Florentin Smarandache
Surapati Pramanik
(Editors)

New Trends in Neutrosophic Theory and Applications

Volume IV

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Emeritus Professor Florentin Smarandache, PhD, Postdoc, Mathematics, Physical and Natural Sciences Division, University of New Mexico, USA. He is the founder of neutrosophy, a new branch of philosophy. He is a Chief Editor, and reviewer for several international journals. He published over 900 scientific papers and over 400 books in mathematical sciences, psychology, sociology, as well as literary works such as poetry, stories, essays, a novel, translations, dramas, plays for children, folklore, and albums of arts. He has been enlisted in the World's Top cited 2% Scientists (prepared by Stanford University, USA in association with Elsevier BV).

Dr. Surapati Pramanik has been working as an assistant professor in mathematics since 2006 at Nandalal Ghosh B.T. College, India. His research focuses on fuzzy Multi-Criteria Decision Making (MCDM), soft computing, and neutrosophic hybrid sets. He is a Chief Editor, Associate Editor, and reviewer for several international journals. His publication includes 37 book chapters, 165 research papers, and 3 editorial books. He received outstanding paper award in WBSSTC 2011 in Mathematics, WBSSTC-2019 in Social Science He is enlisted in the World's Top cited 2% Scientists, 2023 & 2022 (prepared by Stanford University, USA in association with Elsevier BV).

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Aims and Scope

The domain of neutrosophic set theory has been undergoing significant and rapid expansion. New approaches, theories, strategies, and optimization techniques are developed at a swift rate. An important development in neutrosophic theory is its integration with existing set theories, such as soft set theory, rough set theory and more. Neutrosophic set has been extended to neutrosophic hyperset which is vast and extends across multiple disciplines and problem domains, from decision support and Artificial Intelligence (AI) to data analysis and complex systems modelling. A neutrosophic hypersoft set extends the flexibility of neutrosophic logic and soft set theory by considering multidimensional or hypersoft sets to address real-world complexities where information is incomplete, uncertain, or inconsistent. Neutrosophic theories have emerged as essential tools in a diverse range of disciplines, including, MCDM, data mining, biomedical research, social studies, and beyond.

The Book "New Trends in Neutrosophic Theories and Applications, Vol. 4, 2025" focuses on theories, strategies, optimizing techniques for MCDM within neutrosophic frameworks. Some topics deal with introducing of Pythagorean hypersoft sets with possibility degree, quadripartitioned neutrosophic Lie-ideal of Lie-algebra, quadripartitioned neutrosophic quasi coincident topological space, neutrosophic supra β-open set in neutrosophic supra topological space and neutrosophic soft matrices. Some topics deal with medical diagnosis, organ transplantation success using neutrosophic superhyperstructure and artificial intelligence. Some topics deal with revenue management. social situation. Some topics deal with MCDM in single valued neutrosophic set environment, rough set environment, and interval trapezoidal neutrosophic environment.

Florentin Smarandache, Surapati Pramanik

Foreword

Professor Florentin Smarandache grounded the neutrosophic set theory (NST) to deal with uncertainty, indeterminacy, and inconsistent information. NSTs represent a powerful extension of traditional set theory, providing a more nuanced approach to handling various forms of uncertainty and vagueness. NSTs are particularly useful in situations involving indeterminate, imprecise, incomplete, and inconsistent information, making them applicable across several fields.

The key distinction of NST lies in their ability to model not just truth, but also indeterminacy and falsity—allowing them to account for situations where the information is not fully clear or is contradictory. This makes them highly relevant in areas like Artificial Intelligence (AI), cognitive science, and machine learning, where data is often messy and uncertain.

Applications of NSTs in the fields such as data mining, decision analysis, expert systems, and pattern recognition have opened up new possibilities for more robust and flexible systems. In data mining, NSTs can help uncover patterns or relationships in data that traditional methods might miss due to inconsistencies or incomplete information. Similarly, in decision analysis NSTs allow for better handling of contradictory information when making decisions under uncertainty. Due to their broad relevance, the publication of the Journal "*Neutrosophic Sets and Systems*" in 2013 played a key role in sparking global interest and research in NSTs.

Chapter 1 presents the Single-Valued Quadripartitioned Neutrosophic Lie- Algebra (SVQNLA). of Lie-algebra**.** It explores the Single-Valued Quadripartitioned Neutrosophic Lie ideal (SVQNLI) of SVQNLA. It formulates several theorems, and propositions on SVQNLA and SVQNLI.

Chapter 2 investigates the concept of Quadripartitioned Neutrosophic Point (QNP) in a Quadripartitioned Neutrosophic Set (QNS), quadripartitioned neutrosophic quasi coincident with quadripartitioned neutrosophic set and quadripartitioned neutrosophic point. It establishes various properties of quasi coincident in quadripartitioned neutrosophic set relations. It presents the quasi coincident topological property in which the degree of nearness or coincidence between quadripartitioned neutrosophic sets in a Quadripartitioned Neutrosophic Topological (QNT) space.

Chapter 3 presents the Neutrosophic Supra β-Open Set (NS-β-O-S) Via Neutrosophic Supra Topological Space (NSTS) as an expansion of Neutrosophic Supra α-Open Set (NS-α-O-S). It establishes several results on NS-β-O-S via NSTS.

Chapter 4 presents some special neutrosophic soft matrices. **S**ome operations and some associated properties of neutrosophic soft matrices are discussed to make the concept clear.

Chapter 5 introduces a new algebraic structure, the Interval-Valued Neutrosophic Fuzzy *M*-Semigroup (IVNFMS), by merging the notions of Interval-Valued Fuzzy *M*-Semigroups(IVFMSs) and Neutrosophic Fuzzy Sets (NFSs). It deals with direct product, image and inverse image between two IVNFMSs. It establishes some related results.

Chapter 6 presents the concept of independence and dependence among the indices of fuzzy, intuitionistic fuzzy and neutrosophic sets. Further, the degree of dependence is studied that helps to make more informed decisions while modeling real-world problems. These concepts are then extended to define linear dependence and independence of indices in refined neutrosophic sets.

Chapter 7 introduces the Neutro-Genetic Hidden Markov Model (NG-HMM) that combines neutrosophic logic with Hidden Markov Models (HMM) for genomic analysis. The NG-HMM assigns neutrosophic values to genetic states, transition probabilities, and emissions, allowing the model to capture complex genetic interactions and uncertain mutations, often encountered in personalized medicine and risk prediction.

Chapter 8 introduces a neutrosophic framework to evaluate social situations, highlighting the fluid, subjective, and indefinite nature of societal norms and moral assessments. Utilizing neutrosophic measures and statistics, it studies paradoxes in societal norms and argues that the neutrosophic strategy offers a more comprehensive way of modeling social behaviors and examining the evolving and often conflicting nature of social norms across different times, cultures, and populations.

Chapter 9 presents the alpha cut at different levels, including lower, middle, and higher levels, which control the degrees of membership within the neutrosophic set. The distance between the parachute diver and ground level is determined using Harfa analysis and pixel profile to determine the distance and how long it takes to land.

Chapter 10 presents an analysis of Gaussian neutrosophic sets, a mathematical framework for handling uncertainty, Kurtosis which describes the shape of probability distributions especially the tails, and Gaussian semantic security ensuring data confidentiality in cryptographic applications within the context of medical diagnosis.

Chapter 11 presents a new framework combining Neutrosophic SuperHyperStructure with artificial intelligence methods for improved transplant decision-making. The proposed model utilizes long short-term memory networks for organ rejection prediction and reinforcement learning for dynamic optimization of donor-recipient matching. Comparative analysis, and. Sensitivity analysis are presented.

Chapter12 introduces a new approach to quantifying the inherent uncertainty in pre-PhD anxiety among research aspirants using neutrosophic set theory. It develops a neutrosophic anxiety index that captures the multidimensional nature of academic uncertainty, imposter syndrome, and research preparedness concerns. The developed model demonstrates superior representational capacity compared to traditional fuzzy logic approaches when applied to survey data from 245 prospective PhD students across diverse disciplines.

Chapter 13 develops a ranking algorithm within the neutrosophic domain to identify the optimal An Internet Service Provider (ISP). It proposes a mathematical model to determine the most cost-effective policy for an ISP. To validate the model, an illustrative example of an ISP problem is solved.

Chapter 14 conjoins the aspects of Plithogenic hypersoft sets, Pythagorean sets and possibility theory. It proposes an integrated decision model and which is applied to the selection-based decision-making problem of waste management.

Chapter 15 introduces a soft model for supplier selection that integrates multi-attribute decision-making (MADM) with mathematical programming. The proposed model offers a practical and adaptable approach for supplier selection, with potential applications across various industries.

Chapter 16 develops the MABAC strategy in a rough neutrosophic numbers environment. which is termed the RNN-MABAC strategy. The developed strategy is illustrated by solving an illustrative MADM problem.

Chapter 17 develops the TODIM strategy for Multi Criteria Decision Making (MCDM) in Interval Trapezoidal Neutrosophic Number (ITrNN) environment. It defines define a score function and an accuracy function for ITrNNs and prove some of their basic properties. It solves a MCDM problem to illustrate the developed TODIM strategy in ITrNN environment.

Florentin Smarandache, Surapati Pramanik
(Editors)

Chapter 1

Single-Valued Quadripartitioned Neutrosophic Lie-Ideal of Lie-Algebra

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ABSTRACT

This study aims to procure the notion of Single-Valued Quadripartitioned Neutrosophic Lie algebra (SVQNLA). In addition, we explore the Single-Valued Quadripartitioned Neutrosophic Lie ideal (SVQNLI) of SVQNLA. Further, we formulate several interesting results in the form of theorem, proposition, etc. on SVQNLA and SVQNLI.

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Keywords: SVNS, SVQNS, SVQNLA, SVQNLI.

INTRODUCTION

Lie groups were first conceptualized in the nineteenth century by Sophus Lie. Lie algebra (LA) was developed by Sophus Lie as well. The LA representation theory was introduced by Humphreys [1] in 1972. Coelho and Nunes [2] presented a LA application for mobile robot control in 2003. Lie theory has applications across various fields, including physics, cosmology, life sciences, mathematics, and continuum mechanics. The concept of LA may also be used to solve computer vision difficulties. The pioneering work of Zadeh in 1965 on the Fuzzy Set (FS) [3] theory laid the groundwork for applying fuzziness to various mathematical structures. Building on this, Yehia [4] introduced the concepts of Fuzzy Lie-Sub-Algebras and Fuzzy Lie-Ideals in the context of LA in 1996, and expanded further by examining the adjoint representation of Fuzzy LA [5]. In the years following, Akram [6] explored the notion of anti-fuzzy Lie-Ideals in LA, while Akram [7] introduced generalized Fuzzy Lie-Sub-Algebras.

Later, Akram [8] advanced the theory of Fuzzy Lie-Ideals by introducing interval-valued membership functions.

In parallel to fuzzy theories, Atanassov [9] introduced the theory of Intuitionistic Fuzzy Sets (IFS), which incorporated the idea of non-membership, offering a broader framework for dealing with uncertainty in mathematical expressions. This concept was extended to Lie algebra by Akram and Shum [10], and later, Akram [11] developed the theory of Intuitionistic (S, T)-Fuzzy Lie-Ideals. In 2000, Smarandache [12] introduced the Neutrosophic Set (NS) theory, which includes the idea of indeterminacy in membership functions, significantly enhancing the ability to handle uncertainty. Wang et al. [13] further expanded this by developing the Single-Valued Neutrosophic Set (SVNS), which extended the FS and IFS concepts into a more flexible form. Theoretical advancements and practical implementations of NSs [12], SVNSs [13], and associated concepts have been thoroughly explored and documented in various studies [14-20]. Several studiers [21-40] leveraged the SVNS environment for multi-criteria decision-making, demonstrating its applicability in complex decision processes. Following this, Akram et al. [41] proposed the notion of Single-Valued Neutrosophic LA, contributing to the further integration of Neutrosophic Sets with algebraic structures. Das and Khalid [42] also explored d-ideals within the framework of Neutrosophic Sets, leading to the development of new algebraic structures. Chatterjee et al. [43] developed the theory of the Single-Valued Quadripartitioned Neutrosophic Set (SVQNS), expanding the range of Neutrosophic Sets even further. More recently, Das et al. [44] proposed the Pentapartitioned Neutrosophic Q-Ideal, broadening the scope of Neutrosophic algebraic structures in complex environments.

This article introduces the novel concept of SVQNLI of SVQNLA, expanding the applications of Neutrosophic theory within the framework of Lie algebra. We have formulated several significant results about the SVQNLI of SVQNLA, contributing to the growing body of work in this area.

The layout of this article is given below:

Section	Content
1	Introduction.
2	Presents some basic definitions and results on SVNS, SVQNS, LA, Lie-Ideal, Neutrosophic LA, and Neutrosophic Lie-Ideal.
3	Presents the concepts of SVQNLA and SVQNLI, and established some results on them.
4	Conclude the article, and states some directions for further research.

BACKGROUND

Throughout the section, we discuss several preliminary definitions and findings about SVQNLI that will be beneficial when preparing the key findings of this article.

Definition 1. [1] Let F be a field. Consider a vector space $\mathfrak{G}$ over F on which $\mathfrak{G}\times\mathfrak{G}\rightarrow\mathfrak{G}$ is defined by $(q, r) \rightarrow [\hbar, u]$ for all $\hbar, u \in \mathfrak{G}$. Then, $\mathfrak{G}$ is said to be Lie Algebra (LA) provided that the following axioms are satisfied:

(*i*) [q, r] is a bilinear,

(*ii*) $[q, q] = 0$, for all $q \in \mathfrak{G}$,

(*iii*) $[[q, r], s] + [[r, s], q] + [[s, q], r] = 0$, for all $q, r, s \in \mathfrak{G}$.

Remark 1. [1] The associative property was not held in a Lie algebra for the multiplication operation, i.e., $[[q, r], s] = [q, [r, s]]$ is not true in general for a Lie algebra. But the Lie algebra is anti-commutative, i.e., $[q, r] = -[r, q]$. A sub-set N of a Lie algebra $\mathfrak{G}$, which is closed under $[\cdot, \cdot]$ is referred to as a Lie sub-algebra.

Definition 2. [3] A **fuzzy set (FS)** Z in the universal set W is expressed as:

$$Z = \{(\eta, T_Z(\eta)) : \eta \in W\},$$

where $T_Z(\eta)$ denotes the truth-membership value of η within W, constrained by the condition $0 \leq T_Z(\eta) \leq 1$.

Definition 3. [4] A **FS** $Z = \{(\eta, T_Z(\eta)) : \eta \in \mathfrak{G}\}$ is described as a **Fuzzy Lie ideal within a Lie algebra $\mathfrak{G}$ is characterized if the subsequent three criteria** are satisfied:

(*i*) $T_Z(t + s) \geq \min\{T_Z(t), T_Z(s)\}$;

(*ii*) $T_Z(\alpha t) \geq T_Z(t)$;

(*iii*) $T_Z([t, s]) \geq T_Z(t)$, $\forall\, \alpha \in F$ and $t, s \in \mathfrak{G}$.

Definition 4. [13] A Single-Valued Neutrosophic Set (SVNS) Z over W is characterized as $Z = \{(\eta, T_Z(\eta), I_Z(\eta), F_Z(\eta)) : \eta \in W\}$, where T_Z, I_Z, F_Z represent the truth, indeterminacy, and falsity membership functions, respectively. These functions map each element $\eta \in W$ to values in the interval [0, 1], indicating the degrees of truth, uncertainty, and falsity associated with η in the set Z. The membership values are not mutually exclusive and satisfy the condition $0 \leq T_Z(\eta) + I_Z(\eta) + F_Z(\eta) \leq 3$, $\forall\, \eta \in W$.

Definition 5. [13] Let $Y = \{(s, T_Y(s), I_Y(s), F_Y(s)): s \in W\}$ represent an SVNS over W. Then, $Z(T_Y, \alpha) = \{s \in W: T_Y(s) \geq \alpha\}$, $Z(I_Y, \alpha) = \{s \in W: I_Y(s) \leq \alpha\}$, $Z(F_Y, \alpha) = \{s \in W : F_Y(s) \leq \alpha\}$ are respectively referred to as the T-level, I-level, and F-level α-cuts of Y.

Definition 6. [41] A Single-Valued Neutrosophic Set $Z = \{(t, T_Z(t), I_Z(t), F_Z(t)) : t \in \mathfrak{G}\}$ on Lie algebra $\mathfrak{G}$ is said to be a Single-Valued Neutrosophic Lie algebra if the following condition holds:

(*i*) $T_Z(t + s) \geq \min\{T_Z(t), T_Z(s)\}$, $I_Z(t + s) \geq \min\{I_Z(t), I_Z(s)\}$, and $F_Z(t + s) \leq \max\{F_Z(t), F_Z(s)\}$;

(*ii*) $T_Z(\alpha t) \geq T_Z(t)$, $I_Z(\alpha t) \geq I_Z(t)$, and $F_Z(\alpha t) \leq F_Z(t)$;

(*iii*) $T_Z([t, s]) \geq \min\{T_Z(t), T_Z(s)\}$, $I_Z([t, s]) \geq \min\{I_Z(t), I_Z(s)\}$ and $F_Z([t, s]) \leq \max\{F_Z(t), F_Z(s)\}$, $\forall$ t, s ∈ 𝔊 and α ∈F.

Definition 7. [41] Let 𝔊 be a LA over a field F. An SVNS $Z = \{(t, T_Z(t), I_Z(t), F_Z(t)) : t \in 𝔊\}$ on 𝔊 is said to be a single-valued neutrosophic Lie ideal if the following axioms hold:

(i) $T_Z(s + t) \geq \min\{T_Z(s), T_Z(t)\}$, $I_Z(s + t) \geq \min\{I_Z(s), I_Z(t)\}$, and $F_Z(s + t) \leq \max\{F_Z(s), F_Z(t)\}$;

(ii) $T_Z(\alpha t) \geq T_Z(t)$, $I_Z(\alpha t) \geq I_Z(t)$, and $F_Z(\alpha t) \leq F_Z(t)$;

(iii) $T_Z([s, t]) \geq T_Z(s)$, $I_Z([s, t]) \geq I_Z(s)$, and $F_Z([s, t]) \leq F_Z(s)$, $\forall$ s, t ∈ 𝔊.

Remark 2. [41] Let $Z = \{(ɗ, T_Z(ɗ), I_Z(ɗ), F_Z(ɗ)) : ɗ \in 𝔊\}$ be a Single-Valued Neutrosophic LA on a Lie Algebra 𝔊. Then,

(i) $T_Z(0) \geq T_Z(ɗ)$, $I_Z(0) \geq I_Z(ɗ)$, $F_Z(0) \leq F_Z(ɗ)$;

(ii) $T_Z(-ɗ) \geq T_Z(ɗ)$, $I_Z(-ɗ) \geq I_Z(ɗ)$, $F_Z(-ɗ) \leq F_Z(ɗ)$, $\forall$ ɗ ∈ 𝔊.

Definition 8. [43] A Single-Valued Quadripartitioned Neutrosophic Set (SVQNS) Z over the universal set W is defined as follows:

$$Z = \{(ɗ, T_Z(ɗ), C_Z(ɗ), G_Z(ɗ), F_Z(ɗ)) : ɗ \in W\},$$

where $T_Z(ɗ)$, $C_Z(ɗ)$, $G_Z(ɗ)$ and $F_Z(ɗ)$ ($\in [0, 1]$) are the truth, contradiction, ignorance, and false membership values of ɗ ∈ W. So, $0 \leq T_Z(ɗ) + C_Z(ɗ) + G_Z(ɗ) + F_Z(ɗ) \leq 4$, $\forall$ ɗ ∈ W.

Definition 9. [43] Assume that $Z = \{(\eta, T_Z(\eta), C_Z(\eta), G_Z(\eta), F_Z(\eta)): \eta \in W\}$ and $Y = \{(\eta, T_Y(\eta), C_Y(\eta), G_Y(\eta), F_Y(\eta)): \eta \in W\}$ are two Single-Valued Quadripartitioned Neutrosophic Sets over W. Then,

(i) $Z \subseteq Y$ if and only if $T_Z(\eta) \leq T_Y(\eta)$, $C_Z(\eta) \leq C_Y(\eta)$, $G_Z(\eta) \geq G_Y(\eta)$, $F_Z(\eta) \geq F_Y(\eta)$, $\forall \eta \in W$.

(ii) $Z \cup Y = \{(\eta, \max\{T_Z(\eta), T_Y(\eta)\}, \max\{C_Z(\eta), C_Y(\eta)\}, \min\{G_Z(\eta), G_Y(\eta)\}, \min\{F_Z(\eta), F_Y(\eta)\} : \eta \in W\}$.

(iii) $Z^c = \{(\eta, F_Z(\eta), G_Z(\eta), C_Z(\eta), T_Z(\eta)) : \eta \in W\}$.

(iv) $Z \cap Y = \{(\eta, \min\{T_Z(\eta), T_Y(\eta)\}, \min\{C_Z(\eta), C_Y(\eta)\}, \max\{G_Z(\eta), G_Y(\eta)\}, \max\{F_Z(\eta), F_Y(\eta)\}) : \eta \in W\}$.

Single-Valued Quadripartitioned Neutrosophic Lie-Ideal

Throughout this section, we grounded the notion of SVQNLI of SVQNLA. Furthermore, we explore various properties of SVQNLI and establish multiple results related to it. Let us consider $\mathfrak{G}$ be a Lie Algebra over a field F.

Definition 10. An SVQNS $Z = \{(\eta, T_Z(\eta), C_Z(\eta), G_Z(\eta), F_Z(\eta)) : \eta \in W\}$ on $\mathfrak{G}$ is said to be an SVQNLA if the subsequent axioms are satisfied:

(i) $T_Z(\eta + \delta) \geq \min \{T_Z(\eta), T_Z(\delta)\}$, $C_Z(\eta + \delta) \geq \min \{C_Z(\eta), C_Z(\delta)\}$, $G_Z(\eta + \delta) \leq \max \{G_Z(\eta), G_Z(\delta)\}$, and $F_Z(\eta + \delta) \leq \max \{F_Z(\eta), F_Z(\delta)\}$;

(ii) $T_Z(\alpha\eta) \geq T_Z(\eta)$, $C_Z(\alpha\eta) \geq C_Z(\eta)$, $G_Z(\alpha\eta) \leq G_Z(\eta)$, and $F_Z(\alpha\eta) \leq F_Z(\eta)$;

(iii) $T_Z([\eta, \delta]) \geq \min \{T_Z(\eta), T_Z(\delta)\}$, $C_Z([\eta, \delta]) \geq \min \{C_Z(\eta), C_Z(\delta)\}$, $G_Z([\eta, \delta]) \leq \max \{G_Z(\eta), G_Z(\delta)\}$ and $F_Z([\eta, \delta]) \leq \max \{F_Z(\eta), F_Z(\delta)\}$, for all $\eta, \delta \in \mathfrak{G}$ and $\alpha \in F$.

Definition 11. An SVQNS $Z = \{(\eta, T_Z(\eta), C_Z(\eta), G_Z(\eta), F_Z(\eta)) : \eta \in W\}$ on $\mathfrak{G}$ is said to be an SVQNLI if the subsequent axioms are satisfied:

(i) $T_Z(\eta + \delta) \geq \min \{T_Z(\eta), T_Z(\delta)\}$, $C_Z(\eta + \delta) \geq \min \{C_Z(\eta), C_Z(\delta)\}$, $G_Z(\eta + \delta) \leq \max \{G_Z(\eta), G_Z(\delta)\}$ and $F_Z(\eta + \delta) \leq \max \{F_Z(\eta), F_Z(\delta)\}$;

(ii) $T_Z(\alpha\eta) \geq T_Z(\eta)$, $C_Z(\alpha\eta) \geq C_Z(\eta)$, $G_Z(\alpha\eta) \leq G_Z(\eta)$ and $F_Z(\alpha\eta) \leq F_Z(\eta)$;

(iii) $T_Z([\eta, \delta]) \geq T_Z(\eta)$, $C_Z([\eta, \delta]) \geq C_Z(\eta)$, $G_Z([\eta, \delta]) \leq G_Z(\eta)$ and $F_Z([\eta, \delta]) \leq F_Z(\eta)$, for all $\eta, \delta \in \mathfrak{G}$.

Theorem 1. Let $\{Z_i : i \in \Delta\}$ denote the collection of SVQNLIs over $\mathfrak{G}$. Then, $\cap Z_i = \{(\eta, \wedge T_{N_i}(\eta), \wedge C_{N_i}(\eta), \vee G_{N_i}(\eta), \vee F_{N_i}(\eta)) : \eta \in \mathfrak{G}\}$ is also an SVQNLI of $\mathfrak{G}$.

Proof. Let $\{Z_i : i \in \Delta\}$ be the collection of SVQNLIs on $\mathfrak{G}$. It is known that, $\cap Z_i = \{(\eta, \wedge T_{N_i}(\eta), \wedge C_{N_i}(\eta), \vee G_{N_i}(\eta), \vee F_{N_i}(\eta)) : \eta \in \mathfrak{G}\}$.

Now, we have

(i) $\wedge T_{N_i}(\eta + \delta)$

$= \min \{T_{N_i}(\eta + \delta) : i \in \Delta\}$

$\geq \min \{\min \{T_{N_i}(\eta), T_{N_i}(\delta)\} : i \in \Delta\}$

$\geq \min \{\wedge T_{N_i}(\eta), \wedge T_{N_i}(\delta)\}$,

$\wedge C_{N_i}(\eta + \delta)$

$= \min \{C_{N_i}(\eta + \delta) : i \in \Delta\}$

$\geq \min \{\min \{C_{N_i}(\eta), C_{N_i}(\delta)\} : i \in \Delta\}$

$\geq \min \{\wedge C_{N_i}(\eta), \wedge C_{N_i}(\delta)\}$,

$\vee G_{N_i}(\eta + \delta)$

$= \max \{G_{N_i}(\eta + \delta): i \in \Delta\}$

$\leq \max \{\max \{G_{N_i}(\eta), G_{N_i}(\delta)\}: i \in \Delta\}$

$\leq \max \{\vee G_{N_i}(\eta), \vee G_{N_i}(\delta)\}$,

$\vee F_{N_i}(\eta + \delta)$

$= \max \{F_{N_i}(\eta + \delta) : i \in \Delta\}$

$\leq \max \{\max \{F_{N_i}(\eta), F_{N_i}(\delta)\} : i \in \Delta\}$

$\leq \max \{\vee F_{N_i}(\eta), \vee F_{N_i}(\delta)\}$.

(ii) $\wedge T_{N_i}(\alpha\eta) = \min \{T_{N_i}(\alpha\eta) : i \in \Delta\} \geq \min \{T_{N_i}(\eta) : i \in \Delta\} \geq \wedge T_{N_i}(\eta)$,

$\wedge C_{N_i}(\alpha\eta) = \min \{C_{N_i}(\alpha\eta) : i \in \Delta\} \geq \min \{C_{N_i}(\eta) : i \in \Delta\} \geq \wedge C_{N_i}(\eta)$,

$\vee G_{N_i}(\alpha\eta) = \max \{G_{N_i}(\alpha\eta) : i \in \Delta\} \leq \max \{G_{N_i}(\eta) : i \in \Delta\} \leq \vee G_{N_i}(\eta)$,

$\vee F_{N_i}(\alpha\eta) = \max \{F_{N_i}(\alpha\eta) : i \in \Delta\} \leq \max \{F_{N_i}(\eta) : i \in \Delta\} \leq \vee F_{N_i}(\eta)$.

(iii) $\wedge T_{N_i}([\eta, \delta]) = \min \{T_{N_i}([\eta, \delta]) : i \in \Delta\} \geq \min \{T_{N_i}(\eta) : i \in \Delta\} \geq \wedge T_{N_i}(\eta)$,

$\wedge C_{N_i}([\eta, \delta]) = \min \{C_{N_i}([\eta, \delta]) : i \in \Delta\} \geq \min \{C_{N_i}(\eta) : i \in \Delta\} \geq \wedge C_{N_i}(\eta)$,

$\vee G_{N_i}([\eta, \delta]) = \max \{G_{N_i}([\eta, \delta]) : i \in \Delta\} \leq \max \{G_{N_i}(\eta) : i \in \Delta\} \leq \vee G_{N_i}(\eta)$,

$\vee F_{N_i}([\eta, \delta]) = \max \{F_{N_i}([\eta, \delta]) : i \in \Delta\} \leq \max \{F_{N_i}(\eta) : i \in \Delta\} \leq \vee F_{N_i}(\eta)$.

Therefore, $\cap Z_i = \{(\eta, \wedge T_{N_i}(\eta), \wedge C_{N_i}(\eta), \vee G_{N_i}(\eta), \vee F_{N_i}(\eta)) : \eta \in \mathfrak{G}\}$ is an SVQNLI of $\mathfrak{G}$.

Theorem 2. Let $Z = \{(\eta, T_Z(\eta), C_Z(\eta), G_Z(\eta), F_Z(\eta)) : \eta \in \mathfrak{G}\}$ be an SVQNLA on $\mathfrak{G}$. Then,

(i) $T_Z(0) \geq T_Z(\delta)$, $C_Z(0) \geq C_Z(\delta)$, $G_Z(0) \leq G_Z(\delta)$, $F_Z(0) \leq F_Z(\delta)$;

(ii) $T_Z(-\delta) \geq T_Z(\delta)$, $C_Z(-\delta) \geq C_Z(\delta)$, $G_Z(-\delta) \leq G_Z(\delta)$, $F_Z(-\delta) \leq F_Z(\delta)$, for all $\delta \in \mathfrak{G}$.

Proof. This follows straightforwardly from Definition 11.

Remark 3. Every SVQNLI is an SVQNLA.

Theorem 3. Suppose that $Z = \{(\delta, T_Z(\delta), C_Z(\delta), G_Z(\delta), F_Z(\delta)) : \delta \in \mathfrak{G}\}$ be an SVQNLI of $\mathfrak{G}$. Then,

(i) $T_Z([\delta, \eta]) \geq \max \{T_Z(\delta), T_Z(\eta)\}$;

(ii) $C_Z([\delta, \eta]) \geq \max \{C_Z(\delta), C_Z(\eta)\}$;

(iii) $G_Z([\delta, \eta]) \leq \min \{G_Z(\delta), G_Z(\eta)\}$;

(iv) $F_Z([\delta, \eta]) \leq \min \{F_Z(\delta), F_Z(\eta)\}$;

(v) $T_Z([\delta, \eta]) = T_Z(-[\delta, \delta]) = T_Z([\eta, \delta])$;

(vi) $C_Z([\delta, \eta]) = C_Z(-[\delta, \delta]) = C_Z([\eta, \delta])$;

(vii) $G_Z([\delta, \eta]) = G_Z(-[\delta, \delta]) = G_Z([\eta, \delta])$;

(viii) $F_Z([\delta, \eta]) = F_Z(-[\delta, \delta]) = F_Z([\eta, \delta])$, for all $\delta, \eta \in \mathfrak{G}$.

Proof. This follows straightforwardly from Definition 11.

Definition 12. Let $Z = \{(\delta, T_Z(\delta), C_Z(\delta), G_Z(\delta), F_Z(\delta)) : \delta \in \mathfrak{G}\}$ be an SVQNS over $\mathfrak{G}$ and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$. Then, $\mathfrak{G}(T_Z, \alpha_1) = \{\eta \in \mathfrak{G} : T_Z(\eta) \geq \alpha_1\}$, $\mathfrak{G}(C_Z, \alpha_2) = \{\eta \in \mathfrak{G} : C_Z(\eta) \geq \alpha_2\}$, $\mathfrak{G}(G_Z, \alpha_3) = \{\eta \in \mathfrak{G} : G_Z(\eta) \leq \alpha_3\}$, and $\mathfrak{G}(F_Z, \alpha_4) = \{\eta \in \mathfrak{G} : F_Z(\eta) \leq \alpha_4\}$ are respectively said to be T-level α_1-cut, C-level α_2-cut, G-level α_3-cut, F-level α_4-cut of Z.

Definition 13. Let Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)): η ∈ 𝔊} be an SVQNS over 𝔊 and $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$. Then, ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$)-level subset of Z is defined by:

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \{\eta \in \ : T_Z(\eta) \geq \alpha_1, C_Z(\eta) \geq \alpha_2, G_Z(\eta) \leq \alpha_3, G_Z(\eta) \leq \alpha_4\}.$$

Remark 4. If Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ 𝔊} be an SVQNS over 𝔊, then 𝔊($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) = 𝔊($T_Z, \alpha_1$) ∩ 𝔊($C_Z, \alpha_2$) ∩ 𝔊($G_Z, \alpha_3$) ∩ 𝔊($F_Z, \alpha_4$).

Proposition 1. An SVQNS Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ 𝔊} is an SVQNLI of 𝔊 iff 𝔊($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) is a Lie-Ideal of 𝔊 for each $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in [0, 1]$.

Proof. This follows straightforwardly from Definitions 11 and113.

Theorem 4. Assume that Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ 𝔊} be an SVQNLI of 𝔊. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \in [0, 1]$. Then, ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) = ($\beta_1, \beta_2, \beta_3, \beta_4$) iff $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$.

Proof. Suppose that 𝔊 be a LA and Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ 𝔊} be an SVQNLI of 𝔊. Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4 \in [0, 1]$ such that ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) = ($\beta_1, \beta_2, \beta_3, \beta_4$). Therefore, {η ∈ 𝔊: $T_Z(\eta) \geq \alpha_1$, $C_Z(\eta) \geq \alpha_2$, $G_Z(\eta) \leq \alpha_3$, $G_Z(\eta) \leq \alpha_4$} = {η ∈ 𝔊 : $T_Z(\eta) \geq \beta_1$, $C_Z(\eta) \geq \beta_2$, $G_Z(\eta) \leq \beta_3$, $G_Z(\eta) \leq \beta_4$}. This is possible only when $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$. Therefore, ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) = ($\beta_1, \beta_2, \beta_3, \beta_4$) implies $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$.

Conversely, let $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$.

Now, ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$)

= {η ∈ : $T_Z(\eta) \geq \alpha_1$, $C_Z(\eta) \geq \alpha_2$, $G_Z(\eta) \leq \alpha_3$, $G_Z(\eta) \leq \alpha_4$}

= {η ∈ : $T_Z(\eta) \geq \beta_1$, $C_Z(\eta) \geq \beta_2$, $G_Z(\eta) \leq \beta_3$, $G_Z(\eta) \leq \beta_4$}

= ($\beta_1, \beta_2, \beta_3, \beta_4$).

Therefore, $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$, $\alpha_4 = \beta_4$ implies ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$) = ($\beta_1, \beta_2, \beta_3, \beta_4$).

Definition 14. Let $𝔊_1$ and $𝔊_2$ represent two LAs over a field F and ƌ represent a bijective mapping from $𝔊_1$ to $𝔊_2$. If Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ $𝔊_2$} be an SVQNS in $𝔊_2$, then $ƌ^{-1}$(Z) defined by $ƌ^{-1}$(Z) = {(η, $ƌ^{-1}$(T_Z(η)), $ƌ^{-1}$(C_Z(η)), $ƌ^{-1}$(G_Z(η)), $ƌ^{-1}$(F_Z(η))) : η ∈ $𝔊_1$} is also an SVQNS in $𝔊_1$.

Theorem 5. Let $𝔊_1$ and $𝔊_2$ represent two LAs on a field F and ƌ be an onto homomorphism from $𝔊_1$ to $𝔊_2$. If Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ $𝔊_2$} is an SVQNLI of $𝔊_2$, then $ƌ^{-1}$(Z) = {(η, $ƌ^{-1}$(T_Z(η)), $ƌ^{-1}$(C_Z(η)), $ƌ^{-1}$(G_Z(η)), $ƌ^{-1}$(F_Z(η))) : η ∈ $𝔊_1$} is also an SVQNLI of $𝔊_1$.

Proof. This follows straightforwardly from Definitions 11 and 14. By Definition 3.5, $ƌ^{-1}$(Z) is an SVQNS in $𝔊_1$ if Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ $𝔊_2$} is an SVQNS in $𝔊_2$. Now, since Z = {(η, T_Z(η), C_Z(η), G_Z(η), F_Z(η)) : η ∈ $𝔊_2$} is an SVQNLI in $𝔊_2$ (as per the theorem's hypothesis), it satisfies the conditions specified in Definition 11 for $𝔊_2$.

Since f is an onto homomorphism:

1. The operations and scalar multiplication in $𝔊_1$ are preserved under $ƌ^{-1}$, ensuring that the conditions (i), (ii), and (iii) in Definition 3.2 hold for $ƌ^{-1}$(Z) in $𝔊_1$.

2. Specifically:

- For condition (i): The inequalities for T_Z, C_Z, G_Z, and F_Z under addition are preserved due to the homomorphic nature of f.

- For condition (ii): The scalar multiplication properties are maintained because ƌ maps $𝔊_1$ to $𝔊_2$ consistently.

- For condition (iii): The ordering of T_Z, C_Z, G_Z, and F_Z is also preserved via $ƌ^{-1}$, as $ƌ^{-1}$ correctly translates the operations in $𝔊_2$ back to K_1.

Thus, $\mathfrak{d}^{-1}(Z)$ satisfies all the conditions required for it to be an SVQNLI in $\mathfrak{G}_1$. Hence proved.

Proposition 2. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs and $\mathfrak{d}$ be an epimorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. If Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_2$} be an SVQNLI of $\mathfrak{G}_2$, then $\mathfrak{d}^{-1}(Z^c) = (\mathfrak{d}^{-1}(Z))^c$ is also an SVQNLI of $\mathfrak{G}_1$.

Proof. This result is derived from Definitions11 and 14.

Theorem 6. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs and $\mathfrak{d}$ be an epimorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. If Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_2$} be an SVQNLI of $\mathfrak{G}_2$, then $\mathfrak{d}^{-1}(Z)$ = {(η, $\mathfrak{d}^{-1}(T_Z(\eta))$, $\mathfrak{d}^{-1}(C_Z(\eta))$, $\mathfrak{d}^{-1}(G_Z(\eta))$, $\mathfrak{d}^{-1}(F_Z(\eta))$) : η ∈ $\mathfrak{G}_1$} is also an SVQNLI of $\mathfrak{G}_1$.

Proof. This follows straightforwardly from Definitions 11 and 14.

The image of Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_2$} through the mapping f indicated by f(Z) is an SVQNS in $\mathfrak{G}_2$, described as follows if Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_2$} be an SVQNS in $\mathfrak{G}_1$.

Definition 15. Suppose that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs and f is a mapping from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. If Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_1$} be an SVQNS in $\mathfrak{G}_1$, then the image of Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_1$} under f indicated by f(Z) is an SVQNS in $\mathfrak{G}_2$, characterized as follows:

$$f(T_Z)(t) = \begin{cases} \max_{a \in f^{-1}(t)} T_Z(a), & \text{when } f^{-1}(t) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}, \text{ for every } t \in \mathfrak{G}_2$$

$$f(C_Z)(t) = \begin{cases} \max_{a \in f^{-1}(t)} C_Z(a), & \text{when } f^{-1}(t) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}, \text{ for every } t \in \mathfrak{G}_2$$

$$f(G_Z)(t) = \begin{cases} \min_{a \in f^{-1}(t)} G_Z(a), & \text{when } f^{-1}(t) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}, \text{ for every } t \in \mathfrak{G}_2,$$

$$f(F_Z)(t) = \begin{cases} \min_{a \in f^{-1}(t)} F_Z(a), & \text{when } f^{-1}(t) \neq \emptyset \\ 1, & \text{otherwise} \end{cases}, \text{ for every } t \in \mathfrak{G}_2.$$

Theorem 7. Suppose that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs and f : $\mathfrak{G}_1 \rightarrow \mathfrak{G}_2$ is an epimorphism. If Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_1$} is an SVQNLI in $\mathfrak{G}_1$, then the image of Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_1$} i.e., f(Z) is also an SVQNLI in $\mathfrak{G}_2$.

Proof. The result follows directly from the properties outlined in Definition 11, which ensures that the SVQNS Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_1$} satisfies the required conditions on T_Z, C_Z, G_Z, and F_Z. Definition 15 guarantees that these properties are preserved under the mapping f, as the image components $f(T_Z)$, $f(C_Z)$, $f(G_Z)$, and $f(F_Z)$ are constructed accordingly.

Definition 16. Suppose that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs and let f is an onto homomorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. Let Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_2$} be an SVQNS in $\mathfrak{G}_2$. Then, we define $\mathfrak{G}^f = \{(\eta, T_Z^f(\eta), C_Z^f(\eta), G_Z^f(\eta), F_Z^f(\eta)) : \eta \in \mathfrak{G}_1\}$ in $\mathfrak{G}_1$ by $T_Z^f(\eta) = T_Z(f(\eta))$, $C_Z^f(\eta) = C_Z(f(\eta))$, $G_Z^f(\eta) = G_Z(f(\eta))$, $F_Z^f(\eta) = F_Z(f(\eta))$, ∀ η ∈ $\mathfrak{G}_1$. Clearly, $\mathfrak{G}^f$ is an SVQNS in $\mathfrak{G}_1$.

Theorem 8. Suppose that $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs and let f is an onto homomorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. If Z = {(η, $T_Z(\eta)$, $C_Z(\eta)$, $G_Z(\eta)$, $F_Z(\eta)$) : η ∈ $\mathfrak{G}_2$} is an SVQNLI of $\mathfrak{G}_2$, then $\mathfrak{G}^f = \{(\eta, T_Z^f(\eta), C_Z^f(\eta), G_Z^f(\eta), F_Z^f(\eta)) : \eta \in \mathfrak{G}_1\}$ is also an SVQNLI of $\mathfrak{G}_1$.

Proof. Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ represent two LAs on F. Suppose that a ∈ F and η, δ ∈ $\mathfrak{G}_1$. Then,

(i) $T_Z^f(\eta + \delta)$

$= T_Z(f(\eta + \delta))$

$= T_Z(f(\eta) + f(\delta))$

$\geq \min\{T_Z(f(\eta)), T_Z(f(\delta))\}$

$= \min\{T_Z^f(\eta), T_Z^f(\delta)\}$,

$C_Z^f(\eta + \delta)$

$= C_Z(f(\eta + \delta))$

$= C_Z(f(\eta) + f(\delta))$

$\geq \min\{C_Z(f(\eta)), C_Z(f(\delta))\}$

$= \min\{C_Z^f(\eta), C_Z^f(\delta)\}$,

$G_Z^f(\eta + \delta)$

$= G_Z(f(\eta + \delta))$

$= G_Z(f(\eta) + f(\delta))$

$\leq \max\{G_Z(f(\eta)), G_Z(f(\delta))\}$

$= \max\{G_Z^f(\eta), G_Z^f(\delta)\}$,

$F_Z^f(\eta + \delta)$

$= F_Z(f(\eta + \delta))$

$= F_Z(f(\eta) + f(\delta))$

$\leq \max\{F_Z(f(\eta)), F_Z(f(\delta))\}$

$= \max\{F_Z^f(\eta), F_Z^f(\delta)\}$,

(ii) $T_Z^f(a\eta) = T_Z(f(a\eta)) = T_Z(af(\eta)) \geq T_Z(f(\eta)) = T_Z^f(\eta)$,

$C_Z^f(a\eta) = C_Z(f(a\eta)) = C_Z(af(\eta)) \geq C_Z(f(\eta)) = C_Z^f(\eta)$,

$G_Z^f(a\eta) = G_Z(af(\eta)) = G_Z(af(\eta)) \leq G_Z(f(\eta)) = G_Z^f(\eta)$,

$F_Z^f(a\eta) = F_Z(af(\eta)) = F_Z(af(\eta)) \leq F_Z(f(\eta)) = F_Z^f(\eta)$.

(iii) $T_Z^f([\eta, \delta]) = T_Z(f([\eta, \delta])) = T_Z([f(\eta), f(\delta)]) \geq T_Z(f(\eta)) = T_Z^f(\eta)$,

$C_Z^f([\eta, \delta]) = C_Z(f([\eta, \delta])) = C_Z([f(\eta), f(\delta)]) \geq C_Z(f(\eta)) = C_Z^f(\eta)$,

$G_Z^f([\eta, \delta]) = G_Z(f([\eta, \delta])) = G_Z([f(\eta), f(\delta)]) \leq G_Z(f(\eta)) = G_Z^f(\eta)$,

$F_Z^f([\eta, \delta]) = F_Z(f([\eta, \delta])) = F_Z([f(\eta), f(\delta)]) \leq F_Z(f(\eta)) = F_Z^f(\eta)$.

Therefore, $\mathfrak{G}^f = \{(\eta, T_Z^f(\eta), C_Z^f(\eta), G_Z^f(\eta), F_Z^f(\eta)) : \eta \in \mathfrak{G}_1\}$ satisfies all the conditions for being an SVQNLI of $\mathfrak{G}_1$. Hence, $\mathfrak{G}^f$ is an SVQNLI of $\mathfrak{G}_1$.

Theorem 9. Consider $\mathfrak{G}_1$ and $\mathfrak{G}_2$ as two LAs and let f is an onto homomorphism from $\mathfrak{G}_1$ to $\mathfrak{G}_2$. Then, $\mathfrak{G}^f = \{(z, T_Z^f(z), C_Z^f(z), G_Z^f(z), F_Z^f(z)) : z \in \mathfrak{G}_1\}$ is an SVQNLI of $\mathfrak{G}_1$ iff $Z = \{(z, T_Z(z), C_Z(z), G_Z(z), F_Z(z)) : z \in \mathfrak{G}_2\}$ is an SVQNLI of L_2.

Proof. The preceding theorem directly leads to the sufficiency of this one. We now need to prove the theorem's necessity component. Since f is a surjective mapping, there exists $z_1, y_1 \in \mathfrak{G}_2$ such that $z = f(z_1)$,

$y = f(y_1)$ for every $z, y \in \mathfrak{G}_2$. Consequently, $T_Z(z) = T_Z^f(z_1)$, $T_Z(y) = T_Z^f(y_1)$, $C_Z(z) = C_Z^f(z_1)$, $C_Z(y) = C_Z^f(y_1)$, $G_Z(z) = G_Z^f(z_1)$, $G_Z(y) = G_Z^f(y_1)$, $F_Z(z) = F_Z^f(z_1)$, $F_Z(y) = F_Z^f(y_1)$.

Now,

(i) $T_Z(z + y)$

$= T_Z\big(f(z_1) + f(y_1)\big)$

$= T_Z\big(f(z_1 + y_1)\big)$

$= T_Z^f(z_1 + y_1)$

$\geq \min\{T_Z^f(z_1), T_Z^f(y_1)\}$

$= \min\{T_Z(z), T_Z(y)\}$,

$C_Z(z + y)$

$= C_Z\big(f(z_1) + f(y_1)\big)$

$= C_Z\big(f(z_1 + y_1)\big)$

$= C_Z^f(z_1 + y_1)$

$\geq \min\{C_Z^f(z_1), C_Z^f(y_1)\}$

$= \min\{C_Z(z), C_Z(y)\}$,

$G_Z(z + y)$

$= G_Z\big(f(z_1) + f(y_1)\big)$

$= G_Z\big(f(z_1 + y_1)\big)$

$= G_Z^f(z_1 + y_1)$

$\leq \max\{G_Z^f(z_1), G_Z^f(y_1)\}$

$= \max\{G_Z(z), G_Z(y)\}$,

$F_Z(z + y)$

$= F_Z\big(f(z_1) + f(y_1)\big)$

$= F_Z\big(f(z_1 + y_1)\big)$

$= F_Z^f(z_1 + y_1)$

$\leq \max\{F_Z^f(z_1), F_Z^f(y_1)\}$

$= \max\{F_Z(z), F_Z(y)\}$.

(ii) $T_Z(\alpha z) = T_Z\big(\alpha f(z_1)\big) = T_Z\big(f(\alpha z_1)\big) = T_Z^f\big(f(\alpha z_1)\big) \geq T_Z^f(z_1) = T_Z(z)$,

$C_Z(\alpha z) = C_Z\big(\alpha f(z_1)\big) = C_Z\big(f(\alpha z_1)\big) = C_Z^f\big(f(\alpha z_1)\big) \geq C_Z^f(z_1) = C_Z(z)$,

$G_Z(\alpha z) = G_Z\big(\alpha f(z_1)\big) = G_Z\big(f(\alpha z_1)\big) = G_Z^f\big(f(\alpha z_1)\big) \leq G_Z^f(z_1) = G_Z(z)$,

$F_Z(\alpha z) = F_Z\big(\alpha f(z_1)\big) = F_Z\big(f(\alpha z_1)\big) = F_Z^f\big(f(\alpha z_1)\big) \leq F_Z^f(z_1) = F_Z(z)$.

(iii) $T_Z([z, y]) = T_Z([f(z_1), f(y_1)]) = T_Z\big(f([z_1, y_1])\big) = T_Z^f([z_1, y_1]) \geq T_Z(z_1) = T_Z(z)$,

$C_Z([z,y]) = C_Z([f(z_1), f(y_1)]) = C_Z(f([z_1, y_1])) = C^f_Z([z_1, y_1]) \geq C_Z(z_1) = C_Z(z)$,

$G_Z([z,y]) = G_Z([f(z_1), f(y_1)]) = G_Z(f([z_1, y_1])) = G^f_Z([z_1, y_1]) \leq G_Z(z_1) = G_Z(z)$,

$F_Z([z,y]) = F_Z([f(z_1), f(y_1)]) = F_Z(f([z_1, y_1])) = F^f_Z([z_1, y_1]) \leq F_Z(z_1) = F_Z(z)$. Therefore, $\mathfrak{G}^f = \{(z, T^f_Z(z), C^f_Z(z), G^f_Z(z), F^f_Z(z)) : z \in \mathfrak{G}_1\}$ satisfies all the conditions for being an SVQNLI of $\mathfrak{G}_2$.

Conclusions

In this study, we have established the idea of SVQNLI of SVQNLA. Furthermore, we have developed numerous interesting results on SVQNLIs and SVQNLAs. In the future, we anticipate the introduction of new concepts such as Single-Valued Quadripartitioned Neutrosophic Anti-Lie-Ideal and Single-Valued Quadripartitioned Neutrosophic Lie-Topology, building upon the current study of SVQNLA. These new ideas would further extend the framework of SVQNLA, enriching the theoretical landscape and potentially offering novel approaches for solving problems in related areas of algebra and topology.

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Conflicts of Interest

The authors declare no conflicts of interest in the publication of this article. They have disclosed all relevant affiliations and financial relationships, and there are no competing interests that could have influenced the research or its outcomes.

Authors Contributions

All authors have made equal contributions to the conceptualization, research, writing and preparation of this article, and share responsibility for its content

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Chapter 2

Quadripartitioned Neutrosophic Quasi Coincident Topological Space

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ABSTRACT

This chapter investigates the concept of Quadripartitioned Neutrosophic Point (QNP) in a Quadripartitioned Neutrosophic Set (QNS), quadripartitioned neutrosophic quasi coincident with quadripartitioned neutrosophic set and quadripartitioned neutrosophic point. We have established various properties of quasi coincident in quadripartitioned neutrosophic set relations. We also study the quasi coincident topological property in which the degree of nearness or coincidence between quadripartitioned neutrosophic sets in a Quadripartitioned Neutrosophic Topological (QNT) space.

Keywords: Quadripartitioned neutrosophic point, Quadripartitioned neutrosophic quasi coincident, Quasi coincident topology.

INTRODUCTION

A generalized version of the fuzzy set [69], intuitionistic fuzzy set was first proposed by Atanassov [4] in 1986. Intuitionistic fuzzy points were later presented by Coker and Demirci [14]. Neutrosophic Set (NS) was presented and investigated by Smarandache [62, 63, 64]. Later, neutrosophic topology was introduced and researched by Salama and Alblowi [59] and Salama et al. [60]. Since then, additional research has been found in the following areas: topology[3, 19, 20, 27, 30, 34, 35, 36, 37, 38, 40, 41, 47, 48, 57, 58, 59], minimal structure space [18], rough set theory [22], bi-topology [29, 31, 68], infra bi-topology [15], ideals [21], Quadripartitioned Neutrosophic Set (QNS) theory [12, 16, 49], interval QNS [53], interval pentapartitioned NS [54, 32], neutrosophic open set [25], neutrosophic b-open set [26], neutrosophic separation axioms [19] along with hybrid models of intervals and soft sets [33]. Over the years, several terms of open functions have been introduced. "Relation of quasi-coincidence for neutrosophic sets" was defined by Ray and Dey [55] in 2021. Acikgoz and Esenbel [2] investigate neutrosophic connected topological spaces in 2023. Chatterjee et al. [12] defined entropy and a few similarity metrics for quadripartitioned single valued neutrosophic sets in 2016. Das et al. [16] defined topology on quadripartitioned neutrosophic sets in 2021. Das et al. [18] introduced the single-valued quadripartitioned neutrosophic minimal structure space in 2023. Granados et al. [39] established quadripartitioned neutrosophic Q-ideals of Q-algebra in 2023. Das et al. [17] presented neutrosophic D- filter of D-algebra. Aggregation operators of quadripartitioned single-valued neutrosophic Z-numbers were defined by Borah and Dutta [11] in 2023.

NS based game theory was used in 2014 by Pramanik and Roy [52] to study the dispute between India and Pakistan over Jammu and Kashmir. NS theories have been effectively applied to medical diagnostics [46] decision-making issues [1, 5, 7, 8, 9, 10, 42, 45, 51, 61], image processing [13], water quality testing [23, 24], social issues [43, 50], teacher selection [44], and project management research [28]. Applications and theoretical developments of NSs were depicted in [6, 65, 66, 67].In the framework of NS theory, Ray and Dey[56] investigated the concept of Neutrosophic Points (NPs) and the neighbourhood structure in 2021. A connection of quasi coincidence for the NS was later proposed by Ray and Dey [55]. However, the relationship between quasi-coincidence and QNSs or Quadripartitioned NPs (QNPs) has not yet been studied. This chapter defines the quasi-coincidence relation between two QNSs and a QNP. It also looks at several characteristics derived from the quasi-coincidence connection. We then determine a QNP, quadripartitioned neutrosophic quasi-neighbourhood and assess a variety of features. Lastly, we examine whether quadripartitioned neutrosophic quasi-neighbourhoods can be used to describe Quadripartitioned Neutrosophic Topological Space (QNTS).

The structure of this chapter is organized as follows:Section-2 presents the preliminaries and essential definitions, presenting foundational concepts and theorems instrumental to the core findings of the study. Section-3 focuses on the characterization of quadripartitioned neutrosophic quasi neighbourhoods. Section-4 focuses on the quadripartitioned neutrosophic topological space and open sets. The chapter is finally concluded in section 5, which provides a summary of the results and closing thoughts.

BACKGROUND

We go over a few ideas about quadripartitioned neutrosophic sets in this section.

Definition 2.1. [65] Assume V be the universe. A single valued neutrosophic set A over V is stated as $A = \{\langle \varkappa, \mathcal{T}_A(\varkappa), \mathcal{I}_A(\varkappa), \mathcal{F}_A(\varkappa)\rangle : \varkappa \in V\}$, where $\mathcal{T}_A, \mathcal{I}_A, \mathcal{F}_A$ are functions from V to $[0, 1]$and $0 \leq \mathcal{T}_A(\varkappa) + \mathcal{I}_A(\varkappa) + \mathcal{F}_A(\varkappa) \leq 3$. The set of all single valued neutrosophic sets over V is denoted by $\mathcal{N}(V)$. Throughout this article, a single valued neutrosophic set willsimply be called a neutrosophic set (NS, for short).

Definition 2.2. [12] Assume that V be a fixed set. Then, a quadripartitioned neutrosophic set (QPNS) A over V is defined by $A = \{\langle \varkappa, \mathcal{T}_A(\varkappa), \mathcal{C}_A(\varkappa), \mathcal{U}_A(\varkappa), \mathcal{F}_A(\varkappa)\rangle : \varkappa \in V\}$,

where $\mathcal{T}_A, \mathcal{C}_A, \mathcal{U}_A$ and $\mathcal{F}_A \in [0,1]$ are the truth, contradiction, ignorance, and falsity membership values of $\varkappa \in V$.So, $0 \leq \mathcal{T}_A(\varkappa) + \mathcal{C}_A(\varkappa) + \mathcal{U}_A(\varkappa) + \mathcal{F}_A(\varkappa) \leq 4$.

Definition 2.3. [12] Let $A, B \in \mathcal{N}(V)$. Then

(1) If $\mathcal{T}_A(\varkappa) \leq \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \leq \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \geq \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \geq \mathcal{F}_B(\varkappa)$ for all $\varkappa \in V$, then A is referred to as a quadripartitioned neutrosophic sub-set of B and which is indicated by $A \subseteq B$.

(2) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(3) The intersection of A and B, indicated by $A \cap B$, is described as $A \cap B = \{\langle \varkappa, \mathcal{T}_A(\varkappa) \wedge \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \wedge \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \vee \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \vee \mathcal{F}_B(\varkappa)\rangle : \varkappa \in V\}$.

(4) The union of $A$and B, indicated by $A \cup B$, is described as $A \cup B = \{\langle \varkappa, \mathcal{T}_A(\varkappa) \vee \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \vee \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \wedge \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \wedge \mathcal{F}_B(\varkappa)\rangle : \varkappa \in V\}$.

(5) The complement of the QPNS A, indicated byA^c, is described as$A^c = \{\langle \varkappa, \mathcal{F}_A(\varkappa), \mathcal{U}_A(\varkappa), \mathcal{C}_A(\varkappa), \mathcal{T}_A(\varkappa)\rangle : \varkappa \in V\}$

(6) If $\mathcal{T}_A(\varkappa) = 1, \mathcal{C}_A(\varkappa) = 1, \mathcal{U}_A(\varkappa) = 0, \mathcal{F}_A(\varkappa) = 0$ for all $\varkappa \in V$ then A is referred to as neutrosophic universal set and which is indicated by1_{QN}.

(7) If $\mathcal{T}_A(\varkappa) = 0, \mathcal{C}_A(\varkappa) = 0, \mathcal{U}_A(\varkappa) = 1, \mathcal{F}_A(\varkappa) = 1$ for all $\varkappa \in V$ then A is referred to as neutrosophic empty set and which is indicated by$\emptyset$or O_{QN}.

Definition 2.4. [12] Let $\{A_i : i \in \Delta\} \subseteq \mathcal{N}(V)$, where $\triangle$ is an index set. Then,

(1) $\cup_{i\in\Delta} A_i = \{\langle \varkappa, \vee_{i\in\Delta} \mathcal{T}_{A_i}(\varkappa), \vee_{i\in\Delta} \mathcal{C}_{A_i}(\varkappa), \wedge_{i\in\Delta} \mathcal{U}_{A_i}(\varkappa), \wedge_{i\in\Delta} \mathcal{F}_{A_i}(\varkappa)\rangle : \varkappa \in V\}$.

(2) $\cap_{i\in\Delta} A_i = \{\langle \varkappa, \wedge_{i\in\Delta} \mathcal{T}_{A_i}(\varkappa), \wedge_{i\in\Delta} \mathcal{C}_{A_i}(\varkappa), \vee_{i\in\Delta} \mathcal{U}_{A_i}(\varkappa), \vee_{i\in\Delta} \mathcal{F}_{A_i}(\varkappa)\rangle : \varkappa \in V\}$.

Definition 2.5. [16] Let $\tau \subseteq \mathcal{QPN}(V)$. Then τ is referred to as a quadripartitioned neutrosophic topology (QPNT) on V if

(i) $O_{QN}, 1_{QN} \in \tau$
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
(iii) $\cup G_i \in \tau \, \forall\{G_i: i \in J\} \subseteq \tau$

If τ is a quadripartitioned neutrosophic topology on V then the pair (V, τ) is referred to as a quadripartitioned neutrosophic topological space (QPNTS) over$V$. The members of τ are called quadripartitioned neutrosophic open sets in V. For a quadripartitioned neutrosophic set $A \in \tau, A^c$ is referred to as a quadripartitioned neutrosophic closed set in V.

Main Results

Definition 3.1. A QPNS A is referred to as quasi-coincident with a QPNS B at $\varkappa \in V$ or A quasi-coincides with B at$\varkappa \in V$,indicated byAqB at $\varkappa$, iff $\mathcal{T}_A(\varkappa) > \mathcal{T}_{B^c}(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{C}_{B^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{U}_{B^c}(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{F}_{B^c}(\varkappa)$. We say A quasi-coincides with B or A is quasi-coincident with B, indicated byAqB, iff A quasicoincides with B at some point $\varkappa \in V$. Thus A quasi-coincides with B or A is quasi-coincident with B iff there exists an element $\mathcal{T}_A(\varkappa) > \mathcal{T}_{B^c}(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{C}_{B^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{U}_{B^c}(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{F}_{B^c}(\varkappa)$, i.e., $\mathcal{T}_A(\varkappa) > \mathcal{F}_B(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{U}_B(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{C}_B(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{T}_B(\varkappa)$.

If a QNSA, represented by ϰ_(α,β,γ,δ) q ˆA, is not quasi-coincident with the QNP ϰ_(α,β,γ,δ). Likewise, Aq ˆB indicates that the QNS-A is not quasi-coincident with the QNSB. $A\Omega B$ will represent the set of all the points in X at which AqB occurs, indicating that $A\Omega B = \{\varkappa \in V: AqB \ at \ \varkappa\}$.

Definition 3.2. Let $\mathcal{QN}(V)$ be the set of all quadripartitioned neutrosophic sets over$V$. A QPNS $P = \{\langle \varkappa, \mathcal{T}_P(\varkappa), \mathcal{C}_P(\varkappa), \mathcal{U}_P(\varkappa), \mathcal{F}_P(\varkappa)\rangle : \varkappa \in V\}$ is referred to as a quadripartitioned neutrosophic point (QPNP) iff for any element $\mu \in V, \mathcal{T}_P(\mu) = \alpha, \mathcal{C}_P(\mu) = \beta, \mathcal{U}_P(\mu) = \mu, \mathcal{F}_P(\mu) = \delta$ for $\mu = \varkappa$ and $\mathcal{T}_P(\mu) = 0, \mathcal{C}_P(y) = 0, \mathcal{U}_P(\mu) = 1, \mathcal{F}_P(\mu) = 1for\mu \neq \varkappa$, where $0 < \alpha \leq 1, 0 \leq \beta < 1, 0 \leq \gamma < 1, 0 \leq \delta < 1$.

A QPNP $P = \{\langle \varkappa, \mathcal{T}_P(\varkappa), \mathcal{C}_P(\varkappa), \mathcal{U}_P(\varkappa)\mathcal{F}_P(\varkappa)\rangle : \varkappa \in V\}$ will be indicated by$P^{\varkappa}_{\alpha,\beta,\gamma,\delta}$ or $P < \varkappa, \alpha, \beta, \gamma, \delta >$ or simply by $\varkappa_{\alpha,\beta,\gamma,\delta}$. For the QPNP $\varkappa_{\alpha,\beta,\gamma,\delta}, \varkappa$ will be called its support.

The complement of the QPNP $P^{\varkappa}_{\alpha,\beta,\gamma,\delta}$ will be indicated by$\left(P^{\varkappa}_{\alpha,\beta,\gamma,\delta}\right)^c$ or by$\varkappa^c_{\alpha,\beta,\gamma,\delta}$.

Definition 3.3. A QPNP $\varkappa_{\alpha,\beta,\gamma,\delta} \in \mathcal{QN}(V)$ is referred to as quasi-coincident with a QPNS $A \in \mathcal{QN}(V)$ or $\varkappa_{\alpha,\beta,\gamma,\delta} \in \mathcal{QN}(V)$ quasi-coincides with a QPNS $A \in \mathcal{QN}(V)$, indicated by$\varkappa_{\alpha,\beta,\gamma,\delta} qA$, iff $\alpha > \mathcal{T}_{A^c}(\varkappa)$ and $\beta > \mathcal{C}_{A^c}(\varkappa)$ or $\gamma < \mathcal{U}_{A^c}(\varkappa)$ and $\delta < \mathcal{F}_{A^c}(\varkappa)$, i.e., $\alpha > \mathcal{F}_A(\varkappa)$ and $\beta > \mathcal{U}_A(\varkappa)$ or $\gamma < \mathcal{C}_A(\varkappa)$ and $\delta < \mathcal{T}_A(\varkappa)$

Definition 3.4. Let A be a QPNSs over V. Also let $\varkappa_{\alpha,\beta,\gamma,\delta}$ and $\mu_{\alpha',\beta',\gamma',\delta'}$ be two QPNPs in V. Then

(1) $\{\varkappa_{\alpha,\beta,\gamma,\gamma,\delta}\}$is referred to as contained in A, indicated by$\varkappa_{\alpha,\beta,\gamma,\delta} \subseteq A$, iff $\alpha \leq \mathcal{T}_A(\varkappa), \beta \leq \mathcal{C}_A(\varkappa), \gamma \geq \mathcal{U}_A(\varkappa), \delta \geq \mathcal{F}_A(\varkappa)$.

(2) $\varkappa_{\alpha,\beta,\gamma,\gamma,\delta}$is referred to aslong to A, indicated by$\varkappa_{\alpha,\beta,\gamma,\delta} \in A$, iff $\alpha \leq \mathcal{T}_A(\varkappa), \beta \leq \mathcal{C}_A(\varkappa), \gamma \geq \mathcal{U}_A(\varkappa), \delta \geq \mathcal{F}_A(\varkappa)$.

(3) $\varkappa_{\alpha,\beta,\gamma,\delta}$is referred to as contained in $\mu_{\alpha',\beta',\gamma',\delta'}$, indicated by$\varkappa_{\alpha,\beta,\gamma,\delta} \subseteq \mu_{\alpha',\beta',\gamma',\delta'}$, iff $\varkappa = \mu$ and $\alpha \leq \alpha', \beta \leq \beta', \gamma \geq \gamma', \delta \geq \delta'$.

(4) $\varkappa_{\alpha,\beta,\gamma,\delta}$is referred to aslong to $\mu_{\alpha',\beta',\gamma',\delta'}$, indicated by$\varkappa_{\alpha,\beta,\gamma,\delta} \in \mu_{\alpha',\beta',\gamma',\delta'}$, iff $\varkappa = \mu$ and $\alpha \leq \alpha', \beta \leq \beta', \gamma \geq \gamma', \delta \geq \delta'$.

Proposition 3.5. Let $A, B \in \mathcal{N}(V)$. Then $A \subseteq B \Leftrightarrow B^c \subseteq A^c$.

Proof:$A \subseteq B$
$\Leftrightarrow \mathcal{T}_A(\varkappa) \leq \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \leq \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \geq \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \geq \mathcal{F}_B(\varkappa)$ for all $\varkappa \in V$
$\Leftrightarrow \mathcal{F}_B(\varkappa) \leq \mathcal{F}_A(\varkappa), \mathcal{U}_B(\varkappa) \leq \mathcal{U}_A(\varkappa), \mathcal{C}_B(\varkappa) \geq \mathcal{C}_A(\varkappa), \mathcal{T}_B(\varkappa) \geq \mathcal{T}_A(\varkappa)$ for all $\varkappa \in V$
$\Leftrightarrow \mathcal{T}_{B^c}(\varkappa) \leq \mathcal{T}_{A^c}(\varkappa), \mathcal{C}_{B^c}(\varkappa) \leq \mathcal{C}_{A^c}(\varkappa), \mathcal{U}_{B^c}(\varkappa) \geq \mathcal{U}_{A^c}(\varkappa), \mathcal{F}_{B^c}(\varkappa) \geq \mathcal{F}_{A^c}(\varkappa)$ for all $\varkappa \in V \Leftrightarrow B^c \subseteq A^c$

Proposition 3.6. Let A, B, C be three $QPNSs$, and $\varkappa_{\alpha,\beta,\gamma,\delta}$ be a QPNP in V. Then,

(1) $\varkappa_{\alpha,\beta,\gamma,\delta}\hat{q}\tilde{\emptyset}$.

(2) $\varkappa_{\alpha,\beta,\gamma,\delta}q\tilde{X}$.

(3) $\varkappa_{\alpha,\beta,\gamma,\delta} \in A \Leftrightarrow \varkappa_{\alpha,\beta,\gamma,\delta}\hat{q}A^c$.

(4) $\varkappa_{\alpha,\beta,\gamma,\delta}qA \Leftrightarrow \varkappa_{\alpha,\beta,\gamma,\delta} \notin A^c$.

(5) $A \subseteq B \Leftrightarrow A\hat{q}B^c$.

(6) $AqB \Leftrightarrow A \nsubseteq B^c$

(7) $\varkappa_{\alpha,\beta,\gamma,\delta}qAandA \subseteq B$ then $\varkappa_{\alpha,\beta,\gamma,\delta}qB$.

(8) $CqA$and$A \subseteq B$ then CqB.

(9) AqBat$\varkappa \Leftrightarrow BqA$ at $\varkappa$.

(10) $AqB \Leftrightarrow BqA$.

Proof:

(1) The proof is so easy, so omitted.
(2) The proof is so easy, so omitted.
(3) $x_{\alpha,\beta,\gamma,\delta} \in A$

$\Leftrightarrow \alpha \leq \mathcal{T}_A(\varkappa), \beta \leq \mathcal{C}_A(\varkappa), \gamma \geq \mathcal{U}_A(\varkappa), \delta \geq \mathcal{F}_A(\varkappa)$
$\Leftrightarrow \alpha \ngtr \mathcal{T}_A(\varkappa), \beta \ngtr \mathcal{C}_A(\varkappa), \gamma \nless \mathcal{U}_A(\varkappa), \delta \nless \mathcal{F}_A(\varkappa)$
$\Leftrightarrow \alpha \ngtr \mathcal{T}_{(A^c)^c}(\varkappa), \beta \ngtr \mathcal{C}_{(A^c)^c}(\varkappa), \gamma \nless \mathcal{U}_{(A^c)^c}(\varkappa), \delta \nless \mathcal{F}_{(A^c)^c}(\varkappa)$
$\Leftrightarrow \varkappa_{\alpha,\beta,\gamma,\delta,\delta}\hat{q}A^c$

(4)

$\varkappa_{\alpha,\beta,\gamma,\delta}qA$
$\Leftrightarrow \alpha > \mathcal{T}_{A^c}(\varkappa)$ and $\beta > \mathcal{C}_{A^c}(\varkappa)$ or $\gamma < \mathcal{U}_{A^c}(\varkappa)$ and $\delta < \mathcal{F}_{A^c}(\varkappa)$
$\Leftrightarrow \alpha \nleq \mathcal{T}_{A^c}(\varkappa)$ and $\beta \nleq \mathcal{C}_{A^c}(\varkappa)$ or $\gamma \ngeq \mathcal{U}_{A^c}(\varkappa)$ and $\delta \ngeq \mathcal{F}_{A^c}(\varkappa)$
$\Leftrightarrow \varkappa_{\alpha,\beta,\gamma} \notin A^c$

(5)

$A \subseteq B$
$\Leftrightarrow \mathcal{T}_A(\varkappa) \leq \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \leq \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \geq \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \geq \mathcal{F}_B(\varkappa) \forall \varkappa \in V$
$\Leftrightarrow \mathcal{T}_A(\varkappa) \ngtr \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \ngtr \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \nless \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \nless \mathcal{F}_B(\varkappa) \forall \varkappa \in V$
$\Leftrightarrow \mathcal{T}_A(\varkappa) \ngtr \mathcal{T}_{(B^c)^c}(\varkappa), \mathcal{C}_A(\varkappa) \ngtr \mathcal{C}_{(B^c)^c}(\varkappa), \mathcal{U}_A(\varkappa) \nless \mathcal{U}_{(B^c)^c}(\varkappa), \mathcal{F}_A(\varkappa) \nless \mathcal{F}_{(B^c)^c}(\varkappa) \forall \varkappa \in V$
$\Leftrightarrow A\hat{q}B^c$

(6) AqB

$\Leftrightarrow \mathcal{T}_A(\varkappa) > \mathcal{T}_{B^c}(\varkappa)and\mathcal{C}_A(\varkappa) > \mathcal{C}_{B^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{U}_{B^c}(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{F}_{B^c}(\varkappa)$
$\Leftrightarrow \mathcal{T}_A(\varkappa) \nleq \mathcal{T}_{B^c}(\varkappa)$ and $\mathcal{C}_A(\varkappa) \nleq \mathcal{C}_{B^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) \ngeq \mathcal{U}_{B^c}(\varkappa)$ and $\mathcal{F}_A(\varkappa) \ngeq \mathcal{F}_{B^c}(\varkappa) \Leftrightarrow A \nsubseteq B^c$

(7) Since $\varkappa_{\alpha,\beta,\gamma}qA$, so $\alpha > \mathcal{T}_{A^c}(\varkappa)$ and $\beta > \mathcal{C}_{A^c}(\varkappa)$ or $\gamma < \mathcal{U}_{A^c}(\varkappa)$ and $\delta < \mathcal{F}_{A^c}(\varkappa)$. Now
$$A \subseteq B \Rightarrow B^c \subseteq A^c$$
$\Rightarrow \mathcal{T}_{B^c}(\varkappa) \leq \mathcal{T}_{A^c}(\varkappa), \mathcal{C}_{B^c}(\varkappa) \leq \mathcal{C}_{A^c}(\varkappa), \mathcal{U}_{B^c}(\varkappa) \geq \mathcal{U}_{A^c}(\varkappa), \mathcal{F}_{B^c}(\varkappa) \geq \mathcal{F}_{A^c}(\varkappa)$ for all $\varkappa \in X$
$\Rightarrow \mathcal{T}_{A^c}(\varkappa) \geq \mathcal{T}_{B^c}(\varkappa), \mathcal{C}_{A^c}(\varkappa) \geq \mathcal{C}_{B^c}(\varkappa), \mathcal{U}_{A^c}(\varkappa) \leq \mathcal{U}_{B^c}(\varkappa), \mathcal{F}_{A^c}(\varkappa) \leq \mathcal{F}_{B^c}(\varkappa)$ for all $\varkappa \in V \Rightarrow \alpha > \mathcal{T}_{B^c}(\varkappa)$ and $\beta > \mathcal{C}_{B^c}(\varkappa)$ or $\gamma < \mathcal{U}_{B^c}(\varkappa)$ and $\delta < \mathcal{F}_{B^c}(\varkappa)$ for all $\varkappa \in V \Rightarrow \varkappa_{\alpha,\beta,\gamma,\delta}qB$

(8) $CqA \Rightarrow C \nsubseteq A^c \Rightarrow C \nsubseteq B^c[\because A \subseteq B \Rightarrow B^c \subseteq A^c] \Rightarrow CqB$.

(9) AqB at $\varkappa$
$\Leftrightarrow \mathcal{T}_A(\varkappa) > \mathcal{T}_{B^c}(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{C}_{B^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{U}_{B^c}(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{F}_{B^c}(\varkappa)$
$\Leftrightarrow \mathcal{T}_A(\varkappa) > \mathcal{F}_B(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{U}_B(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{C}_B(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{T}_B(\varkappa)$
$\Leftrightarrow \mathcal{T}_B(\varkappa) > \mathcal{F}_A(\varkappa)$ and $\mathcal{C}_B(\varkappa) > \mathcal{U}_A(\varkappa)$ or $\mathcal{U}_B(\varkappa) < \mathcal{C}_A(\varkappa)$ and $\mathcal{F}_B(\varkappa) < \mathcal{T}_A(\varkappa)$

$\Leftrightarrow \mathcal{T}_A(\varkappa) > \mathcal{T}_{B^c}(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{C}_{B^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{U}_{B^c}(\varkappa)$ and $\mathcal{F}_A(\varkappa) < \mathcal{F}_{B^c}(\varkappa)$
$\Leftrightarrow BqA$ at $\varkappa$

(10) It is obvious from (9).

Proposition 3.7. Let $\varkappa_{\alpha,\beta,\gamma,\delta}$ be a QNP in $P, A \in \mathcal{QN}(X)$ and $\{A_i: i \in \Delta\} \subseteq \mathcal{QN}(P)$, where an index set represented by Δ. Then,

(1) $\varkappa_{\alpha,\beta,\gamma,\delta} q \cup_{i\in\Delta} A_i \Leftrightarrow \varkappa_{\alpha,\beta,\gamma,\delta} q A_j$ for some $j \in \Delta$.

(2) $Aq \cup_{i\in\Delta} A_i \Leftrightarrow AqA_j$ for some $j \in \Delta$.

(3) $\varkappa_{\alpha,\beta,\gamma,\delta} q \cap_{i\in\Delta} A_i \Rightarrow \varkappa_{\alpha,\beta,\gamma,\delta q} qA_i$ for all $i \in \Delta$.

(4) $Aq \cap_{i\in\Delta} A_i \Rightarrow AqA_i$ for all $i \in \triangle$.

Proof:

(1)

$$\begin{aligned}
&\varkappa_{\alpha,\beta,\gamma,\delta} q \cup_{i\in\Delta} A_i \\
\Leftrightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} \notin (\cup_{i\in\Delta} A_i)^c \\
\Leftrightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} \notin \cap_{i\in\Delta} A_i^c \\
\Leftrightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} \notin A_j^c \text{ for some } j \in \Delta \\
\Leftrightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} qA_j \text{ for some } j \in \Delta
\end{aligned}$$

(2)

$$\begin{aligned}
&Aq \cup_{i\in\Delta} A_i \\
\Leftrightarrow &A \nsubseteq (\cup_{i\in\Delta} A_i)^c \\
\Leftrightarrow &A \nsubseteq \cap_{i\in\Delta} A_i^c \\
\Leftrightarrow &A \nsubseteq A_j^c \text{ for some } j \in \triangle \\
\Leftrightarrow &AqA_j \text{ for some } j \in \triangle
\end{aligned}$$

(3)

$$\begin{aligned}
&\varkappa_{\alpha,\beta,\gamma,\delta} q \cap_{i\in\Delta} A_i \\
\Rightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} \notin (\cap_{i\in\Delta} A_i)^c \\
\Rightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} \notin \cup_{i\in\Delta} A_i^c \\
\Rightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} \notin A_i^c \text{ for all } i \in \triangle \\
\Rightarrow &\varkappa_{\alpha,\beta,\gamma,\delta} qA_i \text{ for all } i \in \triangle
\end{aligned}$$

(4)

$$\begin{aligned}
&Aq \cap_{i\in\Delta} A_i \\
\Rightarrow &A \nsubseteq (\cap_{i\in\Delta} A_i)^c \\
\Rightarrow &A \nsubseteq \cup_{i\in\Delta} A_i^c \\
\Rightarrow &A \nsubseteq A_i^c \text{ for all } i \in \Delta \\
\Rightarrow &AqA_i \text{ for all } i \in \Delta
\end{aligned}$$

Proposition 3.8.
1. $A\Omega B = B\Omega A$.
2. $AqB \Leftrightarrow A\Omega B \neq \emptyset$.
3. $A \subseteq B \Rightarrow A\Omega C \subseteq B\Omega C$.
4. $A\Omega(\cup_{i\in\Delta} A_i) = \cup_{i\in\Delta} (A\Omega A_i)$.
5. $A\Omega(\cap_{i\in\Delta} A_i) \subseteq \cap_{i\in\Delta} (A\Omega A_i)$.

Proof:

(1) $A\Omega B = \{\varkappa \in V: AqB \text{ at } \varkappa\} = \{\varkappa \in V: BqA \text{ at } \varkappa\} = B\Omega A$.

(2) $AqB \Leftrightarrow AqB$ at some $\varkappa \in V \Leftrightarrow \varkappa \in A\Omega B$. Therefore $AqB \Leftrightarrow A\Omega B \neq \emptyset$.

(3) $A \subseteq B \Rightarrow \mathcal{T}_A(\varkappa) \leq \mathcal{T}_B(\varkappa), \mathcal{C}_A(\varkappa) \leq \mathcal{C}_B(\varkappa), \mathcal{U}_A(\varkappa) \geq \mathcal{U}_B(\varkappa), \mathcal{F}_A(\varkappa) \geq \mathcal{F}_B(\varkappa)$ for all $\varkappa \in V$. Now

$\varkappa \in A\Omega C$
$\Rightarrow AqC$ at $\varkappa \in V$
$\Rightarrow \mathcal{T}_A(\varkappa) > \mathcal{T}_{C^c}(\varkappa)$ and $\mathcal{C}_A(\varkappa) > \mathcal{C}_{C^c}(\varkappa)$ or $\mathcal{U}_A(\varkappa) < \mathcal{U}_{C^c}(\varkappa)$ or $\mathcal{F}_A(\varkappa) < \mathcal{F}_{C^c}(\varkappa)$
$\Rightarrow \mathcal{T}_B(\varkappa) > \mathcal{T}_{C^c}(\varkappa)$ and $\mathcal{C}_B(\varkappa) > \mathcal{C}_{C^c}(\varkappa)$ or $\mathcal{U}_B(\varkappa) < \mathcal{U}_{C^c}(\varkappa)$ or $\mathcal{F}_B(\varkappa) < \mathcal{F}_{C^c}(\varkappa)$
$\Rightarrow BqC$ at $\varkappa \in V$
$\Rightarrow \varkappa \in B\Omega C$
$\therefore A\Omega C \subseteq B\Omega C$.

(4)

$\varkappa \in A\Omega(\cup_{i\in\Delta} A_i)$
$\Rightarrow Aq(\cup_{i\in\Delta} A_i)$ at $\varkappa \in V$
$\Rightarrow \exists j \in \triangle$ such that AqA_j at $\varkappa \in V$
$\Rightarrow \exists j \in \triangle$ such that $\varkappa \in A\Omega A_j$
$\Rightarrow \varkappa \in \cup_{i\in\Delta} (A\Omega A_i)$
$\therefore A\Omega(\cup_{i\in\Delta} A_i) \subseteq \cup_{i\in\Delta} (A\Omega A_i)$.

Again

$\varkappa \in \cup_{i\in\Delta} (A\Omega A_i)$
$\Rightarrow \bigvee_{i\in\Delta} (AqA_i \text{ at } \varkappa \in V)$
$\Rightarrow \bigvee_{i\in\Delta} (A_iqA \text{ at } \varkappa \in V)$
$\Rightarrow \bigvee_{i\in\Delta} \left[\mathcal{T}_{A_i}(\varkappa) > \mathcal{T}_{A^c}(\varkappa) \text{ and } \mathcal{C}_{A_i}(\varkappa) > \mathcal{C}_{A^c}(\varkappa) \text{ or } \mathcal{U}_{A_i}(\varkappa) < \mathcal{U}_{A^c}(\varkappa) \text{ or } \mathcal{F}_{A_i}(\varkappa) < \mathcal{F}_{A^c}(\varkappa)\right]$
$\Rightarrow \sup_{i\in\Delta}\mathcal{T}_{A_i}(\varkappa) > \mathcal{T}_{A^c}(\varkappa)$ and $\sup_{i\in\Delta}\mathcal{C}_{A_i}(\varkappa) > \mathcal{C}_{A^c}(\varkappa)$ or $\inf_{i\in\Delta}\mathcal{U}_{A_i}(\varkappa) < \mathcal{U}_{A^c}(\varkappa)$ or $\inf_{i\in\Delta}\mathcal{F}_{A_i}(\varkappa) < \mathcal{F}_{A^c}(\varkappa)$
$\Rightarrow \mathcal{T}_{\cup A_i}(\varkappa) > \mathcal{T}_{A^c}(\varkappa)$ and $\mathcal{C}_{\cup A_i}(\varkappa) > \mathcal{C}_{A^c}(\varkappa)$ or $\mathcal{U}_{\cup A_i}(\varkappa) < \mathcal{U}_{A^c}(\varkappa)$ or $\mathcal{F}_{\cup A_i}(\varkappa) < \mathcal{F}_{A^c}(\varkappa)$
$\Rightarrow (\cup_{i\in\Delta} A_i)qA$ at $\varkappa \in V$
$\Rightarrow Aq(\cup_{i\in\Delta} A_i)$ at $\varkappa \in V$
$\Rightarrow \varkappa \in A\Omega(\cup_{i\in\Delta} A_i)$
$\therefore \cup_{i\in\Delta} (A\Omega A_i) \subseteq A\Omega(\cup_{i\in\Delta} A_i)$
Hence $A\Omega(\cup_{i\in\Delta} A_i) = \cup_{i\in\Delta} (A\Omega A_i)$.

(5)

$\varkappa \in A\Omega(\cap_{i\in\Delta} A_i)$
$\Rightarrow Aq(\cap_{i\in\Delta} A_i)$ at $\varkappa \in V$
$\Rightarrow AqA_i$ at $\varkappa \in V$ for all $i \in \triangle$
$\Rightarrow \varkappa \in A\Omega A_i$ for all $i \in \triangle$
$\Rightarrow \varkappa \in \cap_{i\in\Delta} (A\Omega A_i)$
$\therefore A\Omega(\cap_{i\in\Delta} A_i) \subseteq \cap_{i\in\Delta} (A\Omega A_i)$

Definition 3.9. Assume that (V, τ) be a QNTS. A QNS A is referred to as a quadripartitioned neutrosophic quasi-neighbourhood or simply Q-neighbourhood (Q-nhbd, for short) of a QNP $\varkappa_{\alpha,\beta,\gamma,\delta}$ iff there exists a QNS $B \in \tau$ such that $\varkappa_{\alpha,\beta,\gamma,\delta}qB \subseteq A$.

The system of Q-neighbourhoods, also known as the Q-neighbourhood system of ϰ_(α,β,γ,δ), is the family that includes all of the Q-neighbourhoods of the QNP ϰ_(α,β,γ,δ). This family is indicated by $N_Q(\varkappa_{\alpha,\beta,\gamma,\delta})$.

Proposition 3.10. Every neutrosophic open set A in a QNTS(V, τ) is a Q-nhbd of every QNP quasi-coincident with A.

Proof: It is evident because for every QNP $\varkappa_{\alpha,\beta,\gamma,\delta}qA$, we have $\varkappa_{\alpha,\beta,\gamma,\delta}qA \subseteq A$.

Properties of Quadripartitioned Neutrosophic Q-Neighbourhoods

Theorem 3.11. Let $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ be the collection of all Q-neighbourhoods of the QNP $\varkappa_{\alpha,\beta,\gamma,\delta}$ in a QNTS (V,τ). Then,
(a) $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \neq \emptyset$ for every $QNP\varkappa_{\alpha,\beta,\gamma,\delta\in\mathcal{QPN}(X)}$.
(b) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow \varkappa_{\alpha,\beta,\gamma,\delta}qP$.
(c) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}), P \subseteq Q \Rightarrow Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$.
(d) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow$ there exists a $Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ such that $Q \subseteq P$ and $Q \in \mathrm{QN_Q}(\mu_{\alpha',\beta',\gamma',\delta'})$ for every QNP $\mu_{\alpha',\beta',\gamma',\delta'}$ quasicoincident with Q.

Proof:(a) Obviously $\tilde{V}$ is a Q-nhbd of every QNP $x_{\alpha,\beta,\gamma,\delta} \in \mathcal{QN}(V)$. Thus there is at least one Q-nhbd for every QNP $\varkappa_{\alpha,\beta,\gamma,\delta} \in \mathcal{QN}(X)$. Therefore $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \neq \emptyset$ for every QNP $\varkappa_{\alpha,\beta,\gamma,\delta} \in \mathcal{QN}(X)$.
(b) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow P$ is a Q-nhbd of $\varkappa_{\alpha,\beta,\gamma,\delta} \Rightarrow \exists$ a $S \in \tau$ such that $\varkappa_{\alpha,\beta,\gamma,\delta q} \subseteq P$. Therefore $\varkappa_{\alpha,\beta,\gamma,\delta}qP$.
(c) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow P$ is a Q-nhbd of $\varkappa_{\alpha,\beta,\gamma,\delta} \Rightarrow \exists$ an open set G such that $\varkappa_{\alpha,\beta,\gamma,\delta}qG \subseteq P \Rightarrow \exists$ an open set G such that $\varkappa_{\alpha,\beta,\gamma,\delta q}G \subseteq Q \Rightarrow Q$ is a Q-nhbd of $\varkappa_{\alpha,\beta,\gamma,\delta} \Rightarrow Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma})$.
(d) Since $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$, so there exists a τ-open set Q such that $\varkappa_{\alpha,\beta,\gamma,\delta}qQ \subseteq P$. Since Q is an open set, so $Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$. Thus $Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ and $Q \subseteq P$.
Again since Q is an open set, so Q is a Q-nhbd of every QNP quasicoincident with Q. Therefore $Q \in \mathrm{QN_Q}(\mu_{\alpha',\beta',\gamma',\delta'})$ for every QNP $\mu_{\alpha',\beta',\gamma',\delta'}$ quasi-coincident with Q.
Hence proved.

Characterization of QNTS in terms of quadripartitioned Neutrosophic Q-Neighbourhoods

Theorem 3.12. Suppose that $V \neq \emptyset$ is any set. Let $x \in X$. Let $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ be a family of all QNSs over X satisfying the following conditions :
(N1) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow \varkappa_{\alpha,\beta,\gamma,\delta q}qP$.
(N2) $P, Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow P \cap Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$.
(N3) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}), P \subseteq Q \Rightarrow Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$.
Then there exists a Quadripartitioned Neutrosophic Topology (QNT)τ on X. Furthermore, if the subsequent condition (N4) is also met then $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ is exactly the Q-neighbourhood system of $\varkappa_{\alpha,\beta,\gamma,\delta}$ in the QNTS(X,τ).
(N4) $P \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow$ there exists a $Q \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ such that $Q \subseteq P$ and $Q\mathrm{QN_Q}(\mu_{\alpha',\beta',\gamma',\delta'})$ for every NP $\mu_{\alpha',\beta',\gamma',\delta'}$ quasicoincident with Q.

Proof: We construct τ in this way:
A QNS $G \in \tau$ iff $G\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ whenever $\varkappa_{\alpha,\beta,\gamma,\delta}qG$.
We claim that τ is a WNT on X.
T1) $\tilde{\emptyset} \in \tau$ as no QNP is quasi-coincident with $\tilde{\emptyset}$. By (N3), $\tilde{X} \in \tau$. Thus $\tilde{\emptyset}, \tilde{X} \in \tau$.

T2) Suppose $G_1, G_2 \in \tau$ and $\varkappa_{\alpha,\beta,\gamma,\delta}q(G_1 \cap G_2)$. Since $\varkappa_{\alpha,\beta,\gamma,\delta}q(G_1 \cap G_2)$, so $\varkappa_{\alpha,\beta,\gamma,\delta}qG_1$ and $\varkappa_{\alpha,\beta,\gamma,\delta}qG_2$. Therefore $G_1, G_2 \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ and so, by (N2), $G_1 \cap G_2 \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$.

T3) Suppose $\{G_i : i \in \Delta\} \subseteq \tau$ and $\varkappa_{\alpha,\beta,\gamma,\delta}q(\cup_{i\in\Delta} G_i)$. We show that $\cup\{G_i : i \in \Delta\} \in \tau$. Now $x_{\alpha,\beta,\gamma,\delta}q(\cup_{i\in\Delta} G_i) \Rightarrow \exists \mathrm{a} j \in \Delta$ such that $\varkappa_{\alpha,\beta,\gamma,\delta}qG_j \Rightarrow \exists \mathrm{a} j \in \Delta$ such that $G_j \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \Rightarrow \cup\{G_i : i \in \Delta\} \in \mathrm{N}(\varkappa_{\alpha,\beta,\gamma,\delta})$[by (N3)] $\Rightarrow \cup\{G_i : i \in \Delta\} \in \tau$.
Therefore τ is a QNT on X.
Assume that (N4) is fulfilled. Assume that all Q-neighborhoods of the QNP ϰ_(α,β,γ,δ) in (V,τ) belong to the family QN_Q^* (ϰ_(α,β,γ,δ)). The equality QN_Q^* (ϰ_(α,β,γ,δ))=QN_Q (ϰ_(α,β,γ,δ)) is demonstrated.
Let $N \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$. By (N4), then, for each QNP μ_(α^',β^',γ^',δ^') quasi-coincident with M, there exists a M∈QN_Q (ϰ_(α,β,γ,δ)) such that M⊆N and M∈QN_Q (μ_(α^',β^',γ^',δ^')). [by (N1)] Now, M∈QN_Q (ϰ_(α,β,γ,δ))⇒ϰ_(α,β,γ,δ) qM. Consequently, M∈τ. M is therefore a τ-open set in which ϰ_(α,β,γ,δ) qM⊆N. Therefore $N \in \mathrm{QN_Q^*}(\varkappa_{\alpha,\beta,\gamma,\delta})$ and so $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) \subseteq \mathrm{QN_Q^*}(\varkappa_{\alpha,\beta,\gamma,\delta})$. Conversely let $N \in \mathrm{QN_Q^*}(\varkappa_{\alpha,\beta,\gamma,\delta})$ so that

N is a Q-nhbd of $\varkappa_{\alpha,\beta,\gamma,\delta}$. Then there exists a τ-open set G such that $\varkappa_{\alpha,\beta,\gamma,\delta} qG \subseteq N$. Therefore $G \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$.

But $G \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$ and $G \subseteq N$ together imply by (N 3) that $N \in \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$. Therefore $\mathrm{QN_Q^*}(\varkappa_{\alpha,\beta,\gamma,\delta}) \subseteq \mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta})$. Consequently $\mathrm{QN_Q}(\varkappa_{\alpha,\beta,\gamma,\delta}) = \mathrm{QN_Q^*}(\varkappa_{\alpha,\beta,\gamma,\delta})$. Thereby, it was demonstrated.

Definition 3.13. Let $\zeta \subseteq \mathcal{QN}(X)$. Then ζ is known as a Quadripartitioned Neutrosophic Quasi Coincident Topology (QNQCT) with $C(\in X)$ on X if

(i) $O_{QN} qC, 1_{QN} qC$

(ii) $(G_1 \cap G_2) qC$ for any $G_1 qC, G_2 qC$

(iii) $\cup G_i qC \ \forall G_i qC$

If ζ is a QNQCT on X then the pair (X, ζ) is known as a QNQCT space over X.

Conclusions

In this chapter, we have established the notion of quadripartitioned neutrosophic quasi coincident with a quadripartitioned neutrosophic set and quadripartitioned neutrosophic quasi coincident with a quadripartitioned neutrosophic point. We also define quadripartitioned neutrosophic point. This chapter contains the various property of quadripartitioned neutrosophic set with some examples. We have defined quadripartitioned neutrosophic quasi coincident topological space. Hopefully, this chapter will helpful for the further investigation on various continuous function in quadripartitioned neutrosophic set.

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Chapter 3

Neutrosophic Supra β-Open Set in Neutrosophic Supra Topological Space

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ABSTRACT

This article aims to present the idea of neutrosophic supra β-open set (NS-β-O-S) via neutrosophic supra topological space (NSTS) as an expansion of neutrosophic supra α-open set (NS-α-O-S). Besides, we establish several results on NS-β-O-S via NSTS. Furthermore, we provide several appropriate examples to support the findings presented in this article.

Keywords: NSTS; NS-α-O-S; NS-β-O-S; Neutrosophic Set

INTRODUCTION

The principle of fuzzy set (FS) [50] was first established by Prof. L.A. Zadeh in 1965, with every component having a membership value lies between zero and one. Atanassov [3] then extended the idea of FS through the addition of Intuitionistic FS (IFS), in which each element has both membership and non-membership values ranging from 0 to 1. In 1998, Smarandache [45] grounded Neutrosophic Set (NS) theory as a further development of FS and IFS theory, with each element having three independent membership values ranging from zero to one, namely truth-membership, indeterminacy-membership and falsity-membership. Information on new neutrosophic theories and their applications are available in the studies [4, 33, 35-36, 46-48].

Till now, many researchers around the globe used the notion NSs in algebra [14] as well as decision making strategies [10-11, 29]. Salama and Alblowi [41] later developed the idea of neutrosophic topological space (NTS) in 2012 as a further development of intuitionistic fuzzy topological space [6]. Salama and Alblowi [42] additionally grounded the generalized NTSs. In the year 2016, Iswarya and Bageerathi [27] introduced neutrosophic semi-open set in NTS. Later, Arokiarani et al. [2] presented the concept of neutrosophic semi-open functions and explored various relationships between them via NTSs. Pushpalatha and Nandhini [39] went on to investigate the neutrosophic generalized closed sets in NTSs. Afterwards, Rao and Srinivasa [40] proposed the neutrosophic pre-open and neutrosophic pre-closed sets using NTS. As a result, Ebenanjar et al. [26] established the *b*-open sets from the standpoint of NTSs. Later on, Maheswari et al. [28] developed the notion of neutrosophic generalized *b*-closed set in NTSs. Das and Pramanik [17] went on to investigate the idea of generalized neutrosophic *b*-open sets in NTSs. Afterwards, the idea of Φ-open sets and Φ-continuous functions was grounded by Das and Pramanik [19] via NTSs in the context of NS. Thereafter, Das and Tripathy [22] proposed the simply *b*-open sets via NTS. Das et al. [9] established the quadripartitioned neutrosophic topology in 2021. Later on, Das and Tripathy [21] studied the notion of pentapartitioned NTS as a further development of NTS. Thereafter, Das and Pramanik [20] presented the notion of neutrosophic tri-topological space in the year 2021.

Mashhour et al. [31] pioneered the ideas of Supra Topology (ST). Devi et al. [23] went on to investigate the idea of supra-open sets and supra-continuous functions using supra topological space (STS). In 1987, Abd El-Monsef and Ramadan [1] established the concept of fuzzy STSs. Turanl [49] then grounded the idea of intuitionistic fuzzy STS (IFSTS). The idea of intuitionistic fuzzy β-supra open set and intuitionistic fuzzy β-supra continuous functions in IFSTS was grounded by Parimala and Indirani [32]. In 2017, Dhavanseelan et al. [25] grounded the concept of neutrosophic STS (NSTS) by extending the notion of NTS and IFSTS, and presented the idea of neutrosophic supra-semi-open set (NSSO-S), neutrosophic supra-pre-open set (NSPO-S), neutrosophic supra-semi continuous function and neutrosophic supra-pre continuous function via NSTSs. The concept of neutrosophic supra-α-open set (NS-α-O-S) and neutrosophic supra-α-continuous functions (NS-α-C-function) via NSTSs was first grounded by Dhavanseelan et al. [24]. Later on, the notion of neutrosophic simply soft open set via neutrosophic soft topological space was presented by Das and Pramanik [18]. In 2021, Das [7] studied the notion of neutrosophic supra simply compactness in the sense of neutrosophic supra simply open set via NSTSs. In 2022, the notion of neutrosophic infi-semi-open set was established by Das et al. [15]. In 2023, Das et al. [12] grounded the notion of single-valued quadripartitioned neutrosophic minimal structure space. In 2024, Das and Das [8] introduced the concept of pairwise neutrosophic infra pre-open set in infra neutrosophic bitopological space. Later on, Poojary et al. [34] grounded the notion of quadripartitioned neutrosophic pre-open set. Thereafter, the notion of neutrosophic supra bitopological spaces was grounded by Das et al. [13]. Recently, Das et al. [16] presented the notion of interval-valued NTS.

In this article, we introduce the concept of NS-β-O-S through NSTS, and derive several results related to NSTSS. Further, we provide several appropriate examples to support the findings presented in this article.

Research Gap: Recent literature lacks research on NS-β-O-S and neutrosophic supra β-continuous function (NS-β-C-function) via NSTS.

Motivation: To address the gap of research, we procure the idea of NS-β-O-S and NS-β-C-function via NSTS, and provide several appropriate examples to support the findings presented in this article.

The layout of this article is given below:

Section	Content
1	Introduction
2	Presents some basic definitions and results on NSTS
3	Presents the notions of NS-β-O-S and NS-β-C-function via NSTS, and established some results on them
4	Conclude the article, and states some directions for further research

BASIC DEFINITIONS AND RESULTS

Throughout the section, we discuss several preliminary definitions and findings about NSTSs that will be beneficial when preparing the key findings of this article.

A collection ɼ of NSs defined over a universal set Ꞓ is referred to as a neutrosophic ST (NST) [25] if the following two conditions hold:

(i) $0_{Ꞓ}$, $1_{Ꞓ}$∈ɼ;

(ii) $\cup_{i\in\Delta}C_i$ ∈ɼ, for every {C_i : i∈Δ}⊆ɼ.

If ɼ is an NST on Ꞓ, then (Ꞓ, ɼ) is referred to as a NSTS. If Ŧ∈ɼ, then Ŧ and $Ŧ^c$ are referred to as a neutrosophic supra open set (NSO-S) and neutrosophic supra closed set (NSC-S) respectively in the NSTS (Ꞓ, ɼ).

The neutrosophic supra interior (N^S_{int}) [25] and the neutrosophic supra closure (N^S_{cl}) of Ǽ in the NSTS (Ꞓ, ɼ) are defined as follows:

(i) N^S_{int}(Ǽ) = ∪ { Ŧ : Ŧ is an NSO-S in (Ꞓ, ɼ) and Ŧ ⊆Ǽ};

(ii) N^S_{cl}(Ǽ) = ∩ { Ŧ : Ŧ is an NSC-S in (Ꞓ, ɼ) and Ǽ⊆ Ŧ }.

Consider a NSTS (Ꞓ, ɼ). An NS Ǽ defined over Ꞓ is referred to as

(i) NS-α-O-S [24] in (Ꞓ, ɼ) if Ǽ ⊆ $N^S_{int}(N^S_{cl}(N^S_{int}$(Ǽ)));

(ii) NSSO-S [25] in (Ꞓ, ɼ) if Ǽ ⊆ $N^S_{cl}(N^S_{int}$(Ǽ));

(iii) NSPO-S [25] in (Ꞓ, ɼ) if Ǽ ⊆ $N^S_{int}(N^S_{cl}$(Ǽ)).

Theorem 2.1. [25] In a NSTS (Ꞓ, ɼ), every NS-α-O-S is NSSO-S (respectively NSPO-S).

Theorem 2.2. [24] Let (Ꞓ, ɼ) be a NSTS.

(i) If Ǽ and Ħ be two NS-α-O-Ss, then Ǽ∪Ħ is also a NS-α-O-S in (Ꞓ, ɼ);

(ii) If Ǽ and Ħ be two NS-α-C-Ss, then Ǽ∩Ħ is also a NS-α-C-S in (Ꞓ, ɼ).

Remark 2.1. [24] Suppose that (Ꞓ, ɼ) be a NSTS.

(i) If Ǽ and Ħ be two NS-α-O-Ss, then Ǽ∩Ħ may not be a NS-α-O-S in (Ꞓ, ɼ);

(ii) If Ǽ and Ħ be two NS-α-C-Ss, then Ǽ∪Ħ may not be a NS-α-C-S in (Ꞓ, ɼ).

The neutrosophic supra α-closure [24] and the neutrosophic supra α-interior of Ǽ in a NSTS (Ꞓ, ɼ) are defined as follows:

(i) $N^S_{\alpha-cl}$(Ǽ) = ∩ { Ŧ : Ŧ is a NS-α-C-S in (Ꞓ, ꞅ) and Ŧ ⊇Ǽ};
(ii) $N^S_{\alpha-int}$(Ǽ) = ∪ { Ŧ : Ŧ is a NS-α-O-S in (Ꞓ, ꞅ) and Ŧ ⊆Ǽ}.

Assume that (Ꞓ, ꞅ) and (Ж, σ) be two NSTSs. A bijective function f from (Ꞓ, ꞅ) into (Ж, σ) is referred to as a neutrosophic supra continuous function [24] if f^{-1}(Ǽ) is a NSO-S in Ꞓ, whenever Ǽ is an NSO-S in Ж.

A bijective function f: (Ꞓ, ꞅ)→(Ж, σ) is referred to as a neutrosophic supra α-continuous function (NS-α-C-f) [24] if f^{-1}(Ǽ) is a NS-α-O-S in Ꞓ, whenever Ǽ is an NSO-S in Ж.

Theorem 2.3. [24] Every NS-C-f is a NS-α-C-f.

Neutrosophic Supra β-Open Set and Neutrosophic Supra β-Continuous Function

Throughout this section, we grounded the idea of NS-β-O-S and NS-β-C-f, and provide several appropriate examples to support the findings presented in this article.

Definition 3.1. A NS Ħ is referred to as a NS-β-O-S in a NSTS (Ꞓ, ꞅ) if Ħ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ħ))).

Example 3.1. Suppose that Ꞓ ={ῃ̃, ẚ}. Clearly, ꞅ={0_N, 1_N, X, Ǽ} is a NST on Ꞓ, where X={(ῃ̃, 0.5, 0.3, 0.1), (ẚ, 0.7, 0.3, 0.2)} and Ǽ={(ῃ̃, 0.5, 0.4, 0.2), (ẚ, 0.7, 0.4, 0.3)} are NSs over Ꞓ. Then, the NS A={(a, 0.6, 0.3, 0.2), (ẚ, 0.4, 0.2, 0.3)} is a NS-β-O-S in (Ꞓ, ꞅ).

Remark 3.1. If G is a NS-β-O-S in (Ꞓ, ꞅ), then G^c will be referred to as a NS-β-C-S in (Ꞓ, ꞅ).

Example 3.2. Let us consider a NSTS (Ꞓ, ꞅ) as shown in Example 3.1. Then, B={(ῃ̃, 0.4, 0.7, 0.8), (ẚ, 0.6, 0.8, 0.7)} is a NS-β-C-S in (Ꞓ, ꞅ), because B^c=A is a NS-β-O-S in (Ꞓ, ꞅ).

Theorem 3.1. Every NSO-S in (Ꞓ, ꞅ) is a NS-β-O-S in (Ꞓ, ꞅ).

Proof. Assume that Ǽ be a NSO-S in (Ꞓ, ꞅ). So, Ǽ⊆$N^S_{cl}(N^S_{int}$(Ǽ)). It is known that, N^S_{int}(Ǽ) ⊆ Ǽ and Ǽ ⊆ N^S_{cl}(Ǽ).

Now, we have N^S_{int}(Ǽ) ⊆ $N^S_{int}(N^S_{cl}$(Ǽ))
⇒ $N^S_{cl}(N^S_{int}$(Ǽ)) ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ǽ)))
⇒ Ǽ ⊆ $N^S_{cl}(N^S_{int}$(Ǽ)) ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ǽ)))
⇒ Ǽ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ǽ)))

Therefore, Ǽ is a NS-β-O-S in (Ꞓ, ꞅ).

Example 3.3. Assume that (Ꞓ, ꞅ) be a NSTS as shown in Example 3.1. Clearly, the NS A={(ῃ̃, 0.6, 0.3, 0.2), (ẚ, 0.4, 0.2, 0.3)} is a NS-β-O-S in (Ꞓ, ꞅ), but it is not a NSO-S in (Ꞓ, ꞅ).

Theorem 3.2. If Ǽ is a NS-α-O-S in the NSTS (Ꞓ, ꞅ), then it is a **NS**-β-O-S in (Ꞓ, ꞅ).

Proof. Assume that Ǽ be a NS-α-O-S in a NSTS (Ꞓ, ꞅ). So, Ǽ⊆$N^S_{int}(N^S_{cl}(N^S_{int}$(Ǽ))). It is known that, N^S_{int}(Ǽ) ⊆ Ǽ and Ǽ ⊆ N^S_{cl}(Ǽ).

Now, we have $N^S_{cl}(N^S_{int}$(Ǽ)) ⊆ N^S_{cl}(Ǽ)
⇒ $N^S_{int}(N^S_{cl}(N^S_{int}$(Ǽ))) ⊆ $N^S_{int}(N^S_{cl}$(Ǽ))
⇒ Ǽ ⊆ $N^S_{int}(N^S_{cl}$(Ǽ)) [Ǽ ⊆ $N^S_{int}(N^S_{cl}(N^S_{int}$(Ǽ)))]
⇒ N^S_{cl}(Ǽ) ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ǽ)))
⇒ Ǽ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ǽ))) [Ǽ ⊆ N^S_{cl}(Ǽ)]

Hence, Ǽ is a NS-β-O-S in (Ꞓ, ꞅ).

Theorem 3.3. If Ħ is a NSPO-S in the NSTS (Ꞓ, ꞅ), then it is a NS-β-O-S in (Ꞓ, ꞅ).

Proof. Suppose that (Ꞓ, ꞅ) be a NSTS. Let Ħ be a NSPO-S in (Ꞓ, ꞅ). Therefore, Ħ⊆$N^S_{int}(N^S_{cl}$(Ħ)). It is known that, Ħ⊆N^S_{cl}(Ħ).

Now, Ħ ⊆ $N^S_{int}(N^S_{cl}$(Ħ))
⇒ N^S_{cl}(Ħ) ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ħ)))
⇒ Ħ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}$(Ħ))) [Ħ ⊆ N^S_{cl}(Ħ)]

So, Ħ is a NS-β-O-S.

Theorem 3.4. Both the NSs $0_Ꞓ$, $1_Ꞓ$ are NS-β-O-Ss in the NSTS (Ꞓ, ꞅ).

Proof. Since the proof is straightforward, it was omitted out.

Theorem 3.5. If $Ħ_1$, $Ħ_2$ be two NS-β-O-Ss in (Ꞓ, ꞅ), then their union $Ħ_1$∪$Ħ_2$ is also a NS-β-O-S.

Proof. Suppose that $Ħ_1$, $Ħ_2$ be two NS-β-O-Ss in (Ꞓ, ꞅ). Therefore,
$Ħ_1$ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_1)))$ and $Ħ_2$ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_2)))$.

Now, $Ħ_1$ ∪ $Ħ_2$ ⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_1)))$∪$N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_2)))$
⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_1))$∪$N^S_{int}(N^S_{cl}(Ħ_2)))$
⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_1)$∪$N^S_{cl}(Ħ_2)))$
⊆ $N^S_{cl}(N^S_{int}(N^S_{cl}(Ħ_1$∪$Ħ_2)))$

Therefore, $Ħ_1 \cup Ħ_2$ is a NS-β-O-S in (Ꞡ, ɍ).
Remark 3.2. The intersection of any two NS-β-O-Ss may not be a NS-β-O-S in general. The example below demonstrates this.
Example 3.4. Suppose that (Ꞡ, ɍ) be a NSTS as shown in Example 3.1. Clearly, P={(η̃, 0.6, 0.7, 0.2), (ẳ, 0.4, 0.7, 0.3)} and Q={(η̃, 0.5, 0.5, 0.9), (ẳ, 0.3, 0.5, 0.8)} are NS-β-O-Ss in (Ꞡ, ɍ), but their intersection i.e., P∩Q is not a NS-β-O-S in (Ꞡ, ɍ).
Theorem 3.6. The NS $0_{Ꞡ}$ is an NS-β-C-S in (Ꞡ, ɍ).
Proof. Let (Ꞡ, ɍ) be a NSTS. It is known that, $(0_W)^c = 1_W$, the neutrosophic whole set. By a known result, we can say 1_W is a NS-β-O-S. This implies, 0_W (complement of 1_W) is a NS-β-C-S in (Ꞡ, ɍ).
Theorem 3.7. If V, U be two NS-β-C-Ss in (Ꞡ, ɍ), then V∩U is also a NS-β-C-S.
Proof. Assume that V and U be two NS-β-C-Ss. This implies, V^c and U^c are NS-β-O-Ss in (Ꞡ, ɍ). By Theorem 3.5., it is clear that $V^c \cup U^c = (V \cap U)^c$ is a NS-β-O-S in (Ꞡ, ɍ). This implies, V∩U is a NS-β-C-S in (Ꞡ, ɍ).
Remark 3.3. In a NSTS (Ꞡ, ɍ), the union of any two NS-β-C-Ss may not be a NS-β-C-S in general. The example below demonstrates this.
Example 3.5. Suppose that (Ꞡ, ɍ) be a NSTS as shown in Example 3.1. Then, R={(η̃, 0.3, 0.4, 0.8), (ẳ, 0.7, 0.4, 0.7)} and S={(η̃, 0.5, 0.6, 0.1), (ẳ, 0.7, 0.6, 0.2)} are NS-β-C-Ss in (Ꞡ, ɍ), because R^c and S^c are NS-β-O-Ss in (Ꞡ, ɍ). We have, R∪S={(η̃, 0.5, 0.4, 0.1), (ẳ, 0.7, 0.4, 0.2)}. This implies, $(R∪S)^c$={(η̃, 0.5, 0.6, 0.9), (ẳ, 0.3, 0.6, 0.8)}. Clearly, $(R∪S)^c$ is not a NS-β-O-S in (Ꞡ, ɍ). Hence, R∪S is not a NS-β-C-S in (Ꞡ, ɍ).
Definition 3.2. The neutrosophic supra β-interior ($N^S_{\beta-int}$) and neutrosophic supra β-closure ($N^S_{\beta-cl}$) of a NS Ǽ are defined as follows:
$N^S_{\beta-int}$(Ǽ) = ∪ {Ä : Ä is a NS-β-O-S in Ꞡ and Ä⊆Ǽ},
and $N^S_{\beta-cl}$(Ǽ) = ∩{ Ŧ: Ŧ is a NS-β-C-S in Ꞡ and Ŧ ⊇Ǽ}.

Here, $N^S_{\beta-int}$(Ǽ) is the smallest NS-β-C-S in (Ꞡ, ɍ) which containing Ǽ and $N^S_{\beta-cl}$(Ǽ) is the largest NS-β-O-S in (Ꞡ, ɍ) which is contained in Ǽ.
Theorem 3.8. Suppose that (Ꞡ, ɍ) be an NSTS. Then,
(i) If $\widehat{\Omega} \subseteq Â$, then $(N^S_{\beta-int}(\widehat{\Omega}) \subseteq (N^S_{\beta-int}(Â)$ and $N^S_{\beta-cl}(\widehat{\Omega}) \subseteq N^S_{\beta-cl}(Â)$;
(ii) $(N^S_{\beta-int}(\widehat{\Omega}))^c = N^S_{\beta-cl}(\widehat{\Omega}^c)$;
(iii) $(N^S_{\beta-cl}(\widehat{\Omega}))^c = (N^S_{\beta-int}(\widehat{\Omega}^c)$.
Proof: The proof is so easy, so omitted.
Theorem 3.9. In an NSTS (Ꞡ, ɍ), the following holds:
(i) the intersection of a NSPO-S and a NS-β-O-S is a NSPO-S;
(ii) the intersection of an NSO-S and a NS-β-O-S is a NS-β-O-S.
Proof: The definition of NS-β-O-S clearly indicates this.
Definition 3.3. Assume that (Ꞡ, ɍ) and (Җ, σ) be two NSTSs. Then, an one to one and onto function f:(Ꞡ, ɍ)→(Җ, σ) is referred to as a NS-β-C-f if f^{-1}(Ħ) is a NS-β-O-S in Ꞡ, whenever Ħ is a NSO-S in Җ.
Theorem 3.10. If f:(Ꞡ, ɍ)→(Җ, σ) is a NS-C-f, then it is also a NS-β-C-f.
Proof: Assume that Ħ be a NSO-S in Җ. By hypothesis, f^{-1}(Ħ) is a NSO-S in Ꞡ. Again since, every NSO-S is a NS-β-O-S, so f^{-1}(Ħ) is a NS-β-O-S in Ꞡ. Therefore, f:(Ꞡ, ɍ)→(Җ, σ) is a NS-β-C-f.
Remark 3.4. Every NS-β-C-f may not be a NS-C-f in general. The example below demonstrates this.
Example 3.6. Suppose that Ꞡ ={η̃, ẳ} and Җ={u, v} be two fixed sets. Then, ɍ={0_N, 1_N, X, Ǽ} and σ={0_N, 1_N, E, L} are NSTs on Ꞡ and Җ respectively such that X={(η̃, 0.5, 0.3, 0.1), (ẳ, 0.7, 0.3, 0.2)}, Ǽ={(η̃, 0.5, 0.4, 0.2), (ẳ, 0.7, 0.4, 0.3)}, E=(u, 0.6, 0.4, 0.4), (v, 0.7, 0.5, 0.6)} and L={(u, 0.7, 0.3, 0.3), (v, 0.7, 0.4, 0.5)}. Define a one to one and onto function f:(Ꞡ, ɍ)→(Җ, σ) such that f(0_N)=0_N, f(1_N)=1_N, f(R)=E, f(S)=L, and so on, where R={(η̃, 0.7, 0.6, 0.2), (ẳ, 0.3, 0.6, 0.3)} and S={(η̃, 0.5, 0.4, 0.9), (ẳ, 0.3, 0.4, 0.8)}. Clearly, the inverse image of the NSO-Ss E, L in (Җ, σ) are NS-β-O-Ss in (Ꞡ, ɍ). Therefore, the function f:(Ꞡ, ɍ)→(Җ, σ) is a NS-β-C-f. But, the function f:(Ꞡ, ɍ)→(Җ, σ) is not a NS-C-f because the NSs R and S are not NSO-Ss in (Ꞡ, ɍ).
Theorem 3.11. A bijective function f: (Ꞡ, ɍ)→(Җ,σ) is a NS-β-C-f iff the inverse image of each NSC-Ss in Җ is a NS-β-C-S in Ꞡ.
Proof: Assume that f: (Ꞡ, ɍ)→(Җ, σ) be a NS-β-C-f. Suppose that $\widehat{\Omega}$ be an arbitrary NSC-S in Җ. Therefore, $\widehat{\Omega}^c$ is an NSO-S in Җ. Since, f is a NS-β-C-f, so $f^{-1}(\widehat{\Omega}^c) = (f^{-1}(\widehat{\Omega}))^c$ is a NS-β-O-S in Ꞡ. This implies, $f^{-1}(\widehat{\Omega})$ is a NS-β-C-S of Ꞡ.
Conversely, let the inverse image of each NSC-Ss in Җ is also a NS-β-C-S in Ꞡ. Let $\widehat{\Omega}$ be an arbitrary NSO-S in Җ. Therefore, $\widehat{\Omega}^c$ is an NSC-S in Җ. By the hypothesis, $f^{-1}(\widehat{\Omega}^c) = (f^{-1}(\widehat{\Omega}))^c$ is a NS-β-C-S in Ꞡ. This implies, $f^{-1}(\widehat{\Omega})$ is a NS-β-O-S in Ꞡ. Therefore, $f^{-1}(\widehat{\Omega})$ is a NS-β-O-S in Ꞡ, whenever $\widehat{\Omega}$ be an arbitrary NSO-S in Җ. Hence, f: (Ꞡ, ɍ)→(Җ, σ) is a NS-β-C-f.
Theorem 3.12. Assume that f: (Ꞡ, ɍ)→(Җ, σ) be a bijective mapping. If the inverse image of each NSC-S in Җ is a NS-β-C-S in Ꞡ, then $N^S_{\beta-cl}(f^{-1}(\widehat{\Omega})) \subseteq f^{-1}(N^S_{cl}(\widehat{\Omega}))$, for each NS $\widehat{\Omega}$ in Җ.

Proof. Assume that f: (Ꞓ, ɼ)→(Ӂ,σ) be a bijective mapping. Suppose that $\widehat{\Omega}$ be a neutrosophic subset of Ӂ. Since, $N^S_{cl}(\widehat{\Omega})$ is an NSC-S in Ӂ, then it follows that $f^{-1}(N^S_{cl}(\widehat{\Omega}))$ is a NS-β-C-S in Ꞓ. Therefore, $f^{-1}(N^S_{cl}(\widehat{\Omega})) = N^S_{\beta-cl}(f^{-1}(N^S_{cl}(\widehat{\Omega}))) \supseteq N^S_{\beta-cl}(f^{-1}(\widehat{\Omega}))$. This implies, $N^S_{\beta-cl}(f^{-1}(\widehat{\Omega})) \subseteq f^{-1}(N^S_{cl}(\widehat{\Omega}))$, for each NS $\widehat{\Omega}$ over Ӂ.

Theorem 3.13. Assume that f: (Ꞓ, ɼ)→(Ӂ, σ) be a bijective mapping. If $N^S_{\beta-cl}(f^{-1}(\widehat{\Omega})) \subseteq f^{-1}(N^S_{cl}(\widehat{\Omega}))$, for each NS $\widehat{\Omega}$ in Ӂ, then $f(N^S_{\beta-cl}(\widehat{\Omega})) \subseteq N^S_{cl}(f(\widehat{\Omega}))$ for each NS $\widehat{\Omega}$ in Ꞓ.

Proof. Assume that f:(Ꞓ, ɼ)→(Ӂ, σ) be a bijective mapping from a NSTS (Ꞓ, ɼ) to another NSTS (Ӂ, σ). Suppose that V be a neutrosophic subset of Ꞓ. By hypothesis, we have $N^S_{\beta-cl}(V) \subseteq N^S_{\beta-cl}(f^{-1}(f(V))) \subseteq f^{-1}(N^S_{cl}(f(V)))$. This implies, $f(N^S_{\beta-cl}(V)) \supseteq N^S_{cl}(f(V))$. Therefore, $f(N^S_{\beta-cl}(V)) \subseteq N^S_{cl}(f(V))$, for every NS V in Ꞓ.

Theorem 3.14. For any bijective function f from (Ꞓ, ɼ) to (Ӂ, σ), if $f^{-1}(N^S_{int}(\widehat{\Omega})) \subseteq N^S_{\beta-int}(f^{-1}(\widehat{\Omega}))$, for each NS $\widehat{\Omega}$ in Ӂ, then f is a NS-β-C-f.

Proof. Suppose that f: (Ꞓ, ɼ)→(Ӂ, σ) be a bijective function from (Ꞓ, ɼ) to (Ӂ, σ). Assume that $\widehat{\Omega}$ be an NSO-S in Ӂ. It is known that, $N^S_{\beta-int}(f^{-1}(\widehat{\Omega})) \subseteq f^{-1}(\widehat{\Omega})$ (1)

Now, by hypothesis $f^{-1}(N^S_{int}(\widehat{\Omega})) \subseteq N^S_{\beta-int}(f^{-1}(\widehat{\Omega}))$. Since, $\widehat{\Omega}$ is a NSO-S, so $N^S_{int}(\widehat{\Omega}) = \widehat{\Omega}$. Therefore, $f^{-1}(\widehat{\Omega}) = f^{-1}(N^S_{int}(\widehat{\Omega})) \subseteq N^S_{\beta-int}(f^{-1}(\widehat{\Omega}))$, which implies, $f^{-1}(\widehat{\Omega}) \subseteq N^S_{\beta-int}(f^{-1}(\widehat{\Omega}))$. (2)

From eq. (1) and eq. (2), we have $f^{-1}(\widehat{\Omega}) = N^S_{\beta-int}(f^{-1}(\widehat{\Omega}))$. Hence, $f^{-1}(\widehat{\Omega})$ is a NS-β-O-S. Therefore, the function f is a NS-β-C-f.

Theorem 3.15. If f: (Ꞓ, ɼ)→(Ӂ, σ) be a NS-β-C-f and g: (Ӂ, σ)→(§, η) be a NS-C-f, then g°f: (Ꞓ, ɼ)→(§, η), the composition function of f: (Ꞓ, ɼ)→(Ӂ, σ) and g: (Ӂ, σ)→(§, η) is a NS-β-C-f.

Proof. Suppose that f: (Ꞓ, ɼ)→(Ӂ, σ) be a NS-β-C-f and g: (Ӂ, σ)→(§, η) be a NS-C-f. Assume that $\widehat{\Omega}$ be a NSO-S in (§, η). Since g: (Ӂ, σ)→(§, η) be a NS-C-f, so $g^{-1}(\widehat{\Omega})$ is a NSO-S in (Ӂ, σ). Further, since f: (Ꞓ, ɼ)→(Ӂ,σ) is a NS-β-C-f, so $(g°f)^{-1}(\widehat{\Omega}) = f^{-1}(g^{-1}(\widehat{\Omega}))$ is a NS-β-O-S in (Ꞓ, ɼ). This implies, the composition function g°f: (Ꞓ, ɼ)→(§, η) is a NS-β-C-f.

Conclusions

We ground the idea of NS-β-O-S and NS-β-C-f via NSTS in this article, and establish several intriguing results on NS-β-O-S and NS-β-C-f in the form of theorems, remarks, etc. Furthermore, we provide several appropriate examples to support the findings.

Future Research Directions

In the future, it is hoped that the idea of NS-β-O-S and NS-β-C-f can also be generalized in the domain of Quadripartitioned Neutrosophic Set (QNS) [5], Quadripartitioned NTS [9], Interval QNS [37], Pentapartitioned Neutrosophic Set (PNS) [30], Interval PNS [38], Pentapartitioned NTS [21], Pentapartitioned Neutrosophic Bitopology [43], Neutrosophic Soft Topological space [18], Pentapartitioned Neutrosophic Soft Set [44], etc.

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Chapter 4

Few Special Types of Neutrosophic Soft Matrices

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ABSTRACT

The main intention of this particular work is to discuss some special neutrosophic soft matrices. These matrices are useful and applicable in situations which are full of uncertainties and imprecision. In this article different types of neutrosophic soft matrices are studied with a view to get rid of some uncertainties which prevails in many cases. Here some operations and some associated properties are discussed to make the concept clear.

Keywords: Neutrosophic sets; neutrosophic soft sets; neutrosophic soft matrix; ToepNSM; TridNSM.

.INTRODUCTION

In day-to-day life, many situations are confronted which are full of uncertainties and imprecisions. Probability, fuzzy set[1], intuitionistic fuzzy se t[2] etc. were the tools to deal with such uncertainties. But soon it was understood that these existing tools are also not sufficient to handle all such situations.

Later on, Molodtsov [3] in due course of time realized that these theories have difficulties in applications and as a result concept of soft set theory was developed. Soft set theory has rich potential for application in solving practical problems in various subject areas. Maji et al. [4], [5] initiated the concept of fuzzy soft set. Later on, intuitionistic fuzzy soft set [6] has been introduced as an extension of the theory of fuzzy soft set.

Smarandache [7] conceptualized neutrosophic sets as a logical method for dealing with certain conditions involving imprecision, inconsistencies and indeterminacy. The newly introduced set was expected to deal with such situations in a more accurate way than those obtained by existing tools such as fuzzy sets or intuitionistic fuzzy sets. The theory of neutrosophic set is extended to neutrosophic soft set [8]by Maji et al. Further Maji et al. [9] applied this theory in decision making process.

Later on many other mathematicians have applied this new concept in various mathematical problems ,as for examples Deli et al ([10], [11],[12]). This concept has been further modified by Deli and Broumi [13] and the idea of neutrosophic soft matrices come into force by successfully utilizing it in many decision making processes. Broumi and Smarandache [14] has introduced the concept of intuitionistic neutrosophic sets and discussed some associated properties therein. Bera and Mahapatra [15] discussed some of the algebraic structure of neutrosophic soft set. Many researchers for example [16] have worked on the theory of neutrosophic soft matrices and applied neutrosophic sets in decision making processes. New development regarding neutrosophic sets and neutrosophic soft matrices can be found in the articles of ([17]-[27] & [29]-[44]). Neutrosophic soft block matrices are nothing but a neutrosophic soft matrix which is a collection smaller neutrosophic soft matrices. Many other researchers as for example ([28], [29]) have studied neutrosophic soft *block* matrices to a certain extent.

Here the main points of discussion are concept of various types of neutrosophic soft matrices and thereafter few operations on these matrices are carried on. Again, the properties of neutrosophic soft matrices are considered for discussion.

BACKGROUND

Definition 1.[7] Neutrosophic sets

Let the universe of discourse be $\hat{U}$. Then the neutrosophic set A on $\hat{U}$ is defined as $A=\{<T_A(u), I_A(u), F_A(u)>: u \in \widehat{U}\}$, where the characteristic functions $T, I, F: \widehat{U} \rightarrow [0,1]$ and $^-0 \le T+I+F \le 3^+$ .Here the three component parts T, I, F describes the degree of membership, indeterminacy and non- membership respectively.

Definition 2.[8] Neutrosophic soft set

Let the initial universe set be $\hat{U}$ and parameters set is E . Let P($\hat{P}(\hat{U})$) denotes the collection of all neutrosophic subsets of $\hat{U}$. Let $\widehat{A} \subseteq \mathrm{E}$. Then $(F_{\hat{A}}, \mathrm{E})$ is called neutrosophic soft set over $\hat{U}$ where the mapping $F_{\hat{A}}$ is defined by $F_{\hat{A}}: \mathrm{E} \rightarrow \widehat{P}(\widehat{U})$

Definition 3.[10] Neutrosophic soft Matrices

Let $(F_{\hat{A}}, \mathrm{E})$ be a neutrosophic soft set over $\hat{U}$. The mapping $F_{\hat{A}}$ defines $F_{\hat{A}}: \mathrm{E} \rightarrow \widehat{P}(\widehat{U})$

The relation form of $(F_{\hat{A}}, \mathrm{E})$ defined by $R_{\hat{A}} = \{(\hat{u}, e), e \in \hat{A}, \hat{u} \in F_{\hat{A}}(e)\}$ represents uniquely the subsets of $(\widehat{U}, \mathrm{E})$ .This characterizes three functions truth, indeterminacy and falsity by $T_{\hat{A}}: \widehat{U} \times \mathrm{E} \rightarrow [0,1]$, $I_{\hat{A}}: \widehat{U} \times \mathrm{E} \rightarrow [0,1]$ and $F_{\hat{A}}: \widehat{U} \times \mathrm{E} \rightarrow [0,1]$ respectively.

Let $\widehat{U} = \{\hat{u}_1, \hat{u}_2, \hat{u}_3, ...\hat{u}_m\}$ be the universe set and the set of parameters be $\mathrm{E} = \{e_1, e_2, e_3, ...e_n\}$. Then tabular form of $R_{\hat{A}}$ is represented by

R_N	e_1	e_2	…..	e_n
$\hat{u}_1$	$(T_{A_{11}}, I_{A_{11}}, F_{A_{11}})$	$(T_{A_{11}}, I_{A_{11}}, F_{A_{11}})$	…..	$(T_{A_{11}}, I_{A_{11}}, F_{A_{11}})$
$\hat{u}_2$	$(T_{A_{21}}, I_{A_{21}}, F_{A_{21}})$	$(T_{A_{22}}, I_{A_{22}}, F_{A_{22}})$	…..	$(T_{A_{2n}}, I_{A_{2n}}, F_{A_{2n}})$
:	………	……	……	
$\hat{u}_m$	$(T_{A_{m1}}, I_{A_{m1}}, F_{A_{m1}})$	$(T_{Am}, I_{Am2}, F_{A_{m2}})$	……..	$(T_{Amn}, I_{A_{1mn}}, F_{A_{mn}})$

Then a neutrosophic soft matrix defined by the above relation can be represented as

$$\tilde{A}_{ij} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix}$$

Here

$$A_{ij} = (T_A(\hat{u}_i, e_j), I_A(\hat{u}_i, e_j), F_A(\hat{u}_i, e_j))$$

Different Types of Neutrosophic Soft Matrices

Definition 4: Trapezoidal neutrosophic soft matrix(TrapNSM)

A neutrosophic soft matrix C is an upper TNSM if the non zero elements exist only in the upper triangular part of the matrix which includes the main diagonal. Then $c_{ij} = (0,0,1), i > j$

A neutrosophic soft matrix D is called a lower TrapNSM if its non- zero elements are found only in the lower triangular part of the matrix in which the main diagonal is also included. That is $d_{ij} = (0,0,1), i < j$

Neutrosophic soft matrices C and D shown below are the examples of upper and lower TrapNSM having order m $\times$ n.

If $m \geq n$;

$$C = \begin{bmatrix} d_{11} & d_{12} & d_{13} & \dots & \dots & d_{1n} \\ (0,0,1) & d_{22} & d_{23} & \dots & \dots & : \\ (0,0,1) & (0,0,1) & u_{33} & \dots & \dots & : \\ : & : & : & \dots & \dots & : \\ (0,0,1) & : & : & : & (0,0,1) & d_{nn} \\ (0,0,1) & : & : & : & & (0,0,1) \\ : & : & : & : & & : \\ (0,0,1) & : & : & : & & (0,0,1) \end{bmatrix}$$

$$D = \begin{bmatrix} l_{11} & (0,0,1) & (0,0,1) & \dots & \dots & (0,0,1) \\ l_{21} & l_{22} & (0,0,1) & \dots & \dots & : \\ l_{31} & l_{32} & l_{33} & \dots & \dots & : \\ : & : & : & \dots & \dots & : \\ & : & : & : & (0,0,1) & (0,0,1) \\ l_{n1} & l_{n2} & : & : & & l_{nn} \\ : & : & : & : & & : \\ l_{m1} & : & : & : & & l_{mn} \end{bmatrix}$$

Definition 5: Toeplitz neutrosophic soft matrix(ToepNSM)

The square neutrosophic soft matrix is called **ToepNSM** which takes the form

$$M^T = \begin{bmatrix} (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (T^A{}_{31}, I^A{}_{31}, F_{31}{}^A) & (T^A{}_{41}, I^A{}_{41}, F_{41}{}^A) \\ (T^A{}_{12}, I^A{}_{12}, F_{12}{}^A) & (T^A_{22}, I^A_{22}, F^A_{22}) & (T^A_{32}, I^A_{32}, F^A_{32}) & (T^A{}_{13}, I^A{}_{13}, F_{13}{}^A) \\ (T^A{}_{13}, I^A_{13}, F_{13}{}^A) & (T^A_{23}, I^A_{23}, F^A_{23}) & (T^A_{33}, I^A_{33}, F^A_{33}) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) \\ (T^A_{14}, I^A_{14}, F^A_{14}) & (T^A_{24}, I^A_{24}, F^A_{24}) & (T \ , I^A{}_{21}, F_{21}{}^A) & (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) \end{bmatrix}$$

Definition 6: Zero neutrosophic soft matrix

Neutrosophic soft matrix having all the entries of the form (0,0,1) is called the **zero neutrosophic soft matrix**.

Definition 7: Sparse Neutrosophic Soft Matrix(SpNSM)

Neutrosophic soft matrices where most of the elements are of the form (0,0,1) is defined as **SpNSM** .For example

$$\tilde{A} = \begin{bmatrix} (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (0,0,1) & (0,0,1)) & (0,0,1) \\ (0.0.1) & (T^A{}_{11}, I^A{}_{11}, F_{11}{}^A) & (0.0.1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) \\ (0,0,1) & (0,0,1) & (T^A{}_{21}, I^A{}_{21}, F_{21}{}^A) & (0,0,1) \end{bmatrix}$$

Here it is to be mentioned that it is not strictly defined how many elements need to be zero for a matrix to be considered as sparse neutrosophic soft matrix though a common method is that the number of non zero elements have to be equal to the number of rows or columns.

Definition 8: Dense Neutrosophic soft matrix

If in a neutrosophic soft matrix , most of the elements are non -zero then that matrix is defined as the dense neutrosophic soft matrix . For example,

$$\tilde{A}=\begin{bmatrix} (T_{11}^A,I_{11}^A,F_{11}^A) & (T_{12}^A,I_{12}^A,F_{12}^A) & (T_{13}^A,I_{13}^A,F_{13}^A) & (T_{14}^A,I_{14}^A,F_{14}^A) \\ (0,0,1) & (T_{22}^A,I_{22}^A,F_{22}^A) & (T^A{}_{23},I^A{}_{23},F_{23}{}^A) & (0.0.1) \\ (T_{31}^A,I_{31}^A,F_{31}^A) & (T_{32}^A,I_{32}^A,F_{32}^A) & (T_{33}^A,I_{33}^A,F_{33}^A) & (T_{34}^A,I_{34}^A,F_{34}^A) \\ (T_{41}^A,I_{41}^A,F_{41}^A) & (0.0.1) & (T_{21}^A,I_{21}^A,F_{21}^A) & (T_{11}^A,I_{11}^A,F_{11}^A) \end{bmatrix}$$

Definition 9: Bidiagonal Neutrosophic soft matrix

Neutrosophic soft matrix having non zero entries along the main diagonal and either the diagonal above or below the main diagonal is called a bidiagonal neutrosophic soft matrix. This states that there exists exactly two non -zero diagonals in the neutrosophic soft matrix.

If the diagonal above the main diagonal contains non-zero entries, then that type of matrix is called upper bidiagonal neutrosophic soft matrix. Similarly, if the diagonal below the main diagonal contains the non- zero entries, then that type of matrix is called lower bidiagonal neutrosophic soft matrix.

For example,

$$A=\begin{bmatrix} (0.6,0.1,0.3) & (0.4,0.2,0.4) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0.6,0.3,0.1) & (0.5,0.2,0.3) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0.5,0.2,0.1) & (0.5,0.2,0.3) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0.5,0.2,0.1) \end{bmatrix}$$

is upper triangular neutrosophic soft matrix and

$$B=\begin{bmatrix} (0.4,0.1,0.2) & (0,0,1) & (0,0,1) & (0,0,1) \\ (0.2,0.6,0.1) & (0.6,0.3,0.1) & (0,0,1) & (0,0,1) \\ (0,0,1) & (0.4,0.5,0.1) & (0.5,0.2,0.1) & (0,0,1) \\ (0,0,1) & (0,0,1) & (0.5,0.4,0.3) & (0.5,0.2,0.1) \end{bmatrix}$$

is lower bidiagonal neutrosophic soft matrix.

Theorem 1: If A and B be two (upper or lower) bidiagonal neutrosophic soft matrices of same order then the sum is also an (upper or lower) bidiagonal neutrosophic soft matrix.

Definition 10: Banded neutrosophic soft matrix

A sparse neutrosophic soft matrix is called a banded or band neutrosophic soft matrix when non zero entries are confined to a diagonal band which comprises of the main diagonal and zero or more diagonals on either side.

Examples of band neutrosophic soft matrices.

i. Banded neutrosophic soft matrix with $k_1 = k_2 = 0$ is called a diagonal neutrosophic soft matrix.
ii. Banded neutrosophic soft matrix with $k_1 = k_2 = 0$ is called a tri diagonal neutrosophic soft matrix.
iii. Banded neutrosophic soft matrix with $k_1 = k_2 = 2$ is called a penta diagonal neutrosophic soft matrix.
iv. Triangular neutrosophic soft matrix
Upper triangular matrix is obtained if $k_1 = 0, k_2 = n-1$.
Lower triangular matrix is obtained if $k_1 = n-1, k_2 = 0$.

Here k_1, k_2 are respectively called lower bandwith and upper bandwith.

Definition 11: Hessenberg Neutrosophic soft matrix

The Hessenberg neutrosophic soft matrix is a square neutrosophic soft matrix of special kind which is almost triangular. These are mainly of two types.

i. Upper Hessenberg Neutrosophic Soft Matrix

A square neutrosophic soft matrix A of order nxn is called upper Hessenberg form or upper Hessenberg matrix if $a_{ij} = 0$ for all i, j with $i > j+1$.

An unreduced upper Hessenberg neutrosophic soft matrix is the one in which subdiagonal entries are nonzero, that is if $a_{i+1,i} \neq 0$ for all $i \in \{1,2,3,......,n-1\}$

ii. Lower Hessenberg Neutrosophic Soft Matrix

A square neutrosophic soft matrix A of order nxn is defined as lower Hessenberg form or lower Hessenberg neutrosophic soft matrix if $a_{ij} = 0$ for all i, j with $j > i+1$.

Unreduced upper Hessenberg neutrosophic soft matrix is the one in which subdiagonal entries are nonzero. i.e if $a_{i,i+1} \neq 0$ for all $i \in \{1,2,3,......,n-1\}$

If all entries below the first subdiagonal has zero entries then that neutrosophic soft matrix is called upper Hessenberg and a lower Hessenberg matrix contains zero entries above the first super diagonal.

For example: If the following neutrosophic soft matrices are taken into consideration

$$A = \begin{bmatrix} (0.4.0.3,0.1) & (0.3,0.1,0.3) & (0.6,0.1,0.2) & (0.4,0.2,0.6) \\ (0.5,0.2,0.3) & (0.6,0.3,0.1) & (0.5,0.2,0.3) & (0.3,0.2,0.4) \\ (0,0,1) & (0.5,0.3,0.2) & (0.5,0.2,0.1) & (0.5,0.2,0.3) \\ (0,0,1) & (0,0,1) & (0.8,0.1,0.1) & (0.5,0.2,0.1) \end{bmatrix}$$

$$B = \begin{bmatrix} (0.5,0.2,0.2) & (0.4,0.2,0.4) & (0,0,1) & (0,0,1) \\ (0.5,0.2,0.3) & (0.6,0.3,0.1) & (0.5,0.2,0.3) & (0,0,1) \\ (0.7,0.2,0.1) & (0.5,0.3,0.2) & (0.5,0.2,0.1) & (0.5,0.2,0.3) \\ (0.4,0.3,0.3) & (0.5,0.3,1) & (0.8,0.1,0.1) & (0.5,0.2,0.1) \end{bmatrix}$$

$$C = \begin{bmatrix} (0.5,0.2,0.2) & (0.4,0.2,0.4) & (0,0,1) & (0,0,1) \\ (0.5,0.2,0.3) & (0.6,0.3,0.1) & (0,0,1) & (0,0,1) \\ (0.7,0.2,0.1) & (0.5,0.3,0.2) & (0.5,0.2,0.1) & (0.5,0.2,0.3) \\ (0.4,0.3,0.3) & (0.5,0.3,1) & (0.8,0.1,0.1) & (0.5,0.2,0.1) \end{bmatrix}$$

Then matrix A is called upper unreduced Hessenberg neutrosopic soft matrix, matrix B is called lower unreduced Hessenberg neutrosopic soft matrix and matrix C is called upper Hessenberg neutrosopic soft matrix but is not unreduced.

Definition 12: Bandwdith of neurosophic soft matrix

Let $A = [a_{ij}]$ be an n $\times$ n neutrosophic soft matrix. If all the elements of this neutrosophic soft matrix is zero outside a diagonally bordered band whose range is determined by constants k_1 and k_2 such that $a_{ij} = 0$ if $j < i - k_1$ or $j < i + k_2$, then the quantities k_1, k_2 are called lower bandwith and upper bandwidth respectively. The bandwidth of a matrix is maximum of k_1, k_2 , in other words it is the number k such $a_{ij} = 0$ whenever $|i-j| > k$. More generally the number of non-zero diagonal above the main diagonal is called the upper bandwith and the number of non-zero elements below the main diagonal is called the lower bandwith.

For example,

$$A = \begin{bmatrix} (0.3,0.2,0.1) & (0.1,0.4,0.3) & (0.4,0.6,0.2) & (0,0,1) & (0,0,1) & ((0,0,1) \\ (0.4,0.5,0.2) & (0.2,0.2,0.6) & (0.6,0.2,0.1) & (0.3,0.2,0.1) & (0,0,1)) & (0,0,1) \\ (0.6,0.2,0.1) & (0.3,0.5,0.1) & (0.3,0.2,0.4) & (0.5,0.2,0.3) & (0.4,0.2,0.4) & (0,0,1) \\ (0,0,1) & (0.2,0.2,0.5) & (0.7,0.1,0.1) & (0.3,0.4,0.3) & (0.5,0.2,0.3) & (0.6,0.2,0.2) \\ (0,0,1) & (0,0,1) & (0.6,0.2,0.2) & (0.6,0.2,0.1) & (0.6,0.3,0.1) & (0.3,0.5,0.1) \\ (0,0,1) & (0,0,1) & (0,0,1) & (0.3,0.5,0.2) & (0.3,0.2,0.3) & (0.4,0.2,0.2) \end{bmatrix}$$

For k=2, whenever $|i-j|>2$ it is seen that $a_{ij}=0$.The pairs (i,j) with $|i-j|>2$ are (1,4),(1,5),(1,6),(2,5),(2,6),(3,6),(4,1),(5,1),(5,2),(6,1),(6,2),(6,3).

For k=1, whenever $|i-j|>1$ it is seen that $a_{ij}\neq 0$. For example (i, j)=(1,3) and $|1-3|>1$ but $a_{13}\neq 0$. Hence the bandwith of the above matrix is 2.

Definition 13: Tri diagonal neutrosophic soft matrix

Neutrosophic soft matrix is called tri diagonal neutrosophic soft matrix if non zero entries are found in the lower diagonal, main diagonal and upper diagonal and all other entries being (0.0.1). This is a special type of neutrosophic soft matrix. Neutrosophic soft tri diagonal matrix takes the form

$$\tilde{A}=\begin{bmatrix} L_1 & M_1 & \ldots & 0 \\ A_1 & L_2 & M_2\ldots & 0 \\ 0 & A_2 & L_3 & M_3 \\ 0 & 0 & A_3 & L_4 \end{bmatrix}$$ where L_i, M_i, N_i are the non zero entries in the lower, main and upper diagonal respectively.

Operations on Neutrosophic Soft Matrices

Here some operations on some special types of neutrosophic soft matrices are discussed.

Summation of neutrosophic soft matrices

If the two neutrosophic soft matrices $M=[(T_{ij}^A, I_{ij}^A, F_{ij}^A)]$ and $N=[(T_{ij}^B, I_{ij}^B, F_{ij}^B)]$ are taken into consideration, then the summation of M and N will be denoted as M+N, and is defined by

$$M+N=[\max(T_{ij}^A, T_{ij}^B), \min(I_{ij}^A, I_{ij}^B), \min(F_{ij}^A, F_{ij}^B)]$$ for all *i* and *j*.

max-min operations on neutrosophic soft matrices

Let $M=[(T_{ij}^A, I_{ij}^A, F_{ij}^A)]$, $N=[(T_{ij}^B, I_{ij}^B, F_{ij}^B)]$ be two neutrosophic soft matrices. Then the max-min operations on the two neutrosophic soft matrices M and N is denoted as $M.N$ and it is defined by

$$M.N=[\max\min(T_{ij}^A, T_{ij}^B), \min\max(I_{ij}^A, I_{ij}^B), \min\max(F_{ij}^A, F_{ij}^B)]$$ for all i and j.

Transpose of neutrosophic soft matrices

Let $\tilde{M}=[(T_{ij}^A, I_{ij}^A, F_{ij}^A)]$ be a neutrosophic soft matrix. Then the transpose of this neutrosophic soft matrix is denoted by M^T and will be is defined by $\tilde{M}^T=[(T_{ji}^A, I_{ji}^A, F_{ji}^A)]$

Some results of the neutrosophic soft matrices of special types

In this section s0me properties of special types of neutrosophic soft matrices of special types discussed in this article are provided.

Properties of the Newly Defined Neutrosophic Soft Matrices

Properties of bidiagonal neutrosophic soft matrices(BdNSM)

Property 1

If two neutrosophic soft upper BdNSMs of same order is added then the resulting matrix is again a BdNSM.

Property 2

If two upper BdNSM are multiplied then the resulting matrix is again a upper BdNSM.

Property 3

If two neutrosophic soft lower BdNSMs are added together then the resulting matrix is again a neutrosophic soft lower BdNSM.

Property 4

If neutrosophic soft lower BdNSMs of same order is multiplied then the resulting matrix is again a neutrosophic soft lower BdNSM.

Properties of tridiagonal neutrosophic soft matrices(TridNSM)

Property 5

If two neutrosophic soft upper TridNSM of same order is added together then the resulting matrix is again a neutrosophic soft upper TridNSM.

Property 6

If two neutrosophic soft upper TridNSM are multiplied then the resulting matrix is again a TridNSM.

Property7

If two neutrosophic soft lower TridNSM are added together then the resulting matrix is again a upper TridNSM.

Property8

If two TridNSM of same order are multiplied then the resulting matrix is again a TridNSM.

Properties of Hassenburg Neutrosophic soft matrices (HaNSM)

Property 9

The product of a neutrosophic soft hassenburgh matrix with a neutrosophic soft triangular matrix is again a hassenburgh matrix. To be more precise, if a neutrosophic soft matrix M is upper HaNSM and the neutrosophic soft matrix T is upper triangular, then MT and TM are upper HaNSM.

Properties of Toepliz neutrosophic soft matrix

Property 10

Summation of two ToepNSM results in a ToepNSM.

Property 11

The transpose of a ToepNSM results in a ToepNSM.

If the matrix M written above is considered then

$$M^T = \begin{bmatrix} (T_{11}^A, I_{11}^A, F_{11}^A) & (T_{21}^A, I_{21}^A, F_{21}^A) & (T_{31}^A, I_{31}^A, F_{31}^A) & (T_{41}^A, I_{41}^A, F_{41}^A) \\ (T_{12}^A, I_{12}^A, F_{12}^A) & (T_{22}^A, I_{22}^A, F_{22}^A) & (T_{32}^A, I_{32}^A, F_{32}^A) & (T_{42}^A, I_{42}^A, F_{42}^A) \\ (T^A{}_{13}, I_{13}^A, F_{13}{}^A) & (T_{23}^A, I_{23}^A, F_{23}^A) & (T_{33}^A, I_{33}^A, F_{33}^A) & (T_{43}^A, I_{43}^A, F_{43}^A) \\ (T_{14}^A, I_{14}^A, F_{14}^A) & (T_{24}^A, I_{24}^A, F_{24}^A) & (T_{34}^A, I_{34}^A, F_{34}^A) & (T_{44}^A, I_{44}^A, F_{44}^A) \end{bmatrix}$$

The above matrix obtained is a ToepNSM

Conclusions

Here few new types of neutrosophic soft matrices are discussed. Thereafter some operations on such types of neutrosophic soft matrices are discussed with some examples. It can be seen from the discussion that the behavior of the different types of neutrosophic soft matrices under consideration in this work is almost the same as those of exists in the literature of matrices. In future works, applications of such neutrosophic soft matrices will be studied.

Future Research Directions

In future applications of such neutrosophic soft matrices will be studied.

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Chapter 5

Interval-Valued Neutrosophic Fuzzy *M*-semigroup

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ABSTRACT

This paper introduces a new algebraic structure, the interval-valued neutrosophic fuzzy *M*-Semigroup (IVNFMS), by merging the notions of interval-valued fuzzy *M*-semigroups(IVFMSs) and Neutrosophic fuzzy sets(NFSs). This study focuses on the sociological and biological applications of an interval-valued Neutrosophic fuzzy *M*-semigroup and numerous algebraic features, including intersection and union are examined. Additionally, we explore direct product, image and inverse image between two interval-valued Neutrosophic fuzzy *M*-Semigroups and present some related results.

Keywords: M-semigroup, fuzzy M-semigroup, interval-valued fuzzy M-semigroup, Neutrosophic fuzzy set, Interval-valued Neutrosophic fuzzy M-semigroup.

INTRODUCTION

Zadeh's idea was to introduce a new way of representing and reasoning about uncertainty, where instead of relying on crisp boundaries, fuzzy sets (FSs) [22]allowed for a more nuanced understanding of membership. By assigning a membership value between 0 and 1 to each element, fuzzy sets provided a flexible framework for capturing the inherent vagueness and ambiguity that often arises in real-world problems. This allowed for a more realistic representation of complex systems, where objects or concepts may not fit neatly into predefined categories. The introduction of fuzzy sets opened up new possibilities in various fields, including artificial intelligence, control systems, pattern recognition, and data analysis. Fuzzy logic, which is based on the principles of fuzzy sets, provided a formal language for expressing and manipulating imprecise information. It enabled the development of fuzzy control systems, which could handle uncertain and nonlinear systems more effectively than traditional control methods. Over the years, fuzzy set theory has been successfully applied to a wide range of fields, including medical diagnosis, image processing, natural language processing and financial modeling. This theory has proven to be a powerful tool for handling uncertainty and imprecision, offering a more robust and flexible approach to real-world challenges. The extension from traditional fuzzy sets to interval-valued fuzzy sets was pioneered by Zadeh [23] in 1975.

Semigroups provide a fundamental framework for studying the properties and behaviour of binary operations [3]. By focusing solely on the associative property, semigroups allow for a simplified analysis of algebraic structures. One important application of semigroups is in the study of algebraic structures. By examining the properties of semigroups, mathematicians can gain insights into more complex structures such as monoids and groups. Monoids, for example, are semigroups that also possess an identity element, while groups are monoids that additionally have inverses for every element. Semigroups can be used to represent the behaviour of finite state machines, where the binary operation represents the composition of transitions [4]. By studying the properties of semigroups, researchers can analyse the behaviour and capabilities of various computational models.

Semigroups play a crucial role in the study of automata theory [4], where they are used to model the behaviour of finite state machines and formal languages. Semigroup theory has found applications in computer science, specifically in the development and analysis of algorithms and data structures [16]. In addition, semigroups are associated with multiple fields of mathematics, including group theory, ring theory, and category theory. They provide a framework for studying algebraic structures and their properties, leading to deeper insights into the underlying mathematical structures.

The algebraic structure known as an *M*-semigroup indeed lies between a semigroup and a monoid, emphasizing the special properties of an identity element. *M*-semigroup theory extends classical semigroup theory by introducing a specific condition on the binary operation within the structure. Lakshmanan's introduction of the concept of *M*-semigroup [5] was a significant contribution to the field of mathematics. *M*-semigroups are a generalization of semigroups, which are algebraic structures consisting of a set and an associative binary operation. Building upon Lakshmanan's work, Narayanan et al. [6] introduced fuzzy *M*-semigroup theory, which further expanded the scope of *M*-semigroups. Fuzzy *M*-semigroups incorporate the concept of fuzziness, which allows for the representation of uncertainty and imprecision in mathematical models.

Anti-fuzzy sets capture the degree to which elements are excluded from a set. They are particularly valuable when the complement of a set is the primary concern.Combining the concepts of *M*-semigroups and anti-fuzzy sets, anti-fuzzy *M*-semigroups extend the traditional *M*-semigroup theory to handle uncertainty and imprecision in the semigroup structure. Anti-fuzzy *M*-semigroup, introduced by Vijayabalji and Sivaramakrishnan [18,19], expand upon classical *M*-semigroup theory by integrating anti-fuzzy sets. They offer a mechanism to address uncertainty and imprecision within semigroup structures and they addressed homomorphism between two anti-fuzzy *M*-semigroups. Moreover, Sivaramakrishnan et al. [17] pioneered the concept of an anti Q-fuzzy *M*-semigroup.

Atanassov's innovative work in 1986 on intuitionistic fuzzy sets (IFSs) [1]revolutionized the field of fuzzy logic by providing a more comprehensive and nuanced approach to dealing with uncertainty. By incorporating the concepts of membership, non-membership, and hesitation or indeterminacy for each element in a set, IFS allowed for a more accurate and flexible representation of real-world problems, leading to significant advancements in decision-making processes and problem-solving strategies.

Neutrosophic sets (NSs), pioneered by Smarandache [9] in the late 1990s, represent a novel approach to extending the traditional notions of classical sets, FSs and IFSs. This new concept aims to capture and represent the inherent indeterminacy, ambiguity, and inconsistency that often arise in real-world situations. Since their introduction, Neutrosophic sets have undergone significant theoretical development and found numerous applications across various fields [10,12,13,14,15]. The fundamental concepts and comprehensive overview of neutrosophic sets have been extensively documented [2], leading to several specialized variants. Notable among these are the single-valued Neutrosophic sets, which provide a more practical framework for real-world applications [8], and rough Neutrosophic sets, which combine the power of rough sets with Neutrosophic theory to handle uncertainty and incompleteness in information systems [7]. The continuous evolution and refinement of Neutrosophic theory have established it as a robust framework for dealing with uncertain, incomplete, and inconsistent information in various domains of science and engineering.

This paper introduces a novel algebraic concept for a IVNFMS by integrating the structures of interval-valued fuzzy *M*-semigroup and Neutrosophic fuzzy set (NFS). We explore the sociological and biological implications of an interval-valued Neutrosophic fuzzy *M*-semigroup, examining their potential applications in these domains. Additionally, we investigate the direct product of these structures and present some related results.

PRELIMINARIES

Definition 1.[5] A *M*-semigroup, denoted as MS, is a *M*-semigroup that fulfills the following conditions:

1. There is at least one left identity e $\in M$ such that e $m = m$, for all $m \in M$.
2. For every $m \in M$, there is a unique left identity, represented as e_m, such that m $e_m = m$, i.e., e_m is a two-sided identity for m.

Definition 2. [6] Consider M be a M-semigroup. Let $\varpi_M: M \to [0,1]$ be a fuzzy set. Then the pair (M, ϖ_M) is referred to as a fuzzy M-semigroup if

1. $\varpi_M(m_1 m_2) \geq \min\{\varpi_M(m_1), \varpi_M(m_2)\}$, for all m_1, m_2 belonging to M,

2. $\varpi_M(e) = 1, \forall$ e in M.

Definition 3.[20] Let M be a M-semigroup and $\overline{\varpi}_M$ be an interval-valued fuzzy M-semigroup. suppose the following conditions hold: For all $x, y \in M$,

1. $\overline{\varpi}_M(xy) \geq min\{\overline{\varpi}_M(x), \overline{\varpi}_M(y)\}$,

2. $\overline{\varpi}_M(e) = \overline{1} = [1,1]$, for every left identity e in M.

Then $\overline{\varpi}_M$ is referred to as anIVFMS on M and is denoted by $(M, \overline{\varpi}_M)$.

Definition 4. [11] Let$\mathbf{X}_{NS}$ be the universal set. A NSis a set of the form Ω= {m,ξ_Ω (m), Ψ_Ω (m), ζ_Ω (m) | m ∈$\mathbf{X}_{NS}$ } and denoted by Ω= (ξ_Ω (m), Ψ_Ω (m), ζ_Ω (m)), where ξ : $\mathbf{X}_{NS}$→[0,1], Ψ : $\mathbf{X}_{NS}$→[0,1] and ζ: $\mathbf{X}_{NS}$→[0,1] represent the degree of truth - membership and indeterminacy - membership and false - membership of the element m ∈$\mathbf{X}_{NS}$ in Ω and 0≤ ξ_Ω (m)+ Ψ_Ω (m)+ ζ_Ω (m)≤ 3.

Interval-Valued Neutrosophic Fuzzy *M*-Semigroup (IVNFMS)

Definition 3.1. Suppose that **MS**. A (**NS**) $\bar{\mathcal{A}} = (\overline{\varpi}_M, \overline{\eta}_M, \overline{\delta}_M)$ is known to be a IVNFMS of MS. If for all $m_1, m_2 \in M$ it holds.

$(i)\overline{\varpi}_M(m_1m_2) \geq min\{\overline{\varpi}_M(m_1), \overline{\varpi}_M(m_2)\}$,

$(ii)\overline{\eta}_M(m_1m_2) \geq min\{\overline{\eta}_M(m_1), \overline{\eta}_M(m_2)\}$,

$(iii)\overline{\delta}_M(m_1m_2) \leq max\{\overline{\delta}_M(m_1), \overline{\delta}_M(m_2)\}$,

$(iv)\overline{\varpi}_M(e) = [1 , 1] = \overline{1}$ for every left identity e in M,

$(v)\overline{\eta}_M(e) = [1 , 1] = \overline{1}$for every left identity e in M,

$(vi)\overline{\delta}_M(e) = [0, 0] = \overline{0}$ for every left identity e in M.

Example 3.2. Consider$M = \{m_1, m_2, m_3, m_4\}$ *as a* MS with the following operation '•'

Table 1: An illustration of the Cayley table for a Neutrosophic fuzzy M-semigroup under the operation '•'

•	m_1	m_2	m_3	m_4
m_1	m_1	m_2	m_3	m_4
m_2	m_1	m_2	m_3	m_4
m_3	m_3	m_4	m_1	m_2
m_4	m_3	m_4	m_1	m_2

Define $IVNFS\ \overline{\mathcal{A}}_M: M \to \mathbf{D}[0,1]$by

$$\overline{\varpi}_M(m) = \begin{cases} [1,1], if & m = m_1, m_2 \\ [0.7,0.81], & \text{otherwise.} \end{cases}$$

$$\overline{\eta}_M(m) = \begin{cases} [1,1], if & m = m_1, m_2 \\ [0.54,0.69], & \text{otherwise.} \end{cases}$$

$$\overline{\delta}_M(m) = \begin{cases} [0,0], if & m = m_1, m_2 \\ [0.4, 0.55], & \text{otherwise}, 0 < \alpha \leq 1. \end{cases}$$

Then$\bar{\mathcal{A}} = (\overline{\varpi}_M, \overline{\eta}_M, \overline{\delta}_M)$ is an IVNFMS.

Applications of an IVNFMS

This section delves into two specific instances. The first example examines the correlation between Kinship relations (KR) and IVNFMS theory. Kinship systems are structured to recognize and define connections based on marriage and other social constructs. For instance, relationships such as "being a father"or "being a mother"can be amalgamated to form more intricate relationships like "being the mother of the father", commonly referred to as "grandmother on the father's side". In certain scenarios, kinship relations like "daughter of a mother"and "daughter of a father"may be deemed identical. By leveraging these concepts, the theory of Neutrosophic fuzzy *M*-semigroup can be intertwined with Kinship systems. Another example involves the association of DNA sequences with Neutrosophic fuzzy *M*-semigroup. In 1953, Watson &Crick [21] unveiled the structure of DNA(Deoxyribonucleic Acid), identifying it as the molecule responsible for carrying genetic information across generations. In 2006, Zhang et al. [24]introduced three techniques for transforming character-based DNA sequences into numerical sequences, one of which utilizes complex number representation. These developments underscore the relevance of Neutrosophic fuzzy *M*-semigroup in DNA analysis, showcasing its potential applicability in this field.

Example 4.1. A kinship system is semigroup $\mathfrak{R}_S = [X_{\mathbf{K}}, L_{\mathbf{R}}]$, where

1. $X_{\mathbf{K}}$ is a set of kinship relationships,
2. $L_{\mathbf{R}}$is a relation on $X_{\mathbf{K}}^*$ $L_{\mathbf{R}}$which expresses equality of kinship relationships**.** $X_{\mathbf{K}}^*$ is a free semigroup in which all formally combined relationships from $X_{\mathbf{K}}$ would be different**.**

Assuming F_M:= "is father of ", M_M:= "is mother of", C_M:= "is Child of", $(FM)_M$:= "is father of the mother", $(MF)_M$:= "is mother of the father", $(CF)_M$:= "is child of the father".

Let $X = \{F_M , M_M , C_M , (FM)_M, (MF)_M ,(CF)_M\}$

The collection of kinship relationships = $L = \{(F_M F_M , F_M),((FM)_M F_M , F_M), (M_M M_M , M_M), ((MF)_M M_M , M_M),((CF)_M , F_M),((CM)_M , F_M), ((CF)_M M_M , (FM)_M), ((CM)_M F_M ,(MF)_M), ((FM)_M M_M ,(FM)_M), ((FM)_M(FM)_M,(FM)_M)((FM)_M (MF)_M , F_M),((MF)_M(MF)_M, (MF)_M),((MF)_M C_M , (MF)_M)\}$.

Using the symbol ◊ to represent the operation of relation product, consider the initial pair of L_R,

where (F_MF_M,F_M) signifies that the relationship "father of the father " is equivalent to the relationship "father". Hence, $F_M◊F_M$ equals F_M. Similarly, $(CM)_M= (CF)_M$, indicating that the children of the mother are identical to the children of the father.
It is clear that $\mathfrak{R}_S = [X_{\mathbf{K}}, L_{\mathbf{R}}]$ constitutes a right *M*-semigroup.

Table2: An illustration of IVNFMS can be represented by Cayley table with Kinship relationship under the operation ◊.

◊	F_M	M_M	C_M	$(FM)_M$	$(MF)_M$	$(CF)_M$
F_M	F_M	$(FM)_M$	F_M	$(FM)_M$	F_M	F_M
M_M	$(MF)_M$	M_M	M_M	M_M	$(MF)_M$	$(MF)_M$
C_M	F_M	F_M	C_M	$(FM)_M$	$(MF)_M$	$(CF)_M$
$(FM)_M$	F_M	$(FM)_M$	$(FM)_M$	$(FM)_M$	F_M	F_M
$(MF)_M$	$(MF)_M$	M_M	$(MF)_M$	M_M	$(MF)_M$	$(MF)_M$
$(CF)_M$	$(CF)_M$	$(MF)_M$	$(CF)_M$	$(MF)_M$	$(MF)_M$	$(CF)_M$

Define the function $\overline{\varpi}_M: \mathfrak{R}_S \to \mathbf{D}[0,1]$as follows:

$\overline{\varpi}_M(C_M) = [1,1]$,

$\overline{\varpi}_M(F_M) = \overline{\varpi}_M((CF)_M) = [0.72, 0.87]$,

$\overline{\varpi}_M(M_M) = \overline{\varpi}_M((MF)_M) = \overline{\varpi}_M((FM)_M) = [0.51, 0.64]$.

Define$\overline{\eta}_M: \mathfrak{R}_S \to \mathbf{D}[0,1]$by

$\overline{\eta}_M(C_M) = [1,1]$,

$\overline{\eta}_M(F_M) = \overline{\eta}_M((CF)_M) = [0.6,0.72]$,

$\overline{\eta}_M(M_M) = \overline{\eta}_M((MF)_M) = \overline{\eta}_M((FM)_M) = [0.43, 0.55]$.

Define $\overline{\delta}_M: \Re_S \to \mathbf{D}[0,1]$by

$\overline{\delta}_M(C_M) = [0,0]$,

$\overline{\delta}_M(F_M) = \overline{\delta}_M((CF)_M) = [0.4, 0.53]$,

$\overline{\delta}_M(M_M) = \overline{\delta}_M((MF)_M) = \overline{\delta}_M((FM)_M) = [0.71, 0.82]$.

We see that, $\overline{\mathcal{A}}_M = (\overline{\varpi}_M, \overline{\eta}_M, \overline{\delta}_M)$ is anIVNFMS of $\Re_S$.

Example 4.2. Zhang et al have defined f(m) as a function that maps the set $f(m)$: $\{A_M, G_M, T_M, C_M\}$ to the set$\{1, -1, i, -i\}$ as

$$f(m) = \begin{cases} 1, & if \ \mathrm{m} = \mathrm{G_M} \\ -1, & if \ \mathrm{m} = \mathrm{T_M} \\ i, & if \ \mathrm{m} = \mathrm{A_M} \\ -i, & if \ \mathrm{m} = \mathrm{C_M} \end{cases}$$

where A_M- Adenine, G_M-Guanine, C_M-Cytosine and T_M-Thymine and m is one of the four nucleotides.

We consider

$A_M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; $T_M = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$; $G_M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $C_M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Let $M = \{A_M, G_M, T_M, C_M\}$ represent a M-semigroup with the following operation$\cdot$

Table3**:** An illustration for an IVNFMS**:** Cayley table with DNA Sequences under the Operation$\cdot$

$\cdot$	A_M	G_M	T_M	C_M
A_M	T_M	A_M	C_M	G_M
G_M	A_M	G_M	T_M	C_M
T_M	C_M	T_M	G_M	A_M
C_M	G_M	C_M	A_M	T_M

Define the mapping $\overline{\varpi}_{M_s}: M \to \mathbf{D}[0,1]$ by

$$\varpi(m) = \begin{cases} [1,1], & if \ m = G_M \\ [0.61, 0.72], & \text{otherwise.} \end{cases}$$

Define the mapping $\overline{\eta}_{M_s}: M \to \mathbf{D}[0,1]$ by

$$\overline{\eta}_{M_s}(m) = \begin{cases} [1,1] & if \ m = G_M \\ [0.49, 0.5] & \text{otherwise,} \end{cases}$$

Define the mapping $\overline{\delta}_{M_s}: M \to \mathbf{D}[0,1]$ by

$$\overline{\delta}_{M_s}(m) = \begin{cases} [0,0], & if \ m = G_M \\ [0.2,0.34], & \text{otherwise.} \end{cases}$$

We see that, $\overline{\mathcal{A}}_M = (\overline{\varpi}_M, \overline{\eta}_M, \overline{\delta}_M)$ is an IVNFMS.

Theorem 4.3. Let $\overline{\mathcal{A}}_{M_1}= (M_1, \overline{\varpi}_{M_1}, \overline{\eta}_{M_1}, \overline{\delta}_{M_1})$ and $\overline{\mathcal{A}}_{M_2}= (M_2, \overline{\varpi}_{M_2}, \overline{\eta}_{M_2}, \overline{\delta}_{M_2})$ be two IVNFMSs. Then their intersection, $(\overline{\mathcal{A}}_{M_1} \cap \overline{\mathcal{A}}_{M_2}) = (\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2} , \overline{\eta}_{M_1} \cap \overline{\eta}_{M_2} , \overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})$ is anIVNFMS**.**

Proof. Given $\overline{\mathcal{A}}_{M_1}= (M_1, \overline{\varpi}_{M_1}, \overline{\eta}_{M_1}, \overline{\delta}_{M_1})$ and $\overline{\mathcal{A}}_{M_2}= \left(M_2, \overline{\varpi}_{M_2}, \overline{\eta}_{M_2}, \overline{\delta}_{M_2}\right)$ be two IVNFMSs then

$\overline{\mathcal{A}}_{M_1} \cap \overline{\mathcal{A}}_{M_2} = \{\langle m, min[\overline{\varpi}_{M_1}(m), \overline{\varpi}_{M_2}(m)], min[\overline{\eta}_{M_1}(m), \overline{\eta}_{M_2}(m)], max[\overline{\delta}_{M_1}(m), \overline{\delta}_{M_2}(m)]\rangle : m \in M\}$

Define $\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2} : M \to \boldsymbol{D}[0,1]$ by $(\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(m) = min\,\{\overline{\varpi}_{M_1}(m), \overline{\varpi}_{M_2}(m)\}$ for all $m \in M$

$$(i)(\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(mn) = min\,\{\overline{\varpi}_{M_1}(mn), \overline{\varpi}_{M_2}(mn)\}$$

$$\geq min\,\{ min\,[\overline{\varpi}_{M_1}(m), \overline{\varpi}_{M_1}(n)], min\,[\overline{\varpi}_{M_2}(m), \overline{\varpi}_{M_2}(n)]\}$$

$$= min\,\{ min\,[\overline{\varpi}_{M_1}(m), \overline{\varpi}_{M_2}(m)], min\,[\overline{\varpi}_{M_1}(n), \overline{\varpi}_{M_2}(n)]\}$$

$$= min\,\{(\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(m), (\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(n)\}$$

$$\Rightarrow (\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(mn) \geq min\,\{(\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(m), (\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(n)\}$$

$$(ii)(\overline{\varpi}_{M_1} \cap \overline{\varpi}_{M_2})(e) = min\,\{\overline{\varpi}_{M_1}(e), \overline{\varpi}_{M_2}(e)\}$$

$$= min\,\{\overline{1}, \overline{1}\}$$

$$= \overline{1}$$

Define

$\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2} : M \to \mathbf{D}[0,1]$by $(\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(m) = min\{\overline{\eta}_{M_1}(m), \overline{\eta}_{M_2}(m)\}$ for all $m \in M$.

$$(iii)(\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(mn) = \min\{\overline{\eta}_{M_1}(mn), \overline{\eta}_{M_2}(mn)\}$$

$$\geq \min\{\min[\overline{\eta}_{M_1}(m), \overline{\eta}_{M_1}(n)], \min[\overline{\eta}_{M_2}(m), \overline{\eta}_{M_2}(n)]\}$$

$$= \min\{\min[\overline{\eta}_{M_1}(m), \overline{\eta}_{M_2}(m)], \min[\overline{\eta}_{M_1}(n), \overline{\eta}_{M_2}(n)]\}$$

$$= \min\{(\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(m), (\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(n)\}$$

$$\Rightarrow (\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(mn) \geq \min\{(\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(m), (\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(n)\}$$

$$(iv)(\overline{\eta}_{M_1} \cap \overline{\eta}_{M_2})(\mathbf{e}) = \min\{\overline{\eta}_{M_1}(\mathbf{e}), \overline{\eta}_{M_2}(\mathbf{e})\}$$

$$= \min\{\overline{1}, \overline{1}\}$$

$$= \overline{1}$$

Define

$\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2} : M \to \mathbf{D}[0,1]$ by $(\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(m) = max\{\overline{\delta}_{M_1}(m), \overline{\delta}_{M_2}(m)\}$ for all $m \in M$.

$$(v)(\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(mn) = \max\{\overline{\delta}_{M_1}(mn), \overline{\delta}_{M_2}(mn)\}$$

$$\leq \max\left\{\max[\overline{\delta}_{M_1}(m), \overline{\delta}_{M_1}(n)], \max[\overline{\delta}_{M_2}(m), \overline{\delta}_{M_2}(n)]\right\}$$

$$= \max\left\{\max[\overline{\delta}_{M_1}(m), \overline{\delta}_{M_2}(m)], \max[\overline{\delta}_{M_1}(n), \overline{\delta}_{M_2}(n)]\right\}$$

$$= \max\{(\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(m), (\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(n)\}$$

$$\Rightarrow (\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(mn) \leq \max\{(\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(m), (\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(n)\}.$$

$$(vi)\ \ (\overline{\delta}_{M_1} \cup \overline{\delta}_{M_2})(\mathrm{e}) = \max\{\overline{\delta}_{M_1}(\mathbf{e}), \overline{\delta}_{M_2}(\mathrm{e})\}$$

$= \max\{\bar{0}, \bar{0}\}$

$= \bar{0}.$

Next we show that the union of two IVNFMSs need not be an IVNFMS of M-semigroup by means of a counter example.

Remark 4.4. The union of two IVNFMSs of M-semigroup need not be an IVNFMSof M-semigroup.

Proof. Let $M = \{e_M, a_M, b_M, (ab)_M\}$ be a M-semigroup, where $a_M^2 = e_M = b_M^2 = (ab)_M^2$ and $(ab)_M = (ba)_M$.

Table 4: Example for M-semigroup - Cayley table under the operation ·

·	a_M	b_M	$(ab)_M$	e_M
a_M	e_M	$(ab)_M$	b_M	a_M
b_M	$(ab)_M$	e_M	a_M	b_M
$(ab)_M$	b_M	a_M	e_M	$(ab)_M$
e_M	a_M	b_M	$(ab)_M$	e_M

Define $\overline{\varpi_1}_M, \overline{\varpi_2}_M$ as follows:

$$\overline{\varpi_1}_M(m) = \begin{cases} \bar{1}, & if m = e_M, \\ [0.71\,, 0.8], & if m = a_M, \\ [0.3, 0.42] & if m = b_M\,, (ab)_M \end{cases}$$

$$\overline{\varpi_2}_M(m) = \begin{cases} \bar{1}, & if m = e_M \\ [0.51, 0.6], & if m = a_M\,, (ab)_M \\ [0.8, 0.9], & if m = b_M \end{cases}$$

We note that $\overline{\varpi_1}_M$, $\overline{\varpi_2}_M$ are two IVNFMSs.

Define ($\overline{\varpi_1}_M \cup \overline{\varpi_2}_M$) (m) = max { $\overline{\varpi_1}_M(m), \overline{\varpi_2}_M(m)$ } for all $m \in M$.

So,

$$(\overline{\varpi_1}_M \cup \overline{\varpi_2}_M)(m) = \begin{cases} \bar{1}, if\ m = e_M, \\ [0.71\,, 0.8], if\ m = a_M, \\ [0.8, 0.9],\ if\ m = b_M, \\ [0.51,\ 0.6], if m = (ab)_M \end{cases} \qquad (1)$$

But $(\overline{\varpi_1}_M \cup \overline{\varpi_2}_M)((ab)_M) \geq \ min\,\{(\overline{\varpi_1}_M \cup \overline{\varpi_2}_M)(a_M), (\overline{\varpi_1}_M \cup \overline{\varpi_2}_M)(b_M)\}$

$= \ min\,\{[0.71, 0.8], [0.8, 0.9]\} = [0.71, 0.8].$

Therefore, $(\overline{\varpi_1}_M \cup \overline{\varpi_2}_M)((ab)_M) \geq [0.71, 0.8]$.

However $(\overline{\varpi_1}_M \cup \overline{\varpi_2}_M)((ab)_M)$ = [0.51, 0.6], by (1) we get [0.51, 0.6] ≥ [0.71, 0.8]. Which is absurd.

The other inequalities are proven in a similar manner.

$\overline{\varpi_1}_M$, $\overline{\varpi_2}_M$ are two IVNFMSs, whereas, $\overline{\varpi_1}_M \cup \overline{\varpi_2}_M$ is not an IVNFMS.

Hence the union of two IVNFMSsof M-semigroup need not be an IVNFMSof M-semigroup.

Definition 4.5. Let $\overline{\mathcal{A}}_{M_1}$= $(M_1, \overline{\varpi}_{M_1}, \overline{\eta}_{M_1}, \overline{\delta}_{M_1})$ and $\overline{\mathcal{A}}_{M_2}$= $(M_2, \overline{\varpi}_{M_2}, \overline{\eta}_{M_2}, \overline{\delta}_{M_2})$ be two Neutrosophic fuzzy subsets of M_1 and M_2 respectively. Then the cartesian product of $\overline{\mathcal{A}}_{M_1}$ and $\overline{\mathcal{A}}_{M_2}$ denoted by

$\overline{\mathcal{A}}_{M_1} \times \overline{\mathcal{A}}_{M_2}$ = { $\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2}$, $\overline{\eta}_{M_1} \times \overline{\eta}_{M_2}$, $\overline{\delta}_{M_1} \times \overline{\delta}_{M_2}$ } and is defined by

1. $(\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})(m_1, m_2) = min\{\overline{\varpi}_{M_1}(m_1), \overline{\varpi}_{M_2}(m_2)\}$, for all $(m_1, m_2) \in M_1 \times M_2$,
2. $(\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})(e, e') = min\{\overline{\varpi}_{M_1}(e), \overline{\varpi}_{M_2}(e')\}$, where e and e' are left identities in M_1 and M_2,
3. $(\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})(m_1, m_2) = min\{\overline{\eta}_{M_1}(m_1), \overline{\eta}_{M_2}(m_2)\}$, for all $(m_1, m_2) \in M_1 \times M_2$,
4. $(\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})(e, e') = min\{\overline{\eta}_{M_1}(e), \overline{\eta}_{M_2}(e')\}$, where e and e' are left identities in M_1 and M_2,
5. $(\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})(m_1, m_2) = min\{\overline{\delta}_{M_1}(m_1), \overline{\delta}_{M_2}(m_2)\}$, for all $(m_1, m_2) \in M_1 \times M_2$,
6. $(\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})(e, e') = min\{\overline{\delta}_{M_1}(e), \overline{\delta}_{M_2}(e')\}$, where e and e' are left identities in M_1 and M_2.

Theorem 4.6. If $\overline{\mathcal{A}}_{M_1}$= $(M_1, \overline{\varpi}_{M_1}, \overline{\eta}_{M_1}, \overline{\delta}_{M_1})$ and $\overline{\mathcal{A}}_{M_2}$= $(M_2, \overline{\varpi}_{M_2}, \overline{\eta}_{M_2}, \overline{\delta}_{M_2})$ are IVNFMSsthen $\overline{\mathcal{A}}_{M_1} \times \overline{\mathcal{A}}_{M_2}$ is an IVNFMS.

Proof. Let $(m_1, m_2), (m_3, m_4) \in M_1 \times M_2$.

$(i)(\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})((m_1, m_2), (m_3, m_4)) = (\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})(m_1m_3, m_2m_4)$

$= min\{\overline{\varpi}_{M_1}(m_1m_3), \overline{\varpi}_{M_2}(m_2m_4)\}$

$\geq min\{min[\overline{\varpi}_{M_1}(m_1), \overline{\varpi}_{M_1}(m_3)], min[\overline{\varpi}_{M_2}(m_2), \overline{\varpi}_{M_2}(m_4)]\}$

$\geq min\{min[\overline{\varpi}_{M_1}(m_1), \overline{\varpi}_{M_2}(m_2)], min[\overline{\varpi}_{M_1}(m_3), \overline{\varpi}_{M_2}(m_4)]\}$

$= min\{(\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})(m_1, m_2), (\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})(m_3, m_4)\}$,

Let e and e' are left identities in M_1 and M_2.

$(ii)(\overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2})(e, e') = min\{\overline{\varpi}_{M_1}(e), \overline{\varpi}_{M_2}(e')\}$

$= \min\{\overline{1}, \overline{1}\}$

$= \overline{1}$,

$(iii)(\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})((m_1, m_2), (m_3, m_4)) = (\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})(m_1m_3, m_2m_4)$

$= min\{\overline{\eta}_{M_1}(m_1m_3), \overline{\eta}_{M_2}(m_2m_4)\}$

$\geq min\{min[\overline{\eta}_{M_1}(m_1), \overline{\eta}_{M_1}(m_3)], min[\overline{\eta}_{M_2}(m_2), \overline{\eta}_{M_2}(m_4)]\}$

$\geq min\{min[\overline{\eta}_{M_1}(m_1), \overline{\eta}_{M_2}(m_2)], min[\overline{\eta}_{M_1}(m_3), \overline{\eta}_{M_2}(m_4)]\}$

$= min\{(\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})(m_1, m_2), (\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})(m_3, m_4)\}$,

Let e and e' are left identities in M_1 and M_2.

$(iv)(\overline{\eta}_{M_1} \times \overline{\eta}_{M_2})(e, e') = min\{\overline{\eta}_{M_1}(e), \overline{\eta}_{M_2}(e')\}$

$= \min\{\overline{1}, \overline{1}\}$

$= \overline{1}$,

$(v)(\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})((m_1, m_2), (m_3, m_4)) = (\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})(m_1m_3, m_2m_4)$

$= max\{\overline{\delta}_{M_1}(m_1m_3), \overline{\delta}_{M_2}(m_2m_4)\}$

$\leq max\{max[\overline{\delta}_{M_1}(m_1), \overline{\delta}_{M_1}(m_3)], max[\overline{\delta}_{M_2}(m_2), \overline{\delta}_{M_2}(m_4)]\}$

$$\leq \ max\ \{ \ min\ [\overline{\delta}_{M_1}(m_1), \overline{\delta}_{M_2}(m_2)], max\ [\overline{\delta}_{M_1}(m_3), \overline{\delta}_{M_2}(m_4)]\}$$

$$= \ max\ \{(\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})(m_1, m_2), (\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})(m_3, m_4)\},$$

Let e and e' are left identities in M_1 and M_2.

$(vi)(\overline{\delta}_{M_1} \times \overline{\delta}_{M_2})(e, e') = \ max\ \{\overline{\delta}_{M_1}(e), \overline{\delta}_{M_1}(e')\}$

$$= \max\{\overline{0}, \overline{0}\}$$

$$= \overline{0}.$$

Hence $\langle \overline{\varpi}_{M_1} \times \overline{\varpi}_{M_2}, \overline{\eta}_{M_1} \times \overline{\eta}_{M_2}, \overline{\delta}_{M_1} \times \overline{\delta}_{M_2} \rangle$ is an IVNFMSof $M_1 \times M_2$.

Definition 4.7. Let $g: M_1 \to M_2$ be a mapping of M-semigroups. If $\overline{\mathcal{A}_M}$ is an IVNFMS in M_2 then the inverse image of $\overline{\mathcal{A}_M}$ under g, denoted by $g^{-1}(\overline{\mathcal{A}_M})$ is IVNFMS in M_1, defined by $g^{-1}(\overline{\mathcal{A}_M})(m) = \overline{\mathcal{A}_M}(g(m))$ for all $m \in M_1$.

Theorem 4.8. Let $g: M_1 \to M_2$ be homomorphism of M-semigroups. If $\overline{\mathcal{A}_{\mathrm{M}}}$ is an IVNFMS in M_2 then the inverse image $g^{-1}(\overline{\mathcal{A}_{\mathrm{M}}})$ of $\overline{\mathcal{A}_{\mathrm{M}}}$ under$g$ is an IVNFMS of M_1.

Proof. Assume that $\overline{\mathcal{A}_{\mathrm{M}}}$ is an IVNFMS in M_2 and $m_1, m_2 \in M_1$.

Then we have

(i) $g^{-1}(\overline{\varpi}_{\mathrm{M}})(\mathrm{m_1 m_2}) = \overline{\varpi}_{\mathrm{M}}(g(\mathrm{m_1 m_2}))$

$$= \overline{\varpi}_{\mathrm{M}}(\mathrm{g(m_1)f(m_2)}) \quad (\text{ since } \ g \ \text{ is homomorphism })$$

$$\geq \ \min\{\overline{\varpi}_{\mathrm{M}}(g(\mathrm{m_1})), \overline{\varpi}_{\mathrm{M}}(g(\mathrm{m_2}))\}$$

$$= \ \min\{g^{-1}(\overline{\varpi}_{\mathrm{M}}(\mathrm{m_1})), g^{-1}(\overline{\varpi}_{\mathrm{M}}(\mathrm{m_2}))\}$$

$\Rightarrow g^{-1}(\overline{\varpi}_{\mathrm{M}})(\mathrm{m_1 m_2}) \geq \ \min\{g^{-1}(\overline{\varpi}_{\mathrm{M}}(\mathrm{m_1})), g^{-1}(\overline{\varpi}_{\mathrm{M}}(\mathrm{m_2}))\}$

(ii)$g^{-1}(\overline{\varpi}_{\mathrm{M}})(\mathrm{e}) = \overline{\varpi}_{\mathrm{M}}(g(\mathrm{e})) = \overline{\varpi}_{\mathrm{M}}(\mathrm{e'}) = \overline{1}$, where e′ is a left identity of M_2.

Therefore $g^{-1}(\overline{\varpi}_{\mathrm{M}})$ is an IVNFMS of M_1.

Similarly, we can prove the remaining results.

Theorem 4.9. Let $\overline{\mathcal{A}_{\mathrm{M}}}$ be an IVNFMS in M and let **g**: $M \to M$be an onto homomorphism. Then the mapping$\overline{\mathcal{A}_{\mathrm{M}}}^{\mathbf{g}}: M \to$ **D**[0,1], is defined by $\overline{\mathcal{A}_{\mathrm{M}}}^{\mathbf{g}}(m_1) = \overline{\mathcal{A}_{\mathrm{M}}}(\mathbf{g}(m_1))$ for all $m_1 \in M$ is an IVNFMSin *M*.

Proof. (i) For any $m_1, m_2 \in M$,

$\overline{\varpi}_M^{\boldsymbol{g}}(m_1 m_2) = \overline{\varpi}_M(\ \boldsymbol{g}(m_1 m_2))$

$$= \overline{\varpi}_M(\boldsymbol{g}(m_1), \boldsymbol{g}(m_2)) \ \ (\text{since } \mathbf{g} \text{ is homomorphism})$$

$$\geq \ \min\{\overline{\varpi}_M(\boldsymbol{g}(m_1)), \overline{\varpi}_M(\boldsymbol{g}(m_2))\}$$

$$= \min\{\ \overline{\varpi}_M^{\mathbf{g}}(m_1), \overline{\varpi}_M^{\mathbf{g}}(m_2)\}$$

$\Rightarrow \overline{\varpi}_M^{\boldsymbol{g}}(m_1 m_2) \geq \min\{\ \overline{\varpi}_M^{\mathbf{g}}(m_1), \overline{\varpi}_M^{\mathbf{g}}(m_2)\}$

(ii) $\overline{\varpi}_M^{\boldsymbol{g}}(e) = \overline{\varpi}_M(\boldsymbol{g}(e)) = \overline{\varpi}_M(e') = \overline{1}$, where e′ is a left identity of *M*.

In the same way, we can prove the other results.

Theorem 4.10. Let **g**: $M_1 \to M_2$ be an epimorhism of M-semigroups. Let $\overline{\mathcal{A}_M}$ be a **g**-invariant IVNFMS of M_1. Then $\mathbf{g}(\overline{\mathcal{A}_M})$ is an IVNFMS of M_2.

Proof. (i) Let $\mathrm{m}_1', \mathrm{m}_2' \in M_2$. Then there exist $\mathrm{m}_1, \mathrm{m}_2 \in M_1$ such that $\mathbf{g}(\mathrm{m}_1) = \mathrm{m}_1'$ and $\mathbf{g}(\mathrm{m}_2) = \mathrm{m}_2'$.

Also $m_1'm_2' = \mathbf{g}(m_1m_2)$ and let $\mathrm{e}' \in M_2$. Then there exists $\mathrm{e} \in M_1$ such that $\boldsymbol{g}(\mathrm{e}) = \mathrm{e}'$, where e and e' are the left identity of M_1 and M_2.

Since $\overline{\mathcal{A}}_{\mathrm{M}}$ is $\mathbf{g}$-invariant,

$$\mathbf{g}(\overline{\varpi}_{\mathrm{M}})(m_1m_2) = \overline{\varpi}_M(m_1'm_2') \geq \min\{\overline{\varpi}_M(\mathrm{m}_1'), \overline{\varpi}_M(\mathrm{m}_2')\}$$

$$= min\{\boldsymbol{g}(\overline{\varpi}_M)(m_1), \boldsymbol{g}(\overline{\varpi}_M)(m_2)\}$$

$$\Rightarrow \boldsymbol{g}(\overline{\varpi}_M)(m_1m_2) \geq min\{\boldsymbol{g}(\overline{\varpi}_M)(m_1), \boldsymbol{g}(\overline{\varpi}_M)(m_2)\}$$

(ii) $g(\overline{\varpi}_M)(e) = \overline{\varpi}_M(e')$

$\Rightarrow g(\overline{\varpi}_M)(e) = \bar{1}.$

Therefore $g(\overline{\varpi}_M)$ is an IVNFMS of M_2.

We can prove the other results in the same manner.

Conclusions

This paper focuses on the theory of an IVNFMS, explore its applications in sociology and biology and examine numerous algebraic features, including intersection and union. Additionally, we introduce a direct product of these **IVNFMSs** and define the image and inverse between twoIVNFMSs.

As future directions, we plan to apply this concept to a range of algebraic structures, such as:

• Neutrosophic cubic M-semigroup,

• Neutrosophic cubic modules,

• Neutrosophic soft modules,

• Neutrosophic cubic ring,

• Neutrosophic interval-valued fuzzy metric space.

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Chapter 6

Relationship among Indices of Fuzzy Sets and Neutrosophic Sets

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ABSTRACT

The relationship among the indices of fuzzy sets (FSs) or neutrosophic sets (NSs) is important to analyze how different aspects of an element are interrelated and the ability to quantify these relationships helps model uncertainty and vagueness amicably. Linear dependence and independence of indices in this context provide critical insights into their structural relationships, which in turn influences their applications. Linear dependence refers to a situation when the indices of an element are not independent of each other i.e., there exist at least one index, which can be articulated as a linear combination of other indices. Whereas, linear independence occurs when the indices are not interrelated, and no index can be articulated as a linear combination of other indices. This chapter explores the concept of independence and dependence among the indices of fuzzy, intuitionistic fuzzy and neutrosophic sets. Further, the degree of dependence is studied that helps to make more informed decisions while modeling real-world problems. These concepts are then extended to define linear dependence and independence of indices in refined neutrosophic sets.

Keywords: Fuzzy set, Intuitionistic fuzzy set, Neutrosophic set, Refined Neutrosophic set.

INTRODUCTION

In the context of fuzzy sets as well as neutrosophic sets, uncertainty and imprecision are pivotal in the analysis and representation of data. These sets extend classical set theory to manage vagueness, ambiguity, and incomplete information that often arise in real world problems. The abstraction of fuzzy sets was established by L.A. Zadeh [28] to overcome limitations of the classical set theory to encompass impreciseness occurring because of inevitable circumstances. A FS is described by a membership function that measures the degree of belongingness or membership of an element in a set unlike the classical set theory where the characteristic or membership function assumes binary values only (0 or 1). The fuzziness allows for partial membership, which is essential when dealing with uncertainty.

Although FS allow for partial belongingness, they do not model the non-belongingness explicitly in a way that captures the degree of non-membership of an element in the considered set. In certain situations, it may be necessary to distinguish between an element being partially in a set and being explicitly excluded from the set. In order to address such situations, Atanassov [1] defined the concept of intuitionistic fuzzy sets (IFS). Intuitionistic fuzzy set theory extended fuzzy set theory by introducing two indices for each element - 'membership' and 'non-membership'. An IFS captures the uncertainty and vagueness present in real world problems more accurately.

However, despite the advancements of intuitionistic fuzzy sets, it still had certain limitations especially in systems where indeterminacy was not fully represented. In response to these limitations, Smarandache [21] introduced neutrosophic set theory, which further generalizes IFS by introducing an additional index - indeterminacy. NS handle uncertainty by incorporating - truth, falsity, and indeterminacy indices. This expansion allows neutrosophic sets to model situations with incomplete, inconsistent, or contradictory information more effectively than either fuzzy or intuitionistic fuzzy sets.

The association among the indices of FS (membership), IFS (membership and non-membership) and NS (truth, falsity, and indeterminacy) is fundamental for analyzing how the different aspects of an element interact. The concept of linear dependence and independence in the context of these sets concern whether membership and non-membership (in the case of IFS) or truth, falsity, and indeterminacy (in case of NS) are interrelated, or they provide independent information about the element's belongingness to the set. Linear dependence refers to a situation when the indices of an element are not independent of each other i.e., there exist an index that can be articulated as a linear combination of other indices. Whereas, linear independence occurs when the indices are not interrelated, and no index can be articulated as a linear combination of other indices. Understanding the linear dependence and independence between these indices is significant in various fields, particularly where uncertainty and imprecision are crucial. This chapter explores the idea of independence and dependence of the indices of fuzzy, intuitionistic fuzzy and neutrosophic sets.

BACKGROUND

Fuzzy set theory is an enhancement of classical set theory, which permits generalization of the concept of set membership. In traditional set theory, an element either belongs to a given set or it does not belong to the given set, i.e., the membership function is either 1 (if the element belongs to the set) or 0 (if the element does not belong to the set). However, in real world, there are sets that do not have a clear boundary. For instance, the set containing dates on which the temperature was hot at a particular place does not have a clear boundary as the boundary of hot depends upon personal interpretation and can lead to disagreeable discontinuity in deciding the dates to be considered in the set. Fuzzy set theory addresses such problems by permitting different degree of membership in the interval [0,1].

Definition 1. [28] Let the universe of discourse be U. Elements of a fuzzy set $\tilde{A}$ are described by ordered pair $(x, m_{\tilde{A}}(x))$ for $x \in U$ where $m_{\tilde{A}}: U \to [0,1]$ is the membership function of set $\tilde{A}$ and $m_{\tilde{A}}(x)$ is the degree of membership or the degree of belongingness of element x in the set $\tilde{A}$.

The membership function can be discrete or parametric (analytic or piecewise continuous). The discrete membership function is a basic type which can be represented using singleton sets. In contrast, parametric memberships are functions that may take many forms including Gaussians, trapezoids and triangular or any smooth or piecewise continuous function.

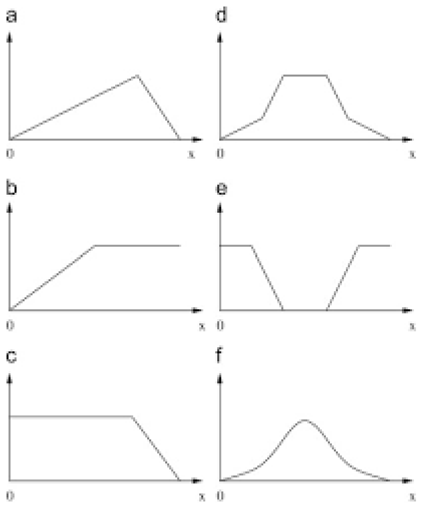

Figure 1. Some types of membership functions

Fuzzy set theory dispenses a sturdy framework for modeling and reasoning under uncertainty, facilitating more significant interpretation of data and decision-making process through its membership functions and several operations.

Operations on Fuzzy Sets

Fuzzy set theory offers various operations [16], which are extension of operations existing in classical set theory.

1. **Equality.** Let *U* be the universe of study. Two fuzzy sets $\tilde{A}$ and $\tilde{B}$ over *U* are said to be equal if $m_{\tilde{A}}(x) = m_{\tilde{B}}(x)$ for all $x \in U$.
2. **Union.** Let *U* be the universe of study. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets over *U*. Union of these fuzzy sets is denoted by $\tilde{A} \cup \tilde{B}$ and is given by
$$\tilde{A} \cup \tilde{B} = \{(x, m_{\tilde{A}\cup\tilde{B}}(x)) \mid m_{\tilde{A}\cup\tilde{B}}(x) = \max\{m_{\tilde{A}}(x), m_{\tilde{B}}(x)\}\}$$
3. **Intersection.** Let *U* be the universe of study. Let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets over *U*. Intersection of these fuzzy sets is denoted by $\tilde{A} \cap \tilde{B}$ and is given by
$$\tilde{A} \cap \tilde{B} = \{(x, m_{\tilde{A}\cap\tilde{B}}(x)) \mid m_{\tilde{A}\cap\tilde{B}}(x) = \min\{m_{\tilde{A}}(x), m_{\tilde{B}}(x)\}\}$$
4. **Complement.** Let *U* be the universe of study. Let $\tilde{A}$ be a fuzzy set over *U*. Complement of this fuzzy set is denoted by $\tilde{A}^C$ and is given by
$$\tilde{A}^C = \{(x, m_{\tilde{A}^C}(x)) \mid m_{\tilde{A}^C}(x) = 1 - m_{\tilde{A}}(x)\}$$

To depict uncertainty and vagueness in a FS, it is crucial to analyze another index of a FS, which is non-membership function or the function representing the degree of non-belongingness of an element to a set. The membership function $m_{\tilde{A}}(x)$ gives the degree to which an element belongs to a fuzzy set $\tilde{A}$, whereas the non-membership function $n_{\tilde{A}}(x)$ indicates the degree to which an element *x* does not belong to $\tilde{A}$. In a FS, $n_{\tilde{A}}(x) = 1 - m_{\tilde{A}}(x)$ i.e., it provides complementary information to the membership function. Collectively they create a complete picture of an element association to the fuzzy set under consideration.

Conventional fuzzy set theory only represents membership value or degree. The non-membership value is implicitly defined as complement of membership value. Traditional Fuzzy set theory does not account for situations where there is degree of hesitation. To overcome this limitation, intuitionistic fuzzy sets were proposed. Intuitionistic fuzzy sets refine the concept of degree of membership and non-membership by introducing a degree of hesitation or indeterminacy.

Definition 2. [1] Let *U* be the universe of discourse. An intuitionistic fuzzy set $\widetilde{A^I}$ is characterized by $(x, m_{\widetilde{A^I}}(x), n_{\widetilde{A^I}}(x))$ for $x \in U$, where $m_{\widetilde{A^I}}: U \to [0,1]$ is the membership function of set $\widetilde{A^I}$ and $m_{\widetilde{A^I}}(x)$ is the membership value of element *x* in set $\widetilde{A^I}$ and $n_{\widetilde{A^I}}: U \to [0,1]$ is the non-membership function of set $\widetilde{A^I}$and $n_{\widetilde{A^I}}(x)$ is the non-membership value of element *x* in set $\widetilde{A^I}$ such that the sum of $m_{\widetilde{A^I}}(x)$ and $n_{\widetilde{A^I}}(x)$ is constrained by $m_{\widetilde{A^I}}(x) + n_{\widetilde{A^I}}(x) \leq 1$ with the hesitation value $i_{\widetilde{A^I}}(x)$ implicitly defined as $i_{\widetilde{A^I}}(x) = 1 - m_{\widetilde{A^I}}(x) - n_{\widetilde{A^I}}(x)$.

The relationship between these three parameters is $m_{\widetilde{A^I}}(x) + n_{\widetilde{A^I}}(x) + i_{\widetilde{A^I}}(x) = 1$, which ensures that the total degree of membership, non-membership and hesitation is always equal to 1. Here, $m_{\widetilde{A^I}}(x)$ defines the degree of certainty about the membership, $n_{\widetilde{A^I}}(x)$ defines the degree of certainty of non-membership, and $i_{\widetilde{A^I}}(x)$ defines the degree of hesitation, i.e. it captures the situation where the decision maker is unsure about the non-membership or membership of an element in the intuitionistic fuzzy set $\widetilde{A^I}$. The additional parameter helps to model a problem more realistically specifically in cases when the relationships between elements are often not clear due to vagueness and subjectivity.

Operations on Intuitionistic Fuzzy Sets

Like Fuzzy set theory, intuitionistic fuzzy set theory offers several operations extending classical operations while incorporating the hesitation margin [2].

1. **Equality**. Two intuitionistic fuzzy sets $\widetilde{A^I}$ and $\widetilde{B^I}$ over the same universe of discourse *U* are said to be equal if $m_{\widetilde{A^I}}(x) = m_{\widetilde{B^I}}(x)$ and $n_{\widetilde{A^I}}(x) = n_{\widetilde{B^I}}(x)$ for every $x \in U$.
2. **Union.** Let $\widetilde{A^I}$ and $\widetilde{B^I}$ be two intuitionistic fuzzy sets over the same universe of discourse *U*. Union of these two intuitionistic fuzzy sets is denoted by $\widetilde{A^I} \cup \widetilde{B^I}$ and is given by

$$\widetilde{A^I} \cup \widetilde{B^I} = \left\{\left(x, m_{\widetilde{A^I} \cup \widetilde{B^I}}(x), n_{\widetilde{A^I} \cup \widetilde{B^I}}(x)\right) \mid x \in U\right\}$$

such that

$$m_{\widetilde{A^I} \cup \widetilde{B^I}}(x) = max\left\{m_{\widetilde{A^I}}(x), m_{\widetilde{B^I}}(x)\right\}$$

and

$$n_{\widetilde{A^I} \cup \widetilde{B^I}}(x) = min\left\{n_{\widetilde{A^I}}(x), n_{\widetilde{B^I}}(x)\right\}$$

3. **Intersection.** Let $\widetilde{A^I}$ and $\widetilde{B^I}$ be two intuitionistic fuzzy sets over the same universe of discourse *U*. Intersection of these two intuitionistic fuzzy sets is denoted by $\widetilde{A^I} \cap \widetilde{B^I}$ and is given by

$$\widetilde{A^I} \cap \widetilde{B^I} = \left\{\left(x, m_{\widetilde{A^I} \cap \widetilde{B^I}}(x), n_{\widetilde{A^I} \cap \widetilde{B^I}}(x)\right) \mid x \in U\right\}$$

such that

$$m_{\widetilde{A^I} \cap \widetilde{B^I}}(x) = min\left\{m_{\widetilde{A^I}}(x), m_{\widetilde{B^I}}(x)\right\}$$

and

$$n_{\widetilde{A^I} \cap \widetilde{B^I}}(x) = max\left\{n_{\widetilde{A^I}}(x), n_{\widetilde{B^I}}(x)\right\}$$

4. **Complement.** Let $\widetilde{A^I}$ be an intuitionistic fuzzy set over the universe of discourse *U*. Complement of this intuitionistic fuzzy set is denoted by $\widetilde{A^I}^{\,C}$ and is given by

$$\widetilde{A^I}^C = \left\{\left(x, m_{\widetilde{A^I}^C}(x), n_{\widetilde{A^I}^C}(x)\right) \mid x \in U\right\}$$

such that

$$m_{\widetilde{A^I}^C}(x) = n_{\widetilde{A^I}}(x)$$

and

$$n_{\widetilde{A^I}^C}(x) = m_{\widetilde{A^I}}(x)$$

Though intuitionistic fuzzy set theory provide more flexibility than standard fuzzy set, but it is unable to handle indeterminacy explicitly, which led to the extension of this theory to Neutrosophic set theory. Neutrosophic set theory is an extension of classical and fuzzy set theory to handle uncertainty, imprecision, vagueness and inconsistency in information.

Definition 3. [22] Let *U* be the universe of discourse. A neutrosophic set $\widetilde{A^N}$ is characterized by $\left(x, m_{\widetilde{A^N}}(x), i_{\widetilde{A^N}}(x), n_{\widetilde{A^N}}(x)\right)$ for $x \in U$, where $m_{\widetilde{A^N}}: U \to [0,1]$ represents the truth degree function, and $m_{\widetilde{A^N}}(x)$ indicates the degree to which *x* is true or belongs to the set $\widetilde{A^N}$; $i_{\widetilde{A^N}}: U \to [0,1]$ represents the indeterminacy degree function, and $i_{\widetilde{A^N}}(x)$ indicates the degree of uncertainty or hesitation about membership or non-membership of *x* in the set $\widetilde{A^N}$; and $n_{\widetilde{A^N}}: U \to [0,1]$ represents the falsity degree function, and $n_{\widetilde{A^N}}(x)$ indicates the degree to which *x* is false or does not belong to the set $\widetilde{A^N}$ such that the sum of $m_{\widetilde{A^N}}(x), i_{\widetilde{A^N}}(x)$ and $n_{\widetilde{A^N}}(x)$ is constrained by $0 \leq m_{\widetilde{A^N}}(x) + i_{\widetilde{A^N}}(x) + n_{\widetilde{A^N}}(x) \leq 3$. Neutrosophic sets are commonly referred as Single-valued Neutrosophic Sets as the indices are single-valued numbers.

The value of $m_{\widetilde{A^N}}(x), i_{\widetilde{A^N}}(x)$ and $n_{\widetilde{A^N}}(x)$ are usually real in the interval $[0,1]$ but may extend beyond this range to $[0^-, 1^+]$ to allow over-estimation or under-estimation. Neutrosophic set theory provides more flexible approach to handle real world problems where classical and fuzzy set theories may fall short.

Operations on Neutrosophic Sets

Neutrosophic set theory offers several operations [22] extending classical operations as in fuzzy set theory.

1. **Equality.** Two neutrosophic sets $\widetilde{A^N}$ and $\widetilde{B^N}$ over the same universe of discourse *U* are said to be equal if $m_{\widetilde{A^N}}(x) = m_{\widetilde{B^N}}(x), i_{\widetilde{A^N}}(x) = i_{\widetilde{B^N}}(x)$ and $n_{\widetilde{A^N}}(x) = n_{\widetilde{B^N}}(x)$ for every $x \in U$.
2. **Union.** Let $\widetilde{A^N}$ and $\widetilde{B^N}$ be two neutrosophic sets over the same universe of discourse *U*. Union of these two neutrosophic sets is denoted by $\widetilde{A^N} \cup \widetilde{B^N}$ and is given by

$$\widetilde{A^N} \cup \widetilde{B^N} = \left\{\left(x, m_{\widetilde{A^N} \cup \widetilde{B^N}}(x), i_{\widetilde{A^N} \cup \widetilde{B^N}}(x), n_{\widetilde{A^N} \cup \widetilde{B^N}}(x)\right) \mid x \in U\right\}$$

such that

$$m_{\widetilde{A^N}\cup\widetilde{B^N}}(x) = max\{m_{\widetilde{A^N}}(x), m_{\widetilde{B^N}}(x)\};$$
$$i_{\widetilde{A^N}\cup\widetilde{B^N}}(x) = min\{i_{\widetilde{A^N}}(x), i_{\widetilde{B^N}}(x)\}$$

and

$$n_{\widetilde{A^N}\cup\widetilde{B^N}}(x) = min\{n_{\widetilde{A^N}}(x), n_{\widetilde{B^N}}(x)\}$$

3. **Intersection.** Let $\widetilde{A^N}$ and $\widetilde{B^N}$ be two neutrosophic sets over the same universe of discourse *U*. Intersection of these two neutrosophic sets is denoted by $\widetilde{A^N} \cap \widetilde{B^N}$ and is given by

$$\widetilde{A^N} \cap \widetilde{B^N} = \left\{\left(x, m_{\widetilde{A^N}\cap\widetilde{B^N}}(x), i_{\widetilde{A^N}\cap\widetilde{B^N}}(x), n_{\widetilde{A^N}\cap\widetilde{B^N}}(x)\right) \mid x \in U\right\}$$

such that

$$m_{\widetilde{A^N}\cap\widetilde{B^N}}(x) = min\{m_{\widetilde{A^N}}(x), m_{\widetilde{B^N}}(x)\};$$
$$i_{\widetilde{A^N}\cap\widetilde{B^N}}(x) = max\{i_{\widetilde{A^N}}(x), i_{\widetilde{B^N}}(x)\}$$

and

$$n_{\widetilde{A^N}\cap\widetilde{B^N}}(x) = max\{n_{\widetilde{A^N}}(x), n_{\widetilde{B^N}}(x)\}$$

4. **Complement.** Let $\widetilde{A^N}$ be a neutrosophic set over the universe of discourse *U*. Complement of this neutrosophic set is denoted by $\widetilde{A^N}^C$ and is given by

$$\widetilde{A^N}^C = \left\{\left(x, m_{\widetilde{A^N}^C}(x), i_{\widetilde{A^N}^C}(x), n_{\widetilde{A^N}^C}(x)\right) \mid x \in U\right\}$$

such that

$$m_{\widetilde{A^N}^C}(x) = n_{\widetilde{A^N}}(x);\ i_{\widetilde{A^N}^C}(x) = i_{\widetilde{A^N}}(x)$$

and

$$n_{\widetilde{A^N}^C}(x) = m_{\widetilde{A^N}}(x)$$

Independence and Dependence of Indices in Fuzzy and Neutrosophic Sets

In classical fuzzy set theory, there is only one index, $m(x)$ so the concept of independence holds no meaning in classical fuzzy set theory. It is presumed that the information about the membership function is completely known [7] so the sum $m(x) + n(x) = 1$ is the only dependence relation in classical fuzzy set theory. In other words, the degree of non-membership is dependent upon the degree of membership and is given as $n(x) = 1 - m(x)$.

However, in Intuitionistic Fuzzy Set theory, there are two indices $m(x)$ and $n(x)$. The hesitation degree or the degree of indeterminacy, $i(x)$ is implicitly derived from $m(x)$ and $n(x)$ by the relation $i(x) = 1 - m(x) - \mathrm{n(x)}$. Thus, $i(x)$ is not an independent parameter, so there is no explicit control over $i(x)$ while modeling the problem [3].

Subsequently, Neutrosophic set theory overcomes the limitation by explicitly defining the indices of a single-valued neutrosophic set $m(x)$, $i(x)$ and $n(x)$. In neutrosophic sets, $m(x)$, $i(x)$ and $n(x)$ are independent indices and the sum of the indices are not constrained to be equal to *1*. In fact, in a neutrosophic set $0 \leq m(x) + i(x) + n(x) \leq 3$. Here, each index can take values independently and hence provides greater flexibility for complex, real world scenarios involving imprecision, vagueness, or inconsistency [25].

Deviating from the classical notion of independence and dependence of indices in intuitionistic fuzzy theory, the concept is reiterated below.

Independence and dependence of indices in an Intuitionistic Fuzzy Set

Suppose $m(x)$ and $n(x)$ are 100% dependent on each other, as in classical intuitionistic fuzzy set theory. Then,

$$0 \le m(x) + n(x) \le 1.$$

In case of 100% dependence of indices when the information on membership and non-membership is complete then,

$$m(x) + n(x) = 1.$$

In case of 100% dependence of indices when the information on membership and non-membership is incomplete then,

$$m(x) + n(x) < 1.$$

Now if, $m(x)$ and $n(x)$ are 100% independent of each other, then,

$$0 \le m(x) + n(x) \le 2.$$

In case of 100% independence of indices when the information on membership and non-membership is complete then,

$$m(x) + n(x) = 2.$$

In case of 100% independence of indices when the information on membership and non-membership is incomplete then,

$$m(x) + n(x) < 2.$$

Independence and dependence of indices in a Neutrosophic Set

In classical Neutrosophic Set Theory, the indices $m(x)$, $i(x)$ and $n(x)$ are assumed to be 100% independent of each other. So,

$$0 \le m(x) + i(x) + n(x) \le 3.$$

In case of 100% independence of indices when the information on truth, indeterminacy and falsity is complete then,

$$m(x) + i(x) + n(x) = 3.$$

In case of 100% independence of indices when the information on truth, indeterminacy and falsity is incomplete then,

$$m(x) + i(x) + n(x) < 3.$$

Suppose two indices are independent, while the third one is dependent upon them. Then,

$$0 \le m(x) + i(x) + n(x) \le 2.$$

In case two indices are independent, while the third one is dependent and the information on truth, indeterminacy and falsity is complete then,

$$m(x) + i(x) + n(x) = 2.$$

In case two indices are independent, while the third one is dependent and the information on truth, indeterminacy and falsity is contradictory then,

$$1 \le m(x) + i(x) + n(x) < 2.$$

In case two indices are independent, while the third one is dependent and the information on truth, indeterminacy and falsity is incomplete then,

$$m(x) + i(x) + n(x) < 2.$$

If all the three indices are dependent upon each other, then

$$0 \leq m(x) + i(x) + n(x) \leq 1.$$

In case of 100% dependence of indices when the information on truth, indeterminacy and falsity is complete then,

$$m(x) + i(x) + n(x) = 1.$$

In case of 100% dependence of indices when the information on truth, indeterminacy and falsity is incomplete then,

$$m(x) + i(x) + n(x) < 1.$$

Bounds of Sum of Indices of Intuitionistic Fuzzy Sets

The degree of independence between the membership and non-membership functions of an IFS refers to how much the two functions vary independently of each other. If $m(x)$ and $n(x)$ vary independently, then change in any one function does not directly affect the other function. In this case, the sum $m(x) + n(x)$ would be far from 1, and the indeterminacy $i(x)$ would be large, which would reflect a high level of uncertainty about the classification of elements in the set. It is apparent that in case $m(x)$ and $n(x)$ are highly correlated, i.e., changes in one function are followed by changes in the other function, then the independence between the functions is low.

It is known that if f and g are indices that vary in the unitary interval $[0,1]$, then the sum is in the following interval

$$0 \leq f + g \leq 2 - d(f, g).$$

where $d(f, g) \in [0,1]$ is the degree of dependence between f and g. Independence can be weighed as the inverse of dependence, so $(1 - d(f, g))$ is the degree of independence between f and g. Here, $d(f, g) = 0$ when f and g are 100% independent and $d(f, g) = 1$ when f and g are 100% dependent.

In an IFS, if $m(x)$ and $n(x)$ are d % dependent, then

$$0 \leq m(x) + n(x) \leq 2 - d/100.$$

Observe that the indeterminacy $I(x)$ varies in relation to $m(x)$ and $n(x)$. A high level of indeterminacy indicates greater independence between the two functions. If $m(x)$ and $n(x)$ are nearly uncorrelated, the element would exhibit a higher degree of indeterminacy, and the set's uncertainty would be more significant.

Example 1: Suppose $\widetilde{A^I}$ is an IFS in which the membership and non-membership functions are 0 % dependent, that is $m(x)$ and $n(x)$ are 100% independent, then $0 \leq m(x) + n(x) \leq 2$. Suppose $\widetilde{A^I}$ is an IFS in which the membership and non-membership functions are 0% dependent, then $0 \leq m(x) + n(x) \leq 1.5$. Suppose $\widetilde{A^I}$ is an IFS in which the membership and non-membership functions are 75% dependent, that is $m(x)$ and $n(x)$ are 25% independent, then $0 \leq m(x) + n(x) \leq 1.25$. Suppose $\widetilde{A^I}$ is an IFS in which the membership and non-membership functions are 100% dependent, that is $m(x)$ and $n(x)$ are 0% independent, then $0 \leq m(x) + n(x) \leq 1$. These relations are consistent with the idea of dependence and independence defined in the previous section.

Bounds of Sum of Indices of Neutrosophic Sets

The degree of dependence and independence between the three indices truth, indeterminacy, and falsity of a NS refers to how these functions relate to each other, i.e., how changes in one affect the other indices. The degree of dependence between the three indices $m(x)$, $i(x)$ and $n(x)$ refers to how closely related these functions are. If these indices are highly correlated, then we can say that they are dependent on each other. Whereas, if the values of one function have little or no impact on the others then it is said that the degree of dependence is low.

If high degree of truth implies a low degree of falsity and vice versa then it is said that there is high dependence between the truth, falsity, and indeterminacy functions. However, this is rarely the case in neutrosophic sets because the functions are generally independent of each other.

As above, if f, g and h are indices that vary in the unitary interval $[0,1]$, if all of these three indices are independent then the sum $f+g+h$ is in the following interval

$$0 \leq f+g+h \leq 3.$$

If all of these indices are 100% dependent upon each other, then

$$0 \leq f+g+h \leq 1.$$

In case of partial dependence, that is in case when f and g are 100% dependent but h is independent, that is $0 \leq f+g \leq 1$ and $0 \leq h \leq 1$, the sum $f+g+h$ is in the following interval

$$1 \leq f+g+h \leq 2.$$

Applying this concept to a NS, in which there are three indices $m(x)$, $i(x)$ and $n(x)$, we get

$$0 \leq m(x)+i(x)+n(x) \leq 3.$$

The information on $m(x)$, $i(x)$ and $n(x)$ are independent if the three sources providing the necessary information do not communicate with each other. Therefore, $\max\{m(x)+i(x)+n(x)\}$ is in between 1 and 3.

Example 2: Let $\widetilde{A^N}$ be a NS in which truth, indeterminacy and falsity are 0 % dependent, i.e., $m(x)$, $i(x)$ and $n(x)$ are 100% independent. Then, $d\big(m(x), i(x)\big) = 0; d\big(m(x), n(x)\big) = 0$ and $d\big(i(x), n(x)\big) = 0$ so $m(x)+i(x) \leq 2$; $m(x)+n(x) \leq 2$ and $i(x)+n(x) \leq 2$. Hence, $0 \leq m(x)+i(x)+n(x) \leq 3$.

If $\widetilde{A^N}$ is a NS in which truth and falsity are 100% dependent and indeterminacy is 100% independent, that is $m(x)$ and $n(x)$ are 100% dependent, and $i(x)$ is 100% independent of them then $0 \leq m(x)+n(x) \leq 1$ and $0 \leq i(x) \leq 1$ so $0 \leq m(x)+i(x)+n(x) \leq 2$.

And if, $\widetilde{A^N}$ is a NS in which truth, falsity and indeterminacy are 100% dependent on each other, that is $m(x)$, $i(x)$ and $n(x)$ balance each other out, which occurs when an increase in the value of truth implies decrease in falsity and the gap is filled with indeterminacy as in the case of intuitionistic fuzzy set theory and hence $0 \leq m(x)+i(x)+n(x) \leq 1$.

These relations are consistent with the idea of dependence and independence defined in the previous section.

Example 3: Suppose $\widetilde{A^N}$ is a NS in which $m(x)$ and $n(x)$ are 20% dependent and $i(x)$ and $n(x)$ are 70% dependent. This is the case of partial dependence and independence and here $\max\{m(x)+i(x)+n(x)\}$ is obtained by observing that $m(x)+n(x) \leq 2-0.2 = 1.8$ and $i(x)+n(x) \leq 2-0.7 = 1.3$ and $\max\{m(x)+i(x)+n(x)\} = 2.3$ as maximum occurs when $m(x)=1, n(x)=0.8, i(x)=0.5$.

Example 4: Suppose $\widetilde{A^N}$ is a NS in which $m(x)$ and $i(x)$ are 100% dependent and $i(x)$ and $n(x)$ are 100% independent. Then, $m(x) + i(x) \leq 1$ and $i(x) + n(x) \leq 2$ so, $\max\{m(x) + i(x) + n(x)\} = 2$ as maximum occurs when $m(x) = 1, n(x) = 1, i(x) = 0$.

Generalizing to Refined Neutrosophic Sets

A Refined Neutrosophic Set [24] and logic [23] is an extension of traditional Neutrosophic set and logic obtained by considering layers of refinement in the indices, truth, indeterminacy, and falsity of a NS. The refined approach allows an even more detailed and precise modeling of a complex real-world problem arising in various fields of decision making. The refined version often involves introducing multiple sub-degrees of truth, indeterminacy, and falsity, breaking them down into refined truth, refined indeterminacy, and refined falsity, which gives a more specific or detailed degree of truth, indeterminacy, and falsity respectively.

Definition 4. [25] Let U be the universe of study. A refined neutrosophic set $\widetilde{A^N}$ is characterized by $\left(x, m_1(x), m_2(x), \ldots, m_p(x), i_1(x), i_2(x), \ldots, i_r(x), n_1(x), n_2(x), \ldots, n_s(x)\right)$ for $x \in U$, where $p, r, s \geq 1$ are integers and $p + r + s = t \geq 3$, where $m_l: U \rightarrow [0,1]$ represent the l^{th} sub-index of truth function and $m_l(x)$ are called the sub-membership degree for $l = 1,2, \ldots, p$; $i_j: U \rightarrow [0,1]$ represent the j^{th} sub-index of indeterminacy function, and $i_j(x)$ indicates the sub-indeterminacy degree for $j = 1,2, \ldots, r$; and $n_k: U \rightarrow [0,1]$ represents the k^{th} sub-index of falsity function, and $n_k(x)$ indicates the sub-falsity degree for k= $1,2, \ldots, s$. Here, $\widetilde{A^N}$ is called *t*-valued refined neutrosophic set as it has *t* sub-indices.

Observe that a single valued neutrosophic set is a special case of refined neutrosophic set, where $t = 3$. In other words, there are only three indices $m(x), i(x)$ and $n(x)$.

Independence and dependence of sub-indices of a refined neutrosophic set

Each of the sub-indices - 'sub-truth, sub-indeterminacy and sub-falsity' - are crisp numbers in the interval $[0,1]$, that is $0 \leq m_l(x) \leq 1$ for $l = 1,2, \ldots, p$; $0 \leq i_j(x) \leq 1$ for $j = 1,2, \ldots, r$ and $0 \leq n_k(x) \leq 1$ for $k = 1,2, \ldots, s$. So, if all these sub-indices are independent, then

$$0 \leq \sum_{l=1}^{p} m_l(x) + \sum_{j=1}^{r} i_j(x) + \sum_{k=1}^{s} n_k(x) \leq t$$

In case all these sub-indices are 100% dependent upon each other, then

$$0 \leq \sum_{l=1}^{p} m_l(x) + \sum_{j=1}^{r} i_j(x) + \sum_{k=1}^{s} n_k(x) \leq 1$$

In case of partial dependence and independence, that is in case *u* indices are 100% dependent and the remaining *(t-u)* indices are 100% independent, then,

$$0 \leq \sum_{l=1}^{p} m_l(x) + \sum_{j=1}^{r} i_j(x) + \sum_{k=1}^{s} n_k(x) \leq t - u + 1$$

as *u* indices are 100% dependent so the sum of these *u* indices lies in the interval $[0,1]$ and each of the remaining *(t-u)* independent indices lie in the interval $[0,1]$.

Example 5: Suppose $\widetilde{A^N}$ is a refined neutrosophic set whose truth function, m splits into m_1, m_2; indeterminacy function, i splits into i_1, i_2, i_3; and falsity function, n does not split, that is there are 6 indices of refined neutrosophic set under consideration. If all the indices are 100% independent of each other, then $0 \leq m_1(x) + m_2(x) + i_1(x) + i_2(x) + i_3(x) + n(x) \leq 6$.

In case, the indices m_1, i_1 F are 100% dependent upon each other, that is $0 \leq m_1(x) + i_1(x) + n(x) \leq 1$ while the rest are totally independent from others that is $0 \leq m_2(x) \leq 1$, $0 \leq i_2(x) \leq 1, 0 \leq i_3(x) \leq 1$ and hence $0 \leq m_1(x) + m_2(x) + i_1(x) + i_2(x) + i_3(x) + n(x) \leq 4$.

Example 6: Suppose $\widetilde{A^N}$ is a refined neutrosophic set whose truth function, m splits into m_1, m_2; indeterminacy function, i splits into i_1, i_2; and falsity function, F splits into n_1, n_2, that is there are 6 indices of refined neutrosophic set under consideration. Suppose m_1 and i_1 are 20% dependent, then $0 \leq m_1(x) + i_1(x) \leq 2 - 0.2 = 1.8$. If each of the other indices are independent of all others, then $0 \leq m_2(x) \leq 1, 0 \leq i_2(x) \leq 1, 0 \leq n_1(x) \leq 1, 0 \leq n_2(x) \leq 1$ and hence $0 \leq m_1(x) + m_2(x) + i_1(x) + i_2(x) + n_1(x) + n_2(x) \leq 4 + 1.8 = 5.8$.

Degree of Independence and Dependence of Indices of Fuzzy and Neutrosophic Sets

The degree of independence or dependence among the indices of fuzzy or neutrosophic set can be measured by using various statistical or information theoretic methods. Methods like covariance, correlation coefficient and graphical methods can be used to examine the dependence between two indices of a fuzzy set or a neutrosophic set. To observe how two indices of a neutrosophic set interact with each other while controlling the effect of the third index, the concept of partial correlation can be put to test.

Covariance

This method enables the decision maker to figure out how the indices vary jointly. The covariance between two discrete indices *f* and g is given by

$$Cov(f, g) = \frac{1}{N} \sum_{x \in U} \left(f(x) - \bar{f}\right)(g(x) - \bar{g})$$

where,

$$\bar{f} = \frac{\sum f(x)}{N};\ \bar{g} = \frac{\sum g(x)}{N}$$

are average of functions *f* and g respectively and *N* is cardinality of universe of study *U*.

And, the covariance between two continuous indices *f* and g is given by

$$Cov(f, g) = \frac{1}{b - a} \int_a^b \left(f(x) - \bar{f}\right)(g(x) - \bar{g}) dx$$

where,

$$\bar{f} = \frac{1}{b - a} \int_a^b f(x) dx;\ \bar{g} = \frac{1}{b - a} \int_a^b g(x) dx$$

are average of functions *f* and g respectively and the universe of study, $U = (a, b)$.

A high value of $Cov(f, g)$ indicates that the two functions under consideration are highly dependent upon each other, whereas a value close to zero suggests that the two functions are independent. In case $Cov(f, g) > 0$, then *f* and g increase or decrease together i.e., they are in direct relation. If $Cov(f, g) < 0$, then *f* and g are in inverse relation i.e., an increase if *f* suggests a decrease in g. If $Cov(f, g) = 0$, then there is no linear dependence between *f* and g suggesting that they are independent.

Covariance between membership function, $m(x)$ and non-membership function, $n(x)$ of a discrete IFS is given by

$$Cov(m,n) = \frac{1}{N}\sum_{x\in U}(m(x) - \bar{m})(n(x) - \bar{n})$$

where,

$$\bar{m} = \frac{\sum m(x)}{N};\ \bar{n} = \frac{\sum n(x)}{N}$$

are average values of membership and non-membership functions respectively and *N* is cardinality of universe of study *U*.

Example 7: Consider the discrete IFS $\widetilde{A^I} = \{(x_1, 0.8, 0.1), (x_2, 0.6, 0.3), (x_3, 0.2, 0.7), (x_4, 0.9, 0.05)\}$. The mean membership value of $\widetilde{A^I}$, $\bar{m} = 0.625$ and the mean non-membership value of $\widetilde{A^I}$, $\bar{n} = 0.288$. Then, the covariance between the membership function and non-membership function of $\widetilde{A^I}$ is -0.068 (Refer to Table 1).

Table 1: Calculation table for Covariance of *m* and *n* of $\widetilde{A^I}$

$m(x)$	$n(x)$	$m(x) - \bar{m}$	$n(x) - \bar{n}$	$(m(x) - \bar{m})(n(x) - \bar{n})$
0.8	0.1	0.175	-0.188	-0.033
0.6	0.3	-0.025	0.012	-0.0003
0.2	0.7	-0.425	0.412	0.175
0.9	0.05	0.275	-0.238	-0.065
$\bar{m} = 0.625$	$\bar{n} = 0.288$			$Cov(m,n) = -0.068$

And, covariance between membership function, $m(x)$ and non-membership function, $n(x)$ of a continuous IFS is given by

$$Cov(m,n) = \frac{1}{b-a}\int_a^b (m(x) - \bar{m})(n(x) - \bar{n})dx$$

where,

$$\bar{m} = \frac{1}{b-a}\int_a^b m(x)dx\,;\ \bar{n} = \frac{1}{b-a}\int_a^b n(x)dx$$

are average values of membership and non-membership functions respectively and the universe of study *U*= (a, b).

Example 8: Consider the continuous IFS $\widetilde{A^I} = \{(x, m(x), n(x)) : x \in [0,1]\}$, where $m(x) = 1 - x$ and $n(x) = 0.5x$. Then, the average membership value and non-membership values are 0.5 and 0.25 respectively and covariance between the membership function and non-membership function is -0.04.

Note that, if $Cov(m,n) > 0$, then m and n increase or decrease together i.e., they are in direct relation. If $Cov(m,n) < 0$, then m and n are in inverse relation i.e., an increase if m suggests a decrease in *n*. If $Cov(m,n) = 0$, then there is no linear dependence between *m* and *n* suggesting that they are independent.

Similarly, covariance between any two discrete neutrosophic indices, say, truth, $m(x)$ and indeterminacy, $i(x)$ of a neutrosophic set is given by

$$Cov(m,i) = \frac{1}{N}\sum_{x\in U}(m(x) - \bar{m})(i(x) - \bar{\iota})$$

where,

$$\bar{m} = \frac{\sum m(x)}{N};\ \bar{\imath} = \frac{\sum i(x)}{N}$$

are average values of truth and indeterminacy functions respectively and *N* is cardinality of universe of study *U*.

Example 9: Consider the NS $\widetilde{A^N} = \{(x_1, 0.8, 0.1, 0.1), (x_2, 0.6, 0.3, 0.1), (x_3, 0.2, 0.5, 0.3), (x_4, 0.9, 0.05, 0.05)\}$. The mean truth value of $\widetilde{A^I}$, $\bar{m} = 0.625$ and the mean indeterminacy value of $\widetilde{A^I}$, $\bar{\imath} = 0237$. Then, the covariance between the truth and indeterminacy of $\widetilde{A^I}$ is -0.047. Similarly, the covariance between the truth and falsity of $\widetilde{A^I}$ is -0.014 and the covariance between the indeterminacy and falsity of $\widetilde{A^I}$ is 0.009.

And, the covariance between any two continuous neutrosophic indices, say, truth, $m(x)$ and indeterminacy, $i(x)$ of a neutrosophic set is given by

$$Cov(m, i) = \frac{1}{b-a}\int_a^b (m(x) - \bar{m})(i(x) - \bar{\imath})dx$$

where,

$$\bar{m} = \frac{1}{b-a}\int_a^b m(x)dx\ ;\ \bar{\imath} = \frac{1}{b-a}\int_a^b i(x)dx$$

are average values of truth and indeterminacy functions respectively and the universe of study $U = (a, b)$.

Example 10: Consider the NS $\widetilde{A^N} = \{(x_1, m(x), i(x), n(x)): x \in [0,1]\}$, where $m(x) = 0.6(1-x)$, $i(x) = 0.3x$, and $n(x) = 0.1(1-x)$. The mean truth value of $\widetilde{A^I}$, $\bar{m} = 0.3$ and the mean indeterminacy value of $\widetilde{A^I}$, $\bar{\imath} = 0.15$. Then, the covariance between the truth and indeterminacy of $\widetilde{A^I}$ is -0.015. Similarly, the covariance between the truth and falsity of $\widetilde{A^I}$ is 0.005, and the covariance between the indeterminacy and falsity of $\widetilde{A^I}$ is -0.07.

Note that, if $Cov(m, i) > 0$, then m and i increase or decrease together i.e., they are in direct relation. If $Cov(m, i) < 0$, then *m* and *i* are in inverse relation i.e., an increase if *m* suggests a decrease in *i*. If $Cov(m, i) = 0$, then there is no linear dependence between *m* and *i* suggesting that they are independent.

To comprehend the dependence between all the indices of a neutrosophic set, one needs to examine $Cov(T, F)$ and $Cov(F, I)$ as well.

Correlation Coefficient

Correlation coefficients are normal forms of covariance that determine the intensity and the direction of the association between the two indices under consideration. The correlation coefficient between two indices *f* and g is given by

$$\rho(f, g) = \frac{Cov(f, g)}{\sqrt{Var(f).Var(g)}}$$

where, $Cov(f, g)$ is the covariance between the indices *f* and g; $Var(f)$ and $Var(g)$ are variance of the indices *f* and g respectively, which are obtained as follows

$Var(f) = \frac{1}{N}\sum_{x\in U}\left(f(x) - \bar{f}\right)^2$; $Var(g) = \frac{1}{N}\sum_{x\in U}(g(x) - \bar{g})^2$, when *f* and g are discrete.

$Var(f) = \int_{x\in U}\left(f(x) - \bar{f}\right)^2 dx$; $Var(g) = \int_{x\in U}(g(x) - \bar{g})^2 dx$, when *f* and g are continuous.

The correlation coefficient ρ lies between -1 and 1. In case $\rho(f,g) > 0$ then there is a positive correlation between f and g i.e., they increase or decrease together. In other words, f and g are in direct relation. If $\rho(f,g) < 0$ then there is a negative correlation between f and g i.e., an increase in f leads to decrease in g and vice versa. In other words, f and g are in inverse relation. If $\rho(f,g) = 0$ then there no correlation between f and g i.e., they are independent.

Correlation coefficient between membership function, $m(x)$ and non-membership function, $n(x)$ of an IFS is given by

$$\rho(m,n) = \frac{Cov(m,n)}{\sqrt{Var(m).Var(n)}}$$

where, $Cov(m,n)$ is the covariance between the indices m and n; $Var(m)$ and $Var(n)$ are variance of the indices m and n respectively, which are obtained as follows

$Var(m) = \frac{1}{N}\sum_{x\in U}(m(x) - \bar{m})^2$; $Var(n) = \frac{1}{N}\sum_{x\in U}(n(x) - \bar{n})^2$, when IFS is discrete.

$Var(m) = \int_{x\in U}(m(x) - \bar{m})^2 dx$; $Var(n) = \int_{x\in U}(n(x) - \bar{n})^2 dx$, when IFS is continuous.

The correlation coefficient ρ lies between -1 and 1. If $\rho(m,n) > 0$ then there is a positive correlation between membership function, m and non-membership function n i.e., they increase or decrease together. In other words, m and n are in direct relation. If $\rho(m,n) < 0$ then there is a negative correlation between m and n i.e., an increase in m leads to decrease in n and vice versa. In other words, m and n are in inverse relation. If $\rho(m,n) = 0$ then there no correlation between m and n i.e., they are independent.

Example 11: Consider the discrete IFS $\widetilde{A^I} = \{(x_1, 0.8,0.1), (x_2, 0.6,0.3), (x_3, 0.2,0.7), (x_4, 0.9,0.05)\}$. The mean membership value of $\widetilde{A^I}$, $\bar{m} = 0.625$; the mean non-membership value of $\widetilde{A^I}$, $\bar{n} = 0.288$; the covariance between the membership function and non-membership function of $\widetilde{A^I}$ is -0.06; the variance of membership value is 0.071; and the variance of non-membership value is 0.065. So, the correlation coefficient between the membership value and non-membership value is -0.99. Thus, the two indices are negatively correlated.

Example 12: Consider the continuous IFS $\widetilde{A^I} = \{(x, m(x), n(x)): x \in [0,1]\}$, where $m(x) = 1 - x$ and $n(x) = 0.5x$. Then, the average membership value and non-membership values are 0.5 and 0.25 respectively; covariance between the membership function and non-membership function is -0.041; and the variance of membership value and non-membership value are 0.16 and 0.021 respectively. So, the correlation coefficient between the membership value and non-membership value is -0.71.

Similarly, correlation coefficient between any two neutrosophic indices [27], say truth, $m(x)$ and indeterminacy, $i(x)$ of a neutrosophic set is given by

$$\rho(m,i) = \frac{Cov(m,i)}{\sqrt{Var(m).Var(i)}}$$

where, $Cov(m,i)$ is the covariance between the indices m and i; $Var(m)$ and $Var(i)$ are variance of the indices m and i respectively, which are obtained as follows:

$Var(m) = \frac{1}{N}\sum_{x\in U}(m(x) - \bar{m})^2$; $Var(i) = \frac{1}{N}\sum_{x\in U}(i(x) - \bar{\iota})^2$, when NS is discrete.

$Var(m) = \int_{x\in U}(m(x) - \bar{m})^2 dx$; $Var(i) = \int_{x\in U}(i(x) - \bar{\iota})^2 dx$, when NS is continuous.

The correlation coefficient ρ lies between -1 and 1. If $\rho(m,i) > 0$ then there is a positive correlation between m and i i.e., they increase or decrease together. In other words, m and i are in direct relation. If $\rho(m,i) < 0$ then there is a negative correlation between m and i i.e., an increase in m leads to decrease in i and vice versa. In other words, m and i are in inverse relation. If $\rho(m, i) = 0$ then there no correlation between m and i i.e., they are independent.

To comprehend the dependence between all the indices of a neutrosophic set, one needs to examine $\rho(m, n)$ and $\rho(n, i)$ as well.

Example 13: Consider the NS $\widetilde{A^N} = \{(x_1, 0.8, 0.1, 0.1), (x_2, 0.6, 0.3, 0.1), (x_3, 0.2, 0.5, 0.3), (x_4, 0.9, 0.05, 0.05)\}$. The mean truth value of $\widetilde{A^I}$, $\bar{m} = 0.625$, the mean indeterminacy value of $\widetilde{A^I}$, $\bar{\iota} = 0237$, the mean falsity value$\widetilde{A^I}$, $\bar{n} = 0.1125$; the covariance between the truth and indeterminacy of $\widetilde{A^I}$ is -0.047, the covariance between the truth and falsity of $\widetilde{A^I}$ is -0.014, the covariance between the indeterminacy and falsity of $\widetilde{A^I}$ is 0.009; and variance of truth, indeterminacy and falsity are 0.07, 0.03 and 0.003 respectively. So, the correlation coefficient between the truth and indeterminacy is -0.988, the correlation coefficient between truth and falsity is -0.963, and the correlation coefficient between indeterminacy and falsity is 0.918.

Graphical Method

In some instances, visual aids such as scatter plots (for pairs of indices) or graphs with the indices plotted against each other, can clarify how the indices relate to one another. A graph like this assumes if there is a strong correlation between two sets of data, no correlation or weak correlation.

Partial Correlation

Partial correlation helps to measure the relationship between two indices f and g while controlling the effect of the third index h. It is given by

$$\rho(f, g|h) = \frac{\rho(f, g) - \rho(f, h).\rho(g, h)}{\sqrt{1 - \rho(f, h)^2}\sqrt{1 - \rho(g, h)^2}}$$

The value of $\rho(f, g|h)$ gives the strength of dependence between f and g when the effect of h is removed.

Partial correlation between any two indices of a neutrosophic set, say truth $m(x)$ and indeterminacy $i(x)$ when the effect of falsity $n(x)$ is removed is given by

$$\rho(m, i|n) = \frac{\rho(m, i) - \rho(m, n).\rho(i, n)}{\sqrt{1 - \rho(m, n)^2}\sqrt{1 - \rho(i, n)^2}}$$

The value of $\rho(m, i|n)$ gives the strength of dependence between truth and indeterminacy when the effect of falsity is removed. To get a more vivid picture other coefficients $\rho(m, n|i)$ and $\rho(n, i|m)$ must also be examined.

Example 14: Consider the NS $\widetilde{A^N} = \{(x_1, 0.8, 0.1, 0.1), (x_2, 0.6, 0.3, 0.1), (x_3, 0.2, 0.5, 0.3), (x_4, 0.9, 0.05, 0.05)\}$. The mean truth value of $\widetilde{A^I}$, $\bar{m} = 0.625$, the mean indeterminacy value of $\widetilde{A^I}$, $\bar{\iota} = 0237$, the mean falsity value$\widetilde{A^I}$, $\bar{n} = 0.1125$; the covariance between the truth and indeterminacy of $\widetilde{A^I}$ is -0.047, the covariance between the truth and falsity of $\widetilde{A^I}$ is -0.014, the covariance between the indeterminacy and falsity of $\widetilde{A^I}$ is 0.009; the variance of truth, indeterminacy and falsity are 0.07, 0.03 and 0.003 respectively; the correlation coefficient between the truth and indeterminacy is -0.988, the correlation coefficient between truth and falsity is -0.963, and the correlation coefficient between indeterminacy and falsity is 0.918. So, partial correlation between truth $m(x)$ and indeterminacy $i(x)$ when the effect of falsity $n(x)$ is removed is -0.975, partial correlation between truth $m(x)$ and falsity $n(x)$ when the effect of indeterminacy $i(x)$ is removed is -0.918, and partial correlation between indeterminacy $i(x)$ and falsity $n(x)$ when the effect of truth value $m(x)$ is removed is -0.813.

Application of Knowledge of Degree of Dependence in Practical Problems

An insight into the degree of independence and dependence between the indices of fuzzy sets and neutrosophic sets helps to accurately model real world complex problems arising in various fields of decision-making, machine learning, data analysis and control system [4,5,10,11,13,15,18].

Decision-making

In decision-making problems, uncertainty and imprecision are often inherent due to incomplete, conflicting, or ambiguous information. Fuzzy and neutrosophic sets provide powerful frameworks for representing this uncertainty. Zhi and Li [29] introduced a novel approach to solve a multi-attribute decision making problem based on intuitionistic fuzzy sets. Mallick et al. [14] proposed a QNN-MAGDM strategy for e-commerce site selection by employing quadripartition neutrosophic neutrality aggregative operators. A key aspect of these frameworks is understanding the degree of dependence and degree of independence between their indices (e.g., membership and non-membership in fuzzy sets, and truth, falsity, and indeterminacy in neutrosophic sets). These relationships help model the interaction between various factors in decision-making scenarios. Understanding how these indices interact is crucial for improving decision-making accuracy, robustness, and reliability.

Machine Learning

Machine learning algorithms are designed to automatically learn patterns and make predictions from data. However, real-world data is often noisy, uncertain, and imprecise. This is where fuzzy sets and neutrosophic sets come into play. These concepts offer powerful tools for dealing with uncertainty, vagueness, and incompleteness in data, which are common challenges in machine learning [9]. Lu et al. [12] gave a thorough review on fuzzy machine learning, which discussed fuzzy machine learning approach to application with insights on recent achievements in the field of fuzzy machine learning. In machine learning, the degree of dependence and degree of independence between the indices of fuzzy and neutrosophic sets can help improve learning algorithms, enhance model robustness, and handle uncertain or ambiguous data more effectively. Understanding these relationships can make machine learning models more interpretable, adaptive, and accurate, especially when dealing with real-world applications where traditional methods fall short.

Data Analysis

In data analysis, understanding the relationships between variables is fundamental to uncovering patterns, drawing conclusions, and making predictions. The degree of dependence and degree of independence between variables (or indices of a dataset) play a crucial role in enhancing the analytical process, especially when the data is uncertain, incomplete, or ambiguous. Gomathy et al. [8] discussed data classification by applying deep learning model based upon optimal neutrosophic rules. Ravi et al. [19] proposed a deep learning framework based on the analysis of e-mail and URLs for cyber threat situational awareness. Thanh et al. [26] gave a novel clustering algorithm for medical diagnosis in a neutrosophic recommender system. Fuzzy sets and neutrosophic sets provide mathematical frameworks for dealing with uncertainty, making them valuable tools for analyzing real-world data [6], where traditional methods may struggle with imprecision.

Control System

Control systems are crucial in engineering, automation, robotics, and various industrial applications, where maintaining desired outputs (such as temperature, speed, or position) despite changing conditions or uncertainties is essential [17]. Control systems need to handle various types of uncertainties, such as Measurement errors, Environmental disturbances, Imprecise or ambiguous data from sensors, Model uncertainties. In classical control theory, controllers like Proportional-Integral-Derivative work well for systems where the variables are well understood and deterministic. However, in systems with uncertain, vague, or incomplete data, traditional methods often fall short. This is where the degree of dependence and degree of independence between variables, modeled using fuzzy sets and neutrosophic sets, come into play. Said et al. [20] enumerated an intelligent traffic control system using fuzzy sets, rough sets, and neutrosophic sets. These frameworks provide a more flexible and robust way to handle uncertainty, making them highly useful for modern fuzzy control systems and adaptive control systems.

Conclusions

In both fuzzy sets and neutrosophic sets, the degree of independence and dependence between the indices—whether membership and non-membership in fuzzy sets, or truth, falsity, and indeterminacy in neutrosophic sets—play a crucial role in modeling uncertainty and vagueness. The ability to quantify these relationships enables better decision-making, data analysis, and system design. Linear independence and dependence of indices in this context provide critical insights into their structural relationships, which in turn influences their applications. Linear dependence refers to a situation when the indices of an element are not independent of each other i.e., there exist an index that can be articulated as a linear combination of other indices. Whereas, linear independence occurs when the indices are not interrelated, and no index can be articulated as a linear combination of other indices. By understanding how these functions interact and affect each other, we can gain deeper insights into the underlying uncertainty of real-world problems and create more effective solutions.

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Chapter 7

Neutrosophic Approaches to Uncertainty Quantification in Biomedical Research: Applications and Future Directions

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ABSTRACT

Neutrosophic logic, which expands classical logic to incorporate truth, falsity, and indeterminacy, offers a powerful framework for addressing uncertainty in biomedical research. This chapter explores the integration of neutrosophic logic with advanced analytical models to improve uncertainty quantification in fields like genetic analysis, disease prediction, and medical imaging. Traditional probabilistic approaches often struggle with ambiguity and conflicting information, whereas neutrosophic logic can manage indeterminacy states effectively, enabling more accurate biomedical insights. Key applications discussed include neutrosophic-enhanced disease prediction models, which improve diagnostic accuracy by considering uncertain inputs, and neutrosophic-based image segmentation techniques that enhance sensitivity in detecting irregularities in diagnostic imaging.

A novel contribution of this chapter is the introduction of the Neutro-Genetic Hidden Markov Model (NG-HMM), which combines neutrosophic logic with Hidden Markov Models (HMM) for genomic analysis. The NG-HMM assigns neutrosophic values to genetic states, transition probabilities, and emissions, allowing the model to capture complex genetic interactions and uncertain mutations, often encountered in personalized medicine and risk prediction. This hybrid model provides a more nuanced approach to representing genetic variability and ambiguity, improving our understanding of gene expression and mutation effects. By expanding the role of indeterminacy in biomedical models, NG-HMM enables better handling of data uncertainties, making it a promising tool in personalized medicine and genetic research.

Future directions include refining neutrosophic methods with machine learning techniques and validating these models in clinical settings to enhance their reliability. This chapter demonstrates that neutrosophic logic, particularly when integrated with existing frameworks like HMM, holds significant potential for advancing uncertainty management in biomedical research and beyond.

Keywords: Neutrosophic logic, uncertainty quantification, genetic analysis, hidden Markov model, personalized medicine.

INTRODUCTION TO NEUTROSOPHIC LOGIC

Neutrosophic logic, first introduced by Smarandache, represents a significant evolution in managing uncertainty in data [1]. Traditional logic frameworks, such as fuzzy logic, allow partial truths but do not adequately address the concept of indeterminacy, which is often crucial in complex, real-world scenarios [2]. Neutrosophic logic addresses this limitation by incorporating three distinct components: truth, falsity, and indeterminacy, each ranging from 0 to 1[3]. This structure allows for a more nuanced understanding of uncertain data, especially when ambiguity and conflicting information are present [4].

Neutrosophic logic’s distinct ability to model indeterminacy makes it particularly suitable for fields like biomedical research [5], where data complexities and uncertainties are inherent. For instance, in genetic research, the presence of genetic mutations or variabilities may be categorized under indeterminacy states rather than strictly true or false outcomes [6].

To understand how neutrosophic logic can be applied effectively, we begin by exploring its core principles and mathematical framework in more detail.

BACKGROUND

Neutrosophic Logic Fundamentals

Neutrosophic logic extends traditional logic by representing information with three components: truth (T), indeterminacy (I), and falsity (F), each with values ranging from 0 to 1 [22]. These components enable a more flexible representation of uncertainty, especially in complex fields like biomedical research [7] where indeterminacy states frequently arise. The neutrosophic value $N(\text{A})$ of any statement A can be represented as:

$$N(\text{A}) = (T, I, F)$$

where $T, I, F \in [0, 1]$ and $0 \leq T + I + F \leq 3$. The conditions $T + I + F \leq 3$ allows for overlapping values of truth, falsity, and indeterminacy, a distinguishing feature of neutrosophic logic.

For example, a genetic mutation could be represented with a neutrosophic triplet $N(\text{mutation}) = (0.6, 0.3, 0.1)$, indicating that the mutation's presence has a 60% degree of truth, 30% indeterminacy, and 10% falsity. This structure supports biomedical contexts where genetic data is inherently uncertain.

Hidden Markov Models (HMMs)

HMMs are statistical models used to describe systems that transition between hidden states over time, based on observed data sequences. HMMs are defined by:

- **A set of hidden states** $S = \{s_1, s_2, \ldots, s_n\}$
- **Transition probabilities** between states $a_{ij} = P(s_j \mid s_i)$
- **Emission probabilities** $b_i(o) = P(o \mid s_i)$, which relate each state to an observed output o
- **Initial state distribution** $\pi = \{\pi_i\}$ where $\pi_i = P(s_i)$

An HMM can be represented by the triplet $\lambda = (A, B, \pi)$, where A is the transition matrix, B is the emission matrix, and π is the initial state distribution [8]. This work on profile HMMs demonstrates how HMMs are applied in bioinformatics to model sequence motifs in protein families, which is highly relevant for understanding genetic patterns and biological sequences where states are not directly observable [9].

Neutro-Genetic Hidden Markov Model (NG-HMM)

In NG-HMM, we integrate neutrosophic logic into the HMM framework to handle indeterminacy states in genetic sequences. NG-HMM modifies the traditional HMM as follows:

1. **Neutrosophic State Representation**: Each hidden state s_i in the NG-HMM has an associated neutrosophic triplet $N(s_i) = (T_i, I_i, F_i)$, enabling the model to account for indeterminacy

information. For example, a gene's activation state could be represented as $N(\text{activation}) = (0.7, 0.2, 0.1)$.

2. **Neutrosophic Transition Probabilities**: Transitions between states are represented by neutrosophic probabilities. For instance, the transition from state s_i to state s_j would have a triplet $a_{ij} = (T_{ij}, I_{ij}, F_{ij})$, where each component reflects the truth, indeterminacy, and falsity associated with moving between these states.

3. **Neutrosophic Emission Probabilities**: Each emission probability $b_i(o)$ is also represented by a triplet $b_i(o) = (T_i(o), I_i(o), F_i(o))$, capturing the uncertainty of observed data relative to the hidden states.

Mathematical Formulation of NG-HMM

Given an observed sequence $O = \{o_1, o_2, \dots, o_T\}$ and a sequence of hidden states $S = \{s_1, s_2, \dots, s_T\}$, the probability of the observed sequence in the context of NG-HMM can be expressed as:

$$P(O \mid \lambda) = \prod_{t=1}^{T} b_{s_t}(o_t$$

where each $b_{s_t}(o_t)$ is a neutrosophic triplet, representing the likelihood of observation o_t given state s_t with truth, indeterminacy, and falsity components.

The NG-HMM parameters are estimated using an adapted Baum-Welch algorithm that iteratively adjusts the neutrosophic values of the transition and emission probabilities to maximize the probability of the observed sequence.

Example Application in Genetic Data Analysis

Consider a sequence of gene expressions where each expression level is uncertain due to variability in environmental factors or experimental conditions. In NG-HMM, each gene expression level could be modeled as a neutrosophic state, with neutrosophic transition and emission probabilities reflecting the inherent uncertainty. For example, transitions in gene expression states under different conditions might be modeled as:

$a_{ij} = (0.5, 0.3, 0.2)$

indicating a 50% chance the transition occurs, with 30% indeterminacy and 20% chance it does not occur. Such representation captures the true complexity of genetic interactions, which are often not strictly binary but probabilistically uncertain. With a foundational understanding of neutrosophic logic and its unique handling of uncertainty, we now examine the biomedical research landscape, where these methods address pressing challenges in data complexity and ambiguity.

Biomedical Research Context and Challenges

Biomedical research frequently encounters challenges in managing uncertainty due to high biological variability, measurement errors, and the inherent unpredictability of biological systems.

Standard probabilistic approaches can quantify variability but fail to capture indeterminacy cases where a definitive classification is impossible [10]. For example, diagnostic imaging often yields results that cannot be classified as purely positive or negative, leading to "gray areas" in diagnosis. Here, neutrosophic logic can serve as a valuable tool, allowing a more flexible categorization of uncertain findings [11].

In clinical diagnosis, indeterminacy arises in patient symptomatology and diagnostic test results, where neutrosophic logic enables handling ambiguous states that fall outside traditional probability frameworks [11]. By representing uncertainty more precisely, researchers and clinicians can improve decision-making in critical areas like disease diagnosis, treatment planning, and patient monitoring. Given these challenges, neutrosophic methods offer promising solutions across several biomedical domains, including disease prediction, medical imaging, and genetic research, each of which we explore in the following sections.

Neutrosophic Methods in Biomedical Applications

Neutrosophic methods have been effectively applied in several biomedical fields to address uncertainty in data interpretation:

- **Disease Prediction and Diagnosis**

Neutrosophic logic has shown promising applications in disease prediction models, where uncertain inputs often lead to mixed diagnoses. For instance, in heart disease prediction, hybrid models that combine neutrosophic sets with machine learning algorithms yield better accuracy by accounting for both uncertain and indeterminacy cases [12, 23]. Additionally, models developed with neutrosophic techniques help minimize diagnostic errors, as they can manage borderline or conflicting data [13].

- **Medical Imaging Analysis**

Medical imaging is another area where neutrosophic approaches have significantly impacted data interpretation[14]. Neutrosophic set theory has been applied to enhance image segmentation in medical scans, where ambiguities in pixel intensity can complicate accurate classification [15]. This application of neutrosophic processing has shown improved sensitivity in detecting irregularities in MRI and CT scans, particularly in oncological and neurological cases [16].

- **Genetic Data Interpretation**

Genetic research deals with the massive complexity of genomic data, where indeterminacy states are common due to unknown gene interactions or mutation effects. Neutrosophic models allow researchers to categorize ambiguous gene expressions as partially known or indeterminacy, leading to more accurate interpretations and predictions [17]. This approach is especially relevant for personalized medicine, where indeterminate genetic markers can have significant implications for individual risk assessments and targeted therapies [18].

To further address genetic uncertainties, the Neutro-Genetic Hidden Markov Model (NG-HMM) incorporates neutrosophic principles, facilitating a refined approach to analyzing genetic data with inherent ambiguity.

Integrating Neutrosophic Logic with Hidden Markov Models: The Neutro-Genetic Hidden Markov Model (NG-HMM)

A novel advancement in neutrosophic approaches to genetic analysis is the **Neutro-Genetic Hidden Markov Model (NG-HMM)**, a theoretical framework combining neutrosophic logic with **Hidden Markov Models (HMM)** to address uncertainty in genomic data. NG-HMM is particularly suited for representing genetic data with high indeterminacy, such as mutations of uncertain significance and variations in gene expression influenced by environmental factors [18].

The NG-HMM builds on the strengths of HMMs, which are commonly used to identify patterns in genetic sequences, and extends them by integrating neutrosophic parameters. Each state and transition in the model is represented by a triplet (*T, I, F*), enabling states to reflect ambiguous genetic states rather than strictly defined ones. This model addresses the following:

- **Neutrosophic State Representation**: Each hidden state in the NG-HMM is expressed as a neutrosophic triplet, such as a gene's activation state represented as 70% true, 20% indeterminacy, and 10% false. This framework allows geneticists to model and manage states that contain indeterminacy or partially unknown characteristics [19].

- **Neutrosophic Transition and Emission Probabilities**: NG-HMM assigns neutrosophic values to both transitions (probabilities of moving between states) and emissions (likelihood of observed data given a state), allowing the model to account for complex transitions that are uncertain or ambiguous. This is particularly useful for cases such as gene expression influenced by uncertain environmental factors, where standard probabilistic models would be insufficient [20].

  The NG-HMM approach thus expands the scope of neutrosophic theories by enabling a more flexible and realistic analysis of genomic sequences. By combining HMM with neutrosophic logic, NG-HMM captures the nuances of genetic data, facilitating more accurate genetic interpretations and paving the way for advanced personalized medicine applications [21].

While the NG-HMM showcases the strength of neutrosophic logic in genetic analysis, further advancements and applications await exploration, as discussed in the following section on future directions and open research questions.

Future Directions and Open Research Questions

Advanced Applications in Genomic and Epigenomic Data Analysis

Future research could extend the Neutro-Genetic Hidden Markov Model (NG-HMM) to analyze epigenomic data, where regulatory elements and histone modifications introduce additional layers of complexity. By expanding the NG-HMM to handle multi-dimensional neutrosophic data, researchers could uncover novel insights into gene regulation and expression patterns influenced by both genetic and epigenetic factors. Such applications could be particularly valuable for complex diseases where gene-environment interactions play a significant role.

Deep Learning with Neutrosophic Logic for Image and Signal Processing

Deep learning is another promising field for neutrosophic logic applications. For example, convolutional neural networks (CNNs) applied in medical imaging, such as MRI or CT scans, could incorporate neutrosophic logic to better manage uncertain image regions, leading to improved diagnostic accuracy. Integrating neutrosophic values could allow CNNs to distinguish between clear and ambiguous regions in medical images, enhancing sensitivity in detecting abnormalities [24]. Similarly, in electroencephalography (EEG) signal analysis, neutrosophic deep learning models could help differentiate between normal and indeterminate brain activity patterns [25, 26].

Optimization of Neutrosophic Parameters

One challenge in applying neutrosophic logic broadly is determining optimal parameter values for truth, indeterminacy, and falsity in different contexts[27]. Future research could explore dynamic parameter optimization methods to automatically adjust these values based on specific dataset characteristics. This approach could lead to more effective neutrosophic applications in domains like clinical decision-making and bioinformatics, where data variability and indeterminacy are especially pronounced [28].

Validation and Standardization in Clinical Settings

For neutrosophic logic to gain traction in biomedical research and healthcare, robust validation studies are essential. Future studies should focus on comparing neutrosophic-enhanced models to traditional probabilistic and fuzzy logic approaches across various datasets and clinical conditions. Standardization of neutrosophic logic applications, especially for use in diagnostic tools, could also be beneficial in increasing clinical acceptance and ensuring reliability in real-world applications [29].

Conclusions

In this chapter, we explored the potential of neutrosophic logic to enhance uncertainty quantification in biomedical research, particularly in areas where ambiguity and complexity are prevalent. Neutrosophic logic, with its unique three-dimensional approach—truth, indeterminacy, and falsity—provides a sophisticated framework for managing uncertain data more flexibly than traditional probabilistic or fuzzy logic systems. This flexibility makes it particularly suited to handle the complexities of genetic analysis, disease prediction, and medical imaging.

The integration of neutrosophic logic with Hidden Markov Models (HMM), resulting in the novel Neutro-Genetic Hidden Markov Model (NG-HMM), represents a significant advancement. The NG-HMM allows for a more nuanced analysis of genetic sequences, accounting for the indeterminacy nature of gene expression and mutation effects in the presence of uncertain environmental influences. This hybrid approach holds promise for applications in personalized medicine, where accurate prediction and interpretation of genetic data are crucial.

Additionally, the chapter outlined potential future directions for neutrosophic logic in biomedical research. The combination of neutrosophic logic with machine learning and deep learning could enable more precise predictive models that account for a range of uncertainties, enhancing diagnostic accuracy and treatment customization. Validation and standardization of these models in clinical settings will be essential for their adoption in real-world medical decision-making.

In conclusion, neutrosophic logic offers transformative possibilities for the biomedical field, paving the way for more adaptive, reliable, and personalized approaches to complex healthcare challenges. By embracing the principles of neutrosophy, researchers and clinicians can develop tools better suited to the uncertain and nuanced nature of biological data, ultimately contributing to improved patient outcomes.

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Chapter 8

An Introduction to Neutrosophic Social Situations

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ABSTRACT

Professor Florentin Smarandache first coined the term Neutrosophic Sociology (or Neutrosociology), which is the study of sociology using neutrosophic scientific methods. His observation that the social data encountered in sociology is full of indeterminacy led to the study of neutrosophic methods and tools such as neutrosophy, neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic statistics, neutrosophic analysis, neutrosophic measure, and more. In the complex landscape of social issues, individuals and cultures often have differing opinions on what is considered good, bad, or ambiguous. Traditional binary views on social situations, which categorize things as either good or evil, do not take into consideration the intricacies and contradictions present in social behaviors and cultural standards. This study introduces a neutrosophic framework for comprehending and evaluating social situations, highlighting the fluid, subjective, and indefinite nature of societal norms and moral assessments. Using neutrosophic measures and statistics, we investigate paradoxes in societal norms and argue that the neutrosophic method provides a more comprehensive way of modeling social behaviors and examining the evolving and often conflicting nature of social norms across different times, cultures, and populations.

Keywords: Neutrosophic sociology**,** Neutrosophic sets, Neutrosophic logic, Neutrosophic statistics, Social situations, Social variance, Neutrosophic measures, Indeterminacy.

INTRODUCTION

The term "sociology" was first introduced by French social scientist Auguste Comte (1798-1857). Comte formed the word by combining the Latin "socius" meaning society, association, unity, or friendship, with the Greek term "logos," which translates to "word" or "to speak about." Though "logos" literally means "word," it is commonly understood in this context as "study" or "science" [8]. Hence, the etymological meaning of sociology is the "study" or "discussion of society." This term was later expanded upon by prominent theorists such as Emile Durkheim, Karl Marx, and Max Weber [22].

In sociology, a range of interdisciplinary research methods are employed to examine and interpret the causes, meanings, and cultural influences behind various human behavioral trends, particularly in contexts of social interaction and shared environments. Yet, the large volumes of social data encountered in sociology are often marked by indeterminacy: they may be imprecise, incomplete, contradictory, mixed, biased, uninformed, redundant, irrelevant, ambiguous, unclear, or lacking meaningful structure. These qualities make it difficult to draw clear conclusions, highlighting the need for methods like neutrosophic sets /logic/probabilities /statistics/analysis/measure that can accommodate and interpret the indeterminate aspects of sociological data [21].

For example, let us consider the classical idea of "equality" in the workplace. In conventional terminology, saying that a workplace W is equal means that W has complete equality across genders, ethnicities, and other social identities, which is regarded as 100% equal. Using neutrosophic notation, we can express this as W is (1, 0, 0), indicating that the workplace is 100% equal, 0% indeterminate-equal, and 0% unequal. However, additional research reveals that specific departments in W still have a wage difference based on gender, impacting around 15% of employees. We then change the description to (0.85, 0, 0.15)-equal, with W being 85%

equal and 15% unequal. Further investigation reveals that promotion policies are fairly ambiguous: around 10% of employees believe these procedures are fair, while others believe they lead to inequality. This raises uncertainty regarding W's equality, therefore we rephrase W's description as (0.75, 0.1, 0.15)-equal, with 75% equality, 10% unclear equality, and 15% inequality. This nuanced (T, I, F)-equality represents the complexity of equality in W significantly better than the first (1, 0, 0)-equal representation.

A social situation is a setting in which individuals or groups interact, communicate, and influence one another's behaviors, perceptions, and attitudes, which are formed by cultural norms, roles, and social dynamics [17]. It is often difficult to determine whether a situation is right or wrong. What may be considered correct by one group of people could be viewed as incorrect by another, and unclear from a third perspective. Definitions of 'right', 'wrong', and 'indeterminate' have evolved throughout time and range from one culture to another. They exhibit neutrosophical dynamics. For example we could consider a situation where a city government installs cameras in public spaces to enhance security and prevent crime. From the perspective of public safety, this action might be viewed as "right" because it helps protect citizens and deter criminal activity. However, from a privacy standpoint, many may see it as "wrong" due to the potential for misuse and the erosion of individual privacy. Meanwhile, some may view it as "unclear" or indeterminate, as it both protects and intrudes, depending on how it is used and regulated.

Due to the great complexity of modern societies, it is virtually impossible to have accurate data or knowledge about any contemporary society. This heterogeneity in social norms often leads to ambiguity, indeterminacy, and inconsistencies, which are difficult to address using typical social research methodologies. Neutrosophic logic, an extension of fuzzy logic, offers a powerful mathematical framework for dealing with these challenges by permitting the representation of truth (T), falsity (F), and indeterminacy (I). In social situations, these three components might indicate whether a social rule or behavior is seen as correct, incorrect, or confusing. It is the main aim of current chapter to explore and establish the concept of neutrosophic social situations and how they can be modeled mathematically within this context. We achieve the aforementioned by using neutrosophic sets and appropriate measures and neutrosophic statistics, demonstrating how social rules may be described as partial truths, partial falsehoods, and partial indeterminacies. In this manner, we are capable of accurately reflecting the complexity and inconsistencies inherent in social reality. Neutrosophic statistics provides a way to analyze social data that includes uncertain or conflicting information. Social surveys, for instance, often have partial responses or contradictory opinions. Neutrosophic statistics accounts for these uncertainties by extending traditional statistical methods. Some researches on neutrosophic statistics can be found in [3-4, 16, 24].

In a neutrosophic social situation, the definitions of right, wrong, and indeterminate are not fixed but rather are subject to subjective interpretation. The neutrosophic model allows us to represent these interpretations as triplets (T, I, F) or neutrosophic appurtenances. A neutrosophic appurtenance of an element x with respect to a given neutrosophic set has the form: *x(*T, I, F*)*, where T is the degree of truth of the element x, I is the degree of indeterminate-truth of *x*, and F the degree of *x*, where T, I, F are independent neutrosophic components, and T, I, F are subsets of the interval [0, 1] [21]. Neutrosophy studies triads, where if <A> is an item or a concept then the triad is (<A>, <neutA>, <antiA>) [21, 25]. In this context, we could state that Neutrosociology is based on triads, e.g. consider the concept <A> = capitalist economy, where the <antiA> = socialist economy, and the <neutA> = mixed economy. In this triad, the capitalist economy prioritizes free markets and private ownership, the socialist economy emphasizes state ownership and equality, and the neutral, mixed economy combines elements of both, balancing market freedoms with social welfare policies.

The chapter is organized as follows: In section 2, introduces some concepts and basic operations are reviewed. In section 3, we present the suggested neutrosophic framework in a controversial social situation like polygamy. In section 4, the results obtained from previous section are discussed and interpreted in perspective of previous studies and of the working hypotheses. Finally, conclusions and further research are highlighted.

BACKGROUND

Definition 1 [20] Let $\mathcal{U}$ be a universe. A neutrosophic set $\mathcal{A}$ over $\mathcal{U}$ is defined by

$$\mathcal{A} = \{\prec u, (\mu_{\mathcal{A}}(u), \nu_{\mathcal{A}}(u), w_{\mathcal{A}}(u)) \succ : u \in \mathcal{U}\}$$

where, $\mu_{\mathcal{A}}(u)$, $v_{\mathcal{A}}(u)$ and $w_{\mathcal{A}}(u)$ are called truth-membership function, indeterminacy-membership function and falsity- membership function, respectively. They are respectively defined by

$$\mu_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \quad v_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[, \quad w_{\mathcal{A}}: \mathcal{U} \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq \mu_{\mathcal{A}}(u)+v_{\mathcal{A}}(u)+w_{\mathcal{A}}(u) \leq 3^{+}$.

Definition 2 [25] Let $\mathcal{U}$ be a universe. An single valued neutrosophic set (SVN-set) over $\mathcal{U}$ is a neutrosophic set over $\mathcal{U}$, but the truth-membership function, indeterminacy-membership function and falsity- membership function are respectively defined by

$$\mu_{\mathcal{A}}: \mathcal{U} \to [0,1], \quad v_{\mathcal{A}}: \mathcal{U} \to [0,1], \quad w_{\mathcal{A}}: \mathcal{U} \to [0,1]$$

such that $0 \leq \mu_{\mathcal{A}}(u)+v_{\mathcal{A}}(u)+w_{\mathcal{A}}(u) \leq 3$.

Definition 3 Let $\mathcal{X}$ be a set of social data points representing individuals' opinions on a social issue. Each data point can be represented as a neutrosophic set $X_i = (T_{x_i}, I_{x_i}, F_{x_i})$ where T_{x_i}, I_{x_i} and F_{x_i} are the truth, indeterminacy, and falsehood components of individual i's opinion. The mean of these neutrosophic values can be computed by:

$$\mu_T = \frac{1}{n} \sum_{i=1}^{n} T_{x_i} \tag{1}$$

$$\mu_I = \frac{1}{n} \sum_{i=1}^{n} I_{x_i} \tag{2}$$

$$\mu_F = \frac{1}{n} \sum_{i=1}^{n} F_{x_i} \tag{3}$$

where n is the number of data points.

Definition 4 The variance of neutrosophic data, as described in the context of social situations, measures the spread or diversity of opinions or perceptions within a population regarding truth, indeterminacy, and falsehood. The variance of the neutrosophic data is calculated as:

$$\text{Var}(T) = \frac{1}{n} \sum_{i=1}^{n} (T_{x_i} - \mu_T)^2 \tag{4}$$

and similarly for I and F.

Low variance values indicate a group's cohesive and common perspective. This consistency may be very useful for decision-makers since it enables them to address the group's preferences efficiently and confidently**.**

Definition 5 (Vector similarity measures for SVN-set) [5] The similarity measure helps assess how closely aligned two individuals' (or groups') views are on a given topic**.** In this chapter we propose a vector similarity measure in the vein of Jaccard similarity measure similar to the one proposed by Ye for SVN-sets [13].

$$\mathcal{S}(A, B) = \frac{1}{3} \left(\frac{T_A \times T_B + I_A \times I_B + F_A \times F_B}{T_A + T_B + I_A + I_B + F_A + F_B} \right) \tag{5}$$

This measure essentially tells us how closely aligned or similar the two sets of values are, typically resulting in a similarity score between 0 (no similarity) and 1 (perfect similarity). It answers the question: "How aligned are the views?"

Definition 6 (Euclidean distance measure) [12] Distance measures can help to assess the divergence between social attitudes**.** Let *A* and *B* be two SVNSs in *X*. Then, the following Euclidean distance measure between *A* and *B* is defined as follows:

$$d(A, B) = \sqrt{(T_A - T_B)^2 + (I_A - I_B)^2 + (F_A - F_B)^2} \tag{6}$$

Instead of suggesting alignment, equation (6) quantifies the degree of difference or separation between two sets of data, with a smaller distance indicating closer similarity and a larger distance indicating more divergence. It answers the question: "How different are the views?"

Assessment of Remote Work Policies Using the Neutrosophic Social Situation Framework

Remote work regulations were quickly established during the COVID-19 epidemic, altering many traditional workplaces. While some workers see remote work as a beneficial move that boosts productivity, decreases commute stress, and enhances work-life balance, others are concerned about isolation, diminished cooperation, and work-life boundaries. The objective is to determine the perceived "truthiness," "falsehood," and "indeterminacy" of remote work regulations in a business setting. Employees' perspectives on the benefits and disadvantages of remote work differ, depending on aspects such as work-life balance, productivity, and team dynamics. This application will use neutrosophic values (T, I, and F) to represent employee opinions. The current scenario studied is a practical application of Neutrosociology, as it embodies the study of social situations involving subjective opinions with inherent uncertainties and partial truths

In this application, the neutrosophic framework is applied as follows:

- **Truth (T)**: Measures the proportion of individuals in each department who feel positively about the remote work policy.
- **Indeterminacy (I)**: Captures the level of ambivalence or neutrality, indicating employees who see both benefits and drawbacks or who are unsure of their standpoint.
- **Falsehood (F)**: Quantifies the proportion of employees who feel negatively about the policy.

Assume we have the following ratings from employees in Departments A, B, and C:

Table 1. Ratings of employees

Department	Employee	Truth (T)	Indeterminacy (I)	Falsehood (F)
A	1	0.8	0.1	0.1
A	2	0.7	0.2	0.1
A	3	0.9	0.05	0.05
A	4	0.75	0.15	0.1
A	5	0.85	0.1	0.05
B	1	0.6	0.3	0.1
B	2	0.5	0.4	0.1
B	3	0.55	0.35	0.1
B	4	0.65	0.25	0.1
B	5	0.6	0.3	0.1
C	1	0.3	0.5	0.2
C	2	0.4	0.4	0.2
C	3	0.35	0.45	0.2
C	4	0.3	0.5	0.2
C	5	0.4	0.4	0.2

Based on the results shown in Table 1 we calculate the average (mean) values of T, I and F for each department of the organization using equations (1), (2) and (3).

For Department A: $\mu_T = \frac{0.8+0.7+0.9+0.75+0.85}{5} = 0.8$, μ_I = 0.12 and μ_F =0.08.

Thus, $Dept_A$ = (0.8, 0.12, 0.08).

In the same way we get $Dept_B$ = (0.5, 0.32, 0.1) and $Dept_C$ = (0.35, 0.45, 0.2).

Next we calculate the variance values for T, I and F for each department by using equation (4).

For Department A: $Var\ (T) = \frac{(0.8-0.8)^2+(0.7-0.8)^2+(0.9-0.8)^2+(0.75-0.8)^2+(0.85-0.8)^2}{5} = 0.006$

$Var\ (I)$ = 0.0022

Var (F) = 0.0012

The above results show that Department A has low variance, indicating general agreement amongst the employees about remote work policies.

Following the same logic, we get the next results:

For Department B: *Var (T) =0.0026 , Var (I)=0.0026 , and Var (F)= 0.0*

For Department C: *Var (T) =0.002 , Var (I)=0.002 , and Var (F)= 0.0*

From the results obtained it can be observed that departments B and **C** exhibit low and nearly identical variances across T, I, and F, indicating a high degree of consensus within each department.

In order to calculate the alignment of viewpoints of employees between the departments of the organization we can calculate their respective similarity measure as expressed in equation (5).

By applying the aforementioned equation we have the following results:

$\mathcal{S}\ (A, B) =$0.837

$\mathcal{S}\ (A, C) =$0.783

$\mathcal{S}\ (B, C) =$0.789

Given the high degree of alignment between departments A and B, policies that promote flexibility and productivity might be implemented universally between the departments. However, for A and C, as well as B and C, policy or collaborative methods may need to be adjusted to suit the different preferences and requirements. In this context, understanding the varied levels of similarity within departments enables managers to customize their leadership strategies, providing both unity and focused interventions to overcome differences.

Using the Euclidean distance measure, we can quantify the difference in perceptions between departments by calculating the distance between their neutrosophic scores in the three-dimensional space defined by Truth (T), Indeterminacy (I), and Falsehood (F).

In our example**:** $Dept_A$ = (0.8, 0.12, 0.08)

$Dept_B$ = (0.5, 0.32, 0.1) and

$Dept_C$ = (0.35, 0.45, 0.2)

Now, we can calculate, for example, the Euclidean distance between departments A and B by using equation (6):

$$d(A, B) = 0.298$$

Next we can calculate the distance measures for departments A and C and B and C respectively:

$$d(A, C) = 0.571$$

$$d(B, C) = 0.282$$

When we apply these findings to different departments' perspectives on remote work, we may conclude that departments with closer views, such as departments A and B (with a distance of 0.298) and departments B and C (0.282), may be better aligned in their acceptance and implementation of remote work. These departments are likely to work together efficiently to design rules that balance remote work with in-office requirements, making transfers easier and reducing possible disputes. Managers will need to engage in greater communication with departments A and C (0.571), which show more varied perspectives, to address problems such as isolation and cooperation challenges.

Conclusions

Sociological studies look at social structures, behaviors, institutions, and interactions within societies. They involve examining how social influences including culture, norms, legislation, and technology shape and are shaped by individuals and communities. A classical sociological *concept*, such as: society, social class, social group, religious community, social relationship, principle, law, welfare, government regulation, political party, sexuality, family, culture, etc. can be represented as a real world neutrosophic triad (<concept>,<neutconcept>,<anticoncept>) and can be neutrosophicated into a (T,I,F)-concept.

The social environment is very subjective, with several opposing trends and perspectives. There is little agreement on social notions, and a significant amount of how-I'd-like-to-be is sadly included into the researcher's social model. Neutrosophic social situations are scenarios within social contexts that involve complex and often ambiguous interactions that incorporate the concept of neutrosophy which acknowledges the presence of indeterminacy, ambiguity, and contradictions in human cognition and perception, recognizing that phenomena often exist in states of partial truth, falsehood, and indeterminacy simultaneously. In this book chapter we suggest for the first time in related literature a neutrosophic mathematical framework to model, in a rigorous way, the complexities present in social interactions.

The contributions of this book chapter can be summarized as follows:

1. The chapter suggests a novel framework for analyzing social situations using neutrosophic logic/, which includes indeterminacy with truth and falsehood. Our approach demonstrates how neutrosophic sets/measures and statistics could be used to model the complexities of social norms and behaviors. The suggested framework, which uses mathematical formulations for truth, indeterminacy and falsehood, represents more accurately the nuanced nature of social perceptions and interactions than traditional sociological models.
2. The chapter provides a foundation for using neutrosophic approaches in multi-criteria decision-making situations in social settings. This is especially effective in circumstances where diverse viewpoints and perspectives must be balanced, such as policymaking and organizational decisions.
3. Contribution to the theoretical foundations of multi-criteria decision making methods by integrating neutrosophic sets/measures into the neutrosophic statistics framework, addressing the limitations of conventional methods in handling ambiguous and indeterminate information.

Future Research Directions

Neutrosophic sociological research can assist organisations, enterprises, governments, and policymakers make judgements regarding their businesses or groups of individuals. By analyzing the degrees of acceptance, rejection, and indifference within societal patterns, neutrosophic models can help anticipate potential conflicts, cooperation, or areas of disengagement, offering valuable insights for strategic planning and policy development. By studying societal patterns, it may be able to predict (to some extent) potential disputes, cooperation, or ignorance among people.

For example, in conflict resolution observed in organizations, we could employ neutrosophic models that could help identify areas of shared understanding and ambiguous beliefs, thus boosting good communication, time management, cooperation and organizational productivity [15]. Expanding our framework to policy analysis may allow for a better understanding of public opinion, which frequently incorporates polarized and unclear perspectives, especially on sensitive matters like healthcare, privacy, and social justice [6, 14, 18-19]. Integrating neutrosophic concepts into policy modelling could assist decision-makers in balancing opposing ideas and making more inclusive judgements that consider uncertain perspectives. Furthermore, merging neutrosophic logic with artificial intelligence may result in more advanced algorithms in sentiment analysis, social behavior prediction, and other areas where data is inherently uncertain [1-2, 9-11, 13, 23]. This fusion could provide a valuable tool for analyzing trends in social media, consumer feedback, and other platforms where opinions are multifaceted and contradictory.

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Chapter 9

Neutrosophic Fuzzy Sets of Interval Analysis in Parachute Dive Using Alpha-Cut and Harfa Fractals

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ABSTRACT

The truth membership function, indefiniteness, and falsification represent degrees of membership within the Neutrosophic set, offering a nuanced approach to characterizing elements based on their association with truth, uncertainty, and falsehood. The Neutrosophic paradox set is introduced as a concept that examines the ambiguity of whether a set is true or false, encompassing all three membership functions. It also has interval analysis to understand the uncertainty complex. This study calculates the alpha cut at different levels, including lower, middle, and higher levels, which control the degrees of membership within the Neutrosophic set. Also, the distance between the parachute diver and ground level was determined using Harfa analysis and pixel profile to determine the distance and how long it took to land.

Keywords: Neutrosophic set, Neutrosophic paradox set, Interval analysis, alpha cut.

AMS classification: 03EXX, 94DXX

INTRODUCTION

Neutrosophy, a term coined by Florentin Smarandache, refers to the study of concepts and ideas that exist in the real between true and false. It explores contradictory, incomplete, or ambiguous concepts that cannot be easily categorized under binary logic [1, 2]. By acknowledging the existence of conflicting, undefined, or ambiguous elements, neutrosophy provides a more nuanced approach to understanding complex real-world scenarios. The origins of neutrosophy are linked to smarandache's work [3] in the late 20th century, where he introduced the concept as a means of addressing the complexities and uncertainties inherent in various fields of knowledge [4, 5]

The crisp logic, fuzzy, and neutrosophic sets are all generalized into pathogenic sets, defined by attribute values for each constituent component. An evaluation and description of the indetermination of each element in a classical set was not possible [6, 7]. The fuzzy set established by Zadeh [8] has been applied in several real-world scenarios to address uncertainty [9]. A powerful and universal formal framework, known as the neutrosophic set [3], extends and simplifies the classic, fuzzy [8], and ambiguous set. This data is frequently ambiguous and lacks a precise definition, often manifesting in forms such as natural language imprecise data or neuromorphic data. This complexity has led to an increased interest in neutrosophic set theory and has lower and upper levels of alpha cut [10], which excels in handling uncertainty and ambiguity in data. Research in this area addresses the challenges of analyzing and interpreting such data accurately [10].

Neutrosophic sets have gained considerable attention in various research areas due to their ability to address uncertainty, impression, and indeterminacy [11, 12]. Nguyen et al. [13] provided an extensive survey on the applications of neutrosophic sets in biomedical diagnoses, emphasizing their utility in managing uncertainty in medical data. Peng and Dai [14] offered a bibliometric analysis that reviewed over two decades of neutrosophic set research, identifying key trends and future directions in the field. Pramanik [15] explored the integration of a rough set theory [16] with neutrosophic sets [3, 17, 18], presenting a new methodology for dealing with vague and imprecise information [19]. The theoretical foundations and applications of neutrosophic sets are further explored in edited volumes by Smarandache and Pramanik [20, 21, 22], emphasizing the most recent inclinations in neutrosophic theories. These volumes highlighted the board applications of neutrosophic sets in areas such as communication, management, and information technology [23].

Section 2 explains the literature review and preliminaries of the Neutrosophic paradox sets, exploring their relevance with examples such as the Parachute dive and voting scenario. In the parachute dive, Neutrosophic sets model the uncertainty surrounding factors like wind speed, altitude, and timing, in the voting scenario, they address conflicting opinions and undecided voters, offering a more nuanced approach to decision-making. This section introduces Neutrosophic Interval Analysis, which examines the interval between Neutrosophic membership functions in truth, indeterminacy, and falsity. Section 3 explains the analysis supports and leads a mid-level of alpha cuts like lower, middle, and higher propositions, also using Harfa software and pixel profile to calculate the distance between the diver and the ground level offering a nuanced approach to understanding and managing the complexities of this scenario.

BACKGROUND

Definition 1. [23] The Significance of Paradoxicity

A suggestion that is both true or false in the same way is termed a paradox. Thus, assume that the assertion is true, it follows that it is false; conversely, assume that the statement is false, it follows that is true.

Definition 2. [23] The Significance of a Semi-Paradox

A semi-paradox is a statement that, if being true, leads to being false(but not reciprocally) or, given being false, leads to being true(but not reciprocally). Accordingly, the statement contains 0.50 (or 50%) of a paradox and 0.50 (or 50%) of a non-paradox.

Definition 3. [24] Neutrosophic Interval Analysis

Neutrosophic interval analysis is an extension of interval analysis that incorporates neutrosophic concepts to handle uncertainty, indeterminacy, and ambiguity. Interval analysis deals with uncertainty by representing variables as intervals rather than precise values. Neutrosophic interval analysis further extends this framework to accommodate neutrosophic membership degrees, which capture the degrees associated with intervals.

Definition 4. Neutrosophic Set of Parachute Dive and Voting

Neutrosophic sets extend classical set theory by incorporating three membership functions: truth, indeterminacy, and falsehood. Each function assigns a value to an element in the set, indicating the set (truth), is uncertain or ambiguous (indeterminacy), or does not belong to the set (falsehood). This approach provides a more nuanced and comprehensive framework for dealing with uncertainty and ambiguity, making neutrosophic sets useful in various fields where traditional binary classifications fall short.

$$0 \leq T_A(z) + I_A(z) + F_A(z) \leq 3$$

In the way of the Parachute Dive scenario, let z represent the diver's position at a given time t. The neutrosophic membership functions can model $T_A(z)$ as representing the likelihood that the diver is at the correct altitude for a safe landing, $I_A(z)$ is uncertainty due to wind speed, atmospheric conditions, or equipment, $F_A(z)$ as the likelihood that the skydiver is not at a safe altitude. In the Voting scenario z is represented by the membership functions like $T_A(z)$ is the degree of support for a candidate or policy, $I_A(z)$ is the degree of indecision or neutrality, $F_A(z)$ is represented as the degree of opposition to the candidate or policy.

Definition 4.1 Illustration- Diver's Free Fall Calculation

Consider the Parachute diving scenario where the diver encounters varying uncertainty about the direction and intensity of the wind. Also, the terrain below having the irregularities represented by the Siegel disc fractals, additionally counts the uncertainty of the landing location. The jumper waits 11.5 seconds before deploying the parachute, during which time they experience free fall without any air resistance. Starting with an initial vertical velocity (z-axis) of zero, the jumper accelerates due to gravity until deploying the parachute.

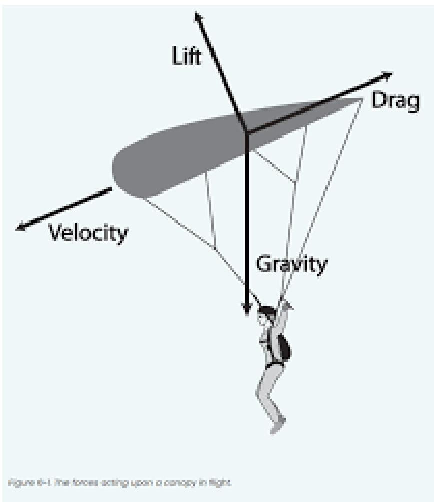

Figure 1 Parachute dive

Gravitational force =-9.80 m/s^2

$$Z(t) = Z_o + V_{Z_o}t + 0.5a_Z t^2$$

$$z(11.5) = 4000 + 0 + 0.5(-9.80)\ (11.5)^2$$

=3351.975m.

Therefore, the parachute diver distance (interval analysis) between the terrain and the diver is to be 3351.975 meters/seconds.

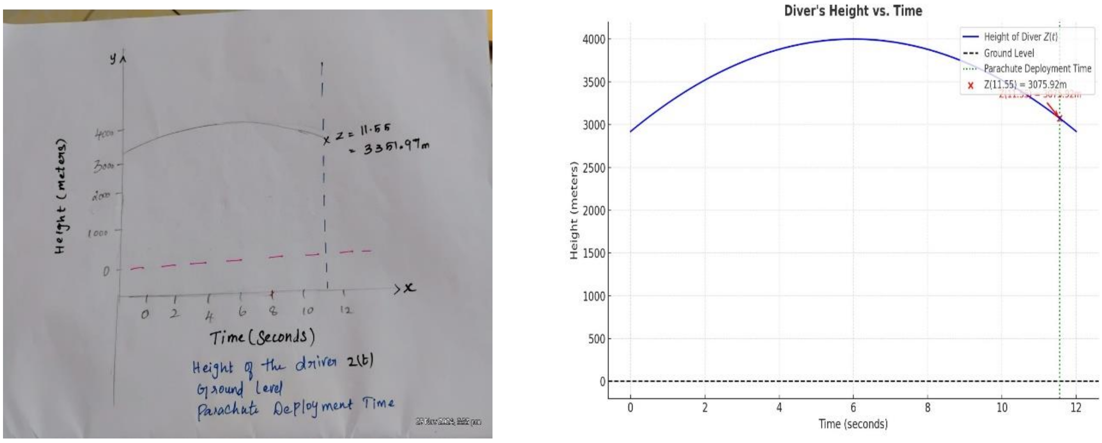

Figure 2 Pictorial Representation of the above example 4.1

Definition 4.2 Illustration- Neutrosophic paradox in voting

Consider a voting scenario where the truth membership of a candidate engaging in an election is high (T), but the indefiniteness membership (I) designates uncertainty due to potential recounts or legal challenges and it is said to be Neutrosophic paradox sets. However, the falsity membership (F) recommends that there is a possibility of the election outcome being contested. By applying neutrosophic alpha cuts at different levels (e.g α_1 lower, α_2 medium, and α_3 upper level), it can identify subsets of the neutrosophic set that represent varying degrees of certainty or ambiguity in the election outcome, helping to resolve the paradox.

Definition 4.3 Alpha Cuts of Lower, Middle, and Upper

- Lower Alpha Cut: For a set P and $\alpha_1 \in [0,1]$, the lower alpha cut is denoted L (P, α_1) and includes elements $x \in P$ for which the membership degree is at least α_1.
- Middle Alpha Cut: For a set P and $\alpha_2 \in [0,1]$, the middle alpha cut is denoted M (P, α_2) and includes elements $x \in P$ for which the membership degree is at least α_2.
- Upper Alpha Cut: For a set P and $\alpha_3 \in [0,1]$, the upper alpha cut is denoted U (P, α_3) and includes elements $x \in P$ for which the membership degree is at least α_3 [10].

Methods

The alpha-cut levels, Harfa fractals, and Pixel profile are used to analyze the distance between the parachute diver and the ground. The diver's location concerning the ground may be determined by analyzing the lower, middle, and upper alpha levels. Better decision-making and safety precautions are made possible by this method's assistance in locating crucial locations and possible hazards throughout the fall.

Lower-Level and Upper- Level $\alpha - CUT$ Neutrosophic Paradox Set

Let P said to be an NS of paradox in a non-void set X. For any $\alpha_{1,2,3} \in [0,1]$, then α_1 lower, α_2 medium and α_3 upper level α of P be noted by $L(P, \alpha_1)$ and $M(P, \alpha_2)$ and $U(P, \alpha_3)$ are defined as

L (P, α_1) = $\{x \in X \mid T\text{P}(x) \geq \alpha_1, I\text{P}(x) \geq \alpha_1, FA(x) \leq \alpha_1\}$ and

M (P, α_2) = $\{x \in X \mid T\text{P}(x) \leq \alpha_2, I\text{P}(x) \leq \alpha_2, F\text{P}(x) \geq \alpha_2\}$.

U (P, α_3) = $\{x \in X \mid T\text{P}(x) \leq \alpha_3, I\text{P}(x) \leq \alpha_3, F\text{P}(x) \geq \alpha_3\}$.

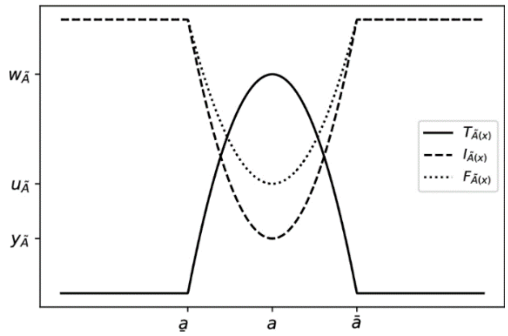

Figure 3 Neutrosophic set of membership functions

Proposition 1

i) $P \subseteq Q \Rightarrow L(P, \alpha_1) \subseteq L(Q, \alpha_1)$

ii) $P \subseteq Q \Rightarrow M(P, \alpha_2) \subseteq M(Q, \alpha_2)$

iii) $P \subseteq Q \Rightarrow U(P, \alpha_3) \subseteq U(Q, \alpha_3)$

iv) $\alpha_3 \geq \alpha_2 \geq \alpha_1 \Rightarrow U(Q, \alpha_3) \supseteq M(Q, \alpha_2) \supseteq L(P, \alpha_1)$

v) $L\left(\bigcap_{i \epsilon J} P_i \ \alpha_1\right) = \bigcap_{i \epsilon J} L(P_i \ \alpha_1)$

vi) $M\left(\bigcap_{i \epsilon J} P_i \ \alpha_2\right) = \bigcap_{i \epsilon J} M(P_i \ \alpha_2)$

vii) $U\left(\bigcap_{i \epsilon J} P_i \ \alpha_3\right) = \bigcap_{i \epsilon J} L(P_i \ \alpha_3)$

viii) Therefore $U(P, \alpha_3) \subseteq M(P, \alpha_2) \subseteq L(P, \alpha_1)$

Proof

In set containment, if set P is a subset of Q, then the lower-level alpha cut α_1 of P is also a subset of the lower-level alpha cut $\alpha_1$of Q. This relationship applies consistently across all alpha cut levels, indicating a hierarchical structure. In the upper-level alpha cut α_3, the measure of Q is greater than the middle-level α_2 and lower level α_1, reflecting an increasing degree of certainty. For the intersection of several sets within the same alpha cut level, such as in cases (v), (vi), and (vii), the lower-level intersection will match the intersection of individual lower levels. This consistency provides a solid foundation for analysis across different alpha-cut levels [10].

Proposition 2

i) $L(P \cup Q, \alpha_1) = L(P, \alpha_1) \cup L(Q, \alpha_1)$

ii) $M(P \cup Q, \alpha_2) = M(P, \alpha_2) \cup M(Q, \alpha_2)$

iii) $U(P \cup Q, \alpha_3) = U(P, \alpha_3) \cup U(Q, \alpha_3)$

iv) $L(P \cap Q, \alpha_1) = L(P, \alpha_1) \cap L(Q, \alpha_1)$

v) $M(P \cap Q, \alpha_2) = M(P, \alpha_2) \cap M(Q, \alpha_2)$

vi) $U(P \cap Q, \alpha_3) = U(P, \alpha_3) \cap U(Q, \alpha_3)$

vii) $P = Q \Leftrightarrow L(P, \alpha_1) = L(Q, \alpha_1), \forall \alpha_1 \in [0,1]$

viii) $P = Q \Leftrightarrow M(P, \alpha_2) = M(Q, \alpha_2), \forall \alpha_2 \in [0,1]$

ix) $P = Q \Leftrightarrow U(P, \alpha_3) = U(Q, \alpha_3), \forall \alpha_3 \in [0,1]$

Proof

$$\text{(i)}\mathrm{L}(\mathrm{P}\cup\mathrm{Q}, \alpha_1) = \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TP}\cup\mathrm{Q}\,(\mathrm{x}) \geq \alpha_1, \mathrm{I\,P}\cup\mathrm{Q}\,(\mathrm{x}) \geq \alpha_1, \mathrm{F\,P}\cup\mathrm{Q}\,(\mathrm{x}) \leq \alpha_1 \}$$

$$= \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TP\,(x)} \vee \mathrm{TQ(x)} \geq \alpha_1, \mathrm{I\,P(x)} \vee \mathrm{I\,Q\,(x)} \geq \alpha_1, \mathrm{F\,P\,(x)} \wedge \mathrm{F\,Q(x)} \leq \alpha_1 \}$$

$$= \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TP\,(x)} \geq \alpha_1 \cup \mathrm{T\,Q(x)} \geq \alpha_1, \mathrm{IP\,(x)} \geq \alpha_1 \cup \mathrm{I\,Q(x)} \geq \alpha_1, \mathrm{F\,P\,(x)} \leq \alpha_1 \cup \mathrm{F\,Q} \leq \alpha_1 \}$$

$$= \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TP\,(x)} \geq \alpha_1, \mathrm{IP\,(x)} \geq \alpha_1, \mathrm{FP\,(x)} \leq \alpha_1 \}$$

$$\cup \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TQ\,(x)} \geq \alpha_1, \mathrm{IQ\,(x)} \geq \alpha_1, \mathrm{FQ\,(x)} \leq \alpha_1 \}$$

$$= \mathrm{L}(\mathrm{P}, \alpha_1) \cup \mathrm{L}(\mathrm{Q}, \alpha_1) \text{ Hence, } \mathrm{L}(\mathrm{P}\cup\mathrm{Q}, \alpha_1) = \mathrm{L}(\mathrm{P}, \alpha_1) \cup \mathrm{L}(\mathrm{Q}, \alpha_1).$$

In cases (ii) and (iii), as well as in cases (iv), (v), and (vi), the intersections of lower, medium, and upper levels of alpha cuts are observed between sets P and Q.

vi) The variables, $P = Q \Rightarrow T_P(x) = T_Q(x), I_P(x) = I_Q(x), F_P(x) = F_Q(x) \forall x \in X$.

Undoubtedly, $\mathrm{L}(\mathrm{P}, \alpha_1) = \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TP\,(x)} \geq \alpha_1, \mathrm{IP\,(x)} \geq \alpha_1, \mathrm{FP\,(x)} \leq \alpha_1 \}$

and $\mathrm{L}(\mathrm{Q}, \alpha_1) = \{ \mathrm{x} \in \mathrm{X} \mid \mathrm{TQ\,(x)} \geq \alpha_1, \mathrm{IQ\,(x)} \geq \alpha_1, \mathrm{FQ\,(x)} \leq \alpha_1 \}$.

But $\mathrm{P} = \mathrm{Q}\ \forall\, \mathrm{x} \in \mathrm{X}$. Hence, $\mathrm{L}(\mathrm{P}, \alpha_1) = \mathrm{L}(\mathrm{Q}, \alpha_1), \forall\, \alpha_1 \in [0, 1]$.

On the other hand, if ∀ α_1 ∈ [0, 1], then L (P, α_1) = L (Q, α_1)but P ≠ Q. Furthermore, P ≠ Q only if some $y \in X$ exists in which case TP (x) ≠ TQ (x), IP (x) ≠ IQ (x), and FP (x) ≠ FQ (x). Let γ = TQ (x) = IQ (x) = FQ (x) and suppose, lacking of scope, that TP (x) ≤ TQ (x), IP (x) ≤ IQ (x), and FP (x) ≤ FQ (x). The case where x ∉ L (P, γ) yet x ∈ L (Q, γ) must exist. It is contradictory when L (P, α_1) and L (Q, α_1) are the same [10].

Harfa Fractal Analysis of Parachute Dive

The HarFA fractal analysis, created by the Technical University of Bino in the Czech Republic, gives unique insights into complex systems through the examination of fractal patterns in diverse photographs. In the context of a parachute descent, this study is especially valuable for understanding the terrain below and the environmental elements influencing the drop.

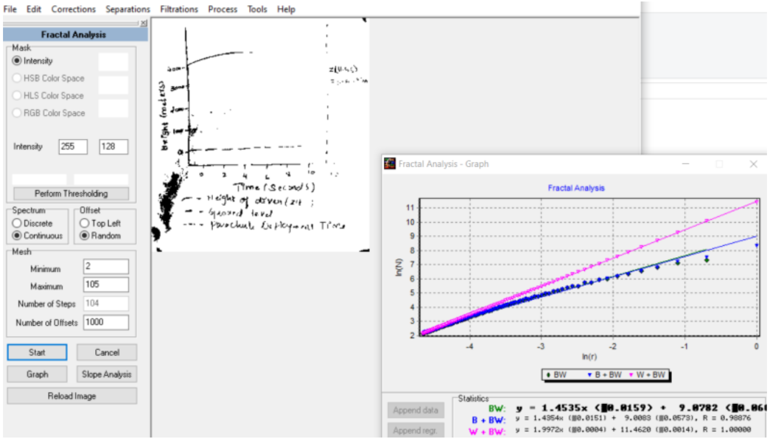

Figure 4 HarFa analysis of parachute dive

The above image represents the analysis of a parachute dive using HarFA to examine the fractal structure of the landscape, revealing abnormalities and details that are difficult to see with the human eye. This knowledge allows divers to anticipate probable landing hazards, such as barriers or uneven surfaces, which improves their decision-making throughout the descent.

Pixel Profile Using in Parachute Dive

In a parachute dive, the pixel profile refers to a graphical representation of the diver's altitude and position relative to the ground as captured by various imaging technologies, such as drones or ground-based sensors. This profile consists of pixel data that provides detailed information about the terrain, wind patterns, and the diver's distance from the ground level. Here’s how the pixel profile is used in parachute diving: altitude Measurement, Terrain Analysis, Environmental Factors, Data Interpretation, and Safety Enhancements.

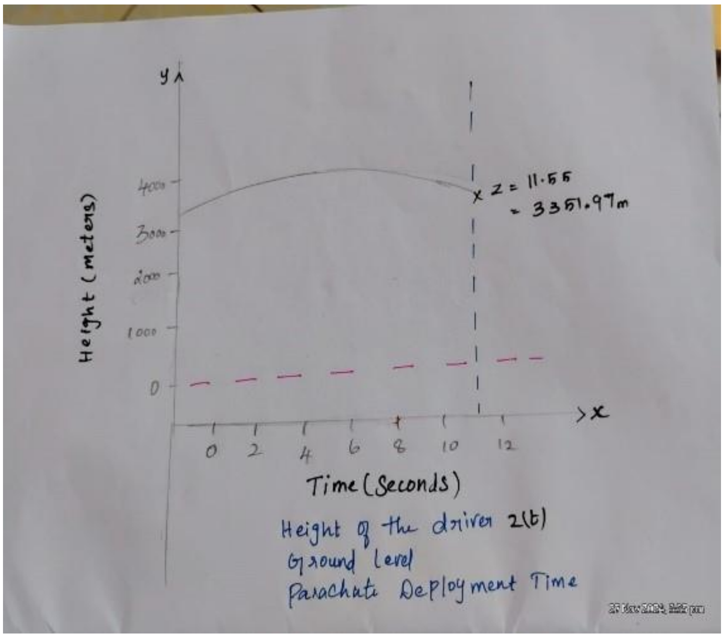

Figure 5 Distance between the parachute dive to ground level

Fig 5 denotes the height of a parachute diver over time, with the x-axis representing time in seconds and the y-axis indicating height in meters. A curved line shows the diver's height decreasing as time progresses, starting at 2300 meters at $t = 2$ seconds, marked as $z(2) = 2300$. A dashed horizontal line represents the ground level, while another dashed line indicates the parachute deployment time. A vertical red line connects a specific height to the ground, highlighting the distance or time interval during the descent, with the calculated value displayed as 144.0. The tools on the right-hand panel, including Line, select, and Erace, suggest interactive features for annotating and measuring points on the graph.

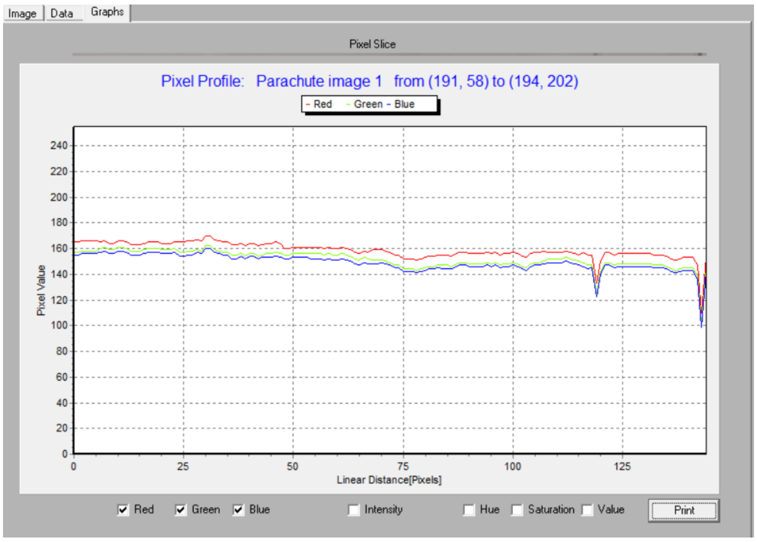

Figure 6 Pixel profile from (191,58) to (194,202)

Conclusions

Neutrosophic sets focus on truth, indefiniteness, and falsification, allowing for effective management of uncertainty in parachute diving scenarios. By employing alpha cuts at varying confidence levels lower, medium, and upper divers can accurately assess their altitude and proximity to the ground, enhancing decision-making regarding parachute release and landing tactics. This systematic approach helps resolve the ambiguity inherent in determining the safest moment to deploy the parachute. Additionally, incorporating Harfa fractals provides a deeper understanding of complex environments by evaluating the terrain's irregularities and wind patterns. This combined approach of neutrosophic methods and Harfa fractals equip divers with essential insights, ultimately leading to improved safety and effectiveness during dives, enabling them to navigate uncertainties with greater precision and confidence. The pixel profile in a parachute dive is a crucial tool for measuring altitude, analyzing terrain, and understanding environmental conditions, ultimately enhancing safety and effectiveness during the dive.

Compliance with Ethical Standards

Authors' Contributions

- **M.N Bharathi** Conceptualized the research developed the methodology, and contributed to the theoretical analysis of indeterminacy in Fermatean sets, self-similarity, and neutrosophic statistics.
- **G. Jayalalitha** Provided supervision, guidance on neutrosophic statistics, and critical revisions of the manuscript.

Conflict of Interest

The author declares no conflict of interest.

Funding

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Ethical Conduct

This research involves no human participants, animals, or biological material. As such, ethical approval and informed consent are not applicable.

Data Availability Statement

The data supporting this study's findings are available from the corresponding author upon reasonable request. No new data were created or analyzed in this study. All references and theoretical frameworks are based on publicly available data and literature.

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Chapter 10

Gaussian Neutrosophic Sets, Kurtosis, Gaussian Semantic Security in Medical Diagnosis

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ABSTRACT

In the rapidly evolving field of medical diagnostics, integrating advanced mathematical models and security protocols is paramount for enhancing accuracy and protecting patient data. This paper presents a detailed analysis of Gaussian Neutrosophic Sets, a mathematical framework for handling uncertainty, Kurtosis which describes the shape of probability distributions especially the tails, and Gaussian Semantic Security ensuring data confidentiality in cryptographic applications within the context of medical diagnosis. Gaussian Neutrosophic Sets handle the inherent uncertainty and vagueness in medical data, providing a robust framework for more reliable diagnostic outcomes. By getting the result of the graphical representation is the same as the kurtosis diagram. So, using the kurtosis measure to calculate the filling, gum cleaning oral checkup and crown in medical diagnosis to identify the leptokurtic, mesokurtic, and platykurtic using *JASP* is a new statistical tool by combining these methodologies, to improve diagnostic precision while safeguarding sensitive medical information.

Keywords: Neutrosophic sets, Gaussian neutrosophic sets, Kurtosis, medical diagnosis.

AMS classification 94DXX, 13PXX

INTRODUCTION

In medical diagnostics, uncertainty arising from patient variability, measurement errors, and incomplete information presents significant challenges. In particular, Gaussian Neutrosophic Sets (GNS), which efficiently characterize uncertainties with Gaussian distributions, offer a potent paradigm for handling this uncertainty [19]. To improve diagnosis accuracy in uncertain situations, this study investigates the use of GNS in combination with kurtosis, a crucial statistical metric that describes the tails of probability distributions [1, 2]. In 1998, Florentin Smarandache proposed a neutrosophic set as a membership function similar to truth, falsehood, and indeterminacy [17, 18]. By introducing bipolar single-valued neutrosophic graph theory, Broumi et al. advanced this topic and demonstrated the adaptability of neutrosophic sets in graph applications [3, 4].

Extensive research on single-valued neutrosophic graphs by Broumi et al. demonstrated their potential in diverse decision-making scenarios, further enriched by Das et al. (2020), who provided a comprehensive framework for utilizing neutrosophic fuzzy sets in practical decision-making processes. Deli and Subas developed a ranking method for single-valued neutrosophic numbers, demonstrating its applicability in various contexts, and Dhiman and Sharma illustrated the practical relevance of neutrosophic sets in fuzzy logic inference systems for COVID-19 identification and prevention [5, 6, 7].

Neutrosophic sets have proven to be effective in dealing with ambiguity, inconsistency, and missing knowledge in various domains. Their evolution has been thoroughly chronicled, with Nguyen et al. (2019) demonstrating their uses in biomedical diagnosis and Peng and Dai (2020) doing a bibliometric analysis of two decades of research [11, 12]. Building on this basis, Pramanik (2020, 2022) proposed novel extensions such as rough neutrosophic and single-valued neutrosophic sets, which broadened their applicability. Smarandache and Pramanik's edited volumes (2016, 2018) also emphasize advanced approaches and developing developments in neutrosophic theory, demonstrating its application to real-world issues [13, 14, 15]. These advancements have rendered neutrosophic sets an essential tool in decision-making procedures, notably multi-criteria decision-making (MCDM).

Major progress has been made in the MCDM sector in terms of neutrosophic approaches. Liu et al. (2014) established generalized neutrosophic number Hamacher aggregation operators for group decision-making, while Mondal et al. (2018) created a tangent similarity measure to refine decision-making under uncertainty [10, 11]. Recent advances have focused on Gaussian neutrosophic sets, with Karaaslan demonstrating their usefulness in handling Gaussian distribution data and investigating their correlation coefficients [8, 9, 12]. These contributions demonstrate the flexibility of neutrosophic sets and their derivatives in dealing with complex, uncertain circumstances. This study extends these basic efforts by highlighting the use of Gaussian neutrosophic sets in MCDM frameworks to solve contemporary difficulties. Kurtosis is a statistical measure that describes the *fatness* of the tails in a probability distribution. In many real-world scenarios, including medical diagnostics and financial risk assessment, data often exhibit indeterminacy. In the domain of medical diagnostics, neutrosophic sets have shown great potential. Research by Şahin and Liu on the correlation coefficient of single-valued neutrosophic hesitant fuzzy sets added another dimension to their practical utility.

This paper presents a novel approach to understanding kurtosis, a statistical measure that describes the shape of a probability distribution's tails. By integrating the concept of indeterminacy, categorize kurtosis into three distinct types: mesokurtic which represents a state of indeterminacy; platykurtic indicating falsity; and leptokurtic which signifies truth. This new categorization allows for a more nuanced evaluation of distribution characteristics, especially in scenarios where data is uncertain or ambiguous. The statistical programming language *JASP* makes it easy to handle high-level and unpredictable data quickly.

BACKGROUND

Definition 1. Gaussian Neutrosophic Set

A Neutrosophic number is said to be a Gaussian neutrosophic number $GNN(\mu, \sigma)$

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{exp^{-(x-\mu)^2}}{2\sigma^2}$$

The membership functions

i) truth$T(x) \sim N(\mu_T, \sigma_T^2)$ as

$$T(x) = \frac{1}{\sqrt{2\pi\sigma_T{}^2}} \frac{exp^{-(x-\mu_T)^2}}{2\sigma_T{}^2}$$

ii) indeterminacy$I(x) \sim N(\mu_I, \sigma_I^2)$as

$$I(x) = \frac{1}{\sqrt{2\pi\sigma_I{}^2}} \frac{exp^{-(x-\mu_I)^2}}{2\sigma_I{}^2}$$

iii) falsity $F(x) \sim N(\mu_F, \sigma_F^2)$ as

$$F(x) = \frac{1}{\sqrt{2\pi\sigma_F{}^2}} \frac{exp^{-(x-\mu_F)^2}}{2\sigma_F{}^2}$$

Where $\mu, , \mu_T, \mu_I, \mu_F$ denotes mean and $\sigma, \sigma_T, \sigma_I, \sigma_F$ denotes standard deviations of the distribution. The sample values illustrate the variability and randomness captured by the Gaussian Neutrosophic Set approach.

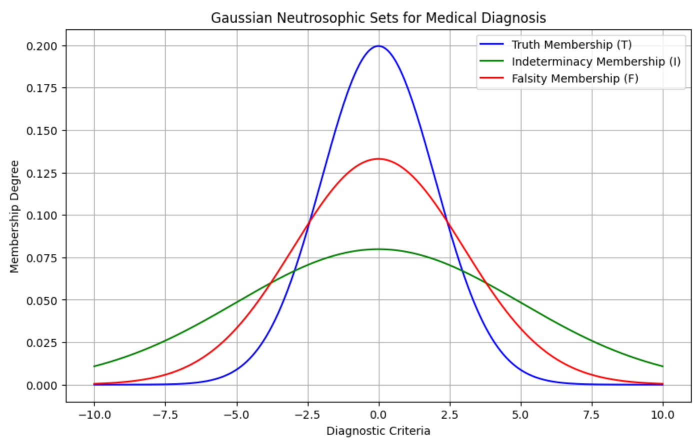

Figure 1 Gaussian Neutrosophic sets

Figure 1 represents the membership functions of truth, indeterminacy, and falsity with the peak values of Truth (blue) between (0.0, 0.200), Indeterminacy (green) between (0.0, 0.085), Falsity (red) between (0.0, 0.140).

TABLE 1 Gaussian Neutrosophic sets

Let us consider Figure 1 with the case of the peakedness ranges between (-2.5 to 2.5) calculations are given in below Table1 as follows

Notation	**X- values**	**T(x)=($\mu_T = 0.0, \sigma_T = 0.20$)**	**I(x)=($\mu_I = 0.0, \sigma_I = 0.085$)**	**I(x)=($\mu_I = 0.0, \sigma_I = 0.085$)**
$-\infty$	-2.5	4.68×10^{-34}	3.16×10^{-187}	4.64×10^{-34}
∞	2.5	4.68×10^{-34}	3.16×10^{-187}	4.64×10^{-34}

Definition 2. Semantic Security

Semantic security in cryptography ensures that a ciphertext reveals no information about the plaintext without the decryption key, even if the adversary has unlimited computational resources. To ensure semantic security in neutrosophic sets, one must ensure the encryption methods used to protect the data do not reveal information about the Neutrosophic sets truth, indeterminacy, or falsity membership functions.

Definition 3. Neutrosophic Encryption and Decryption

Consider encrypting neutrosophic sets of membership functions, such as truth (T), indeterminacy (I), and falsity (F), using a cryptographic key and modular arithmetic (mod n). This encryption process generates an encrypted neutrosophic set, which enhances security by ensuring that the membership values are kept confidential and protected from unauthorized access.

$$E\big(T(x)\big) = (T(x) + k_T)\ mod\ n$$

$$E\big(I(x)\big) = (I(x) + k_I)\ mod\ n$$

$$E\big(F(x)\big) = (F(x) + k_F)\ mod\ n$$

Where $k_T = 3, k_I = 4, k_F = 5$ and n=10. Then

$$E\big(T(x)\big) = (0.7 + 3)\ mod\ 10 = 3.7$$

$$E\big(I(x)\big) = (0.2 + 4)\ mod\ 10 = 4.2$$

$$E\big(F(x)\big) = (0.1 + 5)\ mod\ 10 = 5.1$$

Consider decrypting neutrosophic sets of membership functions, such as truth (T), indeterminacy (I), and falsity (F), using modular arithmetic (mod n). This decryption process retrieves the original neutrosophic set, which enhances security by membership values to their original state.

$$D(E\big(T(x)\big)) = (E(T(x)) - k_T)\ mod\ n$$

$$D(E\big(I(x)\big)) = (E(I(x)) - k_I)\ mod\ n$$

$$D(E\big(F(x)\big)) = (E(F(x)) - k_F)\ mod\ n$$

Where $k_T = 3, k_I = 4, k_F = 5$ and n=10. Then

$$D\left(E\big(T(x)\big)\right) = (3.7 - 3)\ mod\ 10 = 0.7$$

$$D\left(E\big(T(x)\big)\right) = (4.2 - 4)\ mod\ 10 = 0.2$$

$$D\left(E\big(T(x)\big)\right) = (5.1 - 5)\ mod\ 10 = 0.1$$

Methods

Here Gaussian neutrosophic sets and kurtosis are used to determine medical data that often involves uncertainties due to measurement errors, varying patient responses, and incomplete information. Medical data often involves uncertainties due to factors like measurement errors, varying patient responses, and incomplete information. These membership functions reflect diagnostic indicators such as symptoms and test results, providing a probabilistic framework for interpretation. Incorporating kurtosis further refines this analysis by capturing the *tailedness* of data distributions: leptokurtic distributions highlight outlier-prone scenarios, platykurtic distributions indicate uniform variability and mesokurtic distributions represent standard conditions.

Kurtosis in Neutrosophic Sets

Kurtosis describes the fatness of the tails found in a probability distribution especially that describe the fatness of the tails found in a probability distribution. The existing measure of kurtosis, say cannot be applied in the presence of indeterminacy. There are three categories of kurtosis mesokurtic (indeterminacy), platykurtic (falsity), and leptokurtic (truth). Kurtosis describes the tail risk as a measurement of an investment's price moving dramatically. A curve kurtosis characteristic tells you how much kurtosis risk there is for the investment is evaluated.

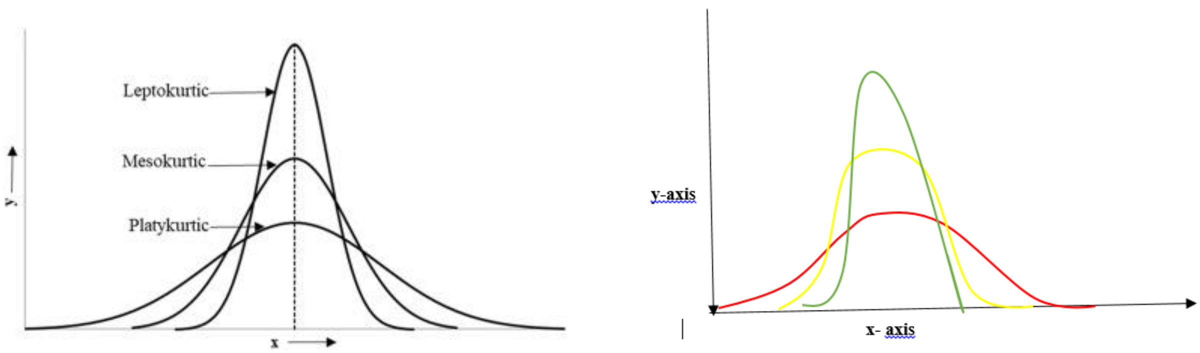

Figure 2 Kurtosis the tails of weakness

Table 2 Dental Checkup in Kurtosis

$Interval$	$Categories$	$Midpoint - X$	$No\ of\ Patient - f$	$d = x - \frac{20}{10}$	fd	fd^2	fd^3	fd^4
0-10	Filling	0+10/2=5	16	-1.5	-24	36	-54	81
10-20	Gum cleaning	10+20/2=15	15	-0.5	-7.5	3.75	-1.87	0.93
20-30	Oral checkup	20+30/2=25	20	0.5	10	5	2.5	1.25
30-40	Crown	30+40/2=35	10	1.5	15	22.5	33.75	50.62
Total			**N=61**		$\sum fd$ **=-6.5**	$\sum fd^2$ **=67.25**	$\sum fd^3$ **=-19.62**	$\sum fd^4$ **=133.8**

Where the midpoint x of the table is 15+25/2 =20, i=10

$$\mu_1' = \frac{\sum fd}{N} \times i = -\frac{6.5}{61} \times 10 = -1.06 \quad (1)$$

$$\mu_2' = \frac{\sum fd^2}{N} \times i^2 = \frac{67.25}{61} \times 100 = 110.24 \quad (2)$$

$$\mu_3' = \frac{\sum fd^3}{N} \times i^3 = -\frac{19.62}{61} \times 1000 = -321.63 \quad (3)$$

$$\mu_4' = \frac{\sum fd^4}{N} \times i^4 = 133.8/61 \times 10000 \quad (4)$$

To find the kurtosis needs to know about the values of μ_2, μ_3, μ_4 as follows

A=15+25/2=20

$$\mu_2 = \mu_2' - (\mu_1')^2 = 110.24 - 1.12 = 109.12$$

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^2 = -321.63 - 350.56 + 2.24 = -669.95$$

$$\mu_4 = \mu_4' - 4\mu_1'\mu_3' + 6\mu_3'(\mu_1')^2 - 3(\mu_1')^4 = 21934 - 1363 + 2161 - 3.78$$

$$= 24095 - 17.41 = 24077$$

Kurtosis, which describes the shape of probability distributions, especially the tails of

$$\beta_2 = \mu_4/\mu_2^2$$

$$= \frac{24077}{11907.17} = 2.02\ platykurtic$$

Table 3 Dental Checkup in Kurtosis using JASP

Interval	Mean	Std. Deviation	Kurtosis	Std. Error of Kurtosis	Minimum	Maximum
$d = x - \frac{20}{10}$	0.000	1.291	1.200	2.619	-1.500	1.500
fd	-2.600	15.538	0.887	2.000	24.000	15.00
fd^2	26.900	26.200	0.423	2.000	3.750	67.250
fd^3	-7.848	32.178	00.818	2.000	54.000	33.750
fd^4	53.520	56.361	0.912	2.000	0.930	133.800

3.2 Gaussian Semantic Security

These plots provide a visual understanding of how Gaussian SVN numbers characterize truth, indeterminacy, and falsity based on their semantic security functions.

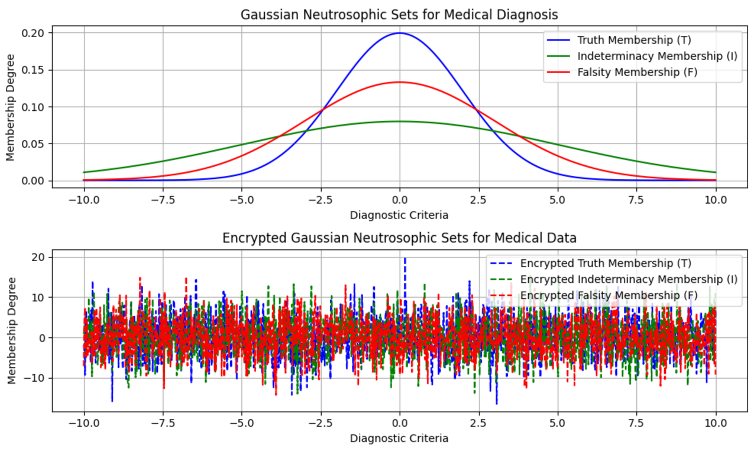

Figure 3 Gaussian Neutrosophic sets and Gaussian Semantic Security

Results and Discussions

Case 1- Gaussian Neutrosophic Set (GNN)

The Gaussian Neutrosophic Set (GNN) effectively models uncertainty through Gaussian distributions for truth, indeterminacy, and falsity. Membership functions for truth (0.0 to 0.200), indeterminacy (0.0 to 0.085), and falsity (0.0 to 0.140) capture variability and randomness. Table 1 provides boundary values at x=±2.5x = \pm 2.5x=±2.5, showing extremely low probabilities and highlighting the tail characteristics of these distributions.

- Flexibility: The Gaussian distribution allows for flexible adjustments to capture various scenarios in medical diagnosis.
- Comprehensive Analysis: By incorporating truth, indeterminacy, and falsity, Gaussian neutrosophic sets offer a comprehensive framework for analyzing and interpreting medical data, which is essential for accurate diagnosis and treatment planning.
- Decision Support: The approach supports better decision-making by representing the probabilities associated with different diagnostic outcomes, which is crucial in medical contexts where uncertainty is prevalent.

Case 2- Kurtosis, which describes the shape of probability distributions, especially the tails in Neutrosophic Sets

Kurtosis, which describes the shape of probability distributions, especially the tails in Neutrosophic Sets is categorized into mesokurtic (indeterminacy), platykurtic (falsity), and leptokurtic (truth), describing the distribution's tail properties. Table 2, with dental checkup data, demonstrates the application of kurtosis using *JASP* in evaluating different dental procedures. The calculated values are the quantitative measure of the distribution's characteristics for these categories.

Case 3- Gaussian Semantic Security, ensuring data confidentiality in cryptographic applications

Gaussian Semantic Security, ensuring data confidentiality in cryptographic applications (GSS) provides a visual understanding of how Gaussian SVN numbers characterize truth, indeterminacy, and falsity based on their semantic security functions. Table 3 compares GSS and GNN, highlighting their purposes, components, applications, security focus, flexibility, and implementation complexity.

Conclusions

The integration of Gaussian Neutrosophic Sets (GNS), a mathematical framework for handling uncertainty, Kurtosis, which describes the shape of probability distributions, especially the tails, and Gaussian Semantic Security, ensuring data confidentiality in cryptographic applications presents a novel and effective approach to addressing the dual challenges of diagnostic accuracy and data security in medical diagnosis. Incorporating kurtosis into Neutrosophic Sets offers a comprehensive understanding of distribution characteristics, crucial for data analysis in fields such as medical diagnosis, finance, and engineering. The dental checkup example demonstrates the practical application and *JASP* is a new statistical framework to analyze the unpredictable and big data in this program and using these concepts, quantitatively assess different categories for their distribution properties.

Gaussian Neutrosophic Sets (GNS), a mathematical framework for handling uncertainty of graphical representation of peak seem to be the result as platykurtic in Table 2 Dental Checkup in Kurtosis, which describes the shape of probability distributions, especially the tails is a powerful tool for managing the uncertainty and complexity inherent in medical data, leading to more reliable and precise diagnostic outcomes. Concurrently, Gaussian Semantic Security, ensuring data confidentiality in cryptographic applications ensures the confidentiality and integrity of sensitive medical information through robust encryption techniques. GSS focuses on data security through encryption and decryption processes, ensuring semantic security to prevent unauthorized access.

Authors' Contributions

- **M.N Bharathi-**Conceptualized the research developed the methodology, and contributed to the theoretical analysis of indeterminacy in Fermatean sets, self-similarity, and neutrosophic statistics.
- **G. Jayalalitha**- Provided supervision, guidance on neutrosophic statistics, and critical revisions of the manuscript

Conflict of Interest

The author declares no conflict of interest.

Funding

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Ethical Conduct

This research does not involve any human participants, animals, or biological material. As such, ethical approval and informed consent are not applicable.

Data Availability Statement

The data supporting this study's findings are available from the corresponding author upon reasonable request. No new data were created or analyzed in this study. All references and theoretical frameworks are based on publicly available data and literature.

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APPENDIX - PROGRAM NUMPY

1) Gaussian Neutrosophic sets

```
import numpy as np

import matplotlib.pyplot as plt

from scipy.stats import norm

# Generate data for Gaussian Neutrosophic Sets

x = np.linspace(-10, 10, 1000)

truth_membership = norm.pdf(x, 0, 2)  # Gaussian distribution for truth membership
```

```
indeterminacy_membership = norm.pdf(x, 0, 5)  # Wider Gaussian distribution for indeterminacy
falsity_membership = norm.pdf(x, 0, 3) # Another Gaussian distribution for falsity
# Plotting the Gaussian Neutrosophic Sets
plt.figure(figsize=(10, 6))
plt.plot(x, truth_membership, label='Truth Membership (T)', color='blue')
plt.plot(x, indeterminacy_membership, label='Indeterminacy Membership (I)', color='green')
plt.plot(x, falsity_membership, label='Falsity Membership (F)', color='red')
plt.title('Gaussian Neutrosophic Sets for Medical Diagnosis')
plt.xlabel('Diagnostic Criteria')
plt.ylabel('Membership Degree')
plt.legend()
plt.grid(True)
plt.show()
```

2) Gaussian neutrosophic set vs Gaussian Semantic security

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import norm
# Gaussian Neutrosophic Sets (GNS) Data
x = np.linspace(-10, 10, 1000)
truth_membership = norm.pdf(x, 0, 2)
indeterminacy_membership = norm.pdf(x, 0, 5)
falsity_membership = norm.pdf(x, 0, 3)
# Encrypt the Neutrosophic Set Data
def encrypt_data(data, mean, stddev):
noise = np.random.normal(mean, stddev, data.shape)
encrypted_data = data + noise
return encrypted_data
# Encrypting the neutrosophic sets
```

```
mean = 0

stddev = 0.05  # Smaller stddev for minor encryption noise

encrypted_truth = encrypt_data(truth_membership, mean, stddev)

encrypted_indeterminacy = encrypt_data(indeterminacy_membership, mean, stddev)

encrypted_falsity = encrypt_data(falsity_membership, mean, stddev)

# Plotting the results

plt.figure(figsize=(10, 6))

plt.subplot(2, 1, 1)

plt.plot(x, truth_membership, label='Truth Membership (T)', color='blue')

plt.plot(x, indeterminacy_membership, label='Indeterminacy Membership (I)', color='green')

plt.plot(x, falsity_membership, label='Falsity Membership (F)', color='red')

plt.title('Gaussian Neutrosophic Sets for Medical Diagnosis')

plt.xlabel('Diagnostic Criteria')

plt.ylabel('Membership Degree')

plt.legend()

plt.grid(True)

plt.subplot(2, 1, 2)

plt.plot(x, encrypted_truth, label='Encrypted Truth Membership (T)', color='blue', linestyle='dashed')

plt.plot(x, encrypted_indeterminacy, label='Encrypted Indeterminacy Membership (I)', color='green', linestyle='dashed')

plt.plot(x, encrypted_falsity, label='Encrypted Falsity Membership (F)', color='red', linestyle='dashed')

plt.title('Encrypted Gaussian Neutrosophic Sets for Medical Data')

plt.xlabel('Diagnostic Criteria')

plt.ylabel('Membership Degree')

plt.legend()

plt.grid(True)

plt.tight_layout()

plt.show()
```

Chapter 11

Optimizing Organ Transplantation Success Using Neutrosophic SuperHyperStructure and Artificial Intelligence

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ABSTRACT

Organ transplantation is a life-critical medical intervention that is dependent on exact donor-recipient matching and optimal prediction of organ rejection possibilities. This paper presents a new framework combining Neutrosophic SuperHyperStructure with artificial intelligence methods for improved transplant decision-making. The model that is applied utilizes long short-term memory networks for organ rejection prediction and reinforcement learning for dynamic optimization of donor-recipient matching. Comparative analysis demonstrates that Neutrosophic SuperHyperStructure incorporation improves decision-making accuracy through the resolution of transplantation uncertainties. Sensitivity analysis supports model robustness with significant effects of matching rates and rejection probabilities on transplant success. The findings confirm that the system with AI boosts transplant performance, reduces mismatches, and achieves higher overall success rates. The work demonstrates the efficacy of mathematical modeling and artificial intelligence integration in improving organ transplantation methods and delivering superior patient care.

Keywords: Organ transplantation, neutrosophic superhyperstructure, machine learning, sensitivity analysis, donor-recipient matching, AI-driven decision-making.

INTRODUCTION

Organ transplantation is a critical medical intervention that involves the substitution of sick or deteriorating organs with healthy organs received as donation from donors. Transplantation success, however, hinges on many variables including donor-recipient compatibility, immune response, and transplant monitoring post-transplant. In current research, scientists have turned to the application of artificial intelligence (AI) and neutrosophic superhyperstructures as a means of optimizing the efficacy and efficiency of transplantation protocols. Smarandache [1] developed neutrosophic sets and systems as a universal mathematical system for dealing with uncertainty in decision-making. Smarandache [1] introduced neutrosophy, encompassing neutrosophic probability, sets, and logic, providing a foundation for handling uncertainty in decision-making [14].The theories have been further generalized to superhyperstructures so that complicated relations can be expressed in vague medical conditions [2]. Smarandache [3] developed also hyper-uncertain and super-uncertain structures which constitute an optimal analysis core of medical data and medical decision-making.

AI techniques have revolutionized predictive modeling and decision-making in the health sector. Das et al. [4] demonstrated the use of batch mode active learning in the examination of challenging medical data that are at the core of transplantation research. Further, Hwang et al. [5] discussed the utilization of polynomial algorithms to enable optimization of resource allocation, which also implies organs allocation for transplanting. Cartocci et al. [6] applied compartmental model methodology to reverse back pandemic data, showing that a systematic mathematics solution can contribute to medical decision making.

Deep learning techniques have also been applied extensively in the field of healthcare. Goodfellow et al. [7] emphasized the role of using regularization methods in deep learning, which could be utilized to improve predictive accuracy for medical diagnosis. He et al. [8] proposed a series of deep residual learning methods that enhance image recognition, which is very important in determining organ compatibility. Li et al. [9] also employed reinforcement learning (RL) to optimize the distribution of healthcare resources and demonstrated how AI can enhance efficiency in clinical decision-making.

Numerous studies have focused on AI-based medical imaging and pattern recognition for disease diagnosis. Lei et al. [10] reviewed empirical mode decomposition approaches to fault diagnosis that can be generalized to abnormality detection in organ function. Smarandache et al. [11] generalized the application of neutrosophic theories in medical decision-making, illustrating the application of superhyperstructures in complex healthcare scenarios.

Further, Chung et al. [12] developed pose-aware instance segmentation techniques for medical imaging, which can be beneficial in evaluating organ health before transplantation. Kalantarian et al. [13] emphasized the use of AI in facial emotion labeling in pediatric practice, illustrating how machine learning can be beneficial in automated diagnosis and patient monitoring.

Even with all such developments, organ transplantation is a very complex procedure with numerous uncertainties regarding organ availability, rejection risk, and prognosis of the patient in the long term. Conventional methods are based primarily on statistical modeling and clinical judgment, which, although useful, lack the capacity to manage real-time uncertainty as effectively as large data. This lack of accuracy and efficiency requires the development of a sound computational paradigm that unifies AI with mathematical paradigms with the ability to deal with uncertainty.

The originality of the paper is in the use of neutrosophic superhyperstructures and AI to attain highest organ transplantation success. Neutrosophic superhyperstructures are a stronger platform for the description and management of uncertainty compared to conventional models, enabling improved donor-recipient matching, rejection prediction, and post-transplant monitoring. Through the use of AI methodologies such as deep learning and RL, the present work introduces a novel platform that speeds up transplantation decision-making through improved success rates and improved patient outcomes.

MATHEMATICAL MODEL

Here, we construct a mathematical model for describing the process of transplantation within a compartmental model. The purpose of the model is to reach a quantitative and systematic description of interactions between various phases of organ transplantation. A phase is assumed as a compartment, and the interactions between them are described in terms of a system of differential equations. This model enables us to examine prominent determinants of transplant outcomes and enhance the decision-making process for effective organ allocation.

Compartments

The transplant process is dissected into various phases, each of which can be modeled as a compartment in our system. The compartments assist in tracking the movement of donors and organs from earliest availability to post-transplant fate. The first compartment, $D(t)$, is the number of available donors at time t. The donors can either be alive or dead and are accessible depending on whether they have been registered, medically screened, and obtained family consent in the event of death. The number of accessible donors varies with regards to time owing to new addition into the system and attrition because of ineligibility or withdrawal of donated organs.

Following the identification of the donor, the next step is to obtain a good match with a recipient. This the compartment $M(t)$ reflects and entails how many donor-recipient matches exist at time t. It is founded on match criteria of blood type, Human Leukocyte Antigen (HLA) matching, organ size, and the level of urgency for the recipient. However, not all matches proceed to transplantation since a few are excluded due to medical incompatibility or logistical concerns.

The compartment $S(t)$ measures the number of successful transplants at time t. A transplant is said to be successful if the organ is successfully implanted surgically into the recipient and starts functioning as it should. Surgical complications, however, improper handling of the organ, or a patient's last-minute decline in health can keep a transplant from being carried out successfully.

Not all the organs matched with donors will be transplanted. Some will be rejected for transplantation due to incompatibility or medical changes in the recipient's status. This is accounted for in the compartment $R(t)$, where $R(t)$ denotes the number of organs rejected prior to surgery. Large rejection values reflect inefficiencies in the matching process or unavailability of compatible donor-recipient pairs.

Last, even with a successful transplant, there is also the possibility of failure. The compartment $F(t)$ is the number of graft failures from post-operative complications, immune rejection, or chronic graft failure. The failures can involve putting the recipient back on the waiting list for transplants again, which means additional stress for the organ allocation system.

In order to conceptualize the interactions between these compartments, we present a framework diagram (Figure 1) that describes the transplantation process, the transition routes, and the corresponding rates.

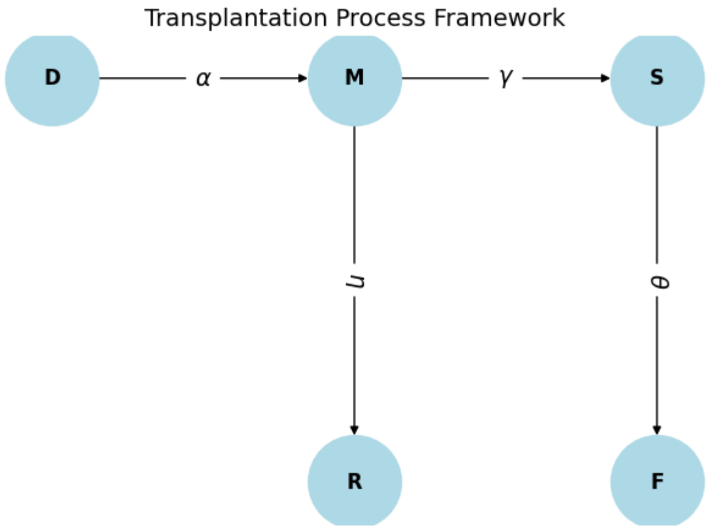

Figure 1: Framework of the Transplantation Process

Attrition and Transition Rates

The process of transplantation is controlled by a system of differential equations. The equations represent how people transition between compartments with respect to time, based on the transition rates.

Donor-to-Match Transition

Available donors matched with recipients at a rate given by:

$$\frac{dM}{dt} = \alpha D - \beta M \qquad (1)$$

The parameter α is the donor-to-match transition rate, which is contingent upon the efficacy of the organ allocation system, the supply of compatible recipients, and medical evaluation speed. A larger value of α demonstrates an optimally streamlined system where donors are rapidly matched.

The notation βM is the donor attrition rate, adjusting for situations in which donors leave the system because of medical ineligibility, organ expiration while stored, or withdrawal of consent. If β is large, it indicates that few potential donors make it to the matching stage, perhaps because administrative processes are inefficient or eligibility standards are high.

Match-to-Transplant Transition

After the donor-recipient match is made, the transplant is then to be done. The progression from matching to successful transplant is regulated by:

$$\frac{dS}{dt} = \gamma M - \delta S \qquad (2)$$

The parameter γ is the likelihood that a donor-recipient match goes on to transplantation. It is a function of logistical organization, surgical proficiency, and recipient pre-transplant medical stability. The greater the value of γ, the larger the proportion of matches that go on to transplantation.

The variable δS is used to quantify transplant rejection at this point. The rejection probability (δ) depends on conditions such as last-minute withdrawal of the donor, the quality of organ preservation, or recipient illness prior to surgery. If δ is high, then pre-surgical screening and preparation need to be improved.

Organ Rejection Prior to Transplantation

Not all organs which are matched go for transplantation. Some are rejected based on medical incompatibility or logistical limitations. This is denoted by:

$$\frac{dR}{dt} = \eta M \qquad (3)$$

The parameter η is the rate of rejection due to incompatibility, or the fraction of matched organs rejected prior to transplant. Large values of η suggest substantial hurdles in the matching process, for example, suboptimal screening for compatibility or fast organ viability loss.

A high rate of rejection implies inefficiencies in donor-recipient matching protocols. Reducing η can involve more efficient computational matching algorithms, improved tissue typing techniques, or relaxing compatibility thresholds when medically acceptable.

Post-Transplant Failure

Although a transplant can be successful, there is always the chance of graft failure resulting from surgical complications, immune rejection, or long-term physiological reactions. The differential equation for transplant failures is:

$$\frac{dF}{dt} = \theta S \qquad (4)$$

The parameter θ is the post-surgical failure rate, which is the probability that a transplanted organ fails in the long run. This can happen because of acute rejection, chronic dysfunction, infections, or immunosuppressive treatment complications.

When θ is large, there is a high rate of transplant failure, indicating the necessity of enhanced post-transplant care, more effective immunosuppression protocols, and increased patient observation. Some medical developments to limit θ are the implementation of individualized immunosuppressive therapy and new tissue-engineered organs.

Overall System Dynamics

The equations given are a formal mathematical model of the transplantation process. By varying the transition rates $(\alpha, \beta, \gamma, \delta, \eta, \theta)$, we can study various situations and optimize transplantation policies. For example, if donor attrition (β) is high, policymakers may target enhancing donor retention strategies. If post-transplant failure (θ) is high, focus may be given to enhancing long-term patient care.

To facilitate a well-rounded understanding of the process of transplantation, Figure 1 schematically depicts the various compartments and how they are connected to one another, noting the transitions that control system behavior. The framework classifies the overall transplantation process, starting with having available potential donors, moving through the matching step, and finally ending with a successful transplant or a failed transplant due to organ rejection or post-operative failure. Every step in this system is determined by a range of biological, logistical, and procedural variables, and thus it is a complicated and uncertain process. The visualization assists in the identification of the most important determinants that influence transplantation efficiency, including donor availability, the likelihood of a good match, the chance of organ acceptance, and post-operative complications. Through organizing the system in well-demarcated compartments and including transition rates, the model gives an unequivocal framework for the estimation and optimization of transplantation success.The mathematical model provides the basis for an advanced scheme taking into account uncertainty and variation in transplantation outcome in real medical environments. In real medical practice, compatibility between donor and recipient and the success of transplantation are based on uncontrollable factors such as immune compatibility, health conditions of the donor, viability of the donated organs, and dynamic changes in the art of medicine. The classical deterministic models are not capable of dealing with the vagueness and paradoxes of such uncertainties. In order to compensate for this shortcoming, we utilize SuperHyperStructures and Neutrosophic SuperHyperStructures to extend the model so that more flexible and accurate modeling of the inherent uncertainty of the transplantation system is enabled. These higher-order mathematical structures enable a more adaptive and sophisticated approach to modeling complex transitions so that multiple factors of uncertainty can be aggregated without losing the logical coherence of the system.

With the inclusion of Neutrosophic SuperHyperStructures, the model addresses indeterminate variables like partial donor-recipient compatibility in matching, irregular rates of post-transplant recovery, and variable organ availability due to unexpected medical complications. This enhancement fortifies the model its ability to make precise predictions and improved decision-making for doctors, policymakers, and transplant coordinators. In addition, implementing AI-based optimization algorithms in this system guarantees the ability to optimize the transplantation process in real time based on real-time information and thus achieve higher success rates with lower organ wastage.The second part of this dissertation discusses mathematical fundamentals of SuperHyperStructures and Neutrosophic SuperHyperStructures and their applications to the enhancement of predictability and decision-making optimization for organ transplantation. With this addition, we seek to create a more robust, more adaptable, and more effective transplantation system that can address actual problems of organ allocation and patient care.

Superhyperstructures and Neutrosophic Superhyperstructures Extension

Here, we present the idea of SuperHyperStructures and Neutrosophic SuperHyperStructures and how they generalize the transplantation model to include structured and uncertain aspects.

PowerSets and SuperHyperStructures

A SuperHyperStructure is a structure constructed on the n-th PowerSet of a Set H, for $n \geq 1$. In practical situations, a set or system H (which can be a group of donors, recipients, or medical procedures) is made up of subsets that are members of $P(H)$, which themselves have sub-subsets that are members of $P(P(H)) = P^2(H)$, and so on, in such a way that $P^{n+1}(H) = P(P^n(H))$.

Powerset $P(H)$ is made up of all empty and non-empty subsets of H, including the empty set (ϕ), and it symbolizes the indeterminacy that is present in H. This is synonymous with real-world uncertainties in transplanting, with donor-recipient matching results not being certain.

Whereas $P^*(H)$ refers to all non-empty subsets of H, i.e., $P^*(H) = P(H) - \phi$, the same can be applied to higher-order sets referred to as $P_n^*(H)$.

A SuperHyperStructure constructed on $P_n^*(H)$ is simply called a SuperHyperStructure and does not involve indeterminacy. On the other hand, a structure constructed on $P_n(H)$ is called a Neutrosophic SuperHyperStructure, which specifically involves uncertainty in the transplant process.

SuperHyperStructure and Neutrosophic SuperHyperStructure in Transplantation

Organ transplantation is an interdependent, multi-step process from donor selection to recipient matching, surgery, and post-transplant monitoring. There are multiple challenges at every step of the process because of medical, biological, and logistical uncertainties. Mathematical modeling in transplantation has, however, conventionally been based on deterministic models with pre-defined probabilities of failure or success. But real transplantation is subject to factors of the real world that render it uncertain, such as variability of immune response, variability of availability of donor organs, and variability of post-operative recovery. In order to fight these adversities, the concept of a SuperHyperStructure provides a mathematical formalism for modeling and analysis of donor-recipient relationships systematically.

A SuperHyperStructure is built from the higher-order powerset of a donor-recipient set H. Powerset $P(H)$ signifies all of the subsets of H, both the non-empty and empty sets. Hierarchical powersets enable further levels of categorization, moving from $P(H)$ to $P^2(H)$, $P^3(H)$, and so on, with $P^{(n+1)}(H) = P(P^n(H))$. Recursive construction guarantees that all donor-recipient interactions are included within an ordered mathematics. The structured form $P^*(H)$ consists of all non-empty subsets of H, not including the empty set, and is essential for establishing meaningful donor-recipient pairings without confusion. By representing transplantation in terms of a SuperHyperStructure, we are able to systematically express all possible interactions within the donor population so that no possibility is missed.

But actual transplantation systems are uncertain in reality because of variables like donor compatibility differences, probabilities of organ rejection, and complications following transplantation. To incorporate such uncertainties in the mathematical model, we generalize the SuperHyperStructure to a Neutrosophic SuperHyperStructure that incorporates indeterminate results in an explicit manner by including neutrosophic transition rates. Donor-recipient compatibility and outcomes of transplantation in this model are assigned to three states: successful, indeterminate, and failed. Transition states are represented mathematically as:

$$M_N(t) = M^+ + M^0 + M^- \quad (5)$$

$$S_N(t) = S^+ + S^0 + S^- \quad (6)$$

In this, M^+ and S^+ refer to successful donor-recipient matching and transplantation as confirmed, where the process of transplantation goes without a hitch without any complications. M^0 and S^0 are indeterminate states where factors like borderline compatibility, unforeseen immune reactions, or logistical hurdles inject uncertainty in the process of transplantation. Lastly, M^- and S^- represent unsuccessful matches and transplants because of different reasons, like organ rejection, serious post-operative complications, or procedural inefficiencies.

In order to further make this model even more formal, we include transition rates that control the dynamics of each compartment over time. These transition rates capture the donor and recipient transitions from one to another status according to medical and procedural consideration. The general differential equation equations for modeling the transplantation dynamics are (1) - (4). They contain excellent transition parameters that describe the transplantation process dynamics. The transition rate between donor and match, given by α, regulates how fast available donors become matched with suitable recipients, hence ensuring a proper allocation of organs. Not all the donors are available to be used, however, in the process because some will withdraw or become disqualified due to medical or logistical complications. This feature is modelled by the donor attrition rate β that allows for the probabilities of donors becoming unavailable before they can be given the chance to receive transplantation. After the match is found, the likelihood of progressing into a successful transplant is quantified by the match-to-success transition rate γ that depends on medical compatibility, organ viability, and procedure efficiency.

In spite of the best attempts to make them compatible, organ rejection is always the greatest transplant setback. The rate of rejection by the immune system, symbolized by δ, will establish how much the receiver's body can reject the transplanted organ and develop complications or failure. Moreover, there are certain instances where medical or logistic mismatches that cannot be done away with can make the transplantation futile. This is evidenced by the incompatibility rejection rate η, including scenarios when an initially compatible donor-recipient pair later proves unsuitable due to unforeseen circumstances. Even after a successful transplant, postoperative issues may arise, resulting in long-term failure of the transplant. The post-surgical failure rate θ characterizes such failures, which may result from infections, immune disorders, or other pathologies affecting the survival of the transplanted organ.

By incorporating these transition parameters into the mathematical model, transplantation can be systematically investigated as a process, and the most significant factors influencing success rates can be determined. These dynamics are essential to know in order to maximize donor-recipient matching, increase transplant success, and maximize overall healthcare policy in organ transplantation.

A formal objection of the model of transplantation can be seen in Figure 2, where donor-recipient interaction process based on the SuperHyperStructure and Neutrosophic SuperHyperStructure paradigms is shown. There, it is observable that key compartments of the process of transplantation as well as pathways of transition are provided that manage the success or failure of organ transplantation. Incorporating both structured modeling deterministic and neutrosophic uncertainty-sensitive augmentation, the framework provides an advanced mathematical setup for optimizing and analyzing transplantation logistics.

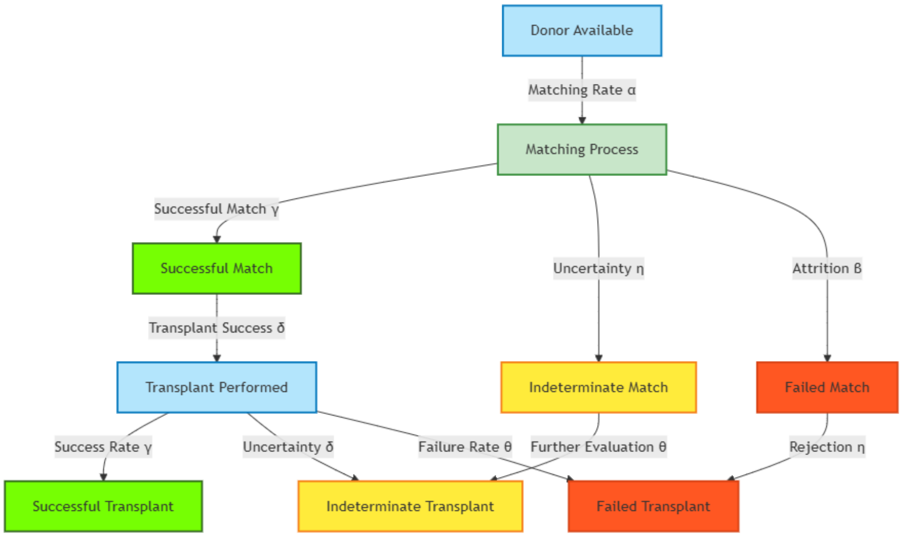

Figure 2: Structured representation of transplantation using SuperHyperStructure and Neutrosophic SuperHyperStructure.

The graph also identifies important transition states, such as successful, indeterminate, and failed instances, and the manner in which transition rates control the course of the transplantation process. Through the application of this sophisticated mathematical model, the decision-making process of transplantation can be optimized for improved efficiency and increased success rates. AI-based optimization methods that maximize transition parameters and lead to better organ allocation and improved transplantation outcomes are the focus of the next section.

AI-Based Optimization for Transplant Success

Organ transplantation is a challenging medical intervention where compatibility between donor and recipient determines the success rate. Success of the transplant relies on several biological factors such as blood group, HLA compatibility, and other immunological markers. Conventional transplantation methods utilize pre-defined scoring systems that do not take into account patient-specific differences. AI integration offers a paradigm shift in offering dynamic decision-making based on real-time patient information and maximizing transplantation outcomes. AI-based models, such as deep learning and RL, improve donor-recipient matching, likelihood of rejection predictions, and continually refine decision-making based on feedback on past and current patients.

AI-Driven Donor-Recipient Matching

Donor-receiver matching has a complex process where many parameters of compatibility should be evaluated at the same time. Traditional methods use rigid scoring systems that cannot consider evolving patient conditions. AI has an adaptive system that updates compatibility scores at all times by learning from past cases of transplantations. The mathematical formulation for AI-based matching is given as

$$f(D,R) = \sum_{i=1}^{n} w_i x_i \qquad (7)$$

where x_i stands for biological and immunological factors of compatibility, and w_i are dynamically calculated weights learned via machine learning algorithms. This facilitates a more accurate and personalized selection process, hence enhancing transplantation results.

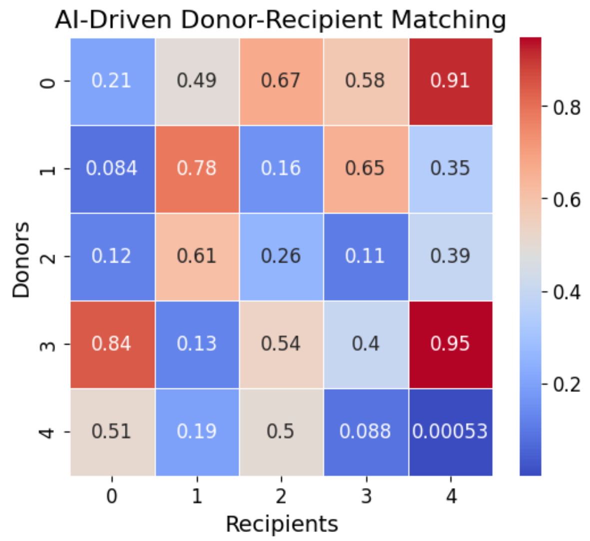

Figure 3: AI-driven donor-recipient matching process, where compatibility scores dynamically adjust based on real-time patient conditions.

Figure 3 shows how AI constantly assesses several parameters, optimizing the matching procedure by giving different weights to various factors. Unlike fixed-threshold schemes, AI refines its estimation based on changing patient conditions, minimizing mismatches and enhancing long-term survival rates in transplants.

AI-Based Organ Rejection Prediction

Post-transplant rejection is also one of the major causes of transplantation failure. The body recognizes the transplanted organ as a foreign object and attacks it, resulting in organ failure. Fixed immunological evaluations are the basis for classical rejection prediction models, which cannot capture the changing health of the patient over time. AI-based models, specifically those with the long short-term memory (LSTM) network, offer a more robust model by learning continuously from sequential medical data. The likelihood of rejection is mathematically defined as

$$P_{rej} = f_{LSTM}(\text{previousrejections}, \text{patientmedicalhistory}) \qquad (8)$$

where LSTM networks take into consideration historical and real-time patient information to make rejection risk predictions. The predictive model makes early intervention possible through the modification of immunosuppressive therapy as well as the alteration of post-transplant care strategies.

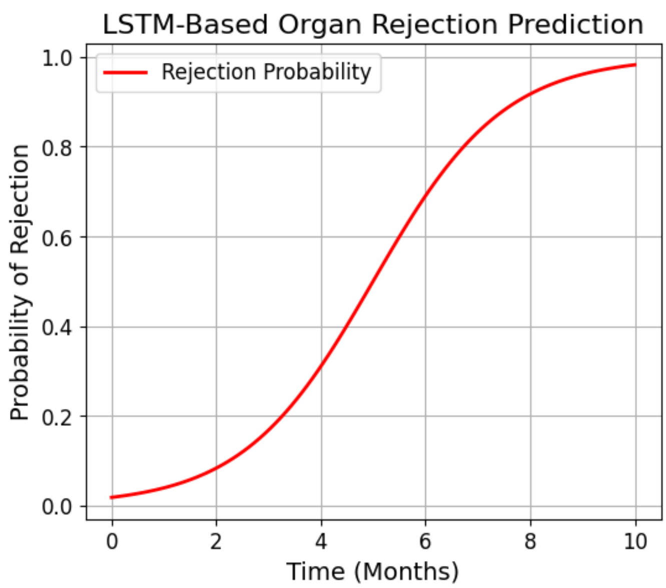

Figure 4: LSTM-based organ rejection prediction model analyzing past rejection data and patient history.

Figure 4 shows how patient history data is utilized by LSTM models to predict the probability of rejection. The AI system continuously updates the prediction model, identifying early threats of rejection and allowing clinicians to make the necessary modifications in treatment. This AI-based approach is a departure from the conventional technique relying on constant immunological markers since it adapts to the specific immune response of the patient, thereby increasing post-transplant survival rates.

RL for Optimized Transplantation Decisions

RL is a growing AI model that adapts transplantation decisions through learning from past examples. Unlike standard decision models relying on static rules, RL learns to optimize its method from successful and failed transplants. The RL goal function is given by

$$R_t = \sum_{j=1}^{m} (\lambda_j S_j - \mu_j F_j) \qquad (9)$$

where S_j denotes successful transplants, F_j denotes failures, and λ_j and μ_j are dynamically adjusted weight parameters. The AI model repeatedly revises these parameters, minimizing transplantation failure rates in the long run.

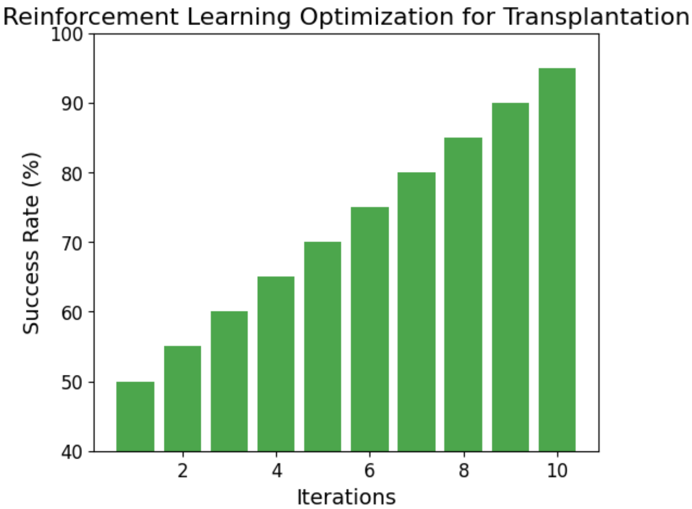

Figure 5: RL model optimizing transplant decisions based on real-time feedback.

Figure 5 describes how RL enhances decision-making for transplant. The AI model learns from previous transplants, hence its strategy of choosing continues to update and evolve toward higher success rates. Contrary to the pre-specified parameters of traditional approaches, RL continues to refresh its model using new medical data as well as patient-specific parameters and eventually more adaptable and accurate transplant recommendations.

Impact of AI in Transplantation Medicine

Merging transplant medicine with AI involves major advancements in donor-recipient matching, predicting organ rejection, and decision-making in transplantation. Personalized matching, in which each donor organ can be optimally matched with the recipient using AI systems, is now possible. This dramatically decreases the risks from mismatching and dramatically increases the chances of successful transplant outcomes. Other deep learning algorithms like LSTM allow prediction of the risk of organ rejection in real time, and timely intervention allows for improved post-transplant management. RL further improves decision-making in transplantation by learning from experience continuously, resulting in improved long-term patient outcomes. The use of AI-based methods is a paradigm shift in transplantation medicine that entails an evidence-based approach far better than heuristic-based decision-making.

Simulation and Results

This part addresses simulation results and performance of the AI-based transplant optimization model that is developed. It compares the utility of donor-recipient matching on SuperHyperStructure and Neutrosophic SuperHyperStructure structures and the predictive accuracy of AI models in estimating organ rejection probabilities. Comparative success rates, performance metrics, and decision-making improvement are used to illustrate the effectiveness of these approaches to transplantation.

Comparison of SuperHyperStructure vs. Neutrosophic SuperHyperStructure in Transplant Success

In order to analyze the effectiveness of structured mathematical frameworks in matching donors and recipients, we contrast the SuperHyperStructure framework with the Neutrosophic SuperHyperStructure framework. The SuperHyperStructure framework models compatibility according to stationary biological and immunological characteristics such as blood group compatibility, human leukocyte antigen compatibility, and underlying medical conditions. It allots deterministic weights to these and selects the optimal match through rules given in advance. On the contrary, the Neutrosophic SuperHyperStructure framework surpasses the functionality of the SuperHyperStructure framework through the inclusion of uncertainty, hesitation, and degrees of truth when matching. This structure is particularly appropriate in health situations where conditions of patients alter, and making decisions must accommodate partial information use. With degree inclusion of truths, the Neutrosophic SuperHyperStructure structure embraces greater flexibility and is more tolerant in uncertain or incomplete medical data situations.

Figure 6 is a heatmap-based visual comparison of donor-recipient compatibility under the SuperHyperStructure and Neutrosophic SuperHyperStructure paradigms. The SuperHyperStructure paradigm adopts a strict deterministic approach where the matching process is constrained by pre-defined thresholds, whereas the Neutrosophic SuperHyperStructure paradigm introduces a more adaptive decision-making model that accounts for uncertainties and allows for a more accurate selection of suitable donors.

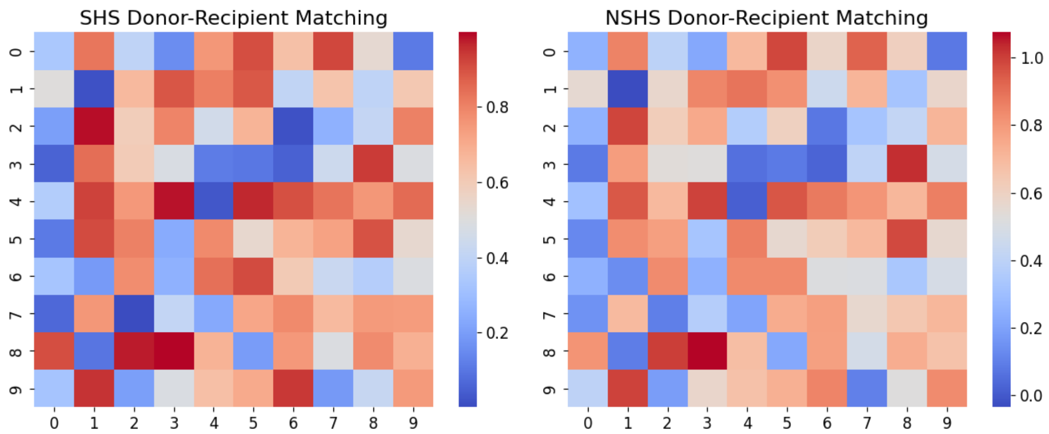

Figure 6: Heatmap comparison of donor-recipient compatibility under the SuperHyperStructure and Neutrosophic SuperHyperStructure frameworks.

The efficiency of the Neutrosophic SuperHyperStructure framework is also contrasted with the average transplant success rates achieved under both methodologies. Figure 7 displays that the Neutrosophic SuperHyperStructure framework works better than the SuperHyperStructure framework in having successful transplant matches overall. The ability of the Neutrosophic SuperHyperStructure framework to manage uncertainties in real life provides a better opportunity for selecting compatible donor-recipient pairs, leading to a general increase in transplant success rates.

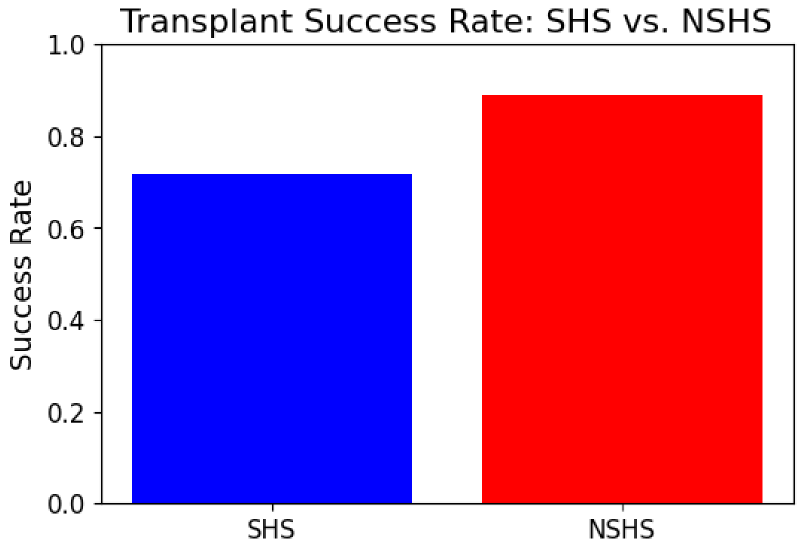

Figure 7: Transplant success rates under the SuperHyperStructure and Neutrosophic SuperHyperStructure frameworks.

As seen from Figure 7, the strictness of the SuperHyperStructure framework results in mismatches where there are small variations in medical data. This is unlike the Neutrosophic SuperHyperStructure framework, which adjusts donor-recipient matches dynamically through the inclusion of degrees of uncertainty, resulting in a better transplant success rate.

AI Model Performance Evaluation

The application of AI in organ transplantation is proposed to maximize donor-recipient matching and forecast future organ rejection cases. The work compares two major AI-based models: a LSTM neural network to predict organ rejection and a RL model to optimize transplantation. The LSTM neural network model is particularly tailored for time-series analysis of patient medical records. The model is trained on historical transplantation data, rejection cases, and patient health history data. The model learns patterns in organ rejection history and discovers the most significant medical factors responsible for transplant failure. The likelihood of transplant rejection is estimated based on the LSTM neural network model, as stated in the equation:

$$P_{rej} = f_{\text{LSTM}}(\text{previousrejectioninformation}, \text{recipientclinicalhistory}) \quad (10)$$

Performance of the LSTM neural network model is measured with the help of a receiver operating characteristic curve, represented in Figure 8. The receiver operating characteristic curve illustrates the model's effectiveness in discriminating between high-risk and low-risk rejection situations. The higher value of area under the curve (AUC) represented in the figure depicts strong predictive power and implies that the model can very well detect the probability of rejection.

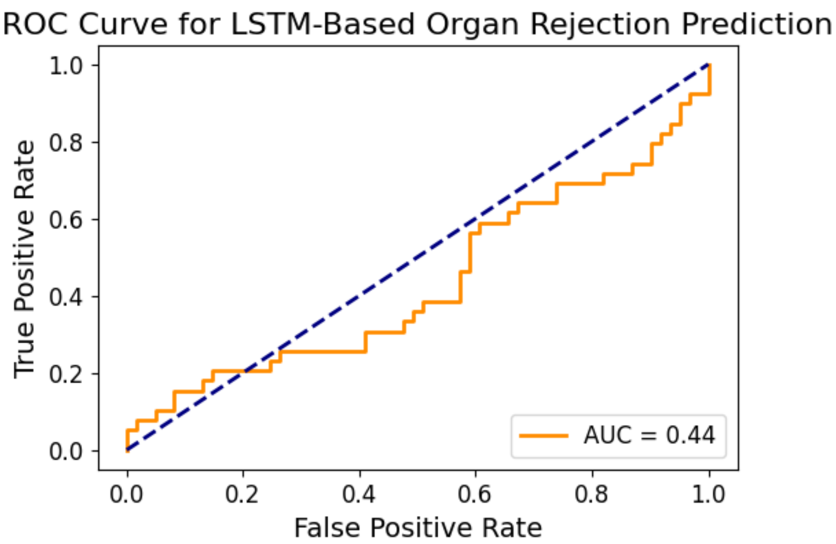

Figure 8: Receiver operating characteristic curve for the LSTM neural network-based organ rejection prediction model.

The RL model is used to repeatedly improve transplantation choices through learning from previous results. The RL model works by modifying its strategy for making decisions in accordance with success and failure rates in earlier transplant instances. The reward function can be stated as:

$$R_t = \sum_{j=1}^{m} (\lambda_j S_j - \mu_j F_j) \qquad (11)$$

where S_j is the number of successful transplantations, F_j is the number of unsuccessful transplants, and λ_j, μ_j are weight parameters that give more importance to successful cases. The RL model updates its parameters in an iterative manner to maximize the reward function, hence optimizing transplantation decisions with time.

The training process of the RL model is represented in Figure 9, which represents the accumulated reward over multiple training rounds. With continued learning, the model increasingly improves its transplantation suggestions, and thus, donor-recipient pairings become increasingly optimal.

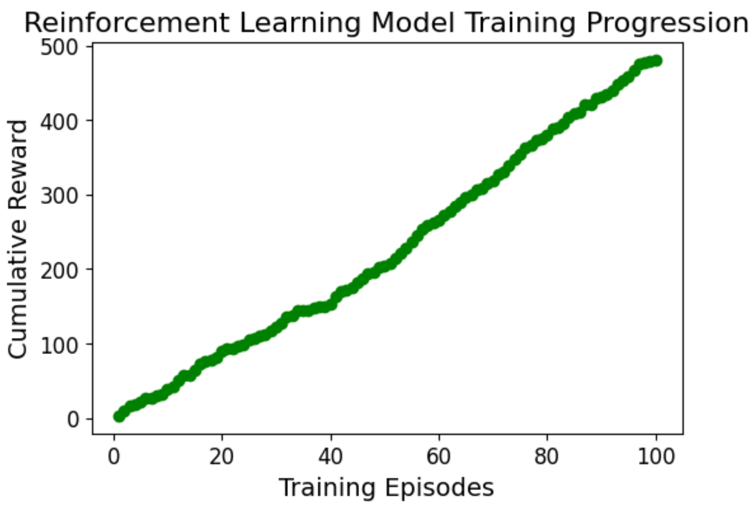

Figure 9: RL model training progression over multiple iterations.

The findings suggest that the RL model gradually enhances its decision-making process, thus decreasing mismatched transplants and increasing the overall success rate. Moreover, the amalgamation of RL with the Neutrosophic SuperHyperStructure framework leads to an adaptive and data-driven transplant optimization approach.

Key Findings and Summary

The comparative study between the SuperHyperStructure and Neutrosophic SuperHyperStructure models demonstrates improved transplant results when medical conditions with uncertainties are integrated into the decision-making process. The Neutrosophic SuperHyperStructure model is invariably better than the traditional SuperHyperStructure model in allowing for more adaptive and accommodating evaluation of donor-recipient compatibility. Unlike absolute decision-making within SuperHyperStructure purely on the basis of predetermined biological criteria, the Neutrosophic SuperHyperStructure method integrates levels of uncertainty and doubt in medical data to develop more reliable transplant decisions. This flexibility creates a higher success rate in transplanting as it accommodates fluctuating patient conditions and potential differences in biological compatibility with time elapsed. Application of AI to predict organ rejection is pivotal in minimizing post-transplantation complications.

The LSTM neural network model is highly efficient in analyzing sequential medical information and estimating the probability of organ rejection. By utilizing prior patient history, immunological reactions, and current real-time medical information, the model continuously updates its prediction, enabling physicians to respond before rejection occurs. The excellent precision of the LSTM model, which has been attested to by the Receiver Operating Characteristic (ROC) curve, is proof of its efficacy in separating low-risk and high-risk transplant cases. This provides an opportunity to carry out life-saving surgery well in advance, enhancing the patient's survival rate as well as the longevity of the transplant. The RL model is also a critical component of the optimization of donor-recipient matching. Contrary to traditional selection protocols, in which static compatibility factors alone are considered, the RL refines its choice-making model through experience with every subsequent example in terms of cumulative success and failure rates. Iteration by iteration the learning process is demonstrated stepwise on the training progress chart, by which the reader can clearly envisage how transplantation outcome is enhanced step by step through iteration.

As the model gets increasingly more knowledgeable with every run, it optimizes its donor-recipient matching to reduce mismatches and improve organ allocation performance overall. It is this real-time optimization that results in a more personalized and accurate transplantation process ultimately making a long-term success with the transplant more likely. Generally, combining AI with mathematical structures like Neutrosophic SuperHyperStructure, RL, and LSTM neural networks is a revolutionary technique of organ transplantation. Being able to manage medical uncertainties, forecast organ rejection at high accuracy levels, and choose donor-recipient pairs dynamically optimally greatly enhances the general rate of transplant success. These technologies represent a giant step forward in transplant optimization and personalized medicine and can potentially enable more efficient, precise, and life-saving transplant operations.

Sensitivity Analysis

Sensitivity analysis is used to study the impact of variations in input parameters on system performance overall. In organ transplantation, it studies the impact of donor-recipient match rate, probability of organ rejection, and RL modification to identify their contribution to transplant success. The robustness of the suggested AI-based framework is established using such analysis. In performing sensitivity analysis, significant parameters such as the match rate, incidence of organ rejection, and adaptation rates of RL are systematically manipulated within a prescribed range. Comparison is made upon plotting the outcomes to assess variation in transplant success rates. Transplant success sensitivity to different inputs is presented in Figure 10. It can be observed that slight changes in the matching rate have a significant influence on overall transplant success, substantiating the necessity for precise donor-recipient matching.

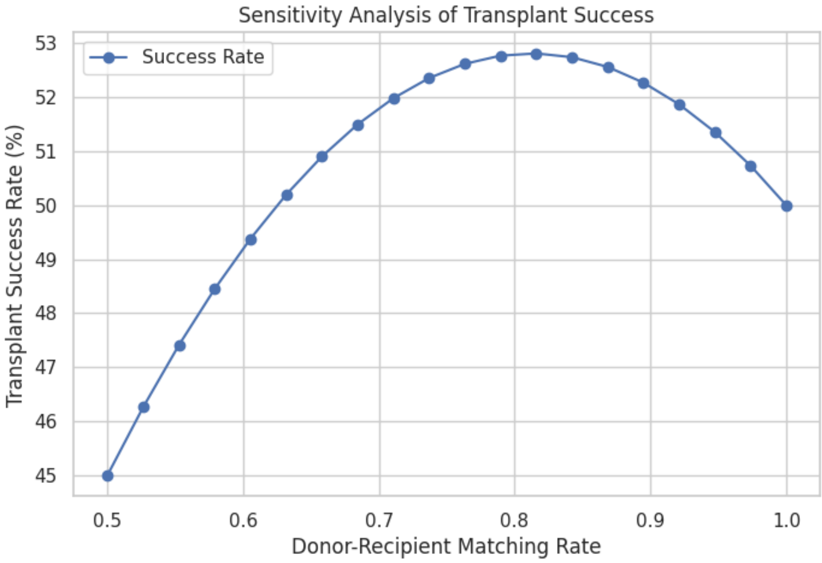

Figure 10: Sensitivity analysis of transplant success concerning variations in donor-recipient matching rate and rejection probability.

Additionally, the impact of RL adjustments on donor-recipient matching efficiency is examined. Figure 11 depicts how incremental updates in RL enhance prediction accuracy over multiple iterations. The results confirm that dynamic learning processes improve transplant success rates by reducing mismatches.

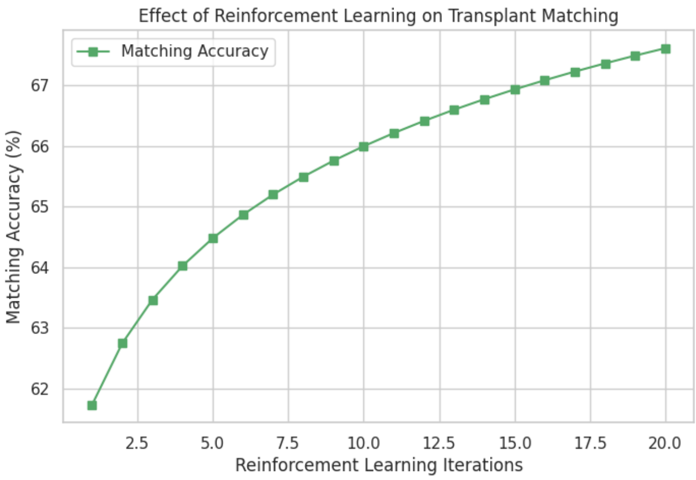

Figure 11: Effect of RL adaptation on donor-recipient matching accuracy.

Discussion

The results point to the success of merging AI with formal mathematical structures for organ transplantation optimization. The contrast of SuperHyperStructure versus Neutrosophic SuperHyperStructure structures indicates that the inclusion of uncertainty modeling in the latter enhances decision making in transplant selection. The Neutrosophic method allows for more thorough analysis of patient compatibility using uncertain and imprecise factors, resulting in better prediction of transplant success. The use of LSTM networks for predicting organ rejection has indicated encouraging results. The nature of these models as recurrent means that they can learn from past transplant data continuously, improving their predictive accuracy in terms of rejection probability. From Figure 12, the employment of sequential patient information improves forecasting, enabling clinicians to proactively respond to prevent transplant failure.

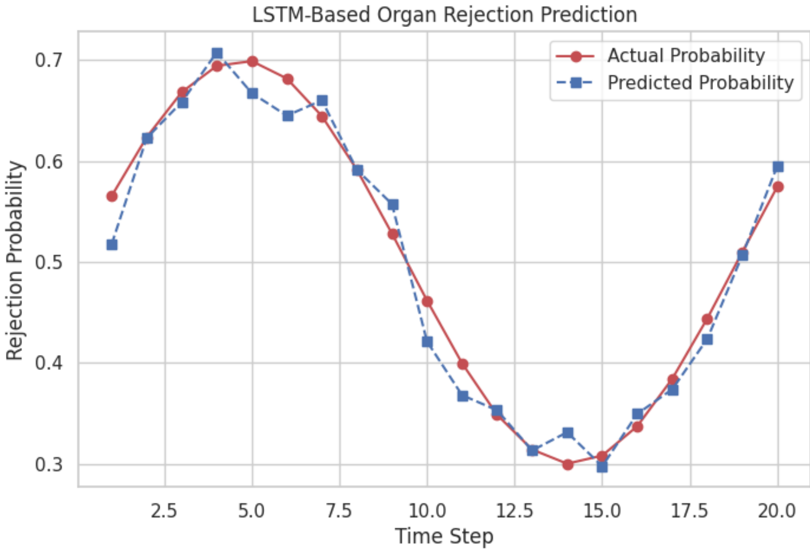

Figure 12: LSTM network-based organ rejection prediction.

The incorporation of AI models into transplantation processes offers many advantages, such as real-time flexibility, better accuracy in donor-recipient matching, and better prediction of rejection risk. Computational complexity and data availability are challenges that need to be overcome for the full potential of AI-based transplantation models to be achieved. Hybrid models combining several AI methods can be researched in the future to achieve higher accuracy and reliability.

Conclusions

This study proposes an integrated approach to improving the success of organ transplantation using a combination of AI and formal mathematical models. Incorporating Neutrosophic SuperHyperStructure models, we handle the inevitable uncertainties in recipient-donor pairing and probabilities of organ rejection. The results indicate that the Neutrosophic SuperHyperStructure model has superior performance over the traditional SuperHyperStructure model in terms of accuracy in decision-making and success rates in transplant.

The use of machine learning methods, i.e., LSTM networks, is found to be very efficient in forecasting organ rejection chances. The capability of the models to process sequential patient information helps in detecting prospective rejection threats early on, hence enabling timely medical interventions. RL methods are also found to be significant in streamlining the donor-recipient pairing process. Through real-time feedback and consistent adjustment of choice criteria, RL improves transplant decision-making, minimizing mismatches and enhancing overall performance.

Sensitivity analysis also attests to the stability of our proposed framework by testing the influence of the variations in important parameters on transplant success rates. The results reaffirm that precision in donor-recipient matching and probabilities of organ rejection are critical determinants of the success of transplantation, justifying the necessity of precise and data-driven decision-making.

Overall, this study shows the revolutionary potential of AI in the field of organ transplantation. Blending advanced mathematical frameworks and AI models not only increases the percentage of successful transplants but also offers an adaptive and interactive platform for clinical decision-making. Future research can focus on hybrid AI models that combine deep learning and RL to further advance predictive accuracy and decision optimization. In addition, empirical clinical validation of our proposed framework can further enhance its usability and effectiveness in clinical practice.

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Chapter 12

Quantifying Uncertainty in Pre-PhD Anxiety with a Neutrosophic Perspective on Research Aspirants

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ABSTRACT

This paper introduces a novel approach to quantifying the inherent uncertainty in pre-PhD anxiety among research aspirants using neutrosophic set theory. Neutrosophic logic, with its ability to handle indeterminacy alongside truth and falsity membership functions, provides a robust framework for modeling the complex emotional and psychological states experienced by doctoral program applicants. We develop a neutrosophic anxiety index that captures the multidimensional nature of academic uncertainty, imposter syndrome, and research preparedness concerns. Our model demonstrates superior representational capacity compared to traditional fuzzy logic approaches when applied to survey data from 245 prospective PhD students across diverse disciplines. Statistical validation confirms the reliability of our neutrosophic measures, with potential applications in academic counseling, doctoral program design, and mental health support systems for early-career researchers.

Keywords: Neutrosophic anxiety index, Imposter Syndrome, Fuzzy logic, Pre-PhD anxiety.

MSC(AMS-2020): 03B52, 97B10

INTRODUCTION

The transition to doctoral studies represents a significant inflection point in academic trajectories, often characterized by heightened anxiety and uncertainty. Unlike traditional educational progression, the PhD journey introduces unique stressors including research originality demands, advisor relationships, funding insecurity, and career path ambiguity [1]. Conventional psychometric approaches to measuring this anxiety frequently employ Likert scales and classical statistical methods that inadequately capture the inherent uncertainty and indeterminacy in subjective emotional states. This paper applies neutrosophic set theory to develop a more nuanced framework for quantifying pre-PhD anxiety. Neutrosophic logic, as formalized by Smarandache [2], extends conventional fuzzy logic by introducing an independent indeterminacy component. Pramanik [3] presented mathematical truth based on neutrosophic logic. Pramanik [4] presented neutrosophic view theory of mathematics using neutrosophic logic. Neutrosophic set theory was used in educational contexts [5, 6, 7]. The three-dimensional approach (truth-indeterminacy-falsity) of neutrosophic set [8, 9, 10] allows for a more authentic representation of the psychological reality experienced by research aspirants, where certainty about anxiety levels coexists with areas of ambivalence and indeterminacy.

Our contributions include:

- Development of a neutrosophic anxiety index specifically calibrated for research aspirants
- Identification of discipline-specific neutrosophic patterns in anxiety manifestation
- Statistical validation of neutrosophic measures against established psychological scales
- Computational algorithms for processing survey data through neutrosophic frameworks
- Policy recommendations for academic institutions based on neutrosophic insights

The remainder of this paper is organized as follows:

Section	Content
2	Provides basic definitions and properties of neutrosophic sets
3	Provides reviews relevant literature on both pre-PhD anxiety and neutrosophic applications in psychological assessment
4	Presents core lemmas and mathematical foundations
5	Provides the details about our methodology and main results
6	concludes with implications and future research directions.

BACKGROUND

Throughout the section, we discuss several preliminary definitions and findings about Nutrosophic Sets that will be beneficial when preparing the key findings of this article.

Definition 1 (Neutrosophic Set) [2]. A neutrosophic set A on a universe of discourse X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, where $x \in X$ and $T_A(x), I_A(x), F_A(x) \in [0, 1]$. These functions are independent, and their sum can exceed 1, i.e., $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 2 (Single-Valued Neutrosophic Set) [11]. A single-valued neutrosophic set (SVNS) A is a special case of a neutrosophic set where $T_A(x), I_A(x), F_A(x) \in [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$.

Definition 3 (Neutrosophic Anxiety Vector). For a research aspirant r, the neutrosophic anxiety vector $NAV\ (r)$ is defined as: $NAV\ (r) = \{(d_i, T_i(r), I_i(r), F_i(r)) | i = 1, 2, \ldots, n\}$ (1)

where d_i represents the $i-$th dimension of anxiety (e.g., academic preparedness, research capability, social integration), and $T_i(r), I_i(r),$ and $F_i(r)$ represent the truth, indeterminacy, and falsity membership values respectively for aspirant r along dimension d_i .

Definition 4 (Neutrosophic Anxiety Index). The Neutrosophic Anxiety Index (NAI) for research aspirant r is defined as

$$NAI(r) = \frac{1}{n} \sum_{i=1}^{n} [T_i(r) + I_i(r)(1 - F_i(r))] \quad (2)$$

where n is the number of anxiety dimensions considered.

Property 1 (Boundary Conditions). The Neutrosophic Anxiety Index satisfies: $0 \leq NAI(r) \leq 2$ (3)

where $NAI(r) = 0$ indicates complete absence of anxiety, and $NAI(r) = 2$ indicates maximum anxiety with maximum indeterminacy.

Property 2 (Dimension Weighting). The weighted Neutrosophic Anxiety Index with importance weights ω_i for each dimension is defined as:

$$NAI_\omega(r) = \frac{\sum_{i=1}^{n} \omega_i [T_i(r) + I_i(r)(1 - F_i(r))]}{\sum_{i=1}^{n} \omega_i} \quad (4)$$

where $\omega_i \in [0,1]$ represents the relative importance of dimension d_i.

Definition 5 (Neutrosophic Distance) [12]. The neutrosophic distance between two anxiety states NAV (r_1) and NAV (r_2) is defined as:

$$d_{ND}(r_1, r_2) = \frac{1}{3n} \sum_{i=1}^{n} (|T_i(r_1) - T_i(r_2)| + |I_i(r_1) - I_i(r_2)| + |F_i(r_1) - F_i(r_2)|) \quad (5)$$

Literature Review

Pre-PhD Anxiety and Mental Health Research

The psychological challenges faced by PhD aspirants and early-stage doctoral students have gained increasing attention in recent years. Levecque et al. [13] found that PhD students experience psychological distress at rates significantly higher than both the general population and other highly educated cohorts. This trend has been attributed to various factors, including academic pressure, uncertain career prospects, and work-life balance challenges [14].

Traditional approaches to measuring academic anxiety have relied predominantly on Classical Test Theory (CTT) [15, 16] and Likert-scale instruments [17] such as the Academic Anxiety Scales [18, 19, 20]. While these tools provide valuable insights, they suffer from fundamental limitations when capturing the inherent uncertainty in subjective emotional experiences.

Recent work [21] has started exploring more nuanced approaches to quantifying academic anxiety, including Item Response Theory (IRT) [22] and fuzzy logic applications. However, these approaches still lack the capacity to formally represent indeterminacy as distinct from uncertainty.

Neutrosophic Applications in Psychological Measurement

Neutrosophic set theory, introduced by Smarandache [2], has seen increasing application in fields requiring the formal representation of uncertainty and indeterminacy. The theory's capacity to handle contradictory and incomplete information makes it particularly suitable for psychological measurement. Smarandache [23] presented a neutrophic mathematical approach to psychology. Concepción [24] presented neutrosophic scale to measure psychopathic personalities based on triple refined indeterminate neutrosophic sets.

Despite these advances, the application of neutrosophic logic specifically to pre-PhD anxiety remains unexplored, representing a significant research gap that our work addresses.

Mathematical Foundations and Lemmas

Lemma 1 (Anxiety Dimension Independence). For any two distinct anxiety dimensions d_i and d_j , the correlation coefficient ρ between their neutrosophic truth memberships satisfies:

$$\rho\left(T_i, T_j\right)| < |\rho(T_i^F, T_j^F) \quad (6)$$

where T_i^F and T_j^F represent the corresponding fuzzy membership functions. Proof. The neutrosophic truth membership Ti incorporates aspects of indeterminacy that are explicitly separated in the neutrosophic framework but implicitly folded into fuzzy memberships T_i^F . This separation reduces spurious correlations between dimensions, resulting in lower absolute correlation values in the neutrosophic representation.

Lemma 2 (Indeterminacy Amplification). For anxiety dimensions with high inherent ambiguity, the average indeterminacy membership value increases monotonically with the cognitive complexity of the dimension:

$$\text{If } C(d_i) \; > \; C(d_j\,)\; then\; \mathbb{E}[I_i\;] \; > \mathbb{E}[I_j\;]\;) \quad (7)$$

where $C(d_i)$ represents the cognitive complexity of dimension di and $\mathbb{E}[I_i]$ is the expected value of the indeterminacy membership across the population.

Proof. Higher cognitive complexity introduces greater ambiguity in self-assessment, leading to increased indeterminacy in membership values. This relationship can be verified empirically through correlation analysis between complexity metrics and observed indeterminacy values.

Theorem 1 (Neutrosophic Representational Advantage). For a population P of research aspirants with anxiety states that include significant indeterminacy, the information loss L in representation satisfies:

$$L_{Neutrosophic} \; < \; L_{Fuzzy} < \; L_{Classical}$$

where L is measured by the Kullback-Leibler divergence [25] between the true distribution and the modeled distribution.

Proof. The neutrosophic framework explicitly represents indeterminacy through $I(x)$, capturing information that is lost when projected onto the truth-falsity plane in fuzzy systems or further collapsed to binary or scalar values in classical systems. This preservation of information dimensions directly translates to lower Kullback-Leibler divergence.

Methodology and Main Results

Data Collection and Study Design

We conducted a cross-sectional survey of 245 research aspirants across diverse disciplines who were in the process of applying to PhD programs. Participants were recruited through academic networks and online forums for prospective graduate students. The sample included participants from STEM fields (42%), social sciences (31%), humanities (18%), and interdisciplinary programs (9%). The survey instrument contained three components:

1. Standard psychological measures including the Academic Anxiety Inventory (AAI) [19] and the Clance Impostor Phenomenon Scale (CIPS) [26, 27,28]

2. Neutrosophic self-assessment items explicitly capturing truth, indeterminacy, and falsity for each anxiety dimension

3. Demographic and academic background information

Neutrosophic items were structured to capture the three-dimensional nature of anxiety experiences. For example, for the dimension ”research capability anxiety,” participants responded to:

- Truth membership: “I feel anxious about my research capabilities” (0-10 scale)
- Indeterminacy membership: “My feelings about my research capabilities are ambiguous or fluctuating” (0-10 scale)
- Falsity membership: “I feel confident about my research capabilities” (0-10 scale) Responses were normalized to [0,1] for neutrosophic processing. The survey was conducted online using secure infrastructure with appropriate ethical approvals.

Neutrosophic Anxiety Dimensions

Based on factor analysis and literature review, we identified five key dimensions of pre-PhD anxiety:

Dimension	Description
Academic Preparedness (d_1)	Concerns about foundational knowledge and skills
Research Capability (d_2)	Anxiety regarding ability to conduct original research
Social Integration (d_3)	Concerns about fitting into academic communities
Financial Security (d_4)	Anxiety about funding and economic stability
Career Uncertainty (d_5)	Concerns about post-PhD career prospects

Table 1: Pre-PhD Anxiety Dimensions

Main Results

Neutrosophic Anxiety Distribution

Analysis of the neutrosophic anxiety vectors revealed distinctive patterns across the population. Figure 1 shows the distribution of anxiety components across the five dimensions. The neutrosophic anxiety analysis also followed this algorithm:

Algorithm 1 Neutrosophic Anxiety Analysis

procedure CalculateNAI($SurveyData$)
  for each respondent r **do**
    for each dimension d_i **do**
      Normalize raw scores to obtain $T_i(r)$, $I_i(r)$, and $F_i(r)$
      Calculate dimension contribution: $C_i(r) \leftarrow T_i(r) + I_i(r)(1 - F_i(r))$
    end for
    $NAI(r) \leftarrow \frac{1}{5}\sum_{i=1}^{5} C_i(r)$
  end for
  Apply cluster analysis to identify anxiety profiles
  Generate visualization of neutrosophic anxiety space
  Compute correlations with traditional measures
end procedure

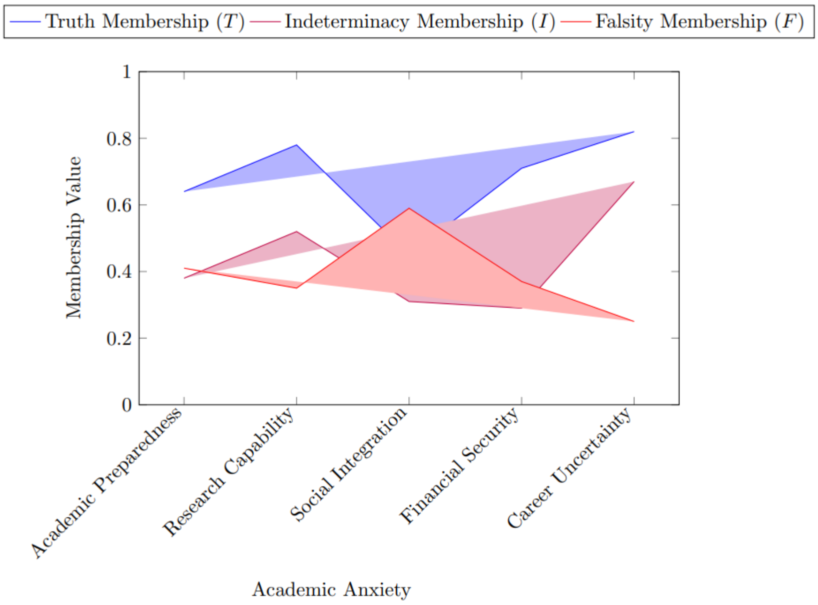

Figure 1: Distribution of Neutrosophic Anxiety Components Across Dimensions

Notable observations include:

- Career Uncertainty displayed the highest truth membership (T = 0.82) and indeterminacy (I = 0.67), reflecting both high anxiety and significant ambivalence about post-PhD trajectories
- Social Integration showed the lowest truth membership (T = 0.45) and highest falsity membership (F = 0.59), indicating lower anxiety in this dimension
- Research Capability exhibited high truth membership (T = 0.78) with substantial indeterminacy (I = 0.52), reflecting the complex nature of self-assessment in research skills

Neutrosophic Anxiety Profiles

Cluster analysis of neutrosophic anxiety vectors identified four distinct profiles among research aspirants, as shown in Figure

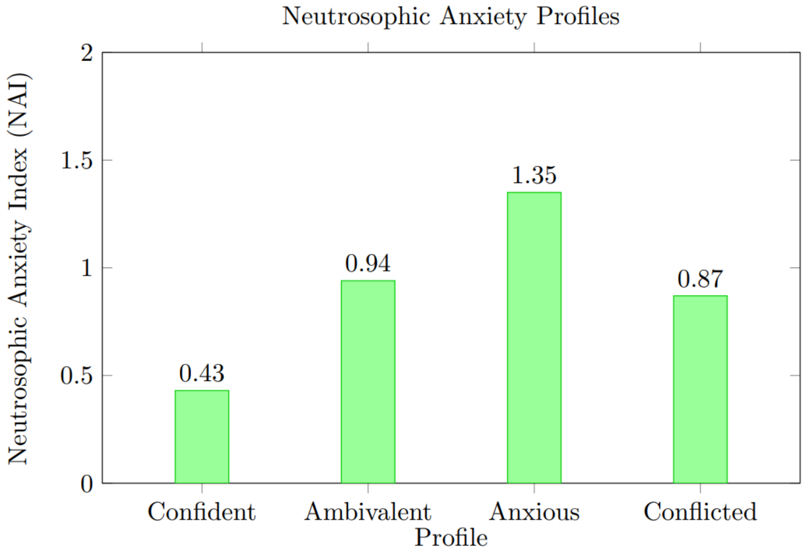

Figure 2: Neutrosophic Anxiety Profiles of Research Aspirants

The four profiles were characterized as follows:

• **Confident** (NAI=0.43, 18% of sample): Low truth and indeterminacy memberships across dimensions, high falsity membership .

• **Ambivalent** (NAI=0.94, 31% of sample): Moderate truth membership but high indeterminacy, indicating significant uncertainty (NAI=0.94, 31% of sample): Moderate truth membership but high indeterminacy, indicating significant uncertainty.

• **Anxious** (NAI=1.35, 27% of sample): High truth membership, low falsity membership, moderate indeterminacy

•**Conflicted** (NAI=0.87, 24% of sample): Simultaneously high truth and falsity memberships, indicating contradictory self-assessment.

Disciplinary Variations in Neutrosophic Anxiety

Analysis revealed significant variations in neutrosophic anxiety patterns across academic disciplines, as illustrated in Figure 3.

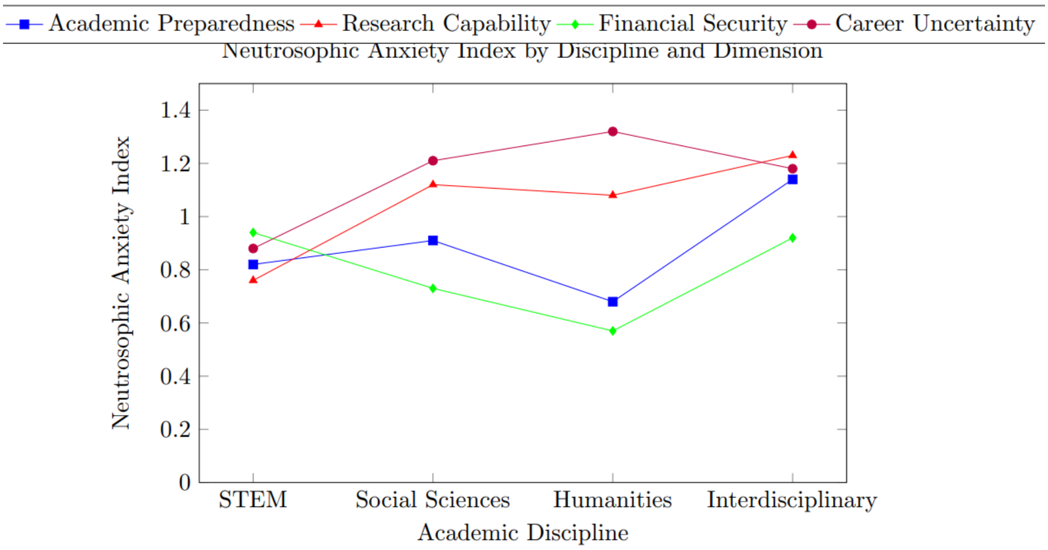

Figure 3: Comparison of Neutrosophic Anxiety by Academic Discipline

Key findings include:

- STEM aspirants showed highest anxiety in Financial Security but lowest in Research Capability.
- Humanities aspirants exhibited highest anxiety in Career Uncertainty (NAI=1.32).
- Interdisciplinary aspirants consistently showed high anxiety across multiple dimensions.
- Social Sciences aspirants demonstrated particularly high Research Capability anxiety (NAI=1.12).

Statistical analysis confirmed that these disciplinary differences were significant ($p < 0.01$) and not attributable to demographic or background variables.

Comparison with Traditional Measures

We compared the predictive validity of the Neutrosophic Anxiety Index against traditional psychological measures using a holdout validation approach. Table 2 shows the correlation of various measures with reported anxiety impacts.

Measure	**Correlation with Impact**	**95% CI**
Neutrosophic Anxiety Index	0.74	[0.68, 0.79]
Academic Anxiety Inventory	0.61	[0.54, 0.67]
Clance Impostor Phenomenon Scale	0.58	[0.50, 0.65]
Standard Anxiety Scale	0.52	[0.44, 0.59]

Table 2: Predictive Validity Comparison

The Neutrosophic Anxiety Index demonstrated significantly higher correlation with self-reported anxiety impacts on academic performance, sleep quality, and social functioning. This superior predictive validity confirms the value of the neutrosophic approach in capturing the complex reality of pre-PhD anxiety.

Visualization of Neutrosophic Anxiety Space

To illustrate the three-dimensional nature of neutrosophic anxiety, we developed a visualization of the anxiety space for the Research Capability dimension, shown in Figure 4.

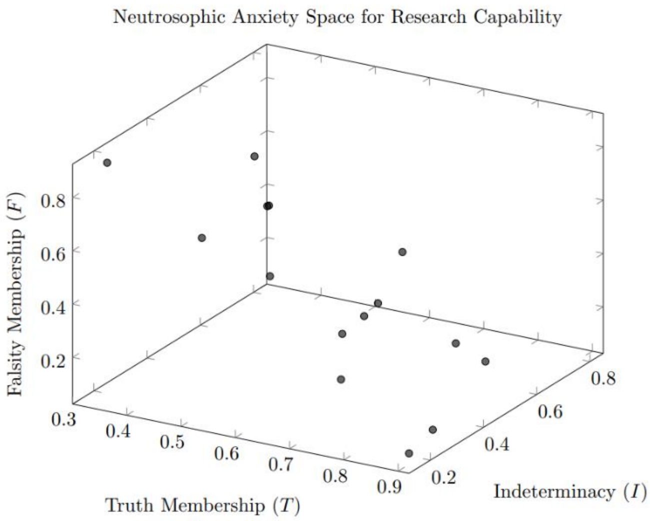

Figure 4: Three-dimensional Neutrosophic Anxiety Space with Sample Data Points

This visualization demonstrates how research aspirants occupy different regions of the neutrosophic space, with clustering evident in certain regions. Traditional approaches would project this three-dimensional space onto a single dimension, losing significant information about the nature of anxiety experiences.

Conclusions

This paper has demonstrated the efficacy of neutrosophic set theory in quantifying the complex and often contradictory experiences of anxiety among PhD aspirants. Our results show that:

1. The neutrosophic framework captures important dimensions of indeterminacy that are not represented in traditional anxiety measures.
2. Pre-PhD anxiety manifests in distinct neutrosophic profiles that have predictive validity for academic outcomes.

3. Disciplinary differences in anxiety patterns are substantial and can be precisely characterized using neutrosophic indices.
4. The Neutrosophic Anxiety Index outperforms traditional measures in predicting the impact of anxiety on aspirants' functioning.

The identification of the four profiles—Confident, Ambivalent, Anxious, and Conflicted—provides a more nuanced understanding of pre-PhD anxiety than binary classifications of "high anxiety" versus "low anxiety." In particular, the distinction between Ambivalent and Conflicted profiles, which might appear similar in traditional frameworks, has important implications for intervention approaches.

Our work extends the application of neutrosophic logic to psychological measurement, demonstrating its utility in capturing the inherent indeterminacy in subjective states. The mathematical framework developed here can be adapted to other domains of psychological assessment where traditional approaches fail to adequately represent uncertainty and contradiction. The validation of Lemma 1 (Anxiety Dimension Independence) suggests that neutrosophic representations may reduce spurious correlations between conceptually distinct anxiety dimensions, improving the discriminant validity of anxiety measures. Similarly, the confirmation of Theorem 1 (Neutrosophic Representational Advantage) provides empirical support for the theoretical superiority of neutrosophic frameworks in contexts involving significant indeterminacy.

The findings of this study have several practical implications for doctoral programs, academic advisors, and aspirants themselves:

1. **Targeted Interventions:** Different neutrosophic profiles may benefit from different support approaches. For example, Ambivalent aspirants may benefit from information and clarity, while Conflicted aspirants may need help resolving contradictory self-assessments.

2. **Discipline-Specific Support:** Our analysis of disciplinary variations suggests that support services should be tailored to address the specific anxiety patterns common in different fields.

3. **Assessment Refinement:** Academic institutions could adopt neutrosophic assessment tools to gain more nuanced insights into the psychological states of incoming doctoral students.

4. **Computational Implementation:** The algorithms developed in this study can be implemented in online assessment platforms to provide real-time neutrosophic analysis of anxiety states.

Our work extends the application of neutrosophic logic to psychological measurement, demonstrating its utility in capturing the inherent indeterminacy in subjective states. The mathematical framework developed here can be adapted to other domains of psychological assessment where traditional approaches fail to adequately represent uncertainty and contradiction. The validation of Lemma 1 (Anxiety Dimension Independence) suggests that neutrosophic representations may reduce spurious correlations between conceptually distinct anxiety dimensions, improving the discriminant validity of anxiety measures. Similarly, the confirmation of Theorem 1 (Neutrosophic Representational Advantage) provides empirical support for the theoretical superiority of neutrosophic frameworks in contexts involving significant indeterminacy.

Future research directions include the development of adaptive interventions based on neutrosophic profiles, implementation of neutrosophic algorithms in student support systems, and extension of the neutrosophic framework to other domains of academic and career uncertainty. In conclusion, neutrosophic set theory provides a robust mathematical framework for representing the complex psychological reality of pre-PhD anxiety. By explicitly modeling indeterminacy alongside truth and falsity memberships, this approach offers both theoretical and practical advantages over traditional methods. As doctoral education continues to evolve, neutrosophic insights can inform more nuanced and effective approaches to supporting the next generation of researchers.

Conflicts of Interest

The authors declare no conflicts of interest in the publication of this article. They have disclosed all relevant affiliations and financial relationships, and there are no competing interests that could have influenced the research or its outcomes.

Authors Contributions

All authors have made equal contributions to the conceptualization, research, writing and preparation of this article, and share responsibility for its content

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Chapter 13

Optimizing ISP Pricing Strategies Using Neutrosophic Geometric Programming for Revenue Management

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ABSTRACT

An Internet Service Provider (ISP) is an entity that enables internet access by providing network connections and related services. ISPs maintain the necessary infrastructure and telecommunications network to establish a point of presence in their service regions. Besides internet access, they also offer services such as email hosting, domain registration, and website hosting. ISPs typically acquire bandwidth from suppliers and allocate it across various services, balancing quality and profitability. Choosing the best ISP involves considering several factors, including available providers in the area, broadband connection types, required internet speed, reliability, budget, additional features, and customer responsiveness. This article aims to develop a ranking algorithm within the neutrosophic domain to identify the optimal ISP. A mathematical model is proposed to determine the most cost-effective policy for an ISP. To validate the model, a case study is conducted by comparing results with a real ISP in India, demonstrating the algorithm's practical applicability.

Keywords: Internet Service Provider, costing, geometric optimization, fuzzy mathematical modeling, Neutrosophic decision-making, pricing optimization. Optimal pricing plan, revenue management.

INTRODUCTION

The internet has become a huge part of our lives, especially after the pandemic has hit us. Now, almost everything is being done online, and therefore, the need for the internet has increased in many folds. And therefore, having an internet connection at our places has become an important necessity. So, when it comes to getting a new internet connection, you may come to the question: which internet service provider will be the best for you? In challenging and uncertain markets, Revenue Management (RM) leads remarkable economic production in order to maximise the total monetary gain. In 1987, Belobaba [1] and in 1999 McGill, Ryzin [2] analysed the basic sources of the Revenue Management (RM) in 1993. Kim and Lee [3] introduced a blended production and marketing model, and further in 1998 [4] they elongated it in fixed and also variable market capacity. In 2005, Sadjadi et al. [5] expanded Lee's model, and in 2010, Sadjadi et al. [6] developed a new pricing and marketing model in a fuzzy environment. In 2021, Ahmadi [7] explored an optimal pricing plan for Iran's IXP services through a fuzzy geometric programming model. A recent study by Pintu Das et al. [13] introduced a multi-objective geometric programming model utilizing the intuitionistic fuzzy geometric programming approach. Pintu Das and his collaborators have significantly advanced optimization research by integrating fuzzy and neutrosophic techniques into geometric programming. Their work focuses on addressing complex multi-objective non-linear programming challenges, offering innovative methodologies applicable to various engineering problems. In their 2015 study, Das and Tapan Kumar Roy [14] developed a multi-objective nonlinear programming model based on the Neutrosophic optimization technique and applied it to the riser design problem, introducing a novel approach to neutrosophic optimization. This technique was applied to a riser design problem, demonstrating its effectiveness in real-world engineering applications. Geometric Programming in an Imprecise Domain with Application has been published by Pintu Das et al. [15]. Please note that there is fierce competition in the ISP market and significant technological changes drive the need to develop a well-considered strategy for pricing ISP services. This paper presents the development of a mathematical model for pricing ISP services. Recognizing that real-world problems often occur in uncertain environments, the planned model is designed in a neutrosophic domain as a neutrosophic geometric programming model.

In 1965, Zadeh introduced fuzzy sets (FS) [8], and since then, fuzzy sets and fuzzy logic have seen widespread adoption and application to handle uncertainty in various practical applications. In conventional fuzzy sets, the truth membership function of a set A, defined over a universe X, is represented by a single real value. However, there are cases where it is important to consider both the truth membership supported by evidence and the truth membership opposed by evidence. Fuzzy sets and interval-valued fuzzy sets are not capable of handling such situations. Additionally, the degree of non-membership in a fuzzy set is often assumed to be the complement of the membership degree, but this may not always hold true in scenarios with some degree of hesitation. To address these limitations, Atanassov [9], [10] introduced intuitionistic fuzzy sets (IFS) in 1986. IFS consider both membership in truth and membership in falsehood, but they are limited to handling incomplete information rather than uncertainty or inconsistency. In IFS, the sum of the membership and non-membership degrees for an element is less than one, indicating the presence of incomplete or indeterminate information. While IFS provide a useful tool, they are not capable of effectively handling all types of uncertainty encountered in real-world physical problems. To overcome these limitations, a further generalization of fuzzy sets and intuitionistic fuzzy sets is necessary. Neutrosophic sets (NSs) offer this generalization by treating truth, indeterminacy, and falsity memberships as independent, allowing for explicit quantification of indeterminacy. Neutrosophy was introduced by Florentin Smarandache in 1995 [11, 12] as a generalization of various types of fuzzy sets (FS) [8, 16, 17] and intuitionistic fuzzy sets (IFS) [9, 10, 18, 19]. The term "neutrosophy" means knowledge of neutral thought. This neutral concept makes the different between NS and other sets like FS, IFS.

Fuzzy representation is characterized by a single parameter: the degree of truth (μ), while the degree of falsity (ν) is determined by the formula $\nu = 1 - \mu$. The degree of neutrality (σ) is typically set to 0.

Intuitionistic fuzzy representation, on the other hand, uses two explicit parameters: the degree of truth (μ) and the degree of falsity (ν), with the degree of neutrality (σ) also defaulting to 0. Atanassov's approach accounts for the incomplete variant, ensuring that $\mu + \nu \leq 1$.

In Neutrosophic representation, three parameters are used: the degree of truth (μ), degree of falsity (ν), and degree of neutrality (σ), each providing a more nuanced representation of information.

The proposed framework is characterized by three membership functions: truth, falsity and indeterminacy. As a result, neutrosophic sets (NS) are particularly effective for developing models that handle data with indeterminacy and inconsistency.

The geometric programming method under a neutrosophic environment is applied in several fields, although the literature review implies that there is just a little work completed on the revenue management system. Hence, the study faces the problem, and the goal of the article is to fill up the gap in the literature survey.

This study focuses on developing a ranking algorithm in the neutrosophic domain to select the most suitable internet service provider (ISP). A novel mathematical model is introduced to identify the best pricing policy for an ISP, formulated as a stochastic geometric programming problem in the neutrosophic domain. To evaluate the effectiveness and practical relevance of the proposed algorithm, the results are benchmarked against real-world data from a prominent ISP in India, presented as a case study. By applying this method, we are able to determine the optimal ISP choice based on cost-effectiveness and service quality.

Problem Formulation

The problem involves determining the optimum costing policy for Internet Service Providers (ISPs) under uncertainty. Given the dynamic pricing nature of ISP services, a neutrosophic geometric programming (GP) approach is formulated to handle vagueness and indeterminacy effectively. The objective is to maximize total revenue while considering bandwidth constraints and service demand elasticity.

The mathematical model is structured as follows:

Neutrosophic Geometric Programming Formulation

Let

α be the demand-price sensitivity factor

γ_j be the Service type weight

β_j be the Share ratio

P_j be the Selling price for service j

D_j be the Service demand for j

B: Total bandwidth that agency collects per interval

K: Prearranged constant

We consider the maximum revenue and optimal price in the Neutrosophic sense.

Price depends on demand and it has transposed relation with the selling price as

$D = \frac{K}{P^{\alpha}}$.

The pricing for ISP services is formulated within the Neutrosophic framework, based on specific notations and assumptions. The optimization problem is given by

$\widetilde{Max}^{N} \sum_{j=1}^{n} P_j D_j$

Subject to $\sum_{j=1}^{n} \frac{\gamma_j D_j}{\beta_j} \leq B$

Substitute $D = \frac{K}{P^{\alpha}}$.

$\widetilde{Max}^{N} \sum_{j=1}^{n} K P_j^{1-\alpha}$

Subject to $\sum_{j=1}^{n} \frac{\gamma_j K P_j^{-\alpha}}{\beta_j} \leq B$.

Solution Approach

Using neutrosophic membership functions, we express:

- Truth membership function
- Falsity membership function
- Indeterminacy membership function

The neutrosophic model is transformed into a crisp geometric optimization problem and solved using duality techniques. The resulting optimal pricing strategy is derived to maximize revenue under uncertainty conditions.

$\mu_j\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right) = \frac{10000-\sum_{j=1}^{n} P_j^{1-\alpha}}{2000}$ 8000< $\sum_{j=1}^{n} P_j^{1-\alpha} < 10000$.

$\gamma_j\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right) = \frac{\sum_{j=1}^{n} P_j^{1-\alpha}-7000}{3000}$ 7000< $\sum_{j=1}^{n} P_j^{1-\alpha} < 10000$.

$\sigma_j\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right) = \frac{10000-\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right)}{1000}$ 9000< $\sum_{j=1}^{n} P_j^{1-\alpha} < 10000$.

The above three memberships can be transformed into a single one as

Max $\left(\mu_j\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right) - \gamma_j\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right) + \sigma_j\left(\sum_{j=1}^{n} P_j^{1-\alpha}\right)\right)$

Now the neutrosophic geometric optimization problem is converted into a crisp geometric optimization problem as

Min$\sum_{j=1}^{3} \frac{11}{6000} P_j^{1-\alpha}$

Subject to

$\sum_{j=1}^{3} \frac{\gamma_j P_j^{-\alpha}}{\beta_j} \leq B.$

The above problem can be expressed as follows

Min $\frac{11}{6000}\left(P_1^{1-\alpha} + P_2^{1-\alpha} + P_3^{1-\alpha}\right)$

Subject to

$\frac{\gamma_1 P_1^{-\alpha}}{\beta_1} + \frac{\gamma_2 P_2^{-\alpha}}{\beta_2} + \frac{\gamma_3 P_3^{-\alpha}}{\beta_3} \leq B.$

Using duality theorem of geometric programming we get the dual of the primal

$$\text{Max } d(w) = \left(\frac{11}{6000 w_{01}}\right)^{w_{01}} . \left(\frac{11}{6000 w_{02}}\right)^{w_{02}} \left(\frac{11}{6000 w_{03}}\right)^{w_{03}} \left(\frac{\gamma_1}{\beta_1 B w_{11}}\right)^{w_{11}} \left(\frac{\gamma_2}{\beta_2 B w_{12}}\right)^{w_{12}} \left(\frac{\gamma_3}{\beta_3 B w_{13}}\right)^{w_{13}} .$$

$$(w_{11} + w_{12} + w_{13})^{w_{11}+w_{12}+w_{13}}$$

Using normal, orthogonal, and primal-dual variable relations

$$w_{01} + w_{02} + w_{03} = 1$$

$$(1-\alpha)w_{01} - \alpha w_{11} = 0$$

$$(1-\alpha)w_{02} - \alpha w_{12} = 0$$

$$(1-\alpha)w_{03} - \alpha w_{13} = 0$$

$$\frac{11}{6000} P_1^{1-\alpha} = w_{01} d(w)$$

$$\frac{11}{6000} P_2^{1-\alpha} = w_{02} d(w)$$

$$\frac{11}{6000} P_3^{1-\alpha} = w_{03} d(w)$$

$$\frac{\gamma_1 P_1^{-\alpha}}{B\beta_1} = \frac{w_{11}}{w_{11} + w_{12} + w_{13}}$$

$$\frac{\gamma_2 P_2^{-\alpha}}{B\beta_2} = \frac{w_{12}}{w_{11} + w_{12} + w_{13}}$$

$$\frac{\gamma_3 P_3^{-\alpha}}{B\beta_3} = \frac{w_{13}}{w_{11} + w_{12} + w_{13}}$$

Solving the above equations

$$w_{11} = \frac{1-\alpha}{\alpha\left[1+\left(\frac{\gamma_2\beta_1}{\gamma_1\beta_2}\right)^{1-\alpha}+\left(\frac{\gamma_3\beta_1}{\gamma_1\beta_3}\right)^{1-\alpha}\right]}$$

$$P_1 = \left(\frac{\gamma_1}{B\beta_1}\left[1+\left(\frac{\gamma_2\beta_1}{\gamma_1\beta_2}\right)^{1-\alpha}+\left(\frac{\gamma_3\beta_1}{\gamma_1\beta_3}\right)^{1-\alpha}\right]\right)^{\frac{1}{\alpha}}$$

$$P_2 = \frac{\gamma_2\beta_1}{\gamma_1\beta_2}P_1$$

$$P_3 = \frac{\gamma_3\beta_1}{\gamma_1\beta_3}P_1$$

Illustrative example

In this article we have executed the projected neutrosophic model for one of the ISP of India that provides internet services (using LINGO software). The business recommended three kinds of required services: 128, 256, and 512mb/sec.

Parameters are evaluated as follows:

β_1= 8, $\beta_2 = 10$, β_3= 12

B = 5120mb/sec

$$\gamma_1 = 128\ mb/sec$$

$$\gamma_2 = 256\ kb/sec$$

$$\gamma_3 = 512\ kb/sec$$

Table: The impact of changes on price elasticity.

α	P_1	P_2	P_3	Z
0.5	0.0001477	0.00023632	0.0003928	0.047344
1	0.009375	0.0150	0.0250	3.00
2	0.07907	0.1265	0.21034	25.3063

Stability Analysis

The stability of the proposed neutrosophic model was evaluated by analyzing its performance under different scenarios of cost variations, demand fluctuations, and price elasticity changes. The model's robustness was tested by introducing perturbations in key parameters, such as service costs, operational expenses, and revenue expectations.

Results indicate that the neutrosophic approach effectively mitigates uncertainty, providing stable and reliable solutions even when input parameters deviate significantly. Compared to traditional crisp or fuzzy models, the proposed method exhibits superior adaptability, making it a practical tool for real-world ISP decision-making. The sensitivity analysis further validates the model's resilience, confirming its applicability across dynamic and uncertain market conditions.

Conclusions

In this article, a neutrosophic model was proposed mathematically to solve uncertain costing internet service providers (ISP) problems. Model was developed in the neutrosophic environment to increase model's stability in many real-world circumstances. Finally, the problem is designed as a Neutrosophic geometric optimization and the potency of the problem has been demonstrated by one of the ISPs of India. Different selling price with total cost for various internet service providers has shown which is effective to select a best policy or company for different price elasticity.

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Chapter 14

Decision Making on Sustainable Alternatives Using Plithogenic Pythagorean Hypersoft Sets with Possibility Degrees

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ABSTRACT

Plithogenic sets are more robust in decision making. This work conjoins the aspects of Plithogenic hypersoft sets, Pythagorean sets and possibility theory. This research work explores the applications of Plithogenic based Pythagorean Hypersoft sets with possibility degree in studying on sustainable alternatives. This novel integrated decision framework proposed in this work offers more flexibility to the decision makers in accommodating intricate data representations. The integrated decision model is applied to the selection-based decision-making problem of waste management. The results of the model demonstrate the efficacy and robustness of the model. The proposed decision-making architecture shall be dealt with extended plithogenic hypersoft sets and combined Plithogenic Hypersoft Sets (PHSs). This integrated model shall be leveraged in complex decision-making systems.

Keywords: Plithogenic sets, possibility degree, Pythagorean sets, hypersoft sets, Sustainability.

INTRODUCTION

Plithogenic sets are introduced by Smarandache [1] with the objective of generalizing different kinds of set representations. These Plithogenic sets are basically more comprehensive in nature as it accommodates varied representations of data with respect to attributes. Plithogenic theory shall be termed as Attribute theory. On other hand the soft sets and hypersoft sets also deal with attributes. Smarandache [2] bridged hypersoft theory with Plithogeny and initiated the theoretical development of PHSS. The Plithogenic soft sets and PHSS are alike in representations and henceforth Smarandache laid a clear differentiation of these two theoretical concepts. Researchers have applied Plithogenic hypersoft sets in several domains. Some of the significant contributions are described as follows. Rana, Sayeed, et al. [3], and Rana, Qayyum, et al. [4] developed Plithogenic whole hypersoft set in a more generalized form and Plithogenic fuzzy whole hypersoft set. Majid et al. [5] discussed the applications of Plithogenic multipolar fuzzy hypersoft sets. Martin and Smarandache [6, 7] introduced combined PHSS and concentric plithogenic hypergraph structures. Gayen et al. [8] discoursed plithogenic hypersoft subgroup. Basumatary et al. [9]) briefed on the properties of plithogenic neutrosophic hypersoft topological group. Dhivya and Lancy [10, 11] briefed on the strong continuity functions in Plithogenic context and also initiated near PHSS. Martin et al. [12] leveraged the applications of extended Plithogenic hypersoft sets in disease diagnostic model. Ahmadet et al. [13] framed Plithogenic hypersoft based decision model.

Researchers have also integrated the theory of PHSS with the theory of possibility. Rahman, Saeed, Khalifa, et al. [14] Rahman, Saeed, Mohammed, Krishnamoorthy, et al. [15] and Rahman, Saeed, Mohammed, Abdulkareem, et al. [16] developed several decision-making techniques, hybrid structures with the implications of possibility integrated hypersoft sets and applied the same in site selection of solid waste management, disease diagnosis, agri-automobile evaluation. Zhao et al. [17] formulated an intelligent based decision making approach in making investment selections.

Rahman, Saeed, & Garg [18], Rahman, Smarandache, Saeed, et al. [19], and Rahman, Saeed, & Abd El-WahedKhalifav [20], developed the hypersoft set with possibility degree using multi attribute decision making based on aggregation and also provide innovative decisive framework.Saeed, Waheb, et al. [21] (2023) presented an innovative approach of quality assessment with possibility q-rung ortho-pair fuzzy hypersoft set. Al-Hijjawi and Alkhazaleh[22] discoursed on possibility neutrosophic hypersoft set. Martin [23] introduced the notion of possibility plithogenic soft sets. In addition to this, researchers have also applied Pythagorean kind of sets in decision making. These sets are a special kind of intuitionistic sets which satisfies certain conditions. Pythagorean kind of hypersoft sets is applied in several decision-making scenario. Zulqarnainet al. [24], Jafar et al. [25], Khan et al.[26], and many other contributed to Pythagorean fuzzy hypersoft sets. Martin et al. [27] developed a decision model with the integration of Plithogenic Pythagorean hypersoft sets. However, from the aforementioned literature it is found that the intersection of PHSS, Pythagorean and Possibility is not found and hence this research work coins a decision model which is based on three Ps. The objective of this research work is to develop an integrated decision model to make optimal decisions on sustainable waste management. However, on other hand the theory of neutrosophy is explored by the researchers in developing decisioning models. Ye [28], Biswas et al. [29], Pramanik et al. [30] developed multi-criteria decisioning models based on single-valued neutrosophic environment. Researchers also developed models based on different decision methods and measures. To mention a few, hybrid vector similarity measures by Pramanik et al. [31], CRITIC-EDAS by Mallick et al. [32]), neutrosophic Delphi by Smarandache et al. [33], TOPSIS by Biswas et al. [34, 35], Neutrosophic logic by Mondal et al. [36], Best Worst Method and TOPSIS by Pramanik, Das, et al. [37]. The developments and extensions of neutrosophic based decision models is high in comparison with Plithogenic based decision models and this research work attempts in augmenting and building more plithogenic based decision models.

The remaining contents of the paper are structured into the following sections. Section 2 comprises the essential preliminaries. Section 3 presents the theoretical developments of Pythagorean Plithogenic Possibility Hypersoft sets (PyPlPoHS). Section 4 sketches the applications of PyPlPoHS. Section 5 discusses the results and the last section concludes the research work.

PRELIMINARIES

This section discusses the fundamental concepts pertinent to this research work.

Definition 1. Plithogenic Set [1]

Smarandache [1] defines Plithogenic set as the generalization of crisp, fuzzy, intuitionistic, and neutrosophic soft set.

The universe of discourse is denoted by U. The plithogenic set is of the form $(P, a, V, d, c)$with set P, the attribute a, the set of attribute values V, the degree of appurtenance d, the degree of contradiction c of the dominant attribute value with respect to other attribute values subjected to a particular attribute i.e. $\forall x \in P, d: P \times V \to \rho([0,1]^z), c: V \times V \to [0,1]^z$.

In this case, if $z = 1$, it is crisp, $z = 2$, it is intuitionistic, $z = 3$, it is neutrosophic.

Definition 2. Plithogenic Hypersoft Set [38]

A PHSS is of the form(X, A, C, d, c). For an n-tuple $(\gamma_1, \gamma_2 \ldots, \gamma_n) \in C, \gamma_i \in A_i, 1 \leq i \leq n$, a plithogenic hypersoft set $F : C \to P(U)$ is mathematically written as

$$F(\gamma_1, \gamma_2 \ldots, \gamma_n) = \{x(d_x(\gamma_1), d_x(\gamma_2), \ldots, d_x(\gamma_n)), x \in X, \gamma_i \in A_i\}.$$

Definition 3. Plithogenic Possibility Hypersoft set [23]

A Plithogenic Possibility hypersoft set over U is a set of ordered pairs of the form (F_{PP}^{z}, Y) defined by $F_{PP}^{z}: \mathrm{Y} \to [0,1]_d \times U^z \times I^U$, where I^U is the assortment of all fuzzy subsets of U.

This can also be represented as

$$F_{PP}^{z} = \left\{\left(e_h, \left\{\left(\frac{(u_l, c_d)}{F_P^z(e_h)(u_l)}, \mu(e_h)(u_l)\right) | u_l \epsilon U\right\}\right) | e_h \epsilon \mathrm{Y}\right\}.$$

In this expression, $\mu(e_h)(u_l)$ signifies the possibility degree of u_l with respect to e_h.

Definition 4. Pythagorean Fuzzy Set [39]

Let X the universal set. The Pythagorean fuzzy set A is a set of ordered pairs over$X$, and it is defined by

$$A = \{\mu_A(x), \vartheta_A(x) | x \in X\}.$$

Where the functions $\mu_A(x): X \to [0,1]$and$\vartheta_A(x): X \to [0,1]$define the degree of membership and the degree of non-membership, respectively of the element $x \in X$to A, which is a subset of X,and for every $x \in X$,

$$0 \leq ((\mu_A(x)))^2 + ((\vartheta_A(x)))^2 \leq 1.$$

$$\pi_A(x) = 1 - \mu_A(x) - \vartheta_A(x)$$

is the intuitionistic fuzzy set index or hesitation margin of$x$in X.The hesitation margin $\pi_A(x)$is the extent of non-determinacy of $x \in X$,to the set$A$ and$\pi_A(x) \epsilon [0,1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$.

Thus,

$$\pi_A(x) + \mu_A(x) + \vartheta_A(x) = 1.$$

Definition 5. Plithogenic Pythagorean Set [40]

The Plithogenic set is said to be Pythagorean Plithogenic set is the degree of appurtenance d is of Pythagorean set. i.e. d: $P \times V \to \rho([0,1]^{2_p})$. In this case if z=2 it is intuitionistic and 2_p stands of Pythagorean sets which is a special kind of Pythagorean sets.

Pythagorean Plithogenic Possibility Hypersoft Set

This section introduces the conceptualization, operations and similarity measures of Pythagorean Plithogenic Possibility Hypersoft Set (***(PyPlPoHS)***.

Definition 6. PyPlPoHS

A PyPlPoHS over U is a set of ordered pairs of the form (F_{PP}^{p}, Y) defined by

$$F_{PP}^{p}: \mathrm{Y} \to [0,1]_d \times U^z \times I^{U_p}.$$

Where I^{U_p} is the pool of all Pythagorean fuzzy subsets of U.

This can also be represented as

$$F_{PP}^{p} = \left\{\left(e_h, \left\{\left(\frac{(u_l, c_d)}{F_P^p(e_h)(u_l)}, \mu(e_h)(u_l)\right) | u_l \epsilon U\right\}\right) | e_h \epsilon \mathrm{Y}\right\}.$$

In this expression, $\mu(e_h)(u_l)$ signifies the possibility degree of u_l with respect to e_h.

Definition 7. Union and Intersection of Two PyPlPoHSs

Let (F_{pp}^{p}, H) and (G_{pp}^{p}, H) be two Pythagorean plithogenic possibility hypersoft sets the union of two Pythagorean plithogenic possibility hypersoft soft set is defined as $(F_{pp}^{p}, H) \vee_{p}^{p} (G_{pp}^{p}, H)$ and the intersection of two PyPlPoHSs is defined as follows: $(F_{pp}^{p}, H) \wedge_{p}^{p} (G_{pp}^{p}, H)$.

The union of a set is defined as

$$(F_{pp}^{p}, H) \vee_{p}^{p} (G_{pp}^{p}, H) = \{(max[M_s(x), N_s(x)], min[M_v(x), N_v(x)]) \mid x \in X\}$$

and the intersection of a set is defined as

$$(F_{pp}^{p}, H) \wedge_{p}^{p} (G_{pp}^{p}, H) = \{(min[M_s(x), N_s(x)], max[M_v(x), N_v(x)]) \mid x \in X\}.$$

Let us understand the union and intersection of PyPlPoHSs with a simple example. Let (F_{PP}^{P}, H) and (G_{PP}^{P}, H) are defined in Tables 1. and2.

Table 1.PyPlPoHS (F_{PP}^{P}, H)

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.1),(0.8,0.1))}, 0.7$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.2),(0.7,0.1))}, 0.6$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.2),(0.6,0.4),(0.8,0.1))}, 0.6$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.7,0.1),(0.7,0.2),(0.6,0.3))}, 0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.3),(0.8,0.1))}, 0.9$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.6,0.4),(0.5,0.4),(0.7,0.1))}, 0.8$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.8,0.1),(0.6,0.1),(0.7,0.1))}, 0.6$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.7,0.4),(0.5,0.2),(0.6,0.1))}, 0.7$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.6,0.3),(0.7,0.1))}, 0.7$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.2),(0.6,0.2))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.7,0.1),(0.8,0.1))}, 0.9$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.5,0.3),(0.7,0.1),(0.5,0.4))}, 0.8$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.7,0.1),(0.8,0.1))}, 0.8$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.3),(0.7,0.1))}, 0.8$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.7,0.1),(0.8,0.1))}, 0.9$

Table 2. PyPlPoHS (G_{PP}^{P}, H)

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.5,0.4),(0.7,0.1))}, 0.9$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.6,0.1))}, 0.8$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.1),(0.5,0.4),(0.6,0.3))}, 0.8$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.5,0.3),(0.7,0.2),(0.8,0.1))}, 0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.6,0.2),(0.8,0.1),(0.5,0.3))}, 0.7$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.6,0.2),(0.8,0.1),(0.8,0.1))}, 0.9$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.7,0.1),(0.8,0.1))}, 0.7$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.5,0.3),(0.7,0.1))}, 0.9$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.7,0.1),(0.7,0.1))}, 0.8$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.4,0.5),(0.6,0.1),(0.6,0.2))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.8,0.1))}, 0.8$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.6,0.1),(0.7,0.1),(0.7,0.1))}, 0.9$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.3),(0.8,0.1),(0.5,0.3))}, 0.6$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.1),(0.6,0.3))}, 0.7$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.7,0.1),(0.6,0.2),(0.5,0.4))}, 0.6$

The union of two PyPlPoHSs of $\left(F_{pp}^{p}, H\right)$ and $\left(G_{pp}^{p}, H\right)$ is $\left(U_{pp}^{p}, H\right)$ and it is obtained in Table 3.

Table 3. PyPlPoHS$\left(U_{pp}^{p}, H\right)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.7,0.1),(0.8,0.1))}, 0.9$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.7,0.1))}, 0.8$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.1),(0.6,0.4),(0.8,0.1))}, 0.8$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.7,0.1),(0.7,0.2),(0.8,0.1))}, 0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.6,0.2),(0.8,0.1),(0.8,0.1))}, 0.9$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.6,0.1),(0.7,0.1),(0.7,0.1))}, 0.8$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.8,0.1),(0.7,0.1),(0.8,0.1))}, 0.7$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.7,0.3),(0.5,0.2),(0.7,0.1))}, 0.9$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.7,0.1),(0.7,0.1))}, 0.8$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.1),(0.6,0.2))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.7,0.1),(0.8,0.1))}, 0.9$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.6,0.1),(0.7,0.1),(0.7,0.1))}, 0.9$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.8,0.1),(0.8,0.1))}, 0.8$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.1),(0.7,0.1))}, 0.8$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.7,0.1),(0.7,0.1),(0.8,0.1))}, 0.9$

To describe the result in the example above let compute

$\left(\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.7,0.1),(0.8,0.1))}, 0.9\right) \vee_p^p \left(\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.5,0.4),(0.7,0.1))}, 0.9\right)$

$\{max(0.6,0.6), min(0.2,0.2)\} = (0.6\,,0.2),$

$\{max(0.7,0.5), min(0.1,0.4)\} = (0.7, 0.1),$

$\{max(0.8,0.7), min(0.1,0.1)\} = (0.8,0.1).$

The intersection of two PyPlPoHSs of $\left(F_{pp}^{p}, H\right)$ and $\left(G_{pp}^{p}, H\right)$ is $\left(V_{pp}^{p}, H\right)$ and it is obtained in Table 4.

Table 4. PyPlPoHS$\left(V_{pp}^{p}, H\right)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.5,0.4),(0.7,0.1))}, 0.7$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.2),(0.6,0.1))}, 0.6$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.2),(0.5,0.4),(0.6,0.3))}, 0.6$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.5,0.3),(0.7,0.2),(0.6,0.3))}, 0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.3),(0.5,0.3))}, 0.7$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.6,0.4),(0.5,0.4),(0.5,0.3))}, 0.7$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.6,0.1),(0.7,0.1))}, 0.6$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.6,0.4),(0.5,0.3),(0.6,0.1))}, 0.7$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.5,0.4),(0.6,0.3),(0.6,0.2))}, 0.7$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.4,0.5),(0.6,0.2),(0.6,0.2))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.8,0.1))}, 0.8$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.5,0.3),(0.6,0.2),(0.5,0.4))},$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.3),(0.7,0.1),(0.5,0.3))}, 0.6$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.3),(0.6,0.3))}, 0.7$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.6,0.2),(0.5,0.4))}, 0.6$

To describe the result in the example above let compute

$$\left(\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.1),(0.8,0.1))},0.9\right)\wedge_p^p\left(\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.5,0.4),(0.7,0.1))},0.9\right)$$

$\{min(0.6,0.6), max(0.4,0.2)\} = (0.6\ ,0.4)$,

$\{min(0.7,0.5), max(0.1,0.4)\}(0.5, 0.4)$,

$\{min(0.8,0.7), max(0.1,0.1)\} = (0.7,0.1)$.

Definition 8. Similarity Measures between Two PyPlPoHSs

Let (F_{pp}^p, H) and (G_{pp}^p, H) be two PyPlPoHSs.

The similarity measure between these two sets is defined as

$$S_p(F_{pp}^p, G_{pp}^p) = \frac{1}{n}\sum_{i=1}^{n} w_i \frac{\left|M_s^2(x_i) - N_s^2(x_i)\right| + \left|M_y^2(x_i) - N_y^2(x_i)\right|}{M_s^2(x_i) + N_s^2(x_i) + M_y^2(x_i) + N_y^2(x_i)}$$

Where $w_i = 1 - \frac{\sum_{j=1}^{|u|}\sum_{i=1}^{|e|}|\mu_{F_{pp}(e_{ik})} - \mu_{F_{pp}(e_{ik})}|}{\sum_{j=1}^{|u|}\sum_{i=1}^{|e|}|\mu_{F_{pp}(e_{ik})} + \mu_{F_{pp}(e_{ik})}|}$.

Let us consider an example. Let (F_{PP}^P, H) and (G_{PP}^P, H) are defined in Tables 5 and 6.

Table 5. PyPlPoHS(F_{PP}^P, H)

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.1),(0.8,0.1))},0.7$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.2),(0.7,0.1))},0.6$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.2),(0.6,0.4),(0.8,0.1))},0.6$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.7,0.1),(0.7,0.2),(0.6,0.3))},0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.3),(0.8,0.1))},0.9$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.6,0.4),(0.5,0.4),(0.7,0.1))},0.8$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.8,0.1),(0.6,0.1),(0.7,0.1))},0.6$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.7,0.4),(0.5,0.2),(0.6,0.1))},0.7$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.6,0.3),(0.7,0.1))},0.7$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.2),(0.6,0.2))},0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.7,0.1),(0.8,0.1))},0.9$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.5,0.3),(0.7,0.1),(0.5,0.4))},0.8$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.7,0.1),(0.8,0.1))},0.8$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.3),(0.7,0.1))},0.8$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.7,0.1),(0.8,0.1))},0.9$

Table 6. PyPlPoHS(G_{PP}^{P}, H)

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.5,0.4),(0.7,0.1))}, 0.9$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.6,0.1))}, 0.8$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.1),(0.5,0.4),(0.6,0.3))}, 0.8$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.5,0.3),(0.7,0.2),(0.8,0.1))}, 0.8$	$\frac{(m_2,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.6,0.1))}, 0.8$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.6,0.1),(0.7,0.1),(0.5,0.3))}, 0.7$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.7,0.1),(0.8,0.1))}, 0.7$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.5,0.3),(0.7,0.1))}, 0.9$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.5,0.4),(0.7,0.1),(0.6,0.2))}, 0.8$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.4,0.5),(0.6,0.1),(0.6,0.2))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.8,0.1))}, 0.8$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.6,0.1),(0.6,0.2),(0.7,0.1))}, 0.9$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.3),(0.8,0.1),(0.5,0.3))}, 0.6$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.1),(0.6,0.3))}, 0.7$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.7,0.1),(0.6,0.2),(0.5,0.4))}, 0.6$

The similarity between these two PyPlPoHSs is computed as follows:

$w_1 = 1 - \frac{0.2+0+0.1+0+0.2}{1.6+1.6+1.3+1.8+1.4} = 0.94$, $w_2 = 0.90$and $w_3 = 0.89$.

$S_p(F_{pp}^{p}, G_{pp}^{p}) = \frac{1}{3}[0.94 \times \frac{3.36}{15.18} + 0.90 \times \frac{2.26}{13.96} + 0.89 \times \frac{3.56}{13.95}] = 0.191.$

Application of Pyplpohs in Optimal Decisioning

This section presents the application of PyPlPoHS in making optimal ranking of the suppliers. Consider a decision-making problem of choosing optimal suppliers of waste management devices. Every industrial sector takes several initiatives in transforming their production process sustainable by using eco-conscious devices. The attributes which are considered with respect to the waste managing devices are as follows:

a_1 – Efficiency A_1= {Very High, High, Average}

a_2 – Cost A_2= {Expensive, Budgetary, Cheap}

a_3 – Environmental Impact A_3={High, Moderate, Low,Very low}

a_4 – Adaptability A_4 = {Exceptional, Advanced, Intermediate}.

Let us consider a manufacturing sector using five waste management devices and these devices are considered to be sustainable in the perspective of the industrial sectors if it possesses the attribute values of very high efficiency, cheap cost, very low environmental impacts and exceptional adaptability. These attribute values are considered to be dominant. i. e.d = {Very High, Cheap, Very Low, Exceptional}. However, the industrial sectors also accept the waste management devices as sustainable if the attribute values belong to the set

$H = \{h_1$ = (High, Budgetary, Very Low, Advanced), h_2 = (Very High, Cheap, Low, Advanced), h_3 = (Very High, Budgetary, Low, Exceptional)}.

The expected standards of the sustainable waste management devices with respect to the aforementioned attribute values is presented with Pythagorean Plithogenic Possibility Hypersoft sets. The decision makers have to make optimal ranking of the companies offering different waste management devices. Let(F_{PP}^{P}, H) be defined in Table 7.

Table 7. PyPlPoHS(F_{PP}^{P}, H)

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.1),(0.8,0.1))}, 0.7$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.2),(0.7,0.1))}, 0.6$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.7,0.2),(0.6,0.4),(0.8,0.1))}, 0.6$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.7,0.1),(0.7,0.2),(0.6,0.3))}, 0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.3),(0.8,0.1))}, 0.9$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.6,0.4),(0.5,0.4),(0.7,0.1))}, 0.8$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.8,0.1),(0.6,0.1),(0.7,0.1))}, 0.6$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.7,0.4),(0.5,0.2),(0.6,0.1))}, 0.7$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.6,0.3),(0.7,0.1))}, 0.7$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.6,0.4),(0.7,0.2),(0.6,0.2))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.7,0.1),(0.8,0.1))}, 0.9$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.5,0.3),(0.7,0.1),(0.5,0.4))}, 0.8$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.7,0.1),(0.8,0.1))}, 0.8$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.3),(0.7,0.1))}, 0.8$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.6,0.2),(0.7,0.1),(0.8,0.1))}, 0.9$

The manufacturing sector receives proposals from different companies and they are also represented as Pythagorean Possibility Plithogenic Hypersoft sets as given below. Now the problem is to choose the optimal suppliers who offers waste management devices meeting the standards of the manufacturing sector. The following Tables are related to $(S_{i\,PP}^{\,P}, H)$, for $i \in \{1, \ldots, 5\}$.

Table 8. PyPlPoHS$(S_{1\,PP}^{\,P}, H)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.5,0.4),(0.5,0.3),(0.8,0.2))}, 0.8$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.1),(0.8,0.1))}, 0.7$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.6,0.2),(0.7,0.2),(0.8,0.3))}, 0.9$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.6,0.2),(0.6,0.3))}, 0.7$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.7,0.3),(0.5,0.4))}, 0.8$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.5,0.4),(0.6,0.1),(0.8,0.1))}, 0.7$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.5,0.3),(0.6,0.3),(0.7,0.1))}, 0.8$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.7,0.3),(0.7,0.1),(0.6,0.3))}, 0.9$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.5,0.4),(0.6,0.2),(0.7,0.1))}, 0.6$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.6,0.3),(0.7,0.1),(0.5,0.3))}, 0.9$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.7,0.2),(0.6,0.2))}, 0.6$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.4,0.3),(0.5,0.1),(0.5,0.2))}, 0.9$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.7,0.1),(0.8,0.3))}, 0.8$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.7,0.4),(0.5,0.3),(0.6,0.1))}, 0.7$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.5,0.2),(0.6,0.1),(0.7,0.1))}, 0.8$

Table 9. PyPlPoHS$\left(S_{2}{}_{PP}^{P}, H\right)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.6,0.3),(0.7,0.2),(0.8,0.1))}, 0.5$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.1),(0.5,0.1))}, 0.7$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.5,0.2),(0.6,0.1),(0.7,0.2))}, 0.7$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.5,0.1),(0.6,0.2),(0.6,0.3))}, 0.6$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.6,0.3),(0.5,0.6),(0.7,0.1))}, 0.8$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.5,0.4),(0.6,0.4),(0.5,0.2))}, 0.5$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.5,0.3),(0.4,0.5),(0.6,0.3))}, 0.7$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.5,0.2),(0.7,0.1))}, 0.6$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.4,0.2),(0.5,0.3),(0.6,0.1))}, 0.6$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.5,0.4),(0.6,0.1),(0.7,0.1))}, 0.5$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.8,0.2),(0.6,0.1),(0.5,0.3))}, 0.9$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.5,0.3),(0.6,0.2),(0.7,0.1))}, 0.5$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.4,0.3),(0.6,0.1),(0.5,0.2))}, 0.8$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.6,0.4),(0.5,0.3),(0.6,0.1))}, 0.5$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.5,0.1),(0.4,0.3),(0.6,0.1))}, 0.6$

Table 10. PyPlPoHS$\left(S_{3}{}_{PP}^{P}, H\right)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.5,0.2),(0.7,0.2),(0.6,0.1))}, 0.6$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.4,0.3),(0.5,0.2),(0.6,0.1))}, 0.7$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.5,0.2),(0.6,0.1),(0.7,0.2))}, 0.8$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.5,0.4),(0.6,0.4),(0.5,0.3))}, 0.8$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.5,0.2),(0.4,0.2),(0.6,0.1))}, 0.8$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.3,0.4),(0.5,0.2),(0.6,0.2))}, 0.9$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.7,0.1),(0.5,0.2),(0.6,0.1))}, 0.7$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.6,0.2),(0.7,0.1))}, 0.7$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.5,0.2),(0.4,0.3),(0.6,0.1))}, 0.6$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.8,0.2),(0.4,0.2))}, 0.8$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.4,0.2),(0.7,0.1))}, 0.6$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.5,0.2),(0.6,0.1),(0.5,0.2))}, 0.7$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.8,0.1),(0.7,0.2))}, 0.9$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.5,0.2),(0.5,0.3),(0.6,0.1))}, 0.8$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.5,0.3),(0.7,0.2),(0.6,0.1))}, 0.6$

Table 11. PyPlPoHS$\left(S_{4\,pp}^{\;P}, H\right)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.4,0.3),(0.5,0.3))}, 0.5$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.4,0.3),(0.5,0.2),(0.6,0.1))}, 0.6$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.5,0.2),(0.3,0.4),(0.6,0.1))}, 0.5$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.6,0.1),(0.5,0.2),(0.4,0.3))}, 0.6$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.4,0.3),(0.5,0.3),(0.6,0.1))}, 0.8$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.5,0.3),(0.6,0.2),(0.3,0.2))}, 0.4$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.7,0.2),(0.5,0.2),(0.6,0.1))}, 0.7$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.6,0.4),(0.4,0.1),(0.5,0.1))}, 0.5$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.4,0.2),(0.5,0.3),(0.6,0.1))}, 0.3$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.5,0.4),(0.6,0.2),(0.5,0.2))}, 0.6$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.1),(0.4,0.2))}, 0.7$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.4,0.3),(0.6,0.1),(0.5,0.3))}, 0.5$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.8,0.1),(0.4,0.2))}, 0.5$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.4,0.2),(0.6,0.1),(0.4,0.3))}, 0.6$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.5,0.2),(0.6,0.1),(0.7,0.1))}, 0.6$

Table 12. PyPlPoHS$\left(S_{5\,pp}^{\;P}, H\right)$

h_1	h_2	h_3
$\frac{(m_1,(0.3,0.3,0,0.3)_d)}{((0.4,0.5),(0.6,0.1),(0.7,0.1))}, 0.5$	$\frac{(m_1,(0,0.4,0,0.4)_d)}{((0.6,0.2),(0.5,0.2),(0.6,0.1))}, 0.7$	$\frac{(m_1,(0.5,0,0.5,0)_d)}{((0.5,0.2),(0.5,0.4),(0.4,0.3))}, 0.4$
$\frac{(m_2,c(0.3,0.3,0,0.3)_d)}{((0.5,0.1),(0.5,0.2),(0.5,0.3))}, 0.4$	$\frac{(m_2,c(0,0.4,0,0.4)_d)}{((0.5,0.2),(0.5,0.3),(0.6,0.1))}, 0.8$	$\frac{(m_2,c(0.5,0,0.5,0)_d)}{((0.5,0.2),(0.4,0.3),(0.6,0.1))}, 0.5$
$\frac{(m_3,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.5,0.3),(0.6,0.1))}, 0.3$	$\frac{(m_3,(0,0.4,0,0.4)_d)}{((0.5,0.4),(0.5,0.2),(0.6,0.1))}, 0.5$	$\frac{(m_3,(0.5,0,0.5,0)d)}{((0.5,0.2),(0.4,0.3),(0.6,0.1))}, 0.6$
$\frac{(m_4,(0.3,0.3,0,0.3)_d)}{((0.5,0.4),(0.6,0.2),(0.5,0.2))}, 0.6$	$\frac{(m_4,(0,0.4,0,0.4)_d)}{((0.5,0.3),(0.6,0.1),(0.7,0.1))}, 0.6$	$\frac{(m_4,(0.5,0,0.5,0)_d)}{((0.6,0.3),(0.7,0.2),(0.4,0.3))}, 0.7$
$\frac{(m_5,(0.3,0.3,0,0.3)_d)}{((0.6,0.2),(0.5,0.2),(0.6,0.1))}, 0.5$	$\frac{(m_5,(0,0.4,0,0.4)_d)}{((0.6,0.2),(0.5,0.3),(0.6,0.1))}, 0.7$	$\frac{(m_5,(0.5,0,0.5,0)d)}{((0.4,0.2),(0.6,0.1),(0.7,0.1))}, 0.5$

$$S_p\left(F_{pp}^{p}, S_{1\,pp}^{\;p}\right) = 0.1336$$

$$S_p\left(F_{pp}^{p}, S_{2\,pp}^{\;p}\right) = 0.1926$$

$$S_p\left(F_{pp}^{p}, S_{3\,pp}^{\;p}\right) = 0.1713$$

$$S_p\left(F_{pp}^{p}, S_{4\,pp}^{\;p}\right) = 0.2024$$

$$S_p\left(F_{pp}^{p}, S_{5\,pp}^{\;p}\right) = 0.1610.$$

From the similarity measures the suppliersS_1, S_2, S_3, S_4andS_5 are ranked accordingly:

$$S_4 > S_2 > S_3 > S_5 > S_1$$

RESULTS AND DISCUSSIONS

The above ranking results is presented graphically in Fig

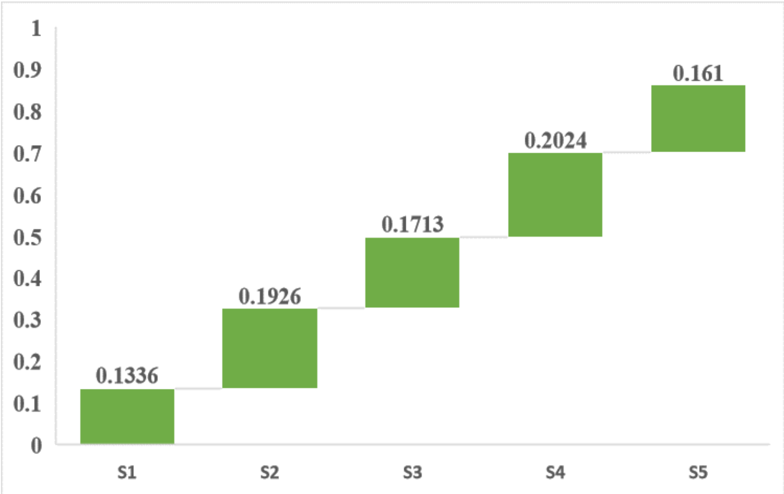

Fig. 1.PyPlPoHS ranking score values of the alternatives

From the ranking results, it is very evident that the supplier S_4 is ranked first and S_1 is ranked last. As Pythagorean sets are competent in facilitating data representations, the ranking result obtained using PyPlPoHS is more comprehensive than fuzzy representations. The same problem is subjected to fuzzy based data representations and the ranking results obtained are same with different score values. The graphical representation of the score value of fuzzy based PlPoHS is presented in Fig.2. and the combined comparison is presented in Fig.3.

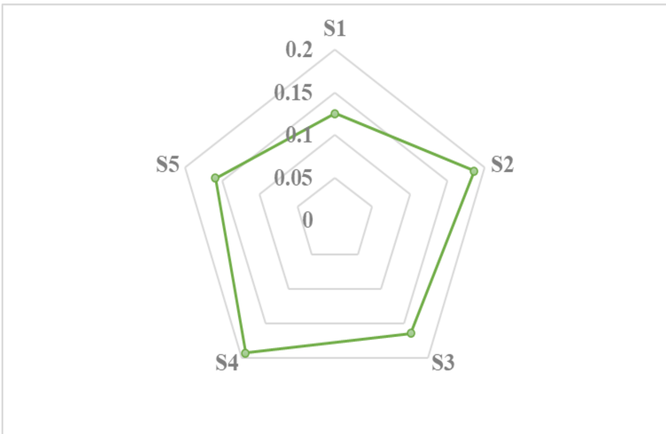

Fig. 2. Fuzzy PlPoHSranking score values of the alternative

The combined representations of the scores are presented in Fig.3.

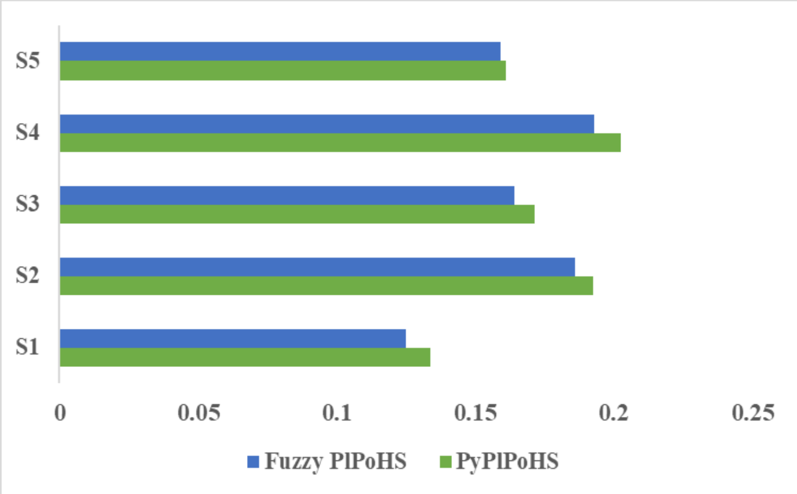

Fig. 3.Comparisons of score values

Conclusions

This paper introduces a novel decision-making method based on PyPlPoHS. The basic operations and the similarity measures are also discussed together with its applications. The efficacy of this representation is validated with PyPlPoHSS and the results are also compared with fuzzy representations. This kind of decision-making method is more comprehensive and henceforth this research work suggests this method for other industrial applications. This work shall be further augmented with other kinds of representations.

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Chapter 15

Supplier Selection Using Advanced Entropy Weight Fuzzy Single Valued Neutrosophic Set

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ABSTRACT

Supplier selection (SS) is a crucial element in the effective management of supply chains, playing a pivotal role in ensuring smooth operations. This paper introduces a robust soft model for supplier selection that integrates multi-attribute decision-making (MADM) with mathematical programming. The proposed model offers a practical and adaptable approach for supplier selection, with potential applications across various industries. By combining these methodologies, the model enables comprehensive evaluation of suppliers based on multiple criteria, addressing the complexities and challenges inherent in real-world supply chain decisions. This study aims to enhance decision-making in supplier selection by utilizing appropriate qualitative techniques. The findings of the proposed model highlight the significance of evaluating and ranking supplier selection criteria, emphasizing that selecting the best supplier is a critical decision for the purchasing department. This process plays a vital role in fostering a sustainable manufacturing environment, demonstrating the importance of a systematic approach to supplier selection in achieving long-term operational success. This chapter presents a novel methodology MADM based on Single-Valued Neutrosophic Sets (SVNS), a specialized extension of Neutrosophic sets that incorporates three distinct membership functions: truth, indeterminacy, and falsehood. The objective is to develop an advanced decision-making framework that improves supplier selection, particularly when handling imprecise, uncertain, and inconsistent information commonly encountered in real-world applications. The proposed method integrates the conversion of crisp and fuzzy data into SVNS and employs an advanced entropy weight approach to enhance the selection process. The effectiveness of the approach is demonstrated through case studies drawn from existing literature on supplier selection, showing that the new methodology yields more accurate results with reduced computational effort when compared to traditional MADM methods. The results validate the superiority of the proposed approach in addressing indeterminacies and uncertainties, offering more reliable and precise decision-making outcomes in complex environments.

Keywords: Multi-Attribute Decision Making (MADM), Single-Valued Neutrosophic Sets (SVNS), Supplier Selection, Entropy Weighting, Uncertainty, Fuzzy Sets, Indeterminacy, Decision Support Systems.

INTRODUCTION

Supplier selection has long been a central topic in the fields of purchasing and supply chain management, with significant contributions from early works in the literature [1], [2], [3, 4] they have insighted empirical study identified 23 key criteria to be considered when selecting suppliers. Further research aimed at determining the weight of these criteria based on the specific needs of suppliers. In the same year, studies focused on three primary criteria—net price, delivery, and quality—highlighting them as the most frequently discussed factors in supplier selection. Additionally, these studies reviewed the use of quantitative methods in vendor selection, with the "linear weighting scheme" being widely discussed. However, only 10 articles incorporated mathematical programming techniques for supplier evaluation. [5] extended the supplier selection framework by examining 85 criteria. Using Grey Analysis and T-Tests, they identified 21 key criteria, ranking them based on mean values, T-test results, and grey numbers. Their research showed that in the early 1980s, price was the dominant selection criterion, while by the early 1990s, time management and customer sensitivity had become important factors alongside product and service cost. By the late 1990s and early 2000s, flexibility emerged as a critical factor, and expert opinions and other variables gained prominence in the decision-making process. Given the complexity of supplier selection, it is not feasible to consider every possible criterion in all situations. Therefore, the most important criteria are often determined through expert judgment. In the literature, supplier selection has primarily been treated as a multi-attribute decision-making (MCDM) problem, with various mathematical methods employed to provide more accurate and effective solutions [5] has shown that combining different multi-attribute decision methods (MADM) enhances ranking accuracy.

EXISTING LITERATURE SURVEY ON SUPPLIER SELECTION PROCESS RANKING USING MULTI ATTRIBUTE DECISION MAKING (MADM)

Analytical hierarchy Process (AHP) method developed by [6] employs a normalization approach that treats both beneficial and non-beneficial criteria in the same manner. It utilizes the Euclidean distance principle for decision-making, but one of its limitations is that it does not account for the potential correlation between attributes. Additionally, this method requires expert or research-based input to determine the weights for each attribute, which can lead to variations in the rankings, particularly if the weightings assigned to the attributes differ. [7] had tried to implement interactive supplier selection model using AHP. [8] had developed the supplier selection model for automotive industry using AHP method. [9] imparted intervalued fuzzy AHP based green supply chain resilience evaluation methodology in post covid era. [10], [11], [12], [13] imparted fuzzy inference system for selection of vendors (Suppliers) for lathe machines bed selection, planner machine bed, shaper machine arm, radial drilling column supplier selection cases respectively to find the effective ranking solution.

VlseKriterijuska OptimizacijaI Komoromisno Resenje (VIKOR) method developed by [14] for outranking method. [15] investigated hierarchical VIKOR methodology with incomplete information for selection of supplier. This method, when extended with linguistic information, helps to address uncertainties and imprecision in evaluating suppliers, making it particularly useful in the highly regulated and risk-sensitive nuclear power sector by [16]. [17] explores sustainable supplier selection using the VIKOR method, integrated with single-valued Neutrosophic sets. This approach addresses uncertainties and imprecision in decision-making, offering a more robust and flexible solution for selecting suppliers in a sustainable context. An interval-valued intuitionistic fuzzy model based on the extended VIKOR and MARCOS methods for sustainable supplier selection in organ transplantation networks investigated by [18].

COmplex PRoportional ASsessment (COPRAS) investigated by [19]. In particular, the focus on wiper insert geometry has garnered attention as it plays a pivotal role in achieving superior surface integrity and minimizing thermal and mechanical stresses during the cutting process. The findings from various studies highlight the effectiveness of the COPRAS method in tailoring these geometries to optimize the overall machining process for OHNS steel. These advances contribute significantly to the understanding and optimization of hard turning processes, making it a promising solution for high-precision machining of hardened steels. Supplier selection using multiple criteria group decision-making based on the COPRAS method with interval type-2 fuzzy sets investigated by [20]. [21] utilized the IFS-TOPSIS method to support supplier selection in a sustainable supply chain. [21] explored the application of An ELECTRE-based multiple criteria decision-making approach for supplier selection utilizing Dempster-Shafer theory. [22] addressed the supplier selection problem using the ELECTRE-I method. [23] worked on novel interval-valued hesitant fuzzy group outranking approach for green supplier evaluation in manufacturing systems. [24] has insighted Neutrosophic set works better in MADM selection methodology.

[25] authors have tried to extend the TOPSIS (Technique for Order Preference by Similarity to Ideal Solution) method to address multi-attribute group decision-making problems in environments characterized by single-valued Neutrosophic sets (SVNS) and interval Neutrosophic sets (INNS). A hybrid approach combining Neutrosophic sets and the DEMATEL method for developing supplier selection criteria carried out by [26]. [27] investigated smart TOPSIS works with a neural network-driven TOPSIS approach using Neutrosophic triplets for green supplier selection in sustainable manufacturing. [28] derived entropy based grey relational analysis method using Neutrosophic set MADM technique. Hybrid vector similarity measures and their applications to multi-attribute decision making under Neutrosophic environment carried out by [29].

[30] derived multi-criteria decision-making for supplier selection under a single-valued Neutrosophic set environment. [31] The TOPSIS method for multi-attribute group decision-making under a single-valued Neutrosophic environment is proposed to effectively handle decision problems characterized by uncertainty and imprecision. In this context, each attribute of an alternative is represented using a single-valued Neutrosophic set,

which includes truth, indeterminacy, and falsity membership functions. This extended approach enables the incorporation of ambiguous or incomplete information, allowing for a more comprehensive and flexible decision-making process.

Proposed Methodology Fuzzy-Single Valued Neutrosophic Set Entropy Weight Based Multi Attribute Decision Making Technique (F-SVNS AEW-MADM)

The Fuzzy Single-Valued Neutrosophic Set Advanced Entropy Weight-based Multi-Attribute Decision Making (F-SVNS AEW-MADM) methodology follows a structured approach to decision-making. The key steps involved in this process are outlined below:

Step 1: Define the objective of the selection process, clearly establishing the goal or desired outcome for the decision-making task.

Step 2: Identify the various alternatives available for selection, along with the relevant attributes (criteria) that will influence the decision. This step involves determining the factors that will be used to evaluate and compare the alternatives in the context of the selection problem.

Step 3: All alternatives and attributes (criteria) in matrix form with comparative performance are known as decision matrix. Let us consider set of alternatives as $A = \{A_i, i = 1, 2, 3, .. m\}$ & set of criteria as $C = \{C_j, j = 1, 2, 3, .. n\}$, X_{ij} is the performance of alternatives A_i for relative criteria C_j. X_{ij} are having qualitative/ quantitative values. The structure of decision matrix is illustrated in Table 1.

Table 1: Decision Matrix for F-SVNS AEW-MADM

Alternatives	C_1	C_2	C_3	…..	C_n
A_1	X_{11}	X_{12}	X_{13}	…..	X_{1n}
A_2	X_{21}	X_{22}	X_{23}	…..	X_{2n}
A_3	X_{31}	X_{32}	X_{33}	…..	X_{3n}
….	…..	…..	…..	…..	…..
A_m	X_{m1}	X_{m2}	X_{m3}	…..	X_{mn}

Step 4: Conversion of Qualitative Data to Quantitative Data

The qualitative (linguistic) information is systematically transformed into quantitative (crisp) values, as outlined in Table 2. This process enables the representation of subjective, descriptive information in a measurable form, facilitating more precise analysis and interpretation

Table 2. Conversion of Linguistic Terms in to Classic (Crisp) Set

Linguistic terms of selection attributes	Fuzzy number	Crisp value of selection attribute
Exceptionally low	M1	0.045
Extremely low	M2	0.135
Very low	M3	0.255
Low	M4	0.335
Below average	M5	0.410
Average	M6	0.500
Above average	M7	0.590
High	M8	0.665
Very high	M9	0.745
Extremely high	M10	0.865

Data collected from sources [13], [14], [16], [17], and [18] were utilized. If the input matrix consists solely of quantitative data, this conversion step can be omitted.

Step 5. Generalization/ normalization of matrix

As outlined in sources [13] and [14], each attribute associated with the alternatives presents distinct values. To ensure consistency, the data is standardized within the range [0, 1] through the application of the Vector Normalization Method (VNM). For beneficial criteria, where higher values are advantageous (such as quality and profit), normalization is achieved using Eq. (1).

$$R_{ij} = \frac{X_{ij}}{\sqrt{\sum_{i=1}^{m} {X_{ij}}^2}}; \ \forall \ i, j \quad (1)$$

For non-beneficial criteria, where lower values are preferable (e.g., price, lead time), normalization is conducted using Eq. (2).

$$R_{ij} = 1 - \frac{x_{ij}}{\sqrt{\sum_{i=1}^{m} {x_{ij}}^2}}; \ \forall \ i, j \quad (2)$$

Normalized decision matrix is shown in Table 3.

Table 3. Normalized Decision Matrix for F-SVNS AEW-MADM

Alternative	C_1	C_2	C_3	…..	C_n
A_1	R_{11}	R_{12}	R_{13}	…..	R_{1n}
A_2	R_{21}	R_{22}	R_{23}	…..	R_{2n}
A_3	R_{31}	R_{32}	R_{33}	…..	R_{3n}
….	…..	…..	…..	…..	…..
A_m	R_{m1}	R_{m2}	R_{m3}	…..	R_{mn}

Step 6: Conversion of Classic Set/Fuzzy Set to Single-Valued Neutrosophic Set (SVNS)

The normalized decision matrix for F-SVNS AEW-MADM is presented in Table 4, where classic or fuzzy sets are converted into a Single-Valued Neutrosophic Set (SVNS). This transformation allows for a more comprehensive representation of uncertainty in decision-making scenarios.

Table 4. SVNS Normalized Decision Matrix for F-SVNS AEW-MADM

Alternative	C_1	C_2	…..	C_n
A_1	$< T_{11}(x), I_{11}(x), F_{11}(x) >$	$< T_{12}(x), I_{12}(x), F_{12}(x) >$	…..	$< T_{1n}(x), I_{1n}(x), F_{1n}(x) >$
A_2	$< T_{21}(x), I_{21}(x), F_{21}(x) >$	$< T_{22}(x), I_{22}(x), F_{22}(x) >$	…..	$< T_{2n}(x), I_{2n}(x), F_{2n}(x) >$
A_3	$< T_{31}(x), I_{31}(x), F_{31}(x) >$	$< T_{32}(x), I_{32}(x), F_{32}(x) >$	…..	$< T_{3n}(x), I_{3n}(x), F_{3n}(x) >$
….	…..	…..	…..	…..
A_m	$< T_{m1}(x), I_{m1}(x), F_{m1}(x) >$	$< T_{m2}(x), I_{m2}(x), F_{m2}(x) >$	…..	$< T_{mn}(x), I_{mn}(x), F_{mn}(x) >$

According to sources [13], [14], and [21], the conversion rules for transforming classic or fuzzy sets to Single-Valued Neutrosophic Sets (SVNS) for both beneficial and non-beneficial criteria are outlined in [14]. For beneficial criteria, where higher values of performance measures are desirable (e.g., profit, quality), the Positive Ideal Solution (PIS) is defined as $< {T_{max}}^*(x), {I_{min}}^*(x), {F_{min}}^*(x) >$ In this context, the normalized input matrix treats the beneficial criteria as the degree of truthfulness, while the degree of indeterminacy and degree of falsehood are represented as $I_A(x) = F_A(x) = 1 - T_A(x)$ respectively. The SVNS conversion is performed using Eq. (3).

$$\langle T_{ij}(x), I_{ij}(x), F_{ij}(x) \rangle = \langle R_{ij}(x), \left(1 - R_{ij}(x)\right), \left(1 - R_{ij}(x)\right) \rangle \quad (3)$$

Step 7: Identification of Ideal Solution for Beneficial and Non-Beneficial Attributes

According to sources [13], [14], and [21], the ideal solution for beneficial attributes—where higher values are advantageous (e.g., quality, profit)—is identified by determining the Positive Ideal Solution (PIS). Eqs. (3,4)

$$BAIS = \langle T^*_{max}(x), I^*_{min}(x), F^*_{min}(x)\rangle = \langle 1,0,0\rangle \tag{4}$$

Non beneficial attributes ideal solution

$$NBAIS = \langle T^*_{min}(x), I^*_{max}(x), F^*_{max}(x)\rangle = \langle 0,1,1\rangle \tag{5}$$

Step 8: Calculation of the entropy value of attribute E_j

Find the entropy value for attribute *with* Eq. (6).

$$E_j = 1 - 1/_n \sum_{i=1}^{m} \left(T_{ij}(x_i) + F_{ij}(x_i)\right) \left|2\left(I_{ij}(x_i)\right) - 1\right| \tag{6}$$

Step 9: Calculation of the entropy weight of attribute W_j

Find the entropy weight of attribute by Eq. (7).

$$W_j = \frac{1 - E_j}{\sum_{j=1}^{n}(1 - E_j)} \tag{7}$$

Here, we get weight vector $W = (w_1, w_{2,} w_{3, \dots\dots\dots\dots} w_n)^T$ of attributes, $C = \{C_j;\ for\ j = 1,2,3, \dots\dots..n\}$ with $W_j \geq 0$ and $\sum_{j=1}^{n} W_j = 1$.

Step 10: Calculate the entropy weight of alterative A_w

Find the alternative weight by Eq. (8).

$$A_w = \sum_{j=1}^{n} W_j * \left((T_{ij}(x) * T_{ij}{}^*(x)) + (I_{ij}(x) * I_{ij}{}^*(x) + (F_{ij}(x) * F_{ij}{}^*(x)\right) \tag{8}$$

Here, for beneficial attributes PIS $= \langle T^*_{max}(x), I^*_{min}(x), F^*_{min}(x)\rangle = \langle 1,0,0\rangle$ and for non-beneficial attributes NIS $= \langle T^*_{min}(x), I^*_{max}(x), F^*_{max}(x)\rangle = \langle 0,1,1\rangle$.

Step 11: Ranking of Alternatives

Upon the calculation of alternative weight A_w, the alternatives are ranked in descending order. The alternative with the highest correlation coefficient A_w is assigned the first rank, while the alternative with the lowest A_w is assigned the last rank. This ranking method ensures that alternatives are ordered based on their relative performance and suitability according to the calculated weights.

Collected Case Example of Supplier Selection

Step 1. One case example of supplier selection was adopted and demonstrate by [32] with DEA. The same case example was further calculated by [33] with DEA non-parametric approach. [34] was calculated the same matrix with GTMA, SAW, WPM, AHP, TOPSIS and modified TOPSIS methodology.

Step 2. Here, [34] explained that in input matrix contains eighteen different alternatives with five attributes. As per [34] attributes measures are C1: supply variety, means the company first listed all parts supplied by each vendor to obtained the supply variety. [34] explained that, if a vendor supplies more than one commodity group, then the supply variety of this vendor in each group is the sum of the number of part in the entire group. C2: aggregate quality with their weighted percentage of non-defective parts supplied by the supplier with regard to alternatives, C3: Distance (in Mile), C4: delivery is represented by percentage of purchase order within the delivery window according to purchase order, C5: price index, average prices were assigned to each part by the material department of the company. Where, [34] considered that beneficial attributes are C1 (Supply variety), C2 (aggregate quality), C4 (delivery) and C5 (price index) are the desirable criteria; whereas Non-beneficial attribute is C3 (distance) is non-desirable/ non beneficial criteria.

Step 3. Decision matrix is collected [32], [33], [34] and [35] shown in Table 5.

Table 5: Supplier Selection Input Matrix (Collected Case Example)

Alternatives (Sr. No.)	C1 (+)	C2 (+)	C3 (-)	C4 (+)	C5 (+)
A1	2	100	249	90	100
A2	13	99.79	643	80	100
A3	3	100	714	90	100
A4	3	100	1809	90	100
A5	24	99.83	238	90	100
A6	28	96.59	241	90	100
A7	1	100	1404	85	100
A8	24	100	984	97	100
A9	11	99.91	641	90	100
A10	53	97.54	588	100	100
A11	10	99.95	241	95	100
A12	7	99.85	567	98	100
A13	19	99.97	567	90	100
A14	12	91.89	967	90	100
A15	33	99.99	635	95	80
A16	2	100	795	95	100
A17	34	99.99	689	95	80
A18	9	99.36	913	85	100

Collected from the Source: [32], [33], [34], [35]

Step 4. Conversion of qualitative data in to quantitative data

Here, the input information contains quantitative information only, so there is no need to convert qualitative value in to quantitative value. So, this step is eliminated in the current case example.

Step 5. Normalization of Table 5 s carried out with the Eq. (1)/ Eq. (2). Supplier selection normalized matrix is shown in Table 6.

Table 6: Supplier Selection Normalized Matrix using VNM

Alternatives (Sr. No.)	C1 (+)	C2 (+)	C3 (-)	C4 (+)	C5 (+)
A1	0.0223	0.2377	0.9283	0.2318	0.2406
A2	0.1450	0.2372	0.8148	0.2060	0.2406
A3	0.0335	0.2377	0.7943	0.2318	0.2406
A4	0.0335	0.2377	0.4788	0.2318	0.2406
A5	0.2676	0.2373	0.9314	0.2318	0.2406
A6	0.3122	0.2296	0.9306	0.2318	0.2406
A7	0.0112	0.2377	0.5955	0.2189	0.2406
A8	0.2676	0.2377	0.7165	0.2498	0.2406
A9	0.1227	0.2375	0.8153	0.2318	0.2406
A10	0.5910	0.2318	0.8306	0.2575	0.2406
A11	0.1115	0.2376	0.9306	0.2447	0.2406
A12	0.0781	0.2373	0.8366	0.2524	0.2406
A13	0.2119	0.2376	0.8366	0.2318	0.2406
A14	0.1338	0.2184	0.7214	0.2318	0.2406
A15	0.3680	0.2377	0.8171	0.2447	0.1925
A16	0.0223	0.2377	0.7710	0.2447	0.2406
A17	0.3791	0.2377	0.8015	0.2447	0.1925

A18	0.1004	0.2362	0.7370	0.2189	0.2406

Step 6. Convert crisp normalized matrix into SVNS decision matrix: Crisp data is converted in SVNS $< T_{ij}(x), I_{ij}(x), F_{ij}(x) >$ degree of truthness, indeterminate and falsehood form by using Eqs. (3)/ (4).

- Beneficial attributes i.e. Alternative A1 and attribute C1 having value 0.0223 converted in SVNS gives the value $\langle 0.0223, (1-0.0223), (1-0.0223)\rangle = \langle 0.0223, 0.9777, 0.9777\rangle$. The same calculation is also carried out for attribute C2, C4 and C5.
- Non-beneficial attributes i.e. Alternative A1 and attribute C3 having value 0.9283 converted in SVNS gives the value $\langle (1-0.9283), 0.9283, 0.9283\rangle = \langle 0.0717, 0.9283, 0.9283\rangle$.
- Find the beneficial attribute ideal solution and non-beneficial attribute ideal solution.

Beneficial attribute ideal solution and non-beneficial attribute ideal solution is discovered with Equation (4)/ Equation (5), where $BAIS\ (A^*) = \langle T^*_{max}(x), I^*_{min}(x),\ F^*_{min}(x)\rangle = \langle 1, 0, 0\rangle$ and $NBAIS\ (A^*) = \langle T^*_{min}(x),\ I^*_{max}(x),\ F^*_{max}(x)\rangle = \langle 0, 1, 1\rangle$.

The rank is calculated with F-SVNS-EW-MADM is as shown in Table 7.

Table 7: F-SVNS EW-MADM Ranking for Supplier Selection

Sr. No.	C1 (+)	C2 (+)	C3 (-)	C4 (+)	C5 (+)	A_w	Rank
A1	<0.0223, 0.9777, 0.9777>	<0.2377, 0.7623, 0.7623>	<0.0717, 0.9283, 0.9283>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.5241	8
A2	<0.1450, 0.8550, 0.8550>	<0.2372, 0.7628, 0.7628>	<0.1852, 0.8148, 0.8148>	<0.2060, 0.7940, 0.7940>	<0.2406, 0.7594, 0.7594>	0.5005	10
A3	<0.0335, 0.9665, 0.9665>	<0.2377, 0.7623, 0.7623>	<0.2057, 0.7943, 0.7943>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.4709	13
A4	<0.0335, 0.9665, 0.9665>	<0.2377, 0.7623, 0.7623>	<0.5212, 0.4788, 0.4788>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.3394	18
A5	<0.2676, 0.7324, 0.7324>	<0.2373, 0.7627, 0.7627>	<0.0686, 0.9314, 0.9314>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.5826	3
A6	<0.3122, 0.6878, 0.6878>	<0.2296, 0.7704, 0.7704>	<0.0694, 0.9306, 0.9306>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.5912	2
A7	<0.0112, 0.9888, 0.9888>	<0.2377, 0.7623, 0.7623>	<0.4045, 0.5955, 0.5955>	<0.2189, 0.7811, 0.7811>	<0.2406, 0.7594, 0.7594>	0.3805	17
A8	<0.2676, 0.7324, 0.7324>	<0.2377, 0.7623, 0.7623>	<0.2835, 0.7165, 0.7165>	<0.2498, 0.7502, 0.7502>	<0.2406, 0.7594, 0.7594>	0.4965	12
A9	<0.1227, 0.8773, 0.8773>	<0.2375, 0.7625, 0.7625>	<0.1847, 0.8153, 0.8153>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.5004	11
A10	<0.5910, 0.4090, 0.4090>	<0.2318, 0.7682, 0.7682>	<0.1694, 0.8306, 0.8306>	<0.2575, 0.7425, 0.7425>	<0.2406, 0.7594, 0.7594>	0.6198	1
A11	<0.1115, 0.8885, 0.8885>	<0.2376, 0.7624, 0.7624>	<0.0694, 0.9306, 0.9306>	<0.2447, 0.7553, 0.7553>	<0.2406, 0.7594, 0.7594>	0.5482	5
A12	<0.0781, 0.9219, 0.9219>	<0.2373, 0.7627, 0.7627>	<0.1634, 0.8366, 0.8366>	<0.2524, 0.7476, 0.7476>	<0.2406, 0.7594, 0.7594>	0.5027	9
A13	<0.2119, 0.7881, 0.7881>	<0.2376, 0.7624, 0.7624>	<0.1634, 0.8366, 0.8366>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.5301	7

A14	<0.1338, 0.8662, 0.8662>	<0.2184, 0.7816, 0.7816>	<0.2786, 0.7214, 0.7214>	<0.2318, 0.7682, 0.7682>	<0.2406, 0.7594, 0.7594>	0.4603	15
A15	<0.3680, 0.6320, 0.6320>	<0.2377, 0.7623, 0.7623>	<0.1829, 0.8171, 0.8171>	<0.2447, 0.7553, 0.7553>	<0.1925, 0.8075, 0.8075>	0.5518	4
A16	<0.0223, 0.9777, 0.9777>	<0.2377, 0.7623, 0.7623>	<0.2290, 0.7710, 0.7710>	<0.2447, 0.7553, 0.7553>	<0.2406, 0.7594, 0.7594>	0.4609	14
A17	<0.3791, 0.6209, 0.6209>	<0.2377, 0.7623, 0.7623>	<0.1985, 0.8015, 0.8015>	<0.2447, 0.7553, 0.7553>	<0.1925, 0.8075, 0.8075>	0.5480	6
A18	<0.1004, 0.8996, 0.8996>	<0.2362, 0.7638, 0.7638>	<0.2630, 0.7370, 0.7370>	<0.2189, 0.7811, 0.7811>	<0.2406, 0.7594, 0.7594>	0.4599	16
A^*	<1.0000,0.0000, 0.0000>	<0.0000, 1.0000, 1.0000>	<0.0000, 1.0000, 1.0000>	<1.0000, 0.0000, 0.0000>	<1.0000, 0.0000, 0.0000>		
E_j	0.3366	0.4713	0.4078	0.4707	0.4704		
W_j	0.2333	0.1860	0.2083	0.1862	0.1862	1	

Performance Measures Comparison: Supplier Selection Ranking

The result of proposed three methodologies is compared with the published results to validate them for supplier selection. To compare the result, all supplier alternatives are ranked according to alternatives weight values is as shown in Table 8. The supplier alternatives are ranked first whose alternative weight value is highest; supplier alternative is ranked second whose alternatives weight values is second highest. Finally, the ranking order obtained by the proposed three methodologies is compared with the ranking order published in the literature and result comparisons are shown in Table 8.

Table 8: Supplier Selection Performance Measures Comparison

Alternatives (Sr. No.)	F-SVNS MADM Advance Entropy Weight	Ranking Solution Collected from [36]	Ranking Solution Collected from [34] [37]	
		DEA*	GTMA	TOPSIS
A1	8	9	7	12
A2	10	10	12	11
A3	13	14	15	15
A4	18	15	17	17
A5	3	3	2	5
A6	2	7	1	4
A7	17	17	18	18
A8	12	6	9	6
A9	11	11	11	10
A10	1	1	3	1
A11	5	5	6	8
A12	9	12	10	9
A13	7	8	8	7
A14	15	18	13	14
A15	4	4	4	3
A16	14	16	16	13
A17	6	2	5	2
A18	16	13	14	16

The result comparisons presented in Table 8 shows that the result obtained from the proposed methodologies are quite similar to the result of reported in the literature. The proposed method suggesting the supplier alternative A10 as the best supplier, which is same as suggests by [36] tried to solve same supplier selection problem by using DEA mathematical technique.

Further, 2nd rank is calculated by proposed methods doesn't match with all published results. While 4th rank is calculated by F-SVVNS AEW-MADM and modified TOPSIS methodologies published results their selves not match among each other, due to different weight criteria calculation/ assumption/ expert opinion or same equation of normalization/ without normalization. 4th rank of proposed F-SVNS AEW-MADM shows A15 which is matched with DEA, GTMA. It shows that the weight criteria and normalization equation/ method make change in rank position in further ranking result, but it holds well for the first ranking purpose.

The proposed methodology is characterized by minimal computational complexity, as they do not require the calculation of attribute weights or resizing of the assignment matrix. Additionally, they offer the flexibility to convert simple sets or linguistic sets into the F-SVNS (Fuzzy Single-Valued Neutrosophic Set) technique, unlike the DEA (Data Envelopment Analysis) approach. In a comparative study, [27] [34] [37] applied the GTMA (Generalized TOPSIS Methodology for Assessment) and TOPSIS methods to the same supplier selection problem. While TOPSIS ranked alternative 10 as the best supplier, the GTMA methodology identified alternative 6 as the best. The difference in rankings was primarily due to the variation in attribute weights.

The proposed methodology, however, operates with minimal calculations, does not require attribute weight determination, and does not need resizing of the assignment matrix. It also offers the advantage of converting simple or linguistic sets into F-SVNS. On the other hand, the F-SVNS EW-MADM method involves the calculation of attribute weights. When compared with other published results, the proposed methods demonstrate their validity, applicability, and reliability in supplier selection for manufacturing environments, ultimately leading to improvements in manufacturing functions.

Conclusions

The proposed methodology offers a streamlined approach with minimal computational complexity, eliminating the need to calculate the relative importance of attributes. It does not require resizing the assignment matrix and is capable of converting simple or linguistic sets into F-SVNS. Furthermore, the suggested methodology outperforms established methods such as GTMA, SAW, WPM, AHP, TOPSIS, Modified TOPSIS, and VIKOR. This chapter introduces the F-SVNS AEW-MADM technique, developed and implemented to evaluate its feasibility in the selection and ranking of supplier selection processes. The key conclusions of the proposed method are as follows:

- The methodology effectively converts both crisp data and fuzzy information provided by the decision maker into SVNS form, resulting in more efficient and reliable ranking solutions.
- It is robust in handling decision-making scenarios involving inconsistent, incomplete, and indeterminate information.
- The proposed approach facilitates more efficient negotiation and selection of the best alternative with reduced computational effort.
- The calculation and normalization process ensures no loss of information, and no attribute value is reduced to zero during the process.
- The F-SVNS AEW-MADM technique incorporates attribute weight calculation, and comparison with existing published results demonstrates its validity, applicability, and reliability in supplier selection for manufacturing environments. This leads to enhanced manufacturing performance and function.

The proposed methodology offers a significant contribution to enhancing decision-making processes for selection methodology, particularly in the context of supplier selection.

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Chapter-16

RNN-COPRAS Strategy for MCDM under Rough Neutrosophic Setting

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ABSTRACT

The purpose of the study is to extend the Complex Proportional Assessment (COPRAS) strategy to the Rough Neutrosophic Number (RNN) setting that we name the RNN-COPRAS strategy. The RNN is the hybrid structure of rough number and neutrosophic number that has emerged as a tool to deal with MCDM problems. In this paper, the COPRAS strategy for MCDM in the RNN setting is developed. To show the applicability of the developed strategy an illustrative MCDM problem is solved and compared with existing method.

Keywords: Neutrosophic set, rough neutrosophic set, MCDM, COPRAS.

INTRODUCTION

Smarandache [1] grounded the Neutrosophic Sets (NSs), building upon the concepts of Fuzzy Sets (FSs) [2] and Intuitionistic FSs (IFSs) [3]. NS provides a more generalized framework for addressing uncertainty and indeterminacy. To make NS more applicable in real-world scenarios, Wang et al. [4] introduced Single-Valued NS (SVNS), a specialized subclass of NSs. Over the years, numerous studies explored theoretical advancements and practical applications of NSs [5-12]. SVNSs and their extensions have been used in decision making [13-31], conflict theory [32] , educational issues [33, 34, 35], social issues [36, 37] and so on.

Pawlak [38] introduced the Rough Set (RS) as a mathematical tool to manage uncertain and incomplete information. To further enhance the ability to handle both incompleteness and uncertainty, Broumi et al. [39, 40] proposed the Rough Neutrosophic Set (RNS), which integrated the principles of RSs and NSs. Recent studies, such as those by Pramanik [41] and Zhang et al. [42] provided comprehensive overviews of RNSs and their extensions and applications. Different MCDM strategies have been grounded under Rough Neutrosophic Number (RNN) environments for handling MCDM problems such as GRA [43], accuracy score-based strategy [44], similarity measure-based strategies [45, 46, 47, 48, 49], TOPSIS [50], aggregation operator-based strategy [51], correlation coefficient -based strategy [52], on projection measures- based strategy [53].

COPRAS (Complex Proportional Assessment) [54] was grounded in 1994. The COPRAS method relies on integrating both the ideal solution and the worst-ideal solution. Bekar et al. [55] developed the COPRAS strategy under FS environment in 2016. Mahdiraji et al. [56] grounded the hybrid BWM-COPRAS strategy in the FS environment in 2018. Using parametric measures for IFSs. Kumari and Mishra [57] proposed the COPRAS strategy using parametric measures for IFSs setting. Hajiagha et al. [58] used COPRAS strategy to interval IFS setting. Sahin [59] presented the COPRAS strategy in interval neutrosophic number [60] setting.

BACKGROUND

Definition 1. Let $\delta_i = (p_i, q_i, r_i)$ be a SVNN with $p_i, q_i, r_i \in [0,1]$, $(p_i + q_i + r_i) \in [0,3]$, i= 1, 2.

The following operations [11] hold.

i. $\delta_1 \oplus \delta_2 = (p_1 + p_2 - p_1 p_2, q_1 q_2, r_1 r_2)$ [Summation] (1)

ii. $\delta_1 \otimes \delta_2 = (p_1 p_2, q_1 + q_2 - q_1 q_2, r_1 + r_2 - r_1 r_2)$ [Multiplication] (2)

iii. $a\delta_1 = (1-(1-p_1)^a, q_1^a, r_1^a)$, $a > 0$ [Scalar multiplication] (3)

iv. $\delta_1^a = (p_1^a, 1-(1-q_1)^a, 1-(1-r_1)^a)), a > 0$ (4)

Definition 2. Euclidean distance function. Euclidean distance [21] between $\delta_1 = (\alpha_1, \beta_1, \gamma_1)$ and $\delta_2 = (\alpha_2, \beta_2, \gamma_2)$ is defined as:

$$d_e = \left[\frac{1}{3} \left\{ (\alpha_1 - \alpha_2)^2 + (\beta_1 - \beta_2)^2 + (\gamma_1 - \gamma_2)^2) \right\} \right]^{\frac{1}{2}} \quad (5)$$

3. Score function.

Score function [61] denoted by $Sc(\delta_1)$ of an SVNN $\delta_1 = (\eta_1, \eta_2, \eta_3)$ is defined as

$$Sc(\delta_1) = (2 + \eta_1 - 0.3 \times \eta_2 - 0.4 \times \eta_3)/3 \quad (6)$$

Definition 3. RNS [39]

Let $\ddot{\Theta}$ be a universe of discourse and $\ddot{R}$ be an equivalence relation over $\ddot{\Theta}$. For a NS $\ddot{\Phi}$, two approximation of $\ddot{\Phi}$ are presented as:

$$\underline{\ddot{\nu}}(\ddot{\Phi}) = \left\langle < \ddot{\theta}, \ddot{\delta}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\varepsilon}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\kappa}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) > / \ddot{\zeta} \in [\ddot{\theta}]_{\ddot{R}}, \ddot{\theta} \in \ddot{\Theta} \right\rangle \quad (7)$$

$$\overline{\ddot{\nu}}(\ddot{\Phi}) = \left\langle < \ddot{\theta}, \ddot{\delta}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\varepsilon}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}), \ddot{\kappa}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) > / \ddot{\zeta} \in [\ddot{\theta}]_{\ddot{R}}, \ddot{\theta} \in \ddot{\Theta} \right\rangle \quad (8)$$

$$\ddot{\delta}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}} \, \ddot{\delta}_{\ddot{\Phi}}(\ddot{\zeta}), \ddot{\varepsilon}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}} \, \ddot{\varepsilon}_{\ddot{\Phi}}(\ddot{\zeta}), \ \ddot{\kappa}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}} \, \ddot{\kappa}_{\ddot{\Phi}}(\ddot{\zeta})$$

$$\ddot{\delta}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}} \, \ddot{\delta}_{\ddot{\Phi}}(\ddot{\zeta}), \ddot{\varepsilon}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}} \, \ddot{\varepsilon}_{\ddot{\Phi}}(\ddot{\zeta}), \ \ddot{\kappa}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) = \vee_{\ddot{\zeta}} \in [\ddot{\theta}]_{\ddot{R}} \, \ddot{\kappa}_{\ddot{\Phi}}(\ddot{\zeta})$$

So, $0 \leq \ddot{\delta}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\varepsilon}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\kappa}_{\underline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) \leq 3$

$0 \leq \ddot{\delta}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\varepsilon}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) + \ddot{\kappa}_{\overline{\ddot{\nu}}(\ddot{\Phi})}(\ddot{\theta}) \leq 3$.

Here, $\vee$ and $\wedge$ present respectively the max and min operator. $\ddot{\delta}_{\ddot{\Phi}}(\ddot{\zeta}), \ddot{\varepsilon}_{\ddot{\Phi}}(\ddot{\zeta})$,and $\ddot{\kappa}_{\ddot{\Phi}}(\ddot{\zeta})$ are the truth Membership Function (MF), indeterminacy MF and falsity MF of $\ddot{\zeta}$ w.r.t. $\ddot{\Phi}$. Clearly, $\underline{\ddot{\nu}}(\ddot{\Phi})$ and $\overline{\ddot{\nu}}(\ddot{\Phi})$ are NSs in $\ddot{\Phi}$.

The NS mapping $\underline{\ddot{\nu}}, \overline{\ddot{\nu}} : \ddot{\nu}(\ddot{\Phi}) \rightarrow \ddot{\nu}(\ddot{\Phi})$ presents as the lower and upper RNS approximation operators. The pair $(\underline{\ddot{\nu}}(\ddot{\Phi}), \overline{\ddot{\nu}}(\ddot{\Phi}))$ is termed as the RNS in $(\ddot{\Theta}, \ddot{R})$.

RNN-COPRAS Strategy in RNN Settings

Consider an MADM problem with n attributes, and m alternatives. Assume that $\dot{C} = \langle \dot{C}_1, \dot{C}_2, ..., \dot{C}_n \rangle$ and $\dot{A} = \langle \dot{A}_1, \dot{A}_2, ..., \dot{A}_m \rangle$ present respectively the set of attributes, and alternatives. Weight w'''_j is allocated to $\dot{C}_j$ with $\sum_{j=1}^{n} w'''_j = 1, w'''_j \geq 0, j = 1\, to\, n.$

RNN-COPRAS strategy is developed utilizing the following steps (see Fig. 1):

Step 1. Formulate the RNN Decision Matrix (RNN-DM)

Utilizing the rating values provided by the expert, RNN-DM $\dot{\Theta}_{\tilde{R}}$ is constructed as:

$$\dot{\Theta}_{\tilde{R}} = \langle \underline{\dot{\varepsilon}}_{ij}, \overline{\dot{\varepsilon}}_{ij} \rangle_{m\times n} = \begin{array}{c|cccc} & \dot{C}_1 & \dot{C}_2 & \cdots & \dot{C}_n \\ \hline \dot{A}_1 & \langle \underline{\dot{\varepsilon}}_{11}, \overline{\dot{\varepsilon}}_{11} \rangle & \langle \underline{\dot{\varepsilon}}_{12}, \overline{\dot{\varepsilon}}_{12} \rangle & \cdots & \langle \underline{\dot{\varepsilon}}_{1n}, \overline{\dot{\varepsilon}}_{1n} \rangle \\ \dot{A}_2 & \langle \underline{\dot{\varepsilon}}_{21}, \overline{\dot{\varepsilon}}_{21} \rangle & \langle \underline{\dot{\varepsilon}}_{22}, \overline{\dot{\varepsilon}}_{22} \rangle & \cdots & \langle \underline{\dot{\varepsilon}}_{2n}, \overline{\dot{\varepsilon}}_{2n} \rangle \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \dot{A}_m & \langle \underline{\dot{\varepsilon}}_{m1}, \overline{\dot{\varepsilon}}_{m1} \rangle & \langle \underline{\dot{\varepsilon}}_{m2}, \overline{\dot{\varepsilon}}_{m2} \rangle & \cdots & \langle \underline{\dot{\varepsilon}}_{mn}, \overline{\dot{\varepsilon}}_{mn} \rangle \end{array} \quad (9)$$

Here, $\langle \underline{\dot{\varepsilon}}_{ij}, \overline{\dot{\varepsilon}}_{ij} \rangle_{m\times n} = \langle \langle \underline{t'}_{ij}, \underline{i'}_{ij}, \underline{\phi'}_{ij} \rangle, \langle \overline{t'}_{ij}, \overline{i'}_{ij}, \overline{\phi'}_{ij} \rangle \rangle_{m\times n}$ indicates the rating value of $\dot{A}_j$ w.r.t. $\dot{C}_j$.

Step 2. Utilizing the Accumulated Geometric Operator (AGO), transform the RNN-DM into a SVNN Decision Matrix (SVNN-DM)

We convert the RNNs into SVNNs by the AGO [51] as follows:

$$\begin{aligned} \langle \underline{\dot{\varepsilon}}_{ij}, \overline{\dot{\varepsilon}}_{ij} \rangle_{AGO} &= \langle \langle \underline{t'}_{ij}, \underline{i'}_{ij}, \underline{\phi'}_{ij} \rangle, \langle \overline{t'}_{ij}, \overline{i'}_{ij}, \overline{\phi'}_{ij} \rangle \rangle_{AG0} \\ &= \left\langle \left(\underline{t'}_{ij} . \overline{t'}_{ij} \right)^{1/2}, \left(\underline{i'}_{ij} . \overline{i'}_{ij} \right)^{1/2}, \left(\underline{\phi'}_{ij} . \overline{\phi'}_{ij} \right)^{1/2} \right\rangle \\ &= \langle \dot{t}_{ij}, \dot{i}_{ij}, \dot{\phi}_{ij} \rangle \end{aligned} \quad (10)$$

The RNN-DM $\dot{\Theta}_{\tilde{R}}$ is converted into the SVNN-DM $\Theta_{\tilde{N}}$

$$\Theta_{\tilde{N}} = \left\langle \dot{t}_{ij}, \mathrm{i}_{ij}, \dot{\phi}_{ij} \right\rangle$$

$$= \begin{array}{c|cccc} & \dot{C}_1 & \dot{C}_2 & \cdots & \dot{C}_n \\ \hline \dot{A}_1 & \left\langle \dot{t}_{11}, \mathrm{i}_{11}, \dot{\phi}_{11} \right\rangle & \left\langle \dot{t}_{12}, \mathrm{i}_{12}, \dot{\phi}_{12} \right\rangle & \cdots & \left\langle \dot{t}_{1n}, \mathrm{i}_{1n}, \dot{\phi}_{1n} \right\rangle \\ \dot{A}_2 & \left\langle \dot{t}_{21}, \mathrm{i}_{21}, \dot{\phi}_{21} \right\rangle & \left\langle \dot{t}_{22}, \mathrm{i}_{22}, \dot{\phi}_{22} \right\rangle & \cdots & \left\langle \dot{t}_{2n}, \mathrm{i}_{2n}, \dot{\phi}_{2n} \right\rangle \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \\ & & & & \cdots \\ . & \cdots & \cdots & \cdots & \\ \dot{A}_2 & \left\langle \dot{t}_{m1}, \mathrm{i}_{m1}, \dot{\phi}_{m1} \right\rangle & \left\langle \dot{t}_{m2}, \mathrm{i}_{m2}, \dot{\phi}_{m2} \right\rangle & \cdots & \left\langle \dot{t}_{mn}, \mathrm{i}_{mn}, \dot{\phi}_{mn} \right\rangle \end{array} \quad (11)$$

Step 3. Standardize the SVNN-DM

We standardize $\Theta_{\tilde{N}}$ using the formula (12) [21].

$$\Theta_{\tilde{N}} = \begin{cases} \left\langle \dot{t}_{ij}, \mathrm{i}_{ij}, \dot{\phi}_{ij} \right\rangle if\ \dot{C}_j\ is\ a\ positive\ criterion \\ \left\langle \dot{\phi}_{ij}, 1-\mathrm{i}_{ij}, \dot{t}_{ij} \right\rangle if\ \dot{C}_j\ is\ a\ negative\ criterion \end{cases} \quad (12)$$

Then the standardized SVNN-DM appears of the form:

$$\dot{\Theta}_{\tilde{N}} = \left\langle \dot{t}'_{ij}, \mathrm{i}'_{ij}, \dot{\phi}'_{ij} \right\rangle_{m\times n} \quad (13)$$

Step 4 . Compute the weighted SVNN-DM

$$\tilde{\dot{\Theta}}_{\tilde{N}} = \left\langle \dot{t}'_{ij}, \mathrm{i}'_{ij}, \dot{\phi}'_{ij} \right\rangle_{m\times n} \otimes \omega'''_j = \left(1-\left(1-\dot{t}'_{ij}\right)^{\omega'''_j}, \mathrm{i}'^{\omega'''_j}_{ij}, \dot{\phi}'^{\omega'''_j}_{ij} \right), i=1,2,...,m;\ j=1,2,...,n$$

(14)

Step 5. Convert the weighted SVNN-DM into Crisp Decision Matrix (CDM) using score values

Using the formula (6), the CDM Θ_C is constructed as follows: .

$$\Theta_C = \begin{array}{c|cccc} & \dot{C}_1 & \dot{C}_2 & \cdots & \dot{C}_n \\ \hline \dot{A}_1 & s_{11} & s_{12} & \cdots & s_{1n} \\ \dot{A}_2 & s_{21} & s_{22} & \cdots & s_{2n} \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \\ & & & & \cdots \\ . & \cdots & \cdots & \cdots & \\ \dot{A}_2 & s_{m1} & s_{m2} & \cdots & s_{mn} \end{array} \quad (15)$$

where $s_{ij} = Sc\left\langle 1-\left(1-\dot{t}'_{ij}\right)^{\omega'''_j}, \mathrm{i}'^{\omega'''_j}_{ij}, \dot{\phi}'^{\omega'''_j}_{ij} \right\rangle = (2+1-\left(1-\dot{t}'_{ij}\right)^{\omega'''_j} - 0.3\times \mathrm{i}'^{\omega'''_j}_{ij} - 0.4\times \dot{\phi}'^{\omega'''_j}_{ij})/3$ (16)

Step 6. Calculate the maximizing and minimizing indices

Utilizing the formula (17) and (18), we obtain the maximizing and minimizing indices of each attribute.

$$\ell_{+i} = \sum_{j=1}^{\theta} s_{ij}, i=1,2,...,m \quad (17)$$

$$\ell_{-i} = \sum_{j=\theta+1}^{n} s_{ij}, i=1,2,...,m \quad (18)$$

where θ indicates the number of benefit attributes and n-g indicates the number of cost (negative) attributes. Here ℓ_{+i} and ℓ_{-i} indicates respectively the maximizing and the minimizing indices of i^{th} alternative.

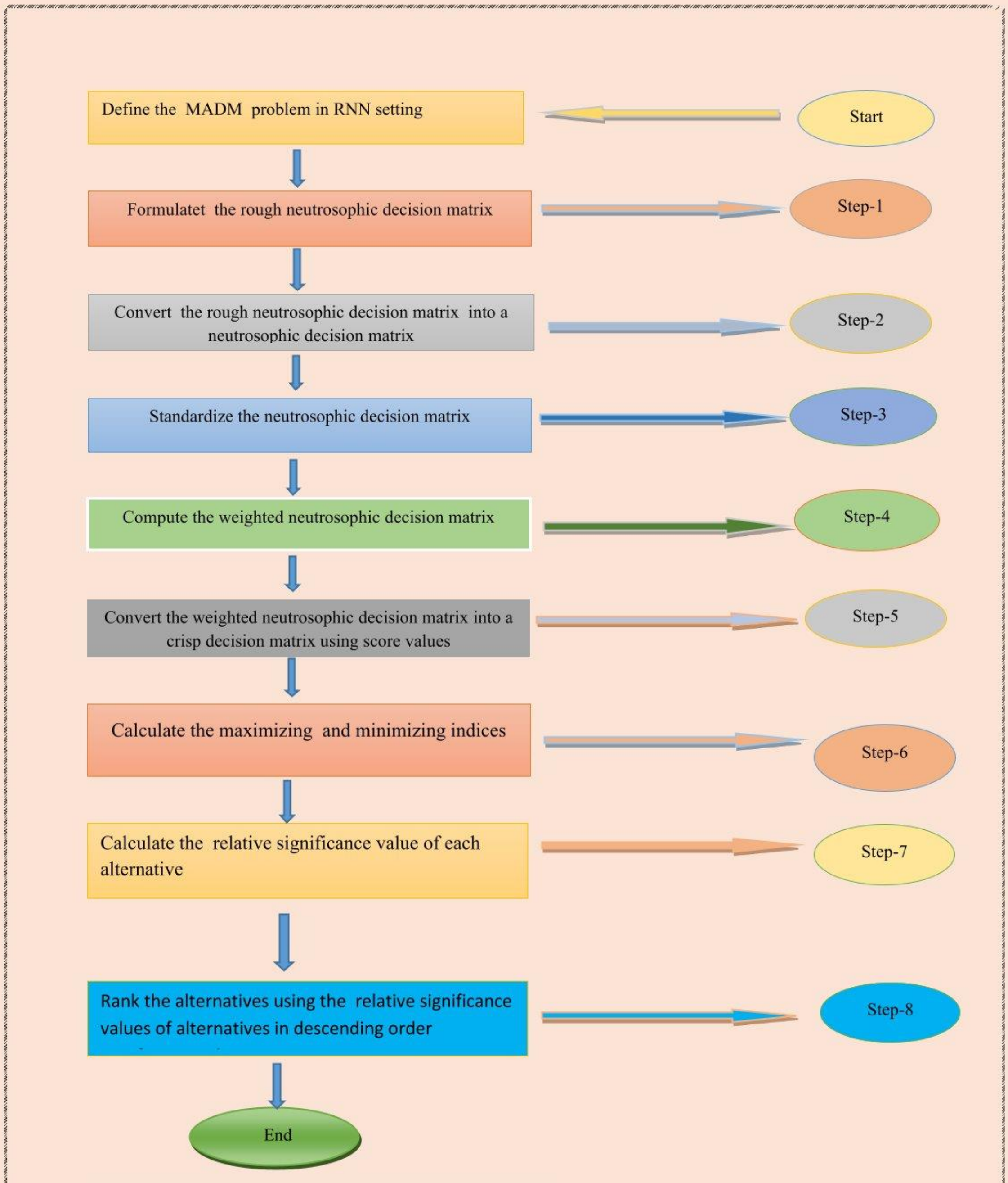

Fig. 1. COPRAS strategy in RNN setting

Step 7. Calculate the 'relative significance value'

The 'relative significance value of each alternative is calculated utilizing the formula (18) or (19).

$$\Psi_i = \ell_{+i} + \frac{\min_i \ell_{-i} \sum_{i=1}^{m} \ell_{-i}}{\ell_{-i} \sum_{i=1}^{m} \frac{\min_i \ell_{-i}}{\ell_{-i}}} \quad (19)$$

$$\Psi_i = \ell_{+i} + \frac{\sum_{i=1}^{m} \ell_{-i}}{\ell_{-i} \sum_{i=1}^{m} \frac{1}{\ell_{-i}}} \quad (20)$$

Step 8. Rank the options/alternatives

The alternatives are ranked based on their relative significance values, listed in descending order. The alternative having the highest final value receives the best rank.

The most desirable alternative is the one with the highest value Ψ_i.

.

Step 9 End.

Illustrative Example

An expert seeks to purchase the most appropriate smartphone from the initially chosen models $\dot{A}_1, \dot{A}_2$, and $\dot{A}_3$. The chosen attributes are namely, characteristics $\dot{C}_1$, cost $\dot{C}_2$, customer care $\dot{C}_3$, and safety factor $\dot{C}_4$.

The weights assigned to the four attributes are 0.3, .03, 0.3, 0.1 respectively.

Using RNN-COPRAS strategy, we solve the problem.

Step 1.

The RNN-DM (refer to Table 1) is constructed based on expert's rating values of alternatives across each criterion

Table 1: RNN-DM

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	positve	*negative*	*positive*	*negative*
$\dot{A}_1$	$\langle(.6,.3,.3),(0.8,0.1,0.1)\rangle$	$\langle(.6,.4,.4),(0.8,0.2,0.2)\rangle$	$\langle(.6,.4,.4),(0.8,0.2,0.4)\rangle$	$\langle(.7,.4,.7),(0.9,0.2,0.1)\rangle$
$\dot{A}_2$	$\langle(0.7,0.3,0.3),(0.9,0.1,0.3)\rangle$	$\langle(0.6,0.3,0.3),(0.8,0.3,0.3)\rangle$	$\langle(0.6,0.2,0.2),(0.8,0.4,0.2)\rangle$	$\langle(0.7,0.3,0.2),(0.9,0.3,0.3)\rangle$
$\dot{A}_2$	$\langle(0.6,0.2,0.2),(0.8,0.0,0.2)\rangle$	$\langle(0.7,0.3,0.2),(0.9,0.1,0.1)\rangle$	$\langle(0.7,0.4,0.6),(0.9,0.2,0.4)\rangle$	$\langle(0.6,0.3,0.2),(0.8,0.1,0.1)\rangle$

Step 2.

By applying formula (10), the RNN- DM is transformed into the SVNN-DM

Table 2: SVNN-DM

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	popsitive	*negative*	*positive*	*negative*
$\dot{A}_1$	$\langle 0.69282, 0.1732051, 0.173205\rangle$	$\langle 0.69282, 0.282843, 0.282843\rangle$	$\langle 0.69282, 0.282843, 0.4\rangle$	$\langle 0.793725, 0.282843, 0.264575\rangle$
$\dot{A}_2$	$\langle 0.793725, 0.1732051, 0.244949\rangle$	$\langle 0.69282, 0.3, 0.3\rangle$	$\langle 0.69282, 0.282843, 0.2\rangle$	$\langle 0.793725, 0.3, 0.244949\rangle$
$\dot{A}_3$	$\langle 0.69282, 0, 0.2\rangle$	$\langle 0.793725, 0.173205, 0.141421\rangle$	$\langle 0.793725, 0.282843, 0.489898\rangle$	$\langle 0.69282, 0.173205, 0.141421\rangle$

Step 3.

The SVNN-DM is normalized (refer to Table 3) by applying formula (12).

Table 3: Standardized SVNN-DM

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	positive	*negative*	*positive*	*negative*
$\dot{A}_1$	$\langle 0.69282, 0.1732051, 0.173205\rangle$	$\langle 0.282843\ \ 0.717157\ \ 0.69282\rangle$	$\langle 0.69282, 0.282843, 0.4\rangle$	$\langle 0.264575\ \ 0.717157\ \ 0.793725\rangle$
$\dot{A}_2$	$\langle 0.793725, 0.1732051, 0.244949\rangle$	$\langle 0.3\ \ 0.7\ \ 0.69282\rangle$	$\langle 0.69282, 0.282843, 0.2\rangle$	$\langle 0.264575\ \ 0.717157\ \ 0.793725\rangle$
$\dot{A}_3$	$\langle 0.69282, 0, 0.2\rangle$	$\langle 0.141421\ \ 0.826795\ \ 0.793725\rangle$	$\langle 0.793725, 0.282843, 0.489898\rangle$	$\langle 0.141421\ \ 0.826795\ \ 0.69282\rangle$

Step 4.

Using the formula (12), the weighted SVNN-DM is formulated (refer to Table 4).

Table 4: Weighted SVNN-DM

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	positive	*negative*	*positive*	*negative*
$\dot{A}$	$\langle 0.298192922\ \ 0.590974\ \ 0.590974\rangle$	$\langle 0.094925509\ \ 0.905074\ \ 0.895749\rangle$	$\langle 0.298193\ \ 0.684642\ \ 0.759658\rangle$	$\langle 0.030263\ \ 0.967301\ \ 0.977163\rangle$
$\dot{A}_2$	$\langle 0.377221329\ \ 0.590974\ \ 0.655726\rangle$	$\langle 0.101476558\ \ 0.898523\ \ 0.895749\rangle$	$\langle 0.298193\ \ 0.684642\ \ 0.617034\rangle$	$\langle 0.027706\ \ 0.964961\ \ 0.977163\rangle$
$\dot{A}_3$	$\langle 0.298192922\ \ 0\ \ 0.617034\rangle$	$\langle 0.044712655\ \ 0.944538\ \ 0.933042\rangle$	$\langle 0.377221\ \ 0.684642\ \ 0.807294\rangle$	$\langle 0.015132\ \ 0.98116\ \ 0.963967\rangle$

Step 5.

Calculate the score values for the weighted SVNN-DM and construct the CDM (refer to Table 5).

Table 5: CDM

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	posiitve	*negative*	*positve*	*negative*
$\dot{A}_1$	0.62817	0.488368	0.596312	0.449736
$\dot{A}_2$	0.64588	0.491207	0.615329	0.449117
$\dot{A}3$	0.683793	0.462712	0.616304	0.445066

Step 6.

We obtain the maximizing and minimizing indices as follows:

$$\Theta_{+1} = 1.224482841, \Theta_{+2} = 0.215971.261208477, \Theta_{+3} = 1.300096851$$

$$\Theta_{-1} = 0.938104, \Theta_{-2} = 0.940324, , \Theta_{-3} = 0.907777$$

Step 7. Calculate the relative significance value

$$\Psi_1 = 10.82078326, \Psi_2 = 10.83485022, \Psi_3 = 11.21698534$$

Step 8. Ranking of the alternatives

Since $\Psi_3 > \Psi_2 > \Psi_1$, we obtain $\dot{A}_3 \succ \dot{A}_2 \succ \dot{A}_1$

So, $\dot{A}_3$ is the most desirable smartphone.

The problem is solved with RNN-MABAC and we obtain the same ranking order (see table 6).

Table 6: Comparison table

Strategy	Ranking order	Best alternative
RNN-MABAC [62]	$\dot{A}_3 \succ \dot{A}_2 \succ \dot{A}_1$	$\dot{A}_3$
RNN-COPRAS (proposed)	$\dot{A}_3 \succ \dot{A}_2 \succ \dot{A}_1$	$\dot{A}_3$

Conclusions

In this study, the RNN-COPRAS approach is developed within the RNN framework. To demonstrate its practicality, an illustrative MCDM problem is solved using the proposed RNN-COPRAS approach. RNN-COPRAS approach suitable to handle other MCDM problems in RNN environments. Furthermore, the RNN-COPRAS strategy has potential for extension into group decision making using suitable aggregation operators.

Future Research Directions

The proposed RNN-COPRAS approach can be utilized to handle different MCDM problems such as clustering analysis [63], green supplier selection [64], E-commerce site selection [65], vaccine selection [66], risk assessment [67], seismic soil liquefaction analysis [68] and so on.

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Chapter 17

ITrNN-TODIM Strategy for MCDM in Interval Trapezoidal Neutrosophic Number Environment

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ABSTRACT

The main purpose of this paper is to develop a TODIM (TOmada de Decisao Interativa Multicriterio) strategy for Multi Criteria Decision Making (MCDM) in Interval Trapezoidal Neutrosophic Number (ITrNN) environment, which we call ITrNN- TODIM. To extend this new MCDM-TODIM strategy, we define a score function and an accuracy function for ITrNNs and prove some of their basic properties. The decision maker evaluates the alternatives with respect to the prescribed criteria and provides his/her rating in terms of ITrNNs. We solve a MCDM problem to reflect the developed TODIM strategy in ITrNN environment.

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Keywords: Trapezoidal Neutrosophic Number, Interval Trapezoidal Neutrosophic Number, Score function, Accuracy Function, Hamming Distance.

INTRODUCTION

Multi-Criteria Decision Making (MCDM) is a significant known branch in decision making analysis. Also, MCDM strategies are very useful on those situations where some alternatives are ranked w.r.t to some conflicting criteria.

Neutrosophic Sets (NSs) [1] offer a robust approach for handling uncertainty involving inconsistent, and indeterminate information. Wang et al. [2] introduced the Single Valued NS (SVNS), where the values of each membership function are confined to the range [0,1]. Interval NS (INS) was presented by Wang et al. [3] by extending NSs. Theoretical developments, as well as practical applications of NSs [4], SVNSs [2], and related concepts, have been extensively studied and documented in the works [5], [6], [7], [8], [9], and [10]. Additionally, several extensions and hybrid versions of NSs have been introduced, including the Quadripartitioned Neutrosophic Set (QNS) [11], Interval QNS (IQNS) [12], Pentapartitioned NS (PNS) [13], Interval PNS (IPNS) [14], Rough NS (RNS) [15], Interval RNS [16], Rough Bipolar NS [17], Tri-complex RNS [18], etc.

Ye [19] and Subas [20] presented the Trapezoidal Neutrosophic Number (TrNN) by extending intuitionistic fuzzy number. On the basis of TrNN, different MCDM strategies have been proposed such as TOPSIS [21], MADM based on expected value of TrNNs [22], VIKOR [23], MULTIMOORA[24], GRA [25], EDAS [26], ARAS [27], etc.

Biswas et al. [28] proposed Interval Trapezoidal Neutrosophic Number (ITrNN) in 2018. In this same paper Biswas et al. developed some operational rules of ITrNN, and developed a new MCDM strategy. Later on, Giri et al. [29] developed the TOPSIS strategy for ITrNN environment. Mallick and Pramanik [30] developed VIKOR strategy in the ItrNN environment.

Here, we explore the development of the TODIM strategy across various environments. Initially, Gomes and Lima [31, 32] introduced the TODIM strategy, which was grounded in prospect theory [33]. The primary aim of TODIM is to establish an optimal ranking order for alternatives. Subsequently, Gomes and Rangel [34] applied the TODIM strategy to assess multicriteria rental evaluations for residential properties. In 2013, Fan et al. [35] extended the TODIM strategy to address hybrid MADM problems. Following this, Wang [36] adapted the TODIM strategy for decision-making under NS environment. In 2016, Zhang et al. [37] advanced the TODIM strategy for Multi-Criteria Group Decision Making (MCGDM) within a Neutrosophic Number (NN) setting. Pramanik et al. [38] introduced the NC-TODIM strategy for MCGDM in a neutrosophic cubic set setting, and in a subsequent study [39], they developed the TODIM approach for MCGDM within a bipolar NS setting. In 2019, Pramanik and Mallick [40] further enhanced the MCGDM-TODIM strategy for the TrNN environment.

Research gap: The TODIM strategy in the ITrNN environment has not been explored in the literature. This approach serves as a broader extension of the TrNN-TODIM method.

Motivation: To take the challenge to develop an ITrNN-TODIM strategy in ITrNN environment.

To demonstrate the proposed TODIM strategy in ITrNN environment, we solve an illustrative MCDM problem.

This paper is structured as follows: The background section presents a brief overview of key definitions, including TrNN, ITrNN, and the Hamming distance between two ITrNNs. The subsequent section introduces the score and accuracy functions. Following that, we discuss the standardization of the decision matrix. In the next section, we present the development of the ITrNN-TODIM strategy within the ITrNN environment. The subsequent section illustrates the proposed TODIM strategy through a numerical example of an MCDM problem. Finally, the paper concludes with a summary of the findings and future research direction.

BACKGROUND

This section presents a brief overview of key definitions, including TrNN, ITrNN, and the Hamming distance between two ITrNNs,

Definition 1: [19, 20] Suppose that γ is an TrNN. Its truth membership, indeterminacy membership and falsity membership functions are defined as:

$$T_{\gamma}(v)=\begin{cases}\dfrac{(v-c_1)p'_{\gamma}}{(c_2-c_1)}, & c_1\le v<c_2\\ p'_{\gamma}, & c_2\le v\le c_3\\ \dfrac{(c_4-v)p'_{\gamma}}{(c_4-c_3)}, & c_3\le v\le c_4\\ 0, & \text{otherwise.}\end{cases} \quad (1)$$

$$I_{\gamma}(v)=\begin{cases}\dfrac{(c_2-v)+(v-c_1)q'_{\gamma}}{(h'_2-h'_1)}, & c_1\le v\le c_2\\ q'_{\gamma}, & c_2\le v\le c_3\\ \dfrac{v-c_3+(c_4-v)q'_{\gamma}}{c_4-c_3}, & c_3\le v\le c_4\\ 0, & \text{otherwise.}\end{cases} \quad (2)$$

$$F_{\gamma}(v)=\begin{cases}\dfrac{c_2-v+(v-c_1)r'_{\gamma}}{c_2-c_1}, & c_1\le v<c_2\\ r'_{\gamma}, & c_2\le v\le c_3\\ \dfrac{v-c_3+(c_4-v)r'_{\gamma}}{c_4-c_3}, & c_3\le v\le c_4\\ 0, & \text{otherwise.}\end{cases} \quad (3)$$

Here $0\le T_{\gamma}(v)\le 1$, $0\le I_{\gamma}(v)\le 1$, and $0\le F_{\gamma}(v)\le 1$ and $0\le T_{\gamma}(v)+I_{\gamma}(v)+F_{\gamma}(v)\le 3; c_1,c_2,c_3,c_4\in R$ .Then TrNN γ is presented as $\gamma=([c_1,c_2,c_3,c_4]; p'_{\gamma},q'_{\gamma},r'_{\gamma})$.

Definition 2: [28] Assume that $p'_{\tilde{\gamma}}=[\underline{p}',\overline{p}'], q'_{\tilde{\gamma}}=[\underline{q}',\overline{q}']$ and $r'_{\tilde{\gamma}}=[\underline{r}',\overline{r}']$. Then an ITrNN denoted by $\tilde{\gamma}=([c_1,c_2,c_3,c_4]; p'_{\tilde{\gamma}},q'_{\tilde{\gamma}},r'_{\tilde{\gamma}})$ is defined as:

$$T_{\tilde{\gamma}}(v)=\begin{cases}\dfrac{(v-c_1)p'_{\tilde{\gamma}}}{(c_2-c_1)}, & c_1\le v<c_2\\ p'_{\tilde{\gamma}}, & c_2\le v\le c_3\\ \dfrac{(c_4-v)p'_{\tilde{\gamma}}}{(c_4-c_3)}, & c_3\le v\le c_4\\ 0, & \text{otherwise.}\end{cases} \quad (4)$$

$$I_{\tilde{\gamma}}(v)=\begin{cases}\dfrac{(c_2-v)+(v-c_1)q'_{\tilde{\gamma}}}{(c_2-c_1)}, & c_1\le v<c_2\\ q'_{\tilde{\gamma}}, & c_2\le v\le c_3\\ \dfrac{v-c_3+(c_4-v)q'_{\tilde{\gamma}}}{c_4-c_3}, & c_3\le v\le c_4\\ 0, & \text{otherwise.}\end{cases} \quad (5)$$

$$F_{\tilde{\gamma}}(v)=\begin{cases}\dfrac{c_2-v+(v-c_1)r'_{\tilde{\gamma}}}{c_2-c_1}, & c_1\le v<c_2\\ r'_{\tilde{\gamma}}, & c_2\le v\le c_3\\ \dfrac{v-c_3+(c_4-v)r'_{\tilde{\gamma}}}{c_4-c_3}, & c_3\le v\le c_4\\ 0, & \text{otherwise.}\end{cases} \quad (6)$$

where the truth, indeterminacy and falsity membership function are $T_{\tilde{\gamma}}, I_{\tilde{\gamma}}$ and $F_{\tilde{\gamma}}$ respectively. Here, $p'_{\tilde{\gamma}}, q'_{\tilde{\gamma}}, r'_{\tilde{\gamma}}$ are subsets of [0, 1] and $0\le \sup(p'_{\tilde{\gamma}})+\sup(q'_{\tilde{\gamma}})+\sup(r'_{\tilde{\gamma}})\le 3$ and $\tilde{\gamma}=([c_1,c_2,c_3,c_4];[\underline{p}',\overline{p}'],[\underline{q}',\overline{q}'],[\underline{r}',\overline{r}'])$ is said to be positive ITrNN if $\tilde{\gamma}\ge 0$ and one of c_1,c_2,c_3,c_4 is not equal to zero.

Definition 3: Let $h_1=([c_{11},c_{12},c_{13},c_{14}];[\underline{p}'_1,\overline{p}'_1],[\underline{q}'_1,\overline{q}'_1],[\underline{r}'_1,\overline{r}'_1])$ and $h_2=([c_{21},c_{22},c_{23},c_{24}];[\underline{p}'_2,\overline{p}'_2],[\underline{q}'_2,\overline{q}'_2],[\underline{r}'_2,\overline{r}'_2])$ be any two ITrNNs and $\mu\ge 0$. Then the following operations hold [28] .

1. $h_1\oplus h_2=([c_{11}+c_{21},c_{12}+c_{22},c_{13}+c_{23},c_{14}+c_{24}];[\underline{p}'_1+\underline{p}'_2-\underline{p}'_1\underline{p}'_2,\overline{p}'_1+\overline{p}'_2-\overline{p}'_1\overline{p}'_2],[\underline{q}'_1\underline{q}'_2,\overline{q}'_1\overline{q}'_2],[\underline{r}'_1\underline{r}'_2,\overline{r}'_1\overline{r}'_2])$

2. $h_1\otimes h_2=([c_{11}c_{21},c_{12}c_{22},c_{13}c_{31},c_{14}c_{24}];[\underline{p}'_1\underline{p}'_2,\overline{p}'_1\overline{p}'_2],[\underline{q}'_1+\underline{q}'_2-\underline{q}'_1\underline{q}'_2,\overline{q}'_1+\overline{q}'_2-\overline{q}'_1\overline{q}'_2],[\underline{r}'_1+\underline{r}'_2-\underline{r}'_1\underline{r}'_2,\overline{r}'_1+\overline{r}'_2-\overline{r}'_1\overline{r}'_2])$

3. $\mu h_1=([\mu c_{11},\mu c_{12},\mu c_{13},\mu c_{14}];[1-(1-\underline{p}'_1)^\mu,1-(1-\overline{p}'_1)^\mu],[(\underline{q}'_1)^\mu,(\overline{q}'_1)^\mu],[(\underline{r}'_1)^\mu,(\overline{r}'_1)^\mu])$

4. $(h_1)^\mu=([(c_{11})^\mu,(c_{12})^\mu,(c_{13})^\mu,(c_{14})^\mu];[(\underline{p}'_1)^\mu,(\overline{p}'_1)^\mu],[1-(1-\underline{q}'_1)^\mu,1-(1-\overline{q}'_1)^\mu],[1-(1-\underline{r}'_1)^\mu,1-(1-\overline{r}'_1)^\mu])$

Definition 7. Hamming distance between two ITrNNs:

Assume that $h_1=([c_{11},c_{12},c_{13},c_{14}];[\underline{p}'_1,\overline{p}'_1],[\underline{q}'_1,\overline{q}'_1],[\underline{r}'_1,\overline{r}'_1])$ and $h_2=([c_{21},c_{22},c_{23},c_{24}];[\underline{p}'_2,\overline{p}'_2],[\underline{q}'_2,\overline{q}'_2],[\underline{r}'_2,\overline{r}'_2])$ are any two ITrNNs. Then the Hamming distance $d(h_1,h_2)$ [28] between them is defined as:

$$d(h_1,h_2)=\frac{1}{24}\left(\begin{array}{l}\left|c_{11}(2+\underline{p}'_1-\underline{q}'_1-\underline{r}'_1)+c_{11}(2+\overline{p}'_1-\overline{q}'_1-\overline{r}'_1)-c_{21}(2+\underline{p}'_2-\underline{q}'_2-\underline{r}'_2)-c_{21}(2+\overline{p}'_2-\overline{q}'_2-\overline{r}'_2)\right|\\ +\left|c_{12}(2+\underline{p}'_1-\underline{q}'_1-\underline{r}'_1)+c_{12}(2+\overline{p}'_1-\overline{q}'_1-\overline{r}'_1)-c_{22}(2+\underline{p}'_2-\underline{q}'_2-\underline{r}'_2)-c_{22}(2+\overline{p}'_2-\overline{q}'_2-\overline{r}'_2)\right|\\ +\left|c_{13}(2+\underline{p}'_1-\underline{q}'_1-\underline{r}'_1)+c_{13}(2+\overline{p}'_1-\overline{q}'_1-\overline{r}'_1)-c_{23}(2+\underline{p}'_2-\underline{q}'_2-\underline{r}'_2)-c_{23}(2+\overline{p}'_2-\overline{q}'_2-\overline{r}'_2)\right|\\ +\left|c_{14}(\underline{p}'_1-\underline{q}'_1-\underline{r}'_1)+c_{14}(2+\overline{p}'_1-\overline{q}'_1-\overline{r}'_1)-c_{24}(2+\underline{p}'_2-\underline{q}'_2-\underline{r}'_2)-c_{24}(2+\overline{p}'_2-\overline{q}'_2-\overline{r}'_2)\right|\end{array}\right) \quad (7)$$

Some important function of ITrNNs

Definition 4: The score function of an ITrNN $\tilde{\gamma}=([c_1,c_2,c_3,c_4];[\underline{p}',\overline{p}'],[\underline{q}',\overline{q}'],[\underline{r}',\overline{r}'])$ is defined by

$$Sc(\tilde{\gamma})=\frac{1}{24}(c_1+c_2+c_3+c_4)(2+\underline{p}'+\overline{p}'+\underline{q}'+\overline{q}'-\underline{r}'-\overline{r}'),Sc(\tilde{\gamma})\in[0,1] \quad (8)$$

Here, we take $0\le c_1\le c_2\le c_3\le c_4\le 1$, $p'_\gamma,q'_\gamma,r'_\gamma$ are subsets of [0, 1] where $p'_{\tilde{\gamma}}=[\underline{p}',\overline{p}'],q'_{\tilde{\gamma}}=[\underline{q}',\overline{q}']$ and $r'_{\tilde{\gamma}}=[\underline{r}',\overline{r}']$.

Property 1: Score function $Sc(\tilde{\gamma})$ is bounded on [0, 1].

Proof: Assume that $\tilde{\gamma}=([c_1,c_2,c_3,c_4];[\underline{p}',\overline{p}'],[\underline{q}',\overline{q}'],[\underline{r}',\overline{r}'])$ be a ITrNN. The value of c_1,c_2,c_3,c_4 lies between [0,1]. Therefore, we can write

$0 \le c_1 \le c_2 \le c_3 \le c_4 \le 1$

$$0 \le c_1 + c_2 + c_3 + c_4 \le 4 \quad (9)$$

Now,

$-2 \le \underline{p}' + \overline{p}' + \underline{q}' + \overline{q}' - \underline{r}' - \overline{r}' \le 4$

$=> 2 - 2 \le 2 + \underline{p}' + \overline{p}' + \underline{q}' + \overline{q}' - \underline{r}' - \overline{r}' \le 2 + 4$

$$=> 0 \le 2 + \underline{p}' + \overline{p}' + \underline{q}' + \overline{q}' - \underline{r}' - \overline{r}' \le 6 \quad (10)$$

Multiplying (9) and (10) we obtain,

$0 \le (c_1 + c_2 + c_3 + c_4)(2 + \underline{p}' + \overline{p}' + \underline{q}' + \overline{q}' - \underline{r}' - \overline{r}') \le 24$

$=> 0 \le \frac{1}{24}(c_1 + c_2 + c_3 + c_4)(2 + \underline{p}' + \overline{p}' + \underline{q}' + \overline{q}' - \underline{r}' - \overline{r}') \le 1$

$=> 0 \le Sc(\tilde{\gamma}) \le 1$

So, $Sc(\tilde{\gamma})$ is bounded.

Example 1: Let $\gamma = ([0.1, 0.2, 0.3, 0.4]; [0.1, 0.2], [0.2, 0.3], [0.4, 0.5])$. Then

$Sc(\gamma) = \frac{1}{24}(0.1 + 0.2 + 0.3 + 0.4)(2 + 0.1 + 0.2 + 0.2 + 0.3 - 0.4 - 0.5)$

$= 0.0792$

Definition 5: The accuracy function of $\tilde{\gamma} = ([c_1, c_2, c_3, c_4]; [\underline{p}', \overline{p}'], [\underline{q}', \overline{q}'], [\underline{r}', \overline{r}'])$ is defined by

$$Ac(\tilde{\gamma}) = \frac{1}{8}(c_3 + c_4 - c_1 - c_2)(2 + \underline{p}' + \overline{p}' - \underline{r}' - \overline{r}') Ac(\tilde{\gamma}) \in [0,1] \quad (11)$$

Here, we consider $0 \le c_1 \le c_2 \le c_3 \le c_4 \le 1$, $p'_\gamma, q'_\gamma, r'_\gamma$ are subsets of [0, 1] where $p'_{\tilde{\gamma}} = [\underline{p}', \overline{p}'], q'_{\tilde{\gamma}} = [\underline{q}', \overline{q}']$ and $r'_{\tilde{\gamma}} = [\underline{r}', \overline{r}']$.

Property 2: Accuracy function $Ac(\tilde{\gamma})$ is bounded on [0, 1].

Proof: Since c_1, c_2, c_3, c_4 lies between [0,1] therefore

$0 \le c_1 \le c_2 \le c_3 \le c_4 \le 1$

$$=> -2 \le c_3 + c_4 - c_1 - c_2 \le 2 \quad (12)$$

Here, $-2 \le \underline{p}' + \overline{p}' - \underline{r}' - \overline{r}' \le 2$

$=> 2 - 2 \le 2 + \underline{p}' + \overline{p}' - \underline{r}' - \overline{r}' \le 2 + 2$

$$=> 0 \le 2 + \underline{p}' + \overline{p}' - \underline{r}' - \overline{r}' \le 4 \quad (13)$$

Multiplying (12) and (13) we obtain,

$0 \le (c_3 + c_4 - c_1 - c_2)(2 + \underline{p}' + \overline{p}' - \underline{r}' - \overline{r}') \le 8$

$$=> 0 \le \frac{1}{8}\left(c_3 + c_4 - c_1 - c_2\right)(2 + \underline{p}' + \overline{p}' - \underline{r}' - \overline{r}') \le 1$$

$$=> 0 \le Ac(\tilde{\gamma}) \le 1$$

Therefore, $Ac(\tilde{\gamma})$ is bounded.

Example 2: Let $\sigma = \left([0.5, 0.6, 0.6, 0.7];[0.7, 0.8],[0.3, 0.4],[0.2, 0.3]\right)$ be an ITrNNs. Then

$$\begin{aligned} Ac(\sigma) &= \frac{1}{8}(0.6 + 0.7 - 0.5 - 0.6)(2 + 0.7 + 0.8 - 0.2 - 0.3) \\ &= 0.075 \end{aligned}$$

Definition 6: Let γ_1, γ_2 be two ITrNN and $Sc(\gamma_1), Sc(\gamma_2), Ac(\gamma_1)$ and $Ac(\gamma_2)$ are the scores and accuracy function of γ_1 and γ_2 respectively.

1. If $Sc(\gamma_1) > Sc(\gamma_2)$, then $\gamma_1 > \gamma_2$
2. If $Sc(\gamma_1) = Sc(\gamma_2)$ and
   (a) If $Ac(\gamma_1) > Ac(\gamma_2)$, then $\gamma_1 > \gamma_2$
   (b) (a) If $Ac(\gamma_1) = Ac(\gamma_2)$ then $\gamma_1 = \gamma_2$;

Example 3: Let, $\gamma_1 = \left([0.4, 0.5, 0.7, 0.7];[0.7, 0.8],[0.3, 0.4],[0.2, 0.3]\right)$ and
$\gamma_2 = \left([0.5, 0.6, 0.6, 0.7];[0.6, 0.7],[0.3, 0.4],[0.2, 0.3]\right)$ be two ITrNNs. Then we obtain

$Sc(\gamma_1) = 0.35, Sc(\gamma_2) = 0.35$

$Ac(\gamma_1) = 0.19, Ac(\gamma_2) = 0.07$

Since $Sc(\gamma_1) = Sc(\gamma_2)$ and $Ac(\gamma_1) > Ac(\gamma_2)$

Therefore, we conclude that $\gamma_1 > \gamma_2$.

Standardize the decision matrix

We consider the following method [30] to obtain the standardized matrix $T = \left(\tilde{t}_{ij}\right)_{yz}$, where
$\tilde{t}_{yz} = ([c_{yz}^1, c_{yz}^2, c_{yz}^3, c_{yz}^4];[\underline{p}_{yz}, \overline{p}_{yz}],[\underline{q}_{yz}, \overline{q}_{yz}],[\underline{r}_{yz}, \overline{r}_{yz}])$ is an ITrNN.

$$\tilde{t}_{yz} = ([\frac{c_{yz}^1}{g_z^+}, \frac{c_{yz}^2}{g_z^+}, \frac{c_{yz}^3}{g_z^+}, \frac{c_{yz}^4}{g_z^+}];[\underline{p}_{yz}, \overline{p}_{yz}],[\underline{q}_{yz}, \overline{q}_{yz}],[\underline{r}_{yz}, \overline{r}_{yz}]) \quad (14)$$

$$\tilde{t}_{yz} = ([\frac{g_z^-}{c_{yz}^4}, \frac{g_z^-}{c_{yz}^3}, \frac{g_z^-}{c_{yz}^2}, \frac{g_z^-}{c_{yz}^1}];[\underline{p}_{yz}, \overline{p}_{yz}],[\underline{q}_{yz}, \overline{q}_{yz}],[\underline{r}_{yz}, \overline{r}_{yz}]) \quad (15)$$

Here, $\tilde{t}_{yz}$ and $\tilde{t}_{yz}$ are respectively benefit type and cost type criteria and

$g_z^+ = \max\{c_{yz}^4 : y = 1, 2, ..., m\}$ and $g_z^- = \min\{c_{yz}^1 : y = 1, 2, ..., m\}$ for v = 1, 2, ..., n.

TODIM strategy for solving MCDM problem ITrNN environment

Assume that $C = \{C_1, C_2, ..., C_p\}$ is a set of p alternatives and $E = \{E_1, E_2, ..., E_r\}$ be a set of r-criteria and $V = \{V_1, V_2, ..., V_r\}$ be the weight vector of the criteria in an MCDM problem. The Decision Maker (DM) ranks the alternatives with respect to (w. r. t.) the criteria. We introduce the ITrNN-TODIM strategy (as illustrated in Figure 1) through the following steps.

Step -1: Assume that q_{pr} is the rating value provided by the DM for the alternative p w. r. t. the criterion E_r. Then, the decision matrix $Q_{p\times r}$ is constructed as:

$$Q_{p\times r} = \begin{pmatrix} & E_1 & E_2 & \cdots & E_r \\ C_1 & q_{11} & q_{12} & \cdots & q_{1r} \\ C_2 & q_{21} & q_{22} & \cdots & q_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_p & q_{p1} & q_{p2} & \cdots & q_{pr} \end{pmatrix} \quad (16)$$

Step-2: Decision matrix (16) is standardized using equations (14) and (15).

Standardized decision matrix is presented as:

$$(\tilde{Q})_{p\times r} = \begin{pmatrix} & E_1 & E_2 & \cdots & E_r \\ C_1 & \tilde{q}_{11} & \tilde{q}_{12} & \cdots & \tilde{q}_{1r} \\ C_2 & \tilde{q}_{21} & \tilde{q}_{22} & \cdots & \tilde{q}_{2r} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_P & \tilde{q}_{p1} & \tilde{q}_{p2} & \cdots & \tilde{q}_{pr} \end{pmatrix} \quad (17)$$

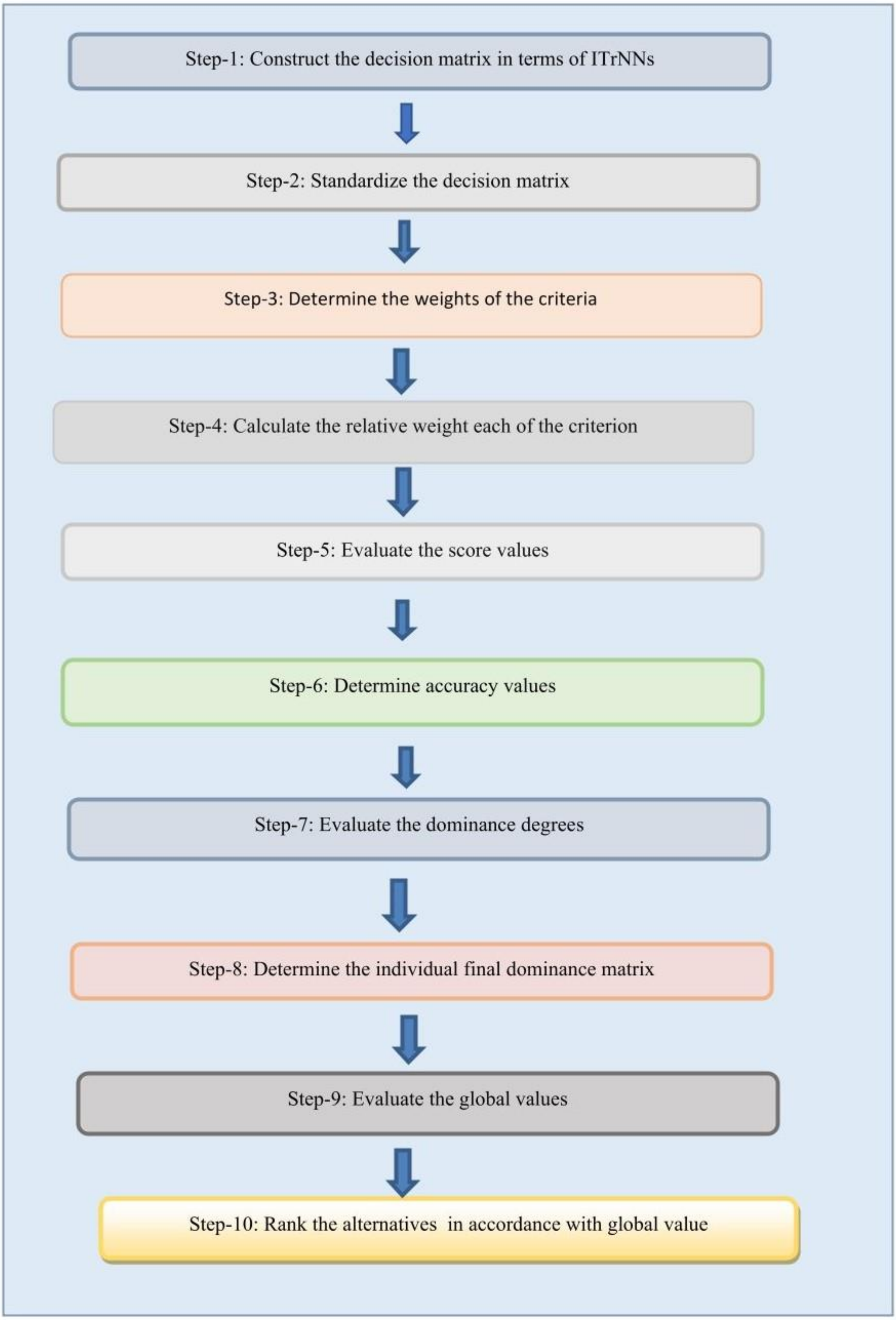

Figure 1: ITrNN-TODIM strategy

Step-3: Calculate the weights of the criteria using entropy.

The criterion vector $\tilde{V}_j$ is expressed in terms of ITrNNs as:

$\tilde{V}_j = \left(\left[v_j^1, v_j^2, v_j^3, v_j^4\right]; \left[\underline{t}_j, \overline{t}_j\right], \left[\underline{i}_j, \overline{i}_j\right]\left[\underline{f}_j, \overline{f}_j\right]\right), j = 1, 2, ..., r.$

Distance vector $d(\tilde{V}_j, I^+)$ is presented as

$d(\tilde{V}) = (d(\tilde{V}_1, I^+), d(\tilde{V}_2, I^+), ..., d(\tilde{V}_r, I^+))$

where

$$d(\tilde{V}_j, I^+) = \frac{1}{24}\left(\begin{array}{l}\left|v_j^1(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+v_j^1(2+\overline{t}_1-\overline{i}_1-\overline{f}_1)-6\right|+\left|v_j^2(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+v_j^2(2+\overline{t}_1-\overline{i}_1-\overline{f}_1)-6\right| \\ +\left|v_j^3(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+v_j^3(2+\overline{t}_1-\overline{i}_1-\overline{f}_1)-6\right|+\left|v_j^4(2+\underline{t}_1-\underline{i}_1-\underline{f}_1)+v_j^4(2+\overline{t}_1-\overline{i}_1-\overline{f}_1)-6\right|\end{array}\right) \quad (18)$$

The normalized distance vector is presented as:

$$\overline{d} = (\overline{d}(\tilde{V}_1, I^+), \overline{d}(\tilde{V}_2, I^+), ..., \overline{d}(\tilde{V}_r, I^+)) \quad (19)$$

where $\overline{d}(\tilde{V}_1, I^+) = \left\lfloor \dfrac{d(\tilde{V}_j, I^+)}{\max\limits_j d(\tilde{V}_j, I^+)} \right\rfloor, j = 1, 2, ..., r$

The measure of the j-th criterion E_j for p feasible alternatives is obtained from

$$e_j = -\frac{1}{\ln(m)}\left\lfloor \frac{\overline{d}(\tilde{V}_j, I^+)}{\sum_{j=1}^{n}\overline{d}(\tilde{V}_j, I^+)} \ln\left(\frac{\overline{d}(\tilde{V}_j, I^+)}{\sum_{j=1}^{n}\overline{d}(\tilde{V}_j, I^+)}\right)\right\rfloor, j = 1, 2, ..., r \quad (20)$$

The normalized weight V_j of the j-th criterion is obtained as:

$$V_j = \frac{1-e_j}{\sum_{j=1}^{n}(1-e_j)} \quad (21)$$

where $0 \le V_r \le 1, j = 1, 2, ..., r.$

Step-4: Evaluate the relative weight V_{Re_i} of each criterion by the formula:

$$V_{\text{Re}_i} = \frac{V_i}{V_C} \quad (22)$$

Here $V_i = \{V_1, V_2, ..., V_r\}$ and $V_C = \max\{V_1, V_2, ..., V_r\}$, is called reference criterion which is chosen by the decision maker.

Step -5: Evaluate the score value for each alternative by applying formula (7) with respect to each criterion in the decision matrix (17).

Step-6: Using equation (10), we evaluate accuracy value of each alternative w. r. t. each criterion.

Step-7: By the following equation, we evaluate the dominance degree C_j of each alternative C_k w. r. t. the criterion E_t

$$\left.\begin{aligned}\vartheta_t(C_j,C_k) &= \sqrt{\frac{V_{\mathrm{Re}_t}}{\sum_{t=1}^{r} V_{\mathrm{Re}_t}} d(q_{jt},q_{kt})}, && if\ q_{jt} > q_{kt} \\ &= 0, && if\ q_{jt} = q_{kt} \\ &= -\frac{1}{\kappa}\sqrt{\frac{\sum_{t=1}^{r} V_{\mathrm{Re}_t}}{V_{\mathrm{Re}_t}} d(q_{jt},q_{kt})}, && if\ q_{jt} < q_{kt}\end{aligned}\right\} \tag{23}$$

where κ denotes the decay factor of losses and $\kappa > 0$. If $\kappa > 1$, losses are diminished, whereas if a $\kappa < 1$ t, losses are magnified Decision maker ranks the alternatives based on κ i.e. according to the gains and the losses. Two cases are to be considered.

Case 1: For higher values of κ, the most favourable alternatives are those that generate the greatest gains.

Case 2: When κ takes on smaller values, the most favourable alternatives are those that minimize losses.

Step 8: Derive the individual final dominance matrix by applying the equation below:

$$\mu(C_j,C_k) = \sum_{t=1}^{r} \vartheta_t(C_j,C_k) \tag{24}$$

Step-9: Using equation (18), we evaluate global value of each alternative,

$$\delta_l = \frac{\sum_{m=1}^{p} \mu(C_l,C_m) - \min_{1\le i\le r}(\sum_{m=1}^{p} \mu(C_l,C_m))}{\max_{1\le l\le r}(\sum_{m=1}^{p} \mu(C_l,C_m)) - \min_{1\le l\le r}(\sum_{m=1}^{p} \mu(C_l,C_m))} \tag{25}$$

Step-10: Sort the alternatives in descending order according to their global values, where the alternative with the highest global value is ranked as the best.

Illustrative Example

We discuss a numerical example to visualize the applicability and utility of the developed ITrNN-TODIM strategy. To illustrate the developed ITrNN-TODIM, we solve an MCDM problem adapted from [19]. A man wants to buy a new laptop. Assume that there are four laptop companies C_1, C_2, C_3 and C_4, each company has to consider three criteria, namely quality, warranty and cost. We consider E_1 for quality, E_2 for warranty and E_3 for cost. The expert or DM assesses the rating values of alternatives $C_i\ (i = 1,2,3,4)$ w. r. t. attributes $E_j\ (j = 1,2,3)$. Now we solve the problem employing the developed TODIM strategy.

Step-1: Construct the decision matrix with four alternatives namely C_1, C_2, C_3 and C_4, and three criteria E_1, E_2 and E_3.

Matrix 1

$$\begin{pmatrix} & E_1 \\ C_1 & ([50,60,70,80];[0.1,0.2],[0.2,0.3],[0.4,0.5]) \\ C_2 & ([30,40,50,60];[0.3,0.4],[0.2,0.3],[0.1,0.2]) \\ C_3 & ([70,80,90,100];[0.6,0.7],[0.2,0.3],[0.4,0.3]) \\ C_4 & ([40,50,60,70];[0.4,0.5],[0.6,0.7],[0.2,0.3]) \end{pmatrix}$$

$$\begin{pmatrix} & E_2 \\ C_1 & ([30,40,50,60];[0.2,0.3],[0.4,0.5],[0.6,0.7]) \\ C_2 & ([10,20,30,40];[0.1,0.2],[0.3,0.4],[0.6,0.7]) \\ C_3 & ([50,60,70,80];[0.1,0.2],[0.3,0.4],[0.6,0.7]) \\ C_4 & ([70,80,90,100];[0.2,0.3],[0.4,0.5],[0.6,0.8]) \end{pmatrix}$$

$$\begin{pmatrix} & E_3 \\ C_1 & ([40,50,60,70];[0.4,0.5],[0.6,0.7],[0.7,0.8]) \\ C_2 & ([20,30,40,50];[0.1,0.2],[0.3,0.4],[0.8,0.9]) \\ C_3 & ([70,80,90,100];[0.3,0.5],[0.4,0.6],[0.7,0.8]) \\ C_4 & ([30,40,50,60];[0.4,0.5],[0.6,0.7],[0.7,0.8]) \end{pmatrix}$$

The importance of attribute E_j is given by

$$\tilde{V} = \begin{Bmatrix} ([0.2,0.3,0.3,0.4];[0.4,0.5],[0.2,0.3],[0.1,0.2]),([0.4,0.4,0.5,0.6];[0.6,0.7],[0.2,0.3],[0.2,0.3]), \\ ([0.6,0.7,0.8,0.8],[0.3,0.4],[0.1,0.2],[0.1,0.2]) \end{Bmatrix}$$

Step-2: First and second criteria are benefit type criteria, and third criterion is cost type criterion. To obtain the standardized decision matrix we use formula (14) and formula (15).

Matrix 2

$$\begin{pmatrix} & E_1 \\ C_1 & ([0.38,0.43,0.50,0.60];[0.1,0.2],[0.2,0.3],[0.4,0.5]) \\ C_2 & ([0.50,0.60,0.75,1];[0.3,0.4],[0.2,0.3],[0.1,0.2]) \\ C_3 & ([0.30,0.33,0.38,0.43];[0.6,0.7],[0.2,0.3],[0.4,0.3]) \\ C_4 & ([0.43,0.50,0.60,0.75];[0.4,0.5],[0.6,0.7],[0.2,0.3]) \end{pmatrix}$$

$$\begin{pmatrix} & E_2 \\ C_1 & ([0.30,0.40,0.50,0.60];[0.2,0.3],[0.4,0.5],[0.6,0.7]) \\ C_2 & ([0.10,0.20,0.30,0.40];[0.1,0.2],[0.3,0.4],[0.6,0.7]) \\ C_3 & ([0.50,0.60,0.70,0.80];[0.1,0.2],[0.3,0.4],[0.6,0.7]) \\ C_4 & ([0.70,0.80,0.90,1];[0.2,0.3],[0.4,0.5],[0.6,0.8]) \end{pmatrix}$$

$$\begin{pmatrix} & E_3 \\ C_1 & ([0.40,0.50,0.60,0.70];[0.4,0.5],[0.6,0.7],[0.7,0.8]) \\ C_2 & ([0.20,0.30,0.40,0.50];[0.1,0.2],[0.3,0.4],[0.8,0.9]) \\ C_3 & ([0.70,0.80,0.90,1];[0.3,0.5],[0.4,0.6],[0.7,0.8]) \\ C_4 & ([0.30,0.40,0.50,0.60];[0.4,0.5],[0.6,0.7],[0.7,0.8]) \end{pmatrix}$$

Step-3: Using equation (19) we obtain $d = (0.795, 0.6596, 0.5046), \overline{d} = (1, 0.8297, 0.6347)$

Using equation (20) we calculate entropy $e = (0.2640, 0.2644, 0.2520)$

Using formula (21) we obtain the normalized weight vector of the criteria as
$V = (0.3316, 0.3314, 0.3370)$

Step-4: To obtain relative weight for each criterion, we use formula (22) and the obtained relative weights are:

$V_{\mathrm{Re}_1} = 0.984, V_{\mathrm{Re}_2} = 0.9834, V_{\mathrm{Re}_3} = 1$.

Step-5: We evaluate score value of each alternative w. r. t. each criterion using equation (8). Matrix 3 represents the score values.

Matrix 3: Score value for M^1

$$\begin{pmatrix} & E_1 & E_2 & E_3 \\ C_1 & 0.1512 & 0.1575 & 0.2475 \\ C_2 & 0.3444 & 0.0708 & 0.0758 \\ C_3 & 0.174 & 0.1845 & 0.3258 \\ C_4 & 0.3515 & 0.2833 & 0.2025 \end{pmatrix}$$

Step-6: To evaluate accuracy value, we use equation (11). Matrix 4 presents accuracy values for M^1.

Matrix 4: Accuracy value for M^1

$$\begin{pmatrix} & E_1 & E_2 & E_3 \\ C_1 & 0.0508 & 0.06 & 0.07 \\ C_2 & 0.195 & 0.05 & 0.03 \\ C_3 & 0.054 & 0.05 & 0.065 \\ C_4 & 0.126 & 0.055 & 0.07 \end{pmatrix}$$

Step-7: We calculate the dominance matrices (Matrices 5-7) (considering κ =1) using equation (23).

Matrix 5: Dominance matrix ϑ_1^1

$$\vartheta_1^1 = \begin{pmatrix} & C_1 & C_2 & C_3 & C_4 \\ C_1 & 0 & -0.8370 & -0.3438 & -0.4383 \\ C_2 & 0.2775 & 0 & 0.2530 & -0.7631 \\ C_3 & 0.1140 & -0.7631 & 0 & -0.2795 \\ C_4 & 0.1453 & 0.2530 & 0.0293 & 0 \end{pmatrix}$$

Matrix 6: Dominance matrix ϑ_2^1

$$\vartheta_2^1 = \begin{pmatrix} & C_1 & C_2 & C_3 & C_4 \\ C_1 & 0 & 0.1594 & -0.4811 & -0.6481 \\ C_2 & -0.4811 & 0 & -0.6801 & -0.8070 \\ C_3 & 0.1594 & 0.2254 & 0 & -0.4343 \\ C_4 & 0.2148 & 0.2674 & 0.1439 & 0 \end{pmatrix}$$

Matrix 7: Dominance matrix ϑ_3^1

$$\vartheta_3^1 = \begin{pmatrix} & C_1 & C_2 & C_3 & C_4 \\ C_1 & 0 & 0.1659 & -0.6289 & 0.1086 \\ C_2 & -0.4923 & 0 & -0.7987 & -0.3723 \\ C_3 & 0.2119 & 0.2692 & 0 & 0.2382 \\ C_4 & -0.3223 & 0.1254 & -0.7067 & 0 \end{pmatrix}$$

Step-8: We determine the final dominance matrix (Matrices 8) using formula (24).

Matrix 8: Final dominance matrix μ

$$\begin{pmatrix} & C_1 & C_2 & C_3 & C_4 \\ C_1 & 0 & -0.5117 & -1.4538 & -0.9778 \\ C_2 & -0.6959 & 0 & -1.2258 & -1.9424 \\ C_3 & 0.4853 & -0.2685 & 0 & -0.4756 \\ C_4 & 0.0378 & 0.6458 & -0.5335 & 0 \end{pmatrix}$$

Step-9: Using formula (25), we calculate global value δ .

$\delta_1 = 0.2294, \delta_2 = 0, \delta_3 = 0.8981, \delta_4 = 1$

Step-10: Arranging the global values, we obtain $\delta_4 > \delta_3 > \delta_1 > \delta_2$.

Therefore, the ranking order of the alternatives is

$C_4 > C_3 > C_1 > C_2$

Therefore, forth alternative C_4 is the best alternative to invest.

Conclusions

To develop the ITrNN-TODIM strategy, we introduced a novel score function and an accuracy function for ITrNNs, and demonstrated their fundamental properties. The steps for applying the extended TODIM strategy were then presented in a simplified manner. As an illustration of the newly developed strategy, a numerical example of an MCDM problem for selecting the best laptop was solved.

The key advantage of this proposed strategy is its effectiveness in addressing MCDM problems involving interval trapezoidal neutrosophic information. Interval trapezoidal neutrosophic numbers are capable of handling indeterminate and inconsistent data, and they represent an extension of both trapezoidal intuitionistic fuzzy numbers and interval neutrosophic numbers. A notable limitation of this MCDM strategy is that, while it is well-suited for the given environment, there is always the potential for developing a more advanced strategy that could outperform the current one.

The proposed strategy can be applied to investigate various MCDM problems, potentially leading to new avenues for research. The ITrNN-TODIM strategy can be utilized to solve MCDM problems such as personnel selection [41, 42], medical diagnosis [43], data mining [44], renewable energy selection [45], green supplier selection [46], E-commerce site selection [47], determining the key water quality parameters in aquaponic systems [48], information retrieval [49], vaccine selection [50], identification of influential parameters in soil liquefaction [51] and so on.

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