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On Symbolic Plithogenic Algebraic Structures and Hyper Structures

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Abstract. The objective of this paper is to study symbolic Plithogenic algebraic structures and hyper structures. In particular, we study symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup.

Keywords: Plithogeny; Plithogenic; Plithogenic Set; Fuzzy Set; Intuitionistic Fuzzy Set; Neutrosophy; Neutrosophic Set; Plithogenic Group; Plithogenic Ring; Plithogenic Hypergroup; Plithogenic canonical Hypergroup.

1. Introduction

The concepts of Plithogeny, Plithogenic logic/set, Plithogenic probability and Plithogenic statistics were introduced by Smarandache in [26]. Plithogenic set/logic is an extension of the classical logic/set, fuzzy logic/set of Zadeh [37], intuitionistic fuzzy logic/set of Atanassov [11], neutrosophic logic/set of Smarandache [30] and quadruple neutrosophic logic/set of Smarandache [29]. Smarandache in [23], [25] and [28] introduced and studied symbolic Plithogenic algebraic structures and hyper structures. In [22], Merkepsi and Abobala studied symbolic 2-Plithogenic rings, in [10], Al-Basheer et al. studied symbolic 3-Plithogenic rings and in [17], Gayen et al. studied Plithogenic Hypersoft Subgroup. Also in [32], Taffach and Hatip studied Symbolic

2-Plithogenic Number Theory And Algebraic Equations, in [33], Taffach and Othman studied Symbolic 2-Plithogenic Modules over Symbolic 2-Plithogenic Rings and in [34], Taffach studied Symbolic 2-Plithogenic Vector Spaces. In [18], [24] and [27], applications of Plithogenic set/logic were presented. In the present paper, we study symbolic Plithogenic algebraic structures and hyper structures. In particular, we study symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup and present their basic properties.

2. Symbolic Plithogenic Set

A symbolic Plithogenic set SPX is defined by

$$SPX = \{(a, a_1P_1, a_2P_2, a_3P_3, \cdots, a_nP_n) : a, a_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (1)$$

where P_i's are the Plithogenic parameters/variables. a is called the non-Plithogenic part of SPX, a_iP_i is called the Plithogenic part of SPX and a_i's are called the coefficients of P_is where $i = 1, 2, 3, \cdots, n$. For a positive integer k, P_i has the following properties :

$$P_i^k = P_i, \quad \forall i \text{ and } k \geq 2, \quad (2)$$

$$kP_i = P_i + P_i + P_i + \cdots + P_i \quad [\text{k summand}] \quad \forall i, \quad (3)$$

$$0P_i = 0 \quad \forall i, \quad (4)$$

$$P_i^{-1} = \frac{1}{P_i} \text{ does not exist } \forall i. \quad (5)$$

when $n = 1$, equation (1) reduces to

$$SPX = \{(a, a_1P_1) : a, a_1 \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (6)$$

and SPX becomes the usual Neutrosophic set with $P_1 = I$.

When $n = 3$, equation (1) reduces to

$$SPX = \{(a, a_1P_1, a_2P_2, a_3P_3) : a, x_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (7)$$

and SPX becomes the usual Neutrosophic Quadruple set with $P_1 = T$, $P_2 = I$ and $P_3 = F$.

When $n = 2$, equation (1) reduces to

$$SPX = \{(a, a_1P_1, a_2P_2) : a, a_i \in \mathbb{R} \text{ or } \mathbb{C} \text{ or any AlgebraicStructure}\} \quad (8)$$

which is called symbolic 2-Plithogenic set.

3. Symbolic Plithogenic Algebraic Structure

All the symbolic Plithogenic sets to be considered in this section and the section after will be symbolic 2-Plithogenic sets of the form given by equation (8) and we are going to assume throughout the prevalence order $P_1 > P_2$ so that

$$P_1P_1 = P_{\min\{1,1\}} = P_1, \tag{9}$$

$$P_2P_2 = P_{\min\{2,2\}} = P_2, \tag{10}$$

$$P_1P_2 = P_2P_1 = P_{\min\{1,2\}} = P_1. \tag{11}$$

Definition 3.1. Let $+$, $-$ and $.$ be the usual arithmetic operations of addition, subtraction and multiplication of numbers respectively and let k be a nonzero scalar. If $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ are arbitrary elements of the symbolic Plithogenic set SPX where $a, b, a_i, b_i \in \mathbb{R}$ or $\mathbb{C}$, then:

$$x \pm y = (a \pm b, (a_1 \pm b_1)P_1, (a_2 \pm b_2)P_2), \tag{12}$$

$$kx = (ka, ka_1P_1, ka_2P_2), \tag{13}$$

$$x.y = (ab, (ab_1 + a_1b + a_1b_1 + a_1b_2 + a_2b_1)P_1, (ab_2 + a_2b + a_2b_2)P_2). \tag{14}$$

When $k = 0$, then we have

$$0x = (0a, 0a_1P_1, 0a_2P_2) = (0, 0P_1, 0P_2) = (0, 0, 0). \tag{15}$$

Notation 3.2. In what follows next, we will use the symbols $SP\mathbb{N}$, $SP\mathbb{Z}$, $SP\mathbb{Q}$, $SP\mathbb{R}$ and $SP\mathbb{C}$ to denote the Plithogenic sets of natural, integer, rational, real and complex numbers respectively.

Example 3.3. $(SP\mathbb{Q}, .)$, $(SP\mathbb{R}, .)$ and $(SP\mathbb{C}, .)$ are symbolic Plithogenic groups.

Definition 3.4. Let $(X, *)$ be any algebraic structure and let SPX be the corresponding symbolic Plithogenic set. The couple $(SPX, *)$ is called a symbolic Plithogenic algebraic structure. SPX will be named according to the name of the underlying algebraic structure X. For instance if X is a group, SPX will be called a symbolic Plithogenic group, if X is a ring, SPX will be called a symbolic Plithogenic ring, if X is a hypergroup, SPX will be called a symbolic Plithogenic hypergroup and so on .

Theorem 3.5. *Let $(G, *)$ be a group and let SPG be the corresponding symbolic Plithogenic group. Then:*

(i) *$G \subset SPG$.*
(ii) *$(SPG, *)$ is a semigroup.*
(iii) *$(SPG, *)$ is not a group.*

Proof. (i) This follows from the definition of SPG.
(ii) Let $x = (a, a_1P_1, a_2P_2)$, $y = (b, b_1P_1, b_2P_2)$ and $z = (c, c_1P_1, c_2P_2)$ be arbitrary elements of SPG. Then:
$x * y = (ab, (ab_1 + a_1b + a_1b_1 + a_1b_2 + a_2b_1)P_1, (ab_2 + a_2b + a_2b_2)P_2) \in SPG$. Now,

$$\begin{aligned}
(x * y) * z &= (abc, (abc_1 + ab_1c + a_1bc + a_1b_1c + a_1b_2c + a_2b_1c + ab_1c_1 + a_1bc_1 + a_1b_1c_1 \\
&\quad + a_1b_2c_1 + a_2b_1c_1 + ab_1c_2 + a_1bc_2 + a_1b_1c_2 + a_1b_2c_2 + a_2b_1c_2 + ab_2c_1 \\
&\quad + a_2bc_1 + a_2b_2c_1)P_1, (abc_2 + ab_2c + a_2bc + a_2b_2c + ab_2c_2 + a_2bc_2 + a_2b_2c_2)P_2) \\
x * (y * z) &= (abc, (abc_1 + ab_1c + ab_1c_1 + ab_1c_2 + ab_2c_1 + a_1bc + a_1bc_1 + a_1b_1c + a_1b_1c_1 \\
&\quad + a_1b_1c_2 + a_1b_2c_1 + a_1bc_2 + a_1b_2c + a_1b_2c_2 + a_2bc_1 + a_2b_1c + a_2b_1c_1 \\
&\quad + a_2b_1c_2 + a_2b_2c_1)P_1, (abc_2 + ab_2c + a_2bc + a_2b_2c + ab_2c_2 + a_2bc_2 + a_2b_2c_2)P_2) \\
&= x * (y * z).
\end{aligned}$$

This shows that $(SPG, *)$ is a semigroup.
(iii) Since P_1^{-1} and P_2^{-1} do not exist, it follows that we cannot find x^{-1}, $\forall x \in SPG$. Hence, $(SPG, *)$ is not a group. □

Remark 3.6. If $(G, +)$ is a group, then the symbolic Plithogenic group $(SPG, +)$ is a group.

Example 3.7. $(SP\mathbb{Z}, +)$, $(SP\mathbb{Q}, +)$, $(SP\mathbb{R}, +)$ and $(SP\mathbb{C}, +)$ are abelian groups.

Theorem 3.8. *Every symbolic Plithogenic group $(SPG, .)$ has at least 2 nontrivial idempotent elements.*

Proof. Since $P_1P_1 = P_1, P_2P_2 = P_2$ in SPG, the required result follows. □

Theorem 3.9. *Let $(G, *)$ be a finite group of order n. Then $(SPG, *)$ is a finite symbolic Plithogenic group of order n^3.*

Example 3.10. Let $\mathbb{Z}_2$ be the group of integers modulo 2. Then

$$SP\mathbb{Z}_2 = \{(0,0,0), (1,0,0), (0,P_1,0,), (0,0,P_2), (0,P_1,P_2), (1,P_1,0), (1,0,P_2), (1,P_1,P_2)\}$$

is a symbolic Plithogenic group of integers modulo 2. The elements $(0, P_1, 0,), (0, 0, P_2)$ and $(1, P_1, P_2)$ of $SP\mathbb{Z}_2$ are nontrivial idempotent elements.

Definition 3.11. Let $\phi : SPG \to SPH$ be a mapping from the symbolic Plithogenic group $(SPG, *)$ to the symbolic Plithogenic group $(SPH, \star)$. ϕ is called a symbolic Plithogenic group homomorphism if the following conditions hold:

(i) $\phi(x * y) = \phi(x) \star \phi(y)$, $\forall x, y \in SPG$,

(ii) $\phi(P_i) = P_i, i = 1, 2$.

The kernel of ϕ denoted by Kerϕ is defined by

$$\text{Ker}\phi = \{x \in SPG : \phi(x) = \text{identity element of } SPH\}.$$

Example 3.12. Let $(G, +)$ be a group and let $\phi : SPG \times SPG \to SPG$ be a mapping defined by

$$\phi(a, b) = a \ \ \forall (a, b) \in SPG \times SPG.$$

Then ϕ is a symbolic Plithogenic group homomorphism.

If $G = \mathbb{Z}_2$, then

$$\begin{aligned} \text{Ker}\phi &= \{((0,0,0),(0,0,0)),((0,0,0),(1,0,0)),((0,0,0),(0,P_1,0,)),((0,0,0),(0,0,P_2)), \\ &\quad ((0,0,0),(0,P_1,P_2)),((0,0,0),(1,P_1,0),((0,0,0),(1,0,P_2)),((0,0,0),(1,P_1,P_2))\} \end{aligned}$$

which is a subgroup of $SP\mathbb{Z}_2 \times SP\mathbb{Z}_2$.

Example 3.13. Let $G = \mathbb{Z}$, let SPG be the corresponding symbolic Plithogenic group of integers and let $G(I)$ be the neutrosophic group of integers. If $\phi : SPG \to G(I)$ is a mapping defined by

$$\phi(x) = (a, (b + c)I), \forall x = (a, bP_1, cP_2) \in SPG,$$

then ϕ is a group homomorphism and Ker$\phi = \{(0, kP_1, -kP_2) : k \in \mathbb{Z}\}$ which is a subgroup of SPG.

Definition 3.14. Let $(R, +, .)$ be any ring. The triple $(SPR, +, .)$ is called a symbolic Plithogenic ring. If R is commutative with unity, so also is SPR.

Theorem 3.15. *Let $(R, +, .)$ be any ring. Then $(SPR, +, .)$ is a ring.*

Proof. Using Definition 3.1, it can easily be shown that $(SPR, +)$ is an abelian group and $(SPR, .)$ is a semigroup. Also, for arbitrary $x, y, z \in SPR$, it can be shown that $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$. Hence, $(SPR, +, .)$ is a ring. $\square$

Theorem 3.16. *Every symbolic Plithogenic ring $(SPR, +, .)$ has at least 2 nontrivial idempotent elements.*

Theorem 3.17. *Let $(R, +, .)$ be a finite ring of order n. Then $(SPR, +, .)$ is a finite symbolic Plithogenic ring of order n^3.*

Example 3.18. Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Then

$$SP\mathbb{Z}_2 = \{(0,0,0),(1,0,0),(0,P_1,0,),(0,0,P_2),(0,P_1,P_2),(1,P_1,0),(1,0,P_2),(1,P_1,P_2)\}$$

is a symbolic Plithogenic ring of integers modulo 2.

Lemma 3.19. *Let $(SPR,+,.)$ be a symbolic Plithogenic ring and let $x=(a,a_1P_1,a_2P_2)$ and $y=(b,b_1P_1,b_2P_2)$ be any two nonzero elements of SPR.*

(a) *x is idempotent if and only if all the following hold:*
(i) *a is idempotent,*
(ii) *$a+a_2$ is idempotent and*
(iii) *$a+a_1+a_2$ is idempotent.*
(b) *x and y are zero divisors if and only if all the following hold:*
(i) *a and b are zero divisors,*
(ii) *$a+a_2$ and $b+b_2$ are zero divisors and*
(iii) *$a+a_1+a_2$ and $b+b_1+b_2$ are zero divisors.*

Example 3.20. Let $SP\mathbb{Z}_6$ be the symbolic Plithogenic ring of integers modulo 6. Then

(i) $(1,3P_1,3P_2),(1,5P_1,3P_2),(3,5P_1,P_2)$ and $(4,P_1,5P_2)$ are idempotent elements.
(ii) $(2,P_1,P_2)$ and $(3,5P_1,P_2)$ are zero divisors.

Definition 3.21. Let $\phi: SPR \to SPS$ be a mapping from the symbolic Plithogenic ring $(SPR,+,.)$ to the symbolic Plithogenic ring $(SPS,+,.)$. ϕ is called a symbolic Plithogenic ring homomorphism if the following conditions hold:

(i) $\phi(x+y)=\phi(x)+\phi(y)$, $\forall x,y\in SPR$,
(ii) $\phi(xy)=\phi(x)\phi(y)$, $\forall x,y\in SPR$,
(iii) $\phi(P_i)=P_i, i=1,2$.

The kernel of ϕ denoted by Kerϕ is defined by

$$\mathrm{Ker}\phi=\{x\in SPR : \phi(x) = \text{identity element of } SPS\}.$$

Example 3.22. Let $(R,+,.)$ be a ring and let $\phi: SPR\times SPR\to SPR$ be a mapping defined by

$$\phi(a,b)=b\ \ \forall (a,b)\in SPR\times SPR.$$

Then ϕ is a symbolic Plithogenic ring homomorphism.

If $R=\mathbb{Z}_2$, then

$$\begin{aligned}\mathrm{Ker}\phi &= \{((0,0,0),(0,0,0)),((1,0,0),(0,0,0)),((0,P_1,0,),(0,0,0)),((0,0,P_2),(0,0,0)),\\ &\quad ((0,P_1,P_2),(0,0,0)),((1,P_1,0),(0,0,0)),((1,0,P_2),(0,0,0)),((1,P_1,P_2),(0,0,0))\}\end{aligned}$$

which is a subring of $SP\mathbb{Z}_2\times SP\mathbb{Z}_2$.

Theorem 3.23. *Let $\psi : R \to S$ be a ring homomorphism and let $\phi : SPR \to SPS$ be a mapping from a symbolic Plithogenic ring SPR into a symbolic Plithogenic ring SPS defined by*

$$\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \quad \forall\ x = (a, bP_1, cP_2) \in SPR.$$

Then ϕ is a ring homomorphism.

Proof. Let $x = (a, bP_1, cP_2)$ and $y = (d, eP_1, fP_2)$ be two arbitrary elements in SPR. Then

$$\begin{aligned}
x + y &= (a + d, (b + e)P_1, (c + f)P_2),\\
xy &= (ad, (ae + bd + be + bf + ce)P_1, (af + cd + cf)P_2),\\
\phi(x) &= (\psi(a), \psi(b)P_1, \psi(c)P_2),\\
\phi(y) &= (\psi(d), \psi(e)P_1, \psi(f)P_2),\\
\therefore\quad \phi(x + y) &= (\psi(a + d), \psi(b + e)P_1, \psi(c + f)P_2),\\
&= (\psi(a) + \psi(d), \psi(b)P_1 + \psi(e)P_1, \psi(c)P_2 + \psi(f)P_2)\\
&= (\psi(a), \psi(b)P_1, \psi(c)P_2) + (\psi(d), \psi(e)P_1, \psi(f)P_2),\\
&= \phi(x) + \phi(y),\\
\phi(xy) &= (\psi(ad), \psi(ae + bd + be + bf + ce)P_1, \psi(af + cd + cf)P_2),\\
&= (\psi(a)\psi(d), (\psi(a)\psi(e) + \psi(b)\psi(d) + \psi(b)\psi(e) + \psi(b)\psi(f) + \psi(c)\psi(e))P_1,\\
&\quad\ (\psi(a)\psi(f) + \psi(c)\psi(d) + \psi(c)\psi(f))P_2),\\
&= [(\psi(a), \psi(b)P_1, \psi(c)P_2)][(\psi(d), \psi(e)P_1, \psi(f)P_2)],\\
&= \phi(x)\phi(y).
\end{aligned}$$

Accordingly, ϕ is a ring homomorphism. □

Example 3.24. Let $R = \mathbb{Z}_6$, $S = \mathbb{Z}_2$ and let $\psi : \mathbb{Z}_6 \to \mathbb{Z}_2$ be a ring homomorphism defined by $\psi(\bar{x}_6) = \bar{x}_2$. Let $\phi : SP\mathbb{Z}_6 \to SP\mathbb{Z}_2$ be a symbolic Plithogenic ring homomorphism defined by

$$\phi((a, bP_1, cP_2)) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \forall (a, bP_1, cP_2) \in SP\mathbb{Z}_6.$$

Then, $\mathrm{Ker}\psi = \{0, 2, 4\}$ and $\mathrm{Ker}\phi = \{(i, jP_1, kP_2) : i, j, k = 0, 2, 4\}$.

4. Symbolic Plithogenic Algebraic Hyper Structure

Definition 4.1. Let H be a nonempty set and $* : H \times H \to \mathbb{P}^{\star}(H)$ be a hyperoperation. The couple $(H, *)$ is called a hypergroupoid.

For any two nonempty subsets A and B of H and $x \in H$, we define

$$\begin{aligned} A * B &= \bigcup_{a\in A, b\in B} a * b, \\ A * x &= A * \{x\} \text{ and} \\ x * B &= \{x\} * B. \end{aligned}$$

A hypergroupoid $(H, *)$ is called a semihypergroup if $\forall\ a, b, c \in H$ we have $(a * b) * c = a * (b * c)$, which means that

$$\bigcup_{u\in a*b} u * c = \bigcup_{v\in b*c} a * v.$$

A hypergroupoid $(H, *)$ is called a quasihypergroup if $\forall a \in H$ we have $a*H = H*a = H$. This condition is also called the reproduction axiom.

If a hypergroupoid $(H, *)$ is both a semihypergroup and a quasihypergroup, then it is called a hypergroup.

Example 4.2.
(i) Let H be a nonempty set and let $x * y = H$, $\forall x, y \in H$. Then $(H, *)$ is a hypergroup called a total hypergroup.
(ii) Let $(H, .)$ be a group and let P be a nonempty subset of H. If $x * y = xPy$, $\forall\ x, y \in H$, then, $(H, *)$ is a hypergroup called a P-hypergroup.
(iii) Let $(H, .)$ be a group. If $x * y =< x, y >$, $\forall\ x, y \in H$, where $< x, y >$ is the subgroup generated by x and y, then $(H, *)$ is a hypergroup.

Definition 4.3. Let $(H, *)$ and $(K, \circ)$ be two hypergroups. A mapping $\phi : H \to K$, is called:

(i) an inclusion homomorphism if $\phi(x * y) \subseteq \phi(x) \circ \phi(y)$, $\forall\ x, y \in H$;
(ii) a good homomorphism if $\phi(x * y) = \phi(x) \circ \phi(y)$, $\forall\ x, y \in H$.

Definition 4.4. Let H be a nonempty set and let $+$ be a hyperoperation on H. The couple $(H, +)$ is called a canonical hypergroup if the following conditions hold:

(i) $x + y = y + x$, $\forall\ x, y \in H$,
(ii) $x + (y + z) = (x + y) + z$, $\forall\ x, y, z \in H$,
(iii) there exists a neutral element $0 \in H$ such that $x + 0 = \{x\} = 0 + x$, $\forall\ x \in H$,
(iv) for every $x \in H$, there exists a unique element $-x \in H$ such that $0 \in x + (-x) \cap (-x) + x$,
(v) $z \in x + y$ implies $y \in -x + z$ and $x \in z - y$, $\forall\ x, y, z \in H$.

Example 4.5. Let $H = \{0, a, b, c\}$ be a set and let $+$ be a hyperoperation on H defined in the Cayley table below.

$+$	0	a	b	c
0	0	a	b	c
a	a	$\{0, b\}$	$\{a, c\}$	b
b	b	$\{a, c\}$	$\{0, b\}$	a
c	c	b	a	0

Then $(H, +)$ is a canonical hypergroup.

Definition 4.6. Let $(H, +)$ and $(K, +)$ be two canonical hypergroups. A mapping $\phi : H \to K$ is called:

(a) a homomorphism if:
(i) $\phi(x + y) \subseteq \phi(x) + \phi(y)$, $\forall\, x, y \in H$ and
(ii) $\phi(0) = 0$.
(b) a good or strong homomorphism if:
(i) $\phi(x + y) = \phi(x) + \phi(y)$, $\forall\, x, y \in H$ and
(ii) $\phi(0) = 0$.

The kernel of ϕ denoted by Kerϕ is the set $\{x \in H : \phi(x) = 0\}$.

Definition 4.7. Let $(H, *)$ be any hypergroup. The couple $(SPH, *)$ is called a symbolic Plithogenic hypergroup. If $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ are any two elements of SPH, the composition of x and y in SPH denoted by $x * y$ is defined as

$$\begin{aligned} x * y &= \{(c, c_1P_1, c_2P_2) : c \in a * b, c_1 \in (a * b_1 \cup a_1 * b \cup a_1 * b_1 \cup a_1 * b_2 \cup a_2 * b_1)P_1), \\ &\quad c_2 \in (a * b_2 \cup a_2 * b \cup a_2 * b_2)P_2)\} \qquad (16) \end{aligned}$$

Example 4.8. (i) Let $(H, *)$ be a total hypergroup. Then $(SPH, *)$ is a symbolic Plithogenic total hypergroup.
(ii) Let $(H, *)$ be a P-hypergroup. Then $(SPH, *)$ is a symbolic Plithogenic P-hypergroup.

Theorem 4.9. *Let $(H, *)$ be a hypergroup and let $(SPH, *)$ be the corresponding symbolic Plithogenic hypergroup. Then:*

(i) *$(SPH, *)$ is a semigroup.*
(ii) *$(SPH, *)$ generally is not a hypergroup.*

Proof. Let $x = (a, a_1P_1, a_2P_2)$, $y = (b, b_1P_1, b_2P_2)$ and $z = (c, c_1P_1, c_2P_2)$ be arbitrary elements of SPH.

(i)

$$\begin{aligned} x*y &= (a,a_1P_1,a_2P_2)*(b,b_1P_1,b_2P_2) \\ &= (a*b,(a*b_1\cup a_1*b\cup a_1*b_1\cup a_1*b_2\cup a_2*b_1)P_1), \\ &\quad (a*b_2\cup a_2*b\cup a_2*b_2)P_2) \\ &\subset SPH. \end{aligned}$$

This shows that $(SPH,*)$ is a groupoid. Next,

$$\begin{aligned} (x*y)*z &= [(a,a_1P_1,a_2P_2)*(b,b_1P_1,b_2P_2)]*(c,c_1P_1,c_2P_2) \\ &= [(a*b,(a*b_1\cup a_1*b\cup a_1*b_1\cup a_1*b_2\cup a_2*b_1)P_1), \\ &\quad (a*b_2\cup a_2*b\cup a_2*b_2)P_2)]*(c,c_1P_1,c_2P_2) \\ &= (a*b*c,(a*b*c_1\cup a*b_1*c\cup a_1*b*c\cup a_1*b_1*c\cup a_1*b_2*c \\ &\quad \cup a_2*b_1*c\cup a*b_1*c_1\cup a_1*b*c_1\cup a_1*b_1*c_1\cup a_1*b_2*c_1\cup a_2*b_1*c_1 \\ &\quad \cup a*b_1*c_2\cup a_1*b*c_2\cup a_1*b_1*c_2\cup a_1*b_2*c_2\cup a_2*b_1*c_2\cup a*b_2*c_1 \\ &\quad \cup a_2*b*c_1\cup a_2*b_2*c_1)P_1,(a*b*c_2\cup a*b_2*c\cup a_2*b*c\cup a_2*b_2*c \\ &\quad \cup a*b_2*c_2\cup a_2*b*c_2\cup a_2*b_2*c_2)P_2) \\ x*(y*z) &= (a*b*c,(a*b*c_1\cup a*b_1*c\cup a*b_1*c_1\cup a*b_1*c_2\cup a*b_2*c_1\cup a_1*b*c \\ &\quad \cup a_1*b*c_1\cup a_1*b_1*c\cup a_1*b_1*c_1\cup a_1*b_1*c_2\cup a_1*b_2*c_1\cup a_1*b*c_2 \\ &\quad \cup a_1*b_2*c\cup a_1*b_2*c_2\cup a_2*b*c_1\cup a_2*b_1*c\cup a_2*b_1*c_1\cup a_2*b_1*c_2 \\ &\quad \cup a_2*b_2*c_1)P_1,(a*b*c_2\cup a*b_2*c\cup a_2*b*c\cup a_2*b_2*c\cup a*b_2*c_2 \\ &\quad \cup a_2*b*c_2\cup a_2*b_2*c_2)P_2 \\ &= x*(y*z). \end{aligned}$$

Accordingly, $(SPH,*)$ is a semigroup.

(ii) For all $x=(a,a_1P_1,a_2P_2)$ in SPH, it can be shown that $x*SPH\neq SPH\neq SPH*x$. This shows that reproduction axiom failed to hold in SPH. Hence, $(SPH,*)$ is not a hypergroup. □

Definition 4.10. Let $(SPH,*)$ and $(SPK,\circ)$ be any two symbolic Plithogenic hypergroups and let $\phi: SPH\to SPK$ be a mapping from SPH into SPK.

(a) ϕ is called a symbolic Plithogenic hypergroup homomorphism if the following conditions hold:

(i) $\phi(x*y)\subseteq\phi(x)\circ\phi(y),\ \forall x,y\in SPH$.

(ii) $\phi(P_i)=P_i$, for $i=1,2$.

(b) ϕ is called a symbolic Plithogenic good hypergroup homomorphism if the following conditions hold:

(i) $\phi(x * y) = \phi(x) \circ \phi(y), \ \forall x, y \in SPH.$

(ii) $\phi(P_i) = P_i$, for $i = 1, 2$.

Theorem 4.11. *Let $\psi : (H, *) \to (K, \circ)$ be a good hypergroup homomorphism from a hypergroup $(H, *)$ into a hypergroup $(K, \circ)$ and let $\phi : SPH \to SPK$ be a mapping from a symbolic Plithogenic hypergroup SPH into a symbolic Plithogenic hypergroup SPK defined by*

$$\phi(x) = (\psi(a), \psi(b)P_1, \psi(c)P_2), \quad \forall\ x = (a, bP_1, cP_2) \in SPH.$$

Then ϕ is a good hypergroup homomorphism.

Proof. Let $x = (a, bP_1, cP_2)$ and $y = (d, eP_1, fP_2)$ be two arbitrary elements in SPR. Then

$$\begin{aligned} x * y &= (a * d, (a * e \cup b * d \cup b * e \cup b * f \cup c * e)P_1, (a * f \cup c * d \cup c * f)P_2), \\ \phi(x) &= (\psi(a), \psi(b)P_1, \psi(c)P_2), \\ \phi(y) &= (\psi(d), \psi(e)P_1, \psi(f)P_2), \\ \therefore\ \phi(x * y) &= (\psi(a * d), \psi((a * e \cup b * d \cup b * e \cup b * f \cup c * e)P_1, \psi(a * f \cup c * d \cup c * f)P_2)), \\ &= (\psi(a) \circ \psi(d), (\psi(a) \circ \psi(e) \cup \psi(b) \circ \psi(d) \cup \psi(b) \circ \psi(e) \cup \psi(b) \circ \psi(f) \cup \psi(c) \circ \psi(e))P_1, \\ &\quad (\psi(a) \circ \psi(f) \cup \psi(c) \circ \psi(d) \cup \psi(c) \circ \psi(f))P_2), \\ &= [(\psi(a), \psi(b)P_1, \psi(c)P_2)] \circ [(\psi(d), \psi(e)P_1, \psi(f)P_2)], \\ &= \phi(x) \circ \phi(y). \end{aligned}$$

Accordingly, ϕ is a good hypergroup homomorphism. □

Definition 4.12. Let $(C, +)$ be any canonical hypergroup. The couple $(SPC, +)$ is called a symbolic Plithogenic canonical hypergroup. If $x = (a, a_1P_1, a_2P_2)$ and $y = (b, b_1P_1, b_2P_2)$ are any two elements of SPC, the composition of x and y in SPC denoted by $x + y$ is defined as

$$x + y = \{(c, c_1P_1, c_2P_2) : c \in a + b, c_1 \in a_1 + b_1, c_2 \in a_2 + b_2\}. \tag{17}$$

Theorem 4.13. *Let $(SPC, +)$ be a symbolic Plithogenic canonical hypergroup. Then $(SPC, +)$ is a canonical hypergroup.*

Proof. Let $x = (a, a_1P_1, a_2P_2)$, $y = (b, b_1P_1, b_2P_2)$ and $z = (c, c_1P_1, c_2P_2)$ be arbitrary elements of SPC. Then

$$\begin{aligned}
x + y &= (a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2)\\
&= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\}\\
&= \{(u, u_1P_1, u_2P_2) : u \in b + a, u_1 \in b_1 + a_1, u_2 \in b_2 + a_2\}\\
&= y + x.
\end{aligned}$$

Next,

$$\begin{aligned}
(x + y) + z &= ((a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2)) + (c, c_1P_1, c_2P_2)\\
&= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\} + (c, c_1P_1, c_2P_2)\\
&= \{(u, u_1P_1, u_2P_2) : u \in a + b + c, u_1 \in a_1 + b_1 + c_1, u_2 \in a_2 + b_2 + c_2\}\\
&= \{(u, u_1P_1, u_2P_2) : u \in a + (b + c), u_1 \in a_1 + (b_1 + c_1), u_2 \in a_2 + (b_2 + c_2)\}\\
&= (a, a_1P_1, a_2P_2) + ((b, b_1P_1, b_2P_2) + (c, c_1P_1, c_2P_2))\\
&= x + (y + z).
\end{aligned}$$

Since SPC is a symbolic Plithogenic canonical hypergroup, it follows that $(0, 0P_1, 0P_2) = (0, 0, 0) \in SPC$ so that

$$\begin{aligned}
x + (0, 0, 0) &= (a, a_1P_1, a_2P_2) + (0, 0, 0)\\
&= \{(u, u_1P_1, u_2P_2) : u \in a + 0, u_1 \in a_1 + 0, u_2 \in a_2 + 0\}\\
&= \{(u, u_1P_1, u_2P_2) : u \in \{a\}, u_1 \in \{a_1\}, u_2 \in \{a_2\}\}\\
&= \{(a, a_1P_1, a_2P_2)\}\\
&= \{x\} \quad \text{and similarly,}\\
(0, 0, 0) + x &= \{x\}.
\end{aligned}$$

Also,

$$\begin{aligned}
x + (-x) \cap (-x) + x &= [(a, a_1P_1, a_2P_2) + (-a, -a_1P_1, -a_2P_2)] \cap [(-a, -a_1P_1, -a_2P_2)\\
&\quad +(a, a_1P_1, a_2P_2)]\\
&= \{(u, u_1P_1, u_2P_2) : u \in a + (-a), u_1 \in a_1 + (-a_1), u_2 \in a_2 + (-a_2)\}\\
&\quad \cap\{(v, v_1P_1, v_2P_2) : v \in (-a) + a, v_1 \in (-a_1) + (a_1), v_2 \in (-a_2) + (a_2)\}\\
&= \{(u, u_1P_1, u_2P_2) : u \in \{0\}, u_1 \in \{0\}, u_2 \in \{0\}\}\\
&\quad \cap\{(v, v_1P_1, v_2P_2) : v \in \{0\}, v_1 \in \{0\}, v_2 \in \{0\}\}\\
\therefore\ (0, 0, 0) &\in x + (-x) \cap (-x) + x
\end{aligned}$$

which shows that $-x$ is the unique inverse of x , $\forall\ x \in SPC$.

Lastly, suppose that $z \in x + y$, then

$$\begin{aligned}
(c, c_1P_1, c_2P_2) &\in (a, a_1P_1, a_2P_2) + (b, b_1P_1, b_2P_2) \\
&= \{(u, u_1P_1, u_2P_2) : u \in a + b, u_1 \in a_1 + b_1, u_2 \in a_2 + b_2\} \\
&= \{(u, u_1P_1, u_2P_2) : b \in -a + u, b_1 \in -a_1 + u_1, b_2 \in -a_2 + u_2\} \\
&= \{(b, b_1P_1, b_2P_2) : b \in -a + u, b_1 \in -a_1 + u_1, b_2 \in -a_2 + u_2\} \\
\therefore\ (b, b_1P_1, b_2P_2) &\in -(a, a_1P_1, a_2P_2) + (c, c_1P_1, c_2P_2) \\
\text{that is } y &\in -x + z \ \text{ and similarly,} \\
z &\in x + y \ \Rightarrow \ x \in z - y.
\end{aligned}$$

Accordingly, $(SPC, +)$ is a canonical hypergroup. □

Example 4.14. Let $\psi, \psi_1, \psi_2 : C_1 \to C_2$ be good canonical hypergroup homomorphisms and let SPC_1 and SPC_2 be two symbolic Plithogenic canonical hypergroups. If $\phi : SPC_1 \to SPC_2$ is a mapping defined by

$$\phi(x) = (\psi(a), \psi_1(a_1)P_1, \psi_2(a_2)P_2),\ \forall\ x = (a, a_1P_1, a_2P_2) \in SPC_1,$$

then ϕ is a good canonical hypergroup homomorphism and

$$\begin{aligned}
\mathrm{Ker}\phi &= \{(a, a_1P_1, a_2P_2) \in SPC_1 : \phi((a, a_1P_1, a_2P_2)) = (0, 0P_1, 0P_2)\} \\
&= \{(a, a_1P_1, a_2P_2) \in SPC_1 : (\psi(a), \psi_1(a_1)P_1, \psi_2(a_2)P_2) = (0, 0P_1, 0P_2)\} \\
&= \{(a, a_1P_1, a_2P_2) \in SPC_1 : \psi(a) = 0, \psi_1(a_1) = 0, \psi_2(a_2) = 0\} \\
&= \{(a, a_1P_1, a_2P_2) \in SPC_1 : a \in \mathrm{Ker}(\psi), a_1 \in \mathrm{Ker}(\psi_1), a_2 \in \mathrm{Ker}(\psi_2)\} \\
&= \{(\mathrm{Ker}\psi, \mathrm{Ker}\psi_1P_1, \mathrm{Ker}\psi_2P_2)\}.
\end{aligned}$$

5. **Conclusion**

We have in this paper studied symbolic Plithogenic algebraic structures and hyper structures. In particular, we studied symbolic Plithogenic group, symbolic Plithogenic ring, symbolic Plithogenic hypergroup and symbolic Plithogenic canonical hypergroup, and we presented their basic properties.

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