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On The Two-Fold Fuzzy n-Refined Neutrosophic Rings For $2 \leq n \leq 3$

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Abstract:

The objective of this paper is to study the two-fold fuzzy algebra based on n-refined neutrosophic rings for some different special values of n, where we study some of the special elements in the case of two-fold 2-refined neutrosophic ring and 3-refined neutrosophic rings such as units, idempotents, and nilpotent elements. Also, we present the concept of two-fold ring homomorphism with its elementary properties.

Keywords: two-fold algebra, 2-refined neutrosophic ring, 3-refined neutrosophic ring, nilpotent

Introduction

The concept of fuzzy algebraic structure is considered a direct application of fuzzy sets and fuzzy mappings [1-2, 4, 6-8], where a fuzzy mapping with truth and falsity values is used to build many algebraic structures.

Also, the concept of neutrosophic set was used by many different authors to generalize classical algebraic structures by using logical conditions instead of algebraic elements [13], where we can see neutrosophic rings, neutrosophic matrices, and neutrosophic mappings [5, 9-12]. The concept of n-refined neutrosophic rings was defined in [18], and then it was studied by many authors in [19], where ideals, Diophantine equations and other related structures were classified and provided [20-21].

Recently, Smarandache in [14] has defined two-fold neutrosophic algebras as novel algebraic structures, and this new concept has been used in [16] to define two-fold fuzzy algebra by combining the standard fuzzy number theoretical system defined in [15], with the concept of two-fold algebraic structure, and many interesting theorems and examples were illustrated about this topic.

On the other hand, Hatip et.al [17], have combined real vector spaces, complex vector spaces, and algebraic modules with a fuzzy well-defined mapping to define and study two-fold fuzzy vector spaces and two-fold fuzzy modules, where they studied many elementary properties of these new structures.

Main Discussion

Definition:

Let $f: \mathbb{R} \rightarrow [0,1]$ with: $\begin{cases} f(0)=0 \\ f(1)=1 \end{cases}$, then f is called a fuzzy mapping.

We use this definition of fuzzy mappings, that is because the property $\begin{matrix} f(0)=0 \\ f(1)=1 \end{matrix}$ is very useful in algebraic structures and operations.

Example:

To understand the concept of fuzzy mapping, we will illustrate two different fuzzy mappings defined on the real field R.

Define: $f, g, h: \mathbb{R} \to [0,1]$ such that:

$$f(x) = \begin{cases} \min(x^2, \frac{1}{x^2}) & if \quad x \neq 0, x \neq -1 \\ 0 & if \quad x = 0 \\ 0.9 & if \quad x = -1 \end{cases} \quad , \quad g(x) = \begin{cases} |x^3| & if -1 < x \leq 1 \\ \frac{1}{|x^3|} & if \quad x > 1 \quad or \quad x < -1 \\ 0.1 & if \quad x = -1 \end{cases}$$

We can see that f and g lie in the closed interval [0,1], with $\mathrm{f}(0) = g(0) = 0, f(1) = g(1) = 1$.

Definition:

Let $R_2(I) = \{a_0 + \sum_{i=1}^{2} a_i I_i \quad ; a_i \in \mathbb{R} \ . I_i I_j = I_{\min(i.j)} \}$

be 2-refined commutative neutrosophic ring with unity, let $f: \mathbb{R} \to [0,1]$ be any fuzzy mapping such that $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$, we define:

$f_2 : R_2(I) \to [0,1] \quad ; \ f_2 \ (a_0 + \sum_{i=1}^{2} a_i I_i \ ) = \max(f(a_i))$, and

$[R_2(I)]_{f_2} = \left\{(a_0 + \sum_{i=1}^{2} a_i I_i \ )_{f_n \ (a_0 + \sum_{i=1}^{2} a_i I_i \ )} \quad ; \ a_i \in \mathbb{R}\right\}$, is called the two-fold fuzzy 2-Refined neutrosophic ring.

Definition:

Operations on $[R_2(I)]_{f_2}$ are define as follows:

*: $[R_2(I)]_{f_2} \times [R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$

$\circ$:$[R_2(I)]_{f_2} \times [R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$

Such that:$\begin{cases} X_{f_2(X)} * Y_{f_2(Y)} = (X + Y)_{f_2(X+Y)} \\ X_{f_2(X)} \circ Y_{f_2(Y)} = (X \cdot Y)_{f_2(XY)} \end{cases}$

Definition:

Let P an ideal of $R_2(I)$, we define the corresponding two-fold fuzzy 2-refined neutrosophic ideal as follows:

$P_{f_2} = \{X_{f_2(X)} \quad ; X \in P\}$

Definition:

Let P_{f_2} be a two-fold fuzzy 2-refined neutrosophic ideal, we define the two-fold fuzzy 2-refined factor as:

$[R_2(I)]_{f_2} / P_{f_2} = XP_{f_2} \quad ; \quad X \in [R_2(I)]_{f_2}$.

Definition:

Let $h : R_2(I) \to R_2(I)$ be a ring homomorphism, we define:

H_n: $[R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$ such that:

$H_2\big(X_{f_2(X)}\big) = (h(X))_{f_2(h(X))}$.

The mapping (H_2) is called two-fold fuzzy 2-refined neutrosophic homomorphism.

The kernel $k_{er}(H_2)$is:

$k_{er}(H_n) = \{X \in [R_2(I)]_{f_2} \quad ; \quad H_2\big(X_{f_2(X)}\big) = 0_0\} = \ (k_{er}(h))_{f_2}$.

The direct image $I_m(H_2)$ is:

$I_m(H_2) = \ (I_m(h))_{f_2}$.

Definition:

Let $H_2 \ .G_2 \ : \ [R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$ be two homomorphisms, then: $H_2 \times G_2$: $[R_2(I)]_{f_2} \to [R_2(I)]_{f_2}$ with:

$(H_2 \times G_2)\big(X_{f_2(X)}\big) = H_2(G_2\big(X_{f_2(X)}\big))$.

Theorem (1):

1] *, ○ are commutative.

2] *, ○ are associative.

3] (○) is distributive on (*).

4] *, ○ has identities.

5] (*) is invertible, i.e any element$X_{f_2(X)} \in [R_2(I)]_{f_2}$ has an iverse with respect to (*).

Theorem (2):

Let P_{f_2} be a two-fold ideal of$[R_2(I)]_{f_2}$, then:

$$\begin{cases} X_{f_2(X)} * Y_{f_2(Y)} \in P_{f_2} \\ r_{f_2(r)} \cdot X_{f_2(X)} \in P_{f_2} \end{cases} \quad ; \quad X.Y \in P \quad .r \in R_2(I)$$

Theorem (3):

$[R_2(I)]_{f_2} /P_{f_2}$ is a commutative ring with unity.

Theorem (4):

Let H_2: $[R_2(I)]_{f_2} \rightarrow [R_2(I)]_{f_2}$ be a homomorphism, then:

1] $H_2\left(X_{f_n(X)} * Y_{f_n(Y)}\right) = H_2\left(X_{f_2(X)}\right) * H_2\left(Y_{f_2(Y)}\right)$

2] $H_2\left(X_{f_2(X)} \circ Y_{f_2(Y)}\right) = H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right)$

3] $k_{er}(H_2)$ is an ideal of $[R_2(I)]_{f_2}$.

4] $I_m(H_2)$ is a subring of $[R_2(I)]_{f_2}$.

5] $[R_2(I)]_{f_2} / k_{er}(H_2) \cong I_m(H_2)$.

6] If P_{f_2} is an ideal of $[R_2(I)]_{f_2}$, then $H_2\left(P_{f_2}\right)$ is an ideal.

7] $H_2(0_0) = 0_0 \quad .H_2(1_1) = 1_1$

Theorem (5):

Let $H_2 \ .G_2 \ : \ [R_2(I)]_{f_2} \rightarrow [R_2(I)]_{f_2}$ be two homomorphisms, then:

1] $H_2\left(-X_{f_2(X)}\right) = -H_2\left(X_{f_2(X)}\right)$.

2] $H_2\left(X^{-1}_{f_2(X^{-1})}\right) = [H_2\left(X_{f_2(X)}\right)]^{-1}$, if X is invertible.

3] $H_2 \times G_2$ is a homomorphism.

Definition:

Let $X_{f_2(X)} \in [R_2(I)]_{f_2}$, then:

1] $X_{f_2(X)}$ is idempotent if $X_{f_2(X)} \circ X_{f_2(X)} = X_{f_2(X)}$.

2] $X_{f_2(X)}$ is a nilpotent if there exists $m \in \mathbb{N}$ such that: $X_{f_2(X)} \circ X_{f_2(X)} \circ \ldots \circ X_{f_2(X)} \ (m - times) = 0_0$

3] $X_{f_2(X)}$ is a zero divisor if there exists $Y_{f_2(Y)}$ such that: $X_{f_2(X)} \circ Y_{f_2(Y)} = 0_0$

Theorem (6):

Let $X_{f_2(X)} \in [R_2(I)]_{f_2}$, then we have:

1] $X_{f_2(X)}$ is idempotent if and only if X is idempotent in $R_2(I)$.

2] $X_{f_2(X)}$ is nilpotent if and only if X is nilpotent in $R_2(I)$.

3] $X_{f_2(X)}$ is a zero divisor if and only if X is a zero diviso in $R_2(I)$.

Theorem (7):

Let H_2: $[R_2(I)]_{f_2} \rightarrow [R_2(I)]_{f_2}$, then:

1] If $X_{f_2(X)} \in [R_2(I)]_{f_2}$ is idempotent, then $H_2\left(X_{f_2(X)}\right)$ is idempotent.

2] If $X_{f_2(X)}$is nilpotent, then $H_2\left(X_{f_2(X)}\right)$ is nilpotent.

3] If $X_{f_2(X)}$is a zero divisor, then $H_2\left(X_{f_2(X)}\right)$ is a zero divisor.

4] If $X_{f_2(X)}$is a unit, then $H_2\left(X_{f_2(X)}\right)$ is a unit.

Proof of theorem (1):

1] $X_{f_2(X)} * Y_{f_2(Y)} = (X+Y)_{f_2(X+y)} = (Y+X)_{f_2(Y+X)} = Y_{f_2(Y)} * X_{f_2(X)}$.

$X_{f_2(X)} \circ Y_{f_2(Y)} = (XY)_{f_2(Xy)} = (YX)_{f_2(YX)} = Y_{f_2(Y)} \circ X_{f_2(X)}$.

2] $X_{f_2(X)} * Y_{f_2(Y)} * Z_{f_{2n}(Z)}) = X_{f_2(X)} * (Y+Z)_{f_2(Y+Z)} = (X+Y+Z)_{f_2(X+Y+Z)} =$

$(X+Y)_{f_n(X+Y)} * Z_{f_2(Z)} = (X_{f_2(X)} * Y_{f_2(Y)}) * Z_{f_2(Z)}$.

$X_{f_2(X)} \circ Y_{f_2(Y)} \circ Z_{f_2(Z)}) = (XYZ)_{f_2(XYZ)} = (XY)_{f_2(XY)} \circ Z_{f_2(Z)} = (X_{f_2(X)} \circ$

$Y_{f_n(Y)}) \circ Z_{f_2(Z)}$.

3] $X_{f_2(X)} \circ (Y_{f_2(Y)} * Z_{f_2(Z)}) = (XY+XZ)_{f_2(XY+XZ)} = (XY)_{f_2(Xy)} * (XZ)_{f_2(XZ)} = (X_{f_2(X)} \circ$

$Y_{f_2(Y)}) * (X_{f_2(X)} \circ Z_{f_2(Z)})$

4] $X_{f_2(X)} * 0_0 = (X+0)_{f_2(X+0)} = X_{f_2(X)}$.

$X_{f_2(X)} \circ 1_1 = (X \cdot 1)_{f_2(X \cdot 1)} = X_{f_2(X)}$.

5] For $X_{f_n(X)}$, we have $(-X)_{f_2(-X)}$ such that:

$$X_{f_2(X)} * (-X)_{f_2(-X)} = (\mathrm{X} - X)_{f_2(X-X)} = 0_0.$$

Proof of theorem (2):

$X_{f_2(X)} * Y_{f_2(Y)} = (X+Y)_{f_2(X+y)} \quad \in P_{f_2}$, that is because $X + Y \quad \in P$.

$r_{f_2(r)} \circ X_{f_2(X)} = (rX)_{f_2(rX)} \in P_{f_2}$, that is because $rX \quad \in P$.

Proof of theorem (3):

Define: $*' : \left([R_2(I)]_{f_2}/P_{f_2}\right) \times \left([R_2(I)]_{f_n} /P_{f_2}\right) \rightarrow [R_2(I)]_{f_2}/P_{f_2}$

$$\circ' : ([R_2(I)]_{f_2}/P_{f_2}) \times ([R_2(I)]_{f_n} /P_{f_2}) \rightarrow [R_2(I)]_{f_2}/P_{f_2}$$

Such that:

$(X_{f_2(X)}P_{f_2}) *' \left(Y_{f_2(Y)}P_{f_2}\right) = \left(X_{f_2(X)} * Y_{f_2(Y)}\right) \; P_{f_2}$

$(X_{f_2(X)}P_{f_2}) \circ' \left(Y_{f_2(Y)}P_{f_2}\right) = (X_{f_2(X)} \circ Y_{f_2(Y)})P_{f_2}$

We have:

$(X_{f_2(X)}P_{f_2}) *' \left(0_0 P_{f_n}\right) = X_{f_2 \;(X)} \; P_{f_2}$,

$(X_{f_2(X)}P_{f_2}) \circ' \left(1_1 P_{f_2}\right) = X_{f_2(X)} \; P_{f_2}$,

$(X_{f_2(X)}P_{f_2}) *' \left((-X)_{f_2(-X)}P_{f_2}\right) = 0_0 \; P_{f_2}$,

$(X_{f_2(X)}P_{f_2}) *' \left[\left(Y_{f_2(Y)}P_{f_2}\right) *' \left(Z_{f_2(Z)}P_{f_2}\right)\right] = \left((X_{f_2(X)}P_{f_2}\right) *' \left[(Y * Z) \; P_{f_2}\right] = [X * Y * Z] \; P_{f_n} =$

$[(X_{f_2(X)}P_{f_2}) *' \left(Y_{f_2(Y)}P_{f_2}\right)] *' \left(Z_{f_2(Z)}P_{f_2}\right)$,

$(X_{f_n(X)}P_{f_n}) \circ' \left[\left(Y_{f_2(Y)}P_{f_2}\right) \circ' \left(Z_{f_2(Z)}P_{f_2}\right)\right] = \left(X \; P_{f_2}\right) \circ' \left[(Y \circ Z) \; P_{f_2}\right] = [X \circ Y \circ Z] \; P_{f_n} =$

$[(X_{f_n(X)}P_{f_2}) \circ' \left(Y_{f_2(Y)}P_{f_2}\right)] \circ' \left(Z_{f_2(Z)}P_{f_2}\right)$.

$$\left((X_{f_2(X)}P_{f_2}\right) \circ' \left[\left(Y_{f_2(Y)}P_{f_2}\right) *' \left(Z_{f_2(Z)}P_{f_2}\right)\right] = [X \circ (Y * Z)] \;\; P_{f_2} = [(X \circ Y) * (X \circ Z)] \;\; P_{f_2}$$

$$= (X_{f_2(X)}P_{f_2}) \circ' \left(Y_{f_2(Y)}P_{f_2}\right)] *' [(X_{f_2(X)}P_{f_2}) \circ' \left(Z_{f_2(Z)}P_{f_2}\right)]$$

Thus, our proof is complete.

Proof of theorem (4):

1] $H_2(X * Y) = (h(X+Y))_{f_2(h(X+Y))} = (h(X) + h(Y))_{f_2((h(X)+h(Y)))} = H_2\left(X_{f_2(X)}\right) * H_2\left(Y_{f_2(Y)}\right)$.

2] $H_2(X \circ Y) = (h(XY))_{f_2(h(XY))} = (h(X)h(Y))_{f_2((h(X)h(Y)))} = H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right)$.

3] since $k_{er}(H_2) = [k_{er}(h)]_{f_2}$, and $k_{er}(h)$ is an ideal of $R_2(I)$, we get: $k_{er}(H_2)$ is an ideal of $[R_2(I)]_{f_2}$.

4] It can be proved by the same.

5] We have that:

$R_2(I)/k_{er}(h) \quad \cong \quad I_m(h)$, thus:

$[R_2(I)]_{f_2}/\ [k_{er}(h)]_{f_2} \quad \cong \quad [I_m(h)]_{f_2}$, therefor:

$[R_2(I)]_{f_2}/\ k_{er}(H_n) \quad \cong \quad I_m(H_n)$.

6] $H_n\left(P_{f_2}\right) = \{\ [h(P)]_{f_2}\}$, and $h(P)$ is an ideal of $R_2(I)$, thus $H_2\left(P_{f_2}\right)$ is an ideal of $[R_2(I)]_{f_2}$.

7] $\begin{cases} H_n(0_0) = (h(0))_{f_2(h(0))} = 0_0 \\ H_n(1_1) = (h(1))_{f_2(h(1))} = 1_1 \end{cases}$.

Proof of theorem (5):

1] $H_2\left(-X_{f_2(X)}\right) = (h(-X))_{f_2(h(-X))} = [-h(X)]_{f_2(-h(X))} = -H_2\left(X_{f_2(X)}\right)$.

2] $H_2\left(X^{-1}_{f_2(X^{-1})}\right) = (h(X^{-1}))_{f_2\left(h(X^{-1})\right)} = [(h(X))^{-1}]_{f_2[(h(X))^{-1}]} = [H_2\left(X_{f_2(X)}\right)]^{-1}$.

3]$(\ H_2 \ \times G_2)\left[X_{f_2(X)} * Y_{f_2(Y)}\right] = (\ H_2 \ \times G_2)[X+Y]_{f_2(X+Y)} = [(h \circ g)(X+Y)]_{f_2[(h \circ g)(X+Y)]}$

$= [(h \circ g)(X) + (h \circ g)(Y)]_{f_2[(h\circ g)(X)+(h\circ g)(Y)]} = (H_2 \times G_2)(X_{f_2(X)}) * (H_2 \times G_2)(Y_{f_2(Y)})$.

$(H_2 \times G_2)\left[X_{f_2(X)} \circ Y_{f_2(Y)}\right] = [(h \circ g)(XY)]_{f_2[(h\circ g)(XY)]} = \left[\left(h \circ g(X)\right)\left(h \circ g(Y)\right)\right]_{f_2[(h\circ g(X))(h\circ g(Y))]}$

$= (H_2 \times G_2)(X_{f_2(X)}) \circ (H_2 \times G_2)(Y_{f_2(Y)})$.

Proof of theorem (6):

1] $X_{f_2(X)} \circ X_{f_2(X)} = X_{f_2(X)} \Leftrightarrow (X^2)_{f_2(X^2)} = X_{f_n(X)} \Leftrightarrow X^2 = X$, and X is idempotent in $R_2(I)$.

2] $X_{f_2(X)} \circ \ldots \circ X_{f_2(X)}$ $(m - times) = 0_0 \Leftrightarrow (X^m)_{f_2 \ (X^m)} = 0_0$, thus $X^m = 0$, and X is nilpotent in $R_2(I)$.

3] Its proof is similar to 1 and 2.

Proof of theorem (7):

1] $H_2\left(X_{f_2(X)}\right) \circ H_2\left(X_{f_2(X)}\right) = [(h(X))^2]_{f_2((h(X))^2)} = (h(X^2))_{f_2(h(X^2))} = (h(X))_{f_2(h(X))} = H_2\left(X_{f_2(X)}\right)$.

2] $H_2\left(X_{f_2(X)}\right)^m = (h(X^m))_{f_2(h(X^m))} = [h(0)]_{f_2(0)} = 0_0$.

3] If $X_{f_2(X)} \circ Y_{f_2(Y)} = 0_0$, then:

$H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right) = (h(XY))_{f_2(h(XY))} = 0_0$.

4] If $X_{f_2(X)} \circ Y_{f_2(Y)} = 1_1$, then:

$H_2\left(X_{f_2(X)}\right) \circ H_2\left(Y_{f_2(Y)}\right) = (h(XY))_{f_2(h(XY))} = (h(1))_{f_2(h(1))} = 1_1$.

Definition:

Let $R_3(I) = \{a_0 + \sum_{i=1}^{3} a_i I_i \quad ; a_i \in \mathbb{R} \ . I_i I_j = I_{\min(i.j)} \}$

be 3-refined commutative neutrosophic ring with unity, let $f: \mathbb{R} \to [0.1]$

Such that $\begin{cases} f(0) = 0 \\ f(1) = 1 \end{cases}$, we define:

$f_3\ :\ R_3(I) \to [0.1] \quad ;\ f_3\ (a_0 + \sum_{i=1}^{3} a_i I_i\ ) = \max(f(a_i))$, and

$[R_3(I)]_{f_3}\ = \left\{(a_0 + \sum_{i=1}^{3} a_i I_i\ )_{f_n\ (a_0+\sum_{i=1}^{3} a_i I_i\ )} \quad ;\ a_i \in \mathbb{R}\right\}$, is called the two-fold fuzzy 3-Refined neutrosophic ring.

Definition:

Operations on $[R_3(I)]_{f_3}$ are define as follows:

$*:\ [R_3(I)]_{f_3}\ \times [R_3(I)]_{f_3}\ \to [R_3(I)]_{f_3}$

$\circ:[R_3(I)]_{f_3}\ \times [R_3(I)]_{f_3}\ \to [R_3(I)]_{f_3}$

Such that:$\begin{cases} X_{f_3\ (X)} * Y_{f_3\ (Y)} = (X+Y)_{f_3\ (X+Y)} \\ X_{f_3\ (X)} \circ Y_{f_3\ (Y)} = (X \cdot Y)_{f_3\ (XY)} \end{cases}$

Definition:

Let P an ideal of $R_3(I)$, we define the corresponding two-fold fuzzy 3-refined neutrosophic ideal as follows:

$P_{f_3}\ = \{X_{f_3\ (X)} \quad ; X \in P\}$

Definition:

Let P_{f_3} be a two-fold fuzzy 3-refined neutrosophic ideal, we define the two-fold fuzzy 3-refined factor as:

$[R_3(I)]_{f_3}\ / P_{f_3}\ = XP_{f_3} \quad ;\ X \in [R_3(I)]_{f_3}$.

Definition:

Let $h : R_3(I) \to R_3(I)$ be a ring homomorphism, we define:

H_n: $[R_3(I)]_{f_3}\ \to [R_3(I)]_{f_3}$ such that:

$H_2(X_{f_3\ (X)}) = (h(X))_{f_3\ (h(X))}$.

The mapping (H_3) is called two-fold fuzzy 3-refined neutrosophic homomorphism.

The kernel $k_{er}(H_3)$is:

$k_{er}(H_n) = \{X \in [R_3(I)]_{f_3} \quad ; \; H_3(X_{f_3\ (X)}) = 0_0\} = \; (k_{er}(h))_{f_3}$.

The direct image $I_m(H_3)$ is:

$I_m(H_3) = \; (I_m(h))_{f_3}$.

Definition:

Let $H_3 \; . G_3 \; : \; [R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$ be two homomorphisms, then: $H_3 \; . G_3 \; :$ $[R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$ with:

$(H_3 \times G_3)(X_{f_3(X)}) = H_3(G_3(X_{f_3(X)}))$.

Theorem (8):

1] *, ○ are commutative.

2] *, ○ are associative.

3] (○) is distributive on (*).

4] *, ○ has identities.

5] (*) is invertible, i.e any element$X_{f_2(X)} \in [R_2(I)]_{f_2}$ has an iverse with respect to (*).

Theorem (9):

Let P_{f_3} be a two-fold ideal of$[R_3(I)]_{f_3}$, then:

$$\begin{cases} X_{f_3(X)} * Y_{f_3(Y)} \in P_{f_3} \\ r_{f_3(r)} \cdot X_{f_3(X)} \in P_{f_3} \end{cases} \quad ; \; X.Y \in P \; .r \in R_3(I)$$

Theorem (10):

$[R_3(I)]_{f_3}/P_{f_3}$ is a commutative ring with unity.

Theorem (11):

Let H_3: $[R_3(I)]_{f_3} \to [R_2(I)]_{f_3}$be a homomorphism, then:

1] $H_2(X_{f_3(X)} * Y_{f_3(Y)}) = \; H_2(X_{f_3(X)}) * H_2\left(Y_{{f_3}_2(Y)}\right)$

2] $H_2(X_{f_3(X)} \circ Y_{f_3(Y)}) = \; H_2(X_{f_3(X)}) \circ H_2(Y_{f_3(Y)})$

3] $k_{er}(H_3)$ is an ideal of $[R_3(I)]_{f_3}$.

4] $I_m(H_3)$ is a subring of $[R_3(I)]_{f_3}$.

5] $[R_3(I)]_{f_3}/\ k_{er}(H_3) \cong I_m(H_3)$.

6] If P_{f_3} is an ideal of $[R_3(I)]_{f_3}$, then $H_3\left(P_{f_3}\right)$ is an ideal.

7] $H_3(0_0) = 0_0$ $. H_3(1_1) = 1_1$

Theorem (12):

Let H_3 $, G_3$ $:$ $[R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$be two homomorphisms, then:

1] $H_3\left(-X_{f_3(X)}\right) = -H_3\left(X_{f_3(X)}\right)$.

2] $H_3\left(X^{-1}_{f_3(X^{-1})}\right) = [H_3\left(X_{f_3(X)}\right)]^{-1}$, if X is invertible.

3] $H_3 \times G_3$ is a homomorphism.

Definition:

Let $X_{f_3(X)} \in [R_2(I)]_{f_2}$, then:

1] $X_{f_3(X)}$ is idempotent if $X_{f_3(X)} \circ X_{f_3(X)} = X_{f_3(X)}$.

2] $X_{f_3(X)}$ is a nilpotent if there exists $m \in \mathbb{N}$ such that: $X_{f_3(X)} \circ X_{f_3(X)} \circ \ldots \circ X_{f_3(X)}$ $(m - times) = 0_0$

3] $X_{f_3(X)}$ is a zero divisor if there exists $Y_{f_3(Y)}$ such that: $X_{f_3(X)} \circ Y_{f_3(Y)} = 0_0$

Theorem (13):

Let $X_{f_3(X)} \in [R_3(I)]_{f_3}$, then we have:

1] $X_{f_3(X)}$ is idempotent if and only if X is idempotent in $R_3(I)$.

2] $X_{f_3(X)}$ is nilpotent if and only if X is nilpotent in $R_3(I)$.

3] $X_{f_3(X)}$ is a zero divisor if and only if X is a zero diviso in $R_3(I)$.

Theorem (14):

Let H_3: $[R_3(I)]_{f_3} \to [R_3(I)]_{f_3}$, then:

1] If $X_{f_3(X)} \in \ [R_3(I)]_{f_2}$ is idempotent, then $H_3\big(X_{f_3(X)}\big)$ is idempotent.

2] If $X_{f_3(X)}$is nilpotent, then $H_3\big(X_{f_3(X)}\big)$ is nilpotent.

3] If $X_{f_3(X)}$is a zero divisor, then $H_3\big(X_{f_3(X)}\big)$ is a zero divisor.

4] If $X_{f_3(X)}$is a unit, then $H_3\big(X_{f_3(X)}\big)$ is a unit.

Proof of theorem (8):

1] $X_{f_3(X)} * Y_{f_3(Y)} = (X+Y)_{f_3(X+y)} = (Y+X)_{f_3(Y+X)} = Y_{f_3(Y)} * X_{f_3(X)}$.

$X_{f_3(X)} \circ Y_{f_3(Y)} = (XY)_{f_3(Xy)} = (YX)_{f_3(YX)} = Y_{f_3(Y)} \circ X_{f_3(X)}$.

2] $X_{f_3(X)} * Y_{f_3(Y)} * Z_{f_3(Z)}) = X_{f_3(X)} * (Y+Z)_{f_3(Y+Z)} = (X+Y+Z)_{f_3(X+Y+Z)} =$

$(X+Y)_{f_3(X+Y)} * Z_{f_3(Z)} = (X_{f_3(X)} * Y_{f_3(Y)}) * Z_{f_3(Z)}$.

$X_{f_3(X)} \circ Y_{f_2(Y)} \circ Z_{f_2(Z)}) = (XYZ)_{f_3(XYZ)} = (XY)_{f_3(XY)} \circ Z_{f_3(Z)} = (X_{f_3(X)} \circ$

$Y_{f_3(Y)}) \circ Z_{f_3(Z)}$.

3] $X_{f_3(X)} \circ (Y_{f_3(Y)} * Z_{f_3(Z)}) = (XY+XZ)_{f_3(XY+XZ)} = (XY)_{f_2(Xy)} * (XZ)_{f_2(XZ)} = (X_{f_2(X)} \circ$

$Y_{f_2(Y)}) * (X_{f_2(X)} \circ Z_{f_2(Z)})$

4] $X_{f_3(X)} * 0_0 = (X+0)_{f_3(X+0)} = X_{f_3(X)}$.

$X_{f_3(X)} \circ 1_1 = (X \cdot 1)_{f_3(X \cdot 1)} = X_{f_3(X)}$.

5] For $X_{f_3(X)}$, we have $(-X)_{f_3(-X)}$ such that:

$$X_{f_3(X)} * (-X)_{f_3(-X)} = (\mathrm{X} - X)_{f_3(X-X)} = 0_0.$$

Proof of theorem (9):

$X_{f_3(X)} * Y_{f_3(Y)} = (X+Y)_{f_3(X+y)} \ \ \in P_{f_3}$, that is because $X+Y \ \ \in P$.

$r_{f_3(r)} \circ X_{f_3(X)} = (rX)_{f_3(rX)} \in P_{f_3}$, that is because $rX \ \ \in P$.

Proof of theorem (10):

Define: $*': (\ [R_3(I)]_{f_3}/P_{f_3}) \times (\ [R_3(I)]_{f_3}/P_{f_3}) \to \ [R_3(I)]_{f_3}/P_{f_{3_2}}$

$$\circ': (\ [R_3(I)]_{f_3}/P_{f_3}) \times (\ [R_3(I)]_{f_3}/P_{f_3}) \to \ [R_3(I)]_{f_3}/P_{f_3}$$

Such that:

$(X_{f_3(X)}P_{f_3}) *' \left(Y_{f_3(Y)}P_{f_3}\right) = \left(X_{f_2(X)} * Y_{f_3(Y)}\right)\ P_{f_3}$

$(X_{f_3(X)}P_{f_3}) \circ' \left(Y_{f_3(Y)}P_{f_3}\right) = (X_{f_3(X)} \circ Y_{f_3(Y)})P_{f_3}$

We have:

$(X_{f_3(X)}P_{f_3}) *' \left(0_0 P_{f_n}\right) = X_{f_2\ \ (X)}\ \ P_{f_2}$,

$(X_{f_3(X)}P_{f_3}) \circ' \left(1_1 P_{f_2}\right) = X_{f_2(X)}\ \ P_{f_2}$,

$(X_{f_3(X)}P_{f_3}) *' \left((-X)_{f_2(-X)}P_{f_2}\right) = 0_0\ \ P_{f_2}$,

$(X_{f_3(X)}P_{f_3}) *' \left[\left(Y_{f_3(Y)}P_{f_3}\right) *' \left(Z_{f_3(Z)}P_{f_3}\right)\right] = \left((X_{f_3(X)}P_{f_3}) *' \left[(Y * Z)\ \ P_{f_3}\right] = [X * Y * Z]\ \ P_{f_3} =\right.$

$(X_{f_3(X)}P_{f_3}) *' \left(Y_{f_3(Y)}P_{f_3}\right)] *' \left(Z_{f_3(Z)}P_{f_3}\right)$,

$(X_{f_3(X)}P_{f_3}) \circ' \left[\left(Y_{f_3(Y)}P_{f_3}\right) \circ' \left(Z_{f_3(Z)}P_{f_3}\right)\right] = \left(X\ \ P_{f_3}\right) \circ' \left[(Y \circ Z)\ \ P_{f_3}\right] = [X \circ Y \circ Z]\ \ P_{f_3} =$

$(X_{f_3(X)}P_{f_3}) \circ' \left(Y_{f_3(Y)}P_{f_3}\right)] \circ' \left(Z_{f_3(Z)}P_{f_3}\right)$.

$\left((X_{f_3(X)}P_{f_3}) \circ' \left[\left(Y_{f_3(Y)}P_{f_3}\right) *' \left(Z_{f_3(Z)}P_{f_3}\right)\right] = [X \circ (Y * Z)]\ \ P_{f_3} = [(X \circ Y) * (X \circ Z)]\ \ P_{f_3}\right.$

$$= (X_{f_3(X)}P_{f_3}) \circ' \left(Y_{f_3(Y)}P_{f_3}\right)] *' [(X_{f_3(X)}P_{f_3}) \circ' \left(Z_{f_3(Z)}P_{f_3}\right)]$$

Thus, our proof is complete.

Proof of theorem (11):

1] $H_3(X * Y) = (h(X+Y))_{f_3(h(X+Y))} = (h(X)+h(Y))_{f_3((h(X)+h(Y)))} = H_3\left(X_{f_3(X)}\right) * H_3\left(Y_{f_{3_2}(Y)}\right)$.

2] $H_3(X \circ Y) = (h(XY))_{f_3(h(XY))} = (h(X)h(Y))_{f_3((h(X)h(Y)))} = H_3\left(X_{f_3(X)}\right) \circ H_3\left(Y_{f_3(Y)}\right)$.

3] since $k_{er}(H_3) = [k_{er}(h)]_{f_2}$, and $k_{er}(h)$ is an ideal of $R_3(I)$, we get: $k_{er}(H_3)$ is an ideal of $[R_3(I)]_{f_2}$.

4] It can be proved by the same.

5] We have that:

$R_3(I)/k_{er}(h) \quad \cong \quad I_m(h)$, thus:

$[R_3(I)]_{f_3}/\ [k_{er}(h)]_{f_2} \quad \cong \quad [I_m(h)]_{f_2}$, therefor:

$[R_3(I)]_{f_3}/\ k_{er}(H_n) \quad \cong \quad I_m(H_n)$.

6] $H_3(P_{f_3}) = \{\ [h(P)]_{f_3}\}$, and $h(P)$ is an ideal of $R_3(I)$, thus $H_3(P_{f_3})$ is an ideal of $[R_3(I)]_{f_3}$.

7] $\begin{cases} H_3(0_0) = (h(0))_{f_3(h(0))} = 0_0 \\ H_3(1_1) = (h(1))_{f_3(h(1))} = 1_1 \end{cases}$.

Proof of theorem (12):

It is similar to that of theorem 5.

Proof of theorem (13):

It holds by a similar argument of theorem 6.

Proof of theorem (14):

It is similar to that of theorem 7.

Conclusion

In this paper we studied the two-fold fuzzy algebra based on n-refined neutrosophic rings for some different special values of n, where we studied some of special elements in the case of two-fold 2-refined neutrosophic ring and 3-refined neutrosophic ring such as units, idempotenets and nilpotent elements. Also, we presented the concept of two-fold ring homomorphism with its elementary properties.

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