NeutroOrderedAlgebra: Theory and Examples

Madeleine Al-Tahan
Lebanese International University
madeline.tahan@liu.edu.lb

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Abstract

In this talk, we firstly review some basic concepts related to neutrosophy. Also, we discuss NeutroAlgebra. Next, we present some of our results related to our new defined concept “NeutroOrderedAlgebra" and compare it to the well known concept of “Ordered Algebra". Finally, we leave with some questions that open new research options in this field.

Neutrosophy [4] is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. While Hegel's and Marx's Dialectics deals only with the dynamics of opposites, Neutrosophy deals with the dynamics of opposites and their neutrals all together.

This mode of thinking

1. proposes new philosophical theses, principles, laws, methods, formulas, movements;
2. interprets the uninterpretable;
3. regards, from many different angles, old concepts, systems: showing that an idea, which is true in a given referential system, may be false in another one, and vice versa;
4. attempts to make peace in the war of ideas, and to make war in the peaceful ideas;
5. measures the stability of unstable systems, and instability of stable systems.

Let's note by <A> an idea or theory or concept, by <Non-A> what is not <A>, and by <Anti-A> the opposite of <A>. Also, <Neut-A> means what is neither <A>, nor <Anti-A>, i.e. neutrality in between the two extremes. And <A'> a version of <A>. <Non-A> is different from <Anti-A>.

Example 1. [4] If <A> = white then <Anti-A> = black (antonym). But <Non-A> = green, red, blue, yellow, black, etc. (any color, except white), while <Neut-A> = green, red, blue, yellow, etc. (any color, except white and black). And <A'> = dark white, etc. (any shade of white).

Example 2. If <A> = love then <Anti-A> = hatred (antonym). But <Non-A> = any feeling except love, while <Neut-A> = any feeling except love and hatred. And <A'> = any type of love.

Let <A> be an idea. Then the following are true.

1. <Neut-A> $\equiv$ <Neut-(Anti-A)>;
2. <Anti-A> $\subseteq$ <Non-A>;
3. <A>, <Neut-A>, and <Anti-A> are pairwise disjoint;
4. <Non-A> is the completitude of <A> with respect to the universal set.

Every idea <A> tends to be neutralized, diminished, balanced by <Non-A> ideas as a state of equilibrium. In between <A> and <Anti-A> there are infinitely many <Neut-A> ideas, which may balance <A> without necessarily any <Anti-A> version. To neuter an idea, we need to discover all its three sides: of sense (truth), of nonsense (falsity), and of undecidability (indeterminacy) - then reverse/combine them. Afterwards, the idea will be classified as neutrality.

In soft sciences the laws are interpreted and re-interpreted; in social and political legislation the laws are flexible; the same law may be true from a point of view, and false from another point of view. Thus, the law is partially true and partially false (it is a Neutrosophic Law). For example, “Wearing clothes from animals’ leather or fur". There are people supporting it because it prevents feeling cold during the winter (and they are right), and people (supporting Animals’ rights) opposing it because they are against killing animals to get their leather or fur (and they are right too). We have two opposite propositions, both of them are true but from different points of view (from different criteria/parameters; plithogenic logic). How can we solve this? Going to the middle, in between opposites (as in neutrosophy): Leather/Fur lovers can wear either leather/fur clothes that are not from animals or from dead animals.

Another example is “Drug legislation". Many people in Lebanon support it for medical and economical purposes (and they are right), and other people in Lebanon oppose it because they want to save their families from drug addiction (and they are right too). We have two opposite propositions, both of them are true but from different points of view (from different criteria/parameters; plithogenic logic). How can we solve this? Going to the middle, in between opposites (as in neutrosophy): The Lebanese government has to support the usage of drugs in medicine and prohibits traders from selling drugs to individuals by monitoring the process starting from planting to distributing and manufacturing.

Why?

"In all classical algebraic structures, the laws of compositions on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call NeutroDefined, or totally undefined (totally false) that we call AntiDefined. Again, in all classical algebraic structures, the Axioms (Associativity, Commutativity, etc.) defined on a set are totally true, but it is again a restrictive case, because similarly there are numerous situations in science and in any domain of knowledge when an Axiom defined on a set may be only partially-true (and partially-false), that we call NeutroAxiom, or totally false that we call AntiAxiom." Based on this, Florentin Smarandache in 2019 opened new fields of research called NeutroStructures and AntiStructures respectively.

Definition (F. Smarandache, 2019)

Let A be any non-empty set and "$\cdot$" be an operation on A. Then "$\cdot$" is called a *NeutroOperation* on A if the following conditions hold.

1. There exist $x, y \in A$ with $x \cdot y \in A$. (This condition is called degree of truth, "T".)
2. There exist $x, y \in A$ with $x \cdot y \notin A$. (This condition is called degree of falsity, "F".)
3. There exist $x, y \in A$ with $x \cdot y$ is indeterminate in A. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical binary closed operation, and from $(0, 0, 1)$ that represents the AntiOperation.

Illustrative Examples

1. We can view the standard division "$\div$" on $\mathbb{R}$, the set of real numbers as NeutroOperation. This is easily seen as

$$x \div y \text{ is } \begin{cases} \in \mathbb{R} & \text{if } y \neq 0; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

2. We can view the standard division "$\div$" on $\mathbb{N}$, the set of positive integers as NeutroOperation. This is easily seen as

$$x \div y \text{ is } \begin{cases} \in \mathbb{N} & \text{if } y \text{ divides } x; \\ \notin \mathbb{N} & \text{otherwise.} \end{cases}$$

3. We can view the standard division "$\div$" on $\mathbb{Z}$, the set of integers as NeutroOperation. This is easily seen as

$$x \div y \text{ is } \begin{cases} \in \mathbb{Z} & \text{if } y \text{ divides } x \text{ and } y \neq 0; \\ \notin \mathbb{Z} & \text{if } y \text{ does not divide } x \text{ and } y \neq 0; \\ \text{undefined} & \text{if } y = 0. \end{cases}$$

Definition (F. Smarandache, 2019)

Let $\boldsymbol{U}$ be a universe of discourse, endowed with some well-defined laws, a non-empty set $\boldsymbol{S} \subseteq \boldsymbol{U}$ and an Axiom α, defined on $\boldsymbol{S}$, using these laws. Then

1. If all elements of $\boldsymbol{S}$ verify the axiom α, we have a Classical Axiom, or simply we say Axiom.
2. If some elements of $\boldsymbol{S}$ verify the axiom α and others do not, we have a NeutroAxiom (which is also called NeutAxiom).
3. If no elements of $\boldsymbol{S}$ verify the axiom α, then we have an AntiAxiom.

The Neutrosophic Triplet Axioms are: (Axiom, NeutroAxiom, AntiAxiom) satisfying the following.

$$\text{NeutroAxiom} \cup \text{AntiAxiom} = \text{NonAxiom},$$
$$\text{NeutroAxiom} \cap \text{AntiAxiom} = \emptyset.$$

Illustration

Let $\boldsymbol{A}$ be any non-empty set and "$\cdot$" be an operation on $\boldsymbol{A}$. Then "$\cdot$" is called a NeutroAssociative on $\boldsymbol{A}$ if there exist $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{e}, \boldsymbol{f}, \boldsymbol{g} \in \boldsymbol{A}$ with the following conditions.

1. $\boldsymbol{x} \cdot (\boldsymbol{y} \cdot \boldsymbol{z}) = (\boldsymbol{x} \cdot \boldsymbol{y}) \cdot \boldsymbol{z}$; (This condition is called degree of truth, "$\boldsymbol{T}$".)
2. $\boldsymbol{a} \cdot (\boldsymbol{b} \cdot \boldsymbol{c}) \neq (\boldsymbol{a} \cdot \boldsymbol{b}) \cdot \boldsymbol{c}$; (This condition is called degree of falsity, "$\boldsymbol{F}$".)
3. $\boldsymbol{e} \cdot (\boldsymbol{f} \cdot \boldsymbol{g})$ is indeterminate or $(\boldsymbol{e} \cdot \boldsymbol{f}) \cdot \boldsymbol{g}$ is indeterminate or we can not find if $\boldsymbol{e} \cdot (\boldsymbol{f} \cdot \boldsymbol{g})$ and $(\boldsymbol{e} \cdot \boldsymbol{f}) \cdot \boldsymbol{g}$ are equal. (This condition is called degree of indeterminacy, "$\boldsymbol{I}$".)

Where $(\boldsymbol{T}, \boldsymbol{I}, \boldsymbol{F})$ is different from $(\mathbf{1}, \mathbf{0}, \mathbf{0})$ that represents the classical associative axiom, and from $(\mathbf{0}, \mathbf{0}, \mathbf{1})$ that represents the AntiAssociativeAxiom.

Definition (F. Smarandache, 2019)

A non-empty set A endowed with n operations "$\star_i$" for $i = 1, \ldots, n$, is called *NeutroAlgebra* if it has at least one NeutroOperation or at least one NeutroAxiom with no AntiOperations nor AntiAxioms.

Remark (F. Smarandache, 2019)

NeutroAlgebra is a generalization of Partial Algebra.

In comparison between the Partial Algebra and the NeutroAlgebra.

- When the NeutroAlgebra has no NeutroAxiom, and no outer-defined operation, it coincides with the Partial Algebra.
- There are NeutroAlgebras that have no NeutroOperations, but have NeutroAxioms. These are different from Partial Algebras.
- There are NeutroAlgebras that have both, NeutroOperations and NeutroAxioms.

Illustrative Examples

(1) Let $S =]0, \infty[$. Then $(S, \div)$ is a NeutroSemigroup [4]. Since "$\div$" is well-defined then we need to show that $(S, \div)$ is a NeutroAssociative.

This is easily seen as $x \div (y \div z)$ is $\begin{cases} = (x \div y) \div z & \text{if } z = 1; \\ \neq (x \div y) \div z & \text{otherwise.} \end{cases}$

(2) Let $S = \{a, b\}$. Then $(S, \odot)$ defined by the following table is a NeutroSemigroup [6].

$\odot$	a	b
a	b	a
b	a	*undefined*

This is clear as "$\odot$" is a NeutroOperation and

$$a \odot (a \odot a) = (a \odot a) \odot a = a.$$

Examples (Cont'd)

(3) Let $S = \{a, b\}$. Then $(S, \cdot)$ defined by the following table is a NeutroSemigroup [6].

$\cdot$	a	b
a	b	a
b	a	$c \notin S$

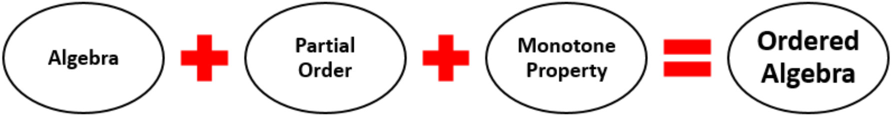
(4) Let $S = \{a, b\}$. Then $(S, +)$ defined by the following table is a NeutroSemigroup [6].

$+$	a	b
a	b	a
b	a	a or b

An ordered algebraic structure A consists of an algebra together with a partial ordering on the underlying set of the algebra. We require that the operations of the algebra are compatible with the partial ordering in that they preserve or reverse order in each coordinate. Ordered algebraic structures occur in a wide variety of areas such as partially ordered vector spaces, lattice ordered groups, Boolean algebras, Heyting algebras, modal algebras, cylindric algebras, relation algebras, etc.

Definition

[3] Let A be an Algebra with n operations "$\star_i$" and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on A. Then $(A, \star_1, \ldots, \star_n, \leq)$ is an Ordered Algebra if the following conditions hold.
If $x \leq y \in A$ then $z \star_i x \leq z \star_i y$ and $x \star_i z \leq y \star_i z$ for all $i = 1, \ldots, n$ and $z \in A$.



Inspired by NeutroAlgebra and ordered Algebra, we introduced **NeutroOrderedAlgebra** in 2021 [1].
Starting with a partial order on a NeutroAlgebra, we get a NeutroStructure.
The latter if it satisfies the conditions of **NeutroOrder**, it becomes a NeutroOrderedAlgebra.

Definition (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

Let A be a NeutroAlgebra with n (Neutro) operations "$\star_i$" and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on A. Then $(A, \star_1, \ldots, \star_n, \leq)$ is a *NeutroOrderedAlgebra* if the following conditions hold.

1. There exist $x \leq y \in A$ with $x \neq y$ such that $z \star_i x \leq z \star_i y$ and $x \star_i z \leq y \star_i z$ for all $z \in A$ and $i = 1, \ldots, n$. (This condition is called degree of truth, "T".)
2. There exist $x \leq y \in A$ and $z \in A$ such that $z \star_i x \not\leq z \star_i y$ or $x \star_i z \not\leq y \star_i z$ for some $i = 1, \ldots, n$. (This condition is called degree of falsity, "F".)
3. There exist $x \leq y \in A$ and $z \in A$ such that $z \star_i x$ or $z \star_i y$ or $x \star_i z$ or $y \star_i z$ are indeterminate, or the relation between $z \star_i x$ and $z \star_i y$, or the relation between $x \star_i z$ and $y \star_i z$ are indeterminate for some $i = 1, \ldots, n$. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ as well from $(0, 0, 1)$.

Definition

Let $(A, \star_1, \dots, \star_n, \leq)$ be a NeutroOrderedAlgebra. If "$\leq$" is a total order on A then A is called *NeutroTotalOrderedAlgebra* [1].

Definition

Let $(A, \star_1, \dots, \star_n, \leq_A)$ be a NeutroOrderedAlgebra and $\emptyset \neq S \subseteq A$. Then S is a *NeutroOrderedSubAlgebra* of A if $(S, \star_1, \dots, \star_n, \leq_A)$ is a NeutroOrderedAlgebra and there exists $x \in S$ with $(x] = \{y \in A : y \leq_A x\} \subseteq S$ [1].

Remark

A NeutroOrderedAlgebra has at least one NeutroOrderedSubAlgebra which is itself.

Definition (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

Let $(S,\cdot)$ be a NeutroSemigroup and "$\leq$" be a partial order (reflexive, anti-symmetric, and transitive) on S. Then $(S,\cdot,\leq)$ is a *NeutroOrderedSemigroup* if the following conditions hold.

1. There exist $x \leq y \in S$ with $x \neq y$ such that $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all $z \in S$. (This condition is called degree of truth, "T".)
2. There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x \nleq z \cdot y$ or $x \cdot z \nleq y \cdot z$. (This condition is called degree of falsity, "F".)
3. There exist $x \leq y \in S$ and $z \in S$ such that $z \cdot x$ or $z \cdot y$ or $x \cdot z$ or $y \cdot z$ are indeterminate, or the relation between $z \cdot x$ and $z \cdot y$, or the relation between $x \cdot z$ and $y \cdot z$ are indeterminate. (This condition is called degree of indeterminacy, "I".)

Where (T, I, F) is different from $(1, 0, 0)$ that represents the classical Ordered Semigroup, and from $(0, 0, 1)$ that represents the AntiOrderedSemigroup.

Example 3. [1] Let $S_1 = \{s, a, m\}$ and $(S_1, \cdot_1)$ be defined by the following table.

$\cdot_1$	s	a	m
s	s	m	s
a	m	a	m
m	m	m	m

Since $s \cdot_1 (s \cdot_1 s) = s = (s \cdot_1 s) \cdot_1 s$ and $s \cdot_1 (a \cdot_1 m) = s \neq m = (s \cdot_1 a) \cdot_1 m$, it follows that $(S_1, \cdot_1)$ is a NeutroSemigroup.
By defining the total order

$$\leq_1 = \{(m, m), (m, s), (m, a), (s, s), (s, a), (a, a)\}$$

on S_1, we get that $(S_1, \cdot_1, \leq_1)$ is a NeutroTotalOrderedSemigroup. This is easily seen as:
$m \leq_1 s$ implies that $m \cdot_1 x \leq_1 s \cdot_1 x$ and $x \cdot_1 m \leq_1 x \cdot_1 s$ for all $x \in S_1$.
And having $s \leq_1 a$ but $s \cdot_1 s = s \not\leq_1 m = a \cdot_1 s$.

Example 4. [1] Let $S_2 = \{0, 1, 2, 3\}$ and $(S_2, \cdot_2)$ be defined by the following table.

$\cdot_2$	0	1	2	3
0	0	0	0	3
1	0	1	1	3
2	0	3	2	2
3	3	3	3	3

Since $0 \cdot_2 (0 \cdot_2 0) = 0 = (0 \cdot_2 0) \cdot_2 0$ and $1 \cdot_2 (2 \cdot_2 3) = 1 \neq 3 = (1 \cdot_2 2) \cdot_2 3$, it follows that $(S_2, \cdot_2)$ is a NeutroSemigroup. By defining the total order "$\leq_2$" on S_2 as follows:

$$\{(0,0),(0,1),(0,2),(0,3),(1,1),(1,2),(1,3),(2,2),(2,3),(3,3)\},$$

we get that $(S_2, \cdot_2, \leq_2)$ is a NeutroTotalOrderedSemigroup. This is easily seen as:
$0 \leq_2 3$ implies that $0 \cdot_2 x \leq_2 3 \cdot_2 x$ and $x \cdot_2 0 \leq_2 x \cdot_2 3$ for all $x \in S_2$.
And having $1 \leq_2 2$ but $2 \cdot_2 1 = 3 \not\leq_2 2 = 2 \cdot_2 2$.

Example 5. [1] Let $S_3 = \{0, 1, 2, 3, 4\}$ and $(S_3, \cdot_3)$ be defined by the following table.

$\cdot_3$	0	1	2	3	4
0	0	0	0	3	0
1	0	1	2	1	1
2	0	4	2	3	3
3	0	4	2	3	3
4	0	0	0	4	0

Since $0 \cdot_3 (0 \cdot_3 0) = 0 = (0 \cdot_3 0) \cdot_3 0$ and $1 \cdot_3 (2 \cdot_3 1) = 1 \neq 4 = (1 \cdot_3 2) \cdot_3 1$, it follows that $(S_3, \cdot_3)$ is a NeutroSemigroup.
By defining the partial order "$\leq_3$" on S_3 as follows

$$\{(0,0),(0,1),(0,3),(0,4),(1,1),(1,3),(1,4),(2,2),(3,3),(3,4),(4,4$$

we get that $(S_3, \cdot_3, \leq_3)$ is a NeutroOrderedSemigroup that is not NeutroTotalOrderedSemigroup as "$\leq_3$" is not a total order on S_3.

Example 5. (Cont'd) This is easily seen as:
$\mathbf{0} \leq_3 \mathbf{4}$ implies that $\mathbf{0} \cdot_3 x \leq_3 \mathbf{4} \cdot_3 x$ and $x \cdot_3 \mathbf{0} \leq_3 x \cdot_3 \mathbf{4}$ for all $x \in S_3$.
And having $\mathbf{0} \leq_3 \mathbf{1}$ but $\mathbf{0} \cdot_3 \mathbf{2} = \mathbf{0} \nleq_3 \mathbf{2} = \mathbf{1} \cdot_3 \mathbf{2}$.

Example 6. [1] Let $\mathbb{Z}$ be the set of integers and define "$\odot$" on $\mathbb{Z}$ as follows: $x \odot y = xy - 1$ for all $x, y \in \mathbb{Z}$. Since $0 \odot (1 \odot 0) = -1 = (0 \odot 1) \odot 0$ and $0 \odot (1 \odot 2) = -1 \neq -3 = (0 \odot 1) \odot 2$, it follows that $(\mathbb{Z}, \odot)$ is a NeutroSemigroup. We define the partial order "$\leq_{\mathbb{Z}}$" on $\mathbb{Z}$ as $-1 \leq_{\mathbb{Z}} x$ for all $x \in \mathbb{Z}$ and for $a, b \geq -1$, $a \leq_{\mathbb{Z}} b$ is equivalent to $a \leq b$ and for $a, b < -1$, $a \leq_{\mathbb{Z}} b$ is equivalent to $a \geq b$. In this way, we get $-1 \leq_{\mathbb{Z}} 0 \leq_{\mathbb{Z}} 1 \leq_{\mathbb{Z}} 2 \leq_{\mathbb{Z}} \ldots$ and $-1 \leq_{\mathbb{Z}} -2 \leq_{\mathbb{Z}} -3 \leq_{\mathbb{Z}} \ldots$. Having $0 \leq_{\mathbb{Z}} 1$ and $x \odot 0 = 0 \odot x = -1 \leq_{\mathbb{Z}} x - 1 = 1 \odot x = x \odot 1$ for all $x \in \mathbb{Z}$ and $-1 \leq_{\mathbb{Z}} 0$ but $(-1) \odot (-1) = 0 \nleq_{\mathbb{Z}} -1 = 0 \odot (-1)$ implies that $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ is a NeutroOrderedSemigroup with -1 as minimum element.

Definition (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

Let $(S, \cdot, \leq)$ be a NeutroOrderedSemigroup and $\emptyset \neq M \subseteq S$. Then

(1) M is a *NeutroOrderedSubSemigroup* of S if $(M, \cdot, \leq)$ is a NeutroOrderedSemigroup and there exist $x \in M$ with $(x] = \{y \in S : y \leq x\} \subseteq M$.

(2) M is a *NeutroOrderedLeftIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ for all $r \in S$.

(3) M is a *NeutroOrderedRightIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $x \cdot r \in M$ for all $r \in S$

(4) M is a *NeutroOrderedIdeal* of S if M is a NeutroOrderedSubSemigroup of S and there exists $x \in M$ such that $r \cdot x \in M$ and $x \cdot r \in M$ for all $r \in S$.

Remark

Unlike the case in Ordered Semigroups, the non-empty intersection of NeutroOrderedSubsemigroups may not be a NeutroOrderedSubsemigroup.

Example 7. [1] Let $(S_3, \cdot_3, \leq_3)$ be the NeutroOrderedSemigroup presented in Example 5. One can easily see that $I = \{0, 1, 2\}, J = \{0, 1, 3\}$ are NeutroOrderedSubsemigroups of S_3. Since $(\{0, 1\}, \cdot_3)$ is a semigroup and not a NeutroSemigroup, it follows that $(I \cap J, \cdot_3, \leq_3)$ is not a NeutroOrderedSubSemigroup of S_3. Here, $I \cap J = \{0, 1\}$.

We present an example on NeutroOrderedIdeal of an infinite NeutroOrderedSemigroup.

Example 8. [1] Let $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ be the NeutroOrderedSemigroup presented in Example 6. Then $I = \{-1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $\mathbb{Z}$. This is clear as:

(1) $0 \odot (1 \odot 0) = -1 = (0 \odot 1) \odot 0$ and $0 \odot (-1 \odot -2) = -1 \neq 1 = (0 \odot -1) \odot -2$;

(2) $g \odot 0 = 0 \odot g = -1 \in I$ for all $g \in \mathbb{Z}$;

(3) $-1 \in I$ and $(-1] = \{-1\} \subseteq I$;

(4) $0 \leq_{\mathbb{Z}} 1 \in I$ implies that $0 \odot x = x \odot 0 = -1 \leq_{\mathbb{Z}} x - 1 = x \odot 1 = 1 \odot x$ for all $x \in I$ and $-1 \leq_{\mathbb{Z}} 0 \in I$ but $-1 \odot -1 = 0 \not\leq_{\mathbb{Z}} -1 = 0 \odot -1$.

Definition (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

Let $(A, \star, \leq_A)$ and $(B, \circledast, \leq_B)$ be NeutroOrderedSemigroups and $\phi : A \to B$ be a function. Then

(1) ϕ is called *NeutroOrderedHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for some $x, y \in A$ and there exist $a \leq_A b \in A$ with $a \neq b$ such that $\phi(a) \leq_B \phi(b)$.

(2) ϕ is called *NeutroOrderedIsomomorphism* if ϕ is a bijective NeutroOrderedHomomorphism.

(3) ϕ is called *NeutroOrderedStrongHomomorphism* if $\phi(x \star y) = \phi(x) \circledast \phi(y)$ for all $x, y \in A$ and $a \leq_A b \in A$ is equivalent to $\phi(a) \leq_B \phi(b) \in B$.

(4) ϕ is called *NeutroOrderedStrongIsomomorphism* if ϕ is a bijective NeutroOrderedStrongHomomorphism.

Lemma (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

Let $(S, \cdot, \leq_S)$ and $(S', \star, \leq_{S'})$ be NeutroOrderedSemigroups and $\phi : S \to S'$ be a NeutroOrderedStrongIsomorphism. If $M \subseteq S$ is a NeutroOrderedSubsemigroup (NeutroOrderedIdeal) of S then $\phi(M)$ is a (NeutroOrderedIdeal) of S'.

Example 9. Let $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ be the NeutroOrderedSemigroups presented in Example 6 and $(\mathbb{Z}, \otimes, \leq_{\otimes})$ be the NeutroOrderedSemigroup defined as follows: $x \otimes y = xy + 1$ for all $x, y \in \mathbb{Z}$ and "$\leq_{\otimes}$" on $\mathbb{Z}$ as $1 \leq_{\otimes} x$ for all $x \in \mathbb{Z}$ and for $a, b \geq 1$, $a \leq_{\otimes} b$ is equivalent to $a \leq b$ and for $a, b \leq 0$, $a \leq_{\otimes} b$ is equivalent to $a \geq b$. Let $\phi : (\mathbb{Z}, \odot, \leq_{\mathbb{Z}}) \to (\mathbb{Z}, \otimes, \leq_{\otimes})$ be defined as $\phi(x) = x + 2$ for all $x \in \mathbb{Z}$. One can easily see that ϕ is a NeutroOrderedStrongIsomorphism. Having $I = \{-1, 0, 1, -2, -3, -4, \ldots\}$ is a NeutroOrderedIdeal of $(\mathbb{Z}, \odot, \leq_{\mathbb{Z}})$ and applying the previous lemma, we get that $\phi(I) = \{1, 2, 3, 0, -1, -2, \ldots\}$ is a NeutroOrderedIdeal of $(\mathbb{Z}, \otimes, \leq_{\otimes})$.

Lemma (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

$(S, \cdot, \leq_S)$ *and* $(S', \star, \leq_{S'})$ *be NeutroOrderedSemigroups and* $\phi : S \to S'$ *be a NeutroOrderedStrongIsomorphism. If* $N \subseteq S'$ *is a NeutroOrderedSubsemigroup (NeutroOrderedIdeal) of* S' *then* $\phi^{-1}(N)$ *is a NeutroOrderedSubsemigroup (NeutroOrderedIdeal) of* S.

Theorem (M. Al-Tahan, F. Smarandache, and B. Davvaz, [1], 2021)

Let $(S, \cdot, \leq_S)$ *and* $(S', \star, \leq_{S'})$ *be NeutroOrderedSemigroups and* $\phi : S \to S'$ *be a NeutroOrderedStrongIsomorphism. Then* $M \subseteq S$ *is a NeutroOrderedSubsemigroup (NeutroOrderedIdeal) of* S *if and only if* $\phi(M)$ *is a NeutroOrderedSubsemigroup (NeutroOrderedIdeal) of* S'.

Let $(A_\alpha, \leq_\alpha)$ be a partial ordered set for all $\alpha \in \Gamma$. We define "$\leq$" on $\prod_{\alpha\in\Gamma} A_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha$,

$$(x_\alpha) \leq (y_\alpha) \iff x_\alpha \leq_\alpha y_\alpha \text{ for all } \alpha \in \Gamma.$$

One can easily see that $(\prod_{\alpha\in\Gamma} A_\alpha, \leq)$ is a partial ordered set.
Let A_α be any non-empty set for all $\alpha \in \Gamma$ and "$\cdot_\alpha$" be an operation on A_α. We define "$\cdot$" on $\prod_{\alpha\in\Gamma} A_\alpha$ as follows: For all $(x_\alpha), (y_\alpha) \in \prod_{\alpha\in\Gamma} A_\alpha$, $(x_\alpha) \cdot (y_\alpha) = (x_\alpha \cdot_\alpha y_\alpha)$.

Theorem (M. Al-Tahan, B. Davvaz, F. Smarandache, and O. Anis, [2], 2021)

Let $(G_1, \leq_1), (G_2, \leq_2)$ be partially ordered sets with operations $\cdot_1, \cdot_2$ respectively. Then $(G_1 \times G_2, \cdot, \leq)$ is a NeutroOrderedSemigroup (NOS) if one of the following statements is true.

1. *G_1 and G_2 are NeutroSemigroups with at least one of them is an NOS.*
2. *One of G_1, G_2 is an NOS and the other is a semigroup.*

Discussion

It is known in ordered algebraic structures that the product $G_1 \times G_2$ of semigroups G_1, G_2 is an ordered semigroup if and only if G_1, G_2 are ordered semigroups. This result in classical algebraic structures is not valid in NeutroAlgebraicStructures. As we have $G_1 \times G_2$ is an NOS if either G_1, G_2 are both NOS, G_1 is an NOS and G_2 is a NeutroSemigroup, G_1 is an NOS and G_2 is a semigroup (or odered semigroup), G_1 is a NeutroSemigroup and G_2 is an NOS, or G_1 is a semigroup (or ordered semigroup) and G_2 is an NOS.

We present some applications of our previous theorem.

Example 9. [2] Let $S_1 = \{s, a, m\}$, $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 3, and "$\leq_1'$" be the trivial order on S_1. Theorem asserts that Cartesian product $(S_1 \times S_1, \cdot, \leq)$ resulting from $(S_1, \cdot_1, \leq_1)$ and $(S_1, \cdot_1, \leq_1')$ is an NOS of order 9.

Example 10. [2] Let $S_1 = \{s, a, m\}$, $(S_1, \cdot_1, \leq_1)$ be the NOS presented in Example 3, and $(\mathbb{R}, \cdot_s, \leq_u)$ be the semigroup of real numbers under standard multiplication and usual order. Theorem asserts that Cartesian product $(\mathbb{R} \times S_1, \cdot, \leq)$ is an NOS of infinite order.

Mostly in idealistic or imaginary or abstract or perfect spaces, we have rigid laws and rigid axioms that totally apply (that are 100 **%** true). But the laws and the axioms should be more flexible in order to deal with our imperfect world. Because of that the introducing of NeutroAlgebras is very important. Such a class of NeutroAlgebras is very large. In this talk, we presented NeutroOrderedAlgebras and we were concerned only about NeutroOrderedSemigroups as a special type of NeutroOrderedAlgebra.

Many other different types of NeutroOrderedAlgebras can be defined. We leave with some related open questions.

1. **Can we find a necessary and a sufficient condition for the product of NOS to be an NOS?**
2. **Given a finite set A. Can we classify the distinct NOS on A (up to NeutroStrongIsomorphism)?**
3. **Can we apply the concept of NeutroOrderedAlgebra to other structures?**

The results in this presentation are from joint work [1, 2] with **Florentin Smarandache** from New Mexico University, USA,

Bijan Davvaz from Yazd University, Iran,

and **Osman Anis** from Lebanese International University, Lebanon.

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Thank you!