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Some Weaker Forms of Bipolar Neutrosophic Nano * Open Sets

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Abstract: In this paper, the weaker forms of open sets in Bipolar Neutrosophic Nano* (BNN*) topology are studied. This topology is defined on a space of bipolar neutrosophic sets with respect to the lower, upper, boundary approximations and the union and intersection of lower and boundary approximations with maximum of 7 elements. The sets $BNN_Q{}^*$ - Preopen, $BNN_Q{}^*$ - Semi open, $BNN_Q{}^*$ - Regular open, $BNN_Q * - \alpha$ - open and $BNN_Q * - \beta$ - open sets are introduced and their properties are investigated in the corresponding topology in detail and the relationships between them are shown diagrammatically. We proved that, in a BNN^* - topological space $(U, \tau_{R_{BNN^*}}(Q))$, the $BNN_Q *$ - open sets of U and for bipolar neutrosophic sets $E \supset \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}*(Q)) = 1_{BNN^*}$ are the only $BNN_Q * - \alpha$ - open sets in U and also the intersection of any two $BNN_Q * - \alpha$ - open sets is $BNN_Q * - \alpha$ - open set in $(U, \tau_{R_{BNN^*}}(Q))$. Moreover it is shown that, in U the $BNN_Q{}^*$ - open sets $0_{BNN^*}, 1_{BNN^*}$, $B_{BN}*(Q), BN_1*(Q)$ and $\overline{BN}*(Q)$ with $BNN_Q{}^*cl(\overline{BN}*(Q)) = (BN_2*(Q))^C$ are the only $BNN_Q{}^*$ - Regular open sets in U.

Keywords: Nano topology; Neutrosophic set; Bipolar Neutrosophic set; Bipolar Neutrosophic nano topology; Bipolar Neutrosophic nano* topology.

1. Introduction

The concept of fuzzy sets was introduced by Zedeh L. in 1965, which has a single membership grade value attached with each element. Further the generalization of the fuzzy set was made by Atanassov [3] in 1986, known as intuitionistic fuzzy sets. In this set, instead of one membership grade, there is also a non- membership grade attached with each element with a restriction that the sum of these two grades is less than or equal to unity. This concept is useful in the situation of insufficient information. This set is extended to interval valued intuitionistic fuzzy set in 1989 by Atanassov and Gargov [4]. The concept of neutrosophic set is initiated by Smarandache [25] in 1998 which is a generalization of fuzzy sets and intuitionistic fuzzy sets and this set becomes a powerful tool to deal the real life problems with incomplete, indeterminate and inconsistent information. It is characterized by Truth, Indeterminacy and False membership functions and these functions are independent. Salama A.A. and Albowli S.A. [23] introduced Neutrosophic topological spaces. Lee [14] gave an extension of fuzzy sets whose range of membership degree is extended from [0,1] to [-1,1], which is named as bipolar fuzzy set. After that, Deli et. al. [9] defined the concept of bipolar neutrosophic set in 2015.

Many researches have been done in neutrosophic set recently such as in application "Toward Sustainable Emerging Economics based on Industry 5.0: Leveraging Neutrosophic Theory in Appraisal Decision Framework" by Mona Mohamed and Abduallah Gamal, "An Integrated Neutrosophic Regional Management Ranking Method for Agricultural Water Management" by A.Abdel-Monem , A.Nabeeh and M.Abouhawwash, "Towards a Responsive Resilient Supply Chain based on Industry 5.0: A Case Study in Healthcare Systems" by Abduallah Gamal, Amal F.Abd El-Gawad and Mohamed Abouhawwash, "Applications of graph complete degree with bipolar fuzzy information" by soumitra Poulik and Ganesh Ghorai, "Bipolar Neutrosophic Sets and Their Application Based on Multi-Criteria Decision Making Problems" by Irfan Deli, Mumtaz Ali and Florentin Smarandache etc. and in theory "Neutrosophic Pre-open Sets and Pre-closed Sets in Neutrosophic Topology" by Vunnam Venkatewra Rao, "Bipolar neutrosophic soft generalized pre-closed sets and pre-open sets in topological space" by Arulpandy P and Trinita Pricilla M, "Bipolar topological pre-closed neutrosophic sets" by G. Upender Reddy, T. Siva Nageswara Rao, N. Srinivasa Rao and V. Venkateswara Rao. "Bipolar neutrosophic soft generalized pre-continuous mappings" by Arulpandy P and Trinita Pricilla M etc.

Neutrosophic sets were widely used in many topological concepts; in particular, general topology. Most of the general topology concepts were combined with neutrosophic sets and some new topologies were proposed. Lellis Thivagar M. [15] proposed the concept of Nano topology which was defined in terms of approximations and boundary region of a subset of a universe using an equivalence relation on it. In 2022, we defined a topology bipolar neutrosophic nano topology as a combination of nano topology and bipolar neutrosophic set. But in this case, we only get topologies for equivalence relations with independent singleton sets of elements of the universe. We decided to construct a definition to find topologies for each bipolar set irrespective of equivalence relation. Thus, we introduced a topology called Bipolar Neutrosophic nano * topology [10] which consist of maximum 7 elements. In this paper, we introduced and studied some weaker forms of Bipolar neutrosophic nano* open sets $(BNN_Q * O)$, namely, $BNN_Q *$ -Preopen sets, $BNN_Q *$ -Regular open sets, $BNN_Q *$ -Semi open sets, BNN_Q * - α open sets and BNN_Q * - β open sets. We found the limitations of these open sets with respect to a particular bipolar neutrosophic set and also investigated the properties of them and the relationships between them in detail.

This manuscript is organized as follows: Section 2 contains some basic definitions related to this manuscript. Section 3 consists of weaker forms of bipolar neutrosophic nano* open sets. Sub section 3.1 consists of the properties and results based on bipolar neutrosophic nano* preopen sets. Sub section 3.2 consists of the properties and results based on bipolar neutrosophic nano* semi open sets. Sub section 3.3 consists of the properties and results based on bipolar neutrosophic nano* α open sets. In particular, we proved that, in a BNN^* - topological space $(U, \tau_{R_{BNN^*}}(Q))$, the BNN_Q * open sets of U and for bipolar neutrosophic sets $E \supset \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = 1_{BNN^*}$ are the only BNN_Q * - α open sets in U and also the intersection of any two BNN_Q * - α open sets is BNN_Q * - α open set in $(U, \tau_{R_{BNN^*}}(Q))$. Sub section 3.4 consists of the properties and results based on bipolar neutrosophic nano* regular open sets. In this section, it is shown that, in U the BNN_Q * - open sets $0_{BNN^*}, 1_{BNN^*}$, $B_{BN}{}^*(Q), BN_1{}^*(Q)$ and $\overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = (BN_2{}^*(Q))^C$ are the only BNN_Q * - Regular open sets in U. Sub section 3.5 consists of the properties and results based on bipolar neutrosophic nano* β open sets. The properties and relationship between the sets are clearly explained with several examples.

2. Preliminaries

Definition: 2.19 [10] Let U be a nonempty set and R be an equivalence relation on U which is indiscernible. Then U can be divided into disjoint equivalence classes. Let Q be a bipolar neutrosophic set (BNS) in U with the positive degree of true membership η_Q^+, indeterminacy ψ_Q^+ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership η_Q^-, indeterminacy ψ_Q^- and the false membership function $\xi_Q{}^-$, where $\eta_Q^+, \psi_Q^+, \xi_Q^+ : U \to [0,1]$, $\eta_Q^-, \psi_Q^-, \xi_Q^- : U \to [-1,0]$. Then the lower, upper and boundary approximations are respectively given as follows:

(i) $\underline{BN}(Q) = \left\{\left\langle q, \left(\eta^+_{\underline{R}(Q)}(q), \psi^+_{\underline{R}(Q)}(q), \xi^+_{\underline{R}(Q)}(q), \eta^-_{\underline{R}(Q)}(q), \psi^-_{\underline{R}(Q)}(q), \xi^-_{\underline{R}(Q)}(q)\right)\right\rangle : z \in [q]_R, q \in U\right\}$.

(ii) $\overline{BN}(Q) = \left\{\left\langle q, \left(\eta^+_{\overline{R}(Q)}(q), \psi^+_{\overline{R}(Q)}(q), \xi^+_{\overline{R}(Q)}(q), \eta^-_{\overline{R}(Q)}(q), \psi^-_{\overline{R}(Q)}(q), \xi^-_{\overline{R}(Q)}(q)\right)\right\rangle : z \in [q]_R, q \in U\right\}$.

(iii) $B_{BN}(Q) = \overline{BN}(Q) - \underline{BN}(Q)$. where,

$\eta_{\underline{R}^*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^+(z)$, $\psi_{\underline{R}^*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^+(z)$, $\xi_{\underline{R}^*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^+(z)$,

$\eta_{\underline{R}^*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^-(z)$, $\psi_{\underline{R}^*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^-(z)$, $\xi_{\underline{R}^*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^-(z)$,

$\eta_{\overline{R}^*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^+(z)$, $\psi_{\overline{R}^*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^+(z)$, $\xi_{\overline{R}^*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^+(z)$,

$\eta_{\overline{R}^*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^-(z)$, $\psi_{\overline{R}^*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^-(z)$, $\xi_{\overline{R}^*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^-(z)$.

Definition: 2.2 [10] Let U be a nonempty set, R be an equivalence relation on U and let Q be a BNS. The collection $\tau_{R_{BNN}}(Q) = \{0_{BNN}, 1_{BNN}, \underline{BN}(Q), \overline{BN}(Q), B_{BN}(Q)\}$ is called the bipolar neutrosophic nano topology (BNN_Q - topology), if it forms a topology. Then the space $(U, \tau_{R_{BNN}}(Q))$ is called the bipolar neutrosophic nano topological space. The elements of $\tau_{R_{BNN}}(Q)$ are called bipolar neutrosophic nano open sets (BNN_QO).

Remark: 2.3 [10] For every bipolar neutrosophic set, we cannot find a corresponding bipolar neutrosophic nano topology in U. So we defined a topology called Bipolar neutrosophic nano * topology which corresponds to any bipolar neutrosophic set in U with respect to its boundary and approximations.

Definition: 2.4 [10] Let U be a nonempty set and R^* be a relation on U, which is indiscernible. Then U can be divided into disjoint equivalence classes. Let Q be a BNS in U with the positive degree of true membership $\eta_Q{}^+$, indeterminacy $\psi_Q{}^+$ and the false membership function $\xi_Q{}^+$ and the negative degree of true membership $\eta_Q{}^-$, indeterminacy $\psi_Q{}^-$ and the false membership function $\xi_Q{}^-$, where, $\eta_Q^+, \psi_Q^+, \xi_Q^+ : U \to [0,1]$, $\eta_Q^-, \psi_Q^-, \xi_Q^- : U \to [-1,0]$. Then

(i) $\underline{BN}^*(Q) = \left\{\left\langle q, \left(\eta^+_{\underline{R}^*(Q)}(q), \psi^+_{\underline{R}^*(Q)}(q), \xi^+_{\underline{R}^*(Q)}(q), \eta^-_{\underline{R}^*(Q)}(q), \psi^-_{\underline{R}^*(Q)}(q), \xi^-_{\underline{R}^*(Q)}(q)\right)\right\rangle : z \in [q]_{R^*}, q \in U\right\}$ is the lower approximation of Q in respect of R^*.

(ii) $\overline{BN}^*(Q) = \left\{\left\langle q, \left(\eta^+_{\overline{R}^*(Q)}(q), \psi^+_{\overline{R}^*(Q)}(q), \xi^+_{\overline{R}^*(Q)}(q), \eta^-_{\overline{R}^*(Q)}(q), \psi^-_{\overline{R}^*(Q)}(q), \xi^-_{\overline{R}^*(Q)}(q)\right)\right\rangle : z \in [q]_{R^*}, q \in U\right\}$ is the upper approximation of Q in respect of R^*.

(iii) $B_{BN}{}^*(Q) = \overline{BN}^*(Q) - \underline{BN}^*(Q)$ is the boundary of Q in respect of R^*.

(iv) $BN_1{}^*(Q) = \underline{BN}^*(Q) \cup B_{BN}{}^*(Q)$.

(v) $BN_2{}^*(Q) = \underline{BN}^*(Q) \cap B_{BN}{}^*(Q)$. where,

$\eta_{\underline{R}^*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^+(z)$, $\psi_{\underline{R}^*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^+(z)$, $\xi_{\underline{R}^*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^+(z)$,

$\eta_{\underline{R}^*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^-(z)$, $\psi_{\underline{R}^*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^-(z)$, $\xi_{\underline{R}^*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^-(z)$,

$\eta_{\overline{R}^*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \eta_Q{}^+(z)$, $\psi_{\overline{R}^*(Q)}{}^+(q) = \bigvee_{z\in[q]_{R^*}} \psi_Q{}^+(z)$, $\xi_{\overline{R}^*(Q)}{}^+(q) = \bigwedge_{z\in[q]_{R^*}} \xi_Q{}^+(z)$,

$\eta_{\overline{R}^*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \eta_Q{}^-(z)$, $\psi_{\overline{R}^*(Q)}{}^-(q) = \bigwedge_{z\in[q]_{R^*}} \psi_Q{}^-(z)$, $\xi_{\overline{R}^*(Q)}{}^-(q) = \bigvee_{z\in[q]_{R^*}} \xi_Q{}^-(z)$.

Then the collection $\tau_{R_{BNN}}*(Q)=\{0_{BNN^*},1_{BNN^*},\underline{BN}*(Q),\overline{BN}*(Q),B_{BN}*(Q),BN_1*(Q),BN_2*(Q)\}$ is a topology which is called a bipolar neutrosophic nano$*$ topology (BNN_Q* - topology). The space $(U,\tau_{R_{BNN^*}}(Q))$ is called a bipolar neutrosophic nano$*$ topological space. The elements of $\tau_{\Re_{BNN}}*(Q)$ are bipolar neutrosophic nano $*$ open sets (BNN_Q*O). The complements of these elements are called bipolar neutrosophic nano $*$ closed sets (BNN_Q*C).

Definition: 2.5[10] Let U be a nonempty universe and K and H be the BNS's, where $K=\{\langle q,(\eta_K^+(q),\psi_K^+(q),\xi_K^+(q),\eta_K^-(q),\psi_K^-(q),\xi_K^-(q))\rangle : q\in U\}$ and $H=\{\langle q,(\eta_H^+(q),\psi_H^+(q),\xi_H^+(q),\eta_H^-(q),\psi_H^-(q),\xi_H^-(q))\rangle : q\in U\}$. Then,

(i) the null bipolar neutrosophic nano set is given by $0_{BNN}=\{\langle q,(0,0,1,0,0,-1)\rangle : q\in U\}$.

(ii) the absolute bipolar neutrosophic nano set is given by $1_{BNN}=\{\langle q,(1,1,0,-1,-1,0)\rangle : q\in U\}$.

(iii) $K\subseteq H$ iff $\eta_K^+(q)\le\eta_H^+(q)$, $\psi_K^+(q)\le\psi_H^+(q)$, $\xi_K^+(q)\ge\xi_H^+(q)$,
$\eta_K^-(q)\ge\eta_H^-(q)$, $\psi_K^-(q)\ge\psi_H^-(q)$, $\xi_K^-(q)\le\xi_H^-(q)$.

(iv) $K=H$ iff $K\subseteq H$ and $H\subseteq K$.

(v) $K^C=\{\langle q,(\xi_K^+(q),1-\psi_K^+(q),\eta_K^+(q),\xi_K^-(q),-1-\psi_K^-(q),\eta_K^-(q))\rangle : q\in U\}$.

(vi) $$K\cap H=\left\{\left\langle q,\begin{pmatrix}\eta_K^+(q)\wedge\eta_H^+(q),\psi_K^+(q)\wedge\psi_H^+(q),\xi_K^+(q)\vee\xi_H^+(q),\\ \eta_K^-(q)\vee\eta_H^-(q),\psi_K^-(q)\vee\psi_H^-(q),\xi_K^-(q)\wedge\xi_H^-(q)\end{pmatrix}\right\rangle : q\in U\right\}.$$

(vii) $$K\cup H=\left\{\left\langle q,\begin{pmatrix}\eta_K^+(q)\vee\eta_H^+(q),\psi_K^+(q)\vee\psi_H^+(q),\xi_K^+(q)\wedge\xi_H^+(q),\\ \eta_K^-(q)\wedge\eta_H^-(q),\psi_K^-(q)\wedge\psi_H^-(q),\xi_K^-(q)\vee\xi_H^-(q)\end{pmatrix}\right\rangle : q\in U\right\}.$$

(viii) $$K-H=\left\{\left\langle q,\begin{pmatrix}\min\{\eta_K^+(q),\xi_H^+(q)\},\min\{\psi_K^+(q),1-\psi_H^+(q)\},\max\{\xi_K^+(q),\eta_H^+(q)\},\\ \max\{\eta_K^-(q),\xi_H^-(q),\max\{\psi_K^-(q),-1-\psi_H^-(q),\min\{\xi_K^-(q),\eta_H^-(q)\}\end{pmatrix}\right\rangle : q\in U\right\}.$$

Remark: 2.6 [10] In a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$, by definition

$BNN_Q*\mathrm{int}\left((\overline{BN}*(Q))^C\right)=BN_2*(Q)$ or 0_{BNN^*},

$BNN_Q*\mathrm{int}((\underline{BN}*(Q))^C)=B_{BN}*(Q)$,

$BNN_Q*\mathrm{int}((B_{BN}*(Q))^C)=BN_1*(Q)$,

$BNN_Q*\mathrm{int}((BN_1*(Q))^C)=B_{BN}*(Q)$,

$BNN_Q*\mathrm{int}((BN_2*(Q))^C)=BN_1*(Q)$.

And

$BNN_Q*\mathrm{cl}(\overline{BN}*(Q))=(BN_2*(Q))^C$ or 1_{BNN^*},

$BNN_Q*\mathrm{cl}(\underline{BN}*(Q))=(B_{BN}*(Q))^C$,

$BNN_Q*\mathrm{cl}(B_{BN}*(Q))=(BN_1*(Q))^C$,

$BNN_Q*\mathrm{cl}(BN_1*(Q))=(B_{BN}*(Q))^C$,

$BNN_Q*\mathrm{cl}(BN_2*(Q))=(BN_1*(Q))^C$.

3. Weaker forms of Bipolar Neutrosophic Nano * Topology

In this section, we are going to introduce some of the weaker forms of open sets in Bipolar Neutrosophic Nano* Topology.

*3.1 Bipolar Neutrosophic Nano * Pre-Open Sets*

Definition: 3.1.1 Let E be a bipolar neutrosophic set in a BNN^*-topological space (BNN^*TS) $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be BNN_Q^* - pre-open set (BNN_Q^*PO set) of U if $E\subseteq BNN_Q^*\ \mathrm{int}(BNN_Q^*\mathrm{cl}(E))$. The complement of BNN_Q^*PO set is called BNN_Q^* - pre-closed set

(BNN_Q^*PC set) of U.

Theorem: 3.1.2 Arbitrary union of BNN_Q^*PO sets in $(U,\tau_{R_{BNN^*}}(Q))$ is BNN_Q^*PO set in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of BNN_Q^*PO sets in $(U,\tau_{R_{BNN^*}}(Q))$. For each $\alpha\in\Omega$, $E_\alpha\subseteq BNN_Q^*\ \mathrm{int}(BNN_Q^*\mathrm{cl}(E_\alpha))$.

$$\begin{aligned}E_1\cup E_2\cup\ldots\ldots &\subseteq BNN_Q^*\ \mathrm{int}(BNN_Q^*\mathrm{cl}(E_1))\cup BNN_Q^*\ \mathrm{int}(BNN_Q^*\mathrm{cl}(E_2))\cup\ldots\ldots\ldots\\ &\subseteq BNN_Q^*\ \mathrm{int}(BNN_Q^*\mathrm{cl}(E_1)\cup BNN_Q^*\mathrm{cl}(E_2)\cup\ldots..)\\ &\subseteq BNN_Q^*\ \mathrm{int}(BNN_Q^*\mathrm{cl}(E_1\cup E_2\cup\ldots.))\end{aligned}$$

Hence $\bigcup_{\alpha\in\Omega}E_\alpha$ is BNN_Q^*PO set in U.

Remark: 3.1.3 The intersection of any two $BNN_Q{}^*PO$ sets need not be a $BNN_Q{}^*PO$ set in U. This is shown in the following example.

Example: 3.1.4 Let $U=\{p_1,p_2\}, U/R=\{\{p_1\},\{p_2\}\}$, $Q=\left\{\begin{matrix}\langle p_1,(0.4,0.8,0.6,-0.5,-0.2,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\}$.

$$\tau_{R_{BNN^*}}(Q)=\left\{0_{BN^*},1_{BN^*},\left\{\begin{matrix}\langle p_1,(0.4,0.8,0.6,-0.5,-0.2,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\},\left\{\begin{matrix}\langle p_1,(0.4,0.2,0.6,-0.1,-0.2,-0.5)\rangle\\ \langle p_2,(0.3,0.5,0.7,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\}\right\}.$$

$$\left(\tau_{R_{BNN^*}}(Q)\right)^C=\left\{0_{BN^*},1_{BN^*},\left\{\begin{matrix}\langle p_1,(0.6,0.2,0.4,-0.1,-0.8,-0.5)\rangle\\ \langle p_2,(0.3,0.5,0.7,-0.6,-0.6,-0.2)\rangle\end{matrix}\right\},\left\{\begin{matrix}\langle p_1,(0.6,0.8,0.4,-0.5,-0.8,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.6,-0.6,-0.2)\rangle\end{matrix}\right\}\right\}.$$

$E_1=\left\{\begin{matrix}\langle p_1,(0.6,0.8,0.3,-0.3,-0.7,-0.4)\rangle\\ \langle p_2,(0.4,0.6,0.6,-0.4,-0.6,-0.6)\rangle\end{matrix}\right\}$, $E_2=\left\{\begin{matrix}\langle p_1,(0.3,0.4,0.6,-0.7,-0.6,-0.3)\rangle\\ \langle p_2,(0.4,0.3,0.5,-0.6,-0.5,-0.3)\rangle\end{matrix}\right\}$ are $BNN_Q{}^*PO$ sets.

$E_1\cap E_2=\left\{\begin{matrix}\langle p_1,(0.3,0.4,0.6,-0.3,-0.6,-0.4)\rangle\\ \langle p_2,(0.4,0.3,0.6,-0.4,-0.5,-0.6)\rangle\end{matrix}\right\}$ and $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1\cap E_2))=\left\{\begin{matrix}\langle p_1,(0.4,0.8,0.6,-0.5,-0.2,-0.1)\rangle\\ \langle p_2,(0.7,0.5,0.3,-0.2,-0.4,-0.6)\rangle\end{matrix}\right\}$.

Clearly $E_1\cap E_2\not\subset BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E_1\cap E_2))$. Hence $E_1\cap E_2$ is not $BNN_Q{}^*PO$ set.

Theorem: 3.1.5 Every $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*PO$ set in U.

Proof. Let E be $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*\operatorname{int}(E)$. Also $E\subseteq BNN_Q{}^*cl(E)$. $BNN_Q{}^*\operatorname{int}(E)\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))$. $E\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))$. E is $BNN_Q{}^*PO$ set in U.

Remark: 3.1.6 The following example shows that the converse of the above theorem is not true.

Example: 3.1.7 From example 3.1.4, $E_1=\left\{\begin{matrix}\langle p_1,(0.6,0.8,0.3,-0.3,-0.7,-0.4)\rangle\\ \langle p_2,(0.4,0.6,0.6,-0.4,-0.6,-0.6)\rangle\end{matrix}\right\}$ is $BNN_Q{}^*PO$ set but not $BNN_Q{}^*O$ set.

Theorem: 3.1.8 If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*PC$ sets in $(U,\tau_{R_{BNN^*}}(Q))$, then $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*PC$ set in U.

Proof. $\{E_\alpha{}^C\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*PO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. By theorem 3.1.2 and De-Morgan's law $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*PC$ set in U.

Remark: 3.1.9 By remark 3.1.3, the union of any two $BNN_Q{}^*PC$ sets need not be a $BNN_Q{}^*PC$ set in U.

Theorem: 3.1.10 Every $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*PC$ set in U.

Proof. Let E be a $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*cl(E)$. Also $BNN_Q{}^*\operatorname{int}(E)\subseteq E$. $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))\subseteq BNN_Q{}^*cl(E)$. $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))\subseteq E$. Hence E is $BNN_Q{}^*PC$ set in U.

*3.2 Bipolar Neutrosophic Nano * Semi Open Sets*

Definition: 3.2.1 Let E be a neutrosophic set in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be $BNN_Q{}^*$ - semi-open set ($BNN_Q{}^*SO$ set) of U if $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. The complement of $BNN_Q{}^*SO$ set is called $BNN_Q{}^*$ - semi-closed set ($BNN_Q{}^*SC$ set) of U.

Theorem: 3.2.2 Arbitrary union of $BNN_Q{}^*SO$ is $BNN_Q{}^*SO$ set in U.

Proof. If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*SO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. For each $\alpha\in\Omega$, $E_\alpha\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_\alpha))$.

$$\begin{aligned}E_1\cup E_2\cup\ldots\ldots &\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1))\cup BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2))\cup\ldots\ldots\ldots\\ &= BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1)\cup BNN_Q{}^*\operatorname{int}(E_2)\cup\ldots\ldots)\\ &\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1\cup E_2\cup\ldots\ldots))\end{aligned}$$

Hence $\bigcup_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*SO$ set in U.

Theorem: 3.2.3 Every $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*SO$ set in U.

Proof. Let E be $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*\operatorname{int}(E)$. $BNN_Q{}^*cl(E)=BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. Also $E\subseteq BNN_Q{}^*cl(E)$. Then $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))$. E is $BNN_Q{}^*SO$ set in U.

Remark: 3.2.4 The following example shows that the converse of the above theorem is not true.

Example: 3.2.5 Let $E=\left\{\begin{matrix}\langle p_1,(0.4,0.2,0.6,-0.4,-0.3,-0.5)\rangle\\ \langle p_2,(0.3,0.5,0.7,-0.4,-0.4,-0.5)\rangle\end{matrix}\right\}$.

From example 3.1.4, $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))=(BN_1{}^*(Q))^C$. Also $E\subseteq(BN_1{}^*(Q))^C$.

$\therefore$ E is $BNN_Q{}^*SO$ set but not $BNN_Q{}^*O$ set.

Theorem: 3.2.6 Arbitrary intersection of $BNN_Q{}^*SC$ set is $BNN_Q{}^*SC$ set in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*SC$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. Then $\{E_\alpha{}^C\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*SO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. By theorem 3.2.2 and by De-Morgan's law, $\bigcap_{\alpha\in\Omega} E_\alpha$ is $BNN_Q{}^*SC$ set in U.

Theorem: 3.2.7 Every $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*SC$ set in U.

Proof. Let E be a $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E = BNN_Q{}^*cl(E)$. Also. $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)) = BNN_Q{}^*\operatorname{int}(E)$. $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)) \subseteq E$. Hence E is $BNN_Q{}^*SC$ set in U.

*3.3 Bipolar Neutrosophic Nano * α Open Sets*

Definition: 3.3.1 Let E be a neutrosophic set in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be $BNN_Q{}^*$ - α -open set ($BNN_Q{}^*\alpha O$ set) of U if $E \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)))$. The complement of $BNN_Q{}^*\alpha O$ set is called $BNN_Q{}^*$ - α -closed set ($BNN_Q{}^*\alpha C$ set) of U.

Theorem: 3.3.2 Arbitrary union of $BNN_Q{}^*\alpha O$ is $BNN_Q{}^*\alpha O$ set in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\alpha O$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. For each $\alpha\in\Omega$, $E_\alpha \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_\alpha)))$.

$$
\begin{aligned}
E_1 \cup E_2 \cup \ldots\ldots &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1))) \cup BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2))) \cup \ldots\ldots\ldots \\
&\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1) \cup BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2)) \cup \ldots\ldots)) \\
&= BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1) \cup BNN_Q{}^*\operatorname{int}(E_2) \cup \ldots\ldots)) \\
&\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1 \cup E_2 \cup \ldots\ldots)))
\end{aligned}
$$

Hence $\bigcup_{\alpha\in\Omega} E_\alpha$ is $BNN_Q{}^*\alpha O$ set in U.

Theorem: 3.3.3 In a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$, the $BNN_Q{}^*$- open sets of U and for sets $E \supset \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = 1_{BNN^*}$ are the only $BNN_Q{}^*\alpha O$ sets in U.

Proof. Since $BNN_Q{}^*O$ sets are $BNN_Q{}^*\alpha O$, then $0_{BNN^*}, 1_{BNN^*}, \overline{BN}^*(Q), \underline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), BN_2{}^*(Q)$ are $BNN_Q{}^*\alpha O$ in U. If $E \neq 0_{BNN^*}$ and $E \subset \underline{BN}^*(Q)$, then $BNN_Q{}^*\operatorname{int}(E) = 0_{BNN^*}$, since 0_{BNN^*} is the only $BNN_Q{}^*O$ subset of E. Therefore $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))) = 0_{BNN^*}$ and hence E is not $BNN_Q{}^*\alpha O$. If $E \subset B_{BN}{}^*(Q)$, then $BNN^*\operatorname{int}(E) = 0_{BNN^*}$ and hence E is not $BNN_Q{}^*\alpha O$. If $E \subset \overline{BN}^*(Q)$, then $E \subset B_{BN}{}^*(Q)$ and $E \subset \underline{BN}^*(Q)$, hence E is not $BNN_Q{}^*\alpha O$. If $E \supset \overline{BN}^*(Q)$, then $BBN_Q{}^*\operatorname{int}(E) = \overline{BN}^*(Q)$ and hence $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))) = BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(\overline{BN}^*(Q)))$ $= BNN_Q{}^*\operatorname{int}(1_{BNN^*}) = 1_{BNN^*}$. Therefore $E \subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E)))$. E is $BNN_Q{}^*\alpha O$. This will exist only in the case if $BNN_Q{}^*cl(\overline{BN}^*(Q)) = 1_{BNN^*}$. If $E \subset \underline{BN}^*(Q)$ and $E \subset B_{BN}{}^*(Q)$, by definition $E \subset BN_1{}^*(Q)$ and $E \subset BN_2{}^*(Q)$, then in both the cases E is not $BNN_Q{}^*\alpha O$.

Remark: 3.3.4 The following example shows that the case $E \supset \overline{BN}^*(Q)$ in the above theorem in which $BNN_Q{}^*cl(\overline{BN}^*(Q)) \neq 1_{BNN^*}$ is not $BNN_Q{}^*\alpha O$.

Example: 3.3.5 Let $U = \{p_1, p_2, p_3\}$, $U/R = \{\{p_1, p_3\}, \{p_2\}\}$, $Q = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.3,-0.4,-0.7)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.2,0.3,0.8,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$.

$\overline{BN}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$, $\underline{BN}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle\end{matrix}\right\}$,

$B_{BN}{}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$, $BN_1{}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle \\ \langle p_2,(0.5,0.6,0.4,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.6,0.5,0.2,-0.5,-0.6,-0.4)\rangle\end{matrix}\right\}$,

$BN_2{}^*(Q) = \left\{\begin{matrix}\langle p_1,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle \\ \langle p_2,(0.4,0.4,0.5,-0.4,-0.5,-0.5)\rangle \\ \langle p_3,(0.2,0.3,0.8,-0.3,-0.4,-0.7)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN^*(Q)}} = \{0_{BNN^*}, \overline{BN}^*(Q), \underline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), BN_2{}^*(Q), 1_{BNN^*}\}$.

Let $E=\left\{\begin{matrix}\langle p_1,(0.7,0.6,0.1,-0.6,-0.7,-0.3)\rangle\\ \langle p_2,(0.6,0.7,0.3,-0.5,-0.5,-0.4)\rangle\\ \langle p_3,(0.8,0.6,0.1,-0.7,-0.7,-0.3)\rangle\end{matrix}\right\}\supset \overline{BN}*(Q)$, then $BNN_Q*\mathrm{int}(E)=\overline{BN}*(Q)$.

$BNN_Q*\mathrm{cl}(BNN_Q*\mathrm{int}(E))=BNN_Q*\mathrm{cl}(\overline{BN}*(Q))=(BN_2*(Q))^C$.

$BNN_Q*\mathrm{int}(BNN_Q*\mathrm{cl}(BNN_Q*\mathrm{int}(E)))=BNN_Q*\mathrm{int}((BN_2*(Q))^C)=\overline{BN}*(Q)$. $E\not\subset\overline{BN}*(Q)$, since it contains $\overline{BN}*(Q)$. Hence E is not $BNN_Q{}^*\alpha O$.

Theorem: 3.3.6 The intersection of any two $BNN_Q{}^*\alpha O$ sets is $BNN_Q{}^*\alpha O$ set in $(U,\tau_{R_{BNN^*}}(Q))$.

Proof. From the above theorem, the $BNN_Q{}^*$ - open sets of U and for sets $E\supset\overline{BN}^*(Q)$ where $BNN_Q{}^*\mathrm{cl}(\overline{BN}*(Q))=1_{BNN^*}$ are the only $BNN_Q{}^*\alpha O$ sets in U. Finite intersection of $BNN_Q{}^*$ - open sets is $BNN_Q{}^*$ - open and hence $BNN_Q{}^*\alpha O$. If $E_1,E_2\supset\overline{BN}^*(Q)$ such that $BNN_Q{}^*\mathrm{int}(E_1)=\overline{BN}*(Q)$, $BNN_Q{}^*\mathrm{int}(E_2)=\overline{BN}*(Q)$ and $BNN_Q{}^*\mathrm{cl}(\overline{BN}*(Q))=1_{BNN^*}$, then

$BNN_Q{}^*\mathrm{int}(E_1\cap E_2)=BNN_Q{}^*\mathrm{int}(E_1)\cap BNN_Q{}^*\mathrm{int}(E_2)=\overline{BN}*(Q)$.

$BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(E_1\cap E_2))=BNN_Q{}^*\mathrm{cl}(\overline{BN}*(Q))=1_{BNN^*}$.

$BNN_Q{}^*\mathrm{int}(BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(E_1\cap E_2)))=BNN_Q{}^*\mathrm{int}(1_{BNN^*})=1_{BNN^*}$.

Hence the intersection of any two $BNN_Q{}^*\alpha O$ sets is $BNN_Q{}^*\alpha O$ set in U.

Theorem: 3.3.7 Every $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*\alpha O$ set in U.

Proof. Let E be $BNN_Q{}^*O$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E=BNN_Q{}^*\mathrm{int}(E)$. $BNN_Q{}^*\mathrm{cl}(E)=BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(E))$. Also $E\subseteq BNN_Q{}^*\mathrm{cl}(E)$. Then $E\subseteq BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(E))$.

Now $BNN_Q{}^*\mathrm{int}(E)\subseteq BNN_Q{}^*\mathrm{int}(BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(E)))$.

Thus $E\subseteq BNN_Q{}^*\mathrm{int}(BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(E)))$. Hence E is $BNN_Q{}^*\alpha O$ set in U.

Example: 3.3.8 The converse of the above theorem need not be true. For example, let

$U=\{p_1,p_2,p_3\}, U/R=\{\{p_1,p_2\},\{p_3\}\}$, $Q=\left\{\begin{matrix}\langle p_1,(0.2,0.5,0.6,-0.7,-0.5,-0.2)\rangle\\ \langle p_2,(0.3,0.4,0.6,-0.6,-0.4,-0.3)\rangle\\ \langle p_3,(0.2,0.3,0.8,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$.

$\overline{BN}*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.5,0.6,-0.7,-0.5,-0.2)\rangle\\ \langle p_2,(0.3,0.5,0.6,-0.7,-0.5,-0.2)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$, $\underline{BN}*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.4,0.6,-0.6,-0.4,-0.3)\rangle\\ \langle p_2,(0.2,0.4,0.6,-0.6,-0.4,-0.3)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$,

$B_{BN}*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.5,0.6,-0.3,-0.5,-0.6)\rangle\\ \langle p_2,(0.3,0.5,0.6,-0.3,-0.5,-0.6)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.4,-0.5,-0.5)\rangle\end{matrix}\right\}$, $BN_1*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.5,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_2,(0.3,0.5,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.5,-0.5,-0.4)\rangle\end{matrix}\right\}$,

$BN_2*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.4,0.6,-0.3,-0.4,-0.6)\rangle\\ \langle p_2,(0.2,0.4,0.6,-0.3,-0.4,-0.6)\rangle\\ \langle p_3,(0.4,0.5,0.5,-0.4,-0.5,-0.5)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN}*(Q)}=\{0_{BNN^*},\overline{BN}*(Q),\underline{BN}*(Q),B_{BN}*(Q),BN_1*(Q),BN_2*(Q),1_{BNN^*}\}$.

Let $E=\left\{\begin{matrix}\langle p_1,(0.4,0.5,0.5,-0.7,-0.6,-0.2)\rangle\\ \langle p_2,(0.4,0.5,0.5,-0.7,-0.6,-0.2)\rangle\\ \langle p_3,(0.5,0.5,0.3,-0.6,-0.5,-0.2)\rangle\end{matrix}\right\}$, then $BNN_Q*\mathrm{int}(BNN_Q*\mathrm{cl}(BNN_Q*\mathrm{int}(E)))=1_{BNN^*}$.

E is $BNN_Q{}^*\alpha O$ but not $BNN_Q{}^*O$ set.

Theorem: 3.3.9 Arbitrary intersection of $BNN_Q{}^*\alpha C$ sets is $BNN_Q{}^*\alpha C$ set in $(U,\tau_{R_{BNN^*}}(Q))$.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\alpha C$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. Then $\{E_\alpha{}^C\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\alpha O$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. By theorem 3.3.2 and De-Morgan's law $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*\alpha C$ set in U.

Remark: 3.3.10 By theorem: 3.3.6, union of two $BNN_Q{}^*\alpha C$ sets is a $BNN_Q{}^*\alpha C$ set in U.

Theorem: 3.3.11 Every $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$ is $BNN_Q{}^*\alpha C$ set in U.

Proof. Let E be a $BNN_Q{}^*C$ set in $(U,\tau_{R_{BNN^*}}(Q))$. Then $BNN_Q{}^*\mathrm{cl}(E)=E$. Also $BNN_Q{}^*\mathrm{int}(E)\subseteq E$.

$BNN_Q{}^*\mathrm{int}(BNN_Q{}^*\mathrm{cl}(E))=BNN_Q{}^*\mathrm{int}(E)\subseteq E$. $BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(BNN_Q{}^*\mathrm{cl}(E)))\subseteq BNN_Q{}^*\mathrm{cl}(E)$.

$BNN_Q{}^*\mathrm{cl}(BNN_Q{}^*\mathrm{int}(BNN_Q{}^*\mathrm{cl}(E)))\subseteq E$. Hence E is $BNN_Q{}^*\alpha C$ set in U.

Remark: 3.3.12 The set of all BNN_Q^* - open sets, BNN_Q^* - pre open sets, BNN_Q^* - semi open sets and BNN_Q^* - α open sets of $\left(U,\tau_{R_{BNN^*}}(Q)\right)$ are denoted by $BNN_Q^*O(U)$, $BNN_Q^*PO(U)$, $BNN_Q^*SO(U)$ and $BNN_Q^*\alpha O(U)$ respectively. The set of all BNN_Q^* - closed sets, BNN_Q^* - pre closed sets, BNN_Q^* - semi closed sets and BNN_Q^* - α closed sets of $\left(U,\tau_{R_{BNN^*}}(Q)\right)$ are denoted by $BNN_Q^*C(U)$, $BNN_Q^*PC(U)$, $BNN_Q^*SC(U)$ and $BNN_Q^*\alpha C(U)$ respectively.

Theorem: 3.3.13 $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*SO(U)$ in a BNN^*TS $\left(U,\tau_{R_{BNN^*}}(Q)\right)$.

Proof. If $E \in BNN_Q^*\alpha O(U)$. $E \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(E)\right)\right) \subseteq BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(E)\right)$.
Then $E \in BNN_Q^*SO(U)$. Hence $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*SO(U)$ in U.

Remark: 3.3.14 The converse of the above theorem need not be true. This is shown in the following example.

Example: 3.3.15 Let $E = \begin{Bmatrix} \langle p_1,(0.4,0.2,0.6,-0.1,-0.3,-0.5)\rangle \\ \langle p_2,(0.3,0.5,0.7,-0.4,-0.4,-0.5)\rangle \end{Bmatrix}$. From example: 3.1.4, E is BNN_Q^*SO but not $BNN_Q^*\alpha O$.

Theorem: 3.3.16 $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*PO(U)$ in a BNN^*TS $\left(U,\tau_{R_{BNN^*}}(Q)\right)$.

Proof. If $E \in BNN_Q^*\alpha O(U)$. $E \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(E)\right)\right)$.
Since $BNN_Q^*\operatorname{int}(E) \subseteq E$, $E \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(E)\right)\right) \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E)\right)$.
Then $E \in BNN_Q^*PO(U)$. Hence $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*PO(U)$ in U.

Remark: 3.3.17 The converse of the above theorem need not be true. This is shown in the following example.

Example: 3.3.18 Let $E = \begin{Bmatrix} \langle p_1,(0.6,0.8,0.3,-0.3,-0.7,-0.4)\rangle \\ \langle p_2,(0.4,0.6,0.6,-0.4,-0.6,-0.6)\rangle \end{Bmatrix}$. From example: 3.1.4, E is BNN_Q^*PO but not $BNN_Q^*\alpha O$.

Theorem: 3.3.19 $BNN_Q^*\alpha O(U) = BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$ in a BNN^*TS $\left(U,\tau_{R_{BNN^*}}(Q)\right)$.

Proof. If $E \in BNN_Q^*\alpha O(U)$, then $E \in BNN_Q^*SO(U)$ and $E \in BNN_Q^*PO(U)$ by theorem 3.3.13 and 3.3.16. This follows that, $E \in BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$. Hence $BNN_Q^*\alpha O(U) \subseteq BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$.
Conversely, if $E \in BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$, then $E \subseteq BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(U)\right)$ and $E \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(U)\right)$.
Consider $E \subseteq BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(U)\right)$,
$BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E)\right) \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(U)\right)\right)\right) = BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(U)\right)\right)$.
Then $E \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(U)\right)\right) \Rightarrow E \in BNN_Q^*\alpha O(U)$.
This gives $BNN_Q^*\alpha O(U) \supseteq BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$.
Hence $BNN_Q^*\alpha O(U) = BNN_Q^*PO(U) \cap BNN_Q^*SO(U)$.

Remark: 3.3.20 The following example shows that the BNN_Q^*PO and BNN_Q^*SO sets are independent of each other.

Example: 3.3.21 From example 3.1.4, $E = \begin{Bmatrix} \langle p_1,(0.6,0.8,0.3,-0.3,-0.7,-0.4)\rangle \\ \langle p_2,(0.4,0.6,0.6,-0.4,-0.6,-0.6)\rangle \end{Bmatrix}$ is BNN_Q^*PO but not BNN_Q^*SO. And $E = \begin{Bmatrix} \langle p_1,(0.4,0.2,0.6,-0.1,-0.3,-0.5)\rangle \\ \langle p_2,(0.3,0.5,0.7,-0.4,-0.4,-0.5)\rangle \end{Bmatrix}$ is BNN_Q^*SO but not BNN_Q^*PO.

Theorem: 3.3.22 The union of BNN_Q^*O sets and $BNN_Q^*\alpha O$ sets of $\left(U,\tau_{R_{BNN^*}}(Q)\right)$ is BNN_Q^*PO.

Proof. Let E be a BNN_Q^*O set and F be a $BNN_Q^*\alpha O$ set in U. Then $BNN_Q^*\operatorname{int}(E) = E$ and $F \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)$. Now

$$\begin{aligned}
E \cup F &\subseteq BNN_Q^*\operatorname{int}(E) \cup BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)\\
&\subseteq BNN_Q^*\operatorname{int}\left(E \cup BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)\\
&\subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E) \cup BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)\\
&\subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E) \cup BNN_Q^*cl(F)\right)\\
&\subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E \cup F)\right).
\end{aligned}$$

Hence $E \cup F$ is BNN_Q^*PO.

Theorem: 3.3.23 The union of BNN_Q^*PO sets and $BNN_Q^*\alpha O$ sets of $\left(U,\tau_{R_{BNN^*}}(Q)\right)$ is BNN_Q^*PO.

Proof. Let E be a BNN_Q^*PO set and F be a $BNN_Q^*\alpha O$ set in U. Then $E \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E)\right)$ and $F \subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)$. Now

$$\begin{aligned}
E \cup F &\subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E)\right) \cup BNN_Q^*\operatorname{int}\left(BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)\\
&\subseteq BNN_Q^*\operatorname{int}\left(BNN_Q^*cl(E) \cup BNN_Q^*cl\left(BNN_Q^*\operatorname{int}(F)\right)\right)
\end{aligned}$$

$\subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E) \cup BNN_Q{}^*cl(F)\right)$

$\subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E \cup F)\right)$. Hence $E \cup F$ is $BNN_Q{}^*PO$.

Theorem: 3.3.24 If E is $BNN_Q{}^*O$ and $BNN_Q{}^*PO$ in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$, then E is $BNN_Q{}^*\alpha O$.

Proof. If E is $BNN_Q{}^*O$ and $BNN_Q{}^*PO$ in U, then $BNN_Q{}^* \operatorname{int}(E) = E$ and $E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E)\right)$.

Consider $E = BNN_Q{}^* \operatorname{int}(E) \Rightarrow BNN_Q{}^*cl(E) = BNN_Q{}^*cl\left(BNN_Q{}^* \operatorname{int}(E)\right) \Rightarrow BNN_Q{}^*\operatorname{int}\left(BNN_Q{}^*cl(E)\right) = BNN_Q{}^*\operatorname{int}$ $\left(BNN_Q{}^*cl\left(BNN_Q{}^* \operatorname{int}(E)\right)\right)$. This implies $E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl\left(BNN_Q{}^* \operatorname{int}(E)\right)\right)$. Hence E is $BNN_Q{}^*\alpha O$.

Theorem: 3.3.25 Let E be a BN set in a BNN^*TS $\left(U, \tau_{R_{BNN^*}}(Q)\right)$. If F is a $BNN_Q{}^*SO$ set such that $F \subseteq E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(F)\right)$, then E is a $BNN_Q{}^*\alpha O$ set.

Proof. Since F is a $BNN_Q{}^*SO$ set, we have $F \subseteq BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(F)\right)$.

We have $E \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl(F)\right) \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(F)\right)\right)\right)$

$= BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(F)\right)\right) \subseteq BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* cl\left(BNN_Q{}^* \operatorname{int}(E)\right)\right)$. Hence E is a $BNN_Q{}^*\alpha O$ set.

*3.4 Bipolar Neutrosophic Nano * Regular Open Sets*

Definition: 3.4.1 Let E be a neutrosophic set in BNN^*TS $\left(U, \tau_{R_{BNN^*}}(Q)\right)$. Then E is said to be $BNN_Q{}^*$ - regular-open set ($BNN_Q{}^*RO$ set) of U if $E = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E)\right)$. The complement of $BNN_Q{}^*RO$ set is called $BNN_Q{}^*$ - regular closed set ($BNN_Q{}^*RC$ set) of U.

Theorem: 3.4.2 Every $BNN_Q{}^*RO$ set is $BNN_Q{}^*O$ set in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$.

Proof. If E is $BNN_Q{}^*RO$ in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$, then $E = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E)\right)$. Now $BNN_Q{}^* \operatorname{int}(E) = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E)\right)\right) = BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E)\right) = E$. Hence E is $BNN_Q{}^*O$ in U.

Remark: 3.4.3 The converse of the above theorem need not be true. A $BNN_Q{}^*O$ set need not be $BNN_Q{}^*RO$ in $\left(U, \tau_{R_{BNN^*}}(Q)\right)$.

Example: 3.4.4 Let $U = \{p_1, p_2, p_3\}, U/R = \{\{p_1, p_3\}, \{p_2\}\}$, $Q = \left\{\begin{array}{l}\langle p_1, (0.6, 0.5, 0.2, -0.3, -0.4, -0.7)\rangle \\ \langle p_2, (0.5, 0.6, 0.4, -0.4, -0.5, -0.5)\rangle \\ \langle p_3, (0.2, 0.3, 0.8, -0.5, -0.6, -0.4)\rangle\end{array}\right\}$.

$\overline{BN}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.6, 0.5, 0.2, -0.5, -0.6, -0.4)\rangle \\ \langle p_2, (0.5, 0.6, 0.4, -0.4, -0.5, -0.5)\rangle \\ \langle p_3, (0.6, 0.5, 0.2, -0.5, -0.6, -0.4)\rangle\end{array}\right\}$, $\underline{BN}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.2, 0.3, 0.8, -0.3, -0.4, -0.7)\rangle \\ \langle p_2, (0.5, 0.6, 0.4, -0.4, -0.5, -0.5)\rangle \\ \langle p_3, (0.2, 0.3, 0.8, -0.3, -0.4, -0.7)\rangle\end{array}\right\}$,

$B_{BN}{}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.6, 0.5, 0.2, -0.5, -0.6, -0.4)\rangle \\ \langle p_2, (0.5, 0.6, 0.4, -0.4, -0.5, -0.5)\rangle \\ \langle p_3, (0.6, 0.5, 0.2, -0.5, -0.6, -0.4)\rangle\end{array}\right\}$, $BN_1{}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.6, 0.5, 0.2, -0.5, -0.6, -0.4)\rangle \\ \langle p_2, (0.5, 0.6, 0.4, -0.4, -0.5, -0.5)\rangle \\ \langle p_3, (0.6, 0.5, 0.2, -0.5, -0.6, -0.4)\rangle\end{array}\right\}$,

$BN_2{}^*(Q) = \left\{\begin{array}{l}\langle p_1, (0.2, 0.3, 0.8, -0.3, -0.4, -0.7)\rangle \\ \langle p_2, (0.4, 0.4, 0.5, -0.4, -0.5, -0.5)\rangle \\ \langle p_3, (0.2, 0.3, 0.8, -0.3, -0.4, -0.7)\rangle\end{array}\right\}$. $\tau_{R_{BNN^*(Q)}} = \left\{0_{BNN^*}, \overline{BN}^*(Q), \underline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), BN_2{}^*(Q), 1_{BNN^*}\right\}$.

$0_{BNN^*}, \overline{BN}^*(Q), B_{BN}{}^*(Q), BN_1{}^*(Q), 1_{BNN}{}^*$ are $BNN_Q{}^*RO$ sets in U.

Theorem: 3.4.5 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*PO$ sets.

Proof. The proof follows from the definitions of $BNN_Q{}^*RO$ and $BNN_Q{}^*PO$ sets.

Remark: 3.4.6 The converse of the above theorem is not true. This is shown in the following example.

Example: 3.4.7 Let $E = \left\{\begin{array}{l}\langle p_1, (0.4, 0.4, 0.5, -0.2, -0.3, -0.8)\rangle \\ \langle p_2, (0.4, 0.5, 0.6, -0.3, -0.3, -0.6)\rangle \\ \langle p_3, (0.1, 0.2, 0.9, -0.4, -0.4, -0.6)\rangle\end{array}\right\}$.

From example: 3.4.4, $BNN_Q{}^* \operatorname{int}\left(BNN_Q{}^*cl(E)\right) = \overline{BN}^*(Q)$. Also $E \subseteq \overline{BN}^*(Q)$.

E is $BNN_Q{}^*PO$, but not $BNN_Q{}^*RO$ set.

Theorem: 3.4.8 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*\alpha O$ sets.

Proof. Since $BNN_Q{}^*RO$ sets are $BNN_Q{}^*O$ and $BNN_Q{}^*O$ sets are $BNN_Q{}^*\alpha O$ sets, the result follows.

Example: 3.4.9 This example shows that the converse of the above theorem is not true.

From example: 3.3.6, E is $BNN_Q{}^*\alpha O$ but not $BNN_Q{}^*RO$.

Theorem: 3.4.10 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*SO$ sets.

Proof. Since $BNN_Q{}^*RO$ sets are $BNN_Q{}^*O$ and $BNN_Q{}^*O$ sets are $BNN_Q{}^*SO$ sets, the result follows.

Example: 3.4.11 This example shows that the converse of the above theorem is not true.

From example: 3.2.5, E is $BNN_Q{}^*SO$ but not $BNN_Q{}^*RO$.

Theorem: 3.4.12 The arbitrary union of $BNN_Q{}^*RO$ sets is $BNN_Q{}^*RO$ in U.

Proof. Let $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*RO$ sets in $(U,\tau_{R_{BNN^*}}(Q))$. Then for each $\alpha\in\Omega$, $E_\alpha \subseteq BNN_Q{}^*\,int(BNN_Q{}^*cl(E_\alpha))$.

$$E_1 \cup E_2 \cup \ldots\ldots \subseteq BNN_Q{}^*\,int(BNN_Q{}^*cl(E_1)) \cup BNN_Q{}^*\,int(BNN_Q{}^*cl(E_2)) \cup \ldots\ldots\ldots$$
$$\subseteq BNN_Q{}^*\,int(BNN_Q{}^*cl(E_1) \cup BNN_Q{}^*cl(E_2) \cup \ldots\ldots)$$
$$\subseteq BNN_Q{}^*\,int(BNN_Q{}^*cl(E_1 \cup E_2 \cup \ldots\ldots))$$

Hence $\bigcup_{\alpha\in\Omega} E_\alpha$ is $BNN_Q{}^*RO$ set in U.

Theorem: 3.4.13 In a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$, the $BNN_Q{}^*$ - open sets $0_{BNN^*}, 1_{BNN^*}$, $B_{BN}{}^*(Q), BN_1{}^*(Q)$ and $\overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = (BN_2{}^*(Q))^C$ are the only $BNN_Q{}^*RO$ sets in U.

Proof.

Table 1. $BNN_Q{}^*$ - interior closure of each $BNN_Q{}^*O$ sets

$BNN_Q{}^*O$ **Set (E)**	$BNN_Q{}^*cl(E)$	$BNN_Q{}^*\,int(BNN_Q{}^*cl(E))$
$\overline{BN}^*(Q)$	$(BN_2{}^*(Q))^C$	$\overline{BN}^*(Q)$
$\underline{BN}^*(Q)$	$(B_{BN}{}^*(Q))^C$	$BN_1{}^*(Q)$
$B_{BN}{}^*(Q)$	$(BN_1{}^*(Q))^C$	$B_{BN}{}^*(Q)$
$BN_1{}^*(Q)$	$(B_{BN}{}^*(Q))^C$	$BN_1{}^*(Q)$
$BN_2{}^*(Q)$	$(BN_1{}^*(Q))^C$	$B_{BN}{}^*(Q)$

Since $BNN_Q{}^*RO$ sets are $BNN_Q{}^*O$, then $0_{BNN^*}, 1_{BNN^*}, B_{BN}{}^*(Q), BN_1{}^*(Q), \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = (BN_2{}^*(Q))^C$ are the only $BNN_Q{}^*RO$ sets in U.

Theorem: 3.4.14 Finite Intersection of $BNN_Q{}^*RO$ sets is $BNN_Q{}^*RO$.

Proof. From theorem 3.4.13, we have $0_{BNN^*}, 1_{BNN^*}, B_{BN}{}^*(Q), BN_1{}^*(Q), \overline{BN}^*(Q)$ with $BNN_Q{}^*cl(\overline{BN}^*(Q)) = (BN_2{}^*(Q))^C$ are the only $BNN_Q{}^*RO$ sets in U. If E is any one of the above $BNN_Q{}^*O$ sets, then $0_{BNN^*} \cap E = 0_{BNN^*}$ and $1_{BNN^*} \cap E = E$ are $BNN_Q{}^*RO$ sets. $B_{BN}{}^*(Q) \cap BN_1{}^*(Q) = B_{BN}{}^*(Q)$, $B_{BN}{}^*(Q) \cap \overline{BN}^*(Q) = B_{BN}{}^*(Q)$, $BN_1{}^*(Q) \cap \overline{BN}^*(Q) = BN_1{}^*(Q)$. Thus finite intersection of $BNN_Q{}^*RO$ sets is $BNN_Q{}^*RO$.

Remark: 3.4.15 The intersection and union of any two $BNN_Q{}^*RC$ sets are $BNN_Q{}^*RC$.

Theorem: 3.4.16 $BNN_Q{}^*RC$ sets are $BNN_Q{}^*C$ sets.

Proof. If E is $BNN_Q{}^*RC$ in $(U,\tau_{R_{BNN^*}}(Q))$, then $E = BNN_Q{}^*cl(BNN_Q{}^*\,int(E))$. Now $BNN_Q{}^*cl(E) = BNN_Q{}^*cl(BNN_Q{}^*cl(BNN_Q{}^*\,int(E))) = BNN_Q{}^*cl(BNN_Q{}^*\,int(E)) = E$. Hence E is $BNN_Q{}^*C$ in U.

Theorem: 3.4.17 $BNN_Q{}^*RC$ sets are $BNN_Q{}^*PC$ sets.

Proof. The proof follows from the definitions of $BNN_Q{}^*RC$ and $BNN_Q{}^*PC$ sets.

*3.5 Bipolar Neutrosophic Nano * β Open Sets*

Definition: 3.5.1 Let E be a BN set in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$. Then E is said to be $BNN_Q{}^*$ - β -open set ($BNN_Q{}^*\beta O$ set) of U if $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(BNN_Q{}^*cl(E)))$. The complement of $BNN_Q{}^*\beta O$ set is called $BNN_Q{}^*$ - β - closed set ($BNN_Q{}^*\beta C$ set) of U.

Theorem: 3.5.2 $BNN_Q{}^*O$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*O$ in $(U,\tau_{R_{BNN^*}}(Q))$. Then $BNN_Q{}^*\,int(E) = E$. We have $E \subseteq BNN_Q{}^*cl(E)$. $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(E)) \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(BNN_Q{}^*cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.3 $BNN_Q{}^*SO$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*SO$ in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(E))$. We have $E \subseteq BNN_Q{}^*cl(E)$. $E \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(E)) \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(BNN_Q{}^*cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.4 $BNN_Q{}^*RO$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*RO$ in $(U,\tau_{R_{BNN^*}}(Q))$. Then $E = BNN_Q{}^*\,int(BNN_Q{}^*cl(E))$.
$E = BNN_Q{}^*\,int(BNN_Q{}^*cl(E)) \subseteq BNN_Q{}^*cl(BNN_Q{}^*\,int(BNN_Q{}^*cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.5 $BNN_Q{}^*\alpha O$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*\alpha O$ in $(U, \tau_{R_{BNN^*}}(Q))$.

Then $E \subseteq BNN_Q{}^* int(BNN_Q{}^* cl(BNN_Q{}^* int(E)))$.

$E \subseteq BNN_Q{}^* int(BNN_Q{}^* cl(BNN_Q{}^* int(E))) \subseteq BNN_Q{}^* int(BNN_Q{}^* cl(E)) \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E)))$.

Hence E is $BNN_Q{}^*\beta O$ in U.

Theorem: 3.5.6 $BNN_Q{}^*PO$ sets are $BNN_Q{}^*\beta O$ sets.

Proof. Let E be a $BNN_Q{}^*PO$ in $(U, \tau_{R_{BNN^*}}(Q))$. Then $E \subseteq BNN_Q{}^* int(BNN_Q{}^* cl(E))$. $E \subseteq BNN_Q{}^* int(BNN_Q{}^* cl(E)) \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E)))$. Hence E is $BNN_Q{}^*\beta O$ in U.

Remark: 3.5.7 The following example shows that the converses of the theorems 3.5.2, 3.5.3, 3.5.4 and 3.5.5 are not true.

Example: 3.5.8 Let $E = \left\{ \begin{matrix} \langle p_1, (0.4, 0.4, 0.5, -0.2, -0.3, -0.8) \rangle \\ \langle p_2, (0.4, 0.5, 0.6, -0.3, -0.3, -0.6) \rangle \\ \langle p_3, (0.1, 0.2, 0.9, -0.4, -0.4, -0.6) \rangle \end{matrix} \right\}$.

From example 3.4.4, $BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E))) = (BN_2{}^*(Q))^C$. Also $E \subseteq (BN_2{}^*(Q))^C$. E is $BNN_Q{}^*\beta O$

And

(i) $BNN_Q{}^* int(E) \neq E$, E is not $BNN_Q{}^*O$ set.

(ii) $BNN_Q{}^* cl(BNN_Q{}^* int(E)) = 0_{BNN^*}$ and $E \not\subset 0_{BNN^*}$. So E is not $BNN_Q{}^*SO$ set.

(iii) $BNN_Q{}^* int(BNN_Q{}^* cl(E)) = \overline{BN}^*(Q) \neq E$. So E is not $BNN_Q{}^*RO$ set.

(iv) $BNN_Q{}^* int(BNN_Q{}^* cl(BNN_Q{}^* int(E))) = 0_{BNN^*}$ and $E \not\subset 0_{BNN^*}$. So E is not $BNN_Q{}^*\alpha O$ set.

Example: 3.5.9 This example shows that the converse of theorem 3.5.6 is not true.

Let $E = \left\{ \begin{matrix} \langle p_1, (0.2, 0.2, 0.8, -0.3, -0.3, -0.7) \rangle \\ \langle p_2, (0.2, 0.3, 0.5, -0.3, -0.3, -0.6) \rangle \\ \langle p_3, (0.2, 0.2, 0.7, -0.3, -0.4, -0.6) \rangle \end{matrix} \right\}$.

$BNN_Q{}^* int(BNN_Q{}^* cl(E)) = BN_2{}^*(Q)$, $E \not\subset BN_2{}^*(Q)$.

$BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E))) = \left(\overline{BN}^*(Q)\right)^C$, $E \subseteq \left(\overline{BN}^*(Q)\right)^C$.

Hence E is $BNN_Q{}^*\beta O$ but not $BNN_Q{}^*PO$.

Theorem: 3.5.10 Arbitrary union of $BNN_Q{}^*\beta O$ sets is $BNN_Q{}^*\beta O$ set.

Proof. If $\{E_\alpha\}_{\alpha \in \Omega}$ is a collection of $BNN_Q{}^*\beta O$ sets in $(U, \tau_{R_{BNN^*}}(Q))$, then for each $\alpha \in \Omega$, $E_\alpha \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E_\alpha)))$.

$$\begin{aligned} E_1 \cup E_2 \cup \ldots\ldots &\subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E_1))) \cup BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E_2))) \cup \ldots\ldots\ldots \\ &= BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E_1)) \cup BNN_Q{}^* int(BNN_Q{}^* cl(E_2)) \cup \ldots) \\ &\subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E_1) \cup BNN_Q{}^* cl(E_2) \cup \ldots\ldots)) \\ &= BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E_1 \cup E_2 \cup \ldots\ldots))) \end{aligned}$$

Hence $\bigcup_{\alpha \in \Omega} E_\alpha$ is $BNN_Q{}^*\beta O$ set in U.

Theorem: 3.5.11 $BNN_Q{}^*SO(U,Q) \cup BNN_Q{}^*PO(U,Q) \subset BNN_Q{}^*\beta O(U,Q)$.

Proof. The proof follows from theorems 3.5.3 and 3.5.4.

Theorem: 3.5.12 If F is BN subset of U and E is $BNN_Q{}^*PO$ in U such that $E \subseteq F \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(E))$, then F is $BNN_Q{}^*\beta O$.

Proof. Since E is $BNN_Q{}^*PO$ in U, $E \subseteq BNN_Q^* int(BNN_Q^* cl(E))$.

Now $F \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(E)) \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* int(BNN_Q{}^* cl(E))))$

$= BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E))) \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(F)))$.

Hence $F \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(F)))$. Then F is $BNN_Q{}^*\beta O$.

Theorem: 3.5.13 Each $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*SC$ is $BNN_Q{}^*SO$.

Proof. Let E be $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*SC$. Then $E \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E)))$ and $BNN_Q{}^* int(BNN_Q{}^* cl(E)) \subseteq E$. Hence $BNN_Q{}^* int(BNN_Q{}^* cl(E)) \subseteq E \subseteq BNN_Q{}^* cl(BNN_Q{}^* int(BNN_Q{}^* cl(E)))$. Since $BNN_Q{}^* int(BNN_Q{}^* cl(E)) = G$ is a $BNN_Q{}^*O$ set in U, that there exists a $BNN_Q{}^*O$ set such that $G \subseteq E \subseteq BNN_Q{}^* cl(G)$. Therefore E is $BNN_Q{}^*SO$ set.

Example: 3.5.14 The statement of the above theorem is shown in this example.

Let $U=\{p_1,p_2,p_3,p_4\}, U/R=\{\{p_1,p_3\},\{p_2,p_4\}\}$, $Q=\left\{\begin{matrix}\langle p_1,(0.2,0.3,0.6,-0.5,-0.6,-0.4)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_3,(0.3,0.3,0.7,-0.6,-0.6,-0.3)\rangle\\ \langle p_4,(0.2,0.4,0.7,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$.

$\overline{BN}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.3,0.6,-0.6,-0.6,-0.3)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.6,-0.5,-0.3)\rangle\\ \langle p_3,(0.3,0.3,0.6,-0.6,-0.6,-0.3)\rangle\\ \langle p_4,(0.4,0.4,0.6,-0.6,-0.5,-0.3)\rangle\end{matrix}\right\}$, $\underline{BN}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.3,0.7,-0.5,-0.6,-0.4)\rangle\\ \langle p_2,(0.2,0.4,0.7,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.2,0.3,0.7,-0.5,-0.6,-0.4)\rangle\\ \langle p_4,(0.2,0.4,0.7,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$,

$B_{BN}{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.3,0.6,-0.4,-0.4,-0.5)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.3,0.3,0.6,-0.4,-0.4,-0.5)\rangle\\ \langle p_4,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$, $BN_1{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.3,0.3,0.6,-0.5,-0.6,-0.4)\rangle\\ \langle p_2,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.3,0.3,0.6,-0.5,-0.6,-0.4)\rangle\\ \langle p_4,(0.4,0.4,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$,

$BN_2{}^*(Q)=\left\{\begin{matrix}\langle p_1,(0.2,0.3,0.7,-0.4,-0.4,-0.5)\rangle\\ \langle p_2,(0.2,0.4,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.2,0.3,0.7,-0.4,-0.4,-0.5)\rangle\\ \langle p_4,(0.2,0.4,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$.

$\tau_{R_{BNN^*}(Q)}=\{0_{BNN^*},\overline{BN}^*(Q),\underline{BN}^*(Q),B_{BN}{}^*(Q),BN_1{}^*(Q),BN_2{}^*(Q),1_{BNN^*}\}$.

Let $E=\left\{\begin{matrix}\langle p_1,(0.5,0.6,0.5,-0.4,-0.4,-0.5)\rangle\\ \langle p_2,(0.6,0.5,0.6,-0.5,-0.5,-0.5)\rangle\\ \langle p_3,(0.5,0.6,0.5,-0.4,-0.4,-0.5)\rangle\\ \langle p_4,(0.6,0.5,0.6,-0.5,-0.5,-0.5)\rangle\end{matrix}\right\}$, $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))=(BN_1{}^*(Q))^C$ and

$BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E))=B_{BN}{}^*(Q)\subseteq E$. Therefore E is both $BNN_Q{}^*\beta O$ and $BNN_Q{}^*SC$. Also $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E))=(BN_1{}^*(Q))^C$. Hence E is $BNN_Q{}^*SO$.

Theorem: 3.5.15 Each $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*\alpha C$ is $BNN_Q{}^*C$.

Proof. Let E be $BNN_Q{}^*\beta O$ set which is $BNN_Q{}^*\alpha C$. Then $E\subseteq BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$ and $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))\subseteq E$.

Hence $BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))\subseteq E\subseteq BNN_Q{}^*cl\ (BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))$.

This implies $E=BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(E)))\Rightarrow E=BNN_Q{}^*cl(E)$. Hence E is $BNN_Q{}^*C$ set.

Theorem: 3.5.16 Arbitrary intersection of $BNN_Q{}^*\beta C$ sets is $BNN_Q{}^*\beta C$ set.

Proof. If $\{E_\alpha\}_{\alpha\in\Omega}$ is a collection of $BNN_Q{}^*\beta C$ sets in $(U,\tau_{R_{BNN^*}}(Q))$, then for each $\alpha\in\Omega$, $BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_\alpha)))\subseteq E_\alpha$.

$$\begin{aligned}E_1\cup E_2\cup\ldots\ldots &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1)))\cap BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2)))\cap\ldots\ldots\ldots\\ &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1))\cap BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_2))\cap\ldots)\\ &= BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1)\cap BNN_Q{}^*\operatorname{int}(E_2)\cap\ldots\ldots))\\ &\subseteq BNN_Q{}^*\operatorname{int}(BNN_Q{}^*cl(BNN_Q{}^*\operatorname{int}(E_1\cap E_2\cap\ldots.)))\end{aligned}$$

Hence $\bigcap_{\alpha\in\Omega}E_\alpha$ is $BNN_Q{}^*\beta C$ set in U.

Remark: 3.5.17 Figure-1 shows the relationships among $BNN_Q{}^*O$, $BNN_Q{}^*PO$, $BNN_Q{}^*\alpha O$, $BNN_Q{}^*RO$, $BNN_Q{}^*\beta O$ and $BNN_Q{}^*SO$ in a BNN^*TS $(U,\tau_{R_{BNN^*}}(Q))$.

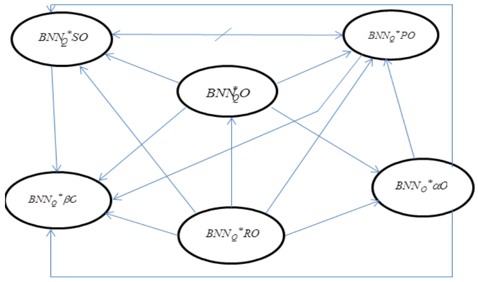

Figure 1. Relationship between the weaker forms of open sets in $BNN_Q{}^*TS$

4. Conclusion

Bipolar neutrosophic set is the base for many topological spaces. In topology, the topological structures such as closedness and openness are the important concepts. It helps to determine the continuity of a mapping between the topologies. Many researchers have proposed various types of topologies with bipolar neutrosophic set. In this paper, we introduced new family of sets namely, bipolar neutrosophic nano* preopen, semi open, $\alpha-$ open , regular open and $\beta-$ open sets in a new topology Bipolar Neutrosophic Nano* topology . Further, some important results based on the corresponding sets are derived and discussed through several examples. As we know neutrosophic sets and nano topology are the roots for many real life applications, we expect that the proposed sets will serve contributions to some future works to the new researchers in real life problems as well as in algebra, geometry and analysis of other sub-branches of mathematics. Our future work will consist of applications of the proposed sets and topology in decision making problems. There are numerous Neutrosophy based decision making algorithms available. In future, we will explore decision making scenarios and try to define novel algorithms by applying proposed concepts. Also, image processing is one of the field which uses neutrosophic logic. We will try to develop image processing algorithms based on proposed neutrosophic topology.

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