_e \in \widetilde{U}$ and $\|u_e\| = \widetilde{0} \Leftrightarrow u_e = \widetilde{\theta}_0$ where $\widetilde{\theta}_0$ denotes the zero element of $\widetilde{U}$.

(ii) $\| \widetilde{\alpha}\, u_e\| = |\widetilde{\alpha}| \|u_e\|$ for all $u_e \in \widetilde{U}$ and for every soft scalar $\widetilde{\alpha}$.

(iii) $\|u_e + v_{e'}\| \leq \|u_e\| + \|v_{e'}\|$ for all $u_e, v_{e'} \in \widetilde{U}$.

(iv) $\|u_e \cdot v_{e'}\| = \|u_e\| \, \|v_{e'}\|, \forall\, u_e, v_{e'} \in \widetilde{U}$.

The soft vector space $\widetilde{U}$ with a soft norm $\| \cdot \|$ on $\widetilde{U}$ is said to be a soft normed linear space and is denoted by $(\widetilde{U}, \| \cdot \|)$.

We now recall the definition of neutrosophic soft normed linear spaces and the convergence structure in these spaces.

Definition 2.8 [29] Let $\widetilde{U}$ be a soft linear space over the field F and $\mathbb{R}(E), \Delta_{\widetilde{U}}$ denote respectively, the set of all soft real numbers and the set of all soft points on $\widetilde{U}$. Then a neutrosophic subset N over $\Delta_{\widetilde{U}} \times \mathbb{R}(E)$ is called a neutrosophic soft norm on $\widetilde{U}$ if for $u_e, v_{e'} \in \widetilde{U}$ and $\widetilde{\alpha} \in F$ ($\widetilde{\alpha}$ being soft scalar), the following conditions hold.

(i) $0 \leq G_N(u_e, \widetilde{\eta_1}), B_N(u_e, \widetilde{\eta_1}), Y_N(u_e, \widetilde{\eta_1}) \leq 1, \forall\, \widetilde{\eta_1} \in \mathbb{R}(E)$.

(ii) $0 \leq G_N(u_e, \widetilde{\eta_1}) + B_N(u_e, \widetilde{\eta_1}) + Y_N(u_e, \widetilde{\eta_1}) \leq 3, \forall\, \widetilde{\eta_1} \in \mathbb{R}(E)$.

(iii) $G_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$.

(iv) $G_N(u_e, \widetilde{\eta_1}) = 1,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(v) $G_N(\widetilde{\alpha}\, u_e, \widetilde{\eta_1}) = G_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall\, \widetilde{\alpha}\, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(vi) $G_N(u_e, \widetilde{\eta_1}) \circ G_N(v_{e'}, \widetilde{\eta_2}) \leq G_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2}), \forall\, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$

(vii) $G_N(u_e, \cdot)$ is continuous non-decreasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} G_N(u_e, \widetilde{\eta_1}) = 1$.

(viii) $B_N(u_e, \widetilde{\eta_1}) = 1$ with $\widetilde{\eta_1} \leq \widetilde{0}$.

(ix) $B_N(u_e, \widetilde{\eta_1}) = 0,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(x) $B_N(\widetilde{\alpha}\, u_e, \widetilde{\eta_1}) = B_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall\, \widetilde{\alpha}\, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(xi) $B_N(u_e, \widetilde{\eta_1}) \diamond B_N(v_{e'}, \widetilde{\eta_2}) \geq B_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2})\ \forall\, \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.

(xii) $B_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

(xiii) $Y_N(u_e, \widetilde{\eta_1}) = 0$ with $\widetilde{\eta_1} \leq \widetilde{0}$.

(xiv)$Y_N(u_e, \widetilde{\eta_1}) = 0,$ with $\widetilde{\eta_1} > \widetilde{0}$ if and only if $u_e = \widetilde{\theta}$, the null soft vector.

(xv) $Y_N(\widetilde{\alpha}\, u_e, \widetilde{\eta_1}) = Y_N\left(u_e, \frac{\widetilde{\eta_1}}{|\widetilde{\alpha}|}\right), \forall\, \widetilde{\alpha}\, (\neq \widetilde{0}), \widetilde{\eta_1} > \widetilde{0}$.

(xvi) $Y_N(u_e, \widetilde{\eta_1}) \diamond Y_N(v_{e'}, \widetilde{\eta_2}) \geq Y_N(u_e \oplus v_{e'}, \widetilde{\eta_1} \oplus \widetilde{\eta_2})\ \forall\ \widetilde{\eta_1}, \widetilde{\eta_2} \in \mathbb{R}(E)$.
(xvii) $Y_N(u_e, \cdot)$ is continuous non-increasing function for $\widetilde{\eta_1} > \widetilde{0}$ and $\lim_{\widetilde{\eta_1} \to \infty} B_N(u_e, \widetilde{\eta_1}) = 0$.

In this case $\mathcal{N} = (G_N, B_N, Y_N)$ is called the neutrosophic soft norm and $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is an neutrosophic soft normed linear space ($NSNLS$ briefly).

Let $(\widetilde{U}, \|\cdot\|)$ be a soft normed space. Take the operations $\circ$ and $\diamond$ as $x \circ y = xy$; $x \diamond y = x + y - xy$. For $\widetilde{\eta} > \widetilde{0}$, define

$$G_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\widetilde{\eta}}{\widetilde{\eta} + \|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$B_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta} + \|u_e\|} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise} \end{cases}$$

$$Y_N(u_e, \widetilde{\eta}) = \begin{cases} \frac{\|u_e\|}{\widetilde{\eta}} \text{ if } \widetilde{\eta} > \|u_e\| \\ 0 \text{ otherwise,} \end{cases}$$

then $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$ is an $NSNLS$. From now onwards, unless otherwise stated by $\widetilde{V}$ we shall denote the $NSNLS$ $(\widetilde{U}(F), G_N, B_N, Y_N, \circ, \diamond)$.

A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be convergent to a soft point $v_e \in \widetilde{V}$ if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}$ $\exists\ n_0 \in \mathbb{N}$ s.t $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \epsilon,\ B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. In this case, we write $\lim_{k \to \infty} v^k_{e_k} = v_e$. A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be Cauchy sequence if for $0 < \epsilon < 1$ and $\widetilde{\eta} > \widetilde{0}$ $\exists\ n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$ $G_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) > 1 - \epsilon,\ B_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon,\ Y_N(v^k_{e_k} \ominus v^p_{e_p}, \widetilde{\eta}) < \epsilon$.

Throughout this paper, $\oplus$ and $\ominus$ denote the sum and difference of soft points respectively.

3. λ-Statistical convergence in NSNLS

In this section, we define λ-statistical convergence in neutrosophic soft normed linear spaces and develop some of its properties.

Definition 3.1 A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be λ-statistical convergent or S_λ-convergent to a soft point v_e in $\widetilde{V}$ if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$,

$$\lim_{n \to \infty} \frac{1}{\lambda_n} \left| \left\{ k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon \right\} \right| = 0,$$

i.e., $\delta_\lambda(K) = 0$ where

$$K = \{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

In this case, we write $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$.

Let, $S_\lambda(G_N, B_N, Y_N)$ denotes the set of all sequences of soft points in $\widetilde{V}$ which are S_λ-convergent with respect to the neutrosophic soft norm (G_N, B_N, Y_N).

Remark 3.1 Since the λ-density of a finite set is zero, therefore, a convergent sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is S_λ-statistical convergent to the same limit. However, the converse may not be true in general.

Remark 3.2 For particular choice $\lambda_n = n$, S_λ-convergence coincides with statistical convergence in neutrosophic soft normed linear space.

Exapmle 3.1 Let $(\widetilde{\mathbb{R}}, \|\cdot\|)$ be a soft normed linear space. For v_e in $\widetilde{\mathbb{R}}$ and $\widetilde{\eta} > \widetilde{0}$, if we define

$$G_N(v_e, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v_e\|},\ B_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta} \oplus \|v_e\|},\ Y_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta}},$$

$x \circ y = xy$ and $x \diamond y = \min\{x + y, 1\}$, then it is easy to see that $\widetilde{V} = (\widetilde{\mathbb{R}}, G_N, B_N, Y_N, \circ, \diamond)$ is a neutrosophic soft normed linear space.

Now define a sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ by

$$v^k_{e_k} = \begin{cases} \widetilde{k} & \text{if } n - [\sqrt{\lambda_n}] + 1 \le k \le n, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

Now, for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, let

$$\begin{aligned} A(\epsilon, \widetilde{\eta}) &= \left\{k \in I_n : G_N(v^k_{e_k}, \widetilde{\eta}) \le 1 - \epsilon \text{ or } B_N(v^k_{e_k}, \widetilde{\eta}) \ge \epsilon, Y_N(v^k_{e_k}, \widetilde{\eta}) \ge \epsilon\right\} \\ &= \left\{k \in I_n : \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \le 1 - \epsilon \text{ or } \frac{\|v^k_{e_k}\|}{\widetilde{\eta} \oplus \|v^k_{e_k}\|} \ge \epsilon, \frac{\|v^k_{e_k}\|}{\widetilde{\eta}} \ge \epsilon\right\} \\ &= \left\{k \in I_n : \|v^k_{e_k}\| \ge \frac{\widetilde{\eta}\,\epsilon}{1 - \epsilon} \text{ or } \|v^k_{e_k}\| \ge \widetilde{\eta}\,\epsilon\right\} \\ &= \left\{k \in I_n : v^k_{e_k} = \widetilde{k}\right\} \\ &= \left\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \le k \le n\right\} \end{aligned}$$

and so we get

$$\frac{1}{\lambda_n}|A(\epsilon, \widetilde{\eta})| = \frac{1}{\lambda_n}|\{k \in I_n : n - [\sqrt{\lambda_n}] + 1 \le k \le n\}| \le \frac{\sqrt{\lambda_n}}{\lambda_n}.$$

Taking $n \to \infty$,

$\lim_{n\to\infty} \frac{1}{\lambda_n}|A(\epsilon, \widetilde{\eta})| \le \lim_{n\to\infty} \frac{\sqrt{\lambda_n}}{\lambda_n} = 0$, i.e., $\delta_\lambda(A(\epsilon, \widetilde{\eta})) = 0$.

This shows that, $v = (v^k_{e_k})$ is λ-statistically convergent to $\widetilde{0}$. But by the structure of the sequence, $v = (v^k_{e_k})$ is not (G_N, B_N, Y_N)-convergent to $\widetilde{0}$.

Lemma 3.1 For any sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$, the following statements are equivalent:

(i) $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$;

(ii) $\delta_\lambda\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon\} = \delta_\lambda\{k \in I_n : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = \delta_\lambda\{k \in I_n : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\} = 0$;

(iii) $\delta_\lambda\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(iv)$\delta_\lambda\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon\} = \delta_\lambda\{k \in I_n : B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = \delta_\lambda\{k \in I_n : Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon\} = 1$;

(v) $S_\lambda - \lim_{k\to\infty} G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 1$ and $S_\lambda - \lim_{k\to\infty} B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$, $S_\lambda - \lim_{k\to\infty} Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) = 0$.

Proof. Omitted.□

Theorem 3.2 For any sequence $v = (v^k_{e_k})$ in $\widetilde{V}$, if $S_\lambda - \lim_{k\to\infty} v^k_{e_k}$ exists, then it is unique.

Proof. Suppose that $S_\lambda - \lim_{n\to\infty} v^k_{e_k} = v_{e_1}$ and $S_\lambda - \lim_{n\to\infty} v^k_{e_k} = v'_{e_2}$, where $v_{e_1} \neq v'_{e_2}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\epsilon_1 > 0$ s.t.

$$(1-\epsilon_1) \circ (1-\epsilon_1) > 1-\epsilon \text{ and } \epsilon_1 \Diamond \epsilon_1 < \epsilon \tag{1}$$

Define the following sets:

$$A_{G_N,1}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : G_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\epsilon_1\right\}.$$

$$A_{G_N,2}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : G_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \leq 1-\epsilon_1\right\}.$$

$$A_{B_N,1}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : B_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

$$A_{B_N,2}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : B_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

$$A_{Y_N,1}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : Y_N\left(v^k_{e_k} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

$$A_{Y_N,2}(\epsilon_1, \widetilde{\eta}) = \left\{k \in I_n : Y_N\left(v^k_{e_k} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

Since $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_{e_1}$, so

$\delta_\lambda\{A_{G_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{B_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{Y_N,1}(\epsilon_1, \widetilde{\eta})\} = 0$ and therefore $\delta_\lambda\{A^C_{G_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{B_N,1}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{Y_N,1}(\epsilon_1, \widetilde{\eta})\} = 1$.

Further, $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v'_{e_2}$, so

$\delta_\lambda\{A_{G_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{B_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A_{Y_N,2}(\epsilon_1, \widetilde{\eta})\} = 0$ and therefore $\delta_\lambda\{A^C_{G_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{B_N,2}(\epsilon_1, \widetilde{\eta})\} = \delta_\lambda\{A^C_{Y_N,2}(\epsilon_1, \widetilde{\eta})\} = 1$ for all $\widetilde{\eta} > \widetilde{0}$. Define

$$\begin{aligned} K_{G_N,B_N,Y_N}(\epsilon, \widetilde{\eta}) = &\{A_{G_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{G_N,2}(\epsilon_1, \widetilde{\eta})\} \\ &\cap\{A_{B_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{B_N,2}(\epsilon_1, \widetilde{\eta})\} \cap \{A_{Y_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{Y_N,2}(\epsilon_1, \widetilde{\eta})\}, \end{aligned}$$

then $\delta_\lambda\{K_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 0$ and therefore, $\delta_\lambda\{K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\} = 1$. Let $m \in K^C_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})$, then we have following possibilities.

1. $m \in \left\{A_{G_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{G_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$;
2. $m \in \left\{A_{B_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{B_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$;
3. $m \in \left\{A_{Y_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{Y_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$.

Case 1: Let $m \in \left\{A_{G_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{G_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$, then $m \in A^C_{G_N,1}(\epsilon_1,\widetilde{\eta})$ and $m \in A^C_{G_N,2}(\epsilon_1,\widetilde{\eta})$ and therefore,

$$G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) > 1-\epsilon_1 \text{ and } G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) > 1-\epsilon_1. \tag{2}$$

Now

$$\begin{aligned} G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= G_N\left(v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &= G_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\epsilon_1) \circ (1-\epsilon_1) \qquad \text{by (2)} \\ &> 1-\epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $G_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 1$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} \ominus v'_{e_2} = \widetilde{\theta}$, i.e., $v_{e_1} = v'_{e_2}$.

Case 2: Let $m \in \left\{A_{B_N,1}(\epsilon_1,\widetilde{\eta}) \cup A_{B_N,2}(\epsilon_1,\widetilde{\eta})\right\}^C$, then $m \in A^C_{B_N,1}(\epsilon_1,\widetilde{\eta}))$ and $m \in A^C_{B_N,2}(\epsilon_1,\widetilde{\eta})$ and therefore,

$$B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1 \text{ and } B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1. \tag{3}$$

Now

$$\begin{aligned} B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= B_N\left(v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &= B_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \qquad \text{by (3)} \\ &< \epsilon. \qquad \text{by (1)} \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $B_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} \ominus v'_{e_2} = \widetilde{\theta}$, i.e., $v_{e_1} = v'_{e_2}$.

Case 3: Let $m \in \left\{A_{Y_N,1}(\epsilon_1, \widetilde{\eta}) \cup A_{Y_N,2}(\epsilon_1, \widetilde{\eta})\right\}^C$, then $m \in A^C_{Y_N,1}(\epsilon_1, \widetilde{\eta})$ and $m \in A^C_{Y_N,2}(\epsilon_1, \widetilde{\eta})$ and therefore,

$$Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1 \text{ and } Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1. \tag{4}$$

Now

$$\begin{aligned}
Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) &= Y_N\left(v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\
&= Y_N\left(v^m_{e_m} \ominus v^m_{e_m} \oplus v_{e_1} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\
&\leq Y_N\left(v^m_{e_m} \ominus v_{e_1}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^m_{e_m} \ominus v'_{e_2}, \frac{\widetilde{\eta}}{2}\right) \\
&< \epsilon_1 \diamond \epsilon_1 \qquad \text{by (4)} \\
&< \epsilon. \qquad\qquad \text{by (1)}
\end{aligned}$$

Since $\epsilon > 0$ is arbitrary, so we have $Y_N(v_{e_1} \ominus v'_{e_2}, \widetilde{\eta}) = 0$ for all $\widetilde{\eta} > \widetilde{0}$, which gives $v_{e_1} \ominus v'_{e_2} = \widetilde{\theta}$, i.e., $v_{e_1} = v'_{e_2}$.

Hence, in all cases we have $v_{e_1} = v'_{e_2}$, i.e., the λ-statistical limit of $(v^k_{e_k})$ is unique.□

Theorem 3.3 Let $u = (u^k_{e_k})$ and $v = (v^k_{e_k})$ be any two sequences in $\widetilde{V}$ s.t $S_\lambda - \lim_{k\to\infty}(u^k_{e_k}) = u_{e_1}$ and $S_\lambda - \lim_{k\to\infty}(v^k_{e_k}) = v_{e_2}$. Then

(i)$S_\lambda - \lim_{k\to\infty}(u^k_{e_k} \oplus\ v^k_{e_k}) = u_{e_1} \oplus\ v_{e_2}$

(ii) $S_\lambda - \lim_{k\to\infty}(\widetilde{\alpha}\ u^k_{e_k}) = \widetilde{\alpha}\ u_{e_1}$, where $\widetilde{0} \neq \widetilde{\alpha} \in F$.

Proof. Omitted.□

Theorem 3.4 A sequence $v = (v^k_{e_k})$ in $\widetilde{V}$ is λ-statistically convergent, if and only if $\exists$ a subset $K = \{k_1, k_2, k_3, ...\}$ of $\mathbb{N}$ s.t $\delta_\lambda(K) = 1$ and $(G_N, B_N, Y_N) - \lim_{\substack{k\in K \\ k\to\infty}} v^k_{e_k} = v_e$.

Proof. First suppose that $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$. For $\widetilde{\eta} > \widetilde{0}$ and $p \in \mathbb{N}$, define the set

$$\begin{aligned}
K_{G_N,B_N,Y_N}(p, \widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1 - \frac{1}{p} \text{ and} \\
B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \frac{1}{p}, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \frac{1}{p}\Big\} \text{ and} \\
K^C_{G_N,B_N,Y_N}(p, \widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \frac{1}{p} \text{ or} \\
B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \frac{1}{p}, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \frac{1}{p}\Big\}.
\end{aligned}$$

Since $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = v_e$, it follows that $\delta_\lambda(K^C_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 0$. Furthermore, for $\widetilde{\eta} > \widetilde{0}$ and $p \in \mathbb{N}$, we observe $K_{G_N,B_N,Y_N}(p,\widetilde{\eta}) \supset K_{G_N,B_N,Y_N}(p+1,\widetilde{\eta})$ and

$$\delta_\lambda(K_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 1. \tag{5}$$

Now, we have to show that, for $k \in K_{G_N,B_N,Y_N}(p,\widetilde{\eta})$, $(G_N,B_N,Y_N) - \lim_{\substack{k\in K\\k\to\infty}} v^k_{e_k} = v_e$. Suppose for $k \in K_{G_N,B_N,Y_N}(p,\widetilde{\eta})$, $(v^k_{e_k})$ is not convergent to v_e w.r.t (G_N,B_N,Y_N). Then $\exists$ some $q > 0$ s.t $\{k \in \mathbb{N} : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-q$ or $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq q, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq q\}$ for infinitely many terms of the sequence $v = (v^k_{e_k})$. If we take

$$K_{G_N,B_N,Y_N}(q,\widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-q \text{ and } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < q, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < q\Big\}$$

and choose $q > \frac{1}{p}$ where $p \in \mathbb{N}$, then we have $\delta_\lambda(K_{G_N,B_N,Y_N}(q,\widetilde{\eta})) = 0$. Further, $K_{G_N,B_N,Y_N}(p,\widetilde{\eta}) \subset K_{G_N,B_N,Y_N}(q,\widetilde{\eta})$ implies that $\delta_\lambda(K_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 0$. In this way we obtained a contradiction to (5) as $\delta_\lambda(K_{G_N,B_N,Y_N}(p,\widetilde{\eta})) = 1$. Hence, $(G_N,B_N,Y_N) - \lim_{\substack{k\in K\\k\to\infty}} v^k_{e_k} = v_e$.

Conversely, Suppose $\exists$ a subset $K = \{k_1,k_2,...,k_j,...\}$ of $\mathbb{N}$ with $\delta_\lambda(K) = 1$ and $(G_N,B_N,Y_N) - \lim_{k\to\infty} v^k_{e_k} = v_e$ over K i.e., $(G_N,B_N,Y_N) - \lim_{\substack{k\in K\\k\to\infty}} v^k_{e_k} = v_e$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists\ k_{j_0} \in \mathbb{N}$ s.t for all $k_j \geq k_{j_0}$, $G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) > 1-\epsilon$ and $B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) < \epsilon$. So if we consider the set

$$T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta}) = \Big\{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1-\epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\Big\},$$

then it is easy to see that $T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta}) \subset \mathbb{N} - \{k_{j_0}, k_{j_0+1}, k_{j_0+2}, ...\}$. This immediately implies that $\delta_\lambda\Big(T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\Big) \leq \delta_\lambda(\mathbb{N}) - \delta_\lambda(\{k_{j_0}, k_{j_0+1}, k_{j_0+2}, ...\}) = 1-1 = 0$ and therefore $\delta_\lambda\Big(T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\Big) = 0$ as $\delta_\lambda\Big(T_{G_N,B_N,Y_N}(\epsilon,\widetilde{\eta})\Big)$ can not be negative. This shows that $v = (v^k_{e_k})$ is λ-statistical convergent to v_e i.e., $S_\lambda - \lim_{n\to\infty} v^k_{e_k} = v_e$.$\square$

4. λ-Statistical Cauchy sequence in NSNLS

Definition 4.1 A sequence $v = (v^k_{e_k})$ of soft points in $\widetilde{V}$ is said to be λ-statistically Cauchy if for each $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$, $\exists\ n_0 \in \mathbb{N}$ s.t for all $k, p \geq n_0$

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{k\in I_n : G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\right\}\right|=0,$$

or equivalently, the λ-density of the set K is zero, i.e., $\delta_\lambda(K)=0$ where

$$K=\{k\in I_n : G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\}.$$

Theorem 4.1 For any sequence $v=(v^k_{e_k})$ in $\widetilde{V}$, the following are equivalent:
(i) $v=(v^k_{e_k})$ is λ-statistically Cauchy.
(ii) $\exists$ a subset $K=\{k_1,k_2,...,k_j,...\}$ of $\mathbb{N}$ with $\delta_\lambda(K)=1$ and $(v^{k_j}_{e_{k_j}})$ is Cauchy sequence over K.

Proof. Omitted.□

Theorem 4.2 Every λ-statistically convergent sequence of soft points in $\widetilde{V}$ is λ-statistically Cauchy.

Proof. Let $v=(v^k_{e_k})$ be any λ-statistically convergent sequence with $S_\lambda-\lim_{k\to\infty} v^k_{e_k}=v_e$. Let $\epsilon>0$ and $\widetilde{\eta}>\widetilde{0}$. Choose $\epsilon_1>0$ s.t (1) is satisfied. Define a set,

$$M(\epsilon_1,\widetilde{\eta})=\left\{k\in I_n : G_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\leq 1-\epsilon_1 \text{ or } B_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\geq\epsilon_1, Y_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})\geq\epsilon_1\right\},$$

then

$$M^C(\epsilon_1,\widetilde{\eta})=\left\{k\in I_n : G_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})> 1-\epsilon_1 \text{ and } B_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})<\epsilon_1, Y_N(v^k_{e_k}\ominus v_e,\frac{\widetilde{\eta}}{2})<\epsilon_1\right\}.$$

Since $S_\lambda-\lim_{n\to\infty} v^k_{e_k}=v_e$, so $\delta_\lambda(M(\epsilon_1,\widetilde{\eta}))=0$ and $\delta_\lambda(M^C(\epsilon_1,\widetilde{\eta}))=1$. Let $p\in M^C(\epsilon_1,\widetilde{\eta})$, then

$$G_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)>1-\epsilon_1 \text{ and } B_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)<\epsilon_1, Y_N\left(v^p_{e_p}\ominus v_e,\frac{\widetilde{\eta}}{2}\right)<\epsilon_1. \quad (6)$$

Now, let $T(\epsilon,\widetilde{\eta})=\{k\in I_n : G_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon$ or $B_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^k_{e_k}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon\}$, then we show that $T(\epsilon,\widetilde{\eta})\subseteq M(\epsilon_1,\widetilde{\eta})$. Let $m\in T(\epsilon,\widetilde{\eta})$, then

$$G_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\leq 1-\epsilon \text{ or } B_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon, Y_N(v^m_{e_m}\ominus v^p_{e_p},\widetilde{\eta})\geq\epsilon. \quad (7)$$

Case 1: If $G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \leq 1-\epsilon$, then $G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1-\epsilon_1$ and therefore $m \in M(\epsilon_1, \widetilde{\eta})$. As otherwise i.e., if $G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) > 1-\epsilon_1$, then by (1), (6) and (7) we get

$$\begin{aligned} 1-\epsilon \geq G_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v_e \oplus \ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1-\epsilon_1) \circ (1-\epsilon_1) > 1-\epsilon, \end{aligned}$$

which is not possible. Thus, $T(\epsilon, \widetilde{\eta}) \subseteq M(\epsilon_1, \widetilde{\eta})$.
Case 2: If $B_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon$, then $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1$ and therefore $m \in M(\epsilon_1, \widetilde{\eta})$. As otherwise i.e., if $B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1$, then by (1), (6) and (7) we get

$$\begin{aligned} \epsilon \leq B_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v_e \oplus \ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 < \epsilon, \end{aligned}$$

which is not possible.
Also, If $Y_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) \geq \epsilon$, then $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1$ and therefore $m \in M(\epsilon_1, \widetilde{\eta})$. As otherwise i.e., if $Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1$, then by (1), (6) and (7) we get

$$\begin{aligned} \epsilon \leq Y_N(v^m_{e_m} \ominus v^p_{e_p}, \widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v_e \oplus \ v_e \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v^m_{e_m} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^p_{e_p} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 < \epsilon, \end{aligned}$$

which is not possible. Thus, $T(\epsilon, \widetilde{\eta}) \subseteq M(\epsilon_1, \widetilde{\eta})$.
Hence in all cases, $T(\epsilon, \widetilde{\eta}) \subseteq M(\epsilon_1, \widetilde{\eta})$. Since $\delta_\lambda(M(\epsilon_1, \widetilde{\eta})) = 0$, so $\delta_\lambda(T(\epsilon, \widetilde{\eta})) = 0$, and therefore $v = (v^k_{e_k})$ is λ-statistically Cauchy.□

Example 4.1 Let $R_1 = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $\| \cdot \| = |\cdot|$ i.e., the usual norm on R_1, then $(R_1, |\cdot|)$ is a normed linear space. For $\widetilde{\eta} > \widetilde{0}$, if we define $G_N(v_e, \widetilde{\eta}) = \frac{\widetilde{\eta}}{\widetilde{\eta} \oplus \|v_e\|}$; $B_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta} \oplus \|v_e\|}$; $Y_N(v_e, \widetilde{\eta}) = \frac{\|v_e\|}{\widetilde{\eta}}$; $x \circ y = xy$ and $x \diamond y = x + y - xy$, then it is easy to see that $(\widetilde{R_1}\ (\mathbb{R}), G_N, B_N, Y_N, \circ, \diamond)$ is a neutrosophic soft normed linear space.
If we define a sequence of soft points $v = (v^k_{e_k})$ by $v^k_{e_k} = \widetilde{\frac{1}{k}}$ and select $\lambda_n = n$ then $(v^k_{e_k})$ is

λ-statistical Cauchy and $S_\lambda - \lim_{k\to\infty} v^k_{e_k} = \widetilde{0}$ but $\widetilde{0}$ is not a member of the space.

Theorem 4.3 If $(v^k_{e_k})$, $(w^k_{e_k})$ are λ-statistical Cauchy sequences of soft vectors and $(\widetilde{\alpha}_k)$ is a λ-statistical Cauchy sequence of soft scalars in $\widetilde{V}$, then $(v^k_{e_k} \oplus \ w^k_{e_k})$ and $(\widetilde{\alpha}_k w^k_{e_k})$ are also λ-statistical Cauchy in $\widetilde{V}$.

Proof. Omitted.□

Definition 4.2 A $NSNLS$ $\widetilde{V}$ is said to be λ-statistically complete if every λ-statistical Cauchy sequence in $\widetilde{V}$ is λ-statistical convergent w.r.t (G_N, B_N, Y_N).

Theorem 4.4 If every λ-statistical Cauchy sequence of soft points in $\widetilde{V}$ has a λ-statistical convergent subsequence then $\widetilde{V}$ is λ-statistically complete.

Proof. Let $v = (v^k_{e_k})$ be any λ-statistically Cauchy sequence of soft points in $\widetilde{V}$ which has a λ-statistical convergent subsequence $(v^{k(j)}_{e_{k(j)}})$ i.e., $S_\lambda - \lim_{j\to\infty} v^{k(j)}_{e_{k(j)}} = v_e$ for some v_e in $\widetilde{V}$. Let $\epsilon > 0$ and $\widetilde{\eta} > \widetilde{0}$. Choose $\epsilon_1 > 0$ s.t (1) is satisfied. Since $v = (v^k_{e_k})$ is λ-statistically Cauchy, so $\exists\ n_0 \in \mathbb{N}$ s.t $\forall\ k, p \geq n_0$, $\delta_\lambda(A) = 0$ where

$$A = \left\{k \in I_n : G_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \epsilon_1 \text{ or } B_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1, Y_N\left(v^k_{e_k} \ominus v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

Again since $S_\lambda - \lim_{j\to\infty} v^{k(j)}_{e_{k(j)}} = v_e$. So we have $\delta_\lambda(B) = 0$, where

$$B = \left\{k(j) \in I_n : G_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \leq 1 - \epsilon_1 \text{ or } B_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1, Y_N\left(v^{k(j)}_{e_{k(j)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \geq \epsilon_1\right\}.$$

Now define

$$D = \{k \in I_n : G_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \leq 1 - \epsilon \text{ or } B_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon, Y_N(v^k_{e_k} \ominus v_e, \widetilde{\eta}) \geq \epsilon\}.$$

We now show that $A^C \cap B^C \subseteq D^C$. Let $m \in A^C \cap B^C$. As $m \in A^C$, so

$$G_N\left(v^m_{e_m} \ominus\ v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) > 1 - \epsilon_1 \text{ and } B_N\left(v^m_{e_m} \ominus\ v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1, Y_N\left(v^m_{e_m} \ominus\ v^p_{e_p}, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1, \tag{8}$$

and since $m \in B^C$, so $m = k(j_0)$ for $j_0 \in \mathbb{N}$ and

$$G_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus \ v_e, \frac{\widetilde{\eta}}{2}\right) > 1 - \epsilon_1 \text{ and}$$
$$B_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus \ v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1, Y_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus \ v_e, \frac{\widetilde{\eta}}{2}\right) < \epsilon_1. \tag{9}$$

Now

$$\begin{aligned} G_N(v^m_{e_m} \ominus v_e, \widetilde{\eta}) &= G_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}} \oplus v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\geq G_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}}, \frac{\widetilde{\eta}}{2}\right) \circ G_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &> (1 - \epsilon_1) \circ (1 - \epsilon_1) \quad \text{for } p = k(j_0) \\ &> 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} B_N(v^m_{e_m} \ominus v_e, \widetilde{\eta}) &= B_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}} \oplus v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq B_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}}, \frac{\widetilde{\eta}}{2}\right) \diamond B_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \quad \text{for } p = k(j_0) \\ &< \epsilon, \end{aligned}$$

$$\begin{aligned} Y_N(v^m_{e_m} \ominus v_e, \widetilde{\eta}) &= Y_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}} \oplus v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2} \oplus \frac{\widetilde{\eta}}{2}\right) \\ &\leq Y_N\left(v^m_{e_m} \ominus v^{k(j_0)}_{e_{k(j_0)}}, \frac{\widetilde{\eta}}{2}\right) \diamond Y_N\left(v^{k(j_0)}_{e_{k(j_0)}} \ominus v_e, \frac{\widetilde{\eta}}{2}\right) \\ &< \epsilon_1 \diamond \epsilon_1 \quad \text{for } p = k(j_0) \\ &< \epsilon, \qquad \text{by (1), (8) and (9)} \end{aligned}$$

which implies that $m \in D^C$, so $A^C \cap B^C \subseteq D^C$ or $D \subseteq A \cup B$. Therefore, $\delta_\lambda(D) \leq \delta_\lambda(A \cup B) = 0$. This shows that $v = (v^k_{e_k})$ is λ-statistically convergent and therefore, $\widetilde{V}$ is λ-statistically complete.□

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