On Symbolic Plithogenic Algebraic Structures and Hyper Structures

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Abstract. The objective of this paper is to study symbolic Plithogenic algebraic structures and hyper structures. In particular, we study symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup.

Keywords: Plithogeny; Plithogenic; Plithogenic Set; Fuzzy Set; Intuitionistic Fuzzy Set; Neutrosophy; Neutrosophic Set; Plithogenic Group; Plithogenic Ring; Plithogenic Hypergroup; Plithogenic canonical Hypergroup.

1. Introduction

The concepts of Plithogeny, Plithogenic logic/set, Plithogenic probability and Plithogenic statistics were introduced by Smarandache in [26]. Plithogenic set/logic is an extension of the classical logic/set, fuzzy logic/set of Zadeh [37], intuitionistic fuzzy logic/set of Atanassov [11], neutrosophic logic/set of Smarandache [30] and quadruple neutrosophic logic/set of Smarandache [29]. Smarandache in [23], [25] and [28] introduced and studied symbolic Plithogenic algebraic structures and hyper structures. In [22], Merkepsi and Abobala studied symbolic 2-Plithogenic rings, in [10], Al-Basheer et al. studied symbolic 3-Plithogenic rings and in [17], Gayen et al. studied Plithogenic Hypersoft Subgroup. Also in [32], Taffach and Hatip studied Symbolic...
2-Plithogenic Number Theory And Algebraic Equations, in [33], Taffach and Othman studied Symbolic 2-Plithogenic Modules over Symbolic 2-Plithogenic Rings and in [34], Taffach studied Symbolic 2-Plithogenic Vector Spaces. In [18], [24] and [27], applications of Plithogenic set/logic were presented. In the present paper, we study symbolic Plithogenic algebraic structures and hyper structures. In particular, we study symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup and present their basic properties.

2. Symbolic Plithogenic Set

A symbolic Plithogenic set \( SPX \) is defined by

\[
SPX = \{(a, a_1P_1, a_2P_2, a_3P_3, \ldots , anP_n) : a, a_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\}
\]  

(1)

where \( P_i \)'s are the Plithogenic parameters/variables. \( a \) is called the non-Plithogenic part of \( SPX \), \( a_iP_i \) is called the Plithogenic part of \( SPX \) and \( a_i \)'s are called the coefficients of \( P_i \)'s where \( i = 1, 2, 3, \ldots , n \). For a positive integer \( k \), \( P_i \) has the following properties:

\[
P_i^k = P_i, \quad \forall i \text{ and } k \geq 2,
\]

(2)

\[
kP_i = P_i + P_i + P_i + \cdots + P_i \text{ [k summand]} \quad \forall i,
\]

(3)

\[
0P_i = 0 \quad \forall i,
\]

(4)

\[
P_i^{-1} = \frac{1}{P_i} \text{ does not exist } \quad \forall i.
\]

(5)

when \( n = 1 \), equation (1) reduces to

\[
SPX = \{(a, a_1P_1) : a, a_1 \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\}
\]

(6)

and \( SPX \) becomes the usual Neutrosophic set with \( P_1 = I \).

When \( n = 3 \), equation (1) reduces to

\[
SPX = \{(a, a_1P_1, a_2P_2, a_3P_3) : a, x_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\}
\]

(7)

and \( SPX \) becomes the usual Neutrosophic Quadruple set with \( P_1 = T, \ P_2 = I \) and \( P_3 = F \).

When \( n = 2 \), equation (1) reduces to

\[
SPX = \{(a, a_1P_1, a_2P_2) : a, a_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\}
\]

(8)

which is called symbolic 2-Plithogenic set.

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3. Symbolic Plithogenic Algebraic Structure

All the symbolic Plithogenic sets to be considered in this section and the section after will be symbolic 2-Plithogenic sets of the form given by equation (8) and we are going to assume throughout the prevalence order $P_1 > P_2$ so that

\[ P_1 P_1 = P_{\min\{1,1\}} = P_1, \quad (9) \]
\[ P_2 P_2 = P_{\min\{2,2\}} = P_2, \quad (10) \]
\[ P_1 P_2 = P_2 P_1 = P_{\min\{1,2\}} = P_1. \quad (11) \]

**Definition 3.1.** Let $+$, $-$ and $.$ be the usual arithmetic operations of addition, subtraction and multiplication of numbers respectively and let $k$ be a nonzero scalar. If $x = (a, a_1 P_1, a_2 P_2)$ and $y = (b, b_1 P_1, b_2 P_2)$ are arbitrary elements of the symbolic Plithogenic set $SPX$ where $a, b, a_i, b_i \in \mathbb{R}$ or $\mathbb{C}$, then:

\[ x \pm y = (a \pm b, (a_1 \pm b_1) P_1, (a_2 \pm b_2) P_2), \quad (12) \]
\[ kx = (k a, k a_1 P_1, k a_2 P_2), \quad (13) \]
\[ x.y = (a b, (a b_1 + a_1 b + a_1 b_1 + a_2 b_1) P_1, (a b_2 + a_2 b + a_2 b_2) P_2). \quad (14) \]

When $k = 0$, then we have

\[ 0x = (0 a, 0 a_1 P_1, 0 a_2 P_2) = (0, 0 P_1, 0 P_2) = (0, 0, 0). \quad (15) \]

**Notation 3.2.** In what follows next, we will use the symbols $SP N$, $SP Z$, $SP Q$, $SP R$ and $SP C$ to denote the Plithogenic sets of natural, integer, rational, real and complex numbers respectively.

**Example 3.3.** $(SP Q, .)$, $(SP R, .)$ and $(SP C, .)$ are symbolic Plithogenic groups.

**Definition 3.4.** Let $(X, \ast)$ be any algebraic structure and let $SPX$ be the corresponding symbolic Plithogenic set. The couple $(SPX, \ast)$ is called a symbolic Plithogenic algebraic structure. $SPX$ will be named according to the name of the underlying algebraic structure $X$. For instance if $X$ is a group, $SPX$ will be called a symbolic Plithogenic group, if $X$ is a ring, $SPX$ will be called a symbolic Plithogenic ring, if $X$ is a hypergroup, $SPX$ will be called a symbolic Plithogenic hypergroup and so on.

**Theorem 3.5.** Let $(G, \ast)$ be a group and let $SPG$ be the corresponding symbolic Plithogenic group. Then:

(i) $G \subset SPG$.

(ii) $(SPG, \ast)$ is a semigroup.

(iii) $(SPG, \ast)$ is not a group.

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Proof. (i) This follows from the definition of $SPG$.
(ii) Let $x = (a, a_1 P_1, a_2 P_2)$, $y = (b, b_1 P_1, b_2 P_2)$ and $z = (c, c_1 P_1, c_2 P_2)$ be arbitrary elements of $SPG$. Then:

$$
x * y = (ab, (ab_1 + a_1 b + a_1 b_1 + a_2 b_1) P_1, (ab_2 + a_2 b + a_2 b_2) P_2) \in SPG.
$$

Now,

$$(x * y) * z = \begin{pmatrix} abc, (abc_1 + ab_1 c + a_1 b_1 c + a_1 b_2 c + a_2 b_1 c + ab_1 c_1 + a_1 b_2 c_1 + a_2 b_1 c_1 + a_2 b_2 c_1) \\
+ (a_1 b_2 c_1 + a_2 b_1 c_1 + ab_1 c_1 + a_1 b_2 c_1 + a_1 b_2 c_1 + a_2 b_1 c_1 + a_2 b_2 c_1 + a_2 b_2 c_1) \end{pmatrix}$$

This shows that $(SPG, *)$ is a semigroup.

(iii) Since $P_1^{-1}$ and $P_2^{-1}$ do not exist, it follows that we cannot find $x^{-1}$, $\forall x \in SPG$. Hence, $(SPG, *)$ is not a group.

**Remark 3.6.** If $(G, +)$ is a group, then the symbolic Plithogenic group $(SPG, +)$ is a group.

**Example 3.7.** $(SPZ, +)$, $(SPQ, +)$, $(SPR, +)$ and $(SPC, +)$ are abelian groups.

**Theorem 3.8.** Every symbolic Plithogenic group $(SPG, \cdot)$ has at least 2 nontrivial idempotent elements.

**Proof.** Since $P_1 P_1 = P_1, P_2 P_2 = P_2$ in $SPG$, the required result follows.

**Theorem 3.9.** Let $(G, *)$ be a finite group of order $n$. Then $(SPG, *)$ is a finite symbolic Plithogenic group of order $n^3$.

**Example 3.10.** Let $Z_2$ be the group of integers modulo 2. Then

$$SPZ_2 = \{(0, 0, 0), (1, 0, 0), (0, P_1, 0), (0, 0, P_2), (0, P_1, P_2), (1, P_1, 0), (1, 0, P_2), (1, P_1, P_2)\}$$

is a symbolic Plithogenic group of integers modulo 2. The elements $(0, P_1, 0), (0, 0, P_2)$ and $(1, P_1, P_2)$ of $SPZ_2$ are nontrivial idempotent elements.

**Definition 3.11.** Let $\phi : SPG \rightarrow SPH$ be a mapping from the symbolic Plithogenic group $(SPG, *)$ to the symbolic Plithogenic group $(SPH, *)$. $\phi$ is called a symbolic Plithogenic group homomorphism if the following conditions hold:

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\[(\phi(x * y) = \phi(x) * \phi(y), \forall x, y \in SPG, \]
\[(\text{ii) } \phi(P_i) = P_i, i = 1, 2.\]

The kernel of \(\phi\) denoted by \(\text{Ker}\phi\) is defined by
\[
\text{Ker}\phi = \{x \in SPG : \phi(x) = \text{identity element of } SPH\}.
\]

**Example 3.12.** Let \((G, +)\) be a group and let \(\phi : SPG \times SPG \to SPG\) be a mapping defined by
\[
\phi(a, b) = a \, \forall (a, b) \in SPG \times SPG.
\]
Then \(\phi\) is a symbolic Plithogenic group homomorphism.

If \(G = \mathbb{Z}_2\), then
\[
\text{Ker}\phi = \{((0,0), (0,0)), ((0,0), (1,0)), ((0,0), (0,P_1,0)), ((0,0), (0,0, P_2)),
((0,0), (0, P_1, P_2)), ((0,0), (1,0), ((0,0), (1,0, P_2)), ((0,0), (1, P_1, P_2))\}
\]
which is a subgroup of \(SP\mathbb{Z}_2 \times SP\mathbb{Z}_2\).

**Example 3.13.** Let \(G = \mathbb{Z}\), let \(SPG\) be the corresponding symbolic Plithogenic group of integers and let \(G(I)\) be the neutrosophic group of integers. If \(\phi : SPG \to G(I)\) is a mapping defined by
\[
\phi(x) = (a, (b+c)I), \forall x = (a, bP_1, cP_2) \in SPG,
\]
then \(\phi\) is a group homomorphism and \(\text{Ker}\phi = \{(0, kP_1, -kP_2) : k \in \mathbb{Z}\}\) which is a subgroup of \(SPG\).

**Definition 3.14.** Let \((R, +, .)\) be any ring. The triple \((SPR, +, .)\) is called a symbolic Plithogenic ring. If \(R\) is commutative with unity, so also is \(SPR\).

**Theorem 3.15.** Let \((R, +, .)\) be any ring. Then \((SPR, +, .)\) is a ring.

**Proof.** Using Definition 3.1, it can easily be shown that \((SPR, +)\) is an abelian group and \((SPR, .)\) is a semigroup. Also, for arbitrary \(x, y, z \in SPR\), it can be shown that \(x(y+z) = xy + xz\) and \((y+z)x = yx + zx\). Hence, \((SPR, +, .)\) is a ring. \(\square\)

**Theorem 3.16.** Every symbolic Plithogenic ring \((SPR, +, .)\) has at least 2 nontrivial idempotent elements.

**Theorem 3.17.** Let \((R, +, .)\) be a finite ring of order \(n\). Then \((SPR, +, .)\) is a finite symbolic Plithogenic ring of order \(n^3\).

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Example 3.18. Let \( \mathbb{Z}_2 \) be the ring of integers modulo 2. Then
\[
SP\mathbb{Z}_2 = \{(0, 0, 0), (1, 0, 0), (0, P_1, 0), (0, 0, P_2), (0, P_1, P_2), (1, P_1, 0), (1, 0, P_2), (1, P_1, P_2)\}
\]
is a symbolic Plithogenic ring of integers modulo 2.

Lemma 3.19. Let \((SPR, +, .)\) be a symbolic Plithogenic ring and let \(x = (a, a_1 P_1, a_2 P_2)\) and \(y = (b, b_1 P_1, b_2 P_2)\) be any two nonzero elements of \(SPR\).

(a) \(x\) is idempotent if and only if all the following hold:
(i) \(a\) is idempotent,
(ii) \(a + a_2\) is idempotent and
(iii) \(a + a_1 + a_2\) is idempotent.

(b) \(x\) and \(y\) are zero divisors if and only if all the following hold:
(i) \(a\) and \(b\) are zero divisors,
(ii) \(a + a_2\) and \(b + b_2\) are zero divisors and
(iii) \(a + a_1 + a_2\) and \(b + b_1 + b_2\) are zero divisors.

Example 3.20. Let \(SP\mathbb{Z}_6\) be the symbolic Plithogenic ring of integers modulo 6. Then
(i) \((1, 3P_1, 3P_2), (1, 5P_1, 3P_2), (3, 5P_1, P_2)\) and \((4, P_1, 5P_2)\) are idempotent elements.
(ii) \((2, P_1, P_2)\) and \((3, 5P_1, P_2)\) are zero divisors.

Definition 3.21. Let \(\phi : SPR \rightarrow SPS\) be a mapping from the symbolic Plithogenic ring \((SPR, +, .)\) to the symbolic Plithogenic ring \((SPS, +, .)\). \(\phi\) is called a symbolic Plithogenic ring homomorphism if the following conditions hold:
(i) \(\phi(x + y) = \phi(x) + \phi(y), \forall x, y \in SPR\),
(ii) \(\phi(xy) = \phi(x)\phi(y), \forall x, y \in SPR\),
(iii) \(\phi(P_i) = P_i, i = 1, 2\).

The kernel of \(\phi\) denoted by \(\text{Ker}\phi\) is defined by
\[
\text{Ker}\phi = \{x \in SPR : \phi(x) = \text{identity element of } SPS\}.
\]

Example 3.22. Let \((R, +, .)\) be a ring and let \(\phi : SPR \times SPR \rightarrow SPR\) be a mapping defined by
\[
\phi(a, b) = b \quad \forall (a, b) \in SPR \times SPR.
\]
Then \(\phi\) is a symbolic Plithogenic ring homomorphism.

If \(R = \mathbb{Z}_2\), then
\[
\text{Ker}\phi = \{((0, 0, 0), (0, 0, 0)), ((1, 0, 0), (0, 0, 0)), ((0, P_1, 0), (0, 0, 0)), ((0, 0, P_2), (0, 0, 0)),
((0, P_1, P_2), (0, 0, 0)), ((1, P_1, 0), (0, 0, 0)), ((1, 0, P_2), (0, 0, 0)), ((1, P_1, P_2), (0, 0, 0))\}
\]
which is a subring of \(SP\mathbb{Z}_2 \times SP\mathbb{Z}_2\).

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Theorem 3.23. Let \( \psi : R \rightarrow S \) be a ring homomorphism and let \( \phi : SPR \rightarrow SPS \) be a mapping from a symbolic Plithogenic ring \( SPR \) into a symbolic Plithogenic ring \( SPS \) defined by

\[
\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \quad \forall \ x = (a, bP_1, cP_2) \in SPR.
\]

Then \( \phi \) is a ring homomorphism.

Proof. Let \( x = (a, bP_1, cP_2) \) and \( y = (d, eP_1, fP_2) \) be two arbitrary elements in \( SPR \). Then

\[
x + y = (a + d, (b + e)P_1, (c + f)P_2),
\]

\[
xy = (ad, (ae + bd + be + bf + ce)P_1, (af + cd + cf)P_2),
\]

\[
\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2),
\]

\[
\phi(y) = (\psi(d), \psi(e)P_1, \psi(f)P_2),
\]

\[
\therefore \phi(x + y) = (\psi(a + d), \psi(b + e)P_1, \psi(c + f)P_2),
\]

\[
= (\psi(a) + \psi(d), \psi(b)P_1 + \psi(e)P_1, \psi(c)P_2 + \psi(f)P_2)
\]

\[
= (\psi(a), \psi(b)P_1, \psi(c)P_2) + (\psi(d), \psi(e)P_1, \psi(f)P_2),
\]

\[
= \phi(x) + \phi(y),
\]

\[
\phi(xy) = (\psi(ad), \psi(ae + bd + be + bf + ce)P_1, \psi(af + cd + cf)P_2),
\]

\[
= (\psi(a)\psi(d), (\psi(a)\psi(e) + \psi(b)\psi(d) + \psi(b)\psi(e) + \psi(b)\psi(f) + \psi(c)\psi(e))P_1,
\]

\[
(\psi(\psi(f)) + \psi(c)\psi(d) + \psi(c)\psi(f))P_2),
\]

\[
= [(\psi(a), \psi(b)P_1, \psi(c)P_2)][(\psi(d), \psi(e)P_1, \psi(f)P_2)],
\]

\[
= \phi(x)\phi(y).
\]

Accordingly, \( \phi \) is a ring homomorphism. \( \blacksquare \)

Example 3.24. Let \( R = Z_6, S = Z_2 \) and let \( \psi : Z_6 \rightarrow Z_2 \) be a ring homomorphism defined by \( \psi(\bar{x}_6) = \bar{x}_2 \). Let \( \phi : SPR_{Z_6} \rightarrow SPR_{Z_2} \) be a symbolic Plithogenic ring homomorphism defined by

\[
\phi((a, bP_1, cP_2)) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \forall (a, bP_1, cP_2) \in SPR_{Z_6}.
\]

Then, \( \text{Ker}\psi = \{0, 2, 4\} \) and \( \text{Ker}\phi = \{(i, jP_1, kP_2) : i, j, k = 0, 2, 4\} \).

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4. Symbolic Plithogenic Algebraic Hyper Structure

**Definition 4.1.** Let $H$ be a nonempty set and $*: H \times H \to \mathbb{P}(H)$ be a hyperoperation. The couple $(H, *)$ is called a hypergroupoid.

For any two nonempty subsets $A$ and $B$ of $H$ and $x \in H$, we define

$$A * B = \bigcup_{a \in A, b \in B} a * b,$$

$$A * x = A * \{x\} \quad \text{and}$$

$$x * B = \{x\} * B.$$

A hypergroupoid $(H, *)$ is called a semihypergroup if $\forall a, b, c \in H$ we have $(a * b) * c = a * (b * c)$, which means that

$$\bigcup_{u \in a * b} u * c = \bigcup_{v \in b * c} a * v.$$

A hypergroupoid $(H, *)$ is called a quasihypergroup if $\forall a \in H$ we have $a * H = H * a = H$. This condition is also called the reproduction axiom.

If a hypergroupoid $(H, *)$ is both a semihypergroup and a quasihypergroup, then it is called a hypergroup.

**Example 4.2.**

(i) Let $H$ be a nonempty set and let $x * y = H$, $\forall x, y \in H$. Then $(H, *)$ is a hypergroup called a total hypergroup.

(ii) Let $(H, .)$ be a group and let $P$ be a nonempty subset of $H$. If $x * y = xPy$, $\forall x, y \in H$, then $(H, *)$ is a hypergroup called a $P$-hypergroup.

(iii) Let $(H, .)$ be a group. If $x * y = < x, y >$, $\forall x, y \in H$, where $< x, y >$ is the subgroup generated by $x$ and $y$, then $(H, *)$ is a hypergroup.

**Definition 4.3.** Let $(H, *)$ and $(K, \circ)$ be two hypergroups. A mapping $\phi: H \to K$, is called:

(i) an inclusion homomorphism if $\phi(x * y) \subseteq \phi(x) \circ \phi(y)$, $\forall x, y \in H$;

(ii) a good homomorphism if $\phi(x * y) = \phi(x) \circ \phi(y)$, $\forall x, y \in H$.

**Definition 4.4.** Let $H$ be a nonempty set and let $+$ be a hyperoperation on $H$. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(i) $x + y = y + x$, $\forall x, y \in H$,

(ii) $x + (y + z) = (x + y) + z$, $\forall x, y, z \in H$,

(iii) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, $\forall x \in H$,

(iv) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,

(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, $\forall x, y, z \in H$. 

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Example 4.5. Let \( H = \{0, a, b, c\} \) be a set and let \(+\) be a hyperoperation on \( H \) defined in the Cayley table below.

\[
\begin{array}{cccc}
+ & 0 & a & b & c \\
0 & 0 & a & b & c \\
a & a & \{0, b\} & \{a, c\} & b \\
b & b & \{a, c\} & \{0, b\} & a \\
c & c & b & a & 0 \\
\end{array}
\]

Then \((H, +)\) is a canonical hypergroup.

Definition 4.6. Let \((H, +)\) and \((K, +)\) be two canonical hypergroups. A mapping \(\phi : H \rightarrow K\) is called:

(a) a homomorphism if:
   (i) \(\phi(x + y) \subseteq \phi(x) + \phi(y), \ \forall \ x, y \in H\) and
   (ii) \(\phi(0) = 0\).

(b) a good or strong homomorphism if:
   (i) \(\phi(x + y) = \phi(x) + \phi(y), \ \forall \ x, y \in H\) and
   (ii) \(\phi(0) = 0\).

The kernel of \(\phi\) denoted by \(\text{Ker}\phi\) is the set \(\{x \in H : \phi(x) = 0\}\).

Definition 4.7. Let \((H, *)\) be any hypergroup. The couple \((SPH, *)\) is called a symbolic Plithogenic hypergroup. If \(x = (a, a_1P_1, a_2P_2)\) and \(y = (b, b_1P_1, b_2P_2)\) are any two elements of \(SPH\), the composition of \(x\) and \(y\) in \(SPH\) denoted by \(x * y\) is defined as

\[
x * y = \{(c, c_1P_1, c_2P_2) : c \in a * b, c_1 \in (a * b_1 \cup a_1 * b \cup a_1 * b_1 \cup a_1 * b_2 \cup a_2 * b_1)P_1, c_2 \in (a * b_2 \cup a_2 * b \cup a_2 * b_2)P_2\}\]

Example 4.8. (i) Let \((H, *)\) be a total hypergroup. Then \((SPH, *)\) is a symbolic Plithogenic total hypergroup.

(ii) Let \((H, *)\) be a P-hypergroup. Then \((SPH, *)\) is a symbolic Plithogenic P-hypergroup.

Theorem 4.9. Let \((H, *)\) be a hypergroup and let \((SPH, *)\) be the corresponding symbolic Plithogenic hypergroup. Then:

(i) \((SPH, *)\) is a semigroup.

(ii) \((SPH, *)\) generally is not a hypergroup.

Proof. Let \(x = (a, a_1P_1, a_2P_2)\), \(y = (b, b_1P_1, b_2P_2)\) and \(z = (c, c_1P_1, c_2P_2)\) be arbitrary elements of \(SPH\).

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This shows that \((SPH, \ast)\) is a groupoid. Next,
\[
(x \ast y) \ast z = [(a, a_1P_1, a_2P_2) \ast (b, b_1P_1, b_2P_2)] \ast (c, c_1P_1, c_2P_2)
\]
\[
= [((a \ast b, (a \ast b_1 \cup a_1 \ast b \cup a_1 \ast b_1 \cup a_1 \ast b_2 \cup a_2 \ast b_1)P_1),
(a \ast b_2 \cup a_2 \ast b \cup a_2 \ast b_2)P_2)] \ast (c, c_1P_1, c_2P_2)
\]
\[
= (a \ast b \ast c, (a \ast b \ast c_1 \cup a \ast b_1 \ast c \cup a_1 \ast b_1 \ast c \cup a_1 \ast b_2 \ast c
\cup a_2 \ast b \ast c_2 \cup a_1 \ast b_1 \ast c_2 \cup a_1 \ast b_2 \ast c_2 \cup a_1 \ast b_1 \ast c_1
\cup a_2 \ast b_1 \ast c_2 \cup a_1 \ast b_2 \ast c_2 \cup a_2 \ast b \ast c_2 \cup a_2 \ast b_1 \ast c_1
\cup a_2 \ast b_2 \ast c_1 \cup a_2 \ast b_2 \ast c \cup a_2 \ast b_2 \ast c_2 \cup a_2 \ast b \ast c_2
\cup a_2 \ast b_2 \ast c \cup a_2 \ast b_2 \ast c_2 \cup a_2 \ast b_2 \ast c \cup a_2 \ast b_2 \ast c_2)
\]
\[
= x \ast (y \ast z).
\]

Accordingly, \((SPH, \ast)\) is a semigroup.

(ii) For all \(x = (a, a_1P_1, a_2P_2)\) in \(SPH\), it can be shown that \(x \ast SPH \neq SPH \neq SPH \ast x\).

This shows that reproduction axiom failed to hold in \(SPH\). Hence, \((SPH, \ast)\) is not a hypergroup. 

**Definition 4.10.** Let \((SPH, \ast)\) and \((SPK, \circ)\) be any two symbolic Plithogenic hypergroups and let \(\phi : SPH \rightarrow SPK\) be a mapping from \(SPH\) into \(SPK\).

(a) \(\phi\) is called a symbolic Plithogenic hypergroup homomorphism if the following conditions hold:

(i) \(\phi(x \ast y) \subseteq \phi(x) \circ \phi(y), \ \forall x, y \in SPH\).

(ii) \(\phi(P_i) = P_i, \ \text{for} \ i = 1, 2.\)
(b) \( \phi \) is called a symbolic Plithogenic good hypergroup homomorphism if the following conditions hold:

(i) \( \phi(x \ast y) = \phi(x) \circ \phi(y), \ \forall x, y \in SPH. \)

(ii) \( \phi(P_i) = P_i, \) for \( i = 1, 2. \)

**Theorem 4.11.** Let \( \psi : (H, \ast) \to (K, \circ) \) be a good hypergroup homomorphism from a hypergroup \( (H, \ast) \) into a hypergroup \( (K, \circ) \) and let \( \phi : SPH \to SPK \) be a mapping from a symbolic Plithogenic hypergroup \( SPH \) into a symbolic Plithogenic hypergroup \( SPK \) defined by

\[
\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \quad \forall \ x = (a, bP_1, cP_2) \in SPH.
\]

Then \( \phi \) is a good hypergroup homomorphism.

**Proof.** Let \( x = (a, bP_1, cP_2) \) and \( y = (d, eP_1, fP_2) \) be two arbitrary elements in \( SPR. \) Then

\[
x \ast y = (a \ast d, (a \ast e \cup b \ast d \cup b \ast e \cup b \ast f \cup c \ast e)P_1, (a \ast f \cup c \ast d \cup c \ast f)P_2),
\]

\[
\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2),
\]

\[
\phi(y) = (\psi(d), \psi(e)P_1, \psi(f)P_2),
\]

\[
\therefore \ \phi(x \ast y) = (\psi(a \ast d), (\psi(a) \circ \psi(e)) \cup \psi(b) \circ \psi(d) \cup \psi(b) \circ \psi(e) \circ \psi(f) \circ \psi(c) P_1, \psi(a \ast f \cup c \ast d \cup c \ast f)P_2),
\]

\[
= (\psi(a) \circ \psi(d), (\psi(a) \circ \psi(e)) \cup \psi(b) \circ \psi(d) \cup \psi(b) \circ \psi(e) \circ \psi(f) \circ \psi(c) P_1, \psi(a \ast f \cup c \ast d \cup c \ast f)P_2),
\]

\[
= [\psi(a), \psi(b)P_1, \psi(c)P_2] \circ [\psi(d), \psi(e)P_1, \psi(f)P_2],
\]

\[
= \phi(x) \circ \phi(y).
\]

Accordingly, \( \phi \) is a good hypergroup homomorphism. \( \Box \)

**Definition 4.12.** Let \( (C, +) \) be any canonical hypergroup. The couple \( (SPC, +) \) is called a symbolic Plithogenic canonical hypergroup. If \( x = (a, a_1P_1, a_2P_2) \) and \( y = (b, b_1P_1, b_2P_2) \) are any two elements of \( SPC, \) the composition of \( x \) and \( y \) in \( SPC \) denoted by \( x + y \) is defined as

\[
x + y = \{(c, c_1P_1, c_2P_2) : c \in a + b, c_1 \in a_1 + b_1, c_2 \in a_2 + b_2\}. \quad (17)
\]

**Theorem 4.13.** Let \( (SPC, +) \) be a symbolic Plithogenic canonical hypergroup. Then \( (SPC, +) \) is a canonical hypergroup.

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Proof. Let \( x = (a, a_1 P_1, a_2 P_2) \), \( y = (b, b_1 P_1, b_2 P_2) \) and \( z = (c, c_1 P_1, c_2 P_2) \) be arbitrary elements of \( SPC \). Then
\[
x + y = (a, a_1 P_1, a_2 P_2) + (b, b_1 P_1, b_2 P_2)
= \{(u, u_1 P_1, u_2 P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\}
= \{(u, u_1 P_1, u_2 P_2) : u \in b + a, u_1 \in b_1 + a_1, u_2 \in b_2 + a_2\}
= y + x.
\]

Next,
\[
(x + y) + z = ((a, a_1 P_1, a_2 P_2) + (b, b_1 P_1, b_2 P_2)) + (c, c_1 P_1, c_2 P_2)
= \{(u, u_1 P_1, u_2 P_2) : u \in a + b + c, u_1 \in a_1 + b_1 + c_1, u_2 \in a_2 + b_2 + c_2\}
= \{(u, u_1 P_1, u_2 P_2) : u \in a + (b + c), u_1 \in a_1 + (b_1 + c_1), u_2 \in a_2 + (b_2 + c_2)\}
= (a, a_1 P_1, a_2 P_2) + ((b, b_1 P_1, b_2 P_2) + (c, c_1 P_1, c_2 P_2))
= x + (y + z).
\]

Since \( SPC \) is a symbolic Plithogenic canonical hypergroup, it follows that \((0, 0 P_1, 0 P_2) = (0, 0, 0) \in SPC \) so that
\[
x + (0, 0, 0) = (a, a_1 P_1, a_2 P_2) + (0, 0, 0)
= \{(u, u_1 P_1, u_2 P_2) : u \in a + 0, u_1 \in a_1 + 0, u_2 \in a_2 + 0\}
= \{(u, u_1 P_1, u_2 P_2) : u \in \{a\}, u_1 \in \{a_1\}, u_2 \in \{a_2\}\}
= \{(a, a_1 P_1, a_2 P_2)\}
= \{x\} \text{ and similarly,}
(0, 0, 0) + x = \{x\}.
\]

Also,
\[
x + (-x) \cap (-x) + x = [(a, a_1 P_1, a_2 P_2) + (-a, -a_1 P_1, -a_2 P_2)] \cap [(a, a_1 P_1, a_2 P_2)]
= \{(u, u_1 P_1, u_2 P_2) : u \in a + (-a), u_1 \in a_1 + (-a_1), u_2 \in a_2 + (-a_2)\}
\cap \{(v, v_1 P_1, v_2 P_2) : v \in (-a) + a, v_1 \in (-a_1) + (a_1), v_2 \in (-a_2) + (a_2)\}
= \{(u, u_1 P_1, u_2 P_2) : u \in \{0\}, u_1 \in \{0\}, u_2 \in \{0\}\}
\cap \{(v, v_1 P_1, v_2 P_2) : v \in \{0\}, v_1 \in \{0\}, v_2 \in \{0\}\}
= \{(0, 0, 0) \in x + (-x) \cap (-x) + x\}
\]

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which shows that \(-x\) is the unique inverse of \(x\), \(\forall x \in SPC\).

Lastly, suppose that \(z \in x + y\), then

\[
(c, c_1P_1, c_2P_2) \in (a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2)
\]

\[
= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\}
\]

\[
= \{(u, u_1P_1, u_2P_2) : b \in -a + u, b_1 \in -a_1 + u_1, b_2 \in -a_2 + u_2\}
\]

\[
= \{(b, b_1P_1, b_2P_2) : b \in -a + u, b_1 \in -a_1 + u_1, b_2 \in -a_2 + u_2\}
\]

\[
\because (b, b_1P_1, b_2P_2) \in -(a, a_1P_1, a_2P_2) + (c, c_1P_1, c_2P_2)
\]

that is \(y \in -x + z\) and similarly,

\[
z \in x + y \implies x \in z - y.
\]

Accordingly, \((SPC, +)\) is a canonical hypergroup. \(\square\)

**Example 4.14.** Let \(\psi, \psi_1, \psi_2 : C_1 \to C_2\) be good canonical hypergroup homomorphisms and let \(SPC_1\) and \(SPC_2\) be two symbolic Plithogenic canonical hypergroups. If \(\phi : SPC_1 \to SPC_2\) is a mapping defined by

\[
\phi(x) = (\psi(a), \psi_1(a_1)P_1, \psi_2(a_2)P_2), \ \forall x = (a, a_1P_1, a_2P_2) \in SPC_1,
\]

then \(\phi\) is a good canonical hypergroup homomorphism and

\[
\text{Ker}\phi = \{(a, a_1P_1, a_2P_2) \in SPC_1 : \phi((a, a_1P_1, a_2P_2)) = (0, 0P_1, 0P_2)\}
\]

\[
= \{(a, a_1P_1, a_2P_2) \in SPC_1 : (\psi(a), \psi_1(a_1)P_1, \psi_2(a_2)P_2) = (0, 0P_1, 0P_2)\}
\]

\[
= \{(a, a_1P_1, a_2P_2) \in SPC_1 : \psi(a) = 0, \psi_1(a_1) = 0, \psi_2(a_2) = 0\}
\]

\[
= \{(a, a_1P_1, a_2P_2) \in SPC_1 : a \in \text{Ker}(\psi), a_1 \in \text{Ker}(\psi_1), a_2 \in \text{Ker}(\psi_2)\}
\]

\[
= \{(\text{Ker}\psi, \text{Ker}\psi_1P_1, \text{Ker}\psi_2P_2)\}.
\]

5. **Conclusion**

We have in this paper studied symbolic Plithogenic algebraic structures and hyperstructures. In particular, we studied symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup, and we presented their basic properties.

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