Florentin Smarandache
Surapati Pramanik
(Editors)

New Trends in Neutrosophic Theory and Applications

Volume III

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2. Chatterjee, T., & Pramanik, S. (2024). Triangular Fuzzy Pentapartitioned Neutrosophic Set and Its Properties. In "New Trends in Neutrosophic Theory and Applications," Volume III. Biblio Publishing, Grandview Heights, OH, United States of America. https://doi.org/10.5281/zenodo.12510801
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Aims and Scope

The field of neutrosophic set theory and its applications has been rapidly expanding, particularly since the introduction of the journal "Neutrosophic Sets and Systems."

New theories, techniques, and algorithms are being developed at a very high rate.

One of the most notable trends in neutrosophic theory is its hybridization with other set theories such as rough set theory, bipolar set theory, soft set theory, hesitant fuzzy set theory, and more.

Various hybrid structures like rough neutrosophic sets, neutrosophic soft set, single valued neutrosophic hesitant fuzzy sets, among others, have been proposed in a short period.

Neutrosophic sets have proven to be crucial tools across a wide array of fields including data mining, decision making, e-learning, engineering, medical diagnosis, social sciences, and beyond.

The third volume in the series "New Trends in Neutrosophic Theories and Applications" focuses on theories, methods, and algorithms for decision making, as well as applications involving neutrosophic information.

Some topics introduce new sets such as the Pythagorean neutrosophic vague soft set, the triangular fuzzy penta-partitioned neutrosophic set, interval-valued neutrosophic b-open sets, and interval-valued neutrosophic b-closed sets.

Other topics present applications in medical diagnosis, non-preemptive neutrosophic priority queues with uneven services (labeled as NM/NM/1), AHP in an interval neutrosophic set environment, MAGDM in a triangular fuzzy neutrosophic number environment, MAGDM in a pentapartitioned neutrosophic environment, the entropy-ARAS strategy in a single-valued neutrosophic number environment, and the MABAC strategy in a rough neutrosophic set environment.

Florentin Smarandache, Surapati Pramanik
(Editors)

Foreword

The Neutrosophic Set Theory (NST) originates from Neutrosophy, a novel branch of philosophy introduced by Professor Florentin Smarandache in 1998. NST adeptly manages uncertainty, indeterminacy, and inconsistent data. NST-based methodologies are ideal for modeling problems where human knowledge and evaluation are indispensable, accommodating situations riddled with uncertainty, indeterminacy, and inconsistent information.

NST is very important because it extends the traditional notions of classical sets and fuzzy sets to handle indeterminate, imprecise, incomplete, and inconsistent information more effectively. NST has garnered significant global attention from researchers and practitioners alike, contributing substantially to its evolution and practical applications after the publication of the journal, "Neutrosophic Sets and Systems" in 2013. Its fundamental significance spans artificial intelligence and cognitive sciences, particularly in domains such as data mining, decision analysis, expert systems, machine learning, intelligent systems, and pattern recognition.

Methods rooted in NST, either independently or in conjunction with complementary approaches, have found extensive application in diverse fields. The versatility and adaptability of NST have thus enabled its widespread adoption across a broad spectrum of scientific and practical domains, facilitating advancements and innovations in each domain.

The present book starts by proposing the Pythagorean neutrosophic vague soft set, the triangular fuzzy penta-partitioned neutrosophic set, interval-valued neutrosophic b-open sets, and interval-valued neutrosophic b-closed sets in the first, second, and third chapters respectively. It then progresses on to topics such as neutrosophic homomorphism in neutrosophic topological spaces, the neutrosophic dimension of a neutrosophic vector space, a comprehensive survey of Q-neutrosophic soft sets in all possible dimensions of the medical diagnosis system, a method for evaluating the performance measures of non-preemptive neutrosophic priority queues with uneven services (labeled as NM/NM/1), interval-valued neutrosophic AHP, MAGDM in a pentapartitioned neutrosophic set environment, MAGDM in a triangular fuzzy neutrosophic number environment, the single-valued neutrosophic entropy ARAS strategy, and the MABAC strategy in a rough neutrosophic numbers environment.

Chapter 1 develops the Pythagorean neutrosophic vague soft set, combining the soft set with the Pythagorean neutrosophic vague set. It presents a decision-making technique based on the Pythagorean neutrosophic vague soft set with a numerical example.

Chapter 2 develops the triangular fuzzy penta-partitioned neutrosophic set by combining the triangular fuzzy number and the penta-partitioned neutrosophic set. It defines some operations on the triangular fuzzy penta-partitioned neutrosophic sets, such as union, intersection, and complement, and establishes some fundamental properties of the developed sets.

Chapter 3 introduces two novel concepts: interval-valued neutrosophic b-open sets and interval-valued neutrosophic b-closed sets. It delves into the concepts of interval-valued neutrosophic b-interior and interval-valued neutrosophic b-closure operators, shedding light on their characteristics and their relationships with other operators in this domain.

Chapter 4 introduces new concepts in N_{eu}-homeomorphism, namely $N_{eu}gs\alpha*$-homeomorphism and $N_{eu}igs\alpha*$-homeomorphism in N_{eu}-topological spaces. Additionally, it presents the characterizations and properties of these functions with already existing N_{eu}-functions.

Chapter 5 presents the neutrosophic dimension of a neutrosophic vector space using a neutrosophic basis. It also discusses some characteristics of these new notions.

Chapter 6 presents a comprehensive survey of Q-neutrosophic soft sets in all possible dimensions of the medical diagnosis system. The survey highlights all possible mathematical frameworks used for medical diagnosis, including their limitations, which encompass fuzzy logic, evidential reasoning, and quantum & machine learning decisions. A comparative analysis of Q-neutrosophic soft sets is presented alongside other mathematical frameworks like neutrosophic soft sets and Q-fuzzy sets.

Chapter 7 develops a novel strategy for evaluating the performance measures of non-preemptive neutrosophic priority queues with uneven services, labeled as NM/NM/1, using the (α, β, γ)-cut approach along with Zadeh's extension principle. The developed strategy comprises a solitary server, where both arrival and service rates are expressed in terms of single-valued trapezoidal neutrosophic numbers. The queueing model involves exponentially distributed service times, arrivals following a Poisson process, and the presence of only one server. The chapter offers a concrete example to elucidate the analytical strategy established within the study.

Chapter 8 determines the criteria that affect franchisee selection in the global cafe chain business. It investigates the franchisee selection problem with interval-valued neutrosophic AHP. In the research, the priorities of the criteria and the scoring of the experts were taken into consideration. According to the results of the analysis, while location was found to be the most important criterion, personal condition was deemed the least important.

Chapter 9 develops a decision-making strategy to solve multi-attribute group decision-making problems under the pentapartitioned neutrosophic number environment. An illustrative example of a multi-attribute group decision-making problem is provided to show the applicability of the developed strategy.

Chapter 10 develops two multi-criteria group decision-making strategies using the proposed Triangular Fuzzy Neutrosophic Number Einstein's Ordered Weighted Average (TFNNEOWA) operator and Triangular Fuzzy Neutrosophic Number Ordered Weighted Geometric Average (TFNNEOWGA) operator. The chapter uses Shannon's entropy to determine the weights of the criteria and the decision-makers.

Chapter 11 develops the SVNN-E-ARAS strategy using the arithmetic averaging aggregation operator in single-valued neutrosophic number settings. It covers the group popularity ranking criteria and provides weight to each ranking component individually based on user evaluation using the developed approach.

Chapter 12 develops the MABAC strategy in a rough neutrosophic numbers environment, termed the RNN-MABAC strategy. The developed strategy is illustrated by solving an illustrative MADM problem.

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Application of Pythagorean Neutrosophic Vague Soft on Decision Making Problem

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ABSTRACT

Decision making problems often involve uncertainty and vagueness, which require effective mathematical models to handle these complexities. In recent years, several hybrid fuzzy set theories have been proposed to address these challenges, such as Pythagorean fuzzy sets, neutrosophic sets, and vague sets. However, each of these theories has its own limitations in representing uncertainty and vagueness adequately. To overcome these limitations, this study introduces a novel approach called Pythagorean Neutrosophic Vague Soft (PNVS) sets.

The PNVS sets integrate the concepts of Pythagorean fuzzy sets, neutrosophic sets, and vague sets to provide a comprehensive framework for decision making under uncertainty. The proposed methodology allows decision makers to express their opinions using three membership functions: truth, indeterminacy, and falsity. Moreover, the PNVS sets incorporate the notion of vagueness, enabling decision makers to express their uncertainty through vague membership degrees.

To demonstrate the applicability of the PNVS sets, a decision making problem is formulated and solved using the proposed methodology. The decision making problem involves evaluating potential investment options based on multiple criteria. The PNVS sets are used to model the uncertainties and vagueness associated with the criteria and their relative importance. The proposed approach provides a systematic and flexible framework for decision making, allowing decision makers to consider multiple perspectives and adequately handle uncertainties and vagueness in various problem of decision-making systems.

The experimental results demonstrate the effectiveness of the PNVS approach in capturing the uncertainties and vagueness inherent in decision making problems. The proposed methodology allows decision makers to make informed decisions by considering multiple criteria and their associated uncertainties. The PNVS sets provide a robust and intuitive framework for decision making, enhancing the decision-making process in various domains.

KEYWORDS: **Decision making, Pythagorean neutrosophic vague soft sets, uncertainty, vagueness, fuzzy sets, neutrosophic sets, vague sets**.

*1. **INTRODUCTION***

Pythagorean Neutrosophic Vague Soft Sets (PNVSS) is an extension of the neutrosophic vague soft set theory that combines the concepts of neutrosophic sets, vague sets, and soft sets. PNVSS provides a flexible and comprehensive framework for dealing with uncertainty, vagueness, and indeterminacy in decision-making problems. It incorporates the Pythagorean fuzzy set theory, which allows for the representation of membership, non-membership, and indeterminacy degrees in a more intuitive and realistic manner.

The PNVSS model consists of three components: the membership degree, non-membership degree, and indeterminacy degree, each represented by a Pythagorean fuzzy number. These components can be used to describe the uncertainty associated with the elements of a set, allowing decision makers to capture various degrees of belief, disbelief, and uncertainty in a unified manner.

The application of PNVSS in decision-making problems involves the following steps:

1. Problem formulation: Clearly define the decision problem and identify the criteria and alternatives involved. Determine the degree of uncertainty and vagueness associated with the problem.

2. Data collection: Gather the necessary data and information related to the decision problem. This may include expert opinions, historical data, or other relevant sources.

3. Representation: Represent the collected data and information using Pythagorean fuzzy numbers to express the membership, non-membership, and indeterminacy degrees associated with each element.

4. Aggregation: The Pythagorean fuzzy numbers should be combined to reflect the decision problem as a whole. The Pythagorean weighted average, Pythagorean weighted geometric mean, or other appropriate aggregation operators can be used to accomplish this.

5. Ranking and selection: Use appropriate ranking methods to prioritize the alternatives based on their aggregated Pythagorean fuzzy numbers. This can involve comparing the membership, non-membership, or indeterminacy degrees of the alternatives.

6. Decision analysis: Analyze the results obtained from the ranking process and make a decision based on the desired criteria. Consider the trade-offs between different factors and the decisionmaker's preferences.

7. Sensitivity analysis: Assess the sensitivity of the decision to changes in the input data and aggregation methods. This step helps evaluate the robustness of the decision and identify potential risks or uncertainties.

The application of PNVSS in decision making offers several advantages. It provides a comprehensive framework that can handle various types of uncertainty and vagueness simultaneously. The Pythagorean fuzzy numbers enable a more flexible and intuitive representation of uncertain information. Moreover, the aggregation and ranking methods used in PNVSS allow decision-makers to incorporate their preferences and subjective judgments in a systematic manner.

Overall, Pythagorean Neutrosophic Vague Soft Sets offer a promising approach to decision making under uncertainty, particularly when dealing with complex and ambiguous situations where traditional crisp models may fall short.

Yager and Abbasov (2013) first proposed the novel idea of Pythagorean fuzzy sets. Gau and Buehrer (1993) made the initial proposal for the theory of the vague set. Molodtsov (1999) first proposed the idea of a soft set. In this essay, we explore the idea of Pythagorean Neutrosophic Vague Soft (PNVS) sets. There have been some proposed definitions and operations. It combines Pythagorean neutrosophic vague set and soft set. The following notions have also been used to a decision-making dilemma. It could be used with realistic data to apply to real-world issues for further research.

The following describes the format of this study: in section 2, we quickly present some basic definitions and findings. Section 3 introduces the concept of PNVS sets. A few definitions and conclusions have been established. In section 4, a decision making problem application is demonstrated. Numerous researchers have made contributions to this topic.

Maji (2013) presented the neutrosophic soft set. Shil et al. (2024) presented single-valued pentapartitioned neutrosophic soft set. Das, Das, and Pramanik (2022a, 2022b) employed neutrosophic sets in developing single valued bipolar pentapartitioned neutrosophic set and single valued pentapartitioned neutrosophic graphs respectively. Neutrosophic vague set theory was studied by Alkhazaleh (2015). Das et al. (2022) Application of neutrosophic similarity measures in Covid-19. Das, Mukherjee, and Tripathy (2022) presented an application of neutrosophic similarity measure in COVID-19. Jansi et al. (2019) studied on correlation measure for Pythagorean neutrosophic sets with and as dependent neutrosophic components. Mukherjee (2015) presented a generalized rough set and its application. Mukherjee and Das (2020) presented the neutrosophic bipolar vague soft set and its application to decision making problems. Smarandache (1998), Smarandache (2005) did the most significant work on Neutrosophic Sets and Systems and generalized the thoughts. Xu et al. (2013) study the vague soft sets and their properties. Zadeh (1965) introduced the Fuzzy sets. So many authors have given significant efforts to establishing the neutrosophic idea. Development of neutrosophic theories and their applications were depicted in the studies (Broumi et al., 2018; Pramanik et al., 2018; Peng & Dai, 2020; Pramanik, 2020, 2022; Smarandache, & Pramanik, 2016, 2018; Delcea et al, 2023).

2 PRELIMINARIES

We recall some basic notions for future work.

Definition 2.1 Gau and Buehrer (1993). Let X be a non-empty set. Let A and B be two VSs in the form $A = \{< x, t_A, 1 - f_A > x \in X\}, B = \{< x, t_B, 1 - f_B > x \in X\}$. Then

(i) $A \subseteq B$ if and only if $t_A \leq t_B$ and $1 - f_A \leq 1 - f_B$.

(ii) $A \cup B = \{< x, \max(t_A(x), t_B(x)), \max(1 - f_A(x), 1 - f_B(x)) > x \in X\}$

(iii) $A \cap B = \{< x, \min(t_A(x), t_B(x)), \min(1 - f_A(x), 1 - f_B(x)) > x \in X\}$

(iv) $A^e = \{< x, f_A, 1 - t_A > x \in X\}$.

Definition 2.2. (Alkhazaleh, 2015). For any two NVSs A_{NV} and B_{NV} the union is a $NVSC_{NV}$, written as $C_{NV} = A_{NV} U\ B_{NV}$, whose truth, indeterminacy and false-membership functions are related to those of A_{NV} and $\mathrm{B_{NV}}$ given by

$$T_{C_{NV}}(x) = \left[\max\left(T^-_{A_{NV_x}}, T^-_{B_{NV_x}}\right), \max\left(T^+_{A_{NV_x}}, T^+_{B_{NV_x}}\right)\right]$$

$$I_{C_{NV}}(x) = \left[\min\left(I^-_{A_{NV}}, I^-_{B_{NV}}\right), \min\left(I^+_{A_{NV_x}}, I^+_{B_{NV_x}}\right)\right] \text{and}$$

$$F_{C_{NV}}(x) = \left[\min\left(F^-_{A_{NV}}, F^-_{B_{NV}}\right), \min\left(F^+_{A_{NV}}, F^+_{B_{NV}}\right)\right]$$

Definition 2.3. (Alkhazaleh, 2015). For any NVSs $\mathrm{A_{NV}}$ and $\mathrm{B_{NV}}$ the intersection is $NVSC_{NV}$, Known as $\mathrm{H_{NV}} = \mathrm{A_{NV}} \cap B_{NV}$, whose membership functions are related to those of A_{NV} and B_{NV} given by

$$T_{H_{NV}}(x) = \left[\min\left(T^-_{A_{NV_x}}, T^-_{B_{NV_x}}\right), \min\left(T^+_{A_{NV_x}}, T^+_{B_{NV_x}}\right)\right]$$

$$I_{H_{NV}}(x) = \left[\max\left(I^-_{A_{NV_x}}, I^-_{B_{NV_x}}\right), \max\left(I^+_{A_{NV_x}}, I^+_{B_{NV_x}}\right)\right] \text{and}$$

$$F_{H_{NV}}(x) = \left[\max\left(F^-_{A_{NV}}, F^-_{B_{NV_x}}\right), \max\left(F^+_{A_{NV_x}}, F^+_{B_{NV_x}}\right)\right]$$

Definition 2.4. **(**Yager & Abbasov, 2013). Consider X be a nonempty set and I the unite interval [0,1]. A Pythagorean fuzzy set is an object having the form $A = \{(x, \mu_A(x), v_A(x)) : x \in X\}$, where the function $\mu_A : X \to [0,1]$ and $v_A : X \to [0,1]$ denote the respectively degree of membership and degree of non-membership of each element $x \in X$ to the set A and $0 \le (\mu_A(x))^2 + (\gamma_A(x))^2 \le 1$ for each $x \in X$. Supposing, $0 \le (\mu_A(x))^2 + (\gamma_A(x))^2 \le 1$ then the degree of indeterminacy of $x \in X$ to A is denoted by $\pi_A(x) = \sqrt{(\mu_A(x))^2 + (y_A(x))^2} \& \pi_A(x) \in [0,1]$.

Definition 2.5. **(**Yager & Abbasov, 2013). Suppose X be a nonempty Universal set. A Pythagorean neutrosophic set with
truth, falsity an dependent neutrosophic components [PNSet] an a non-empty set X is an object of

the form $A = \{(x, \mu_A(x), v_A(x), \delta_A(x)) : x \in X\}$ where $\mu_A(x), v_A(x), \delta_A(x) \in [0,1]$, $0 \le (\mu_A(x))^2 + (v_A(x))^2 + (\delta_A(x))^2 \le 2$ for all $x \in X$. Where $\mu_A(x)$ is the degree of membership, $v_A(x)$ degree of indeterminacy and, $\delta_A(x)$ degree of non-membership. Here $\mu_A(x)$ and $\delta_A(x)$ are dependent component and $v_A(x)$ is independent component.

Definition 2.6 (Yager, 2013). Let X be a nonempty set and I be the unit interval [0,1]. A Pythagorean neutrosophic set with T and F are dependent neutrosophic components [PNSet]A and B of the form $A = \{(x, \mu_A(x), v_A(x), \delta_A(x)) : x \in X\}$ and $B = \{(x, \mu_B(x), v_B(x), \delta_B(x)) : x \in X\}$ then

1. $A^e = \{(x, \delta_A(x), v_A(x), \mu_A(x)) : x \in X\}$
2. $A \cup B = \{(x, \max\{\mu_A(x), \mu_B(x)\}, \max\{v_A(x), v_B(x)\}, \min\{\delta_A(x), \delta_B(x)\}) : x \in X\}$
3. $A \cap B = \{(x, \min\{\mu_A(x), \mu_B(x)\}, \min\{v_A(x), v_B(x)\}, \max\{\delta_A(x), \delta_B(x)\}) : x \in X\}$

3. PYTHAGOREAN NEUTROSOPHIC VAGUE (PNV) SET.

Definition 3.1 Consider X be a nonempty set. A PNVS with T and F are dependent neutrosophic components

$A_{PNV} = \left\{\left(x, T_{A_{PNV}}(x), I_{A_{PNV}}(x), F_{A_{PNV}}(x)\right) : x \in X\right\}$ where the definition of the truth, indeterminacy, and falsity membership functions is $T_{APNV}(x) = [T^+, T^-], I_{APNV}(x) = [I^+, I^-]$and $F_{APNV}(x) = [F^+, F^-]$

Where 1)$T^+ = 1 - F^-$,

2)$F^+ = 1 - T^-$and

3) $0 \le (T^-)^2 + (I^-)^2 + (F^-)^2 \le 2, 4)^- 0 \le \mathrm{T}^+ + \mathrm{I}^+ + \mathrm{F}^+ \le 2^+$.

Example 3.2. Let $X = \{u_1, u_2, u_3\}$ be a set of universe. Then the PNV set A_{PNV} is as follows $A_{PNV} = \{\frac{u_1}{[0.3,0.5],[0.5,0.5],[0.5,0.7]}, \frac{u_2}{[0.3,0.7],[0.4,0.6],[0.3,0.7]}, \frac{u_3}{[0.4,0.7],[0.4,0.6],[0.3,0.6]}\}$ satisfies (1), (2) and (3) of definition 3.1

(a) $0 \le (0.3)^2 + (0.5)^2 + (0.5)^2 = 0.09 + 0.25 + 0.25 = 0.59 \le 2.$

(b) $0 \le (0.3)^2 + (0.4)^2 + (0.3)^2 = 0.09 + 0.16 + 0.09 = 0.34 \le 2.$

(c) $0 \le (0.4)^2 + (0.4)^2 + (0.3)^2 = 0.16 + 0.16 + 0.09 = 0.41 \le 2.$

Note: In particular, PNV set A_{PNV} may be as follows

$$A_{PNV} = \left\{\frac{u_1}{[0,1][0,1][0,1]}, \frac{u_2}{[0,1][0,1][0,1]}, \frac{u_3}{[0,1][[0,1][0,1]}\right\}$$

Then we have the conditions $0 \le (T^-)^2 + (I^-)^2 + (F^-)^2 \le 2$ and $0 \le (T^+)^2 + (I^+)^2 + (F^+)^2 \le 2.$

Definition 3.3 Let A_{PNV} and B_{PNV} be two PNV sets of the universal set U. If $\forall u_i \in U$

1. $T_{APNV}(u_i) = T_{B_{PNV}}(u_i)$
2. $I_{A_{PNV}}(u_i) = I_{B_{PNV}}(u_i)$ and
3. $F_{APNV}(u_i) = F_{B_{PNV}}(u_i)$

Then the PNV sets A_{PNV} is equals to PNV set B_{PNV}, denoted by $A_{PNV} = B_{PNV}$, where $1 \le i \le n$

Definition 3.4 Let A_{PNV} and B_{PNV} be two PNV sets of the universal set U. If $\forall u_i \in U$

1. $T_{APNV}(u_i) \le T_{\overline{BP}_{PVV}}(u_i)$
2. $l_{APNV}(u_i) \ge I_{\overline{BPNVV}^P}(u_i)$
3. $F_{APNV}(u_i) \ge F_{B_{PNV}}(u_i)$

Then the PNV sets A_{PNV} is included in B_{PNV}; denoted by $A_{PNV} \subseteq B_{PNV}$, where $1 \le i \le n$

Definition 3.5 A^e_{PNV} represents the complement of a PNV set A_{PNV}, which is defined as $T_{A\Gamma_{NV}}(x) = [1 - T^+, 1 - T^-], l_{A\Gamma_{NV}}(x) = [1 - I^-, 1 - I^+]$ and $F_{A\Gamma_{NV}}(x) = [1 - F^-, 1 - F^+].$

Example 3.6 Take example 3.2 into consideration.

Then

$$A^c_{PNV} = \left\{\frac{u_1}{[0.5, 0.7][0.5, 0.5][0.3, 0.5]}, \frac{u_2}{[0.7, 0.8][0.4, 0.6][0.3, 0.7]}, \frac{u_3}{[0.3, 0.6][[0.4, 0.6][0.4, 0.7]}\right\}$$

Note: Under the given conditions, example 3.6 meets the requirements of definition 3.5.

$0 \le (T^-)^2 + (I^-)^2 + (F^-)^2 \le 2.$

$0 \le (T^+)^2 + (I^+)^2 + (F^+)^2 \le 2.$

Definition 3.7. PNVS Set.

Let U be the universal set and let E be the parameter set. The set of all PNV sets of U is denoted as $A \subseteq E, \mathrm{PNVset}(U)$. The pair (f, A) over U is thus referred to as the $PNVS$ set ($PNVS$ set in short). f in this case is a mapping $f: A \to PNV$ set (u). $PNVS$ set(U) is the collection of all $PNVS$ sets over U.

Example 3.8. Let $U = \{u_1, u_2, u_2\}$ and $E = \{e_1, e_2\}$. Next, over U, $PNVS$ sets A_1 and A_2 are as follows:

$$A_1\left[\left(e_1, \begin{cases}(u_1, [0.3,0.5], [0.5,0.5], [0.5,0.7]), (u_2, [0.2,0.6], [0.6,0.7], [0.4,0.8]), \\ (u_2, [0.4,0.6], [0.3,0.4], [0.4,0 \end{cases}\right.\right.$$

$(u_2, [0.5,0.6], [0.7,0.8], [0.4,0.5])\}]$.

$A_2 = [(e_1, \{(u_1, [0.4,0.5], [0.3,0.4], [0.5,0.6],), (u_2, [0.3,0.7], [0.5,0.6], [0.3,0.7]),$

$(u_3, [0.5,0.7], [0.2,0.3], [0.3,0.5],)\}), (e_2, \{(u_1, [0.6,0.7], [0.2,0.4], [0.3,0.4],), (u_2, [0.4,0.5], [0.5,0.7], [0.5,0.6])$

$(u_3, [0.6,0.7], [0.5,0.7], [0.3,0.4])\}]$.

Definition 3.9. $\emptyset = \{(e, \{(u, [0,0], [0,0], [1,1])\}: e \in \mathrm{E}$ and $u \in U\}$ is the definition of an empty $PNVS$ set $\emptyset$ in U.

Definition 3.10. $I = \{(e_s\{(u_s[1,1], [1,1], [0,0])\}: e \in E$ and $u \in U\}$ is the definition of an absolute PNVS set I in U.

Example 3.11. If $E = \{e_1, e_2\}$ and $U = \{u_1, u_2, u_2\}$, then

(a) $\emptyset = \{(e_1, (u_1, [0,0], [0,0], [1,1]), (u_2, [0,0], [0,0], [1,1]), (u_3, [0,0], [0,0], [1,1]),$
$e_2, (u_1, [0,0], [0,0], [1,1]), (u_2, [0,0], [0,0], [1,1],), (u_2, [0,0], [0,0], [1,1],)\}$ is the definition of the empty PNVS set $\emptyset$ in U. (Page 7)

(b) $I = \{(e_1, (u_1, [1,1], [1,1], [0,0]), (u_2, [1,1], [1,1], [0,0]), (u_2, [1,1], [1,1], [0,0])$

$(e_2, (u_1, [1,1], [1,1], [0,0]), (u_2, [1,1], [1,1], [0,0]), (u_2, [1,1], [1,1], [0,0])\}$ is the definition of Absolute PNVS set I in U.

Definition 3.12. $C^i = \left\{\mathrm{e}, \left(\mathrm{u}, T_{q_{NVS}}, I_{c_{NVS}}, F_{c\beta_{NVS}}\right): \mathrm{u} \in U_s \in\in E\right\}$ be the Pythagorean neutrosophic vague soft set over U, with i=1,2. Afterward, by $c^1 \subseteq c^2$ defines the c^1 I PNVS sub-set of C^2 in the cases where
$T_{C^1_{PNVS}} \le T_{C^2_{PNVS}}$, $I_{C^1_{PNVS}} \ge I_{C^2_{PNVS}}$, $F_{C^1_{PNVS}} \ge F_{C^2_{PNVS}}$.

Example 3.13. According to our definition 3.12, we have the observation $A_1 \subseteq A_2$ from case 3.8.

Definition 3.14. Assume that A is a PNVS set over U. Next, A^c defines A's complement, which is given by
$A^C = \left\{e_i\left(\mathrm{u}, T_{A^c_{PNVS}}, I_{A^c_{PNVS}}, F_{A^c_{PNVS}}\right): \mathrm{u} \in U, e \in E\right\}$

$T_{A\Gamma_{\mathrm{NVS}}}(\mathrm{u}) = [(1 - T^+(u)), (1 - T^-(u))]$

$I_{A\rho_{NVS}}(\mathrm{u}) = [(1 - I^{+}(u)), (1 - I^{-}(u))] F_{A\rho_{NVS}}(\mathrm{u}) = [(1 - F^{+}(u)), (1 - F^{-}(u))]$

Example 3.15. Let $U = \{u_1, u_2\}$ and $\mathrm{E} = \{e_1, e_2\}$ then the PNVS set A is

$A = [(e_1, \{(u_1, [0.1,0.3], [0.2,0.4], [0.7,0.9])\}, \{(u_2, [0.6,0.8], [0.3,0.5], [0.2,0.4])\}, (e_2, \{(u_1, [0.7,0.9], [0.2,0.5], [0.1,0.3])\}, \{(u_2, [0.8,0.9], [0.5,0.6], [0.1,0.2])\}]$ Then the compliment of A is defined by $\mathrm{A^c}$ is as follows

$A^c = [(e_1, \{(u_1, [0.7,0.9], [0.6,0.8], [0.1,0.3])\}, \{(u_2, [0.2,0.4], [0.5,0.7], [0.6,0.8])\}, (e_2, \{(u_1, [0.1,0.3], [0.5,0.8], [0.7,0.9])\}, \{(u_2, [0.1,0.2], [0.4,0.5], [0.8,0.9])\}]$

Definition 3.16. $A^i = \left\{\mathrm{e}, \left(\mathrm{u}, T_{A\rho_{NVS}}, I_{A\rho_{NVS}}, F_{A\rho_{NVS}}\right) : \mathrm{u} \in \mathrm{U}, e \in E\right\}$ where $\mathrm{i} = 1, 2$ denotes the two PNVS sets over U. The union and intersection of A^1 and A^2 of two PNVS sets are defined as follows:

(a) $A^1 \cup A^2 = A^3 = \left\{e_i \left(\mathrm{u}, T_{A^3_{PNVS}}, I_{A^3_{PNVS}}, F_{A^3_{PNVS}}\right)\right\}$ where,

$$T_{A^3_{PNVS}}(u) = \left[\left(T^-_{A^1_{PNVS}}(u)\right) \vee \left(T^-_{A^2_{PNVS}}(u)\right), \left(T^+_{A^1_{PNVS}}(u)\right) \vee \left(T^+_{A^2_{PNVS}}(u)\right)\right]$$

$$I_{A^3_{PNVS}}(u) = \left[\left(I^-_{A^1_{PNVS}}(u)\right) \wedge \left(I^-_{A^2_{PNVS}}(u)\right), \left(I^+_{A^1_{PNVS}}(u)\right) \wedge \left(I^+_{A^2_{PNVS}}(u)\right)\right]$$

$$F_{A^3_{PNVS}}(u) = \left[\left(F^-_{A^1_{PNVS}}(u)\right) \wedge \left(F^-_{A^2_{PNVS}}(u)\right), \left(F^+_{A^1_{PNVS}}(u)\right) \wedge \left(F^+_{A^2_{PNVS}}(u)\right)\right]$$

(b) $A^1 \cap A^2 = A^4 = \left\{e_i \left(u_s T_{A^4_{PNVS}}, I_{A^4_{PNVS}}, F_{A^4_{PNVS}}\right)\right\}$ where,

$$T_{A^4_{PNVS}}(u) = \left[\left(T^-_{A^1_{PNVS}}(u)\right) \wedge \left(T^-_{A^2_{PNVS}}(u)\right), \left(T^+_{A^1_{PNVS}}(u)\right) \wedge \left(T^+_{A^2_{PNVS}}(u)\right)\right]$$

$$I_{A^4_{PNVS}}(u) = \left[\left(I^-_{A^1_{PNVS}}(u)\right) \vee \left(I^-_{A^2_{PNVS}}(u)\right), \left(I^+_{A^1_{PNVS}}(u)\right) \vee \left(I^+_{A^2_{PNVS}}(u)\right)\right]$$

$$F_{A^4_{PNVS}}(u) = \left[\left(F^-_{A^1_{PNVS}}(u)\right) \vee \left(F^-_{A^2_{PNVS}}(u)\right), \left(F^+_{A^1_{PNVS}}(u)\right) \vee \left(F^+_{A^2_{PNVS}}(u)\right)\right]$$

Definition 3.17

Let $\mathrm{A} = \left\{\mathrm{e}, \left(\mathrm{u}, T_{A_{PNV}}(u), I_{A_{PNV}}(u), F_{A_{PNV}}(u)\right) : \mathrm{u} \in U_s \in E\right\}$ be a PNVS set over U. Then aggregation PNVS operator denoted by A_{agg} is denoted as

$$A_{\mathrm{agg}} = \left\{\frac{[\mathrm{E_R} + \mathrm{E_A}]}{u} : u \in U\right\}$$

Where $[\delta^+_A, \delta^-_A]$

$$= \frac{1}{2|E \times U|} [\Sigma_{\varepsilon \in E}([1,1] - I_e(u)[T_e - F_e(u)]$$

Where $I_e(u) = [I^+_e(u) - I^-_e(u)]$

$$T_e(u) = \left[T^+_e(u) - T_e^{-(w)}\right]$$

$F_e(u) = [F_e^+(u) - F_e^-(u)]$

$|E \times U|$ is the cardinality of $E \times U$.

4. APPLICATION OF PYTHAGOREAN NEUTROSOPHIC VAGUE SOFT (PNVS) SET

In our daily lives, we face decision-making challenges in the areas of politics, management, the economy, education, and technology use. The academic results reflect which college education is the best. A range of professional standards are used to evaluate teacher preparation while selecting a college teaching curriculum. We identify a factor that is believed to affect parental judgment: The campus environment, academic quality, and career opportunities are the three components of the academic factor that have been found. We wish to select the finest solution from a range of options by comparing expert evaluations with the standards.

The goal of the parent committee is to select a popular college programmer. In this instance, the committee plans to select three institutions, $U = \{u_1, u_2, u_3\}$. $E = \{e_1 =$ Popular Environments, $e_2 =$ Academic quality, $e_3 =$ Career Opportunity $\}$ is the expert evaluation score for a college education. Algorithm

1. First, on U, we construct the Pythagorean Neutrosophic Soft Set.

2. A computation is made of the Pythagorean Neutrosophic Soft Set Aggregation Operator.

3. Calculate $|A_{\text{agg}}|$ by taking the average of each intervals. (The numerical value)

4. Determine the optimum value on U. Let $U = \{u_1, u_2, u_3\}$ be the set of colleges. These colleges can be described by a set of parameters $\mathrm{E} = \{e_1, e_2, e_3\}$.

(a) The parents committee construct a PNVS set A over $\mathbf{U}$ as

$\mathrm{A} = [\{(e_1, (u_1, [0.8,0.9], [0.5,0.7], [0.1,0.2]), (u_2, [0.5,0.7], [0.4,0.6], [0.3,0.5]), (u_3, [0.7,0.9], [0.2,0.4], [0.1,0.3])\}, \{e_2, (u_1, [0.5,0.7], [0.4,0.6], [0.3,0.5]), (u_2, [0.7,0.9], [0.4,0.6], [0.1,0.3]), (u_3, [0.6,0.8], [0.8,0.9], [0.2,0.4])\}, \{e_3, (u_1, [0.7,0.9], [0.2,0.4], [0.1,0.3]), (u_2, [0.6,0.8], [0.4, 0.6], [0.2,0.4]), (u_3, [0.5,0.7], [0.5,0.7], [0.3,0.5])\}]$

(b) Then we find the PNVS set aggregation operator A_{agg} of A as follows:

For u_1

$\frac{1}{18}[[1,1] - [0.5,0.7]([0.8,0.9] - [0.1,0.2]) + [1,1] - [0.4,0.6]([0.5,0.7] - [0.3,0.5]) + [1,1] - [0.2,0.4]([0.7$

,

$0.9] - [0.1,0.3])]$

For u_2

$\frac{1}{18}[[1,1]-[0.4,0.6]([0.5,0.7]-[0.3,0.5])+[1,1]-[0.4,0.6]([0.7,0.9]-[0.1,0.3])+[1,1]-[0.4,0.6]([0.6$

,

$0.8]-[0.2,0.4])]$

For u_3

$\frac{1}{18}[[1,1]-[0.2,0.4]([0.7,0.3]-[0.1,0.3])+[1,1]-[0.8,0.9]([0.6,0.8]-[0.2,0.4])+[1,1]-[0.5,0.7]([0.5$
$,0.7]-[0.3,0.5])]$

(c) Each interval's average is calculated i.e $[1,1]\&(u)=[T^{-}(u)-T^{+}(u)]I(u)=[I^{-}(u)-I^{+}(u)], F(w)=[F^{-}(u)-F^{+}(u)]$

(d) Then $|A_{\text{agg}}|=\frac{0.1277}{u_1},\frac{0.1331}{u_2},\frac{0.1311}{u_3}$

(e) Because |Agg| has the highest degree of 0.1333 among the colleges, the family board eventually decides on college u_2. To get our outcome in this case, we only need to perform a few simple calculations. The validity of this approach is higher than that of earlier research.

5. FUTURE VISION OF PYTHAGOREAN NEUTROSOPHIC VAGUE SOFT

Pythagorean Neutrosophic Vague Soft (PNVS) sets can be defined as a framework that unifies several vague and uncertain elements. It allows for a more thorough representation and handling of uncertainty, ambiguity, and vagueness in decision-making and reasoning processes by integrating Pythagorean fuzzy sets, neutrosophic sets, and vague sets.

It's crucial to remember that, as an AI language model, I am only able to speculate on the future and cannot foretell Pythagorean Neutrosophic Vague Soft. Thus, the following vision is entirely speculative and ought to be considered an artistic rendering rather than an exact prediction.

The Pythagorean Neutrosophic Vague Soft framework might see substantial developments and applications in a number of fields in the future. The following are some possible developments:

Decision-Making: By simultaneously taking into account several dimensions of uncertainty, ambiguity, and vagueness, PNVS sets can improve decision-making processes. Future work might concentrate on creating increasingly complex algorithms and processes for generating decisions in PNVS environments, utilizing cutting-edge computational intelligence methods like evolutionary computation, deep learning, and machine learning.

Expert Systems: Multiple sources of uncertainty in expert knowledge can be modelled and captured by expert systems using PNVS. These technologies, by taking into account the fuzzy, neutrosophic, and ambiguous characteristics of experts' knowledge domains, could help them make more informed and nuanced decisions.

Data Analysis and Mining: When dealing with datasets that contain ambiguity and uncertainty, PNVS might be used. It may be possible to manage PNVS data using sophisticated methods and algorithms, which would allow for more precise and perceptive examination of complicated and uncertain datasets.

Artificial Intelligence and Robotics: PNVS may help robots and AI systems become more capable of making decisions. Artificial intelligence (AI) systems can more effectively adapt to real-world situations and make more informed decisions if they can handle uncertainty, ambiguity, and vagueness.

Risk Assessment and Management: In a variety of industries, including banking, engineering, and healthcare, PNVS sets can offer a strong foundation for assessing and managing risks. Effective risk mitigation techniques and more accurate forecasts may be provided by PNVS-based risk assessment models that incorporate the uncertainty related to risk components.

Multi-Criteria Decision Analysis: To handle a variety of competing criteria involving fuzzy, neutrosophic, and vague information, PNVS sets can be incorporated into multi-criteria decision analysis frameworks. Future developments could concentrate on creating effective algorithms for prioritizing and rating options in PNVS environments.

All things considered, Pythagorean Neutrosophic Vague Soft has a bright future ahead of it, with possible uses in many different domains where vagueness and uncertainty are present. We may anticipate greater developments in theory, methodologies, and real-world applications as this field of study develops, which will make it possible to make more thorough and comprehensive decision-making in challenging and uncertain environments.

6. CONCLUSIONS

We propose the Pythagorean neutrosophic vague soft set. It combines the soft set with the Pythagorean neutrosophic vague set. In the present article, we develop a decision-making technique based on the Pythagorean Neutronic VFS. A numerical example has been presented. The Pythagorean neutrosophic vague soft set has been subjected to multiple novel techniques. It can be applied to real-world problems for additional study when given realistic data.

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Triangular Fuzzy Pentapartitioned Neutrosophic Set and Its Properties

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ABSTRACT

The main objective of the paper is to hybridize the triangular fuzzy number and the penta-partitioned neutrosophic set and develop the triangular fuzzy penta-partitioned neutrosophic set. The triangular fuzzy number has great potential to express uncertainty systematically. So, the combination of the triangular fuzzy number and pentapartitioned neutrosophic set is an intelligent mathematical tool that will be a helpful mathematical tool for decision-making. We define some operations on the triangular fuzzy penta-partitioned neutrosophic sets such as union, intersection, and complement. We establish some fundamental properties of the developed triangular fuzzy penta-partitioned neutrosophic sets.

KEYWORDS: **Fuzzy set, triangular fuzzy number, neutrosophic set, pentapartitioned neutrosophic set.**

1. INTRODUCTION

NS was first developed by Smarandache (1998) by exploring the properties of Fuzzy Set (FS) (Zadeh, 1965) and Intuitionistic FS (IFS) (Atanassov, 1986) by initiating indeterminacy and falsity as independent membership components. Wang et al. (2010) defined Single- Valued NS (SVNS) by confining the "truth", "indeterminacy'' and "falsity'' membership degrees in the unit interval [0, 1]. An overview of SVNS was documented by Pramanik (2022). Quardripartitioned SVNS (QSVNS) was defined by Chatterjee et al. (2016) with the introduction of "truth", "falsity", "unknown "and "contradiction" as four independent membership functions using four-valued logic (Belnap, 1977), and refined neutrosophic logic (Smarandache, 2013). Pramanik developed the interval Quardripartitioned NS by exploring interval NS (INS) (Wang et al., 2005) and QSVNS (Chatterjee et al., (2016). Chatterjee and Pramanik (2024) presented the triangular fuzzy quardripartitioned neutrosophic sets by combining QSVNS and triangular fuzzy number.

The theory of PNS was developed by Mallick and Pramanik (2020) by splitting the indeterminacy membership component into "contradiction", "ignorance", and "unknown". Pramanik (2023) presented the Interval PNS (IPNS) by combining PNS and INS (Wang et al., 2005). PNSs became popular and were employed in Multi Criteria Decision Making (MCDM) (Das et al., 2022a; Shil et al., 2022; Pramanik 2023, Majumder et al., 2023] and graph theory (Das et al., 2022b; Broumi et al.,2022). Triangular Fuzzy Number (TFN) (Arora, & Naithani, 2023) is an important mathematical tool for decision making. Biswas et al. (2016) combined the TFN and SVNS and developed the Triangular Fuzzy Neutrosophic Set (TFNS). TFNSs have been utilized in MCDM and different MCDM strategies were developed such as the EDAS method (Fan et al., 2020), GRA method (Xie, 2023; Yao and Ran, 2023), cross-entropy strategy (Wang et al., 2023). TFNS is an important mathematical tool for decision making. So, the combination of TFNS and PNS will be an effective tool for decision-making. TFNS is not explored in the PNS environment.

Research gap: No study combining the TFN and PNS has been reported in the literature.

Motivation: The research gap motivates us to study by combining the concepts of TFN and TFPNS and develop the theory of Triangular Fuzzy Penta-partitioned Neutrosophic Set (TFPNS).

The TFPNS is a breakthrough in the field of NS. Since the TFPNS is a hybrid structure, it is well capable of expressing uncertainty comprehensively and precisely. TFPNS has more advantages for dealing with uncertainty as it can utilize the advantages of TFN and PNS. The computational techniques based on TFN or PNS alone may not always produce the best results but the hybrid structure TFPNS may yield the best result.

We also investigate some fundamental properties of the newly introduced set.

The paper has four sections given as follows: Section 2 is dedicated to presenting some existing preliminary concepts of NSs. Section 3 represents the concept of TFPNS and some important mathematical operations on TFPNS. Section 4 presents a possible future research direction. Section 5 presents a discussion. Section 6 concludes the study by indicating some future scope of research in some emerging fields of study.

2. *PRELIMINARIES*

1. Preliminary

Definition 2.1. (Smarandache, 1998) An NS $\ddot{\Theta}$ in the "universe of discourse" Ъ is represented as

$\ddot{\Theta} = \{(\sigma, (T\tilde{T}_{\ddot{\Theta}}(\sigma), I\tilde{I}_{\ddot{\Theta}}(\sigma), U\tilde{U}_{\ddot{\Theta}}(\sigma)) : \sigma \in \tau\}$ where, $T\tilde{T}_{\ddot{\Theta}}(\sigma), I\tilde{I}_{\ddot{\Theta}}(\sigma), F\tilde{F}_{\ddot{\Theta}}(\sigma)$: Ъ→ [0,1] and we have,

$0 \le (T\tilde{T}_{\ddot{\Theta}}(\sigma) + I\tilde{I}_{\ddot{\Theta}}(\sigma) + F\tilde{F}_{\ddot{\Theta}}(\sigma)) \le 3$

where $T\tilde{T}_{\ddot{\Theta}}(\sigma), I\tilde{I}_{\ddot{\Theta}}(\sigma), F\tilde{F}_{\ddot{\Theta}}(\sigma)$ represents Truth worthy membership function (TMF), indeterminacy membership function (IMF), Falsehood membership function(FMF).

Definition 2.2. (Biswas et al., 2016) Assume that $\bar{\bar{\chi}}$ is a definite set. A TFNS $a\ddot{\zeta}$ in $\bar{\bar{\chi}}$ is represented as:

$$a\ddot{\zeta} = \{(\sigma, ((T\tilde{T}^{(l)}_{a\ddot{\zeta}}(\sigma), T\tilde{T}^{(m)}_{a\ddot{\zeta}}(\sigma), T\tilde{T}^{(u)}_{a\ddot{\zeta}}(\sigma)), (I\tilde{I}^{(l)}_{a\ddot{\zeta}}(\sigma), I\tilde{I}^{(m)}_{a\ddot{\zeta}}(\sigma)(\sigma), I\tilde{I}^{(u)}_{a\ddot{\zeta}}(\sigma)),$$
$$(F\tilde{F}^{(l)}_{a\ddot{\zeta}}(\sigma), F\tilde{F}^{(m)}_{a\ddot{\zeta}}(\sigma), F\tilde{F}^{(u)}_{a\ddot{\zeta}}(\sigma)))) : \sigma \in \ddot{\bar{\chi}}\}$$
$$0 \le T\tilde{T}^{(l)}_{a\ddot{\zeta}}(\sigma) + I\tilde{I}^{(l)}_{a\ddot{\zeta}}(\sigma) + F\tilde{F}^{(l)}_{a\ddot{\zeta}}(\sigma) \le 3 \ \& \ 0 \le T\tilde{T}^{(l)}_{a\ddot{\zeta}}(\sigma) \le T\tilde{T}^{(m)}_{a\ddot{\zeta}}(\sigma) \le T\tilde{T}^{(u)}_{a\ddot{\zeta}}(\sigma) \le 3$$
$$\& 0 \le T\tilde{T}^{(\alpha)}_{a\ddot{\zeta}}(\sigma) + I\tilde{I}^{(\alpha)}_{a\ddot{\zeta}}(\sigma) + F\tilde{F}^{(\alpha)}_{a\ddot{\zeta}}(\sigma) \le 3, \text{for} \forall \alpha = l, m, u$$

Definition 2.3. A PNS $a\ddot{\vartheta}$ in the universe of discourse $\ddot{\bar{\chi}}$ (a fixed set) may be expressed as,

$$a\ddot{\vartheta} = \{(\sigma, (T\tilde{T}_{a\ddot{\vartheta}}(\sigma), C\tilde{C}_{a\ddot{\vartheta}}(\sigma), I\tilde{I}_{a\ddot{\vartheta}}(\sigma), U\tilde{U}_{a\ddot{\vartheta}}(\sigma),$$
$$F\tilde{F}_{a\ddot{\vartheta}}(\sigma))) : \sigma \in \ddot{\bar{\chi}}\}$$

where, $T\tilde{T}_{a\ddot{\vartheta}}(\sigma), C\tilde{C}_{a\ddot{\vartheta}}(\sigma), I\tilde{I}_{a\ddot{\vartheta}}(\sigma), U\tilde{U}_{a\ddot{\vartheta}}(\sigma), F\tilde{F}_{a\ddot{\vartheta}}(\sigma)$ express truthworthiness membership function (TMF), contradiction membership function (CMF), ignorance Membership function (IMF), unknown membership function (UMF), falsihood membership function (FMF) with,

$$0 \le T\tilde{T}_{a\ddot{\vartheta}}(\sigma) + C\tilde{C}_{a\ddot{\vartheta}}(\sigma) + I\tilde{I}_{a\ddot{\vartheta}}(\sigma) + U\tilde{U}_{a\ddot{\vartheta}}(\sigma) + F\tilde{F}_{a\ddot{\vartheta}}(\sigma) \le 5$$

And for, $\forall \sigma \in \ddot{\bar{\chi}}, T\tilde{T}_{a\ddot{\vartheta}}(\sigma), C\tilde{C}_{a\ddot{\vartheta}}(\sigma), I\tilde{I}_{a\ddot{\vartheta}}(\sigma), U\tilde{U}_{a\ddot{\vartheta}}(\sigma), F\tilde{F}_{a\ddot{\vartheta}}(\sigma) : \ddot{\bar{\chi}} \to [0,1]$

3. THE FUNDAMENTAL THEORY OF TFPNS

Definition 3.1. TFPNS

Assume that $\ddot{\bar{\chi}}$ represents a particular set. We define a TFPNS $a\ddot{G}$ over $\ddot{\bar{\chi}}$ and is presented by

$$a\ddot{G} = \{(\sigma, ((T\tilde{T}^{(l)}_{a\ddot{G}}(\sigma), T\tilde{T}^{(m)}_{a\ddot{G}}(\sigma), T\tilde{T}^{(u)}_{a\ddot{G}}(\sigma)), (C\tilde{C}^{(l)}_{a\ddot{G}}(\sigma), C\tilde{C}^{(m)}_{a\ddot{G}}(\sigma), C\tilde{C}^{(u)}_{a\ddot{G}}(\sigma)),$$
$$(I\tilde{I}^{(l)}_{a\ddot{G}}(\sigma), I\tilde{I}^{(m)}_{a\ddot{G}}(\sigma), I\tilde{I}^{(u)}_{a\ddot{G}}(\sigma)), (U\tilde{U}^{(l)}_{a\ddot{G}}(\sigma), U\tilde{U}^{(m)}_{a\ddot{G}}(\sigma), U\tilde{U}^{(u)}_{a\ddot{G}}(\sigma)), (F\tilde{F}^{(l)}_{a\ddot{G}}(\sigma), F\tilde{F}^{(m)}_{a\ddot{G}}(\sigma), F\tilde{F}^{(u)}_{a\ddot{G}}(\sigma)))) : \sigma \in \ddot{\bar{\chi}}\rangle$$

where, $\forall \sigma, 0 \le T\tilde{T}^{(l)}_{a\ddot{G}}(\sigma) + C\tilde{C}^{(l)}_{a\ddot{G}}(\sigma) + I\tilde{I}^{(l)}_{a\ddot{G}}(\sigma) + U\tilde{U}^{(l)}_{a\ddot{G}}(\sigma) + F\tilde{F}^{(l)}_{a\ddot{G}}(\sigma), F\tilde{F}^{(m)}_{a\ddot{G}}(\sigma) \le 5$

and, $0 \le T\tilde{T}^{(\varepsilon)}_{a\ddot{G}}(\sigma) + C\tilde{C}^{(\varepsilon)}_{a\ddot{G}}(\sigma) + I\tilde{I}^{(\varepsilon)}_{a\ddot{G}}(\sigma) + U\tilde{U}^{(\varepsilon)}_{a\ddot{G}}(\sigma) + F\tilde{F}^{(\varepsilon)}_{a\ddot{G}}(\sigma), F\tilde{F}^{(\varepsilon)}_{a\ddot{G}}(\sigma) \le 5 \text{for}, \forall \varepsilon = l, m, u$

or, $a\ddot{G} = \{(\sigma, ((T\tilde{T}_{a\ddot{G}}(\sigma), C\tilde{C}_{a\ddot{G}}(\sigma), I\tilde{I}_{a\ddot{G}}(\sigma), U\tilde{U}_{a\ddot{G}}(\sigma), F\tilde{F}_{a\ddot{G}}(\sigma))) : \sigma \in \ddot{\bar{\chi}}\}$, is a TFPNS.

and, $T\tilde{T}_{a\ddot{G}}(\sigma), C\tilde{C}_{a\ddot{G}}(\sigma), I\tilde{I}_{a\ddot{G}}(\sigma), U\tilde{U}_{a\ddot{G}}(\sigma), F\tilde{F}_{a\ddot{G}}(\sigma) : \ddot{\bar{\chi}} \to [0,1]$.

where, $T\tilde{T}_{a\ddot{G}}(\sigma), C\tilde{C}_{a\ddot{G}}(\sigma), I\tilde{I}_{a\ddot{G}}(\sigma), U\tilde{U}_{a\ddot{G}}(\sigma), F\tilde{F}_{a\ddot{G}}(\sigma)$ represents TMF,CMF,IMF,UMF,FMF respectively with

$T\tilde{T}_{a\ddot{G}}(\sigma) = (T\tilde{T}^{(l)}_{a\ddot{G}}(\sigma), T\tilde{T}^{(m)}_{a\ddot{G}}(\sigma), T\tilde{T}^{(u)}_{a\ddot{G}}(\sigma))$, $C\tilde{C}_{a\ddot{G}}(\sigma) = (C\tilde{C}^{(l)}_{a\ddot{G}}(\sigma), C\tilde{C}^{(m)}_{a\ddot{G}}(\sigma), C\tilde{C}^{(u)}_{a\ddot{G}}(\sigma))$,

$I\tilde{I}_{a\ddot{G}}(\sigma) = (I\tilde{I}^{(l)}_{a\ddot{G}}(\sigma), I\tilde{I}^{(m)}_{a\ddot{G}}(\sigma), I\tilde{I}^{(u)}_{a\ddot{G}}(\sigma))$, $U\tilde{U}_{a\ddot{G}}(\sigma) = (U\tilde{U}^{(l)}_{a\ddot{G}}(\sigma), U\tilde{U}^{(m)}_{a\ddot{G}}(\sigma), U\tilde{U}^{(u)}_{a\ddot{G}}(\sigma))$,.

$F\tilde{F}_{a\ddot{G}}(\sigma) = ((F\tilde{F}^{(l)}_{a\ddot{G}}(\sigma), (F\tilde{F}^{(m)}_{a\ddot{G}}(\sigma), (F\tilde{F}^{(u)}_{a\ddot{G}}(\sigma))$.

Definition 3.2. We introduce the notion $\hat{0}$ and $\hat{1}$ as follows:

$\hat{0} = \langle (0,0,0), (0,0,0), (1,1,1), (1,1,1), (1,1,1) \rangle$ and $\hat{1} = \langle (1,1,1), (1,1,1), (0,0,0), (0,0,0), (0,0,0) \rangle$ **as** null and unity of TFPNS triangular fuzzy Penta partitioned neutrosophic set.

Definition 3.3. Union of any two TFPNSs $\tilde{\kappa}_1, \tilde{\kappa}_2$ is a TFPNS $\tilde{\kappa}_3$ written as $\tilde{\kappa}_3 = \tilde{\kappa}_1 \cup \tilde{\kappa}_2$,whose

MF of truth, MF of contradiction, MF of ignorance, MF of unknown, MF of falsity are linked to corresponding MFs of $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ by,

$$T\tilde{T}_{\tilde{\kappa}_3}(\sigma) = (\max(T\tilde{T}^{(l)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\kappa}_2}(\sigma)),$$
$$C\tilde{C}_{\tilde{\kappa}_3}(\sigma) = (\max(C\tilde{C}^{(l)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(C\tilde{C}^{(u)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\kappa}_1}(\sigma)),$$
$$I\tilde{I}_{\tilde{\kappa}_3}(\sigma) = (\min(I\tilde{I}^{(l)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(I\tilde{I}^{(m)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(I\tilde{I}^{(u)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\kappa}_2}(\sigma)),$$
$$U\tilde{U}_{\tilde{\kappa}_3}(\sigma) = (\min(U\tilde{U}^{(l)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(U\tilde{U}^{(m)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\kappa}_2}(\sigma)),$$
$$F\tilde{F}_{\tilde{\kappa}_3}(\sigma) = (\min(F\tilde{F}^{(l)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\kappa}_2}(\sigma))\forall\sigma \in \ddot{\bar{\chi}}$$

Therefore, $\tilde{\kappa}_3 = \{(\sigma, (T\tilde{T}_{\tilde{\kappa}_3}, C\tilde{C}_{\tilde{\kappa}_3}, I\tilde{I}_{\tilde{\kappa}_3}, U\tilde{U}_{\tilde{\kappa}_3}, F\tilde{F}_{\tilde{\kappa}_3})) : \sigma \in \ddot{\bar{\chi}}\}$

$$= \{(\sigma, \langle(\max(T\tilde{T}^{(l)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\kappa}_2}(\sigma))\rangle,$$
$$\langle\max(C\tilde{C}^{(l)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(C\tilde{C}^{(u)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\kappa}_1}(\sigma))\rangle,$$
$$\langle\min(I\tilde{I}^{(l)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(I\tilde{I}^{(m)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(I\tilde{I}^{(u)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\kappa}_2}(\sigma)\rangle, \langle\min(U\tilde{U}^{(l)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\kappa}_2}(\sigma)),$$
$$\min(U\tilde{U}^{(m)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\kappa}_2}(\sigma))\rangle,$$
$$\langle\min(F\tilde{F}^{(l)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\kappa}_2}(\sigma))\rangle\rangle) : \sigma \in \ddot{\bar{\chi}}\}$$

represents a triangular fuzzy penta partitioned neutrosophic set.

Example 1. Consider two TFPNSs as

$$\bar{\Theta}_1 = \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.3, 0.4), (0.2,0.2,0.2),$$
$$(0.3,0.3,0.3)\rangle_{\bar{\delta}_1} + \langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.4, 0.5, 0.6), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2} +$$
$$+\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.5, 0.6, 0.7), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3},$$
$$\bar{\Theta}_2 = \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +$$
$$\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2}$$
$$+\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.4, 0.5, 0.6), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}$$

So, $\bar{\Theta}_1 \cup \bar{\Theta}_2 = \langle(0.6, 0.6, 0.8), (0.4, 0.5, 0.6), (0.2, 0.3, 0.4), (0.2, 0.2, 0.2), (0.3, 0.3, 0.3)\rangle_{\bar{\delta}_1} +$
$\langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.3, 0.4, 0.5), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2} +$
$\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3}$

Example of Triangular fuzzy penta-partitioned neutrosophic number:

Consider a real-world scenario where we just want to express uncertainty and indeterminacy associated with the completion time of a project using triangular fuzzy membership functions, namely neutrosophic numbers.

Assume that three executive engineers of a construction company are present in a meeting room called by the managing director of the company to discuss a time frame that should be required for the completion of a new project. The managing director has raised a question before the three engineers, what should be the time frame for an important construction project, that the company will undertake. The objective of the company is to provide the minimum time of completion for the project with assured quality of work. The company aims to complete the project within the shortest possible time. Three engineers present here constitute the universe of discourse.

Now let us consider about first engineer's ($\bar{\delta}_1$) assessment regarding the expected time of project completion. He thinks that the time frame of completion as expected by the company is correct and he is optimistic about the time frame where everything goes smoothly. This constitutes the Truth membership function. He is quite confident in completing the project within the desired time and as per him, the time of completion is 6-8 years. In 0-1 scale, truth membership function can be presented as (0.6,0,6,0.8) as a TFNN rating. But at the same time, he has some contradiction whether the project can be completed in between 4-6 years taking into account potential delays and uncertainties that may come into play. In 0-1 scale, contradiction membership function may be expressed as (0.4,0.5,0.6) as a TFNN rating. He is completely ignorant about the fact that the project can be completed within most 2-4 years. This constitutes the ignorance membership function. In 0-1 scale, ignorance membership function may be expressed as (0.2,0.3,0.4) as a TFNN rating. He is completely unknown upon the fact that that the project can be completed within 2 years. This constitutes the unknown membership function. In 0-1 scale, In 0-1 scale, unknown membership function may be expressed as (0.2,0.2,0.2) as a TFNN rating. He never relies upon the fact that the project can be completed within 3 years. This constitutes the falsity membership function. In 0-1 scale, falsity membership function may be expressed as (0.3,0.3,0.3). So, his overall rating is expressed as a Triangular Fuzzy Pentapartitioned Neutrosophic Number (TFPNN) as: $\langle(0.6,0.6,0.8),(0.4,0.5,0.6),(0.2,0.3,0.4),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\bar{\delta}_1}$. Similarly, second engineer's assessment ($\bar{\delta}_2$) regarding time frame of completion is presented by TFPNN as $\langle(0.8,0.7,0.6),(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\bar{\delta}_2}$. Third engineer ($\bar{\delta}_3$) gives his assessment rating regarding probable time of completion of project by a TFPNN rating represented as, $\langle(0.7,0.8,0.9),(0.6,0.7,0.8),(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\bar{\delta}_3}$. All these three TFPNN ratings are elements of Triangular fuzzy Penta-partitioned Neutrosophic Set (TFPNS)

represented as, $\bar{\Theta}_1=\langle(0.6,0.6,0.8),(0.4,0.5.0.6),(0.2,0.3,0.4),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\bar{\delta}_1}$ $+\langle(0.8,0.7,0.6),(0.5,0.6,0.7),(0,4,0.5,0.6),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\bar{\delta}_2}+\langle(0.7,0.8,0.9),(0.6,0.7,0.8),$ $(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\bar{\delta}_3}$.

Definition 3.4. Intersection of two TFPNSs $\tilde{\kappa}_1,\tilde{\kappa}_2$ is represented as $\tilde{\kappa}_4$ and is expressed as $\tilde{\kappa}_4=\tilde{\kappa}_1\cap\tilde{\kappa}_2$, such that its truth , contradiction, ignorance, unknown and falsity components are presented as,

$$T\tilde{T}_{\tilde{\kappa}_4}(\sigma)=(\min(T\tilde{T}^{(l)}_{\tilde{\kappa}_1}(\sigma),T\tilde{T}^{(l)}_{\tilde{\kappa}_2}(\sigma)),\min(T\tilde{T}^{(m)}_{\tilde{\kappa}_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\kappa}_2}(\sigma)),\min(T\tilde{T}^{(u)}_{\tilde{\kappa}_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\kappa}_2}(\sigma))$$
$$C\tilde{C}_{\tilde{\kappa}_4}(\sigma)=(\min(C\tilde{C}^{(l)}_{\tilde{\kappa}_1}(\sigma),C\tilde{C}^{(l)}_{\tilde{\kappa}_2}(\sigma)),\min(C\tilde{C}^{(m)}_{\tilde{\kappa}_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\kappa}_2}(\sigma)),\min(C\tilde{C}^{(u)}_{\tilde{\kappa}_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\kappa}_2}(\sigma)),$$
$$I\tilde{I}_{\tilde{\kappa}_4}(\sigma)=\max(I\tilde{I}^{(l)}_{\tilde{\kappa}_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\kappa}_2}(\sigma)),\max(I\tilde{I}^{(m)}_{\tilde{\kappa}_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\kappa}_2}(\sigma)),\max(I\tilde{I}^{(u)}_{\tilde{\kappa}_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\kappa}_2}(\sigma))$$
$$U\tilde{U}_{\tilde{\kappa}_4}=\max(U\tilde{U}^{(l)}_{\tilde{\kappa}_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\kappa}_2}(\sigma)),\max(U\tilde{U}^{(m)}_{\tilde{\kappa}_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\kappa}_2}(\sigma)),\max(U\tilde{U}^{(u)}_{\tilde{\kappa}_1}(\sigma),U\tilde{U}^{(u)}_{\tilde{\kappa}_2}(\sigma))$$
$$F\tilde{F}_{\tilde{\kappa}_4}=\max(F\tilde{F}^{(l)}_{\tilde{\kappa}_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\kappa}_2}(\sigma)),\max(F\tilde{F}^{(m)}_{\tilde{\kappa}_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\kappa}_2}(\sigma)),\max(F\tilde{F}^{(u)}_{\tilde{\kappa}_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\kappa}_2}(\sigma))\text{for},\forall\Upsilon\in\ddot{\chi}$$

So,

$$\tilde{\kappa}_4 = \tilde{\kappa}_1 \cap \tilde{\kappa}_2 = \{(\sigma, T\tilde{T}_{\tilde{\kappa}_4}(\sigma), C\tilde{C}_{\tilde{\kappa}_4}(\sigma), I\tilde{I}_{\tilde{\kappa}_4}(\sigma), U\tilde{U}_{\tilde{\kappa}_4}, F\tilde{F}_{\tilde{\kappa}_4}) : \sigma \in \ddot{\overline{\chi}}\}$$
$$= \{(\sigma, ((\min(T\tilde{T}^{(l)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(T\tilde{T}^{(m)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(T\tilde{T}^{(u)}_{\tilde{\kappa}_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\kappa}_2}(\sigma)))$$
$$(\min(C\tilde{C}^{(l)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \min(C\tilde{C}^{(m)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \min(C\tilde{C}^{(u)}_{\tilde{\kappa}_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\kappa}_2}(\sigma))),$$
$$(\max(I\tilde{I}^{(l)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(I\tilde{I}^{(m)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(I\tilde{I}^{(u)}_{\tilde{\kappa}_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\kappa}_2}(\sigma))),$$
$$(\max(U\tilde{U}^{(l)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(U\tilde{U}^{(m)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(U\tilde{U}^{(u)}_{\tilde{\kappa}_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\kappa}_2}(\sigma))),$$
$$(F\tilde{F}^{(l)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\kappa}_2}(\sigma)), \max(F\tilde{F}^{(m)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\kappa}_2}(\sigma)), \max(F\tilde{F}^{(u)}_{\tilde{\kappa}_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\kappa}_2}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$$

Example 2. Example of intersection

Consider two TFPNS as

$$\overline{\Theta}_1 = \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.3, 0.4), (0.2,0.2,0.2),$$
$$(0.3,0.3,0.3)\rangle_{\bar{\delta}_1} + \langle(0.8, 0.7, 0.6), (0.5, 0.6, 0.7), (0.4, 0.5, 0.6), (0.3, 0.3, 0.3), (0.2, 0.2, 0.2)\rangle_{\bar{\delta}_2} +$$
$$+\langle(0.7, 0.8, 0.9), (0.6, 0.7, 0.8), (0.5, 0.6, 0.7), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5)\rangle_{\bar{\delta}_3},$$
$$\overline{\Theta}_2 = \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +$$
$$\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2}$$
$$+\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.4, 0.5, 0.6), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}$$

$$So,\ \overline{\Theta}_1 \cap \overline{\Theta}_2 = \langle(0.4, 0.5, 0.6), (0.3, 0.4, 0.5), (0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_1} +$$
$$\langle(0.3, 0.4, 0.5), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.4, 0.5, 0.6), (0.7, 0.8, 0.9)\rangle_{\bar{\delta}_2} +$$
$$\langle(0.3, 0.4, 0.5), (0.2, 0.3, 0.4), (0.5, 0.6, 0.7), (0.5, 0.6, 0.7), (0.6, 0.7, 0.8)\rangle_{\bar{\delta}_3}$$

Definition 3.5. Complement of a TFPNS

Consider a TFPNS $\tilde{\Im}a$ with its representation as,

$$\tilde{\Im}a = \{(\sigma, ((T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$$

The complement of a TFPNS $\tilde{\Im}a$ is expressed as $(\tilde{\Im}a)^{cmt}$ and is represented as,

$$(\tilde{\Im}a)^{cmt} = \{(\sigma, ((F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(1 - I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma), 1 - I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma), 1 - I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$$

Example 3. Assume a TFPNS of the form:

$$\overline{\Theta}_1 = \langle(0.6, 0.6, 0.8), (0.4, 0.5.0.6), (0.2, 0.3, 0.4), (0.2,0.2,0.2),$$

$(0.3,0.3,0.3)\rangle / \bar{\delta}_1 +$
$\langle(0.8,0.7,0.6),(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\bar{\delta}_2} + \langle(0.7,0.8,0.9),(0.6,0.7,0.8),$
$(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\bar{\delta}_3},$

Accordingly, $(\bar{\Theta}_1)^{cmt} =$
$\langle(0.3,0.3,0.3),(0.2,0.2,0.2),(0.8,0.7,0.6),(0.4,0.5,0.6),(0.6,0.6,0.8)\rangle |_{\bar{\delta}_1}$
$+\langle(0.2,0.2,0.2),(0.3,0.3,0.3),(0.6,0.5,0.4),(0.5,0.6,0.7),(0.8,0.7,0.6)\rangle |_{\bar{\delta}_2}$
$+\langle(0.3,0.4,0.5),(0.4,0.5,0.6),(0.5,0.4,0.3),(0.6,0.7,0.8),(0.7,0.8,0.9)\rangle |_{\bar{\delta}_3}$

Example 3. Assume a TFPNS $\bar{\Theta}_1$ of the form:

$\bar{\Theta}_1 = \langle(0.6,0.6,0.8),(0.4,0.5.0.6),(0.2,0.3,0.4),(0.2,0.2,0.2),(0.3,0.3,0.3)\rangle_{\bar{\delta}_1}$
$\langle(0.8,0.7,0.6),(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.3,0.3),(0.2,0.2,0.2)\rangle_{\bar{\delta}_2} +$
$\langle(0.7,0.8,0.9),(0.6,0.7,0.8),(0.5,0.6,0.7),(0.4,0.5,0.6),(0.3,0.4,0.5)\rangle_{\bar{\delta}_3}$

Accordingly, $(\bar{\Theta}_1)^{cmt} =$
$\langle(0.3,0.3,0.3),(0.2,0.2,0.2),(0.8,0.7,0.6),(0.4,0.5,0.6),(0.6,0.6,0.8)\rangle_{\bar{\delta}_1}$
$+\langle(0.2,0.2,0.2),(0.3,0.3,0.3),(0.6,0.5,0.4),(0.5,0.6,0.7),(0.8,0.7,0.6)\rangle_{\bar{\delta}_2}$
$+\langle(0.3,0.4,0.5),(0.4,0.5,0.6),(0.5,0.4,0.3),(0.6,0.7,0.8),(0.7,0.8,0.9)\rangle_{\bar{\delta}_3}.$

Definition 3.6. Containment

A TFPNS $\tilde{\Im}a_1$ can be defined to be contained in another TFPNS $\tilde{\Im}a_2$ and is denoted by $\tilde{\Im}a_1 \subseteq \tilde{\Im}a_2$

if, $T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma) \le T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma) \le T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma) \le T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma);$
$C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma) \le C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma) \le C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma) \le C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma);$
$I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma) \ge I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma) \ge I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma) \ge I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma);$
$U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma) \ge U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma) \ge U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma) \ge U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma);$
$F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma) \ge F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma) \ge F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma) \ge F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma); \text{for} \forall \sigma \in \ddot{\bar{\chi}}$

Theorem 3.1.

If $\tilde{\Im}a$ represents a TFPNS, then a) $\tilde{\Im}a \cup \tilde{\Im}a = \tilde{\Im}a$ b) $\tilde{\Im}a \cap \tilde{\Im}a = \tilde{\Im}a$

Proof.

$$\text{a) Now, } \tilde{\Im}a \cup \tilde{\Im}a = \{(\sigma, ((\max(T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma)), \max(C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$(\min(I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma)), \min(I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma)), \min(I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma)), \min(U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$= \{(\sigma, (T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma)), (F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$= \tilde{\Im}a$$

$$\text{b) } \tilde{\Im}a \cap \tilde{\Im}a$$
$$= \{(\sigma, ((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma)), \min(T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma)), \min(T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma)), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma)), \min(C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$(\max(I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma)), \max(I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma)), \max(I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$(\max(U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma)), \max(U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma)), \max(U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma))),$$
$$\langle \max(F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma)), \max(F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma)), \max(F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$= \{(\sigma, (T\tilde{T}^{(l)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a}(\sigma)),$$
$$(I\tilde{I}^{(l)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a}(\sigma)), (F\tilde{F}^{(l)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a}(\sigma))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$= \tilde{\Im}a$$

Theorem 3.2. For any two TFPNSs $\tilde{\Im}a_1 \,\&\, \tilde{\Im}a_2$, law of commutation holds:

Law of commutation

a) $\tilde{\Im}a_1 \cup \tilde{\Im}a_2 = \tilde{\Im}a_2 \cup \tilde{\Im}a_1$

b) $\tilde{\Im}a_1 \cap \tilde{\Im}a_2 = \tilde{\Im}a_2 \cap \tilde{\Im}a_1$

a) Proof. We have , $\tilde{\Im}a_1 \cup \tilde{\Im}a_2 =$

$$\{(\sigma, ((\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\}$$

$$=\{(\sigma,((\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma)))):\sigma\in\overline{\ddot{\chi}}\}$$

$$=\tilde{\Im}a_2\cup\tilde{\Im}a_1$$

b) $\tilde{\Im}a_1\cap\tilde{\Im}a_2=\{(\sigma,((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$

$$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\max(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma)),$$

$$(\max(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\max(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma)))):\sigma\in\overline{\ddot{\chi}}\}$$

$$=\{(\sigma,((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma)),\min(T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma)),\min(T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma))),$$

$$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma)),\min(C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma)),\min(C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma))),$$

$$(\max(I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma)),\max(I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma)),\max(I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma))),$$

$$(\max(U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma)),\max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma),U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma))),$$

$$(\max(F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma)),\max(F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma)),\max(F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))):\sigma\in\overline{\ddot{\chi}}\}$$

$$=\tilde{\Im}a_2\cap\tilde{\Im}a_1$$

Theorem 3.3. For any three TFPNSs, $\tilde{\Im}a_1,\tilde{\Im}a_2,\tilde{\Im}a_3$,

$$\tilde{\Im}a_1\cup(\tilde{\Im}a_2\cup\tilde{\Im}a_3)=(\tilde{\Im}a_1\cup\tilde{\Im}a_2)\cup\tilde{\Im}a_3$$

Proof. $\tilde{\Im}a_1\cup(\tilde{\Im}a_2\cup\tilde{\Im}a_3)=\{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma)),(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma))))\sigma:\in\overline{\ddot{\chi}}\}$$

$$\cup$$

$$\{(\sigma,(\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\max(T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\max(T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_3}(\sigma))),$$

$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_3}(\sigma))),$$

$$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_3}(\sigma),\min(U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_3}(\sigma))),$$

$$(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_3}(\sigma)),\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_3}(\sigma)))):\sigma\in\overline{\ddot{\chi}}\}$$

$$
\begin{aligned}
&=\{(\sigma,(\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma))),\\
&(\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma))),\\
&(\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma))),\\
&(\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma))),\\
&(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma)))):\sigma\in\ddot{\bar{\chi}}\}\cup\\
&\{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_3}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_3}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_3}(\sigma)),\\
&(C\tilde{C}^{(l)}_{\tilde{\Im}a_3}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_3}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_3}(\sigma)),(I\tilde{I}^{(l)}_{\tilde{\Im}a_3}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_3}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\\
&(U\tilde{U}^{(l)}_{\tilde{\Im}a_3}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_3}(\sigma),\Psi\ddot{\bar{\delta}}^{(u)}_{\Omega_3}(\sigma)),(F\tilde{F}^{(l)}_{\tilde{\Im}a_3}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_3}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_3}(\sigma)))):\sigma\in\ddot{\bar{\chi}}\}\\
&=(\tilde{\Im}a_1\cup\tilde{\Im}a_2)\cup\tilde{\Im}a_3
\end{aligned}
$$

Theorem 3.4. For any three TFPNS $\tilde{\Im}a_1,\tilde{\Im}a_2,\tilde{\Im}a_3$

$$\tilde{\Im}a_1\cup(\tilde{\Im}a_2\cap\tilde{\Im}a_3)=(\tilde{\Im}a_1\cup\tilde{\Im}a_2)\cap(\tilde{\Im}a_1\cup\tilde{\Im}a_3)$$

$$
\begin{aligned}
&\text{Proof}:\tilde{\Im}a_1\cup(\tilde{\Im}a_2\cap\tilde{\Im}a_3)=\{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),\\
&(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma))):\sigma\in\ddot{\bar{\chi}}\}\\
&\cup\{(\sigma,((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\min(T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_3}(\sigma),\min(T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_3}(\sigma)),\\
&(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\min(C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_3}(\sigma)),\min(C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_3}(\sigma))),\\
&(\max(I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\max(I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)),I\tilde{I}^{(m)}_{\tilde{\Im}a_3}(\sigma))),\max(I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma)),I\tilde{I}^{(u)}_{\tilde{\Im}a_3}(\sigma)))),\\
&(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_3}(\sigma)),\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_3}(\sigma)),\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_3}(\sigma)))):\sigma\in\ddot{\bar{\chi}}\}\\
&=\{(\sigma,((\min((\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(l)}_{\tilde{\Im}a_3}(\sigma)))),\min((\max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\\
&\max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(m)}_{\tilde{\Im}a_3}(\sigma)))).\min((\max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma)),\max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma),T\tilde{T}^{(u)}_{\tilde{\Im}a_3}(\sigma)))),\\
&(\min((\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(l)}_{\tilde{\Im}a_3}(\sigma)))),\min((\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\\
&\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(m)}_{\tilde{\Im}a_3}(\sigma)))),\min((\max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma)),\max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma),C\tilde{C}^{(u)}_{\tilde{\Im}a_3}(\sigma)))),\\
&(\max((\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(l)}_{\tilde{\Im}a_3}(\sigma)))),\max((\min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\\
&\min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(m)}_{\tilde{\Im}a_3}(\sigma)))),\max((\min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma)),\min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma),I\tilde{I}^{(u)}_{\tilde{\Im}a_3}(\sigma)))),\\
&(\max((\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(l)}_{\tilde{\Im}a_3}(\sigma)))),(\max((\min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\\
&\min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(m)}_{\tilde{\Im}a_3}(\sigma)))),\max((\min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma)),\min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma),U\tilde{U}^{(u)}_{\tilde{\Im}a_3}(\sigma)))),\\
&(\max((\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(l)}_{\tilde{\Im}a_3}(\sigma)))),\max((\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\\
&\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(m)}_{\tilde{\Im}a_3}(\sigma)))),\max((\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma)),\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma),F\tilde{F}^{(u)}_{\tilde{\Im}a_3}(\sigma))))):\sigma\in\ddot{\bar{\chi}}\}\\
&=(\tilde{\Im}a_1\cup\tilde{\Im}a_2)\cap(\tilde{\Im}a_1\cup\tilde{\Im}a_3)
\end{aligned}
$$

Theorem 3.5. For any two TFPNS
$\tilde{\Im}a_1, \tilde{\Im}a_2$, the following properties hold.

a) $\tilde{\Im}a_1 \cup (\tilde{\Im}a_1 \cap \tilde{\Im}a_2) = \tilde{\Im}a_1$,

b) $\Im a_1 \cap (\tilde{\Im}a_1 \cup \tilde{\Im}a_2) = \tilde{\Im}a_1$

a) Proof : $\Im a_1 \cup (\Im a_1 \cap \Im a_2) = \{(\sigma, ((\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), \min(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma))), \max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),$

$\min(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma))), \max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), \min(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$

$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), \min(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma))),$

$\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma))), \max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma), \min(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$

$(\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), \max(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma))), \min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), \max(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma))),$

$\min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), \max(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma)))), (\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), \max(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma))),$

$\min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), \max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma))), \min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), \max(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma)))),$

$(\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), \max(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma))), \min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), \max(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma))),$

$\min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), \max(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma))))) : \sigma \in \overline{\ddot{\chi}}\}$

$= \{(\sigma, ((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \overline{\ddot{\chi}}\}$

$= \tilde{\Im}a_1$

b) Proof : $\tilde{\Im}a_1 \cap (\tilde{\Im}a_1 \cup \tilde{\Im}a_2) = \{(\sigma, ((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), \max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma))),$

$\min(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), \max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma))), \min(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), \max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma)))),$

$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), \max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma))), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), \max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma))),$

$\min(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma), \max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_2}(\sigma)))), (\max(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), \min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma))),$

$\max(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), \min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma))), \max(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), \min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)))),$

$(\max(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), \min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma))), \max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), \min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)),$

$\max(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), \min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma)))), \langle \max(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), \min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma))),$

$\max(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), \min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma))), \max(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), \min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)))\rangle) : \sigma \in \overline{\ddot{\chi}}\}$

$= \{(\sigma, \langle (T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \overline{\ddot{\chi}}\}$

$= \tilde{\Im}a_1$

Theorem 3.6. For any TFPNS $\tilde{\Im}a_1$, $(\tilde{\Im}a_1{}^{Cmt})^{Cmt} = \tilde{\Im}a_1$

Proof : $\tilde{\Im}a_1 = \{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$

So, $(\tilde{\Im}a_1)^{cmt} = \{(\sigma,((F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(1-I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1-I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), 1-I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$

$\therefore (\tilde{\Im}a_1{}^{Cmt})^{Cmt} = \{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$

$= \tilde{\Im}a_1$

Theorem 3.7. For any TFPNS $\tilde{\Im}a_1$, represented as ,

$\tilde{\Im}a_1 = \{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$,

Following relations holds-

(a) $\tilde{\Im}a_1 \cap \hat{0}$

b) $\tilde{\Im}a_1 \cup \hat{1} = \hat{1}$

Proof (a) $\tilde{\Im}a_1 \cap \hat{0} = \{(\sigma,((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)),(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$

$\cap \{(\sigma,(0,0,0),(0,0,0),(1,1,1),(1,1,1),(1,1,1)) : \sigma \in \ddot{\overline{\chi}}\}$

$= \{(\sigma,((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma),0), \min(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma),0), \min(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma),0)),$

$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma),0), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma),0), \min(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma),0)),$

$(\max(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma),1), \max(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma),1), \max(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma),1)),$

$(\max(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma),1), \max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma),1), \max(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma),1)),$

$(\max(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma),1), \max(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma),1), \max(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma),1)))) : \sigma \in \ddot{\overline{\chi}}\}$

$= \{(\sigma,((0,0,0),(0.0.0),(1,1,1),(1,1,1),(1,1,1))) : \sigma \in \ddot{\overline{\chi}}\}$

$= \hat{0}$

$\Pr oof.(b)\ \tilde{\Im}a_1 \cup \hat{1} = \{(\sigma, ((T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma)),$

$(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\bar{\chi}}\}$

$\cup \{(\sigma, (1,1,1), (1,1,1), (0,0,0), (0,0,0), (0,0,0)) : \sigma \in \ddot{\bar{\chi}}\}$

$= \{(\sigma, ((\max(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1), \max(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), 1), \max(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), 1)), (\max(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1),$

$\max(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), 1), \max(C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma), 1)), (\min(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), 0), \min(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), 0), \min(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), 0)),$

$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), 0), \min(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), 0), \min(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), 0)), (\min(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), 0),$

$\min(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), 0), \min(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), 0))) : \sigma \in \ddot{\bar{\chi}}\}$

$= \{(\sigma, ((1,1,1), (1,1,1), (0,0,0), (0,0,0), (0,0,0))) : \sigma \in \ddot{\bar{\chi}}\}$

$= \hat{1}$

Theorem 3.8. For any two TFPNSs $\tilde{\Im}a_1$ & $\tilde{\Im}a_2$, $(\tilde{\Im}a_1 \cap \tilde{\Im}a_2)^{Cmt} = \tilde{\Im}a_1{}^{Cmt} \cup \tilde{\Im}a_2{}^{Cmt}$

$(\tilde{\Im}a_1 \cap \tilde{\Im}a_2) = \{(\sigma, ((\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$

$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma)), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma))),$

$(\max(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma)), \max(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$

$(\max(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$

$(\max(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma)))) : \sigma \in \ddot{\bar{\chi}}\}$

$so, (\tilde{\Im}a_1 \cap \tilde{\Im}a_2)^{Cmt} = \{(\sigma, (\max(F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_2}(\sigma)),$

$\max(F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_2}(\sigma))), (\max(U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}a_2}(\sigma)),$

$\max(U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_2}(\sigma))), (1 - \max(I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}a_2}(\sigma)),$

$1 - \max(I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}a_2}(\sigma)), 1 - \max(I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}a_2}(\sigma))), (\min(C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(C$

$(\min(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_2}(\sigma)),$

$\min(T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_2}(\sigma)))) : \sigma \in \ddot{\bar{\chi}}\} ..(1)$

Again, $\tilde{\Im}a_1{}^{Cmt} \cup \tilde{\Im}a_2{}^{Cmt} = \{(\sigma, ((F\tilde{F}^{(l)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}a_1}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}a_1}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(1 - I\tilde{I}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1 - I\tilde{I}^{(m)}_{\tilde{\Im}a_1}(\sigma), 1 - I\tilde{I}^{(u)}_{\tilde{\Im}a_1}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}a_1}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}a_1}(\sigma)),$

$(T\tilde{T}^{(l)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}a_1}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}a_1}(\sigma)))) : \sigma \in \ddot{\bar{\chi}}\}$

$\cup$

$$\{(\sigma,((\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_2}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_2}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_2}(\sigma)),(\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_2}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_2}(\sigma),$$
$$\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_2}(\sigma)),(1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)),$$
$$(\mathrm{C}\tilde{\mathrm{C}}^{(l)}_{\tilde{\Im}a_2}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(m)}_{\tilde{\Im}a_2}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(u)}_{\tilde{\Im}a_2}(\sigma)),(\mathrm{T}\tilde{\mathrm{T}}^{(l)}_{\tilde{\Im}a_2}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(m)}_{\tilde{\Im}a_2}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(u)}_{\tilde{\Im}a_2}(\sigma)))):\sigma\in\bar{\ddot{\chi}}\}$$
$$=\{(\sigma,(((\max(\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(\mathrm{C}\tilde{\mathrm{C}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{C}\tilde{\mathrm{C}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{C}\tilde{\mathrm{C}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(\mathrm{T}\tilde{\mathrm{T}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{T}\tilde{\mathrm{T}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{T}\tilde{\mathrm{T}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(u)}_{\tilde{\Im}a_2}(\sigma)))):\sigma\in\bar{\ddot{\chi}}\}..(2)$$

We consider the following cases**.**

Case (a). if, $\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma)\geq\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma)$ then, $1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma)\leq 1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma)$ Then, $\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma))$ $=1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma)=1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma))$...(3) if, $\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma)\geq\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)$ then, $1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma)\leq 1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)$

Then, $\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma))=1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma)=1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma))$...(4)

if, $\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma)\geq\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)$ then, $1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma)\leq 1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)$

So, $\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma))=1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)=1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma))$ **… (6)**

And, if, $\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma)\leq\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)$ we have, $1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma)\geq 1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)$

So, $\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma))=1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)=1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma))$...(7)

Also, $\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma)\leq\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)$, we have, $1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma)\geq 1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)$

So, $\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma))=1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma)=1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma))$ **… (8)**

So, we obtain using the above relations

(3),(4),(5),(6),(7),(8) $\tilde{\Im}a_1{}^{Cmt}\cup\tilde{\Im}a_2{}^{Cmt}=\{(\sigma,(\max(\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_1}(\sigma),$

$$\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),(\max(\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),$$
$$\max(\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_2}(\sigma))\max(\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_1}(\sigma))),(\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),$$
$$\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma),1-\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(\mathrm{C}\tilde{\mathrm{C}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{C}\tilde{\mathrm{C}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{C}\tilde{\mathrm{C}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{C}\tilde{\mathrm{C}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(\mathrm{T}\tilde{\mathrm{T}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{T}\tilde{\mathrm{T}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\min(\mathrm{T}\tilde{\mathrm{T}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{T}\tilde{\mathrm{T}}^{(u)}_{\tilde{\Im}a_2}(\sigma)))):\sigma\in\bar{\ddot{\chi}}\}$$
$$=\{(\sigma,(\max(\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{F}\tilde{\mathrm{F}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),\max(\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{U}\tilde{\mathrm{U}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$((1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(l)}_{\tilde{\Im}a_2}(\sigma)),(1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(m)}_{\tilde{\Im}a_2}(\sigma)),(1-\max(\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_1}(\sigma),\mathrm{I}\tilde{\mathrm{I}}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$

$$(\min(C\tilde{C}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(C\tilde{C}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(C\tilde{C}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\min(T\tilde{T}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(T\tilde{T}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(T\tilde{T}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$= (\tilde{\Im}a_1 \cap \tilde{\Im}a_1)^{Cmt}$$

Theorem 3.9. For any two TFPNS $\tilde{\Im}a_1 \,\&\, \tilde{\Im}a_2, (\tilde{\Im}a_1 \cup \tilde{\Im}a_2)^{Cmt} = \tilde{\Im}a_1^{Cmt} \cap \tilde{\Im}a_2^{Cmt}$

Proof.

$$(\tilde{\Im}a_1 \cup \tilde{\Im}a_2) = \{(\sigma, ((\max(T\tilde{T}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\min(I\tilde{I}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma)), \min(I\tilde{I}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(I\tilde{I}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))), (\min(U\tilde{U}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)),$$
$$\min(U\tilde{U}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\min(F\tilde{F}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\}$$

So,

$$(\tilde{\Im}a_1 \cup \tilde{\Im}a_2)^{Cmt} = \{(\sigma, (\min(F\tilde{F}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(U\tilde{U}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(1-\min(I\tilde{I}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), I\tilde{I}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), 1-\min(I\tilde{I}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), I\tilde{I}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), 1-\min(I\tilde{I}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), I\tilde{I}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(C\tilde{C}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\max(T\tilde{T}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\} \;..(1)$$

Again, $\tilde{\Im}a_1^{Cmt} \cap \tilde{\Im}a_2^{Cmt} = \{(\sigma, ((F\tilde{F}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma)),$

$$(1-I\tilde{I}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), 1-I\tilde{I}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), 1-I\tilde{I}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma)),$$
$$(T\tilde{T}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$\cap\{(\sigma, ((F\tilde{F}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)), (U\tilde{U}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)), (1-I\tilde{I}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma), 1-I\tilde{I}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma),$$
$$1-I\tilde{I}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)), (C\tilde{C}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)), (T\tilde{T}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\}$$
$$= \{(\sigma, (((\min(F\tilde{F}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(F\tilde{F}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(F\tilde{F}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), F\tilde{F}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\min(U\tilde{U}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(U\tilde{U}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \min(U\tilde{U}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), U\tilde{U}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\max(1-I\tilde{I}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), 1-I\tilde{I}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(1-I\tilde{I}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), 1-I\tilde{I}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(1-I\tilde{I}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), 1-I\tilde{I}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\max(C\tilde{C}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(C\tilde{C}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(C\tilde{C}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), C\tilde{C}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma))),$$
$$(\max(T\tilde{T}^{(l)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(l)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(T\tilde{T}^{(m)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(m)}_{\tilde{\Im}_{a_2}}(\sigma)), \max(T\tilde{T}^{(u)}_{\tilde{\Im}_{a_1}}(\sigma), T\tilde{T}^{(u)}_{\tilde{\Im}_{a_2}}(\sigma)))) : \sigma \in \bar{\ddot{\chi}}\} \;..(2)$$

We consider the following cases.

$$\text{Case (a). if}, \tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma) \geq \tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma), \text{then}, 1-\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma) \leq 1-\tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma) \text{ or}, \min(1-\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma))$$
$$= 1-\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma) = 1-\max(\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma))...(3) \;\&\; \max(1-\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma)) = 1-\tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma)$$
$$= 1-\min(\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma))...(4). \text{If}, \tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma) \geq \tilde{II}^{(m)}_{\tilde{\Im}a_2}(\sigma), \text{then}, 1-\tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma) \leq 1-\tilde{II}^{(m)}_{\tilde{\Im}a_2}(\sigma) \text{ or},$$
$$\min(1-\tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(m)}_{\tilde{\Im}a_2}(\sigma)) = 1-\tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma) = 1-\max(\tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(m)}_{\tilde{\Im}a_2}(\sigma))...(5)$$
$$\text{if}, \tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma) \geq \tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma) \text{ then}, 1-\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma) \leq 1-\tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma), \text{so} \max(1-\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma)) = 1-\tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma)$$
$$= 1-\min(\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma))...(6) \;\&\; \min(1-\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma)) = 1-\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma) = 1-\max(\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma))...(7)$$

Using equations $(3)-(7)$, we obtain

$$\tilde{\Im}a_1^{Cmt} \cap \tilde{\Im}a_2^{Cmt} = \{(\sigma, (((\min(\tilde{FF}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{FF}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{FF}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{FF}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{FF}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{FF}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(\tilde{UU}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{UU}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{UU}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{UU}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{UU}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{UU}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(1-\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(1-\tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(1-\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma), 1-\tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(\tilde{CC}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{CC}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{CC}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{CC}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{CC}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{CC}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(\tilde{TT}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{TT}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{TT}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{TT}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{TT}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{TT}^{(u)}_{\tilde{\Im}a_2}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$$
$$= \{(\sigma, (\min(\tilde{FF}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{FF}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{FF}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{FF}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{FF}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{FF}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\min(\tilde{UU}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{UU}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{UU}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{UU}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \min(\tilde{UU}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{UU}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(1-\min(\tilde{II}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(l)}_{\tilde{\Im}a_2}(\sigma)), 1-\min(\tilde{II}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(m)}_{\tilde{\Im}a_2}(\sigma)), 1-\min(\tilde{II}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{II}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(\tilde{CC}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{CC}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{CC}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{CC}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{CC}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{CC}^{(u)}_{\tilde{\Im}a_2}(\sigma))),$$
$$(\max(\tilde{TT}^{(l)}_{\tilde{\Im}a_1}(\sigma), \tilde{TT}^{(l)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{TT}^{(m)}_{\tilde{\Im}a_1}(\sigma), \tilde{TT}^{(m)}_{\tilde{\Im}a_2}(\sigma)), \max(\tilde{TT}^{(u)}_{\tilde{\Im}a_1}(\sigma), \tilde{TT}^{(u)}_{\tilde{\Im}a_2}(\sigma)))) : \sigma \in \ddot{\overline{\chi}}\}$$
$$= (\tilde{\Im}a_1 \cup \tilde{\Im}a_2)^{Cmt}$$

Hence the theorem is proved.

4. DISCUSSION

In this paper, the notion of TFPNS is introduced by combining the TFN and the PNS to utilize the advantages of TFN and PNS. The significance of introducing the hybrid set structure TFPNS is that the computational techniques based on TFN or PNS alone may not always produce the best results. But a fusion of them may produce better results. We have presented a real-world example of which is elegant to express uncertainty by utilizing triangular fuzzy numbers which was not possible using PNS alone.

5. CONCLUSIONS

In this paper, we have developed a new notional concept of TFPNS and proved its important properties like union, intersection, complement, etc. We hope that this treatment will show a future scope of development of logical systems in information sciences. We further hope that

TFPNSs will be helpful in decision-making, information retrieval systems, etc. In the future, aggregation operators and other set-theoretic operations and their important properties will be explored. TFPNS is more advantageous than PNS.

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Through Interval-Valued Neutrosophic Topological Space, Interval-Valued Neutrosophic b-Open Set

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ABSTRACT

In this chapter, we delve into the fascinating realm of interval-valued neutrosophic sets by introducing two novel concepts: interval-valued neutrosophic b-open sets and interval-valued neutrosophic b-closed sets. These sets bring forth intriguing properties which we thoroughly explore. Additionally, we delve into the concept of interval-valued neutrosophic b-interior and interval-valued neutrosophic b-closure operators, shedding light on their characteristics and delving into their relationships with other operators in this domain.

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KEYWORDS: **Interval-valued neutrosophic b-open, neutrosophic b-closure operator, neutrosophic b-closure operator, neutrosophic topology**.

1. INTRODUCTION

Interval-Valued Neutrosophic (IVN) b-open sets are a concept in mathematical set theory that combines the notions of IVN Sets (IVNSs) (Wang et al., 2005) and b-open sets (Ebenanjar et al., 2018). To understand this term, let's break it down:

1. IVNS: An IVNS is a mathematical representation that extends classical sets to accommodate uncertain or indeterminate information. It introduces three components for each element: truth, indeterminacy, and falsity membership, each represents an interval. This allows for a more nuanced description of uncertainty and vagueness in sets.

2. b-Open Sets: In topology, a set is called "b-open" (Andrijevic, 996) if it satisfies the conditions of both openness and closed-ness. This concept generalizes the notion of Open Sets (OSs)in topology.

Combining these two concepts, "IVN b-OSs" likely refer to sets that have properties of both IVNSs and b-OSs. These sets would describe elements with uncertain and imprecise truth values using interval membership values, while also exhibiting properties of openness and closed-ness in the context of a specific Topological Space (TS).

An IVN Topological Space (IVNTS) is an extension of the traditional concept of a topological space that incorporates interval-valued neutrosophic sets to account for uncertainty and indeterminacy. This concept is rooted in both topology and NS theory and aims to provide a framework for handling complex and uncertain information within the context of open and closed sets.

In mathematics, a TS is a fundamental concept in topology. It consists of a set of points along with a collection of open sets, which are subsets of the space satisfying specific properties (such as being closed under finite intersections and arbitrary unions). These open sets define the concepts of continuity, neighborhood, convergence, and various other important concepts in topology.

By combining the concepts of TS and IVNS, an IVNTS is developed which generalizes the notion of a TS by incorporating IVNSs as elements. In this context, open sets and closed sets are defined using these IVNSs.

Uncertainties are a major part of business, engineering, finance, medical, and social science challenges. Traditional mathematical models have trouble resolving the uncertainties in these data. Fuzzy Sets (FSs) (Zadeh, 1965), extensions of FSs that are intuitionistic FSs (Atanassov, 1986), rough sets (Pawlak, 1982) are some sets that can be used as mathematical tools to get around problems involving unclear data. However, due to the limitations of parametrization tools, all of these approaches have an underlying problem when trying to solve issues with uncertainty. Neutrosophic Sets (NSs) were examined by Smarandache (1998, 2005) as a strategy for resolving problems involving unreliable, indeterminate, and inconsisitent data.

Wang et al. (2010) introduced a novel approach called Single Valued NS (SVNS). IVNSs (Wang et al., 2005) are an extension of interval valued FS (Turksen, 1986) and NSs. SVNS was further extended to Quadripartitioned NS (QNS) (Chatterjee et al., 2016) and Pentapartitioned NS (Mallick, & Pramanik, 2020). Using INS (Wang et al., 2005) and QNS (Chatterjee et al., 2016), Interval QNS (IQNS) was proposed by Pramanik (2022b. Using IVNS (Wang et al., 2005) and PNS (Mallick, & Pramanik, 2020), Interval QNS (IPNS) was developed by Pramanik (2023). Theories of NSs and their applications are depicted in the studies (Broumi et al., 2018; Otay, & Kahraman, 2019; Pramanik et al., 2018; Peng & Dai, 2020; Pramanik, 2020, 2022a; Smarandache, & Pramanik, 2016, 2028; Delcea et al, 2023), Salama and Alblowi (2012a) introduced the Neutrosophic TS (NTS) in 2012. NTSs were further examined by the studies (Salama, & Alblowi, 2012b; Salama et al., 2014; Das & Tripathy, 2020).

Iswarya and Bageerathi (2016) explored the Neutrosophic SO (NSO)) set and

Neutrosophic Semi-Closed (NSC) set. Imran et al. (2017) grounded the NSO in 2017 and investigated their basic characteristics. The NSO function was defined by Arokiarani et al. (2017). Neutrosophic pre-OSs were first introduced by Rao and Rao (2017).
Andrijevic (1996) presented b-open sets. Dutta, & Tripathy (2017) presented the fuzzy b-OS. Das, & Pramanik (2020) grounded the generalized neutrosophic b-OSs in NTS. Das, & Tripathy (2020) presented the pairwise neutrosophic-b-OS in neutrosophic bi-TSs.

The concept of IVN b-OS via IVNTS combines the ideas of interval-valued membership degrees and NSs within the framework of TSs.

Research Gap: There hasn't been any new research on interval-valued neutrosophic b-open set and interval-valued neutrosophic b-continuous mapping and their properties via IVNTS.

Motivation: We introduce the concept of IVN b-open sets and IVN b-continuous mappings, along with their respective properties, to address the existing research gap.

The following parts have been created from the remaining text of this article:
We reviewed some pertinent definitions and findings on IVNS and IVNTS in the next section. Section 3 introduces the idea of IVN b-OS and IVN b-continuous mapping, and proves their properties. Section 4 wraps up the paper by outlining avenues for future research.

2. PRELIMINARIES

We review some previous definitions and findings about IVNS and IVNTS, which are highly beneficial for the presentation of the article's primary findings. For the definition of union, intersection, and complement we have used the article (Wang et al., 2005).

Definition 2.1. Consider $X\neq\varnothing$ be a set of objects. An IVNS (Wang et al., 2005)) D in X is characterized by truth-T_D, indeterminacy- I_D and falsity–F_D membership functions. For each point$x\in X$, $T_D(x)$, $I_D(x)$, $F_D(x)\subseteq[0,1]$.

Example 2.1. Let X={x_1, x_2} be a fixed set. Then, D={(x_1, [0.2, 0.4], [0.4, 0.6], [0.2, 0.3]), (x_2, [0.4, 0.6], [0.3, 0.5], [0.2, 0.4])} is an IVNS over X.

Definition2.2 An IVNS (Wang et al., 2005) Θ is called as

(i) null IVN set denoted by 0_{IVN} if for each point$x\in X$, inf T_Θ (x) = sup T_Θ (x)= 0, inf I_Θ (x) = sup I_Θ (x) = 1, and inf F_Θ (x) = sup F_Θ = 0.

(ii) absolute IVNS denoted by (1_{IVN}) if for each point $x\in X$,
inf T_Θ (x) = sup T_Θ (x)= 1, inf I_Θ (x) = sup I_Θ (x) = 0, and inf F_Θ (x) = sup F_Θ = 1.

Remark 2.2. Suppose that A and B are two IVNSs over X. Then, their union $A\cup B$ is the smallest

IVNS containing both A and B.

Definition 2.3. Let X be a non-empty set, and τ_{IVN} be a family of IVNSs defined over X. Then, (X,τ_{IVN}) is called an IVN Topology (IVNT) if it satisfies the following axioms:

(i) 0_{IVN}, $1_{IVN}\in\tau_{IVN}$,

(ii) $A,B\in \tau_{IVN} \Rightarrow A\cap B\in \tau_{IVN}$,

(iii) $A_i\in \tau_{IVN}$,$\in I\Rightarrow \cup_{i\in I}A_i\in \tau_{IVN}$.

In that case, the pair (X,τ_{IVN}) is referred to as an IVNTS. All the members of (X,τ_{IVN}) are said to be an IVN OS (IVNOS], and their complement is said to be an IVN Closed Set (CS) (IVNCS).

Definition 2.4. (Salama & Alblowi, 2012) Let (X,τ_{IVN}) be an IVNTS, and U be an IVNS over X. Then, the IVN closure and IVN interior of U are defined as follows:

$IVN_{cl}(U)=\cap\{D:D$ is an IVNCS in X and $U\subseteq D\}$

$IVN_{int}(U)=\cup\{E:E$ is an IVNOS in X and $E\subseteq U\}$.

3. *B*-OPEN SET IN IVNS

The ideas of IVN b-OS and IVN b-CS are here introduced. Their properties are characterized.

Definition 3.1. Let (X,τ_{IVN}) be an IVNTS and U is an IVNS. Then, U is called as

(i) IVN α-OS (IVN-α-OS) if $U \subseteq IVN_{int}(IVN_{cl}(IVN_{int}(U)))$;

(ii) IVN Semi-OS (IVNSOS) if $U \subseteq IVN_{cl}(IVN_{int}(U))$;

(iii) IVN Pre-OS (IVNPOS) if $U \subseteq IVN_{int}(IVN_{cl}(U))$.

Remark 3.1. The complement of IVN-α-OS, IVNSOS and IVNPOS in an IVNTS (X,τ_{IVN}) are called IVN α-CS (IVN-α-CS), IVN Semi-CS (IVNSCS) and IVN Pre-CS(IVNPCS) respectively.

Theorem 3.1. Let (X,τ_{IVN}) be an IVNTS. Then,

(i) each IVNOS is an IVNSOS,

(ii) each IVNOS is an IVNPOS.

Proof. (i) Let (X,τ_{IVN}) be an IVNTS. Let A be an IVNOS. Therefore, $A=IVN_{int}(A)$. It is known that $A\subseteq IVN_{cl}(A)$. This implies, $A \subseteq IVN_{cl}(IVN_{int}(A))$. Therefore, A is an IVNSOS in (X, τ_{IVN}).

(ii) Let (X,τ_{IVN}) be an IVNTS. Let A be an IVNOS. Therefore, $A=IVN_{int}(A)$. It is known that $A \subseteq IVN_{cl}(A)$. This implies, $IVN_{int}(A) \subseteq IVN_{int}(IVN_{cl}(A))$ i.e. $A = IVN_{int}(A) \subseteq IVN_{int}(IVN_{cl}(A))$. Therefore, $A \subseteq IVN_{int}(IVN_{cl}(A))$. Hence, A is a IVNPOS in (X,τ_{IVN}) .

Theorem 3.2. In an IVNTS (X,τ_{IVN}), the union of any two IVNSOSs is an IVNSOS.

Proof. Let P and Q be two IVNSOSs in an IVNTS (X,τ_{IVN}). Therefore,

$$P \subseteq IVN_{cl}(IVN_{int}(P)) \tag{1}$$

and $Q \subseteq IVN_{cl}(IVN_{int}(Q))$ (2)

Using the relations (1) and (2), we obtain

$P \cup Q \subseteq IVN_{cl}(IVN_{int}(P)) \cup IVN_{cl}(IVN_{int}(Q))$
$=IVN_{cl}(IVN_{int}(P) \cup IVN_{int}(Q))$
$\subseteq IVN_{cl}(IVN_{int}(P \cup Q))$.
Therefore, $P \cup Q \subseteq IVN_{cl}(IVN_{int}(P \cup Q))$. Hence, $P \cup Q$ is an IVNSOS in (X, τ_{IVN}).

Theorem 3.3. In an IVNTS (X, τ_{IVN}), the union of any two IVNPOSs is also an IVNPOS.
Proof. Let P and Q be any two IVNPOSs in an IVNTS (X, τ_{IVN}).
Therefore,
$P \subseteq IVN_{int}(IVN_{cl}(P))$ (3)
and $Q \subseteq IVN_{int}(IVN_{cl}(Q))$ (4)
Using the relations (3) and (4), we obtain
$P \cup Q \subseteq IVN_{int}(IVN_{cl}(P)) \cup IVN_{int}(IVN_{cl}(Q))$
$\subseteq IVN_{int}(IVN_{cl}(P) \cup IVN_{cl}(Q))$
$= IVN_{int}(IVN_{cl}(P \cup Q))$.
Therefore, $P \cup Q \subseteq IVN_{int}(IVN_{cl}(P \cup Q))$. Hence, $P \cup Q$ is an IVNPOS in (X, τ_{IVN}).

Lemma 3.1. In an IVNTS (X, τ_{IVN}), every IVNOS is an IVN-α-OS.

Theorem3.4 In an IVNTS (X, τ_{IVN}),
(*i*) Every IVN-α-OS is an IVNSOS.
(*ii*) Every IVN-α-OS is an IVNSOS.
Proof. (*i*) Let Q be an IVN-α-OS in (X, τ_{IVN}). Therefore, $Q \subseteq IVN_{int}(IVN_{cl}(IVN_{int}(Q)))$. It is known that $IVN_{int}(IVN_{cl}(IVN_{int}(Q))) \subseteq IVN_{cl}(IVN_{int}(Q))$. Thus, we have, $Q \subseteq IVN_{cl}(IVN_{int}(Q))$. Hence, Q is an IVNSOS. Therefore, every IVN-α-OS is an IVNSOS.
(*ii*) Let (X, τ_{IVN}) be an IVNTS. Let Q be an IVN-α-OS in (X, τ_{IVN}). Therefore, $Q \subseteq IVN_{int}(IVN_{cl}(IVN_{int}(Q)))$. It is known that $IVN_{int}(Q) \subseteq Q$. This implies, $IVN_{cl}(IVN_{int}(Q)) \subseteq IVN_{cl}(Q)$. Which implies $IVN_{int}(IVN_{cl}(IVN_{int}(Q))) \subseteq IVN_{int}(IVN_{cl}(Q)$. Therefore, $Q \subseteq IVN_{int}(IVN_{cl}(Q)$. Hence, $Q$is an IVNPOS. Therefore, every IVN-α-OS is an IVNPOS in (X, τ_{IVN}).
Definition 3.2. An IVN set U in an IVNTS(X, τ_{IVN}) is referred to as an IVN b-open set [in short IVN-b-OS] if $U \subseteq IVN_{int}(IVN_{cl}(U)) \cup IVN_{cl}(IVN_{int}(U))$. If U is an IVN-b-OS, then U^C is said to be an IVN b-closed set [in short IVN-b-CS].

Remark 3.2. An IVNSU is called IVN-b-CS iff $U \supseteq IVN_{int}(IVN_{cl}(U)) \cap IVN_{cl}(IVN_{int}(U))$.

Theorem3.5 In an IVNTS(X, τ_{IVN}),
(*i*) Every IVNPOS is an IVN-b-OS.
(*ii*) Every IVNSOS is an IVN-b-OS.
Proof. Let Q be an IVNPOS in an IVNTS (X, τ_{IVN}). Therefore, $Q \subseteq IVN_{int}(IVN_{cl}(Q))$.

This implies, $Q \subseteq IVN_{int}(IVN_{cl}(Q)) \cup IVN_{cl}(IVN_{int}(Q))$. Hence, Q is an IVN-b-OS.
Therefore, every IVNPOS is an IVN-b-OS.
Similarly, it can be proved that every IVNSOS is an IVN-b-OS.

Theorem 3.6. The union of any two IVN-b-Os in an IVNTS (X, τ_{IVN}) is also an IVN-b-OS.
Proof. Let P and Q be two IVN-b-OSs in an IVNTS (X, τ_{IVN}).
Therefore, $P \subseteq IVN_{int}(IVN_{cl}(P)) \cup IVN_{cl}(IVN_{int}(P))$ (5)
and $Y \subseteq IVN_{int}(IVN_{cl}(Q)) \cup IVN_{cl}(IVN_{int}(Q))$ (6)
It is known that, $P \subseteq P \cup Q$ and $Y \subseteq P \cup Q$.
Now, $P \subseteq P \cup Q$
$\Rightarrow IVN_{int}(P) \subseteq IVN_{int}(P \cup Q)$
$\Rightarrow IVN_{cl}(IVN_{int}(P)) \subseteq IVN_{cl}(IVN_{int}(P \cup Q))$ (7)
and $P \subseteq P \cup Q$
$\Rightarrow IVN_{cl}(P) \subseteq IVN_{cl}(P \cup Q)$
$\Rightarrow IVN_{int}(IVN_{cl}(P)) \subseteq IVN_{int}(IVN_{cl}(P \cup Q))$ (8)
Similarly, it can be shown that
$IVN_{cl}(IVN_{int}(Q)) \subseteq IVN_{cl}(IVN_{int}(P \cup Q))$ (9)
$IVN_{int}(IVN_{cl}(Q)) \subseteq IVN_{int}(IVN_{cl}(P \cup Q))$ (10)
Using, eq. (5) and eq. (6), we obtain,
$P \cup Q \subseteq IVN_{cl}(IVN_{int}(P)) \cup IVN_{int}(IVN_{cl}(P)) \cup IVN_{cl}(IVN_{int}(Q)) \cup IVN_{int}(IVN_{cl}(Q))$
$\subseteq IVN_{cl}(IVN_{int}(P \cup Q)) \cup IVN_{int}(IVN_{cl}(P \cup Q)) \cup IVN_{cl}(IVN_{int}(P \cup Q)) \cup IVN_{int}(IVN_{cl}(P \cup Q))$
[by eqs (7), (8), (9), & (10)]
$= IVN_{cl}(IVN_{int}(P \cup Q)) \cup IVN_{int}(IVN_{cl}(P \cup Q))$
$\Rightarrow P \cup Q \subseteq IVN_{cl}(IVN_{int}(P \cup Q)) \cup IVN_{int}(IVN_{cl}(P \cup Q))$.
Therefore, $P \cup Q$ is an IVN-b-OS.
Hence, the union of two IVN-b-OSs is an IVN-b-OS.
Theorem 3.7. In an IVNTS (X, τ_{IVN}), the intersection of two IVN-b-CSs is also an IVN-b-CS.
Proof. Let (X, τ_{IVN}) be an IVNTS. Let P and Q be two IVN-b-CSs in (X, τ_{IVN}). Therefore,
$IVN_{int}(IVN_{cl}(P)) \cap IVN_{cl}(IVN_{int}(P)) \subseteq P$ (11)
and $IVN_{int}(IVN_{cl}(Q)) \cap IVN_{cl}(IVN_{int}(Q)) \subseteq Q$ (12)
Since, $P \cap Q \subseteq P$ and $P \cap Q \subseteq Q$, so we get
$IVN_{int}(P \cap Q) \subseteq IVN_{int}(P) \Rightarrow IVN_{cl}(IVN_{int}(P \cap Q)) \subseteq IVN_{cl}(IVN_{int}(P))$; (13)
$IVN_{cl}(P \cap Q) \subseteq IVN_{cl}(P) \Rightarrow IVN_{int}(IVN_{cl}(P \cap Q)) \subseteq IVN_{int}(IVN_{cl}(P))$ (14)
$IVN_{int}(P \cap Q) \subseteq IVN_{int}(Q) \Rightarrow IVN_{cl}(IVN_{int}(P \cap Q)) \subseteq IVN_{cl}(IVN_{int}(Q))$ (15)
and $IVN_{cl}(P \cap Q) \subseteq IVN_{cl}(Q) \Rightarrow IVN_{int}(IVN_{cl}(P \cap Q)) \subseteq IVN_{int}(IVN_{cl}(Q))$ (16)
From eq. (11) and eq. (12) we get,
$P \cap Q \supseteq IVN_{int}(IVN_{cl}(P)) \cap IVN_{cl}(IVN_{int}(P)) \cap IVN_{int}(IVN_{cl}(Q)) \cap IVN_{cl}(IVN_{int}(Q))$
$\supseteq IVN_{int}(IVN_{cl}(P \cap Q)) \cap IVN_{cl}(IVN_{int}(P \cap Q)) \cap IVN_{int}(IVN_{cl}(P \cap Q)) \cap IVN_{cl}(IVN_{int}(P \cap Q))$

[by eqs (13), (14), (15) & (16)]

$= IVN_{int}(IVN_{cl}(P \cap Q)) \cap IVN_{cl}(IVN_{int}(P \cap Q))$

$\Rightarrow P \cap Q \supseteq IVN_{cl}(IVN_{int}(P \cap Q)) \cap IVN_{int}(IVN_{cl}(P \cap Q))$.

Hence, $P \cap Q$ is an IVN-b-CS in (X, τ_{IVN}).

Therefore, the intersection of two IVN-b-CSs is again an IVN-b-CS.

Definition3.3. Let (X, τ_{IVN}) be an IVNTS. Let U be an IVNS over X. Then, the

(i) IVN b-interior of U[in short $IVN_{b\text{-}int}(U)$] is the union of all IVN-b-OSs of X contained in U, i.e.,$IVN_{b\text{-}int}(U)=\cup\{G:G$ is an IVN-b-OS in X and $G \subseteq U\}$.

(ii) IVN b-closure *of* U[in short $IVN_{b\text{-}cl}(U)$] is the intersection of all IVN-b-CSs of X contained in U,i.e.,$IVN_{b\text{-}cl}(U) = \cap\{H:H$ is an IVN-b-OS in X and $K \supseteq U\}$.

Remark 3.3. From the above definition, it is clear that $IVN_{b\text{-}cl}(U)$ is the smallest IVN-b-CS over X which contains U, and $IVN_{b\text{-}int}(U)$ is the largest IVN-b-OS over X which is contained in U.

Theorem 3.8 Assume that U is an IVNS in an IVNTS(X, τ_{IVN}). Then,

(i) $(IVN_{b\text{-}int}(U))^C = IVN_{b\text{-}cl}(U^C)$;

(ii) $(IVN_{b\text{-}cl}(U))^C = IVN_{b\text{-}int}(U^C)$.

Proof:(i) Assume that U be an IVNS in an IVNTS(X, τ_{IVN}).Now, $IVN_{b\text{-}int}(U) = \cup\{D:D$ is an IVN-b-OS in$(X, \tau_{IVN})$and $D \subseteq U\}$.

Then, $(IVN_{b\text{-}int}(U))^C=\{\cup \{D:D$ is an IVN-b-OS in (X, τ_{IVN}) and $D \subseteq U\}]^C=\cap\{D^C:D^C$ is an IVN-b-CS in (X, τ_{IVN}) and $U^C \subseteq D^C\}$. Replacing D^C by M, we obtain $(IVN_{b\text{-}int}(U))^C=\cap\{M:M$ is an IVN-b-CS in (X, τ_{IVN}) and $M \supseteq U^C\}$. Therefore, $(IVN_{b\text{-}int}(U))^C= IVN_{b\text{-}cl}((U)^C)$.

Analogously, we can prove(ii).

Definition 3.4. Let (X, τ_{IVN}) and (Y, δ_{IVN}) be any two IVNTSs. Then, a bijective mapping $(X, \tau_{IVN}))\rightarrow(Y, \delta_{IVN}))$ is referred to as

(i) IVN Continuous (IVN-C) mapping iff $\xi^{-1}(L)$ is an IVNOS in X, whenever L is an IVNOS in Y;

(ii) IVN Semi-Continuous (IVNS-C) mapping iff $\xi^{-1}(L)$ is an IVNSOS in X, whenever L is an IVNOS in Y;

(iii) IVN Pre-Continuous (IVNP-C) iff $\xi^{-1}(L)$ is an IVNPOS in X, whenever L is an IVNOS in Y;

(iv) IVN b-Continuous (IVN-b-C) mapping iff $\xi^{-1}(L)$ is an IVN-b-OS in X, whenever L is an IVNOS in Y.

Theorem 3.9. Let (X, τ_{IVN}) and (Y, δ_{IVN}) any two IVNTSs. Then, every IVN-C mapping from (X, τ_{IVN}) to (Y, δ_{IVN}) is an IVNP-C mapping (resp. IVNS-C mapping).

Proof. Let $\xi: (X, \tau_{IVN}) \rightarrow (Y, \delta_{IVN})$ be an IVN-C mapping. Let L be an IVNOS in (Y, δ_{IVN}). Therefore, $\xi^{-1}(L)$ is an IVNOS in (X, τ_{IVN}). It is known that every IVNOS is an IVNPOS (resp. IVNSOS). Therefore, $\xi^{-1}(L)$ is an IVNPOS (resp. IVNSOS) in (X, τ_{IVN}). Hence, $\xi: (X, \tau_{IVN}) \rightarrow (Y, \delta_{IVN})$ is an

IVNP-C mapping (resp. IVNS-C mapping).

Theorem 3.8. Let (X,τ_{IVN}) and (Y,δ_{IVN}) be any two IVNTSs. Then, every IVNS-C mapping (resp. IVNP-C mapping) from (X,τ_{IVN}) to (Y,δ_{IVN}) is an IVN-*b*-C mapping (resp. IVNS-C mapping).

Proof. Let $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ be an IVNS-C mapping (resp. IVNP-C mapping). Let L be an IVNOS in (Y,δ_{IVN}). Therefore, $\xi^{-1}(L)$ is an IVNSOS (resp. IVNPOS) in (X, τ_{IVN}). It is known that every IVNSOS (resp. IVNPOS) is an IVN-*b*-OS. Therefore, $\xi^{-1}(L)$ is an IVN-*b*-OS in (X,τ_{IVN}). Hence, $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ is an IVN-*b*-C mapping.

Theorem 3.9. Assume that Let (X,τ_{IVN}) and (Y,δ_{IVN}) are any two IVNTSs. Then, every IVN-C mapping from (X, τ_{IVN}) to (Y, δ_{IVN}) is an IVN-*b*-C mapping.

Proof. Let $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ be an IVN-C mapping. Let L be an IVNOS in (Y,δ_{IVN}).

Therefore, $\xi^{-1}(L)$ is an IVNOS in (X,τ_{IVN}). It is known that, every IVNOS is also an IVN-*b*-OS. Therefore, $\xi^{-1}(L)$ is an IVN-*b*-OS in (X,τ_{IVN}). Hence, $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ is an IVN-*b*-C mapping.

Theorem 3.10. If $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ and $\chi:(Y,\delta_{IVN})\rightarrow(Z,\theta_{IVN})$ be any two IVN-C mappings, then the composition mapping $\chi\circ\xi:(X,\tau_{IVN})\rightarrow(Z,\theta_{IVN})$ is also an IVN-C mapping.

Proof.

Let $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ and $\chi:(Y,\delta_{IVN})\rightarrow(Z,\theta_{IVN})$ be two IVN-C-mappings.

Let L be an IVNOS in (Z,θ_{IVN}). Since, $\chi:(Y,\delta_{IVN})\rightarrow(Z,\theta_{IVN})$ is an IVN-C mapping, so $\chi^{-1}(L)$ is an IVNOS in Y. Since, $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ is an IVN-C mapping, so $\xi^{-1}(\chi^{-1}(L))= (\chi\circ\xi)^{-1}(L)$ is an IVNOS in X. Therefore, $(\chi\circ\xi)^{-1}(L)$ is an IVNOS in X, whenever L is an IVNOS in Z. Hence, $\chi\circ\xi:(X,\tau_{IVN})\rightarrow(Z,\theta_{IVN})$ is also an IVN-C mapping.

Theorem 3.11. If $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ is an IVN-*b*-C mapping and $\chi:(Y,\delta_{IVN})\rightarrow(Z,\theta_{IVN})$ ise an IVN-C mapping, then the composition mapping $\chi\circ\xi:(X,\tau_{IVN})\rightarrow(Z, \theta_{IVN})$ is an IVN-*b*-C mapping.

Proof. Let $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ be an IVN-*b*-C mapping and $\chi:(Y,\delta_{IVN})\rightarrow(Z,\theta_{IVN})$ be an IVN-C mapping. Let L be an IVNOS in (Z,θ_{IVN}). Since, $\chi:(Y,\delta_{IVN})\rightarrow(Z,\theta_{IVN})$ is an IVN-C mapping, so $\chi^{-1}(L)$ is an IVNOS in Y. Since, $\xi:(X,\tau_{IVN})\rightarrow(Y,\delta_{IVN})$ is an IVN-*b*-C mapping, so $\xi^{-1}(\chi^{-1}(L))=(\chi\circ\xi)^{-1}(L)$ is an IVN-*b*-OS in X. Therefore, $(\chi\circ\xi)^{-1}(L)$ is an IVN-*b*-OS in X, whenever L is an IVNOS in Z. Hence, $\chi\circ\xi:(X,\tau_{IVN})\rightarrow(Z,\theta_{IVN})$ is an IVN-*b*-C mapping.

4. CONCLUSIONS

In this paper, the idea of IVN-b-OS and IVN-b-C-mapping has been introduced. Additionally, we looked into the concepts of an IVNS's IVN b-closure and b-interior. Additionally, using IVNTS, we produced several intriguing conclusions on them in the form of theorems, propositions, lemma, etc.

The idea of different open sets, such as IVNSOS, IVNPOS, IVN-b-OS, etc., in IVNTS is hoped to be applied in the future to the direction of IVN supra-topological space, IVN bitopological space, IVN tri-topological space, etc.

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Neutrosophic gsα* - Homeomorphism in Neutrosophic Topological Spaces

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ABSTRACT

The main aim of this paper is to introduce a new concept in $N_{eu}-$ homeomorphism namely $N_{eu}gs\alpha^{*}-$ homeomorphism and $N_{eu}igs\alpha^{*}-$ homeomorphism in $N_{eu}-$ Topological Spaces. In additionally, we discussed the characterizations and properties of these functions with already existing $N_{eu}-$ functions.

KEYWORDS: $\mathbf{N_{eu}gs\alpha^{*}-}$ **closed set,** $\mathbf{N_{eu}gs\alpha^{*}-}$ **open set,** $\mathbf{N_{eu}gs\alpha^{*}-}$ **continuous,** $\mathbf{N_{eu}gs\alpha^{*}-}$ **irresolute,** $\mathbf{N_{eu}gs\alpha^{*}-}$ **homeomorphism and** $\mathbf{N_{eu}igs\alpha^{*}-}$ **homeomorphism.**

1. INTRODUCTION

As a generalization of Fuzzy Sets (FSs) introduced by Zadeh (1965)] and intuitionistic FSs introduced by Atanassov (1986), the Neutrosophic set (shortly, $N_{eu}-$ set) theory was introduced by Smarandache (1998, 2010). Overview of $N_{eu}-$ sets and their developments. extensions and applications are depicted in the studies (Broumi et al., 2018; Otay, & Kahraman, 2019; Pramanik et al., 2018; Peng & Dai, 2020; Pramanik, 2020, 2022; Smarandache, & Pramanik, 2016, 2028; Delcea et al, 2023). It consists of three components namely truth, indeterminacy, and false membership function. Dhavaseelan and Jafari (2018) introduced the idea of $N_{eu}g-CS$ and its continuity. Page and Imran (2020) introduced the concept of $N_{eu}g-h_{om}$.

The real-life application of $N_{eu}-$ topology spans various fields, including information systems, applied mathematics, neutrosophic logic, decision-making systems, and more. These applications often involve dealing with uncertain, incomplete, or inconsistent information, where traditional mathematical tools may fall short. We introduce some new concepts in $N_{eu}-$ topological spaces such as $N_{eu}gs\alpha^{*}-h_{om}$ and $N_{eu}igs\alpha^{*}-h_{om}$.

2. PRELIMINARIES

Definition 2.1 (Sreeja & Sarankumar**,** 2018): Let $\mathbb{P}$ be a non-empty fixed set. A $N_{eu}-$ set Ƕ on the universe $\mathbb{P}$ is defined as Ƕ$=\left\{\left\langle \wp,\left(t_{Ƕ}(\wp), i_{Ƕ}(\wp), f_{Ƕ}(\wp)\right)\right\rangle : \wp \in \mathbb{P}\right\}$ where $t_{Ƕ}(\wp), i_{Ƕ}(\wp), f_{Ƕ}(\wp)$ represent the degree of membership, indeterminacy, non-membership function $t_{Ƕ}(\wp)$, $i_{Ƕ}(\wp)$ and $f_{Ƕ}(\wp)$ respectively for each element $\wp \in \mathbb{P}$ to the set Ƕ. Also, $t_{Ƕ}$, $i_{Ƕ}$, $f_{Ƕ}$: $\mathbb{P} \rightarrow]^{-}0, 1^{+}[$ and $^{-}0 \leq t_{Ƕ}(\wp) + i_{Ƕ}(\wp) + f_{Ƕ}(\wp) \leq 3^{+}$. Set of all $N_{eu}-$ set over $\mathbb{P}$ is denoted by $N_{eu}(\mathbb{P})$.

Definition 2.2 (Sreeja & Sarankumar**,** 2018): Let $\mathbb{P}$ be a non-empty set. $\mathbb{A} = \left\{\langle\, \wp, \left(t_{\mathbb{A}}(\wp), i_{\mathbb{A}}(\wp), f_{\mathbb{A}}(\wp)\right)\rangle : \wp \in \mathbb{P}\right\}$ and $\mathbb{B} = \{\langle\, \wp, \left(t_{\mathbb{B}}(\wp), i_{\mathbb{B}}(\wp), f_{\mathbb{B}}(\wp)\right)\rangle : \wp \in \mathbb{P}\}$ are $N_{eu} -$ sets, then

(i) $\mathbb{A} \subseteq \mathbb{B}$ if $t_{\mathbb{A}}(\wp) \leq t_{\mathbb{B}}(\wp), i_{\mathbb{A}}(\wp) \leq i_{\mathbb{B}}(\wp), f_{\mathbb{A}}(\wp) \geq f_{\mathbb{B}}(\wp)$ for all $\wp \in \mathbb{P}$.

(ii) $\mathbb{A} \cap \mathbb{B} = \left\{\langle\, \wp, \left(\min\left(t_{\mathbb{A}}(\wp), t_{\mathbb{B}}(\wp)\right), \min\left(i_{\mathbb{A}}(\wp), i_{\mathbb{B}}(\wp)\right), \max\left(f_{\mathbb{A}}(\wp), f_{\mathbb{B}}(\wp)\right)\right)\rangle : \wp \in \mathbb{P}\right\}$.

(iii) $\mathbb{A} \cup \mathbb{B} = \left\{\langle\, \wp, \left(\max\left(t_{\mathbb{A}}(\wp), t_{\mathbb{B}}(\wp)\right), \max\left(i_{\mathbb{A}}(\wp), i_{\mathbb{B}}(\wp)\right), \min\left(f_{\mathbb{A}}(\wp), f_{\mathbb{B}}(\wp)\right)\right)\rangle : \wp \in \mathbb{P}\right\}$.

(iv) $\mathbb{A}^c = \left\{\langle\, \wp, \left(f_{\mathbb{A}}(\wp), 1 - i_{\mathbb{A}}(\wp), t_{\mathbb{A}}(\wp)\right)\rangle : \wp \in \mathbb{P}\right\}$.

(v) $0_{N_{eu}} = \{\langle\, \wp, (0,0,1)\rangle : \wp \in \mathbb{P}\}$ and $1_{N_{eu}} = \{\langle\, \wp, (1,1,0)\rangle : \wp \in \mathbb{P}\}$.

Definition 2.3 (Sreeja & Sarankumar**,** 2018): A $N_{eu} -$ topology (N_{eu}T) on a non-empty set $\mathbb{P}$ is a family $\tau_{N_{eu}}$ of $N_{eu} -$ sets in $\mathbb{P}$ satisfying the following axioms,

(i) $0_{N_{eu}}, 1_{N_{eu}} \in \tau_{N_{eu}}$.

(ii) $\mathbb{A}_1 \cap \mathbb{A}_2 \in \tau_{N_{eu}}$ for any $\mathbb{A}_1, \mathbb{A}_2 \in \tau_{N_{eu}}$.

(iii) $\bigcup \mathbb{A}_i \in \tau_{N_{eu}}$ for every family $\{\mathbb{A}_i \,/\, i \in \Omega\} \subseteq \tau_{N_{eu}}$.

In this case, the ordered pair $\left(\mathbb{P}, \tau_{N_{eu}}\right)$ or simply $\mathbb{P}$ is called a neutrosophic topological space (N_{eu}TS). The elements of $\tau_{N_{eu}}$ is neutrosophic open set ($N_{eu} - OS$) and $\tau_{N_{eu}}{}^c$ is neutrosophic closed set ($N_{eu} - CS$) .

Definition 2.4 (Rodrigo & Maheswari, 2021a): A $N_{eu} -$ set $\mathbb{A}$ in a N_{eu}TS $\left(\mathbb{P}, \tau_{N_{eu}}\right)$ is called a neutrosophic generalized semi-alpha star closed set ($N_{eu}gs\alpha^* - CS$) if $N_{eu}\alpha - int\left(N_{eu}\alpha - cl(\mathbb{A})\right) \subseteq N_{eu} - int(\mathcal{G})$, whenever $\mathbb{A} \subseteq \mathcal{G}$ and $\mathcal{G}$ is $N_{eu}\alpha^* - OS$.

Definition 2.5 (Rodrigo & Maheswari, 2021b):] A N_{eu}TS $\left(\mathbb{P}, \tau_{N_{eu}}\right)$ is called a $N_{eu}gs\alpha^* - T_{1/2}$ space if every $N_{eu}gs\alpha^* - CS$ in $\left(\mathbb{P}, \tau_{N_{eu}}\right)$ is a $N_{eu} - CS$ in $\left(\mathbb{P}, \tau_{N_{eu}}\right)$.

Definition 2.6: A $N_{eu} -$ function $f_N : \left(\mathbb{P}, \tau_{N_{eu}}\right) \to \left(\mathbb{Q}, \sigma_{N_{eu}}\right)$ is

(1) $N_{eu} -$ continuous (Blessie & Shalini, 2019). if $f_N{}^{-1}$ of $N_{eu} - CS$ in $\left(\mathbb{Q}, \sigma_{N_{eu}}\right)$ is a $N_{eu} - CS$ in $\left(\mathbb{P}, \tau_{N_{eu}}\right)$.

(2) $N_{eu}\alpha -$ continuous (Arokiarani et al., 2017) if $f_N{}^{-1}$ of $N_{eu} - CS$ in $\left(\mathbb{Q}, \sigma_{N_{eu}}\right)$ is a $N_{eu}\alpha - CS$ in $\left(\mathbb{P}, \tau_{N_{eu}}\right)$.

(3) $N_{eu}R -$ continuous (Nandhini & Vigneshwaran, 2019) if $f_N{}^{-1}$ of $N_{eu} - CS$ in $\left(\mathbb{Q}, \sigma_{N_{eu}}\right)$ is a $N_{eu}R - CS$ in $\left(\mathbb{P}, \tau_{N_{eu}}\right)$.

(4) $N_{eu}gs\alpha^* -$ continuous (Rodrigo & Maheswari, 2021b) if $f_N{}^{-1}$ of $N_{eu} - CS$ in $\left(\mathbb{Q}, \sigma_{N_{eu}}\right)$ is a $N_{eu}gs\alpha^* - CS$ in $\left(\mathbb{P}, \tau_{N_{eu}}\right)$.

(5) $N_{eu}gs\alpha^* -$ irresolute map (Rodrigo & Maheswari, 2021b) if ${f_N}^{-1}$ of $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$.

(6) $N_{eu}gs\alpha^* -$ closed map (Rodrigo & Maheswari, 2021c) if f_N of every $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

(7) $N_{eu}gs\alpha^* -$ open map (Rodrigo & Maheswari, 2021c) if f_N of every $N_{eu} - OS$ in $(\mathbb{P}, \tau_{N_{eu}})$ is a $N_{eu}gs\alpha^* - OS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Definition 2.7: A $N_{eu} -$ bijection function $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ is

(1) $N_{eu} - h_{om}$ (Parimala et al., 2018) if f_N and ${f_N}^{-1}$ are $N_{eu} -$ continuous.

(2) $N_{eu}\alpha - h_{om}$ (Priya et al., 2020) if f_N and ${f_N}^{-1}$ are $N_{eu}\alpha -$ continuous.

(3) $N_{eu}R - h_{om}$ (Savithiri & Janaki, 2021) if f_N and ${f_N}^{-1}$ are $N_{eu}R -$ continuous.

3. NEUTROSOPHIC $gs\alpha^*$ – HOMEOMORPHISM

Definition 3.1: A $N_{eu} -$ bijection function $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - h_{om}$ if f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ continuous.

Theorem 3.2: Every $N_{eu} - h_{om}$ is $N_{eu}gs\alpha^* - h_{om}$, but not conversely.

Proof: Let a bijection mapping $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be any $N_{eu} -$ function. Given f_N is $N_{eu} - h_{om}$, then f_N and ${f_N}^{-1}$ are $N_{eu} -$ continuous $\Rightarrow f_N$ and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ continuous $\Rightarrow f_N$ is $N_{eu}gs\alpha^* - h_{om}$.

Example 3.3: Let $\mathbb{P} = \{\wp\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle \wp, (0.2, 0.5, 0.4)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.4, 0.2, 0.7)\rangle\}$ are $N_{eu}(\mathbb{P})$ and $N_{eu}(\mathbb{Q})$. Define a map $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(\wp) = q$. Let $\mathbb{B}^c = \{\langle q, (0.7, 0.8, 0.4)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Then ${f_N}^{-1}(\mathbb{B}^c) = \{\langle \wp, (0.7, 0.8, 0.4)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}^c\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left({f_N}^{-1}(\mathbb{B}^c)\right)\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever ${f_N}^{-1}(\mathbb{B}^c) \subseteq 1_{N_{eu}} \Rightarrow {f_N}^{-1}(\mathbb{B}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$ $\Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ continuous $\to$ 1. Also, let $\mathbb{A}^c = \{\langle \wp, (0.4, 0.5, 0.2)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.4, 0.5, 0.2)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}^c\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left(f_N(\mathbb{A}^c)\right)\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever $f_N(\mathbb{A}^c) \subseteq 1_{N_{eu}} \Rightarrow f_N(\mathbb{A}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^*$ –continuous $\to$ 2. From 1 and 2, f_N is $N_{eu}gs\alpha^* - h_{om}$. But f is not $N_{eu} - h_{om}$, because f_N and ${f_N}^{-1}$ are not $N_{eu} -$ continuous, $N_{eu} - cl\left({f_N}^{-1}(\mathbb{B}^c)\right) = 1_{N_{eu}} \neq {f_N}^{-1}(\mathbb{B}^c)$ and $N_{eu} - cl\left(f_N(\mathbb{A}^c)\right) = 1_{N_{eu}} \neq f_N(\mathbb{A}^c)$.

Theorem 3.4: Every $N_{eu}\alpha - h_{om}$ is $N_{eu}gs\alpha^* - h_{om}$, but not conversely.

Proof: Let a bijection mapping $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be any $N_{eu} -$ function . Given f_N is $N_{eu}\alpha - h_{om}$, then f_N and f_N^{-1} are $N_{eu}\alpha -$ continuous $\Rightarrow f_N$ and f_N^{-1} are $N_{eu}gs\alpha^* -$ continuous $\Rightarrow f_N$ is $N_{eu}gs\alpha^* - h_{om}$.

Example 3.5: Let $\mathbb{P} = \{\wp\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle \wp, (0.2,0.4,0.8)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.3, 0.1, 0.6)\rangle\}$ are $N_{eu}(\mathbb{P})$ and $N_{eu}(\mathbb{Q})$. Define a map $f_N: (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(\wp) = q$. Let $\mathbb{B}^c = \{\langle q, (0.6, 0.9, 0.3)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Then $f_N^{-1}(\mathbb{B}^c) = \{\langle \wp, (0.6, 0.9, 0.3)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}^c\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left(f_N^{-1}(\mathbb{B}^c)\right)\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever $f_N^{-1}(\mathbb{B}^c) \subseteq 1_{N_{eu}} \Rightarrow f_N^{-1}(\mathbb{B}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ continuous $\to$ 1 . Also, let $\mathbb{A}^c = \{\langle \wp, (0.8, 0.6, 0.2)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.8, 0.6, 0.2)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}^c\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl(f_N(\mathbb{A}^c))\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever $f_N(\mathbb{A}^c) \subseteq 1_{N_{eu}} \Rightarrow f_N(\mathbb{A}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N^{-1}$ is $N_{eu}gs\alpha^* -$ continuous $\to$ 2 . From 1 and 2 , f_N is $N_{eu}gs\alpha^* - h_{om}$. But f_N is not $N_{eu}\alpha - h_{om}$, because f_N and f_N^{-1} are not $N_{eu}\alpha -$ continuous. $N_{eu} - cl\left(N_{eu} - int\left(N_{eu} - cl\left(f_N^{-1}(\mathbb{B}^c)\right)\right)\right) = 1_{N_{eu}} \nsubseteq f_N^{-1}(\mathbb{B}^c)$ and $N_{eu} - cl\left(N_{eu} - int\left(N_{eu} - cl(f_N(\mathbb{A}^c))\right)\right) = 1_{N_{eu}} \nsubseteq f_N(\mathbb{A}^c)$.

Theorem 3.6: Every $N_{eu}R - h_{om}$ is $N_{eu}gs\alpha^* - h_{om}$, but not conversely.

Proof: Let a bijection mapping $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be any $N_{eu} -$ function. Given f_N is $N_{eu}R - h_{om}$, then f_N and f_N^{-1} are $N_{eu}R -$ continuous $\Rightarrow f_N$ and f_N^{-1} are $N_{eu}gs\alpha^* -$ continuous $\Rightarrow f_N$ is $N_{eu}gs\alpha^* - h_{om}$.

Example 3.7: Let $\mathbb{P} = \{\wp\}$ and $\mathbb{Q} = \{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$, $\mathbb{A} = \{\langle \wp, (0.1,0.4,0.6)\rangle\}$ and $\mathbb{B} = \{\langle q, (0.2, 0.2, 0.8)\rangle\}$ are $N_{eu}(\mathbb{P})$ and $N_{eu}(\mathbb{Q})$. Define a map $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ by $f_N(\wp) = q$. Let $\mathbb{B}^c = \{\langle q, (0.8, 0.8, 0.2)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Then $f_N^{-1}(\mathbb{B}^c) = \{\langle \wp, (0.8, 0.8, 0.2)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}^c\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left(f_N^{-1}(\mathbb{B}^c)\right)\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever $f_N^{-1}(\mathbb{B}^c) \subseteq 1_{N_{eu}} \Rightarrow f_N^{-1}(\mathbb{B}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$ $\Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ continuous $\to$ 1 . Also , let $\mathbb{A}^c = \{\langle \wp, (0.6, 0.6, 0.1)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q, (0.6, 0.6, 0.1)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{B}^c\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl(f_N(\mathbb{A}^c))\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever $f_N(\mathbb{A}^c) \subseteq 1_{N_{eu}} \Rightarrow f_N(\mathbb{A}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N^{-1}$ is $N_{eu}gs\alpha^* -$ continuous $\to$ 2 . From 1 and 2 , f_N is $N_{eu}gs\alpha^* - h_{om}$. But f_N is not $N_{eu}R - h_{om}$, because f_N and f_N^{-1} are not $N_{eu}R -$

continuous . $N_{eu}-cl\left(N_{eu}-int\left({f_N}^{-1}(Ƀ^c)\right)\right)=Ⱥ^c \nsubseteq {f_N}^{-1}(Ƀ^c)$ and $N_{eu}-cl\left(N_{eu}-int\left(f_N(Ⱥ^c)\right)\right)=Ƀ^c \nsubseteq f_N(Ⱥ^c)$.

Remark 3.8: Let $f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ and $g_N:\left(\mathbb{Q},\sigma_{N_{eu}}\right)\to\left(\mathbb{R},\gamma_{N_{eu}}\right)$ be $N_{eu}gs\alpha^*-h_{om}$, then $g_N o f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{R},\gamma_{N_{eu}}\right)$ need not be $N_{eu}gs\alpha^*-h_{om}$.

Example 3.9: Let $\mathbb{P}=\{\wp\}$ and $\mathbb{Q}=\{q\}$. $\tau_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},Ⱥ\}$ and $\sigma_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},Ƀ\}$ are N_{eu}TS on $\left(\mathbb{P},\tau_{N_{eu}}\right)$ and $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$. $Ⱥ=\{\langle\wp,(0.4,0.6,0.8)\rangle\}$ and $Ƀ=\{\langle q,(0.6,0.5,0.7)\rangle\}$ are $N_{eu}(\mathbb{P})$ and $N_{eu}(\mathbb{Q})$. Define a map $f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ by $f_N(\wp)=q-0.1 \Rightarrow f_N$ is $N_{eu}gs\alpha^*-h_{om}$. Let $\mathbb{R}=\{r\}$ and $Ȼ=\{\langle r,(0.3,0.4,0.5)\rangle\}$ is $N_{eu}(\mathbb{R})$ and $\gamma_{N_{eu}}=\{0_{N_{eu}},1_{N_{eu}},Ȼ\}$ is N_{eu}TS on $\left(\mathbb{R},\gamma_{N_{eu}}\right)$. Define a map $g_N:\left(\mathbb{Q},\sigma_{N_{eu}}\right)\to\left(\mathbb{R},\gamma_{N_{eu}}\right)$ by $g_N(q)=r\Rightarrow g_N$ is $N_{eu}gs\alpha^*-h_{om}$. Define a map $g_N o f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{R},\gamma_{N_{eu}}\right)$ by $g_N o f_N(\wp)=r-0.1\Rightarrow g_N o f_N$ is not $N_{eu}gs\alpha^*-h_{om}$.

Theorem 3.10: Let $f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ and $g_N:\left(\mathbb{Q},\sigma_{N_{eu}}\right)\to\left(\mathbb{R},\gamma_{N_{eu}}\right)$ be $N_{eu}gs\alpha^*-h_{om}$. Also , $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then $g_N o f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{R},\gamma_{N_{eu}}\right)$ is $N_{eu}gs\alpha^*-h_{om}$.

Proof: Given f_N is $N_{eu}gs\alpha^*-h_{om}$, then f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^*-$ continuous. Given g_N is $N_{eu}gs\alpha^*-h_{om}$, then g_N and ${g_N}^{-1}$ are $N_{eu}gs\alpha^*-$ continuous . Let $Ⱥ$ be a $N_{eu}-CS$ in $\left(\mathbb{R},\gamma_{N_{eu}}\right)$. Given g_N is $N_{eu}gs\alpha^*-$ continuous , then ${g_N}^{-1}(Ⱥ)$ is a $N_{eu}gs\alpha^*-CS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$. Given $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then ${g_N}^{-1}(Ⱥ)$ is a $N_{eu}-CS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$. Given f_N is $N_{eu}gs\alpha^*-$ continuous , then ${f_N}^{-1}\left({g_N}^{-1}(Ⱥ)\right)=(g_N o f_N)^{-1}(Ⱥ)$ is a $N_{eu}gs\alpha^*-CS$ in $\left(\mathbb{P},\tau_{N_{eu}}\right)\Rightarrow g_N o f_N$ is $N_{eu}gs\alpha^*-$ continuous → 1 . Similarly , let $Ƀ$ be a $N_{eu}-CS$ in $\left(\mathbb{P},\tau_{N_{eu}}\right)$. Given ${f_N}^{-1}$ is $N_{eu}gs\alpha^*-$ continuous , then $\left({f_N}^{-1}\right)^{-1}(Ƀ)=f_N(Ƀ)$ is a $N_{eu}gs\alpha^*-CS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$. Given $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ is $N_{eu}gs\alpha^*-T_{1/2}$ space , then $f_N(Ƀ)$ is a $N_{eu}-CS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)$. Given ${g_N}^{-1}$ is $N_{eu}gs\alpha^*-$ continuous , then $({g_N}^{-1})^{-1}\left(f(Ƀ)\right)=g_N\left(f_N(Ƀ)\right)=g_N o f_N(Ƀ)$ is a $N_{eu}gs\alpha^*-CS$ in $\left(\mathbb{R},\gamma_{N_{eu}}\right)\Rightarrow(g_N o f_N)^{-1}$ is $N_{eu}gs\alpha^*-$ continuous → 2 . From 1 and 2 , $g_N o f_N$ is $N_{eu}gs\alpha^*-h_{om}$.

Theorem 3.11: Let $f_N:\left(\mathbb{P},\tau_{N_{eu}}\right)\to\left(\mathbb{Q},\sigma_{N_{eu}}\right)$ be a bijective mapping . If f_N is $N_{eu}gs\alpha^*-$ continuous , then the following statements are equivalent .

(1) f_N is $N_{eu}gs\alpha^*-$ open mapping .

(2) f_N is $N_{eu}gs\alpha^*-h_{om}$.

(3) f_N is $N_{eu}gs\alpha^*-$ closed mapping .

Proof: (1) ⇒ (2) , Consider a bijective $N_{eu}gs\alpha^*-$ open mapping . Let $Ⱥ$ be a $N_{eu}-CS$ in $\left(\mathbb{P},\tau_{N_{eu}}\right)$. Then $Ⱥ^c$ is a $N_{eu}-OS$ in $\left(\mathbb{P},\tau_{N_{eu}}\right)$. Given f_N is $N_{eu}gs\alpha^*-$ open map , then $f_N(Ⱥ^c)=\left(f_N(Ⱥ)\right)^c$ is a $N_{eu}gs\alpha^*-OS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)\Rightarrow f_N(Ⱥ)$ is a $N_{eu}gs\alpha^*-CS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)\Rightarrow\left({f_N}^{-1}\right)^{-1}(Ⱥ)$ is a $N_{eu}gs\alpha^*-CS$ in $\left(\mathbb{Q},\sigma_{N_{eu}}\right)\Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^*-$ continuous. Also , f_N is $N_{eu}gs\alpha^*-$ continuous , then f_N is $N_{eu}gs\alpha^*-h_{om}$.

(2) ⇒ (3) , Suppose f_N is $N_{eu}gs\alpha^* - h_{om}$. Then f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ continuous . Let $\mathbb{A}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ continuous , then $({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow$ f_N is $N_{eu}gs\alpha^* -$ closed map .

(3) ⇒ (1) , Let $\mathbb{A}$ be a $N_{eu} - OS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $\mathbb{A}^c$ is a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ closed map , then $f_N(\mathbb{A}^c) = (f_N(\mathbb{A}))^c$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow$ $f_N(\mathbb{A})$ is a $N_{eu}gs\alpha^* - OS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ open map .

Theorem 3.12: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* - h_{om}$. Then f_N is $N_{eu} - h_{om}$, if $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$ are $N_{eu}gs\alpha^* - T_{1/2}$ space .

Proof: Let $\mathbb{A}$ be a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* - h_{om}$, then f_N is

$N_{eu}gs\alpha^* -$ continuous ⇒ ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given $(\mathbb{P}, \tau_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow f_N$ is $N_{eu} -$ continuous → ① . Similarly , let $\mathbb{A}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* - h_{om}$, then ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ continuous ⇒ $({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then $({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$ ⇒ ${f_N}^{-1}$ is $N_{eu} -$ continuous → ② . From ① and ② , f_N is $N_{eu} - h_{om}$.

Theorem 3.13: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* - h_{om}$. Then f_N is $N_{eu}\alpha - h_{om}$, if $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$ are $N_{eu}gs\alpha^* - T_{1/2}$ space .

Proof: Let $\mathbb{A}$ be a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* - h_{om}$, then f_N is $N_{eu}gs\alpha^* -$ continuous ⇒ ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given $(\mathbb{P}, \tau_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow {f_N}^{-1}(\mathbb{A})$ is a $N_{eu}\alpha - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}\alpha$ –continuous → ① . Similarly , let $\mathbb{A}$ be a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* - h_{om}$, then ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ continuous ⇒ $({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then $({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$ ⇒ $({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu}\alpha - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$ ⇒ ${f_N}^{-1}$ is $N_{eu}\alpha -$ continuous → ② . From ① and ② , f_N is a $N_{eu}\alpha - h_{om}$.

Remark 3.14: If we replace f_N is $N_{eu}\alpha - h_{om}$ by $N_{eu}R - h_{om}$, then the theorem 3.13 is true .

Theorem 3.15: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* - h_{om}$ iff ${f_N}^{-1}: (\mathbb{Q}, \sigma_{N_{eu}}) \to (\mathbb{P}, \tau_{N_{eu}})$ is $N_{eu}gs\alpha^* - h_{om}$.

Proof: Given f_N is $N_{eu}gs\alpha^* - h_{om}$, then f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ continuous . Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ continuous , then $f_N(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow ({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ continuous → ① . Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is

$N_{eu}gs\alpha^* -$ continuous , then ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P},\tau_{N_{eu}}) \Rightarrow \left(\left({f_N}^{-1}\right)^{-1}\right)^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P},\tau_{N_{eu}})$ $\Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ continuous → ② . From ① and ② , ${f_N}^{-1}$ is $N_{eu}gs\alpha^* - h_{om}$. Converse is similar .

Section 4. NEUTROSOPHIC $igs\alpha^*$ – HOMEOMORPHISM

Definition 4.1: A $N_{eu} -$ bijection function $f_N : (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}\,,\,\sigma_{N_{eu}})$ is $N_{eu}igs\alpha^* - h_{om}$ if f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ irresolute mappings .

Theorem 4.2: Every $N_{eu}igs\alpha^* - h_{om}$ is $N_{eu}gs\alpha^* - h_{om}$, but not conversely .

Proof: Let a bijection mapping $f_N : (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}\,,\,\sigma_{N_{eu}})$ be any $N_{eu} -$ function . Given f_N is $N_{eu}igs\alpha^* - h_{om}$, then f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ irresolute mapping $\Rightarrow f_N$ and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ continuous $\Rightarrow$ f_N is $N_{eu}gs\alpha^* - h_{om}$.

Example 4.3: Let $\mathbb{P}=\{\wp\}$ and $\mathbb{Q}=\{q\}$. $\tau_{N_{eu}} = \{0_{N_{eu}}, 1_{N_{eu}}, \mathbb{A}\}$ and $\sigma_{N_{eu}} = \{0_{N_{eu}}\,, 1_{N_{eu}}, \mathbb{B}\}$ are N_{eu}TS on $(\mathbb{P},\tau_{N_{eu}})$ and $(\mathbb{Q}\,,\,\sigma_{N_{eu}})$. $\mathbb{A}= \{\langle\wp,(0.3,0.8,0.6)\rangle\}$ and $\mathbb{B} = \{\langle q\,,(0.6\,,0.5\,,0.8)\rangle\}$ are $N_{eu}(\mathbb{P})$ and $N_{eu}(\mathbb{Q})$. Define a map $f_N : (\mathbb{P},\tau_{N_{eu}}) \to (\mathbb{Q}\,,\,\sigma_{N_{eu}})$ by $f_N(\wp) = q - 0.3$. Let $\mathbb{B}^c = \{\langle q\,,(0.8\,,0.5,0.6)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{Q}\,,\,\sigma_{N_{eu}})$. Then ${f_N}^{-1}(\mathbb{B}^c) = \{\langle\wp\,,(0.5\,,\ 0.2\,,\ 0.3)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}},\ 1_{N_{eu}},\ \mathbb{A},\ D\,,\ E\,,\ F\,\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}\,,\ 1_{N_{eu}},\ \mathbb{A}^c,\ L\,,\ M\,,N\}$, where $D = \{\langle\,\wp\,,\ ([0.6\ ,1]\ ,\ [0.8\,,1]\ ,\ [0,\ 0.3])\rangle\}$, $E = \{\langle\,\wp\,,\ ([0.6\,,1]\,,\ [0.8\,,1]\,,\ [0.4\,,0.6])\rangle\}$, $F = \{\langle\,\wp\,,([0.3\,,0.5]\,,\ [0.8\,,1]\,,\ [0,\ 0.6])\rangle\}$, $L = \{\langle\,\wp\,,\ ([0,\ 0.3]\,,\ [0\,,0.2]\,,\ [0.6\,,1])\rangle\}$, $M = \{\langle\,\wp\,,\ ([0\,,0.6]\ ,\ [0\,,0.2]\,,\ [0.3\,,0.5])\rangle\}$, $N = \{\langle\,\wp\ ,\ ([0.4\,,0.6]\ ,\ [0\,,0.2]\ ,\ [0.6\,,1])\rangle\}$. Now, $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left({f_N}^{-1}(\mathbb{B}^c)\right)\right) = 0_{N_{eu}} \subseteq N_{eu} - int(D)\,,\ N_{eu} - int(J)\,,\ N_{eu} - int(1_{N_{eu}}) = \mathbb{A}\ ,\ 1_{N_{eu}}$ whenever ${f_N}^{-1}(\mathbb{B}^c) \subseteq D\ ,\ J\ ,\ 1_{N_{eu}} \Rightarrow {f_N}^{-1}(\mathbb{B}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P},\tau_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ continuous → ① . let $\mathbb{A}^c = \{\langle\wp\,,(0.6\,,\ 0.2\,,\ 0.3)\rangle\}$ be a $N_{eu} - CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(\mathbb{A}^c) = \{\langle q\,,(0.9\,,0.5\,,0.6)\rangle\}$. $N_{eu}\alpha^* - OS = N_{eu}\alpha - OS = \{0_{N_{eu}}\,,\ 1_{N_{eu}},\ \mathbb{B}\,\}$ and $N_{eu}\alpha - CS = \{0_{N_{eu}}\,,\ 1_{N_{eu}},\ \mathbb{B}^c\}$. Now , $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left(f_N(\mathbb{A}^c)\right)\right) = 1_{N_{eu}} \subseteq N_{eu} - int(1_{N_{eu}}) = 1_{N_{eu}}$, whenever $f_N(\mathbb{A}^c) \subseteq 1_{N_{eu}} \Rightarrow f_N(\mathbb{A}^c)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}\ ,\ \sigma_{N_{eu}}) \Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ continuous → ② . From ① and ② , f_N is $N_{eu}gs\alpha^* - h_{om}$. Let $\mathbb{E}= \{\langle\wp\,,(0.2\,,\ 0.1\,,\ 0.7)\rangle\}$ be any $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P},\tau_{N_{eu}})$. Then $f_N(\mathbb{E}) = \{\langle q\,,(0.5\,,0.4\,,1)\rangle\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left(f_N(\mathbb{E})\right)\right) = \mathbb{B} \subseteq N_{eu} - int(\mathbb{B})\,,\ N_{eu} - int(1_{N_{eu}}) = \mathbb{B}\,, 1_{N_{eu}}$, whenever $f_N(\mathbb{E}) \subseteq \mathbb{B}\,, 1_{N_{eu}} \Rightarrow f_N(\mathbb{E})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}\ ,\ \sigma_{N_{eu}})$ $\Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ irresolute → ③ . Let $\mathbb{C} = \{\langle q\,,(0.6\,,0.6\,,0.9)\rangle\}$ be any $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}\ ,\ \sigma_{N_{eu}})$. Then ${f_N}^{-1}(\mathbb{C}) = \{\langle\wp\,,(0.3,\ 0.3\,,\ 0.6)\rangle\}$. $N_{eu}\alpha - int\left(N_{eu}\alpha - cl\left({f_N}^{-1}(\mathbb{C})\right)\right) = 1_{N_{eu}} \nsubseteq N_{eu} - int(\mathbb{A}) = \mathbb{A}$, whenever ${f_N}^{-1}(\mathbb{C}) \subseteq \mathbb{A} \Rightarrow {f_N}^{-1}(\mathbb{C})$ is not a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P},\tau_{N_{eu}})$ $\Rightarrow f_N$ is not $N_{eu}gs\alpha^* -$ irresolute → ④ . From ③ and ④ , f_N is not $N_{eu}igs\alpha^* - h_{om}$.

Theorem 4.4: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}gs\alpha^* - h_{om}$ and $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then f_N is $N_{eu}igs\alpha^* - h_{om}$.

Proof: Given f_N is $N_{eu}gs\alpha^* - h_{om}$, then f_N and f_N^{-1} are $N_{eu}gs\alpha^* -$ continuous . Also , $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space , then f_N and f_N^{-1} are $N_{eu}gs\alpha^* -$ irresolute $\Rightarrow f_N$ is $N_{eu}igs\alpha^* - h_{om}$.

Theorem 4.5: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ and $g_N : (\mathbb{Q}, \sigma_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ be $N_{eu}igs\alpha^* - h_{om}$. Then $g_N of_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{R}, \gamma_{N_{eu}})$ is $N_{eu}igs\alpha^* - h_{om}$.

Proof: Given g_N is $N_{eu}igs\alpha^* - h_{om}$, then g_N and g_N^{-1} are $N_{eu}gs\alpha^* -$ irresolute . Given f_N is $N_{eu}igs\alpha^* - h_{om}$, then f_N and f_N^{-1} are $N_{eu}gs\alpha^* -$ irresolute . Let $\mathbb{A}$ be a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{R}, \gamma_{N_{eu}})$. Given g_N is $N_{eu}gs\alpha^* -$ irresolute , then $g_N^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ irresolute , then $f_N^{-1}(g_N^{-1}(\mathbb{A})) = (g_N of_N)^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow g_N of_N$ is $N_{eu}gs\alpha^* -$ irresolute $\to$ ① . Let $\mathbb{B}$ be any $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given f_N^{-1} is $N_{eu}gs\alpha^* -$ irresolute , then $f_N(\mathbb{B})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given g_N^{-1} is $N_{eu}gs\alpha^* -$ irresolute , then $g_N(f_N(\mathbb{B})) = g_N of_N(\mathbb{B}) = ((g_N of_N)^{-1})^{-1}(\mathbb{B})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{R}, \gamma_{N_{eu}}) \Rightarrow (g_N of_N)^{-1}$ is $N_{eu}gs\alpha^* -$ irresolute $\to$ ② . From ① and ② , $g_N of_N$ is $N_{eu}igs\alpha^* - h_{om}$.

Theorem 4.6: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}igs\alpha^* - h_{om}$, then $N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right) = f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A}))$ for each $N_{eu} -$ set $\mathbb{A}$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Proof: Given f_N is $N_{eu}igs\alpha^* - h_{om}$, then f_N is $N_{eu}gs\alpha^* -$ irresolute . Let $\mathbb{A}$ be any $N_{eu} -$ set in $(\mathbb{Q}, \sigma_{N_{eu}})$. Clearly , $N_{eu}gs\alpha^* - cl(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. By hypothesis , $f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A}))$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given $f_N^{-1}(\mathbb{A}) \subseteq f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A})) \Rightarrow N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right) \subseteq N_{eu}gs\alpha^* - cl\left(f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A}))\right) = f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A})) \Rightarrow N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right) \subseteq f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A})) \to$ ① . Given f_N is $N_{eu}igs\alpha^* - h_{om}$, then f_N^{-1} is $N_{eu}gs\alpha^* -$ irresolute . Let $f_N^{-1}(\mathbb{A})$ be any $N_{eu} -$ set in $(\mathbb{P}, \tau_{N_{eu}})$. Clearly , $N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. By hypothesis , $(f_N^{-1})^{-1}\left(N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)\right) = f_N\left(N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)\right) \to$ ② is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow \mathbb{A} = (f_N^{-1})^{-1}\left(f_N^{-1}(\mathbb{A})\right) \subseteq (f_N^{-1})^{-1}\left(N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)\right) = f_N\left(N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)\right) \Rightarrow N_{eu}gs\alpha^* - cl(\mathbb{A}) \subseteq N_{eu}gs\alpha^* - cl\left(f_N\left(N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)\right)\right) = f_N\left(N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right)\right)$ (by②) $\Rightarrow f_N^{-1}\left(N_{eu}gs\alpha^* - cl(\mathbb{A})\right) \subseteq N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right) \to$ ③ . From ① and ③ , $N_{eu}gs\alpha^* - cl\left(f_N^{-1}(\mathbb{A})\right) = f_N^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A}))$.

Theorem 4.7: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \to (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}igs\alpha^* - h_{om}$. Then f_N is $N_{eu} - h_{om}$ if $(\mathbb{P}, \tau_{N_{eu}})$ and $(\mathbb{Q}, \sigma_{N_{eu}})$ are $N_{eu}gs\alpha^* - T_{1/2}$ space .

Proof: Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Then $\mathbb{A}$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ irresolute, then ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given $(\mathbb{P}, \tau_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space, then ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow f_N$ is $N_{eu} -$ continuous $\rightarrow$ ① . Let $\mathbb{A}$ be any $N_{eu} - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Then $\mathbb{A}$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ irresolute, then $f_N(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given $(\mathbb{Q}, \sigma_{N_{eu}})$ is $N_{eu}gs\alpha^* - T_{1/2}$ space, then $f_N(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow {f_N}^{-1}$ is $N_{eu} -$ continuous $\rightarrow$ ② . From ① and ② , f_N is $N_{eu} - h_{om}$.

Theorem 4.8: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \rightarrow (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}igs\alpha^* - h_{om}$, then $N_{eu}gs\alpha^* - cl\left({f_N}^{-1}(\mathbb{A})\right) \subseteq {f_N}^{-1}(N_{eu} - cl(\mathbb{A}))$ for each $N_{eu} -$ set $\mathbb{A}$ in $(\mathbb{Q}, \sigma_{N_{eu}})$.

Proof: Let $\mathbb{A}$ be any $N_{eu} -$ set in $(\mathbb{Q}, \sigma_{N_{eu}})$. Then $N_{eu} - cl(\mathbb{A})$ is a $N_{eu} - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$ $\Rightarrow N_{eu} - cl(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ irresolute map, then ${f_N}^{-1}(N_{eu} - cl(\mathbb{A}))$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow N_{eu}gs\alpha^* - cl\left({f_N}^{-1}(N_{eu} - cl(\mathbb{A}))\right) = {f_N}^{-1}(N_{eu} - cl(\mathbb{A}))$. Given $\mathbb{A} \subseteq N_{eu} - cl(\mathbb{A}) \Rightarrow {f_N}^{-1}(\mathbb{A}) \subseteq {f_N}^{-1}(N_{eu} - cl(\mathbb{A})) \Rightarrow N_{eu}gs\alpha^* - cl\left({f_N}^{-1}(\mathbb{A})\right) \subseteq N_{eu}gs\alpha^* - cl\left({f_N}^{-1}(N_{eu} - cl(\mathbb{A}))\right) \Rightarrow N_{eu}gs\alpha^* - cl\left({f_N}^{-1}(\mathbb{A})\right) \subseteq {f_N}^{-1}(N_{eu} - cl(\mathbb{A}))$.

Theorem 4.9: Let $f_N : (\mathbb{P}, \tau_{N_{eu}}) \rightarrow (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}igs\alpha^* - h_{om}$ iff ${f_N}^{-1} : (\mathbb{Q}, \sigma_{N_{eu}}) \rightarrow (\mathbb{P}, \tau_{N_{eu}})$ is $N_{eu}igs\alpha^* - h_{om}$.

Proof: Given f_N is $N_{eu}igs\alpha^* - h_{om}$, then f_N and ${f_N}^{-1}$ are $N_{eu}gs\alpha^* -$ irresolute . Let $\mathbb{A}$ be any $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}})$. Given ${f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ irresolute, then $f_N(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow ({f_N}^{-1})^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}}) \Rightarrow {f_N}^{-1}$ is $N_{eu}gs\alpha^* -$ irresolute $\rightarrow$ ① . Let $\mathbb{A}$ be any $N_{eu}gs\alpha^* - CS$ in $(\mathbb{Q}, \sigma_{N_{eu}})$. Given f_N is $N_{eu}gs\alpha^* -$ irresolute, then ${f_N}^{-1}(\mathbb{A})$ is a $N_{eu}gs\alpha^* - CS$ in $(\mathbb{P}, \tau_{N_{eu}}) \Rightarrow f_N$ is $N_{eu}gs\alpha^* -$ irresolute $\rightarrow$ ② . From ① and ② , ${f_N}^{-1}$ is $N_{eu}igs\alpha^* - h_{om}$. Converse is similar .

Theorem 4.10: Let $f : (\mathbb{P}, \tau_{N_{eu}}) \rightarrow (\mathbb{Q}, \sigma_{N_{eu}})$ be $N_{eu}igs\alpha^* - h_{om}$, then $N_{eu}gs\alpha^* - cl(f_N(\mathbb{A})) = f_N(N_{eu}gs\alpha^* - cl(\mathbb{A}))$ for each $N_{eu} -$ set $\mathbb{A}$ in $(\mathbb{P}, \tau_{N_{eu}})$.

Proof: Given f_N is $N_{eu}igs\alpha^* - h_{om}$, then ${f_N}^{-1}$ is $N_{eu}igs\alpha^* - h_{om}$. Let $\mathbb{A}$ be any $N_{eu} -$ set in $(\mathbb{P}, \tau_{N_{eu}})$. By theorem 4.6 , $N_{eu}gs\alpha^* - cl\left(({f_N}^{-1})^{-1}(\mathbb{A})\right) = ({f_N}^{-1})^{-1}(N_{eu}gs\alpha^* - cl(\mathbb{A})) \Rightarrow N_{eu}gs\alpha^* - cl(f_N(\mathbb{A})) = f_N(N_{eu}gs\alpha^* - cl(\mathbb{A}))$.

5. CONCLUSIONS

We have discussed some new concepts in Neutrosophic Topological spaces. We defined a new definition $N_{eu}gs\alpha^* -$ closed sets. Especially we discussed about $N_{eu}gs\alpha^* -$ homeomorphism and $N_{eu}igs\alpha^* -$ homeomorphism in this topological space. Further in the future, we will discuss its application in the decision-making domain.

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Neutrosophic Dimension of the Neutrosophic Vector Space

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ABSTRACT

In this article, neutrosophic dimension of a neutrosophic vector space has been discussed by using neutrosophic basis. Some characteristics of the new notions are discussed.

KEYWORDS: **Neutrosophic set, neutrosophic vector space, neutrosophic dimension.**

1. INTRODUCTION

One of the sets with a great deal of applications is the neutrosophy concept which was initiated by F. Smarandache (Smarandache, 1998, 2005). The notion of neutrosophic vector space (Agboola, & Akinleye, 2014) was initiated in 2014. The authors (Broumi et al., 2018; Pramanik, 2022, Smarandache & Pramanik, 2018) have contributed many articles in neutrosophic sets and their applications. In this work, we develop the notion of neutrosophic dimension of a neutrosophic vector space, and some properties are interpreted.

2. PRELIMINARIES

Definition 2.1 (Elrawy, 2022**):** Neutrosophic vector space is a quaternary $\bar{V} = (V, \mu, \gamma, \varsigma)$ where V is a vector space over arbitrary field K with

$$\mu: V \to [0,1],$$

$$\gamma: \mathrm{V} \to [0,1],$$

$$\varsigma: \mathrm{V} \to [0,1],$$

with the following properties

$$\mu(au + bv) \geq \mu(u) \wedge \mu(v),$$

$$\gamma(au + bv) \leq \gamma(u) \vee \gamma(v),$$

$$\varsigma(au + bv) \leq \varsigma(u) \vee \varsigma(v),$$

$$\text{where } u, v \in V \text{ and } a, b \in \mathrm{K}$$

Definition 2.2 (Elrawy, 2022**):** If $\bar{V} = (V, \mu, \gamma, \varsigma)$ is a neutrosophic vector space over a field K, then

I. $\mu(au) = \mu(u),\ \forall\, a \in \mathrm{K} - \{0\},$
II. $\gamma(au) = \gamma(u),\ \forall\, a \in \mathrm{K} - \{0\},$
III. $\varsigma(au) = \varsigma(u),\ \forall\, a \in \mathrm{K} - \{0\},$
IV. If $u, v \in V$ and $\mu(u) > \mu(v), then\ \mu(u + v) = \mu(v).$
V. If $u, v \in V$ and $\mu(u) < \mu(v), then\ \mu(u + v) = \mu(v).$
VI. If $u, v \in V$ and $\mu(u) < \mu(v), then\ \mu(u + v) = \mu(v).$

Definition 2.3 (Elrawy, 2022)**:** Let W be a subspace of a vector space V. Then, $(W, \mu W, \gamma W, \varsigma W)$ is called neutrosophic subspace of a neutrosophic vector $(V, \mu, \gamma, \varsigma)$ if the following conditions are satisfied:

I. $\mu w(x - y) \geq \mu w(x) \wedge \mu w(y)$
II. $\mu w(cx) = \mu w(x)$
III. $\gamma w(x - y) \leq \gamma \mathrm{w(x)} \vee \gamma w(y)$
IV. $\gamma w(cx) = \gamma w(x)$
V. $\varsigma w(x - y) \leq \varsigma \mathrm{w(x)} \vee \varsigma w(y)$
VI. $\varsigma w(cx) = \varsigma w(x)$

Definition 2.4 (Elrawy, 2022)**:** Let $\overline{V_1} = (V, \mu_1, \gamma_1, \varsigma_1)$ and $\overline{V_2} = (V, \mu_2, \gamma_2, \varsigma_2)$ be two neutrosophic vector spaces over K, then

The intersection of $\overline{V_1}$ and $\overline{V_2}$ define as follows: $\overline{V_1} \cap \overline{V_2} = \left(V, \mu_1 \wedge \mu_{2,} \gamma_1 \vee \gamma_{2,} \varsigma_1 \vee \varsigma_2\right)$

The sum of $\overline{V_1}$ and $\overline{V_2}$ define as follows: $\overline{V_1} + \overline{V_2} = \left(V, \mu_1 + \mu_{2,} \gamma_1 + \vee\, \gamma_{2,} \varsigma_1 + \varsigma_2\right),$ where

$(\mu_1 + \mu_2)(a) = sup\{\mu_1(a) \wedge \mu_2(a - v)\}, (\gamma_1 + \gamma_2)(a) = inf\{\gamma_1(a) \vee \gamma_2(a - v)\},$

$(\varsigma_1 + \varsigma_2)(a) = inf\{\varsigma_1(a) \vee \varsigma_2(a - v)\}$ and $a = u + v$

3. *NEUTROSOPHIC DIMENSION OF A NEUROSOPHIC VECTOR SPACE*

Definition 3.1: For a neutrosophic set (NS in short) $\hat{A} \subseteq \hat{X}$. Then for $\delta, \rho, \sigma \in [0,1]$ with $\delta + \rho + \sigma \leq 1$, the set $\hat{A}^{[\delta,\rho,\sigma]} = \{x \in X\colon \mu_{\hat{A}}(x) \geq \delta, \gamma_{\hat{A}}(x) \geq \rho, \varsigma_{\hat{A}}(x) \leq \sigma\}$ is called (δ, ρ, σ) −level subset of $\hat{A}$.

Definition 3.2: For a NS $\hat{A} \subseteq \hat{X}$ and $(\delta_1, \rho_1, \sigma_1), (\delta_2, \rho_2, \sigma_2) \in Im(\hat{A})$, If $\delta_1 \geq \delta_2,\ \rho_1 \geq \rho_2,$ $\sigma_1 \leq \sigma_2$, then $\hat{A}^{[\delta_1,\rho_1,\sigma_1]} \supseteq \hat{A}^{[\delta_2,\rho_2,\sigma_2]}$.

Definition 3.3: For a NS $\hat{A} \subseteq \hat{X}$, define a map $|\hat{A}|$: $\mathbb{N} \rightarrow$ [0,1]*[0,1] $\forall\, n \in \mathbb{N}$,

$\mu_{|\hat{A}|}(n) = \vee \{p: (\delta, \rho, \sigma) \in [0,1] \times [0,1] \times [0,1] \backslash \{[0,1]\}$ with $\delta + \rho + \sigma \leq 1$ and $\left|\widehat{A}^{[\delta,\rho,\sigma]}\right| \geq n\}$,

$\gamma_{\hat{A}}(n) = \vee \{q: (\delta, \rho, \sigma) \in [0,1] \times [0,1] \times [0,1] \backslash \{[0,1]\}$ with $\delta + \rho + \sigma \leq 1$ and $\left|\widehat{A}^{[\delta,\rho,\sigma]}\right| \geq n\}$,
$\varsigma_{|\hat{A}|}(n) = \wedge \{r: (\delta, \rho, \sigma) \in [0,1] \times [0,1] \times [0,1] \backslash \{[0,1]\}$ with $\delta + \rho + \sigma \leq 1$ and $\left|\widehat{A}^{[\delta,\rho,\sigma]}\right| \geq n\}$

Then, $|\hat{A}|$ is called as a Neutrosophic set over $\mathbb{N}$ where $|\hat{A}|$ is the cardinality of $\hat{A}$.

Definition 3.4: For two NSs $\hat{A}$ and $\hat{B}$, the addition of the cardinalities is defined as for any $n \in \mathbb{N}$,

$$\mu_{(|\hat{A}|+|\hat{B}|)}(n) = \vee_{n_1+n_2=n}\left(\mu_{|\hat{A}|}(n_1) \wedge \mu_{|\hat{B}|}(n_2)\right)$$

$$\gamma_{(|\hat{A}|+|\hat{B}|)}(n) = \vee_{n_1+n_2=n}\left(\gamma_{|\hat{A}|}(n_1) \wedge \gamma_{|\hat{B}|}(n_2)\right)$$

$$\varsigma_{(|\hat{A}|+|\hat{B}|)}(n) = \wedge_{n_1+n_2=n}\left(\varsigma_{|\hat{A}|}(n_1) \vee \varsigma_{|\hat{B}|}(n_2)\right)$$

Preposition 3.5: For two NSs $\hat{A}$ and $\hat{B}$ over $\mathbb{N}$, for any $(\delta, \rho, \sigma) \in [0,1] \times [0,1] \times [0,1]$ with $\delta + \rho + \sigma \leq 1$,

$$\mu_{(|\hat{A}|+|\hat{B}|)}(n) = \mu_{|\hat{A}|}^{[\delta]} + \mu_{|\hat{B}|}^{[\delta]}$$

$$\gamma_{(|\hat{A}|+|\hat{B}|)}(n) = \gamma_{|\hat{A}|}^{[\delta]} + \gamma_{|\hat{B}|}^{[\delta]}$$

$$\varsigma_{(|\hat{A}|+|\hat{B}|)}(n) = \varsigma_{|\hat{A}|}^{[\delta]} + \varsigma_{|\hat{B}|}^{[\delta]}$$

Proof:

Suppose $n \in \mu_{|\hat{A}|}^{[\delta]} + \mu_{|\hat{B}|}^{[\delta]}$, then there exist n_1, n_2 with $n_1 + n_2 = n$ with $n_1 \in \mu_{|\hat{A}|}^{[\delta]}$, $n_2 \in \mu_{|\hat{B}|}^{[\delta]}$.

Then, $\mu_{|\hat{A}|}^{[\delta]} \geq \delta, \mu_{|\hat{B}|}^{[\delta]} \geq \delta$. By definition, $\mu_{(|\hat{A}|+|\hat{B}|)}(n) = \vee_{n_1+n_2=n}\left(\mu_{|\hat{A}|}(n_1) \wedge \mu_{|\hat{B}|}(n_2)\right) \geq \delta$.

Therefore, $n \in \mu_{(|\hat{A}|+|\hat{B}|)}^{[\delta]}$. Hence, $\mu_{|\hat{A}|}^{[\delta]} + \mu_{|\hat{B}|}^{[\delta]} \subseteq \mu_{(|\hat{A}|+|\hat{B}|)}^{[\delta]}$.

Conversely, let $n \in \mu_{(|\hat{A}|+|\hat{B}|)}^{[\delta]}$. Then $\mu_{(|\hat{A}|+|\hat{B}|)}(n) = \vee_{n_1+n_2=n}\left(\mu_{|\hat{A}|}(n_1) \wedge \mu_{|\hat{B}|}(n_2)\right) \geq \delta$.

Hence, one can find n_1, n_2 with $n_1 + n_2 = n$, and $\mu_{|\hat{A}|}(n_1) \wedge \mu_{|\hat{B}|}(n_2) \geq \delta$. Then, $n_1 \in \mu_{|A|}^{[\delta]}, n_2 \in \mu_{|\hat{B}|}^{[\delta]}$,

that is, $n = n_1 + n_2 \in \mu_{|\hat{A}|}^{[\delta]} + \mu_{|\hat{B}|}^{[\delta]}$. Thus, $\mu_{(|\hat{A}|+|\hat{B}|)}^{[\delta]} \subseteq \mu_{|\hat{A}|}^{[\delta]} + \mu_{|\hat{B}|}^{[\delta]}$.

Definition 3.6: Let $\hat{V} \in N(\hat{X})$ with a neutrosophic basis $\hat{B}$. Then $D(\hat{V})$ is the neutrosophic dimension of $\hat{V}$.

Preposition 3.7: Let $\hat{B}$ and $\hat{B}'$ be two neutrosophic bases of a neutrosophic vector space $\hat{V} \in N(\hat{X})$. Then, $|\hat{B}| = |\hat{B}'|$.

Proof:

Both $\hat{B}^{[\delta,\rho,\sigma]}$ and $\hat{B}^{[\delta,\rho,\sigma]}$ are bases of $\hat{V}^{[\delta,\rho,\sigma]}$ for $a \in (0,1], b \in (0,1], c \in (0,1]$ with $\delta + \rho + \sigma \leq 1$.
Therefore, $\left|\hat{B}^{[\delta,\rho,\sigma]}\right| = \left|\hat{B}'^{[\delta,\rho,\sigma]}\right|$. Hence

$$\mu_{|\hat{B}|}(n) = \vee \{p: (\delta,\rho,\sigma) \in [0,1] \times [0,1] \times [0,1] \backslash \{[0,1]\} \text{ with } \delta + \rho + \sigma \leq 1 \text{ and } \left|\hat{B}^{\delta,\rho,\sigma]}\right| \geq n\}$$

$$= \vee \{p: (\delta,\rho,\sigma) \in [0,1] \times [0,1] \times [0,1] \backslash \{[0,1]\} \text{ with } \delta + \rho + \sigma \leq 1 \text{ and } \left|\hat{B}'^{[\delta,\rho,\sigma]}\right| \geq n\}$$

$$= \mu_{|\hat{B}'|}(n).$$

Similarly, $\gamma_{|\hat{B}|}(n) = \gamma_{|\hat{B}'|}(n)$, $\varsigma_{|\hat{B}|}(n) = \varsigma_{|\hat{B}'|}(n)$ holds.

Preposition 3.8: Let $\hat{X}$ be a vector space with $D(\hat{X}) = m$ and $\hat{V} \in N(\hat{X})$. Then, for any $\delta, \rho, \sigma \in [0,1] \times [0,1] \times [0,1]$ with $\delta + \rho + \sigma \leq 2$, and $n \in \mathbb{N}$, $n \in \mu_{D(\hat{V})}^{[\delta]} \Leftrightarrow n \leq D\left(\mu_{\hat{V}}^{[\delta]}\right)$ and $n \in \mu_{D(\hat{V})}^{[\rho]} \Leftrightarrow n \leq D\left(\mu_{\hat{V}}^{[\rho]}\right)$.

Proof: Suppose that $Im(\hat{V}) = \{(p_0, q_0, r_0), (p_1, q_1, r_1), \ldots, (p_k, q_k, r_k)\}, k \leq m$ such that $(1,1,0) \geq (p_0, q_0, r_0) > (p_1, q_1, r_1) \ldots . > (p_k, q_k, r_k) \geq (0,0,1)$. Then there exists a nested collection of subspaces of $\hat{X}$ as $\{\Delta\} \subseteq \hat{V}^{[p_0,q_0,r_0]} \subsetneq \hat{V}^{[p_1,q_1,r_1]} \subsetneq \cdots \subsetneq \hat{V}^{[p_k,q_k,r_k]} = \hat{X}$. Let $\hat{B}_{\hat{V}_i}$ be the basis of $\hat{V}^{[p_i,q_i,r_i]}, i = 0,1,\ldots,k$ such that $\hat{B}_{\hat{V}_0} \subsetneq \hat{B}_{\hat{V}_1} \subsetneq \cdots \ldots \ldots \subsetneq \hat{B}_{\hat{V}_k}$.

Let $\mathcal{B}$ be a neutrosophic basis and let $n \in \mu_{D(\hat{V})}^{[\delta]} \Rightarrow \mu_{D(\hat{V})}^{(n)} \geq \delta \Rightarrow \vee \{\sigma_1: (\delta_1, \rho_1, \sigma_1) \in (0,1] \times (0,1] \times (0,1] \; with \; \delta_1 + \rho_1 + \sigma_1 \leq 2 \; and \; \left|\mathcal{B}^{[\delta_1,\rho_1,\sigma_1]}\right| \geq n\} \geq \delta$. Then there exists $(\delta_1, \rho_1, \sigma_1) \in [0,1] \times [0,1] \times [0,1] \backslash \{(0,1)\}$ with $\delta_1 + \rho_1 + \sigma_1 \leq 2$ such that $\delta_1 \geq \delta$ and $\left|\mathcal{B}^{[\delta_1,\rho_1,\sigma_1]}\right| \geq n$. Now, $D\left(\mu_{\hat{V}}^{[\delta]}\right) = \left|\mu_{\mathcal{B}}^{[\delta]}\right| \geq \left|\mu_{\mathcal{B}}^{[\delta_1]}\right| \geq \left|\mathcal{B}^{[\delta_1,\rho_1,\sigma_1]}\right| \geq n$.

Conversely, suppose that $n \leq D\left(\mu_{\hat{V}}^{[\delta]}\right) = \left|\mu_{\mathcal{B}}^{[\delta]}\right|$. Now $a \in (p_{i+1}, p_i]$, for some i. Hence $\left|\mu_{\mathcal{B}}^{[\delta]}\right| = \left|\mu_{\mathcal{B}}^{[p_i]}\right| = \left|B_{\hat{V}_i}\right| = \left|\mathcal{B}^{[\delta_1,b_1,\sigma_1]}\right|$. Then $\mu_{D(\hat{V})}(n) = \vee \{\{\sigma_1: (\delta_1, \rho_1, \sigma_1) \in (0,1] \times (0,1] \times (0,1] \; with \; \delta_1 + \rho_1 + c_1 \leq 2 \; and \; \left|\mathcal{B}^{[\delta_1,\rho_1,\sigma_1]}\right| \geq n\} \geq p_i \geq \delta \Rightarrow n \in \mu_{D(\hat{V})}^{[\delta]}$. Hence $n \in \mu_{D(\hat{V})}^{[\delta]}$ if and only if $n \leq D\left(\mu_{\hat{V}}^{[\delta]}\right)$. Similarly for, $n \in \mu_{D(\hat{V})}^{[\rho]} \Leftrightarrow n \leq D\left(\mu_{\hat{V}}^{[\rho]}\right)$.

Preposition 3.9: Let $\hat{X}$ be a vector space with $D(\hat{X}) = m$ and $\hat{V}_1, \hat{V}_2 \in N(\hat{X})$. Then, we have the following results:

- **A.** For all $(\delta, \rho, \sigma) \in [0,1] \times [0,1] \times [0,1]$ with $\delta + \rho + \sigma \leq 2$, $\mu_{\hat{V}_1 \cap \hat{V}_2}^{[\delta]} = \mu_{\hat{V}_1}^{[\delta]} \cap \mu_{\hat{V}_2}^{[\delta]}$, $\gamma_{\hat{V}_1 \cap \hat{V}_2}^{[\rho]} = \gamma_{\hat{V}_1}^{[\rho]} \cap \gamma_{\hat{V}_2}^{[\rho]}$ and $\varsigma_{\hat{V}_1 \cap \hat{V}_2}^{[\sigma]} = \varsigma_{\hat{V}_1}^{[\sigma]} \cap \varsigma_{\hat{V}_2}^{[\sigma]}$.
- **B.** For all $(\delta, \rho, \sigma) \in [0,1] \times [0,1] \times [0,1]$ with $\delta + \rho + \sigma \leq 2$, $\mu_{\hat{V}_1 + \hat{V}_2}^{[\delta]} = \mu_{\hat{V}_1}^{[\delta]} + \mu_{\hat{V}_2}^{[\delta]}$, $\gamma_{\hat{V}_1 + \hat{V}_2}^{[\rho]} = \gamma_{\hat{V}_1}^{[\rho]} + \gamma_{\hat{V}_2}^{[\rho]}$ and $\varsigma_{\hat{V}_1 + \hat{V}_2}^{[\sigma]} = \varsigma_{\hat{V}_1}^{[\sigma]} + \varsigma_{\hat{V}_2}^{[\sigma]}$.

Proof:

Proof of (A) is straight forward.

(B): For all $(\delta,\rho,\sigma) \in [0,1] \times [0,1] \times [0,1]$ with $\delta+\rho+\sigma \le 2$, we have $x \in \mu^{[\delta]}_{(\hat{V}_1+\hat{V}_2)} \Leftrightarrow \sup_{x=x_1+x_2}\{\mu_{\hat{V}(x_1)} \wedge \mu_{\hat{V}(x_2)}\} \ge \delta$

$\Leftrightarrow$ there exists x_1, x_2 such that $x_1+x_2 = x$ and $\mu_{\hat{V}(x_1)} \wedge \mu_{\hat{V}(x_2)} \ge \delta$

$\Leftrightarrow$ there exists x_1, x_2 such that $x_1+x_2 = x$ and $x_1 \in \mu^{[\delta]}_{\hat{V}_1}$ and $x_2 \in \mu^{[\delta]}_{\hat{V}_2}$

The proof for $\gamma^{[\rho]}_{\hat{V}_1+\hat{V}_2} = \gamma^{[\rho]}_{\hat{V}_1} + \gamma^{[\rho]}_{\hat{V}_2}$ and $\varsigma^{[\sigma]}_{\hat{V}_1+\hat{V}_2} = \varsigma^{[\sigma]}_{\hat{V}_1} + \varsigma^{[\sigma]}_{\hat{V}_2}$ are similar.

Preposition 3.10: Let $\hat{X}$ be a vector space with $D(\widehat{X}) = m$ and $\hat{V}_1, \hat{V}_2 \in N(\hat{X})$.

Then, $D(\hat{V}_1+\hat{V}_2) + D(\hat{V}_1 \cap \hat{V}_2) = D(\hat{V}_1) + D\ (\hat{V}_2)$

Proof: For all $(\delta,\rho,\sigma) \in [0,1] \times [0,1] \times [0,1]$ with $\delta+\rho+\sigma \le 2$, let $n \in \mu^{[\delta]}_{D(\hat{V}_1+\hat{V}_2)+D(\hat{V}_1\cap\hat{V}_2)}$.

Then, there exists a n_1, n_2 such that $n = n_1+n_2$ and $n_1 \in \mu^{[\delta]}_{D(\hat{V}_1+\hat{V}_2)}$ and $n_2 \in \mu^{[\delta]}_{D(\hat{V}_1\cap\hat{V}_2)}$.

Then by preposition 3.6, $n_1 \le D\left(\mu^{[\delta]}_{(\hat{V}_1+\hat{V}_2)}\right) = D\ (\mu^{[\delta]}_{\hat{V}_1} + \mu^{[\delta]}_{\hat{V}_2})$ and $n_2 \le D\left(\mu^{[\alpha]}_{(\hat{V}_1\cap\hat{V}_2)}\right) = D\ (\mu^{[\alpha]}_{\hat{V}_1} \cap \mu^{[\alpha]}_{\hat{V}_2})$.

Then, $n \le D\left(\mu^{[\delta]}_{\hat{V}_1} + \mu^{[\delta]}_{\hat{V}_2}\right) + D\left(\mu^{[\delta]}_{\hat{V}_1} \cap \mu^{[\delta]}_{\hat{V}_2}\right) = D\left(\mu^{[\delta]}_{(\hat{V}_1)}\right) + D\left(\mu^{[\delta]}_{(\hat{V}_2)}\right)$.

Then there exists $n_1{}'$ and $n_2{}'$ such that $n = n_1{}' + n_2{}'$ and $n_1{}' \le D\left(\mu^{[\delta]}_{(\hat{V}_1)}\right)$ and $n_2{}' \le D\left(\mu^{[\delta]}_{(\hat{V}_2)}\right)$.

Now, by preposition 3.6, $n_1{}' \le \mu^{[\delta]}_{D(\hat{V}_1)}$ and $n_2{}' \le \mu^{[\delta]}_{D(\hat{V}_2)}$.

Therefore, $n = n_1{}' + n_2{}' \in \mu^{[\delta]}_{D(\hat{V}_1)} + \mu^{[\delta]}_{D(\hat{V}_2)} = \mu^{[\delta]}_{D(\hat{V}_1+\hat{V}_2)}$.

Thus, $\mu^{[\delta]}_{\mathrm{D}(\hat{V}_1+\hat{V}_2)+\mathrm{D}(\hat{V}_1\cap\hat{V}_2)} \subseteq \mu^{[\delta]}_{D(\hat{V}_1)} + \mu^{[\delta]}_{D(\hat{V}_2)}$.

Similarly, $\gamma^{[\rho]}_{D(\hat{V}_1+\hat{V}_2)+D(\hat{V}_1\cap\hat{V}_2)} \subseteq \gamma^{[\rho]}_{D(v_1)} + \gamma^{[\rho]}_{D(\hat{V}_2)}$.

Also, the reverse inclusion relationship can be proved.

Hence, for all $(\delta,\rho,\sigma) \in [0,1] \times [0,1] \times [0,1]$ with $\delta+\rho+\sigma \le 2$,

$\mu^{[\delta]}_{D(\hat{V}_1+\hat{V}_2)+D(\hat{V}_1\cap\hat{V}_2)} = \mu^{[\delta]}_{D(\hat{V}_1)} + \mu^{[\delta]}_{D(\hat{V}_2)}$, $\gamma^{[\delta]}_{D(\hat{V}_1+\hat{V}_2)+D(\hat{V}_1\cap\hat{V}_2)} = \gamma^{[\delta]}_{D(\hat{V}_1)} + \gamma^{[\delta]}_{D(\hat{V}_2)}$ and

$\varsigma^{[\rho]}_{D(\hat{V}_1+\hat{V}_2)+D(\hat{V}_1\cap\hat{V}_2)} = \varsigma^{[\rho]}_{D(\hat{V}_1)} + \varsigma^{[\rho]}_{D(\hat{V}_2)}$.

Thus, $D(\widehat{V}_1 + \widehat{V}_2) + D(\widehat{V}_1 \cap \widehat{V}_2) = D(\widehat{V}_1) + D\ (\widehat{V}_2)$.

4. CONCLUSIONS

In this article, the idea of neutrosophic dimension in a neutrosophic vector space is discussed. This idea can be extended by interpreting some examples which will be appended in future work.

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Comprehensive Survey of Recent Applications of Q-Neutrosophic Soft Set in Medical Diagnosis System

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ABSTRACT

Neutrosophic Soft Set (NSS) is one of the potential mathematical model for handling parametric uncertainties in dynamic environment. Q-NSS is extended version of NSS which incorporates the features of both NSS and Q-fuzzy set in handling uncertainty. Nowadays medical diagnosis system is prone to varieties of uncertainty in terms of uncertain disease symptoms, processing logic, and even uncertain clinical decisions. Handling the uncertainties is important before arriving at meaningful inferences. Hence in this chapter a comprehensive survey is carried out towards Q-NSS in all possible dimensions of medical diagnosis system. The survey highlights all possible mathematical frameworks used for medical diagnosis along with their limitations which include fuzzy logic, evidential reasoning, and quantum & machine learning decisions. The main focus of the paper is to perform early diagnosis of diseases, decision making under uncertainty, solutions for multi-attribute decision making problems, arriving at best decisions from several alternatives, and many more. A comparative analysis of Q-NSS is carried out with other mathematical frameworks like Neutrosophic Soft Set (NSS), and Q-Fuzzy set. It is inferred that the performance of Q-NSS is satisfactory towards performance metrics like error rate, throughput, latency, and resource utilization.

KEYWORDS**: Uncertainty, neutrosophic set, neutrosophic soft set, medical diagnosis.**

1. INTRODUCTION

Neutrosophic Soft Set (NSS) is a form of mathematical model that is used to handle parameter uncertainties by making use of three different types of membership functions. The membership functions considered are truth membership, false membership function, and indeterminacy membership function. In many critical real-time applications such as military, medical science, astrology, and so on, the incomplete input information is handled efficiently using NSS theory (Evanzalin et al., 2020). The extended version of NSS is Q-NSS which is a hybrid form of NSS it preserves the characteristics of both NSS and Q-fuzzy set. The characteristics of NSS is useful in handing the information uncertainty and similarly, characteristics of Q-fuzzy set is useful in handling the information which is in a two-dimensional format. The Q-NSS extends support for numerous operators which include union, intersection, OR, and AND operations. The mathematical definition of Q-NSS is as follows: Consider U as a universal set, the Q is taken as a nonempty set. Suppose $\mu' Q - NSS(U)$ is the set composed of multiple Q-NSSs on the universal set U over the pair (Γ_Q, A). Where $\Gamma_Q = A \rightarrow \mu' Q - NSS(U)$, such that the $\Gamma_Q(e) = \phi$, provided $e \notin A$ (Abu Qamar et al., 2019; Dalkılıç & Demirtaş, 2023; Qamar et al.,2020). The comparison between NSS, Q-fuzzy set, and Q- NSS is shown in Table 1 (Uluçay, 2021; Abuqamar, & Abd Ghafur Ahmad, 2022).

Table 1: Comparison between Neutrosophic soft set, Q-fuzzy set, and Q- Neutrosophic soft set

Sl. No	Neutrosophic soft set	Q-fuzzy set	Q- Neutrosophic soft set
1	Applied to universe of discourse domain	Applied to universe of discourse domain	Applied to universe of discourse domain
2	Co-domain of application is $[0,1]^3$	Co-domain of application is [0,1]	Co-domain of application is $[0,1]^3$
3	Truth membership function is present	Truth membership function is present	Truth membership function is present
4	False membership function is present	False membership function is not absent	False membership function is present
5	Inderminacy membership function is present	Inderminacy membership function is absent	Inderminacy membership function is present
6	Q-function is absent	Q-function is present	Q-function is present
7	Able to handle uncertainty in the computing domain	Unable to handle uncertainty in the computing domain	Able to handle uncertainty in the computing domian
8	Unable to handle information in two-dimensional format	Able to handle information in two-dimensional format	Able to handle information in two-dimensional format

Q-NSS is used to in a variety of applications which include game theory, measurement theory, logical rules and relationships representation, economics, medical diagnosis, agriculture, transportation, analysis of food grain items, pattern recognition, industrial automation, share market prediction, and so on. One of the promising application areas of Q-NSS is medical diagnosis, where the Q-NSS can handle uncertainty in every stage of diagnosis which includes patient observation, data preparation, data categorization, and data planning.

2. *PRELIMINARIES*

2.1. *Fuzzy Logic*

Fuzzy logic is being applied in day-to-day life. It is being used in a variety of applications which include aerospace, highway systems, air condition systems, underwater vehicles, transportation, radiology diagnosis, modeling neurological findings, crime investigation, and so on.

The literature review of works carried out for performing medical diagnosis using fuzzy logic is discussed below.

Bany Domi presents a fuzzy logic-based framework that is applied for medical diagnosis applications. Fuzzy logic is used in a variety of applications which include Asthma disease diagnosis, metabolic sickness determination, bacterial disease identification, finding irregularities in cell development, periodontal disease recognition, and so on. The process followed by the fuzzy system includes the following steps that are feeding in the crisp input, fuzzification, feeding of fuzzy inputs, evaluation of fuzzy rules, and generation of fuzzy outputs, defuzzification, and crisp output generation. The developed fuzzy logic framework is used to identify the risks of heart disease among individuals. The framework is tested by experts which represents an accuracy of 94%. The most important benefit offered by the framework is any individual patient can self-diagnose himself for heart disease without the need for any doctors (Khawla, 2021).

Bartczuk, and Rutkowska (2019) discussed the type-2 fuzzy decision tree approach for medical diagnosis. The decision tree is composed of several attribute values which are categorized using type-2 soft set theory. After experimenting, the results obtained are tested using three benchmark datasets namely, heart disease, breast cancer disease, and Pima Indian diabetes, which are found to be satisfactory. The well-known method for the development of a crisp decision tree is ID3 which is combined with fuzzy logic for the classification of medical diagnosis. The decision tree is developed by considering an array of decision rules in which every rule represents a leaf node of the tree. The reason for using a type-2 soft set over the attribute value is words can give different meanings in an expert system. So, associating an expert value with each value helps in arriving at exact inferences (Bartczuk, &Rutkowska, 2019).

Ejegwa (2019) described the application of an advanced Pythagorean fuzzy set in the medical diagnosis field. Uncertainty plays an important role in medical applications which influences on decision making process. Here Pythagorean fuzzy set which is one of the recently developed mathematical frameworks is applied for medical diagnosis which helps in quick decision making ability. Pythagorean fuzzy set is a generalized form of intuitionistic fuzzy set. The performance of Pythagorean fuzzy set overcomes the composite max-min-max relation of Pythagorean fuzzy set. It achieves sustainable performance while solving multiple criteria and multiple attribute, and pattern recognition decision making problems (Ejegwa, 2019).

2.2. Evidential Reasoning

A high-level view of inference drawn using evidential reasoning is shown in Figure 1. Evidential reasoning mechanism draws an automated inference from the evidence. Meaningful inferences are drawn from several factors like inherent factors, internal controls, analytical procedures, and tests of details. It is a generic form of multi-criteria-based decision-making approach that addresses the computation problem considering both qualitative and quantitative parameters considering randomness and ignorance-related parameters. The recent works carried out for decision making using evidential reasoning are given below.

Chang et al. (2021) presented evidential reasoning based on belief rule mining for diagnosis of medical applications. A set of multiple models consisting of belief rules with varying weights are initialized. During the mining of belief rules the reliability and weights of the models are determined and a customized set is generated. In this work thyroid disease dataset is considered and the correctness of medical decisions are determined. The belief rule mining approach is composed of several stages which include optimization of sub-model, calculation of sub-model weight, calculation of sub-model reliability, mining of belief rules, and validation of the results obtained. Initially, the beliefs generated by the belief rule mining approach are inaccurate but over a period of time accuracy improves (Chang et al., 2021).

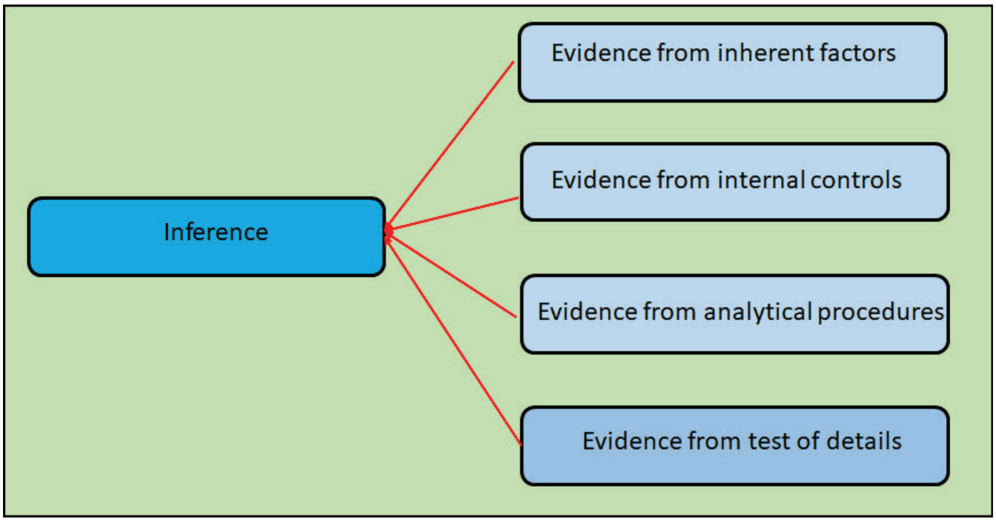

Figure 1: Inference from evidential reasoning

Liao et al. (2022) presented an evidential reasoning approach based on linguistic belief for medical disease diagnosis. The approach is tested over the lung cancer disease diagnosis. The traditional evidential reasoning approach is extended using a linguistic-belief system which allocates hesitancy degree-based weights for the experts. It is applied to problems involving multiple criteria and multiple expert decision-making problems. It works in several stages which include an invitation for Q-experts to evaluate the alternatives, calculation of hesitancy degree for alternatives, calculation of weight vector, combining the belief degree of several alternatives, and rank the alternatives to generate utility values (Liao et al., 2022).

Fu et al. (2021) discussed an evidential reasoning approach based on a driven drive approach driven by machine learning algorithms. The advantage of both evidential reasoning and machine learning is combined with the interpretability feature for multiple criteria-based decision-making applications. The hybrid approach is tested over the thyroid module of the tertiary hospital to achieve high-performance results. The proposed method works in several stages which include a collection of historical data, a comparison pf performance attained by machine learning algorithms, and exploratory decision-making based on evidential reasoning and machine learning algorithms. A set of machine learning algorithms is considered, out of which one best machine learning algorithm is chosen and is tied up with evidential reasoning to generate accurate exploratory solutions for multiple criteria decision-making problems (Fu et al., 2021)

2.3. Quantum and Machine Learning decisions

Quantum machine learning is one of the powerful approaches for decision making which is based on data constraints. The efficiency of quantum machine learning improves for episodic kind of tasks and decision-making games. A high-level view of decision-making using quantum enriched machine-learning approach is shown in Figure 2. The decisions are made using two approaches, namely, model-based reinforcement learning, and model-free reinforcement learning. In model-based reinforcement learning, the model represents the varying dynamic states of the environment. Here the agent is enabled with prior knowledge of the real world to develop an exact representation of the functional state of the computing environment. Whereas model-free reinforcement learning is exactly the opposite of model-based reinforcement learning, which does not use transition probability and reward function to solve computation-oriented problems. The recent works carried out for decision-making using quantum enriched machine learning approach are given below.

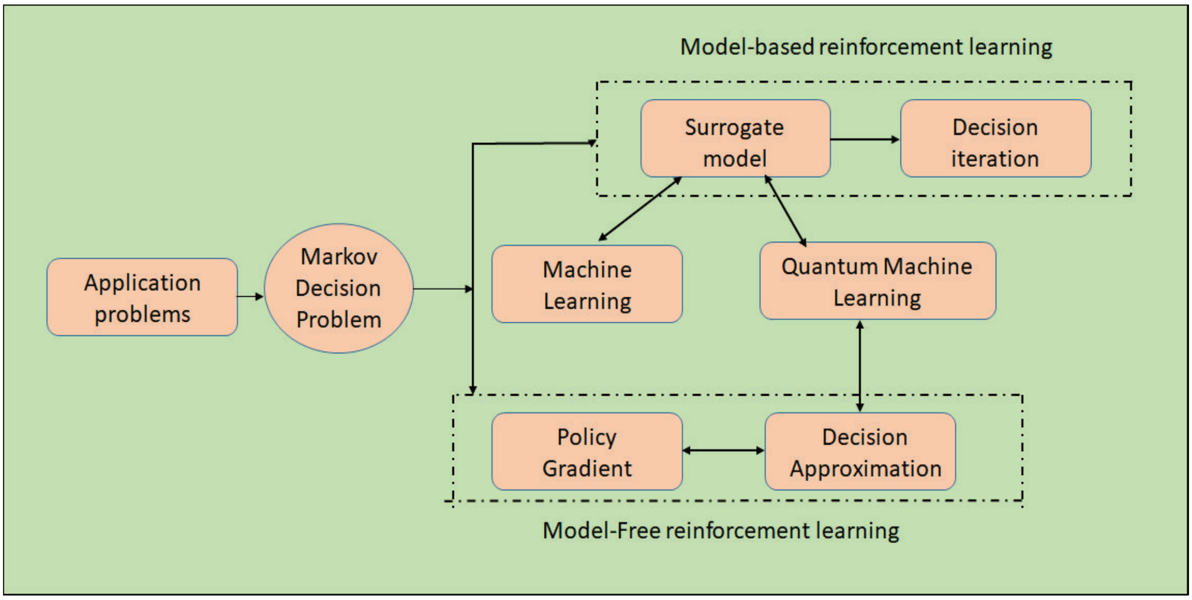

Figure 2: Decision making using quantum enriched machine learning approach

Njafa and Engo (2018) discussed the application of quantum mechanics for medical diagnosis. Quantum-enhanced associative memory is useful for untrained medical staff to identify dengue, malaria, and many more which exhibit similar kinds of signs and symptoms. The associative memory can classify between single infection and poly infection. A hybrid model is designed that combines two algorithms linear quantum retrieving algorithm and non-linear quantum search algorithm which perform precise medical diagnosis. The user interface is very much friendly and the cost of operation is less (Njafa & Engo, 2018)

Solenov et al. (2018) explained the potential features of quantum computing and machine learning which enhances the approach of clinical research and medical practice followed in modern days. The computational power of quantum computing combined with complexity feature of machine learning helps in delivering on time results in real world. Because of the availability of huge amount of data models enhanced with the quantum computing power, the medical expert is able to determine the therapy suitable for any individual patients. The treatment plan is updated to determine treatment response by considering various characteristics of patient including genetic, age, race, gender, and so on (Solenov et al., 2018).

Kumar et al. (2021) detected the chances of heart failure among adolescents using machine learning enhanced with quantum computing technology. The features related to heart failure is normalized by combining the algorithms min-max, scalar, and pipelining techniques. The comparison is performed between quantum random forest, quantum K-Nearest neighbor, quantum decision tree, and quantum random forest. The performance of the above quantum-enriched machine learning algorithms is found to be better compared to traditional machine learning algorithms. The execution time encountered between quantum-enriched machine learning algorithms is very much less, 150 ms (Kumar et al. (2021)

However, the existing works dealing with the application of fuzzy sets, rough sets, evidential reasoning, quantum decisions, and machine learning have several limitations in decision-making for medical diagnosis problems. The limitations observed with the use of fuzzy sets are, namely, the loss of valuable information from the available dataset, compromised accuracy of the system, the reasoning is not precise, and many more. Limitations with the use of evidential reasoning are, namely, rule-based decision-making has practical implementation limitations, decision-making becomes difficult under uncertainties, and many more. Limitations while making quantum decisions are: quantum systems are highly sensitive to noise and errors, the quality of computation degrades over a period of time, the error correction process is tedious, etc. Similarly, limitations of machine learning are: high chances of errors in results interpretation, lack of trust over the inference drawn, complex and difficult interpretation, and so on.

3. MAIN FOCUS OF THE ARTICLE

The main focus of the article is the design and development of Q-NSS framework for medical diagnosis application. The medical diagnosis domain is composed of four important components planning phase, patient observation and measurement, data interpretation, and data categorization. The decision-making under uncertain medical diagnosis problems becomes easier with the use of Q-NSS framework, as it involves four important components that are Q-function, false-function, truth-function, and indeterminacy-function. A high-level view of Q-NSS and its application in medical diagnosis is shown in Figure 3.

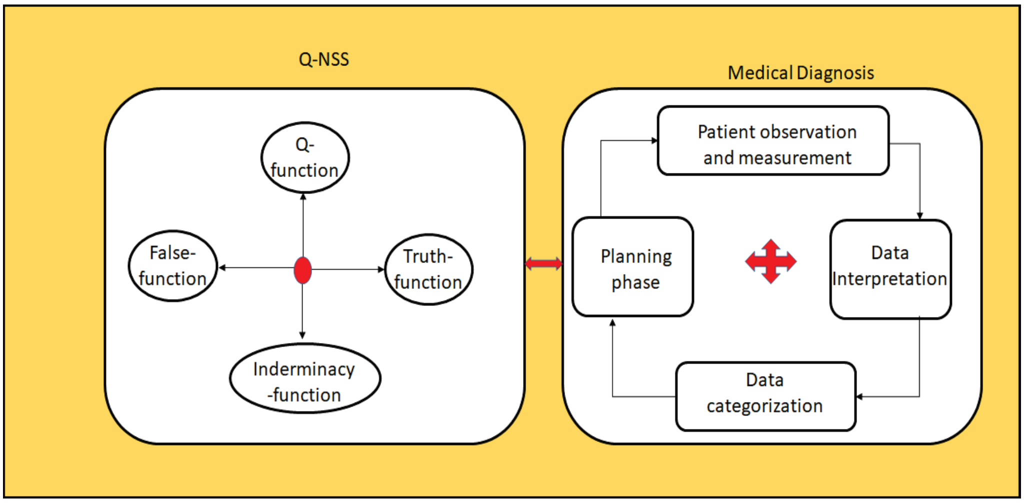

Figure 3: High level view of Q-NSS and its application in medical diagnosis

3.1. Blurred and Hazed Information: Three different Perspectives of Human Disease

Many times, the decisions taken by clinical experts fail as they fail to handle uncertainty caused because of the blurred and hazed information associated with patient records. The hazed information in the medical system is broadly classified into five types. They are disciplinary, ontological, conceptual, epistemic, and vagueness. The probable reasons for the haziness of medical information are medical systems' lacks of precise boundaries, inability to manage indistinct phenomena, availability of uncertain knowledge about the diseases, and a wide variety of fact-value interactions between the patient and doctor. The blurred and hazed information is obtained because of the three perspectives of the human disease diagnosis process which is shown in Figure 2. It is observed from Figure 2 that the symptoms exhibited by illness, disease, and sickness are overlapping in nature. This creates lot of ambiguity while processing the patient information and arriving at particular clinical decisions.

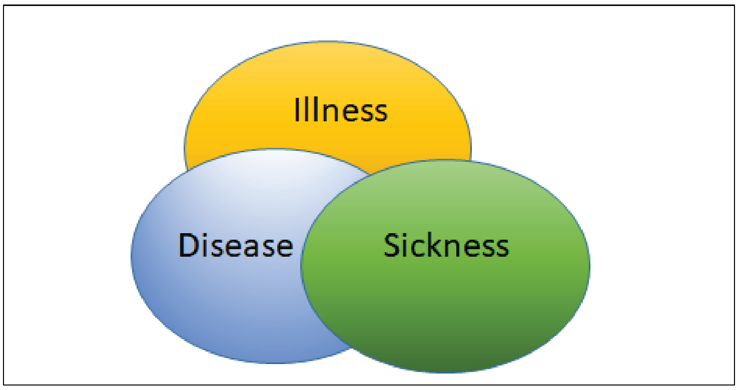

Figure 2: Three different perspectives of human disease

Comparatively, the overlapping between illness, disease, and sickness is high for chronic diseases and critical diseases like diabetes, heart disease, stroke, cancer, variation in blood pressure, asthma, and many more. Whereas the overlapping between illness, disease, and sickness is low for non-chronic diseases like fever, diarrhea, headache, acute illness, gastric, and many more. The inability to handle the overlapping characteristics of three different perspectives of human disease leads to wrong clinical decisions.

3.2. Uncertainty in Medical Diagnosis System

Medical diagnosis systems are inherently prone to a variety of uncertainties in the field of medicine. There are several sources of uncertainties which include incompleteness in the voice of medicine suggested, ambiguity in the symptoms conveyed by the patients, inability to arrive at the best decision that works well for the patient, and complexity that arises from collaborative communication between multiple clinical hospitals. Out of all sources of uncertainty, uncertainty that arises from patient symptoms is the most common. In most of the cases the patient exhibit symptoms which cannot be differentiated from one another concerning time. Because of undifferentiated symptoms, doctors/clinicians find it difficult to precisely identify the disease and give suggest proper medicine. The uncertainty in diagnosis is associated with lot of other diagnostic variations which include unnecessary hospitalization of patients, increased treatment cost, conflict between patient and clinician, overutilization of healthcare resources, excessive contribution to generation of diagnostic errors, and many more. According to a recent survey, it is predicted that one among twenty patients suffers from diagnostic error which causes fatal consequences. If the diagnostic uncertainties are handled with less care it results in significant effects on both the diagnosis system and outcomes generated by the patients. Aggregating multiple QNSSs is found to be one of the promising mathematical frameworks to arrive at accurate decisions by handling a variety of uncertainties (Alizadehsani et al., 2021). A pictorial representation of decision-making framework using QNSS over a patient exhibiting uncertain symptoms is shown in Figure 3.

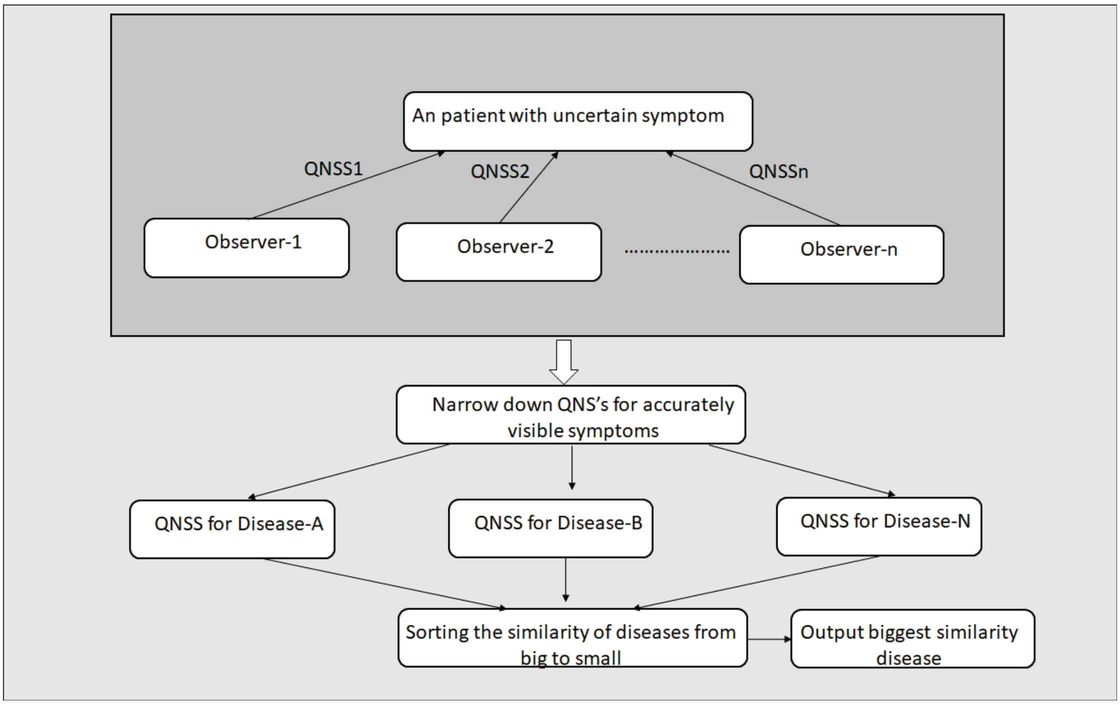

Figure 3: Decision-making framework using QNSS

Example: One of the useful applications of QNSS in the medical system is decision-making under uncertain situations. Consider the situation in which a patient is exhibiting uncertain symptoms, then aggregation operation combined with the QNSS is very useful in arriving at exact decisions. N observers will be spooled across the patient and each observer will employ a QNSS then aggregation operation is employed over the set of QNSS outputs generated by the N observers. The QNSS towards all probable diseases from disease-A to disease-N is generated based on the similarity matching factor. Further, the diseases are sorted from bigger values to smaller values based on their similarity index. Finally, the disease with the biggest similarity is output.

3.3. Early Detection of Disease

For chronic diseases like cancer, HIV, tuberculosis, influenza, and heart disease, a special diagnosis is required to identify the disease in its early stages even though symptoms are present or absent. Early detection of disease offers several advantages in terms of early treatment and intervention, improving the quality of life of the patient, longer survival of patients, changing the treatment plans, preventing the spread of disease to various parts of the body, saving lives, preventing the complications in the disease, and many more. Hence there is a necessity to identify the disease in early stage and prevent it from propagation to further parts of the body.

Nowadays vast amount of medical data is available over the Internet for analysis purposes. Several machine learning algorithms are available in the literature which are extensively used for disease prediction. However, for properly assessing the available patient's data, early detection of disease is possible. Many mathematical frameworks like probability theory, rough sets, fuzzy sets, and soft sets are used to deal with parametric uncertainty. However decision-making system based on Q-NSS aids in processing huge amounts of available patient data for proper diagnosis and prediction of disease. As the Q-NSS offers many advantages in terms of denoising the gathered information, proper segmentation of the large volume of preprocessed information, and precise classification of segments (Abbosh et al., 2020).

3.4. Solving Multi-Attribute Decision-Making Problems

Developing an ideal decision-making system for the medical field in real-life scenarios is very challenging in nature. First, the complex medical form is examined carefully, and all characteristics in terms of uncertainty, conflicting objectives set, inappropriate perspectives, and varying interests. The multi-attribute decision process is composed of several factors that are different scenarios, criteria, actions, and alternatives. These factors are interdependent hence they need to be handled with care for choosing the best alternative among the available alternatives of treatments.

The Q-NSS uses truth membership, falsity membership, and Indeterminacy membership functions combined with the Q-function to precisely evaluate the group of alternative treatments available for the disease. Then select the best treatment that satisfies the requirement exhibited by the multiple attributes of disease associated with the patient. The application of Q-NSS aids in proper treatment selection because of several advantages which include prioritizing the decision attributes, establishing the tradeoff between the conflicting attributes, performing proper decision analysis, choosing the appropriate utility function for parameter selection, analysis of the applicability of various solutions, and many more (Ullah t al., 2020)

3.5. Prediction of best treatment using Q-NSS

The Q-NSS is useful in choosing the best treatment action to improve the quality of treatment and improve the life expectancy of the patient. The multiple valued Q-NSS are capable enough of precisely extracting the inherent rules and useful patterns from the historical data which increases the prediction accuracy of the model. The useful information is not lost even when a fluctuating pattern appears in the time series model. Even when different neutrosophic sets exhibit the same values, a similarity measure is applied over the sets using different distance functions to arrive at a meaningful conclusion. The characteristic function of Q-NSS with hyper compositional structures expands Newton's mechanics with a neutrosophic set to choose the best course of action among the available set of actions (Jamshidi, 2020).

3.6. Performance analysis

Three different mathematical frameworks are Neutrosophic Soft Set (NSS), Q-Fuzzy Set (QFS), and Q-Neutrosophic Soft Set (QNSS) for medical diagnosis purposes. Table 2 provides a comparison of performance achieved in handling medical diagnosis uncertainty. The performance of QNSS overtakes other two popular mathematical frameworks that are QFS and NSS.

Table2: Comparison of performance achieved by three potential mathematical frameworks i.e. Neutrosophic soft set, Q-fuzzy set, and Q-Neutrosophic soft set.

Mathematical Frameworks	**Error rate**	**Throughput**	**Latency**	**Resource utilization**
NSS	High	Medium	High	Medium
QFS	High	Low	Medium	Medium
QNSS	Less	High	Low	High

4. *FUTURE RESEARCH DIRECTIONS*

In future work, all probable sources causing uncertainty in high computing domains like healthcare and the military will be discussed. Further in-depth analysis of all probable applications of QNSS to other application areas like transportation, education, academics, and entertainment will be carried out. Mathematical modeling of QNSS is performed by considering various performance metrics like latency, delay, jitter, and throughput.

5. *CONCLUSION*

This chapter considers the medical diagnosis system as one of the uncertainty-prone applications. Q-NSS is considered as a potential mathematical framework to handle the uncertainty in the medical diagnosis domain. The sources of uncertainties are identified and useful applications of Q-NSS like early disease detection, multi-attribute decision-making, and handling hazed information are discussed in detail.

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Study on Neutrosophic Non- Preemtive Prority Queue with Uneven Service Rate

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ABSTRACT

This chapter presents a practical method for evaluating the performance measures of non-preemptive neutrosophic priority queues with uneven services, labeled as NM/NM/1. This system comprises a solitary server, where both arrival and service rates are expressed using a single-valued trapezoidal neutrosophic number (SVTNN). The queueing model involves exponentially distributed service times, arrivals following a Poisson process, and the presence of only one server. To simplify the neutrosophic queueing model into a more straightforward form, the (α, β, γ)-cut approach along with Zadeh's extension principle are employed, and the results are presented. Moreover, a concrete example is offered to elucidate the analytical methodology established within this study.

KEYWORDS: **Neutrosophic set, single valued trapezoidal neutrosophic number, on-preemptive priority queue with uneven services, queueing models, arrival rate, service rate.**

1. INTRODUCTION

Fundamental queuing systems consist of orderly queues where the sequence of waiting and the rates of client arrival are carefully managed. However, in real-world circumstances, the majority of queueing models involve priority discipline since the most important activity must be given preference. The usage of priority queueing models is beneficial in many different contexts. In priority queues, clients receive service according to the priority of their requests. Customers with the highest priorities receive service first, while those with lower priorities receive service with less urgency. Priority queues are used in communication, and engineering to examine networks with varying levels of service quality.

Preemptive priority and non-preemptive priority are both common types of priority control. Consider a queueing system with two types of customers: when a first-class client arrives at the server and discovers that the server is serving a second -class customer, he squeezes the customer-in-service out and obtains service at once. Customers belonging to the same class follow the FCFS discipline at the same time; this method is known as preemptive priority queueing. If a first -class arriving customer discovers that the server is serving a second-class customer, he should wait until the customer-in -service finishes its service before beginning to receive service; customers of the same kind obey the FCFS discipline; this mechanism is known as non-preemptive priority queueing.

The inception of queueing theory can be dated back to the early 1900s through the examination of the Copenhagen telephone exchange by Agner Kraup Erlang, a Danish engineer, statistician, and mathematician. Erlang's thorough investigations into wait times in automated telephone services and his suggestions for enhancing network efficiency gained widespread acceptance among telephone companies. By 1963, his work had led to the exploration of preemptive priority queues involving K-class clients, as well as preemptive repeat and preemptive resume techniques.

The concept of fuzzy sets (Zadeh, 1965) which makes an element belongs to the set partially using the membership function that takes the value in the range [0,1]. The applications of M/M/c model are in decision making for reducing the waiting time for the customers in the queue (Zadeh, 1965).

Atanassov introduced the concept of intuitionistic fuzzy sets in 1986, which expands upon Zadeh's fuzzy set notation. In intuitionistic fuzzy sets, elements are characterized by degrees of both membership and non-membership.

The notion of neutrosophic probability, set, and logic was pioneered, presenting a broader framework beyond fuzzy logic and intuitionistic fuzzy logic, known as neutrosophic logic. When the parameters of a queueing system are represented by neutrosophic numbers, it qualifies as a neutrosophic queue (Smarandache, 1998).

Pardo and De La Fuente (2007) explored the optimization of a priority-discipline queueing model utilizing fuzzy set theory, incorporating both preemptive and non-preemptive priority systems. Additionally, Rashad and Mohamed (2021) conducted a case study investigating neutrosophic theory and its utilization across different queueing models.

In their study, Parimala and Palaniammal (2014) concentrated on the single-server delayed vacation aspect of the M/M (a, b)/1 queueing system, specifically examining the switchover state. They derived steady-state solutions and analyzed the system's characteristics, providing numerical illustrations for various parameter values.

Smarandache (2016) provided a critical examination of neutrosophic numbers, where he introduced the methodologies for subtracting and dividing neutrosophic single-valued numbers. Furthermore, he elucidated the constraints associated with these operations for neutrosophic single-valued numbers, along with those for neutrosophic single-valued over numbers, under numbers, and off numbers.

Sumathi and Antony Crispin Sweety (2019) introduced a novel method for handling differential equations using trapezoidal neutrosophic numbers. Neutrosophic Little's formulas played a crucial role in addressing queueing system challenges within a neutrosophic framework, as observed in the Erlang service queueing model with neutrosophic parameters (Zeina, 2020b).

Zeina (2020a) presented a neutrosophic event-based queueing model. An interpretation of a non-preemptive priority queueing system in a fuzzy environment with asymmetrical service rates was presented by (Karupothu et al., 2021). Heba and Mohame (2021)] examined the performance metrics of the neutrosophic NM/NM/1, NM/NM/s, and NM/NM/1/b queueing systems (Zeina, 2020c).

In their research, Zeina and Hatip (2021) put forth an extensive characterization of neutrosophic random variables, exploring their characteristics and applications across various fields like quality control, stochastic modeling, reliability theory, queueing theory, decision-making, and electrical engineering, prioritization mechanisms determine the sequence in which tasks or requests are handled. They also advanced the concept of linguistic single-valued neutrosophic M/M/1 queues, In this context, the rates of arrival and departure are represented using single-valued neutrosophic numbers denoted by $\tilde{A} = (T, I, F)$, with T indicating truth, I representing indeterminacy, and F signifying falsity.

Aarthi et al. (2022) undertook a research endeavor that involved evaluating the efficiency of non-preemptive priority queueing systems by examining both fuzzy queueing and intuitionistic fuzzy queuing models across a range of service rates. In parallel, Suvitha et al. (2023) delved into exploring neutrosophic priority discipline within queueing models.

2. *PRELIMINARIES*

Definition 1

A neutrosophic set (Smarandache, 1998) N is given as $N = \{r, (TA(r), IA(r), FA(r)) / r \in r\}$ where $TA(r), IA(r), FA(r): r \to]0^-, 1^+[$ are the degree of truth value, indeterminacy value and falsity value such that $0^- \leq \sup TA(r) + \sup IA(r) + \sup FA(r) \leq 3^+$

Definition 2

A single valued neutrosophic set (SVNS) (Wang et al., 2010) N in r is stated as $N = \{r, (T_A(r), I_A(r), F_A(r)) / r \in r\}$, where, $T_A(r), I_A(r), F_A(r) \in [0,1]$ and $0 \leq \sup T_A(r) + \sup I_A(r) + \sup F_A(r) \leq 3$.

Definition 3

A single valued trapezoidal neutrosophic number (SVTNN) (Sumathi et al., 2019) A is defined as

$$T_A(r) = \begin{cases} \dfrac{r^T - q_1^T}{q_2^T - q_1^T} & \text{for } q_1^T \leq r^T \leq q_2^T \\ 1 & \text{for } q_2^T \leq r^T \leq q_3^T \\ \dfrac{q_4^T - r^T}{q_4^T - q_3^T} & \text{for } q_3^T \leq r^T \leq q_4^T \\ 0 & \text{otherwise} \end{cases} \quad \text{where } q_1^T \leq q_2^T \leq q_3^T \leq q_4^T$$

$$I_A(r) = \begin{cases} \dfrac{q_2^I - r^I}{q_2^I - q_1^I} & \text{for } q_1^I \leq r^I \leq q_2^I \\ 0 & \text{for } q_2^I \leq r^I \leq q_3^I \\ \dfrac{r^I - q_3^I}{q_4^I - q_3^I} & \text{for } q_3^I \leq r^I \leq q_4^I \\ 1 & \text{otherwise} \end{cases} \quad \text{where } q_1^I \leq q_2^I \leq q_3^I \leq q_4^I.$$

$$F_A(r) = \begin{cases} \dfrac{q_2^F - r^F}{q_2^F - q_1^F} & \text{for } q_1^F \leq r^F \leq q_2^F \\ 0 & \text{for } q_2^F \leq r^F \leq q_3^F \\ \dfrac{r^F - q_3^F}{q_4^F - q_3^F} & \text{for } q_3^F \leq r^F \leq q_4^F \\ 1 & \text{otherwise} \end{cases} \quad \text{where } q_1^F \leq q_2^F \leq q_3^F \leq q_4^F.$$

Definition 4 (Sumathi et al., 2019)

(α, β, γ)-cut of a SVTNN is defined as follows:

$$B_{\alpha,\beta,\gamma} = [B_1(\alpha), B_2(\alpha)]; [B_1^{'}(\beta), B_2^{'}(\beta)]; [B_1^{''}(\gamma), B_2^{''}(\gamma)], 0 \leq \alpha + \beta + \gamma \leq 3$$

$$[B_1(\alpha), B_2(\alpha)] = [(q_1^T + \alpha(q_2^T - q_1^T)), (q_4^T + (q_4^T - q_3^T))],$$

$$[B_1^{'}(\beta), B_2^{'}(\beta)] = [(q_2^I - \beta(q_2^I - q_1^I)), (q_3^I + \beta(q_4^I - q_3^I))],$$

$$[B_1^{''}(\gamma), B_2^{''}(\gamma)] = [(q_2^F - \gamma(q_2^F - q_1^F)), (q_3^F + \gamma(q_4^F - q_3^F))].$$

Definition 5 (Sumathi et al., 2019)

Consider two closed and bounded real intervals denoted as $[c_1, c_2]$ and $[c_3, c_4]$. If $*$ represents addition, subtraction, multiplication or division, then $[c_1, c_2] * [c_3, c_4] = [\alpha, \beta]$. For division, it is presupposed that the divisor does not belong to the closed interval $[c_3, c_4]$. Utilizing fundamental operations, the development proceeds as follows:

i. $[c_1, c_2] + [c_3, c_4] = [c_1 + c_3, c_2 + c_4]$

ii. $[c_1, c_2] - [c_3, c_4] = [c_1 - c_4, c_2 - c_3]$

iii. $[c_1, c_2].[c_3, c_4] = [\min\{c_1c_3, c_1c_4, c_2c_3, c_2c_4\}, \max\{c_1c_3, c_1c_4, c_2c_3, c_2c_4\}]$

iv. $\frac{[c_1, c_2]}{[c_3, c_4]} = \left[\min\left\{\frac{c_1}{c_3}, \frac{c_1}{c_4}, \frac{c_2}{c_3}, \frac{c_2}{c_4}\right\}, \max\left\{\frac{c_1}{c_3}, \frac{c_1}{c_4}, \frac{c_2}{c_3}, \frac{c_2}{c_4}\right\}\right]$

3. NEUTROSOPHIC NON PREEMPTIVE PRIORITY QUEUEING MODEL

The following section discusses the examination of a single server queue with non-preemptive priority within a neutrosophic framework.

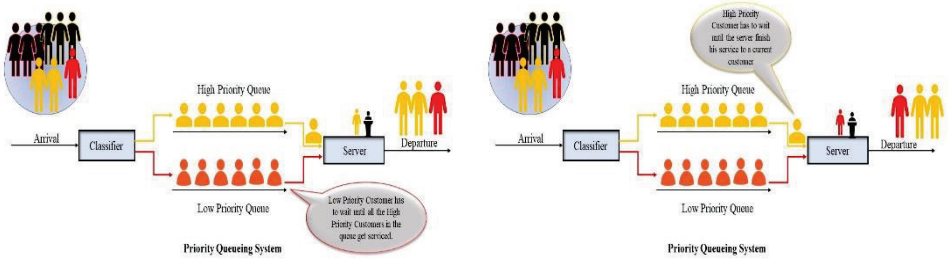
3.1 A standard M/M/1 queue with a non-preemptive priority scheme:

Take into account a queue with a single server, where non-preemptive priority is applied. In this scenario, two distinct client arrival streams are observed: one with higher priority and the other with lower priority. These streams adhere to separate Poisson processes characterized by parameters $\lambda_1$and λ_2, respectively. A single server tends to these clients, and service times follow an exponential distribution governed by rates μ_1 andμ_2. Clients with higher priority are granted immediate service precedence over others. The system's capacity is infinite, and a first-come, first-served principle is upheld within each priority group.

Several aspects of system performance measures include:

- The mean queue length for higher-priority tasks: $L_{q_1} = \frac{\lambda_1\left(\frac{\lambda_1}{\mu_1^2}+\frac{\lambda_2}{\mu_2^2}\right)}{(1-\rho_1)}$
- Mean queue length for lower priority: $L_{q_2} = \frac{\lambda_2\left(\frac{\lambda_1}{\mu_1^2}+\frac{\lambda_2}{\mu_2^2}\right)}{(1-\rho_1)(1-\rho)}$
- Mean waiting duration for the higher priority queue: $W_{q_1} = \frac{L_{q_1}}{\lambda_1}$
- Mean waiting duration for the lower priority queue: $W_{q_2} = \frac{L_{q_2}}{\lambda_2}$

where $\lambda = \lambda_1 + \lambda_2$ and traffic intensity $\rho_1 = \frac{\lambda_1}{\mu_1}, \rho_2 = \frac{\lambda_2}{\mu_2}, \rho = \rho_1 + \rho_2$



(a) Customers with higher priority being served (b) Customers with low priority being served

Figure 2.1: Priority queue structured as M/M/1 system

3.2 The construction of an NM/NM/1 queue model with non-preemptive priority and dynamic service rates

Consider a non -preemptive priority queueing system with a single server, operating under an NM/NM/1 configuration, where service times are uneven. The inter-arrival times for units with neutrosophic characteristics, denoted as $\tilde{A}_n$, n=1,2 as well as the service times $\tilde{S}_n$, n=1,2 for units with first and second priority ,are approximately determined and expressed as follows:

$$\tilde{A}_n = \left\{\left(a, T_{\tilde{A}_n}(a), I_{\tilde{A}_n}(a), F_{\tilde{A}_n}(a)\right) / a \in U\right\}; n = 1,2$$

$$\tilde{S}_n = \left\{\left(s, T_{\tilde{S}_n}(s), I_{\tilde{S}_n}(s), F_{\tilde{S}_n}(s)\right) / s \in V\right\}; n = 1,2$$

where U and V are the universal crisp sets of the neutrosophic inter arrival and neutrosophic service times and $\mu_{\tilde{A}_n}(a); n = 1,2$,$\mu_{\tilde{S}_n}(s); n = 1,2$ are the corresponding membership functions. The (α,β,γ)-cuts of $\tilde{A}_n$, n=1,2 and $\tilde{S}_n$, n=1,2 are

$$\tilde{A}_n(\alpha,\beta,\gamma) = \{a \in U/\, T_{\tilde{A}_n}(a) \geq \alpha, I_{\tilde{A}_n}(a) \leq \beta, F_{\tilde{A}_n}(a) \leq \gamma\}; n = 1,2$$

$$\tilde{S}_n(\alpha,\beta,\gamma) = \{(s \in V/T_{\tilde{S}_n}(s) \geq \alpha, I_{\tilde{S}_n}(s) \leq \beta, F_{\tilde{S}_n}(s) \leq \gamma)/\}; n = 1,2$$

where the $\tilde{A}_n(\alpha,\beta,\gamma)$ and $\tilde{S}_n(\alpha,\beta,\gamma)$ are the crisp subsets of U and V respectively. By employing (α,β,γ)-cuts, it's possible to represent neutrosophic interarrivals and service times through various confidence interval levels. Consequently neutrosophic queue can be simplified into a series of crisp sets ,each with unique (α,β,γ)-cuts.

$$\{\tilde{A}_n(\alpha,\beta,\gamma): 0<\alpha\leq 1, 0<\beta\leq 1, 0<\gamma\leq 1\} \text{ and}$$
$$\{\tilde{S}_n(\alpha,\beta,\gamma): 0<\alpha\leq 1, 0<\beta\leq 1, 0<\gamma\leq 1\}$$

In this proposal, a non-preemptive queueing model is introduced where both inter-arrival time $\tilde{A}_n$, n=1,2 and service times $\tilde{S}_n$, n=1,2 are depicted as SVTNN. Confidence levels for these parameters are denoted by of $\tilde{A}_n$ and $\tilde{S}_n$ by $[l_{\tilde{A}_n(\alpha,\beta,\gamma)}, u_{\tilde{A}_n(\alpha,\beta,\gamma)}]$ and $[l_{\tilde{S}_n(\alpha,\beta,\gamma)}, u_{\tilde{S}_n(\alpha,\beta,\gamma)}]$.

The performance metric, symbolized as $p(\tilde{A}_n, \tilde{S}_n)$ can be articulated employing Zadeh's extension principle where in membership functions for truth, indeterminacy and the falsity of $p(\tilde{A}_n, \tilde{S}_n)$ and are defined as follows.

$$T_{p(\tilde{A}_n,\tilde{S}_n)}(x) = sup\{min_{b\in X,b'\in Y}\,(\mu_{\tilde{A}_n(b)}, T_{\tilde{S}_n(b')}): x = p(b,b')\}$$

and

$$I_{p(\tilde{A}_n,\tilde{S}_n)}(x) = inf\{min_{b\in X,b'\in Y}\,(\mu_{\tilde{A}_n(b)}, T_{\tilde{S}_n(b')}): x = p(b,b')\}$$

and

$$I_{p(\tilde{A}_n,\tilde{S}_n)}(x) = inf\{min_{b\in X,b'\in Y}\,(\mu_{\tilde{A}_n(b)}, T_{\tilde{S}_n(b')}): x = p(b,b')\}$$

We can define the lower and upper boundaries of the (α,β,γ)-cuts of $\tilde{A}_n, \tilde{S}_n$ as follows:

$$l_{p(\alpha,\beta,\gamma)} = \min p(b,b') \text{ such that } l_{\tilde{A}_n(\alpha,\beta,\gamma)} \leq b \leq u_{\tilde{A}_n(\alpha,\beta,\gamma)}, l_{\tilde{S}_n(\alpha,\beta,\gamma)} \leq b' \leq u_{\tilde{S}_n(\alpha,\beta,\gamma)} \quad (1)$$

$$u_{p(\alpha,\beta,\gamma)} = \max p(b,b') \text{ such that } l_{\tilde{A}_n(\alpha,\beta,\gamma)} \leq b \leq u_{\tilde{A}_n(\alpha,\beta,\gamma)}, l_{\tilde{S}_n(\alpha,\beta,\gamma)} \leq b' \leq u_{\tilde{S}_n(\alpha,\beta,\gamma)} \quad (2)$$

where $b \in \tilde{A}_n(\alpha,\beta,\gamma)$ and $b' \in \tilde{S}_n(\alpha,\beta,\gamma)$.

If both $l_{p(\alpha,\beta,\gamma)}$ and $u_{p(\alpha,\beta,\gamma)}$ are reversible with respect to (α,β,γ) then the left and right shape functions are $L_T(x) = (l_{p(\alpha,\beta,\gamma)})^{-1}$ and $R_T(x) = (u_{p(\alpha,\beta,\gamma)})^{-1}$ respectively ,the resulting in the the truth membership function (z) as $\mu_{p(\tilde{A}_n,\tilde{S}_n)}(x)$ is expressed as

$$T_{p(\tilde{A}_n,\tilde{S}_n)}(x) = \begin{cases} L_T(x); & x_1^T \leq x \leq x_2^T \\ R_T(x); & x_3^T \leq x \leq x_4^T \\ 0; & otherwise \end{cases}$$

where $x_1^T \le x \le x_4^T$ and $L_T(x_1^T) = R_T(x_4^T) = 0$ for the SVTNN.

$$I_{p(\tilde{A}_n,\tilde{S}_n)}(x) = \begin{cases} L_I(x); & x_1^I \le x \le x_2^I \\ R_I(x); & x_3^I \le x \le x_4^I \\ 0; & otherwise \end{cases}$$

where $x_1^I \le x \le x_4^I$ and $L_I(x_1^I) = R_I(x_4^I) = 0$ for the SVTNN.

$$F_{p(\tilde{A}_n,\tilde{S}_n)}(x) = \begin{cases} L_F(x); & x_1^F \le x \le x_2^F \\ R_F(x); & x_3^F \le x \le x_4^F \\ 0; & otherwise \end{cases}$$

where $x_1^F \le x \le x_4^F$ and $L_F(x_1^F) = R_F(x_4^F) = 0$ for the SVTNN.

By employing the (α, β, γ)-cut approach, the suggested NM/NM/1 priority queue can be simplified to the conventional M/M/1 queue with non-preemptive priority.

4. NUMERICAL EXAMPLE

In this section, a practical example is presented to clarify the introduced NM/NM/1 queueing concept with non-preemptive priority. The arrival rates and service rates of first and second priority are denoted by SVTNN.

$\tilde{A}_1 = [(3,4,5,6), (2,5,8,11), (2,4,6,8)]$,

$\tilde{A}_2 = [(4,5,6,7), (3,4,5,6), (5,6,7,8)]$,

$\tilde{S}_1 = [(16,17,18,19), (18,20,22,24), (17,19,21,23)]$,

$\tilde{S}_2 = [(17,18,19,20), (16,17,18,19), (18,19,20,21)]$ per hour respectively.

The (α, β, γ)-cut of $\tilde{A}_n, n = 1,2$ and $\tilde{S}_n, n = 1,2$ are

$\tilde{A}_1 = [(3 + \alpha, 6 - \alpha), (5 - 3\beta, 8 + 3\beta), (4 - 2\gamma, 6 + 2\gamma)]$,

$\tilde{A}_2 = [(4 + \alpha, 7 - \alpha), (4 - \beta, 5 + \beta), (6 - \gamma, 7 + \gamma)]$,

$\tilde{S}_1 = [(16 + \alpha, 19 - \alpha), (20 - 2\beta, 22 + 2\beta), (19 - 2\gamma, 21 + 2\gamma)]$,

$\tilde{S}_2 = [(17 + \alpha, 20 - \alpha), (17 - \beta, 18 + \beta), (19 - \gamma, 20 + \gamma)]$

The formulation of parametric programming problems to derive the membership functions $\bar{L}_{q_1}, \bar{L}_{q_2}, \bar{W}_{q_1}, \bar{W}_{q_2}$ is based on equations (1) and (2), and their computation is outlined below.

The performance metrices of

i. $\bar{L}_{q_1}$- The mean length of a higher priority queue.

ii. $\bar{L}_{q_2}$- The mean length of the queue for tasks with lower priority.

iii. $\overline{W}_{q_1}$- The average time spent waiting in the queue with higher priority

iv. $\overline{W}_{q_2}$- The mean waiting duration in the queue with lower priority is determined by the corresponding parametric programs.

These variations are solely distinguished by their objective functions and are outlined as follows:

$$l_{L_{q_1}}(\alpha) = min\left(\frac{e_1\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{f_1(f_1-e_1)}\right), u_{L_{q_2}}(\alpha) = max\left(\frac{e_1\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{f_1(f_1-e_1)}\right)$$

such that

$$\begin{gathered} 3+\alpha < e_1 < 6-\alpha \\ 4+\alpha < e_2 < 7-\alpha \\ 16+\alpha < f_1 < 19-\alpha \\ 17+\alpha < f_2 < 20-\alpha \end{gathered} \tag{3}$$

where $0 < \alpha \leq 1$. $l_{L_{q_1}}(\alpha)$ is found when e_1, e_2 approaches its lower bounds (l. b) and f_1, f_2 approaches its upper bound (u. b) and $u_{L_{q_1}}(\alpha)$ is found whene_1,e_2 approaches its upper bound (u. b) and f_1,f_2 approaches its lower bound (l. b).Therefore the optimal solution for (3) are

$$l_{L_{q_1}}(\alpha)=\frac{21156-4228\alpha-201\alpha^2+41\alpha^3+2\alpha^4}{11837440+5979648\alpha+1163232\alpha^2+115050\alpha^3+6198\alpha^4+174\alpha^5+2\alpha^6}$$ and

$$u_{L_{q_1}}(\alpha)=\frac{7932+4111\alpha+276\alpha^2-65\alpha^3+2\alpha^4}{43897600-16808160\alpha+2582784\alpha^2-206166\alpha^3+9078\alpha^4-210\alpha^5+2\alpha^6}$$

$$T_{\tilde{L}_{q_1}}(x) = \begin{cases} L_T(x); & \left[l_{L_{q_1}}(\alpha)\right]_{\alpha=0} \leq x \leq \left[l_{L_{q_1}}(\alpha)\right]_{\alpha=1} \\ R_T(x); & \left[u_{L_{q_1}}(\alpha)\right]_{\alpha=1} \leq x \leq \left[u_{L_{q_1}}(\alpha)\right]_{\alpha=0} \\ 0; & otherwise \end{cases}$$

which is estimated as

$$T_{\tilde{L}_{q_1}}(x) = \begin{cases} L_T(x); & 0.00018 \leq x \leq 0.00041 \\ R_T(x); & 0.00178 \leq x \leq 0.00087 \\ 0; & otherwise \end{cases}$$

$$l_{L_{q_1}}(\beta) = min\left(\frac{e_1\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{f_1(f_1-e_1)}\right), u_{L_{q_1}}(\beta) = max\left(\frac{e_1\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{f_1(f_1-e_1)}\right)$$

such that

$$\begin{gathered} 4-\beta < e_2 < 5+\beta \\ 5-3\beta < e_1 < 8+3\beta \end{gathered}$$

(4)

$$20 - 2\beta < f_1 < 22 + 2\beta$$

$$17 - \beta < f_2 < 18 + \beta$$

where $0 < \beta \leq 1$. $l_{L_{q_1}}(\beta)$ is found when e_1, e_2 approaches its lower bounds (l. b) and f_1, f_2 approaches its upper bound (u. b) and $u_{L_{q_1}}(\beta)$ is found when e_1,e_2 approaches its upper bound (u. b) and f_1,f_2 approaches its lower bound (l. b).Therefore the optimal solution for (4) are

$$l_{L_{q_1}}(\beta) = \frac{17780 - 15288\beta + 1897\beta^2 + 490\beta^3 + 21\beta^4}{58649184 + 39761568\beta + 10033496\beta^2 + 1258640\beta^3 + 84384\beta^4 + 2896\beta^5 + 40\beta^6} \text{ and}$$

$$u_{L_{q_1}}(\beta) = \frac{34496 + 17696\beta + 553\beta^2 - 406\beta^3 + 21\beta^4}{27744000 - 23147200\beta + 6735520\beta^2 - 949264\beta^3 + 70504\beta^4 - 2656\beta^5 + 40\beta^6}$$

The indeterminacy membership function is

$$I_{L_{q_1}}(x) = \begin{cases} L_I(x); & \left[l_{L_{q_1}}(\beta)\right]_{\beta=1} \leq x \leq \left[l_{L_{q_1}}(\beta)\right]_{\beta=0} \\ R_I(x); & \left[u_{L_{q_1}}(\beta)\right]_{\beta=0} \leq x \leq \left[u_{L_{q_1}}(\beta)\right]_{\beta=1} \\ 0; & otherwise \end{cases}$$

which is estimated as

$$I_{L_{q_1}}(x) = \begin{cases} L_I(x); & 0.00004 \leq x \leq 0.00030 \\ R_I(x); & 0.00124 \leq x \leq 0.00501 \\ 0; & otherwise \end{cases}$$

$$l_{L_{q_1}}(\gamma) = min\left(\frac{e_1\left(\frac{e_1}{f_1^2} + \frac{e_2}{f_2^2}\right)}{f_1(f_1 - e_1)}\right), u_{L_{q_1}}(\gamma) = max\left(\frac{e_1\left(\frac{e_1}{f_1^2} + \frac{e_2}{f_2^2}\right)}{f_1(f_1 - e_1)}\right)$$

such that

$$\begin{aligned} &4 - 2\gamma < e_1 < 6 + 2\gamma \\ &6 - \gamma < e_2 < 7 + \gamma \\ &19 - 2\gamma < f_1 < 21 + 2\gamma \\ &19 - \gamma < f_2 < 20 + \gamma \end{aligned} \tag{5}$$

where $0 < \gamma \leq 1$. $l_{L_{q_1}}(\gamma)$ is found when e_1, e_2 approaches its lower bounds (l. b) and f_1, f_2 approaches its upper bound (u. b) and $u_{L_{q_1}}(\gamma)$ is found when e_1,e_2 approaches its upper bound (u. b) and f_1,f_2 approaches its lower bound (l. b). Therefore the optimal solution for (5) are

$$l_{L_{q_1}}(\gamma) = \frac{16984 - 10800\gamma + 610\gamma^2 + 248\gamma^3 + 12\gamma^4}{62974800 + 39107880\gamma + 9385677\gamma^2 + 1134346\gamma^3 + 73420\gamma^4 + 2424\gamma^5 + 32\gamma^6} \text{ and}$$

$$u_{L_{q_1}}(\gamma)=\frac{28158+11324\gamma-62\gamma^2-200\gamma^3+12\gamma^4}{32189287-23457780\gamma+6399447\gamma^2-864234\gamma^3+61788\gamma^4-2232\gamma^5+32\gamma^6}$$

The Falsity membership function is

$$F_{\bar{L}_{q_1}}(x)=\begin{cases} L_F(x); & \left[l_{L_{q_1}}(\gamma)\right]_{\gamma=1} \le x \le \left[l_{L_{q_1}}(\gamma)\right]_{\gamma=0} \\ R_F(x); & \left[u_{L_{q_1}}(\gamma)\right]_{\gamma=0} \le x \le \left[u_{L_{q_1}}(\gamma)\right]_{\gamma=1} \\ 0; & otherwise \end{cases}$$

which is estimated as

$$F_{\bar{L}_{q_1}}(x)=\begin{cases} L_F(x); & 0.00006 \le x \le 0.00026 \\ R_F(x); & 0.00087 \le x \le 0.00273 \\ 0; & otherwise \end{cases}$$

For different values of $\alpha,\beta,\gamma \in [0,1]$, the mean length of queue for higher priority $\bar{L}_{q_1}$ is calculated and shown in table 4.1. Moreover, there is a graphical representation that illustrates the concepts of truth, indeterminacy, and falsity concerning the mean queue length of higher priority is shown in figure 4.2, 4.3 and 4.4.

Table 4.1 α,β,γ-cut for $\bar{L}_{q_1}$

α	$l_{L_{q_1}}(\alpha) \times 10^{-4}$	$u_{L_{q_1}}(\alpha) \times 10^{-3}$	β	$l_{L_{q_1}}(\beta) \times 10^{-5}$	$u_{L_{q_1}}(\beta) \times 10^{-3}$	γ	$l_{L_{q_1}}(\gamma) \times 10^{-5}$	$u_{L_q}(\gamma) \times 10^{-3}$
0	1.81	1.78	0	30.32	1.24	0	27.00	0.87
0.1	1.98	1.66	0.1	25.94	1.42	0.1	23.75	0.97
0.2	2.16	1.55	0.2	22.10	1.62	0.2	20.86	1.09
0.3	2.35	1.44	0.3	18.71	1.86	0.3	18.26	1.22
0.4	2.56	1.34	0.4	15.74	2.13	0.4	15.93	1.37
0.5	2.78	1.25	0.5	13.14	2.44	0.5	13.84	1.53
0.6	3.02	1.16	0.6	10.90	2.81	0.6	11.96	1.72
0.7	3.28	1.08	0.7	8.91	3.23	0.7	10.28	1.93
0.8	3.55	1.01	0.8	7.20	3.73	0.8	8.79	2.16
0.9	3.84	0.94	0.9	5.73	4.31	0.9	7.45	2.43
1	4.16	0.87	1	4.46	5.01	1	6.26	2.73

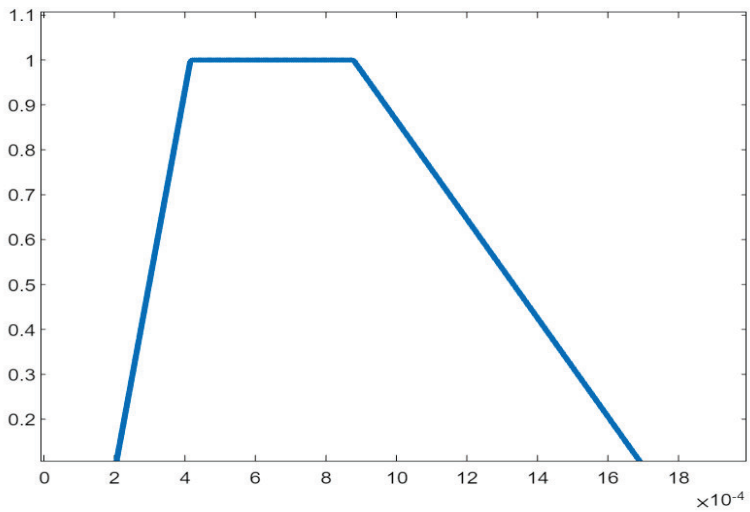


Figure 4.1 Truth value for $\bar{L}_{q_1}$

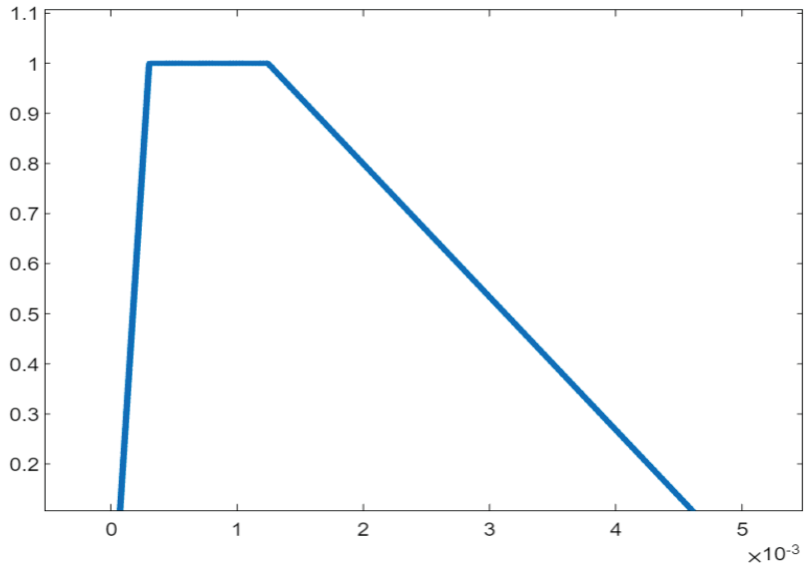


Figure 4.2 Indeterminacy value for $\bar{L}_{q_1}$

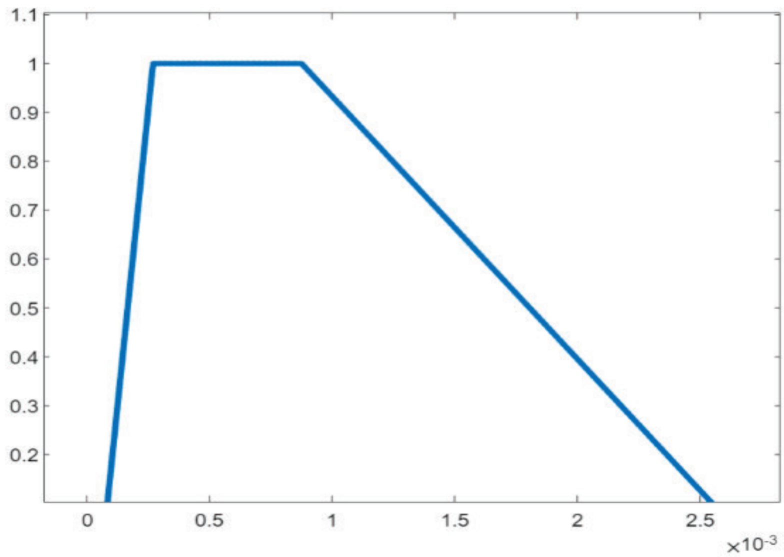


Figure 4.3 Falsity value for $\bar{L}_{q_1}$

The performance function of $\bar{L}_{q_2}$ of α is listed as follows.

$$l_{L_{q_2}}(\alpha) = min\left(\frac{e_2\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{\left(1-\frac{e_1}{f_1}\right)\left(1-\frac{e_1}{f_1}-\frac{e_2}{f_2}\right)}\right) \tag{6}$$

$$u_{L_{q_2}}(\alpha) = max\left(\frac{e_2\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{\left(1-\frac{e_1}{f_1}\right)\left(1-\frac{e_1}{f_1}-\frac{e_2}{f_2}\right)}\right) \tag{7}$$

Equation (6) and (7) with (3) give the following results:

$l_{L_{q_2}}(\alpha)=\frac{10576+4600\alpha+205\alpha^2-63\alpha^3+2\alpha^4}{78080-36384\alpha+5424\alpha^2-310\alpha^3+6\alpha^4}$ and

$u_{L_{q_2}}(\alpha)=\frac{24682+4345\alpha-254\alpha^2+39\alpha^3+2\alpha^4}{9860+11562\alpha+2958\alpha^2+238\alpha^3+6\alpha^4}$

$$T_{\bar{L}_{q_2}}(x)=\begin{cases} L_T(x); & 0.13545 \le x \le 0.32723 \\ R_T(x); & 2.50324 \le x \le 0.81725 \\ 0; & otherwise \end{cases}$$

The performance function of $\bar{L}_{q_2}$ of β is listed as follows.

$$l_{L_{q_2}}(\beta)=min\left(\frac{e_2\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{\left(1-\frac{e_1}{f_1}\right)\left(1-\frac{e_1}{f_1}-\frac{e_2}{f_2}\right)}\right) \quad (8)$$

$$u_{L_{q_2}}(\beta)=max\left(\frac{e_2\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{\left(1-\frac{e_1}{f_1}\right)\left(1-\frac{e_1}{f_1}-\frac{e_2}{f_2}\right)}\right) \quad (9)$$

Equation (8) and (9) with (4) give the following result:

$l_{L_{q_2}}(\beta)=\frac{14224-7252\beta+224\beta^2+147\beta^3+7\beta^4}{66708+60352\beta+16179\beta^2+1354\beta^3+35\beta^4}$ and

$u_{L_{q_2}}(\beta)=\frac{21560+7287\beta-175\beta^2-119\beta^3+7\beta^4}{21216-31916\beta+12327\beta^2-1214\beta^3+35\beta^4}$

$$I_{\bar{L}_{q_2}}(x)=\begin{cases} L_I(x); & 0.05082 \le x \le 0.21322 \\ R_I(x); & 1.01621 \le x \le 63.75 \\ 0; & otherwise \end{cases}$$

The performance function of $\bar{L}_{q_2}$ of γ is listed as follows.

$$l_{L_{q_2}}(\gamma)=min\left(\frac{e_2\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{\left(1-\frac{e_1}{f_1}\right)\left(1-\frac{e_1}{f_1}-\frac{e_2}{f_2}\right)}\right) \quad (10)$$

$$u_{L_{q_2}}(\gamma)=max\left(\frac{e_2\left(\frac{e_1}{f_1^2}+\frac{e_2}{f_2^2}\right)}{\left(1-\frac{e_1}{f_1}\right)\left(1-\frac{e_1}{f_1}-\frac{e_2}{f_2}\right)}\right) \quad (11)$$

Equation (10) and (11) with (5) give the following results:

$l_{L_{q_2}}(\gamma)=\frac{25476-7708\gamma-239\gamma^2+100\gamma^3+6\gamma^4}{72760+56789\gamma+13178\gamma^2+1006\gamma^3+24\gamma^4}$ and

$u_{L_{q_2}}(\gamma)=\frac{32851+6954\gamma-503\gamma^2-76\gamma^3+6\gamma^4}{28158-33364\gamma+10304\gamma^2-910\gamma^3+24\gamma^4}$

$$F_{\bar{L}_{q_2}}(x) = \begin{cases} L_F(x); & 0.12266 \leq x \leq 0.35013 \\ R_F(x); & 1.16666 \leq x \leq 9.31433 \\ 0; & otherwise \end{cases}$$

For varying values of $\alpha, \beta, \gamma \in [0,1]$, the average length of the queue for tasks with lower priority $\bar{L}_{q_2}$ is determined and shown in table 4.2. Additionally, a graphical depiction illustrating the concepts of truth, uncertainty, and falsehood concerning the mean queue length of lower priority is presented in figures 4.4, 4.5, and 4.6.

Table 4.2 $\boldsymbol{\alpha, \beta, \gamma}$-cut for $\bar{L}_{q_2}$

α	$l_{L_{q_2}}(\alpha)$	$u_{L_{q_2}}(\alpha)$	Β	$l_{L_{q_2}}(\beta)$	$u_{L_{q_2}}(\beta)$	γ	$l_{L_{q_2}}(\gamma)$	$u_{L_{q_2}}(\gamma)$
0	0.1354	2.5032	0	0.2132	1.0162	0	0.3501	1.1666
0.1	0.1481	2.2735	0.1	0.1851	1.2281	0.1	0.3144	1.3457
0.2	0.1619	2.0777	0.2	0.1609	1.5022	0.2	0.2826	1.5633
0.3	0.1769	1.9089	0.3	0.1398	1.8656	0.3	0.2543	1.8313
0.4	0.1933	1.7619	0.4	0.1215	2.3624	0.4	0.2289	2.1670
0.5	0.2110	1.6328	0.5	0.1055	3.0701	0.5	0.2063	2.5963
0.6	0.2304	1.5186	0.6	0.0915	4.1374	0.6	0.1859	3.1595
0.7	0.2515	1.4170	0.7	0.0793	5.8876	0.7	0.1676	3.9229
0.8	0.2745	1.3260	0.8	0.0685	9.1795	0.8	0.1510	5.0038
0.9	0.2997	1.2441	0.9	0.0591	17.2781	0.9	0.1361	6.6307
1	0.3272	1.1701	1	0.0508	63.75	1	0.1226	9.3143

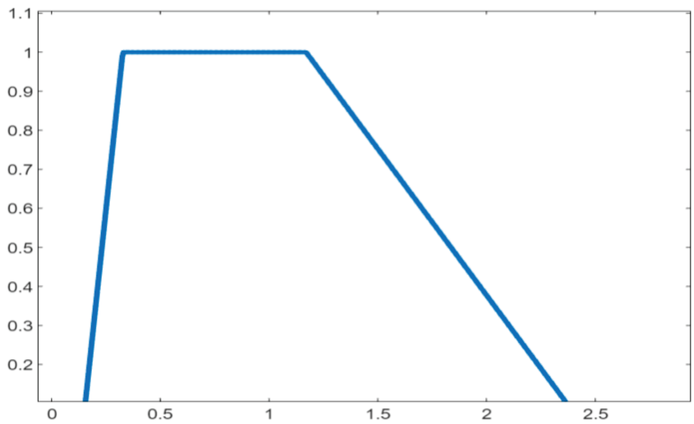


Figure 4.4 Truth value for $\bar{L}_{q_2}$

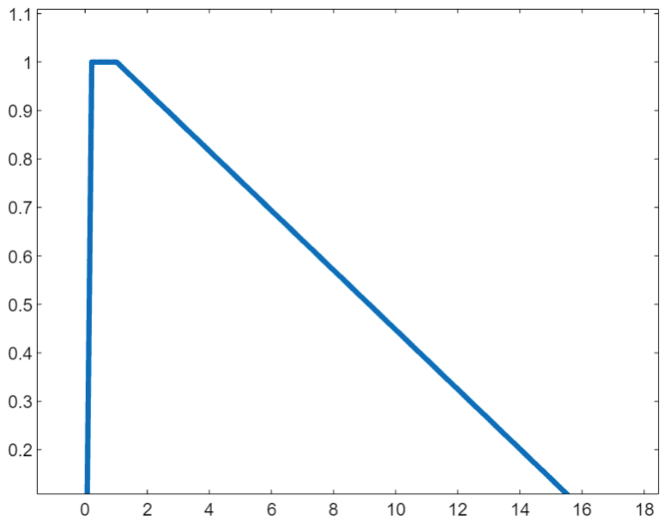


Figure 4.5 Indeterminacy value for $\bar{L}_{q_2}$

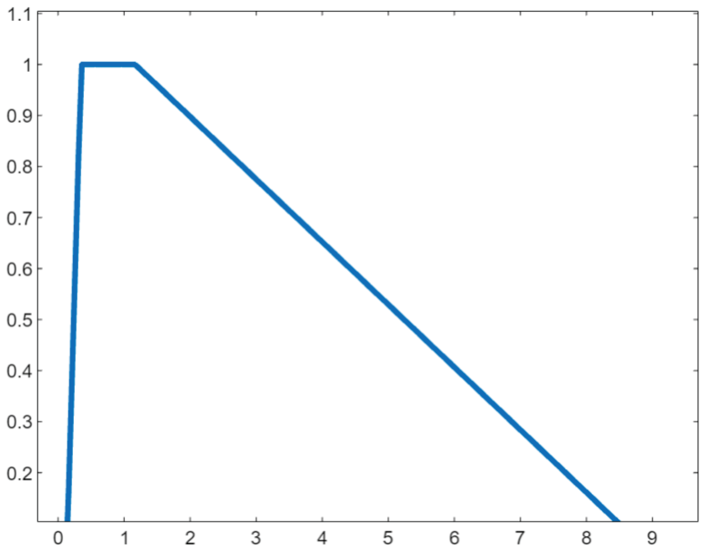


Figure 2.7 Falsity value for $\bar{L}_{q_2}$

The performance function of $\overline{W}_{q_1}$ of α is listed as follows.

$$l_{W_{q_1}}(\alpha) = min\left(\frac{L_{q_1}(\alpha)}{e_1}\right) \tag{12}$$

$$u_{W_{q_1}}(\alpha) = max\left(\frac{L_{q_1}(\alpha)}{e_1}\right) \tag{13}$$

Equation (12) and (13) with (3) give the following results:

$$l_{W_{q_1}}(\alpha)=\frac{7932+4111\alpha+276\alpha^2-65\alpha^3+2\alpha^4}{263385600-144746560\alpha+32304864\alpha^2-3819780\alpha^3+260634\alpha^4-10338\alpha^5+222\alpha^6-2\alpha^7}$$

and

$$u_{W_{q_1}}(\alpha)=\frac{21156-4228\alpha-201\alpha^2+41\alpha^3+2\alpha^4}{35512320+29776384\alpha+9469344\alpha^2+1508382\alpha^3+133644\alpha^4+6720\alpha^5+180\alpha^6+2\alpha^7}$$

$$T_{\overline{W}_{q_1}}(x) = \begin{cases} L_T(x); & 0.00003 \le x \le 0.00008 \\ R_T(x); & 0.00021 \le x \le 0.00059 \\ 0; & otherwise \end{cases}$$

The performance function of $\overline{W}_{q_1}$ of β is listed as follows.

$$l_{W_{q_1}}(\beta) = min\left(\frac{L_{q_1}(\beta)}{e_1}\right) \quad (14)$$

$$u_{W_{q_1}}(\beta) = max\left(\frac{L_{q_1}(\beta)}{e_1}\right) \quad (15)$$

Equation (14) and (15) with (4) give the following result:

$$l_{W_{q_1}}(\beta)=$$

$$\frac{17780-15288\beta+1897\beta^2+490\beta^3+21\beta^4}{469193472+494040096\beta+199552672\beta^2+40169608\beta^3+4450992\beta^4+276320\beta^5+9008\beta^6+120\beta^7}$$

and

$$u_{W_{q_1}}(\beta)=$$

$$\frac{34496+17696\beta+553\beta^2-406\beta^3+21\beta^4}{138720000-198968000\beta+103119200\beta^2-24952880\beta^3+3200312\beta^4-224792\beta^5+8168\beta^6-120\beta^7}$$

$$I_{\overline{W}_{q_1}}(x) = \begin{cases} L_I(x); & 0.000004 \le x \le 0.00003 \\ R_I(x); & 0.00024 \le x \le 0.00250 \\ 0; & otherwise \end{cases}$$

The performance function of $\overline{W}_{q_1}$ of γ is listed as follows.

$$l_{W_{q_1}}(\gamma) = min\left(\frac{L_{q_1}(\gamma)}{e_1}\right) \quad (16)$$

$$u_{W_{q_1}}(\gamma) = max\left(\frac{L_{q_1}(\gamma)}{e_1}\right) \quad (17)$$

Equation (16) and (17) with (5) give the following results:

$$l_{W_{q_1}}(\gamma)=$$

$$\frac{16984-10800\gamma+610\gamma^2+248\gamma^3+12\gamma^4}{377848800+360596880\gamma+134529822\gamma^2+25577430\gamma^3+2709212\gamma^4+161384\gamma^5+5040\gamma^6+64\gamma^7}$$ and

$$u_{W_{q_1}}(\gamma)=$$

$$\frac{28158+11324\gamma-62\gamma^2-200\gamma^3+12\gamma^4}{128757148-158209694\gamma+72513348\gamma^2-16255830\gamma^3+1975620\gamma^4-132504\gamma^5+4592\gamma^6-64\gamma^7}$$

$$F_{\overline{W}_{q_1}}(x) = \begin{cases} L_F(x); & 0.000007 \le x \le 0.00004 \\ R_F(x); & 0.00021 \le x \le 0.00136 \\ 0; & otherwise \end{cases}$$

For varying parameters of $\alpha, \beta, \gamma \in [0,1]$, The typical waiting duration in the queue with higher priority $\overline{W}_{q_1}$ is calculated and shown in table 4.3. Moreover, there is a graphical depiction illustrating the concepts of truth, uncertainty, and falsity regarding the mean waiting time in the queue with higher priority, as depicted in figures 4.7, 4.8, and 4.9.

Table 4.3 α, β, γ-cut for $\overline{W}_{q_1}$

α	$l_{W_{q_1}}(\alpha) \times 10^{-5}$	$u_{W_{q_1}}(\alpha) \times 10^{-4}$	β	$l_{W_{q_1}}(\beta) \times 10^{-5}$	$u_{W_{q_1}}(\beta) \times 10^{-4}$	γ	$l_{W_{q_1}}(\gamma) \times 10^{-5}$	$u_{W_{q_1}}(\gamma) \times 10^{-4}$
0	3.01	5.95	0	3.78	2.48	0	4.49	2.18
0.1	3.34	5.37	0.1	3.12	3.02	0.1	3.83	2.57
0.2	3.71	4.85	0.2	2.56	3.69	0.2	3.25	3.04
0.3	4.12	4.38	0.3	2.10	4.54	0.3	2.76	3.60
0.4	4.57	3.96	0.4	1.71	5.61	0.4	2.34	4.29
0.5	5.05	3.58	0.5	1.38	6.99	0.5	1.97	5.12
0.6	5.59	3.24	0.6	1.11	8.78	0.6	1.66	6.15
0.7	6.18	2.94	0.7	0.88	11.15	0.7	1.39	7.43
0.8	6.82	2.66	0.8	0.69	14.35	0.8	1.15	9.03
0.9	7.53	2.41	0.9	0.53	18.76	0.9	0.95	11.07
1	8.31	2.19	1	0.40	25.05	1	0.78	13.69

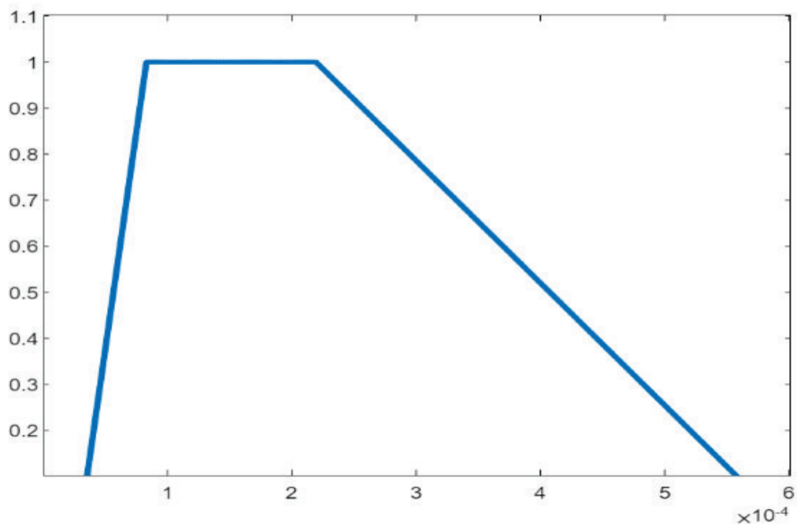


Figure 4.7 Truth value for $\overline{W}_{q_1}$

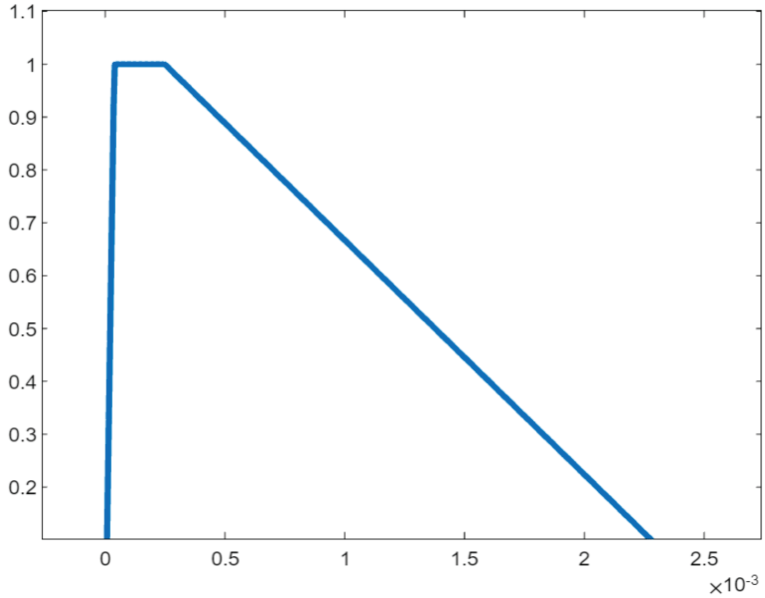


Figure 4.8 Indeterminacy value for $\overline{W}_{q_1}$

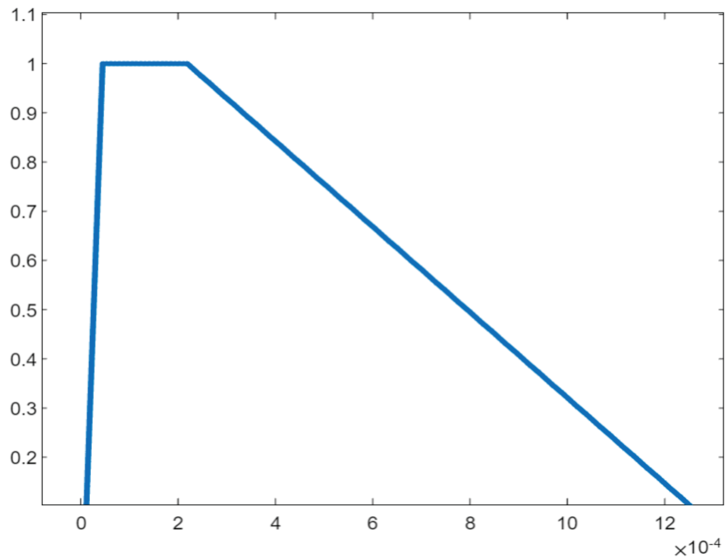


Figure 4.9 Falsity value for $\overline{W}_{q_1}$

The performance function of $\overline{W}_{q_2}$ of α is listed as follows.

$$l_{W_{q_2}}(\alpha) = min\left(\frac{L_{q_2}(\alpha)}{e_2}\right) \tag{18}$$

$$u_{W_{q_2}}(\alpha) = max\left(\frac{L_{q_2}(\alpha)}{e_2}\right) \tag{19}$$

Equation (18) and (19) with (3) give the following results:

$$l_{W_{q_2}}(\alpha) = \frac{10576+4600\alpha+205\alpha^2-63\alpha^3+2\alpha^4}{546560-332768\alpha+74352\alpha^2-7594\alpha^3+352\alpha^4-6\alpha^5}$$

and

$$u_{W_{q_2}}(\alpha)=\frac{24682+4345\alpha-254\alpha^2+39\alpha^3+2\alpha^4}{39440+56108\alpha+23394\alpha^2+3910\alpha^3+262\alpha^4+6\alpha^5}$$

$$T_{\overline{W}_{q_2}}(x)=\begin{cases} L_T(x); & 0.01935 \le x \le 0.05453 \\ R_T(x); & 0.16362 \le x \le 0.62581 \\ 0; & otherwise \end{cases}$$

The performance function of $\overline{W}_{q_2}$ of β is listed as follows.

$$l_{W_{q_2}}(\beta)=min\left(\frac{L_{q_2}(\beta)}{e_2}\right) \tag{20}$$

$$u_{W_{q_2}}(\beta)=max\left(\frac{L_{q_2}(\beta)}{e_2}\right) \tag{21}$$

Equation (20) and (21) with (4) give the following result:

$$l_{W_{q_2}}(\beta)=\frac{14224-7252\beta+224\beta^2+147\beta^3+7\beta^4}{333540+368468\beta+141247\beta^2+22949\beta^3+1529\beta^4+35\beta^5}$$

and

$$u_{W_{q_2}}(\beta)=\frac{21560+7287\beta-175\beta^2-119\beta^3+7\beta^4}{84864-148880\beta+81224\beta^2-17183\beta^3+1354\beta^4-35\beta^5}$$

$$I_{\overline{W}_{q_2}}(x)=\begin{cases} L_I(x); & 0.00847 \le x \le 0.04264 \\ R_I(x); & 0.2540 \le x \le 21.25 \\ 0; & otherwise \end{cases}$$

The performance function of $\overline{W}_{q_2}$ of γ is listed as follows.

$$l_{W_{q_2}}(\gamma)=min\left(\frac{L_{q_2}(\gamma)}{e_2}\right) \tag{22}$$

$$u_{W_{q_2}}(\gamma)=max\left(\frac{L_{q_2}(\gamma)}{e_2}\right) \tag{23}$$

Equation (22) and (23) with (5) give the following results:

$$l_{W_{q_2}}(\gamma)=\frac{25476-7708\gamma-239\gamma^2+100\gamma^3+6\gamma^4}{509320+470346\gamma+149044\gamma^2+20220\gamma^3+1174\gamma^4+24\gamma^5}$$

and

$$u_{W_{q_2}}(\gamma)=\frac{32851+6954\gamma-503\gamma^2-76\gamma^3+6\gamma^4}{168948-228342\gamma+95188\gamma^2-15764\gamma^3+1054\gamma^4-24\gamma^5}$$

$$F_{\overline{W}_{q_2}}(x) = \begin{cases} L_F(x); & 0.05001 \le x \le 0.15333 \\ R_F(x); & 0.19444 \le x \le 1.86286 \\ 0; & otherwise \end{cases}$$

For varying values of $\alpha, \beta, \gamma \in [0,1]$, the mean waiting duration in the queue with lower priority queue $\overline{W}_{q_2}$ is determined and shown in table 4.4. Additionally, there is a graphical depiction illustrating the concepts of truth, uncertainty, and falsity regarding the mean waiting time in the queue with lower priority, as displayed in figures 4.10, 4.11, and 4.12.

Table 4.4 α, β, γ-cut for $\overline{W}_{q_2}$

α	$l_{W_{q_2}}(\alpha)$	$u_{W_{q_2}}(\alpha)$	β	$l_{W_{q_2}}(\beta)$	$u_{W_{q_2}}(\beta)$	γ	$l_{W_{q_2}}(\gamma)$	$u_{W_{q_2}}(\gamma)$
0	0.0193	0.6258	0	0.0426	0.2540	0	0.0500	0.1944
0.1	0.0214	0.5545	0.1	0.0363	0.3149	0.1	0.0442	0.2280
0.2	0.0238	0.4947	0.2	0.0309	0.3953	0.2	0.0392	0.2695
0.3	0.0264	0.4439	0.3	0.0263	0.5042	0.3	0.0348	0.3212
0.4	0.0292	0.4004	0.4	0.0225	0.6562	0.4	0.0309	0.3869
0.5	0.0324	0.3628	0.5	0.0191	0.8771	0.5	0.0275	0.4720
0.6	0.0360	0.3301	0.6	0.0163	1.2169	0.6	0.0244	0.5851
0.7	0.0399	0.3015	0.7	0.0139	1.7841	0.7	0.0217	0.7401
0.8	0.0442	0.2762	0.8	0.0118	2.8685	0.8	0.0193	0.9622
0.9	0.0491	0.2539	0.9	0.0100	5.5736	0.9	0.0172	1.3001
1	0.0545	0.2340	1	0.0084	21.25	1	0.0153	1.8628

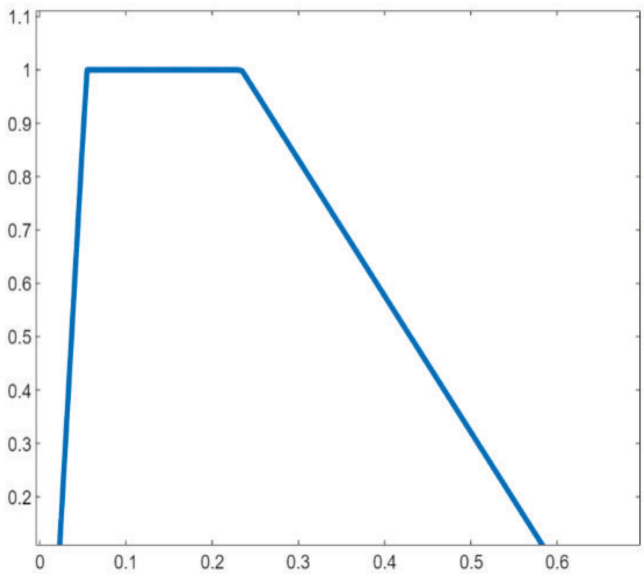


Figure 4.10 Truth value for $\overline{W}_{q_2}$

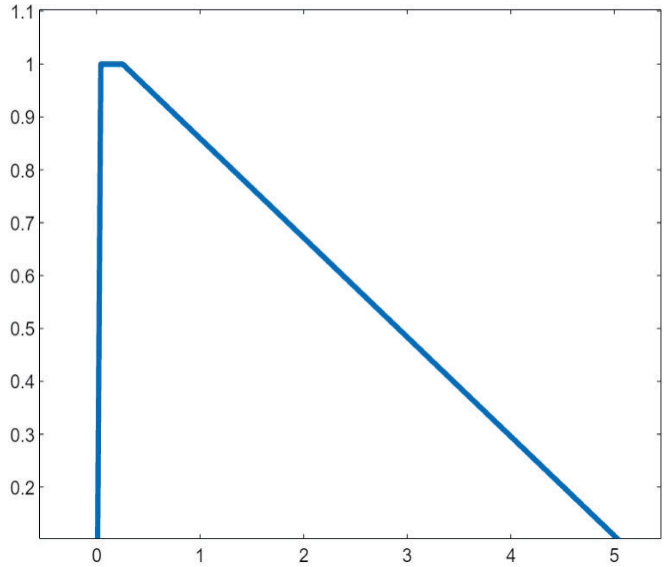


Figure 4.11 Indeterminacy value for $\overline{W}_{q_2}$

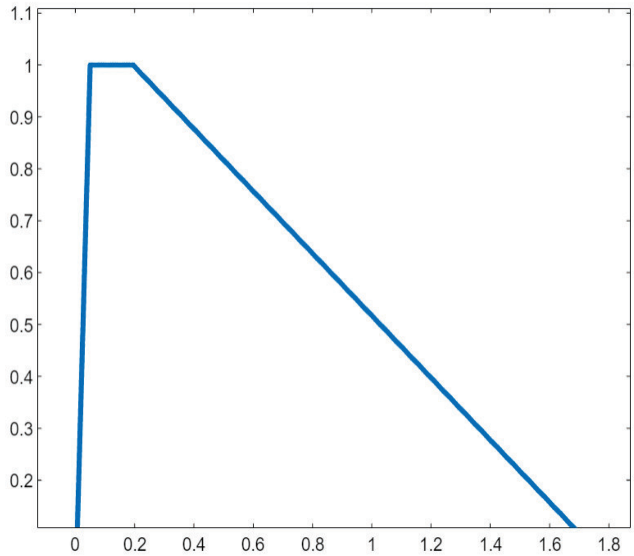


Figure 4.12 Falsity value for $\overline{W}_{q_2}$

4. FUTURE RESEARCH DIRECTIONS

As a future work, other important performance measures can be analysed. Ranking technique could be employed with this proposed work for analysing decision -making problem.

5. CONCLUSIONS

Models of queueing with priority find application in various real-world scenarios, including urgency management in hospitals, communication networks, and other scenarios. The parameters used in queueing decision models may often be uncertain, leading to imprecise system performance measures. This paper introduces and outlines a single-server queueing model employing a non-preemptive priority discipline. The model's service time and arrival time are articulated through a single-valued trapezoidal neutrosophic numbers. An illustration is given to demonstrate the efficiency assessment of the proposed model, integrating the membership degrees of truth, uncertainty, and falsehood of SVTNN. This method illustrates enhanced efficiency.

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Analyzing Franchisee Selection Problem Via Interval-Valued Neutrosophic Sets: Case of Cafe Chain

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ABSTRACT

In the changes in the economy and dynamic market environment, franchisee business partnership is an important factor in the growth and strengthening of businesses. Technological developments and uncertain conditions are increasing this importance. We are faced with uncertainty in solving real life problems. Interval-valued neutrosophic set is an effective method used to solve problems with uncertainty and complexity. The aim of this study is to determine the criteria that affect franchisee selection in the global cafe chain business. Franchisee selection problem has been investigated with interval-valued neutrosophic AHP. In the research, the priorities of the criteria and the scoring of the experts were taken into consideration. According to the results of the analysis, while the location was found to be the most important criteria, personal condition was obtained as the least important one.

KEYWORDS: **Franchisee selection, cafe chain, ınterval-valued neutrosophic set, ınterval-valued neutrosophic AHP.**

1. INTRODUCTION

The number of cafe businesses in the service sector is increasing day by day. This increase, especially in cafe chain businesses, attracts the attention of investors in this direction. Investors who want to become dealers of chain businesses with the franchisee system become a problem that needs to be carefully decided in terms of franchisor businesses. Because the right choice of business partner eliminates the negative monetary and strategic effects on the brand and increases success. At the same time, franchisee partner selection is an important issue in the growth and strengthening of businesses.

We have to struggle with many uncertainties in the decision problems we encounter in daily life. Scientists have presented theories such as mathematics, probability and fuzzy sets from past to present in solving such problems with uncertainty. Fuzzy set theory developed by Zadeh (1965) has been frequently used in solving problems involving uncertainty. Zadeh defined a fuzzy set as a membership function taking values in the interval [0,1], which is a set different from the empty set. Later, fuzzy sets appear in different structures such as intuitionistic fuzzy sets proposed by Atanassov (1986) and neutrosophic sets proposed by Smarandache (1998). In intuitionistic fuzzy set theory, uncertainty is analysed with membership and non-membership functions. Neutrosophic sets are a general version of fuzzy sets. In the case of neutrosophic sets, the uncertainty function is considered a separate term, and each element x is characterized by a truth membership function $T_A(x)$, an uncertainty membership function $I_A(x)$, and a falsity membership function $F_A(x)$. Wang et al. (2010) defined single-valued neutrosophic sets. Single-valued neutrosophic sets can be used feasibly to deal with real world decision problems. Neutrosophic sets were later included in the literature with different extensions. One of these extensions is interval valued neutrosophic sets.

The purpose of this article is to present a model for identifying important criteria for franchisors to select the most suitable franchisees. Decision makers often have to make their choices under the influence of multiple conflicting criteria. In such cases, multi-criteria decision making gives the opportunity to choose the best among multiple alternatives. AHP is one of the multi-criteria decision making techniques. In this study, interval-valued neutrosophic AHP multi-criteria decision making approach is used to analyze the franchisee selection problem. The study was conducted in a global cafe chain. The study is structured as follows. The second section includes a literature review on franchisee selection. The third section includes the preliminary part consisting of fuzzy, intuitionistic fuzzy, neutrosophic, interval-valued neutrosophic sets and the application technique interval-valued neutrosophic AHP technique. The fourth part of the study covers the analysis of franchisee selection criteria with IVN-AHP. The study concludes with findings and conclusion.

2. LITERATURE REVIEW

Various studies on franchisee selection have been observed in the literature. Franchisee, which basically means concession holder, also appears with different words such as dealer and distributor in studies. Tatham et al (1972), examined the franchisor and franchisee selection processes. It has taken the criteria in the educational background, personality (the ability to meet the public and win respect), health, past work experience, credit, or financial standing, the franchisee would personally manage operations at the restaurant franchisor's selection. In the selection of franchisees, it has taken the franchisor’s capital requirements, franchisor’s training program, franchisee agreement’s fairness, franchisor’s reputation and progressiveness, franchisor’s demonstrated profitability, recognized demand for the franchisor’s product criteria. These criteria were analyzed with testing the hypothesis, Kolmogorov Smirnov One Sample Test. Watson et al., (2016) studied franchisee selection theory. Criteria namely franchisee age, number of franchisees and sector were examined by hypothesis test. Ramirez-Hurtado et al (2011), identified the franchisee profiles that franchisors prefer. Characteristics related to franchisee profile in terms of the review of literature can be stated as follows: shrewdness, self-esteem, management ability, human relations ability, entrepreneurial character, ethical behaviour, creativity, need of achievement, willingness to work hard, communication, age, emotional stability, marital status. Brookes and Altınay (2011), determined how different selection criteria affect the selection process with data analysis. Ramdhani et al. (2021), analyzed the franchisee selection process, capital, sales points, BEP (Break even point), franchisee fees and profit criteria with the smart technique. Traneva and Tranev (2022), considered franchisee selection problem by using intuitionistic fuzzy sets. Studies on franchisee selection are summarized in Table 1.

Table1: **Literature on franchisee selection**

Year	Author(s)	Method(s)	Objecives	Criteria
2006	Clarkin and Swavely	Statistical analysis	Determine the criteria for franchisors to evaluate franchisees	Financial net worth, general business experience, industry experience, formal educations, psychological profile, personal interview.
2008	Hsu and Chen	AHP and ENTROPY	Determine the essential criteria releated to franchisee selection	Personal location, personal background, financial situation,business ability, location condition, area, traffic, consumer.
2011	Faradillah et al	AHP, decision support system	Franchisee outlet selection	Franchisee fee, continuing franchisee fee, franchisor size, franchisor reputation
2011	Sivakumar and Schoormans	Social and commercial franchisee impact on franchisee selection	Application analysis of commercial franchisee selection criteria in social franchisee selection	Financial net worth, business experience, formal education, local market knowledge, personal profile,
2013	Karaca	ELECTRE I, TOPSIS	Dealer selection	Prestige, Location, professionalism, potential customer, financial status, Service area adequacy, Experience in the sector, Land situation
2014	Gaul	Literature review and proposal selection model	Examined fit between franchisor and franchisee.	Internationalization fit, interpersonal fit, objective fit
2020	Urevic	DEA, AIM	Determine the franchisee selection criteria for restaurant businesses	Brand name/ reputation, brand age, recognition, Franchisee support, training, consultancy, call centre availability, scaling, geographical suitability, regional agreements, growth options, Operational processes, quality, monetary conditions
2020	Kıran et al.	Content analysis	Determine the factors that franchisees take into account in the selection of the franchisor	Product diversity, bilateral relations, brand, company potential, professionalism, product and service standard, suitability of investment conditions, profitability rate, logistic support
2020	Metin	TOPSIS	Determine the models and criteria used in internationalisation	Cost, time, support, trust, ease
2021	Calderon-Monge, Sariz and Garcia	AHP	Design a model proposal that franchisors can objectively evaluate franchisees in a selection process.	Talent, respect for the customer (friendship), good public relation, Behaviour, belief in the product concept, motivation, interest in healthy lifestyle, Past experience, location, commercial vision, sectoral experience, management ability, business capacity, entrepreneurial spirit.

Besides studies related to MADM in interval neutrosophic set environment can be summarized as below:

Mondal et al. (2018) proposed tangent similarity measure of interval valued neutrosophic sets and presented a MADM strategy based on this similarity measure namely the selection of a suitable sector for money investment of a government employee for a financial year. Dalapati et al. (2017) defined a new cross-entropy measure namely IN-cross entropy under the interval neutrosophic set environment and developed a novel MAGDM strategy. Dey et al. (2016) examined an extended grey relational analysis method for MADM problems under the interval neutrosophic uncertain linguistic environment. Pramanik and Mondal (2015) introduced MADM based on interval neutrosophic sets and extended the single-valued neutrosophic grey relational analysis to an interval neutrosophic environment.

3. PRELIMINARIES

In this section, we will give basic definitions of fuzzy set, intuitionistic fuzzy set, neutrosophic set, single-valued neutrosophic set, interval-valued neutrosophic set, interval valued neutrosophic AHP.

3.1. Fuzzy set

Let E be a universal set and let x be a general element in this set. Fuzzy set $\tilde{A}$ defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\} \tag{1}$$

The degree of the membership function $\mu_{\tilde{A}}(x)$ is also called the degree of accuracy. The degree of membership function takes values between 0 and 1 and is defined as $\mu_{\tilde{A}}(x)$: X→[0,1] (Bhattacharyya et al. 2018).

3.2. Intuitionistic fuzzy set

Let E be a universal set and let x be a general element in this set. Intuitionistic fuzzy set $\tilde{A}$ defined as:

$$A^{\tilde{I}} = \{(x, \mu_{\tilde{A}}(x), \nu_{\tilde{A}}(x)) \mid x \in X\} \tag{2}$$

The degree of membership $\mu_{A^{\tilde{I}}}$ and the degree of non-membership $\upsilon_{A^{\tilde{I}}}$ take values between 0 and 1 , and they are defined as : $\mu_{A^{\tilde{I}}} : X \to [0,1]$ ve $\upsilon_{A^{\tilde{I}}} : X \to [0,1]$.

3.3. Neutrosophic Set

Let E be a universal set and let x be a general element in this set. The neutrosophic set A defined in E is characterized by truth $T_A(x)$, indeterminacy $I_A(x)$ and falsity $F_A(x)$ membership functions. These membership functions take values T_A: E $\to]0^-, 1^+[$, I_A: E $\to]0^-, 1^+[$, F_A: E $\to]0^-, 1^+[$ and sum of them ; $\forall x \in E, 0^- \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$. A neutrosophic set is defined as:

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in E\} \tag{3}$$

3.4. Single Valued Neutrosophic Set

Wang et al. (2010) developed single valued neutrosophic sets to be applied to real life problems. Single valued neutrosophic set is characterized by, truth-membership function $T_A(x)$: X→[0,1], indeterminacy-membership function $I_A(x)$:X→[0,1] and falsity-membership function $F_A(x)$:X→[0,1]. There is not restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$,

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

The single valued neutrosophic set is defined as:

$$A = \{x, T_A(x),\ I_A(x), F_A(x)\rangle : x \in E\} \tag{4}$$

3.5. Interval Valued Neutrosophic Set

Let E be a universal set and let x be a general element in this set. The interval-valued neutrosophic set A defined in E is characterized by truth $T_A(x)$, indeterminacy $I_A(x)$ and falsity$F_A(x)$ membership functions. Where with the condition;

$T_A(x) = [T_A^L(x), T_A^U(x)] \subseteq [0,1]$,
$I_A(x) = [I_A^L(x), I_A^U(x)] \subseteq [0,1]$,
$F_A(x) = [F_A^L(x), F_A^U(x)] \subseteq [0,1]$ the interval-valued neutrosophic set is defined as:

$$A = \{x, [T_A^L(x), T_A^U(x)], [_A^L(x), I(x)], [F_A^L(x), F_A^U(x)]\ x\ \in E\ \} \tag{5}$$

3.6. Interval Valued Neutrosophic AHP

Saaty (1998) developed the Analytic Hierarchy Process (AHP) method. It is one of the most inclusive methods in solving multi criteria decision making problems. This method deals with problems in a hierarchical structure. At the top level of the hierarchy is the goal and at the bottom level are the different alternatives that need to be decided (Arquero et al. 2009). The AHP method is then used to solve different problems with different structures of fuzzy sets. One of these is the interval-valued neutrosophic AHP.

Interval- valued neutrosophic AHP method is similar to AHP method and is simple to implement. In the following the steps of the interval-valued neutrosophic AHP method are presented (Ricardo et al., 2021):

Step 1: The pairwise comprasion matrix ($\ddot{\tilde{P}}$) is constructed. To construct the matrix, the linguistic terms given in Table 2 were used.

Step 2: ($\ddot{\tilde{P}}$) pairwise comparison matrix is converted into the interval -valued neutrosophic comparison matrix constructed using Table 2.

$$(\ddot{\tilde{P}}) = \begin{bmatrix} [T_{11}^L, T_{11}^U], [I_{11}^L, I_{11}^U][F_{11}^L, F_{11}^U] & [T_{11}^L, T_{11}^U], [I_{11}^L, I_{11}^U][F_{11}^L, F_{11}^U] & \dots & [T_{1n}^L, T_{1n}^U], [I_{1n}^L, I_{1n}^U][F_{1n}^L, F_{1n}^U] \\ [T_{21}^L, T_{21}^U], [I_{21}^L, I_{21}^U][F_{21}^L, F_{21}^U] & [T_{22}^L, T_{22}^U], [I_{22}^L, I_{22}^U][F_{22}^L, F_{22}^U] & \dots & \dots \\ \dots & \dots & \dots & \dots \\ [T_{n1}^L, T_{n1}^U], [I_{n1}^L, I_{n1}^U][F_{n1}^L, F_{n1}^U] & [T_{n2}^L, T_{n2}^U], [I_{n2}^L, I_{n2}^U][F_{n2}^L, F_{n2}^U] & \dots & [T_{nn}^L, T_{nn}^U], [I_{nn}^L, I_{nn}^U][F_{nn}^L, F_{nn}^U] \end{bmatrix} \tag{6}$$

Table 2: Scale of Interval-Valued Neutrosophic AHP

Linguistic Term	T^L	T^U	I^L	I^U	F^L	F^U
Absolutely more important	0.9	0.95	0	0.05	0.05	0.15
Strongly more important	0.8	0.9	0.05	0.1	0.1	0.2
More important	0.7	0.8	0.15	0.25	0.2	0.3
Weakly more important	0.6	0.7	0.25	0.35	0.3	0.4
Equal importance	0.5	0.5	0.5	0.5	0.5	0.5
Weakly less important	0.4	0.5	0.55	0.65	0.5	0.6
Less important	0.3	0.4	0.65	0.75	0.6	0.7
Strongly less important	0.2	0.3	0.75	0.85	0.7	0.8
Absolutely less important	0.1	0.2	0.9	0.95	0.8	0.9

Step 3: $\tilde{\tilde{P}}$ is deneutrosophicated to check for consistency. The neutrosophic matrix is evaluated as consistent only if the deneutrosophicated matrix is determined to be consistent.
$\tilde{\tilde{N}}$ is deneutrosophicated as below (Bolturk and Kahraman, 2018):

$$Đ(\tilde{\tilde{N}})=\left(\frac{T_N^L(X)+T_N^U(X)}{2}+(I_N^U(X))\right)\left((1-\frac{I_N^L(X)+I_N^U(X)}{2}\right)-\left(1-F_N^U(X)(\frac{F_N^L(X)+F_N^U(X)}{2})\right) \quad (7)$$

Step 4: The criteria weights are normalised .

$$\tilde{\tilde{N}}_{ij}=\left[\frac{T_{kj}^L}{\sum_{k=1}^n T_{kj}^U},\frac{T_{kj}^U}{\sum_{k=1}^n T_{kj}^L}\right],\left[\frac{I_{kj}^L}{\sum_{k=1}^n I_{kj}^U},\frac{I_{kj}^U}{\sum_{k=1}^n I_{kj}^L}\right],\left[\frac{F_{kj}^L}{\sum_{k=1}^n F_{kj}^U},\frac{F_{kj}^U}{\sum_{k=1}^n F_{kj}^L}\right] \quad (8)$$

Where n indicates the number of criteria.

Step 5: The neutrosophic weight vector ($\tilde{\tilde{W}}_j$) is determined by taking the mean of each row.

$$\tilde{\tilde{W}}_j=\left[\frac{\sum_{k=1}^n\frac{T_{1j}^L}{\sum_{k=1}^n T_{kj}^U}}{n},\frac{\sum_{k=1}^n\frac{T_{1j}^U}{\sum_{k=1}^n T_{kj}^U}}{n}\right],\left[\frac{\sum_{k=1}^n\frac{I_{1j}^L}{\sum_{k=1}^n I_{kj}^U}}{n},\frac{\sum_{k=1}^n\frac{I_{1j}^U}{\sum_{k=1}^n I_{kj}^U}}{n}\right],\left[\frac{\sum_{k=1}^n\frac{F_{1j}^L}{\sum_{k=1}^n F_{kj}^U}}{n},\frac{\sum_{k=1}^n\frac{F_{1j}^U}{\sum_{k=1}^n F_{kj}^U}}{n}\right]$$

(9)

Step 6: The criteria weights are determined as given in Equation 7.
Step 7: The weights are normalized to determine final weights.

4. *CASE STUDY*

Main criteria and sub-criteria that are considered for this study are shown in Table 3.

Table 3: Main criteria and sub-criteria

Main criteria	**Code**	**References**	**Sub-criteria**	**Code**	**References**
Financial Condition	C1	Tatham et al. 1972; Karaman and Yıldız, 2021; Valeri, 2020; Hsu and Chen, 2008	Size of the store	C11	Hsu and Chen, 2008
			Targeted profitability	C12	Valeri, 2020
			Covering the franchisee fee	C13	Valeri, 2020
Location	C2	Hsu and Chen, 2008; Calderon-Monge, Sariz and Garcia, 2021	Accessibility	C21	Valeri, 2020
			Geographical suitability	C22	Valeri, 2020
			Closeness to center	C23	Karaman and Yıldız, 2021
Personnel	C3	Sivakumar and Schoormans, 2011; Hsu and Chen, 2008	Education	C31	Hsu and Chen,2008
			Knowledge	C32	Hsu and Chen,2008
			Social relationship	C33	Hsu and Chen,2008
Reputation	C4	Valeri, 2020	Awareness	C41	Valeri, 2020
			Trustworthiness	C42	Metin, 2020
Personal condition	C5	Hsu and Chen, 2008	Fiscal status	C51	Hsu and Chen,2008
			Famousness	C52	Valeri,2020
			Education level	C53	Hsu and Chen, 2008; Gaul, 2014; Valeri, 2020
			Experience	C54	Hsu and Chen, 2008; Caldeon-Monge, Sariz and Garcia, 2021

Pairwise comparison of main criteria for DM1 are given in Table 4.

Table4: Pairwise comparison of main criteria for DM1

DM1	**C1**	**C2**	**C3**	**C4**	**C5**
C1	EI	WLI	WMI	WMI	MI
C2	WMI	EI	WMI	MI	MI
C3	WLI	WLI	EI	WLI	MI
C4	WLI	LI	WMI	EI	WMI
C5	LI	LI	LI	WLI	EI

Similarly, pairwise comparisons of the main criteria for other DMs are given in the following tables.

Table 5: Pairwise comparison of main criteria for DM2

DM2	C1	C2	C3	C4	C5
C1	EI	LI	MI	WMI	SMI
C2	MI	EI	MI	SMI	SMI
C3	LI	LI	EI	WMI	SMI
C4	WLI	SLI	WLI	EI	MI
C5	SLI	SLI	SLI	LI	EI

Table6: Pairwise comparison of main criteria for DM3

DM3	C1	C2	C3	C4	C5
C1	EI	WLI	WMI	WLI	WMI
C2	WMI	EI	SMI	EI	AMI
C3	WLI	SLI	EI	LI	WMI
C4	WMI	EI	MI	EI	SMI
C5	WLI	ALI	WLI	SLI	EI

All CR values are smaller than the threshold so consistency of pairwise comparisons related to main criteria is consistent. Following to that interval valued neutrosophic evaluation matrix by taking linguistic terms given in Table 2 into the account. Interval valued neutrosophic evaluation matrix of main criteria for DM1,DM2 and DM3 are given in Tables 7,8 and 9 respectively.

Table7: Interval valued neutrosophic evaluation matrix of main criteria for DM1

DM1	C1	C2	C3	C4	C5
C1	([0.50,0.50], [0.50,0.50], [0.50,0.50])	([0.40,0.50], [0.55,0.65], [0.50,0.60])	([0.60,0.70], [0.25,0.35], [0.30,0.40])	([0.60,0.70], [0.25,0.35], [0.30,0.40])	([0.70,0.80], [0.15,0.25], [0.20,0.30])
C2	([0.60,0.70], [0.25,0.35], [0.30,0.40])	([0.50,0.50], [0.50,0.50], [0.50,0.50])	([0.60,0.70], [0.25,0.35], [0.30,0.40])	([0.70,0.80], [0.15,0.25], [0.20,0.30])	([0.70,0.80], [0.15,0.25], [0.20,0.30])
C3	([0.40,0.50], [0.55,0.65], [0.50,0.60])	([0.40,0.50], [0.55,0.65], [0.50,0.60])	([0.50,0.50], [0.50,0.50], [0.50,0.50])	([0.40,0.50], [0.55,0.65], [0.50,0.60])	([0.70,0.80], [0.15,0.25], [0.20,0.30])
C4	([0.40,0.50], [0.55,0.65], [0.50,0.60])	([0.30,0.40], [0.65,0.75], [0.60,0.70])	([0.60,0.70], [0.25,0.35], [0.30,0.40])	([0.50,0.50], [0.50,0.50], [0.50,0.50])	([0.60,0.70], [0.25,0.35], [0.30,0.40])
C5	([0.30,0.40], [0.65,0.75], [0.60,0.70])	([0.30,0.40], [0.65,0.75], [0.60,0.70])	([0.30,0.40], [0.65,0.75], [0.60,0.70])	([0.40,0.50], [0.55,0.65], [0.50,0.60])	([0.50,0.50], [0.50,0.50], [0.50,0.50])

Table 8: Interval valued neutrosophic evaluation matrix of main criteria for DM2

DM2	C1	C2	C3	C4	C5
C1	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.65,0.75],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.15,0.25],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.80,0.90],\\ [0.05,0.10],\\ [0.10,0.20]\end{pmatrix}$
C2	$\begin{pmatrix}[0.70,0.80],\\ [0.15,0.25],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.15,0.25],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.80,0.90],\\ [0.05,0.10],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.80,0.90],\\ [0.05,0.10],\\ [0.10,0.20]\end{pmatrix}$
C3	$\begin{pmatrix}[0.30,0.40],\\ [0.65,0.75],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.65,0.75],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.80,0.90],\\ [0.05,0.10],\\ [0.10,0.20]\end{pmatrix}$
C4	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.75,0.85],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.15,0.25],\\ [0.20,0.30]\end{pmatrix}$
C5	$\begin{pmatrix}[0.20,0.30],\\ [0.75,0.85],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.75,0.85],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.75,0.85],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.65,0.75],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$

Table9: Interval valued neutrosophic evaluation matrix of main criteria for DM3

DM3	C1	C2	C3	C4	C5
C1	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$
C2	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.80,0.90],\\ [0.05,0.10],\\ [0.10,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.90,0.95],\\ [0.00,0.05],\\ [0.05,0.15]\end{pmatrix}$
C3	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.75,0.85],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.30,0.40],\\ [0.65,0.75],\\ [0.60,0.70]\end{pmatrix}$	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$
C4	$\begin{pmatrix}[0.60,0.70],\\ [0.25,0.35],\\ [0.30,0.40]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.70,0.80],\\ [0.15,0.25],\\ [0.20,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$	$\begin{pmatrix}[0.80,0.90],\\ [0.05,0.10],\\ [0.10,0.20]\end{pmatrix}$
C5	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.20],\\ [0.90,0.95],\\ [0.80,0.90]\end{pmatrix}$	$\begin{pmatrix}[0.40,0.50],\\ [0.55,0.65],\\ [0.50,0.60]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.30],\\ [0.75,0.85],\\ [0.70,0.80]\end{pmatrix}$	$\begin{pmatrix}[0.50,0.50],\\ [0.50,0.50],\\ [0.50,0.50]\end{pmatrix}$

After that normalization process is applied for main criteria. The normalized pairwise comparison matrix for main criteria in terms of DM1,DM2 and DM3 are given in Tables 10,11 and 12 respectively

Table10: The normalized pairwise comparison matrix for main criteria in terms of DM1

DM1	C1	C2	C3	C4	C5
C1	([0.19,0.19], [0.17,0.17], [0.18,0.18])	([0.17,0.22], [0.17,0.20], [0.16,0.19])	([0.20,0.23], [0.11,0.15], [0.12,0.17])	([0.20,0.23], [0.10,0.14], [0.12,0.17])	([0.19,0.22], [0.09,0.16], [0.11,0.17])
C2	([0.23,0.27], [0.09,0.12], [0.11,0.14])	([0.22,0.22], [0.15,0.15], [0.16,0.16])	([0.20,0.23], [0.11,0.15], [0.12,0.17])	([0.23,0.27], [0.06,0.10], [0.08,0.12])	([0.19,0.22], [0.09,0.16], [0.11,0.17])
C3	([0.15,0.19], [0.19,0.22], [0.18,0.21])	([0.17,0.22], [0.17,0.20], [0.16,0.19])	([0.17,0.17], [0.22,0.22], [0.21,0.21])	([0.13,0.17], [0.23,0.27], [0.21,0.25])	([0.19,0.22], [0.09,0.16], [0.11,0.17])
C4	([0.15,0.19], [0.19,0.22], [0.18,0.21])	([0.13,0.17], [0.20,0.23], [0.19,0.23])	([0.20,0.23], [0.11,0.15], [0.12,0.17])	([0.17,0.17], [0.21,0.21], [0.21,0.21])	([0.17,0.19], [0.16,0.22], [0.17,0.22])
C5	([0.12,0.15], [0.22,0.26], [0.21,0.25])	([0.13,0.17], [0.20,0.23], [0.19,0.26])	([0.10,0.13], [0.28,0.33], [0.25,0.29])	([0.13,0.17], [0.23,0.27], [0.21,0.25])	([0.14,0.14], [0.31,0.31], [0.28,0.28])

Table11: The normalized pairwise comparison matrix for main criteria in terms of DM2

DM2	C1	C2	C3	C4	C5
C1	([0.20,0.20], [0.17,0.17], [0.17,0.17])	([0.16,0.21], [0.18,0.20], [0.17,0.20])	([0.24,0.28], [0.06,0.10], [0.08,0.12])	([0.19,0.22], [0.12,0.17], [0.14,0.18])	([0.20,0.22], [0.05,0.10], [0.07,0.14])
C2	([0.28,0.32], [0.05,0.08], [0.07,0.10])	([0.26,0.26], [0.13,0.13], [0.14,0.14])	([0.24,0.28], [0.06,0.10], [0.08,0.12])	([0.25,0.28], [0.02,0.05], [0.05,0.09])	([0.20,0.22], [0.05,0.10], [0.07,0.14])
C3	([0.12,0.16], [0.22,0.25], [0.21,0.24])	([0.16,0.21], [0.18,0.20], [0.17,0.20])	([0.17,0.17], [0.20,0.20], [0.20,0.20])	([0.19,0.22], [0.12,0.17], [0.14,0.18])	([0.20,0.22], [0.05,0.10], [0.07,0.14])
C4	([0.16,0.20], [0.18,0.22], [0.17,0.21])	([0.11,0.16], [0.20,0.23], [0.20,0.23])	([0.14,0.17], [0.22,0.26], [0.20,0.24])	([0.16,0.16], [0.24,0.24], [0.23,0.23])	([0.17,0.20], [0.14,0.24], [0.14,0.21])
C5	([0.08,0.12], [0.25,0.28], [0.24,0.28])	([0.11,0.16], [0.20,0.23], [0.20,0.23])	([0.07,0.10], [0.30,0.34], [0.28,0.32])	([0.09,0.12], [0.32,0.37], [0.27,0.32])	([0.12,0.12], [0.48,0.48], [0.36,0.36])

Table 12: The normalized pairwise comparison matrix for main criteria in terms of DM3

DM3	C1	C2	C3	C4	C5
C1	$\begin{pmatrix}[0.17,0.17],\\ [0.20,0.20],\\ [0.20,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.20,0.25],\\ [0.16,0.19],\\ [0.15,0.18]\end{pmatrix}$	$\begin{pmatrix}[0.18,0.21],\\ [0.14,0.19],\\ [0.15,0.20]\end{pmatrix}$	$\begin{pmatrix}[0.18,0.23],\\ [0.17,0.20],\\ [0.16,0.19]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.19],\\ [0.19,0.26],\\ [0.18,0.24]\end{pmatrix}$
C2	$\begin{pmatrix}[0.21,0.24],\\ [0.10,0.14],\\ [0.12,0.16]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.25],\\ [0.14,0.14],\\ [0.15,0.15]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.26],\\ [0.02,0.05],\\ [0.05,0.10]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.23],\\ [0.15,0.15],\\ [0.16,0.16]\end{pmatrix}$	$\begin{pmatrix}[0.24,0.25],\\ [0.00,0.04],\\ [0.03,0.09]\end{pmatrix}$
C3	$\begin{pmatrix}[0.14,0.17],\\ [0.22,0.26],\\ [0.20,0.24]\end{pmatrix}$	$\begin{pmatrix}[0.10,0.15],\\ [0.22,0.25],\\ [0.21,0.24]\end{pmatrix}$	$\begin{pmatrix}[0.15,0.15],\\ [0.27,0.27],\\ [0.25,0.25]\end{pmatrix}$	$\begin{pmatrix}[0.14,0.18],\\ [0.20,0.23],\\ [0.19,0.23]\end{pmatrix}$	$\begin{pmatrix}[0.16,0.19],\\ [0.19,0.26],\\ [0.18,0.24]\end{pmatrix}$
C4	$\begin{pmatrix}[0.21,0.24],\\ [0.10,0.14],\\ [0.12,0.16]\end{pmatrix}$	$\begin{pmatrix}[0.25,0.25],\\ [0.14,0.14],\\ [0.15,0.15]\end{pmatrix}$	$\begin{pmatrix}[0.21,0.24],\\ [0.08,0.14],\\ [0.10,0.15]\end{pmatrix}$	$\begin{pmatrix}[0.23,0.23],\\ [0.15,0.15],\\ [0.16,0.16]\end{pmatrix}$	$\begin{pmatrix}[0.21,0.24],\\ [0.04,0.07],\\ [0.06,0.12]\end{pmatrix}$
C5	$\begin{pmatrix}[0.14,0.17],\\ [0.22,0.26],\\ [0.20,0.24]\end{pmatrix}$	$\begin{pmatrix}[0.05,0.10],\\ [0.26,0.28],\\ [0.24,0.27]\end{pmatrix}$	$\begin{pmatrix}[0.12,0.15],\\ [0.30,0.35],\\ [0.25,0.30]\end{pmatrix}$	$\begin{pmatrix}[0.09,0.14],\\ [0.23,0.26],\\ [0.23,0.26]\end{pmatrix}$	$\begin{pmatrix}[0.13,0.13],\\ [0.37,0.37],\\ [0.30,0.30]\end{pmatrix}$

The neutrosophic importance weights related to main criteria in terms of DM1,DM2 and DM3 are computed and given in Tables 13, 14 and 15 respectively.

Table13: IVN importance weights for main criteria in terms of DM1

DM1	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.19213	0.21971	0.12913	0.16472	0.14019	0.17442
C2	0.21518	0.24176	0.10053	0.13695	0.11757	0.15249
C3	0.16444	0.19305	0.17932	0.21311	0.17352	0.20656
C4	0.16352	0.19213	0.17198	0.20613	0.17442	0.20746
C5	0.12360	0.15333	0.24907	0.27906	0.22878	0.25905

Table14: IVN importance weights for main criteria in terms of DM2

DM2	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.19735	0.22602	0.11438	0.14706	0.12632	0.16341
C2	0.24690	0.27305	0.06342	0.09249	0.08174	0.12001
C3	0.16756	0.19733	0.15238	0.18373	0.15722	0.19321
C4	0.14688	0.17731	0.19855	0.23767	0.18850	0.22340
C5	0.09459	0.12626	0.30919	0.33902	0.27025	0.29995

Table15:IVN importance weights for main criteria in terms of DM3

DM3	T^L	T^U	I^L	I^U	F^L	F^U
C1	0.17814	0.20844	0.16979	0.20737	0.16892	0.20355
C2	0.23189	0.24733	0.08516	0.10597	0.10262	0.13274
C3	0.13627	0.16759	0.21856	0.25333	0.20749	0.24013
C4	0.22067	0.23878	0.10337	0.12959	0.11868	0.14880
C5	0.10596	0.13783	0.27586	0.30372	0.24425	0.27476

Crisp weights for main criteria in terms of DM1, DM2 and DM3 are obtained via Eq.(7) and shown as Table 16. The main criteria weights for all DMs are aggregated via geometric mean. Then normalization process is applied and final weights related to main criteria are computed.

Table16: The weights related to main criteria

Main criteria	DM1	DM2	DM3	Final weight
C1	0.216579	0.218336	0.213228	0.215966
C2	0.234731	0.256496	0.233400	0.241236
C3	0.199254	0.193943	0.175422	0.189196
C4	0.193661	0.187992	0.230392	0.203114
C5	0.163108	0.139990	0.149402	0.150487

According to Table 16 while location (C2) was found as the most important criterion having with the value of 0.241236, personal condition (C5) was acquired as the least important one with the value of 0.150487.

Similarly, all the above steps are applied for each sub-criteria and crisp weights related to sub-criteria in terms of DM1, DM2 and DM3 are given in Table 17.

Table17: The weights related to sub-criteria

Sub-criteria	DM1	DM2	DM3	Final weight
C11	0.308890	0.290151	0.310343	0.311659
C12	0.247932	0.261562	0.397909	0.303963
C13	0.426989	0.427536	0.285838	0.384379
C21	0.398004	0.344704	0.308992	0.351303
C22	0.262654	0.344704	0.379211	0.327461
C23	0.334149	0.313920	0.308992	0.321236
C31	0.318410	0.310343	0.308992	0.315105
C32	0.265964	0.285838	0.308992	0.285437
C33	0.412249	0.397909	0.379211	0.399458
C41	0.452750	0.500000	0.400738	0.452362
C42	0.545238	0.500000	0.590415	0.547638
C51	0.245700	0.231712	0.244616	0.240362
C52	0.245700	0.236291	0.244616	0.241935
C53	0.291011	0.280958	0.233130	0.266871
C54	0.220591	0.254794	0.281586	0.250832

After that local and global importance weights of criteria/sub-criteria are obtained and shown in Table 18.

Table18: Local and global weights of criteria/sub-criteria

Main criteria	Weight	Sub-criteria	Local weight	Sub-criteria	Global weight	Rank
C1	0.215966	C11	0.311659	C11	0.067308	8
		C12	0.303963	C12	0.065646	9
		C13	0.384379	C13	0.083013	4
C2	0.241236	C21	0.351303	C21	0.084747	3
		C22	0.327461	C22	0.078995	5
		C23	0.321236	C23	0.077494	6
C3	0.189196	C31	0.315105	C31	0.059617	10
		C32	0.285437	C32	0.054004	11
		C33	0.399458	C33	0.075576	7
C4	0.203114	C41	0.452362	C41	0.091881	2
		C42	0.547638	C42	0.111233	1
C5	0.150487	C51	0.240362	C51	0.036171	15
		C52	0.241935	C52	0.036408	14
		C53	0.266871	C53	0.040161	12
		C54	0.250832	C54	0.037747	13

According to Table 18 while trustworthiness (C42) was found as the most important sub-criterion with a value of 0.111233, fiscal status (C51) was acquired as the least important one having a value of 0.036171. The ranking of other sub-criteria can be stated as:

C41> C21 > C13 > C22 > C23 > C33 > C11 > C12 > C31 > C32 > C53 > C54 > C52.

5. CONCLUSIONS

As explained earlier, the purpose of this study is to provide a perspective on the criteria that are important in franchisee selection. The study presents the criteria that are important in the selection of franchisees of a global cafe chain business by taking into account the criteria in the context of the studies in the literature.

The study has three specific purposes: 1-to identify the criteria for franchisors to evaluate the franchisee, 2-to rank the importance of the criteria considered by the franchisors, 3-to demonstrate the use of the IVN-AHP technique in determining the selection criteria. In the analysis, it is concluded that location (C2) is the most important criterion for decision makers in franchisee selection. The second most important criterion is financial condition (C1). These criteria are followed by reputation (C4), personnel (C3), and personal condition (C5).

The analysis of this study is limited to the franchisee selection of a global cafe chain business. However, the study reveals that other criteria apart from the financial criteria are taken into account in the selection criteria as in the studies of Valeri (2020), Hsu and Chen (2008) etc.

Franchisors' selection of franchisees based on these and similar criteria will allow them to avoid future problems and make a quality selection.

Researchers and decision makers can consider the selection criteria in more detail in their future studies and determine the selection criteria in different sectors and fields. In addition, IVN-AHP and other multi-criteria decision analysis methods can be used effectively in similar and different selection problems. In the future, researches specific to different businesses that examine subgroups of the service sector can also be conducted.

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Weighted Geometric Aggregation Operator Based MAGDM Strategy for Pentapartitioned Neutrosophic Numbers

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ABSTRACT

In this paper, we define pentapartitioned number and the geometric aggregation operator. Some of their basic properties are established. Then, we develop a decision making strategy to solve multi-attribute group decision making under the pentapartitioned neutrosophic number environment. An illustrative example of multi attribute group decision making problem is solved to show the applicability of the developed strategy.

KEYWORDS: **Neutrosophic set, Single valued neutrosophic set, pentapartitioned neutrosophic set, pentapartitioned neutrosophic number, geometric aggregative operator.**

1. INTRODUCTION

Smarandache (1998) defined the Neutrosophic Set (NS) by extending the Fuzzy Set (FS) (Zadeh, 1965) and the Intuitionistic FS (IFS) (Atanassov, 1986). Single Valued NS (SVNS) (Wang et al., 2010) was proposed as a simple form of NS. Based on our-valued logic (Belnap, 1977) and multi-valued refined neutrosophic logic (Smarandache, 2013), Quadripartition SVNS (QSVNS) (Chatterjee et al., 2016) was introduced. Pramanik (2022) presented the Interval Quardiparitioned NS (IQNS). In 2020, Mallick and Pramanik (2020) defined the Pentapartitioned Neutrosophic Set (PNS) using multi-valued logic (Smarandache, 2013) by replacing indeterminacy with three independent components. Pentapartitioned neutrosophic graph was developed by Das et al. (2022) and Quek et al. (2022). Pramanik (2023a) developed interval PNS (IPNS) using PNS and Interval NS (INS) (Wang et al., 2005). Broumi et al. (2018), Pramanik (2020), and Pramanik (2022) presented an overview of NS, rough NS, and SVNS respectively. Ye (2014) developed the Single-Valued Neutrosophic (SVN) Weighted Averaging (SVNWA) and SVN Weighted Geometric (SVNWG) operators. Liu et al. (2014) developed the SVN Hamacher weighted averaging (SVNHWA), SVN Hamacher ordered weighted averaging (SVNHOWA), SVN Hamacher weighted geometric (SVNHWG) and SVN Hamacher ordered weighted geometric (SVNHOWG). Peng et al. (2016) characterized the operations of SVN Set (SVNS) and developed the SVN Ordered Weighted Average (SVNOWA) and SVN Ordered Weighted Geometric (SVNOWG) operators. Nancy and Garg (2016) proposed the Frank norm-based weighted averaging and geometric operators namely, SVN Frank weighted averaging and geometric operators denoted by SVNFWA and SVNFWG respectively. Pramanik (2023) developed pentapartitioned neutrosophic average operating operator.

Multi-Attribute Decision Making (MADM) (Ye, 2013; Dey, Pramanik, & Giri, 2015; Pramanik, Dalapati, Alam, & Roy, 2018; Pramanik, Dalapati, Alam, Smarandache, & Roy, 2018, Mondal, Pramanik, & Giri, 2018a, 2018b; Pramanik, Dey, Smarandache, & Ye, 2018; Mallick & Pramanik, 2019, 2020, 2021a, 2021b, Pramanik & Mallick, 2018, 2019; Pramanik & Mondal, 2015b; Smarandache & Pramanik, 2016, 2018) is a branch of operational research that deals with the structure of decision making involving conflicting criteria and chooses the best alternative from a set of feasible alternatives. To deal with group decision making, MADM is extended to Multi-Attribute Group Decision Making (MAGDM). There exists a vast literature on MAGDM (Pramanik, Banerjee, & Giri, 2016; Dalapati, Pramanik, Alam, Smarandache, & Roy, 2017; Mondal, Pramanik, & Giri, 2018c; Pramanik, & Dalapati, 2018).) in neutrosophic environments. Different weighted average operators were defined in different fuzzy and neutrosophic environments to solve the MAGDM problems. PNS (Mallick & Pramanik, 2020) is a newly developed set and its number, and aggregation operators are to be developed. Das, Shil, and Pramanik (2021) developed the Grey Relational Analysis (GRA) based MADM strategy in the Pentapartitioned Neutrosophic Number (PNN) environment by extending the GRA (Biswas et al., 2014a, 2014b) based MADM strategy in the SVNS environment. Das, Shil, and Tripathy (2021) presented the tangent similarity based MADM strategy in the PNN environment by extending the work of Pramanik and Mondal (2015a). Saha et al. (2022) presented the Dice similarity-based MADM strategy in the PNN environment. Das, Shil, and Pramanik (2022) developed the hyperbolic sine similarity measure based MADM strategy in the PNN environment. Majumderet al. (2023) presented the hyperbolic tangent similarity measure based MADM strategy. Pramanik (2023b) presented the ARAS strategy based on the PNN weighted averaging operator in the PNN environment.

Research gap: PN Geometric Average (PNWGA) operator is not proposed in the literature and MAGDM strategy based on the PNWGA operator is not developed .

Motivation: To fill the research gap, we initiate to study the MAGDM strategy using PNNWGA operator.

The main contributions of this work are outlined as follows:

(1) Pentapartitioned Neutrosophic Number (PNN) is introduced using five independent components.
(2) PNN geometric average operator is introduced and its desirable properties are established.
(3) MAGDM strategy using the PNWGA operator with PNNs is developed.
(4) Applicability of the developed strategy is shown by solving a green supplier selection problem.

The remainder of this paper unfolds as follows: Section 2 presents the PNN, operation laws for PNNs. Section 3 presents the PNGWA operator and their basic properties and proofs of the related theorems. Section 4 develops a MAGDM strategy based on the PNGWA operator under PNN environment. In Section 5, a green supplier selection problem is solved. Section 6 presents the future scope of research band concluding remarks.

2. *PENTAPARTITIONED NEUTROSOPHIC NUMBERS*

We introduce the notion of PNN and study some of their properties.
A pentapartitioned neutrosophic number is defined as follows:
Definition 1: An element of $[0,1]^5$, denoted by $\eta=\langle t_\eta,c_\eta,g_\eta,u_\eta,f_\eta\rangle$, where t_η denotes the truth - membership degree of η, c_η denotes contradiction membership degree, g_η denotes an ignorance membership degree, u_η denotes unknown membership degree and f_η denotes a falsity membership degree such that for each $p\in P$, $t_\eta,c_\eta,g_\eta,u_\eta,f_\eta\in[0,1]$ and $0\le t_\eta(p)+c_\eta(p)+g_\eta(p)+u_\eta(p)+f_\eta(p)\le 5$.
This collection of elements is said to be Pentapartitioned Neutrosophic Number (PNNs).
Definition 2: Assume that $\eta_1,\eta_2\in PNN$. Then the addition and multiplication of two PNNs are defined as follows:

$$\eta_1+\eta_2=\langle t_{\eta_1}+t_{\eta_2}-t_{\eta_1}.t_{\eta_2},c_{\eta_1}+c_{\eta_2}-c_{\eta_1}.c_{\eta_2},g_{\eta_1}.g_{\eta_2},u_{\eta_1}.u_{\eta_2},f_{\eta_1}.f_{\eta_2}\rangle \quad (1)$$

$$\eta_1.\eta_2=\langle t_{\eta_1}.t_{\eta_2},c_{\eta_1}.c_{\eta_2},g_{\eta_1}+g_{\eta_2}-g_{\eta_1}.g_{\eta_2},u_{\eta_1}+u_{\eta_2}-u_{\eta_1}.u_{\eta_2},f_{\eta_1}+f_{\eta_2}-f_{\eta_1}.f_{\eta_2}\rangle \quad (2)$$

Proposition 1: For any $\eta_1,\eta_2,\eta_3\in PNN$, the following operations hold:

i. $\eta_1+\eta_2=\eta_2+\eta_1$
ii. $(\eta_1+\eta_2)+\eta_3=\eta_1+(\eta_2+\eta_3)$
iii. $\eta_1.\eta_2=\eta_2.\eta_1$
iv. $(\eta_1.\eta_2)\eta_3=\eta_1(\eta_2.\eta_3)$
v. $s\eta=\langle 1-(1-t_\eta)^s,1-(1-c_\eta)^s,(g_\eta)^s,(u_\eta)^s,(f_\eta)^s\rangle, c\in N$
vi. $\eta^s=\langle (t_\eta)^s,(c_\eta)^s,1-(1-g_\eta)^s,1-(1-u_\eta)^s,1-(1-f_\eta)^s\rangle, s\in N$
vii. $s(\eta_1+\eta_2)=s\eta_2+s\eta_2, s\in N$
viii. $(s_1+s_2)\eta=s_1\eta+s_2\eta, s_1,s_2\in N$

Proof: Assume that, $\eta_1=\langle t_{\eta_1},c_{\eta_1},g_{\eta_1},u_{\eta_1},f_{\eta_1}\rangle$, $\eta_2=\langle t_{\eta_2},c_{\eta_2},g_{\eta_2},u_{\eta_2},f_{\eta_2}\rangle$ and $\eta_3=\langle t_{\eta_3},c_{\eta_3},g_{\eta_3},u_{\eta_3},f_{\eta_3}\rangle$

$$\begin{aligned}&(i)\eta_1+\eta_2\\&=\langle t_{\eta_1}+t_{\eta_2}-t_{\eta_1}.t_{\eta_2},c_{\eta_1}+c_{\eta_2}-c_{\eta_1}.c_{\eta_2},g_{\eta_1}.g_{\eta_2},u_{\eta_1}.u_{\eta_2},f_{\eta_1}.f_{\eta_2}\rangle\\&=\langle t_{\eta_2}+t_{\eta_1}-t_{\eta_2}.t_{\eta_1},c_{\eta_2}+c_{\eta_1}-c_{\eta_2}.c_{\eta_1},g_{\eta_2}.g_{\eta_1},u_{\eta_2}.u_{\eta_1},f_{\eta_2}.f_{\eta_1}\rangle\\&=\eta_2+\eta_1\end{aligned}$$

$\eta_1+\eta_2=\eta_2+\eta_1(proved)$

$$\begin{aligned}&(ii)(\eta_1+\eta_2)+\eta_3\\&=\langle t_{\eta_1}+t_{\eta_2}-t_{\eta_1}.t_{\eta_2},c_{\eta_1}+c_{\eta_2}-c_{\eta_1}.c_{\eta_2},g_{\eta_1}.g_{\eta_2},u_{\eta_1}.u_{\eta_2},f_{\eta_1}.f_{\eta_2}\rangle+\langle t_{\eta_3},c_{\eta_3},g_{\eta_3},u_{\eta_3},f_{\eta_3}\rangle\\&=\left\langle\begin{matrix}(t_{\eta_1}+t_{\eta_2}-t_{\eta_1}.t_{\eta_2})+t_{\eta_3}-(t_{\eta_1}+t_{\eta_2}-t_{\eta_1}.t_{\eta_2}).t_{\eta_3},(c_{\eta_1}+c_{\eta_2}-c_{\eta_1}.c_{\eta_2})+c_{\eta_3}-(c_{\eta_1}+c_{\eta_2}-c_{\eta_1}.c_{\eta_2}).c_{\eta_3},\\(g_{\eta_1}.g_{\eta_2}).g_{\eta_3},(u_{\eta_1}.u_{\eta_2}).u_{\eta_3},(f_{\eta_1}.f_{\eta_2}).f_{\eta_3}\end{matrix}\right\rangle\\&=\left\langle\begin{matrix}t_{\eta_1}+t_{\eta_2}+t_{\eta_3}-t_{\eta_1}.t_{\eta_2}-t_{\eta_1}.t_{\eta_3}-t_{\eta_2}.t_{\eta_3}+t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}+c_{\eta_2}+c_{\eta_3}-c_{\eta_1}.c_{\eta_2}-c_{\eta_1}.c_{\eta_3}-c_{\eta_2}.c_{\eta_3}+c_{\eta_1}.c_{\eta_2}.c_{\eta_3},\\g_{\eta_1}.g_{\eta_2}.g_{\eta_3},u_{\eta_1}.u_{\eta_2}.u_{\eta_3},f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{matrix}\right\rangle\\&=\left\langle\begin{matrix}t_{\eta_1}+t_{\eta_2}+t_{\eta_3}-t_{\eta_1}.t_{\eta_2}-t_{\eta_2}.t_{\eta_3}-t_{\eta_3}.t_{\eta_1}+t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}+c_{\eta_2}+c_{\eta_3}-c_{\eta_1}.c_{\eta_2}-c_{\eta_2}.c_{\eta_3}-c_{\eta_3}.c_{\eta_1}+c_{\eta_1}.c_{\eta_2}.c_{\eta_3},\\g_{\eta_1}.g_{\eta_2}.g_{\eta_3},u_{\eta_1}.u_{\eta_2}.u_{\eta_3},f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{matrix}\right\rangle \quad (3)\end{aligned}$$

$$\eta_1+(\eta_2+\eta_3)$$
$$=\left\langle t_{\eta_1},c_{\eta_1},g_{\eta_1},u_{\eta_1},f_{\eta_1}\right\rangle+\left\langle t_{\eta_2}+t_{\eta_3}-t_{\eta_2}.t_{\eta_3},c_{\eta_2}+c_{\eta_3}-c_{\eta_2}.c_{\eta_3},g_{\eta_2}.g_{\eta_3},u_{\eta_2}.u_{\eta_3},f_{\eta_2}.f_{\eta_3}\right\rangle$$
$$=\left\langle\begin{array}{l}t_{\eta_1}+\left(t_{\eta_2}+t_{\eta_3}-t_{\eta_2}.t_{\eta_3}\right)-t_{\eta_1}.\left(t_{\eta_2}+t_{\eta_3}-t_{\eta_2}.t_{\eta_3}\right),c_{\eta_1}+\left(c_{\eta_2}+c_{\eta_3}-c_{\eta_2}.c_{\eta_3}\right)-c_{\eta_1}.\left(c_{\eta_2}+c_{\eta_3}-c_{\eta_2}.c_{\eta_3}\right),\\ g_{\eta_1}.\left(g_{\eta_2}.g_{\eta_3}\right),u_{\eta_1}.\left(u_{\eta_2}.u_{\eta_3}\right),f_{\eta_1}.\left(f_{\eta_2}.f_{\eta_3}\right)\end{array}\right\rangle$$
$$=\left\langle\begin{array}{l}t_{\eta_1}+t_{\eta_2}+t_{\eta_3}-t_{\eta_2}.t_{\eta_3}-t_{\eta_1}.t_{\eta_2}-t_{\eta_1}.t_{\eta_3}+t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}+c_{\eta_2}+c_{\eta_3}-c_{\eta_2}.c_{\eta_3}-c_{\eta_1}.c_{\eta_2}-c_{\eta_1}.c_{\eta_3}+c_{\eta_1}.c_{\eta_2}.c_{\eta_3},\\ g_{\eta_1}.g_{\eta_2}.g_{\eta_3},u_{\eta_1}.u_{\eta_2}.u_{\eta_3},f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{array}\right\rangle$$
$$=\left\langle\begin{array}{l}t_{\eta_1}+t_{\eta_2}+t_{\eta_3}-t_{\eta_1}.t_{\eta_2}-t_{\eta_2}.t_{\eta_3}-t_{\eta_3}.t_{\eta_1}+t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}+c_{\eta_2}+c_{\eta_3}-c_{\eta_1}.c_{\eta_2}-c_{\eta_2}.c_{\eta_3}-c_{\eta_3}.c_{\eta_1}+c_{\eta_1}.c_{\eta_2}.c_{\eta_3},\\ g_{\eta_1}.g_{\eta_2}.g_{\eta_3},u_{\eta_1}.u_{\eta_2}.u_{\eta_3},f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{array}\right\rangle(4)$$

Therefore, from (3) and (4), $(\eta_1+\eta_2)+\eta_3=\eta_1+(\eta_2+\eta_3)(proved)$

$$(iii)\eta_1.\eta_2=\left\langle t_{\eta_1}.t_{\eta_2},c_{\eta_1}.c_{\eta_2},g_{\eta_1}+g_{\eta_2}-g_{\eta_1}.g_{\eta_2},u_{\eta_1}+u_{\eta_2}-u_{\eta_1}.u_{\eta_2},f_{\eta_1}+f_{\eta_2}-f_{\eta_1}.f_{\eta_2}\right\rangle$$
$$=\left\langle t_{\eta_2}.t_{\eta_1},c_{\eta_2}.c_{\eta_1},g_{\eta_2}+g_{\eta_1}-g_{\eta_2}.g_{\eta_1},u_{\eta_2}+u_{\eta_1}-u_{\eta_2}.u_{\eta_1},f_{\eta_2}+f_{\eta_1}-f_{\eta_2}.f_{\eta_1}\right\rangle$$
$$=\eta_2.\eta_1$$
$$\Rightarrow\eta_1.\eta_2=\eta_2.\eta_1(proved)$$

$$(iv)(\eta_1.\eta_2).\eta_3=\left\langle t_{\eta_1}.t_{\eta_2},c_{\eta_1}.c_{\eta_2},g_{\eta_1}+g_{\eta_2}-g_{\eta_1}.g_{\eta_2},u_{\eta_1}+u_{\eta_2}-u_{\eta_1}.u_{\eta_2},f_{\eta_1}+f_{\eta_2}-f_{\eta_1}.f_{\eta_2}\right\rangle.\left\langle t_{\eta_3},c_{\eta_3},g_{\eta_3},u_{\eta_3},f_{\eta_3}\right\rangle$$
$$=\left\langle\begin{array}{l}\left(t_{\eta_1}.t_{\eta_2}\right).t_{\eta_3},\left(c_{\eta_1}.c_{\eta_2}\right).c_{\eta_3},\left(g_{\eta_1}+g_{\eta_2}-g_{\eta_1}.g_{\eta_2}\right)+g_{\eta_3}-\left(g_{\eta_1}+g_{\eta_2}-g_{\eta_1}.g_{\eta_2}\right).g_{\eta_3},\\ \left(u_{\eta_1}+u_{\eta_2}-u_{\eta_1}.u_{\eta_2}\right)+u_{\eta_3}-\left(u_{\eta_1}+u_{\eta_2}-u_{\eta_1}.u_{\eta_2}\right).u_{\eta_3},\left(f_{\eta_1}+f_{\eta_2}-f_{\eta_1}.f_{\eta_2}\right)+f_{\eta_3}-\left(f_{\eta_1}+f_{\eta_2}-f_{\eta_1}.f_{\eta_2}\right).f_{\eta_3}\end{array}\right\rangle$$
$$=\left\langle\begin{array}{l}t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}.c_{\eta_2}.c_{\eta_3},g_{\eta_1}+g_{\eta_2}+g_{\eta_3}-g_{\eta_1}.g_{\eta_2}-g_{\eta_1}.g_{\eta_3}-g_{\eta_2}.g_{\eta_3}+g_{\eta_1}.g_{\eta_2}.g_{\eta_3},\\ u_{\eta_1}+u_{\eta_2}+u_{\eta_3}-u_{\eta_1}.u_{\eta_2}-u_{\eta_1}.u_{\eta_3}-u_{\eta_2}.u_{\eta_3}+u_{\eta_1}.u_{\eta_2}.u_{\eta_3}\\ ,f_{\eta_1}+f_{\eta_2}+f_{\eta_3}-f_{\eta_1}.f_{\eta_2}-f_{\eta_1}.f_{\eta_3}+f_{\eta_2}.f_{\eta_3}+f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{array}\right\rangle$$
$$\left\langle\begin{array}{l}t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}.c_{\eta_2}.c_{\eta_3},g_{\eta_1}+g_{\eta_2}+g_{\eta_3}-g_{\eta_1}.g_{\eta_2}-g_{\eta_2}.g_{\eta_3}-g_{\eta_3}.g_{\eta_1}.+g_{\eta_1}.g_{\eta_2}.g_{\eta_3},\\ u_{\eta_1}+u_{\eta_2}+u_{\eta_3}-u_{\eta_1}.u_{\eta_2}-u_{\eta_2}.u_{\eta_3}-u_{\eta_3}.u_{\eta_1}+u_{\eta_1}.u_{\eta_2}.u_{\eta_3}\\ ,f_{\eta_1}+f_{\eta_2}+f_{\eta_3}-f_{\eta_1}.f_{\eta_2}-f_{\eta_2}.f_{\eta_3}-f_{\eta_3}.f_{\eta_1}+f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{array}\right\rangle(5)$$

$$\eta_1.(\eta_2.\eta_3)=$$
$$\left\langle t_{\eta_1},c_{\eta_1},g_{\eta_1},u_{\eta_1},f_{\eta_1}\right\rangle.\left\langle t_{\eta_2}.t_{\eta_3},c_{\eta_2}.c_{\eta_3},g_{\eta_2}+g_{\eta_3}-g_{\eta_2}.g_{\eta_3},u_{\eta_2}+u_{\eta_3}-u_{\eta_2}.u_{\eta_3},f_{\eta_2}+f_{\eta_3}-f_{\eta_2}.f_{\eta_3}\right\rangle$$
$$=\left\langle\begin{array}{l}t_{\eta_1}.\left(t_{\eta_2}.t_{\eta_3}\right),c_{\eta_1}.\left(c_{\eta_2}.c_{\eta_3}\right),g_{\eta_1}+\left(g_{\eta_2}+g_{\eta_3}-g_{\eta_2}.g_{\eta_3}\right)-g_{\eta_1}.\left(g_{\eta_2}+g_{\eta_3}-g_{\eta_2}.g_{\eta_3}\right),\\ u_{\eta_1}+\left(u_{\eta_2}+u_{\eta_3}-u_{\eta_2}.u_{\eta_3}\right)-u_{\eta_1}.\left(u_{\eta_2}+u_{\eta_3}-u_{\eta_2}.u_{\eta_3}\right),f_{\eta_1}+\left(f_{\eta_2}+f_{\eta_3}-f_{\eta_2}.f_{\eta_3}\right)-f_{\eta_1}.\left(f_{\eta_2}+f_{\eta_3}-f_{\eta_2}.f_{\eta_3}\right)\end{array}\right\rangle$$
$$=\left\langle\begin{array}{l}t_{\eta_1}.t_{\eta_2}.t_{\eta_3},c_{\eta_1}.c_{\eta_2}.c_{\eta_3},g_{\eta_1}+g_{\eta_2}+g_{\eta_3}-g_{\eta_1}.g_{\eta_2}-g_{\eta_2}.g_{\eta_3}-g_{\eta_3}.g_{\eta_1}.+g_{\eta_1}.g_{\eta_2}.g_{\eta_3},\\ u_{\eta_1}+u_{\eta_2}+u_{\eta_3}-u_{\eta_1}.u_{\eta_2}-u_{\eta_2}.u_{\eta_3}-u_{\eta_3}.u_{\eta_1}+u_{\eta_1}.u_{\eta_2}.u_{\eta_3},f_{\eta_1}+f_{\eta_2}+f_{\eta_3}-f_{\eta_1}.f_{\eta_2}-f_{\eta_2}.f_{\eta_3}-f_{\eta_3}.f_{\eta_1}+f_{\eta_1}.f_{\eta_2}.f_{\eta_3}\end{array}\right\rangle(6)$$

Therefore from (5) and (6), $(\eta_1.\eta_2).\eta_3=\eta_1.(\eta_2.\eta_3)(proved)$

(v) Let $\eta=\left\langle t_\eta,c_\eta,g_\eta,u_\eta,f_\eta\right\rangle\in PNN$

By definition,

$$1\eta=\left\langle 1-(1-t_\eta)^1,1-(1-c_\eta)^1,(g_\eta)^1,(u_\eta)^1,(f_\eta)^1\right\rangle$$

Suppose that the result holds for $s=k,k\in N$. Therefore,

$$k\eta = \left\langle 1-(1-t_\eta)^k, 1-(1-c_\eta)^k, (g_\eta)^k, (u_\eta)^k, (f_\eta)^k \right\rangle \quad (7)$$

$$\therefore (k+1)\eta = k\eta + \eta$$

$$= \left\langle 1-(1-t_\eta)^k, 1-(1-c_\eta)^k, (g_\eta)^k, (u_\eta)^k, (f_\eta)^k \right\rangle + \left\langle t_\eta, c_\eta, g_\eta, u_\eta, f_\eta \right\rangle$$

$$= \left\langle t_\eta + \left\{1-(1-t_\eta)^k\right\} - t_\eta\left\{1-(1-t_\eta)^k\right\}, c_\eta + \left\{1-(1-c_\eta)^k\right\} - c_\eta\left\{1-(1-c_\eta)^k\right\}, (g_\eta)^k.g_\eta, (u_\eta)^k.u_\eta, (f_\eta)^k.f_\eta \right\rangle$$

$$= \left\langle 1-(1-t_\eta)^{k+1}, 1-(1-c_\eta)^{k+1}, (g_\eta)^{k+1}, (u_\eta)^{k+1}, (f_\eta)^{k+1} \right\rangle (8)$$

Thus, from equation (7) and (8), by principal of mathematical induction,

$$s\eta = \left\langle 1-(1-t_\eta)^s, 1-(1-c_\eta)^s, (g_\eta)^s, (u_\eta)^s, (f_\eta)^s \right\rangle, \forall c \in N$$

(v) Let $\eta = \left\langle t_\eta, c_\eta, g_\eta, u_\eta, f_\eta \right\rangle \in PNN$

By definition,

$$\eta^1 = \left\langle (t_\eta)^1, (c_\eta)^1, 1-(1-g_\eta)^1, 1-(1-u_\eta)^1, 1-(1-f_\eta)^1 \right\rangle$$

Suppose that the result holds for $\eta = k, k \in N$. Therefore,

$$\eta^k = \left\langle (t_\eta)^k, (c_\eta)^k, 1-(1-g_\eta)^k, 1-(1-u_\eta)^k, 1-(1-f_\eta)^k \right\rangle \quad (9)$$

$$\therefore \eta^{k+1} = \eta^k.\eta$$

$$= \left\langle (t_\eta)^k, (c_\eta)^k, 1-(1-g_\eta)^k, 1-(1-u_\eta)^k, 1-(1-f_\eta)^k \right\rangle . \left\langle t_\eta, c_\eta, g_\eta, u_\eta, f_\eta \right\rangle$$

$$= \left\langle (t_\eta)^{k+1}, (c_\eta)^{k+1}, 1-(1-g_\eta)^k + g_\eta - (1-(1-g_\eta)^k).g_\eta, 1-(1-u_\eta)^k + u_\eta - (1-(1-u_\eta)^k).u_\eta, 1-(1-f_\eta)^k + f_\eta - \left(1-(1-f_\eta)^k\right).f_\eta \right\rangle$$

$$== \left\langle (t_\eta)^{k+1}, (c_\eta)^{k+1}, 1-(1-g_\eta)^{k+1}, 1-(1-u_\eta)^{k+1}, 1-(1-f_\eta)^{k+1} \right\rangle \quad (10)$$

Thus, from equation (9) and (10), by principal of mathematical induction,

$$\eta^s = \left\langle (t_\eta)^s, (c_\eta)^s, 1-(1-g_\eta)^s, 1-(1-u_\eta)^s, 1-(1-f_\eta)^s \right\rangle, s \in N$$

(vii) $\eta_1 + \eta_2 = \left\langle t_{\eta_1} + t_{\eta_2} - t_{\eta_1}.t_{\eta_2}, c_{\eta_1} + c_{\eta_2} - c_{\eta_1}.c_{\eta_2}, g_{\eta_1}.g_{\eta_2}, u_{\eta_1}.u_{\eta_2}, f_{\eta_1}.f_{\eta_2} \right\rangle$

$L.H.S$

$$s(\eta_1 + \eta_2)$$

$$= \left\langle 1-\left\{1-\left(t_{\eta_1} + t_{\eta_2} - t_{\eta_1}.t_{\eta_2}\right)\right\}^s, 1-\left\{1-\left(c_{\eta_1} + c_{\eta_2} - c_{\eta_1}.c_{\eta_2}\right)\right\}^s, \left(g_{\eta_1}.g_{\eta_2}\right)^s, \left(u_{\eta_1}.u_{\eta_2}\right)^s, \left(f_{\eta_1}.f_{\eta_2}\right)^s \right\rangle \quad (7)$$

R.H.S

$$s\eta_2 + s\eta_1$$

$$= \left\langle 1-\left(1-t_{\eta_1}\right)^s, 1-\left(1-c_{\eta_1}\right)^s, \left(g_{\eta_1}\right)^s, \left(u_{\eta_1}\right)^s, \left(f_{\eta_1}\right)^s \right\rangle + \left\langle 1-\left(1-t_{\eta_2}\right)^s, 1-\left(1-c_{\eta_2}\right)^s, \left(g_{\eta_2}\right)^s, \left(u_{\eta_2}\right)^s, \left(f_{\eta_2}\right)^s \right\rangle$$

$$= \left\langle \begin{array}{l} 1-\left(1-t_{\eta_1}\right)^s + 1-\left(1-t_{\eta_2}\right)^s - \left\{1-\left(1-t_{\eta_1}\right)^s\right\}\left\{1-\left(1-t_{\eta_2}\right)^s\right\}, 1-\left(1-c_{\eta_1}\right)^s + 1-\left(1-c_{\eta_2}\right)^s \\ -\left\{1-\left(1-c_{\eta_2}\right)^s\right\}\left\{1-\left(1-c_{\eta_1}\right)^s\right\}, \left(g_{\eta_1}\right)^s.\left(g_{\eta_2}\right)^s, \left(u_{\eta_1}\right)^s.\left(u_{\eta_2}\right)^s, \left(f_{\eta_1}\right)^s.\left(f_{\eta_2}\right)^s \end{array} \right\rangle \quad (11)$$

Now,

$$1-\left(1-t_{\eta_1}\right)^s + 1-\left(1-t_{\eta_2}\right)^s - \left\{1-\left(1-t_{\eta_1}\right)^s\right\}\left\{1-\left(1-t_{\eta_2}\right)^s\right\}$$

$$= 2-\left(1-t_{\eta_1}\right)^s - \left(1-t_{\eta_2}\right)^s - \left\{1-\left(1-t_{\eta_1}\right)^s - \left(1-t_{\eta_2}\right)^s + \left(1-t_{\eta_1}\right)^s\left(1-t_{\eta_2}\right)^s\right\}$$

$$= 2-\left(1-t_{\eta_1}\right)^s - \left(1-t_{\eta_2}\right)^s - 1 + \left(1-t_{\eta_1}\right)^s + \left(1-t_{\eta_2}\right)^s - \left\{1-\left(t_{\eta_2} + t_{\eta_1} - t_{\eta_1}t_{\eta_2}\right)\right\}^s$$

$$= 1-\left\{1-\left(t_{\eta_2} + t_{\eta_1} - t_{\eta_1}t_{\eta_2}\right)\right\}^s$$

Similarly,

$$1-\left(1-c_{\eta_1}\right)^s+1-\left(1-c_{\eta_2}\right)^s-\left\{1-\left(1-c_{\eta_2}\right)^s\right\}\left\{1-\left(1-c_{\eta_1}\right)^s\right\}$$
$$=2-\left(1-c_{\eta_1}\right)^s-\left(1-c_{\eta_2}\right)^s-\left\{1-\left(1-c_{\eta_1}\right)^s-\left(1-c_{\eta_2}\right)^s+\left(1-c_{\eta_1}\right)^s\left(1-c_{\eta_2}\right)^s\right\}$$
$$=2-\left(1-c_{\eta_1}\right)^s-\left(1-c_{\eta_2}\right)^s-1+\left(1-c_{\eta_1}\right)^s+\left(1-c_{\eta_2}\right)^s-\left\{1-(c_{\eta_2}+c_{\eta_1}-c_{\eta_1}.c_{\eta_2})\right\}^s$$
$$=1-\left\{1-(c_{\eta_2}+c_{\eta_1}-c_{\eta_1}.c_{\eta_2})\right\}^s$$

R.H.S

$$\left\langle 1-\left\{1-\left(t_{\eta_2}+t_{\eta_1}-t_{\eta_1}t_{\eta_2}\right)\right\}^s,1-\left\{1-(c_{\eta_2}+c_{\eta_1}-c_{\eta_1}.c_{\eta_2})\right\}^s,\left(g_{\eta_1}g_{\eta_2}\right)^s,\left(u_{\eta_1}u_{\eta_2}\right)^s,\left(f_{\eta_1}f_{\eta_2}\right)^s\right\rangle$$

= L.H.S

$(viii)\, R.H.S$

$s_1\eta+s_2\eta$

$$=\left\langle 1-\left(1-t_\eta\right)^{s_1},1-\left(1-c_\eta\right)^{s_1},\left(g_\eta\right)^{s_1},\left(u_\eta\right)^{s_1},\left(f_\eta\right)^{s_1}\right\rangle+\left\langle 1-\left(1-t_\eta\right)^{s_2},1-\left(1-c_\eta\right)^{s_2},\left(g_\eta\right)^{s_2},\left(u_\eta\right)^{s_2},\left(f_\eta\right)^{s_2}\right\rangle$$
$$=\left\langle \begin{array}{l} 1-\left(1-t_\eta\right)^{s_1}+1-\left(1-t_\eta\right)^{s_2}-\left\{1-\left(1-t_\eta\right)^{s_1}\right\}.\left\{1-\left(1-t_\eta\right)^{s_2}\right\},1-\left(1-c_\eta\right)^{s_1}+1-\left(1-c_\eta\right)^{s_2} \\ -\left\{1-\left(1-c_\eta\right)^{s_1}\right\}\left\{1-\left(1-c_\eta\right)^{s_2}\right\},\left(g_\eta\right)^{s_1}\left(g_\eta\right)^{s_2},\left(u_\eta\right)^{s_1}\left(u_\eta\right)^{s_2},\left(f_\eta\right)^{s_1}\left(f_\eta\right)^{s_2} \end{array}\right\rangle$$
$$=\left\langle 1-\left(1-t_\eta\right)^{s_1+s_2},1-\left(1-c_n\right)^{s_1+s_2},\left(g_\eta\right)^{s_1+s_2},\left(u_\eta\right)^{s_1+s_2},\left(f_\eta\right)^{s_1+s_2}\right\rangle$$
$$=\left(s_1+s_2\right)\eta$$
$$=L.H.S$$

3. PENTAPARTITIONED NEUTROSOPHIC NUMBER WEIGHTED GEOMETRIC AGGREGATIVE OPERATOR

Definition 3.1. Let $\eta_i=\left\langle t_{\eta_i},c_{\eta_i},g_{\eta_i},u_{\eta_i},f_{\eta_i}\right\rangle(i=1,2,....m)$ be a collection of PNNs. A Pentapartitioned Neutrosophic Weighted Geometric Aggregation (PNWGA) operator is defined by:

$$PNWGA\left(\eta_1,\eta_2,....,\eta_m\right)=\prod_{i=1}^{m}\left(\eta_i\right)^{w_i} \tag{12}$$

where $w=\left(w_1,w_2,....,w_m\right)^T$ is the weight of $\eta_i\left(i=1,2,....,m\right)$ with $0\le w_i\le 1$ and $\sum_{i=1}^{n}w_i=1$

Theorem 1: Assume that $\eta_i=\left\langle t_{\eta_i},c_{\eta_i},g_{\eta_i},u_{\eta_i},f_{\eta_i}\right\rangle(i=1,2,....m)$ is a collection of PNNs and $w=\left(w_1,w_2,....,w_m\right)^T$ is the weight vector, where $0\le w_i\le 1$ and $\sum_{i=1}^{n}w_i=1$. Then

$$PNWGA\left(\eta_1,\eta_2,....,\eta_m\right)=\prod_{i=1}^{m}\left(\eta_i\right)^{w_i}$$
$$=\left(\eta_1\right)^{w_1}\otimes\left(\eta_2\right)^{w_2}\otimes....\otimes\left(\eta_m\right)^{w_m}$$
$$=\left\langle \prod_{i=1}^{m}\left(t_{\eta_i}\right)^{w_i},\prod_{i=1}^{m}\left(c_{\eta_i}\right)^{w_i},1-\prod_{i=1}^{m}\left(1-g_{\eta_i}\right)^{w_i},1-\prod_{i=1}^{m}\left(1-u_{\eta_i}\right)^{w_i},1-\prod_{i=1}^{m}\left(1-f_{\eta_i}\right)^{w_i}\right\rangle \tag{13}$$

Proof: By definition, for $w_1\in w$ and $\eta_1\in PNN$

$$\left(\eta_1\right)^{w_1}=\left\langle\left(t_{\eta_1}\right)^{w_1},\left(c_{\eta_1}\right)^{w_1},1-\left(1-g_{\eta_1}\right)^{w_1},1-\left(1-u_{\eta_1}\right)^{w_1},1-\left(1-f_{\eta_1}\right)^{w_1}\right\rangle$$

Thus, the expression trivially holds for n=1. Similarly, for $w_2 \in w$ and $\eta_2 \in PNN$

$$(\eta_2)^{w_2}$$
$$=\left\langle \left(t_{\eta_2}\right)^{w_2}, \left(c_{\eta_2}\right)^{w_2}, 1-\left(1-g_{\eta_2}\right)^{w_2}, 1-\left(1-u_{\eta_2}\right)^{w_2}, 1-\left(1-f_{\eta_2}\right)^{w_2} \right\rangle$$

Therefore, we can write,

$$PNNWGA(\eta_1,\eta_2) = (\eta_1)^{w_1} \otimes (\eta_2)^{w_2}$$
$$=\left\langle \begin{array}{l} \left\{\left(t_{\eta_1}\right)^{w_1}\left(t_{\eta_2}\right)^{w_2}\right\}, \left\{\left(c_{\eta_1}\right)^{w_1}\left(c_{\eta_2}\right)^{w_2}\right\}, \left\{1-\left(1-g_{\eta_1}\right)^{w_1}\right\} + \left\{1-\left(1-g_{\eta_2}\right)^{w_2}\right\} - \left\{1-\left(1-g_{\eta_1}\right)^{w_1}\right\}\left\{1-\left(1-g_{\eta_2}\right)^{w_2}\right\}, \left\{1-\left(1-u_{\eta_1}\right)^{w_1}\right\} \\ +\left\{1-\left(1-u_{\eta_2}\right)^{w_2}\right\} - \left\{1-\left(1-u_{\eta_1}\right)^{w_1}\right\}\left\{1-\left(1-u_{\eta_2}\right)^{w_2}\right\}, \left\{1-\left(1-f_{\eta_1}\right)^{w_1}\right\} + \left\{1-\left(1-f_{\eta_2}\right)^{w_2}\right\} - \left\{1-\left(1-f_{\eta_1}\right)^{w_1}\right\} + \left\{1-\left(1-f_{\eta_2}\right)^{w_2}\right\} \end{array} \right\rangle$$
$$=\left\langle \begin{array}{l} \left\{\left(t_{\eta_1}\right)^{w_1}\left(t_{\eta_2}\right)^{w_2}\right\}, \left\{\left(c_{\eta_1}\right)^{w_1}\left(c_{\eta_2}\right)^{w_2}\right\}, \left\{1-\left(1-g_{\eta_1}\right)^{w_1}\left(1-g_{\eta_2}\right)^{w_2}\right\}, \left\{1-\left(1-u_{\eta_1}\right)^{w_1}\left(1-u_{\eta_2}\right)^{w_2}\right\}, \\ \left\{1-\left(1-u_{\eta_1}\right)^{w_1}\left(1-u_{\eta_2}\right)^{w_2}\right\}, \left\{1-\left(1-f_{\eta_1}\right)^{w_1}\left(1-f_{\eta_2}\right)^{w_2}\right\} \end{array} \right\rangle \quad (14)$$

Thus, the expression holds true for n = 1, 2. Further suppose that the expression holds for n = k , $k \in N$. Then it follows that,

$$PNWGA(\eta_1,\eta_2,....,\eta_k)$$
$$=\left\langle \prod_{i=1}^{k}\left(t_{\eta_i}\right)^{w_i}, \prod_{i=1}^{k}\left(c_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k}\left(1-g_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k}\left(1-u_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k}\left(1-f_{\eta_i}\right)^{w_i} \right\rangle \quad (15)$$

Now, for n = k+1, we obtain,

$$PNWGA(\eta_1,\eta_2,....,\eta_k,\eta_{k+1}) = \prod_{i=1}^{k+1}(\eta_i)^{w_i}$$
$$=\sum_{i=1}^{k}(\eta_i)^{w_i} \otimes (\eta_{k+1})^{w_{k+1}}$$
$$=\left\langle \prod_{i=1}^{k}\left(t_{\eta_i}\right)^{w_i}, \prod_{i=1}^{k}\left(c_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k}\left(1-g_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k}\left(1-u_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k}\left(1-f_{\eta_i}\right)^{w_i} \right\rangle$$
$$+\left\langle \left(t_{\eta_{k+1}}\right)^{w_{k+1}}, \left(c_{\eta_{k+1}}\right)^{w_{k+1}}, 1-\left(1-g_{\eta_{k+1}}\right)^{w_{k+1}}, 1-\left(1-u_{\eta_{k+1}}\right)^{w_{k+1}}, 1-\left(1-f_{\eta_{k+1}}\right)^{w_{k+1}} \right\rangle$$
$$=\left\langle \begin{array}{l} \left\{\prod_{i=1}^{n}\left(t_{\eta_i}\right)^{w_i}\right\}\left\{\left(t_{\eta_{k+1}}\right)^{w_{k+1}}\right\}, \left\{\prod_{i=1}^{n}\left(c_{\eta_i}\right)^{w_i}\right\}\left\{\left(c_{\eta_{k+1}}\right)^{w_{k+1}}\right\}, \left\{1-\prod_{i=1}^{n}\left(1-g_{\eta_i}\right)^{w_i}\right\} \\ +\left\{1-\left(1-g_{\eta_{k+1}}\right)^{w_{k+1}}\right\} - \left\{1-\prod_{i=1}^{n}\left(1-g_{\eta_i}\right)^{w_i}\right\}\left\{1-\left(1-g_{\eta_{k+1}}\right)^{w_{k+1}}\right\}, \\ \left\{1-\prod_{i=1}^{n}\left(1-u_{\eta_i}\right)^{w_i}\right\} + \left\{1-\left(1-u_{\eta_{k+1}}\right)^{w_{k+1}}\right\} - \left\{1-\prod_{i=1}^{n}\left(1-u_{\eta_i}\right)^{w_i}\right\}\left\{1-\left(1-u_{\eta_{k+1}}\right)^{w_{k+1}}\right\}, \\ \left\{1-\prod_{i=1}^{n}\left(1-f_{\eta_i}\right)^{w_i}\right\} + \left\{1-\left(1-f_{\eta_{k+1}}\right)^{w_{k+1}}\right\} - \left\{1-\prod_{i=1}^{n}\left(1-f_{\eta_i}\right)^{w_i}\right\}\left\{1-\left(1-f_{\eta_{k+1}}\right)^{w_{k+1}}\right\} \end{array} \right\rangle$$
$$=\left\langle \prod_{i=1}^{k+1}\left(t_{\eta_i}\right)^{w_i}, \prod_{i=1}^{k+1}\left(c_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k+1}\left(1-g_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{k+1}\left(1-u_{\eta_i}\right)^{w_i}, \prod_{i=1}^{k+1}1-\left(1-f_{\eta_i}\right)^{w_i} \right\rangle \quad (16)$$

Hence, in general, by mathematical induction, the expression

$$PNWGA(\eta_1,\eta_2,....,\eta_m)$$
$$=\left\langle \prod_{i=1}^{m}\left(t_{\eta_i}\right)^{w_i}, \prod_{i=1}^{m}\left(c_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{m}\left(1-g_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{m}\left(1-u_{\eta_i}\right)^{w_i}, 1-\prod_{i=1}^{m}\left(1-f_{\eta_i}\right)^{w_i} \right\rangle \quad (17)$$

holds true $\forall n \in N$.

This completes the proof.

Theorem 2. The PNWGA operator satisfies the following properties:

i. Consistency: $PNWGA(\eta_1,\eta_2,.....,\eta_m) \in PNN$

ii. Idempotency: $PNWGA(\eta,\eta,.....,\eta) = \eta$

iii. $PNWGA(\eta_1,\eta_2,.....,\eta_m) = PNWGA(\eta_m,\eta_{m-1},.....,\eta_1)$

iv. Let ϕ be the permutation on (1, 2…., m) then

$PNWGA(\eta_{\phi(1)},\eta_{\phi(2)},.....,\eta_{m(\phi)}) = PNWGA(\eta_1,\eta_2,.....,\eta_m)$

Proof: Proof: (i) Assume that $\eta_i = \langle t_{\eta_i}, c_{\eta_i}, g_{\eta_i}, u_{\eta_i}, f_{\eta_i} \rangle (i = 1,2,.....m) \in PNN$

Since

$$PNWGA(\eta_1,\eta_2,....,\eta_m)$$

$$= \left\langle \prod_{i=1}^{n}(t_{\eta_i})^{w_i}, \prod_{i=1}^{n}(c_{\eta_i})^{w_i}, 1-\prod_{i=1}^{n}(1-g_{\eta_i})^{w_i}, 1-\prod_{i=1}^{n}(1-u_{\eta_i})^{w_i}, 1-\prod_{i=1}^{n}(1-f_{\eta_i})^{w_i} \right\rangle$$

Obviously $PNWGA(\eta_1,\eta_2,.....,\eta_m) \in PNN$

(ii)

$$PNWGA(\eta,\eta,...,\eta) = \eta$$

$$PNWGA(\eta,\eta,...,\eta) = \prod_{i=1}^{m}(\eta)^{w_i} = (\eta)^{w_1} \otimes (\eta)^{w_2} \otimes ... \otimes (\eta)^{w_m}$$

$$= \left\langle \prod_{i=1}^{m}(t_\eta)^{w_i}, \prod_{i=1}^{m}(c_\eta)^{w_i}, 1-\prod_{i=1}^{m}(1-g_\eta)^{w_i}, 1-\prod_{i=1}^{m}(1-u_\eta)^{w_i}, 1-\prod_{i=1}^{m}(1-f_\eta)^{w_i} \right\rangle$$

$$= \left\langle (t_\eta)^{\sum_{i=1}^{m} w_i}, (c_\eta)^{\sum_{i=1}^{m} w_i}, 1-(1-g_\eta)^{\sum_{i=1}^{m} w_i}, 1-(1-u_\eta)^{\sum_{i=1}^{m} w_i}, 1-(1-f_\eta)^{\sum_{i=1}^{m} w_i} \right\rangle$$

$$= \langle t_\eta, c_\eta, g_\eta, u_\eta, f_\eta \rangle, \text{ Since} \sum_{i=1}^{m} w_i = 1$$

$$= \eta$$

(iii)Since

$$PNWGA(\eta_1,\eta_2,...,\eta_m) = \prod_{i=1}^{m}(\eta_i)^{w_i}$$

$$= (\eta_1)^{w_1} \otimes (\eta_2)^{w_2} \otimes ... + (\eta_m)^{w_m} = (\eta_p)^{w_m} \otimes (\eta_{p-1})^{w_{m-1}} \otimes ... \otimes (\eta_1)^{w_1}$$

$$= PNWGA(\eta_m,\eta_{m-1},...,\eta_1)$$

(iv) Suppose that ϕ is a permutation on (1,2, …., m). Then,

$$PNWAA(\eta_{\phi(1)},\eta_{\phi(2)},...,\eta_{mp(\phi)}) = \prod_{i=1}^{m}(\eta_{\phi(i)})^{w_i(\eta_{\phi(i)})}$$

$$= (\eta_{\phi(1)})^{w_1(\eta_{\phi(1)})} \otimes (\eta_{\phi(2)})^{w_2(\eta_{\phi(2)})} \otimes ... \otimes (\eta_{\phi(m)})^{w_p(\eta_{\phi(m)})}$$

$$= (\eta_1)^{w_1} \otimes (\eta_2)^{w_2} \otimes ... \otimes (\eta_m)^{w_m} \left(\text{using}(vi) of \text{ proposition 1}\right)$$

$$= PNWGA(\eta_1,\eta_2,...,\eta_m)$$

This completes the proof.

Theorem 3. (Monotonicity) Consider sequence of PNNs $(\eta_1,\eta_2,....,\eta_m)$ and $(\gamma_1,\gamma_2,....,\gamma_m)$ such that $t_{\eta_i} \le t_{\gamma_i}, c_{\eta_i} \le c_{\gamma_i}, g_{\eta_i} \ge g_{\gamma_i}, u_{\eta_i} \ge u_{\gamma_i}$ and $f_{\eta_i} \ge f_{\gamma_i} \forall i (i = 1,2,...,m)$. Then,

$PNWGA(\eta_1,\eta_2,....,\eta_m) \le PNWGA(\gamma_1,\gamma_2,....,\gamma_m)$.

Proof: We know

$$PNWGA(\eta_1,\eta_2,....,\eta_m)=\left\langle \prod_{i=1}^{m}(t_{\eta_i})^{w_i},\prod_{i=1}^{m}(c_{\eta_i})^{w_i},1-\prod_{i=1}^{m}(1-g_{\eta_i})^{w_i},1-\prod_{i=1}^{m}(1-u_{\eta_i})^{w_i},1-\prod_{i=1}^{m}(1-f_{\eta_i})^{w_i}\right\rangle \quad (18)$$

$$PNWGA(\gamma_1,\gamma_2,....,\gamma_m)=\left\langle \prod_{i=1}^{m}(t_{\gamma_i})^{w_i},\prod_{i=1}^{m}(c_{\gamma_i})^{w_i},1-\prod_{i=1}^{m}(1-g_{\gamma_i})^{w_i},1-\prod_{i=1}^{m}(1-u_{\gamma_i})^{w_i},1-\prod_{i=1}^{m}(1-f_{\gamma_i})^{w_i}\right\rangle \quad (19)$$

Case 1: Suppose that $t_{\eta_i}<t_{\gamma_i},c_{\eta_i}<c_{\gamma_i},g_{\eta_i}>g_{\gamma_i},u_{\eta_i}>u_{\gamma_i}$ and $f_{\eta_i}>f_{\gamma_i}$

Then

$$\prod_{i=1}^{p}(t_{\eta_i})^{w_i}<\prod_{i=1}^{p}(t_{\gamma_i})^{w_i} \quad (20)$$

$$\prod_{i=1}^{p}(c_{\eta_i})^{w_i}<\prod_{i=1}^{p}(c_{\gamma_i})^{w_i} \quad (21)$$

$$g_{\eta_i}<g_{\gamma_i}$$

$$or, 1-g_{\eta_i}>1-g_{\gamma_i}$$

$$or, 1-\prod_{i=1}^{m}(1-g_{\eta_i})^{w_i}<1-\prod_{i=1}^{m}(1-g_{\gamma_i})^{w_i} \quad (22)$$

Similarly, one obtains,

$$1-\prod_{i=1}^{m}(1-u_{\eta_i})^{w_i}<1-\prod_{i=1}^{m}(1-u_{\gamma_i})^{w_i} \quad (23)$$

$$1-\prod_{i=1}^{m}(1-f_{\eta_i})^{w_i}<1-\prod_{i=1}^{m}(1-f_{\gamma_i})^{w_i} \quad (24)$$

From (15),

$$Sc(PNWGA(\eta_1,\eta_2,....,\eta_m))$$

$$=Sc\left(\left\langle \prod_{i=1}^{m}(t_{\eta_i})^{w_i},\prod_{i=1}^{m}(c_{\eta_i})^{w_i},1-\prod_{i=1}^{m}(1-g_{\eta_i})^{w_i},1-\prod_{i=1}^{m}(1-u_{\eta_i})^{w_i},1-\prod_{i=1}^{m}(1-f_{\eta_i})^{w_i}\right\rangle\right) \quad (25)$$

and

$$Sc(PNWGA(\gamma_1,\gamma_2,....,\gamma_m))$$

$$=Sc\left(\left\langle \prod_{i=1}^{m}(t_{\gamma_i})^{w_i},\prod_{i=1}^{m}(c_{\gamma_i})^{w_i},1-\prod_{i=1}^{m}(1-g_{\gamma_i})^{w_i},1-\prod_{i=1}^{m}(1-u_{\gamma_i})^{w_i},1-\prod_{i=1}^{m}(1-f_{\gamma_i})^{w_i}\right\rangle\right) \quad (26)$$

From equation (18)-(25), we obtain,

$$Sc(PNWAA(\eta_1,\eta_2,....,\eta_m))<Sc(PNWAA(\gamma_1,\gamma_2,....,\gamma_m)) \quad (27)$$

Finally, from equation (27), we obtain

$$PNWAA(\eta_1,\eta_2,....,\eta_m)<PNWAA(\gamma_1,\gamma_2,....,\gamma_m) \quad (28)$$

Case 2: Assume that, $t_{\eta_i}=t_{\gamma_i},c_{\eta_i}=c_{\gamma_i},g_{\eta_i}=g_{\gamma_i},u_{\eta_i}=u_{\gamma_i}$ and $f_{\eta_i}=f_{\gamma_i}$.

Therefore, $t_{\eta_i}>t_{\gamma_i}$, for each i,

$$\prod_{i=1}^{m}(t_{\eta_i})^{w_i}=\prod_{i=1}^{m}(t_{\gamma_i})^{w_i} \quad (29)$$

Similarly,

$$\prod_{i=1}^{m}(c_{\eta_i})^{w_i}=\prod_{i=1}^{m}(c_{\gamma_i})^{w_i} \quad (30)$$

and

$$1-\prod_{i=1}^{m}\left(1-g_{\eta_i}\right)^{w_i}=1-\prod_{i=1}^{m}\left(1-g_{\gamma_i}\right)^{w_i} \tag{31}$$

$$1-\prod_{i=1}^{m}\left(1-u_{\eta_i}\right)^{w_i}=1-\prod_{i=1}^{m}\left(1-u_{\gamma_i}\right)^{w_i} \tag{32}$$

$$1-\prod_{i=1}^{m}\left(1-f_{\eta_i}\right)^{w_i}=1-\prod_{i=1}^{m}\left(1-f_{\gamma_i}\right)^{w_i} \tag{33}$$

From equation (29) and (33),

$$PNWGA\left(\eta_1,\eta_2,....,\eta_m\right)=PNWGA\left(\gamma_1,\gamma_2,....,\gamma_m\right) \tag{34}$$

Therefore, finally, we obtain from equation (27) and (34)

$$PNWGA(\eta_1,\eta_2,....,\eta_m)\leq PNWGA\left(\gamma_1,\gamma_2,....,\gamma_m\right)$$

This completes the proof.

Theorem 8 (Boundedness) Consider sequence of PNNs $(\eta_1,\eta_2,...,\eta_m)\left(i=1,2,...,m\right)$ then

$$\underline{\eta}\leq PNWGA(\eta_1,\eta_2,...,\eta_m)\leq\overline{\eta}$$

where,

$$\underline{\eta}=\left\langle \min_i\left(t_{\eta_i}\right),\min_i\left(c_{\eta_i}\right),\max_i\left(g_{\eta_i}\right),\max_i\left(u_{\eta_i}\right)\max_i\left(f_{\eta_i}\right)\right\rangle \text{ and}$$

$$\overline{\eta}=\left\langle \max_i\left(t_{\eta_i}\right),\max_i\left(c_{\eta_i}\right),\min_i\left(g_{\eta_i}\right),\min_i\left(u_{\eta_i}\right)\min_i\left(f_{\eta_i}\right)\right\rangle$$

Proof:

By definition $\forall i=1,2,...,m$

$t_{\underline{\eta}}\leq t_{\eta_i}, c_{\underline{\eta}}\leq c_{\eta_i}, g_{\underline{\eta}}\geq g_{\eta_i}, u_{\underline{\eta}}\geq u_{\eta_i}$ and $f_{\underline{\eta}}\geq f_{\eta_i}$ and

$t_{\overline{\eta}}\geq t_{\eta_i}, c_{\overline{\eta}}\geq c_{\eta_i}, g_{\overline{\eta}}\leq g_{\eta_i}, u_{\overline{\eta}}\leq u_{\eta_i}$ and $f_{\overline{\eta}}\leq f_{\eta_i}$

$$PNWGA(\underline{\eta},\underline{\eta},....,\underline{\eta})\leq PNWGA(\eta_1,\eta_2,....,\eta_p)\leq PNWGA(\overline{\eta},\overline{\eta},....,\overline{\eta})$$

$$\Rightarrow\underline{\eta}\leq PNWGA(\eta_1,\eta_2,....,\eta_m)\leq\overline{\eta}$$

4. MAGDM STRATEGY FOR SELECTION OF THE MOST SUITABLE ALTERNATIVE USING PENTAPARTITIONED NEUTROSOPHIC WEIGHTED GEOMETRIC AGGREGATION (PNWGA) OPERATOR

Let, $T=\left\{T_1,T_2,....,T_l\right\}$ and $C=\left\{C_1,C_2,....,C_m\right\}$ be a set of l alternatives and m attributes. Suppose that the "l" alternatives are subjected to the judgement of m number of decision makers based on the prefixed judging parameters. The weight vector of the decision makers $v=\left\{v_1,v_2,.....,v_l\right\}$ Further suppose that the weight vector assigned to the attributes is $w\left(C\right)\in\left[0,1\right]$ and $\sum_{i=1}^{l}w\left(C_i\right)=1$.

Step-1: Define the decision matrix.

Suppose that $A^p=(a_{rs}^p)_{l\times m}$ is the p-th decision matrix where information about the alternative T_r is provided by the p-th decision maker with respect to the attribute C_s. The p-th decision matrix is defined as follows:

$$A^p=(a_{rs}^p)_{l\times m}=\begin{pmatrix} a_{11}^p & a_{12}^p & a_{1m}^p \\ a_{21}^p & a_{22}^p\cdots & a_{2m}^p \\ \vdots & \ddots & \vdots \\ a_{l1}^p & a_{l2}^p\cdots & a_{lm}^p \end{pmatrix} \tag{35}$$

where p = 1, 2, …P.

Step- 2: Standardize the decision matrix.

Assume that in the neutrosophic decision matrix (35), $\eta_i = \langle t_{\eta_i}, c_{\eta_i}, g_{\eta_i}, u_{\eta_i}, f_{\eta_i} \rangle (i = 1, 2, ...m)$ is the rating value of alternative L_r provided by the p-th decision maker with respect to attribute E_s such that $0 \le c_{n_{rs}^p} \le 1, 0 \le d_{n_{rs}^p} \le 1, 0 \le e_{n_{rs}^p} \le 1, 0 \le c_{n_{rs}^p} + c_{n_{rs}^p} + c_{n_{rs}^p} \le 3$

To remove the effects derived from different physical dimensions, the decision matrix $(n_{rs}^p)_{y\times z}$ is standardized. To obtain the standardized decision matrix $X^p = (x_{rs}^p)_{y\times z}$, in which the component x_{rs}^p of the entry $\eta_i = \langle t_{\eta_i}, c_{\eta_i}, g_{\eta_i}, u_{\eta_i}, f_{\eta_i} \rangle (i = 1, 2, ...m)$ in the matrix X^p is considered as:

i. For benefit criterion

$$\eta_i = \langle t_{\eta_i}, c_{\eta_i}, g_{\eta_i}, u_{\eta_i}, f_{\eta_i} \rangle (i = 1, 2, ...m) \tag{36}$$

ii. For cost criterion

$$\eta_i = \langle f_{\eta_i}, u_{\eta_i}, g_{\eta_i}, c_{\eta_i}, t_{\eta_i} \rangle (i = 1, 2, ...m) \tag{37}$$

Here $k_s^+ = \max\{n_{rs}^{p4} | r = 1, 2, ..., y\}$ and $k_s^- = \min\{n_{rs}^{p1} | r = 1, 2, ..., y\}$ for s =1, 2, …, z.

Then we obtain the following standardized decision matrix:

$$\eta^p = (\eta_{rs}^p)_{l\times m} = \begin{pmatrix} \eta_{11}^p & \eta_{12}^p & \eta_{1m}^p \\ \eta_{21}^p & \eta_{22}^p \cdots & \eta_{2m}^p \\ \vdots & \ddots & \vdots \\ \eta_{l1}^p & \eta_{l2}^p \cdots & \eta_{lm}^p \end{pmatrix} \tag{38}$$

Step-3: Aggregate the decision matrix using the weights of the decision makers

$$PNWGA(\eta_1, \eta_2, ..., \eta_l)$$
$$= \left\langle \prod_{i=1}^m (t_{\eta_i})^{v_i}, \prod_{i=1}^m (c_{\eta_i})^{v_i}, 1 - \prod_{i=1}^m (1 - g_{\eta_i})^{v_i}, 1 - \prod_{i=1}^m (1 - u_{\eta_i})^{v_i}, 1 - \prod_{i=1}^m (1 - f_{\eta_i})^{v_i} \right\rangle \tag{39}$$

The decision matrix reduces to

$$\eta^{*p} = (\eta_{rs}^{*p})_{l\times m} = \begin{pmatrix} \eta_{11}^{*p} & \eta_{12}^{*p} & \eta_{1m}^{*p} \\ \eta_{21}^{*p} & \eta_{22}^{*p} \cdots & \eta_{2m}^{*p} \\ \vdots & \ddots & \vdots \\ \eta_{l1}^{*p} & \eta_{l2}^{*p} \cdots & \eta_{lm}^{*p} \end{pmatrix} \tag{40}$$

Step-4: Construct the final decision matrix using weights of the attributes

$$PNWGA(\eta_1^*, \eta_2^*, ..., \eta_l^*)$$
$$= \left\langle \prod_{i=1}^m (t_{\eta_i})^{w_i}, \prod_{i=1}^m (c_{\eta_i})^{w_i}, 1 - \prod_{i=1}^m (1 - g_{\eta_i})^{w_i}, 1 - \prod_{i=1}^m (1 - u_{\eta_i})^{w_i}, 1 - \prod_{i=1}^m (1 - f_{\eta_i})^{w_i} \right\rangle \tag{41}$$

$$n^{p} = (n_{rs}^{p})_{l\times m} = \begin{pmatrix} n_{11}^{p} & n_{12}^{p} & n_{1m}^{p} \\ n_{21}^{p} & n_{22}^{p} \cdots & n_{2m}^{p} \\ \vdots & \ddots & \vdots \\ n_{l1}^{p} & n_{l2}^{p} \cdots & n_{lm}^{p} \end{pmatrix} \tag{42}$$

Step-5: Calculate the score value and accuracy (Pramanik, in press) value of the final decision matrix.

$$Sc\left(n^{2}\right) = \frac{t_{n^2} + c_{n^2}}{2} + \frac{g_{n^2} + u_{n^2} + f_{n^2}}{3} \tag{43}$$

$$Ac\left(n^{2}\right) = \frac{t_{n^2} + c_{n^2} + g_{n^2} - u_{n^2} - f_{n^2}}{2} \tag{44}$$

Step-6: Rank the alternative using the score value and accuracy value of the alternatives.

Step-7: End.

5. ILLUSTRATIVE EXAMPLE OF SUPPLIER SELECTION PROBLEM

This section uses a green supplier selection problem adapted from (Wan & Dong , 2015) to demonstrate the applicability of the proposed method. Shanghai General Motors Company Limited (SGM) is planning to incorporate environmentally friendly features into the product design stage to protect the environment and achieve sustainable development of the social economy. For this reason, SGM wishes to select the most appropriate green supplier for one of the key elements in its manufacturing process. After pre-evaluation, four suppliers remain as candidates for further evaluation. They are Howden Hua Engineering Company (T_1), Sino Trunk (T_2), Taikai Electric Group Company Limited (T_3), and Shantui Construction Machinery Company Limited (T_4). SGM employs four experts to form a group of DMs coming from four consultancy departments: DM (P_1) is from the production department; DM (P_2) is from the purchasing department; DM (P_3) is from the quality inspection department; DM (P_4) is from the engineering department. The attributes for evaluating suppliers are important because they obviously influence the selection result. Utilizing principal component analysis, the experts choose the following three independent criteria as evaluation principles: product quality (C_1), pollution control (C_2), and environment management (C_3). According to historical data, the weight vector of the three criteria is $w = (0.4, 0.35, 0.25)$ and weight of the decision maker is $v = (0.38, 0.30, 0.32)$.

Step 1: Decision matrix

$$A^{1} = \begin{pmatrix} & C_1 & C_2 & C_3 \\ T_1 & \langle 0.580, 0.320, 0.450, 0.210, 0.370\rangle & \langle 0.430, 0.520, 0.480, 0.560, 0.340\rangle & \langle 0.840, 0.750, 0.560, 0.450, 0.230\rangle \\ T_2 & \langle 0.740, 0.520, 0.420, 0.470, 0.280\rangle & \langle 0.450, 0.320, 0.710, 0.580, 0.290\rangle & \langle 0.540, 0.750, 0.560, 0.480, 0.310\rangle \\ T_3 & \langle 0.710, 0.530, 0.800, 0.670, 0.750\rangle & \langle 0.730, 0.450, 0.750, 0.580, 0.590\rangle & \langle 0.740, 0.527, 0.621, 0.320, 0.480\rangle \\ T_4 & \langle 0.410, 0.570, 0.640, 0.520, 0.480\rangle & \langle 0.620, 0.450, 0.620, 0.710, 0.550\rangle & \langle 0.870, 0.425, 0.358, 0.690, 0.340\rangle \end{pmatrix}$$

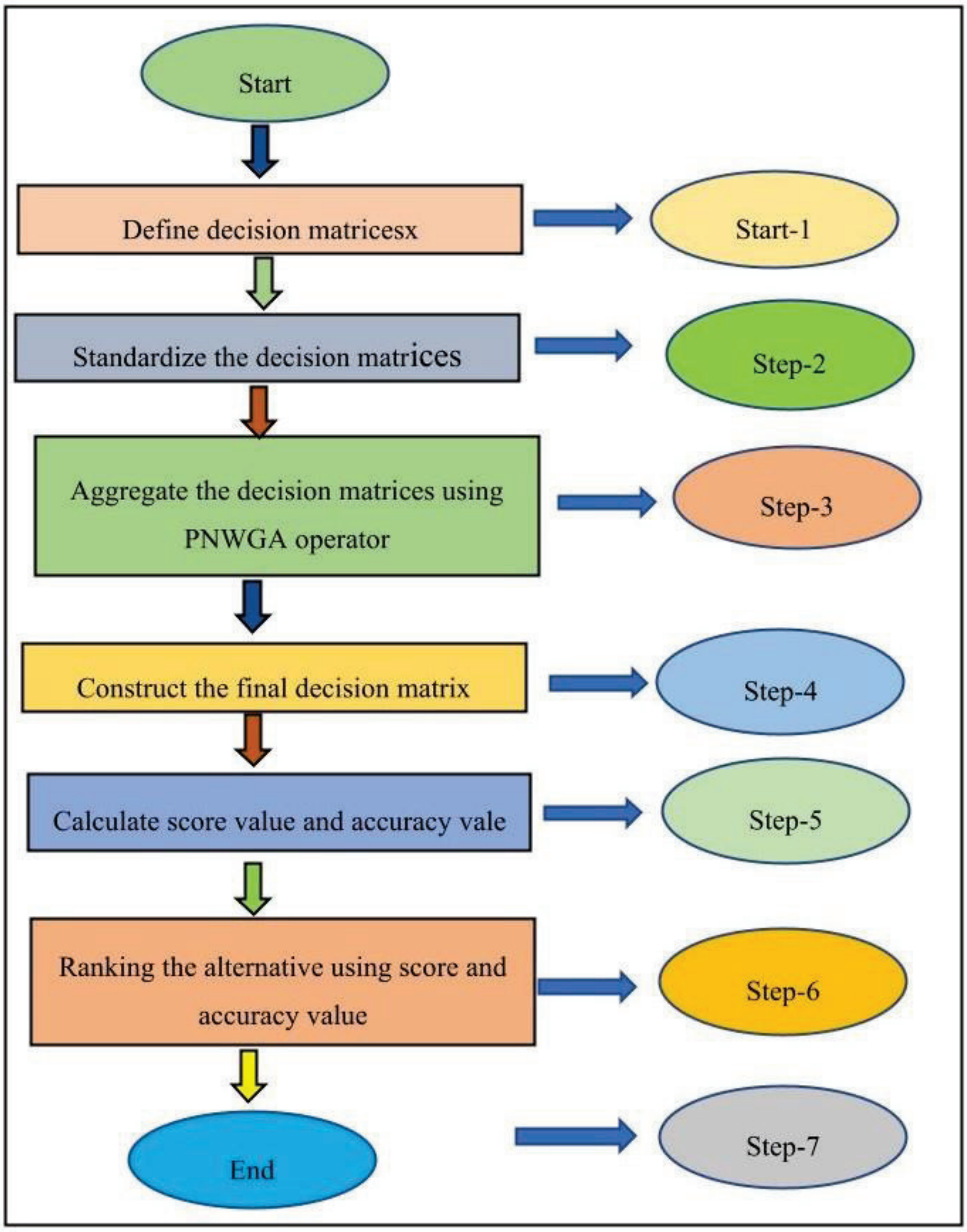

Figure 1. MAGDM strategy based on PNWGA operator

$$A^2 = \begin{array}{cccc} & C_1 & C_2 & C_3 \\ \end{array}$$
$$A^2 = \left(\begin{array}{cccc} & C_1 & C_2 & C_3 \\ T_1 & \langle 0.680,0.720,0.350,0.410,0.360\rangle & \langle 0.450,0.720,0.520,0.620,0.240\rangle & \langle 0.640,0.820,0.920,0.550,0.430\rangle \\ T_2 & \langle 0.640,0.480,0.580,0.750,0.280\rangle & \langle 0.850,0.620,0.820,0.520,0.390\rangle & \langle 0.740,0.550,0.850,0.230,0.410\rangle \\ T_3 & \langle 0.610,0.870,0.580,0.370,0.350\rangle & \langle 0.360,0.710,0.420,0.580,0.400\rangle & \langle 0.740,0.553,0.560,0.420,0.260\rangle \\ T_4 & \langle 0.520,0.620,0.440,0.890,0.250\rangle & \langle 0.820,0.560,0.780,0.500,0.320\rangle & \langle 0.650,0.560,0.348,0.230,0.340\rangle \end{array}\right)$$

$$A^3 = \left(\begin{array}{cccc} & C_1 & C_2 & C_3 \\ T_1 & \langle 0.780,0.650,0.450,0.310,0.270\rangle & \langle 0.530,0.620,0.430,0.260,0.320\rangle & \langle 0.740,0.820,0.420,0.250,0.330\rangle \\ T_2 & \langle 0.840,0.560,0.250,0.370,0.380\rangle & \langle 0.850,0.520,0.210,0.380,0.390\rangle & \langle 0.840,0.650,0.560,0.380,0.310\rangle \\ T_3 & \langle 0.810,0.620,0.560,0.270,0.350\rangle & \langle 0.530,0.650,0.330,0.380,0.290\rangle & \langle 0.640,0.827,0.521,0.320,0.280\rangle \\ T_4 & \langle 0.610,0.770,0.240,0.320,0.220\rangle & \langle 0.920,0.750,0.520,0.410,0.250\rangle & \langle 0.670,0.725,0.458,0.290,0.340\rangle \end{array}\right)$$

Step-2: Because all the criteria are of the benefit type, the decision information does not need to be normalized

Step-3: Evaluating decision matrix using PNWGA equation (69)

$$\eta^{*2} = \left(\begin{array}{cccc} & C_1 & C_2 & C_3 \\ T_1 & \langle 0.6688,0.5120,0.4217,0.3069,0.3364\rangle & \langle 0.4661,0.6065,0.4772,0.5027,0.3048\rangle & \langle 0.7434,0.7927,0.7118,0.4281,0.3271\rangle \\ T_2 & \langle 0.7378,0.5198,0.4284,0.5529,0.3136\rangle & \langle 0.6675,0.4558,0.6536,0.5048,0.3538\rangle & \langle 0.6837,0.6528,0.6814,0.3811,0.3417\rangle \\ T_3 & \langle 0.5330,,0.6466,0.6784,0.4835,0.5479\rangle & \langle 0.5330,0.5804,0.5588,0.5242,0.4521\rangle & \langle 0.7064,0.6176,0.5728,0.3517,0.3585\rangle \\ T_4 & \langle 0.5000,0.6436,0.4780,0.6551,0.3392\rangle & \langle 0.7650,0.5658,0.6524,0.5714,0.4002\rangle & \langle 0.7332,0.5477,0.3890,0.4690,0.34\rangle \end{array}\right)$$

Step-4: Construct the decision matrix using attribute weights. By equation (71) the decision matrix is

$$n^2 = \left(\begin{array}{cc} T_1 & \langle 0.6052,0.6060,0.5310,0.4119,0.3232\rangle \\ T_2 & \langle 0.6990,0.5256,0.5855,0.4973,0.3349\rangle \\ T_3 & \langle 0.5718,0.6155,0.6144,0.4688,0.4722\rangle \\ T_4 & \langle 0.6385,0.5909,0.5290,0.5854,0.3614\rangle \end{array}\right)$$

Step-5: Evaluated score value and accuracy value using equation (73) and (74)

$$Sc(n^2) = \begin{pmatrix} T_1 & 1.0276 \\ T_2 & 1.0849 \\ T_3 & 1.1122 \\ T_4 & 1.1067 \end{pmatrix} \quad Ac(n^2) = \begin{pmatrix} T_1 & 0.2014 \\ T_2 & 0.1955 \\ T_3 & 0.1721 \\ T_4 & 0.1623 \end{pmatrix}$$

Step-6: Ranking of the alternative

$T_3 > T_4 > T_2 > T_1$

Therefore 3rd alternative is the best option.

Step-7: End.

Table 1.Comparison between the results that are obtained from two strategies

Operator Name	**Rank of the alternative**
PNWAA operator (Pramanik, 2023b)	$T_3 > T_2 > T_4 > T_1$
PNWAG operator (proposed)	$T_3 > T_4 > T_2 > T_1$

Ranking order of the alternatives are different for these two operators (see Table 1). Best alternative is same for both the operator. Using PNWAA operator the best alternative is 3rd alternative and using PNWAG operator the best alternative is also 3rd.

6. CONCLUSIONS

In this paper, we have defined pentapartitioned neutrosophic number and aggregated operator. A decision making strategy is developed to solve MCGDM in PNN environment. A green supplier selection problem is solved to show the applicability of the strategy. Though the green supplier selection example is used to illustrate the application and validation of the proposed methods. The proposed method is are very suitable for the decision-making problems in many areas, especially in situations where the problems involve multiple different attributes with different dimensions and neutrosophic information. It is expected that the developed strategy is applicable to the water resource assessment, risk investment, performance evaluation of military system, engineering management, library and information science (Sahoo, Panigrahi, & Pramanik, 2023; 2023, Sahoo, Pramanik& Panigrahi, 2023), etc.

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Aggregation Operators Based TFNN-MCGDM Strategies under Triangular Fuzzy Neutrosophic Number Environment

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ABSTRACT

This paper aims to develop an aggregation operator in the triangular fuzzy neutrosophic number environment as a breakthrough in aggregation operators by utilizing Einstein operations. This paper proposes the Triangular Fuzzy Neutrosophic Number Einstein's Ordered Weighted Average (TFNNEOWA) operator and Triangular Fuzzy Neutrosophic Number Ordered Weighted Geometric Average (TFNNEOWGA) operator and we prove some basic interesting properties of the proposed aggregation operators. Using Shanon's entropy, the weights of the criteria and decision makers are determined. We develop two Multi-Criteria Group Decision-Making (MCGDM) strategies using the TFNNEOWA and TFNNEOWGA operators. Lastly, by utilizing the newly introduced aggregation operators, a sales manager selection problem is solved.

KEYWORDS: **Entropy, fuzzy set, neutrosophic set, triangular fuzzy neutrosophic number, multi-criteria group decision making, triangular fuzzy neutrosophic Einstein's ordered weighted arithmetic operator, triangular fuzzy neutrosophic Einstein's ordered weighted, geometric operator**.

1. INTRODUCTION

Smarandache (1998) grounded indeterminacy as an independent membership function and grounded the Neutrosophic Set (NS) by extending the Fuzzy Set (FS) (Zadeh, 1965) and Intuitionistic FS (Atanassov, 1986). To easily understand NS, Single-Valued NS (SVNS) (Wang et al., 2010) was proposed. The development of NSs and their extensions and applications have been depicted in (El-Hefenawy, 2016; Smarandache & Pramanik, 2016, 2018; Pramanik, Mallick & Dasgupta, 2018; Broumi et al., 2018; Nguyen et al., 2019; Pramanik, 2020, 2022; Peng & Dai, 2020). NSs and their extensions have a huge contribution to several research topics like medical diagnosis (Ye & Ye, 2014; Ye, 2015), Multi-Criteria Decision Making (MCDM) and Multi-Criteria Group Decision Making(MCGDM) (Ye, 2013, 2014a, 2014b; Biswas et al., 2014a, 2014b, 2015, 2016a, 2016b, 2016c, 2016d, 2016e, 2018a, 2018b, 2019a, 2019b; Majumder, Paul, & Pramanik, 2023; Mondal & Pramanik, 2014, 2015a, 2015b, 2015c, 2015d, 2015e, 2015f; Mondal, Pramanik, & Giri, 2018a, 2018b, 2018c, 2018d; Mondal, Pramanik, & Smarandache,2016a, 2016b, 2016c, 2016d, 2018; Mallick & Pramanik, 2019, 2021a, 2021b; Mallick & Pramanik, & Giri (2023,in press); Sodenkamp et al., 2018; Liu &Wang,2014; Kharal,2014; Sahin & Liu, 2015; Das, Shil, & Pramanik, 2021, 2022, Das, Das, & Pramanik, 2022; Dey et al. (2015a, 2015b, 2015b, 2016a, 2016b, 2016c, 2016d, 2016e; Banerjee et al., 2017; Pramanik, Biswas, & Giri, 2017; Pramanik, Roy, Roy & Smarandache, 2017, 2018a, 2018b; Pramanik et al., 2018a, 2018b; Deli & Subas, 2017, Zavadskas et al., 2020; Stanujkić et al., 2021; Pramanik & Dalapati, 2023; Pramanik, Das, Das, & Tripathy, 2023a, 2023b; Pramanik & Mallick, 2018, 2019, 2020a, 2020b), conflict resolution (Pramanik and Roy, 2014), education (Pramanik, 2013, 2023a, 2023b), etc.

In dealing with practical problems, the aggregation of different scores presented in terms of Neutrosophic Numbers (NNs) is very important for MCGDM. Ye (2014a) presented the strategic conception of the "weighted arithmetic mean operator" along with the "weighted geometric mean operator" under the Single-Valued NN (SVNN) environment. Later some important simplified neutrosophic aggregation operators like "simplified neutrosophic number weighted geometric averaging operator", "simplified neutrosophic number ordered weighted arithmetic averaging operator" and the important hybrid operator "hybrid arithmetic ordered weighted operator" were introduced (Peng et al., 2016). In some critical neutrosophic MCGDM problems, DMs may not be able to express their ratings using SVNNs. To deal with the issue, the combination of triangular fuzzy numbers with SVNS is a great help. Biswas et al. (2016b) developed the aggregation operators for the Triangular Fuzzy NNs (TFNNs) and employed them to solve an MCGDM problem.

Wang and Liu (2012) introduced Einstein's aggregation operators for aggregating triangular IFS information. Li et al. (2018) presented Einstein's operators and investigated the properties of these operators for SVNNs. Different decision-making strategies have valuable contributions to MCGDM problems. MCGDM problem tackles the problems of logically selecting the best alternative in the prevailing environment of many conflicting criteria. Extensive research in the domain of MCGDM in the NS environments has been done. Jana et al. (2021) presented Dombi aggregation operators for the MCDM strategy using Single Valued Trapezoidal Neutrosophic Numbers (SVTrNNs). Several important types of research have been conducted by several researchers in MCDM fields exploring several operators in the domain with the introduction of different methods like entropy (Biswas, Pramanik, & Giri, 2014a), cross-entropy (Pramanik et al., 2018), similarity measures (Pramanik, Biswas, & Giri, 2017), etc. Sahin et al. (2018) explored the generalized single valued TFNNs and applied them to solve MCDM problems. Fan, Jia, and Wu (2019) used Dombi prioritized Bonferroni mean operator with TFNNs for green supplier selection. Irvanizam et al. (2020) investigated the extended MABAC method based on TFNNs for MCGDM problems. Meng et al. (2020) presented the TFNN preference relations and utilized it software selection problems. Fan, Jia, and Wu (2020) solved a new MCGDM model based on TFNNs and the EDAS method. Zhang, Zhou, Pan, and Wei (2022) investigated the MCDM method with TFNNs based on regret theory and the catastrophe progression method. Yao and Ran (2023) studied the operational efficiency evaluation of Urban and rural residents' basic pension insurance system based on the triangular fuzzy neutrosophic Grey Relational Analysis (GRA) strategy. Xie (2023) presented the modified GRA strategy under the TFNN environment for blended teaching effect evaluation of college English courses. Wang, Yan, Wang, and Ouyang (2023) presented the cross-entropy strategy for MADM under the TFNN environment.

Research gap:

However, no such strategy to solve MCGDM problems using Einstein's operations in the TFNN environment is reported. We find a research gap in with dealing MCGDM problems in the TFNN environment, especially under Einstein's operations. In this chapter, we have proposed Triangular Fuzzy Neutrosophic Einstein's Ordered Weighted Arithmetic (TFNEOWA) operator and Triangular Fuzzy Neutrosophic Einstein's Ordered Weighted Geometric (TFNEOWG) operator to aggregate information expressed in TFNNs to deal with MCGDM problems.

The objectives of the study include:

1) to present the aggregation operators namely, TFNEOWA operator and TFNEOWG operator.
2) to prove some of the basic properties of the TFNEOWA Operator and TFNEOWG operator.
3) to develop two MCGDM strategies based on TFNEOWA operator and TFNEOWG operator
4) to discuss the developed strategies for solving MCGDM problems with illustrative examples.

The rest of the chapter is organized as follows: Section 2 presents the preliminaries regarding TFNSs. Section 3 presents the formulation of TFNEOWA and TFNEOWG operators. Section 4 presents the entropy formulation for TFNNs. Section 5 deals with MCGDM strategy based on TFNEOWA and TFNEOWG Operators. Section 6 presents a numerical example of MCGDM strategy of sales manager selection in a pharmaceutical company. Section 7 includes the chapter.

2. PRELIMINARIES

Vital definitions of TFNS with their basic operational underlying principles are elaborately discussed in this section. Some basic Einstein principles of operations are also mentioned.

2 .1 TFNSs (Biswas, Pramanik, & Giri, 2016b)

Definition 2.1. Let Ψ be a finite domain of definition (a fixed set) and $\breve{\ddot{\phi}}$ [0,1] is a set of all TFNNs on [0, 1]. A TFNS $\bar{\Theta}$ in the set of real numbers is expressed as:

$$\bar{\Theta} = \{ \theta, (\xi_{\bar{\Theta}}(\theta), \eta_{\bar{\Theta}}(\theta), \delta_{\bar{\Theta}}(\theta)) \mid \theta \in \Psi \}$$

where $\xi_{\bar{\Theta}}(\theta): \Psi \to [0,1]$, $\eta_{\bar{\Theta}}(\theta): \Psi \to [0,1]$,

$\delta_{\bar{\Theta}}(\theta): \Psi \to [0,1]$, where $\xi_{\bar{\Theta}}(\theta) = (\xi_{\bar{\Theta}}^{(1)}(\theta), \xi_{\bar{\Theta}}^{(2)}(\theta), \xi_{\bar{\Theta}}^{(3)}(\theta))$

$\eta_{\bar{\Theta}}(\theta) = (\eta_{\bar{\Theta}}^{(1)}(\theta), \eta_{\bar{\Theta}}^{(2)}(\theta), \eta_{\bar{\Theta}}^{(3)}(\theta))$,

where $\delta_{\bar{\Theta}}^{(1)}(\theta)$, $\delta_{\bar{\Theta}}^{(2)}(\theta)$, $\delta_{\bar{\Theta}}^{(3)}(\theta)$ respectively represents the "degree of truth", " degree of indeterminacy" and "degree of falsity" and $0 \le \xi_{\bar{\Theta}}^{(1)}(\theta) + \eta_{\bar{\Theta}}^{(1)}(\theta) + \delta_{\bar{\Theta}}^{(1)}(\theta) \le 3$. For other membership degrees, we have similar results.

For symbolical convenience, we take $(\xi_{\bar{\Theta}}^{(1)}(\theta), \xi_{\bar{\Theta}}^{(2)}(\theta), \xi_{\bar{\Theta}}^{(3)}(\theta)) = (\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3)$

$(\eta_{\bar{\Theta}}^{(1)}(\theta), \eta_{\bar{\Theta}}^{(2)}(\theta), \eta_{\bar{\Theta}}^{(3)}(\theta)) = (\bar{v}_1, \bar{v}_2, \bar{v}_3)$ and $(\delta_{\bar{\Theta}}^{(1)}(\theta), \delta_{\bar{\Theta}}^{(2)}(\theta), \delta_{\bar{\Theta}}^{(3)}(\theta))) = (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3)$

So $\bar{\Theta} = ((\bar{\mu}_1, \bar{\mu}_2, \bar{\mu}_3), (\bar{v}_1, \bar{v}_2, \bar{v}_3), (\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3))$ is a TFNN.

Definition 2.2. Hamming distance between two TFNNs (Wang, Yan, Wang, & Ouyang (2023)
Let $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ be two TFNNs presented as
$\tilde{A}^{(1)} = \langle (p\tilde{a}^1, p\tilde{b}^1, p\tilde{c}^1), (p\tilde{e}^1, p\tilde{f}^1, p\tilde{g}^1), (p\tilde{r}^1, p\tilde{s}^1, p\tilde{t}^1) \rangle$ & $\tilde{A}^2 = \langle (p\tilde{a}^2, p\tilde{b}^2, p\tilde{c}^2), (p\tilde{e}^2, p\tilde{f}^2, p\tilde{g}^2), (p\tilde{r}^2, p\tilde{s}^2, p\tilde{t}^2) \rangle$.

The "normalized Hamming distance" (Wang, Wei, & Lu, 2018) is presented as:

$$\delta_N(\tilde{A}^1, \tilde{A}^2) = \frac{1}{9}[|p\tilde{a}^1 - p\tilde{a}^2| + |p\tilde{b}^1 - p\tilde{b}^2| + |p\tilde{c}^1 - p\tilde{c}^2| + |p\tilde{e}^1 - p\tilde{e}^2| + |p\tilde{f}^1 - p\tilde{f}^2| + |p\tilde{g}^1 - p\tilde{g}^2| + |p\tilde{r}^1 - p\tilde{r}^2| + |p\tilde{s}^1 - p\tilde{s}^2| + |p\tilde{t}^1 - p\tilde{t}^2|] \quad (1)$$

3. TFNEOWA AND TFNEOWG OPERATORS

In this part, we define Einstein's operations for TFNN. We formulate two operators namely, TFNEOWA and TFNEOWG operators and establish some of their necessary properties.

Definition 3.1. Let $\tilde{\alpha}_1 = (\tilde{\mu}_1, \tilde{v}_1, \tilde{\lambda}_1)$ and $\tilde{\alpha}_2 = (\tilde{\mu}_2, \tilde{v}_2, \tilde{\lambda}_2)$ be any two SVNNs. λ>0. Einstein operations (Li et al., 2018) are defined as follows:

$$1)\tilde{\alpha}_1 \otimes \tilde{\alpha}_2 = \left(\frac{\tilde{\mu}_1.\tilde{\mu}_2}{1+(1-\tilde{\mu}_1)(1-\tilde{\mu}_2)}, \frac{\tilde{v}_1.\tilde{v}_2}{1+\tilde{v}_1.\tilde{v}_2}, \frac{\tilde{\lambda}_1.\tilde{\lambda}_2}{1+\tilde{\lambda}_1.\tilde{\lambda}_2}\right) \tag{2}$$

$$2)\tilde{\alpha}_1 \oplus \tilde{\alpha}_2 = \left(\frac{\tilde{\mu}_1+\tilde{\mu}_2}{1+\tilde{\mu}_1.\tilde{\mu}_2}, \frac{\tilde{v}_1.\tilde{v}_2}{1+(1-\tilde{v}_1)(1-\tilde{v}_2)}, \frac{\tilde{\lambda}_1.\tilde{\lambda}_2}{1+(1-\tilde{\lambda}_1)(1-\tilde{\lambda}_2)}\right) \tag{3}$$

$$3)\lambda\tilde{\alpha}_1 = \left(\frac{(1+\tilde{\mu}_1)^\lambda-(1-\tilde{\mu}_1)^\lambda}{(1+\tilde{\mu}_1)^\lambda+(1-\tilde{\mu}_1)^\lambda}, \frac{2(\tilde{v}_1)^\lambda}{(2-\tilde{v}_1)^\lambda+(\tilde{v}_1)^\lambda}, \frac{2(\tilde{\lambda}_1)^\lambda}{(2-\tilde{\lambda}_1)^\lambda+(\tilde{\lambda}_1)^\lambda}\right), \lambda>0 \tag{4}$$

$$4)(\alpha_1^{\sim})^\lambda = \left(\frac{2(\tilde{\mu}_1)^\lambda}{(2-\tilde{\mu}_1)^\lambda+(\tilde{\mu}_1)^\lambda}, \frac{(1+\tilde{v}_1)^\lambda-(1-\tilde{v}_1)^\lambda}{(1+\tilde{v}_1)^\lambda+(1+\tilde{v}_1)^\lambda}, \frac{(1+\tilde{\lambda}_1)^\lambda-(1-\tilde{\lambda}_1)^\lambda}{(1+\tilde{\lambda}_1)^\lambda+(1+\tilde{\lambda}_1)^\lambda}\right), \lambda>0 \tag{5}$$

Let $\tilde{A} = <(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3),(\bar{v}_1,\bar{v}_2,\bar{v}_3),(\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3)>, \tilde{B} = <((\bar{\mu}_1',\bar{\mu}_2',\bar{\mu}_3'),(\bar{v}_1',\bar{v}_2',\bar{v}_3'),(\bar{\lambda}_1',\bar{\lambda}_2',\bar{\lambda}_3'))$ be two TFNNs. Then, we define the following, mathematical operations with the equivalence symbolic representation as: $(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3) = (p\tilde{a}_1, p\tilde{b}_1, p\tilde{c}_1)$, $(\bar{v}_1,\bar{v}_2,\bar{v}_3) = (p\tilde{e}_1, p\tilde{f}_1, p\tilde{g}_1)$, $(\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3) = (p\tilde{r}_1, p\tilde{s}_1, p\tilde{t}_1)$,
$(\bar{\mu}_1',\bar{\mu}_2',\bar{\mu}_3') = (p\tilde{a}_2, p\tilde{b}_2, p\tilde{c}_2), (\bar{v}_1',\bar{v}_2',\bar{v}_3') = (p\tilde{e}_2, p\tilde{f}_2, p\tilde{g}_2), (\bar{\lambda}_1',\bar{\lambda}_2',\bar{\lambda}_3') = (p\tilde{r}_2, p\tilde{s}_2, p\tilde{t}_2)$.
Then,

$$5.A_1^{\sim} \otimes A_2^{\sim} = \langle\left(\frac{p\tilde{a}_1.p\tilde{a}_2}{1+(1-p\tilde{a}_1)(1-p\tilde{a}_2)}, \frac{p\tilde{b}_1.p\tilde{b}_2}{1+(1-p\tilde{b}_1)(1-p\tilde{b}_2)}, \frac{p\tilde{c}_1.p\tilde{c}_2}{1+(1-p\tilde{c}_1)(1-p\tilde{c}_2)}\right), \left(\frac{p\tilde{e}_1.p\tilde{e}_2}{1+p\tilde{e}_1.p\tilde{e}_2}, \frac{p\tilde{f}_1.p\tilde{f}_2}{1+p\tilde{f}_1.p\tilde{f}_2} \frac{p\tilde{g}_1.p\tilde{g}_2}{1+p\tilde{g}_1.p\tilde{g}_2}\right), \left(\frac{p\tilde{r}_1.p\tilde{r}_2}{1+p\tilde{r}_1.p\tilde{r}_2}, \frac{p\tilde{s}_1.p\tilde{s}_2}{1+p\tilde{s}_1.p\tilde{s}_2}, \frac{p\tilde{t}_1.p\tilde{t}_2}{1+p\tilde{t}_1.p\tilde{t}_2}\right)\rangle \tag{6}$$

$$6.\tilde{A}_1 \oplus \tilde{A}_2 = \langle\left(\frac{p\tilde{a}_1+p\tilde{a}_2}{1+(p\tilde{a}_1.p\tilde{a}_2)}, \frac{p\tilde{b}_1+p\tilde{b}_2}{1+(p\tilde{b}_1.p\tilde{b}_2)}, \frac{p\tilde{c}_1+p\tilde{c}_2}{1+(p\tilde{c}_1.p\tilde{c}_2)}\right), \left(\frac{p\tilde{e}_1.p\tilde{e}_2}{1+(1-p\tilde{e}_1)(1-p\tilde{e}_2)}, \frac{p\tilde{f}_1.p\tilde{f}_2}{1+(1-p\tilde{f}_1)(1-p\tilde{f}_2)}, \frac{(p\tilde{g}_1.p\tilde{g}_2)}{1+(1-p\tilde{g}_1)(1-p\tilde{g}_2)}\right), \left(\frac{(p\tilde{r}_1.p\tilde{r}_2)}{1+\{(1-pr_1^{\sim})(1-pr_2^{\sim})\}}, \frac{(p\tilde{s}_1.p\tilde{s}_2)}{\{(1-p\tilde{s}_1)(1-p\tilde{s}_2)\}+1}, \frac{p\tilde{t}_1.p\tilde{t}_2}{\{(1-p\tilde{t}_1)(1-p\tilde{t}_2)\}+1}\right)\rangle \tag{7}$$

$$7.\lambda\tilde{A}_1 = \langle\left(\frac{(1+p\tilde{a}_1)^\lambda-(1-p\tilde{a}_1)^\lambda}{(1+p\tilde{a}_1)^\lambda+(1-p\tilde{a}_1)^\lambda}, \frac{(1+p\tilde{b}_1)^\lambda-(1-p\tilde{b}_1)^\lambda}{(1+p\tilde{b}_1)^\lambda+(1-p\tilde{b}_1)^\lambda}, \frac{(1+p\tilde{c}_1)^\lambda-(1-p\tilde{c}_1)^\lambda}{(1+p\tilde{c}_1)^\lambda+(1-p\tilde{c}_1)^\lambda}\right), \left(\frac{2(p\tilde{e}_1)^\lambda}{(2-p\tilde{e}_1)^\lambda+(p\tilde{e}_1)^\lambda}, \frac{2(p\tilde{f}_1)^\lambda}{(2-p\tilde{f}_1)^\lambda+(p\tilde{f}_1)^\lambda}, \frac{2(p\tilde{g}_1)^\lambda}{(-p\tilde{g}_1+2)^\lambda+(p\tilde{g}_1)^\lambda}\right), \left(\frac{2(p\tilde{r}_1)^\lambda}{(-p\tilde{r}_1+2)^\lambda+(p\tilde{r}_1)^\lambda}, \frac{2(p\tilde{s}_1)^\lambda}{(-p\tilde{s}_1+2)^\lambda+(p\tilde{s}_1)^\lambda} \frac{2(p\tilde{t}_1)^\lambda}{(-p\tilde{t}_1+2)^\lambda+(p\tilde{t}_1)^\lambda}\right)\rangle \tag{8}$$

$$8).(\tilde{A}_1)^\lambda = \langle\left(\frac{2(p\tilde{a}_1)^\lambda}{(2-p\tilde{a}_1)^\lambda+(p\tilde{a}_1)^\lambda}, \frac{2(p\tilde{b}_1)^\lambda}{(2-p\tilde{b}_1)^\lambda+(p\tilde{b}_1)^\lambda}, \frac{2(p\tilde{c}_1)^\lambda}{(2-p\tilde{c}_1)^\lambda+(p\tilde{c}_1)^\lambda}\right), \left(\frac{(1+p\tilde{e}_1)^\lambda-(1-p\tilde{e}_1)^\lambda}{(1+p\tilde{e}_1)^\lambda+(1-p\tilde{e}_1)^\lambda}, \frac{(1+p\tilde{f}_1)^\lambda-(1-p\tilde{f}_1)^\lambda}{(1+p\tilde{f}_1)^\lambda+(1-p\tilde{f}_1)^\lambda}, \frac{(1+p\tilde{g}_1)^\lambda-(1-p\tilde{g}_1)^\lambda}{(1+p\tilde{g}_1)^\lambda+(1-p\tilde{g}_1)^\lambda}\right), \left(\frac{(1+p\tilde{r}_1)^\lambda-(1+p\tilde{r}_1)^\lambda}{(1+p\tilde{r}_1)^\lambda+(1-p\tilde{r}_1)^\lambda}, \frac{(1+p\tilde{s}_1)^\lambda-(1-p\tilde{s}_1)^\lambda}{(1+p\tilde{s}_1)^\lambda+(1-p\tilde{s})^\lambda}, \frac{(1+p\tilde{t}_1)^\lambda-(1-p\tilde{t}_1)^\lambda}{(1+p\tilde{t}_1)^\lambda+(1-p\tilde{t}_1)^\lambda}\right)\rangle, \lambda>0. \tag{9}$$

Definition 3.2. 'Score Function' and 'Accuracy function' of TFNN

Let $\tilde{A} = \langle(\bar{\mu}_1,\bar{\mu}_2,\bar{\mu}_3),(\bar{v}_1,\bar{v}_2.\bar{v}_3),(\bar{\lambda}_1,\bar{\lambda}_2,\bar{\lambda}_3)\rangle$ be a TFNN. Its score function Scr ($\tilde{A}$) is defined as:

$$\mathrm{Scr}(\tilde{A}) = \frac{1}{9}[6+(\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3)-(\bar{v}_1+\bar{v}_2+\bar{v}_3)-(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)] \tag{10}$$

and $\mathrm{Scr}(\tilde{A}) \in [0,1]$.

3.1. Some properties of score function and accuracy function

3.1.1. Boundedness.

Max value of $\mathrm{Scr}(\tilde{A}) = \frac{1}{9}[6+\max(\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3)-\min(\bar{v}_1+\bar{v}_2+\bar{v}_3)-\min(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)]$

$=\frac{1}{9}[6+3-0-0]$
$=1$

Similarly, Min of $Scr(\tilde{A})=\frac{1}{9}[6+\min(\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3)-\max(\bar{\nu}_1+\bar{\nu}_2+\bar{\nu}_3)-\max(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)]$

$=\frac{1}{9}[6+0-3-3]$
$=0$

So, $Scr(\tilde{A})\in[0,1]$ i.e. boundedness property of the $Scr(\tilde{A})$ is proved.

Accuracy function of TFNN is denoted as H ($\tilde{A}$) and is presented as:

$$H(\tilde{A})=\frac{1}{3}[(\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3)-(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)] \tag{11}$$

and $H(\tilde{A}) \in [-1,+1]$

Maximum value of $H(\tilde{A})=$

$\frac{1}{3}[\max(\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3)-\min(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)]$
$=\frac{1}{3}[3-0]$
$=1$

Similarly, minimum value of $H(\tilde{A})=\frac{1}{3}[\min(\bar{\mu}_1+\bar{\mu}_2+\bar{\mu}_3)-\max(\bar{\lambda}_1+\bar{\lambda}_2+\bar{\lambda}_3)]$

$=\frac{1}{3}[0-3]$
$=-1.$

$\therefore H(\tilde{A}) \in [-1,+1]$

So, the boundedness property of H ($\tilde{A}$) is also proved.

3.1.2 Monotonicity

Let

$\bar{A}^{(1)}=\langle(\bar{\mu}^{(1)},\bar{\mu}^{(2)},\bar{\mu}^{(3)}),(\bar{\nu}^{(1)},\bar{\nu}^{(2)},\bar{\nu}^{(3)}),(\bar{\lambda}^{(1)},\bar{\lambda}^{(2)},\bar{\lambda}^{(3)})\rangle$
$\bar{A}^{(2)}=\langle(\bar{\mu}^{/(1)},\bar{\mu}^{/(2)},\bar{\mu}^{/(3)}),(\bar{\nu}^{/(1)},\bar{\nu}^{/(2)},\bar{\nu}^{/(3)}),(\bar{\lambda}^{/(1)},\bar{\lambda}^{/(2)},\bar{\lambda}^{/(3)})\rangle$

If, $\bar{A}^{(1)}\le\bar{A}^{(2)}$, we have $\bar{\mu}^{(1)}\le\bar{\mu}^{/(1)}$, $\bar{\mu}^{(2)}\le\bar{\mu}^{/(2)}$, $\bar{\mu}^{(3)}\le\bar{\mu}^{/(3)}$

$\bar{\nu}^{(1)}\ge\bar{\nu}^{/(1)}$, $\bar{\nu}^{(2)}\ge\bar{\nu}^{/(2)}$, $\bar{\nu}^{(3)}\ge\bar{\nu}^{/(3)}$ and, $\bar{\lambda}^{(1)}\ge\bar{\lambda}^{/(1)}$, $\bar{\lambda}^{(2)}\ge\bar{\lambda}^{/(2)}$, $\bar{\lambda}^{(3)}\ge\bar{\lambda}^{/(3)}$

So, $Scr(\bar{A}^{(1)})=\frac{1}{9}[6+(\bar{\mu}_1^{(1)}+\bar{\mu}_2^{(2)}+\bar{\mu}_3^{(3)})-(\bar{\nu}_1^{(1)}+\bar{\nu}_2^{(2)}+\bar{\nu}_3^{(3)})-(\bar{\lambda}_1^{(1)}+\bar{\lambda}_2^{(2)}+\bar{\lambda}_3^{(3)})]$

$\le\frac{1}{9}[6+(\bar{\mu}_1'^{(1)}+\bar{\mu}_2^{/(2)}+\bar{\mu}_3^{/(3)})-(\bar{\nu}_1^{/(1)}+\bar{\nu}_2^{/(2)}+\bar{\nu}_3^{/(3)})-(\bar{\lambda}_1^{/(1)}+\bar{\lambda}_2^{/(2)}+\bar{\lambda}_3^{/(3)})]$

$=Scr(\bar{A}^{(2)})$

$\therefore Scr(\bar{A}^{(1)})\le Scr(\bar{A}^{(2)})$.

So, monotonicity of $Scr(\tilde{A})$ is proved.

Similarly, $H(\tilde{A}^{(1)})=\frac{1}{3}[(\bar{\mu}_1^{(1)}+\bar{\mu}_2^{(2)}+\bar{\mu}_3^{(3)})-(\bar{\lambda}_1^{(1)}+\bar{\lambda}_2^{(2)}+\bar{\lambda}_3^{(3)})]$

$\le\frac{1}{3}[(\bar{\mu}_1^{/(1)}+\bar{\mu}_2^{/(2)}+\bar{\mu}_3^{/(3)})-(\bar{\lambda}_1^{/(1)}+\bar{\lambda}_2^{/(2)}+\bar{\lambda}_3^{/(3)})]$

$=H(\tilde{A}^{(2)})$

$\therefore H((\tilde{A}^{(1)})\le H(\tilde{A}^{(2)})$

So, monotonicity property for the H ($\tilde{A}$) is proved.

3.2. Aggregation of TFNNs

We first remind few important definitions of Arithmetic Operations (AOs) applicable for “real numbers”. The weighted averaging operator of a “collection of real n umbers” $p\tilde{a}_i$ (i=1, 2, …, n-1,

n) is defined as: WAi $(p\tilde{a}_1, p\tilde{a}_2, p\tilde{a}_3, ..., p\tilde{a}_n) = \sum_{i=1}^{i=n} p\hat{\delta}_{w_i} p\tilde{a}_i$ (12)

Here, $p\hat{\delta}_{w_i} = (p\hat{\delta}_{w_1}, p\hat{\delta}_{w_2}, p\hat{\delta}_{w_3}, ..., p\hat{\delta}_{w_n})^T$ represents the" weight vector" of pai, $p\hat{\delta}_{w_i}$ Є [0,1] and $\sum_{i=1}^{n} p\hat{\delta}_{w_i} = 1$. In similar way, assuming $p\hat{\delta}_{w_i}$: $(Re)^n \to Re$ and for an assemblage of real numbers pai (i=1, 2, …, n-1, n), we can define the weighted geometric operator WTGn as

WTGn $(p\tilde{a}_1, p\tilde{a}_2, p\tilde{a}_3, ..., p\tilde{a}_n) = \prod_{i=1}^{n} (p\tilde{a}_i)^{p\hat{\delta}_{w_i}}$ (13)

3.3. TFNEOWA

Let $\bar{A}_i = <(p\tilde{a}_i, p\tilde{b}_i, p\tilde{c}_i), (p\tilde{e}_i, p\tilde{f}_i, p\tilde{g}_i), (p\tilde{r}_i, p\tilde{s}_i, p\tilde{t}_i)>$ (i=1, 2, …, n-1, n) be an ordered collection of TFNNs in the fixed set or well-defined accumulation of "real numbers". Then TFNEOWA $(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, ..., \tilde{A}_{n-1}, \tilde{A}_n)$ is defined in the following way:

$$\text{TFNEOWA}(\tilde{A}_1, \tilde{A}_2, ..., \bar{A}_{n-1}, \tilde{A}_n) = (p\hat{\delta}_{w_1})\bar{A}_1 \oplus (p\hat{\delta}_{w_2})\bar{A}_2 \oplus (p\hat{\delta}_{w_3})\bar{A}_3 \oplus ... \oplus (p\hat{\delta}_{w_n})\bar{A}_n = \bigoplus_{i=1}^{i=n} (p\hat{\delta}_{w_i}\bar{A}_i) \quad (14)$$

where $p\hat{\delta}_{w_i} \in [0,1]$ is regarded as "weight vector "of $\bar{A}_i$ and $\sum_{i=1}^{i=n} p\hat{\delta}_{w_i} = 1$

Theorem 3.3.1. Let $\bar{A}_i = <(p\tilde{a}_i, p\tilde{b}_i, p\tilde{c}_i), (p\tilde{e}_i, p\tilde{f}_i, p\tilde{g}_i), (p\tilde{r}_i, p\tilde{s}_i, p\tilde{t}_i)>$ (i=1, 2, 3…n-1, n) be an ordered collection of TFNNs in the well-defined accumulation of "real numbers". Then TFNEOWA $(\tilde{A}_1, \tilde{A}_2, ..., \tilde{A}_n)$ can be written in the following way:

$$\text{TFNEOWA } (\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 ..., \tilde{A}_n) = pw_1\tilde{A}_1 \oplus pw_2\tilde{A}_2 \oplus pw_3\tilde{A}_3 \oplus ... \oplus pw_n\tilde{A}_n = \bigoplus_{i=1}^{i=n} (pw_n\tilde{A}_i) \quad (15)$$

$$= < (\frac{\prod_{i=1}^{n}(p\tilde{a}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{a}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}, \frac{\prod_{i=1}^{n}(p\tilde{b}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{b}_i+1)^{pw_i} + \prod_{i=}^{n}(1-p\tilde{b}_i)^{pw_i}}, \frac{\prod_{i=1}^{n}(p\tilde{c}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}),$$
$$(\frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}, \frac{2\prod_{1}^{n}(p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{f}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{f}_i)^{pw_i}}, \frac{2\prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{g}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}),$$
$$(\frac{2\prod_{i=1}^{i=n}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{r}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}}, \frac{2\prod_{i=1}^{n}(p\tilde{s}_i)^{pw_i}}{\prod_{i=}^{n}(2-p\tilde{s}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{s}_i)^{pw_i}}, \frac{2\prod_{i=1}^{n}(p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{t}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{t}_i)^{pw_i}})\rangle, \quad (16)$$

where for symbolical simplicity for representation, we represent the "weight vector" as follows: $p\hat{\delta}_{w_1} = pw_1, p\hat{\delta}_{w_2} = pw_2, p\hat{\delta}_{w_3} = pw_3, ..., p\hat{\delta}_{w_i} = pw_i$

Proof: We make use of the mathematical induction method to prove the theorem

1. Take n = 1, the case is trivial.

TFNEOWA $(\tilde{A}_1)$=

$$< (\frac{\prod_{i=1}^{1}(p\tilde{a}_i+1)^{pw_i}-\prod_{i=1}^{1}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{1}(p\tilde{a}_i+1)^{pw_i}+\prod_{i=1}^{1}(1-p\tilde{a}_i)^{pw_i}},\frac{\prod_{i=1}^{1}(p\tilde{b}_i+1)^{pw_i}-\prod_{i=1}^{1}(1-p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{1}(p\tilde{b}_i+1)^{pw_i}+\prod_{i=1}^{1}(1-p\tilde{b}_i)^{pw_i}},\frac{\prod_{i=1}^{1}(p\tilde{c}_i+1)^{pw_i}-\prod_{i=1}^{1}(1-p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{1}(p\tilde{c}_i+1)^{pw_i}+\prod_{i=1}^{1}(1-p\tilde{c}_i)^{pw_i}}),$$
$$(\frac{2\prod_{i=1}^{1}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}},\frac{2\prod_{i=1}^{1}(p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{f}_i)^{pw_i}},\frac{2\prod_{i=1}^{1}(p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}),$$
$$(\frac{2\prod_{i=1}^{i=1}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{1}(2-p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{1}(p\tilde{r}_i)^{pw_i}},\frac{2\prod_{i=1}^{1}(p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{1}(2-p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{1}(p\tilde{s}_i)^{pw_i}},\frac{2\prod_{i=1}^{1}(p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{1}(2-p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{1}(p\tilde{t}_i)^{pw_i}})\rangle,$$
$$= pw_1\tilde{A}_1 \qquad (16)$$

2. For n = 2, then we have

$$\oplus_{i=1}^{i=2}(pw_i\tilde{A}_i) = pw_1\tilde{A}_1 \oplus pw_2\tilde{A}_2$$
$$pw_1\tilde{A}_1 = \langle(\frac{(1+p\tilde{a}_1)^{pw_1}-(1-p\tilde{a}_1)^{pw_1}}{(1+p\tilde{a}_1)^{pw_1}+(1-p\tilde{a}_1)^{pw_1}},\frac{(1+p\tilde{b}_1)^{pw_1}-(1-p\tilde{b}_1)^{pw_1}}{(1+p\tilde{b}_1)^{pw_1}+(1-p\tilde{b}_1)^{pw_1}},\frac{(1+p\tilde{c}_1)^{pw_1}-(1-p\tilde{c}_2)^{pw_1}}{(1+p\tilde{c}_1)^{pw_1}+(1-p\tilde{c}_1)^{pw_1}}),$$
$$(\frac{2(p\tilde{e}_1)^{pw_1}}{(2-p\tilde{e}_1)^{pw_1}+(p\tilde{e}_1)^{pw_1}},\frac{2(p\tilde{f}_1)^{pw_1}}{(2-p\tilde{f}_1)^{pw_1}+(p\tilde{f}_1)^{pw_1}},\frac{2(p\tilde{g}_1)^{pw_1}}{(2-p\tilde{g}_1)^{pw_1}+(p\tilde{g}_1)^{pw_1}}),$$
$$(\frac{2(p\tilde{r}_1)^{pw_1}}{(2-p\tilde{r}_1)^{pw_1}+(p\tilde{r}_1)^{pw_1}},\frac{2(p\tilde{s}_1)^{pw_1}}{(2-p\tilde{s}_1)^{pw_1}+(p\tilde{s}_1)^{pw_1}},\frac{2(p\tilde{t}_1)^{pw_1}}{(2-p\tilde{t}_1)^{pw_1}+(p\tilde{t}_1)^{pw_1}})\rangle$$
$$pw_2A_{\tilde{2}} = \langle(\frac{(1+p\tilde{a}_2)^{pw_2}-(1-p\tilde{a}_2)^{pw_2}}{((1+p\tilde{a}_2)^{pw_2}+(1-p\tilde{a}_2)^{pw_2}},\frac{(1+p\tilde{b}_1)^{pw_2}-(1-p\tilde{b}_1)^{pw_2}}{(1+p\tilde{b}_1)^{pw_2}+(1-p\tilde{b}_1)^{pw_2}},\frac{(1+p\tilde{c}_1)^{pw_2}-(1-p\tilde{c}_1)^{pw_2}}{(1+p\tilde{c}_1)^{pw_2}-(1-p\tilde{c}_1)^{pw_2}}),$$
$$(\frac{2(p\tilde{e}_2)^{pw_2}}{(2-p\tilde{e}_2)^{pw_2}+(p\tilde{e}_2)^{pw_2}},\frac{2(p\tilde{f}_2)^{pw_2}}{(2-p\tilde{f}_2)^{pw_1}+(p\tilde{f}_2)^{pw_2}},\frac{2(p\tilde{g}_2)^{pw_2}}{(2-p\tilde{g}_2)^{pw_1}+(p\tilde{g}_2)^{pw_1}}),$$
$$(\frac{2(p\tilde{r}_2)^{pw_2}}{(2-p\tilde{r}_2)^{pw_1}+(p\tilde{r}_2)^{pw_1}},\frac{2(p\tilde{s}_2)^{pw_2}}{(2-p\tilde{s}_2)^{pw_2}+(p\tilde{s}_2)^{pw_2}},\frac{2(p\tilde{t}_2)^{pw_2}}{(2-p\tilde{t}_2)^{pw_2}+(p\tilde{t}_2)^{pw_2}})\rangle$$

,

$$pw_1\tilde{A}_1 \oplus pw_2\tilde{A}_2 = \langle(\frac{(1+p\tilde{a}_1)^{pw_1}-(1-p\tilde{a}_1)^{pw_1}}{(1+p\tilde{a}_1)^{pw_1}+(1-p\tilde{a}_1)^{pw_1}},\frac{(1+p\tilde{b}_1)^{pw_1}-(1-p\tilde{b}_1)^{pw_1}}{(1+p\tilde{b}_1)^{pw_1}+(1-p\tilde{b}_1)^{pw_1}},\frac{(1+p\tilde{c}_1)^{pw_1}-(1-p\tilde{c}_1)^{pw_1}}{(1+p\tilde{c}_1)^{pw_1}+(1-p\tilde{c}_1)^{pw_1}}),$$
$$(\frac{2(p\tilde{e}_1)^{pw_1}}{(2-p\tilde{e}_1)^{pw_1}+(p\tilde{e}_1)^{pw_1}},\frac{2(p\tilde{f}_1)^{pw_1}}{(2-p\tilde{f}_1)^{pw_1}+(p\tilde{f}_1)^{pw_1}},\frac{2(p\tilde{g}_1)^{pw_1}}{(2-p\tilde{g}_1)^{pw_1}+(p\tilde{g}_1)^{pw_1}}),$$
$$(\frac{2(p\tilde{r}_1)^{pw_1}}{(2-p\tilde{r}_1)^{pw_1}+(p\tilde{r}_1)^{pw_1}},\frac{2(p\tilde{s}_1)^{pw_1}}{(2-p\tilde{s}_1)^{pw_1}+(p\tilde{s}_1)^{pw_1}},\frac{2(p\tilde{t}_1)^{pw_1}}{(2-p\tilde{t}_1)^{pw_1}+(p\tilde{t}_1)^{pw_1}})\rangle \oplus$$
$$\langle(\frac{(1+p\tilde{a}_2)^{pw_2}-(1-p\tilde{a}_2)^{pw_2}}{(1+p\tilde{a}_2)^{pw_2}+(1-p\tilde{a}_2)^{pw_2}},\frac{(1+p\tilde{b}_2)^{pw_2}-(1-p\tilde{b}_2)^{pw_2}}{(1+p\tilde{b}_2)^{pw_2}+(1-p\tilde{b}_2)^{pw_2}},\frac{(1+p\tilde{c}_2)^{pw_2}-(1-p\tilde{c}_2)^{pw_2}}{(1+p\tilde{c}_2)^{pw_2}+(1-p\tilde{c}_2)^{pw_2}}),$$
$$(\frac{2(p\tilde{e}_2)^{pw_2}}{(2-p\tilde{e}_2)^{pw_2}+(p\tilde{e}_2)^{pw_2}},\frac{2(p\tilde{f}_2)^{pw_2}}{(2-p\tilde{f}_2)^{pw_2}+(p\tilde{f}_2)^{pw_2}},\frac{2(p\tilde{g}_2)^{pw_2}}{(2-p\tilde{g}_2)^{pw_2}+(p\tilde{g}_2)^{pw_2}}), \qquad (17)$$
$$(\frac{2(p\tilde{r}_2)^{pw_2}}{(2-p\tilde{r}_2)^{pw_2}+(p\tilde{r}_2)^{pw_2}},\frac{2(p\tilde{s}_2)^{pw_2}}{(2-p\tilde{s}_2)^{pw_2}+(p\tilde{s}_2)^{pw_2}},\frac{2(p\tilde{t}_2)^{pw_2}}{(2-p\tilde{t}_2)^{pw_2}+(p\tilde{t}_2)^{pw_2}})\rangle$$

We use the following abbreviations.

$(1+p\tilde{a}_1)^{pw_1}=\alpha_1,(-p\tilde{a}_1+1)^{pw_1}=\alpha_1{}',(p\tilde{a}_2+1)^{pw_2}=\delta_1,(-p\tilde{a}_2+1)^{pw_2}=\delta_1{}',(p\tilde{r}_2)^{pw_1}=\lambda_1,(p\tilde{s}_1)^{pw_1}=\lambda_2,(p\tilde{t}_1)^{pw_1}=\lambda_3$
$(1+p\tilde{b}_1)^{w_1}=\beta_1,(1-p\tilde{b}_1)^{pw_1}=\beta_1{}',(1+p\tilde{b}_2)^{pw_2}=\delta,(1-p\tilde{b}_2)^{pw_2}=\delta',(p\tilde{e}_1)^{pw_1}=\lambda_1',(p\tilde{f}_1)^{pw_1}=\lambda_2',(p\tilde{g}_1)^{pw_1}=\lambda_3'$
$(1+p\tilde{c}_1)^{pw_1}=\gamma_1,(1-p\tilde{c}_1)^{pw_1}=\gamma_1{}',(1+p\tilde{c}_2)^{pw_2}=\sigma,(1-p\tilde{c}_2)^{pw_2}=\sigma',(2-p\tilde{e}_1)^{pw_1}=\lambda,(2-p\tilde{f}_1)^{pw_1}=\lambda'$
$(2-p\tilde{g}_1)^{pw_1}=\lambda'',(2-p\tilde{r}_1)^{pw_1}=\eta,(2-p\tilde{s}_1)^{pw_1}=\eta',(2-p\tilde{t}_1)^{pw_1}=\eta'',(2-p\tilde{e}_2)^{pw_2}=\phi_1',(2-p\tilde{f}_2)^{pw_2}=\phi_2'$
$(2-p\tilde{g}_2)^{pw_2}=\phi_3',(2-p\tilde{r}_2)^{pw_2}=\eta_1,(2-p\tilde{s}_2)^{pw_2}=\eta_1{}',(2-p\tilde{t}_2)^{pw_2}=\eta_1{}'',(p\tilde{e}_2)^{pw_2}=\phi_1,(p\tilde{f}_2)^{pw_2}=\phi_2,(p\tilde{g}_2^{w_2})=\phi_3$
$(p\tilde{r}_2)^{w_2}=\varepsilon_1,(p\tilde{s}_2)^{w_2}=\varepsilon_2,(p\tilde{t}_2)^{w_2}=\varepsilon_3$

$$\therefore w_1\tilde{A}_1 \oplus w_2\tilde{A}_2 = \langle(\frac{\alpha_1-\alpha_1{}'}{\alpha_1+\alpha_1{}'},\frac{\beta_1-\beta_1{}'}{\beta_1+\beta_1{}'},\frac{\gamma_1-\gamma_1{}'}{\gamma_1+\gamma_1{}'}),(\frac{2\lambda_1'}{\lambda+\lambda_1'},\frac{2\lambda_2'}{\lambda'+\lambda_2'},\frac{2\lambda_3'}{\lambda''+\lambda_3'}),(\frac{2\lambda_1}{\eta+\lambda_1},\frac{2\lambda_2}{\eta'+\lambda_2},\frac{2\lambda_3}{\eta''+\lambda_3})\rangle$$
$$\oplus\langle(\frac{\delta_1-\delta_1'}{\delta_1+\delta_1'},\frac{\delta-\delta'}{\delta+\delta'},\frac{\sigma-\sigma'}{\sigma+\sigma'}),(\frac{2\phi_1}{\phi_1'+\phi_1},\frac{2\phi_2}{\phi_2'+\phi_2},\frac{2\phi_3}{\phi_3'+\phi_3}),(\frac{2\varepsilon_1}{\eta_1+\varepsilon_1},\frac{2\varepsilon_2}{\eta_1'+\varepsilon_2},\frac{2\varepsilon_3}{\eta_1''+\varepsilon_3})\rangle$$

$$=\langle(\frac{\frac{\alpha_1-\alpha_1^{/}}{\alpha_1+\alpha_1^{/}}+\frac{\delta_1-\delta_1^{/}}{\delta_1+\delta_1^{/}}}{1+(\frac{\alpha_1-\alpha_1^{/}}{\alpha_1+\alpha_1^{/}}\frac{\delta_1-\delta_1^{/}}{\delta_1+\delta_1^{/}})},\frac{\frac{\beta_1-\beta_1^{/}}{\beta_1+\beta_1^{/}}+\frac{\delta-\delta^{/}}{\delta+\delta^{/}}}{1+(\frac{\beta_1-\beta_1^{/}}{\beta_1+\beta_1^{/}}\frac{\delta-\delta^{/}}{\delta+\delta^{/}})},\frac{\frac{\gamma_1-\gamma_1^{/}}{\gamma_1+\gamma_1^{/}}+\frac{\sigma-\sigma^{/}}{\sigma+\sigma^{/}}}{1+(\frac{\gamma_1-\gamma_1^{/}}{\gamma_1+\gamma_1^{/}}\frac{\sigma-\sigma^{/}}{\sigma+\sigma^{/}})}),$$

$$(\frac{\frac{2\lambda_1^{/}}{\lambda+\lambda_1^{/}}\cdot\frac{2\phi_1}{\phi_1^{/}+\phi_1}}{1+(1-\frac{2\lambda_1^{/}}{\lambda+\lambda_1^{/}})(1-\frac{2\phi_1}{\phi_1^{/}+\phi_1})},\frac{\frac{2\lambda_2^{/}}{\lambda^{/}+\lambda_2^{/}}\cdot\frac{2\phi_2}{\phi_2^{/}+\phi_2}}{1+(1-\frac{2\lambda_2^{/}}{\lambda^{/}+\lambda_2^{/}})(1-\frac{2\phi_2}{\phi_2^{/}+\phi_2})},\frac{\frac{2\lambda_3^{/}}{\lambda^{//}+\lambda_3^{/}}\cdot\frac{2\phi_3}{\phi_3^{/}+\phi_3}}{1+(1-\frac{2\lambda_3^{/}}{\lambda^{//}+\lambda_3^{/}})(1-\frac{2\phi_3}{\phi_3^{/}+\phi_3})}),$$

$$(\frac{\frac{2\lambda_1}{\eta_1+\lambda_1}\cdot\frac{2\varepsilon_1}{\eta_1+\varepsilon_1}}{1+(1-\frac{2\lambda_1}{\eta_1+\lambda_1})(1-\frac{2\varepsilon_1}{\eta_1+\varepsilon_1})},\frac{\frac{2\lambda_2}{\eta^{/}+\lambda_2}\cdot\frac{2\varepsilon_2}{\eta_1^{/}+\varepsilon_2}}{1+(1-\frac{2\lambda_2}{\eta^{/}+\lambda_2})(1-\frac{2\varepsilon_2}{\eta_1^{/}+\varepsilon_2})},\frac{\frac{2\lambda_3}{\eta^{//}+\lambda_3}\cdot\frac{2\varepsilon_3}{\eta_1^{//}+\varepsilon_3}}{1+(1-\frac{2\lambda_3}{\eta^{//}+\lambda_3})(1-\frac{2\varepsilon_3}{\eta_1^{//}+\varepsilon_3})})>$$

$$=\langle(\frac{\alpha_1\delta_1-\alpha_1{}^{/}\delta_1^{/}}{\alpha_1\delta_1+\alpha_1{}^{/}\delta_1^{/}},\frac{\beta_1\delta-\beta_1{}^{/}\delta^{/}}{\beta_1\delta+\beta_1{}^{/}\delta^{/}},\frac{\gamma_1\sigma-\gamma_1^{/}\sigma^{/}}{\gamma_1\sigma+\gamma_1^{/}\sigma^{/}}),(\frac{2\lambda_1^{/}\phi_1}{\lambda\phi_1^{/}+\lambda_1^{/}\phi_1},\frac{2\lambda_2^{/}\phi_2}{\lambda^{/}\phi_2^{/}+\lambda_2^{/}\phi_2},\frac{2\lambda_3^{/}\phi_3}{\lambda^{//}\phi_3^{/}+\lambda_3^{/}\phi_3}),$$

$$(\frac{2\lambda_1\varepsilon_1}{\eta\eta_1+\lambda_1\varepsilon_1},\frac{2\lambda_2\varepsilon_2}{\eta^{/}\eta_1^{/}+\lambda_2\varepsilon_2},\frac{2\lambda_3\varepsilon_3}{\eta^{//}\eta_1^{//}+\lambda_3\varepsilon_3})\rangle$$

$$=\langle(\frac{(1+p\tilde{a}_1)^{pw_1}(1+p\tilde{a}_2)^{pw_2}-(1-p\tilde{a}_1)^{pw_1}(1-p\tilde{a}_2)^{pw_2}}{(1+p\tilde{a}_1)^{pw_1}(1+p\tilde{a}_2)^{pw_2}+(1-p\tilde{a}_1)^{pw_1}(1-p\tilde{a}_2)^{pw_2}},\frac{(1+p\tilde{b}_1)^{pw_1}(1+p\tilde{b}_2)^{pw_2}-(1-p\tilde{b}_1)^{pw_1}(1-p\tilde{b}_2)^{pw_2}}{(1+p\tilde{b}_1)^{pw_1}(1+p\tilde{b}_2)^{pw_2}+(1-p\tilde{b}_1)^{pw_1}(1-p\tilde{b}_2)^{pw_2}},$$

$$\frac{(1+p\tilde{c}_1)^{pw_1}(1+p\tilde{c}_2)^{pw_2}-(1-p\tilde{c}_1)^{pw_1}(1-p\tilde{c}_2)^{pw_2}}{(1+p\tilde{c}_1)^{pw_1}(1+p\tilde{c}_2)^{pw_2}+(1-p\tilde{c}_1)^{pw_1}(1-p\tilde{c}_2)^{pw_2}}),(\frac{2(p\tilde{e}_1)^{pw_1}(p\tilde{e}_2)^{pw_2}}{(2-p\tilde{e}_1)^{pw_1}(2-p\tilde{e}_2)^{pw_2}+(p\tilde{e}_1)^{pw_1}(p\tilde{e}_2)^{pw_2}},$$

$$\frac{2(p\tilde{f}_1)^{pw_1}(p\tilde{f}_2)^{pw_2}}{(2-p\tilde{f}_1)^{pw_1}(2-p\tilde{f}_2)^{pw_2}+(p\tilde{f}_1)^{pw_1}(p\tilde{f}_2)^{pw_2}},\frac{2(p\tilde{g}_1)^{pw_1}(p\tilde{g}_2)^{pw_2}}{(2-p\tilde{g}_1)^{pw_1}(2-p\tilde{g}_2)^{pw_2}+(p\tilde{g}_1)^{pw_1}(p\tilde{g}_2)^{pw_2}})$$

$$(\frac{2(p\tilde{r}_1)^{pw_1}(p\tilde{r}_2)^{pw_2}}{(2-p\tilde{r}_1)^{pw_1}(2-p\tilde{r}_2)^{pw_2}+(p\tilde{r}_1)^{pw_1}(p\tilde{r}_2)^{pw_2}},\frac{2(p\tilde{s}_1)^{pw_1}(p\tilde{s}_2)^{pw_2}}{(2-p\tilde{s}_1)^{pw_1}(2-p\tilde{s}_2)^{pw_2}+(p\tilde{s}_1)^{pw_1}(p\tilde{s}_2)^{pw_2}},$$

$$\frac{2(p\tilde{t}_1)^{pw_1}(p\tilde{t}_2)^{pw_2}}{(2-p\tilde{t}_1)^{pw_1}(2-p\tilde{t}_2)^{pw_2}+(p\tilde{t}_1)^{pw_1}(p\tilde{t}_2)^{pw_2}})\rangle$$

$$=\langle(\frac{\prod_{i=1}^{2}(1+p\tilde{a}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{a}_i)^{pw_i}},\frac{\prod_{i=1}^{2}(1+p\tilde{b}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{b}_i)^{pw_i}},\frac{\prod_{i=1}^{2}(1+p\tilde{c}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{c}_i)^{pw_i}}),$$

$$(\frac{2\prod_{i=1}^{2}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{e}_i)^{pw_i}+\prod_{1}^{2}(p\tilde{e}_i)^{pw_i}},\frac{2\prod_{1}^{2}(p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{2}(p\tilde{f}_i)^{pw_i}},\frac{2\prod_{1}^{2}(p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{2}(p\tilde{g}_i)^{pw_i}})$$

$$(\frac{2\prod_{i=1}^{2}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{2}(2-pr_i^{\sim})^{pw_i}+\prod_{i=1}^{2}(pr_i^{\sim})^{pw_i}},\frac{2\prod_{i=1}^{2}(p\tilde{s}_1)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{s}_1)^{pw_i}+\prod_{i=1}^{2}(p\tilde{s}_1)^{pw_i}},\frac{2\prod_{i=1}^{2}(p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{2}(p\tilde{t}_i)^{pw_i}})\rangle \quad (18)$$

Assume that the result is valid for i= n, i.e.

$$TFNEWAA(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,...,\tilde{A}_n)=\langle(\frac{\prod_{i=1}^{n}(p\tilde{a}_i+1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{b}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{b}_i)^{pw_i}}$$

$$\frac{\prod_{i=1}^{n}(1+p\tilde{c}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}),(\frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{f}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}),$$

$$(\frac{2\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{s}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{t}_i)^{pw_i}})\rangle \quad (19)$$

Then, for i= n+1, we have,

$$\mathrm{TFNEWA}(\tilde{A}_1,\tilde{A}_2,A_{\tilde{3}},...,\tilde{A}_{n+1}) = (pw_1\tilde{A}_1)\oplus(pw_2\tilde{A}_2)\oplus...\oplus(pw_{n+1}\tilde{A}_{n+1})$$

$$= \mathrm{TFNEWA}(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,...,\tilde{A}_n)\oplus pw_{n+1}\tilde{A}_{n+1}$$

$$=\langle\left(\frac{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{b}_i)^{pw_i}-\prod_{i=1}^{n}(-p\tilde{b}_i+1)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{n}(-p\tilde{b}_i+1)^{pw_i}},\right.$$

$$\left.\frac{\prod_{i=1}^{n}(1+p\tilde{c}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}\right),\left(\frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{f}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{g}_i)^{pw_i}}\right),$$

$$\left(\frac{2\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(-p\tilde{r}_i+2)^{pw_i}+\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{n}(-p\tilde{s}_i+2)^{pw_i}+\prod_{i=1}^{n}(p\tilde{s}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{n}(-p\tilde{t}_i+2)^{pw_i}+\prod_{i=1}^{n}(p\tilde{t}_i)^{pw_i}}\right)\rangle$$

$$\oplus\langle\left(\frac{(1+p\tilde{a}_{n+1})^{pw_{n+1}}-(1-p\tilde{a}_{n+1})^{pw_{n+1}}}{(1+p\tilde{a}_{n+1})^{pw_{n+1}}+(1-p\tilde{a}_{n+1})^{pw_{n+1}}},\frac{(1+p\tilde{b}_{n+1})^{pw_{n+1}}-(1-p\tilde{b}_{n+1})^{pw_{n+1}}}{(1+p\tilde{b}_{n+1})^{pw_{n+1}}+(1-p\tilde{b}_{n+1})^{pw_{n+1}}},\frac{(1+p\tilde{c}_{n+1})^{pw_{n+1}}-(1-p\tilde{c}_{n+1})^{pw_{n+1}}}{(1+p\tilde{c}_{n+1})^{pw_{n+1}}+(1-p\tilde{c}_{n+1})^{pw_{n+1}}}\right),$$

$$\left(\frac{2(p\tilde{e}_{n+1})^{pw_{n+1}}}{(2-p\tilde{e}_{n+1})^{pw_{n+1}}+(p\tilde{e}_{n+1})^{pw_{n+1}}},\right.$$

$$\left.\frac{2(p\tilde{f}_{n+1})^{pw_{n+1}}}{(2-pf_{\tilde{n+1}})^{pw_{n+1}}+(p\tilde{f}_{n+1})^{pw_{n+1}}},\frac{2(p\tilde{g}_{n+1})^{pw_{n+1}}}{(2-p\tilde{g}_{n+1})^{pw_{n+1}}+(p\tilde{g}_{n+1})^{pw_{n+1}}}\right),\left(\frac{2(p\tilde{r}_{n+1})^{pw_{n+1}}}{(2-p\tilde{r}_{n+1})^{pw_{n+1}}+(p\tilde{r}_{n+1})^{pw_{n+1}}},\frac{2(p\tilde{s}_{n+1})^{pw_{n+1}}}{(2-p\tilde{s}_{n+1})^{pw_{n+1}}+(p\tilde{s}_{n+1})^{pw_{n+1}}},\right.$$

$$\left.\frac{2(p\tilde{t}_{n+1})^{pw_{n+1}}}{(2-p\tilde{t}_{n+1})^{pw_{n+1}}+(p\tilde{t}_{n+1})^{pw_{n+1}}}\right)\rangle \tag{20}$$

Therefore, the theorem stands valid for i= n+1, when it is assumed that the theorem is true for n= m.

Therefore, by mathematical induction, the theorem is proved.

3.4. Some properties of TFNEOWA operator

Property 3.4.1. TFNEOWA $(A_{\tilde{1}},A_{\tilde{2}},A_{\tilde{3}},...,A_{\tilde{n}})$ is a TFNN.

Proof:

As, $\{\prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i}+\prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i}\}\geq\{\prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i}-\prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i}\}$

so, $\dfrac{\prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i}-\prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i}}{\prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i}+\prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i}}\leq 1$

Again, $\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}\geq\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}$

So, $\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}\geq 2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}$

$\therefore \dfrac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}\leq 1$,

similarly, $\dfrac{2\prod_{i=1}^{n}(pr_{\tilde{i}})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}}\leq 1$

$$\text{so, }\frac{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}+\frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}+\frac{2\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{r}_i)^{pw_i}}\leq 3 \tag{21}$$

Therefore, TFNEOWA ($(\tilde{A}_1,\tilde{A}_2,A_{\tilde{3}},...,A_{\tilde{n}})$ is definitely a TFNN.

Property 3.4.2. Idempotency

$$\text{TFNEOWA}(\tilde{A},\tilde{A},...,\tilde{A}) = pw_1\tilde{A} \oplus pw_2\tilde{A} \oplus p\,w_3\tilde{A} \oplus ... \oplus pw_n\tilde{A} = \oplus_{i=1}^{n} pw_i\tilde{A}$$

$$= \tilde{A}, \because \sum_{i=1}^{n} pw_i = 1$$

$$\text{TFNEOWA}(\tilde{A},\tilde{A},...,\tilde{A}) = \langle (\frac{\prod_{i=1}^{n}(1+p\tilde{a})^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{a})^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a})^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{a})^{pw_i}}, \frac{\prod_{i=1}^{n}(1+p\tilde{b})^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{b})^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{b})^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{b})^{pw_i}}, \frac{\prod_{i=1}^{n}(1+p\tilde{c})^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{c})^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{c})^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{c})^{pw_i}}),$$

$$(\frac{2\sum_{i=1}^{n}(p\tilde{e})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e})^{pw_i} + \prod_{i=1}^{n}(p\tilde{e})^{pw_i}}, \frac{2\sum_{i=1}^{n}(p\tilde{f})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{f})^{pw_i} + \prod_{i=1}^{n}(p\tilde{f})^{pw_i}}, \frac{2\sum_{1}^{n}(p\tilde{g})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{f})^{pw_i} + \prod_{i=1}^{n}(p\tilde{f})^{pw_i}}),$$

$$(\frac{2\sum_{i=1}^{n}(p\tilde{r})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{r})^{pw_i} + \prod_{i=1}^{n}(p\tilde{r})^{pw_i}}, \frac{2\sum_{1}^{n}(p\tilde{s})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{s})^{pw_i} + \prod_{i=1}^{n}(p\tilde{s})^{pw_i}}, \frac{2\sum_{1}^{n}(p\tilde{t})^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{t})^{pw_i} + \prod_{i=1}^{n}(p\tilde{t})^{pw_i}}))\rangle$$

$$= \langle (\frac{(1+p\tilde{a})^{\sum_{i=1}^{n}pw_i} - (1-p\tilde{a})^{\sum_{i=1}^{n}pw_i}}{(1+p\tilde{a})^{\sum_{i=1}^{n}pw_i} + (1-p\tilde{a})^{\sum_{i=1}^{n}pw_i}}, \frac{(1+p\tilde{b})^{\sum_{i=1}^{n}pw_i} - (1-p\tilde{b})^{\sum_{i=1}^{n}pw_i}}{(1+p\tilde{b})^{\sum_{i=1}^{n}w_i} + (1-p\tilde{b})^{\sum_{i=1}^{n}pw_i}}, \frac{(1+p\tilde{c})^{\sum_{i=1}^{n}pw_i} - (1-p\tilde{c})^{\sum_{i=1}^{n}pw_i}}{(1+p\tilde{c})^{\sum_{i=1}^{n}pw_i} + (1-p\tilde{c})^{\sum_{i=1}^{n}pw_i}}),$$

$$(\frac{2(p\tilde{e})^{\sum_{1}^{n}pw_i}}{(2-p\tilde{e})^{\sum_{i=1}^{n}pw_i} + (p\tilde{e})^{\sum_{i=1}^{n}pw_i}}, \frac{2(p\tilde{f})^{\sum_{1}^{n}pw_i}}{(2-p\tilde{f})^{\sum_{i=1}^{n}pw_i} + (p\tilde{f})^{\sum_{i=1}^{n}pw_i}}, \frac{2(p\tilde{g})^{\sum_{i=1}^{n}pw_i}}{(2-p\tilde{g})^{\sum_{i=1}^{n}pw_i} + (p\tilde{g})^{\sum_{i=1}^{n}pw_i}}),$$

$$(\frac{2(p\tilde{r})^{\sum_{i=1}^{n}pw_i}}{(2-p\tilde{r})^{\sum_{i=1}^{n}pw_i} + (p\tilde{r})^{\sum_{i=1}^{n}pw_i}}, \frac{2(p\tilde{s})^{\sum_{i=1}^{n}pw_i}}{(2-p\tilde{s})^{\sum_{i=1}^{n}w_i} + (p\tilde{s})^{\sum_{i=1}^{n}pw_i}}, \frac{2(p\tilde{t})^{\sum_{i=1}^{n}pw_i}}{(2-p\tilde{t})^{\sum_{i=1}^{n}pw_i} + (p\tilde{t})^{\sum_{i=1}^{n}pw_i}}))\rangle$$

$$= \langle (\frac{1+p\tilde{a}-1+p\tilde{a}}{1+p\tilde{a}+1-p\tilde{a}}, \frac{1+p\tilde{b}-1+p\tilde{b}}{1+p\tilde{b}+1-p\tilde{b}}, \frac{1+p\tilde{c}-1+p\tilde{c}}{1+p\tilde{c}+1-p\tilde{c}}), (\frac{2p\tilde{e}}{2-p\tilde{e}+p\tilde{e}}, \frac{2p\tilde{f}}{2-p\tilde{f}+p\tilde{f}}, \frac{2p\tilde{g}}{2-p\tilde{g}+p\tilde{g}}), (\frac{2p\tilde{r}}{2-p\tilde{r}+p\tilde{r}},$$

$$\frac{2p\tilde{s}}{2-p\tilde{s}+p\tilde{s}}, \frac{2p\tilde{t}}{2-p\tilde{t}+p\tilde{t}}))\rangle$$

$$= \langle (p\tilde{a}, p\tilde{b}, p\tilde{c}), (p\tilde{e}, p\tilde{f}, p\tilde{g}), (p\tilde{r}, p\tilde{s}, p\tilde{t})\rangle = \tilde{A} \tag{22}$$

which is a TFNN.

Hence property of idempotency is proved.

Property 3.4.3. Boundedness

Let

$$\tilde{A}^{(-)} = \langle (\min_i(p\tilde{a}_i, p\tilde{b}_i, p\tilde{c}_i), \max_i(p\tilde{e}_i, p\tilde{f}_i, p\tilde{g}_i), \max_i(p\tilde{r}_i, p\tilde{s}_i, p\tilde{t}_i))\rangle$$

$$\text{and}, \tilde{A}^{(+)} = \langle (\max_i(p\tilde{a}_i, p\tilde{b}_i, p\tilde{c}_i), \min_i(p\tilde{e}_i, p\tilde{f}_i, p\tilde{g}_i), \min_i(p\tilde{r}_i, p\tilde{s}_i, p\tilde{t}_i))\rangle.$$

Then

$$A^{\sim(-)} \le \text{TFNEOWA}(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3 ...., \tilde{A}_{n-1}, \tilde{A}_n) \le \tilde{A}^{(+)} \tag{23}$$

Proof: We already have

$$\prod_{i=1}^{n}(\max_i(p\tilde{a}_i)+1)^{pw_i} \geq \prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(-\max_i(p\tilde{a}_i)+1)^{pw_i}$$

$$\text{also,} \prod_{i=1}^{n}(1-\min_i(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(1-\max_i(p\tilde{a}_i))^{pw_i},$$

$$\text{and,} \prod_{i=1}^{n}(1-\max_i(p\tilde{a}_i))^{pw_i} \leq \prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i} \leq \prod_{i=1}^{n}(1-\min_i(p\tilde{a}_i))^{pw_i}$$

$$\text{so,} \prod_{i=1}^{n}(1+\max_i(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(1+\min_i(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(1-\min_i(p\tilde{a}_i))^{pw_i}$$

$$\geq \prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i} \geq \prod_{i=1}^{n}(1-\max_i(p\tilde{a}_i))^{pw_i}$$

$$\text{so,} \frac{\prod_{i=1}^{n}(1+\max_i(p\tilde{a}_i))^{pw_i} - \prod_{i=1}^{n}(1-\max_i(p\tilde{a}_i))^{pw_i}}{\prod_{i=1}^{n}(1+\max_i(p\tilde{a}_i))^{pw_i} + \prod_{i=1}^{n}(1-\max_i(p\tilde{a}_i))^{pw_i}} \geq \frac{\prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i} - \prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i}}{\prod_{i=1}^{n}(1+(p\tilde{a}_i))^{pw_i} + \prod_{i=1}^{n}(1-(p\tilde{a}_i))^{pw_i}} \geq$$

$$\frac{\prod_{i=1}^{n}(1+\min_i(p\tilde{a}_i))^{pw_i} - \prod_{i=1}^{n}(1-\min_i(p\tilde{a}_i))^{pw_i}}{\prod_{i=1}^{n}(1+\min_i(p\tilde{a}_i))^{pw_i} + \prod_{i=1}^{n}(1-\min_i(p\tilde{a}_i))^{pw_i}}$$

$$\text{or,} \frac{(1+\max_i(p\tilde{a}_i))^{\sum_{i=1}^{n} pw_i} - (1-\max_i(p\tilde{a}_i))^{\sum_{i=1}^{n} pw_i}}{(1+\max_i(p\tilde{a}_i))^{\sum_{i=1}^{n} pw_i} + (1-\max_i(p\tilde{a}_i))^{\sum_{i=1}^{n} pw_i}} \geq \frac{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i} - \prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i}}$$

$$\geq \frac{(1+\min_i(p\tilde{a}_i))^{\sum_{i=1}^{n} pw_i} - (1-\min_i(p\tilde{a}_i))^{\sum_{i=1}^{n} pw_i}}{(\min_i(p\tilde{a}_i)+1)^{\sum_{i=1}^{n} pw_i} + (-\min_i(p\tilde{a}_i)+1)^{\sum_{i=1}^{n} pw_i}}$$

$$\text{or,} \max_i(p\tilde{a}_i) \geq \frac{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{a}_i)^{pw_i}} \geq \min_i(p\tilde{a}_i) \tag{24}$$

Similarly, we can prove,

$$\text{or,} \max_i(p\tilde{b}_i) \geq \frac{\prod_{i=1}^{n}(1+p\tilde{b}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{b}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{b}_i)^{pw_i}} \geq \min_i(p\tilde{b}_i) \tag{25}$$

$$\& \max_i(p\tilde{c}_i) \geq \frac{\prod_{i=1}^{n}(p\tilde{c}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{c}_i)^{pw_i}} \geq \min_i(p\tilde{c}_i) \tag{26}$$

$$\text{Now,} \frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}} = \frac{2}{\prod_{i=1}^{n}(\frac{2}{p\tilde{e}_i}-1)^{pw_i}+1}$$

$$\text{now,} \frac{2}{\min_i(p\tilde{e}_i)} \geq \frac{2}{(p\tilde{e}_i)} \geq \frac{2}{\max_i(p\tilde{e}_i)}$$

$$(\frac{2}{\min_i(p\tilde{e}_i)}-1) \geq (\frac{2}{p\tilde{e}_i}-1) \geq (\frac{2}{\max_i(p\tilde{e}_i)}-1)$$

$$\text{or,} 1+\prod_{i=1}^{n}(\frac{2}{\min_i(p\tilde{e}_i)}-1)^{pw_i} \geq 1+\prod_{i=1}^{n}(\frac{2}{(p\tilde{e}_i)}-1)^{pw_i} \geq 1+\prod_{i=1}^{n}(\frac{2}{\max_i(p\tilde{e}_i)}-1)^{pw_i}$$

$$\text{or,} \frac{1}{\prod_{i=1}^{n}(\frac{2}{\min_i(p\tilde{e}_i)}-1)^{pw_i}+1} \leq \frac{1}{\prod_{i=1}^{n}(\frac{2}{(p\tilde{e}_i)}-1)^{pw_i}+1} \leq \frac{1}{\prod_{i=1}^{n}(\frac{2}{\max_i(p\tilde{e}_i)}-1)^{pw_i}+1}$$

$$\text{or,}\ \frac{1}{(\frac{2}{\min_i(p\tilde{e}_i)}-1)^{\sum_{i=1}^{n} pw_i}+1} \le \frac{1}{\prod_{i=1}^{n}(\frac{2}{p\tilde{e}_i}-1)^{pw_i}+1} \le \frac{1}{(\frac{2}{\max_i(p\tilde{e}_i)}-1)^{\sum_{i=1}^{n} pw_i}+1}$$

$$\text{or,}\ \frac{\min_i(p\tilde{e}_i)}{2} \le \frac{1}{\prod_{i=1}^{n}(\frac{2}{p\tilde{e}_i}-1)^{pw_i}+1} \le \frac{\max_i(p\tilde{e}_i)}{2}$$

$$\text{or,}\ \min_i(p\tilde{e}_i) \le \frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+1} \le \max_i(p\tilde{e}_i)$$

$$\min_i(p\tilde{e}_i) \le \frac{2\prod_{i=1}^{n}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{e}_i)^{pw_i}+1} \le \max_i(p\tilde{e}_i) \quad (27)$$

Similarly, we can show,

$$\min_i(p\tilde{f}_i) \le \frac{2\prod_{i=1}^{n}(p\tilde{f}_i)^{w_i}}{\prod_{i=1}^{n}(2-p\tilde{f}_i)^{w_i}+\prod_{i=1}^{n}(p\tilde{f}_i)^{w_i}} \le \max_i(p\tilde{f}_i)$$

Similarly, $$\min_i(p\tilde{g}_i) \le \frac{2\prod_{i=1}^{n}(p\tilde{g}_i)^{w_i}}{\prod_{i=1}^{n}(2-p\tilde{g}_i)^{w_i}+\prod_{i=1}^{n}(p\tilde{g}_i)^{w_i}} \le \max_i(p\tilde{g}_i) \quad (28)$$

Same type of proof of inequalities can be shown for other falsity components also.
Combining the results (24)-(28), it follows,

$$\tilde{A}^{(-)} \le \mathrm{TFNEOWA}(\tilde{A}_1,\tilde{A}_2,...,\tilde{A}_n) \le \tilde{A}^{(+)}$$

Value of scr($\tilde{A}^{(-)}$)

$$= \mathrm{Scr}(\tilde{A}^{(-)}) = \frac{1}{9}[6+(\min_i(p\tilde{a}_i)+\min_i(p\tilde{b}_i)+\min_i(p\tilde{c}_i))-(\max_i(p\tilde{e}_i)+\max_i(p\tilde{f}_i)+\max_i(p\tilde{g}_i)$$
$$-(\max_i(p\tilde{r}_i)+\max_i(p\tilde{s}_i)+\max_i(p\tilde{t}_i))]$$

$$\le \frac{1}{9}[6+(p\tilde{a}+p\tilde{b}+p\tilde{c})-(p\tilde{e}+p\tilde{f}+p\tilde{g})-(p\tilde{r}+p\tilde{s}+p\tilde{t})]$$
$$\le \frac{1}{9}[(\max_i(p\tilde{a}_i)+\max_i(p\tilde{b}_i)+\max_i(p\tilde{c}_i))-(\min_i(p\tilde{e}_i)+\min_i(p\tilde{f}_i)+\min_i(p\tilde{g}_i))-(\min_i(\tilde{r}_i)+\min_i(\tilde{s}_i)+$$
$$+\min(\tilde{t}_i))+6]$$

So, $\mathrm{Scr}(\tilde{A}^{(-)}) \le \mathrm{Scr}(\tilde{A}) \le \mathrm{Scr}(\tilde{A}^{(+)})$

Property 3.4.4. Monotonicity

Let $\tilde{A}_i^{(1)}$ and $\tilde{A}_i^{(2)}$ be two TFNNs in the defined set of “real numbers” and

$\tilde{A}_i^{(1)} \le \tilde{A}_i^{(2)} \text{for}, \forall i = n, n-1, n-2, ..., 3, 2, 1$.

Then we have, $\mathrm{TFNEOWA}(\tilde{A}_1^{(1)},\tilde{A}_2^{(1)},\tilde{A}_3^{(1)},...,\tilde{A}_n^{(1)}) \le \mathrm{TFNEOWA}(\tilde{A}_1^{(2)},\tilde{A}_2^{(2)},\tilde{A}_3^{(2)},...,\tilde{A}_n^{(2)})$

Proof:

$$\text{Let}, \tilde{A}_i^{(1)} = \langle (p\tilde{a}_i^{(1)}, p\tilde{b}_i^{(1)}, p\tilde{c}_i^{(1)}), (p\tilde{e}_i^{(1)}, p\tilde{f}_i^{(1)}, p\tilde{g}_i^{(1)}), (p\tilde{r}_i^{(1)}, p\tilde{s}_i^{(1)}, p\tilde{t}_i^{(1)})\rangle$$
$$\tilde{A}_i^{(2)} = \langle (p\tilde{a}_i^{(2)}, p\tilde{b}_i^{(2)}, p\tilde{c}_i^{(2)}), (p\tilde{e}_i^{(2)}, p\tilde{f}_i^{(2)}, p\tilde{g}_i^{(2)}), (p\tilde{r}_i^{(2)}, p\tilde{s}_i^{(2)}, p\tilde{t}_i^{(3)})\rangle$$
$$\text{for}, \tilde{A}_i^{(1)} \le \tilde{A}_i^{(2)}$$

We assume $\tilde{a}_i^{(1)} \le \tilde{a}_i^{(2)}, \tilde{e}_i^{(1)} \ge \tilde{e}_i^{(2)}, \tilde{r}_i^{(1)} \ge \tilde{r}_i^{(2)}$ for, $i = n, n-1, ..., 3, 2, 1$ (29)

$$\prod_{i=1}^{n}(1+p\tilde{a}_i^{(1)})^{pw_i} \le \prod_{i=1}^{n}(1+p\tilde{a}_i^{(2)})^{pw_i}$$
$$\&\prod_{i=1}^{n}(1-p\tilde{a}_i^{(1)})^{pw_i} \ge \prod_{i=1}^{n}(1-p\tilde{a}_i^{(2)})^{pw_i}, \forall i = n, n-1, n-2, ..., 3, 2, 1$$

And $\prod_{i=1}^{n}(1+p\tilde{a}_i^{(2)})^{pw_i} \ge \prod_{i=1}^{n}(1+p\tilde{a}_i^{(1)})^{pw_i} \ge \prod_{i=1}^{n}(1-p\tilde{a}_i^{(1)})^{pw_i} \ge \prod_{i=1}^{n}(1-p\tilde{a}_i^{(2)})^{pw_i}, \forall i = 1, 2, 4, ..., n$

$$\text{and,}\prod_{i=1}^{n}(1+p\tilde{a}_i^{(2)})^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i^{(2)})^{pw_i}\ge\prod_{i=1}^{n}(1+p\tilde{a}_i^{(1)})^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i^{(1)})^{pw_i}$$
$$\text{also,}\prod_{i=1}^{n}(1+p\tilde{a}_i^{(2)})^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i^{(2)})^{pw_i}\le\prod_{i=1}^{n}(1+p\tilde{a}_i^{(1)})^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i^{(1)})^{pw_i}$$

$$\frac{\prod_{i=1}^{n}(1+p\tilde{a}_i^{(2)})^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i^{(2)})^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i^{(2)})^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i^{(2)})^{pw_i}}\ge\frac{\prod_{i=1}^{n}(1+p\tilde{a}_i^{(1)})^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}_i^{(1)})^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}_i^{(1)})^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}_i^{(1)})^{pw_i}}$$

$$\text{now,}\,p\tilde{e}_i^{(1)}\ge p\tilde{e}_i^{(2)}\rightarrow\frac{2}{p\tilde{e}_i^{(1)}}\le\frac{2}{p\tilde{e}_i^{(2)}}$$
$$\text{or,}\prod_{i=1}^{n}\left(\frac{2}{p\tilde{e}_i^{(1)}}-1\right)^{pw_i}\le\prod_{i=1}^{n}\left(\frac{2}{p\tilde{e}_i^{(2)}}-1\right)^{pw_i}$$
$$\text{or,}\,1+\prod_{i=1}^{n}\left(\frac{2}{p\tilde{e}_i^{(1)}}-1\right)^{pw_i}\le 1+\prod_{i=1}^{n}\left(\frac{2}{p\tilde{e}_i^{(2)}}-1\right)^{pw_i}$$
$$\text{or,}\frac{1}{1+\prod_{i=1}^{n}\left(\frac{2}{p\tilde{e}_i^{(1)}}-1\right)^{pw_i}}\ge\frac{1}{1+\prod_{i=1}^{n}\left(\frac{2}{p\tilde{e}_i^{(2)}}-1\right)^{pw_i}}$$
$$\text{or,}\frac{2\prod_{i=1}^{n}(p\tilde{e}_i^{(1)})^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}_i^{(1)})^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{e}_i^{(1)})^{pw_i}}\ge\frac{2\prod_{i=1}^{n}(p\tilde{e}_i^{(2)})^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}_i^{(2)})^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{e}_i^{(2)})^{pw_i}}$$

Similarly, for $p\tilde{r}_i^{(1)}\ge p\tilde{r}_i^{(2)}$

$$\Rightarrow\frac{2\prod_{i=1}^{n}(p\tilde{r}_i^{(1)})^{pw_i}}{\prod_{i=1}^{n}(p\tilde{r}_i^{(1)})^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{r}_i^{(1)})^{pw_i}}\ge\frac{2\prod_{i=1}^{n}(p\tilde{r}_i^{(2)})^{pw_i}}{\prod_{i=1}^{n}(p\tilde{r}_i^{(2)})^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{r}_i^{(2)})^{pw_i}}.$$

Now, let $\tilde{A}^{(j)}=TFNEOWA(\tilde{A}_1^{(j)},\tilde{A}_2^{(j)},\tilde{A}_3^{(j)}....,\tilde{A}_n^{(j)})=\langle(p\tilde{a}^{(J)},p\tilde{b}^{(J)},p\tilde{c}^{(J)}),(p\tilde{e}^{(J)},p\tilde{f}^{(J)},p\tilde{g}^{(J)}),(p\tilde{r}^{(J)},p\tilde{s}^{(J)},p\tilde{t}^{(J)})\rangle$

$$\text{where,}\,p\tilde{a}^{(j)}=\frac{\prod_{i=1}^{n}(1+p\tilde{a}^{(J)})^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{a}^{(J)})^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{a}^{(J)})^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{a}^{(J)})^{pw_i}},\ p\tilde{e}^{(J)}=\frac{2\prod_{i=1}^{n}(p\tilde{e}^{(J)})^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}^{(J)})^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{e}^{(J)})^{pw_i}},\ \forall i=1,2,3,..,n$$

Now score function value of two aggregated TFNNs $\tilde{A}^{(1)}$ and $\tilde{A}^{(2)}$ are computed.
Using relations eqn (29), we redefine $\{\bar{A}^{(1)}=\langle(\bar{\mu}^{(1)},\bar{\mu}^{(2)},\bar{\mu}^{(3)}),(\bar{\nu}^{(1)},\bar{\nu}^{(2)},\bar{\nu}^{(3)}),(\bar{\lambda}^{(1)},\bar{\lambda}^{(2)},\bar{\lambda}^{(3)})\rangle\}$
$\bar{A}^{(2)}=\langle(\bar{\mu}^{/(1)},\bar{\mu}^{/(2)},\bar{\mu}^{/(3)}),(\bar{\nu}^{/(1)},\bar{\nu}^{/(2)},\bar{\nu}^{/(3)}),(\bar{\lambda}^{/(1)},\bar{\lambda}^{/(2)},\bar{\lambda}^{/(3)})\rangle$

$$Scr(\tilde{A}^{(1)})=\frac{1}{9}[6+(\bar{\mu}^{(1)}+\bar{\mu}^{(2)}+\bar{\mu}^{(3)})-(\bar{\nu}^{(1)}+\bar{\nu}^{(2)}+\bar{\nu}^{(3)})-(\bar{\lambda}^{(1)}+\bar{\lambda}^{(2)}+\bar{\lambda}^{(3)})],$$
$$Scr(\tilde{A}^{(2)})=\frac{1}{9}[6+(\bar{\mu}^{/(1)}+\bar{\mu}^{/(2)}+\bar{\mu}^{/(3)})-(\bar{\nu}^{/(1)}+\bar{\nu}^{/(2)}+\bar{\nu}^{/(3)})-(\bar{\lambda}^{/(1)}+\bar{\lambda}^{/(2)}+\bar{\lambda}^{/(3)})]$$

$$\text{and,}\,Scr(\tilde{A}^{(1)})=\frac{1}{9}[6+(\bar{\mu}^{(1)}+\bar{\mu}^{(2)}+\bar{\mu}^{(3)})-(\bar{\nu}^{(1)}+\bar{\nu}^{(2)}+\bar{\nu}^{(3)})-(\bar{\lambda}^{(1)}+\bar{\lambda}^{(2)}+\bar{\lambda}^{(3)})]\le$$
$$\frac{1}{9}[6+(\bar{\mu}^{/(1)}+\bar{\mu}^{/(2)}+\bar{\mu}^{/(3)})-(\bar{\nu}^{/(1)}+\bar{\nu}^{/(2)}+\bar{\nu}^{/(3)})-(\bar{\lambda}^{/(1)}+\bar{\lambda}^{/(2)}+\bar{\lambda}^{/(3)})]=scr(\tilde{A}^{(2)})$$

If $Scr(\tilde{A}^{(1)})\le Scr(\tilde{A}^{(2)})$,we intend to prove

$TFNEOWA(\tilde{A}_1^{(1)},\tilde{A}_2^{(1)},\tilde{A}_3^{(1)},\tilde{A}_4^{(1)},...,\tilde{A}_n^{(1)})<TFNEOWA(\tilde{A}_1^{(2)},\tilde{A}_2^{(2)},\tilde{A}_3^{(2)},\tilde{A}_4^{(2)},...,\tilde{A}_n^{(2)})$

The following cases are considered.

Case 1: If $Scr(\tilde{A}^{(1)})=Scr(\tilde{A}^{(2)})$ then it follows,

$$\frac{1}{9}[6+(\bar{\mu}^{(1)}+\bar{\mu}^{(2)}+\bar{\mu}^{(3)})-(\bar{\nu}^{(1)}+\bar{\nu}^{(2)}+\bar{\nu}^{(3)})-(\bar{\lambda}^{(1)}+\bar{\lambda}^{(2)}+\bar{\lambda}^{(3)})]$$
$$=\frac{1}{9}[6+(\bar{\mu}^{/(1)}+\bar{\mu}^{/(2)}+\bar{\mu}^{/(3)})-(\bar{\nu}^{/(1)}+\bar{\nu}^{/(2)}+\bar{\nu}^{/(3)})-(\bar{\lambda}^{/(1)}+\bar{\lambda}^{/(2)}+\bar{\lambda}^{/(3)})$$
$$\text{so,}\,(\bar{\mu}^{(1)}+\bar{\mu}^{(2)}+\bar{\mu}^{(3)})=(\bar{\mu}^{/(1)}+\bar{\mu}^{/(2)}+\bar{\mu}^{/(3)}),(\bar{\nu}^{(1)}+\bar{\nu}^{(2)}+\bar{\nu}^{(3)})=(\bar{\nu}^{/(1)}+\bar{\nu}^{/(2)}+\bar{\nu}^{/(3)}),$$
$$(\bar{\nu}^{/(1)}+\bar{\nu}^{/(2)}+\bar{\nu}^{/(3)})=(\bar{\lambda}^{/(1)}+\bar{\lambda}^{/(2)}+\bar{\lambda}^{/(3)})$$

Now for accuracy function of $\tilde{A}^{(1)}$, we have, $H(\tilde{A}^{(1)}) = \frac{1}{3}[(\bar{\mu}^{(1)} + \bar{\mu}^{(2)} + \bar{\mu}^{(3)}) - (\bar{\lambda}^{(1)} + \bar{\lambda}^{(2)} + \bar{\lambda}^{(3)})] = \frac{1}{3}[(\bar{\mu}^{/(1)} + \bar{\mu}^{/(2)} + \bar{\mu}^{/(3)}) - (\bar{\lambda}^{/(1)} + \bar{\lambda}^{/(2)} + \bar{\lambda}^{/(3)})] = H(\tilde{A}^{(2)})$

Therefore, $TFNEOWA(\tilde{A}_1^{(1)}, \tilde{A}_2^{(1)}, \tilde{A}_3^{(1)} ..., \tilde{A}_n^{(1)}) = TFNEOWA(\tilde{A}_1^{(2)}, \tilde{A}_2^{(2)}, \tilde{A}_3^{(2)} ..., \tilde{A}_n^{(2)})$ **(**30)

Case2: if $Scr(\tilde{A}^{(1)}) < Scr(\tilde{A}^{(2)})$, we have

$$\frac{1}{9}[6 + (\bar{\mu}^{(1)} + \bar{\mu}^{(2)} + \bar{\mu}^{(3)}) - (\bar{\nu}^{(1)} + \bar{\nu}^{(2)} + \bar{\nu}^{(3)}) - (\bar{\lambda}^{(1)} + \bar{\lambda}^{(2)} + \bar{\lambda}^{(3)})]$$
$$\leq \frac{1}{9}[6 + (\bar{\mu}^{/(1)} + \bar{\mu}^{/(2)} + \bar{\mu}^{/(3)}) - (\bar{\nu}^{/(1)} + \bar{\nu}^{/(2)} + \bar{\nu}^{/(3)}) - (\bar{\lambda}^{/(1)} + \bar{\lambda}^{/(2)} + \bar{\lambda}^{/(3)})]$$

Or, $\bar{\mu}_i^{(1)} \leq \bar{\mu}_i^{/(1)}, \bar{\mu}_i^{(2)} \leq \bar{\mu}_i^{/(2)}, \bar{\mu}_i^{(3)} \leq \bar{\mu}_i^{/(3)}, \bar{\nu}_i^{(1)} \geq \bar{\nu}_i^{/(1)}, \bar{\nu}_i^{(2)} \geq \bar{\nu}_i^{/(2)}$ ...etc, for $\forall i = 1, 2, 3, 4..., n-1, n$

$H(\tilde{A}^{(1)}) \leq H(\tilde{A}^{(2)})$

So, $TFNEOWA(\tilde{A}_i^{(1)}) < TFNEOWA(\tilde{A}_i^{(2)})$ for, $\forall i = 1, 2, 3, 4, ..., n-1, n$

or, $TFNEOWA(\tilde{A}_1^{(1)}, \tilde{A}_2^{(1)}, \tilde{A}_3^{(1)}, \tilde{A}_4^{(1)}, ..., \tilde{A}_n^{(1)}) < TFNEOWA(\tilde{A}_1^{(2)}, \tilde{A}_2^{(2)}, \tilde{A}_3^{(2)}, \tilde{A}_4^{(2)}, .., \tilde{A}_n^{(1)})$

So, we have

$$TFNEOWA(\tilde{A}_1^{(1)}, \tilde{A}_2^{(1)}, \tilde{A}_3^{(1)}, \tilde{A}_4^{(1)}, ...\tilde{A}_n^{(1)}) \leq TFNEOWA(\tilde{A}_1^{(2)}, \tilde{A}_2^{(2)}, \tilde{A}_3^{(2)}, \tilde{A}_4^{(2)}, .., \tilde{A}_n^{(2)}) \quad (31)$$

So, monotonicity is proved.

Theorem 3.4. 1. $TFNEOWGA\,(\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4, .., \tilde{A}_n) = \tilde{A}_1^{pw_1} \otimes \tilde{A}_2^{pw_2} \otimes ... \otimes \tilde{A}_n^{pw_n}$ (32)

$$=<(\frac{2\prod_{i=1}^{n}(p\tilde{a}_i)^{w_i}}{\prod_{i=1}^{n}(2-p\tilde{a}_i)^{w_i} + \prod_{i=1}^{n}(p\tilde{a}_i)^{w_i}}, \frac{2\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{b}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}}, \frac{2\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{c}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}), (\frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}},$$
$$\frac{\prod_{i=1}^{n}(1+p\tilde{f}_i)^{w_i} - \prod_{i=1}^{n}(1-p\tilde{f}_i)^{w_i}}{\prod_{i=1}^{n}(1+p\tilde{f}_i)^{w_i} + \prod_{i=1}^{n}(1-p\tilde{f}_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}}), (\frac{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}}, \frac{\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i}}, \frac{\prod_{1}^{n}(1+p\tilde{t}_i)^{pw_i} - \prod_{1}^{n}(1-p\tilde{t}_i)^{pw_i}}{\prod_{1}^{n}(1+p\tilde{t}_i)^{pw_i} + \prod_{1}^{n}(1-p\tilde{t}_i)^{pw_i}}) >$$

Proof: Mathematical inductive method is used to present the proof of the theorem.

Case1. For i=1

$TFNEOWGA\,(\tilde{A}_1) =$

$$=<(\frac{2\prod_{i=1}^{1}(p\tilde{a}_i)^{w_i}}{\prod_{i=1}^{1}(2-p\tilde{a}_i)^{w_i} + \prod_{i=1}^{1}(p\tilde{a}_i)^{w_i}}, \frac{2\prod_{i=1}^{1}(p\tilde{b}_i)^{w_i}}{\prod_{i=1}^{1}(2-p\tilde{b}_i)^{pw_i} + \prod_{i=1}^{1}(p\tilde{b}_i)^{pw_i}}, \frac{2\prod_{i=1}^{1}(p\tilde{c}_i)^{w_i}}{\prod_{i=1}^{1}(2-p\tilde{c}_i)^{pw_i} + \prod_{i=1}^{1}(p\tilde{c}_i)^{pw_i}}), (\frac{\prod_{i=1}^{1}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{1}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{1}(1+p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{1}(1-p\tilde{e}_i)^{pw_i}},$$
$$\frac{\prod_{i=1}^{1}(1+p\tilde{f}_i)^{w_i} - \prod_{i=1}^{1}(1-p\tilde{f}_i)^{w_i}}{\prod_{i=1}^{1}(1+p\tilde{f}_i)^{w_i} + \prod_{i=1}^{1}(1-p\tilde{f}_i)^{w_i}}, \frac{\prod_{i=1}^{1}(1+p\tilde{g}_i)^{pw_i} - \prod_{i=1}^{1}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{1}(1+p\tilde{g}_i)^{pw_i} + \prod_{i=1}^{1}(1-p\tilde{g}_i)^{pw_i}}), (\frac{\prod_{i=1}^{1}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{1}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{1}(1+p\tilde{r}_i)^{pw_i} + \prod_{i=1}^{1}(1-p\tilde{r}_i)^{pw_i}}, \frac{\prod_{i=1}^{1}(1+p\tilde{s}_i)^{pw_i} - \prod_{i=1}^{1}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{1}(1+p\tilde{s}_i)^{pw_i} + \prod_{i=1}^{1}(1-p\tilde{s}_i)^{pw_i}}, \frac{\prod_{i=1}^{1}(1+p\tilde{t}_i)^{pw_i} - \prod_{i=1}^{1}(1-p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{1}(1+p\tilde{t}_i)^{pw_i} + \prod_{i=1}^{1}(1-p\tilde{t}_i)^{pw_i}}) >$$

$= p\tilde{A}_1^{pw_1}$

Case2. For i= 2, we have,

TFNEOWGA

($(\tilde{A}_1, \tilde{A}_2)$

$$\tilde{A}_1^{pw_1} \otimes \tilde{A}_2^{pw_2} =<(\frac{2(p\tilde{a}_1)^{pw_1}}{(2-p\tilde{a}_1)^{pw_1}+(p\tilde{a}_1)^{pw_1}},\frac{2(p\tilde{b}_1)^{pw_1}}{(2-p\tilde{b}_1)^{w_1}+(p\tilde{b}_1)^{w_1}},\frac{2(p\tilde{c}_1)^{pw_1}}{(2-p\tilde{c}_1)^{pw_1}+(p\tilde{c}_1)^{pw_1}}),(\frac{(1+p\tilde{e}_1)^{pw_1}-(1-p\tilde{e}_1)^{pw_1}}{(1+p\tilde{e}_1)^{pw_1}+(1-p\tilde{e}_1)^{pw_1}},$$
$$\frac{(1+p\tilde{f}_1)^{pw_1}-(1-p\tilde{f}_1)^{pw_1}}{(1+p\tilde{f}_1)^{pw_1}+(1-p\tilde{f}_1)^{pw_1}},\frac{(1+p\tilde{g}_1)^{pw_1}-(1-p\tilde{g}_1)^{pw_1}}{(1+\tilde{g}_1)^{pw_1}+(1-\tilde{g}_1)^{pw_1}}),$$
$$(\frac{(1+p\tilde{r}_1)^{pw_1}-(1-p\tilde{r}_1)^{pw_1}}{(1+p\tilde{r}_1)^{pw_1}+(1-p\tilde{r}_1)^{pw_1}},\frac{(1+p\tilde{s}_1)^{pw_1}-(1-p\tilde{s}_1)^{pw_1}}{(1+p\tilde{s}_1)^{pw_1}+(1-p\tilde{s}_1)^{pw_1}},\frac{(1+p\tilde{t}_1)^{pw_1}-(1-p\tilde{t}_1)^{pw_1}}{(1+p\tilde{t}_1)^{pw_1}+(1-p\tilde{t}_1)^{pw_1}})>$$
$$\otimes<(\frac{2(p\tilde{a}_2)^{pw_2}}{(2-p\tilde{a}_2)^{pw_2}+(p\tilde{a}_2)^{pw_2}},\frac{2(p\tilde{b}_2)^{pw_2}}{(2-p\tilde{b}_2)^{pw_2}+(p\tilde{b}_2)^{pw_2}},\frac{2(p\tilde{c}_2)^{pw_2}}{(2-p\tilde{c}_2)^{pw_2}+(p\tilde{c}_2)^{pw_2}}),(\frac{(1+p\tilde{e}_2)^{pw_2}-(1-p\tilde{e}_2)^{pw_2}}{(1+p\tilde{e}_2)^{w_2}+(1-p\tilde{e}_2)^{w_2}},$$
$$\frac{(1+p\tilde{f}_2)^{pw_1}-(1-p\tilde{f}_2)^{pw_1}}{(1+p\tilde{f}_2)^{pw_1}+(1-p\tilde{f}_2)^{pw_1}},\frac{(1+p\tilde{g}_2)^{pw_1}-(1-p\tilde{g}_2)^{pw_1}}{(1+p\tilde{g}_2)^{pw_1}+(1-p\tilde{g}_2)^{pw_1}}),$$
$$(\frac{(1+p\tilde{r}_2)^{pw_2}-(1-p\tilde{r}_2)^{pw_2}}{(1+p\tilde{r}_2)^{pw_2}+(1-p\tilde{r}_2)^{pw_2}},\frac{(1+p\tilde{s}_2)^{pw_2}-(1-p\tilde{s}_2)^{pw_2}}{(1+p\tilde{s}_2)^{pw_2}+(1-p\tilde{s}_2)^{pw_2}},\frac{(1+p\tilde{t}_2)^{pw_2}-(1-p\tilde{t}_2)^{pw_2}}{(1+p\tilde{t}_2)^{pw_2}+(1-p\tilde{t}_2)^{pw_2}})>$$

$$=<(\frac{2(p\tilde{a}_1)^{pw_2}(p\tilde{a}_2)^{pw_2}}{(2-p\tilde{a}_1)^{pw_1}(2-p\tilde{a}_2)^{pw_2}+(p\tilde{a}_1)^{pw_2}(p\tilde{a}_2)^{pw_2}},\frac{2(p\tilde{b}_1)^{pw_1}(p\tilde{b}_2)^{pw_2}}{(2-p\tilde{b}_1)^{pw_1}(2-p\tilde{b}_2)^{pw_2}+(p\tilde{b}_1)^{pw_1}(p\tilde{b}_2)^{pw_2}},$$
$$\frac{2(p\tilde{c}_1)^{pw_1}(p\tilde{c}_2)^{pw_2}}{(2-p\tilde{c}_1)^{pw_1}(2-p\tilde{c}_2)^{pw_2}+(p\tilde{c}_1)^{pw_1}(p\tilde{c}_2)^{pw_2}}),$$
$$(\frac{(1+p\tilde{e}_1)^{pw_1}(1+p\tilde{e}_2)^{pw_2}-(1-p\tilde{e}_1)^{pw_1}(1-p\tilde{e}_2)^{pw_2}}{(1+p\tilde{e}_1)^{pw_1}(1+p\tilde{e}_2)^{pw_2}+(1-p\tilde{e}_1)^{pw_1}(1-p\tilde{e}_2)^{pw_2}},\frac{(1+p\tilde{f}_1)^{pw_1}(1+p\tilde{f}_2)^{pw_2}-(1-p\tilde{f}_1)^{pw_1}(1-p\tilde{f}_2)^{pw_2}}{(1+p\tilde{f}_1)^{pw_1}(1+p\tilde{f}_2)^{pw_2}+(1-p\tilde{f}_1)^{pw_1}(1-p\tilde{f}_2)^{pw_2}},$$
$$\frac{(1+p\tilde{g}_1)^{pw_1}(1+p\tilde{g}_2)^{pw_2}-(1-p\tilde{g}_1)^{pw_1}(1-p\tilde{g}_2)^{pw_2}}{(1+p\tilde{g}_1)^{pw_1}(1+p\tilde{g}_2)^{pw_2}+(1-p\tilde{g}_1)^{pw_1}(1-p\tilde{g}_2)^{pw_2}}),(\frac{(1+p\tilde{r}_1)^{pw_1}(1+p\tilde{r}_2)^{pw_2}-(1-p\tilde{r}_1)^{pw_1}(1-p\tilde{r}_2)^{pw_2}}{(1+p\tilde{r}_1)^{pw_1}(1+p\tilde{r}_2)^{pw_2}+(1-p\tilde{r}_1)^{pw_1}(1-p\tilde{r}_2)^{pw_2}},$$
$$\frac{(1+p\tilde{s}_1)^{pw_1}(1+p\tilde{s}_2)^{pw_2}-(1-p\tilde{s}_1)^{pw_1}(1-p\tilde{s}_2)^{pw_2}}{(1+p\tilde{s}_1)^{pw_1}(1+p\tilde{s}_2)^{pw_2}+(1-p\tilde{s}_1)^{pw_1}(1-p\tilde{s}_2)^{pw_2}},\frac{(1+p\tilde{t}_1)^{pw_1}(1+p\tilde{t}_2)^{pw_2}-(1-p\tilde{t}_1)^{pw_1}(1-p\tilde{t}_2)^{pw_2}}{(1+p\tilde{t}_1)^{pw_1}(1+p\tilde{t}_2)^{pw_2}+(1-p\tilde{t}_1)^{pw_1}(1-p\tilde{t}_2)^{pw_2}})>$$

$$=<(\frac{2\prod_{i=1}^{2}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{a}_i)^{w_i}+\prod_{i=1}^{2}(p\tilde{a}_i)^{pw_i}},\frac{2\prod_{i=1}^{2}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{2}(p\tilde{b}_i)^{pw_i}},\frac{2\prod_{i=1}^{2}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{2}(2-p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{2}(p\tilde{c}_i)^{pw_i}}),(\frac{\prod_{i=1}^{2}(1+p\tilde{e}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{e}_i)^{pw_i}},$$
$$\frac{\prod_{i=1}^{2}(1+p\tilde{f}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{f}_i)^{pw_i}},\frac{\prod_{i=1}^{2}(1+p\tilde{g}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{g}_i)^{pw_i}}),(\frac{\prod_{i=1}^{2}(1+p\tilde{r}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{r}_i)^{pw_i}}$$
$$\frac{\prod_{i=1}^{2}(1+p\tilde{s}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{s}_i)^{pw_i}},\frac{\prod_{i=1}^{2}(1+p\tilde{t}_i)^{pw_i}-\prod_{i=1}^{2}(1-p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{2}(1+p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{2}(1-p\tilde{t}_i)^{pw_i}})> \quad (33)$$

Now we assume that the theorem is proved for i= n, i.e.

$$TFNEOWGA(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,..,\tilde{A}_n)$$
$$=\tilde{A}_1^{pw_1}\otimes\tilde{A}_2^{pw_2}\otimes...\otimes\tilde{A}_n^{pw_n}$$
$$=<(\frac{2\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}}$$
$$\frac{2\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}),(\frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{f}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}}$$
$$(\frac{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{t}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i}})> \quad (34)$$

For i = n+1. we hav**e**

TFNEOWG A ($\tilde{A}_1^{pw_1}\otimes\tilde{A}_2^{pw_2}\otimes...\otimes\tilde{A}_n^{pw_n}\otimes\tilde{A}_{n+1}^{pw_{n+1}}$) $=\tilde{A}_1^{pw_1}\otimes\tilde{A}_2^{pw_2}...\otimes\tilde{A}_n^{pw_n}\otimes\tilde{A}_{n+1}^{pw_{n+1}}=$

$$
\begin{aligned}
&TFNEOWG(\tilde{A}_1,\tilde{A}_2,..,\tilde{A}_n)\otimes \tilde{A}_{n+1}^{w_{n+1}}\\
&=<(\frac{2\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}},\frac{2\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}),\\
&(\frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{f}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i}-\prod_{i=1}^{n}(1-pg_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}}),\\
&(\frac{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i}},\frac{\prod_{i=1}^{n}(1+p\tilde{t}_i)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i}})>
\end{aligned}
$$

$$
\begin{aligned}
&\otimes<(\frac{2(p\tilde{a}_{n+1})^{pw_{n+1}}}{(2-p\tilde{a}_{n+1})^{pw_{n+1}}+(p\tilde{a}_{n+1})^{pw_{n+1}}},\frac{2(p\tilde{b}_{n+1})^{pw_{n+1}}}{(2-p\tilde{b}_{n+1})^{pw_{n+1}}+(p\tilde{b}_{n+1})^{pw_{n+1}}},\frac{2(p\tilde{c}_{n+1})^{pw_{n+1}}}{(2-p\tilde{c}_{n+1})^{pw_{n+1}}+(p\tilde{c}_{n+1})^{pw_{n+1}}}),\\
&(\frac{(1+p\tilde{e}_{n+1})^{pw_{n+1}}-(1-p\tilde{e}_{n+1})^{pw_{n+1}}}{(1+p\tilde{e}_{n+1})^{pw_{n+1}}+(1-p\tilde{e}_{n+1})^{pw_{n+1}}},\\
&\frac{(1+p\tilde{f}_{n+1})^{pw_{n+1}}-(1-p\tilde{f}_{n+1})^{pw_{n+1}}}{(1+p\tilde{f}_{n+1})^{pw_{n+1}}+(1-p\tilde{f}_{n+1})^{pw_{n+1}}},\frac{(1+p\tilde{g}_{n+1})^{pw_{n+1}}-(1-p\tilde{g}_{n+1})^{pw_{n+1}}}{(1+p\tilde{g}_{n+1})^{pw_{n+1}}+(1-p\tilde{g}_{n+1})^{pw_{n+1}}}),\\
&(\frac{(1+p\tilde{r}_{n+1})^{pw_{n+1}}-(1-p\tilde{r}_{n+1})^{pw_{n+1}}}{(1+p\tilde{r}_{n+1})^{pw_{n+1}}+(1-p\tilde{r}_{n+1})^{pw_{n+1}}},\frac{(1+p\tilde{s}_{n+1})^{pw_{n+1}}-(1-p\tilde{s}_{n+1})^{pw_{n+1}}}{(1+p\tilde{s}_{n+1})^{pw_{n+1}}+(1-p\tilde{s}_{n+1})^{pw_{n+1}}},\\
&\frac{(1+p\tilde{t}_{n+1})^{pw_{n+1}}-(1-p\tilde{t}_{n+1})^{pw_{n+1}}}{(1+p\tilde{t}_{n+1})^{pw_{n+1}}+(1-p\tilde{t}_{n+1})^{pw_{n+1}}})>
\end{aligned}
$$

$$
\begin{aligned}
&=<(\frac{2(\prod_{i=1}^{n}p\tilde{a}_i)^{pw_i}(p\tilde{a}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}p\tilde{a}_i)^{pw_i}(p\tilde{a}_{n+1})^{pw_{n+1}}+\prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i}(2-p\tilde{a}_{n+1})^{pw_{n+1}}},\frac{2(\prod_{i=1}^{n}p\tilde{b}_i)^{pw_i}(p\tilde{b}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}p\tilde{b}_i)^{pw_i}(p\tilde{b}_{n+1})^{pw_{n+1}}+\prod_{i=1}^{n}(2-p\tilde{b}_i)^{pw_i}(2-p\tilde{b}_{n+1})^{pw_{n+1}}},\\
&\frac{2(\prod_{i=1}^{n}p\tilde{c}_i)^{pw_i}(p\tilde{c}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}p\tilde{c}_i)^{pw_i}(p\tilde{c}_{n+1})^{pw_{n+1}}+\prod_{i=1}^{n}(2-p\tilde{c}_i)^{pw_i}(2-p\tilde{c}_{n+1})^{pw_{n+1}}}),(\frac{(\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i})(1+p\tilde{e}_{n+1})^{pw_{n+1}}-(\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i})(1-p\tilde{e}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i})(1+p\tilde{e}_{n+1})^{pw_{n+1}}+(\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i})(1-p\tilde{e}_{n+1})^{pw_{n+1}}}\\
&(\frac{(\prod_{i=1}^{n}(1+p\tilde{f}_i)^{pw_i})(1+p\tilde{f}_{n+1})^{pw_{n+1}}-(\prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i})(1-p\tilde{f}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}(1+p\tilde{f}_i)^{w_i})(1+p\tilde{f}_{n+1})^{pw_{n+1}}+(\prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i})(1-p\tilde{f}_{n+1})^{pw_{n+1}}},\frac{(\prod_{i=1}^{n}(1+p\tilde{g}_i)^{pw_i})(1+p\tilde{g}_{n+1})^{pw_{n+1}}-(\prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i})(1-p\tilde{g}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}(1+p\tilde{f}_i)^{pw_i})(1+p\tilde{f}_{n+1})^{pw_{n+1}}+(\prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i})(1-p\tilde{g}_{n+1})^{pw_{n+1}}})\\
&(\frac{(\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i})(1+p\tilde{r}_{n+1})^{pw_{n+1}}-(\prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i})(1-p\tilde{r}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i})(1+p\tilde{r}_{n+1})^{pw_{n+1}}+(\prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i})(1-p\tilde{r}_{n+1})^{pw_{n+1}}},\frac{(\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i})(1+p\tilde{s}_{n+1})^{pw_{n+1}}-(\prod_{i=1}^{n}(1-p\tilde{s}_i)^{pw_i})(1-p\tilde{s}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}(1+p\tilde{s}_i)^{pw_i})(1+p\tilde{s}_{n+1})^{pw_{n+1}}+(\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i})(1-p\tilde{t}_{n+1})^{pw_{n+1}}}\\
&\frac{(\prod_{i=1}^{n}(1+p\tilde{t}_i)^{pw_i})(1+p\tilde{t}_{n+1})^{pw_{n+1}}-(\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i})(1-p\tilde{t}_{n+1})^{pw_{n+1}}}{(\prod_{i=1}^{n}(1+p\tilde{t}_i)^{pw_i})(1+p\tilde{t}_{n+1})^{pw_{n+1}}+(\prod_{i=1}^{n}(1-p\tilde{t}_i)^{pw_i})(1-p\tilde{t}_{n+1})^{pw_{n+1}}})>
\end{aligned}
$$

$$
\begin{aligned}
&=<(\frac{2\prod_{i=1}^{n+1}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n+1}(p\tilde{a}_i)^{pw_i}+\prod_{i=1}^{n+1}(2-p\tilde{a}_i)^{pw_i}},\frac{2\prod_{i=1}^{n+1}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n+1}(p\tilde{b}_i)^{pw_i}+\prod_{i=1}^{n+1}(2-p\tilde{b}_i)^{pw_i}},\frac{2\prod_{i=1}^{n+1}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n+1}(p\tilde{c}_i)^{pw_i}+\prod_{i=1}^{n+1}(2-p\tilde{c}_i)^{pw_i}}),(\frac{\prod_{i=1}^{n+1}(1+p\tilde{e}_i)^{pw_i}-\prod_{i=1}^{n+1}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n+1}(1+p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{n+1}(1-p\tilde{e}_i)^{pw_i}}\\
&\frac{\prod_{i=1}^{n+1}(1+p\tilde{f}_i)^{pw_i}-\prod_{i=1}^{n+1}(1-p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n+1}(1+p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{n+1}(1-p\tilde{f}_i)^{pw_i}},\frac{\prod_{i=1}^{n+1}(1+p\tilde{g}_i)^{pw_i}-\prod_{i=1}^{n+1}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n+1}(1+p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{n+1}(1-p\tilde{g}_i)^{pw_i}}),(\frac{\prod_{i=1}^{n+1}(1+p\tilde{r}_i)^{pw_i}-\prod_{i=1}^{n+1}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n+1}(1+p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{n+1}(1-p\tilde{r}_i)^{pw_i}},\frac{\prod_{i=1}^{n+1}(1+p\tilde{s}_i)^{pw_i}-\prod_{i=1}^{n+1}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{n+1}(1+p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{n+1}(1-p\tilde{s}_i)^{pw_i}}\\
&\frac{\prod_{i=1}^{n+1}(1+p\tilde{t}_i)^{pw_i}-\prod_{i=1}^{n+1}(1-p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{n+1}(1+p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{n+1}(1-p\tilde{t}_i)^{pw_i}})> \qquad (35)
\end{aligned}
$$

So, the theorem is true for i = n+1.

Therefore, by mathematical induction, the theorem is proved for any positive integer n.

3.5. Some properties of TFNEOWG operator

Property 3.5.1. TFNEOWG operator is a TFNN

Proof. Since, $0 \le p\tilde{a}_i \le 1$, we have $0 \le \prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i} \le 1$

$$\text{so,} \prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i} \le \prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i}$$

$$\text{or,} \{\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}\} \le \{\prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}\}$$

$$\text{or,} \frac{2\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}} \le 1$$

$$\text{now,} \frac{\prod_{i=1}^{n}(p\tilde{e}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}} = \frac{-\prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i}+1}{\prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i}+1}$$

$$\text{now,} \frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{w_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{w_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{w_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{w_i}} = \frac{-\prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i}+1}{\prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i}+1}$$

$$\text{as,} 0 \le p\tilde{e}_i \le 1, \text{so,} 0 \le \prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i} \le 1$$

$$0 \le 1-\prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i} \le 1 \,\&\, 1+\prod_{i=1}^{n}(\frac{1-p\tilde{e}_i}{1+p\tilde{e}_i})^{pw_i} \ge 1,$$

$$\therefore 0 \le \frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{w_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{w_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{w_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{w_i}} \le 1, \text{similarly,} 0 \le \frac{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}} \le 1.$$

$$\text{Therefore,} 0 \le \frac{2\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{n}(2-p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}} + \frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}} + \frac{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{r}_i)^{pw_i}} \le 3$$

Similarly, for other truth, indeterminacy and falsity components, we can show similar inequality relations hold. So, TFNEOWG operator is a TFNN.

Property 3.5.2. Idempotency

Let $\tilde{A}_i =< ((p\tilde{a}, p\tilde{b}, p\tilde{c}), (p\tilde{e}, p\tilde{f}, p\tilde{g}), (p\tilde{r}, p\tilde{s}, p\tilde{t}) >$ be a TFNN and $\tilde{A}_i$ be equal for all i=1,2, 3...**n. Then,**

$\text{TFNEOWGA}(\tilde{A}, \tilde{A}, \tilde{A}, ..., \tilde{A}) = \tilde{A}$

Proof.

$$=< (\frac{2(p\tilde{a})^{\sum_{i=1}^{n} pw_i}}{(p\tilde{a})^{\sum_{i=1}^{n} pw_i} + (2-p\tilde{a})^{\sum_{i=1}^{n} pw_i}}, \frac{2(p\tilde{b})^{\sum_{i=1}^{n} pw_i}}{(p\tilde{b})^{\sum_{i=1}^{n} pw_i} + (2-p\tilde{b})^{\sum_{i=1}^{n} w_i}}, \frac{2(p\tilde{c})^{\sum_{i=1}^{n} pw_i}}{(p\tilde{c})^{\sum_{i=1}^{n} pw_i} + (2-p\tilde{c})^{\sum_{i=1}^{n} pw_i}}), (\frac{(1+p\tilde{e})^{\sum_{i=1}^{n} pw_i} - (1-p\tilde{e})^{\sum_{i=1}^{n} pw_i}}{(1+p\tilde{e})^{\sum_{i=1}^{n} pw_i} + (1-p\tilde{e})^{\sum_{i=1}^{n} pw_i}},$$

$$\frac{(1+p\tilde{f})^{\sum_{i=1}^{n} pw_i} - (1-p\tilde{f})^{\sum_{i=1}^{n} pw_i}}{(1+p\tilde{f})^{\sum_{i=1}^{n} pw_i} + (1-p\tilde{f})^{\sum_{i=1}^{n} pw_i}},$$

$$\frac{(1+p\tilde{g})^{\sum_{i=1}^{n} pw_i} - (1-p\tilde{g})^{\sum_{i=1}^{n} pw_i}}{(1+p\tilde{g})^{\sum_{i=1}^{n} pw_i} + (1-p\tilde{g})^{\sum_{i=1}^{n} pw_i}}), (\frac{(1+p\tilde{r})^{\sum_{i=1}^{n} pw_i} - (1-p\tilde{r})^{\sum_{i=1}^{n} w_i}}{(1+p\tilde{r})^{\sum_{i=1}^{n} pw_i} + (1-p\tilde{r})^{\sum_{i=1}^{n} pw_i}}, \frac{(1+p\tilde{s})^{\sum_{i=1}^{n} pw_i} - (1-p\tilde{s})^{\sum_{i=1}^{n} pw_i}}{(1+p\tilde{s})^{\sum_{i=1}^{n} pw_i} + (1-p\tilde{s})^{\sum_{i=1}^{n} pw_i}},$$

$$\frac{(1+p\tilde{t})^{\sum_{i=1}^{n} pw_i} - (1-p\tilde{t})^{\sum_{i=1}^{n} pw_i}}{(1+p\tilde{t})^{\sum_{i=1}^{n} pw_i} + (1-p\tilde{t})^{\sum_{i=1}^{n} pw_i}}) >$$

$$=< (p\tilde{a}, p\tilde{b}, p\tilde{c}), (p\tilde{e}, p\tilde{f}, p\tilde{g}), (p\tilde{r}, p\tilde{s}, p\tilde{t}) >, \because \sum_{i=1}^{n} pw_i = 1$$

$$= \tilde{A} \tag{36}$$

So, property of idempotency is completely proved.

Property 3.5.3. Boundedness

Let $\tilde{A}_i = <(p\tilde{a}_i, p\tilde{b}_i, p\tilde{c}_i), (p\tilde{e}_i, p\tilde{f}_i, p\tilde{g}_i), (p\tilde{r}_i, p\tilde{s}_i, p\tilde{t}_i)>$ i= n, n-1,…,3, 2,1 be an assemblage of TFNNs, in the set of "real numbers". We assume,

$$\tilde{A}^{(+)} = <(\underset{i}{Max}(p\tilde{a}_i), \underset{i}{Max}(p\tilde{b}_i), \underset{i}{Max}(p\tilde{c}_i)), (\underset{i}{Min}(p\tilde{e}_i), \underset{i}{Min}(p\tilde{f}_i), \underset{i}{Min}(p\tilde{g}_i)), (\underset{i}{Min}(p\tilde{r}_i), \underset{i}{Min}(p\tilde{s}_i), \underset{i}{Min}(p\tilde{t}_i))>$$

$$\& \tilde{A}^{(-)} = <(\underset{i}{Min}(p\tilde{a}_i), \underset{i}{Min}(p\tilde{b}_i), \underset{i}{Min}(p\tilde{c}_i)), (\underset{i}{Max}(p\tilde{e}_i), \underset{i}{Max}(p\tilde{f}_i), \underset{i}{Max}(p\tilde{g}_i)), (\underset{i}{Max}(p\tilde{r}_i), \underset{i}{Max}(p\tilde{s}_i), \underset{i}{Max}(p\tilde{t}_i))>$$

Then $\tilde{A}^{(-)} \le TFNEOWGA(\tilde{A}_i) \le \tilde{A}^{(+)}$ (37)

Proof.

We have $\underset{i}{min}(p\tilde{a}_i) \le p\tilde{a}_i \le \underset{i}{max}(p\tilde{a}_i)$ for, i=1, 2, …, n-1, n.

$$\text{or,} \frac{1}{\underset{i}{max}(p\tilde{a}_i)} \le \frac{1}{p\tilde{a}_i} \le \frac{1}{\underset{i}{min}(p\tilde{a}_i)}$$

$$\text{or,} \frac{2}{\underset{i}{max}(p\tilde{a}_i)} \le \frac{2}{p\tilde{a}_i} \le \frac{2}{\underset{i}{min}(p\tilde{a}_i)}$$

$$\text{or,} \left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right) \le \left(\frac{2}{p\tilde{a}_i} - 1\right) \le \left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)$$

$$\text{or,} \left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)^{pw_i} \ge \left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i} \ge \left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right)^{pw_i}$$

$$\therefore \frac{2}{\underset{i}{max}(p\tilde{a}_i)} \le \frac{2}{p\tilde{a}_i} \le \frac{2}{\underset{i}{min}(p\tilde{a}_i)}$$

$$\text{or,} \left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right)^{pw_i} \le \left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i} \le \left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)^{pw_i}$$

$$\text{or,} 1 + \left(\prod_{i=1}^{n}\left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right)^{pw_i}\right) \le 1 + \left(\prod_{i=1}^{n}\left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i}\right) \le 1 + \left(\prod_{i=1}^{n}\left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)^{pw_i}\right)$$

$$\text{or,} \frac{1}{1 + \left(\prod_{i=1}^{n}\left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right)^{pw_i}\right)} \ge \frac{1}{1 + \left(\prod_{i=1}^{n}\left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i}\right)} \ge \frac{1}{1 + \left(\prod_{i=1}^{n}\left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)^{pw_i}\right)}$$

$$\text{or,} \frac{2}{1 + \left(\prod_{i=1}^{n}\left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right)^{pw_i}\right)} \ge \frac{2}{1 + \left(\prod_{i=1}^{n}\left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i}\right)} \ge \frac{2}{1 + \left(\prod_{i=1}^{n}\left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)^{pw_i}\right)}$$

$$\text{or,} \frac{2}{1 + \left(\frac{2}{\underset{i}{max}(p\tilde{a}_i)} - 1\right)^{\sum_{i=1}^{n} pw_i}} \ge \frac{2}{1 + \prod_{i=1}^{n}\left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i}} \ge \frac{2}{1 + \left(\frac{2}{\underset{i}{min}(p\tilde{a}_i)} - 1\right)^{\sum_{i=1}^{n} pw_i}}$$

$$\text{or,} \underset{i}{max}(p\tilde{a}_i) \ge \frac{2}{1 + \prod_{i=1}^{n}\left(\frac{2}{p\tilde{a}_i} - 1\right)^{pw_i}} \ge \underset{i}{min}(p\tilde{a}_i)$$

$$\text{or,} \underset{i}{max}(p\tilde{a}_i) \ge \frac{2\prod_{i=1}^{n}(p\tilde{a}_i)^{pw_i}}{\prod_{1}^{n}(p\tilde{a}_i)^{pw_i} + \prod_{1}^{n}(2 - p\tilde{a}_i)^{pw_i}} \ge \underset{i}{min}(p\tilde{a}_i)$$

Similarly, $\underset{i}{max}(p\tilde{b}_i) \ge \dfrac{2\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{b}_i)^{pw_i} + \prod_{i=1}^{n}(2 - p\tilde{b}_i)^{pw_i}} \ge \underset{i}{min}(p\tilde{b}_i) \& \underset{i}{max}(p\tilde{c}_i) \ge \dfrac{2\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i)^{pw_i} + \prod_{i=1}^{n}(2 - p\tilde{c}_i)^{pw_i}} \ge \underset{i}{min}(p\tilde{c}_i)$

$$\text{again,} 0 \le p\tilde{e}_i \le 1 \Rightarrow \underset{i}{max}(p\tilde{e}_i) \ge p\tilde{e}_i \ge \underset{i}{min}(p\tilde{e}_i)$$

$$\text{so,} 1 - \underset{i}{max}(p\tilde{e}_i) \le 1 - p\tilde{e}_i \le 1 - \underset{i}{min}(p\tilde{e}_i) \& 1 + \underset{i}{max}(p\tilde{e}_i) \ge (1 + p\tilde{e}_i) \ge 1 + \underset{i}{min}(p\tilde{e}_i)$$

$$\Rightarrow \prod_{i=1}^{n}(1-\max_i(p\tilde{e}_i))^{pw_i} \le \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i} \le \prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{pw_i}$$

$$\text{also,} \prod_{i=1}^{n}(\max_i(p\tilde{e}_i)+1)^{pw_i} \ge \prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} \ge \prod_{i=1}^{n}(\min_i(p\tilde{e}_i)+1)^{pw_i}$$

$$\text{so,} \{\prod_{i=1}^{n}(1-\max_i(p\tilde{e}_i))^{pw_i} + \prod_{i=1}^{n}(1+\max_i(p\tilde{e}_i))^{pw_i}\} \le \{\prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i}\} \le \{\prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{pw_i} + \prod_{i=1}^{n}(1+\min_i(p\tilde{e}_i))^{pw_i}\}$$

$$\text{and,} \{\prod_{i=1}^{n}(1+\max_i(p\tilde{e}_i))^{pw_i} - \prod_{i=1}^{n}(1-\max_i(p\tilde{e}_i))^{pw_i}\} \ge \{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}\} \ge \{\prod_{i=1}^{n}(1+\min_i(p\tilde{e}_i))^{pw_i} - \prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{pw_i}\}$$

$$\frac{\prod_{i=1}^{n}(1+\max_i(p\tilde{e}_i))^{pw_i} - \prod_{i=1}^{n}(1-\max_i(p\tilde{e}_i))^{pw_i}}{\prod_{i=1}^{n}(1+\max_i(p\tilde{e}_i))^{pw_i} + \prod_{i=1}^{n}(-\max_i(p\tilde{e}_i)+1)^{pw_i}} \ge \frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}} \ge \frac{\prod_{i=1}^{n}(\min_i(p\tilde{e}_i)+1)^{pw_i} - \prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{pw_i}}{\prod_{i=1}^{n}(\min_i(p\tilde{e}_i)+1)^{pw_i} + \prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{pw_i}}$$

$$\text{or,} \frac{(1+\max_i(p\tilde{e}_i))^{\sum_{i=1}^{n}pw_i} - (1-\max_i(p\tilde{e}_i))^{\sum_{i=1}^{n}pw_i}}{(\max_i(p\tilde{e}_i)+1)^{\sum_{i=1}^{n}pw_i} + (-\max_i(p\tilde{e}_i)+1)^{\sum_{i=1}^{n}pw_i}} \ge \frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}} \ge \frac{\prod_{i=1}^{n}(1+\min_i(p\tilde{e}_i))^{\sum_{i=1}^{n}pw_i} - \prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{\sum_{i=1}^{n}pw_i}}{\prod_{i=1}^{n}(1+\min_i(p\tilde{e}_i))^{\sum_{i=1}^{n}pw_i} + \prod_{i=1}^{n}(1-\min_i(p\tilde{e}_i))^{\sum_{i=1}^{n}pw_i}}$$

$$\text{as,} \sum_{i=1}^{n}pw_i = 1$$

So

$$\frac{(1+\max_i(p\tilde{e}_i))-(1-\max_i(p\tilde{e}_i))}{(1+\max_i(p\tilde{e}_i))+(1-\max_i(p\tilde{e}_i))} \ge \frac{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}} \ge \frac{(1+\min_i(p\tilde{e}_i))-(1-\min_i(p\tilde{e}_i))}{(1+\min_i(p\tilde{e}_i))+(1-\min_i(p\tilde{e}_i))}$$

$$\text{or,} \max_i(p\tilde{e}_i) \ge \frac{\prod_{i=1}^{n}(p\tilde{e}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{e}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{e}_i)^{pw_i}} \ge \min_i(p\tilde{e}_i)$$

$$\text{similarly,} \max_i(p\tilde{f}_i) \ge \frac{\prod_{i=1}^{n}(p\tilde{f}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{f}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{f}_i)^{pw_i}} \ge \min_i(p\tilde{f}_i) \,\&$$

$$\max_i(p\tilde{g}_i) \ge \frac{\prod_{i=1}^{n}(p\tilde{g}_i+1)^{pw_i} - \prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{g}_i+1)^{pw_i} + \prod_{i=1}^{n}(1-p\tilde{g}_i)^{pw_i}} \ge \min_i(p\tilde{g}_i)$$

Similarly, we can show, $\max_i(p\tilde{r}_i) \ge \frac{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i} + \prod_{i=1}^{n}(1+p\tilde{r}_i)^{pw_i}} \ge \min_i(p\tilde{r}_i)$ and same type of inequality is valid for other falsity components also, for all values of i=1,2, 3..., n.

So, we conclude that $\tilde{A}^{(-)} \le TFNEOWGA(\tilde{A}_1, \tilde{A}_1, \tilde{A}_1, .., \tilde{A}_1) \le \tilde{A}^{(+)}$ (38)

So, the operator is bounded.

To reconfirm the result of eqn(38), we consider the inequalities between $Scr(\tilde{A}^{(+)})$, $Scr(\tilde{A}^{(-)})$, $Scr(\tilde{A})$. and corresponding accuracy functions.

We have already defined "score function" (see eqn (10) and "accuracy function" (see eqn (11) which are presented as:

$$Scr(\tilde{A}_i) = \frac{1}{9}[6+(p\tilde{a}_i+p\tilde{b}_i+p\tilde{c}_i)-(p\tilde{e}_i+p\tilde{f}_i+p\tilde{g}_i)-(p\tilde{r}_i+p\tilde{s}_i+p\tilde{t}_i)]$$

$$\therefore Scr^{(max)}(\tilde{A}) = 1 \,\&\, Scr^{(min)}(\tilde{A}) = 0 \Rightarrow Scr \in [0,1], \forall i = (1,2,3,...,n)$$

$$H(\tilde{A}) = \frac{1}{3}[(p\tilde{a}+p\tilde{b}+p\tilde{c})-(p\tilde{r}+p\tilde{s}+p\tilde{t})] \,\&\, H(\tilde{A}) \in [-1,1]$$

Now,

$$\operatorname{Scr}(\tilde{A}^{(+)})=\frac{1}{9}[(\underset{i}{\operatorname{Max}}(p\tilde{a}_i)+\underset{i}{\operatorname{Max}}(p\tilde{b}_i)+\underset{i}{\operatorname{Max}}(p\tilde{c}_i)-(\underset{i}{\operatorname{Min}}(p\tilde{e}_i)+\underset{i}{\operatorname{Min}}(p\tilde{f}_i)+\underset{i}{\operatorname{Min}}(p\tilde{g}_i))-(\underset{i}{\operatorname{Min}}(p\tilde{r}_i)+\underset{i}{\operatorname{Min}}(p\tilde{s}_i)+\underset{i}{\operatorname{Min}}(p\tilde{t}_i))+6]$$
$$\geq\frac{1}{9}[(p\tilde{a}_i+p\tilde{b}_i+p\tilde{c}_i)-(p\tilde{e}_i+p\tilde{f}_i+p\tilde{g}_i)-(p\tilde{r}_i+p\tilde{s}_i+p\tilde{t}_i)+6]=\operatorname{scr}(\tilde{A})$$
$$\text{similarly, }\operatorname{Scr}(\tilde{A}^{(-)})=\frac{1}{9}[(\underset{i}{\operatorname{Min}}(p\tilde{a}_i)+\underset{i}{\operatorname{Min}}(p\tilde{b}_i)+\underset{i}{\operatorname{Min}}(p\tilde{c}_i))-(\underset{i}{\operatorname{Max}}(p\tilde{e}_i)+\underset{i}{\operatorname{Max}}(p\tilde{f}_i)+\underset{i}{\operatorname{Max}}(p\tilde{g}_i))-$$
$$(\underset{i}{\operatorname{Max}}(p\tilde{r}_i)+\underset{i}{\operatorname{Max}}(p\tilde{s}_i)+\underset{i}{\operatorname{Max}}(p\tilde{t}_i)+6]$$
$$\leq\frac{1}{9}[(p\tilde{a}_i+p\tilde{b}_i+p\tilde{c}_i)-(p\tilde{e}_i+p\tilde{f}_i+p\tilde{g}_i)-(p\tilde{r}_i+p\tilde{s}_i+p\tilde{t}_i)+6]$$
$$=\operatorname{Scr}(\tilde{A})$$
$$\therefore \operatorname{Scr}(\tilde{A}^{(-)})\leq\operatorname{Scr}(\tilde{A})\leq\operatorname{Scr}(\tilde{A}^{(+)}) \tag{39}$$

Now we consider the two cases:

Case 1. If $\operatorname{Scr}(\tilde{A})=\operatorname{Scr}(\tilde{A}^{(+)})\,\&\,\operatorname{Scr}(\tilde{A})=\operatorname{Scr}(\tilde{A}^{(-)})$

Then,

$$\operatorname{Scr}(\tilde{A})=\operatorname{Scr}(\tilde{A}^{(+)})\Rightarrow$$
$$\frac{1}{9}[6+(p\tilde{a}_i+p\tilde{b}_i+p\tilde{c}_i)-(p\tilde{e}_i+p\tilde{f}_i+p\tilde{g}_i)-(p\tilde{r}_i+p\tilde{s}_i+p\tilde{t}_i)]=\frac{1}{9}[6+(\underset{i}{\operatorname{Max}}(p\tilde{a}_i)+\underset{i}{\operatorname{Max}}(p\tilde{b}_i)+\underset{i}{\operatorname{Max}}(p\tilde{c}_i))-(\underset{i}{\operatorname{Min}}(p\tilde{e}_i)+\underset{i}{\operatorname{Min}}(p\tilde{f}_i)+\underset{i}{\operatorname{Min}}(p\tilde{g}_i)$$
$$-(\underset{i}{\operatorname{Min}}(p\tilde{r}_i)+\underset{i}{\operatorname{Min}}(p\tilde{s}_i)+\underset{i}{\operatorname{Min}}(p\tilde{t}_i)]$$

So, we have,

$$p\tilde{a}_i+p\tilde{b}_i+p\tilde{c}_i=(\underset{i}{\operatorname{Max}}(p\tilde{a}_i)+\underset{i}{\operatorname{Max}}(p\tilde{b}_i)+\underset{i}{\operatorname{Max}}(p\tilde{c}_i))\,\&\,(p\tilde{e}_i+p\tilde{f}_i+p\tilde{g}_i)=(\underset{i}{\operatorname{Min}}(p\tilde{e}_i)+\underset{i}{\operatorname{Min}}(p\tilde{f}_i)+\underset{i}{\operatorname{Min}}(p\tilde{g}_i)$$
$$\&p\tilde{r}_i+p\tilde{s}_i+p\tilde{t}_i=(\underset{i}{\operatorname{Min}}(p\tilde{r}_i)+\underset{i}{\operatorname{Min}}(p\tilde{s}_i)+\underset{i}{\operatorname{Min}}(p\tilde{t}_i)$$

Therefore,

$$H(\tilde{A})=\frac{1}{3}[(p\tilde{a}_i+p\tilde{b}_i+p\tilde{c}_i)-(p\tilde{r}_i+p\tilde{s}_i+p\tilde{t}_i)]=\frac{1}{3}[(\underset{i}{\operatorname{Max}}(p\tilde{a}_i)+(\underset{i}{\operatorname{Max}}(p\tilde{b}_i)+(\underset{i}{\operatorname{Max}}(p\tilde{c}_i))-(\underset{i}{\operatorname{Min}}(p\tilde{r}_i)+(\underset{i}{\operatorname{Min}}(p\tilde{s}_i)+(\underset{i}{\operatorname{Min}}(p\tilde{t}_i))]$$
$$=H(\tilde{A}^{(+)})$$

Similarly, taking $\operatorname{Scr}(\tilde{A}^{(-)})=\operatorname{Scr}(\tilde{A})$, we have $H(\tilde{A})=H(\tilde{A}^{(-)})$

Case 2.Similarly from $\operatorname{Scr}(\tilde{A}^{(-)})\prec\operatorname{Scr}(\tilde{A})\prec\operatorname{Scr}(\tilde{A}^{(+)})$**,** we obtain $H(\tilde{A}^{(-)})\prec H(\tilde{A})\prec H(\tilde{A}^{(+)})$

So, we have $H(\tilde{A}^{(-)})\leq H(\tilde{A})\leq H(\tilde{A}^{(+)})$

So, we have $\tilde{A}^{(-)}\leq \operatorname{TFNEOWGA}(\tilde{A}_1,\tilde{A}_2,\tilde{A}_3,...,\tilde{A}_n)\leq\tilde{A}^{(+)}$.

Property 3.5.4. Monotonicity

Assume that $\tilde{A}_i^{(1)}\,\&\,\tilde{A}_i^{(2)}$ are any two TFNNs such that $\tilde{A}_i^{(1)}\leq\tilde{A}_i^{(2)}$ for,i=1, 2, …, n.

Then, $\operatorname{TFNEOWGA}(\tilde{A}_1^{(1)},\tilde{A}_2^{(1)},\tilde{A}_3^{(1)},..,\tilde{A}_n^{(1)})\leq\operatorname{TFNEOWGA}(\tilde{A}_1^{(2)},\tilde{A}_2^{(2)},\tilde{A}_3^{(2)},...,\tilde{A}_n^{(2)})$

Now

for, $\tilde{A}_i^{(1)}=<(p\tilde{a}_i^1,p\tilde{b}_i^1,p\tilde{c}_i^1),(p\tilde{e}_i^1,p\tilde{f}_i^1,p\tilde{g}_i^1),(p\tilde{r}_i^1,p\tilde{s}_i^1,p\tilde{t}_i^1)>\&\tilde{A}_i^{(2)}=<(p\tilde{a}_i^2,p\tilde{b}_i^2,p\tilde{c}_i^2),(p\tilde{e}_i^2,p\tilde{f}_i^2,p\tilde{g}_i^2),(p\tilde{r}_i^2,p\tilde{s}_i^2,p\tilde{t}_i^2)>$

We consider $p\tilde{c}_i^1,p\tilde{g}_i^1,p\tilde{t}_i^1$ of $\tilde{A}_i^1\,\&\,p\tilde{c}_i^2,p\tilde{g}_i^2,p\tilde{t}_i^2$ of $\tilde{A}_i^2$ for, $\tilde{A}_i^1\leq\tilde{A}_i^2\,(i=1,2,3,...,n)$

$$0\leq p\tilde{c}_i^1\leq 1\,\&\,0\leq p\tilde{c}_i^2\leq 1$$
$$\text{also, }p\tilde{c}_i^1\leq p\tilde{c}_i^2$$
$$\text{so, }\frac{2}{p\tilde{c}_i^1}\geq\frac{2}{p\tilde{c}_i^2}\Rightarrow(\frac{2}{p\tilde{c}_i^1}-1)\geq(\frac{2}{p\tilde{c}_i^2}-1),$$
$$\text{or, }(\frac{2}{p\tilde{c}_i^1}-1)^{pw_i}\geq(\frac{2}{p\tilde{c}_i^2}-1)^{pw_i},$$
$$\text{or, }\prod_{i=1}^{n}(\frac{2}{p\tilde{c}_i^1}-1)^{pw_i}\geq\prod_{i=1}^{n}(\frac{2}{p\tilde{c}_i^2}-1)^{pw_i}\Rightarrow\{\prod_{i=1}^{n}(\frac{2}{p\tilde{c}_i^1}-1)^{pw_i}\}+1\geq\{\prod_{i=1}^{n}(\frac{2}{p\tilde{c}_i^2}-1)^{pw_i}\}+1$$
$$\text{or, }\frac{1}{\prod_{i=1}^{n}(\frac{2}{p\tilde{c}_i^1}-1)^{pw_i}+1}\leq\frac{1}{\prod_{1}^{n}(\frac{2}{p\tilde{c}_i^2}-1)^{pw_i}+1}$$

$$\text{or,}\frac{\prod_{i=1}^{n}(p\tilde{c}_i^1)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i^1)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{c}_i^1)^{pw_i}}\le\frac{\prod_{i=1}^{n}(p\tilde{c}_i^2)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i^2)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{c}_i^2)^{pw_i}}$$

$$\text{or,}\frac{2\prod_{i=1}^{n}(p\tilde{c}_i^1)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i^1)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{c}_i^1)^{pw_i}}\le\frac{2\prod_{i=1}^{n}(p\tilde{c}_i^2)p^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i^2)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{c}_i^2)^{pw_i}}$$

$$\text{similarly,}\frac{2\prod_{i=1}^{n}(p\tilde{a}_i^1)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{a}_i^1)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{a}_i^1)^{pw_i}}\le\frac{2\prod_{i=1}^{n}(p\tilde{a}_i^2)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{a}_i^2)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{a}_i^2)^{pw_i}}\ \&$$

$$\frac{2\prod_{i=1}^{n}(p\tilde{b}_i^1)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{b}_i^1)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{b}_i^1)^{pw_i}}\le\frac{2\prod_{i=1}^{n}(p\tilde{b}_i^2)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{b}_i^2)^{pw_i}+\prod_{i=1}^{n}(2-p\tilde{b}_i^2)^{pw_i}}$$

$$\text{for,}\ p\tilde{t}_i^1\ge p\tilde{t}_i^2\Rightarrow(1+p\tilde{t}_i^1)\ge(1+p\tilde{t}_i^2)$$

$$\text{or,}\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}\ge\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i},\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}\le\prod_{i=1}^{n}(1-p\tilde{t}_i^2)^{pw_i}\ \text{for,}\ i=(1,2,3,....,n)$$

$$\text{also,}\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}\ge\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}\Rightarrow\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}\ge\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}\ge\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}$$

$$\&\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}\ge\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}\ge\prod_{i=1}^{n}(1-p\tilde{t}_i^2)^{pw_i}\ge\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}$$

$$\text{so,}\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}\ge\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{t}_i^2)^{pw_i}$$

$$\&\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}\le\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{t}_i^2)^{pw_i}$$

$$\text{so,}\frac{\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{t}_i^1+1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{t}_i^1)^{pw_i}}\ge\frac{\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{t}_i^2)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{t}_i^2+1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{t}_i^2)^{pw_i}}$$

For, $p\tilde{g}_i^1>p\tilde{g}_i^2$ we can show,

$$\frac{\{\prod_{i=1}^{n}(1+p\tilde{g})^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{g}_i^1)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{g}_i^1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{g}_i^1)^{pw_i}\}}\ge\frac{\{\prod_{i=1}^{n}(1+p\tilde{g}_i^2)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{g}_i^2)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{g}_i^2)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{g}_i^2)^{pw_i}\}}\ \& \tag{40}$$

$$\frac{\{\prod_{i=1}^{n}(1+p\tilde{e}_i^1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{e}_i^1)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{e}_i^1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{e}_i^1)^{pw_i}\}}\ge\frac{\{\prod_{i=1}^{n}(1+p\tilde{e}_i^2)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{e}_i^2)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{e}_i^2)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{e}_i^2)^{pw_i}\}} \tag{41}$$

Similarly, we can show that

$$\frac{\{\prod_{i=1}^{n}(1+p\tilde{r}_i^1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{r}_i^1)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{r}_i^1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{r}_i^1)^{pw_i}\}}\ge\frac{\{\prod_{i=1}^{n}(1+p\tilde{r}_i^2)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{r}_i^2)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{r}_i^2)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{r}_i^2)^{pw_i}\}}\ \&$$

$$\frac{\{\prod_{i=1}^{n}(1+p\tilde{s}_i^1)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{s}_i^1)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{s}_i^1)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{s}_i^1)^{pw_i}\}}\ge\frac{\{\prod_{i=1}^{n}(1+p\tilde{s}_i^2)^{pw_i}-\prod_{i=1}^{n}(1-p\tilde{s}_i^2)^{pw_i}\}}{\{\prod_{i=1}^{n}(1+p\tilde{s}_i^2)^{pw_i}+\prod_{i=1}^{n}(1-p\tilde{s}_i^2)^{pw_i}\}}$$

$$\text{let}, \tilde{A}_i^j = TFNEOWGA(\tilde{A}_1^j, \tilde{A}_2^j, \tilde{A}_3^j, .., \tilde{A}_n^j) = <(p\tilde{a}^j, p\tilde{b}^j, p\tilde{c}^j), (p\tilde{e}^j, p\tilde{f}^j, p\tilde{g}^j), (p\tilde{r}^j, p\tilde{s}^j, p\tilde{t}^j)>$$

$$\text{where}, p\tilde{a}^j = \frac{2\prod_{i=1}^{n}(p\tilde{a}_i^j)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{a}_i^j)^{pw_i} + \prod_{i=1}^{n}(2 - p\tilde{a}_i^j)^{pw_i}}, p\tilde{b}^j = \frac{2\prod_{i=1}^{n}(p\tilde{b}_i^j)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{b}_i^j)^{pw_i} + \prod_{i=1}^{n}(2 - p\tilde{b}_i^j)^{pw_i}}, p\tilde{c}^j = \frac{2\prod_{i=1}^{n}(p\tilde{c}_i^j)^{pw_i}}{\prod_{i=1}^{n}(p\tilde{c}_i^j)^{pw_i} + \prod_{i=1}^{n}(2 - p\tilde{c}_i^j)^{pw_i}}$$

$$\& p\tilde{e}^j = \frac{\prod_{i=1}^{n}(1 + p\tilde{e}_i^j)^{pw_i} - \prod_{i=1}^{n}(1 - p\tilde{e}_i^j)^{pw_i}}{\prod_{i=1}^{n}(1 + p\tilde{e}_i^j)^{pw_i} + \prod_{i=1}^{n}(1 - p\tilde{e}_i^j)^{pw_i}}, p\tilde{f}^j = \frac{\prod_{i=1}^{n}(1 + p\tilde{f}_i^j)^{pw_i} - \prod_{i=1}^{n}(1 - p\tilde{f}_i^j)^{pw_i}}{\prod_{i=1}^{n}(1 + p\tilde{f}_i^j)^{pw_i} + \prod_{i=1}^{n}(1 - p\tilde{f}_i^j)^{pw_i}}, p\tilde{g}^j = \frac{\prod_{i=1}^{n}(1 + p\tilde{g}_i^j)^{pw_i} - \prod_{i=1}^{n}(1 - p\tilde{g}_i^j)^{pw_i}}{\prod_{i=1}^{n}(1 + p\tilde{g}_i^j)^{pw_i} + \prod_{i=1}^{n}(1 - p\tilde{g}_i^j)^{pw_i}}$$

$$p\tilde{r}^j = \frac{\prod_{i=1}^{n}(1 + p\tilde{r}_i^j)^{pw_i} - \prod_{i=1}^{n}(1 - p\tilde{r}_i^j)^{pw_i}}{\prod_{i=1}^{n}(1 + p\tilde{r}_i^j)^{pw_i} + \prod_{i=1}^{n}(1 - p\tilde{r}_i^j)^{pw_i}}, p\tilde{s}^j = \frac{\prod_{i=1}^{n}(1 + p\tilde{s}_i^j)^{pw_i} - \prod_{i=1}^{n}(1 - p\tilde{s}_i^j)^{pw_i}}{\prod_{i=1}^{n}(1 + p\tilde{s}_i^j)^{pw_i} + \prod_{i=1}^{n}(1 - p\tilde{s}_i^j)^{pw_i}}, p\tilde{t}^j = \frac{\prod_{i=1}^{n}(1 + p\tilde{t}_i^j)^{pw_i} - \prod_{i=1}^{n}(1 - p\tilde{t}_i^j)^{pw_i}}{\prod_{i=1}^{n}(1 + p\tilde{t}_i^j)^{pw_i} + \prod_{i=1}^{n}(1 - p\tilde{t}_i^j)^{pw_i}},$$

Now we consider score function of $\tilde{A}^{(1)}$

$$scr(\tilde{A}^1) = \frac{1}{9}[6 + (p\tilde{a}^1 + p\tilde{b}^1 + p\tilde{c}^1) - (p\tilde{e}^1 + p\tilde{f}^1 + p\tilde{g}^1) - (p\tilde{r}^1 + p\tilde{s}^1 + p\tilde{t}^1)]$$

We assume, $\tilde{A}^2 = TFNEOWGA(\tilde{A}_1^2, \tilde{A}_2^2, \tilde{A}_3^2, ..., \tilde{A}_n^2) = <(p\tilde{a}^2, \tilde{b}^2, \tilde{c}^2), (p\tilde{e}^2, p\tilde{f}^2, p\tilde{g}^2), (p\tilde{r}^2, p\tilde{s}^2, p\tilde{t}^2)>$

$$\& scr(\tilde{A}^2) = \frac{1}{9}[6 + (p\tilde{a}^2 + p\tilde{b}^2 + p\tilde{c}^2) - (p\tilde{e}^2 + p\tilde{f}^2 + p\tilde{g}^2) - (p\tilde{r}^2 + p\tilde{s}^2 + p\tilde{t}^2)] \geq$$
$$\frac{1}{9}[6 + (p\tilde{a}^1 + p\tilde{b}^1 + p\tilde{c}^1) - (p\tilde{e}^1 + p\tilde{f}^1 + p\tilde{g}^1) - (p\tilde{r}^1 + p\tilde{s}^1 + p\tilde{t}^1)] = scr(\tilde{A}^1)$$
$$[\text{as}, \tilde{A}^2 \geq \tilde{A}^1 \Rightarrow p\tilde{a}^2 \geq p\tilde{a}^1, ... \& p\tilde{e}^2 \leq p\tilde{e}^1, ..., p\tilde{r}^2 \leq p\tilde{r}^1, \text{etc.}] \tag{42}$$

We consider the following cases:

Case1. $scr(\tilde{A}^1) < scr(\tilde{A}^2)$

Then, $TFNEOWGA(\tilde{A}_1^1.\tilde{A}_2^1.\tilde{A}_3^1, ..., \tilde{A}_n^1) < TFNEOWGA(\tilde{A}_1^2.\tilde{A}_2^2.\tilde{A}_3^2 ..., \tilde{A}_n^2)$

Case 2: If $scr(\tilde{A}^1) = scr(\tilde{A}^2)$ then,

$$\frac{1}{9}[6 + (p\tilde{a}^1 + p\tilde{b}^1 + p\tilde{c}^1) - (p\tilde{e}^1 + p\tilde{f}^1 + p\tilde{g}^1) - (p\tilde{r}^1 + p\tilde{s}^1 + p\tilde{t}^1)] =$$
$$\frac{1}{9}[6 + (p\tilde{a}^2 + p\tilde{b}^2 + p\tilde{c}^2) - (p\tilde{e}^2 + p\tilde{f}^2 + p\tilde{g}^2) - (p\tilde{r}^2 + p\tilde{s}^2 + p\tilde{t}^2)]$$

Or, $p\tilde{a}^1 = p\tilde{a}^2, p\tilde{b}^1 = p\tilde{b}^2, p\tilde{c}^1 = p\tilde{c}^2 \,\&\, p\tilde{e}^1 = p\tilde{e}^2, p\tilde{f}^1 = p\tilde{f}^2,\ p\tilde{g}^1 = p\tilde{g}^2 \,\&\, p\tilde{r}^1 = p\tilde{r}^2, p\tilde{s}^1 = p\tilde{s}^2, p\tilde{t}^1 = p\tilde{t}^2$

Now, for accuracy function of $\tilde{A}^1$, we have,

$$H(\tilde{A}^1) = \frac{1}{3}[(p\tilde{a}^1 + p\tilde{b}^1 + p\tilde{c}^1) - (p\tilde{r}^1 + p\tilde{s}^1 + p\tilde{t}^1)] = \frac{1}{3}[(p\tilde{a}^2 + p\tilde{b}^2 + p\tilde{c}^2) - (p\tilde{r}^2 + p\tilde{s}^2 + p\tilde{t}^2)] = H(\tilde{A}^2)$$

So, we have, $TFNEOWGA(\tilde{A}_1{}^1, \tilde{A}_2{}^1\tilde{A}_3{}^1, ..., \tilde{A}_n{}^1) = TFNEOWGA(\tilde{A}_1{}^2, \tilde{A}_2{}^2\tilde{A}_3{}^2, ..., \tilde{A}_n{}^2)$

Finally, $TFNEOWG(\tilde{A}_1{}^1, \tilde{A}_2{}^1\tilde{A}_3{}^1, .., \tilde{A}_n{}^1) \leq TFNEOWG(\tilde{A}_1{}^2, \tilde{A}_2{}^2\tilde{A}_3{}^2, ..., \tilde{A}_n{}^2)$ **(**43)

So, monotonicity property is proved.

4. ENTROPY FOR TFNNS

Assume that k-th decision matrix $(\bar{D}_{mn}{}^k)_{M\times N}$ is constructed based on the rating values of k-th Decision Maker (DM) as follows:

$$(\bar{D}_{mn}{}^k)_{M\times N} = \begin{pmatrix} \tilde{x}_{11}^k & \tilde{x}_{12}^k \cdots & \tilde{x}_{1N}^k \\ \vdots & \ddots & \vdots \\ \tilde{x}_{M1}^k & \cdots & \tilde{x}_{MN}^k \end{pmatrix}$$

where, $\tilde{x}_{mn}^k = <(p\tilde{a}_{mn}^k, p\tilde{b}_{mn}^k, p\tilde{c}_{mn}^k), (p\tilde{e}_{mn}^k, p\tilde{f}_{mn}^k, p\tilde{g}_{mn}^k), (p\tilde{r}_{mn}^k, p\tilde{s}_{mn}^k, p\tilde{t}_{mn}^k)>$ represents rating value provided by the k-th DM in terms of TFNN.

Consider j-th criterion or attribute. Its average rating value is presented by

$$\bar{\tilde{x}}_{ij}^k = \frac{1}{M}\sum_{i=1}^{M}(\tilde{x}_{ij}^k) \tag{44}$$

Entropy of a particular criterion: Let there be m alternatives. The Hamming distances of the different TFNN ratings under the same criterion from average TFNN rating are calculated. Let $\hat{\delta}(\tilde{x}_{mj}^{k}, \tilde{x}_{ij}^{k*})$ be difference rating of the TFNN rating of m-th alternative under j-th criterion from the average rating.

The Hamming distance of the j-th criterion is calculated as follows: -

$$\hat{H}_{ij} = \frac{1}{9} < \hat{\delta}(\tilde{x}_{1j}^{k}, \tilde{x}_{ij}^{k*}) + \hat{\delta}(\tilde{x}_{2j}^{k}, \tilde{x}_{ij}^{k*}) + \ldots + \hat{\delta}(\tilde{x}_{mj}^{k}, \tilde{x}_{ij}^{k*}) > \tag{45}$$

The normalized Hamming distance is $\breve{\hat{H}}_{ij} = \dfrac{\hat{H}_{ij}}{\max(\hat{H}_{ij})}$

Entropy of the j-th criterion is presented by $\ddot{\varepsilon}_j = -\dfrac{1}{\ln(M)} \sum_{i=1}^{M} \left(\dfrac{\breve{\hat{H}}_{ij}}{\sum_{i=1}^{N} \breve{\hat{H}}_{ij}}\right) \ln\left(\dfrac{\breve{\hat{H}}_{ij}}{\sum_{i=1}^{N} \breve{\hat{H}}_{ij}}\right)$ (46)

Dispersion associated with the j-th criterion is given by $\tilde{d}_j = (1 - \ddot{\varepsilon}_j)$.

So, the weight of j-th criterion is given by $\tilde{w}_j = \dfrac{(1 - \ddot{\varepsilon}_j)}{\sum_{j=1}^{N} (1 - \ddot{\varepsilon}_j)}$ (47)

Entropy function $\ddot{\varepsilon}_j(\tilde{x})$ satisfies the following properties:

1. $\ddot{\varepsilon}_j(\tilde{x}, \tilde{x}) = 0$, if $\tilde{x}$ is a crisp set.
2. $\ddot{\varepsilon}_j(\tilde{x}, \tilde{x}_{11}) \geq \ddot{\varepsilon}_j(\tilde{x}, \tilde{x}_{12})$ if, $\tilde{x}_{11} \leq \tilde{x}_{12}$

$$\ddot{\varepsilon}_j(\tilde{x}, \tilde{x}_{11}) = -\frac{1}{\ln(M)} \sum_{i=1}^{M} \left(\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}}\right) \ln\left(\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}}\right) \;\&\; \ddot{\varepsilon}_j(\tilde{x}, \tilde{x}_{12}) = -\frac{1}{\ln(M)} \sum_{i=1}^{M} \left(\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}\right) \ln\left(\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}\right)$$

$$\text{as, } \frac{1}{\ln(M)} \sum_{i=1}^{M} \left(\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}}\right) \ln\left(\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}}\right) \leq \frac{1}{\ln(M)} \sum_{i=1}^{M} \left(\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}\right) \ln\left(\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}\right) [\because \tilde{x}_{12} > \tilde{x}_{11} \Rightarrow ((\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}) > (\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}})) \;\&\; m > 1]$$

$$\text{or, } -\frac{1}{\ln(m)} \sum_{i=1}^{M} \left(\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}}\right) \ln\left(\frac{\breve{\hat{H}}_{i1}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i1}}\right) \geq -\frac{1}{\ln(m)} \sum_{i=1}^{M} \left(\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}\right) \ln\left(\frac{\breve{\hat{H}}_{i2}}{\sum_{i=1}^{N} \breve{\hat{H}}_{i2}}\right)$$

or $\ddot{\varepsilon}_j(\tilde{x}, \tilde{x}_{12}) \leq \ddot{\varepsilon}_j(\tilde{x}, \tilde{x}_{11})$

3. $\ddot{\varepsilon}_j(\tilde{x}_{11}, \tilde{x}_{13}) = \bar{\ddot{\varepsilon}}_j(\tilde{x}_{11}^{c}, \tilde{x}_{13}^{c})$

Where $\tilde{X}_{11} = <(\bar{\mu}_1^{(1)}, \bar{\mu}_1^{(2)}, \bar{\mu}_1^{(3)}), (\bar{v}_1^{(1)}, \bar{v}_1^{(2)}, \bar{v}_1^{(3)}), (\bar{\lambda}_1^{(1)}, \bar{\lambda}_1^{(2)}, \bar{\lambda}_1^{(3)})> \;\&\; \tilde{X}_{13} = <(\bar{\mu}_3^{(1)}, \bar{\mu}_3^{(2)}, \bar{\mu}_3^{(3)}), (\bar{v}_3^{(1)}, \bar{v}_3^{(2)}, \bar{v}_3^{(3)}), (\bar{\lambda}_3^{(1)}, \bar{\lambda}_3^{(2)}, \bar{\lambda}_3^{(3)})>$
$\tilde{X}_{13}^{c} = <(\bar{\lambda}_3^{(1)}, \bar{\lambda}_3^{(2)}, \bar{\lambda}_3^{(3)}), (\bar{v}_3^{(1)}, \bar{v}_3^{(2)}, \bar{v}_3^{(3)}), (\bar{\mu}_3^{(1)}, \bar{\mu}_3^{(2)}, \bar{\mu}_3^{(3)})> \;\&\; \tilde{X}_{11}^{c} = <(\bar{\lambda}_1^{(1)}, \bar{\lambda}_1^{(2)}, \bar{\lambda}_1^{(3)}), (\bar{v}_1^{(1)}, \bar{v}_1^{(2)}, \bar{v}_1^{(3)}), (\bar{\mu}_1^{(1)}, \bar{\mu}_1^{(2)}, \bar{\mu}_1^{(3)})>$

$\because \tilde{d}_{\underline{N}}(\tilde{X}_{11}^{c}, \tilde{X}_{13}^{c}) = \tilde{d}_{\underline{N}}(\tilde{X}_{11}, \tilde{X}_{13})$
so, $\ddot{\varepsilon}_j(\tilde{X}_{11}^{c}, \tilde{X}_{13}^{c}) = \ddot{\varepsilon}_j(\tilde{X}_{11}, \tilde{X}_{13})$ also holds.

Weight of the DM: Weight of a DM is calculated using the formula:

$$w^{(D)} = \frac{\sum_{j=1}^{M} \tilde{w}_j}{M} \tag{48}$$

4.1 Determination of the weight of the criteria and DM by Shannon's entropy method (Liang, 2013)

Step 1: Construct the decision matrices

Let there be U number of decision- makers.

Assume that $(\bar{D}_{mn}{}^{k})_{M \times N}$ is the decision matrix from the k^{th} DM , where ($\tilde{x}_{mn}{}^{k}$) represents the TFNN rating of the alternative ς_m over the attribute β_n provided by k^{th} DM in terms of

TFNNs. Then we get, $(\bar{D}_{mn}{}^{k})_{M\times N}=\begin{pmatrix} \tilde{x}^{k}_{11} & \cdots & \tilde{x}^{k}_{1N} \\ \vdots & \ddots & \vdots \\ \tilde{x}^{k}_{M1} & \cdots & \tilde{x}^{k}_{MN} \end{pmatrix}$

Step2: Calculate the average TFNN rating for each attribute under a decision- matrix using the formula $(\bar{\tilde{x}}^{k}_{mn})=\frac{\sum_{m=1}^{M}\tilde{x}^{k}_{mn}}{M}, n=1,2,...,N$; m=1,2,…,M; k=1, 2,…,U (49)

Step 3: Average Hamming distance for each criterion

Hamming distance of j-th criterion is calculated by the formula

$$\hat{H}_{ij}=\frac{1}{9}<\hat{\delta}(\tilde{x}^{k}_{1j},\bar{\tilde{x}}_{j}{}^{K})+\hat{\delta}(\tilde{x}^{k}_{2j},\bar{\tilde{x}}_{j}{}^{K})+...+\hat{\delta}(\tilde{x}^{k}_{mj},\bar{\tilde{x}}_{j}{}^{K})> \quad (50)$$

where $\hat{\delta}(\tilde{x}^{k}_{mj},\bar{\tilde{x}}_{j}{}^{K})$ = Hamming distance of m-th TFNN rating under k-th decision matrix, m=1,2, 3…, M is presented by $\hat{\delta}(\tilde{x}^{k}_{mj},\bar{\tilde{x}}_{j}{}^{K})=\frac{1}{9}[(p\tilde{a}_{mj}{}^{k}-p\bar{\tilde{a}}^{K}_{j})+...+(p\tilde{r}_{mj}{}^{k}-p\bar{\tilde{r}}^{K}_{j})]$ (51)

Average normalized Hamming distance is calculated using $\breve{H}_{ij}=\frac{\hat{H}_{ij}}{\max(\hat{H}_{ij})}$ (52)

Entropy of j-th criteria is calculated using $\ddot{\varepsilon}_{j}=-\frac{1}{\ln(m)}\sum_{i=1}^{m}(\frac{\breve{H}_{ij}}{\sum_{i=1}^{n}\breve{H}_{ij}})\ln(\frac{\breve{H}_{ij}}{\sum_{i=1}^{n}\breve{H}_{ij}})$ (53)

Weight of j-th criteria is calculated using $\tilde{w}_{j}=\frac{(1-\ddot{\varepsilon}_{j})}{\sum_{j=1}^{n}(1-\ddot{\varepsilon}_{j})}$ (54)

Average weight of j-th criteria is calculated using the formula $\bar{\tilde{w}}_{j}=\frac{\sum_{j=1}^{U}\tilde{w}_{j}}{U}$ (55)

U= Number of DMs.

Step 4: DM's weight is calculated taking average value of weights of all the criteria under the decision matrix using $w_{k}{}^{(D)}=\frac{\sum_{j=1}^{N}\tilde{w}_{j}}{N}$ (56)

5. MCGDM STRATEGY UNDER TFNN ENVIRONMENT USING TFNEOWA AND TFNEOWG OPERATOR

5.1 MCGDM Strategy under TFNN Environment Using TFNEOWA

Assume that $\beta=\{\beta_1,\beta_2,...,\beta_N\}$ is a set of N attributes and $\varsigma=\{\varsigma_1,\varsigma_2,...,\varsigma_M\}$ is the set of M alternatives, and pw= $\{pw_1,pw_2,...,pw_U\}^{T}$ represents the weight vector of U DMs satisfying $0\le pw_i\le 1$ and $\sum_{i=1}^{U}pw_i=1$. Furthermore, let $(\bar{\delta}_w)_n$ be the weight assigned to the attribute ς_n with $0\le(\bar{\delta}_w)_n\le 1$ and $\sum_{1}^{N}(\bar{\delta}_w)_n=1$. The proposed MCGDM strategy is developed using the following steps.

Step 1: Construct the decision matrices

Consider that $(\bar{D}^{k}_{mn})_{M\times N}$ is the decision matrix from the k^{th} DM , where ($\tilde{x}_{mn}{}^{k}$) represents the TFNN rating of the alternative ς_m over the attribute β_n provided by k^{th} DM in terms of

TFNNs. Then we obtain, $(\bar{D}^k_{mn})_{M\times N} = \begin{pmatrix} \tilde{x}^k_{11} & \cdots & \tilde{x}^k_{1N} \\ \vdots & \ddots & \vdots \\ \tilde{x}^k_{M1} & \cdots & \tilde{x}^k_{MN} \end{pmatrix}$

Where $\tilde{x}^k_{mn} = <(p\tilde{a}^k_{mn}, p\tilde{b}^k_{mn}, p\tilde{c}^k_{mn}),(p\tilde{e}^k_{mn}, p\tilde{f}^k_{mn}, p\tilde{g}^k_{mn}),(p\tilde{r}^k_{mn}, p\tilde{s}^k_{mn}, p\tilde{t}^k_{mn})>$ where n=1, 2…, N; m=1, 2,…,M; k=1, 2,…,U. (57)

Step 2: Standardize the decision matrices

We remove the effect of different types of physical dimensions and corresponding measurements by standardizing the decision matrices $(\tilde{x}_{mn}{}^k)_{M\times N}$ in the following way. For the TFNN entry $\tilde{x}^k_{mn} = <(p\tilde{a}^k_{mn}, p\tilde{b}^k_{mn}, p\tilde{c}^k_{mn}),(p\tilde{e}^k_{mn}, p\tilde{f}^k_{mn}, p\tilde{g}^k_{mn}),(p\tilde{r}^k_{mn}, p\tilde{s}^k_{mn}, p\tilde{t}^k_{mn})>$ in the decision matrix $(\bar{D}^k_{mn})_{M\times N}$ is implemented as,

1. If the criterion is of benefit type, then there will be no change in TFNN rating.
$$\tilde{x}^k_{mn} = <(p\tilde{a}^k_{mn}, p\tilde{b}^k_{mn}, p\tilde{c}^k_{mn}),(p\tilde{e}^k_{mn}, p\tilde{f}^k_{mn}, p\tilde{g}^k_{mn}),(p\tilde{r}^k_{mn}, p\tilde{s}^k_{mn}, p\tilde{t}^k_{mn})> \quad (58)$$
2. If the criterion is of cost type, complement of the TFNN is considered.
$$\tilde{x}^k_{mn} = (p\tilde{r}^k_{mn}, p\tilde{s}^k_{mn}, p\tilde{t}^k_{mn}),(p\tilde{e}^k_{mn}, p\tilde{f}^k_{mn}, p\tilde{g}^k_{mn}),(p\tilde{a}^k_{mn}, p\tilde{b}^k_{mn}, p\tilde{c}^k_{mn})>$$

With n=1, 2,…,N; m=1, 2,…,M; k=1, 2,…,U. (59)

Then the standardized decision matrix is given by

$$(\bar{D}^k_{mn})_{M\times N} = \begin{pmatrix} \tilde{x}^k_{11} & \cdots & \tilde{x}^k_{1N} \\ \vdots & \ddots & \vdots \\ \tilde{x}^k_{M1} & \cdots & \tilde{x}^k_{MN} \end{pmatrix} \quad (60)$$

Step 3: Determine of the weights of the criteria and the DMs

Weights of the criteria and DMs are calculated using eqn (55) and eqn (56).

Step 4: Aggregate the decision matrices

The decision matrices are fused into a single decision matrix using DMs weights using eqn (62).

The Aggregated Decision Matrix (ADM) $(\bar{\delta}_{mn})_{M\times N}$ is constructed using TFNEOWA operator

$$\text{TFNEOWA}\left((\tilde{x}_{mn})^1,(\tilde{x}_{mn})^2,\ldots,(\tilde{x}_{mn})^U\right)$$
$$= \bigoplus_{k=1}^{U} pw_k(\tilde{x}_{mn})^k \quad (61)$$
$$= pw_1(\tilde{x}_{mn})^1 \oplus pw_2(\tilde{x}_{mn})^2 \oplus \ldots \oplus pw_k(\tilde{x}_{mn})^U$$

$$=<\left(\frac{\prod_{i=1}^{U}(p\tilde{a}_i+1)^{pw_i}-\prod_{i=1}^{U}(1-p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{U}(p\tilde{a}_i+1)^{pw_i}+\prod_{i=1}^{U}(1-p\tilde{a}_i)^{pw_i}},\frac{\prod_{i=1}^{U}(p\tilde{b}_i+1)^{pw_i}-\prod_{i=1}^{U}(1-p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{U}(p\tilde{b}_i+1)^{pw_i}+\prod_{i=1}^{U}(1-p\tilde{b}_i)^{pw_i}},\frac{\prod_{i=1}^{U}(p\tilde{c}_i+1)^{pw_i}-\prod_{i=1}^{U}(1-pc\tilde{}_i)^{pw_i}}{\prod_{i=1}^{U}(p\tilde{c}_i+1)^{pw_i}+\prod_{i=1}^{U}(1-pc\tilde{}_i)^{pw_i}}\right),$$
$$\left(\frac{2\prod_{i=1}^{U}(p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{e}_i)^{pw_i}+\prod_{i=1}^{U}(p\tilde{e}_i)^{pw_i}},\frac{2\prod_{1}^{U}(p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{f}_i)^{pw_i}+\prod_{i=1}^{U}(p\tilde{f}_i)^{pw_i}},\frac{2\prod_{i=1}^{U}(p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{g}_i)^{pw_i}+\prod_{i=1}^{U}(p\tilde{g}_i)^{pw_i}}\right),$$
$$\left(\frac{2\prod_{i=1}^{U}(p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{r}_i)^{pw_i}+\prod_{i=1}^{U}(p\tilde{r}_i)^{pw_i}},\frac{2\prod_{i=1}^{U}(p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{s}_i)^{pw_i}+\prod_{i=1}^{U}(p\tilde{s}_i)^{pw_i}},\frac{2\prod_{i=1}^{U}(p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{t}_i)^{pw_i}+\prod_{i=1}^{U}(p\tilde{t}_i)^{pw_i}}\right)\rangle \quad (62)$$

Then we obtain $(\bar{\delta}_{mn})_{M\times N} = \begin{pmatrix} \bar{\delta}_{11} & \cdots & \bar{\delta}_{1N} \\ \vdots & \ddots & \vdots \\ \bar{\delta}_{M1} & \cdots & \bar{\delta}_{MN} \end{pmatrix}$ (63)

Step 5: We now calculate aggregated TFNN rating against each alternative using criteria weights using eqn (62). Let the aggregated TFNN rating be represented by $\langle \tilde{x}_{mn} \rangle = \langle (p\tilde{a}_{mn}, p\tilde{b}_{mn}, p\tilde{c}_{mn}), (p\tilde{e}_{mn}, p\tilde{f}_{mn}, p\tilde{g}_{mn}), (p\tilde{r}_{mn}, p\tilde{s}_{mn}, p\tilde{t}_{mn}) \rangle$

Step 6: Calculate the $Scr(\tilde{x}_{mn})$ and H ($\tilde{x}_{mn}$) value of the aggregated TFNNV of each alternative using eqn (64) and eqn (65)

$$Scr(\tilde{x}_{mn}) = \frac{1}{9}[6 + (p\tilde{a}_{mn} + p\tilde{b}_{mn} + p\tilde{c}_{mn}) - (p\tilde{e}_{mn} + p\tilde{f}_{mn} + p\tilde{g}_{mn}) - (p\tilde{r}_{mn} + p\tilde{s}_{mn} + p\tilde{t}_{mn})] \tag{64}$$

$$H\,(\tilde{x}_{mn}) = \frac{1}{3}[(p\tilde{a}_{mn} + p\tilde{b}_{mn} + p\tilde{c}_{mn}) - (p\tilde{r}_{mn} + p\tilde{s}_{mn} + p\tilde{t}_{mn})] \tag{65}$$

Step 7: Rank of the alternatives

Ranking is made on the basis of the descending value of $Scr(\tilde{x}_{mn})$ and H ($\tilde{x}_{mn}$).

5.2 MCGDM Strategy under TFNN Environment Using TFNEOWG

Step 1 -Step 3 are same as that of 5.1

Step 4: The aggregated decision matrix (ADM) $(\bar{\delta}_{mn})_{M\times N}$ is constructed using TFNEOWG operator presented as:

$$\text{TFNEOWG}\ ((\tilde{x}_{mn})^1, (\tilde{x}_{mn})^2, ..., (\tilde{x}_{mn})^k) = \{(\tilde{x}_{mn})^1\}^{pw_1} \otimes \{(\tilde{x}_{mn})^2\}^{pw_2} \otimes ... \otimes \{(\tilde{x}_{mn})^k\}^{pw_U} \tag{66}$$

$$=< (\frac{2\prod_{i=1}^{U}(p\tilde{a}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{a}_i)^{pw_i} + \prod_{i=1}^{U}(p\tilde{a}_i)^{pw_i}}, \frac{2\prod_{i=1}^{U}(p\tilde{b}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{b}_i)^{pw_i} + \prod_{i=1}^{U}(p\tilde{b}_i)^{pw_i}}, \frac{2\prod_{i=1}^{U}(p\tilde{c}_i)^{pw_i}}{\prod_{i=1}^{U}(2-p\tilde{c}_i)^{pw_i} + \prod_{i=1}^{U}(p\tilde{c}_i)^{pw_i}}), (\frac{\prod_{i=1}^{U}(1+p\tilde{e}_i)^{pw_i} - \prod_{i=1}^{U}(1-p\tilde{e}_i)^{pw_i}}{\prod_{i=1}^{U}(1+p\tilde{e}_i)^{pw_i} + \prod_{i=1}^{U}(1-p\tilde{e}_i)^{pw_i}},$$

$$\frac{\prod_{i=1}^{U}(1+p\tilde{f}_i)^{pw_i} - \prod_{i=1}^{U}(1-p\tilde{f}_i)^{pw_i}}{\prod_{i=1}^{U}(1+p\tilde{f}_i)^{pw_i} + \prod_{i=1}^{U}(1-p\tilde{f}_i)^{pw_i}}, \frac{\prod_{i=1}^{U}(1+p\tilde{g}_i)^{pw_i} - \prod_{i=1}^{U}(1-p\tilde{g}_i)^{pw_i}}{\prod_{i=1}^{U}(1+p\tilde{g}_i)^{pw_i} + \prod_{i=1}^{U}(1-p\tilde{g}_i)^{pw_i}}), (\frac{\prod_{i=1}^{U}(1+p\tilde{r}_i)^{pw_i} - \prod_{i=1}^{U}(1-p\tilde{r}_i)^{pw_i}}{\prod_{i=1}^{U}(1+p\tilde{r}_i)^{pw_i} + \prod_{i=1}^{U}(1-p\tilde{r}_i)^{pw_i}}, \frac{\prod_{i=1}^{U}(1+p\tilde{s}_i)^{pw_i} - \prod_{i=1}^{U}(1-p\tilde{s}_i)^{pw_i}}{\prod_{i=1}^{U}(1+p\tilde{s}_i)^{pw_i} + \prod_{i=1}^{U}(1+p\tilde{s}_i)^{pw_i}}, \frac{\prod_{i=1}^{U}(1+p\tilde{t}_i)^{pw_i} - \prod_{i=1}^{U}(1-p\tilde{t}_i)^{pw_i}}{\prod_{i=1}^{U}(1+p\tilde{t}_i)^{pw_i} + \prod_{i=1}^{U}(1-p\tilde{t}_i)^{pw_i}}) > \tag{67}$$

Then, we obtain $$(\bar{\delta}_{mn})_{M\times N} = \begin{pmatrix} \bar{\delta}_{11} & \cdots & \bar{\delta}_{12} \\ \vdots & \ddots & \vdots \\ \bar{\delta}_{M1} & \cdots & \bar{\delta}_{MN} \end{pmatrix} \tag{68}$$

Step 5: We now calculate aggregated TFNN rating against each alternative using criteria weights using formula (66). Let the aggregated TFNN rating be represented as:

$\langle \tilde{x}_{mn} \rangle = \langle (p\tilde{a}_{mn}, p\tilde{b}_{mn}, p\tilde{c}_{mn}), (p\tilde{e}_{mn}, p\tilde{f}_{mn}, p\tilde{g}_{mn}), (p\tilde{r}_{mn}, p\tilde{s}_{mn}, p\tilde{t}_{mn}) \rangle$

Step 6: Calculate the $Scr(\tilde{x}_{mn})$ and $H(\tilde{x}_{mn})$ value of the aggregated TFNNV of each alternative using

$$Scr(\tilde{x}_{mn}) = \frac{1}{9}[6 + (p\tilde{a}_{mn} + p\tilde{b}_{mn} + p\tilde{c}_{mn}) - (p\tilde{e}_{mn} + p\tilde{f}_{mn} + p\tilde{g}_{mn}) - (p\tilde{r}_{mn} + p\tilde{s}_{mn} + p\tilde{t}_{mn})] \tag{69}$$

$$H(\tilde{x}_{mn}) = \frac{1}{3}[(p\tilde{a}_{mn} + p\tilde{b}_{mn} + p\tilde{c}_{mn}) - (p\tilde{r}_{mn} + p\tilde{s}_{mn} + p\tilde{t}_{mn})] \tag{70}$$

Step 7: Rank of the alternatives

Ranking is made on the basis of the descending value of score function and accuracy function.

6. A NUMERICAL EXAMPLE OF MCGDM STRATEGY OF SALES MANAGER SELECTION IN A PHARMACEUTICAL COMPANY

This section presents an illustrative numerical example of selection of a sales manager in a pharmaceutical company reflecting the relevancy of the proposed MCGDM strategy. Selecting an effective sales manager in a pharmaceutical company is crucial because they play a key role in drawing revenue and ensuring compliance with industry regulations. A skilled sales manager can lead and motivate the sales team, navigate complex healthcare environments, and can maintain ethical standards in promoting pharmaceutical products. Their strategic decisions impact sales performance, market share, and overall success in the reputed and competitive industry. The pharmaceutical company appoints four DMs as eminent experts in the pharmaceutical field. After primary detailed scrutiny four candidates $\zeta_i (i = 1, 2, 3, \&4)$ (four alternatives) are selected for further assessment under five criteria $\beta_m (m = 1, 2, 3, 4, \&5)$ namely,

1.Verbal interaction skill (β_1)

2.Past field work experience (β_2)

3. General aptitude (β_3)

4. Willingness of hard labor (β_4)

5. Self-determination and instant decisive capacity (β_5)

The criteria are very much crucial, judging upon which best alternative is to be chosen.

As example, verbal interaction skills are crucial for a sales manager in a pharmaceutical company due to nature of the role. Effective communication enables sales manager to articulate complex information, build relationships with healthcare professionals and convey the value of pharmaceutical products. Past field work experience and general aptitude are crucial and important for a pharmaceutical sales manager as they offer practical insights into the dynamics of industry. Willingness of hard labour is very important for all round growth of the company. Self-determination and instant decisive capacity are crucial for human resource development and financial growth of the company.

Step 1: Construct the decision matrices

Four decision matrices are shown in Table 1-Table 4.

Table 1: Decision matrix DM-1

	β_1	β_2	β_3	β_4	β_5
ζ_1	$\langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$
ζ_2	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.80, 0.85, 0.90), (0.10, 0.15, 0.20), (0.05, 0.10, 0.15) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$
ζ_3	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$
ζ_4	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.50, 0.55, 0.60), (0.25, 0.30, 0.35), (0.20, 0.25, 0.30) \rangle$	$\langle (0.40, 0.45, 0.50), (0.40, 0.45, 0.50), (0.35, 0.40, 0.45) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$	$\langle (0.70, 0.75, 0.80), (0.15, 0.20, 0.25), (0.10, 0.15, 0.20) \rangle$

Table-2: Decision matrix DM-2

	β_1	β_2	β_3	β_4	β_5
ζ_1	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.80,0.85,0.90),(0.10,0.15,0.20),(0.05,0.10,0.15)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$
ζ_2	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.80,0.85,0.90),(0.10,0.15,0.20),(0.05,0.10,0.15)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$
ζ_3	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$
ζ_4	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$

Table-3: Decision matrix DM-3

	β_1	β_2	β_3	β_4	β_5
ζ_1	$\langle(0.80,0.85.0.90),(0.10,0.15,0.20),(0.05,0.10,0.15)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$
ζ_2	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.80,0.85,0.90),(0.10,0.15,0.20),(0.05,0.10,0.15)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$
ζ_3	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$
ζ_4	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$

Table 4: Decision matrix DM-4

	β_1	β_2	β_3	β_4	β_5
ζ_1	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0,70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0,70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$
ζ_2	$\langle(0.80,0,85,0.90),(0.10,0.15,0.20),(0,05,0.10,0.15)\rangle$	$\langle(0.80,0,85,0.90),(0.10,0.15,0.20),(0,05,0.10,0.15)\rangle$	$\langle(0.80,0,85,0.90),(0.10,0.15,0.20),(0,05,0.10,0.15)\rangle$	$\langle(0,70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$
ζ_3	$\langle(0,40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0,40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0,40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$
ζ_4	$\langle(0.40,0.45,0.50),(0.40,0.45,0.50),(0.35,0.40,0.45)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0,20,0.25,0.30)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$	$\langle(0.70,0.75,0.80),(0.15,0.20,0.25),(0.10,0.15,0.20)\rangle$	$\langle(0.50,0.55,0.60),(0.25,0.30,0.35),(0.20,0.25,0.30)\rangle$

Step 2: Standardization of decision matrices
All the criteria are of benefit type, so there is no need to standardize them.

Step 3: Calculate the weights of criteria and DMs
Average TFNN rating of criteria under DM-1 is shown in table 5. Entropy of criteria under DM-1 shown in table 6, weights of criteria under DM-1 is shown in table 7, average TFNN ratings of criteria under DM-2 is shown in table 8. Entropy of criteria under DM-2 is shown in table 9, weights of criteria under DM-2 and weight of DM-2 are shown in table 10, average TFNN ratings of criteria under DM-3 are shown in table 11. Entropy of criteria under DM-3 is shown in table

12, weights of criteria under DM-3 and weight of DM-3 are shown in table 13, average TFNN ratings of criteria under DM-4 is shown in table 14. Entropy of criteria under DM-4 is shown in table 15, weights of criteria under DM-4 and weight of DM-4 are shown in table 16. Weights of all the DMa are listed in table 17. Average weights and entropy of all criteria are represented in table 18.

Table 5: Average TFNN ratings of criteria under DM-1

Criterion (β_i)	Aggregated TFNN rating ($\bar{\bar{x}}_{ij}^{K}$)
β_1	$\left\langle \begin{matrix}(0.5250, 0.5750, 0.6250)\\(0.2875, 0.3375, 0.3875)\\(0.2375, 0.2875, 0.3375)\end{matrix} \right\rangle$
β_2	$\left\langle \begin{matrix}(0.5500, 0.6000, 0.6500)\\(0.2250, 0.2750, 0.3250)\\(0.1750, 0.2250, 0.2750)\end{matrix} \right\rangle$
β_3	$\left\langle \begin{matrix}(0.5750, 0.6250, 0.6750)\\(0.2625, 0.3125, 0.3625)\\(0.2125, 0.2625, 0.3125)\end{matrix} \right\rangle$
β_4	$\left\langle \begin{matrix}(0.6500, 0.7000, 0.7500)\\(0.2000, 0.2500, 0.3000)\\(0.1500, 0.2000, 0.2500)\end{matrix} \right\rangle$
β_5	$\left\langle \begin{matrix}(0.6500, 0.7000, 0.7500)\\(0.1750, 0.2250, 0.2750)\\(0.1250, 0.1750, 0.2250)\end{matrix} \right\rangle$

Table 6: Entropy of criteria under DM-1

Criterion (β_i)	Hamming distance	Normalized Hamming distance $\breve{\hat{H}}_{ij}$	$\sum \hat{H}_{ij}^{(N)}$	$\frac{\hat{H}_{ij}^{(N)}}{\sum \hat{H}_{ij}^{(N)}}$	Entropy of the criterion
β_1	(0.2167,0.0333,0.1167,0.1167)	(1,0.1537,0.5385,0.5385)	2.2307	(0.4482,0.0689,0.2414.0.2414)	0.8874
β_2	(0.0333,0.1000,0.0333,0.0333)	(0.333,1,0.333, 0.333)	1.999	(0.1665,0.5002,0.1665,0.1665)	0.8959
β_3	(0.1167,0.1833, 0.1500,0.1500)	(0.6367,1,0.8183, 0.8183)	3.2733	(0.1945,0.3055,0.2499,0.2499)	0.9909
β_4	(0.1167,0.0333,0.2167,0.0333)	(0.5385,0.1537, 1, 0.1537)	1.8459	(0.2917,0.08326,0.5417,0.0833)	0.7972
β_5	(0.0497,0.0497,0.1175,0.0183)	(0.4230,0.4230,1,0.1557)	2.0017	(0.2113,0.2113,0.4996,0.0778)	0.9161

Table 7: Weight of DM-1

Attribute/ Criterion (β_i)	Entropy ($\ddot{\varepsilon}_j$)	Dispersion ($\tilde{d}_j$)	Weight of the criterion ($\tilde{w}_j$)	Average weight of the criteria	Weight of DM-1
β_i	0.8874	0.1126	0.2167	0.1999	0.1999
β_2	0.8959	0.1041	0.2003		
β_3	0.9909	0.0091	0.0175		
β_4	0.7902	0.2095	0.4038		
β_5	0.9161	0.0839	0.1615		

Table: 8: Avg TFNN ratings of criteria under DM-2

Criterion (β_i)	TFNN Ratings
β_1	$\langle(0.45, 0.50, 0.55), (0.3250, 0.3750, 0.4240), (0.2750, 0.3250, 0.3750)\rangle$
β_2	$\langle(0.6000, 0.6500, 0.7000), (0.2000, 0.2500, 0.3000), (0.1500, 0.2000, 0.2500)\rangle$
β_3	$\langle(0.5250, 0.5750, 0.6250), (0.2875, 0.3375, 0.3875)(0.2375, 0.2875, 0.3375)\rangle$
β_4	$\langle(0.6750, 0.7250, 0.7750), (0.1625, 0.2125, 0.2625), (0.1125, 0.1625, 0.2125)\rangle$
β_5	$\langle(0.6500, 0.7000, 0.7500), (0.1750, 0.2250, 0.2750), (0.1250, 0.1750, 0.2250)\rangle$

Table 9: Entropy of different criterions under DM-2

Criterion (β_i)	Hamming distance rating	Normalized Hamming distance $\breve{\hat{H}}_{ij}$	$\sum \hat{H}_{ij}^{(N)}$	$\frac{\breve{\hat{H}}_{ij}}{\sum \breve{\hat{H}}_{ij}}$	Entropy ($\ddot{\varepsilon}_j$)
β_1	(0.0666,0.0666, 0.0666,0.0666)	(1,1,1,1)	4	(0.25,0.25,0.25,0.25)	1
β_1	(0.0666,0.0666, 0.0666,0.0666)	(1,1,1,1)	4	(0.25,0.25, 0.25,0.25)	1
β_1	(0.0666,0.0666, 0.0666,0.0666)	(0.1537,1.000,0.6925,0.6925)	2.5387	(0.0605,0.3939,0.2727,0.2727)	0.8982
β_1	(0.0833,0.0166, 0.1166,0.0166)	(0.1537,1.000,0.6925,0.6925)	1.9991	(0.3573,0.0712,0.5025,0.0712)	0.7861
β_1	(0.0333,0.0333, 0.1000,0.0333)	(0.333,0.333,1,0.333)	1.9990	(0.1666,0.1666,0.5002,0.1666)	0.8960

Table 10: Weight of DM-2

Attribute/Criteria (β_i)	Entropy ($\ddot{\varepsilon}_j$)	Dispersion ($\tilde{d}_j$)	$\sum \tilde{d}_j$	criterion/attribute weight ($\tilde{w}_j$)	Weight of DM-2
β_1	1	0	0.4117	O	0.2022
β_2	1	0		0	
β_3	0.8982	0.1018		0.2437	
β_4	0.7861	0.2119		0.5146	
β_5	0.8960	0.1040		0.2526	

Table 11: Average TFNN ratings of the criteria under DM -3

Criterion	Average TFNN Rating
β_1	$\left\langle \begin{matrix} (0.6500, 0.7000, 0.7500) \\ (0.1750, 0.2250, 0.2750) \\ (0.1250, 0.1750, 0.2250) \end{matrix} \right\rangle$
β_2	$\left\langle \begin{matrix} (0.5500, 0.6000, 0.6500) \\ (0.2250, 0.2750, 0.3250) \\ (0.1750, 0.2250, 0.2750) \end{matrix} \right\rangle$
β_3	$\left\langle \begin{matrix} (0.6750, 0.7250, 0.7750) \\ (0.1625, 0.2125, 0.2625) \\ (0.1125, 0.1625, 0.2125) \end{matrix} \right\rangle$
β_4	$\left\langle \begin{matrix} (0.5250, 0.5750, 0.6250) \\ (0.2625, 0.3125, 0.3625) \\ (0.2125, 0.2625, 0.3125) \end{matrix} \right\rangle$
β_5	$\left\langle \begin{matrix} (0.5750, 0.6250, 0.6750) \\ (0.2375, 0.2875, 0.3375) \\ (0.1875, 0.2375, 0.2875) \end{matrix} \right\rangle$

Table 12: Entropy of different criteria under DM-3

Criterion (β_1)	Hamming distance	Normalized Hamming distance ($\breve{\tilde{H}}_{ij}$)	$\sum \breve{\tilde{H}}_{ij}$	$\frac{\breve{\tilde{H}}_{ij}}{\sum \breve{\tilde{H}}_{ij}}$	Entropy ($\ddot{\varepsilon}_j$)
β_1	(0.1000,0.0333, 0.0333,0.0333)	(1, 0.333, 0.333, 0.333)	2.332	(0.4288, 0.1427, 0.1427, 0.1427)	0.8631
β_2	(0.0333,0.1000,0.0333,0.0333)	(0.333,1,0.333, 0.333)	1.999	(0.1665,0,5002,0.1665, 0.1665)	0.8959
β_3	(0.1166,0.0500,0.0166,0.0166)	(1,0.4288,0.1424,0.1424)	1.7136	(0.5835,0.2502,0.0831, 0.0831)	0.7750
β_4	(0.0166,0.1333,0.1333,0.0166)	(0.1245,1,1,0.1245)	2.249	(0.055, 0.4446, 0.4446, 0.055)	0.7501
β_5	0.1000,0.1000, 0.1666,0.0333)	0.6002,0.6002,1,0.1999)	2.4003	0.2501,0.2501, 0.4146,0.0829)	0.9121

Table 13: Weight of DM-3

Criterion (β_i)	Entropy ($\ddot{\varepsilon}_j$)	Dispersion ($\tilde{d}_j$)	$\sum_j \tilde{d}_j$	Weight of Criterion ($\tilde{w}_j$)	Weight of DM-3
β_1	0.8631	0.1369	0.8767	0.1562	0.1909
β_2	0.8959	0.1041		0.1187	
β_3	0.7750	0.2250		0.2566	
β_4	0.7501	0.2499		0.2850	
β_5	0.8392	0.1608		0.1834	

Table:14 Average TFNN ratings of criteria under DM-4

Criterion (β_i)	Average TFNN rating
β_1	$\left\langle \begin{matrix} (0.5250, 0.5750, 0.6250) \\ (0.2875, 0.3375, 0.3875) \\ (0.2375, 0.2875, 0.3375) \end{matrix} \right\rangle$
β_2	$\left\langle \begin{matrix} (0.6250, 0.6750, 0.7250) \\ (0.1875, 0.2375, 0.2875) \\ (0.1375, 0.1875, 0.2375) \end{matrix} \right\rangle$
β_3	$\left\langle \begin{matrix} (0.5250, 0.5750, 0.6250) \\ (0.2875, 0.3375, 0.3875) \\ (0.2375, 0.2875, 0.3375) \end{matrix} \right\rangle$
β_4	$\left\langle \begin{matrix} (0.6500, 0.7000, 0.7500) \\ (0.1750, 0.2250, 0.2750) \\ (0.1250, 0.1750, 0.2250) \end{matrix} \right\rangle$
β_5	$\left\langle \begin{matrix} (0.4750, 0.5250, 0.5750) \\ (0.2875, 0.3375, 0.3875) \\ (0.2375, 0.2875, 0.3375) \end{matrix} \right\rangle$

Table-15: Entropy of different criteria under DM-4

Criterion (β_i)	**Hamming distance**	**Normalized Hamming Distance** $\breve{\tilde{H}}_{ij}$	$\sum \breve{\tilde{H}}_{ij}$	$\frac{\breve{\tilde{H}}_{ij}}{\sum \breve{\tilde{H}}_{ij}}$	**Entropy** ($\ddot{\varepsilon}_j$)
β_1	.0333,0.2166,0.1500,0.1500)	(0.1537,1.000,0.6925,0.6925)	2.5387	(0.0605,0.3939, 0.2727,0.2727)	0.8982
β_2	(0.05,0.1167,0.0833,0.0833)	(0.4284,1,0.7138,0.7138)	2.856	(0.1500,0.3501,0.2499,0.2499)	0.9702
β_3	(0.0333,0.2166,0.1500,0.1500)	(0.1537,1.000, 0.6925,0.6925)	2.5387	(0.0605,0.3939,0.2727, 0.2727)	0.8982
β_4	(0.0333,0.0333,0.1000,0.0333)	(0.333,0.333, 1,0.333)	1.999	(0.1666, 0.1666,0.5002, 0.1666)	0.8960
β_5	(0.0333,0.0333,0.1000,0.0333)	(0.333,0.333,1,0.333)	1.999	(0.1666,0.1666,0.5002,0.1665)	0.8960

Table 16: Weight of DM-4

CRITERION/ATTRIBUTE (β_i)	Entropy ($\ddot{\varepsilon}_j$)	Dispersion ($\tilde{d}_j$)	Weight of Criterion ($\tilde{w}_j$)	Weight of DM-4
β_1	0.8982	0.1018	0.2306	0.2000
β_2	0.9702	0.0298	0.0674	
β_3	0.8982	0.1018	0.2306	
β_4	0.8960	0.1040	0.2357	
β_5	0.8959	0.1041	0.2357	

Table17: Weight of all DMs

DM	Weight(w)	Normalized weight= $\frac{w}{\sum(w)}$ (approx.)
DM-1	0.1999	0.2526
DM-2	0.2002	0.2530
DM-3	0.1909	0.2412
DM-4	0.2000	0.2530

So, normalized weight of decision makers, $w^{(D)} = (0.25,0.25,0.24,0.25)^T$

Table18: **Average entropy and average weights of different criteria**

Criterion β_i)	Avg TFNN rating	Hamming Distance	Normalized Hamming Distance $\tilde{\breve{H}}_{ij}$	Entropy ($\ddot{\varepsilon}_j$)	Dispersion ($\tilde{d}_j$)	Weight ($\tilde{w}_j$)	Normalized weight of criterion
β_1	$\left\langle \begin{matrix} (0.55,0.61,0.66) \\ (0.25,0.31,0.36) \\ (0.20,0.25,0.31) \end{matrix} \right\rangle$	(0.05,0.08, 0.07, 0.07)	(0.19,0.32, 0.25,0.25)	0.99	0.01	0.021	0.021
β_2	$\left\langle \begin{matrix} (0.59,0.64,0.69) \\ (0.21,0.26,0.31) \\ (0.15,0.20,0.26) \end{matrix} \right\rangle$	(0.02,0.065 0.065, 0.022)	(0.12,0.38, 0.38,0.12)	0.90	0.10	0.182	0.18
β_3	$\left\langle \begin{matrix} (0.58,0.64,0.69) \\ (0.23,0.28,0.33) \\ (0.18,0.23,0.28) \end{matrix} \right\rangle$	(0.10,1.00, 0.60, 0.36)	(0.05,0.48, 0.29,0.17)	0.84	0.16	0.30	0.30
β_4	$\left\langle \begin{matrix} (0.63,0.68,0.73) \\ (0.19,0.25,0.30) \\ (0.14,0.19,0.25) \end{matrix} \right\rangle$	(0.07,0.05, 0.14, 0.02)	(0.23,0.18, 0.50,0.084)	0.87	0.13	0.25	0.25
β_5	$\left\langle \begin{matrix} (0.59,0.64,0.69) \\ (0.21,0.26,0.32) \\ (0.16,0.21,0.27) \end{matrix} \right\rangle$	(0.05,0.05, 0.12,0.02)	(0.21,0.21, 0.50,0.077)	0.87	0.13	0.25	0.25

Step 4: Fusion of decision matrices by TFNEOWA operator
Decision matrices, DM-1, DM-2, DM-3, DM-4 are fused or aggregated using eqn (62) shown in table 19.

Step 5. Calculation of aggregated TFNN rating against each alternative.
We now calculate aggregated TFNN rating against each alternative using criteria weights using equation (61). The aggregated TFNN ratings are shown in table 20.

Table-19: fused decision matrix using TFNEOWA operator

Alternative / Criterion	β_1	β_2	β_3	β_4	β_5
ς_1	$\left\langle\begin{matrix}0.6720, 0.7288, 0.7874\\ 0.1634, 0.2179, 0.2707\\ 0.1045, 0.1634, 0.2179\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5539, 0.6054, 0.6576\\ 0.2240, 0.2752, 0.3260\\ 0.1718, 0.2240, 0.2752\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5539, 0.5907, 0.6575\\ 0.2240, 0.2752, 0.3261\\ 0.1718, 0.2240, 0.2753\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.7156, 0.7700, 0.8268\\ 0.1422, 0.1952, 0.2471\\ 0.0864, 0.1422, 0.1952\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6528, 0.7043, 0.7564\\ 0.1743, 0.2256, 0.2765\\ 0.1222, 1743, 0.2256\end{matrix}\right\rangle$
ς_1	$\left\langle\begin{matrix}0.6403, 0.6953, 0.7530\\ 0.1797, 0.2330, 0.2851\\ 0.1230, 0.1797, 0.2331\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6847, 0.7383, 0.7936\\ 0.1578, 0.2103, 0.2618\\ 0.1031, 0.1579, 0.2103\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.7965, 0.8469, 0.8975\\ 0.1024, 0.1531, 0.2035\\ 0.0516, 0.1025, 0.1531\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6961, 0.7463, 0.7965\\ 0.1530, 0.2035, 0.2537\\ 0.1023, 0.1530, 0.2035\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6528, 0.7043, 0.7593\\ 0.1743, 0.2256, 0.2765\\ 0.1222, 0.1743, 0.2256\end{matrix}\right\rangle$
ς_1	$\left\langle\begin{matrix}0.4831, 0.5358, 0.5896\\ 0.3233, 0.3777.0.4305\\ 0.2663, 0.3233, 0.3777\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5539, 0.6054, 0.6576\\ 0.2240, 0.2752, 0.3260\\ 0.1718, 0.2240, 0.2752\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4831, 0.5358, 0.5896\\ 0.3233, 0.3776, 0.4305\\ 0.2663, 0.3233, 0.3776\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4485, 0.4985, 0.5487\\ 0.3211, 0.3724, 0, 4233\\ 0.2693, 0.3211, 0.3724\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4585, 0.4985, 0.5486\\ 0.3211, 0.3724, 0, 4233\\ 0.2693, 0.3211, 0.3724\end{matrix}\right\rangle$
ς_1	$\left\langle\begin{matrix}0.4830, 0.5358, 0.5896\\ 0.3233, 0.3776, 0.4305\\ 0.2663, 0.3233, 0.3776\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5539, 0.6054, 0.6576\\ 0.2240, 0.2752, 0.3260\\ 0.1718, 0.2240, 0.2752\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5068, 0.5592, 0.6126\\ 0.2872, 0.3409, 0.3936\\ 0.2311, 0.2872, 0.3409\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6544, 0.7059, 0.7579\\ 0.1735, 0.2248, 0.2756\\ 0.1215, 0.1735, 0.2248\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6066, 0.6586, 0.7113\\ 0.1973, 0.2489, 0.2999\\ 0.1446, 0.1973, 0.2489\end{matrix}\right\rangle$

Table-20: **Aggregated TFNN ratings using TFNEOWA operator**

Alternative	*TFNN rating*
ζ_1	$\left\langle\begin{matrix}(0.6269, 0.6770, 0.7362)\\ (0.1865, 0.2391, 0.2907)\\ (0.1316, 0.1865, 0.2391)\end{matrix}\right\rangle$
ζ_2	$\left\langle\begin{matrix}(0.7180, 0.7707, 0.8255)\\ (0.1414, 0.1933, 0.2444)\\ (0.0876, 0.1414, 0.1933)\end{matrix}\right\rangle$
ζ_3	$\left\langle\begin{matrix}(0.4799, 0.5313, 0.5833)\\ (0.3017, 0.3543, 0.4061)\\ (0.2475, 0.3017, 0.3543)\end{matrix}\right\rangle$
ζ_4	$\left\langle\begin{matrix}(0.5801, 0.6326, 0.6860)\\ (0.2212, 0.2741, 0.3261)\\ (0.1666, 0.2212, 0.2741)\end{matrix}\right\rangle$

Step 6: Calculation of score and accuracy values of different alternatives

Score and accuracy values of different alternatives are calculated using eqn (69), eqn (70) and the results are listed in table 21.

Step 7: Ranking of the alternatives

Ranking of the alternatives on the basis of accuracy value and score value is shown in table 21

Table-21: Score values and Accuracy values of different alternatives

Alternative	Score value	Accuracy value	Remark
ζ_1	0.7539	0.4984	Depending on score and accuracy values, we have $\zeta_2 > \zeta_1 > \zeta_4 > \zeta_3$ ζ_2 is the best alternative.
ζ_2	0.8151	0.6359	
ζ_3	0.6289	0.2372	
ζ_4	0.7173	0.4190	

Step 8: Fusion of decision matrices by TFNEOWG operator Decision matrices, DM-1, DM-2, DM-3, DM-4 are fused or aggregated using eqn (67) shown in Table 22.

Table 22: Fused decision matrix by TFNEOWG OPERATOR

Alternative / criterion	β_1	β_2	β_3	β_4	β_5
ς_1	$\left\langle\begin{matrix}0.6422, 0.6935, 0.7446\\ 0.1752, 0.2251, 0.2751\\ 0.1253, 0.1753, 0.2251\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5501, 0.6004, 0.6506\\ 0.2233, 0.2730, 0.3228\\ 0.1859, 0.2232, 0.2730\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5501, 0.6004, 0.6506\\ 0.2254, 0.2754, 0.3257\\ 0.2233, 0.2730, 0.3228\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6984, 0.7491, 0.7997\\ 0.1485, 0.1983, 0.2481\\ 0.0987, 0.1485, 0.1983\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6488, 0.6991, 0.7492\\ 0.1741, 0.2238, 0.2736\\ 0.1244, 0.1741, 0.2238\end{matrix}\right\rangle$
ς_1	$\left\langle\begin{matrix}0.6201, 0.6710, 0.7217\\ 0.1870, 0.2369, 0.2867\\ 0.1372, 0.1870, 0.2369\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6724, 0.7230, 0.7734\\ 0.1618, 0.2115, 0.2613\\ 0.1120, 0.1618, 0.2115\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.8017, 0.8513, 0.9009\\ 0.0991, 0.1487, 0.1983\\ 0.0496, 0.0991, 0.1487\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.7024, 0.7521, 0.8017\\ 0.1487, 0.1983, 0.2479\\ 0.0991, 0.1487, 0.1983\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6488, 0.6991, 0.7492\\ 0.1741, 0.2238, 0.2736\\ 0.1244, 0.1741, 0.2238\end{matrix}\right\rangle$
ς_1	$\left\langle\begin{matrix}0.4668, 0.5181, 0.5692\\ 0.3401, 0.3905, 0.4411\\ 0.2898, 0.3401, 0.3905\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5501, 0.6004, 0.6506\\ 0.2233, 0.2730, 0.3228\\ 0.1736, 0.2233, 0.2730\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4668, 0.5181, 0.5692\\ 0.3401, 0.3905, 0.4411\\ 0.2898, 0.3401, 0.3905\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4522, 0.5024, 0.5524\\ 0.3237, 0.3738, 0.4240\\ 0.2737, 0.3237, 0.3738\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4522, 0.5024, 0.5524\\ 0.3238, 0.3739, 0.4240\\ 0.2738, 0.3238, 0.3739\end{matrix}\right\rangle$
ς_1	$\left\langle\begin{matrix}0.4668, 0.5181, 0.5692\\ 0.3401, 0.3905, 0.4411\\ 0.2898, 0.3401, 0.3905\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5501, 0.6004, 0.6506\\ 0.2233, 0.2730, 0.3228\\ 0.1863, 0.2361, 0.2730\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.4930, 0.5442, 0.5951\\ 0.3024, 0.3528, 0.4033\\ 0.2520, 0.3024, 0.3528\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.6436, 0.7009, 0.7511\\ 0.1732, 0.2229, 0.2727\\ 0.1235, 0.1732, 0.2229\end{matrix}\right\rangle$	$\left\langle\begin{matrix}0.5991, 0.6496, 0.6999\\ 0.1984, 0.2482, 0.2979\\ 0.1487, 0.1984, 0.2482\end{matrix}\right\rangle$

Step 9: Calculation of aggregated TFNN rating against each alternative

We now calculate aggregated TFNN rating against each alternative using criteria weights using eqn (67). The aggregated TFNN ratings are shown in table 23.

Table 23: Aggregated TFNN ratings of different alternatives based on TFNEOWG Operator

Alternative	Aggregated TFNN rating
ζ_1	$\left\langle\begin{matrix}(0.6115, 0.6623, 0.7129)\\ (0.1917, 0.2416, 0.2914)\\ (0.1441, 0.1917, 0.2416)\end{matrix}\right\rangle$
ζ_2	$\left\langle\begin{matrix}(0.7102, 0.7605, 0.8107)\\ (0.1436, 0.1934, 0.2431)\\ (0.0939, 0.1436, 0.1934)\end{matrix}\right\rangle$
ζ_3	$\left\langle\begin{matrix}(0.4732, 0.5239, 0.5745)\\ (0.3118, 0.3621, 0.4125)\\ (0.2615, 0.3118, 0.3621)\end{matrix}\right\rangle$
ζ_4	$\left\langle\begin{matrix}(0.5645, 0.6172, 0.6680)\\ (0.2317, 0.2819, 0.3321)\\ (0.1839, 0.2340, 0.2819)\end{matrix}\right\rangle$

Step 10: Calculation of score and accuracy values of different alternatives

Score and accuracy values of different alternatives are calculated using eqn (68) and eqn (69) and results are presented in table 24.

Step 11: Ranking of the alternatives
Ranking of the alternatives on the basis of accuracy values and score value is shown in table 24

Table-24: Score value and accuracy value of different alternatives based on TFNEOWGA Operator

Alternative	Score value	Accuracy value	Remark
ζ_1	0.7427	0.4698	Depending on the value of score and accuracy function Alternative -2(ζ_2) has the highest value of both score function and accuracy function. So, Alternative-2 is the best alternative. Ranking of the alternatives is $\zeta_2 > \zeta_1 > \zeta_4 > \zeta_3$
ζ_2	0.8078	0.6168	
ζ_3	0.6167	0.2121	
ζ_4	0.7005	0.3833	

So, using both TFNEOWA and TFNEOWGA operator, we conclude that alternative-2 is the best alternative

7. CONCLUSIONS

MCDM and MCGDM related problems are generally observed in quite complex environments and are mostly linked with incomplete and uncertain information. TFNNs are very useful tools to tackle the incompleteness and inaccuracy of DMs assessments for the selection of best alternatives among the group of alternatives on the basis of different criteria involved. We have defined the score function and accuracy function for TFNNs and established some of their basic properties. We have also introduced two operators namely TFNEOWA and TFNEOWGA operators and proved some of their basic properties. Finally, two numerical examples regarding sales manager selection in a pharmaceutical company have been provided to reflect the applicability of the developed strategies. We hope that the developed strategies will help deal with other MCDM problems such as the library and information system (Sahoo, Panigrahi, & Pramanik, 2023, Sahoo, Pramanik, Panigrahi, 2023), supplier selection (Abdel-Baset et al., 2019), diagnosing COVID-19 cases (Alsattar et al., 2024), COVID-19 vaccine selection (Mallick, Pramanik, Giri, 2024), sustain route selection of petroleum transportation (Simić et al, 2023), tourist destination choice problems (Lan et al., 2023), etc.

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SVNN- E-ARAS Strategy Based Information Retrieval Considering Popularity Ranking Factors: An MCGDM Framework

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ABSTRACT

Mostly library and information systems fail to provide satisfactory search results, and exhibit poor performance regarding ranking factors, and do not use user-centered searching approaches. As a result, getting information through such a library and information system needs improvement to get satisfactory results. Six categories of ranking factors—"Text Statistics, Popularity, Freshness, Locality and Availability, Content Properties, and User Background"—are used to determine rankings. To rank search results using Single Valued Neutrosophic Numbers (SVNNs), the study aims to provide the elements influencing the ranking of search results in library and information systems, assigning weight to the major broad attributes of the popularity group according to the opinions of experts. The literature study shows that no studies have ever used the entropy-Additive Ratio Assessment (ARAS) and ordered search results taking popularity ranking variables. The study is innovative in all these ways as well as the elements and weighting strategy help in developing Web-scale Discovery Tools (DTs), Integrated Library Management Systems (ILMSs), and any other Information Retrieval (IR) system.

KEYWORDS: **Information retrieval, relevance ranking, OPAC, ranking factors, single valued neutrosophic number, MCGDM, ARAS, entropy, search result.**

1. INTRODUCTION

The software assists us in locating a library's collection using its Online Public Access Catalogue (OPAC), or its web version known as Web-OPAC. There exists a tonne of Integrated Library Management Systems (ILMSs) that are open-source and free, as well as numerous paid options. However, due to a lack of user-centeredness and presentational sophistication, the OPAC search results have several drawbacks (Lewandowski, 2010). To produce their search results in a relevant order, today's library and information systems take into account a meagre number of criteria, as well as weak principles and tactics, which is why they provide such subpar results. Additionally, users' preferences are not taken into account, which is more important now than ever.

Due to its lack of user-centeredness and consideration for a wide range of elements, Discovery tools are also unable to satisfy users (Sahoo & Panigrahi, 2022). To satisfy consumers and keep the search results ordered while retaining relevance, the best search results in a ranking carried out by web search engines may be notable examples for any other information systems like a Library and Information System (LIS). To meet consumers' expectations for information search and discovery, search engine technologies have been deployed (Breeding, 2006; Antelman, Lynema, & Pace, 2006; Niu & Hemminger, 2011; Connaway & Dickey, 2010).

Behnert & Lewandowski, (2015) categorized the Ranking Factors (RFs) into six groups: i. "text statistics", ii. "popularity", iii. "freshness", iv. "locality and availability", v. "content properties" and vi. "user background". Various elements can be taken into account under each group to rank library items while keeping the relevant order of search results. Only a few criteria are used by LIS in their system, but we need to strive to add more for better outcomes. Many popularity criteria are appropriate for LIS, but in this case, we have simply taken ten (10) large sub-groups under group popularity to demonstrate how to implement them in the system (Sahoo, Pramanik, & Panigrahi, 2023).

Real-life problems are mostly uncertain. Uncertainty, indeterminacy, and inconsistent results are fundamental characteristics of ranking factors. Zadeh (1965) proposed the Fuzzy Sets (FSs) to deal with uncertainty. The Neutrosophic Set (NS) (Smarandache, 1998), which is an extension of various FSs and Intuitionistic FSs (IFSs) (Atanossov,1986), is competent to deal with uncertainty comprehensively. A truth Membership Function (MF), an indeterminacy MF, and a falsity MF are independent components of an NS (Smarandache, 1998). As a subclass of NS that is more common in MCDM situations, Single-Valued NS (SVNS) (Wang, Smarandache, Zhang, &Sunderraman (2010)) was introduced. SVNS is further extended to the quadripartitioned NS (Chatterjee et al.(2016), interval quadripartitioned NS (Pramanik, 2022b), Pentapartitioned NS (PNS) (Mallick & Pramanik, 2020), and interval PNS (Pramanik, 2023a). The studies (Peng & Dai, 2020, Pramanik 2020, 2022a; Broumi et al., 2018; Smarandache & Pramanik 2016, 2018; Pramanik, Mallick, & Dasgupta, 2018) provide specifics on the evolution of neutrosophic theories and implementations.

For the current study, we opt for the SVNN environment. We refer to the hybrid approach in this environment as the SVNN-E -ARAS strategy as a combination of the entropy strategy and group decision-making utilizing Additive Ratio ASsessment (ARAS). Based on the recommendations of the domain experts, the factors are given weights using the entropy technique.

In the actual world, Decision-Makers (DMs) prefer to use linguistic variables to evaluate the significance of traits in a flexible manner. This is due to a variety of factors, including incomplete knowledge of the attributes or criteria, a lack of information processing skills in the field, the presence of specialists, and more (Sahoo, Panigrahi, & Pramanik, 2023). Our framework is developed using a user-centered approach and the SVNS theory, which is more suited to reflect reality than the conventional approach.

Research gap: No research work has been developed using an entropy technique with ARAS method for information retrieval in an SVNS environment to incorporate RFs considered for the relevance ranking of search results in LIS.

Motivation: To fill the research gap, we initiate to develop a new strategy, namely SVNN-E-ARAS in the SVNS environment.

2. *LITERATURE REVIEW*

We present a literature search on library materials ranking factors, popularity group ranking factors, SVNS, the process of assigning weights to the criteria, the entropy strategy, and the ARAS strategy. Freshness was the most-used ranking criterion (Lewandowski, 2009) in catalogues. For a real ranking (Dellit & Boston, 2007), OPACs usually employ only standard text matching. There are some ideas to improve the relevance ranking that goes beyond unblended text matching. Flimm (2007) proposed the popularity RFs in catalogues for relevance ranking. According to Mercun & Zumer (2008) and Sadeh (2007) ranking search results in the LIS include “circulation statistics, book review data, the number of downloads, and the number of print copies owned by the institutions”.

It may happen that users are not interested or they are not able to look through the whole result sets. So quality ranking reduces to a crucial factor (Lewandowski, 2009). Behnert and Lewandowski (2015) categorized all RFs into six (6) groups. Plassmeier et.al. (2015) stated in their study “Catalogues rank usually search results based on the date of publication but the additional inclusion of popularity-based factors was highly promiiswassing to yield valuable benefits” and “popularity-based relevance ranking can be determined by citation counts, author metrics, and usage data, while we also consider other popularity data in our complete relevance model”. Bornmann, Mutz, and Daniel (2008) mentioned that the h-index and m-index were more important to reflect the impact of the work of a researcher. Glanzel and Schubert (1988) introduced the Characteristic Scores and Scales (CSS) technique which helps in finding the characteristic partitions for citation distributions of papers that are interpreted as “poorly cited”, “fairly cited”’ “remarkably cited”, or “outstandingly cited”. Plassmeier et al. (2015) stated that “the effectiveness of CSS scores as utilities in the overall relevance model must still be evaluated in user studies”.

There are many established criterion weighting procedures found in the literature (Peng, 2019) for the MCDM process such as CRiteria Importance Through Intercriteria Correlation (CRITIC) method (Diakoulaki et al., 1995), entropy method (Majumder & Samanta, 2014), maximizing deviation method (Wu & Chen, 2007), optimization method (Wang & Zhang, 2009; Biswas, Pramanik & Giri, 2014b). The information entropy method was used by Biswas, Pramanik, and Giri (2014a) to determine the unknown attribute weights in the SVNN environment.

Zavadskas and Turskis (2010) developed the ARAS strategy to deal with MCDM problems. Stanujkic (2015) developed the ARAS strategy for Multi-Criteria Group Decision Making (MCGDM) using linguistic variables. Koçak, Kazaz, and Ulubeyli (2018) used the ARAS strategy in the subcontractor selection problem. Büyüközkan, and Göçer (2018) presented the ARAS strategy in an interval-valued IFS environment. Ghram and Frikha (2019) presented the hierarchical ARAS strategy to rank the websites of tourist destination brands. Liu and Cheng (2019) developed the ARAS strategy under a probability multi-valued NS environment. Mallick and Pramanik (2021) presented the ARAS strategy for MCGDM in the trapezoidal NS environment. Adali et al. (2023) presented the ARAS strategy using CRITIC in the SVNN settings. Pramanik (2023b) developed the SVPNN- ARAS strategy for the MCGDM in the PNS environment. An overview of the ARAS strategy was documented by Liu and Xu (2021).

No research work has been developed to use the entropy-ARAS strategy for information retrieval in the SVNS environment to incorporate RFs considered for the relevance ranking of search results in LIS.

3. OBJECTIVES OF THE STUDY

The primary goals are listed below.

i. To determine the group ranking criteria

ii. To develop a unique entropy-ARAS strategy for MCGDM in the SVNN environment,

which we refer to as the SVNN-E-ARAS strategy.
iii. To develop a framework using the developed SVNN-E-ARAS technique that incorporates a few ranking elements of the group popularity for the relevancy order of search results in LIS.

4. METHODOLOGY

All the data used here were collected from the research of Sahoo, Pramanik, & Panigrahi (2023). A new MCGDM strategy is developed by incorporating SVNNs, Entropy, and ARAS for the study which is shown in section 5.

5. A NEW INTEGRATED SVNN MCGDM METHODOLOGY : SVNN-ENTROPY ARAS (SVNN-E-ARAS) STRATEGY FOR MCGDM IN SVNN ENVIRONMENT

Using the following steps, the proposed MCGDM strategy (refer to Figure 1) is developed.

Step 1: Construct the DM (Expert) Committee

Formulate a committee of P ($\geq$ 2) DMs.

Step 2: Define the objective, criteria, and alternatives

P DMs evaluate the alternative A_r (r= 1, 2, …, m), (m$\geq$ 2) with respect to n criteria F_s (s=1, 2, …, n), (n $\geq$ 2)

Step 3: Define the linguistic terms (LT scales to weigh DMs and criteria)

The weights of the DMs and criteria are presented in linguistic terms and the conversion formulae between linguistic terms and SVNNs are shown in Table 1.

Table 1: Conversion between LT and SVNN for weighting of attributes and DMs (Biswas, Pramanik, & Giri, 2016)

LTs	SVNNs
Extremely Important (EI)	$\langle 0.90, 0.10, 0.10 \rangle$
Very Important (VI)	$\langle 0.80, 0.20, 0.15 \rangle$
Important (I)	$\langle 0.50, 0.40, 0.45 \rangle$
Very Unimportant (VU)	$\langle 0.35, 0.60, 0.70 \rangle$
Extremely Unimportant (EU),	$\langle 0.10, 0.80, 0.90 \rangle$

Step 4: Formulate the Single Valued Neutrosophic Decision Matrices (SVNDMs)

We assume that the rating of alternative A_r (r= 1, 2, …, m) concerning criterion F_s (s=1, 2, …,n) offered by the p-th DM is a linguistic term α_{rs}^{p} that can be expressed by SVNN (Biswas, Pramanik, & Giri, 2016) (See Table 1).

Then the p-th decision matrix is constructed as:

$$\Delta^{p} = (\alpha^{p}_{rs})_{m\times n} = \begin{pmatrix} \alpha^{p}_{11} & \alpha^{p}_{12} & \cdots & \alpha^{p}_{1n} \\ \alpha^{p}_{21} & \alpha^{p}_{22} & \cdots & \alpha^{p}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \alpha^{p}_{m1} & \alpha^{p}_{m2} & \cdots & \alpha^{p}_{mn} \end{pmatrix} \tag{1}$$

After converting the LTs into SVNNs, the p-th SVNDM reduces to

$$G^{p} = (g^{p}_{rs})_{m\times n} = \begin{pmatrix} g^{p}_{11} & g^{p}_{12} & \cdots & g^{p}_{1n} \\ g^{p}_{21} & g^{p}_{22} & \cdots & g^{p}_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ g^{p}_{m1} & g^{p}_{m2} & \cdots & g^{p}_{mn} \end{pmatrix} \tag{2}$$

where $g^{p}_{rs} = \langle a^{p}_{rs}, b^{p}_{rs}, c^{p}_{rs} \rangle$

where p =1, 2, … P, r = 1, 2, …, m and s = 1, 2, …, n

Step 5: Normalize individual SVNDMs

Normalization is done using the following rule ((Biswas et al., 2016)

$$d^{p}_{rs} = \begin{cases} g^{p}_{rs}, \text{for benefit critrion} \\ (g^{p}_{rs})', \text{for cost criterion} \end{cases} \tag{3}$$

and the matrix G^{p} is converted into the matrix $D^{p}_{rs} = (d^{p}_{rs})_{m\times n}$ where $(g^{p}_{rs})' = (c^{p}_{rs}, 1-b^{p}_{rs}, a^{p}_{rs})$ is the complement of SVNN $g^{p}_{rs} = \langle a^{p}_{rs}, b^{p}_{rs}, c^{p}_{rs} \rangle$.

Then the normalized SVNDM appears as:

$$D^{p} = \begin{pmatrix} d^{p}_{11} & d^{p}_{12} & \cdots & d^{p}_{1n} \\ d^{p}_{21} & d^{p}_{22} & \cdots & d^{p}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ d^{p}_{m1} & d^{p}_{m2} & \cdots & d^{p}_{mn} \end{pmatrix}, \text{ p= 1, 2, ..., P.} \tag{4}$$

Step 6: Determine the weights of the DMs

Assume that $\varphi_p = \langle T_p(\omega), I_p(\omega), F_p(\omega) \rangle$ is rating for the p-th DM. Then, φ_P, weight of the pth DM =

$$\frac{1-\sqrt{\left\{(1-T_p(\omega))^2 + (I_p(\omega))^2 + (F_p(\omega))^2\right\}/3}}{\sum_{p=1}^{P}(1-\sqrt{\left\{(1-T_p(\omega))^2 + (I_p(\omega))^2 + (F_p(\omega))^2\right\}/3})} \tag{5}$$

and $$\sum_{p=1}^{P} \varphi_p = 1 \tag{6}$$

Step 7: Aggregate the SVNDMs using the weights of the DMs

Utilizing $D^{p}_{rs} = (d^{p}_{rs})_{m\times n}$, $\varphi = (\varphi_1, \varphi_2, ..., \varphi_P)^T$, $\varphi_p \in [0,1]$ and $\sum_{p=1}^{P} \varphi_p = 1$, the aggregated SVNDM is formed by employing the Single- Valued Neutrosophic Weighted

Arithmetic Averaging Aggregation (SVNWAAA) operator (Ye, 2014) as follows:

$$\begin{aligned} &\mathrm{SVNWAAA}_{\varphi}(d_{rs}^{1}, d_{rs}^{2}, \dots, d_{rs}^{P}) \\ &= \varphi_1 d_{rs}^{1} \oplus \varphi_2 d_{rs}^{2} \oplus \dots \oplus \varphi_P d_{rs}^{P} \\ &= \left\langle 1-\prod_{p=1}^{P}(1-T_{rs}^{(p)})^{\varphi_p}, \prod_{p=}^{P}(I_{rs}^{(p)})^{\varphi_p}, \prod_{p=}^{P}(F_{rs}^{(p)})^{\varphi_p} \right\rangle \end{aligned} \tag{7}$$

Then the aggregated SVNDM is obtained as:

$$D' = \left(d'_{rs}\right)_{m\times n} = \\ = \begin{pmatrix} d'_{11} & d'_{12} & \dots & d'_{1n} \\ d'_{21} & d'_{22} & \dots & d'_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ d'_{m1} & d'_{m2} & \dots & d'_{mm} \end{pmatrix} \tag{8}$$

where $d'_{rs} = \langle T'_{rs}, I'_{rs}, F'_{rs} \rangle$. (9)

Step 8: Determine the weights of the attributes

The entropy value (Majumder &Samanta, 2014) E_s of the t^h attribute F_s (s=1, 2, …, n), is obtained using the formula

$$E_s = 1 - \frac{1}{n}\sum_{r=1}^{m}(T'_{rs} + F'_{rs})(I_{rs} - I'_{rs}) \tag{10}$$

For r =1, 2, …, m; s =1, 2, …, n.

The entropy weight (Hwang &Yoon, 1981; Wang & Zhang, 2009) ω_s of the s-th attribute F_s is presented by

$$\omega_s = \frac{1-E_s}{\sum_{s=1}^{n}(1-E_s)} \tag{11}$$

We obtain the weight vector $\omega = (\omega_1, \omega_2, ,, \omega_n)'$ with $\omega_s \in [0,1]$ and $\sum_{s=1}^{n}\omega_s = 1.$

Step 9: Formulate the weighted aggregated SVNDM

The weighted aggregated SVNDM is presented as follows:

$$\theta_{rs} = \begin{pmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1n} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2n} \\ \vdots & \ddots & & \vdots \\ \theta_{m1} & \theta_{m2} & \cdots & \theta_{mn} \end{pmatrix}$$

where $\theta_{rs} = d'_{rs} * \omega_s$, r =1, 2,…, m; s=1, 2,…, n (12)

ω_s refers to the weight of the attribute F_s and $\sum_{s=1}^{n}\omega_s = 1$.

Step 10: Determine the optimal function values

To calculate the optimal values of the weighted aggregated SVNDM, we can use the equation (13).

$$\begin{aligned} \psi_r &= \theta_{r1} \oplus \theta_{r2} \oplus \dots \oplus \theta_{m} \\ &= \left\langle 1-\prod_{s=1}^{n}(1-T'_{rs}), \prod_{s=1}^{n}(I'_{rs}), \prod_{s=1}^{n}(F'_{rs}) \right\rangle \end{aligned} \quad \text{where } r = 1, 2, \dots, m \tag{13}$$

Step 11: Deneutrosophication

We calculate the score values of the elements of (13) using the formula of score function

$$Sc(\psi_r) = \frac{2 + T'_{rs} - I'_{rs} - F'_{rs}}{3} \tag{14}$$

Step 12: Evaluate the alternative utility degree

The degree of alternative utility determined by contrasting the score value $Sc(S_r)$ with the best suited S^*. The alternative's utility degree of Ξ_r is given below.

$$\Xi_r = \frac{Sc(\psi_r)}{\psi^*}; r = 1, 2, ..., m. \tag{15}$$

Step 13: Rank the alternatives

The descending order of can be used to identify the relative priority of workable alternatives Ξ_r. That is the alternative with the highest value of Ξ_r is the best choice.

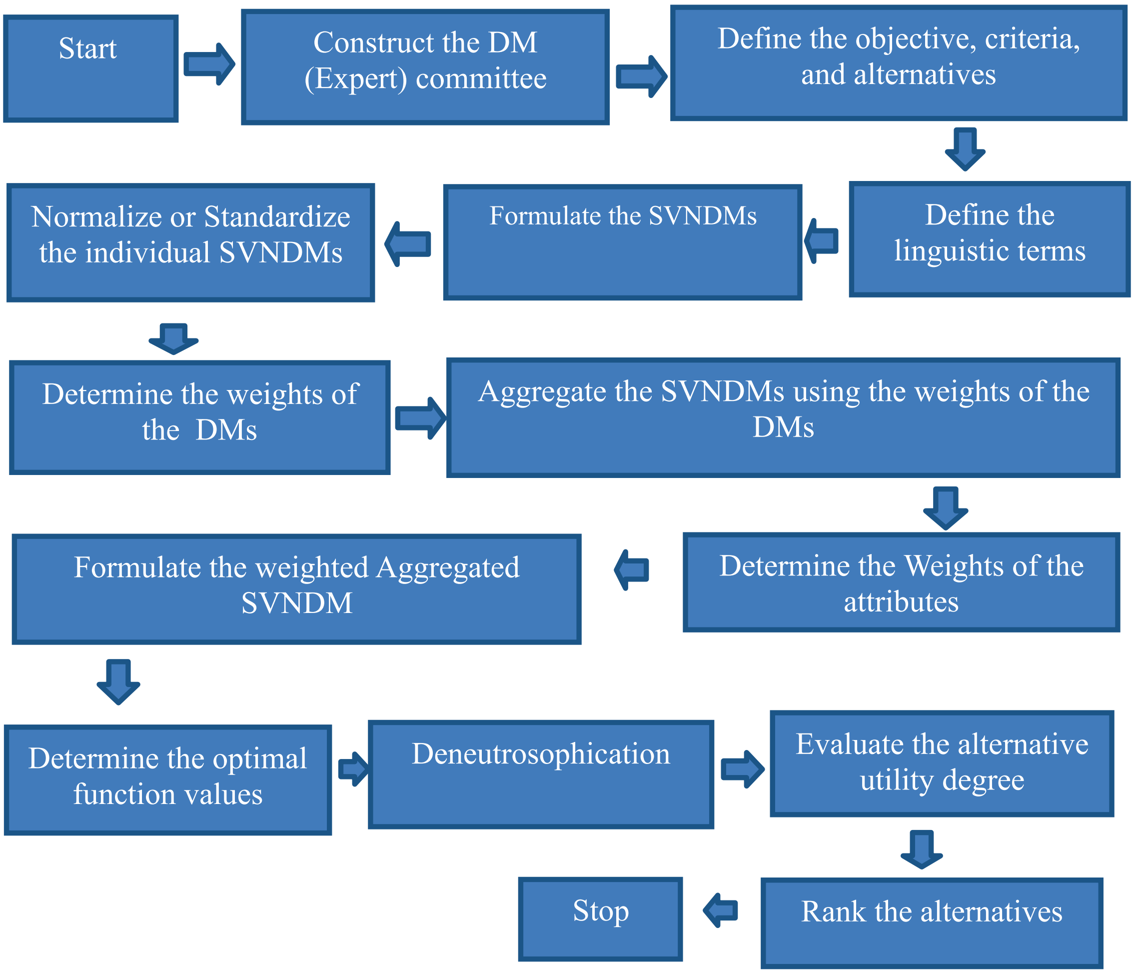

Fig.1: Flowchart of the SVNN-E-ARAS strategy for MCGDM

6. DATA, CALCULATIONS AND RESULTS

. The following steps of SVNN-E-ARAS are used to resolve the problem under consideration based on the suggested strategy discussed:

Step 1: Construct the DM Committee

We have considered five experts cum users as DMs (DM_1, DM_2, DM_3 , DM_4, DM_5) in the study.

Step 2: Define the objective, criteria, and alternatives

At first, we elaborately define the objectives of the study to the experts. Then briefly explained the definition, scope and coverage of all criteria. A group of five DMs (DM_1, DM_2, DM_3 , DM_4, DM_5) has provided their opinions about the importance of each particular ranking factors under the group popularity mentioned in the questionnaire on the basis of five-point Likert scale. The factors are Subject (F_1), Circulation (F_2), Language (F_3), Number of published edition (F_4), Number of Copies (F_5), Bibliometric Methods (F_6), Publisher Authority (F7), Purchasing Behaviour (F8), Ratings (F9) and Enriched Metadata (F10). The factors are related to the documents denoted as A_1 , A_2 , A_3, A_4 and we want to design a framework to order the documents according to their relevancy.

Step 3: Define the linguistic terms for the weights DMs and the criteria

Depending on their position, the five DMs may not be equally important. Table 1 represents the importance of the DMs Table 2 displays the significance of each DM as indicated by the LTs.

Table 2: Importance of DMs and Criteria

DM	**DM1**	**DM2**	**DM3**	**DM4**	**DM5**
LT	EI	VI	VI	EI	EI
SVNN	$\langle 0.90, 0.10, 0.10\rangle$	$\langle 0.80, 0.20, 0.15\rangle$	$\langle 0.80, 0.20, 0.15\rangle$	$\langle 0.90, 0.10, 0.10\rangle$	$\langle 0.90, 0.10, 0.10\rangle$

Step4: Construction of the decision matrices

Based on the rating values in terms of linguistic terms, the decision matrices are formed (see Table 3, Table 4, Table 5, Table 6, Table 7),

Table 3: Decision matrix $P^{(1)}$

A_i	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
A_1	VI	VI	VI	VI	VI	VI	VI	VI	EI	EI
A_2	EI	VI	I	EI	VI	VI	VI	EI	I	VU
A_3	VI	VI	VI	VU	VI	VU	I	I	I	I
A_4	VI	VI	VI	VI	VI	VI	VU	VU	I	I

Table 4: Decision matrix $P^{(2)}$

A_i	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
A_1	VI	VU	I	I	I	I	EI	I	EI	VI
A_2	VI	I	VU	I	VI	VI	VI	I	VI	VI
A_3	I	I	I	VI	VI	I	I	VU	I	VI
A_4	VI	VI	VI	VU	VU	VU	VU	VI	VU	I

Table 5: Decision matrix $P^{(3)}$

A_i	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
A_1	VI	I	VU	I	I	I	VI	I	VI	VI
A_2	VI	VI	VI	I	VI	I	I	VI	VI	VI
A_3	I	VI	VI	VI	VI	VI	I	I	I	I
A_4	VI	I	I	VU	I	VI	VU	I	I	VI

Table 6: Decision matrix $P^{(4)}$

A_i	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
A_1	VI	VI	VI	VI	I	VI	VI	I	EI	EI
A_2	I	I	VI	EI	VI	I	I	VI	VI	VI
A_3	VI	VI	I	I	I	I	VI	EI	I	I
A_4	I	VI	VI	I	EI	VI	I	I	EI	I

Table 7: Decision matrix $P^{(5)}$

A_i	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}
A_1	VI	I	VI	VU	VI	VI	I	EI	I	VI
A_2	I	VI	VU	I	VI	VU	VU	I	VI	I
A_3	I	I	I	I	I	I	I	I	I	I
A_4	VI	VI	VI	VI	VI	VI	VI	VI	VU	VU

The decision matrices are converted into SVNDMs $P^{(i)}$ (i= 1, 2, 3, 4, 5).

$P^{(1)}=$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.90,0.10,0.10\rangle \\ \langle 0.90,0.10,0.10,\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.35,0.60,0.70\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle \end{pmatrix} \tag{16}$$

$P^{(2)}=$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.80,0.20,0.15,\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle \end{pmatrix} \tag{17}$$

$P^{(3)}=$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle \end{pmatrix} \quad (18)$$

$P^{(4)}=$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.90,0.10,0.10\rangle \\ \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle \\ \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.50,0.40,0.45\rangle \end{pmatrix} \quad (19)$$

$P^{(5)}=$

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.90,0.10,0.10\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle \\ \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.50,0.40,0.45\rangle \\ \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle & \langle 0.50,0.40,0.45\rangle \\ \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.80,0.20,0.15\rangle & \langle 0.35,0.60,0.70\rangle & \langle 0.35,0.60,0.70\rangle \end{pmatrix} \quad (20)$$

Step 5: Normalize the SVNDMs

The considered criteria are benefit type. So, no normalization technique is required.

Step 6: Determine the weights of the DMs

Using the formula described in eq. (5), we obtain the weights of the DMs (see Table 8).

Table 8: Weight of the DMs

DM	φ_1	φ_2	φ_3	φ_4	φ_5
Weight	0.207837	0.188244	0.188244	0.207837	0.207837

Step 7: Construction of the aggregated SVNDM

Using the formula (eq. (7)) and decision matrices (see eq. (16), eq. (17), eq. (18), eq. (19), and eq. (20)), we obtain the aggregated SVNDM (see eq. (21)).

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.80,0.20,0.15\rangle & \langle 0.64,0.32,0.31\rangle & \langle 0.7,0.28,0.25\rangle & \langle 0.64,0.33,0.31\rangle & \langle 0.66,0.3,0.29\rangle & \langle 0.72,0.26,0.23\rangle & \langle 0.79,0.2,0.17\rangle & \langle 0.7,0.26,0.26\rangle & \langle 0.84,0.15,0.15\rangle & \langle 0.85,0.15,0.13\rangle \\ \langle 0.75,0.23,0.22\rangle & \langle 0.71,0.26,0.23\rangle & \langle 0.61,0.36,0.35\rangle & \langle 0.74,0.23,0.24\rangle & \langle 0.8,0.2,0.15\rangle & \langle 0.63,0.33,0.32\rangle & \langle 0.63,0.33,0.32\rangle & \langle 0.75,0.23,0.21\rangle & \langle 0.76,0.23,0.19\rangle & \langle 0.69,0.29,0.26\rangle \\ \langle 0.66,0.3,0.29\rangle & \langle 0.71,0.26,0.23\rangle & \langle 0.65,0.31,0.29\rangle & \langle 0.63,0.34,0.33\rangle & \langle 0.71,0.27,0.24\rangle & \langle 0.56,0.38,0.4\rangle & \langle 0.59,0.35,0.36\rangle & \langle 0.62,0.32,0.36\rangle & \langle 0.5,0.4,0.45\rangle & \langle 0.58,0.35,0.37\rangle \\ \langle 0.76,0.23,0.19\rangle & \langle 0.76,0.23,0.18\rangle & \langle 0.76,0.23,0.18\rangle & \langle 0.62,0.35,0.34\rangle & \langle 0.74,0.24,0.23\rangle & \langle 0.75,0.25,0.2\rangle & \langle 0.52,0.44,0.46\rangle & \langle 0.65,0.3,0.29\rangle & \langle 0.6,0.35,0.39\rangle & \langle 0.56,0.38,0.4\rangle \end{pmatrix} \quad (21)$$

Step 8: Determine the weights of the attributes

To determine the weights of the 10 attributes, we have calculated the entropy value of each attribute using eq. (10) . The obtained entropy values are tabulated in the Table 9.

Table 9: Entropy value for the attributes

E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}
0.8013	0.8248	0.8448	0.8553	0.8109	0.8516	0.8698	0.8292	0.8307	0.8400

After the calculation of the entropy values of all ten attributes, we calculate the weight of each attribute (see Table 10) using eq. (11).

Table 10: Weights of the attributes

W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}
0.1210	0.1067	0.0945	0.0882	0.1152	0.0904	0.0793	0.1040	0.1031	0.0975

Step 9: Construct the weighted aggregated SVNDM

Using the formula (see eq. (12)), the weighted aggregated SVNDM (see eq. (22)). is constructed.

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.18,0.82,0.79\rangle\langle 0.1,0.86,0.88\rangle\langle 0.12,0.89,0.88\rangle\langle 0.09,0.91,0.9\rangle\langle 0.12,0.87,0.87\rangle\langle 0.12,0.89,0.88\rangle\langle 0.12,0.88,0.87\rangle\langle 0.12,0.87,0.87\rangle\langle 0.17,0.82,0.82\rangle\langle 0.17,0.83,0.82\rangle \\ \langle 0.15,0.84,0.83\rangle\langle 0.12,0.87,0.85\rangle\langle 0.09,0.91,0.91\rangle\langle 0.11,0.88,0.88\rangle\langle 0.17,0.83,0.8\rangle\langle 0.09,0.9,0.9\rangle\langle 0.08,0.92,0.92\rangle\langle 0.13,0.86,0.85\rangle\langle 0.14,0.86,0.84\rangle\langle 0.12,0.89,0.88\rangle \\ \langle 0.12,0.86,0.86\rangle\langle 0.12,0.87,0.85\rangle\langle 0.09,0.9,0.89\rangle\langle 0.08,0.91,0.91\rangle\langle 0.13,0.86,0.85\rangle\langle 0.07,0.92,0.92\rangle\langle 0.07,0.92,0.92\rangle\langle 0.1,0.89,0.9\rangle\langle 0.07,0.91,0.92\rangle\langle 0.08,0.9,0.91\rangle \\ \langle 0.16,0.84,0.82\rangle\langle 0.14,0.85,0.83\rangle\langle 0.13,0.87,0.85\rangle\langle 0.08,0.91,0.91\rangle\langle 0.14,0.85,0.84\rangle\langle 0.12,0.88,0.86\rangle\langle 0.06,0.94,0.94\rangle\langle 0.1,0.88,0.88\rangle\langle 0.09,0.9,0.91\rangle\langle 0.08,0.91,0.91\rangle \end{pmatrix} \quad (22)$$

Step 10: Determine the optimal function values

Using eq. (13), we obtain the optimal function values (see eq. (23)).

$$\begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array}\begin{pmatrix} \langle 0.7459, 0.2361, 0.2145\rangle \\ \langle 0.7184, 0.2602, 0.2364\rangle \\ \langle 0.6298, 0.3223, 0.3213\rangle \\ \langle 0.6889, 0.28680.2637\rangle \end{pmatrix} \quad (23)$$

Step 11: Deneutrosophication

We calculate the score values (see Table 11) using the formula (see eq. (14)).

Table 11: Score values of the alternatives

Alternatives	Sc1	Sc2	Sc3	Sc4
Values	0.7651	0.7406	0.6621	0.7128

Step 12: Evaluate alternative utility degree

The values of the alternative utility degree Ξ_r are shown in Table 12.

Table 12: Utility degree of the alternatives

Alternatives	Ξ_1	Ξ_2	Ξ_3	Ξ_4
Utility degree	1	0.9680	0.8653	0.9316
Relevancy Position	1st	2nd	4th	3rd

Step13: Rank the alternatives

The ranking order is done in descending order of utility degree. The final relevancy ranking order is $A_1>A_2>A_4>A_3$.

7. CONCLUSIONS

This chapter develops the SVNN-E-ARAS strategy using the SVNNWAAA operator in SVNN settings. The developed strategy has the advantage of handling uncertainty using neutrosophic number with respect to other methods.

The chapter covers the group popularity ranking criteria and weights each ranking component individually based on user evaluation using the developed approach. The alternatives, or documents, were ranked using the ARAS methodology. Here, we have taken into account the popularity-related ranking variables and created a framework to include the components after determining weights. This is the first information retrieval strategy to take into account an SVNN environment using contemporary techniques and a created Entropy-ARAS strategy. For better and more precise results in the future, more RFs can be added. Additionally, it is useful for creating discovery tools, coming up with a ranking model for a library and information system, or conversing with ILMS vendors.

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RNN-MABAC Strategy for MADM in Rough Neutrosophic Number Environment

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ABSTRACT

The Rough Neutrosophic Set (RNS) has emerged as a hybrid structure to deal with decision-making involving uncertainty. The MABAC (Multi-Attributive Border Approximation Area Comparison) strategy deals with decision-making issues by utilizing the distance between each alternative and the Border Approximation Area (BAA). In the article, the MABAC strategy has been developed using rough neutrosophic numbers (RNNs) which we call RNN-MABAC strategy. The developed strategy is illustrated by solving a numerical example of MADM problem.

KEYWORDS: **Fuzzy set, neutrosophic set, rough set, rough neutrosophic number, MADM MABAC.**

1. INTRODUCTION

Smarandache (1998) introduced Neutrosophic Sets (NS), which extend the foundational ideas of Fuzzy Set (FSs) by Zadeh (1965) and Intuitionistic FSs by Atanassov (1986) to encompass a more comprehensive treatment of uncertainty. Subsequently, Wang et al. (2010) developed Single-Valued Neutrosophic Set (SVNS) as a specific subclass of NS tailored for practical applications. Theoretical improvements and various applications of NSs have been depicted by several studies (Broumi et al., 2018; Pramanik et al., 2018; Otay, & Kahraman, 2019; Peng & Dai, 2020; Pramanik, 2020, 2022; Smarandache, & Pramanik, 2016, 2028; Delcea et al, 2023). Pawlak (1982) introduced the Rough Set (RS) to handle uncertain and incomplete information. Rough NS (RNS) (Broumi, Smarandache, & Dhar, 2014) was proposed by combining the RS and NS to handle incompleteness and uncertainty. An overview of RNSs has been documented in the studies (Pramanik, 2020; Zhang et al., 2020).

Multi-Attribute Decision Making (MADM) strategy selects the best option or makes a preference list of options subject to a list of conflicting criteria. Several MADM strategies have been developed in the Rough Neutrosophic Number (RNN) environment. Mondal and Pramanik (2015) developed a grey relational analysis (Deng, 1989) based MADM strategy in the RNN environment. Several similarity measures (Mondal, Pramanik, & Smarandache, 2016a) in RNN environments were investigated. and their properties were established. Mondal, Pramanik, and

Smarandache (2016b) developed the TOPSIS strategy in the RNS environment. Pramanik, Roy, Roy, and Smarandache (2017) developed a MADM strategy in the RNS environment using the correlation coefficient measure in the RNS setting. Pramanik, Roy, and Roy 2018) developed the projection and bidirectional measured-based MADM strategy in the RNN environments. Mondal, Pramanik, and Giri (2018) developed four MADM strategies using arithmetic and geometric mean operators.

In 2015, Pamucar and Ćirović (2015) developed the Multi-attributive Border Approximation Area Comparison (MABAC) strategy for MADM in a crisp environment. In 2016, Peng and Yang (2016) presented the MABAC strategy in the Pythagorean FS environment using Choquet integral. Jia et al. (2019) presented MABAC strategy under the intuitionistic fuzzy rough number setting. Gigović et al. (2017) presented an application of MABAC strategy in locating wind farms The BMW and MABAC in modified form were presented in the study (Pamučar, Petrović, & Ćirović, 2018). The interval rough AHP and MABAC strategies were integrated in the study (Pamučar, Stević, & Zavadskas, 2018). Peng, and Dai (2018) presented the MABAC strategy in the SVNN environment. In 2022, Jiang et al. (2022) presented MABAC strategy in the picture FS setting. Tan et al. (2023) presented MABAC strategy based on prospect theory in Fermatean FS environment. In 2023, a literature review of MABAC strategy was documented by Torkayesh et al. (2023).
Research gap: No studies have been proposed using the MABAC strategy in the RNN settings.

Motivation: The gap in research motivates us to explore the RNN-MABAC strategy.

Objectives: To present the MABAC strategy in the RNN settings which we name the RNN-MABAC strategy.

The rest of the paper is presented as follows. Preliminaries of the SVNSs and RNSs are presented in Section 2. RNN-MABAC strategy is developed in Section 3. A numerical example of a MADM is solved using the RNN-MABAC strategy. Section 5 provides insights into future research directions, summarizing the paper's conclusions.

2. PRELIMINARIES

An SVNS (Wang et al., 2010) χ in a universal set Ω is characterized by a truth-MF $\xi_\chi(\omega)$, an indeterminacy-MF $\psi_\chi(\omega)$, and a falsity-MF $\psi_\chi(\omega)$ with $\xi_\chi(\omega), \psi_\chi(\omega), \zeta_\chi(\omega) \in [0,1], \forall \omega \in \Omega$.
When Ω is continuous, an SNVS χ can be presented as:

$$\chi = \int_{\omega} \left\langle \xi_\chi(\omega), \psi_\chi(\omega), \zeta_\chi(\omega) \right\rangle / \omega, \forall \omega \in \Omega$$

and when Ω is discrete, an SVNS χ can be presented as:

$$\chi = \Sigma \left\langle \xi_\chi(\omega), \psi_\chi(\omega), \zeta_\chi(\omega) \right\rangle / \omega, \forall \omega \in \Omega$$

with $0 \leq \sup \xi_\chi(\omega) + \sup \psi_\chi(\omega) + \zeta_\chi(\omega) \leq 3, \omega \forall \in \Omega$

An SVNS χ is also presented as:

$\chi = \left\langle \omega, \xi_\chi(\omega), \psi_\chi(\omega), \zeta_\chi(\omega) > /, \omega \in \Omega \right\rangle$, where $\xi_\chi(\omega), \psi_\xi(\omega), \zeta_\xi(\omega) \in [0, 1]$, for each ω in Ω. Therefore, $0 \leq \sup \xi_\chi(\omega) + \sup \psi_\chi(\omega) + \sup \zeta_\chi(\omega) \leq 3$.

The triplet $\left(\xi_\chi(\omega), \psi_\chi(\omega), \zeta_\chi(\omega)\right)$ is termed as the Single-Valued Neutrosophic Number (SVNN) and presented as $\left(\xi_\chi, \psi_\chi, \zeta_\chi\right)$.

2.1.1 Let $\eta_1 = (\alpha_1, \beta_1, \gamma_1)$ and $\eta_2 = (\alpha_2, \beta_2, \gamma_2)$ be any two SVNNs with $\alpha_1, \beta_1, \gamma_1, \alpha_2, \beta_2, \gamma_2 \in [0,1]$, $(\alpha_1 + \beta_1 + \gamma_1) \in [0,3]$ and $(\alpha_2 + \beta_2 + \gamma_2) \in [0,3]$

Then, some selected operations involving SVNNs (Peng & Dai, 2018) are stated as follows;

i. $\eta_1 \oplus \eta_2 = (\alpha_1 + \alpha_2 - \alpha_1\alpha_2, \beta_1, \beta_2, \gamma_1\gamma_2)$ [Summation] (1)

ii. $\eta_1 \otimes \eta_2 = (\alpha_1\alpha_2, \beta_1 + \beta_2 - \beta_1\beta_2, \gamma_1 + \gamma_2 - \gamma_1\gamma_2)$ [Multiplication] (2)

iii. $\lambda\eta_1 = (1-(1-\alpha_1)^\lambda, \beta_1^\lambda, \gamma_1^\lambda)$, $\lambda > 0$ [Scalar multiplication] (3)

iv. $\eta_1^\lambda = (\alpha_1^\lambda, 1-(1-\beta_1)^\lambda, 1-(1-\gamma_1)^\lambda)), \lambda > 0$ (4)

2.2. Euclidean distance function. Euclidean distance Biswas et al., 20[16] between $\eta_1 = (\alpha_1, \beta_1, \gamma_1)$ and $\eta_2 = (\alpha_2, \beta_2, \gamma_2)$ is defined as:

$$d_e = \left[\frac{1}{3}\left\{(\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2)\right\}\right]^{\frac{1}{2}} \tag{5}$$

2.4 Score function.

Score function (Peng & Dai, 2018) denoted by $Sc(n_1)$ of an SVNN $n_1 = (\eta_1, \eta_2, \eta_3)$ is defined as $Sc(n_1) = (2 + \eta_1 - 0.3 \times \eta_2 - 0.4 \times \eta_3)/3$ (6)

.

Definition 2.5 (Broumi, Smarandache, & Dhar, 2014)

Assume that $\ddot{\Theta}$ *is* a nonvoid set. Assume that $\ddot{R}$ is an equivalence relation on $\ddot{\Theta}$. Let $\ddot{\Phi}$ be an NS in $\ddot{\Theta}$ with the truth Membership Function (MF) $\dot{T}_{\dot{\Phi}}$, indeterminacy MF $\dot{I}_{\dot{\Phi}}$, and falsity MF $\dot{F}_{\dot{\Phi}}$. The lower and the upper approximations of $\dot{\Phi}$ in the approximation ($\ddot{\Theta}$, $\ddot{R}$) presented by $\underline{\ddot{\nu}}\left(\ddot{\Phi}\right)$ and $\overline{\ddot{\nu}}\left(\ddot{\Phi}\right)$ are presented as:

$$\underline{\ddot{\nu}}\left(\ddot{\Phi}\right) = \left\langle < \ddot{\theta}, \ddot{\delta}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\varepsilon}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\varphi}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) > / \zeta \in [\ddot{\theta}]_{\ddot{R}}, \ddot{\theta} \in \ddot{\Theta} \right\rangle \tag{7}$$

$$\overline{\ddot{\nu}}\left(\ddot{\Phi}\right) = \left\langle < \ddot{\theta}, \ddot{\delta}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\varepsilon}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\varphi}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) > / \ddot{\zeta} \in [\ddot{\theta}]_{\ddot{R}}, \ddot{\theta} \in \ddot{\Theta} \right\rangle \tag{8}$$

$$\ddot{\delta}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}}\, \ddot{\delta}_{\ddot{\Phi}}(\ddot{\zeta}), \ddot{\varepsilon}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}}\, \ddot{\varepsilon}_{\ddot{\Phi}}(\ddot{\zeta}),\ \ddot{\varphi}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}}\, \ddot{\varphi}_{\ddot{\Phi}}(\ddot{\zeta})$$

$$\ddot{\delta}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}}\, \ddot{\delta}_{\ddot{\Phi}}(\ddot{\zeta}), \ddot{\varepsilon}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}}\, \ddot{\varepsilon}_{\ddot{\Phi}}(\ddot{\zeta}),\ \ddot{\varphi}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}}\, \ddot{\varphi}_{\ddot{\Phi}}(\ddot{\zeta})$$

So, $0 \leq \ddot{\delta}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\varepsilon}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\varphi}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) \leq 3$

$0 \leq \ddot{\delta}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\varepsilon}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\varphi}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) \leq 3$.

Here, $\vee$ and $\wedge$ present respectively the max and "min'' operator. $\ddot{\delta}_{\ddot{\Phi}}(\ddot{\zeta}), \ddot{\varepsilon}_{\ddot{\Phi}}(\ddot{\zeta})$,and $\ddot{\varphi}_{\ddot{\Phi}}(\ddot{\zeta})$ are the truth MF, indeterminacy MF, and falsity MF of $\ddot{\zeta}$ w.r.t. $\ddot{\Phi}$. Here, $\underline{\ddot{\nu}}(\ddot{\Phi})$ and $\overline{\ddot{\nu}}(\ddot{\Phi})$ are NSs in $\ddot{\Phi}$. The NS mapping $\underline{\ddot{\nu}}, \overline{\ddot{\nu}} : \ddot{\nu}\left(\ddot{\Phi}\right) \rightarrow \ddot{\nu}(\ddot{\Phi})$ denote as the lower and upper RNS approximation operators. The pair $(\underline{\ddot{\nu}}\left(\ddot{\Phi}\right), \overline{\ddot{\nu}}\left(\ddot{\Phi}\right))$ is called the RNS in ($\ddot{\Theta}, \ddot{R}$).

3. *RNN-MABAC STRATEGY IN RNN SETTINGS*

Consider a MADM problem having n attributes, $C = \langle C_1''', C_2''', ..., C_n''' \rangle$ and m alternatives $A = \langle A_1''', A_2''', ..., A_m''' \rangle$. The weight w_j''' (j= 1, 2, …, n) is assigned to C_j''' such that $w_j''' > 0$, and $\sum_{j=1}^{n} w_j''' = 1$.

Utilizing the following steps, the RNN-MABAC strategy is developed (see Fig. 1):

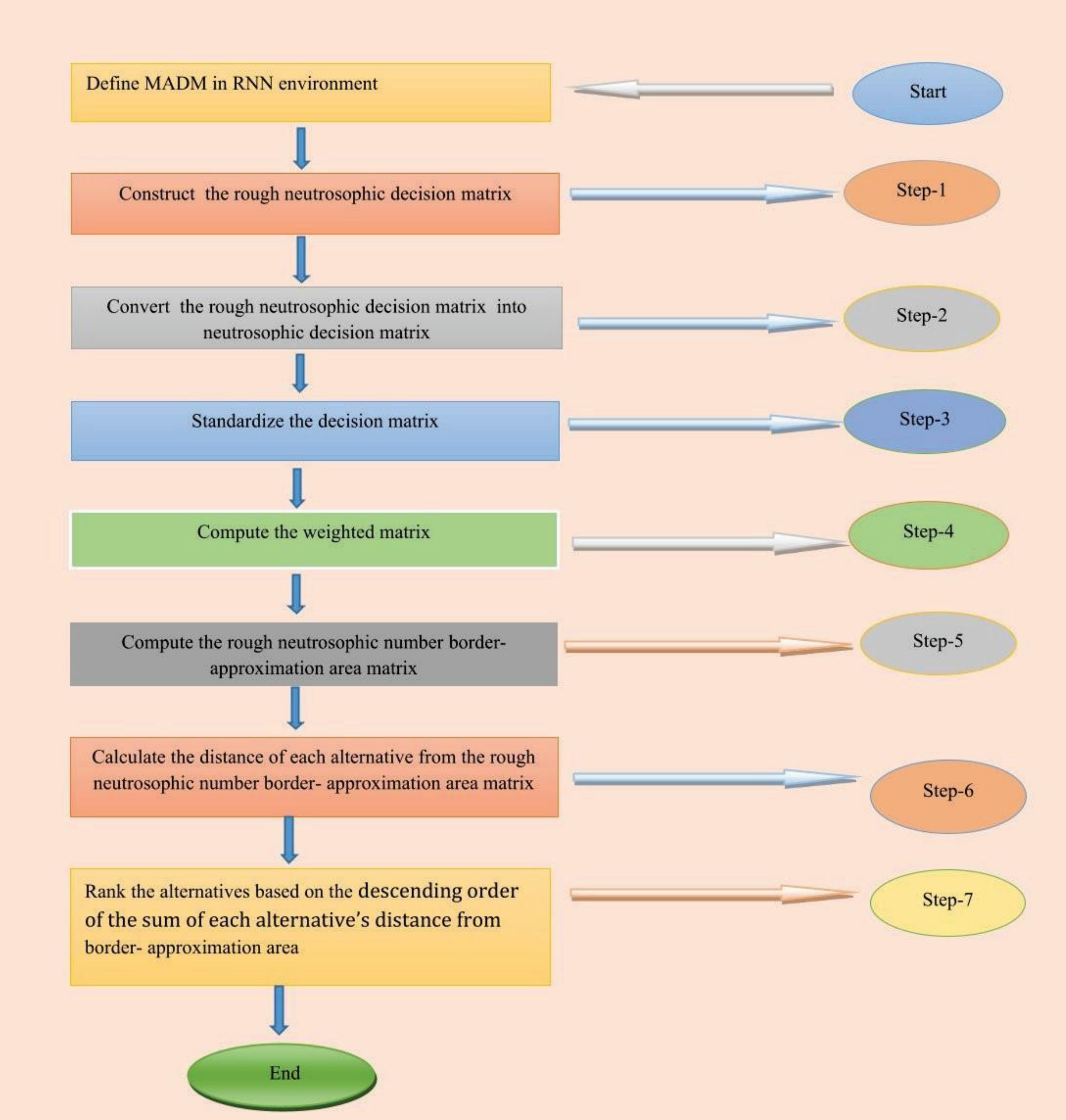

Fig. 1. MABAC strategy in RNN environment

Step 1. Formulate the decision matrix

Decision matrix $\dddot{D}$ is formulated using the RNN rating values of the alternatives provided by the decision maker (DM).

$$\dddot{D} = \langle \underline{\dddot{x}}_{ij}, \overline{\dddot{x}}_{ij} \rangle_{m\times n} =$$

$$\begin{array}{c|cccc} & C_1''' & C_2''' & \cdots & C_n''' \\ \hline A_1''' & \langle \underline{\dddot{x}}_{11}, \overline{\dddot{x}}_{11} \rangle & \langle \underline{\dddot{x}}_{12}, \overline{\dddot{x}}_{12} \rangle & \cdots & \langle \underline{\dddot{x}}_{1n}, \overline{\dddot{x}}_{1n} \rangle \\ A_2''' & \langle \underline{\dddot{x}}_{21}, \overline{\dddot{x}}_{21} \rangle & \langle \underline{\dddot{x}}_{22}, \overline{\dddot{x}}_{22} \rangle & \cdots & \langle \underline{\dddot{x}}_{2n}, \overline{\dddot{x}}_{2n} \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ A_m''' & \langle \underline{\dddot{x}}_{m1}, \overline{\dddot{x}}_{m1} \rangle & \langle \underline{\dddot{x}}_{m2}, \overline{\dddot{x}}_{m2} \rangle & \cdots & \langle \underline{\dddot{x}}_{mn}, \overline{\dddot{x}}_{mn} \rangle \end{array} \tag{9}$$

Here, $\langle \underline{\dddot{x}}_{ij}, \overline{\dddot{x}}_{ij} \rangle = \langle \langle \underline{\dddot{\tau}}_{ij}, \underline{\dddot{\iota}}_{ij}, \underline{\dddot{\varphi}}_{ij} \rangle, \langle \overline{\dddot{\tau}}_{ij}, \overline{\dddot{\iota}}_{ij}, \overline{\dddot{\varphi}}_{ij} \rangle \rangle$ denotes the RNN rating value of A_i''' w.r.t. c_j''' provided by the DM.

Step 2. Convert the decision matrix into a neutrosophic decision matrix using the Accumulated Geometric Operator (AGO).

We convert the RNN to SVNN by the AGO (Mondal & Pramanik, 2015) as follows:

$$\langle \underline{\dddot{x}}_{ij}, \overline{\dddot{x}}_{ij} \rangle_{AGO} = \langle \langle \underline{\dddot{\tau}}_{ij}, \underline{\dddot{\iota}}_{ij}, \underline{\dddot{\varphi}}_{ij} \rangle, \langle \overline{\dddot{\tau}}_{ij}, \overline{\dddot{\iota}}_{ij}, \overline{\dddot{\varphi}}_{ij} \rangle \rangle_{AG0}$$
$$= \langle (\underline{\dddot{\tau}}_{ij} . \overline{\dddot{\tau}}_{ij})^{0.5}, (\underline{\dddot{\iota}}_{ij} . \overline{\dddot{\iota}}_{ij})^{0.5}, (\underline{\dddot{\varphi}}_{ij} . \overline{\dddot{\varphi}}_{ij})^{0.5} \rangle \tag{10}$$
$$= \langle \dddot{\tau}_{ij}', \dddot{\iota}_{ij}', \dddot{\varphi}_{ij}' \rangle$$

The decision matrix is transformed to neutrosophic decision matrix $d_{\tilde{N}}$

$$\dddot{d}_{\tilde{N}} = \langle \dddot{\tau}_{ij}', \dddot{\iota}_{ij}', \dddot{\varphi}_{ij}' \rangle$$

$$= \begin{array}{c|cccc} & C_1''' & C_2''' & \cdots & C_n''' \\ \hline A_1''' & \langle \dddot{\tau}_{11}', \dddot{\iota}_{11}' \dddot{\varphi}_{11}' \rangle & \langle \dddot{\tau}_{12}', \dddot{\iota}_{12}' \dddot{\varphi}_{12}' \rangle & \cdots & \langle \dddot{\tau}_{1n}', \dddot{\iota}_{1n}' \dddot{\varphi}_{1n}' \rangle \\ A_2''' & \langle \dddot{\tau}_{21}', \dddot{\iota}_{21}' \dddot{\varphi}_{21}' \rangle & \langle \dddot{\tau}_{22}', \dddot{\iota}_{22}' \dddot{\varphi}_{22}' \rangle & \cdots & \langle \dddot{\tau}_{2n}', \dddot{\iota}_{2n}' \dddot{\varphi}_{2n}' \rangle \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \\ & & & & \cdots \\ . & \cdots & \cdots & \cdots & \\ A_m''' & \langle \dddot{\tau}_{m1}', \dddot{\iota}_{m1}' \dddot{\varphi}_{m1}' \rangle & \langle \dddot{\tau}_{m2}', \dddot{\iota}_{m2}' \dddot{\varphi}_{m2}' \rangle & \cdots & \langle \dddot{\tau}_{mn}', \dddot{\iota}_{mn}' \dddot{\varphi}_{mn}' \rangle \end{array} \tag{11}$$

Step 3. Standardize the decision matrix

Since criteria are two types, namely, benefit or cost, then there is a need to standardize them using formula (12) (Biswas et al., 2016)

$$\dddot{D}_{ij} = \begin{cases} \langle \langle \dddot{\tau}_{ij}', \dddot{\iota}_{ij}', \dddot{\varphi}_{ij}' \rangle \rangle, C_j''' \text{ is a benefit critrion} \\ \langle \dddot{\varphi}_{ij}', 1 - \dddot{\iota}_{ij}', \dddot{\tau}_{ij}' \rangle, C_j''' \text{ is a cost critrion} \end{cases} \tag{12}$$

Then the standardized decision matrix appears in the form:

$$\dddot{d}_{\tilde{N}} = \left\langle \dddot{\tau}_{ij}'', \dddot{\iota}_{ij}'' \dddot{\varphi}_{ij}'' \right\rangle_{m\times n} \tag{13}$$

Step 4 . Construct the weighted decision matrix

$$Y_{ij} = \left\langle \dddot{\tau}_{m1}''', \dddot{\iota}_{m1}''' \dddot{\varphi}_{m1}''' \right\rangle = \omega_j''' \otimes \left\langle \dddot{\tau}_{ij}'', \dddot{\iota}_{ij}'' \dddot{\varphi}_{ij}'' \right\rangle = \left(1-\left(1-\dddot{\tau}_{ij}''\right)^{\omega_j'''}, \dddot{\iota}_{ij}''^{\omega_j'''}, \dddot{\varphi}_{ij}''^{\omega_j'''} \right). \tag{14}$$

Step 5. Compute the RNN BAA (RNN-BAA) matrix G obtained by formula (15).

$$y_j = \left(\prod_{i=1}^{m} Y_{ij}\right)^{1/m} = \left(\left(\prod_{i=1}^{m}\left(\dddot{\tau}_{ij}'''\right)\right)^{1/m}, 1-\left(\prod_{i=1}^{m}\left(\dddot{\iota}_{ij}'''\right)^{1/m}\right), 1-\left(\prod_{i=1}^{m}\left(\dddot{\varphi}_{ij}'''\right)^{1/m}\right) \right) \tag{15}$$

Step 6. Determine the distance of each alternative from BAA. Reckon the distance matrix $\Delta = \left(p_{ij}\right)_{m\times n}$ by the formula (16)

$$p_{ij} = \begin{cases} \delta_e\left(Y_{ij}, n_j\right), if\ Y_{ij} > y_j \\ 0, if\ Y_{ij} = y_j \\ -\delta_e\left(Y_{ij}, y_j\right), if\ Y_{ij} < y_j \end{cases} \tag{16}$$

where Euclidean distance measure $\delta_e\left(Y_{ij}, y_j\right)$ means the distance from Y_{ij} to y_j. It is defined by the formula (5).

Particular case: Alternative A_m''' will pertain to BAA (G) if $p_{ij} = 0$, upper Approximation Area (AA) (G^+), if $p_{ij} > 0$, and lower AA (G^-) if $p_{ij} < 0$.

The upper AA (G^+) refers to the area that includes the ideal alternative (A^+). The lower AA (G^-) refers to the area that includes the anti-ideal alternative (A^-) (see Fig.2.) (Pamučar, Petrović, & Ćirović, 2018). To select A_i''' as the best alternative, it is requisite for it to have as many attributes as possible pertaining to the upper AA (G^+).

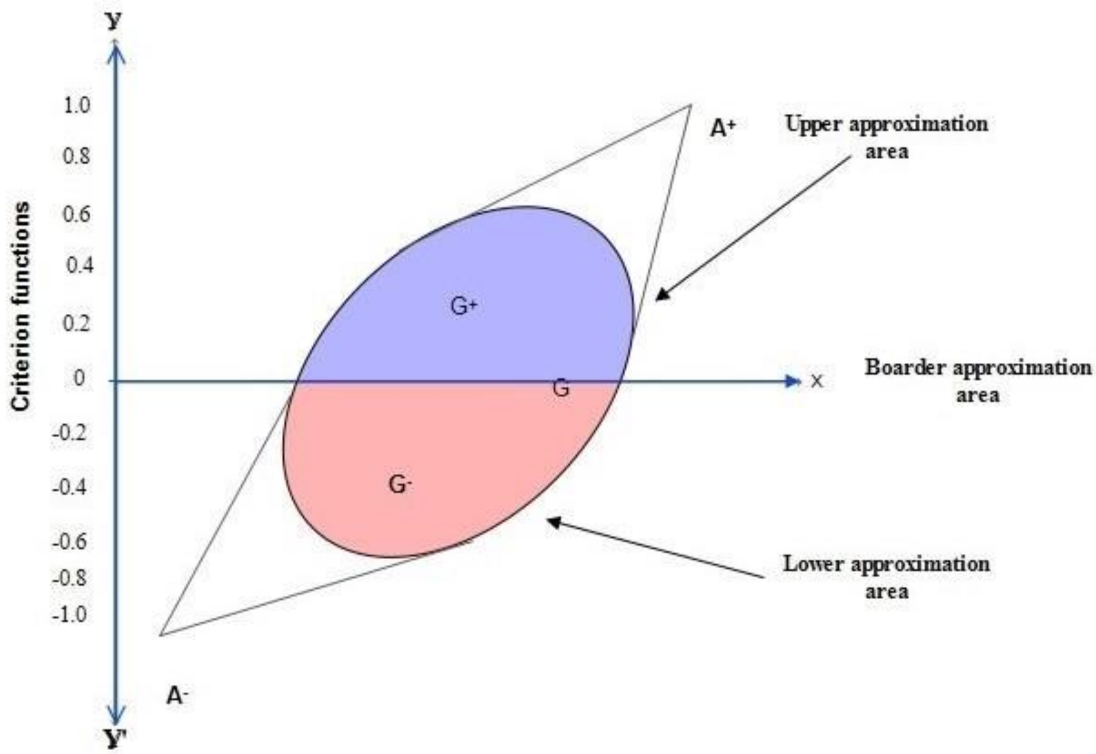

Fig.2. Presentation of the upper (G^+), lower (G^-), and border approximation (G) areas

Step 7. Sort the alternatives by the descending order of the sum of each alternative's distance from BAA

Calculate the sum of the elements of matrix Δ by row. The final evaluating value S_i of alternative A_i''' can be obtained by the formula (17).

$$S_i = \sum_{j=1}^{n} p_{ij}, i = 1,2,...,m \text{ ; j = 1, 2, …, n.} \qquad (17)$$

The ranking of alternatives is done according to the descending order of S_i. The highest value of S_i corresponds to the most desired alternative.

Step 8. End.

4. ILLUSTRATIVE EXAMPLE

Assume that an expert intends to buy the most suitable smartphone from the initially selected smartphones ($\dot{\alpha}_1, \dot{\alpha}_2, \dot{\alpha}_3$). The attributes are:

I. Features $\dot{\chi}_1$,

II. price $\dot{\chi}_2$,

III. customer support $\dot{\chi}_3$ and

IV. risk factor $\dot{\chi}_4$.

Weights of the four attributes are considered as 0.3, .03, 0.3, 0.1 respectively. Based on the developed RNN-MABAC strategy, the problem is solved as follows:

Step 1.

The RNN decision matrix (see Table 1) is formulated based on the rating value of the alternative over the criterion.

Table 1. RNN decision matrix

	$\dot{\chi}_1$ *benefit type*	$\dot{\chi}_2$ *cost type*	$\dot{\chi}_3$ *benefit type*	$\dot{\chi}_4$ *cost type*
$\dot{\alpha}_1$	$\langle (.6,.3,.3), (0.8,0.1,0.1) \rangle$	$\langle (.6,.4,.4), (0.8,0.2,0.2) \rangle$	$\langle (.6,.4,.4), (0.8,0.2,0.4) \rangle$	$\langle (.7,.4,.7), (0.9,0.2,0.1) \rangle$
$\dot{\alpha}_2$	$\langle (0.7,0.3,0.3), (0.9,0.1,0.3) \rangle$	$\langle (0.6,0.3,0.3), (0.8,0.3,0.3) \rangle$	$\langle (0.6,0.2,0.2), (0.8,0.4,0.2) \rangle$	$\langle (0.7,0.3,0.2), (0.9,0.3,0.3) \rangle$
$\dot{\alpha}_3$	$\langle (0.6,0.2,0.2), (0.8,0.0,0.2) \rangle$	$\langle (0.7,0.3,0.2), (0.9,0.1,0.1) \rangle$	$\langle (0.7,0.4,0.6), (0.9,0.2,0.4) \rangle$	$\langle (0.6,0.3,0.2), (0.8,0.1,0.1) \rangle$

Step 2.
Using the formula (10), the RNN decision matrix is converted to the SVNN decision matrix.

Table 2. SVNN decision matrix

	$\dot{\chi}_1$ *benefit type*	$\dot{\chi}_2$ *cost type*	$\dot{\alpha}_3$ *benefit type*	$\dot{\alpha}_4$ *cost type*
$\dot{\alpha}_1$	$\langle 0.69282, 0.1732051, 0.173205 \rangle$	$\langle 0.69282, 0.282843, 0.282843 \rangle$	$\langle 0.69282, 0.282843, 0.4 \rangle$	$\langle 0.793725, 0.282843, 0.264575 \rangle$
$\dot{\alpha}_2$	$\langle 0.793725, 0.1732051, 0.244949 \rangle$	$\langle 0.69282, 0.3, 0.3 \rangle$	$\langle 0.69282, 0.282843, 0.2 \rangle$	$\langle 0.793725, 0.3, 0.244949 \rangle$
$\dot{\alpha}_3$	$\langle 0.69282, 0, 0.2 \rangle$	$\langle 0.793725, 0.173205, 0.141421 \rangle$	$\langle 0.793725, 0.282843, 0.489898 \rangle$	$\langle 0.69282, 0.173205, 0.141421 \rangle$

Step 3.

The SVNN decision matrix is standardized (see Table 3) using the formula (12)

Table 3. Standardized decision matrix

	$\dot{\chi}_1$ *benefit type*	$\dot{\chi}_2$ *cost type*	$\dot{\chi}_3$ *benefit type*	$\dot{\chi}_4$ *cost type*
$\dot{\alpha}_1$	$\langle 0.69282, 0.1732051, 0.173205 \rangle$	$\langle 0.282843 \ 0.717157 \ 0.69282 \rangle$	$\langle 0.69282, 0.282843, 0.4 \rangle$	$\langle 0.264575 \ 0.717157 \ 0.793725 \rangle$
$\dot{\alpha}_2$	$\langle 0.793725, 0.1732051, 0.244949 \rangle$	$\langle 0.3 \ 0.7 \ 0.69282 \rangle$	$\langle 0.69282, 0.282843, 0.2 \rangle$	$\langle 0.264575 \ 0.717157 \ 0.793725 \rangle$
$\dot{\alpha}_3$	$\langle 0.69282, 0, 0.2 \rangle$	$\langle 0.141421 \ 0.826795 \ 0.793725 \rangle$	$\langle 0.793725, 0.282843, 0.489898 \rangle$	$\langle 0.141421 \ 0.826795 \ 0.69282 \rangle$

Step 4.
Using the formula (12), and standardized matrix, the weighted decision matrix is formulated (see table 4).
Table 4. Weighted decision matrix

	$\dot{\chi}_1$ benefit type	$\dot{\chi}_2$ cost type	$\dot{\chi}_3$ benefit type	$\dot{\chi}_4$ cost type
$\dot{\alpha}_1$	$\langle 0.298192922 \quad 0.590974 \quad 0.590974 \rangle$	$\langle 0.094925509 \quad 0.905074 \quad 0.895749 \rangle$	$\langle 0.298193 \quad 0.684642 \quad 0.759658 \rangle$	$\langle 0.030263 \quad 0.967301 \quad 0.977163 \rangle$
$\dot{\alpha}_2$	$\langle 0.377221329 \quad 0.590974 \quad 0.655726 \rangle$	$\langle 0.101476558 \quad 0.898523 \quad 0.895749 \rangle$	$\langle 0.298193 \quad 0.684642 \quad 0.617034 \rangle$	$\langle 0.027706 \quad 0.964961 \quad 0.977163 \rangle$
$\dot{\alpha}_3$	$\langle 0.298192922 \quad 0 \quad 0.617034 \rangle$	$\langle 0.044712655 \quad 0.944538 \quad 0.933042 \rangle$	$\langle 0.377221 \quad 0.684642 \quad 0.807294 \rangle$	$\langle 0.015132 \quad 0.98116 \quad 0.963967 \rangle$

Step 5.

The values of BAA are shown in the BAA matrix $(Z)=\left[\zeta_j\right]_{1\times 4}$ (See Table 5)

Table 5. Computed values of BAA

ζ_1	$\langle 0.322500432, 0.322500432, 0.622191 \rangle$
ζ_2	$\langle 0.075519605, 0.918857, 0.910053 \rangle$
ζ_3	$\langle 0.322500432, 0.684642, 0.739207 \rangle$
ζ_4	$\langle 0.023323601, 0.972156, 0.973414 \rangle$

Step 6. Reckon the distance matrix

Compute the distance matrix (see Table 6) using the formula (16), and score function (6).

For example:

Sc(0.298192922, 0.590974, 0.590974)= 0.62817

Sc(g_1)= 0.64631

Since Sc(g_1)> Sc(η_{ij}), so $g_1 > \eta_{11}$, and $\delta_{11} =$ -0.0851

Table 6.

	$\dot{\chi}_1$ benefit type	$\dot{\chi}_2$ cost type	$\dot{\chi}_3$ benefit type	$\dot{\chi}_4$ cost type
$\dot{\alpha}_1$	-0.085103	0.016033	-0.01834	-0.005348
$\dot{\alpha}_2$	0.089965	0.020751	0.071919	0.005324
$\dot{\alpha}_3$	0.259615	-0.02669	0.050432	0.008896

Step 7. Sort the alternatives

The sum of values of each alternative's δ_{ij} is calculated by the formula (17).

$$S_1 = \sum_{j=1}^{4} \delta_{1j} \; j = 1,2,3,4$$
$$= \delta_{11} + \delta_{12} + \delta_{13} + \delta_{14}$$
$$= (-0.0851) + 0.016033 + (-0.01834) + 0.005348$$
$$= -0.08206$$

Similarly, we derive the other computing results and obtain

S_2=0.18796, S_3=0.283357

So, $S_3 > S_2 > S_1$

Hence, $\dot{\alpha}_3 \succ \dot{\alpha}_2 \succ \dot{\alpha}_1$

So, 3rd alternative is the most suitable smartphone.

5. CONCLUSIONS

In this paper, the RNN-MABAC strategy in the RNN environment is developed. The developed RNN-MABAC strategy can be effectively used to solve real-world MADM problems with inconsistent and incomplete information. We hope that this paper will inspire researchers to conduct research in the field of MADM. The developed RNN- MABAC strategy can be explored for group decision-making strategy using a suitable aggregation operator which we shall do in the future.

The developed RNN-MABAC can be used to solve other MADM problems such as E-commerce site selection (Mallick, Pramanik, & Giri, 2024a), COVID-19 vaccine selection (Mallick, Pramanik, & Giri, 2024b), green supplier selection problem (Pramanik, 2023), etc.

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The field of neutrosophic set theory and its applications has been rapidly expanding, particularly since the introduction of the journal "Neutrosophic Sets and Systems." New theories, techniques, and algorithms are being developed at a very high rate. One of the most notable trends in neutrosophic theory is its hybridization with other set theories such as rough set theory, bipolar set theory, soft set theory, hesitant fuzzy set theory, and more. Various hybrid structures like rough neutrosophic sets, neutrosophic soft set, single valued neutrosophic hesitant fuzzy sets, among others, have been proposed in a short period. Neutrosophic sets have proven to be crucial tools across a wide array of fields including data mining, decision making, e-learning, engineering, medical diagnosis, social sciences, and beyond.

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