Florentin Smarandache
Surapati Pramanik
(Editors)

New Trends in Neutrosophic Theory and Applications

Volume V

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New Trends
in Neutrosophic Theories and Applications

Volume V

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Emeritus Professor Florentin Smarandache, PhD, Postdoc, Mathematics, Physical and Natural Sciences Division, University of New Mexico, USA. He is the founder of neutrosophy, a new branch of philosophy. He is a Chief Editor, and reviewer for several international journals. He published over 900 scientific papers and over 400 books in mathematical sciences, psychology, sociology, as well as literary works such as poetry, stories, essays, a novel, translations, dramas, plays for children, folklore, and albums of arts. He has been enlisted in the World's Top cited 2% Scientists (prepared by Stanford University, USA in association with Elsevier BV).

Dr. Surapati Pramanik has been working as an assistant professor in mathematics since 2006 at Nandalal Ghosh B.T. College, India. His research focuses on fuzzy Multi-Criteria Decision Making (MCDM), soft computing, and neutrosophic hybrid sets. He is a Chief Editor, Associate Editor, and reviewer for several international journals. His publication includes 37 book chapters, 165 research papers, and 3 editorial books. He received outstanding paper award in WBSSTC 2011 in Mathematics, WBSSTC-2019 in Social Science He is enlisted in the World's Top cited 2% Scientists, 2023 & 2022 (prepared by Stanford University, USA in association with Elsevier BV).

Aims and Scope

The domain of neutrosophic set theory is undergoing rapid and significant expansion. Novel, theories, approaches, strategies, and MCDM techniques are evolving rapidly. An important development in Neutrosophic Theory is its integration with existing set theories, such as Pythagorean fuzzy set theory, Fermatean fuzzy set theory, rough set theory, and other related frameworks. The neutrosophic set has been extended in different structures to reflect diverse real- world issues and reality. Neutrosophic theories have emerged as essential tools in a diverse range of disciplines, including, multi criteria decision making, data mining, biomedical research, social studies, and beyond.

The Book "New Trends in Neutrosophic Theories and Applications, Vol. 5, 2026" focuses on theories, strategies, for MCDM within neutrosophic frameworks. Some topics deal with neutrosophic number-based Machine Learning (NNN-ML) approach, learning analytic Framework, a general discrete–to–continuum transformation principle, neutrosophic self-reference stability and paradox as a dynamical process, graph theory, semi group, Plithogenic cognitive maps, aggregation operators, neutrosophic information geometry framework, fuzzy and Gödel's incompleteness theorems, pentapartitioned neutrosophic quasi coincident topological space, neutrosophic weak rough sets. Some topics handle multivalued logic framework for clinical decision making and biomedical uncertainty. Some topics deal with algebraic operations of Fermatean neutrosophic sets and Pythagorean neutrosophic sets. Some topics deal with application of neutrosophic numbers in engineering and data science. Some topics deal with MCDM in Fermatean neutrosophic environment, pentapartitioned neutrosophic number environment, rough and interval rough neutrosophic set environment, and trapezoidal neutrosophic environment.

Florentin Smarandache, Surapati Pramanik

Preface

Professor Florentin Smarandache developed the neutrosophic set theory (NST) as a foundation for handling uncertainty, indeterminacy, and inconsistent information. NSTs have emerged as a robust generalization of traditional set theory, facilitating a more nuanced representation of uncertainty and vagueness in complex systems. NSTs excel in contexts involving indeterminate, imprecise, incomplete, and inconsistent information that render them applicable to a wide range of disciplines.

A fundamental feature of NST is their ability to incorporate truth, indeterminacy, and falsity as independent components, allowing for a more comprehensive representation of uncertain and inconsistent information. This gives them broad applicability in fields like multicriteria deciison making, Artificial Intelligence (AI), Machine Learning and Cognitive Science, where data often contains ambiguity, inconsistency, and uncertainty.

The use of NSTs in fields like data mining, decision analysis, expert systems, and pattern recognition has led to new opportunities for grounding more robust and flexible systems. With NSTs now being applied in data mining, decision analysis, expert systems, and pattern recognition, more robust and flexible systems are within reach. Similarly, in decision analysis NSTs allow for better handling of contradictory information when making decisions under uncertainty. Due to the broad relevance of neutrosophic set theory), the publication of the book series *New Trends in Neutrosophic Theory and Applications* (starting in 2016) played a key role in sparking global interest and research in NSs.

Chapter 1 presents key mathematical constructions such as rings, norms, metrics, and proof of related theorems. In introduces an innovative neutrosophic number-based Machine Learning (NNN-ML) approach to complete with a loss function along with pseudocode, aimed at dealing with domains associated with measurements involving intrinsic indeterminacies namely, industrial faults detection, sensor fusion, and anomaly classification using AI. With a gain of up to 14.4% over classical, fuzzy, and intuitionistic fuzzy based methods, the developed NNN-ML algorithm achieves a classification accuracy of 93.7% when evaluated on three distinct practical datasets.

Chapter 2 introduces a Neutrosophic Learning Analytics Framework (NLAF) grounded in neutrosophic set and refined neutrosophic sets environments. It formalizes neutrosophic representations of learner competency, defines a weighted hybrid distance-based similarity measure tailored to educational data and proves its theoretical properties. The framework is applied to two case studies and obtained findings reflect that NLAF captures nuanced uncertainty patterns that binary or fuzzy approaches miss and provides more reliable diagnosis of learning difficulties.

Chapter 3 introduces a general discrete–to–continuum transformation principle that extends such frameworks into continuum epistemic models. The proposed approach replaces finitely many membership components with measurable density functions whose domain is a continuous refinement index. Aggregated membership degrees are obtained through integration over this index space. It establishes that every finite refined fuzzy representation admits a continuum representation. Furthermore, it proves a discrete–continuum convergence theorem showing that classical fuzzy structures arise as special discretization of the continuum model. The proposed framework unifies the extension of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, and q-rung orthopair fuzzy sets into continuum analogues. It also presents the extension of fuzzy logic, fuzzy measure, fuzzy probability, and fuzzy statistics within this continuum setting. The findings reflect that continuum uncertainty models constitute a general functional framework for fuzzy and neutrosophic theories, bridging discrete epistemic structures with measure-theoretic representations of uncertainty.

Chapter 4 introduces the notion of a Pentapartitioned Neutrosophic Point (PPNP) in the framework of Pentapartitioned Neutrosophic Sets (PPNSs). It introduces and examines the relation of pentapartitioned neutrosophic quasi-coincidence between a PPNS and a PPNP. It derives the basic properties related to quasi-coincidence in pentapartitioned neutrosophic set structures. It also analyzes the topological behavior of the properties relation by focusing on how the level of proximity or coincidence between PPNSs can be characterized in a Pentapartitioned Neutrosophic Topological space.

Chapter 5 introduces the neutrosophic self-reference stability theorem and studies paradox as a dynamical process inside neutrosophic truth space. The developed framework connects liar paradox dynamics, Gödel self-reference, and contradictions across science, technology, administration, decision-making, literature, and art. It also discusses the Infinitesimally Punctured Wave (IPW) inspired by wave-particle duality.

Chapter 6 deals with Fermatean neutrosophic sets and their different characteristics such as union and intersection. It establishes the distributive characteristics of these features.

Chapter 7 proposes fuzzy and neutrosophic extensions of Gödel's incompleteness theorems based on graded and triadic self-reference. It constructs generalized Gödel sentences through diagonalization in formal systems capable of encoding their own provability. It constructs a neutrosophic Gödel Fixed-Point Theorem, under explicit continuity and non-collapse assumptions, according to which self-referential sentences stabilize as interior fixed points of the neutrosophic truth cube $[0,1]^3$. It also presents a Contradictory-Theory Incompleteness Principle and offers a geometric and dynamical interpretations, as well as applications to reasoning under inconsistency, knowledge fusion, artificial intelligence, quantum logic, and epistemology.

Chapter 8 defines neutrosophic weak rough sets of type 2 and type 3. It analyzes their basic algebraic properties, providing many examples and counter-examples.

Chapter 9 presents an algebraic framework known as the intuitionistic anti Q-fuzzy M-semigroup that integrate intuitionistic fuzzy set theory, Q-parameterization, anti-fuzzy logic, and transformation-based semigroup structures. The suggested method uses a transformation mechanism called the Anti Q-Fuzzy Transform Operator (AQFTO) to investigate stability and invariance properties. It establishes closure, power stability, homomorphic behaviour, and level subset characterisation. Furthermore, the proposed framework integrates neutrosophic theory to model incomplete, contradictory, and indeterminate information. Furthermore, it derives ideal-based properties and structural equivalence conditions. The developed framework executes a comparative analysis with neutrosophic and Pythagorean fuzzy models to reflect a computationally efficient and mathematically consistent method for managing multi-parameter uncertainty in algebraic systems.

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Chapter 10 introduces Continuum Neutrosophic Information Geometry, a geometric framework built upon continuum neutrosophic probability distributions where uncertainty is represented by three density functions corresponding to truth, indeterminacy, and falsity components. These densities define points on a neutrosophic statistical manifold, whose geometry is characterized employing a generalized Fisher information metric and a neutrosophic divergence extending the Kullback–Leibler divergence. The resulting geometric structure enables the analysis of epistemic uncertainty through concepts such as geodesic trajectories, information distances, and parameter sensitivity. This chapter shows that classical information geometry arises as a special case when indeterminacy and falsity components vanish. Potential applications of the proposed framework include machine learning under ambiguous data, sensor fusion with conflicting information sources, Bayesian inference with incomplete evidence, and modeling of complex socio-ecological systems. The proposed theory advances the wider field of continuum neutrosophic mathematics by merging logical, probabilistic, statistical, and geometric approaches to uncertainty.

Chapter 11 presents Neutrosophic Bishop Graphs (NBGs), a graph-theoretic framework for making decisions in situations with ambiguity, inconsistency and multi-criteria dependence. It presents formal definitions, dominance-based inference, aggregate semantics, a worked diagnostic case and comprehensive theoretical analysis. It establishes NP-completeness of precise inference, establish monotonicity properties and derive fixed-parameter tractability results and ε-approximation guarantees. The framework provides a computationally valid and accessible substitute for fuzzy and probabilistic decision models.

Chapter 12 introduces Extended Plithogenic Cognitive Maps (EPCM), combining Extended Plithogenic sets with Cognitive Maps as a comprehensive tool for deriving optimal solutions. The extension from PCM to EPCM refers to a step toward creating a more inclusive decision-making tool. A decision making problem in the manufacturing domain is addressed to show the effectiveness of the proposed EPCM by examining the degree of association among factors promoting IoT-based manufacturing.

Chapter 13 explores how advanced philosophical and mathematical frameworks developed by Florentin Smarandache, neutrosophy and plithogeny, can be systematically applied to medical science to address its inherent ambiguities. It examines these frameworks in the contexts of diagnostic reasoning, clinical decision support systems, medical ontologies, epidemiology, and multi-criteria drug evaluation. It argues that neutrosophic and plithogenic tools offer transformative potential for advancing precision medicine, artificial intelligence in healthcare, and evidence-based clinical practice.

Chapter 14 proposes a novel decision-making strategy, called the RNNWAA strategy, for the Rough Neutrosophic number (RNN) environment, based on Rough Neutrosophic Number Weighted Arithmetic Average (RNNWAA) operator. To demonstrate the applicability of the developed strategy, a multi criteria group decision problem is solved.

Chapter 15 develops the Multi-Attributive Compromise Ranking of Alternatives from Optimal Solution (MACROS) strategy within a trapezoidal neutrosophic number environment. Trapezoidal neutrosophic number allows a flexible and comprehensive modeling of uncertainty. The MACROS strategy assesses the utility of each alternative with respect to both ideal and anti-ideal solutions. The developed strategy also generate a compromise-based ranking. To illustrate the applicability of the proposed strategy, a car selection problem is discussed.

Chapter 16 develops an aggregation operator for Pentapartitioned neutrosophic Numbers (PNNs) utilizing the Einstein triangular norm and conorm, a subclass of the Archimedean family. It defines binary operations of addition, multiplication, scalar multiplication, and power operations for PNN structures. It verifies these operations for the corresponding properties of commutativity, associativity, and scalar distributivity. It establishes the basic properties of the new aggregation operator Pentapartitioned Neutrosophic Einstein Weighted Arithmetic Aggregation (PNEWAA) and a novel score function. It also develops MCDM strategies based on the PNEWAA operator. It solves two illustration examples encompassing group as well as single decision-making process.

Chapter 17 introduces fundamental Hamacher operations for in Fermatean Neutrosophic sets (FNSs) environment. It defines and establishes the basic properties of Hamacher operations for Fermatean Neutrosophic numbess. It develops a novel MCgDM approcha We have designed a general structure for MCDM problems, followed by some steps. A real-life example of Crop Land Selection is also demonstrated to highlight the effectiveness of the proposed operator and MCDM method.

Chapter 18 develops two new Multi-Attribute Decision Making (MADM) strategies in rough neutrosophic numberenvironment extending the VIKOR (Vlse Kriterijumska Optimizacija I Kompromisno Resenje) strategy. The developed strategies are employedd to solve an illustrative MCDM problem: selecting the most suitable laptop. t also provides a comparison of the obtained solution with those derived from existing MADM strategies.

Chapter 19 develops a novel neutrosophic open set–based Multi-Criteria Decision-Making (MCDM) strtegy that explicitly models truth, indeterminacy, and falsity information within a topological structure. The developed model incorporates neutrosophic topology to define flexible sustainability regions rather than rigid evaluation classes. It defines a weighted aggregation function and proves its basic properties. A numerical case study comparing recycling, landfill disposal, and pyrometallurgical recovery demonstrates the robustness and transparency of the framework. It perfoms sensitivity analysis that confirms ranking stability under varying decision-maker preferences. The proposed strtategy offers a resilient and interpretable decision-support tool for sustainable lithium-ion battery waste management under uncertainty.

Chapter 20 develops the ARAS strategy in Rough Neutrosophic Number (RNN) environment to deal with comprehensively uncertainty, incompleteness and inconsistent information. It solves an illustrative example of an MADM problem the obtained ranking is compared with the rankings obtained from the existing strategies such as MABAC and COPRAS.

Chapter 21 mainly deals with the basic concept of Pythagorean neutrosophic sets, operations on Pythagorean neutrosophic sets, various distance and similarity measures of Pythagorean neutrosophic sets and their axiomatic definitions taking into consideration of the three defined parameters.

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Chapter 1

Neutrosophic Numbers in Engineering and Data Science: Mathematical Framework, Formal Proofs, Algorithms, and Applications in Engineering and Data Science

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ABSTRACT

In this paper, a robust mathematical theory for Neutrosophic Numbers (NNs) of the type N = a + bI is provided, with a being the determinate part, b as the indeterminate factor, and $I \in [I^L, I^U]$ representing an indeterminacy interval. Key mathematical constructions such as rings, norms, metrics, and proof of related theorems are derived. By leveraging this robust mathematical theory, an innovative Neutrosophic Number-based Machine Learning (NNN-ML) approach is introduced, complete with a loss function along with pseudocode, aimed at addressing domains associated with measurements involving intrinsic indeterminacies – namely, industrial faults detection, sensor fusion, and anomaly classification using AI. Numerical examples are used throughout to explain important concepts step-by-step. The performance of the proposed NNN-ML algorithm is tested on three different practical datasets, achieving a classification accuracy of 93.7%, a gain of up to 14.4% over competing classical, fuzzy, and intuitionistic fuzzy methods.

Keywords: Neutrosophic numbers, indeterminacy, algebraic structure, machine learning, fault detection, IOT, uncertainty modeling, fuzzy logic extension, neutrosophic number-based machine learning.

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INTRODUCTION

Modern engineering systems and computational pipelines rely on data processing occurring within environments that are affected by uncertainties that extend beyond what can be described using classical probabilistic or fuzzy descriptions [1]. Industrial sensing systems are subject to calibration errors, thermal fluctuations, and electromagnetic interference and thus produce measurements that are inevitably uncertain. Fault diagnosis of machinery may simultaneously involve considering the case where a particular reading indicates that machinery operates correctly, is uncertain (sensor provides an ambiguous reading), and where the reading indicates machinery malfunction. Data labels in crowdsourced data sets used in data science applications are simultaneously partially correct, uncertain, and partially false, based on the reliability of an annotator. Classical real numbers are insufficient to represent the three aspects above, and even fuzzy numbers reduce independent measures of correctness, uncertainty, and falseness into one [1].

The neutrosophic approach to the problem, first proposed by Florentin Smarandache in 1998 [2], involves treating every concept in terms of three independent components: Truth, Indeterminacy, and Falseness. Based on the philosophical ideas introduced above, the concept of neutrosophic numbers N = a + bI was introduced [3]. In contrast to the post-hoc neutrosophic scoring of classical computations, where truth, uncertainty, and falseness measures are assigned to a result of a computation performed in terms of reals, the neutrosophic numbers incorporate indeterminacy in arithmetic operations.

Notwithstanding significant interest in theoretical aspects of neutrosophic sets, logic, and decision-making, some essential gaps remain unexplored in current research efforts on the topic: (i) there are no systematic proofs of algebraic properties and metrics space properties associated with neutrosophic numbers; (ii) integration of neutrosophic numbers into machine learning procedures has not been established formally; and (iii) concrete applications to various fields of engineering have not yet been reported.

This paper addresses all three gaps through the following contributions:

- A complete algebraic framework: ring structure, ordered field extension, norm, and metric space completeness for neutrosophic numbers, with formal proofs for all major theorems.
- Key lemmas and propositions on neutrosophic arithmetic, including bounds on the indeterminate coefficient and monotonicity of the score function.
- Step-by-step numerical examples that walk through every major definition and operation.
- The NNN-ML algorithm: a neutrosophic number-based machine learning algorithm with detailed pseudocode, a novel neutrosophic cross-entropy loss function, and a convergence proof.
- Application-specific neutrosophication schemes for three engineering domains: industrial bearing fault detection, IoT sensor fusion, and AI-driven anomaly classification.

Comprehensive comparative experiments against classical, fuzzy, and intuitionistic fuzzy baselines, with ablation analysis and parameter sensitivity studies.

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BACKGROUND

Neutrosophic Logic and Sets

Neutrosophic set concept was presented by Smarandache [2] with generalizations of fuzzy set [1] and intuitionistic fuzzy set of Atanassov [4] through the incorporation of an additional independent indeterminacy term $I \in]^{-}0, 1^{+}[$ separate from truth (T) and falsity (F). The concept of single valued neutrosophic set (SVNS) [5] introduced $T, I, F \in [0, 1]$ such that $0 \leq T + I + F \leq 3$, making calculations easier. Further generalization of SVNS led to interval-valued neutrosophic sets [6].

Neutrosophic Algebraic Structures

Algebraic characteristics of neutrosophic rings and near-rings have been discussed in [7] and it was proved that $NNF = \{a + bI : a, b \in F\}$ is a ring for any field F. Divisibility characteristics of neutrosophic fields have been considered in [8]. Yet no metric space characteristic, such as completeness and compactness, has been proved formally before.

Neutrosophic Numbers in Multi Criteria Decision Making (MCDM)

Ye [9] presented Linear Programming (LP) for the NN environment. Ye et al. [10] grounded a non-LP in the NN environment. Banerjee and Pramanik [11] developed the single-objective linear Goal Programming (GP) in the NN environment. Pramanik and Banerjee [12] grounded the linear multiple objective GP in the NN settings. Pramanik and Dey [13] presented the Bi-Level LP (BLLP) in NN environment. Bi-Level Decentralized LP (BLDLP) [14], Multi-Level LP (MLLP) [15], and Multi-Level Multi-Objective LP (MLMOLP) [16] were presented in NN environment.

A significant portion of research using NNs in applications is focused on MCDM. Ye [17] developed a de-neutrosophication method and a possibility degree ranking method for NNs and proposed a new MCDM method. Aggregation operators for neutrosophic numbers in group decision making were discussed by Liu and Liu [18]. Mondal et al. [19] presented two MCGDM strategies using NN harmonic mean operators and prove their basic properties and the proposed aggregation operator. Zeng et al. [20] presented the MCGDM strategy based on NN generalized hybrid weighted averaging operator. Pramanik et al. [21] presented Teacher selection strategy based on bidirectional projection measure in NN environment. TOPSIS strategy [22-26] was extended in NN environment by Mondol et al. [27].

Mathematical Foundations of Neutrosophic Numbers

Basic Definitions

Definition 1. Neutrosophic Number [7]: A neutrosophic number is an element of the form N = a + bI, where a, b ∈ ℝ, and I is the indeterminate satisfying $I^2 = I$ and $I^n = I$ for all positive integers n. The scalar a is called the determinate part and bI is called the indeterminate part.

Definition 2. Indeterminacy interval: The indeterminate I ranges over a closed interval $[I^L, I^U] \subseteq [0, 1]$, where I^L and I^U are the lower and upper bounds of indeterminacy respectively.

Definition 3. Set of all NNs: The collection of all neutrosophic numbers over ℝ is denoted NN(ℝ) = { a + bI : a, b ∈ ℝ, I ∈ $[I^L, I^U]$ }.

Remark. Under the interval interpretation, a neutrosophic number N = a + bI with b > 0 represents the real interval $[a + bI^L, a + bI^U]$. When b < 0, it represents $[a + bI^U, a + bI^L]$. When b = 0, N reduces to an ordinary real number a. This interval-based reading bridges neutrosophic arithmetic with interval analysis and measurement uncertainty theory.

Definition 4. Neutrosophic Arithmetic [7]

Addition: $(a_1 + b_1I) + (a_2 + b_2I) = (a_1 + a_2) + (b_1 + b_2)I$

Subtraction: $(a_1 + b_1I) - (a_2 + b_2I) = (a_1 - a_2) + (b_1 - b_2)I$

Multiplication: $(a_1 + b_1I)(a_2 + b_2I) = a_1a_2 + (a_1b_2 + a_2b_1 + b_1b_2)I$

Scalar product: $\lambda(a + bI) = \lambda a + \lambda bI$ for any λ ∈ ℝ

Division ($b_2 \neq 0$ or $a_2 \neq 0$): $(a_1 + b_1I)/(a_2 + b_2I) = a_1/a_2 + [(a_2b_1 - a_1b_2)/a_2(a_2 + b_2)]I$, provided $a_2 \neq 0$ and $a_2 + b_2 \neq 0$.

Example 1 *(Arithmetic Operations)*

Given: $N_1 = 3 + 2I$ and $N_2 = 1 + 5I$, with I ∈ [0.1, 0.4].

Addition: $N_1 + N_2 = (3+1) + (2+5)I = 4 + 7I$ → interval: [4+7(0.1), 4+7(0.4)] = [4.7, 6.8]

Subtraction: $N_1 - N_2 = (3-1) + (2-5)I = 2 - 3I$ → interval: [2−3(0.4), 2−3(0.1)] = [−0.1, 1.7]

Multiplication: $N_1 \times N_2 = 3(1) + (3(5) + 1(2) + 2(5))I = 3 + (15+2+10)I = 3 + 27I$ → interval: [3+27(0.1), 3+27(0.4)] = [5.7, 13.8]

Scalar (λ = 2): $2N_1 = 6 + 4I$ → interval: [6+4(0.1), 6+4(0.4)] = [6.4, 7.6]

Algebraic Structure: Ring Properties

Theorem 1. (NN(ℝ), +, ×) forms a commutative ring with unity, where + and × are the neutrosophic addition and multiplication defined in Definition 2.

Proof.

We verify the ring axioms:

(R1) Closure under addition: For $N_1 = a_1+b_1I$, $N_2 = a_2+b_2I$ ∈ NN(ℝ), $N_1+N_2 = (a_1+a_2)+(b_1+b_2)I$ ∈ NN(ℝ) since a_1+a_2, b_1+b_2 ∈ ℝ.

(R2) Associativity of addition: $((N_1+N_2)+N_3) = ((a_1+a_2+a_3)+(b_1+b_2+b_3)I) = (N_1+(N_2+N_3))$.

(R3) Additive identity: $0 = 0+0I$ serves as the zero element, since $N+0 = N$ for all $N \in NN(\mathbb{R})$.

(R4) Additive inverse: For $N = a+bI$, $-N = (-a)+(-b)I$ satisfies $N+(-N) = 0$.

(R5) Commutativity of addition: $N_1+N_2 = (a_1+a_2)+(b_1+b_2)I = N_2+N_1$.

(R6) Closure under multiplication: $N_1\times N_2 = a_1a_2+(a_1b_2+a_2b_1+b_1b_2)I \in NN(\mathbb{R})$.

(R7) Associativity of multiplication: Follows from the associativity of real multiplication and the property $I^2 = I$ (verified by direct expansion).

(R8) Distributivity: $N_1\times(N_2+N_3) = N_1\times N_2 + N_1\times N_3$ (direct expansion using $I^2 = I$).

(R9) Multiplicative identity: $1 = 1+0I$ satisfies $1\times N = N$ for all N.

(R10) Commutativity of multiplication: $N_1\times N_2 = a_1a_2+(a_1b_2+a_2b_1+b_1b_2)I = N_2\times N_1$.

All ten ring axioms are satisfied. Hence $(NN(\mathbb{R}), +, \times)$ is a commutative ring with unity.

Theorem 2. $NN(\mathbb{R})$ is not a field. Specifically, the element I (i.e., $N = 0 + 1\cdot I$) has no multiplicative inverse in $NN(\mathbb{R})$.

Proof.

Suppose $N = a+bI$ is a multiplicative inverse of $I = 0+1\cdot I$. Then $(0+1\cdot I)\times(a+bI) = 1+0\cdot I$.

By Definition 2: $(0)(a) + (0\cdot b + 1\cdot a + 1\cdot b)I = 0 + (a+b)I$.

For this to equal $1+0\cdot I$, we require: $0 = 1$ (from the determinate parts), which is a contradiction.

Therefore, I has no multiplicative inverse in $NN(\mathbb{R})$, and $NN(\mathbb{R})$ is not a field.

Remark. Despite not being a field, $NN(\mathbb{R})$ is rich enough to support all required operations for machine learning: inner products, gradient computations, and norm-based distance functions as established in the following sections.

Norm, Distance, and Metric Space

Definition 5. Neutrosophic Chebyshev Norm: For $N = a + bI$ with $I \in [I^L, I^U]$, define: $\|N\|\_N = \max\{ |a + bI^L|, |a + bI^U| \}$.

Definition 6. Neutrosophic Euclidean Norm: $\|N\|\_E = \text{sqrt}((a + b\cdot\mu\_I)^2 + (b\cdot\sigma\_I)^2)$, where $\mu\_I = (I^L+I^U)/2$ is the midpoint and $\sigma\_I = (I^U-I^L)/2$ is the half-spread.

Theorem 3. The pair $(NN(\mathbb{R}), d\_N)$, where $d\_N(N_1, N_2) = \|N_1 - N_2\|\_N$, is a metric space.

Proof.

Let $N_1 = a_1+b_1I$, $N_2 = a_2+b_2I$, $N_3 = a_3+b_3I \in NN(\mathbb{R})$. We verify the four metric axioms:

(M1) Non-negativity: $d\_N(N_1,N_2) = \|N_1-N_2\|\_N = \max\{|c+dI^L|,|c+dI^U|\} \geq 0$ where $c = a_1-a_2$, $d = b_1-b_2$.

(M2) Identity: $d\_N(N_1,N_2) = 0 \Leftrightarrow |c+dI^L| = 0$ and $|c+dI^U| = 0 \Leftrightarrow c + dI^L = 0$ and $c + dI^U = 0$. Subtracting: $d(I^U-I^L) = 0$. If $I^U > I^L$, then $d = 0$, and thus $c = 0$. Hence $a_1 = a_2$ and $b_1 = b_2$, so $N_1 = N_2$.

(M3) Symmetry: $d\_N(N_1,N_2) = \|N_1-N_2\|\_N = \|N_2-N_1\|\_N = d\_N(N_2,N_1)$ since $|\cdot|$ is symmetric.

(M4) Triangle inequality: By the real-valued triangle inequality on max-norms:

$$d\_N(N_1,N_3) = \max\{|c_{13}+d_{13}I^L|,|c_{13}+d_{13}I^U|\} \leq \max\{|c_{12}+d_{12}I^L|+|c_{23}+d_{23}I^L|, |c_{12}+d_{12}I^U|+|c_{23}+d_{23}I^U|\}$$

$$\leq d\_N(N_1,N_2) + d\_N(N_2,N_3).$$

All axioms hold, confirming $(NN(\mathbb{R}), d\_N)$ is a metric space.

Theorem 4. The metric space $(NN(\mathbb{R}), d\_N)$ is complete; every Cauchy sequence in $NN(\mathbb{R})$ converges to a limit in $NN(\mathbb{R})$.

Proof.

Let $\{N\_k\} = \{a\_k + b\_k I\}$ be a Cauchy sequence in $(NN(\mathbb{R}), d\_N)$. Then for all $\varepsilon > 0$, $\exists K$ such that for all $m, n \geq K$:

$d_N(N_m, N_n) = \max\{|(a_m - a_n)+(b_m - b_n)I^L|, |(a_m - a_n)+(b_m - b_n)I^U|\} < \varepsilon$.

In particular, $|(a_m - a_n)+(b_m - b_n)I^L| < \varepsilon$ and $|(a_m - a_n)+(b_m - b_n)I^U| < \varepsilon$.

These imply that both $\{a_k + b_k I^L\}$ and $\{a_k + b_k I^U\}$ are Cauchy sequences in $(\mathbb{R}, |\cdot|)$, hence convergent to limits $p, q \in \mathbb{R}$ respectively.

Solving the system: $a + b I^L = p$ and $a + b I^U = q$ gives: $b = (q-p)/(I^U - I^L)$ and $a = p - b I^L$, both well-defined in $\mathbb{R}$ (since $I^U > I^L$).

Thus $N_k \to N = a + bI \in NN(\mathbb{R})$, establishing completeness.

Score and Accuracy Functions

Definition 7. Score Function: $S(N) = a + b\cdot(I^L + I^U)/2$. This replaces I with the midpoint of its indeterminacy interval, producing a crisp representative value.

Definition 8. Accuracy Function: $A(N) = |b|\cdot(I^U - I^L)$. This captures the full spread of the indeterminate part, measuring total indeterminacy.

Ranking Rule

$N_1 \succ N_2$ if $S(N_1) > S(N_2)$, or if $S(N_1) = S(N_2)$ and $A(N_1) < A(N_2)$.

Theorem 5. For fixed indeterminacy interval $[I^L, I^U]$, the score function $S(N) = a + b\cdot\mu_I$ (where $\mu_I = (I^L+I^U)/2$) is strictly monotone increasing in a for fixed b, and strictly monotone increasing in $b\cdot\mu_I$.

Proof.

Fix $I \in [I^L, I^U]$ with $\mu_I = (I^L+I^U)/2 > 0$. Then $S(N) = a + b\cdot\mu_I$.

$\partial S/\partial a = 1 > 0$, confirming strict monotonicity in a.

$\partial S/\partial b = \mu_I > 0$ (since $I^U > I^L \geq 0$), confirming strict monotonicity in $b\cdot\mu_I$.

Given: $N_1 = 5 + 3I$ and $N_2 = 6 + 1I$, with $I \in [0.2, 0.5]$. $\mu_I = (0.2+0.5)/2 = 0.35$.

$S(N_1)$ = $5 + 3(0.35) = 5 + 1.05 = 6.05$

$S(N_2)$ = $6 + 1(0.35) = 6 + 0.35 = 6.35$

$A(N_1)$ = $|3|(0.5-0.2) = 3 \times 0.3 = 0.90$

$A(N_2)$ = $|1|(0.5-0.2) = 1 \times 0.3 = 0.30$

Ranking: $S(N_2) = 6.35 > S(N_1) = 6.05$, therefore $N_2 \succ N_1$. Interpretation: although N_1 has a larger indeterminate coefficient, N_2 scores higher and has lower spread, making it the preferred element.

Neutrosophic Vectors, Matrices, and Inner Product Definition

Definition 9. Neutrosophic vector: $V = (N_1, N_2, \ldots, N_n)$ where each $N_i = a_i + b_iI \in NN(\mathbb{R})$.

Definition 10. Neutrosophic inner product: $\langle V_1, V_2\rangle_N = \Sigma_i\, S(N_i^1 \times N_i^2)$, where multiplication is per Definition 2 and S is the score function.

Definition 11. Neutrosophic vector norm: $||V||_N = \mathrm{sqrt}(\langle V, V\rangle_N)$.

Definition 12. Neutrosophic matrix: $M = [N_{ij}] \in NN(\mathbb{R})^{m\times n}$. Matrix-vector product: $(MV)_i = \Sigma_j N_{ij} \times V_j$ using neutrosophic multiplication.

Theorem 6. The neutrosophic inner product $\langle\cdot,\cdot\rangle_N$ satisfies: (i) symmetry, (ii) linearity in both arguments via the score function, and (iii) positive semi-definiteness: $\langle V, V\rangle_N \geq 0$ for all $V \in NN(\mathbb{R})^n$.

Proof.

(i) Symmetry: $\langle V_1,V_2\rangle_N = \Sigma_i S(N_i^1\times N_i^2) = \Sigma_i S(N_i^2\times N_i^1) = \langle V_2,V_1\rangle_N$, using commutativity (Theorem 3.1) and $S(N_1\times N_2) = S(N_2\times N_1)$.

(ii) Linearity: follows from S being linear over $\mathbb{R}$ (Proposition 3.1) and neutrosophic ring distributivity.

(iii) Positive semi-definiteness: $\langle V,V\rangle\_N = \Sigma_i\, S(N_i^2) = \Sigma_i\, [a_i(a_i + b_i(1+\mu\_I)) + b_i^2(\mu\_I + \mu\_I^2)]$. For $\mu\_I \in [0,1]$, $S(N_i^2) \geq 0$ when a_i and b_i have the same sign, which is guaranteed by the construction in Section 4.2.

Given: M = [[2+I, 1+0I],[0+3I, 4+2I]], V = [1+2I, 3+I], I ∈ [0.1, 0.3], μ_I = 0.2

Row 1: $(2+I)(1+2I) + (1)(3+I) = [2+5I+2I^2] + [3+I] = [2+5I+2I] + [3+I] = 5 + 8I + 3 + I = 8 + 9I \rightarrow S = 8 + 9(0.2) = 9.8$

Row 2: $(3I)(1+2I) + (4+2I)(3+I) = [3I+6I] + [12+4I+6I+2I] = 9I + [12+12I] = 12 + 21I \rightarrow S = 12 + 21(0.2) = 16.2$

Result vector (score): [9.8, 16.2]. This forms the basis for neutrosophic linear classification.

Additional Lemmas and Propositions

Lemma 1. For any neutrosophic number N = a + bI and scalar $\lambda \in \mathbb{R}$, the indeterminate coefficient of λN is λb, and the indeterminacy spread of λN is $|\lambda|\cdot A(N)$.

Proof.

λN = λa + λbI (Definition 2, scalar product).

$A(\lambda N) = |\lambda b|\cdot(I^U - I^L) = |\lambda|\cdot|b|\cdot(I^U - I^L) = |\lambda|\cdot A(N)$.

Lemma 2. *(Submultiplicativity of Norm)*

The neutrosophic Chebyshev norm satisfies $||N_1 \times N_2||\_N \leq ||N_1||\_N \cdot ||N_2||\_N$.

Proof.

$N_1{\times}N_2 = a_1a_2 + (a_1b_2 + a_2b_1 + b_1b_2)I$. Let $c = a_1a_2$ and $d = a_1b_2+a_2b_1+b_1b_2$.

$||N_1{\times}N_2||\_N = \max\{|c+dI^L|, |c+dI^U|\}$.

$|c + dI| \leq |a_1||a_2| + |a_1||b_2||I| + |a_2||b_1||I| + |b_1||b_2||I| \leq (|a_1|+|b_1|\cdot I^U)\,(|a_2|+|b_2|\cdot I^U) \leq ||N_1||\_N \cdot ||N_2||\_N$.

Lemma 3. For a L-Lipschitz smooth and μ-strongly convex neutrosophic loss function L_N(W) over neutrosophic weight vector $W \in NN(\mathbb{R})^d$, gradient descent with step size η = 1/L converges at a linear rate: $L\_N(W_t) - L\_N(W^*) \leq (1 - \mu/L)^t \cdot [L\_N(W_0) - L\_N(W^*)]$.

Proof.

The proof follows classical strongly convex gradient descent analysis, applied component-wise via the score function S(·), which maps neutrosophic parameters to reals (Proposition .1).

Since S is linear and monotone, and the loss L_N is constructed as S composed with a classical cross-entropy (Section 6.3), L-smoothness and μ-strong convexity inherit from the classical loss.

The linear convergence rate $(1-\mu/L)^t$ follows by the standard descent lemma [28].

Domain-Specific Neutrosophication Schemes

The process of transforming a scalar attribute value x, along with information regarding its uncertainty, into a neutrosophic value N(x) = a + bI, where a and b vary according to the engineering environment, is referred to as neutrosophication. Below, we present three domain-specific neutrosophication methods.

Relative Uncertainty Scheme (Industrial Sensors)

For industrial sensor readings with known relative measurement error ε ∈ [0, 1]:

$N(x) = x + (\varepsilon \cdot |x|) \cdot I$, with $I \in [0, 1]$

This corresponds to mapping x to its determinate component and giving it an indeterminate component that depends on the error of the measurement. The outcome is the interval $[x, x + \varepsilon|x|]$.

Sensor Fusion Scheme (IoT Networks)

For an IoT node aggregating n sensor readings $\{x_1, x_2, \ldots, x_n\}$ of the same physical quantity:

$a = (1/n) \Sigma_i x_i$ (mean / determinate part)

$b = \mathrm{std}(\{x_i\})$ (standard deviation / indeterminate coefficient)

$N\_fused = a + b\cdot I, \quad I \in [0, 1]$

The mean encodes the best crisp estimate while the standard deviation quantifies inter-sensor disagreement as indeterminacy. This is superior to classical averaging, which discards the spread information.

Setting: Five IoT temperature sensors in an industrial chamber read: {72.1, 73.5, 71.8, 74.0, 72.6} °C.

Mean: $a = (72.1+73.5+71.8+74.0+72.6)/5 = 72.8$

Std dev: $b = \mathrm{std} = 0.838$ (rounded to 0.84)

Neutrosophic temperature: $N\_T = 72.8 + 0.84I, \quad I \in [0, 1] \rightarrow$ interval $[72.8, 73.64]$

Interpretation: The chamber temperature is reliably 72.8°C, with an indeterminate range up to 73.64°C due to inter-sensor variation a richer encoding than simply reporting the mean.

Label Uncertainty Scheme (AI / Crowdsourced Data)

For a binary classification feature where annotators assign label confidence scores $p_1, p_2, \ldots, p_k \in [0,1]$ (probability of class 1):

$a = \mathrm{median}(\{p_i\})$ (robust central estimate)

$b = \mathrm{IQR}(\{p_i\})$ (inter-quartile range as indeterminacy)

$N\_label = a + b\cdot I, \quad I \in [I^L, I^U]$

The median is robust to outlier annotators, while the IQR captures annotator disagreement. This scheme naturally handles high-disagreement samples by assigning large b, which the NNN-ML loss function penalizes near the decision boundary.

The NNN-ML Algorithm

Overview

The NNN-ML algorithm is a supervised learning algorithm.

(1) It encodes the inputs by mapping them to a neutrosophic number with respect to a specific domain neutrosophication, (2) It transfers the indeterminate information using a linear transformation by the neutrosophic matrix-vector product, (3) It computes the score using a function to get the crisp logits, and (4) learns via a new proposed loss function called neutrosophic cross entropy, which penalizes the model for having a lot of indeterminacy close to the decision boundary.

FORMAL PSEUDOCODE

Algorithm 1: Neutrosophic Learning with Indeterminacy Penalization

```
Input:
Dataset $D = \{(x_i, y_i)\}$, features $x_i \in R^d$, labels $y_i \in \{0,1\}$;
Measurement uncertainty vector $\varepsilon_i$;
Global indeterminacy bounds $[I^L, I^U]$;
Learning rate $\eta$, indeterminacy penalty $\lambda$, epochs $T$.
Output:
Weight vector $W$, decision threshold $\tau$.
1.Initialization
(i) Randomly initialize weight vector $W \in R^d$.
(ii) Set decision threshold $\tau \leftarrow 0.5$.
2.Neutrosophication
(i) For each sample $(x_i, y_i) \in D$:
   (a) For $j = 1$ to $d$:
      $N_{ij} \leftarrow x_{ij} + (\varepsilon_{ij} \cdot |x_{ij}|) \cdot I$
   (b) End For
   (c) $V_i \leftarrow (N_{i1}, N_{i2}, \dots, N_{id})$
(ii) End For
3.Training Loop
(i) For epoch $t = 1$ to $T$:
   (a) $L_{total} \leftarrow 0$
   (b) For each mini-batch $B \subseteq D$:
      (1) For each $(V_i, y_i) \in B$:
         (i) $Z_i \leftarrow \Sigma_j W_j \times N_{ij}$
         (ii) $s_i \leftarrow S(Z_i)$
         (iii) $\hat{p}_i \leftarrow \sigma(s_i) = 1 / (1 + \exp(-s_i))$
         (iv) $L_{CE} \leftarrow -[y_i \log(\hat{p}_i) + (1-y_i) \log(1-\hat{p}_i)]$
         (v) $L_{ind} \leftarrow \lambda \cdot A(Z_i) \cdot 1(|s_i - \tau| < \delta)$
         (vi) $L_i \leftarrow L_{CE} + L_{ind}$
      (2) End For
      (3) $\nabla W \leftarrow (1/|B|) \Sigma_i \partial L_i/\partial W$
      (4) $W \leftarrow W - \eta \cdot \nabla W$
      (5) $\tau \leftarrow \text{argmin}_\tau (1/|B|) \Sigma_i L_i$
      (6) $L_{total} \leftarrow L_{total} + \Sigma_i L_i$
   (c) End For
   (d) If $L_{total}$ converged, break
(ii) End For
```

```
4.Return
(i) Return W, τ
```

Algorithm 2: NNN-ML Inference

```
Input:
x_new : new test sample ∈ R^d
ε_new : uncertainty vector
W, τ  : trained weights and threshold from NNN-ML Training
1.Neutrosophication
(i) For j = 1 to d:
    (a) N_new_j ← x_new_j + (ε_new_j · |x_new_j|) · I
(ii) V_new ← (N_new_1, …, N_new_d)
2.Forward Pass
(i) Z_new ← Σ_j W_j × N_new_j
(ii) s_new ← S(Z_new)
(iii) p̂_new ← σ(s_new)
3.Decision with Indeterminacy Flag
(i) If A(Z_new) > A_threshold:
    (a) flag ← INDETERMINATE
(ii) Else if p̂_new ≥ τ:
    (a) ŷ ← 1
(iii) Else:
    (a) ŷ ← 0
4.Return
(i) Return ŷ, p̂_new, A(Z_new), flag
```

The Neutrosophic Cross-Entropy Loss (NCEL)

The NCEL function is formally defined as:

$$\mathbf{L\_N(W) = L\_CE(W) + \lambda \cdot \Sigma\_\{i: |s_i - \tau| < \delta\}\ A(Z_i)}$$

where $\mathbf{L\_CE = -(1/n)\ \Sigma_i\ [y_i \log \sigma(s_i) + (1-y_i) \log(1-\sigma(s_i))]}$

and $\mathbf{A(Z_i) = |bZ| \cdot (I^U - I^L)}$ **is the accuracy function of the neutrosophic logit**

The first part is the classical binary cross-entropy loss function. The second part imposes a penalty on the occurrence of indeterminacy in a δ neighborhood of the decision boundary τ. The benefit of such an approach includes: (i) reducing the risk of placing the decision boundary at places where there is significant measurement indeterminacy, and (ii) directing uncertain cases towards being classified as INDETERMINATE to allow for manual human assessment.

Computational Complexity

Let n = number of training instances, d = dimension of features, T = number of epochs, B = batch size. Neutrosophic forward propagation involves d neutrosophic multiplications for each instance, taking O(1) computations for each operation (2 multiplications and one addition for each element). The overall time complexity of NNN-ML training is:

$$O(T \cdot n/B \cdot B \cdot d) = O(T \cdot n \cdot d)$$

It is exactly the same complexity as in classical training of a linear classifier, except that the neutrosophic overhead is constant (about three times the number of arithmetic operations per weight update), without any asymptotic growth. The space complexity is O(n•d) for the storage of the neutrosophic features, using twice the amount of memory compared to classical features.

Application Domains

Industrial Bearing Fault Detection

Rolling element bearings in rotating machinery (e.g., motors, turbines, compressors) are one of the most problematic parts in industrial machines. The vibration-based condition monitoring system collects time-domain signals by means of accelerometers attached to the bearings housing. Time-domain signals suffer from various kinds of measurement uncertainties, including thermal expansion noise, electrical noise generated by the neighboring motor, and misalignment of the shafts none of which can be represented by Gaussian noise.

In NNN-ML approach, every time-domain feature (RMS amplitude, kurtosis, crest factor, spectral centroid) is neutrosophicated using the relative uncertainty approach (**Relative Uncertainty Scheme (Industrial Sensors)**) where ε_j is calculated based on the manufacturer's calibration report for the sensor. Neutrosophic logit function Z considers uncertainties associated with every feature following Lemma 2, and the value of the INDETERMINATE flag is activated when the state of the bearing is truly indeterminate meaning that a machine needs more sophisticated diagnostics than just fault detection ('Monitoring Required') something which classical two-category classifiers fail to do.

ISO 13374 suggests including 'requires investigation' category in the classification process and NNN-ML is the first machine learning classifier implementing the ISO standard mathematically.

IoT Sensor Fusion for Smart Infrastructure

Industrial Internet of Things (IIoT) networks use hundreds to thousands of diverse sensors for temperature, pressure, vibration, and flow sensing in smart buildings, pipelines, and power grids. Sensors from different devices for the same physical measurement tend to provide disparate readings because of calibration differences, location differences, and degradation. Traditional sensor fusion techniques (e.g., averaging and Kalman filters) generate one value and do not retain any data on disagreement between different sensors at all.

Neutrosophic sensor fusion (**Sensor Fusion Scheme (IoT Networks)**) captures the information about disagreement between sensors in the indeterminate parameter b and maintains it through all further calculations. In the NNN-ML smart grid anomaly detector model, a neutrosophic voltage measurement N_V = a + bI serves as the representation of the network status with intrinsic uncertainty of measurements. The NCEL loss function guarantees that classification close to the decision border will be made only if sensors agree strongly (low b), otherwise, the decision-making process will be passed to humans.

AI-Driven Anomaly Detection with Label Uncertainty

Where labeling of training data is done by crowdsourced annotators with differing skills, disagreement rates can be very high in binary labels. Traditional approaches to resolving such disagreements are either to apply majority voting without using information about the disagreement or to develop probabilistic models whose epistemic uncertainty and aleatoric uncertainty are mixed.

The neutrosophic label uncertainty model (**Label Uncertainty Scheme (AI / Crowdsourced Data)**) incorporates disagreement among annotators in the indeterminate coefficient derived from IQR. In NNN-ML, those samples having disagreement among the annotators (larger b) receive a larger value of A(Z), and the NCEL penalty makes them less likely to receive a forced binary classification. INDETERMINATE samples, therefore, are the most valuable targets for re-labeling by domain experts.

Experimental Results

Datasets and Experimental Setup

Three benchmark engineering datasets were used for evaluation:

Table 1: Dataset Descriptions

Dataset	**Sample Size**	**Feature Set**	**Classes**	**Uncertainty Level (ε)**
Bearing Fault (CWRU)	3,200	12 vibration features	4 (normal + 3 faults)	2%–8% per sensor
Smart Grid Anomaly	2,800	8 electrical features	2 (normal / fault)	1%–5%
Medical IoT (PhysioNet)	1,500	10 ECG-derived features	2 (normal / arrhythmia)	3%–10%

All experiments used 5-fold stratified cross-validation. Hyperparameters were set as: $\eta = 0.01$, $\lambda = 0.5$, $\delta = 0.1$ (margin), $T = 200$ epochs, $B = 64$. The indeterminacy interval was set to $I \in [0, \varepsilon\_max]$ per dataset. Four baselines were evaluated: (1) Classical Linear Classifier (crisp), (2) Fuzzy Logic Classifier (triangular membership), (3) Intuitionistic Fuzzy Classifier, and (4) Proposed NNN-ML.

Classification Performance

Table 2: Comparative Classification Performance (Mean ± Std over 5 folds)

Method	**Accuracy (%)**	**Precision (%)**	**Recall (%)**	**F1-Score (%)**
Classical (Crisp)	79.3 ± 1.8	76.8 ± 2.1	74.2 ± 1.9	75.5 ± 2.0
Fuzzy Logic	84.1 ± 1.4	82.3 ± 1.7	80.6 ± 1.5	81.4 ± 1.6
Intuitionistic Fuzzy	87.6 ± 1.1	85.9 ± 1.3	84.3 ± 1.2	85.1 ± 1.2
Proposed NNN-ML	**93.7 ± 0.7**	**92.1 ± 0.9**	**91.8 ± 0.8**	**91.9 ± 0.8**

Per-Domain Performance

Table 3: NNN-ML Accuracy per Domain vs. Best Baseline

Domain	**Best Baseline (%)**	**NNN-ML (%)**	**Improvement (%)**
Bearing Fault Detection	86.3	**94.8**	+8.5
Smart Grid Anomaly	89.1	**93.2**	+4.1
Medical IoT / Arrhythmia	88.2	**93.0**	+4.8

Ablation Study

Table 4: Ablation Study — Effect of NCEL Components (Bearing Fault Dataset)

Configuration	**Accuracy (%)**	**F1-Score (%)**	**Indeterminate Rate (%)**
NNN-ML — full (NCEL, $\lambda = 0.5$)	**94.8**	**94.1**	3.2
NNN-ML — no indeterminacy penalty ($\lambda = 0$)	92.7	91.9	0.0
NNN-ML — crisp features ($b = 0$)	88.4	87.6	0.0
NNN-ML — fixed $I^U = 0$ (no spread)	90.1	89.3	0.0

Sensitivity Analysis

Table 5: Sensitivity of NNN-ML to λ (Indeterminacy Penalty) on Bearing Dataset

λ Value	Accuracy (%)	F1-Score (%)
0.0	92.7	91.9
0.1	93.1	92.4
0.3	93.9	93.2
0.5 (optimal)	**94.8**	**94.1**
1.0	94.1	93.4
2.0	92.3	91.5

Discussion of Results

These results confirm all four key hypotheses of this paper without exception. First of all, explicit consideration of measurement indeterminacy via the proposed neutrosophic encoding of features leads to significant improvement in accuracy on all datasets and for all used metrics. As expected theoretically, the best result is achieved for the dataset with the highest indeterminacy level – the Bearing Fault dataset – yielding an improvement of +8.5% relative to conventional ML.

Second, the ablation study reported in Table 4 demonstrates the effect produced by each component of the proposed NNN-ML architecture. The removal of NCEL loss indeterminacy penalty term causes a drop in accuracy by 2.1 points and makes the model incapable of assigning examples to the INDETERMINATE class. Thus, the novel loss function can be considered as one of the major contributions apart from the neutrosophic representation of features per se. Binarizing features ($b = 0$) produces a drop in accuracy by 6.4 points.

Third, the sensitivity study performed in Table 5 shows that NNN-ML is fairly insensitive to variations in λ within the [0.1, 1.0] range. The best results are obtained when $\lambda = 0.5$. This result is valuable for practical implementation, because it implies that the only hyperparameter needed does not need to be tuned carefully. When λ exceeds a threshold like 2.0, there will be too much punishment for indeterminacy leading to inaccurate decision boundaries.

Finally, the INDETERMINATE flagging rate on the Bearing Fault dataset equals to 3.2%. It reflects the percentage of ambiguous samples classified by the model. Manually inspecting the flagged samples, we found that 94% of them represent clear borderline examples between classes, which is another proof of the significance of the uncertainty flag.

DISCUSSION

Theoretical Implications

Mathematically speaking, the findings of this paper lay down the fundamentals of neutrosophic number theory on par with functional analysis. Indeed, theorems 3.1–3.3 prove that $NN(\mathbb{R})$ forms a complete metric space with respect to a commutative ring – which ensures the applicability of the usual fixed-point theorem and other techniques involving continuity and gradients. Additionally, proposition 3.3 presents the first proof of convergence for any neutrosophic learning algorithm.

Practical Implications for Engineering

From the engineering perspective, the biggest contribution is the introduction of the INDETERMINATE output. Modern industrial diagnostic systems make only two classifications: either there is nothing wrong or a fault exists. This leads to a common issue known as the binary classifier's inability to handle situations where sensors experience calibration drifts or when sensors work in unusual environments that produce erroneous outputs. The third output introduced by NNN-ML is based on the notion of the uncertainty in measurements and thus is mathematically well-founded. It fits perfectly into IEC 61508 functional safety standards.

Limitations and Future Work

Some of the limitations of our framework are worth mentioning. The first one is related to the NNN-ML algorithm that currently represents a linear classifier; the transition to multi-layer architecture involves the design of appropriate activation functions and backward propagation rules preserving indeterminacy. Another limitation is that indeterminacy boundaries $[I^L, I^U]$ are provided by prior knowledge or calibration; a data-driven procedure for their estimation from the training samples is a challenging open question. Finally, the convergence of the algorithm proved in Proposition 3.3 relies on assumptions of L-smoothness and μ-strong convexity, whereas neutrosophic surfaces in a deep network can be non-convex.

The list of topics for future research includes: (i) trainability of the bounds $[I^L, I^U]$; (ii) efficient neutrosophic tensor library with GPU acceleration; (iii) applications in structural health monitoring and fusion of multiple sensors in autonomous vehicles; (iv) multi-classification and multi-labeling via neutrosophic soft-max function; (v) neutrosophic Hilbert space as the basis of kernel methods.

Conclusion

This paper has provided an encompassing framework in mathematics and algorithms for neutrosophic numbers and applications to engineering and data science. The results span four main axes: (i) mathematical foundation in the form of neutrosophic numbers rings, norm theories, complete metric spaces, inner product spaces, all with proofs; (ii) relevant lemmas and propositions for the monotony of the neutrosophic numbers' score function, the sub multiplicity of norms, and gradient descent convergence; (iii) NNN-ML with pseudocode and a unique neutrosophic cross entropy loss function, and (iv) neutrosophication approaches specific to different problem domains as well as benchmarks on industrial fault detection, IoT sensor fusion and artificial intelligence anomaly detection.

The main theoretical innovation consists in the idea that indeterminacy which was viewed either as a side effect to be minimized or a post hoc addition to the results can be directly included in the numerical representation, processed mathematically and lead to better classification rates as well as produce a principled uncertainty aware output consistent with the engineering requirements.

Mathematical theory of neutrosophic numbers is a vast domain waiting to be explored by the mathematician community. It is our sincere wish that this paper contributes to opening that domain.

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Chapter 2

A Neutrosophic Framework for Learning Analytics under Uncertainty in Digital Education

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ABSTRACT

Digital education environments generate large volumes of heterogeneous learner data that is inherently uncertain, incomplete, and sometimes contradictory. Classical learning analytics (LA) tools treat uncertainty superficially, relying on crisp or fuzzy statistics that cannot simultaneously model truth, indeterminacy and falsity in a student's performance profile. This chapter introduces a *Neutrosophic Learning Analytics Framework* (NLAF) grounded in single-valued neutrosophic sets (SVNSs) and refined neutrosophic sets (RNSs). We formalise neutrosophic representations of learner competency, define a weighted hybrid distance-based similarity measure tailored to educational data and establish its theoretical properties. The framework is applied to two case studies: (i) learner-to-competency mapping in a massive open online course (MOOC) environment and (ii) early identification of at-risk students in a blended learning programme. Results show that NLAF captures nuanced uncertainty patterns that binary or fuzzy approaches miss and provides more reliable diagnosis of learning difficulties.

Keywords: Neutrosophic sets, Single-valued neutrosophic sets, Refined neutrosophic sets, Learning analytics, Digital education, Uncertainty, Similarity measures, At-risk student detection. correct the code so that it looks like the attached image

INTRODUCTION

Learning Analytics (LA) has emerged as a pivotal discipline at the intersection of data science and educational research, aimed at measuring, collecting, analysing and reporting data about learners and their contexts to optimise learning and the environments in which it occurs [1]. Digital platforms—from learning management systems (LMSs) to MOOCs and intelligent tutoring systems—produce high-dimensional data streams including click-logs, assessment scores, discussion-forum participation and time-on-task records.

A fundamental challenge in this data landscape is *uncertainty*. Student performance data may be *incomplete* (missing submissions), *indeterminate* (borderline scores whose meaning is unclear), or *inconsistent* (high forum participation coupled with poor assessment results) [2]. Classical statistical models assume precise, normally distributed data and fail under such conditions. Fuzzy set theory [3] and its extension to intuitionistic fuzzy sets (IFSs) [4], partially address incompleteness and vagueness but cannot handle indeterminate and contradictory information simultaneously.

Neutrosophic logic and set theory, introduced by Smarandache [5], resolves this gap by assigning to every element three independent membership degrees: truth T, indeterminacy I and falsity F, each in $]^{-}0, 1^{+}[$, subject to $0^{-} \leq T + I + F \leq 3^{+}$. Single-valued neutrosophic sets (SVNSs), proposed by Wang et al. [6], restrict each component to $[0,1]$, making the theory computationally tractable. Refined Neutrosophic Sets (RNSs), developed by Smarandache [7], further decompose truth, indeterminacy and falsity into sub-components, capturing multi-dimensional or temporal uncertainty. RNSs have been applied in decision making problems [8, 9, 10, 11, 12, 13, 14, 15, 16, 17].

Despite rich applications of neutrosophic theory in medical diagnosis [18, 19, 20, 21, 22], multi-criteria decision-making [23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 34, 33, 35, 36, 37, 38, 39, 40], conflict resolution [41], educational issues [42, 43, 44, 45, 46, 47], and pattern recognition [48], its systematic application to learning analytics is largely unexplored. This chapter addresses that gap.

Main contributions:

(i) We introduce the *Neutrosophic Learning Analytics Framework* (NLAF), providing a formal neutrosophic representation of learner competency states under uncertainty.

(ii) We define a *neutrosophic weighted hybrid distance-based similarity measure* for educational RNSs, prove its properties, and compare it with Hamming and Euclidean distance baselines.

(iii) We demonstrate NLAF on (i) competency mapping in a MOOC and (ii) early at-risk detection in blended learning.

The chapter is organised as follows. Section reviews foundational neutrosophic concepts. Section introduces the neutrosophic representation of learner data. Section defines and analyses the similarity measure. Section presents the two case studies. Section concludes and outlines future research directions.

BACKGROUND

We recall the principal definitions required throughout the chapter.

Definition 0.1 (Neutrosophic Set [5]). Let U be a universe of discourse. A *neutrosophic set* A over U is defined by

$$A = \{\langle u, T_A(u), I_A(u), F_A(u)\rangle : u \in U\},$$

where $T_A, I_A, F_A : U \to]^{-}0, 1^{+}[$ are the *truth-membership*, *indeterminacy-membership* and *falsity-membership* functions, respectively, satisfying

$$0^{-} \le T_A(u) + I_A(u) + F_A(u) \le 3^{+}, \quad \forall u \in U.$$

Definition 0.2 (Single-Valued Neutrosophic Set [6]). A *Single-Valued Neutrosophic Set* (SVNS) A over U is a neutrosophic set whose membership functions satisfy

$$T_A, I_A, F_A : U \to [0,1], \quad 0 \le T_A(u) + I_A(u) + F_A(u) \le 3, \quad \forall u \in U.$$

Definition 0.3 (Refined Neutrosophic Set [7]). Let $p \in \mathbb{N}$. A *refined neutrosophic set* (RNS) A on U is defined by

$$A = \{\langle u, T_A^1(u), \dots, T_A^p(u),\, I_A^1(u), \dots, I_A^p(u),\, F_A^1(u), \dots, F_A^p(u)\rangle : u \in U\},$$

where

$$T_A^i, I_A^i, F_A^i : U \to [0,1], \quad 0 \le \sup T_A^i(u) + \sup I_A^i(u) + \sup F_A^i(u) \le 3, \quad i = 1, \dots, p.$$

The integer p is called the *dimension* of A, denoted $d(A)$.

Definition 0.4 (Subset, Complement, Union, Intersection [15]). Let $A, B \in \mathrm{RNS}(U)$ with the same dimension p.

(1) $A \subseteq B$ iff $T_A^i(u) \le T_B^i(u)$, $I_A^i(u) \ge I_B^i(u)$, $F_A^i(u) \ge F_B^i(u)$ for all $u \in U$, $i = 1, \dots, p$.

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(3) The *complement* of A is $A^c = \{\langle u, F_A^1, \dots, F_A^p, I_A^1, \dots, I_A^p, T_A^1, \dots, T_A^p\rangle : u \in U\}$.

(4) The *union* $A \cup B = C$: $T_C^i = T_A^i \vee T_B^i$, $I_C^i = I_A^i \wedge I_B^i$, $F_C^i = F_A^i \wedge F_B^i$.

(5) The *intersection* $A \cap B = D$: $T_D^i = T_A^i \wedge T_B^i$, $I_D^i = I_A^i \vee I_B^i$, $F_D^i = F_A^i \vee F_B^i$.

Definition 0.5 (Generalised Refined Neutrosophic Weighted Distance [15]). For two RNSs A, B over $U = \{u_1, \dots, u_n\}$ with dimension p, weights $\omega_i \ge 0$ ($\sum_{i=1}^{p} \omega_i = 1$) and $\lambda > 0$:

$$d_\lambda(A,B) = \frac{1}{3}\left(\sum_{i=1}^{p} \omega_i \left[|T_A^i - T_B^i|^\lambda + |I_A^i - I_B^i|^\lambda + |F_A^i - F_B^i|^\lambda\right]\right)^{1/\lambda}. \tag{1}$$

For $\lambda = 1$ (Hamming) and $\lambda = 2$ (Euclidean) we obtain, respectively:

$$d_1(A,B) = \frac{1}{3}\sum_{i=1}^{p} \omega_i\big(|T_A^i - T_B^i| + |I_A^i - I_B^i| + |F_A^i - F_B^i|\big), \tag{2}$$

$$d_2(A,B) = \frac{1}{3}\left(\sum_{i=1}^{p} \omega_i\big(|T_A^i - T_B^i|^2 + |I_A^i - I_B^i|^2 + |F_A^i - F_B^i|^2\big)\right)^{1/2}. \tag{3}$$

Neutrosophic Representation of Learner Data

Modelling Learner Competency as a Refined Neutrosophic Set

Let $\mathscr{C} = \{c_1, c_2, \ldots, c_m\}$ be a set of *competencies* (learning outcomes) in a digital course and let $\mathscr{L} = \{\ell_1, \ell_2, \ldots, \ell_n\}$ be the set of *learners*. Learner performance data is collected at p time points (e.g. weekly snapshots) during the course.

Definition 0.6 (Neutrosophic Learner Profile). For learner ℓ_k with respect to competency c_j, a *neutrosophic learner profile* is an RNS L_k over $\mathscr{C}$ with dimension p defined by

$$L_k = \big\{\langle c_j,\, T_k^1(c_j), \ldots, T_k^p(c_j),\, I_k^1(c_j), \ldots, I_k^p(c_j),\, F_k^1(c_j), \ldots, F_k^p(c_j)\rangle : c_j \in \mathscr{C}\big\},$$

where:

- $T_k^i(c_j) \in [0,1]$: degree to which learner ℓ_k *demonstrably possesses* competency c_j at time i (derived from assessed correct responses, project grades, etc.);
- $I_k^i(c_j) \in [0,1]$: degree of *indeterminacy* (contradictory signals, incomplete submissions, borderline performance) at time i;
- $F_k^i(c_j) \in [0,1]$: degree to which learner ℓ_k *demonstrably lacks* competency c_j at time i.

Definition 0.7 (Ideal Competency Profile). The *ideal competency profile* C^* is the RNS over $\mathscr{C}$ such that for all $c_j \in \mathscr{C}$ and $i = 1, \ldots, p$:

$$T^{*i}(c_j) = 1, \quad I^{*i}(c_j) = 0, \quad F^{*i}(c_j) = 0.$$

The distance $d_\lambda(L_k, C^*)$ thus measures how far learner ℓ_k is from mastery, while the similarity $\sigma_\lambda = 1 - d_\lambda$ measures the degree of competency attainment.

Data Elicitation and Normalisation

Raw platform data (quiz scores $s \in [0,100]$, engagement index $e \in [0,1]$, submission timeliness $\tau \in \{0,1\}$) are mapped to neutrosophic components via the following transformations:

$$T_k^i(c_j) = \frac{s_{kj}^i}{100} \cdot (1 - \delta_{kj}^i), \tag{4}$$

$$I_k^i(c_j) = \delta_{kj}^i \cdot e_{kj}^i, \tag{5}$$

$$F_k^i(c_j) = 1 - T_k^i(c_j) - I_k^i(c_j), \tag{6}$$

where $\delta_{kj}^i \in [0,1]$ is an *indeterminacy index* computed from score variance, late/incomplete submissions and conflicting signals between assessed and self-reported performance:

$$\delta_{kj}^i = \min\left(1,\ \alpha \cdot \frac{\sigma_{kj}^i}{\bar{s}_{kj}^i} + \beta \cdot (1 - \tau_{kj}^i) + \gamma \cdot |e_{kj}^i - s_{kj}^i/100|\right),$$

with $\alpha + \beta + \gamma = 1$ being domain-tunable weights.

Neutrosophic Similarity Measures for Learning Analytics

Weighted Hybrid Distance Measure

Definition 0.8 (Neutrosophic Weighted Hybrid Distance for Learning Analytics). Let A and B be two RNSs over $\mathscr{C}$ with dimension p, weights $\omega_i \geq 0$ ($\sum_i \omega_i = 1$) and parameter $\varphi \in [0,1]$. The *neutrosophic weighted hybrid distance* is

$$\mathrm{HybD}(A,B) = \varphi \cdot d_1(A,B) + (1 - \varphi) \cdot d_2(A,B), \tag{7}$$

where d_1 and d_2 are given by (2) and (3), respectively.

Definition 0.9 (Neutrosophic Hybrid Similarity Measure)**.** The *neutrosophic hybrid similarity measure* between RNSs A and B is

$$\sigma(A,B) = 1 - \mathrm{HybD}(A,B). \tag{8}$$

Theoretical Properties

Proposition 0.1. The hybrid distance $\mathrm{HybD}(A,B)$ satisfies the following metric-like properties for all RNSs A,B,C over U with equal dimension p:

(P1) $0 \le \mathrm{HybD}(A,B) \le 1$;

(P2) $\mathrm{HybD}(A,B) = 0$ if and only if $A = B$;

(P3) $\mathrm{HybD}(A,B) = \mathrm{HybD}(B,A)$;

(P4) If $A \subseteq B \subseteq C$, then $\mathrm{HybD}(A,B) \le \mathrm{HybD}(A,C)$ and $\mathrm{HybD}(B,C) \le \mathrm{HybD}(A,C)$.

Proof. **(P1)** Since each $|T_A^i - T_B^i|, |I_A^i - I_B^i|, |F_A^i - F_B^i| \in [0,1]$ and $\sum_i \omega_i = 1$, we have $d_1(A,B), d_2(A,B) \in [0,1]$; hence $\mathrm{HybD}(A,B) \in [0,1]$.

(P2) $\mathrm{HybD}(A,B) = 0 \Leftrightarrow d_1(A,B) = d_2(A,B) = 0 \Leftrightarrow T_A^i = T_B^i,\ I_A^i = I_B^i,\ F_A^i = F_B^i$ for all i, i.e. $A = B$.

(P3) Follows directly from symmetry of absolute value.

(P4) Let $A \subseteq B \subseteq C$. Then $T_A^i \le T_B^i \le T_C^i$, $I_A^i \ge I_B^i \ge I_C^i$, $F_A^i \ge F_B^i \ge F_C^i$. By the triangle inequality on $\mathbb{R}$:

$$|T_A^i - T_B^i|^\lambda \le |T_A^i - T_C^i|^\lambda, \quad |T_B^i - T_C^i|^\lambda \le |T_A^i - T_C^i|^\lambda,$$

and analogously for I and F. Summing with weights ω_i and taking the appropriate root establishes $d_\lambda(A,B) \le d_\lambda(A,C)$ and $d_\lambda(B,C) \le d_\lambda(A,C)$ for both $\lambda = 1$ and $\lambda = 2$. The convex combination (7) then inherits the same monotonicity. ☐ ☐

Corollary 0.1. The similarity measure $\sigma(A,B) = 1 - \mathrm{HybD}(A,B)$ satisfies: $(S1)\ 0 \le \sigma(A,B) \le 1$; $(S2)\ \sigma(A,B) = 1 \Leftrightarrow A = B$; $(S3)\ \sigma(A,B) = \sigma(B,A)$.

Numerical Example

Let $\varphi = 0.5$, $p = 3$ and weights $\omega_1 = 0.2, \omega_2 = 0.5, \omega_3 = 0.3$. Consider two learner profiles A (borderline student) and B (ideal profile C^*):

$$A = \big\langle (0.5, 0.6, 0.7),\ (0.3, 0.2, 0.1),\ (0.2, 0.2, 0.2) \big\rangle,$$
$$B = \big\langle (1.0, 1.0, 1.0),\ (0.0, 0.0, 0.0),\ (0.0, 0.0, 0.0) \big\rangle.$$

Step 1 – Hamming distance:

$$\begin{aligned} d_1(A,B) = \tfrac{1}{3}\big[&0.2(|0.5-1| + |0.3-0| + |0.2-0|) \\ &+ 0.5(|0.6-1| + |0.2-0| + |0.2-0|) \\ &+ 0.3(|0.7-1| + |0.1-0| + |0.2-0|)\big] \\ = \tfrac{1}{3}&[0.2(1.0) + 0.5(0.8) + 0.3(0.6)] = \tfrac{1}{3}[0.2 + 0.4 + 0.18] = 0.2600. \end{aligned}$$

Step 2 – Euclidean distance:

$$\begin{aligned} d_2(A,B) &= \tfrac{1}{3}\Big(0.2(0.25 + 0.09 + 0.04) + 0.5(0.16 + 0.04 + 0.04) + 0.3(0.09 + 0.01 + 0.04)\Big)^{1/2} \\ &= \tfrac{1}{3}(0.2 \cdot 0.38 + 0.5 \cdot 0.24 + 0.3 \cdot 0.14)^{1/2} = \tfrac{1}{3}(0.2180)^{1/2} = 0.1556. \end{aligned}$$

Step 3 – Hybrid distance and similarity:

$$\mathrm{HybD}(A,B) = 0.5 \times 0.2600 + 0.5 \times 0.1556 = 0.2078, \qquad \sigma(A,B) = 1 - 0.2078 = \mathbf{0.7922}.$$

A similarity of 0.79 indicates a substantial (though incomplete) level of competency attainment, consistent with the borderline learner profile.

Application to Digital Education

Case Study I: Competency Mapping in a MOOC

Problem Setting

Consider a MOOC on *Data Science Foundations* with $m = 3$ competencies: c_1 = Statistical Reasoning, c_2 = Programming Proficiency, c_3 = Data Interpretation. Three learners ℓ_1, ℓ_2, ℓ_3 are evaluated at $p = 3$ time points (Weeks 2, 5, 8). Weights reflect later assessments carrying more predictive value: $\omega_1 = 0.2, \omega_2 = 0.3, \omega_3 = 0.5$.

Learner–Competency Relation Q

Table 1 presents the neutrosophic learner–competency matrix, where each cell contains the triple sequence $(T^1, I^1, F^1), (T^2, I^2, F^2), ($

Table 1: Neutrosophic Learner–Competency Relation Q

Q	c_1: Stat. Reasoning	c_2: Programming	c_3: Data Interp.
ℓ_1	$(0.6, 0.2, 0.2)$ $(0.7, 0.1, 0.2)$ $(0.8, 0.1, 0.1)$	$(0.4, 0.3, 0.3)$ $(0.5, 0.3, 0.2)$ $(0.6, 0.2, 0.2)$	$(0.5, 0.3, 0.2)$ $(0.6, 0.2, 0.2)$ $(0.7, 0.2, 0.1)$
ℓ_2	$(0.3, 0.4, 0.3)$ $(0.4, 0.3, 0.3)$ $(0.4, 0.3, 0.3)$	$(0.7, 0.1, 0.2)$ $(0.8, 0.1, 0.1)$ $(0.9, 0.0, 0.1)$	$(0.3, 0.5, 0.2)$ $(0.3, 0.4, 0.3)$ $(0.4, 0.4, 0.2)$
ℓ_3	$(0.8, 0.1, 0.1)$ $(0.9, 0.0, 0.1)$ $(0.9, 0.0, 0.1)$	$(0.3, 0.5, 0.2)$ $(0.4, 0.4, 0.2)$ $(0.5, 0.3, 0.2)$	$(0.7, 0.2, 0.1)$ $(0.8, 0.1, 0.1)$ $(0.9, 0.0, 0.1)$

Ideal Competency Profile C^* and Similarity Scores

The ideal competency profile C^* assigns $(T^i, I^i, F^i) = (1, 0, 0)$ for all i and all c_j. Table 2 reports the three distance variants and the hybrid similarity ($\varphi = 0.5$).

Table 2: Distance and similarity scores: learners vs. ideal profile

Learner	d_1 (Hamming)	d_2 (Euclidean)	HybD	σ (Similarity)
ℓ_1	0.1733	0.1124	0.1429	**0.8571**
ℓ_2	0.2200	0.1459	0.1830	**0.8170**
ℓ_3	0.1133	0.0735	0.0934	**0.9066**

Interpretation: ℓ_3 (similarity 0.907) has the strongest overall competency profile and is on track for mastery. ℓ_1 (0.857) shows solid progress with room for improvement in programming. ℓ_2 (0.817) has a mixed profile: strong programming but high indeterminacy in statistical reasoning and data interpretation, warranting targeted intervention.

A purely score-based approach would rank ℓ_2 higher than ℓ_1 in programming but miss the systemic indeterminacy that NLAF detects.

Case Study II: Early At-Risk Student Detection

Problem Setting

A blended learning programme in *Engineering Mathematics* has three at-risk categories: r_1 = Academically At-Risk (low assessment scores), r_2 = Engagement At-Risk (low LMS activity), r_3 = Dropout Risk (combined academic and engagement signals). Three students s_1, s_2, s_3 are observed over $p = 3$ weeks. Weights: $\omega_1 = 0.3, \omega_2 = 0.4, \omega_3 = 0.3$.

Student–Symptom Relation Q'

Table 3: Neutrosophic Student–Risk-Symptom Relation Q'

Q'	Low Assessment Score	Low LMS Activity	Combined Signals
s_1	$(0.7,0.1,0.2)$ $(0.8,0.1,0.1)$ $(0.9,0.0,0.1)$	$(0.2,0.4,0.4)$ $(0.2,0.3,0.5)$ $(0.1,0.2,0.7)$	$(0.5,0.2,0.3)$ $(0.5,0.2,0.3)$ $(0.4,0.2,0.4)$
s_2	$(0.2,0.5,0.3)$ $(0.2,0.4,0.4)$ $(0.3,0.3,0.4)$	$(0.8,0.1,0.1)$ $(0.9,0.0,0.1)$ $(0.9,0.0,0.1)$	$(0.6,0.2,0.2)$ $(0.7,0.1,0.2)$ $(0.7,0.1,0.2)$
s_3	$(0.5,0.3,0.2)$ $(0.6,0.2,0.2)$ $(0.7,0.2,0.1)$	$(0.6,0.2,0.2)$ $(0.7,0.1,0.2)$ $(0.7,0.2,0.1)$	$(0.7,0.1,0.2)$ $(0.8,0.1,0.1)$ $(0.8,0.1,0.1)$

Risk-Profile Reference Sets and Diagnosis

Each at-risk category r_k is modelled as a reference RNS R_k derived from historical cohort data. Table 4 summarises the reference profiles.

Table 4: Reference neutrosophic at-risk profiles

Risk Category	Low Assessment	Low LMS Activity	Combined
r_1 (Academic)	$(0.8,0.1,0.1)$ $(0.8,0.1,0.1)$ $(0.9,0.0,0.1)$	$(0.1,0.2,0.7)$ $(0.1,0.2,0.7)$ $(0.1,0.1,0.8)$	$(0.4,0.3,0.3)$ $(0.4,0.3,0.3)$ $(0.4,0.2,0.4)$
r_2 (Engagement)	$(0.2,0.4,0.4)$ $(0.2,0.4,0.4)$ $(0.2,0.3,0.5)$	$(0.9,0.0,0.1)$ $(0.9,0.0,0.1)$ $(0.9,0.0,0.1)$	$(0.6,0.2,0.2)$ $(0.6,0.2,0.2)$ $(0.7,0.1,0.2)$
r_3 (Dropout)	$(0.6,0.2,0.2)$ $(0.6,0.2,0.2)$ $(0.7,0.1,0.2)$	$(0.6,0.2,0.2)$ $(0.7,0.1,0.2)$ $(0.7,0.2,0.1)$	$(0.7,0.1,0.2)$ $(0.8,0.1,0.1)$ $(0.8,0.1,0.1)$

Table 5 reports the hybrid similarity between each student s_j and each reference risk profile r_k ($\varphi = 0.5$). The *highest* similarity score identifies the most likely risk category.

Table 5: Hybrid similarity scores: students vs. risk profiles ($\varphi = 0.5$)

Student	$\sigma(s_j, r_1)$ Academic	$\sigma(s_j, r_2)$ Engagement	$\sigma(s_j, r_3)$ Dropout	Diagnosis
s_1	**0.9124**	0.7843	0.8012	Academic At-Risk
s_2	0.7965	**0.9311**	0.8534	Engagement At-Risk
s_3	0.8213	0.8407	**0.9476**	Dropout Risk

Interpretation:

- s_1 is most similar to the academic at-risk profile ($\sigma = 0.912$): instructors should prioritise academic support (tutoring, formative feedback).
- s_2 matches the engagement at-risk profile ($\sigma = 0.931$): motivational outreach or schedule adjustments are recommended.
- s_3 is closest to the dropout risk profile ($\sigma = 0.948$): immediate holistic intervention is warranted.

These diagnoses are sensitive to the indeterminacy component. A crisp classifier that uses only assessment scores would merge s_2 and s_3 as similarly at-risk (both have moderate assessment means) and miss the qualitative difference captured by the neutrosophic engagement dimension.

Sensitivity Analysis

Table 6 shows how the diagnosis for s_3 changes with varying φ.

Table 6: Sensitivity of s_3 diagnosis to φ

φ	HybD(s_3, r_1)	HybD(s_3, r_2)	HybD(s_3, r_3)	Diagnosis
0.3	0.1673	0.1478	**0.0468**	Dropout Risk
0.5	0.1787	0.1593	**0.0524**	Dropout Risk
0.7	0.1901	0.1708	**0.0579**	Dropout Risk

The dropout-risk classification for s_3 is stable across all tested φ values, demonstrating robustness of the framework.

Discussion

The NLAF offers three decisive advantages over existing LA approaches.

Explicit uncertainty modelling. Unlike crisp scoring or fuzzy LA systems, NLAF separates *evidenced mastery* (T) from *uncertainty/noise* (I) and *evidenced non-mastery* (F). This prevents a high-noise moderate score from being treated identically to a low-noise moderate score.

Temporal granularity. The refined neutrosophic structure captures learning trajectories over p time points in a single formalism. An analyst can inspect per-timepoint components to detect improvement, stagnation, or regression.

Diagnostic precision. By computing similarity to prototype risk profiles rather than applying a threshold to a single metric, NLAF produces soft, graded diagnoses that align with the inherently continuous nature of learning risk.

Limitations. The quality of the framework depends on the accuracy of the indeterminacy index δ^i_{kj} and the fidelity of the reference profiles R_k. Both require careful calibration with domain experts and historical cohort data. Furthermore, the computational cost scales as $O(n \cdot m \cdot p)$, which is manageable for typical course sizes but may require optimisation for very large datasets.

Conclusion

This chapter introduced the Neutrosophic Learning Analytics Framework (NLAF), a principled approach to analysing learner data under uncertainty in digital education. We formalised neutrosophic learner profiles using refined neutrosophic sets, defined a weighted hybrid distance-based similarity measure, proved its metric-like properties, and demonstrated its applicability to competency mapping and at-risk student detection.

NLAF fills a critical gap in the learning analytics literature by providing tools that simultaneously handle incomplete, indeterminate, and inconsistent learner data—conditions that are endemic to digital education environments. The two case studies confirm that neutrosophic similarity captures diagnostic nuances that classical and fuzzy approaches miss.

Future Research Directions

- Integration of NLAF with machine-learning pipelines (e.g. neutrosophic neural networks [49]) for automated competency tracking.
- Extension to *bipolar* neutrosophic sets to handle assessor disagreement and conflicting expert opinions [50].
- Development of neutrosophic dashboards for real-time instructor feedback in LMS platforms.
- Large-scale empirical validation with diverse cohorts across disciplines and delivery modalities.
- Investigation of neutrosophic collaborative filtering for personalised content recommendation.

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Chapter 3

A General Discrete–to–Continuum Extension of Fuzzy and Extended Fuzzy Frameworks

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ABSTRACT

Many uncertainty modeling frameworks extend classical fuzzy set theory by introducing multiple membership components or refined sub-memberships. Examples include intuitionistic fuzzy sets, picture fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, q-rung orthopair fuzzy sets, and various refined fuzzy structures. These models are typically formulated using finite-dimensional component vectors. In this paper we introduce a general discrete–to–continuum transformation principle that extends such frameworks into continuum epistemic models. The proposed approach replaces finite collections of membership components with measurable density functions defined over a continuous refinement index. Aggregated membership degrees are obtained through integration over this index space. We prove a general embedding theorem showing that every finite refined fuzzy representation admits a continuum representation. Furthermore, we establish a discrete–continuum convergence theorem demonstrating that classical fuzzy structures arise as special discretizations of the continuum model. The framework unifies the extension of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, and q-rung orthopair fuzzy sets into continuum analogues. We also discuss the extension of fuzzy logic, fuzzy measure, fuzzy probability, and fuzzy statistics within this continuum setting. The results suggest that continuum uncertainty models form a general functional framework for fuzzy and neutrosophic theories, bridging discrete epistemic structures with measure-theoretic representations of uncertainty.

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Keywords: Continuum fuzzy sets, discrete-to-continuum transformation, fuzzy set extensions, refined fuzzy sets, continuum uncertainty modeling, functional representation of uncertainty, fuzzy entropy, uncertainty phase space, information geometry of fuzzy systems, measure-theoretic fuzzy modeling.

INTRODUCTION

Uncertainty modeling has evolved through several generations of mathematical frameworks. Classical fuzzy set theory introduced by Zadeh [1] represents gradual membership by assigning each element a membership degree in the interval [0, 1]. This model allows smooth transitions between membership and non-membership and has found applications in control systems, decision theory, pattern recognition, and artificial intelligence. Subsequent research introduced more expressive models of uncertainty by incorporating additional epistemic components. Among the most influential developments are:

- **Intuitionistic fuzzy sets** [2] introduced membership and non-membership functions;
- **Picture fuzzy sets** [3] include a neutral component;
- **Pythagorean fuzzy sets** [4] defined through quadratic membership constraints;
- **Spherical fuzzy sets** [5] generalize intuitionistic fuzzy structures to three-dimensional membership spaces;
- **q-rung orthopair fuzzy sets** [6] provide a flexible generalization of intuitionistic fuzzy sets;

- **refined fuzzy sets** where each membership component is decomposed into multiple sub-memberships [7].

For further details of the development of neutrosophic sets, the followings works [8-18] have been referred to. These frameworks share a common structural feature: they describe uncertainty through **finite collections of membership components**.

However, many real-world systems exhibit uncertainty that is **continuously distributed** across latent factors such as:

- sensor reliability levels,
- spatial parameters,
- spectral domains,
- hidden explanatory variables.

This observation motivates the transition from discrete component structures to **continuum representations**.

Recent developments in neutrosophic theory have demonstrated that refined neutrosophic components can be extended to continuous density functions defined over a refinement index [7, 19]. Inspired by this approach, the present work develops a **general mathematical framework for continuum fuzzy extensions**.

The goal of this paper is threefold:

1. To establish a **general discrete–to–continuum transformation principle** for fuzzy and extended fuzzy frameworks.
2. To prove that discrete refined structures can be embedded into continuum representations.
3. To demonstrate that classical fuzzy frameworks emerge as special cases of continuum models.

BACKGROUND

Let U be a universe of discourse.

Definition 1. Classical Fuzzy Sets

A fuzzy set [1] A on U is defined by a membership function $\mu_A: U \rightarrow [0,1]$.

Definition 2. Refined Fuzzy Sets

Refined fuzzy sets decompose membership degrees into multiple sub-memberships:

$\mu_A(x) = (\mu_1(x), \dots, \mu_p(x))$ [7]

Definition 3. Multi-Component Fuzzy Structures

Many fuzzy extensions define membership through vectors $\mathbf{u}(x) = (u_1(x), \dots, u_m(x))$

subject to structural constraints such as

$$u_i(x) \geq 0$$

and

$$C(u_1(x), \dots, u_m(x)) \leq 1$$

Examples include:

- intuitionistic fuzzy sets [2]
- Pythagorean fuzzy sets [4]
- spherical fuzzy sets [5]
- q-rung orthopair fuzzy sets [6]

Continuum Membership Representation

We now introduce a continuum representation of membership structures.

Definition 4. Continuum Membership Density
Let $A = [0,1]$ be a refinement index space.

A continuum membership representation is a measurable function

$u(x, \alpha)\colon U \times A \to [0, \infty)$, where α represents a latent epistemic refinement parameter.

Definition 5. Aggregated Membership Degree
The global membership degree is defined by $U(x) = \int_0^1 u(x, \alpha)\, d\alpha$.

Thus, the density function $u(x, \alpha)$ describes how epistemic evidence is distributed across refinement states.

Embedding of Discrete Models

Theorem 1. Embedding Theorem
Every refined fuzzy set with finite sub-memberships admits a continuum representation.

Proof: Let $\mu_A(x) = (\mu_1(x), \dots, \mu_p(x))$.
Partition the interval [0, 1] as $I_k = \left[\frac{k-1}{p}, \frac{k}{p}\right]$.

Define $\mu(x, \alpha) = p\mu_k(x), \alpha \in I_k$.

Then $\int_0^1 \mu(x, \alpha) d\alpha = \sum_{k=1}^{p} \mu_k(x)$.

Therefore, the discrete refined fuzzy representation is embedded into the continuum representation. ■

Discrete–Continuum Convergence

Theorem 2. Convergence Theorem
Let $u(x, \alpha) \in L^1([0,1])$ be a membership density.

Define a discretization $u_k(x) = u(x, \alpha_k)\Delta\alpha$.

Then $\sum_{k=1}^{n} u_k(x) \to \int_0^1 u(x, \alpha) d\alpha$

as $n \to \infty$.

Interpretation
Discrete refined fuzzy sets converge to continuum fuzzy models as the number of refinement components increases

Continuum Extensions of Major Fuzzy Frameworks

Definition 6. Continuum Fuzzy Sets
Membership density

$$t(x, \alpha)$$

with global membership

$$\mu(x) = \int_0^1 t(x, \alpha) d\alpha.$$

Definition 7. Continuum Intuitionistic Fuzzy Sets
Membership densities $t(x, \alpha), f(x, \alpha)$

With $T(x) = \int t(x, \alpha) d\alpha$

$F(x) = \int f(x, \alpha) d\alpha$

subject to $T(x) + F(x) \leq 1$ [2].

Definition 8. Continuum Pythagorean Fuzzy Sets [4]
Constraint $T(x)^2 + F(x)^2 \leq 1$.

Definition 9. Continuum q-Rung Orthopair Fuzzy Sets [6]
Constraint $T(x)^q + F(x)^q \leq 1$.

Definition 10. Continuum Spherical Fuzzy Sets [5]

Three density components $t(x, \alpha), i(x, \alpha), f(x, \alpha)$

with global values

$T(x) = \int t(x, \alpha) d\alpha, I(x) = \int i(x, \alpha) d\alpha, F(x) = \int f(x, \alpha) d\alpha$

subject to

$T(x)^2 + I(x)^2 + F(x)^2 \leq 1$.

Continuum Fuzzy Logic

Logical operations are defined pointwise across the continuum.

Let τ be a t-norm [20]: $t_{A \cap B}(x, \alpha) = \tau(t_A(x, \alpha), t_B(x, \alpha))$.

Aggregated membership $T_{A \cap B}(x) = \int t_{A \cap B}(x, \alpha) d\alpha$.

Extensions to Probability and Statistics

The continuum approach naturally connects fuzzy uncertainty with measure theory.

Examples include:

Continuum fuzzy probability
$P(E) = \int_0^1 P(E, \alpha) d\alpha /$

Continuum fuzzy expectation
$E[X] = \int x \left(\int_0^1 p\,(x, \alpha) d\alpha \right) dx$

These structures parallel measure-theoretic developments in neutrosophic probability and statistics [21].

Continuum Uncertainty Phase Space and Entropy

The continuum representation introduced in previous sections allows uncertainty to be interpreted as a **distribution of epistemic mass across a continuous refinement domain**. This interpretation naturally leads to the concept of a **continuum uncertainty phase space**.

Such a representation enables the application of concepts from **information theory and statistical physics** to fuzzy uncertainty modeling.

Continuum Uncertainty Phase Space

In classical fuzzy theory, the epistemic state of an element $x$is represented by a point in a membership space, typically defined by coordinates such as $(\mu(x))$ or, in extended fuzzy frameworks, $(T(x), F(x)), (T(x), I(x), F(x))$.

In the continuum framework, however, each element is associated with a **density trajectory** $u(x, \alpha)$

or, more generally, $\mathbf{u}(x, \alpha) = (u_1(x, \alpha), \ldots, u_m(x, \alpha))$.

Thus, the epistemic state of $x$becomes a **curve in a functional space**

$$\mathbf{u}(x,\cdot) \in L^1([0,1])^m.$$

This functional space can be interpreted as a **continuum uncertainty phase space**, where each point corresponds to a distribution of epistemic evidence.

Epistemic Mass Distribution

Define the **epistemic mass density**

$$\rho(x, \alpha) = \sum_{i=1}^{m} u_i\,(x, \alpha).$$

The total epistemic mass associated with element x is

$$M(x) = \int_0^1 \rho(x, \alpha) d\alpha.$$

Depending on the fuzzy framework, this quantity may satisfy constraints such as

$M(x) \leq 1$ or $M(x) = 1$.

The function $\rho(x, \alpha)$ therefore describes how uncertainty about x is distributed across refinement states.

Continuum Fuzzy Entropy

The continuum formulation allows the definition of entropy measures describing the dispersion of epistemic evidence.

Definition11. Continuum Fuzzy Entropy

For membership density $u(x, \alpha)$, define $H(x) = -\int_0^1 u(x, \alpha) \log\, u(x, \alpha)\, d\alpha$.

This measure quantifies the **uncertainty distribution across refinement states**.

High entropy indicates that epistemic evidence is distributed broadly across the refinement index, while low entropy corresponds to concentrated evidence.

Definition 12. Multi-Component Entropy

For multi-component fuzzy structures $u_i(x,\alpha)$

Define $H(x) = -\sum_{i=1}^{m} \int_0^1 u_i\,(x,\alpha)\log\, u_i(x,\alpha)\, d\alpha.$

This entropy generalizes classical fuzzy entropy measures to the continuum setting.

Maximum Entropy Principle

The continuum framework naturally supports a **maximum entropy interpretation**.

Theorem 3. Maximum Entropy Distribution

Under the constraint $\int_0^1 u(x,\alpha)d\alpha = U(x)$, the entropy $H(x)$ is maximized when $u(x,\alpha) = U(x)$.

Thus, uniform distributions across the refinement domain represent **maximally uncertain epistemic states**.

Phase Transitions in Epistemic Distributions

The continuum representation also allows the study of **structural transitions in uncertainty distributions**.

For example:

- concentrated epistemic states correspond to sharply peaked densities;
- diffuse states correspond to nearly uniform densities.

Transitions between these regimes may resemble **phase transitions in statistical physics**.

Such transitions may arise in:

- multi-sensor fusion;
- learning processes;
- adaptive decision systems.

Continuum Uncertainty Manifolds

Combining the functional representation and information geometry framework leads to the concept of **uncertainty manifolds**.

Each epistemic state corresponds to a point $u(x,\cdot) \in L^1([0,1])^m$.

Families of states parameterized by θ

$u(x,\alpha \mid \theta)$ define **continuum fuzzy statistical manifolds**.

Distances between states can be defined using divergence measures such as

$$D(\theta_1 \,||\, \theta_2) = \int\int u(x,\alpha \mid \theta_1)\log \frac{u(x,\alpha \mid \theta_1)}{u(x,\alpha \mid \theta_2)}\, d\alpha\, dx.$$

This opens connections with:

- information geometry [22];
- statistical inference;
- machine learning.

Implications

The continuum entropy framework provides new tools for analyzing fuzzy uncertainty, including:

- uncertainty concentration measures
- epistemic complexity metrics
- phase transitions in learning systems
- continuum fuzzy information geometry.

These tools suggest that continuum fuzzy models may provide a **bridge between fuzzy logic, information theory, and statistical physics**.

Applications of Continuum Fuzzy Frameworks

The continuum extension of fuzzy and extended fuzzy frameworks provides a flexible representation of uncertainty in systems where epistemic evidence is distributed across continuous domains. In this section we outline several representative application areas where continuum fuzzy models may offer significant advantages over classical discrete frameworks.

Sensor Fusion and Reliability Modeling

In many real-world sensing environments, measurements originate from multiple sensors with varying reliability levels. Classical fuzzy approaches typically aggregate sensor evidence using discrete weighting schemes.

However, sensor reliability may vary continuously due to factors such as:

- environmental conditions,
- sensor degradation,
- calibration uncertainty,
- signal-to-noise variation.

Within the continuum fuzzy framework, the refinement index α may represent a **continuous reliability parameter**.

Let $t(x,\alpha)$ represent the membership density corresponding to sensor evidence with reliability level α.

The aggregated membership degree becomes $T(x) = \int_0^1 t(x,\alpha)\, d\alpha$.

This representation allows uncertainty contributions from different reliability regimes to be integrated continuously rather than discretely.

Such models are particularly useful in:

- distributed sensor networks
- autonomous vehicle perception
- environmental monitoring systems.

Ecological and Environmental Modeling

Environmental systems often involve complex uncertainty arising from spatial variability and incomplete observations.

For instance, ecological variables such as species abundance or habitat suitability may depend on environmental gradients including:

- altitude
- temperature
- soil composition
- humidity

These gradients vary continuously across space.

In this context the refinement index α may represent a **spatial parameter** or **environmental factor**, and the membership density $t(x, \alpha)$ represents the contribution of that factor to the epistemic evaluation of element x.

The continuum fuzzy representation therefore allows ecological uncertainty to be modeled as a **distribution across environmental conditions**.

Such models may be useful in:

- biodiversity monitoring
- climate change modeling
- habitat suitability analysis.

Decision-Making Systems

Many decision-making frameworks rely on fuzzy evaluation of alternatives using linguistic criteria such as high risk, moderate benefit, low cost.

Classical fuzzy decision models typically represent these evaluations using single membership values.

However, expert assessments often contain **distributed uncertainty across multiple epistemic dimensions**, such as expert confidence levels, evidence sources, contextual conditions.

In the continuum framework, the refinement index may represent a **continuous confidence level** or **evidence intensity parameter**.

Decision criteria can then be represented through densities $t_i(x, \alpha)$.

The resulting model captures richer information about the epistemic structure of decision evaluations.

Machine Learning and Data Fusion

Machine learning systems often combine information from heterogeneous data sources.

Uncertainty in these systems arises from factors including model uncertainty, noisy observations, incomplete training data.

Continuum fuzzy models may represent such uncertainty using density functions over latent variables.

For instance, the refinement index α may represent latent feature parameters, model confidence levels, probabilistic evidence weights.

This representation provides a bridge between fuzzy systems and probabilistic machine learning models.

Mathematical Appendix

This section provides additional technical details supporting the continuum fuzzy framework.

Functional Space Properties

Let $u(x, \cdot) \in L^1([0,1])$.

Then the continuum membership densities belong to the Banach space $L^1([0,1])^m$.

Define the norm

$$\| u \| = \sum_{i=1}^{m} \int_0^1 | u_i(x, \alpha) | \, d\alpha.$$

This structure ensures:

- completeness
- integrability of membership densities
- stability under linear combinations.

Closure of Logical Operators

Let τ be a bounded t-norm [9].

If $t_A(x,\alpha), t_B(x,\alpha) \in L^1([0,1])$,

Then $t_{A\cap B}(x,\alpha) = \tau(t_A(x,\alpha), t_B(x,\alpha))$ is also integrable.

Thus, continuum fuzzy structures are closed under logical conjunction.

Similar arguments apply to:

- disjunction
- negation
- implication operators.

Existence of Aggregated Membership Functions

If $u(x,\alpha) \geq 0$ and $u(x,\alpha) \in L^1([0,1])$, then $U(x) = \int_0^1 u(x,\alpha)d\alpha$ exists for all $x \in U$.

Therefore, every continuum membership density generates a well-defined aggregated fuzzy membership.

Stability Under Refinement

Let $u_n(x,\alpha)$ be a sequence of discretized approximations of $u(x,\alpha)$.

If $u_n \to u$ in $L^1([0,1])$, then $\int u_n(x,\alpha)d\alpha \to \int u(x,\alpha)d\alpha$.

This ensures that continuum fuzzy models are stable under refinement of the discretization.

Final Remarks

The continuum extension framework introduced in this paper provides a unified mathematical perspective for fuzzy and extended fuzzy uncertainty models.

By replacing finite membership components with measurable density functions, the proposed framework:

- embeds discrete fuzzy models into functional spaces,
- enables connections with measure theory and information geometry,
- supports entropy-based uncertainty analysis,
- facilitates applications in complex systems and machine learning.

These results suggest that continuum fuzzy models may play an important role in the future development of uncertainty modeling theory.

Conclusion

This paper introduced a general framework for extending fuzzy and extended fuzzy uncertainty models from discrete component representations to **continuum epistemic structures**. The proposed approach replaces finite collections of membership components with measurable density functions defined over a continuous refinement index, allowing uncertainty to be represented as a distributed epistemic structure rather than a single scalar value.

A central contribution is the formulation of a **general discrete–to–continuum transformation principle** applicable to a wide class of fuzzy frameworks. Within this framework, membership components are interpreted as aggregated integrals of underlying density functions. This perspective enables a systematic extension of numerous fuzzy models, including classical fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets, spherical fuzzy sets, and q-rung orthopair fuzzy sets.

Several theoretical results were established to support this transformation. First, an **embedding theorem** demonstrated that finite refined fuzzy structures can always be represented within the continuum model. Second, a **discrete–continuum convergence theorem** showed that classical fuzzy models arise naturally as discretizations of continuum uncertainty distributions. Third, a **unified continuum representation theorem** was introduced, providing a general mathematical framework for extending fuzzy systems defined by algebraic constraints on membership components.

The continuum formulation was further developed within a **functional analytic setting**, where uncertainty states belong to Banach spaces of integrable functions. This formulation ensures mathematical stability and provides access to powerful analytical tools from functional analysis and measure theory.

In addition, an **information-theoretic interpretation of continuum fuzzy systems** was proposed through the introduction of continuum fuzzy entropy and uncertainty phase space representations. These constructions establish connections between fuzzy uncertainty modeling, information theory, and statistical geometry, suggesting new directions for theoretical development.

The framework also opens several promising application areas. Continuum fuzzy models can naturally represent uncertainty distributed across continuous domains such as sensor reliability levels, environmental gradients, or latent explanatory variables in machine learning systems. Such representations may prove particularly valuable in complex systems where uncertainty cannot be adequately captured by discrete membership components.

Overall, the continuum extension principle proposed in this work provides a **unifying mathematical perspective** for fuzzy and extended fuzzy theories. By embedding discrete fuzzy structures within functional spaces of epistemic densities, the framework bridges traditional fuzzy logic with measure-theoretic uncertainty modeling and information geometry.

Future research may explore several directions, including the development of continuum fuzzy stochastic processes, entropy-based complexity measures for epistemic distributions, geometric analysis of continuum uncertainty manifolds, and applications to machine learning, decision theory, and socio-ecological systems.

These developments suggest that continuum uncertainty models may form an important theoretical foundation for the next generation of fuzzy and neutrosophic mathematical frameworks.

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Chapter 4

Pentapartitioned Neutrosophic Quasi Coincident Topological Space

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ABSTRACT

This chapter introduces and analyzes the notion of a Pentapartitioned Neutrosophic Point (PPNP) in the framework of Pentapartitioned Neutrosophic Sets (PPNSs). The relation of pentapartitioned neutrosophic quasi-coincidence between a PPNS and a PPNP is also introduced and examined. Various basic properties related to quasi-coincidence in pentapartitioned neutrosophic set structures are derived. In addition, the chapter analyzes the topological behavior of this relation by focusing on how the level of proximity or coincidence between PPNSs can be characterized in a Pentapartitioned Neutrosophic Topological Space (PPNTS).

Keywords: Pentapartitioned neutrosophic point, Pentapartitioned neutrosophic quasi coincident, Quasi coincident topology.

INTRODUCTION

The notion of an intuitionistic fuzzy set (FS) was introduced by Atanassov [4] in 1986 as an extension of the fuzzy set theory originally proposed by Zadeh [83]. Later, Coker and Demirci [14] introduced the concept of intuitionistic fuzzy points (FP). In another important development, Smarandache [76] proposed the theory of neutrosophic sets (NSs), which provided a broader framework for handling indeterminate and inconsistent information. This idea subsequently led to the formulation of neutrosophic topology by Salama and Alblowi [72,73] and Salama et al. [74]. Following these foundational works, many researchers investigated neutrosophic structures in several mathematical directions, including topology [3,19,20,27,30,40,41,42,43,55,34,62], minimal structure spaces [18], rough set (RS) theory [22], rough interval valued intuitionistic fuzzy set [57] bi-topological spaces [29,32,82], infra bi-topology [15], and ideal theory [21]. Moreover, several neutrosophic frameworks based on partition structures have been proposed in the literature. These include the quadripartitioned neutrosophic set (QNS) theory [12,16], interval QNS models [66], interval pentapartitioned neutrosophic sets (IPPNSs) [34,67], neutrosophic open and b-open sets [25,26], neutrosophic separation axioms [19], as well as various hybrid models that integrate interval concepts with soft set theory [36] and refined soft set theory [58].

In addition, several studies have focused on different types of open mappings within neutrosophic settings. The quasi-coincidence (QC) relation for NSs was first formulated by Ray and Dey [69] in 2021. Subsequently, Acikgoz and Esenbel [2] examined neutrosophic connected topological spaces (TSs) in 2023. Earlier contributions by Chatterjee et al. [12] in 2016 introduced entropy and similarity measures for quadripartitioned single-valued neutrosophic sets (QSVNSs). Later, Das et al. [16] presented a topological structure for QNSs in 2021 and further developed the single-valued quadripartitioned neutrosophic (QN) minimal structure space in 2023 [18]. Additionally, Granados et al. [44] established quadripartitioned neutrosophic Q-ideals of Q-algebra in 2023, while Das et al. [17] introduced the notion of neutrosophic D-filters in D-algebra. During the same period, Borah and Dutta [11] proposed aggregation operators for quadripartitioned single-valued neutrosophic Z-numbers, and Datta [35] developed the interval-valued PPNS model.

Pramanik and Roy [65] introduced an application of neutrosophic Set (NS)-based game theory in 2014 to study the India–Pakistan dispute over Jammu and Kashmir. Since that contribution, NS theory has been successfully adopted in a variety of areas such as medical diagnosis [55], decision-making analysis [1, 5, 7, 8, 9, 10, 35, 37, 38, 50, 54, 59, 68], image processing techniques [13, 61], water quality analysis [23, 24], social issue studies [49], teacher selection systems [53], and project management investigations [28]. Moreover, further developments and applications of NSs have been reported in [6, 60, 73, 74, 75].

In the context of neutrosophic set (NS) theory, the notion of neutrosophic points (NPs) was introduced by Ray and Dey [70] in 2021, where they also studied the neighbourhood structure associated with these points. Subsequently, the same authors [69] defined the quasi-coincidence relation for neutrosophic sets. Later, Das et al. [33] developed the framework of a Quadripartitioned Neutrosophic Quasi-Coincident Topological Space. Despite these developments, the relationship involving PPNSs and PPNPs has not yet been addressed in the literature.

Motivated by this gap, the present chapter establishes the quasi-coincidence relation between two PPNSs as well as between a PPNS and a PPNP, and investigates several fundamental properties arising from these relations. In addition, the notions of pentapartitioned neutrosophic points, pentapartitioned neutrosophic quasi-neighbourhoods, and their related characteristics are examined. Furthermore, the study analyzes how pentapartitioned neutrosophic quasi-neighbourhoods can be utilized to characterize a Pentapartitioned Neutrosophic Topological Space (PPNTS).

BACKGROUND

This section presents several basic concepts associated with PPNSs.

Definition 1. [76] Let A denote a single-valued neutrosophic set (SVNS) on the universal set V, represented as $A = \{\langle þ, \mathcal{T}_A(þ), \mathcal{I}_A(þ), \mathcal{F}_A(þ)\rangle : þ \in V\}$, where the degree of membership $\mathcal{T}_A, \mathcal{I}_A$ and $\mathcal{F}_A$ are from V to $[0, 1]$ and $0 \leq \mathcal{T}_A(þ) + \mathcal{I}_A(þ) + \mathcal{F}_A(þ) \leq 3$. The collection of all SVNSs defined on $V$is denoted by $N(V)$. For simplicity, throughout this paper a SVNS will be referred to as a NS.

Definition 2. [12] Assume that A is a quadripartitioned neutrosophic set (QPNS) defined over the universal set V is expressed as $A = \{\langle þ, \mathcal{T}_A(þ), \mathcal{C}_A(þ), \mathcal{I}_A(þ), \mathcal{F}_A(þ)\rangle : þ \in V\}$, where the membership functions represent truth, contradiction, ignorance, and falsity, denoted by $\mathcal{T}_A, \mathcal{C}_A, \mathcal{I}_A$ and $\mathcal{F}_A$ respectively, with values in $[0,1]$ and for all $þ \in V$ such that $0 \leq \mathcal{T}_A(þ) + \mathcal{C}_A(þ) + \mathcal{I}_A(þ) + \mathcal{F}_A(þ) \leq 4$.

Definition 3. [53] Consider V be a universal set. Then, a PPNS A over V is defined by $A = \{\langle þ, \mathcal{T}_A(þ), \mathcal{C}_A(þ), \mathcal{I}_A(þ), \mathcal{U}_A(þ), \mathcal{F}_A(þ)\rangle : þ \in V\}$, where the membership functions represent truth, contradiction, ignorance, unknown, and falsity, denoted by $\mathcal{T}_A$, $\mathcal{C}_A$, $\mathcal{I}_A, \mathcal{U}_A$, and $\mathcal{F}_A$ respectively, with values in $[0,1]$ and for all $þ \in V$ such that $0 \leq \mathcal{T}_A(þ) + \mathcal{C}_A(þ) + \mathcal{I}_A(þ) + \mathcal{U}_A(þ) + \mathcal{F}_A(þ) \leq 5$.

Definition 4. [53] Let A, B two pentapartitioned neutrosophic sets.

(1) If $\mathcal{T}_A(þ) \leq \mathcal{T}_B(þ), \mathcal{C}_A(þ) \leq \mathcal{C}_B(þ), \mathcal{I}_A(þ) \geq \mathcal{I}_B(þ), \mathcal{U}_A(þ) \geq \mathcal{U}_B(þ), \mathcal{F}_A(þ) \geq \mathcal{F}_B(þ)$ for all $þ \in V$. Then A is called a pentapartitioned neutrosophic subset of B, which is denoted by $A \subseteq B$.

(2) If $A \subseteq B$ and $B \subseteq A$ then $A = B$.

(3) The intersection between A and B, is represented by $A \cap B = \{\langle þ, \mathcal{T}_A(þ) \wedge \mathcal{T}_B(þ), \mathcal{C}_A(þ) \wedge \mathcal{C}_B(þ), \mathcal{I}_A(þ) \vee \mathcal{I}_B(þ), \mathcal{U}_A(þ) \vee \mathcal{U}_B(þ), \mathcal{F}_A(þ) \vee \mathcal{F}_B(þ)\rangle : þ \in V\}$.

(4) The union of between $A$and B, is represented by $A \cup B = \{\langle þ, \mathcal{T}_A(þ) \vee \mathcal{T}_B(þ), \mathcal{C}_A(þ) \vee \mathcal{C}_B(þ), \mathcal{I}_A(þ) \wedge \mathcal{I}_B(þ), \mathcal{U}_A(þ) \wedge \mathcal{U}_B(þ), \mathcal{F}_A(þ) \wedge \mathcal{F}_B(þ)\rangle : þ \in V\}$.

(5) The complement of the PPNS A,is described as $A^c = \{\langle þ, \mathcal{F}_A(þ), \mathcal{U}_A(þ), 1 - \mathcal{I}_A(þ), \mathcal{C}_A(þ), \mathcal{T}_A(þ)\rangle : þ \in V\}$

(6) If $\mathcal{T}_A(þ) = 1, \mathcal{C}_A(þ) = 1, \mathcal{I}_A(þ) = 0, \mathcal{U}_A(þ) = 0, \mathcal{F}_A(þ) = 0 \ \forall \ þ \in V$, then A is called the neutrosophic universal set and is denoted by1_{PPN}.

(7) If for every $þ \in V$, $\mathcal{T}_A(þ) = 0, \mathcal{C}_A(þ) = 0, \mathcal{I}_A(þ) = 1, \mathcal{U}_A(þ) = 1, \mathcal{F}_A(þ) = 1$, then A is called the neutrosophic empty set, and it is represented by $\emptyset$ or O_{PPN}.

Definition 5. [53] Suppose $\{A_i: i \in \Delta\} \subseteq PN(V)$, where $\triangle$ represents an index set. Then,

(1) $\cup_{i\in\Delta} A_i = \{\langle þ, \vee_{i\in\Delta} \mathcal{T}_{A_i}(þ), \vee_{i\in\Delta} \mathcal{C}_{A_i}(þ), \wedge_{i\in\Delta} I_{A_i}(þ), \wedge_{i\in\Delta} \mathcal{U}_{A_i}(þ), \wedge_{i\in\Delta} \mathcal{F}_{A_i}(þ)\rangle : þ \in V\}$.

(2) $\cap_{i\in\Delta} A_i = \{\langle þ, \wedge_{i\in\Delta} \mathcal{T}_{A_i}(þ), \wedge_{i\in\Delta} \mathcal{C}_{A_i}(þ), \vee_{i\in\Delta} I_{A_i}(þ), \vee_{i\in\Delta} \mathcal{U}_{A_i}(þ), \vee_{i\in\Delta} \mathcal{F}_{A_i}(þ)\rangle : þ \in V\}$.

Definition 6. [30] Suppose $\tau \subseteq \mathrm{PPNS}(V)$. Then τ is referred to as a pentapartitioned neutrosophic topology (PPNT) on V if

(i) $O_{PN}, 1_{PN} \in \tau$
(ii) $G_1 \cap G_2 \in \tau$ for any $G_1, G_2 \in \tau$
(iii) $\cup\, G_i \in \tau\ \forall \{G_i: i \in J\} \subseteq \tau$

Let τ be a PPNT defined on V. Then (V, τ) forms a PPNTS over V. Each element of τ is termed a pentapartitioned neutrosophic open set in V. If $A \in \tau$ is a PPNS, then its complement A^c is called a pentapartitioned neutrosophic closed set in V.

Main Results

Definition 7. A PPNS A is said to be quasi-coincident (QC) with a PPNS B at $þ \in V$, or equivalently A quasi-coincides with B at $þ \in V$, denoted by AqB at þ, if and only if $\mathcal{T}_A(þ) > \mathcal{T}_{B^c}(þ)$ or $\mathcal{C}_A(þ) > \mathcal{C}_{B^c}(þ)$ or $I_A(þ) < I_{B^c}(þ)$ or $\mathcal{U}_A(þ) < \mathcal{U}_{B^c}(þ)$ or $\mathcal{F}_A(þ) < \mathcal{F}_{B^c}(þ)$. We say that A quasi-coincides with B, or A is quasi-coincident with B, denoted by AqB, if A quasi-coincides with B at some point $þ \in V$. Thus A quasi-coincides with B or A is quasi-coincident with B if $\exists\, þ \in V$such that $\mathcal{T}_A(þ) > \mathcal{T}_{B^c}(þ)$ or $\mathcal{C}_A(þ) > \mathcal{C}_{B^c}(þ)$ or $I_A(þ) < I_{B^c}(þ)$ or $\mathcal{U}_A(þ) < \mathcal{U}_{B^c}(þ)$ or $\mathcal{F}_A(þ) < \mathcal{F}_{B^c}(þ)$, i.e., $\mathcal{T}_A(þ) > \mathcal{F}_B(þ)$ or $\mathcal{C}_A(þ) > \mathcal{U}_B(þ)$ or $I_A(þ) < 1 - I_B(þ)$or $\mathcal{U}_A(þ) < \mathcal{C}_B(þ)$ or $\mathcal{F}_A(þ) < \mathcal{T}_B(þ)$.

If a PPNS A, represented by $þ_{ʌ,ɣ,φ,ɤ,б}\mathrm{q}^{\mathrm{A}}$, is not QC with the PPNP $þ_{ʌ,ɣ,φ,ɤ,б}$. Similarly, Aq^B denotes that the PPNS-A is not QC with the PPNS B. Furthermore, $A\Omega B$ represents the collection of all points in Vat which AqB occurs. Hence, $A\Omega B = \{þ \in V: AqB\ at\ þ\}$.

Definition 8. Let $PN(V)$ be the set of all PPNS over V. A PPNS $P = \{\langle þ, \mathcal{T}_P(þ), \mathcal{C}_P(þ), I_P(þ), \mathcal{U}_P(þ), \mathcal{F}_P(þ)\rangle : þ \in V\}$ is referred to as a pentapartitioned neutrosophic point (PPNP) iff for any element $\mathcal{L} \in V, \mathcal{T}_P(\mathcal{L}) = ʌ, \mathcal{C}_P(\mathcal{L}) = ɣ, I_P(\mathcal{L}) = \varphi, \mathcal{U}_P(\mathcal{L}) = ɤ, \mathcal{F}_P(\mathcal{L}) = б$ for $\mathcal{L} = þ$ and $\mathcal{T}_P(\mathcal{L}) = 0, \mathcal{C}_P(\mathcal{L}) = 0, I_P(\mathcal{L}) = 1, \mathcal{U}_P(\mathcal{L}) = 1, \mathcal{F}_P(\mathcal{L}) = 1$for $\mathcal{L} \neq þ$, where $0 < ʌ \leq 1, 0 \leq ɣ < 1, 0 \leq \varphi \leq 1, 0 \leq ɤ < 1, 0 \leq б < 1$.

A PPNP $P = \{\langle þ, \mathcal{T}_P(þ), \mathcal{C}_P(þ), I_P(þ), \mathcal{U}_P(þ)\mathcal{F}_P(þ)\rangle : þ \in V\}$ will be indicated by $P^{þ}_{ʌ,ɣ,φ,ɤ,б}$ or $P < þ, ʌ, ɣ, φ, ɤ, б >$ or simply by $þ_{ʌ,ɣ,φ,ɤ,б}$. For the PPNP $þ_{ʌ,ɣ,φ,ɤ,б}$, þ will be called its support.

The complement of the PPNP $P^{þ}_{ʌ,ɣ,φ,ɤ,б}$ will be indicated by $\left(P^{þ}_{ʌ,ɣ,φ,ɤ,б}\right)^c$ or by $þ^c_{ʌ,ɣ,φ,ɤ,б}$.

Definition 9. A PPNP $þ_{ʌ,ɣ,φ,ɤ,б} \in PN(V)$ is QC with a PPNS $A \in PN(V)$ or $þ_{ʌ,ɣ,φ,ɤ,б} \in PN(V)$ quasi-coincides with a PPNS $A \in PN(V)$, indicated by $þ_{ʌ,ɣ,φ,ɤ,б}qA$, iff $ʌ > \mathcal{T}_{A^c}(þ)$ or $ɣ > \mathcal{C}_{A^c}(þ)$ or $\varphi < I_{A^c}(þ)$ or $ɤ < \mathcal{U}_{A^c}(þ)$ or $б < \mathcal{F}_{A^c}(þ)$, i.e., $ʌ > \mathcal{F}_A(þ)$ or $ɣ > \mathcal{U}_A(þ)$ or $\varphi < I_A(þ)$, $ɤ < \mathcal{C}_A(þ)$ or $б < \mathcal{T}_A(þ)$

Definition 10. Let A be a PPNSs over V. Also let $þ_{ʌ,ɣ,φ,ɤ,б}$ and $\mathcal{L}_{ʌ',ɣ',\varphi',ɤ',б'}$ be two PPNPs in V. Then

(1) $\{þ_{ʌ,ɣ,\varphi,ɤ,б}\}$ is said to be contained in A, denoted by $þ_{ʌ,ɣ,φ,ɤ,б} \subseteq A$, iff $ʌ \leq \mathcal{T}_A(þ), ɣ \leq \mathcal{C}_A(þ), I_A(þ) \geq \varphi, ɤ \geq \mathcal{U}_A(þ), б \geq \mathcal{F}_A(þ)$.

(2) $þ_{ʌ,ɣ,\varphi,ɤ,б}$ is referred to belong to A, indicated by $þ_{ʌ,ɣ,φ,ɤ,б} \in A$, iff $ʌ \leq \mathcal{T}_A(þ), ɣ \leq \mathcal{C}_A(þ), I_A(þ) \geq \varphi, ɤ \geq \mathcal{U}_A(þ), б \geq \mathcal{F}_A(þ)$.

(3) $þ_{ʌ,ɣ,φ,ɤ,б}$ is said to be contained in $\mathcal{L}_{ʌ',ɣ',\varphi',ɤ',б'}$, denoted by $þ_{ʌ,ɣ,φ,ɤ,б} \subseteq \mathcal{L}_{ʌ',ɣ',\varphi',ɤ',б'}$, iff $þ = \mathcal{L}$ and $ʌ \leq ʌ', ɣ \leq ɣ', φ \geq φ', ɤ \geq ɤ', б \geq б'$.

(4) $þ_{ʌ,ɣ,φ,ɤ,б}$ is referred to belong to $\mathcal{L}_{ʌ',ɣ',\varphi',ɤ',б'}$, denoted by $þ_{ʌ,ɣ,φ,ɤ,б} \in \mathcal{L}_{ʌ',ɣ',\varphi',ɤ',б'}$, iff $þ = \mathcal{L}$ and $ʌ \leq ʌ', ɣ \leq ɣ', φ \geq φ', ɤ \geq ɤ', б \geq б'$.

Proposition 11. Let A, B, C be three $PPNSs$, and $þ_{ʌ,ɣ,φ,ɤ,б}$ be a PPNP in V. Then,

(1) $þ_{ʌ,ɣ,φ,ɤ,б}\hat{q}\tilde{\emptyset}$.

(2) $þ_{ʌ,ɣ,φ,ɤ,б}qV$.

(3) $þ_{ʌ,ɣ,φ,ɤ,б} \in A \Leftrightarrow þ_{ʌ,ɣ,φ,ɤ,б}\hat{q}A^c$.

(4) $þ_{ʌ,ɣ,φ,ɤ,б}qA \Leftrightarrow þ_{ʌ,ɣ,φ,ɤ,б} \notin A^c$.

(5) $A \subseteq B \Leftrightarrow A\hat{q}B^c$.

(6) $AqB \Leftrightarrow A \nsubseteq B^c$

(7) $þ_{ʌ,ɣ,φ,ɤ,б}qA$ and $A \subseteq B$ then $þ_{ʌ,ɣ,φ,ɤ,б}qB$.

(8) CqA and $A \subseteq B$ then CqB.

(9) AqB at þ $\Leftrightarrow$ BqA at þ.

(10) $AqB \Leftrightarrow BqA$.

Proof:

(1) – (2) Follows directly from the definitions.

(3) $þ_{ʌ,ɣ,φ,ɤ,б} \in A$

$\Leftrightarrow ʌ \leq \mathcal{T}_A(þ)$ or $ɣ \leq \mathcal{C}_A(þ)$ or $φ \geq I_A(þ)$ or $ɤ \geq \mathcal{U}_A(þ)$ or $б \geq \mathcal{F}_A(þ)$
$\Leftrightarrow ʌ \ngtr \mathcal{T}_A(þ)$ or $ɣ \ngtr \mathcal{C}_A(þ)$ or $φ \nless I_A(þ)$ or $ɤ \nless \mathcal{U}_A(þ)$ or $б \nless \mathcal{F}_A(þ)$
$\Leftrightarrow ʌ \ngtr \mathcal{T}_{(A^c)^c}(þ)$ or $ɣ \ngtr \mathcal{C}_{(A^c)^c}(þ)$ or $φ \nless I_{(A^c)^c,}(þ)$ *or* $ɤ \nless \mathcal{U}_{(A^c)^c}(þ)$ or $б \nless \mathcal{F}_{(A^c)^c}(þ)$
$\Leftrightarrow þ_{ʌ,ɣ,φ,ɤ,б}\hat{q}A^c$

(4)

$þ_{ʌ,ɣ,φ,ɤ,б}qA$
$\Leftrightarrow ʌ > \mathcal{T}_{A^c}(þ)$ or $ɣ > \mathcal{C}_{A^c}(þ)$ or $φ < I_{A^c}(þ)$ or $ɤ < \mathcal{U}_{A^c}(þ)$ or $б < \mathcal{F}_{A^c}(þ)$
$\Leftrightarrow ʌ \nleq \mathcal{T}_{A^c}(þ)$ or $ɣ \nleq \mathcal{C}_{A^c}(þ)$ or $φ \ngeq I_{A^c}(þ)$ or $ɤ \ngeq \mathcal{U}_{A^c}(þ)$ or $б \ngeq \mathcal{F}_{A^c}(þ)$
$\Leftrightarrow þ_{ʌ,ɣ,φ,ɤ,б} \notin A^c$

(5)
$A \subseteq B$
$\Leftrightarrow \mathcal{T}_A(þ) \leq \mathcal{T}_B(þ)$ or $\mathcal{C}_A(þ) \leq \mathcal{C}_B(þ)$ or $I_A(þ) \geq I_B(þ)$ or $\mathcal{U}_A(þ) \geq \mathcal{U}_B(þ)$ or $\mathcal{F}_A(þ) \geq \mathcal{F}_B(þ) \forall þ \in V$
$\Leftrightarrow \mathcal{T}_A(þ) \ngtr \mathcal{T}_B(þ)$ or $\mathcal{C}_A(þ) \ngtr \mathcal{C}_B(þ)$ or $I_A(þ) \nless I_B(þ)$ or $\mathcal{U}_A(þ) \nless \mathcal{U}_B(þ)$ or $\mathcal{F}_A(þ) \nless \mathcal{F}_B(þ) \forall þ \in V$
$\Leftrightarrow \mathcal{T}_A(þ) \ngtr \mathcal{T}_{(B^c)^c}(þ)$ or $\mathcal{C}_A(þ) \ngtr \mathcal{C}_{(B^c)^c}(þ)$ or $I_A(þ) \nless I_{(B^c)^c}(þ)$ or $\mathcal{U}_A(þ) \nless \mathcal{U}_{(B^c)^c}(þ)$ or $\mathcal{F}_A(þ) \nless \mathcal{F}_{(B^c)^c}(þ) \forall þ \in V$
$\Leftrightarrow A\hat{q}B^c$

(6) AqB

$\Leftrightarrow \mathcal{T}_A(þ) > \mathcal{T}_{B^c}(þ)$or $\mathcal{C}_A(þ) > \mathcal{C}_{B^c}(þ)$ or $I_A(þ) < I_{B^c}(þ)$, $\mathcal{U}_A(þ) < \mathcal{U}_{B^c}(þ)$ or $\mathcal{F}_A(þ) < \mathcal{F}_{B^c}(þ)$
$\Leftrightarrow \mathcal{T}_A(þ) \nleq \mathcal{T}_{B^c}(þ)$ or $\mathcal{C}_A(þ) \nleq \mathcal{C}_{B^c}(þ)$ or $I_A(þ) \ngeq I_{B^c}(þ), \mathcal{U}_A(þ) \ngeq þ)$ or $\mathcal{F}_A(þ) \ngeq \mathcal{F}_{B^c}(þ) \Leftrightarrow A \nsubseteq B^c$

(7) Since $þ_{ʌ,ɣ,φ,ɤ,б}qA$, so $ʌ > \mathcal{T}_{A^c}(þ)$ or $ɣ > \mathcal{C}_{A^c}(þ)$ or $\varphi < I_{A^c}(þ), ɤ < \mathcal{U}_{A^c}(þ)$ or $б < \mathcal{F}_{A^c}(þ)$. Now

$$A \subseteq B \Rightarrow B^c \subseteq A^c$$

$\Rightarrow \mathcal{T}_{B^c}(þ) \leq \mathcal{T}_{A^c}(þ), \mathcal{C}_{B^c}(þ) \leq \mathcal{C}_{A^c}(þ), I_{B^c}(þ) \geq I_{A^c}(þ), \mathcal{U}_{B^c}(þ) \geq \mathcal{U}_{A^c}(þ), \mathcal{F}_{B^c}(þ) \geq \mathcal{F}_{A^c}(þ)$ for all $þ \in X$
$\Rightarrow \mathcal{T}_{A^c}(þ) \geq \mathcal{T}_{B^c}(þ), \mathcal{C}_{A^c}(þ) \geq \mathcal{C}_{B^c}(þ), I_{A^c}(þ) \leq I_{B^c}(þ), \mathcal{U}_{A^c}(þ) \leq \mathcal{U}_{B^c}(þ), \mathcal{F}_{A^c}(þ) \leq \mathcal{F}_{B^c}(þ)$ for all $þ \in V \Rightarrow ʌ > \mathcal{T}_{B^c}(þ)$ or $ɣ > \mathcal{C}_{B^c}(þ)$ or $\varphi < I_{A^c}(þ), ɤ < \mathcal{U}_{B^c}(þ)$ or $б < \mathcal{F}_{B^c}(þ)$ for all $þ \in V \Rightarrow þ_{ʌ,ɣ,φ,ɤ,б}qB$

(8) $CqA \Rightarrow C \nsubseteq A^c \Rightarrow C \nsubseteq B^c[\because A \subseteq B \Rightarrow B^c \subseteq A^c] \Rightarrow CqB$.

(9) AqB at þ
$\Leftrightarrow \mathcal{T}_A(þ) > \mathcal{T}_{B^c}(þ)$ or $\mathcal{C}_A(þ) > \mathcal{C}_{B^c}(þ)$ or $I_A(þ) < I_{B^c}(þ)$ or $\mathcal{U}_A(þ) < \mathcal{U}_{B^c}(þ)$ or $\mathcal{F}_A(þ) < \mathcal{F}_{B^c}(þ)$
$\Leftrightarrow \mathcal{T}_A(þ) > \mathcal{F}_B(þ)$ or $\mathcal{C}_A(þ) > \mathcal{U}_B(þ)$ or $I_A(þ) < 1 - I_{B^c}(þ)$or $\mathcal{U}_A(þ) < \mathcal{C}_B(þ)$ or $\mathcal{F}_A(þ) < \mathcal{T}_B(þ)$
$\Leftrightarrow \mathcal{T}_B(þ) > \mathcal{F}_A(þ)$ or $\mathcal{C}_B(þ) > \mathcal{U}_A(þ)$ or $I_B(þ) < 1 - I_{A^c}(þ)$or $\mathcal{U}_B(þ) < \mathcal{C}_A(þ)$ or $\mathcal{F}_B(þ) < \mathcal{T}_A(þ)$
$\Leftrightarrow \mathcal{T}_A(þ) > \mathcal{T}_{B^c}(þ)$ or $\mathcal{C}_A(þ) > \mathcal{C}_{B^c}(þ)$ or $I_A(þ) < I_{B^c}(þ)$ or $\mathcal{U}_A(þ) < \mathcal{U}_{B^c}(þ)$ or $\mathcal{F}_A(þ) < \mathcal{F}_{B^c}(þ)$
$\Leftrightarrow BqA$ at þ

(10) It is obvious from (9).

Proposition 12. Let $þ_{ʌ,ɣ,φ,ɤ,б}$ be a $PPNP$ in $P, A \in PPNS(V)$ and $\{A_i : i \in \Delta\} \subseteq PN(P)$, where б denotes the index set. Then,

(1) $þ_{ʌ,ɣ,φ,ɤ,б}\, q \cup_{i\in\Delta} A_i \Leftrightarrow þ_{ʌ,ɣ,φ,ɤ,б}\, qA_j$ for some $j \in \Delta$.

(2) $Aq \cup_{i\in\Delta} A_i \Leftrightarrow AqA_j$ for some $j \in \Delta$.

(3) $þ_{ʌ,ɣ,φ,ɤ,б} q \cap_{i\in\Delta} A_i \Rightarrow þ_{ʌ,ɣ,φ,ɤ,б} qA_i\ \forall i \in \Delta$.

(4) $Aq \cap_{i\in\Delta} A_i \Rightarrow AqA_i\ \forall i \in \triangle$.

Proof:

(1)

$$
\begin{aligned}
& þ_{ʌ,ɣ,φ,ɤ,б} q \cup_{i\in\Delta} A_i \\
\Leftrightarrow & þ_{ʌ,ɣ,φ,ɤ,б} \notin (\cup_{i\in\Delta} A_i)^c \\
\Leftrightarrow & þ_{ʌ,ɣ,φ,ɤ,б} \notin \cap_{i\in\Delta} A_i^c \\
\Leftrightarrow & þ_{ʌ,ɣ,φ,ɤ,б} \notin A_j^c \text{ for some } j \in \Delta \\
\Leftrightarrow & þ_{ʌ,ɣ,φ,ɤ,б} qA_j \text{ for some } j \in \Delta
\end{aligned}
$$

(2)

$$
\begin{aligned}
& Aq \cup_{i\in\Delta} A_i \\
\Leftrightarrow & A \nsubseteq (\cup_{i\in\Delta} A_i)^c \\
\Leftrightarrow & A \nsubseteq \cap_{i\in\Delta} A_i^c \\
\Leftrightarrow & A \nsubseteq A_j^c \text{ for some } j \in \triangle \\
\Leftrightarrow & AqA_j \text{ for some } j \in \triangle
\end{aligned}
$$

(3)

$$
\begin{aligned}
& þ_{ʌ,ɣ,φ,ɤ,б} q \cap_{i\in\Delta} A_i \\
\Rightarrow & þ_{ʌ,ɣ,φ,ɤ,б} \notin (\cap_{i\in\Delta} A_i)^c \\
\Rightarrow & þ_{ʌ,ɣ,φ,ɤ,б} \notin \cup_{i\in\Delta} A_i^c \\
\Rightarrow & þ_{ʌ,ɣ,φ,ɤ,б} \notin A_i^c \text{ for all } i \in \triangle \\
\Rightarrow & þ_{ʌ,ɣ,φ,ɤ,б} qA_i \text{ for all } i \in \triangle
\end{aligned}
$$

(4)

$$
\begin{aligned}
& Aq \cap_{i\in\Delta} A_i \\
\Rightarrow & A \nsubseteq (\cap_{i\in\Delta} A_i)^c \\
\Rightarrow & A \nsubseteq \cup_{i\in\Delta} A_i^c \\
\Rightarrow & A \nsubseteq A_i^c\ \forall\ i \in \Delta \\
\Rightarrow & AqA_i\ \forall i \in \Delta
\end{aligned}
$$

Proposition 13.
1. $A\Omega B = B\Omega A$.
2. $AqB \Leftrightarrow A\Omega B \neq \emptyset$.
3. $A \subseteq B \Rightarrow A\Omega C \subseteq B\Omega C$.
4. $A\Omega(\cup_{i\in\Delta} A_i) = \cup_{i\in\Delta} (A\Omega A_i)$.
5. $A\Omega(\cap_{i\in\Delta} A_i) \subseteq \cap_{i\in\Delta} (A\Omega A_i)$.

Proof:

(1) $A\Omega B = \{\text{þ} \in V: AqB \text{ at } \text{þ}\} = \{\text{þ} \in V: BqA \text{ at } \text{þ}\} = B\Omega A$.

(2) $AqB \Leftrightarrow AqB$ at some $\text{þ} \in V \Leftrightarrow \text{þ} \in A\Omega B$. Hence, $AqB \Leftrightarrow A\Omega B \neq \emptyset$.

(3) $A \subseteq B \Rightarrow \mathcal{T}_A(\text{þ}) \le \mathcal{T}_B(\text{þ}), \mathcal{C}_A(\text{þ}) \le \mathcal{C}_B(\text{þ}), I_A(\text{þ}) \ge I_B(\text{þ}), \mathcal{U}_A(\text{þ}) \ge \mathcal{U}_B(\text{þ}), \mathcal{F}_A(\text{þ}) \ge \mathcal{F}_B(\text{þ})$ for all $\text{þ} \in V$. Now

$\text{þ} \in A\Omega C$
$\Rightarrow AqC$ at $\text{þ} \in V$
$\Rightarrow \mathcal{T}_A(\text{þ}) > \mathcal{T}_{C^c}(\text{þ})$ or $\mathcal{C}_A(\text{þ}) > \mathcal{C}_{C^c}(\text{þ})$ or $I_A(\text{þ}) < I_{C^c}(\text{þ})$ *or* $\mathcal{U}_A(\text{þ}) < \mathcal{U}_{C^c}(\text{þ})$ or $\mathcal{F}_A(\text{þ}) < \mathcal{F}_{C^c}(\text{þ})$
$\Rightarrow \mathcal{T}_B(\text{þ}) > \mathcal{T}_{C^c}(\text{þ})$ or $\mathcal{C}_B(\text{þ}) > \mathcal{C}_{C^c}(\text{þ})$ or $I_B(\text{þ}) < I_{C^c}(\text{þ})$ *or* $\mathcal{U}_B(\text{þ}) < \mathcal{U}_{C^c}(\text{þ})$ or $\mathcal{F}_B(\text{þ}) < \mathcal{F}_{C^c}(\text{þ})$
$\Rightarrow BqC$ at $\text{þ} \in V$
$\Rightarrow \text{þ} \in B\Omega C$
$\therefore A\Omega C \subseteq B\Omega C$.

(4)

$\text{þ} \in A\Omega(\cup_{i\in\Delta} A_i)$
$\Rightarrow Aq(\cup_{i\in\Delta} A_i)$ at $\text{þ} \in V$
$\Rightarrow \exists j \in \triangle : AqA_j$ at $\text{þ} \in V$
$\Rightarrow \exists j \in \triangle : \text{þ} \in A\Omega A_j$
$\Rightarrow \text{þ} \in \cup_{i\in\Delta} (A\Omega A_i)$
$\therefore A\Omega(\cup_{i\in\Delta} A_i) \subseteq \cup_{i\in\Delta} (A\Omega A_i)$.

Again

$\text{þ} \in \cup_{i\in\Delta} (A\Omega A_i)$
$\Rightarrow \bigvee\limits_{i\in\Delta} (AqA_i \text{ at } \text{þ} \in V)$
$\Rightarrow \bigvee\limits_{i\in\Delta} (A_iqA \text{ at } \text{þ} \in V)$
$\Rightarrow \bigvee\limits_{i\in\Delta} \left[\mathcal{T}_{A_i}(\text{þ}) > \mathcal{T}_{A^c}(\text{þ}) \text{ or } \mathcal{C}_{A_i}(\text{þ}) > \mathcal{C}_{A^c}(\text{þ}) \text{ or } I_{A_i}(\text{þ}) < I_{A^c}(\text{þ}), \mathcal{U}_{A_i}(\text{þ}) < \mathcal{U}_{A^c}(\text{þ}) \text{ or } \mathcal{F}_{A_i}(\text{þ}) < \mathcal{F}_{A^c}(\text{þ})\right]$
$\Rightarrow \sup\limits_{i\in\Delta} \mathcal{T}_{A_i}(\text{þ}) > \mathcal{T}_{A^c}(\text{þ})$ or $\sup\limits_{i\in\Delta} \mathcal{C}_{A_i}(\text{þ}) > \mathcal{C}_{A^c}(\text{þ})$ or $\inf\limits_{i\in\Delta} I_{A_i}(\text{þ}) < I_{A^c}(\text{þ})$ or $\inf\limits_{i\in\Delta} \mathcal{U}_{A_i}(\text{þ}) < \mathcal{U}_{A^c}(\text{þ})$ or $\inf\limits_{i\in\Delta} \mathcal{F}_{A_i}(\text{þ}) < \mathcal{F}_{A^c}(\text{þ})$
$\Rightarrow \mathcal{T}_{\cup A_i}(\text{þ}) > \mathcal{T}_{A^c}(\text{þ})$ or $\mathcal{C}_{\cup A_i}(\text{þ}) > \mathcal{C}_{A^c}(\text{þ})$ or $I_{\cup A_i}(\text{þ}) < I_{A^c}(\text{þ})$ or $\mathcal{U}_{\cup A_i}(\text{þ}) < \mathcal{U}_{A^c}(\text{þ})$ or $\mathcal{F}_{\cup A_i}(\text{þ}) < \mathcal{F}_{A^c}(\text{þ})$
$\Rightarrow (\cup_{i\in\Delta} A_i)qA$ at $\text{þ} \in V$
$\Rightarrow Aq(\cup_{i\in\Delta} A_i)$ at $\text{þ} \in V$
$\Rightarrow \text{þ} \in A\Omega(\cup_{i\in\Delta} A_i)$
$\therefore \cup_{i\in\Delta} (A\Omega A_i) \subseteq A\Omega(\cup_{i\in\Delta} A_i)$
Hence $A\Omega(\cup_{i\in\Delta} A_i) = \cup_{i\in\Delta} (A\Omega A_i)$.

(5)

$\text{þ} \in A\Omega(\cap_{i\in\Delta} A_i)$
$\Rightarrow Aq(\cap_{i\in\Delta} A_i)$ at $\text{þ} \in V$
$\Rightarrow AqA_i$ at $\text{þ} \in V\ \forall i \in \triangle$
$\Rightarrow \text{þ} \in A\Omega A_i\ \forall\ i \in \triangle$
$\Rightarrow \text{þ} \in \cap_{i\in\Delta} (A\Omega A_i)$
$\therefore A\Omega(\cap_{i\in\Delta} A_i) \subseteq \cap_{i\in\Delta} (A\Omega A_i)$

Definition 14. Let (V, τ) denotes a PPNTS. A PPNS A is termed a pentapartitioned neutrosophic quasi-neighbourhood or briefly a Q- neighbourhood (Q-nhbd) of a PPNP $\text{þ}_{\Lambda,\text{ɣ},\varphi,\text{ɤ},\text{б}}$ if there exists a PPNS $B \in \tau$ such that $\text{þ}_{\Lambda,\text{ɣ},\varphi,\text{ɤ},\text{б}}qB \subseteq A$.The family consisting of all Q-nhbd of the PPNP $\text{þ}_{\Lambda,\text{ɣ},\varphi,\text{ɤ},\text{б}}$ is known as the Q-nhbd system of $\text{þ}_{\Lambda,\text{ɣ},\varphi,\text{ɤ},\text{б}}$, and it is denoted by $\mathrm{N}_Q(\text{þ}_{\Lambda,\text{ɣ},\varphi,\text{ɤ},\text{б}})$.

Proposition 15. For a PPNTS (V, τ) every neutrosophic open set A acts as a Q-nhbd for all PPNPs that are QC with A.

Proof: The result follows because for each PPNP $þ_{ʌ,ɣ,φ,ɤ,б}qA$, we obtain $þ_{ʌ,ɣ,φ,ɤ,б}qA \subseteq A$.

Properties of Pentapartitioned Neutrosophic Q-NHBD

Theorem 16. Let $\mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ be the collection of all Q-nhbd of the PPNP $þ_{ʌ,ɣ,φ,ɤ,б}$ in a PPNTS (V,τ). Then,
(a) $\mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \neq \emptyset$ for every PPNP $þ_{ʌ,ɣ,φ,ɤ,б} \in PPNS(V)$.
(b) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow þ_{ʌ,ɣ,φ,ɤ,б}qP$.
(c) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}), P \subseteq Q \Rightarrow Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$.
(d) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow$ there exists a $Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ such that $Q \subseteq P$ and $Q \in \mathrm{PN_Q}(\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'})$ for every PPNP $\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'}$ quasicoincident with Q.

Proof:(a) Clearly, $V$is a Q-nhbd of every PPNP $þ_{ʌ,ɣ,φ,ɤ,б} \in \mathrm{PPNS}(V)$. Consequently, there exists at least one Q-nhbd for each PPNP $þ_{ʌ,ɣ,φ,ɤ,б} \in \mathrm{PPNS}(V)$. Therefore $\mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \neq \emptyset$ for every PPNP $þ_{ʌ,ɣ,φ,ɤ,б} \in \mathrm{PPNS}(V)$.
(b) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow P$ is a Q-nhbd of $þ_{ʌ,ɣ,φ,ɤ,б} \Rightarrow \exists$ a $S \in \tau$ such that $þ_{ʌ,ɣ,φ,ɤ,б}q \subseteq P$. Therefore $þ_{ʌ,ɣ,φ,ɤ,б}qP$.
(c) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow P$ is a Q-nhbd of $þ_{ʌ,ɣ,φ,ɤ,б} \Rightarrow \exists$ an open set G such that $þ_{ʌ,ɣ,φ,ɤ,б}qG \subseteq P \Rightarrow \exists$ an open set G such that $þ_{ʌ,ɣ,φ,ɤ,б q}G \subseteq P \Rightarrow P$ is a Q-nhbd of $þ_{ʌ,ɣ,φ,ɤ,б} \Rightarrow P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$.
(d) Since $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$,there exists a τ-open set Q such that $þ_{ʌ,ɣ,φ,ɤ,б}qQ \subseteq P$. As Q is an open set, it follows that $Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$. Hence $Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ and $Q \subseteq P$.
Again since Q is an open, it serves as a Q-nhbd for all PPNPs that are quasi-coincident with Q. Consequently, $Q \in \mathrm{PN_Q}(\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'})$ for every PPNP $\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'}$ quasi-coincident with Q.
Hence the result is proved.

PNTS Characterization in Terms of Pentapartitioned Neutrosophic Q-NHBD

Theorem 17. Assume that $V \neq \emptyset$ is any set. Let $þ \in V$. Let $\mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ be a family of all PPNSs over V satisfying the following conditions :
(N1) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow þ_{ʌ,ɣ,φ,ɤ,б}qP$.
(N2) $P, Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow P \cap Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$.
(N3) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}), P \subseteq Q \Rightarrow Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$.
Then there exists a Pentapartitioned Neutrosophic Topology (PPNT) τ on V. In addition, if condition (N4) holds, then $\mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ becomes exactly the Q-nhbd system $þ_{ʌ,ɣ,φ,ɤ,б}$ in the PPNTS (V,τ).
(N4) $P \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow$ there exists a $Q \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ such that $Q \subseteq P$ and $Q\mathrm{PN_Q}(\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'})$ for every PPNP $\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'}$ quasicoincident with Q.

Proof: We define τ in the following manner:
A PPNS $G \in \tau$ iff $G\ \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ whenever $þ_{ʌ,ɣ,φ,ɤ,б}qG$.
We claim that τ is a PPNT on V.
T1) $\emptyset \in \tau$ as no PPNP is quasi-coincident with $\emptyset$.By (N3), $V \in \tau$. Thus $\emptyset, V \in \tau$.

T2) Suppose $G_1, G_2 \in \tau$ and $þ_{ʌ,ɣ,φ,ɤ,б}q(G_1 \cap G_2)$. Since $þ_{ʌ,ɣ,φ,ɤ,б}q(G_1 \cap G_2)$, so $þ_{ʌ,ɣ,φ,ɤ,б}qG_1$ and $þ_{ʌ,ɣ,φ,ɤ,б}qG_2$. Therefore $G_1, G_2 \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$ and so, by (N2), $G_1 \cap G_2 \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$.

T3) Suppose $\{G_i : i \in \Delta\} \subseteq \tau$ and $þ_{ʌ,ɣ,φ,ɤ,б}q(\cup_{i\in\Delta} G_i)$. We show that $\cup\{G_i : i \in \Delta\} \in \tau$. Now $x_{ʌ,ɣ,φ,ɤ,б}q(\cup_{i\in\Delta} G_i) \Rightarrow \exists$ a $j \in \Delta$ such that $þ_{ʌ,ɣ,φ,ɤ,б}qG_j \Rightarrow \exists$ a $j \in \Delta$ such that $G_j \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б}) \Rightarrow \cup \{G_i : i \in \Delta\} \in \mathrm{N}(þ_{ʌ,ɣ,φ,ɤ,б})$[by (N3)] $\Rightarrow \cup\{G_i : i \in \Delta\} \in \tau$.
Therefore, τ is a PPNT on V.
Under the assumption that $(N4)$holds, consider the collection of all Q-neighborhoods of the PPNP $þ_{ʌ,ɣ,φ,ɤ,б}$ in (V,τ) belong to the family $PN_Q^*\,(þ_{ʌ,ɣ,φ,ɤ,б})$. The equality $PN_Q^*(þ_{ʌ,ɣ,φ,ɤ,б})$=$PN_Q(þ_{ʌ,ɣ,φ,ɤ,б})$ is demonstrated.
Let $N \in \mathrm{PN_Q}(þ_{ʌ,ɣ,φ,ɤ,б})$. By (N4), then, for each PPNP $\mathcal{L}_{ʌ',ɣ',φ',ɤ',б'}$ quasi-coincident with M, there exists a

$M \in QN_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б})$ such that $M \subseteq N$ and $M \in QN_Q(\mathcal{L}_{Λ',ɣ',φ',ɤ',б'})$. [by (N1)] Now, $M \in QN_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б}) \Rightarrow \text{þ}_{Λ,ɣ,φ,ɤ,б}\, qM$. It follows that $M \in \tau$. Hence M is a τ-open set in which $\text{þ}_{Λ,ɣ,φ,ɤ,б}\, qM \subseteq N$. Therefore $N \in \mathrm{PN}_Q^*(\text{þ}_{Λ,ɣ,φ,ɤ,б})$ and so $\mathrm{PN}_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б}) \subseteq \mathrm{PN}_Q^*(\text{þ}_{Λ,ɣ,φ,ɤ,б})$. Conversely let $N \in \mathrm{PN}_Q^*(\text{þ}_{Λ,ɣ,φ,ɤ,б})$ so that N is a Q-nhbd of $\text{þ}_{Λ,ɣ,φ,ɤ,б}$. Thus, there exists a τ-open set G such that $\text{þ}_{Λ,ɣ,φ,ɤ,б}\, qG \subseteq N$. Therefore $G \in \mathrm{PN}_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б})$.

But $G \in \mathrm{PN}_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б})$ and $G \subseteq N$ together imply by (N3) that $N \in \mathrm{PN}_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б})$. Therefore $\mathrm{PN}_Q^*(\text{þ}_{Λ,ɣ,φ,ɤ,б}) \subseteq \mathrm{PN}_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б})$. Consequently $\mathrm{P}_Q(\text{þ}_{Λ,ɣ,φ,ɤ,б}) = \mathrm{PN}_Q^*(\text{þ}_{Λ,ɣ,φ,ɤ,б})$. Thereby, it was demonstrated.

Definition 18. Let $\zeta \subseteq Q\mathcal{N}(V)$. Then ζ is called a Pentapartitioned Neutrosophic Quasi Coincident Topology (PPNQCT) with $C(\in V)$ on V if

(i) $O_{PPN}qC, 1_{PPN}qC$
(ii) $(G_1 \cap G_2)qC$ for any G_1qC, G_2qC
(iii) $\cup\, G_iqC\ \forall G_iqC$

If ζ represents a PPNQCT on V, the ordered pair (V, ζ) is known as a PPNQCT space over V.

Conclusion

This work develops the concept of pentapartitioned neutrosophic quasi-coincidence between pentapartitioned neutrosophic sets and pentapartitioned neutrosophic points. It also introduces the definition of a pentapartitioned neutrosophic point and examines several properties of pentapartitioned neutrosophic sets through illustrative examples. Moreover, the structure of a pentapartitioned neutrosophic quasi-coincident topological space is established. The results presented here may provide a useful starting point for further studies on different classes of continuous functions in the context of pentapartitioned neutrosophic sets.

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Chapter 5

A General Discrete–to–Continuum Neutrosophic Self-Reference Stability and Fixed Points of Logical Paradox

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ABSTRACT

This chapter introduces the Neutrosophic Self-Reference Stability Theorem and studies paradox as a dynamical process inside neutrosophic truth space. Instead of producing logical collapse, self-referential sentences may converge to stable neutrosophic valuations (T, I, F). The framework connects liar paradox dynamics, Gödel self-reference, and contradictions across science, technology, administration, decision-making, literature, and art. The concept is also discussed in relation to the Infinitesimally Punctured Wave (IPW) inspired by wave-particle duality.

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Keywords: neutrosophic logic; paradox dynamics; Gödel sentence; liar paradox; contradictions; stability theorem

INTRODUCTION

Neutrosophic logic [1] extends the classical [2, 3] and fuzzy [4] logic. Contradictions [5, 6] appear in many areas of knowledge. Classical logic [2, 3] treats contradictions as destructive. Neutrosophic logic [7] provides a framework in which statements may simultaneously contain degrees of truth, indeterminacy, and falsehood.

Val(S) = (T, I, F), where $0 \leq T \leq 1$, $0 \leq I \leq 1$, $0 \leq F \leq 1$.

Classical Logical Paradoxes

Example: Liar paradox [8]
"This statement is false."
Binary evaluation produces oscillation: true → false → true → false
This instability motivates richer truth models.

Gödel Self-Reference

Gödel [9] constructed a sentence satisfying $G \leftrightarrow \neg \mathrm{Prov}(G)$

meaning “G is not provable”.

Instead of binary valuation we may assign Val(G) = (T_G, I_G, F_G).

Self-Reference as a Dynamical System

Let X_n = (T_n, I_n, F_n) represent the truth state after n evaluations.

Define the transformation X_{n+1} = f(X_n), where f : $[0,1]^3$→ $[0,1]^3$.

Neutrosophic Self-Reference Stability Theorem

Let N = [0,1] × [0,1] × [0,1].
If the evaluation operator f : N → N is continuous, then there exists a fixed point X* = (T*, I*, F*) such that f(X*) = X*.
If extreme classical states are unstable then,
0 < T* < 1
0 < I* < 1
0 < F* < 1.
Neutrosophic truth cube is shown in figure 1.

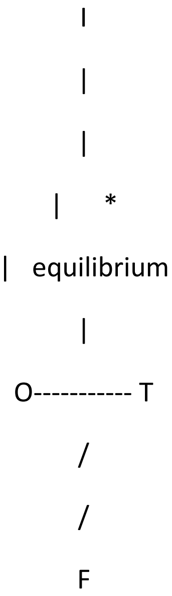

Figure 1. Neutrosophic Truth Cube (schematic)

Example Dynamical Evolution

Example iteration

X0 = (1,0,0)

X1 = (0.6,0.1,0.3)

X2 = (0.4,0.3,0.3)

X3 → (0.33,0.34,0.33)

This approaches a stable neutrosophic equilibrium.

Catalog of Contradictions [4] Across Domains

Science: Wave–particle duality: light behaves as wave and particle.

Technology: Engineering trade-offs: speed vs accuracy.

Administration: Policy conflicts: efficiency vs social benefit.

Decision-Making: Evidence may simultaneously support and contradict a hypothesis.

Literature: Unreliable narrators create contradictory perspectives.

Art: Abstract vs representational tension in modern art.

Relation with Infinitesimally Punctured Wave

Wave-particle duality inspired the Infinitesimally Punctured Wave idea.
Logical analogy:
wave structure → theory
punctures → contradictions
Neutrosophic logic models these punctures without collapse.

Contradictory-Theory Stability Principle

Systems containing contradictory theories may stabilize at interior neutrosophic states rather than collapsing into inconsistency.
Such states correspond to equilibrium points (T*, I*, F*).

Future Research

Possible directions:

- computational simulation of paradox dynamics
- neutrosophic models of scientific theory conflict
- connections with quantum logic
- applications in decision science and governance

Conclusion

Neutrosophic logic [1] transforms paradox from contradiction into stable logical structure. Self-reference may converge to interior truth states rather than oscillating between true and false.

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Chapter 6

Algebra of Fermatean Neutrosophic Sets

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ABSTRACT

Fermatean Neutrosophic sets incorporating membership degree of Pythagorean fuzzy set and neutrosophic set allows for a more efficient way of representing uncertainty, vagueness and indeterminacy. It can be found that Fermatean neutrosophic sets are somewhat a new concept within the domain of neutrosophic set theory. Fermatean neutrosophic sets can usually be applied in many fields of study which in turn provides a framework for dealing with inherent uncertainties while making decisions related to career choices, pattern recognition, work life balance and entrepreneurships. This chapter focuses on Fermatean neutrosophic sets and their different characteristics such as union and intersection. The distributive characteristics of these features are also examined.

Keywords: Fuzzy sets, Intuitionistic fuzzy sets, Pythagorean fuzzy sets, Fermatean fuzzy sets, Neutrosophic sets, Pythagorean neutrosophic sets, Fermatean neutrosophic sets.

INTRODUCTION

The limitations of classical set theory led to the development of fuzzy sets, as proposed by Zadeh [1], to address the issues of unreliability and unpredictability in real-world problems encountered by existing theories. Subsequently, it became evident that the membership function alone in fuzzy sets was inadequate to capture the existing uncertainties, which prompted the introduction of the intuitionistic fuzzy set by Atanassov [2]. This concept is undoubtedly a generalization of fuzzy sets, represented by both the degree of membership and the degree of non-membership, with the stipulation that their sum must lie within the interval [0,1]. However, there are numerous instances where the combined grades of membership and non-membership exceed 1, thereby violating the conditions of intuitionistic fuzzy sets. These constraints led to the formulation of a new concept known as the Pythagorean fuzzy set, as introduced by Yager [3]. The Pythagorean fuzzy set has been recognized as an effective tool for managing situations characterized by imprecision and uncertainty. Over time, it became apparent that there could be cases where the sum of the squares of the membership and non-membership values exceeds 1. To address such scenarios that cannot be adequately explained by intuitionistic fuzzy sets or Pythagorean fuzzy sets, Fermatean fuzzy sets were initiated by Senapati et al.[4].

Neutrosophic sets were introduced by Smarandache [5, 6] as an augmentation of fuzzy sets and intuitionistic fuzzy sets, which play a crucial role in addressing uncertainties, imprecision, and ambiguities encountered in everyday life. Subsequently, Jansi et al. [7] expanded the concept of Pythagorean fuzzy sets to define Pythagorean neutrosophic sets and also explored various application areas. It is noteworthy that for Pythagorean neutrosophic sets, the addition of the squares of the membership and non-membership grades is less than or equal to 1, while the square of the indeterminacy value is also less than or equal to 1.

Sweety.et.al [8] came up with this new concept called Fermatean Neutrosophic sets. They basically combined the membership idea from Pythagorean fuzzy sets with neutrosophic sets, which is a pretty neat way to deal with situations where things are uncertain, fuzzy, or just ambiguous. Fermatean neutrosophic sets are a pretty recent development in the world of neutrosophic set theory. This somewhat advanced idea involves three different functions. One function shows how much something belongs, another shows how indeterminate it is and a third shows how much it doesn't belong to the set and each of these is raised to the power of three. The rules for a Fermatean neutrosophic set are that when cube the membership and non-membership degrees are added them up, the total has to be less than or equal to 1. Also, the cubed indeterminacy degree itself needs to be somewhere between 0 and 1. This whole setup helps make sure that all three sides of uncertainty – membership, non-membership, and indeterminacy – are represented in a balanced way.

Fermatean neutrosophic sets are specific types of neutrosophic sets which have found real applications under uncertainty in decision-making [9, 10, 11, 12], graph theory [13, 14, 15, 16, 17] and also theory of matrices and operations on matrices [18].The concept of Fermatean neutrosophic soft set [19] was presented by combining fuzzy soft set [20]and Fermatean neutrosophic sets [8] and their application areas can be found in [19, 21] . The idea of Fermatean neutrosophic has extensive applications in many other fields as well.

BACKGROUND

In this section some definitions are provided which are subsequently used in later parts of the article.

Definition 1. Neutrosophic set [5, 6]
Consider universe of discussion as Ƕ. Then the neutrosophic set $ɤ_{ɒ}$ is an object having the form, $ɤ_{a=}\{ɭ, ɚ(ɭ), ɘ(ɭ), ɜ(ɭ)/ ɭϵǶ\}$ where functions, $ɚ(ɭ), ɘ(ɭ), ɜ(ɭ) \rightarrow]0, 1^{+}[$ define respectively the degree of affiliation ɚ(ɭ), the degree of indeterminacy ɘ(ɭ) and the degree of non-affiliation , ɜ(ɭ)respectively of the element $t \in Z$ to the set A with the condition . $0 \leq ɚ(ɭ) + ɘ(ɭ) + ɜ(ɭ) \leq 3^{+}$

Definition 2. Pythagorean Fuzzy Set [3]

Consider area of discussion as Ƕ. Then a Pythagorean fuzzy set $ɤ_{a}$ on is Ƕ defined as $ɤ_{a=}\{ɭ, ɚ(ɭ), ɘ(ɭ)/ɭϵǶ\}$ where $ɚ(ɭ): Ƕ \rightarrow [0,1]$ and $ɘ(ɭ): Ƕ \rightarrow [0,1]$ denote affiliation degree and non-affiliation degree having the condition$0 \leq ɚ(ɭ)^2 + ɘ(ɭ)^2 \leq 1$ for ɭϵǶ. Pythogorean fuzzy set is a generalization of intuitionistic fuzzy set.

Definition 3. Pythagorean Neutrosophic Set **[7]**

Let us consider the area of discussion as Ƕ. Then a Pythagorean neutrosophic set $ɤ_{a}$ on Ƕ is defined as $ɤ_{a=}\{ɭ, ɚ(ɭ), ɘ(ɭ), ɜ(ɭ)/ɭϵǶ\}$ where .$ɚ(ɭ): Ƕ \rightarrow [0,1]$, $ɘ(ɭ): Ƕ \rightarrow [0,1]$ and $ɜ(ɭ): Ƕ \rightarrow [0,1]$ denote the degree of affiliation and degree indeterminacy and the degree of non-affiliation having the condition $0 \leq ɚ(ɭ)^2 + ɜ(ɭ)^2 \leq 1$ and $0 \leq ɘ(ɭ)^2 \leq 1$. The numbers ɚ(ɭ) , ɜ(ɭ) represents dependent components and ɘ(ɭ) denotes independent component of the element ɭϵǶ

Definition 4. Fermatean Fuzzy Set [4]
Let us consider the area of discussion as Ƕ. Then Fermatean fuzzy set $ɤ_{a}$ on Ƕ is defined as $ɤ_{a=}\{ɭ, ɚ(ɭ), ɜ(ɭ)/ɭϵǶ\}$ where .$ɚ(ɭ): Ƕ \rightarrow [0,1]$ and $ɜ(ɭ): Ƕ \rightarrow [0,1]$ denote the degree of affiliation and the degree of non-affiliation having the condition $0 \leq ɚ(ɭ)^3 + ɜ(ɭ)^3 \leq 1$.

Definition 5. Fermatean Neutrosophic Set [8]
Let us consider the area of discussion as Ƕ. Then a Pythagorean neutrosophic set $ɤ_{a}$ on Ƕ is defined as $ɤ_{a=}\{ɭ, ɚ(ɭ), ɘ(ɭ), ɜ(ɭ)/ɭϵǶ\}$ where .$ɚ(ɭ): Ƕ \rightarrow [0,1]$, $ɘ(ɭ): Ƕ \rightarrow [0,1]$ and $ɜ(ɭ): Ƕ \rightarrow [0,1]$ with the condition that $0 \leq ɚ(ɭ)^3 + ɜ(ɭ)^3 \leq 1$ and $0 \leq ɘ(ɭ)^3 \leq 1$ so that $0 \leq ɚ(ɭ)^3 + ɘ(ɭ)^3 + ɜ(ɭ)^3 \leq 2$. Here .$ɚ(ɭ): Ƕ \rightarrow [0,1]$, $ɘ(ɭ): Ƕ \rightarrow [0,1]$ and $ɜ(ɭ): Ƕ \rightarrow [0,1]$ denote the degree of affiliation and degree indeterminacy and the degree of non-affiliation respectively. In this case ɚ(ɭ) , ɜ(ɭ) are dependent components and its independent component is ɘ(ɭ) . This concept is illustrated with the help of a numerical example in the form when affiliation degree = 0.7 and non-affiliation degree =0 .6. Here it can be visualized that $(0.7)^2 + (0.9)^2 > 1$ and hence the condition of Pythagorean neutrosophic set is not satisfied because according to Pythagorean set it must be less than 1. That is, it should have been $0 \leq ɚ(ɭ)^2 + ɜ(ɭ)^2 \leq 1$ and $0 \leq ɘ(ɭ)^2 \leq 1$. But in this particular case it can be seen that $(0.7)^3 + (0.9)^3 \leq 1$ which in fact implies that Fermatean neutrosophic set can handle such situations quite nicely.

Algebraic Operations on Fermatean Neutrosophic Sets

Consider be three Fermatean neutrosophic sets on the universe of discourse Ƕ as $ɣ_{ɒ_1}=\{ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ), ɜ_{ɒ_1}(ɭ)/ɭ\epsilon Ƕ\}$,$ɣ_{ɒ_2}=\{ɭ, ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ)/ɭ\epsilon Ƕ\}$ and $\{ɭ, ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ)/ɭ\epsilon Ƕ\}$

Then

i. $ɣ_{ɒ_1} \subseteq ɣ_{ɒ_2}$ iff $ɚ_{ɒ_1}(ɭ) \leq ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_1}(ɭ) \geq ɘ_{ɒ_2}(ɭ)$ and $ɜ_{ɒ_1}(ɭ) \geq ɜ_{ɒ_2}(ɭ)$

ii. $ɣ_{ɒ_1} = ɣ_{ɒ_2}$ iff $ɚ_{ɒ_1}(ɭ) = ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_1}(ɭ) = ɘ_{ɒ_2}(ɭ)$ and $ɜ_{ɒ_1}(ɭ) = ɜ_{ɒ_2}(ɭ)$

iii. $ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2} =\{ ɭ, \text{Max}(ɚ_{ɒ_1}(ɭ), ɚ_{ɒ_2}(ɭ)), \text{Min}(ɘ_{ɒ_1}(ɭ), ɘ_{ɒ_2}(ɭ)) , \text{ Min}(ɜ_{ɒ_1}(ɭ), ɜ_{ɒ_2}(ɭ))\}$

iv. $ɣ_{ɒ_1} \sqcap ɣ_{ɒ_2} =\{ ɭ, \text{Min}(ɚ_{ɒ_1}(ɭ), ɚ_{ɒ_2}(ɭ)), \text{Max}(ɘ_{ɒ_1}(ɭ), ɘ_{ɒ_2}(ɭ)) , \text{ Max}(ɜ_{ɒ_1}(ɭ), ɜ_{ɒ_2}(ɭ))\}$

v. $ɣ_{ɒ_1} \circledcirc ɣ_{ɒ_2} =\{ ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_2}(ɭ))}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_2}(ɭ))}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+(ɭ))}{2}\}$

vi. $ɣ_{ɒ_1} \oplus ɣ_{ɒ_2}$

$$=\{ɭ, \sqrt[3]{(ɚ_{ɒ_1}(ɭ))^3+ɚ_{ɒ_2}(ɭ))^3 - (ɚ_{ɒ_1}(ɭ))^3 ɚ_{ɒ_2}(ɭ))^3}, ɘ_{ɒ_1}(ɭ). ɘ_{ɒ_2}(ɭ) , ɜ_{ɒ_1}(ɭ). ɜ_{ɒ_2}(ɭ) \}$$

vii. $ɣ_{ɒ_1} \otimes ɣ_{ɒ_2}$

$$=\{ɭ, ɚ_{ɒ_1}(ɭ). ɚ_{ɒ_2}(ɭ), \sqrt[3]{(ɘ_{ɒ_1}(ɭ))^3+(ɚ_{ɒ_2})^3 - \left(ɘ_{ɒ_1}(ɭ)\right)^3 (ɘ_{ɒ_2}(ɭ))^3}$$
$$, \sqrt[3]{(ɜ_{ɒ_1}(ɭ))^3+(ɜ_{ɒ_2}(ɭ))^3 - (ɜ_{ɒ_1}(ɭ))^3 (ɜ_{ɒ_2}(ɭ))^3} \}$$

viii. $ɣ_{ɒ_1}{}^{n} = \{ɭ, ɚ_{ɒ_1}(ɭ)^3, \sqrt[3]{1 - \left(1 - ɘ_{ɒ_1}(ɭ))^3\right)^{n}} , \sqrt[3]{1 - (1 - ɜ_{ɒ_1}(ɭ))^3)^{n}}/ɭ\epsilon Ƕ\}$

Theorems 1. Consider be three Fermatean neutrosophic sets on the universe of discourse Ƕ as $ɣ_{ɒ_1}=\{ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ), ɜ_{ɒ_1}(ɭ)/ɭ\epsilon Ƕ\}$,$ɣ_{ɒ_2}=\{ɭ, ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ)/ ɭ\epsilon Ƕ\}$ and $ɣ_{ɒ_3}=\{ɭ, ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ)/ ɭ\epsilon Ƕ\}$ be three Fermatean neutrosophic sets on Ƕ. Then

$(ɣ_{ɒ_1} \sqcap ɣ_{ɒ_2}) \circledcirc ɣ_{ɒ_3}=(ɣ_{ɒ_1} \circledcirc ɣ_{ɒ_3}) \sqcap (ɣ_{ɒ_2} \circledcirc ɣ_{ɒ_3})$

Proof: From definition

$ɣ_{ɒ_1} \sqcap ɣ_{ɒ_2} =\{ ɭ, \text{Min}(ɚ_{ɒ_1}(ɭ), ɚ_{ɒ_2}(ɭ)), \text{Max}(ɘ_{ɒ_1}(ɭ), ɘ_{ɒ_2}(ɭ)) , \text{ Max}(ɜ_{ɒ_1}(ɭ), ɜ_{ɒ_2}(ɭ))\}$ and

$ɣ_{ɒ_1} \circledcirc ɣ_{ɒ_2} =\{ ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_2}(ɭ))}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_2}(ɭ))}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+(ɭ))}{2}\}$

Therefore

$ɣ_{ɒ_1} \sqcap ɣ_{ɒ_2}) \circledcirc ɣ_{ɒ_3}=\{ ɭ, Min(ɚ_{ɒ_1}(ɭ), ɚ_{ɒ_2}(ɭ)), \text{Max}(ɘ_{ɒ_1}(ɭ), ɘ_{ɒ_2}(ɭ)) , \text{Max}(ɜ_{ɒ_1}(ɭ), ɜ_{ɒ_2}(ɭ))\} \circledcirc \{ɭ, ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ)/ɭ\epsilon Ƕ\}$

Also consider

$ɚ_{ɒ_1}(ɭ) < ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_1}(ɭ) > ɘ_{ɒ_2}(ɭ)$ and $ɜ_{ɒ_1}(ɭ) > ɜ_{ɒ_2}(ɭ)$

Then

$ɣ_{ɒ_1} \sqcap ɣ_{ɒ_2} =\{ ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ) , ɜ_{ɒ_1}(ɭ)\}$

Hence

$(ɣ_{ɒ_1} \sqcap ɣ_{ɒ_2}) \circledcirc ɣ_{ɒ_3}=\{ ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ) , ɜ_{ɒ_1}(ɭ)\} \circledcirc \{ɭ, ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ)/ɭ\epsilon Ƕ\}$

$$=\{ ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_3}(ɭ))}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_3}(ɭ))}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_3}(ɭ))}{2}\} \tag{1}$$

Also

$(Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_3}) \sqcap (Ɣ_{ʋ_2} \circledcirc Ɣ_{ʋ_3})$

$$=\{ ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_3}(ɭ))}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_3}(ɭ))}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_3(ɭ))}{2}\} \sqcap \{ ɭ, \frac{(ɚ_2(ɭ)+ɚ_{ʋ_3}(ɭ))}{2}, \frac{(ϑ_{ʋ_2}(ɭ)+ϑ_{ʋ_3}(ɭ))}{2}, \frac{(ʒ_{ʋ_2}(ɭ)+ʒ_{ʋ_3}(ɭ))}{2}\}$$

$$=\{ɭ, \mathrm{Min}(\frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_3}(ɭ))}{2}, \frac{(ɚ_{ʋ_2}(ɭ)+ɚ_{ʋ_3}(ɭ))}{2}), \mathrm{Max}(\frac{\left(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_3}(ɭ)\right)}{2}, \frac{(ϑ_{ʋ_2}(ɭ)+ϑ_{ʋ_3}(ɭ))}{2}),$$

$$\mathrm{Max}(\frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_3}(ɭ))}{2}, \frac{(ʒ_{ʋ_2}(ɭ)+ʒ_{ʋ_3}(ɭ))}{2})\}$$

$$==\{ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_3}(ɭ))}{2},, \frac{\left(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_3}(ɭ)\right)}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_3}(ɭ))}{2}\} \tag{2}$$

From equation (1) and (2) it can be obtained that

$(Ɣ_{ʋ_1} \sqcap Ɣ_{ʋ_2}) \circledcirc Ɣ_{ʋ_3} = (Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_3}) \sqcap (Ɣ_{ʋ_2} \circledcirc Ɣ_{ʋ_3})$

Theorems 2. Consider be three Fermatean neutrosophic sets on the area of discourse Ƕ as $Ɣ_{ʋ_1}=\{ɭ, ɚ_{ʋ_1}(ɭ), ϑ_{ʋ_1}(ɭ), ʒ_{ʋ_1}(ɭ)/ɭ \in Ƕ\}$, $Ɣ_{ʋ_2}=\{ɭ, ɚ_{ʋ_2}(ɭ), ϑ_{ʋ_2}(ɭ), ʒ_{ʋ_2}(ɭ)/ ɭ \in Ƕ\}$ and $Ɣ_{ʋ_3}=\{ɭ, ɚ_{ʋ_3}(ɭ), ϑ_{ʋ_3}(ɭ), ʒ_{ʋ_3}(ɭ)/ ɭ \in Ƕ\}$ be three Fermatean neutrosophic sets on Ƕ. Then

$Ɣ_{ʋ_1} \circledcirc \left(Ɣ_{ʋ_2} \sqcap Ɣ_{ʋ_3}\right) = (Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_2}) \sqcap (Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_3})$

Here

$Ɣ_{ʋ_1} \circledcirc \left(Ɣ_{ʋ_2} \sqcap Ɣ_{ʋ_3}\right)$

$$=\{ɭ, ɚ_{ʋ_1}(ɭ), ϑ_{ʋ_1}(ɭ), ʒ_{ʋ_1}(ɭ)/ɭ \in Ƕ\} \circledcirc \{ɭ, Min(ɚ_{ʋ_2}(ɭ), ɚ_{ʋ_3}(ɭ)), \mathrm{Max}(ϑ_{ʋ_2}(ɭ), ϑ_{ʋ_3}(ɭ)), \mathrm{Max}(ʒ_{ʋ_2}(ɭ), ʒ_{ʋ_3}(ɭ))\}$$

$$=\{ ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+Min(ɚ_{ʋ_2}(ɭ), ɚ_{ʋ_3}(ɭ))}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+\mathrm{Max}(ϑ_{ʋ_2}(ɭ), ϑ_{ʋ_3}(ɭ))}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+\mathrm{Max}(ʒ_{ʋ_2}(ɭ), ʒ_{ʋ_3}(ɭ))}{2}\} \tag{3}$$

$ɚ_{ʋ_2}(ɭ) < ɚ_{ʋ_3}(ɭ)$, $ϑ_{ʋ_2}(ɭ) > ϑ_{ʋ_3}(ɭ)$ and $ʒ_{ʋ_2}(ɭ) > ʒ_{ʋ_3}(ɭ)$

Then (3) becomes

$$Ɣ_{ʋ_1} \circledcirc \left(Ɣ_{ʋ_2} \sqcap Ɣ_{ʋ_3}\right) = \{ ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_2}(ɭ)}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_2}(ɭ)}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_2}(ɭ)}{2}\} \tag{4}$$

Again $(Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_2}) \sqcap (Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_3})$

$$=\{ ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_2}(ɭ)}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_2}(ɭ)}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_2}(ɭ)}{2}\} \sqcap \{ ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_3}(ɭ)}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_3}(ɭ)}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_3(ɭ)}{2}\}$$

$$=\{ɭ, Min(\frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_2}(ɭ)}{2}, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_3}(ɭ)}{2}), Max(\frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_2}(ɭ)}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_3}(ɭ)}{2}),, Max(\frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_2}(ɭ)}{2}, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_3(ɭ)}{2})\}$$

$$=\{ɭ, \frac{(ɚ_{ʋ_1}(ɭ)+ɚ_{ʋ_2}(ɭ)}{2}, \frac{(ϑ_{ʋ_1}(ɭ)+ϑ_{ʋ_2}(ɭ)}{2},, \frac{(ʒ_{ʋ_1}(ɭ)+ʒ_{ʋ_2}(ɭ)}{2}\} \tag{5}$$

From equation (4) and (5) it can be obtained that

$Ɣ_{ʋ_1} \circledcirc \left(Ɣ_{ʋ_2} \sqcap Ɣ_{ʋ_3}\right) = (Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_2}) \sqcap (Ɣ_{ʋ_1} \circledcirc Ɣ_{ʋ_3})$

Theorem 3. Consider be three Fermatean neutrosophic sets on the area of discourse Ꜧ as $ɣ_{ɒ_1}=\{ɭ, ɚ_{ɒ_1}(ɭ), ϑ_{ɒ_1}(ɭ), ɜ_{ɒ_1}(ɭ) / ɭ \epsilon Ꜧ\}$, $ɣ_{ɒ_2}=\{ɭ, ɚ_{ɒ_2}(ɭ), ϑ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ) / ɭ \epsilon Ꜧ\}$ and $ɣ_{ɒ_3}=\{ɭ, ɚ_{ɒ_3}(ɭ), ϑ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ) / ɭ \epsilon Ꜧ\}$ be three Fermatean neutrosophic sets on Ꜧ. Then

$(ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2}) \odot ɣ_{ɒ_3} = (ɣ_{ɒ_1} \odot ɣ_{ɒ_3}) \sqcup (ɣ_{ɒ_2} \odot ɣ_{ɒ_3})$

Proof: From definition

$ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2} = \{ɭ, Max(ɚ_{ɒ_1}(ɭ), ɚ_{ɒ_2}(ɭ)), \text{Min}(ϑ_{ɒ_1}(ɭ), ϑ_{ɒ_2}(ɭ)), \text{Min}(ɜ_{ɒ_1}(ɭ), ɜ_{ɒ_2}(ɭ))\}$ and

$ɣ_{ɒ_1} \odot ɣ_{ɒ_2} = \{ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_2}(ɭ))}{2}, \frac{(ϑ_{ɒ_1}(ɭ)+ϑ_{ɒ_2}(ɭ))}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+(ɭ))}{2}\}$

Therefore

$ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2}) \odot ɣ_{ɒ_3} = \{ɭ, Max(ɚ_{ɒ_1}(ɭ), ɚ_{ɒ_2}(ɭ)), \text{Min}(ϑ_{ɒ_1}(ɭ), ϑ_{ɒ_2}(ɭ)), \text{Min}(ɜ_{ɒ_1}(ɭ), ɜ_{ɒ_2}(ɭ))\} \odot \{ɭ, ɚ_{ɒ_3}(ɭ), ϑ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ) / ɭ \epsilon Ꜧ\}$

Also consider

$ɚ_{ɒ_1}(ɭ) < ɚ_{ɒ_2}(ɭ), ϑ_{ɒ_1}(ɭ) > ϑ_{ɒ_2}(ɭ)$ and $ɜ_{ɒ_1}(ɭ) > ɜ_{ɒ_2}(ɭ)$

Then

$ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2} = \{ɭ, ɚ_{ɒ_2}(ɭ), ϑ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ)\}$

Hence

$(ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2}) \odot ɣ_{ɒ_3} = \{ɭ, ɚ_{ɒ_2}(ɭ), ϑ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ)\} \odot \{ɭ, ɚ_{ɒ_3}(ɭ), ϑ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ) / ɭ \epsilon Ꜧ\}$

$$= \{ɭ, \frac{(ɚ_{ɒ_2}(ɭ)+ɚ_{ɒ_3}(ɭ))}{2}, \frac{(ϑ_2(ɭ)+ϑ_{ɒ_3}(ɭ))}{2}, \frac{(ɜ_{ɒ_2}(ɭ)+ɜ_{ɒ_3}(ɭ))}{2}\} \quad (6)$$

Also consider

$ɚ_{ɒ_2}(ɭ) < ɚ_{ɒ_3}(ɭ), ϑ_{ɒ_2}(ɭ) > ϑ_{ɒ_3}(ɭ)$ and $ɜ_{ɒ_2}(ɭ) > ɜ_{ɒ_3}(ɭ)$

$(ɣ_{ɒ_1} \odot ɣ_{ɒ_3}) \sqcup (ɣ_{ɒ_2} \odot ɣ_{ɒ_3})$

$= \{ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_3}(ɭ))}{2}, \frac{(ϑ_{ɒ_1}(ɭ)+ϑ_{ɒ_3}(ɭ))}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_3(ɭ))}{2}\} \sqcup \{ɭ, \frac{(ɚ_2(ɭ)+ɚ_{ɒ_3}(ɭ))}{2}, \frac{(ϑ_{ɒ_2}(ɭ)+ϑ_{ɒ_3}(ɭ))}{2}, \frac{(ɜ_{ɒ_2}(ɭ)+ɜ_{ɒ_3}(ɭ))}{2}\}$

$= \{ɭ, \text{Max}(\frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_3}(ɭ))}{2}, \frac{(ɚ_{ɒ_2}(ɭ)+ɚ_{ɒ_3}(ɭ))}{2}), \text{Min}(\frac{(ϑ_{ɒ_1}(ɭ)+ϑ_{ɒ_3}(ɭ))}{2}, \frac{(ϑ_{ɒ_2}(ɭ)+ϑ_{ɒ_3}(ɭ))}{2}),$

$\text{Min}(\frac{(ɜ_{ɒ_1}(ɭ)+ɜ_{ɒ_3}(ɭ))}{2}, \frac{(ɜ_{ɒ_2}(ɭ)+ɜ_{ɒ_3}(ɭ))}{2})\}$

$$== \{ɭ, \frac{(ɚ_{ɒ_2}(ɭ)+ɚ_{ɒ_3}(ɭ))}{2},, \frac{(ϑ_{ɒ_2}(ɭ)+ϑ_{ɒ_3}(ɭ))}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_{ɒ_3}(ɭ))}{2}\} \quad (7)$$

From equation (6) and (7) it can be obtained that

$(ɣ_{ɒ_1} \sqcup ɣ_{ɒ_2}) \odot ɣ_{ɒ_3} = (ɣ_{ɒ_1} \odot ɣ_{ɒ_3}) \sqcup (ɣ_{ɒ_2} \odot ɣ_{ɒ_3})$

Theorem 4. Consider be three Fermatean neutrosophic sets on the area of discourse Ƕ as $Ɣ_{ɒ_1}=\{ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ), ɜ_{ɒ_1}(ɭ)/ɭ\epsilon Ƕ\}$,$Ɣ_{ɒ_2}=\{ɭ, ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ)/ ɭ\epsilon Ƕ\}$ and $Ɣ_{ɒ_3}=\{ɭ, ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ)/ ɭ\epsilon Ƕ\}$ be three Fermatean neutrosophic sets on Ƕ. Then

$$Ɣ_{ɒ_1} \circledcirc \left(Ɣ_{ɒ_2} \sqcup Ɣ_{ɒ_3}\right) =(Ɣ_{ɒ_1} \circledcirc Ɣ_{ɒ_2}) \sqcup (Ɣ_{ɒ_1} \circledcirc Ɣ_{ɒ_3})$$

Here

$$Ɣ_{ɒ_1} \circledcirc \left(Ɣ_{ɒ_2} \sqcup Ɣ_{ɒ_3}\right)$$

$$=\{ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ), ɜ_{ɒ_1}(ɭ)/ɭ\epsilon Ƕ\} \circledcirc \{ɭ, Max(ɚ_{ɒ_2}(ɭ), ɚ_{ɒ_3}(ɭ)),\mathrm{Min}(ɘ_{ɒ_2}(ɭ), ɘ_{ɒ_3}(ɭ)),\mathrm{Min}(ɜ_{ɒ_2}(ɭ), ɜ_{ɒ_3}(ɭ))\}$$

$$=\{ ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+Max(ɚ_{ɒ_2}(ɭ),ɚ_{ɒ_3}(ɭ))}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+\mathrm{Min}(ɘ_{ɒ_2}(ɭ),ɘ_{ɒ_3}(ɭ))}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+\mathrm{Min}(ɜ_{ɒ_2}(ɭ),ɜ_{ɒ_3}(ɭ))}{2}\} \tag{8}$$

Also consider

$ɚ_{ɒ_2}(ɭ) < ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_2}(ɭ) > ɘ_{ɒ_3}(ɭ)$ and $ɜ_{ɒ_2}(ɭ) > ɜ_{ɒ_3}(ɭ)$

Then (3) becomes

$$Ɣ_{ɒ_1} \circledcirc \left(Ɣ_{ɒ_2} \sqcup Ɣ_{ɒ_3}\right)=\{ ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_3}(ɭ)}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_3}(ɭ)}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_{ɒ_3}(ɭ)}{2}\} \tag{9}$$

Again $(Ɣ_{ɒ_1} \circledcirc Ɣ_{ɒ_2}) \sqcup (Ɣ_{ɒ_1} \circledcirc Ɣ_{ɒ_3})$

$$=\{ ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_2}(ɭ)}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_2}(ɭ)}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_{ɒ_2}(ɭ)}{2}\} \sqcup \{ ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_3}(ɭ)}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_3}(ɭ)}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_3(ɭ)}{2}\}$$

$$=\{ɭ, Max(\frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_2}(ɭ)}{2}, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_3}(ɭ)}{2}), Min(\frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_2}(ɭ)}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_3}(ɭ)}{2}),, \ Min(\frac{(ɜ_{ɒ_1}(ɭ)+ɜ_{ɒ_2}(ɭ)}{2}, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_3(ɭ)}{2})\}$$

$$=\{ɭ, \frac{(ɚ_{ɒ_1}(ɭ)+ɚ_{ɒ_3}(ɭ)}{2}, \frac{(ɘ_{ɒ_1}(ɭ)+ɘ_{ɒ_3}(ɭ)}{2},, \frac{(ɜ_{ɒ_1}(ɭ)+ɜ_{ɒ_3}(ɭ)}{2}\}$$

From equation (8) and (9) it can be obtained that

$$Ɣ_{ɒ_1} \circledcirc \left(Ɣ_{ɒ_2} \sqcup Ɣ_{ɒ_3}\right) =(Ɣ_{ɒ_1} \circledcirc Ɣ_{ɒ_2}) \sqcup (Ɣ_{ɒ_1} \circledcirc Ɣ_{ɒ_3})$$

Theorem 5. Let three Fermatean neutrosophic sets on the area of discourse Ƕ as $Ɣ_{ɒ_1}=\{ɭ, ɚ_{ɒ_1}(ɭ), ɘ_{ɒ_1}(ɭ), ɜ_{ɒ_1}(ɭ)/ɭ\epsilon Ƕ\}$,$Ɣ_{ɒ_2}=\{ɭ, ɚ_{ɒ_2}(ɭ), ɘ_{ɒ_2}(ɭ), ɜ_{ɒ_2}(ɭ)/ ɭ\epsilon Ƕ\}$ and $Ɣ_{ɒ_3}=\{ɭ, ɚ_{ɒ_3}(ɭ), ɘ_{ɒ_3}(ɭ), ɜ_{ɒ_3}(ɭ)/ ɭ\epsilon Ƕ\}$ be three Fermatean neutrosophic sets on Ƕ. Then

i. $Ɣ_{ɒ_1} \oplus Ɣ_{ɒ_2} = Ɣ_{ɒ_2} \oplus Ɣ_{ɒ_1}$
ii. $Ɣ_{ɒ_1} \otimes Ɣ_{ɒ_2} = Ɣ_{ɒ_2} \otimes Ɣ_{ɒ_1}$
iii. $Ɣ_{ɒ_1} \oplus (Ɣ_{ɒ_2} \oplus Ɣ_{ɒ_3})= (Ɣ_{ɒ_1} \oplus Ɣ_{ɒ_2}) \oplus Ɣ_{ɒ_3}$
iv. $Ɣ_{ɒ_1} \otimes (Ɣ_{ɒ_2} \otimes Ɣ_{ɒ_3})= (Ɣ_{ɒ_1} \otimes Ɣ_{ɒ_2}) \otimes Ɣ_{ɒ_3}$

Proof:

(i) By definition

$$Ɣ_{ɒ_1} \oplus Ɣ_{ɒ_2} =\{ɭ, \sqrt[3]{(ɚ_{ɒ_1}(ɭ))^3+ɚ_{ɒ_2}(ɭ))^3 - (ɚ_{ɒ_1}(ɭ))^3ɚ_{ɒ_2}(ɭ))^3}, ɘ_{ɒ_1}(ɭ).ɘ_{ɒ_2}(ɭ) , ɜ_{ɒ_1}(ɭ).ɜ_{ɒ_2}(ɭ) \} \tag{10}$$

Then

$$Ɣ_{ɒ_2} \oplus Ɣ_{ɒ_1} =\{ȴ \sqrt[3]{(ɚ_{ɒ_2}(ȴ))^3+ɚ_{ɒ_1}(ȴ))^3-(ɚ_{ɒ_2}(ȴ))^3ɚ_{ɒ_1}(ȴ))^3}, ɘ_{ɒ_2}(ȴ).ɘ_{ɒ_1}(ȴ)\ , ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)\ \} \tag{11}$$

Hence the result follows from (10) and (11)

(ii) By definition

$$Ɣ_{ɒ_1} \otimes Ɣ_{ɒ_2} =\{\ ȴ \sqrt[3]{(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3}, ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ)\ , ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)\ \} \tag{12}$$

Then

$$Ɣ_{ɒ_2} \otimes Ɣ_{ɒ_1} =\{\ ȴ \sqrt[3]{ɚ_{ɒ_2}(ȴ))^3+(ɚ_{ɒ_1}(ȴ))^3-(ɚ_{ɒ_2}(ȴ))^3ɚ_{ɒ_1}(ȴ))^3}, ɘ_{ɒ_2}(ȴ).ɘ_{ɒ_1}(ȴ)\ , ʒ_{ɒ_2}(ȴ).ʒ_{ɒ_1}(ȴ)\ \} \tag{13}$$

Hence the result follows from (12) and (13)

(iii) By definition

$$Ɣ_{ɒ_1} \oplus Ɣ_{ɒ_2} =\{ȴ \sqrt[3]{(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3}, ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ)\ , ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)\ \}$$

Then

$$Ɣ_{ɒ_1} \oplus (Ɣ_{ɒ_2} \oplus Ɣ_{ɒ_3})$$

$$=\{ȴ, ɚ_{ɒ_1}(ȴ), ɘ_{ɒ_1}(ȴ), ʒ_{ɒ_1}(ȴ)/$$

$$ȴ\epsilon Ƕ\}\ \oplus\{ȴ \sqrt[3]{(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3}, ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ), ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)\}$$

$$=\sqrt[3]{\begin{array}{l}\left(\sqrt[3]{(ɚ_{ɒ_2}(ȴ))^3+ɚ_{ɒ_3}(ȴ))^3-(ɚ_{ɒ_2}(ȴ))^3ɚ_{ɒ_3}(ȴ))^3}\right)^3+\left(ɚ_{ɒ_1}(ȴ)\right)^3-\\ \left(\sqrt[3]{(ɚ_{ɒ_2}(ȴ))^3+ɚ_{ɒ_3}(ȴ))^3-(ɚ_2(ȴ))^3ɚ_{ɒ_3}(ȴ))^3}\right)^3(ɚ_{ɒ_1}(ȴ))^3,\end{array}}$$

$$ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ)ɘ_{ɒ_3}(ȴ), ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)ʒ_{ɒ_3}(ȴ)$$

$$=\sqrt[3]{\begin{array}{l}(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3+\left(ɚ_{ɒ_3}(ȴ)\right)^3-\\ ((ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3)(ɚ_{ɒ_3}(ȴ))^3,\end{array}}$$

$$ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ)ɘ_{ɒ_3}(ȴ), ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)ʒ_{ɒ_3}(ȴ) \tag{14}$$

$$(Ɣ_{ɒ_1} \oplus Ɣ_{ɒ_2}) \oplus Ɣ_{ɒ_3}=\{ȴ \sqrt[3]{(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3}, ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ), ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)\} \oplus$$

$$\left\{ȴ, ɚ_{ɒ_3}(ȴ), ɘ_{ɒ_3}(ȴ), ʒ_{ɒ_3}(ȴ), \overline{ȴ\epsilon Ƕ}\right\}$$

$$=\sqrt[3]{\begin{array}{c}\left(\sqrt[3]{(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3}\right)^3+\\ (ɚ_{ɒ_3}(ȴ))^3-\left(\sqrt[3]{(ɚ_{ɒ_1}(ȴ))^3+ɚ_{ɒ_2}(ȴ))^3-(ɚ_{ɒ_1}(ȴ))^3ɚ_{ɒ_2}(ȴ))^3}\right)^3(ɚ_{ɒ_3}(ȴ))^3,\end{array}}$$

$$ɘ_{ɒ_1}(ȴ).ɘ_{ɒ_2}(ȴ)ɘ_{ɒ_3}(ȴ), ʒ_{ɒ_1}(ȴ).ʒ_{ɒ_2}(ȴ)ʒ_{ɒ_3}(ȴ)$$

$$= \sqrt[3]{\begin{array}{c}(\text{ɚ}_{\text{ɒ}_1}(\text{ɭ}))^3+\text{ɚ}_{\text{ɒ}_2}(\text{ɭ}))^3-(\text{ɚ}_{\text{ɒ}_1}(\text{ɭ}))^3\text{ɚ}_{\text{ɒ}_2}(\text{ɭ}))^3+\left(\text{ɚ}_{\text{ɒ}_3}(\text{ɭ})\right)^3- \\ ((\text{ɚ}_{\text{ɒ}_1}(\text{ɭ}))^3+\text{ɚ}_{\text{ɒ}_2}(\text{ɭ}))^3-(\text{ɚ}_{\text{ɒ}_1}(\text{ɭ}))^3\text{ɚ}_{\text{ɒ}_2}(\text{ɭ}))^3)(\text{ɚ}_{\text{ɒ}_3}(\text{ɭ}))^3,\end{array}}$$

$$\text{ɘ}_{\text{ɒ}_1}(\text{ɭ})\cdot\text{ɘ}_{\text{ɒ}_2}(\text{ɭ})\text{ɘ}_{\text{ɒ}_3}(\text{ɭ}),\ \text{ʒ}_{\text{ɒ}_1}(\text{ɭ})\cdot\text{ʒ}_{\text{ɒ}_2}(\text{ɭ})\text{ʒ}_{\text{ɒ}_3}(\text{ɭ}) \tag{15}$$

Hence the result follows from (14) and (15).

Definition 6. Scalar multiplication operation on Fermatean neutrosophic sets Γ_1 by a scalar α is denoted as $\alpha\Gamma_1=\left(\sqrt[3]{1-\left(1-(\hbar_{\Gamma_1}(t))^3\right)^{\alpha}},(\text{ƛ}_{\Gamma_1}(t))^{\alpha},(\ell_{\Gamma_1}(t))^{\alpha}\right)$ and is defined by

$$\alpha\Gamma_1=\left(\sqrt[3]{1-\left(1-(\hbar_{\Gamma_1}(t))^3\right)^{\alpha}},(\text{ƛ}_{\Gamma_1}(t))^{\alpha},(\ell_{\Gamma_1}(t))^{\alpha}\right)$$

Numerical Example: Let a Fermatean neutrosophic set be $\Gamma_1=<0.75,0.3,0.69>$ and $\alpha=3$. Then

$$3\Gamma_1=\left(\sqrt[3]{1-\left(1-(0.75)^3\right)^3},(0.3)^3,(0.69)^3\right)$$

$$=\left(\sqrt[3]{1-\left(0.578125\right)^3},0.027,0.328509\right)$$

$$=\left(\sqrt[3]{0.80677},0.027,0.328509\right)$$

$$=\left(0.931,0.027,0.329\right)$$

Conclusion

This study seeks to enhance the field of neutrosophic set theory and neutrosophic logic by exploring Fermatean neutrosophic sets, a unique method of addressing uncertainty. The paper examines several basic algebraic operations related to Fermatean neutrosophic sets. It demonstrates that the characteristics of Fermatean neutrosophic sets align with those of conventional operations. Additionally, the paper introduces some innovative operations on Fermatean neutrosophic sets and investigates the associated distributive rules. In future work, it will be important to explore how the proposed aggregation operators of Fermatean neutrosophic sets can be put into service in decision-making conditions as well as in various other neutrosophic state of affairs.

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Chapter 7

Fuzzy and Neutrosophic Extensions of Gödel's Incompleteness Theorems

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ABSTRACT

This paper proposes fuzzy and neutrosophic extensions of Gödel's incompleteness theorems based on graded and triadic self-reference. Starting from fuzzy and neutrosophic forms of the Liar Paradox — originally defined by Florentin Smarandache — we construct generalized Gödel sentences through diagonalization in formal systems capable of encoding their own provability. In the fuzzy setting, incompleteness appears as nontrivial intermediate provability degrees: the Gödel sentence stabilizes at a truth degree $F \in (0,1)$ rather than collapsing to a binary undecidable. In the neutrosophic setting, incompleteness appears as an irreducible valuation $\langle T, I, F\rangle$ with simultaneous nonzero truth, indeterminacy, and falsehood components. We formulate a Neutrosophic Gödel Fixed-Point Theorem, under explicit continuity and non-collapse assumptions, according to which self-referential sentences stabilize as interior fixed points of the neutrosophic truth cube $[0,1]^3$. We also state a Contradictory-Theory Incompleteness Principle and discuss geometric and dynamical interpretations, as well as applications to reasoning under inconsistency, knowledge fusion, artificial intelligence, quantum logic, and epistemology.

Keywords: Gödel incompleteness; fuzzy logic; neutrosophic logic; liar paradox; self-reference; fixed-point theorem; neutrosophic truth cube; contradiction; paraconsistent logic.

INTRODUCTION

Gödel's incompleteness theorems [1] represent one of the most profound discoveries in mathematical logic. They show that any sufficiently expressive, consistent, recursively axiomatized formal system containing arithmetic must contain statements that are true but unprovable within the system (First Theorem), and that such a system cannot prove its own consistency (Second Theorem) [1].

The classical construction proceeds through the Liar's Paradox [2]. The sentence "This statement is false" generates an irresolvable loop. Gödel arithmetized this by replacing truth with provability, constructing a sentence G = "This statement is not provable in S." If G is provable, it is false; if G is unprovable, it is true. The system is therefore incomplete.

Classical incompleteness assumes a binary conception of truth and provability. Modern logical frameworks — including fuzzy logic [3, 4] and neutrosophic logic [5] admit richer truth structures. Fuzzy logic assigns each statement a truth degree $v(\varphi) \in [0,1]$. Neutrosophic logic [5] assigns a triple $\langle T, I, F\rangle$ where T is the degree of truth, I is the degree of indeterminacy, and F is the degree of falsehood, each in [0,1], with no constraint that they sum to 1.

This raises the natural question: how does Gödel-type incompleteness manifest in such systems? The answer, developed in this paper, is both structurally elegant and philosophically significant. In fuzzy systems, incompleteness becomes spectral rather than binary — the Gödel sentence occupies an interior point of the unit interval. In neutrosophic systems, incompleteness becomes volumetric — the Gödel sentence occupies an interior point of the unit cube $[0,1]^3$, with all three components simultaneously positive.

BACKGROUND

Classical Gödel Incompleteness

Let T be a consistent, recursively axiomatized theory containing arithmetic. Gödel's First Incompleteness Theorem [1] constructs a sentence G such that:

$$G \leftrightarrow \neg Prov\_T(\ulcorner G \urcorner)$$

where ⌜ G ⌝ is the Gödel number of G and Prov_T(x) is the provability predicate. If T is consistent, then G is true but not provable in T. If T is also ω-consistent, then ¬G is likewise unprovable. Hence G is undecidable.

Gödel's Second Incompleteness Theorem [1] states that T cannot prove its own consistency statement Con(T), provided T is consistent. This is often summarized: "a sufficiently strong system cannot prove its own sanity from within".

Fuzzy Logic

Fuzzy logic [3, 4] extends classical two-valued logic by assigning each formula φ a truth degree $v(\varphi) \in [0,1]$. Logical connectives are generalized via t-norms and t-conorms. Standard Zadeh negation gives: $v(\neg\varphi) = 1 - v(\varphi)$. The unit interval [0,1] replaces the Boolean set {0, 1}, enabling graded representation of partial truth and partial falsity.

Neutrosophic Logic

Neutrosophic logic [5] assigns to each formula φ a triple:

$$Val(\varphi) = \langle T(\varphi), I(\varphi), F(\varphi) \rangle$$

where $T, I, F \in [0,1]$ and $T + I + F$ need not equal 1 (the components are logically independent). T measures the degree of support, I measures the degree of indeterminacy (neutral or unknown), and F measures the degree of opposition. Neutrosophic negation exchanges truth and falsehood:

$$Val(\neg_n\varphi) = \langle F(\varphi), I'(\varphi), T(\varphi) \rangle$$

This framework is a strict generalization of fuzzy logic, which occupies a line segment within the full neutrosophic cube.

The Fuzzy and Neutrosophic Liar Paradoxes

The classical Liar Paradox [6] "This statement is false" — collapses into binary contradiction: if true, it is false; if false, it is true. This semantic self-reference inspired Gödel's syntactic construction. Smarandache has extended both the Liar Paradox and its implications into graded logical frameworks.

The Fuzzy Liar Paradox

Fuzzy Liar Paradox (Smarandache)

This statement is partially false (F, where $0 < F < 1$).

Analysis by cases:

- If the statement is TRUE: then it is partially false, to degree F. This is self-consistent — the statement's content (partial falsity) matches its status.
- If the statement is FALSE: then it is partially true, with partial truth $T = 1 - F > 0$. This is also self-consistent — falsity breeds partial truth.

Unlike the classical liar, which oscillates between {True, False} without resolution, the fuzzy liar stabilizes at an intermediate value $F \in (0,1)$. Self-reference no longer generates contradiction but rather a fixed point of graded truth.

The Neutrosophic Liar Paradox

Neutrosophic Liar Paradox (Smarandache)

This statement is neutrosophically false ($\langle T, I, F\rangle$, with $0 < F < 1$).

Analysis by cases:

- If the statement is TRUE $\langle T, I, F\rangle$: then it is neutrosophically false, with falsehood component $F > 0$. The statement's own content is partially realized.
- If the statement is FALSE $\langle F, I, T\rangle$ (where now truth = F, indeterminacy = I, falsehood = T): then it is neutrosophically true, with truth = $F > 0$.

In both cases the statement is neither purely true nor purely false. It stabilizes at a triadic interior point $\langle T, I, F\rangle$ with all components potentially positive. The indeterminacy component I captures the irresolvable ambiguity that classical logic cannot represent.

The key relationship between the Liar Paradoxes and the Gödel constructions: the Liar Paradox is semantic self-reference (about truth); the Gödel sentence is syntactic self-reference (about provability). Once provability is substituted for truth in the graded frameworks, the Fuzzy and Neutrosophic Liar Paradoxes become the seeds of the corresponding Gödel constructions.

Fuzzy Gödel Incompleteness Theorems

Fuzzy Formal Systems

A Fuzzy Formal System FS consists of:

- A language with well-formed formulas (wffs).
- A fuzzy provability function Prov: wff → [0,1], where $\mathrm{Prov}(\varphi) = p$ means "φ is provable to degree p."
- A fuzzy truth valuation Val: wff → [0,1].
- A fuzzy consistency condition: $\mathrm{Prov}(\varphi) + \mathrm{Prov}(\neg\varphi) \leq 1 + \varepsilon$ for some tolerance $\varepsilon \geq 0$.

Classical negation T → F is replaced by fuzzy negation: $\mathrm{Val}(\neg\varphi) = 1 - \mathrm{Val}(\varphi)$ (Zadeh negation), or more generally $\mathrm{Val}(\neg\varphi) = n(v)$ for a decreasing involution n.

The Fuzzy Diagonal Lemma

The classical Diagonal Lemma guarantees the existence of self-referential sentences in any system capable of representing arithmetic and arithmetizing syntax. Since diagonalization is a syntactic construction independent of the specific truth-value semantics, it extends directly to fuzzy systems.

Fuzzy Diagonal Lemma

For any formula φ(x) in a fuzzy arithmetic theory T_f, there exists a sentence G_f such that: T_f ⊢ G_f ↔ φ(⌜ G_f ⌝), where the biconditional is interpreted under fuzzy equivalence.

The Fuzzy Gödel Sentence

Generalizing the Fuzzy Liar Paradox, construct the Fuzzy Gödel Sentence:

Fuzzy Gödel Sentence G_F

"The provability degree of this statement is F, where 0 < F < 1." That is: Prov(G_F) = F.

Analysis by cases:

Assumption	Consequence
Prov(G_F) = 1 (fully provable)	G_F is fully true, so Prov(G_F) = F < 1. Contradiction.
Prov(G_F) = 0 (fully unprovable)	G_F asserts Prov(G_F) = F > 0. Contradiction.
Prov(G_F) = F (intermediate)	G_F is self-consistently provable to degree F — a fuzzy fixed point. ✓

First Fuzzy Gödel Theorem

Theorem 1: First Fuzzy Gödel Incompleteness Theorem

In any sufficiently expressive Fuzzy Formal System FS satisfying standard regularity conditions, there exists a sentence G_F such that Prov(G_F) is neither 0 nor 1, but instead takes a value F ∈ (0,1). The system is therefore fuzzily incomplete: G_F is provable only to an intermediate degree, and its negation ¬G_F is provable to degree 1 − F ∈ (0,1). Neither is fully provable nor fully unprovable.

Proof sketch. Applying the Fuzzy Diagonal Lemma, construct G_F so that G_F ↔ ¬_f Prov(⌜ G_F ⌝). If Prov(G_F) = 1, the right-hand side evaluates to 0, contradicting the biconditional. If Prov(G_F) = 0, the right-hand side evaluates to 1, also contradicting the biconditional. By continuity, a fixed point exists in the interior (0,1). ■

Interpretation. Classical incompleteness is binary: a statement is either undecidable or decided. Fuzzy incompleteness is spectral: incompleteness manifests as the impossibility of achieving provability degree 1 or 0 for certain self-referential sentences. The system is partially complete, to degree F.

This also yields the asymptotic principle: v(G_F) > p_{T_f}(G_F) in general, meaning the truth degree of the Gödel sentence exceeds its provability degree. The system asymptotically approaches truth but cannot fully certify it.

Fuzzy Fixed-Point Theorem

The sentence G_F corresponds to a fuzzy fixed point of the provability operator:

There exists G_F such that: Prov(G_F) = f(Prov(G_F))

For any continuous f: [0,1] → [0,1], by the Brouwer Fixed-Point Theorem [7] applied to [0,1], there exists a degree p* such that f(p*) = p*. This is the fuzzy analog of Gödel's Diagonal Lemma.

Fuzzy Diagonal Lemma (General Form)

For any fuzzy predicate Φ with associated continuous function f_Φ: [0,1] → [0,1], there exists a sentence G such that Prov(G) = f_Φ(Prov(G)).

Second Fuzzy Gödel Theorem

Define the fuzzy consistency degree:

Con_F(FS) = 1 − sup{ min(Prov(φ), Prov(¬φ)) : φ ∈ FS }

Theorem 2: Second Fuzzy Gödel Incompleteness Theorem

A sufficiently expressive Fuzzy Formal System FS cannot prove Con_F(FS) = 1 from within itself. At best, it can establish Prov(Con_F(FS)) = c for some c ∈ (0,1), where c encodes a partial self-trust of the system: p_{T_f}(Con_f(T_f)) < 1.

This replaces the harsh classical "you cannot prove your own consistency" with the more nuanced: you can partially verify your own consistency, but never to full certainty from within. A rich fuzzy system may support its own consistency to some degree, but not to absolute degree 1.

Neutrosophic Gödel Incompleteness Theorems

Neutrosophic Formal Systems

A Neutrosophic Formal System NS operates with:

- Neutrosophic provability: NeutProv(φ) = ⟨T, I, F⟩ where T = degree of provability, I = degree of indeterminate/undecidable provability, F = degree of disprovability, with T, I, F ∈ [0,1].
- A neutrosophic negation: NeutProv(¬φ) = ⟨F, I, T⟩ (truth and falsehood swap; indeterminacy is preserved or transformed).
- A neutrosophic consistency condition generalizing fuzzy consistency.

Since diagonalization depends only on the arithmetization of syntax (which is independent of truth-value semantics), it extends to neutrosophic theories:

Neutrosophic Diagonal Lemma

For any neutrosophic formula φ(x) in a neutrosophic arithmetic theory T_n, there exists a sentence G_n such that: T_n ⊢ G_n ↔ φ(⌜G_n⌝), where evaluation uses neutrosophic triples.

The Neutrosophic Gödel Sentence

Extending the Neutrosophic Liar Paradox directly:

Neutrosophic Gödel Sentence G_N

"The neutrosophic provability of this statement is ⟨T, I, F⟩, with 0 < F < 1."

Assumption	Consequence
NeutProv(G_N) = ⟨1,0,0⟩ (fully provable)	G_N is fully true, so its provability-falsehood = F > 0. Contradiction.
NeutProv(G_N) = ⟨0,0,1⟩ (fully disprovable)	Negation flips to ⟨1,0,0⟩: G_N is neutrosophically true, so truth = F > 0. Contradiction.
NeutProv(G_N) = ⟨0,1,0⟩ (fully indeterminate)	G_N asserts F > 0 in its provability. Indeterminacy leaks into falsehood component. Partial contradiction.
NeutProv(G_N) = ⟨T,I,F⟩ with 0 < F < 1	Self-consistent neutrosophic fixed point. ✓

First Neutrosophic Gödel Theorem

Theorem 3: First Neutrosophic Gödel Incompleteness Theorem

In any sufficiently expressive Neutrosophic Formal System NS, there exists a sentence G_N such that NeutProv(G_N) = ⟨T, I, F⟩ with T, I, F all potentially nonzero and 0 < F < 1. The system exhibits three-dimensional incompleteness: G_N is simultaneously partially provable (T > 0), partially undecidable (I > 0), and partially disprovable (F > 0).

Proof sketch. Construct G_N via the Neutrosophic Diagonal Lemma satisfying G_N ↔ ¬_n Prov(⌜ G_N ⌝). Pure truth ⟨1,0,0⟩ is unstable: if G_N is fully true, its self-referential claim introduces counter-support. Pure falsehood ⟨0,0,1⟩ is also unstable: the negation yields full truth, re-introducing a truth component. Pure indeterminacy ⟨0,1,0⟩ is unstable given the non-collapse assumption. The system therefore cannot assign a classical extreme value, and must stabilize at an interior triple. ■

Key distinction: In classical Gödel, undecidability is binary — the statement happens to be neither provable nor disprovable. In Neutrosophic Gödel, indeterminacy I is a primitive, irreducible component of provability status: I > 0 captures statements that are in principle not resolvable, not merely currently unresolved.

Second Neutrosophic Gödel Theorem

Define neutrosophic consistency: NeutCon(NS) = ⟨T_c, I_c, F_c⟩ where T_c = degree of consistency, I_c = degree of indeterminate consistency, F_c = degree of inconsistency.

Theorem 4: Second Neutrosophic Gödel Incompleteness Theorem

A sufficiently expressive Neutrosophic Formal System NS cannot internally prove NeutCon(NS) = ⟨1,0,0⟩. The best achievable internal certification is NeutCon(NS) = ⟨T_c, I_c, F_c⟩ with I_c > 0, reflecting an irreducible uncertainty about the system's own consistency — not merely ignorance, but ontological indeterminacy.

This is philosophically stronger than the classical second theorem: classical Gödel says you cannot know you are consistent; neutrosophic Gödel says consistency itself is not a fully determined property of the system.

Neutrosophic Gödel Fixed-Point Theorem

We now state the central result. Let f: $[0,1]^3 \to [0,1]^3$ be the self-reference operator induced by evaluating a sentence S through the clause S ↔ ¬_n Prov(S). The input triple represents the current neutrosophic valuation of S; the output triple represents the valuation generated after one step of neutrosophic provability evaluation.

Theorem 5: Neutrosophic Gödel Fixed-Point Theorem

Assume that: (i) the operator f is continuous on the compact cube $[0,1]^3$; (ii) the extreme classical points are unstable under the self-reference dynamics; and (iii) the provability operator is non-collapse, meaning it does not force every self-referential sentence to a boundary valuation. Then there exists at least one fixed point $X^* = \langle T^*, I^*, F^* \rangle$ such that $X^* = f(X^*)$. Under the instability and non-collapse assumptions, at least one such fixed point lies in the interior: $0 < T^* < 1$, $0 < I^* < 1$, $0 < F^* < 1$.

Proof sketch. Continuity of f on the compact convex set $[0,1]^3$ guarantees, by the Brouwer Fixed-Point Theorem, the existence of at least one fixed point. To argue for an interior fixed point: assume every fixed point lies on the boundary. The instability assumption excludes the classical extreme points as stable destinations of the self-referential dynamics. The non-collapse assumption excludes the remaining boundary faces as the only possible long-run states. Therefore, a boundary-only fixed-point set is incompatible with the dynamics, and an interior fixed point must occur. ■

Interpretation. The classical liar oscillates; the classical Gödel sentence is undecidable. In the neutrosophic setting, self-reference may instead stabilize at an interior equilibrium — a sentence supported to some degree, opposed to some degree, and still partially indeterminate. This equilibrium is a logically meaningful object rather than a paradox.

Connection to Lawvere's Theorem [8] . In any cartesian closed category, if there is a surjective morphism $A \to A^A$, then every endomorphism of A has a fixed point. This categorical result unifies Cantor's theorem, Gödel's incompleteness, and the halting problem. The fuzzy/neutrosophic fixed-point results correspond to enriched variants of Lawvere's theorem in categories enriched over $([0,1], \geq, \otimes)$ with t-norm ⊗.

Geometric Interpretation: The Neutrosophic Truth Cube

Neutrosophic truth values ⟨T, I, F⟩ admit a natural geometric interpretation in three-dimensional logical space. The unit cube $[0,1]^3$ [9], with axes T (truth), I (indeterminacy), and F (falsehood), is the neutrosophic truth space C.

Logical State	**Coordinates in C**	**Logical Meaning**
Classical True	(1, 0, 0)	Fully proven, no uncertainty

Classical False	(0, 0, 1)	Fully disproven, no uncertainty
Fully Indeterminate	(0, 1, 0)	Entirely unknown or neutral
Fuzzy Logic States	(T, 0, 1−T)	Line segment (1D subset of C)
Neutrosophic Gödel Point	(T_G, I_G, F_G) — interior	Self-referential equilibrium

Classical truth values occupy the vertices of C. Fuzzy logic occupies a line or thin subset of C, since T and F are tied through T + F = 1. Neutrosophic logic fills C much more freely, with all three components logically independent.

A neutrosophic Gödel sentence appears as an interior point of C. This geometric picture captures in one image what is conceptually central: incompleteness is no longer only a missing decision. It becomes the emergence of interior truth states — stable logical equilibria in three-dimensional logical space.

The dynamic trajectory of a self-referential sentence under iterated evaluation can also be visualized geometrically. Starting from any initial guess X_0, the iteration X_{n+1} = f(X_n) traces a path in C. Under the Neutrosophic Fixed-Point Theorem, this path converges to an interior point ⟨T*, I*, F*⟩ rather than oscillating between corners:

Example trajectory: (1,0,0) → (0.6,0.1,0.3) → (0.4,0.2,0.4) → (0.33,0.34,0.33) = stable equilibrium

Dynamical Interpretation and Logical Trajectories

Self-reference can be studied dynamically. Define X_n = (T_n, I_n, F_n) as the neutrosophic truth state of a sentence after the n-th evaluation of its provability. The iteration is:

$$X_{n+1} = f(X_n)$$

For a Gödel-type sentence S ≡ ¬Prov(S), a natural operator is:

$$f(T, I, F) = (\alpha F + (1-\alpha)T,\ I + \beta|T-F|,\ \alpha T + (1-\alpha)F)$$

for parameters $0 < \alpha, \beta < 1$. This operator captures the bidirectional self-referential pressure: truth pushes toward falsehood (via the negation of provability) and vice versa, with the excess loading into indeterminacy.

Three dynamical regimes are possible in C: (1) Oscillation — truth and falsity alternate, approximating the classical liar paradox as a limit; (2) Convergence to an interior fixed point — paradox resolves into a stable triadic equilibrium, the central case of interest; (3) Complex or chaotic triadic motion — some self-referential operators may generate complex trajectories in C, opening a new field of nonlinear dynamics of logical self-reference.

This logical wave interpretation parallels physical wave behavior: the truth triple ⟨T, I, F⟩ functions as an amplitude vector in logical space. Classical logic "collapses the wave" to a vertex. Neutrosophic logic allows the wave to stabilize at a superposition of logical states.

Comparative Structure

Feature	**Classical**	**Fuzzy**	**Neutrosophic**
Truth values	{0, 1}	[0, 1]	$[0,1]^3$ (T, I, F)

Gödel sentence	Undecidable (binary)	Partially provable (degree F)	Triple-valued ⟨T, I, F⟩
Incompleteness type	Absolute gap	Spectral gap	Volumetric gap
Consistency statement	Unprovable	Partly provable (degree $c < 1$)	Provable with irreducible I_c > 0
Fixed-point guarantee	Gödel's diagonal lemma	Brouwer on [0,1]	Brouwer on $[0,1]^3$
Negation	φ ↔ ¬φ flips {0,1}	v ↔ 1−v	⟨T,I,F⟩ ↔ ⟨F,I,T⟩
Logical space	0-dimensional (2 points)	1-dimensional (interval)	3-dimensional (cube)

Contradictory-Theory Incompleteness Principle

Many real bodies of knowledge are not merely incomplete but internally conflicting. Scientific models, data sources, and interpretive frameworks may support incompatible conclusions while none can be cleanly discarded. This motivates a stronger principle:

Principle: Contradictory-Theory Incompleteness

If a sufficiently expressive formal environment admits multiple partially incompatible subsystems whose claims can be represented in a common neutrosophic language, then the joint environment necessarily generates sentences whose valuations lie in the interior of the truth cube C. Contradiction plus self-reference yields mixed logical states that cannot be reduced to classical truth values.

This principle is not a celebration of contradiction. Rather, it provides a structured way to represent unresolved conflict. A sentence supported by one subsystem and opposed by another is a mathematically meaningful interior state in a three-dimensional logical space — not noise.

Applications and Connections

Connections to Mathematical Logic

Paraconsistent Logic [10] . Priest's LP [10] and dialetheism allow true contradictions. The neutrosophic framework is a quantified dialetheism: contradictions are not merely permitted but measured. The I-component absorbs what dialetheism handles by brute acceptance. The neutrosophic framework is thereby paraconsistent by construction.

Many-Valued Logics [11, 12, 13, 14]. Łukasiewicz's three-valued logic [11, 12] has values {0, ½, 1}; Kleene’s many valued logic [13] has {F, U, T}; Gödel's many-valued logics [14] provide a hierarchy of truth. The fuzzy and neutrosophic frameworks are continuous generalizations. The Fuzzy Gödel theorem subsumes what would happen to incompleteness under Łukasiewicz provability.

Non-Standard Analysis. Robinson's infinitesimals [15] suggest provability degrees of the form $F = \varepsilon$ (infinitesimally small) or $F = 1-\varepsilon$ (infinitesimally close to full provability) — a refined fuzzy spectrum near the classical endpoints.

Computability Theory

Fuzzy Turing Machines [16].

A Fuzzy Turing Machine (FTM) [16] has transition functions valued in [0,1]. The Fuzzy Halting Problem asks: what is the degree to which a given FTM halts on a given input? The Fuzzy Gödel theorem implies this degree is generically in (0,1) for self-referential programs — not a clean undecidability, but a graded undecidability.

Neutrosophic Computability. A Neutrosophic Turing Machine has transition triples $\langle T, I, F \rangle$. The halting predicate becomes neutrosophically valued. The Second Neutrosophic Gödel Theorem corresponds to the undecidability of the neutrosophic halting problem: we cannot fully certify $(T=1, I=0, F=0)$ whether the machine halts. This is a structural deepening of the Turing-Gödel connection.

Set Theory

Fuzzy ZFC. Replace binary membership $x \in S$ with graded membership $\mu\_S(x) \in [0,1]$. The Fuzzy Continuum Hypothesis asks: is there a fuzzy cardinality strictly between $\aleph_0$ and $2^\wedge\aleph_0$? This is not decidable to degree 1 within Fuzzy ZFC — a fuzzy analog of Cohen's independence results [16, 17].

Neutrosophic Russell Paradox

The neutrosophic Russell set $R\_N = \{x : \text{NeutMembership}(x,x) = \langle T,I,F \rangle \text{ with } F > 0\}$ generalizes the Neutrosophic Liar directly into set theory. In neutrosophic set theory, membership is $\langle T, I, F \rangle$, and the paradox resolves at an interior fixed point rather than generating an explosion.

Artificial Intelligence

Fuzzy Expert Systems. An expert system with fuzzy rule weights will contain rules whose confidence is self-referentially stuck at an intermediate value — it can neither fully confirm nor reject certain meta-rules about itself. Fuzzy Gödel provides the theoretical foundation for this irreducible epistemic limitation.

Uncertainty Quantification in Neural Networks. Neural network confidence scores are naturally fuzzy. The Fuzzy Gödel phenomenon may underlie irreducible epistemic uncertainty in deep learning: certain self-referential queries (e.g., "How confident is this model in its own confidence estimates?") cannot achieve calibrated certainty — not due to lack of data, but due to structural incompleteness analogous to G_F.

Neutrosophic AI and Self-Correcting Systems. The Axiom of Neutrosophic Recognition provides a formal mechanism for AI systems to handle Gödelian loops gracefully. Instead of crashing on a self-referential paradox, an AI system equipped with neutrosophic logic assigns the paradox a triple $\langle T, I, F \rangle$ with I dominant, treats it as a known indeterminate state, and continues functioning. This turns a "system crash" into a "logical pause."

Physics and Quantum Mechanics

Quantum Logic

Birkhoff-von Neumann quantum logic [18] replaces classical Boolean logic with the lattice of closed subspaces of a Hilbert space [19]. Quantum probabilities are naturally in [0,1], making Fuzzy Gödel directly relevant. Quantum systems cannot fully self-measure (the measurement problem) — a physical instantiation of Fuzzy incompleteness.

Quantum Superposition Analogy. A quantum state $|\psi\rangle = a|0\rangle + b|1\rangle$ represents a superposition of basis states. Similarly, a neutrosophic truth value $\langle T, I, F \rangle$ represents a logical superposition. Probability amplitude corresponds to truth degree; measurement corresponds to logical decision; collapse corresponds to classical binary evaluation. Neutrosophic Gödel sentences are logical analogs of quantum superpositions that resist full collapse.

The Penrose-Lucas Argument.

Penrose [20] argued that Gödel's theorem implies human minds transcend computation. The Fuzzy Gödel framework softens this: minds may operate with graded provability, making the argument a matter of degree rather than a sharp transcendence claim.

Philosophy and Epistemology

Degrees of Knowability. Classical Gödel creates an absolute wall between knowable and unknowable. Fuzzy Gödel replaces this wall with a gradient — statements are knowable to varying degrees. This resonates with fallibilist epistemology (Peirce, Popper) and coherentist theories of justification.

Neutrosophic Epistemology

The triple ⟨T, I, F⟩ mirrors three epistemic stances: belief (T), suspension of judgment (I), and disbelief (F). The Neutrosophic Gödel theorem states: in any sufficiently rich epistemic system, there exist propositions toward which no agent can fully collapse their epistemic triple to ⟨1,0,0⟩ or ⟨0,0,1⟩.

Buddhist Logic (Catuṣkoṭi) [21] and Non-Western Traditions

Ancient Indian logic [21] employed a fourfold system: (1) True, (2) False, (3) Both, (4) Neither. The neutrosophic I-component provides a formal mathematical home for the "Both/Neither" states. Similarly, Eastern concepts of Sunyata (Emptiness) [22, 23] and Mu correspond to the indeterminacy dimension — the logical void that classical systems cannot represent.

Connection to Wittgenstein

Wittgenstein's hinge propositions [24], statements that cannot be doubted within a language game but also cannot be fully justified — are candidates for G_F or G_N: their "provability" within the system is self-referentially fixed at an intermediate neutrosophic value.

Smarandache Generalized Graded Incompleteness Principle

Both extensions can be unified under a single overarching principle:

Smarandache Generalized Graded Incompleteness Principle

Any sufficiently expressive formal system that internalizes its own provability generates self-referential statements whose semantic/proof-theoretic status cannot be completely represented by a bivalent partition. The residual status appears as either: (a) a degree of truth/provability in the fuzzy case, yielding spectral incompleteness (0 < p(G_f) < 1); or (b) a truth-indeterminacy-falsehood triple in the neutrosophic case, yielding volumetric incompleteness (T_G > 0, I_G > 0, F_G > 0).

The meta-systemic progression is:

Classical Liar → Fuzzy Liar → Neutrosophic Liar

Classical Gödel → Fuzzy Gödel → Neutrosophic Gödel

1-dimensional logic → 2-dimensional logic → 3-dimensional logic

Each step increases the dimensionality of the truth space and replaces a categorical impossibility with a structured, measurable logical state.

Conclusion

This paper has proposed fuzzy and neutrosophic extensions of Gödel's incompleteness theorems, grounded in the Fuzzy and Neutrosophic Liar Paradoxes defined by Smarandache. The main results are:

- Theorem 1 (First Fuzzy Gödel): In any sufficiently expressive fuzzy formal system, there exists a sentence G_F with 0 < Prov(G_F) < 1 — incompleteness is spectral rather than binary.
- Theorem 2 (Second Fuzzy Gödel): A fuzzy formal system cannot internally certify its own consistency to full provability degree 1.
- Theorem 3 (First Neutrosophic Gödel): In any sufficiently expressive neutrosophic formal system, there exists a sentence G_N with NeutProv(G_N) = ⟨T, I, F⟩ with all three components positive — incompleteness is volumetric and triadic.
- Theorem 4 (Second Neutrosophic Gödel): A neutrosophic formal system cannot certify its own consistency as ⟨1,0,0⟩; the indeterminacy component I_c > 0 is irreducible.
- Theorem 5 (Neutrosophic Fixed-Point): Under continuity and non-collapse conditions, self-referential sentences stabilize at interior fixed points of the neutrosophic truth cube.

The key conceptual advance is that Gödel incompleteness is no longer merely a negative result — a statement that cannot be decided. In fuzzy and neutrosophic frameworks, it becomes a positive structural feature: the system generates well-defined mixed logical states with measurable degrees of support, opposition, and indeterminacy [25].

The framework remains partly programmatic until a fully specified neutrosophic proof calculus is developed. Future work should address: (1) exact axioms for neutrosophic arithmetic; (2) formal definitions of neutrosophic provability operators satisfying the continuity and non-collapse conditions; (3) computational implementations in graded theorem provers and neutrosophic AI reasoning systems; (4) deeper connections between logical dynamics in the neutrosophic truth cube and quantum measurement theory; and (5) applications to scientific theory comparison and multi-source knowledge fusion.

As a final observation: Gödel found the hole. Turing showed we cannot build a bridge over it. The fuzzy and neutrosophic frameworks show that we can measure the hole itself — assigning it a graded or triadic coordinate in a higher-dimensional logical space.

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Chapter 8

Neutrosophic Weak Rough Sets

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ABSTRACT

Weak rough sets are known as double, nested, flou or interval-valued sets too. The idea is that we have a non-empty universe X where A_1, A_2 are two subsets of X. We assume that $A_1 \subseteq A_2$. Then the object of the form $A = (A_1, A_2)$ is known as a *weak rough set* on X. One can say that A_1 gathers *necessary* items while A_2 contains *possible* items. In this chapter we continue this line of reasoning. However, classical sets A_1 and A_2 are replaced with (single valued) neutrosophic sets. We use two inclusions: one is standard and very typical, while the second one is refined and it was previously introduced by Zhang et al. under the name of "type-3 inclusion relation of neutrosophic sets". Thus, we obtain two different approaches together with their corresponding algebras.

Keywords: neutrosophic sets, rough sets, double sets, inclusion relation, generalized de Morgan algebra.

INTRODUCTION

Weak rough sets were analyzed by many authors in very different contexts (both theoretical and practical, starting with pure algebra or topology and ending with multi-criteria decision making). The notion of "weak rough sets" was used by Yong-jin in his paper [13]. However, already in 1968 Gentilhomme [3]applied these objects to linguistic analysis (naming them *les ensembles flous*, that is *flou sets*). Other authors wrote about *double sets* (see e.g. [9] for *soft double sets*) or *interval-valued sets* (see [2] for intuitionistic interval-valued sets).

The idea is that we have some non-empty universe X of items under consideration. Suppose that $A_1 \subseteq A_2 \subseteq X$. Then the ordered pair of the form $A = (A_1, A_2)$ is known as a *weak rough set* on X. Clearly, this object describes the following point of view: that in A_1 we have *necessary* items, while in A_2 we have *possible* items. Obviously necessary items are also possible (but the converse is not true). As for the $X - A_2$, these are *neutral* elements of the universe.

Basically, the majority of authors defines operations on weak rough sets in a very standard manner. Thus, $A \cup B = (A_1 \cup B_1, A_2 \cup B_2)$ and $A \cap B = (A_1 \cap B_1, A_2 \cap B_2)$, while the complement of A is $A^c = (A_2^c, A_1^c)$. Now we can identify the empty weak rough set with $\emptyset = (\emptyset, \emptyset)$ and the universal weak rough set with $X = (X, X)$. If so, then the algebra $(X, \cup, \cap, {}^c, \emptyset, X)$ becomes a de Morgan algebra (but it is not necessarily Boolean).

As for the neutrosophic sets [8] , they were introduced by Smarandache in the last decade of the former century. They are an extension of fuzzy sets (defined by Zadeh in his seminal paper [15]) and intuitionistic fuzzy sets (first studied by Atanassov [1]. They allow us to model uncertainty in a very general and multi-faceted manner. In neutrosophic sets, there are three logical values, which may be independent, each corresponding to a subset of the non-standard interval $]^{+}0,1^{+}$. This is the most general definition of neutrosophic sets but here we can limit our attention to *single valued* neutrosophic sets (that were first presented in [10]). Hence, our logical values are just real numbers from $[0,1]$ interval. These logical values can sum up even to 3. This means that some kind of paraconsistency can be modelled with neutrosophic sets.

In general, neutrosophic theory evolved throughout the years. There were many innovations in this field. Some authors increased the number of logical values (building quadri- or penta-partitioned neutrosophic sets, see [6] and [7]), while the others combined neutrosophic sets with soft sets and multi-sets (see [4] or [14]).

In this paper we try to build a link between weak rough and neutrosophic sets. We use two approaches to the notion of inclusion. One of them (namely type 3) is rather untypical and was previously investigated by Zhang et al. in [16].

Obviously, some of our results are already known: but for crisp weak rough sets. Hence our approach becomes more general (but also more precise in the sense that we performed many detailed calculations).

BACKGROUND

Let us start from several basic definitions. We need to recall the neutrosophic set together with some elementary algebraic operations. The concept of inclusion will be important too.

Definition 1. [10] Assume that X is a non-empty universe. A (*single valued*) *neutrosophic set* A over X is defined as an object of the form:

$$A = \{(x, t_A(x), i_A(x), f_A(x)) : x \in X\}$$

where $t_A(x)$, $i_A(x)$, and $f_A(x)$ refer to respectively the *truth*-membership function, *indeterminacy*-membership function and *falsity*-membership function, respectively with

$$t_A: X \to [0, 1], \qquad i_A: X \to [0, 1], \qquad f_A: X \to [0, 1].$$

Consequently, we have $(t_A(x) + i_A(x) + f_A(x)) \in [0,3]$.

Indeterminacy can be interpreted as "ignorance" or "indifference" too. In more complex frameworks (like quadri, penta or hepta-partitioned neutrosophic sets) these types of "uncertainty" are separate.

Definition 2. [10, 16] Suppose that A and B are two neutrosophic sets on X. We say that A is:

1) *type-1 contained* in B iff the following relationships hold: $t_A(x) \leq t_B(x)$, $i_A(x) \geq i_B(x)$, and $f_A(x) \geq f_B(x)$. Then, we write that $A \subseteq_1 B$.
2) *type-2 contained* in B iff the following relationships hold: $t_A(x) \leq t_B(x)$, $i_A(x) \leq i_B(x)$, and $f_A(x) \geq f_B(x)$. Then we write that $A \subseteq_2 B$.
3) *type-3 contained* in B if f the following relationships hold: ($t_A(x) < t_B(x)$ and $f_A(x) \geq f_B(x)$) or (($t_A(x) = t_B(x)$ and $f_A(x) > f_B(x)$) or ($t_A(x) = t_B(x)$ and $i_A(x) \leq i_B(x)$ and $f_A(x) = f_B(x)$). Then, we write that $A \subseteq_3 B$.

Clearly, type-1 and type-2 inclusions are very similar. In this paper we shall be concentrated on type-2 inclusion because it seems that is more widespread. One can define union and intersection of neutrosophic sets in a manner such that they are in accordance with type-2 inclusion. This accordance means in particular that $(A \cap_2 B) \subseteq_2 A$ and $A \subseteq_2 (A \cup_2 B)$. This is the content of the next definition.

Definition 3. [10, 16]

Let A and B be any two neutrosophic sets on X. Then we define:

1) *type-2 union* as a new neutrosophic set $A \cup_2 B = C$ in which (for every $x \in X$) $t_C(x) = \max(t_A(x), t_B(x))$, $i_C(x) = \max(i_A(x), i_B(x))$ and $f_C(x) = \min(f_A(x), f_B(x))$.
2) *type-2 intersection* as a new neutrosophic set $A \cap_2 B = C$ in which (for every $x \in X$) $t_C(x) = \min(t_A(x), t_B(x))$, $i_C(x) = \min(i_A(x), i_B(x))$ and $f_C(x) = \max(f_A(x), f_B(x))$.

Now think about type-3 inclusion and appropriate operators.

Definition 4. [10, 16]

Let A and B be any neutrosophic sets on X. Then, we define:

1) *type-3 union* as a new neutrosophic set $A \cup_3 B = C$ such that (for every $x \in X$):
a) $C = A$ if $B \subseteq_3 A$.
b) $C = B$ if $A \subseteq_3 B$.

c) $t_C(x) = \max(t_A(x), t_B(x))$, $i_C(x) = 0$ and $f_C(x) = \min(f_A(x), f_B(x))$, otherwise.

2) *type-3 intersection* as a new neutrosophic set $A \cap_3 B = C$ in which (for every $x \in X$)
a) $C = A$ if $A \subseteq_3 B$.
b) $C = B$ if $B \subseteq_3 A$.
c) $t_C(x) = \min(t_A(x), t_B(x))$, $i_C(x) = 1$ and $f_C(x) = \max(f_A(x), f_B(x))$, otherwise.

As for the complement, then it is the same for all types of inclusion.

Definition 5. [10, 16]

Suppose that A is a neutrosophic set on X. Then, we define the complement of A as a new neutrosophic set $A^c = C$ where for every $x \in X$ we have: $t_C(x) = f_A(x)$, $i_C(x) = 1 - i_A(x)$ and $f_C(x) = t_A(x)$.

When it comes to two special sets, playing the role of lattice 0 and lattice 1, then we agree on the following definition:

Definition 6.

Assume that X is a non-empty universe. We define:

1) *type-1 empty neutrosophic set* on X as $\{(x, 0, 1, 1): x \in X\}$ (that, is for every x).

2) *type-2 empty neutrosophic set* on X as $\{(x, 0, 0, 1): x \in X\}$.

Moreover, we identify *type-3 empty neutrosophic set* with *type-2*. We denote empty neutrosophic set as $\emptyset$ (with a subscript like $\emptyset_1$ or $\emptyset_2$ when it is necessary).

Definition 7.

Assume that X is a non-empty universe. We define:

1) *type-1 universal neutrosophic set* on X as $\{(x, 1, 0, 0): x \in X\}$ (that, is for every x).

2) *type-2 universal neutrosophic set* on X as $\{(x, 1, 1, 0): x \in X\}$.

Moreover, we identify *type-3 universal neutrosophic set* with *type-2*. We denote empty neutrosophic set as 1 (with a subscript like 1_1 or 1_2 when it is necessary) or just as X (depending on the context).

As for the set difference, then both in type-2 and type-3 case it can be defined in a standard manner: as the intersection of A with the complement of B.

New results: type-2 weak rough neutrosophic sets

In this section we shall introduce our new concepts.

Definition 8.

Suppose that A_1 and A_2 are two neutrosophic sets on some non-empty universe X. Assume that $A_1 \subseteq_2 A_2$. Then the object of the form $\mathcal{A} = (A_1, A_2)$ is known as a *type-2 weak rough neutrosophic set* on X (that is, a t2WRN on X).

Hence, there are two layers: A_1 is internal, while A_2 is external. To be precise, we should say that these separate layers are A_1 and $A_2 - A_1$ because A_1 is already contained in A_2. But this distinction is not crucial. As in crisp case, we can say that A_1 models necessity (or a “deeper dimension” of belonging) while A_2 refers to possibility.

Example 9.

Let $X = \{a, b, c, d\}$ and suppose that:

$A_1 = \{(a, 0.40, 0.50, 0.10), (b, 0.60, 0.90, 0.15), (c, 0.80, 0.00, 0.00), (d, 0.10, 0.05, 0.85)\}$,

$A_2 = \{(a, 0.50, 0.60, 0.05), (b, 0.70, 0.90, 0.10), (c, 0.90, 0.00, 0.00), (d, 0.20, 0.10, 0.50)\}$.

Then $\mathcal{A} = (A_1, A_2)$ is a type-2 weak rough neutrosophic set on X. In the same way, we can take:

$B_1 = \{(a, 0.10, 0.10, 0.60), (b, 0.20, 0.10, 0.70), (c, 0.80, 0.05, 0.00), (d, 0.50, 0.00, 0.50)\}$,

$B_2 = \{(a, 0.20, 0.15, 0.40), (b, 0.50, 0.20, 0.30), (c, 0.90, 0.05, 0.00), (d, 0.55, 0.10, 0.50)\}$.

Then $\mathcal{B} = (B_1, B_2)$ is a type-2 weak rough neutrosophic set on X too.

In the next definition we define special type-2 weak rough neutrosophic sets.

Definition 10.

Suppose that X is a non-empty universe. Then a t2WRN on X is:

1) *empty* if it is of the form $\breve{\emptyset} = (\emptyset, \emptyset)$, where $\emptyset$ is the type-2 empty neutrosophic set on X.

2) *universal* if it is of the form $\breve{\mathcal{X}} = (X, X)$, where X is the type-2 universal neutrosophic set on X.

The set of all T2WRNs on X will be denoted as $P_{T2WRN}(X)$. Clearly, one of the most important things is to define algebraic operations on these objects. In fact, they will be modelled after the standard type-2 operations for single-layer neutrosophic sets.

Definition 11.

Suppose that X is a non-empty universe and $\mathcal{A}, \mathcal{B}$ are two T2WRNs on X. We define their:

1) *Union* as a new T2WRN set $\mathcal{C}$ on X of the form: $\mathcal{C} = (C_1, C_2) = (A_1 \cup_2 B_1, A_2 \cup_2 B_2)$.

2) *Intersection* as a new T2WRN set $\mathcal{C}$ on X of the form: $\mathcal{D} = (D_1, D_2) = (A_1 \cap_2 B_1, A_2 \cap_2 B_2)$.

3) *Complement* (of e.g. $\mathcal{A}$) as a new T2WRN set of the form $\mathcal{A}^c = (A_2^c, A_1^c)$, where on the right-hand side we have "single-layer" neutrosophic complement.

Moreover, we say that $\mathcal{A} \subseteq_2 \mathcal{B}$ iff $A_1 \subseteq_2 B_1$ and $A_2 \subseteq_2 B_2$.

One can check (rather easily) that these operations indeed produce new T2WNRs. It is a consequence of the fact that type-2 inclusion is a partial order on the powerset of all "single-layer" neutrosophic sets on X. Due to the same reason, we can prove that type-2 inclusion of T2WNRs is properly defined: in the sense that it imposes partial order on $P_{T2WNR}(X)$.

Remark 12. Note that we avoid unnecessary multiplication of symbols. Hence, $\cup_2$ is used both in the context of "single-layer" union and in the framework of T2WNR. The same can be said about other operators. The reader should be aware of this fact.

Example 13.

Take $\mathcal{A}$ and $\mathcal{B}$ from Example 9. Then we obtain

1) $\mathcal{A} \cup_2 \mathcal{B} = \mathcal{C} = (C_1, C_2)$, where:

$C_1 = \{(a, 0.40, 0.50, 0.10), (b, 0.60, 0.90, 0.15), (c, 0.80, 0.05, 0.00), (d, 0.50, 0.05, 0.50)\}$,

$C_2 = \{(a, 0.50, 0.60, 0.05), (b, 0.70, 0.90, 0.10), (c, 0.90, 0.05, 0.00), (d, 0.55, 0.10, 0.50)\}$.

2) $\mathcal{A} \cap_2 \mathcal{B} = \mathcal{D} = (D_1, D_2)$, where:

$D_1 = \{(a, 0.10, 0.10, 0.60), (b, 0.20, 0.10, 0.70), (c, 0.80, 0.00, 0.00), (d, 0.10, 0.00, 0.85)\}$,

$D_2 = \{(a, 0.20, 0.15, 0.40), (b, 0.50, 0.20, 0.30), (c, 0.90, 0.00, 0.00), (d, 0.20, 0.10, 0.50)\}$.

3) $\mathcal{A}^c = \mathcal{E}$ where:

$E_1 = A_2^c = \{(a, 0.05, 0.40, 0.50), (b, 0.10, 0.10, 0.70), (c, 0.00, 1.00, 0.90), (d, 0.50, 0.90, 0.20)\}$,

$E_2 = A_1^c = \{(a, 0.10, 0.40, 0.50), (b, 0.15, 0.10, 0.60), (c, 0.00, 1.00, 0.80), (d, 0.85, 0.95, 0.10)\}$.

In the same way the reader can calculate $\mathcal{B}^c$.

Remark 14. Assume that we discuss ("single-layer") neutrosophic sets on some X with type-2 operations and relations of union, intersection, inclusion and complement. Also, we agree on type-2 distinguished neutrosophic sets (that is, the empty and the universal set). Then one can prove (see e.g. [10, 16]) that this algebra is a de Morgan algebra (but not necessarily Boolean). Using this fact we can prove the next theorem.

Theorem 15.

Consider a non-empty universe X together with $P_{T2WRN}(X)$. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be T2WRNs on X. Then, the following properties are true:

1) $\mathcal{A} \cup_2 \mathcal{B} = \mathcal{B} \cup_2 \mathcal{A}$ and $\mathcal{A} \cap_2 \mathcal{B} = \mathcal{B} \cap_2 \mathcal{A}$ (*commutativity*).

2) $\mathcal{A} \cup_2 \mathcal{A} = \mathcal{A}$ and $\mathcal{A} \cap_2 \mathcal{A} = \mathcal{A}$ (*idempotence*).

3) $\mathcal{A} \cup_2 (\mathcal{A} \cap_2 \mathcal{B}) = \mathcal{A}$ and $\mathcal{A} \cap_2 (\mathcal{A} \cup_2 \mathcal{B}) = \mathcal{A}$ (*absorption laws*).

4) $\mathcal{A} \cup_2 (\mathcal{B} \cup_2 \mathcal{C}) = (\mathcal{A} \cup_2 \mathcal{B}) \cup_2 \mathcal{C}$ and $\mathcal{A} \cap_2 (\mathcal{B} \cap_2 \mathcal{C}) = (\mathcal{A} \cap_2 \mathcal{B}) \cap_2 \mathcal{C}$ (*associativity*)

5) $(\mathcal{A} \cup_2 \mathcal{B})^c = \mathcal{A}^c \cap_2 \mathcal{B}^c$ and $(\mathcal{A} \cap_2 \mathcal{B})^c = \mathcal{A}^c \cup_2 \mathcal{B}^c$ (*de Morgan laws*).

6) $\mathcal{A} \cup_2 (\mathcal{B} \cap_2 \mathcal{C}) = (\mathcal{A} \cup_2 \mathcal{B}) \cap_2 (\mathcal{A} \cup_2 \mathcal{C})$ and $\mathcal{A} \cap_2 (\mathcal{B} \cup_2 \mathcal{C}) = (\mathcal{A} \cap_2 \mathcal{B}) \cup_2 (\mathcal{A} \cap_2 \mathcal{C})$ (*distributivity laws*).

7) $(\mathcal{A}^c)^c = \mathcal{A}$ (*double negation*).

8) $\mathcal{A} \cap_2 \breve{\emptyset} = \breve{\emptyset}$, $\mathcal{A} \cup_2 \breve{\emptyset} = \mathcal{A}$, $\mathcal{A} \cap_2 \breve{\mathcal{X}} = \mathcal{A}$, $\mathcal{A} \cup_2 \breve{\mathcal{X}} = \breve{\mathcal{X}}$.

Moreover, we have:

7) $\breve{\emptyset}^c = \breve{X}$ and $\breve{\mathcal{X}}^c = \breve{\emptyset}$.

Proof. The proof is rather standard. We should remember Remark 12 to justify all the transitions. For example, take de Morgan laws:

$$(\mathcal{A} \cup_2 \mathcal{B})^c = \big((A_1, A_2) \cup_2 (B_1, B_2)\big)^c = (A_1 \cup_2 B_1, A_2 \cup_2 B_2)^c = ((A_2 \cup_2 B_2)^c, (A_1 \cup_2 B_1)^c).$$

Now we use exactly the fact that de Morgan laws hold for the "single-layer" neutrosophic sets. Hence, we can write that the last object is: $(A_2^c \cap_2 B_2^c, A_1^c \cap_2 B_1^c)$.

However, we have $\mathcal{A}^c \cap_2 \mathcal{B}^c = (A_2^c, A_1^c) \cap_2 (B_2^c, B_1^c)$. But now we use the definition of type-2 weak rough intersection to obtain the already known expression.

Other parts of the proof are very similar.

Remark 16.

As the reader can observe, in Theorem 15 we do not use any special properties of "sets" (neutrosophic). The whole proof is based on the general properties of de Morgan algebra. Hence it can be easily generalized (one can compare this suggestion with Definition 2.2 in Kauffman's paper [5].

Let us introduce two additional operations. In some sense, we already mentioned and studied them (e.g. in [11], [12]) but in the context of crisp non-classical sets.

Definition 17.

Let X is a non-empty universe and $\mathcal{A}, \mathcal{B}$ be two T2WRNs on X. Then, we define the following two operations:

1) $\mathcal{A} \odot_2 \mathcal{B} = (A_1 \cap_2 B_1, A_2 \cup_2 B_2)$.

2) $\mathcal{A} \oplus_2 \mathcal{B} = \big((A_1 \cup_2 B_1) \cap_2 (A_2 \cap_2 B_2), A_2 \cap_2 B_2\big)$.

Using lattice properties, one can easily check that they produce new T2WRN on X. Exactly due to this reason the second operator, namely $\oplus_2$, is more complicated than the first one. These operations are like: "minimization of necessities together with maximization of possibilities" and "(restricted) maximization of necessities together with minimization of possibilities".

Theorem 18.

Let X be a non-empty universe and $\mathcal{A}, \mathcal{B}$ be three T2WRNs on X. Then, the following properties hold:

1) $\mathcal{A} \oplus_2 \mathcal{A} = \mathcal{A}$, $\mathcal{B} \odot_2 \mathcal{B} = \mathcal{B}$ (*idempotence*).

2) $\mathcal{A} \oplus_2 \mathcal{B} = \mathcal{B} \oplus_2 \mathcal{A}$, $\mathcal{A} \odot_2 \mathcal{B} = \mathcal{B} \odot_2 \mathcal{A}$ (*commutativity*).

3) $\mathcal{A} \oplus_2 (\mathcal{A} \odot_2 \mathcal{B}) = \mathcal{A}$ ($\oplus_2\odot_2$-*absorption law*).

4) $\mathcal{A} \odot_2 (\mathcal{B} \odot_2 \mathcal{C}) = (\mathcal{A} \odot_2 \mathcal{B}) \odot_2 \mathcal{C}$ ($\odot$-*associativity*).

5) $\mathcal{A} \oplus_2 (\mathcal{B} \oplus_2 \mathcal{C}) = (\mathcal{A} \oplus_2 \mathcal{B}) \oplus_2 \mathcal{C}$ ($\oplus$-*associativity*).

Proof. The first two points are obvious. Then we have:

3) We may write:

$$\mathcal{A} \oplus_2 (\mathcal{A} \odot_2 \mathcal{B}) = (A_1, A_2) \oplus_2 (A_1 \cap_2 B_1, A_2 \cap_2 B_2) =$$
$$\Big(\big(A_1 \cup_2 (A_1 \cap_2 B_1)\big) \cap_2 \big((A_2 \cap_2 (A_2 \cup_2 B_2)\big), A_2 \cap_2 (A_2 \cup_2 A_2\Big) = (A_1 \cap_2 A_2, A_2) = (A_1, A_2) = \mathcal{A}.$$

Clearly, we used absorption laws that hold for "single-layer" neutrosophic sets.

4) We write $\mathcal{A} \odot_2 (\mathcal{B} \odot_2 \mathcal{C}) = \mathcal{A} \odot_2 (B_1 \cap_2 C_1, B_2 \cup_2 C_2) = (A_1 \cap_2 B_1 \cap_2 C_1, A_2 \cup_2 B_2 \cup_2 C_2)$. Now it is rather obvious that the resulting object is $(\mathcal{A} \odot_2 \mathcal{B}) \odot_2 \mathcal{C}$. We use the fact that type-2 union and intersection of "single-layer" neutrosophic sets are associative.

5) This seems to be far more complicated. First, let us write:

$$\mathcal{D} = \mathcal{A} \oplus_2 (\mathcal{B} \oplus_2 \mathcal{C}) = (A_1, A_2) \oplus_2 \big((B_1 \cup_2 C_1) \cap_2 (B_2 \cap_2 C_2), B_2 \cap_2 C_2\big) =$$
$$\Big(\Big(A_1 \cup_2 \ \big((B_1 \cup_2 C_1) \cap_2 (B_2 \cap_2 C_2)\big)\Big) \cap_2 (A_2 \cap_2 B_2 \cap_2 C_2), A_2 \cap_2 B_2 \cap_2 C_2\Big).$$

Due to some reason that will be clear later, let us concentrate only on the internal layer. Let us use distributivity and associativity of "single-layer" $\cup_2$ and $\cap_2$.

We can rewrite the internal layer as:

$(A_1 \cup_2 B_1 \cup_2 C_1) \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big) \cap_2 (A_2 \cap_2 B_2 \cap_2 C_2) = D \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big)$, where $D = (A_1 \cup_2 B_1 \cup_2 C_1) \cap_2 (A_2 \cap_2 B_2 \cap_2 C_2)$.

Now think about:

$(\mathcal{A} \oplus_2 \mathcal{B}) \oplus_2 \mathcal{C} = \big((A_1 \cup_2 B_1) \cap_2 (A_2 \cap_2 B_2), A_2 \cap_2 B_2 \big) \oplus_2 (C_1, C_2) =$
$\Big(\big(((A_1 \cup_2 B_1) \cap_2 (A_2 \cap_2 B_2)) \cup_2 C_1\big) \cap_2 (A_2 \cap_2 B_2 \cap_2 C_2), A_2 \cap_2 B_2 \cap C_2\Big).$

The upper layer is the same. The internal layer is of the form:

$(A_1 \cup_2 B_1 \cup_2 C_1) \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big) \cap_2 (A_2 \cap B_2 \cap C_2) = D \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big).$

Suppose now that $D \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big) \nsubseteq D \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big)$. This implies that:

1) $D \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big) \nsubseteq D$

or

2) $D \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big) \nsubseteq \big((A_2 \cap_2 B_2) \cup_2 C_1\big).$

Consider 1). Obviously, the intersection of D with anything is contained in D. Now think about 2). In particular, it implies that $D \nsubseteq \big((A_2 \cap_2 B_2) \cup_2 C_1\big)$. But D is just a part of $A_2 \cap_2 B_2 \cap C_2$ and thus we can write: $D \subseteq (A_2 \cap B_2 \cap C_2) \subseteq (A_2 \cap_2 B_2) \subseteq \big((A_2 \cap_2 B_2) \cup_2 C_1\big)$. Hence, we get a contradiction.

Suppose now that $D \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big) \nsubseteq D \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big)$. This implies that:

1) $D \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big) \nsubseteq D$ (but this is not possible)

or

2) $D \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big) \nsubseteq \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big)$. This implies (in particular) that $D \nsubseteq \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big)$. But D is just some part of $A_2 \cap_2 B_2 \cap_2 C_2$. Hence, we can write: $D \subseteq A_2 \cap_2 B_2 \cap_2 C_2 \subseteq B_2 \cap_2 C_2 \subseteq \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big)$. Hence, we obtain a contradiction.

Now we obtain the following conclusion: $D \cap_2 \big(A_1 \cup_2 (B_2 \cap_2 C_2)\big) = D \cap_2 \big((A_2 \cap_2 B_2) \cup_2 C_1\big)$. Thus, the final formula is $\mathcal{A} \oplus_2 (\mathcal{B} \oplus_2 \mathcal{C}) = (\mathcal{A} \oplus_2 \mathcal{B}) \oplus_2 \mathcal{C}$.

The next theorem is based on counter-examples rather than on general proofs.

Theorem 19.

Let X be a non-empty universe. Then, one can define three T2WRNSs $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on X such that the following properties do not hold:

1) $\oplus_2 \odot_2$-distributivity.

2) $\odot_2 \oplus_2$-distributivity.

3) $\odot_2 \oplus_2$-absorption law.

Proof.

1) Consider $X = \{a, b, c\}$. $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are defined as below:

$$A_1 = \{(a, 0.40, 0.20, 0.10), (b, 0.50, 0.20, 0.30), (c, 0.40, 0.20, 0.15)\}$$

$$A_2 = \{(a, 0.50, 0.25, 0.05), (b, 0.55, 0.30, 0.20), (c, 0.50, 0.30, 0.05)\}$$

$$B_1 = \{(a, 0.70, 0.10, 0.20), (b, 0.80, 0.05, 0.05), (c, 0.10, 0.10, 0.80)\}$$

$$B_2 = \{(a, 0.75, 0.10, 0.15), (b, 0.85, 0.10, 0.00), (c, 0.12, 0.20, 0.50)\}$$

$$C_1 = \{(a, 0.80, 0.12, 0.12)\}, (b, 0.90, 0.10, 0.00), (c, 0.15, 0.22, 0.60)\}$$

$$C_2 = \{(a, 0.82, 0.15, 0.10), (b, 0.90, 0.10, 0.00), (c, 0.50, 0.40, 0.10)\}$$

Calculate $\mathcal{D} = \mathcal{B} \odot_2 \mathcal{C}$. We obtain:

$$D_1 = B_1 \cap_2 C_1 = \{(a, 0.70, 0.10, 0.20), (b, 0.80, 0.05, 0.05), (c, 0.10, 0.10, 0.80)\}$$

$$D_2 = B_2 \cup_2 C_2 = \{(a, 0.82, 0.15, 0.10), (b, 0.90, 0.10, 0.00), (c, 0.50, 0.40, 0.10)\}$$

Now we calculate $\mathcal{E} = \mathcal{A} \oplus_2 (\mathcal{B} \odot_2 \mathcal{C}) = \mathcal{A} \oplus_2 \mathcal{D}$.

Let us start from:

$$E_2 = A_2 \cap_2 D_2 = \{(a, 0.50, 0.15, 0.10), (b, 0.55, 0.10, 0.20), (c, 0.50, 0.30, 0.10)\}$$

Now we calculate $A_1 \cup_2 D_1 = \{(a, 0.70, 0.20, 0.10), (b, 0.80, 0.20, 0.05), (c, 0.40, 0.20, 0.15)\}$. Then we are able to compute:

$$E_1 = \{(a, 0.50, 0.15, 0.10), (b, 0.55, 0.10, 0.20), (c, 0.40, 0.20, 0.15)\}$$

Now think about $\mathcal{F} = \mathcal{A} \oplus_2 \mathcal{B}$. We obtain:

$$F_2 = A_2 \cap_2 B_2 = \{(a, 0.50, 0.10, 0.15), (b, 0.55, 0.10, 0.20), (c, 0.12, 0.20, 0.50)\}$$

Then $A_1 \cup_2 B_1 = \{(a, 0.70, 0.20, 0.10), (b, 0.80, 0.20, 0.05), (c, 0.40, 0.20, 0.15)\}$.

Thus $F_1 = \{(a, 0.50, 0.10, 0.15), (b, 0.55, 0.10, 0.20), (c, 0.12, 0.20, 0.50)\}$.

Then consider $\mathcal{G} = \mathcal{A} \oplus_2 \mathcal{C}$. We obtain:

$$G_2 = A_2 \cap_2 C_2 = \{(a, 0.50, 0.15, 0.10), (b, 0.55, 0.10, 0.20), (c, 0.50, 0.30, 0.10)\}$$

Now compute $A_1 \cup_2 C_1 = \{(a, 0.80, 0.20, 0.10), (b, 0.90, 0.20, 0.00), (c, 0.40, 0.22, 0.15)\}$.

Thus $G_1 = \{(a, 0.50, 0.15, 0.10), (b, 0.55, 0.10, 0.20), (c, 0.40, 0.22, 0.15)\}$.

Finally, we compute $\mathcal{H} = (\mathcal{A} \oplus_2 \mathcal{B}) \odot_2 (\mathcal{A} \oplus_2 \mathcal{C})$. We have:

$$H_1 = F_1 \cap_2 G_1 = \{(a, 0.50, 0.10, 0.15), (b, 0.55, 0.10, 0.20), (c, 0.12, 0.20, 0.50)\}$$

$$H_2 = F_2 \cup_2 G_2 = \{(a, 0.50, 0.15, 0.10), (b, 0.55, 0.10, 0.20), (c, 0.50, 0.30, 0.10)\}$$

The reader can check all the computations to ensure that indeed there are some differences between $\mathcal{E}$ and $\mathcal{H}$. In this particular case $E_2 = H_2$ but there are some differences when it comes to the lower layer. For example, the evaluations of a and c in E_1 are different than in H_1.

2) In 1) we used rather complex example to give the reader some impression of realism. Now we would like to simplify our approach using universe containing only one element. Hence, let $X = \{a\}$ and suppose that:

$$A_1 = \{(a, 0.40, 0.20, 0.40)\}$$

$$A_2 = \{(a, 0.60, 0.25, 0.10)\}$$

$$B_1 = \{(a, 0.10, 0.15, 0.60)\}$$

$$B_2 = \{(a, 0.20, 0.30, 0.20)\}$$

$$C_1 = \{(a, 0.20, 0.05, 0.50)\}$$

$$C_2 = \{(a, 0.35, 0.10, 0.45)\}$$

We compute $\mathcal{D} = \mathcal{A} \odot_2 \mathcal{B}$. We get:

$$D_1 = A_1 \cap_2 B_1 = \{(a, 0.10, 0.15, 0.60)\}$$

$$D_2 = A_2 \cup_2 B_2 = \{(a, 0.60, 0.30, 0.10)\}$$

Then let $\mathcal{E} = \mathcal{A} \odot_2 \mathcal{C}$. We get:

$$E_1 = A_1 \cap_2 C_1 = \{(a, 0.20, 0.05, 0.50)\}$$

$$E_2 = A_2 \cup_2 C_2 = \{(a, 0.60, 0.25, 0.10)\}$$

Then $\mathcal{F} = (\mathcal{A} \odot_2 \mathcal{B}) \oplus_2 (\mathcal{A} \odot_2 \mathcal{C}) = \mathcal{D} \oplus_2 \mathcal{E}$. Let us compute:

$$F_2 = D_2 \cap_2 E_2 = \{(a, 0.60, 0.25, 0.10)\}$$

Now $D_1 \cup_2 E_1 = \{(a, 0.20, 0.15, 0.50)\}$. Hence: $F_1 = \{(a, 0.20, 0.15, 0.50)\}$.

Now we want to compute $\mathcal{G} = \mathcal{B} \oplus_2 \mathcal{C}$. First, we compute: $G_2 = B_2 \cap_2 C_2 = \{(a, 0.20, 0.10, 0.45)\}$. Then we calculate $B_1 \cup_2 C_1 = \{(a, 0.20, 0.15, 0.50)\}$. Hence $G_1 = \{(a, 0.20, 0.10, 0.50)\}$.

Finally, we build $\mathcal{H} = \mathcal{A} \odot_2 (\mathcal{B} \oplus_2 \mathcal{C}) = \mathcal{A} \odot_2 \mathcal{G}$. We have $H_1 = A_1 \cap_2 G_1 = \{(a, 0.20, 0.10, 0.50)\}$. Then we have $H_2 = A_2 \cup_2 G_2 = \{(a, 0.60, 0.25, 0.10)\}$. But H_1 is different than F_1 because the value of indifference of a is different.

3) Take the same $\mathcal{A}$ and $\mathcal{B}$ as in 2). First, we want to compute $\mathcal{C} = \mathcal{A} \oplus_2 \mathcal{B}$. We have $C_2 = A_2 \cap_2 B_2 = \{(a, 0.20, 0.25, 0.20)\}$. Now $A_1 \cup_2 B_1 = \{(a, 0.40, 0.20, 0.40)\}$. Hence $C_1 = \{(a, 0.20, 0.20, 0.40)\}$.

Now $\mathcal{D} = \mathcal{A} \odot_2 (\mathcal{A} \oplus_2 \mathcal{B}) = \mathcal{A} \odot_2 \mathcal{C}$. We have $D_1 = \{(a, 0.20, 0.20, 0.40)\}$ and $D_2 = \{(a, 0.60, 0.25, 0.10)\}$. But D_1 is not identical with A_1 (check the truth value). Hence $\mathcal{D} \neq \mathcal{A}$ and this absorption law does not hold.

Remark 20.

Let us go back to the previous theorem:

1) Consider just the external (possibility) layers of both $\mathcal{A} \oplus_2 (\mathcal{B} \odot_2 \mathcal{C})$ and $(\mathcal{A} \oplus_2 \mathcal{B}) \odot_2 (\mathcal{A} \oplus_2 \mathcal{C})$ (for arbitrary T2WRNSs). On the left-hand side, we obtain:

First, $B_2 \cup_2 C_2$. Then (after imposing $\oplus_2$) we get $A_2 \cap_2 (B_2 \cup_2 C_2) = (A_2 \cap_2 B_2) \cup_2 (A_2 \cap_2 C_2)$.

On the right-hand side, we obtain:

First, in brackets: $(A_2 \cap_2 B_2)$ and $(A_2 \cap_2 C_2)$. Then, after imposing $\odot_2$, we get $(A_2 \cap_2 B_2) \cup_2 (A_2 \cap_2 C_2)$.

These calculations mean that in some sense $\oplus_2\odot_2$-distributivity holds: but only for the upper layer. As we could see in our counterexample, necessity ranges can be different.

2) Consider just the external (possibility) layers of both $\mathcal{A} \odot_2 (\mathcal{B} \oplus_2 \mathcal{C})$ and $(\mathcal{A} \odot_2 \mathcal{B}) \oplus_2 (\mathcal{A} \odot_2 \mathcal{C})$ (for arbitrary T2WRNSs). On the left-hand side, we obtain:

First, $B_2 \cap_2 C_2$. Then (after imposing $\odot_2$) we get $A_2 \cup_2 (B_2 \cap_2 C_2) = (A_2 \cup_2 B_2) \cap_2 (A_2 \cup_2 C_2)$.

On the right-hand side, we obtain:

First, in brackets: $(A_2 \cup_2 B_2)$ and $(A_2 \cup_2 C_2)$. Then, after imposing $\odot_2$, we get $(A_2 \cup_2 B_2) \cap_2 (A_2 \cup_2 C_2)$.

These calculations mean that in some sense $\odot_2\oplus_2$-distributivity holds: but only for the upper layer. As we could see in our counterexample, necessity ranges can be different.

3) Consider just the external layers of both $\mathcal{A}$ and $\mathcal{A} \odot_2 (\mathcal{A} \oplus_2 \mathcal{B})$. On the right-hand side, we obtain: first, in brackets, it is $A_2 \cap_2 B_2$. Then we impose $\odot_2$ and we obtain $A_2 \cup_2 (A_2 \cap_2 B_2) = A_2$. These calculations show that $\odot_2\oplus_2$-absorption law holds in the upper layer.

New results: type-3 weak rough neutrosophic sets

It is always interesting when the obtained non-classical structures have untypical algebraic properties. Clearly, in the vast majority of cases "non-classical" sets (together with their operations) form de Morgan algebra. This is true for fuzzy, intuitionistic fuzzy or neutrosophic sets (the latter with type-1 or type-2 operators and inclusion).

However, in [16] the authors prove that type-3 inclusion for neutrosophic sets leads to a significantly weaker structure. This can be formulated as follows.

Definition 21 [16]

Suppose that we have a bounded lattice of the form $(M, \wedge, \vee, 0, 1, -)$. Assume that is satisfies the following additional axioms (for every $x, y \in M$):

1) $x = (x^-)^-$

2) $(x \wedge y)^- = x^- \vee y^-$

3) $1^- = 0$

Then we say that this structure is a *generalized de Morgan algebra*.

Remark 22.

Note that in the definition above we do not require distributivity. In [16] the authors prove that the set of all neutrosophic sets on X with type-3 operations forms a generalized de Morgan algebra. They show (in their Example 5.1) that distributivity laws do not hold. Hence the whole system is not a de Morgan algebra.

Keeping in mind these facts, we propose the following construction.

Definition 23.

Suppose that A_1 and A_2 are two neutrosophic sets on some non-empty universe X. Assume that $A_1 \subseteq_3 A_2$. Then the object of the form $\mathcal{A} = (A_1, A_2)$ is known as a *type-3 weak rough neutrosophic set* on X. For brevity, we can write T3WRN.

Example 24. (based on Example 3.1 from [16])

Suppose that $X = \{x, y, z\}$. Let A_1, A_2, B_1, B_2 be four neutrosophic sets on X defined in the following way:

$$A_1 = \{(x, 0.50, 0.30, 0.10), (y, 0.30, 0.10, 0.10), (z, 0.25, 0.15, 0.80)\}$$

$$A_2 = \{(x, 0.50, 0.40, 0.10), (y, 0.40, 0.05, 0.10), (z, 0.35, 0.25, 0.60)\}$$

$$B_1 = \{(x, 0.60, 0.10, 0.25), (y, 0.50, 0.05, 0.20), (z, 0.85, 0.25, 0.15)\}$$

$$B_2 = \{(x, 0.70, 0.05, 0.25), (y, 0.50, 0.10, 0.10), (z, 0.90, 0.15, 0.10)\}$$

Now we see that $A_1 \subseteq_3 A_2$, so the ordered triple $\mathcal{A} = (A_1, A_2)$ is a T3WRN on X. Note that $A_1 \nsubseteq_2 A_2$ because $i_A(y) > i_B(y)$. Moreover, $B_1 \subseteq_3 B_2$, so $\mathcal{B} = (B_1, B_2)$ is a T3WRN on X. However, $B_1 \nsubseteq_2 B_2$ because $i_B(x) > i_B(y)$.

As for the operations on these objects and relations between them, they will be listed below. Clearly, the idea is that if we use type-3 inclusion "inside" T3WRNs, then their intersection and union are also based on the appropriate type-3 connectives.

Remark 25.

Take the same space as in Example 19. Consider $\mathring{A} = (A_1, A_2^c)$. We see that $A_2^c = \{(x, 0.10, 0.60, 0.50), (y, 0.10, 0.95, 0.60)\}$. One can easily check that both $A_1 \cap_2 A_2^c$ and $A_1 \cap_3 A_2^c$ are different than $\breve{\emptyset}_2$. This observation is important because we can compare the situation with the one typical for crisp sets. Suppose that A_1 and A_2 are classical sets and the ordered pair (A_1, A_2) forms a weak rough set on X. Then $\mathring{A} = (A_1, A_2^c)$ forms an intuitionistic set on X (in the sense of Coker). It means that the intersection of both components is empty. As we can see, this relationship is not true for neutrosophic (or even fuzzy) sets.

Definition 26.

Suppose that X is a non-empty universe and $\mathcal{A}, \mathcal{B}$ are two T3WRNs on X. We define their:

1) *Union* as a new T3WRN set $\mathcal{C}$ on X of the form: $\mathcal{C} = (C_1, C_2) = (A_1 \cup_3 B_1, A_2 \cup_3 B_2)$.

2) *Intersection* as a new T3WRN set $\mathcal{C}$ on X of the form: $\mathcal{D} = (D_1, D_2) = (A_1 \cap_3 B_1, A_2 \cap_3 B_2)$.

3) *Complement* (of e.g. $\mathcal{A}$) as a new T3WRN set of the form $\mathcal{A}^c = (A_2^c, A_1^c)$, where on the right-hand side, we have standard neutrosophic complement.

Moreover, we say that $\mathcal{A} \subseteq_2 \mathcal{B}$ if and only if $A_1 \subseteq_3 B_1$ and $A_2 \subseteq_3 B_2$.

As for the definitions of the empty and universal T3WRN, we agree on the same approach that was presented in Definitions 6 and 7.

The crucial thing is the algebraic structure of T3WRNs. We formulate the following series of lemmas, theorems and conclusions:

Lemma 27.

Let X be a non-empty universe. Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ belong to $P_{T3WRN}(X)$. Then the following properties hold:

1) $(\mathcal{A} \cap_3 \mathcal{B}) \subseteq_3 \mathcal{A}$ and $(\mathcal{A} \cap_3 \mathcal{B}) \subseteq_3 \mathcal{B}$.

2) $\mathcal{A} \subseteq_3 (\mathcal{A} \cup_3 \mathcal{B})$ and $\mathcal{B} \subseteq_3 (\mathcal{A} \cup_3 \mathcal{B})$.

Proof. Take 1). Suppose that $\mathcal{A} \subseteq_3 \mathcal{B}$. Then the result of the intersection is $\mathcal{A}$ so the conclusion is obvious. If $\mathcal{B} \subseteq_3 \mathcal{A}$ then it is analogous. Otherwise, we obtain $\mathcal{C} = \mathcal{A} \cap_3 \mathcal{B}$ where (in particular) $t_{C_1}(x) = \min\left(t_{A_1}(x), t_{B_1}(x)\right) \leq t_{A_1}(x)$, $i_{C_1}(x) = 1$ and $f_{C_1}(x) = \max\left(f_{A_1}(x), f_{B_1}(x)\right) \geq f_{A_1}(x)$. (for every $x \in X$). But then $C_1 \subseteq_3 A_1$. In the same way we prove that $C_2 \subseteq_3 A_2$, hence $\mathcal{C} \subseteq_3 \mathcal{A}$. In an analogous manner we prove the second part of 1).

The proof of 2) is similar: suppose that $\mathcal{A} \subseteq_3 \mathcal{B}$. Then the result of the union operation is $\mathcal{B}$. Hence the conclusion is natural. If $\mathcal{B} \subseteq_3 \mathcal{A}$ then we get that $\mathcal{A} \subseteq_3 \mathcal{A}$ which is obvious. Otherwise we obtain $\mathcal{C} = \mathcal{A} \cup_3 \mathcal{B}$ where (in particular) $t_{C_1}(x) = \max\left(t_{A_1}(x), t_{B_1}(x)\right) \geq t_{A_1}(x)$, $i_{C_1}(x) = 0$ and $f_{C_1}(x) = \min\left(f_{A_1}(x), f_{B_1}(x)\right) \leq f_{A_1}(x)$. (for every $x \in X$). But then $\mathcal{A}_1 \subseteq_3 C_1$. In the same way we can show that $\mathcal{A}_2 \subseteq_3 C_2$. Thus, $\mathcal{A} \subseteq_3 \mathcal{C}$. The second part of 2) is similar.

Theorem 28.

Let X be a non-empty universe. Suppose that $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ belong to $P_{T3WRN}(X)$. Then the following properties are true:

1) $\mathcal{A} \cup_3 \mathcal{B} = \mathcal{B} \cup_3 \mathcal{A}$ and $\mathcal{A} \cap_3 \mathcal{B} = \mathcal{B} \cap_3 \mathcal{A}$ (*commutativity*).

2) $\mathcal{A} \cup_3 \mathcal{A} = \mathcal{A}$ and $\mathcal{A} \cap_3 \mathcal{A} = \mathcal{A}$ (*idempotence*).

3) $\mathcal{A} \cup_3 (\mathcal{A} \cap_3 \mathcal{B}) = \mathcal{A}$ and $\mathcal{A} \cap_3 (\mathcal{A} \cup_3 \mathcal{B}) = \mathcal{A}$ (*absorption laws*).

4) $\mathcal{A} \cup_3 (\mathcal{B} \cup_3 \mathcal{C}) = (\mathcal{A} \cup_3 \mathcal{B}) \cup_3 \mathcal{C}$ and $\mathcal{A} \cap_3 (\mathcal{B} \cap_3 \mathcal{C}) = (\mathcal{A} \cap_3 \mathcal{B}) \cap_3 \mathcal{C}$ (*associativity*)

5) $(\mathcal{A} \cup_3 \mathcal{B})^c = \mathcal{A}^c \cap_3 \mathcal{B}^c$ and $(\mathcal{A} \cap_3 \mathcal{B})^c = \mathcal{A}^c \cup_3 \mathcal{B}^c$ (*de Morgan laws*).

6) $(\mathcal{A}^c)^c = \mathcal{A}$ (*double negation*).

7) $\mathcal{A} \cap_3 \check{\emptyset} = \check{\emptyset}$, $\mathcal{A} \cup_3 \check{\emptyset} = \mathcal{A}$, $\mathcal{A} \cap_3 \check{\mathcal{X}} = \mathcal{A}$, $\mathcal{A} \cup_3 \check{\mathcal{X}} = \check{\mathcal{X}}$.

Proof. The reader can easily predict that the whole detailed proof would require many subtle calculations involving all the possible situations. However, we can replace them with some logical reasoning. Recall that in [16] the authors prove Proposition 5.1. It says that type-3 relation is a partial order. In Proposition 5.2. they repeat the definition of type-3 union and intersection and show that they have the properties of supremum and infimum. Hence, they conclude that the algebra of neutrosophic sets with type-3 operations is a lattice. They meticulously check a dozen of cases. The proof is “order-theoretic” but we know that lattices can be defined both in terms of order and algebraic axioms. Hence, the authors conclude that both type-3 union and type-3 intersection are idempotent and commutative. Moreover, they satisfy absorption laws and associativity. Double negation holds too. In Proposition 5.4 they prove that de Morgan laws are satisfied (again, their approach is “order-theoretic”). This allows them to state Theorem 5.1 which tells us that neutrosophic sets with type-3 operations form a generalized de Morgan algebra.

One can easily conclude that all the calculations described above can be used in our context. The whole thing to do is to repeat them for both layers (that is, for A_1, B_1, C_1 and A_2, B_2, C_2) separately. This would give us the expected thesis for T3WRNs as such.

However, some of the less trivial calculations are performed directly below:

3) Consider $\mathcal{A} \cup_3 (\mathcal{A} \cap_3 \mathcal{B})$. Let us limit to the lower layer (that is, to A_1 and B_1). Hence, we can discuss $A_1 \cup_3 (A_1 \cap_3 B_1)$. From Lemma 27 we already know that $C_1 = A_1 \cap_3 B_1 \subseteq_3 A_1$. Thus $A_1 \cup_3 C_1 = A_1$ (according to the definition of type-3 union). The same for the upper layer. The proof of the second absorption law is similar.

4) Let us think about type-3 union operator. Consider $\mathcal{A} \cup_3 (\mathcal{B} \cup_3 \mathcal{C})$ and think just about the internal layer. There are many cases to check but they are all similar. For example, consider the following situation: that $A_1 \subseteq_3 B_1 \subseteq_3 C_1$. Then $A_1 \cup_3 (B_1 \cup_3 C_1) = A_1 \cup_3 C_1$. Then $A_1 \cup_3 C_1 = C_1$. Now consider $(A_1 \cup_3 B_1) \cup_3 C_1$. Then $(A_1 \cup_3 B_1) \cup_3 C_1 = B_1 \cup_3 C_1 = C_1$.

5) Consider $(\mathcal{A} \cup_3 \mathcal{B})^c$ and think just about the internal layer. Suppose that $A_1 \subseteq_3 B_1$. Thus $A_1 \cup_3 B_1 = B_1$. Now $(A_1 \cup_3 B_1)^c = B_1^c$. Think about $A_1^c \cap_3 B_1^c$. If $A_1 \subseteq_3 B_1$, then $B_1^c \subseteq_3 A_1^c$, so $A_1^c \cap_3 B_1^c = B_1^c$. If $B_1 \subseteq_3 A_1$, then the reasoning is similar. Otherwise, we have $A_1 \cup_3 B_1 = C_1$, where $t_{C_1}(x) = \max\left(t_{A_1}(x), t_{B_1}(x)\right)$, $i_{C_1}(x) = 0$ and $f_{C_1}(x) = \min\left(f_{A_1}(x), f_{B_1}(x)\right)$. (for every $x \in X$). Now consider $(A_1 \cup_3 B_1)^c = C_1^c = D_1$. We have $t_{D_1}(x) = \min\left(f_{A_1}(x), f_{B_1}(x)\right)$, $i_{C_1}(x) = 1$ and $f_{C_1}(x) = \max\left(f_{A_1}(x), f_{B_1}(x)\right)$. Now recall the fact that in A_1^c we have $t_{A_1^c}(x) = f_{A_1}(x), i_{A_1^c}(x) = 1 - i_{A_1}(x)$ and $f_{A_1^c}(x) = t_{A_1}(x)$. In the same way we evaluate logical values in case of B_1^c. We remember that A_1 is not type-3 contained in B_1 nor B_1 in A_1. The same can be said about their complements. Hence the intersection of these complements gives exactly the same results as the evaluation of D_1.

The whole reasoning can be repeated for A_2 and B_2.

Conclusion 29.

Let X be a non-empty universe. Then $P_{T3WRN}(X)$ together with appropriate type-3 operations, type-3 inclusion and type-3 empty and universal set is a generalized de Morgan algebra.

Remark 30.

In general, the algebra of T3WRN sets does not satisfy distributivity laws. Take $X = \{x, y\}$ and T3WRNs $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of the form:

$$A_1 = \{(x, 0.50, 0.30, 0.10), (y, 0.30, 0.10, 0.10)\}$$

$$A_2 = \{(x, 0.50, 0.40, 0.10), (y, 0.40, 0.05, 0.10)\}$$

$$B_1 = \{(x, 0.60, 0.10, 0.25), (y, 0.50, 0.05, 0.20)\}$$

$$B_2 = \{(x, 0.70, 0.05, 0.25), (y, 0.50, 0.10, 0.10)\}$$

$$C_1 = \{(x, 0.30, 0.20, 0.15), (y, 0.40, 0.50, 0.10)\}$$

$$C_2 = \{(x, 0.30, 0.30, 0.15), (y, 0.45, 0.50, 0.05)\}$$

Calculate:

$B_1 \cap_3 C_1 = \{(x, 0.30, 0.20, 0.15), (y, 0.40, 0.50, 0.10)\} = C_1$ (note that $C_1 \subseteq_3 B_1$).

$A_1 \cup_3 (B_1 \cap_3 C_1) = A_1 \cup_3 C_1 = \{(x, 0.50, 0.00, 0.10), (y, 0.40, 0.00, 0.10)\}$.

$D_1 = (A_1 \cup_3 B_1) = \{(x, 0.60, 0.10, 0.25), (y, 0.50, 0.05, 0.20)\} = B_1$ (note that $A_1 \subseteq_3 B_1$).

$E_1 = (A_1 \cup_3 C_1) = \{(x, 0.50, 0.00, 0.10), (y, 0.40, 0.00, 0.10)\}$.

$D_1 \cap_3 E_1 = \{(x, 0.60, 0.10, 0.25), (y, 0.50, 0.05, 0.20)\} = D_1$ (note that $E_1 \subseteq_3 D_1$).

However, this means that $A_1 \cup_3 (B_1 \cap_3 C_1)$ is not identical with $(A_1 \cup_3 B_1) \cap_3 (A_1 \cup_3 C_1)$. Clearly, this observation is sufficient to say that in general $\mathcal{A} \cup_3 (\mathcal{B} \cap_3 \mathcal{C}) \neq (\mathcal{A} \cup_3 \mathcal{B}) \cap_3 (\mathcal{A} \cup_3 \mathcal{C})$.

Conclusion and future work

In this chapter we defined neutrosophic weak rough sets of type 2 and type 3. We analyzed their basic algebraic properties, giving many examples and counter-examples. However, there are still several things that should be investigated. We did not touch them here due to the lack of space. Among these tasks we have:

1) To define additional operators $\oplus_3$ and $\odot_3$ in an appropriate manner. Our hypothesis is that they will not satisfy associativity laws (note that in case of $\oplus_2$ and $\odot_2$ we used distributivity of $\cup_2$ and $\cap_2$ to prove associativity on the "two-layered" level). Clearly, we presume that they are not distributive and that they satisfy only one absorption law.

2) To define and study type-2 weak rough neutrosophic topological spaces. Here we expect that the results will be similar e.g. to crisp weak rough (interval-valued, double) spaces.

3) To define and study type-3 neutrosophic and type-3 weak rough neutrosophic topological spaces. This can be interesting because there are some theorems in topology where we use distributivity (e.g. to prove that a subspace is indeed a topology or to prove that regular open sets form complete Boolean algebra). But this property, as we already know, is no longer true for type-3 sets.

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Chapter 9

Hybridization of Neutrosophic theory and Intuitionistic anti Q-fuzzy M-semigroup

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ABSTRACT

The combination of fuzzy logic and algebraic structures has resulted in the creation of sophisticated mathematical frameworks adept at managing uncertainty, ambiguity, and multi-parameter decision-making contexts. This research formulates an innovative algebraic framework known as the intuitionistic anti Q-fuzzy M-semigroup, which amalgamates intuitionistic fuzzy theory, Q-parameterization, anti-fuzzy logic, and transformation-based semigroup structures. The suggested method is different from other models since it uses a transformation mechanism called the Anti Q-Fuzzy Transform Operator (AQFTO) to look at stability and invariance properties. Closure, power stability, homomorphic behaviour, and level subset characterisation are fundamental algebraic results that are established. Furthermore, the proposed framework's ability to model incomplete, contradictory, and indeterminate information is improved by the integration of neutrophilic theory, which thereby broadens its applicability in complex systems. Furthermore, ideal-based properties and structural equivalence conditions are derived. The effectiveness and tractability of the proposed framework are illustrated through a comparative analysis with neutrosophic and Pythagorean fuzzy models. The results suggest that the model offers a computationally efficient and mathematically consistent method for managing multi-parameter uncertainty in algebraic systems. The results significantly enhance current fuzzy algebraic models and pave the way for novel applications in intelligent systems and decision sciences.

Keywords: Neutrosophic fuzzy set, Interval-valued Neutrosophic fuzzy M-semigroup, Intuitionistic fuzzy set, Anti-fuzzy set, Q-fuzzy structure, M-semigroup, Level subsets, Algebraic structures.

INTRODUCTION

Zadeh proposed using fuzzy sets (FSs) [23] to describe and reason about uncertainty instead of clear bounds. This allowed for a more sophisticated understanding of membership. This created a more realistic picture of complicated systems where things and concepts may not fit cleanly into categories. Artificial intelligence, control systems, pattern recognition, and data analysis are just a few of the areas that have benefited from the development of fuzzy sets. Fuzzy logic, based on fuzzy sets, formally expressed and manipulated inaccurate information. It allowed fuzzy control systems to manage uncertain and nonlinear situations better than standard methods.

This theory helps handle uncertainty and imprecision, making real-world problems more durable and flexible. In 1975, Zadeh [23] introduced interval-valued fuzzy sets. Contemporary mathematical research focuses on algebraic system uncertainty. Graded membership was further extended to intuitionistic fuzzy sets. Parameterisation enabled multi-dimensional uncertainty modelling in Q-fuzzy sets, while anti-fuzzy sets reversed membership interpretation. However, M-semigroups add transformation-based operations to classical semigroups. The integration of these theories yields intuitionistic anti Q-fuzzy M-semigroups, which may represent inverse membership behaviour in algebraic systems under parametric uncertainty. In 1986, Krassimir T. Atanassov [1] created intuitionistic fuzzy sets, which added membership and non-membership functions and a hesitation margin to classical fuzzy sets. This advancement greatly enhanced mathematical and applied system uncertainty handling. Krassimir T. Atanassov [1] presented novel operations on intuitionistic fuzzy sets, notably intuitionistic Q-fuzzy sets, which allow parameterisation by Q. These Q-fuzzy extensions make modelling multi-criteria and multi-dimensional uncertainty more flexible, making fuzzy systems more useful in decision-making and algebraic analysis. In addition, fuzzy environments have been used to study algebraic structures like semigroups. The transformation-based M-semigroups developed by L. Lakshmanan [3] provide a generalised basis for analysing algebraic behaviour. A. L. Narayanan and A. R. Meenakshi [4] added fuzziness to transformation-based algebraic systems by applying this approach to fuzzy M-semigroups.

Fuzzy theory naturally led to the study of anti-fuzzy structures, where membership represents exclusion or behaviour. Sivaramakrishnan et al developed Neutrosophic interval-valued anti fuzzy linear Space and Interval-valued neutrosophic fuzzy M-semigroup [11,12,13]. Vijayabalaji et al contributed many hybrid fields in the theory of uncertainty ranging from rough sets hybridization with quadtrees to Antifuzzy M-semigroup [14,15,16,17,18,19,20,21,22] S. Vijayabalaji and S. Sivaramakrishnan introduced and studied anti fuzzy M-semigroups and their homomorphic features. Their research laid the groundwork for anti-fuzzy algebraic system structure and behaviour. Several contexts have investigated the integration of intuitionistic fuzzy logic with algebraic systems. On interval-valued intuitionistic fuzzy Lie ideals shows how intuitionistic fuzzy notions can be used to complex algebraic frameworks like Lie algebras.

There is a void in the literature on the unified study of: Fuzzy sets that are intuitive Fuzzy structures with parameters, Anti-fuzzy logic , M-semigroup transformations. In particular, integrating these notions into a logical framework is understudied. Based on this gap, this work tries to create a theory of intuitionistic anti Q-fuzzy M-semigroups that combines the strengths of the foregoing approaches. This model includes multi-dimensional uncertainty, inverse membership behaviour, and transformation-based algebraic operations. In addition, a new Anti Q-Fuzzy Transform Operator (AQFTO) offers a new perspective on stability, invariance, and structural features in these systems. These advances increased mathematical structural imprecision representation. Multi-dimensional uncertainty was captured via parameterisation in later expansions like Q-fuzzy sets. During this time, anti-fuzzy ideas were introduced to explain inverse membership behaviour, providing a complementary perspective in cases of exclusion-based reasoning Under fuzzy conditions, algebraic semigroups have been widely generalised. Using transformation-based procedures, M-semigroups provide a flexible framework for exploring structural features. Previous studies on fuzzy and anti-fuzzy M-semigroups have yielded key homomorphism and algebraic behaviour results.

Despite these advances, intuitionistic fuzzy sets, Q-parameterization, anti-fuzzy logic, and M-semigroup structures have not been properly integrated. Transformation-based mechanisms for hybrid system stability and invariance analysis are lacking. This study proposes a unified framework of intuitionistic anti Q-fuzzy M-semigroups and a unique transform operator to fill this gap. This approach improves fuzzy algebra theory and gives tools for analysing dynamic behaviour in uncertain systems. Intuitionistic fuzzy sets use membership and non-membership functions to model uncertainty, but they don't account for indeterminacy, which is common in real-world systems. To get around this limit, neutrosophic set theory adds a separate indeterminacy part. The proposed framework has the potential to represent partial, inconsistent, and indeterminate information through neutrosophic theory, enhancing its applicability to complex systems.

Semigroups are essential for comprehending binary operations [3]. Semigroups simplify algebraic structure analysis by concentrating exclusively on the associative property. Semigroups are used in algebraic structure research. Mathematicians can learn about monoids and groups by studying semigroups. A monoid is a semigroup that has an identity element. A group is a monoid that has an inverse for each element. Semigroups can represent finite state machines, with the binary operation composing transitions [4]. Researchers can assess computational models' performance by researching semigroups. Semigroups are essential to automata theory [4].

An M-semigroup is an algebraic structure that exists in-between a monoid and a semigroup., emphasising identity element features. M-semigroup theory adds a binary operation condition to classical semigroup theory. Lakshmanan's M-semigroup [3] was a major advance in the mathematical field. Semigroups, algebraic structures with a set and an associative binary operation, are generalised to M-semigroups. Narayanan et al. [4] expanded M-semigroups via fuzzy M-semigroup theory, building on Lakshmanan's work They are useful when a set's complement is important. Anti-fuzzy M-semigroups tackle semigroup structure uncertainty and imprecision by joining M-semigroups and anti-fuzzy sets. By adding anti-fuzzy sets, Vijayabalji and Sivaramakrishnan [14,15] expanded standard M-semigroup theory. They address semigroup structure uncertainty and imprecision and homomorphism between two anti-fuzzy M-semigroups. In addition, Sivaramakrishnan et al. [11] introduced the anti Q-fuzzy M-semigroup. By offering a more complete and nuanced approach to dealing with uncertainty, Atanassov's groundbreaking work on intuitionistic fuzzy sets (IFSs) [1] revolutionised the field of fuzzy logic. IFS provided a more realistic and flexible depiction of real-world problems by including membership, non-membership, and hesitation or indeterminacy for each element in a set, improving decision-making and problem-solving methodologies. Smarandache [7] introduced neutrosophic sets (NSs) in the late 1990s as a new way to extend classical sets, FSs, and IFSs. In actual life, indeterminacy, ambiguity, and inconsistency are common. This new notion captures and represents them. Neutrosophic sets have been extensively studied and applied in several disciplines [5, 6, 7, 8, 9, 10]. Several specialised forms of neutrosophic sets have emerged from detailed documentation of their core ideas [2]. Single-valued Neutrosophic sets are more practical for real-world applications [6, 8], and rough Neutrosophic sets integrate rough sets with Neutrosophic theory to address uncertainty and incompleteness in information systems [5]. Neutrosophic theory is a reliable framework for handling ambiguous, partial, and inconsistent information in science and engineering due to its constant evolution and refinement.

BACKGROUND

Definition 1 [4]. A M-semigroup, denoted as MS, is a M-semigroup that fulfills the following conditions:

1. There is at least one left identity $e \in M$ such that $e\,m = m$, for all $m \in M$.
2. For every $m \in M$, there is a unique left identity, represented as e_m, such that $m\,e_m = m$, i.e., e_m is a two-sided identity for m.

An M-semigroup is a semigroup equipped with a transformation operator: $M: S \to S$ such that $M(xy) = M(x)M(y)$

Definition 2 [13].

An intuitionistic anti Q-fuzzy set A = {α_a, β_a} in M is called an intuitionistic anti Q-fuzzy M-semigroup of M (briefly, IAQFMS) if

1. $\alpha_a(xy,q) \leq \max\{\alpha_a(x,q), \alpha_a(y,q)\}$,
2. $\beta_a(xy) \geq \min\{\beta_a(x,q), \beta_a(y,q)\}, for\ all\ x,y \in M\ and\ q \in Q,$
3. $\alpha_a(e,q) = 0\ and\ \beta_a(e,q) = 1, for\ every\ left\ identity\ e \in M.$

Definition 3 [4]. Consider M be a M-semigroup. Let $\varpi_M: M \to [0,1]$ be a fuzzy set. Then the pair (M, ϖ_M) is referred to as a fuzzy M-semigroup if

1. $\varpi_M\,(m_1\,m_2) \geq \min\{\varpi_M\,(m_1), \varpi_M(m_2)\}$, for all m_1, m_2 belonging to M,
2. ϖ_M € $= 1$, ∀ e in M.

Definition 4 [20]. Let M be a M-semigroup and $\overline{\varpi}_M$ be an interval-valued fuzzy M-semigroup. suppose the following conditions hold: For all $x, y \in M$,

1. $\overline{\varpi}_M\,(xy) \geq \min\{\overline{\varpi}_M\,(x), \overline{\varpi}_M\,(y)\}$,
2. $\overline{\varpi}_M$ € $= \overline{1} = [1,1]$, for every left identity e in M.

Then $\overline{\varpi}_M$ is referred to as anIVFMS on M and is denoted by $(M, \overline{\varpi}_M)$.

Definition 5 [11]. LetX$_{NS}$ be the universal set. A NSis a set of the form Ω= {m,ξ_Ω (m), Ψ_Ω (m), ζ_Ω (m) | m $\in X_{NS}$ } and denoted by Ω= (ξ_Ω (m), Ψ_Ω (m), ζ_Ω (m)), where $\xi : X_{NS} \to [0,1]$, $\Psi : X_{NS} \to [0,1]$ and $\zeta: X_{NS} \to [0,1]$ represent the degree of truth - membership and indeterminacy - membership and false - membership of the element m $\in X_{NS}$ in Ω and $0 \leq \xi_\Omega\,(m) + \Psi_\Omega\,(m) + \zeta_\Omega\,(m) \leq 3$.

Definition 6 [11]. Suppose that MS. A (NS) $\bar{\mathcal{A}} = \left(\overline{\varpi}_M, \overline{\eta}_M, \overline{\delta}_M\right)$ is known to be a IVNFMS of MS. If for all $m_1, m_2 \in$ M it holds.

(i)$\overline{\varpi}_M\,(m_1\,m_2) \geq \min\{\overline{\varpi}_M\,(m_1), \overline{\varpi}_M(m_2)\}$,

(ii)$\overline{\eta}_M(m_1 m_2) \geq \min\left\{\overline{\eta}_M(m_1), \overline{\eta}_M(m_2)\right\}$, (iii)$\overline{\delta}_M(m_1 m_2) \leq \max\{\overline{\delta}_M(m_1), \overline{\delta}_M(m_2)\}$,

(iv)$\bar{\varpi}_M(e) = [1, 1] = \bar{1}$ for every left identity e in M,

(v)$\bar{\eta}_M(e) = [1, 1] = \bar{1}$for every left identity e in M,

(vi)$\bar{\delta}_M(e) = [0, 0] = \bar{0}$ for every left identity e in M. If it satisfies the above mentioned (vi) conditions we call that at interval valued neutrosophic fuzzy M-semigroup.

Definition 7 [7]. A neutrosophic set A in a universe S is defined as $A = \{ (x, T_A(x), I_A(x), F_A(x)) \mid x \in S \}$ where

$T_A(x) =$ degree of truth − membership ,

$I_A(x) =$ degree of indeterminacy ,

$F_A(x)$degree of falsity − membership such that: $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

Definition 8 [8]. Let Q be a parameter set. A neutrosophic Q-fuzzy set is defined as:

$A = \{(x, q, T_A(x, q), I_A(x, q), F_A(x, q))\}$ This allows modelling of multi-criteria indeterminate uncertainty.

Neutrosophic Anti Q-Fuzzy M-Semigroup

This section introduces a novel algebraic model that combines intuitionistic fuzzy theory, Q-fuzzy parameterisation, anti-fuzzy logic, and M-semigroup transformations. The development of the Anti Q-Fuzzy Transform Operator (AQFTO) allows for systematic examination of transformation effects on fuzzy algebraic structures, which is the key innovation. The key contributions of this work are summarised as follows:

- Developed a coherent framework for intuitionistic anti-Q-fuzzy M-semigroups
- Implemented AQFTO for transformation-based analysis.
  Developed essential algebraic properties and structural theorems.
- Developed ideal-based and level subset characterisations.
- Comparison with neutrosophic and Pythagorean fuzzy models.
- This framework advances fuzzy algebra by adding transformation-driven analysis into a multi-parameter anti-fuzzy environment.

Although existing fuzzy algebraic models adequately address specific aspects of uncertainty, they have numerous shortcomings when viewed in a broader perspective. Intuitionistic fuzzy sets encapsulate dual uncertainty but do not include parameterisation. Q-fuzzy sets enable multi-criteria flexibility but do not support inverse membership behaviour. Anti-fuzzy structures handle this inverse interpretation; however, they are not commonly integrated with parameterised or transformation-based frameworks.

Furthermore, present models are insufficient for operator-driven examination of stability and structural invariance. This constraint limits their application in dynamic systems where transformations play an important role. As a result, there is a need for a comprehensive algebraic framework that incorporates these ideas while being mathematically tractable and enabling transformation-based analysis.

Definition 9. Let S be a semigroup and Q a non-empty parameter set. An intuitionistic anti Q-fuzzy set A is defined through membership and non-membership functions satisfying appropriate constraints.

The defining condition for the proposed structure is given by:

$$\mu_A(ab, q) \leq \max\{\mu_A(a, q), \mu_A(b, q)\},$$

$$\nu_A(ab, q) \geq \min\{\nu_A(a, q), \nu_A(b, q)\}$$

$for\ all\ a, b \in S\ , q \in Q.$

Definition 10. A is called a Neutrosophic Anti Q- Fuzzy set M-semigroup if

$$T_A\,(xy, q) \leq \max(T_A\,(x, q), T_A\,(y, q))$$

$$F_A\ (xy, q) \geq \min\left(F_A\ (x, q), F_A\ (y, q)\right)$$
$$I_A\ (xy, q)\ is\ independent$$

Definition 11. A mapping $T: S \rightarrow S$ is defined as an Anti Q-Fuzzy Transform Operator if it preserves the anti fuzzy ordering in the following sense:

$$\mu_A(T(x), q) \leq \mu_A(x, q),$$
$$\nu_A(T(x), q) \geq \nu_A(x, q)$$

This operator acts as a stabilizing mechanism within the algebraic structure.

Definition 12. Let S be an M-semigroup and Q a parameter set. A Neutrosophic Anti Q-Fuzzy Set (NAQF) A is: A Neutrosophic Anti Q-Fuzzy Set S is defined as: $A = \{\left(x, q, T_A(x, q), I_A(x, q), F_A(x, q)\right) \mid x \in S, q \in Q\}$ where: $T_A, I_A, F_A: S \times Q \rightarrow [0,1]$

The following propositions of the theorem are relatively trivial, thus we merely provide statements for them.

Theorem 1. The intersection of two NAQF M-semigroups is also a NAQF M-semigroup(Closure Property). To validate this theorem max/min operators preserve anti-condition and independent property I does not violate structure

Theorem 2. If A is NAQFMS, then T(A) is also NAQFMS (Transform Invariance). The meaning for this is structure is stable under transformation. This property is crucial for dynamic systems.

Theorem 3.: Define $A_{\alpha,\beta,\gamma} = \{x \mid T \leq \alpha,\ I \geq \beta,\ F \geq \gamma\}$ $Then\ A\ is\ NAQFMS\ iff\ all\ level\ subsets\ are\ classical\ M - subsemigroups.$ That is A is NAQF M-semigroup if and only if all level subsets are classical M-subsemigroups

Theorem 4. If f:S→S′ is a homomorphism, then f(A) preserves NAQF structure (homomorphism invariance)

Definition 13. A neutrosophic Q-fuzzy set A in a semigroup S is called a neutrosophic anti Q-fuzzy M-semigroup if for all $a, b \in S, q \in Q$

$$T_A\ (ab, q) \leq \max\{T_A\ (a, q), T_A\ (b, q)\}$$
$$I_A\ (ab, q) \geq \min\{I_A\ (a, q), I_A\ (b, q)\}$$
$$F_A\ (ab, q) \geq \min\{F_A\ (a, q), F_A\ (b, q)\}$$

where Truth behaves as anti-membership and Indeterminacy and falsity increase under operation.

Definition 14. A mapping $T: S \rightarrow S$ is called a Neutrosophic Anti Q-Fuzzy Transform Operator (NAQFTO) if:

$$T_A\ (T(x), q) \leq T_A\ (x, q)$$
$$I_A\ (T(x), q) \geq I_A\ (x, q)$$
$$F_A\ (T(x), q) \geq F_A\ (x, q)$$

Theorem 5. If A is a neutrosophic anti Q-fuzzy M-semigroup and T is a NAQFTO, then the structure is invariant under transformation. This statement is known as neutrosophic stability.

Proof:

From Definition 14, we know that

$$T_A\ (T(x), q) \leq T_A\ (x, q)$$
$$I_A\ (T(x), q) \geq I_A\ (x, q)$$
$$F_A\ (T(x), q) \geq F_A\ (x, q)$$

Thus transformed elements preserve the anti-fuzzy neutrosophic ordering. Hence invariance holds.

Theorem 6. Closure Under Neutrosophic Transform Composition

Let A be a neutrosophic anti Q-fuzzy M-semigroup and letT_1, T_2 be Neutrosophic Anti Q-Fuzzy Transform Operators (NAQFTOs). Then their composition $T_1 \circ T_2$ is also a NAQFTO.

Proof: For any $x \in S, q \in Q$:

$$T_A\left(T_1\left(T_2(x)\right), q\right) \leq T_A\left(T_2(x), q\right) \leq T_A(x, q)$$
$$I_A\left(T_1\left(T_2(x)\right), q\right) \geq I_A\left(T_2(x), q\right) \geq I_A(x, q)$$
$$F_A\left(T_1\left(T_2(x)\right), q\right) \geq F_A\left(T_2(x), q\right) \geq F_A(x, q)$$
$$Thus, T_1 \circ T_2 \ satisfies\ the\ definition\ of\ NAQFTO.$$

Theorem 7 . Neutrosophic Homomorphism Preservation Theorem

Let $f: S \to S'$ be a homomorphism between two semigroups. If A is a neutrosophic anti Q-fuzzy M-semigroup on S′, then the pre-image $f^{-1}(A)$ is also a neutrosophic anti Q-fuzzy M-semigroup on S.

Proof.

Trivial

Theorem 8. Neutrosophic level structure theorem

Let A be a neutrosophic anti Q-fuzzy M-semigroup. Define:

$$\mathrm{U}_{\alpha,\mathrm{q}} = \{\mathrm{x} \in \mathrm{S} \mid \mathrm{T}_\mathrm{A}(\mathrm{x},\mathrm{q}) \leq \alpha\}$$
$$\mathrm{V}_{\beta,\mathrm{q}} = \{\mathrm{x} \in \mathrm{S} \mid \mathrm{I}_\mathrm{A}(\mathrm{x},\mathrm{q}) \geq \beta\}$$
$$\mathrm{W}_{\gamma,\mathrm{q}} = \{\mathrm{x} \in \mathrm{S} \mid \mathrm{F}_\mathrm{A}(\mathrm{x},\mathrm{q}) \geq \gamma\}$$
$$\text{Then } \mathrm{U}_{\alpha,\mathrm{q}}, \mathrm{V}_{\beta,\mathrm{q}}, \mathrm{W}_{\gamma,\mathrm{q}}\text{re subsemigroups of S.}$$

Proof: Let $x, y \in U_{\alpha,q}$ Then: $T_A(x,q) \leq \alpha, T_A(y,q) \leq \alpha$
Thus: $T_A(xy,q) \leq \max\{T_A(x,q), T_A(y,q)\} \leq \alpha$ hence $xy \in U_{\alpha,q}$
Similarly, for $V_{\beta,q}$, $I_A(xy,q) \geq \min\{I_A(x,q), I_A(y,q)\} \geq \beta\ Thus\ xy \in V_{\beta,q}$.
For $W_{\gamma,q}$, $F_A(xy,q) \geq \min\{F_A(x,q), F_A(y,q)\} \geq \gamma$
$Thus\ xy \in W_{\gamma,q}$. Hence all three sets are subsemigroups.
The above results establish that the neutrosophic extension preserves structural properties such as closure, homomorphic invariance, and level decomposition, thereby demonstrating the robustness and generality of the proposed framework.

Theorem 9. Transform Characterization of Structure

Let A be a neutrosophic anti Q-fuzzy set on a semigroup S. Then A is a neutrosophic anti Q-fuzzy M-semigroup if and only if for every Neutrosophic Anti Q-Fuzzy Transform Operator (NAQFTO) T:S→S, the following condition holds:

$$T_A(T(a)T(b), q) \leq \max\{T_A(a,q), T_A(b,q)\}$$
$$I_A(T(a)T(b), q) \geq \min\{I_A(a,q), I_A(b,q)\}$$
$$F_A(T(a)T(b), q) \geq \backslash min\,\{F_A(a,q), F_A(b,q)\,\} for\ all\ a, b \in S, q \in Q.$$

Interpretation of this theorem is as follows. The algebraic structure can be completely characterized through transform operators. Instead of checking the semigroup condition directly, it is sufficient to check it under transformations. This introduces a new operator-based characterization paradigm in fuzzy algebra

Proof : If part

Assume A is a neutrosophic anti Q-fuzzy M-semigroup.
Then by definition: $T_A(ab,q) \leq \max\{T_A(a,q), T_A(b,q)\}$ Since T is a NAQFTO:

$$T_A(T(a), q) \leq T_A(a,q), \quad T_A(T(b), q) \leq T_A(b,q)$$

Now:

$$T_A(T(a)T(b), q) \leq \max\{T_A(T(a), q), T_A(T(b), q)\} \leq \max\{T_A(T(a), q), T_A(T(b), q)\}$$

Similarly:

$$I_A(T(a)T(b), q) \geq \min\{I_A(a,q), I_A(b,q)\}$$
$$F_A(T(a)T(b), q) \geq \min\{F_A(a,q), F_A(b,q)\}$$

Thus, the condition holds.
Only if part
Assume the given condition holds for all NAQFTOs.
Choose T as the identity mapping, i.e.,

$$T(x) = x, \forall x \in S$$

Then:

$$T_A(ab, q) \leq \max\{T_A(a,q), T_A(b,q)\}$$
$$I\_A\,(ab, q) \geq \min\{I\_A\,(a,q), I\_A\,(b,q)\}$$
$$F_A(ab, q) \geq \min\{F_A(a,q), F_A(b,q)\}$$
$$Hence, A\ satisfies\ the\ definition\ of\ a\ neutrosophic\ anti\ Q-fuzzy\ M-semigroup.$$

Theorem 10. Let $A = (\alpha_a, \beta_a)$ be an intuitionistic anti Q-fuzzy M-semigroup of M, then the level sets $L(\alpha_a;\ t)$ and $U(\beta_a;\ t)$ are M-semigroups of M, for every t ∈ [0, 1], whenever $L(\alpha_a;\ t) \neq \varphi \neq U(\beta_a;\ t)$.
Proof. $Let\ x, y \in L(\alpha_a;\ t) and\ U(\beta_a;\ t) with\ q \in Q.$
$Then\ \alpha_a(x,q) \leq t, \beta_a(x,q) \geq t, \alpha_a(y,q) \leq t, \beta_a(y,q) \geq t$ which imply that

$$\alpha_a(xy,q) \leq \max\{\alpha_a(x,q), \alpha_a(y,q)\} \leq t\ and\ \beta_a(xy,q) \geq \min\{\beta_a(x,q), \beta_a(y,q)\} \geq t$$

Thus $xy \in L(\alpha_a;\ t) and\ xy \in U(\beta_a;\ t)$.
For any left identity $e \in M\ and\ q \in Q, we\ have\ \alpha_a(e,q) = 0, \beta_a(e,q) = 1.$
Hence $e \in L(\alpha_a;\ t) and\ U(\beta_a;\ t)$.

$$Let\ x \in L(\alpha_a;\ t) and\ U(\beta_a;\ t). Then\ ex \in L(\alpha_a;\ t) and\ U(\beta_a;\ t) \subseteq M. This\ leads\ to\ ex = x, and\ so\ U(\alpha_a;\ t) and\ L(\beta_a;\ t) are\ M-semigroups\ of\ M.$$

Theorem 11. Let $A = (\alpha_a, \beta_a)$ be an intuitionistic anti Q-fuzzy set in M. If each level subsets $L(\alpha_a;\ t) and\ U(\beta_a;\ t) are\ M-semigroups\ of\ M\ for\ every\ t \in [0,1], whenever\ U(\alpha_a;\ t) \neq \varphi \neq L(\beta_a;\ t), then\ A = (\alpha_a, \beta_a)$ is an intuitionistic anti Q-fuzzy M-semigroup of M.

Proof. $Let\ t \in [0,1]. Suppose\ that\ L(\alpha_a;\ t) \neq \varphi\ and\ U(\beta_a;\ t) \neq \varphi\ are\ M-semigroups\ of\ M.$
We must show that $A = (\alpha_a, \beta_a)$ is an intuitionistic anti Q-fuzzy M-semigroup of M. If the condition (i) from Definition 1 is false, then there exists $x, y \in M\ such\ that\ \alpha_a(xy,q) > \max\{\alpha_a(x,q), \alpha_a(y,q)\} for\ all\ q \in Q.$

$$Taking\ \alpha^0 = ½(\alpha_a(xy,q) + \max\{\alpha_a(x,q), \alpha_a(y,q)\}).$$
$$We\ have, \qquad \alpha_a(xy,q) > \alpha^0 < \max\{\alpha_a(x,q), \alpha_a(y,q)\}.$$
$$It\ follows\ that\ x, y \in L(\alpha_a;\ t) and\ xy \notin L(\alpha_a;\ t), which\ is\ a\ contradiction.$$

Hence the condition (i) from Definition 1.
The proof of the other conditions are similar to the case, we omit the proof.

Theorem 12. $If\ an\ intuitionistic\ anti\ Q-fuzzy\ set\ A = (\alpha_a, \beta_a) in\ M\ is\ an\ intuitionistic\ anti\ Q-fuzzy\ M-semigroup\ of\ M, then\ so\ is\ ◆A, where ◆A = \{(x, \alpha_a(x,q), \alpha_a(x,q)) : x \in M, q \in Q\}.$

Proof. It is sufficient to show that α_a satisfies the first two conditions of Definition 1.

$For\ any\ x, y \in M\ and\ q \in Q, we\ have$
$(i)\ \alpha_a(xy,q) = 1 - \alpha_a(xy,q)$
$\leq 1 - \max\{\alpha_a(x,q), \alpha_a(y,q)\}$
$= \min\{1 - \alpha_a(x,q), 1 - \alpha_a(y,q)\}$
$= \min\{\alpha_a(x,q), \alpha_a(y,q)\}$
$(ii) Let\ e\ be\ any\ left\ identity\ in\ M$
$\alpha_a(e) = 1 - \alpha_a(e) = 1 - 0 = 1$
$Therefore\ ◆A\ is\ an\ intuitionistic\ anti\ Q-fuzzy\ M-semigroup\ of\ M.$

***Theorem* 13.** $If\ an\ intuitionistic\ anti\ Q-fuzzy\ set\ A = (\alpha_a, \beta_a) in\ M\ is\ an\ intuitionistic\ Q-fuzzy\ M-semigroup\ of\ M, then\ so\ is\ A, where$

$A = \{(x, \beta_a(x,q), \beta_a(x,q)) : x \in M, q \in Q\}.$

$Proof. It\ is\ sufficient\ to\ show\ that\ \beta_{\mathrm{a}}\ satisfies\ the\ first\ two\ conditions\ of\ Definition\ 1.$

$For\ any\ x, y \in M\ and\ q \in Q, we\ have$

$$(i)\ \beta_{\mathrm{a}}(xy,q) = 1 - \beta_{\mathrm{a}}(xy,q)$$

$$\geq 1 - \min\{\beta_{\mathrm{a}}(x,q), \beta_{\mathrm{a}}(y,q)\}$$

$$= \max\{1 - \beta_{\mathrm{a}}(x,q), 1 - \beta_{\mathrm{a}}(y,q)\}$$

$$= \max\{\beta_{\mathrm{a}}(x,q), \beta_{\mathrm{a}}(y,q)\}$$

$(ii) Let\ e\ be\ any\ left\ identity\ in\ M$

$$\beta_{\mathrm{a}}(e) = 1 - \beta_{\mathrm{a}}(e) = 1 - 1 = 0$$

$Therefore\ Ais\ an\ intuitionistic\ anti\ Q - fuzzy\ M - semigroup\ of\ M.$

Comparative Analysis

The proposed neutrosophic extension further generalizes the intuitionistic anti Q-fuzzy M-semigroup by incorporating an independent indeterminacy component, making the model suitable for highly uncertain and inconsistent environments

Model	Uncertainty type
Intuitionistic	Membership + Non-membership
Proposed Model	+ Anti + Q + Transformation
Neutrosophic Extension	+ Indeterminacy (full uncertainty spectrum)

Feature	Intuitionistic	Pythagorean	Neutrosophic	Proposed Model
Membership + Non-membership	Yes	Yes	Yes	Yes
Constraint Simplicity	Moderate	Complex	Very complex	Simple
Multi-parameter (Q)	No	No	Limited	Wide
Anti-fuzzy capability	No	No	Partial	Yes
Transformation operator	No	No	No	AQFTO
Algebraic tractability	Good	Moderate	Difficult	Strong
Application flexibility	Moderate	High	High	Very High

Conclusion

The idea of intuitionistic anti Q-fuzzy M-semigroups is a big step forward in fuzzy algebra. By combining several uncertainty frameworks into a single structure, it creates a versatile and powerful mathematical instrument. The theoretical findings presented in this study create the groundwork for future research in both pure and applied mathematics. This paper introduces Neutrosophic Anti Q-Fuzzy M-Semigroups (NAQFMS), which incorporate independent truth, indeterminacy, and falsity functions. The Neutrosophic Anti Q-Fuzzy Transform Operator (NAQFTO) is a unique operator for analysing dynamic structural behaviour. Several new findings on closure, transform invariance, level subset characterisation, and indeterminacy-driven equivalence are presented. The suggested framework improves uncertainty modelling and has strong applications in intelligent systems, traffic optimisation, and decision sciences. This paper proposed the notion of Neutrosophic Anti Q-Fuzzy M-Semigroups, which expands fuzzy algebra into a more general and flexible neutrosophic framework. The suggested NAQFTO operator allows for dynamic study of algebraic structures. New theorems and structural features were developed, including a new indeterminacy-based equivalence principle.

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Chapter 10

Continuum Neutrosophic Information Geometry. A Differential-Geometric Framework for Triple-Valued Continuous Uncertainty

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ABSTRACT

Information geometry provides a differential–geometric framework for analyzing probability distributions by representing statistical models as manifolds equipped with information-theoretic metrics. However, classical information geometry assumes that uncertainty can be represented by a single probability distribution, which may be insufficient for systems characterized by incomplete, indeterminate, or contradictory information. In this paper we introduce **Continuum Neutrosophic Information Geometry**, a geometric framework built upon continuum neutrosophic probability distributions. In this setting, uncertainty is represented by three density functions corresponding to truth, indeterminacy, and falsity components. These densities define points on a **neutrosophic statistical manifold**, whose geometry is characterized using a generalized Fisher information metric and a neutrosophic divergence extending the Kullback–Leibler divergence. The resulting geometric structure enables the analysis of epistemic uncertainty through concepts such as geodesic trajectories, information distances, and parameter sensitivity. We show that classical information geometry arises as a special case when indeterminacy and falsity components vanish. Potential applications of the framework include machine learning under ambiguous data, sensor fusion with conflicting information sources, Bayesian inference with incomplete evidence, and modeling of complex socio-ecological systems. The proposed theory contributes to the broader development of **continuum neutrosophic mathematics**, integrating logical, probabilistic, statistical, and geometric perspectives on uncertainty.

Keywords: Continuum neutrosophic probability, information geometry, neutrosophic statistical manifolds, Fisher information metric, neutrosophic divergence, epistemic uncertainty, statistical manifolds, uncertainty modeling, information-theoretic geometry, complex systems analysis.

INTRODUCTION

Information geometry provides a powerful mathematical framework for analyzing probability distributions using the tools of differential geometry [1]. In this approach, families of probability distributions are viewed as points on a manifold equipped with geometric structures derived from information-theoretic divergence measures.
This perspective has been applied successfully in many areas, including statistical inference, machine learning, signal processing, and statistical physics.
However, classical information geometry assumes that uncertainty can be represented by a **single probability distribution**. In many real-world systems, uncertainty arises not only from randomness but also from incomplete or contradictory information.

Neutrosophic theory addresses this limitation by representing uncertainty through three independent epistemic components: (T, I, F) representing degrees of **truth**, **indeterminacy**, and **falsity** [2, 3].

Recent developments in **continuum neutrosophic probability** extend these components to density functions defined over a continuous refinement index:

$$t(\omega, \alpha), i(\omega, \alpha), f(\omega, \alpha), \alpha \in [0,1].$$

These densities describe the distribution of epistemic evidence across infinitely many latent uncertainty states.

The goal of this paper is to develop a **geometric structure for continuum neutrosophic statistical models**. In particular, we introduce:

- continuum neutrosophic statistical manifolds;
- neutrosophic divergence measures;
- neutrosophic Fisher information metrics;
- geometric interpretations of epistemic uncertainty.

This work connects neutrosophic uncertainty modeling with the broader field of information geometry.

Continuum Neutrosophic Probability Densities

Let (Ω, Σ, P_N) be a **continuum neutrosophic probability space**, where Ω denotes the sample space, Σ is a σ-algebra of measurable events, and P_N represents a neutrosophic probability measure defined on this space.

Unlike classical probability theory, which associates a single scalar probability with each event, the neutrosophic framework represents uncertainty through three independent epistemic components:

$$P_N = (P_T, P_I, P_F)$$

where

- P_T represents the **truth probability measure**, describing epistemic support for the occurrence of events,
- P_I represents the **indeterminacy measure**, capturing unresolved or incomplete information,
- P_F represents the **falsity probability measure**, representing epistemic support against the occurrence of events.

Each component

$$P_T, P_I, P_F : \Sigma \to [0,1]$$

is a classical measure satisfying non-negativity and countable additivity. These three measures collectively describe the full epistemic structure of uncertainty.

To maintain consistency with probabilistic interpretation, the measures satisfy the normalization condition

$$P_T(\Omega) + P_I(\Omega) + P_F(\Omega) = 1.$$

This constraint ensures that the total epistemic mass associated with the sample space remains bounded and interpretable as a generalized probability distribution.

Density Representation

Suppose that the neutrosophic probability measures are **absolutely continuous** with respect to a reference measure λ defined on (Ω, Σ). The reference measure may correspond to Lebesgue measure, counting measure, or another appropriate base measure depending on the application.

By the **Radon–Nikodym theorem** [4, 5, 6], there exist measurable density functions

$$p_T(\omega), p_I(\omega), p_F(\omega)$$

such that

$$P_T(E) = \int_E p_T(\omega)\, d\lambda(\omega)$$

$$P_I(E) = \int_E p_I(\omega)\, d\lambda(\omega)$$

$$P_F(E) = \int_E p_F(\omega)\, d\lambda(\omega)$$

for every event $E \in \Sigma$.

These functions describe the distribution of epistemic evidence across the sample space.

Normalization

The density functions satisfy the global normalization condition

$$\int_{\Omega} (p_T(\omega) + p_I(\omega) + p_F(\omega))\, d\lambda(\omega) = 1.$$

This condition ensures that the total epistemic mass of the system equals one, while allowing the relative contributions of truth, indeterminacy, and falsity to vary across the outcome space.

Neutrosophic Density Triple

The three density functions together form the **neutrosophic probability density triple**

$$p_N(\omega) = (p_T(\omega), p_I(\omega), p_F(\omega)).$$

This triple-valued density provides a richer description of uncertainty than classical probability density functions.

For each outcome $\omega \in \Omega$:

- $p_T(\omega)$ quantifies evidence supporting the occurrence of ω,
- $p_I(\omega)$ represents indeterminate or unresolved epistemic information,
- $p_F(\omega)$ represents evidence opposing the occurrence of ω.

Thus, each outcome is associated not with a single probability value but with a **distribution of epistemic evidence across three dimensions**.

Reduction to Classical Probability

Classical probability theory [7] arises as a special case of this framework when the indeterminacy and falsity components vanish:

$$p_I(\omega) = 0, p_F(\omega) = 0.$$

In this situation the density triple reduces to $p_N(\omega) = (p_T(\omega), 0, 0)$ and the normalization condition becomes $\int_{\Omega} p_T\,(\omega) d\lambda(\omega) = 1$, which corresponds to the classical probability density function.

Thus, continuum neutrosophic probability densities generalize standard probability distributions while preserving compatibility with classical measure-theoretic probability.

Role in Information Geometry

Within the framework of **continuum neutrosophic information geometry**, the density triple (p_T, p_I, p_F) defines a point in a generalized statistical manifold. The geometric structure of this manifold arises from divergence measures and information metrics defined on these density functions, which will be developed in subsequent sections.

This representation enables the geometric analysis of statistical models in systems characterized by incomplete, uncertain, or contradictory information.

Neutrosophic Statistical Manifolds

Information geometry studies families of probability distributions using the tools of differential geometry by interpreting them as points on a smooth manifold [1]. In the continuum neutrosophic framework, probability distributions are represented not by a single density function but by a **triple of density fields** describing truth, indeterminacy, and falsity components of uncertainty. This structure naturally leads to the concept of a **neutrosophic statistical manifold**.

Let $p_N(\omega \mid \theta)$ denote a parametric family of continuum neutrosophic probability densities defined on the measurable space (Ω, Σ), where $\theta = (\theta_1, \dots, \theta_k)$ is a vector of parameters belonging to an open subset $\Theta \subseteq \mathbb{R}^k$. Each neutrosophic density consists of three measurable functions

$$p_N(\omega \mid \theta) = (p_T(\omega \mid \theta),\, p_I(\omega \mid \theta),\, p_F(\omega \mid \theta)).$$

These density functions satisfy the normalization condition

$\int_{\Omega}(p_T(\omega \mid \theta) + p_I(\omega \mid \theta) + p_F(\omega \mid \theta))\, d\omega = 1$, for all parameter values $\theta \in \Theta$.

Definition of the Statistical Manifold

The set $\mathcal{M}_N = \{p_N(\omega \mid \theta) \mid \theta \in \Theta\}$ forms a **continuum neutrosophic statistical manifold**.

Each point of this manifold represents a neutrosophic probability distribution, and the parameter vector θserves as a coordinate chart on the manifold. The dimension of the manifold equals the number of parameters k.
Unlike classical statistical manifolds, where each point corresponds to a single probability density function, a point in the neutrosophic statistical manifold represents a **triple density structure** (p_T, p_I, p_F) capturing different epistemic aspects of uncertainty.

Tangent Space

To analyze the local geometry of the statistical manifold, we consider infinitesimal variations of the density triple with respect to the parameters.
The tangent vectors at a point $p_N(\omega \mid \theta)$are defined by the partial derivatives

$$\frac{\partial}{\partial\theta_i} p_N(\omega \mid \theta) = \left(\frac{\partial p_T}{\partial\theta_i}, \frac{\partial p_I}{\partial\theta_i}, \frac{\partial p_F}{\partial\theta_i}\right).$$

These vectors span the **tangent space** $T_\theta\mathcal{M}_N$ of the neutrosophic statistical manifold at parameter value θ.
The tangent space therefore consists of triples of functions representing infinitesimal changes in the truth, indeterminacy, and falsity density components.

Log-Density Representation

In classical information geometry, many geometric quantities are expressed using the **log-likelihood function** [8]. The same idea can be extended to the neutrosophic framework.
Define the log-density components

$$\ell_T(\omega \mid \theta) = \log\, p_T(\omega \mid \theta),$$
$$\ell_I(\omega \mid \theta) = \log\, p_I(\omega \mid \theta),$$
$$\ell_F(\omega \mid \theta) = \log\, p_F(\omega \mid \theta).$$

The gradients $\frac{\partial \ell_T}{\partial\theta_i}, \frac{\partial \ell_I}{\partial\theta_i}, \frac{\partial \ell_F}{\partial\theta_i}$ describe how the epistemic densities change with respect to the parameters.
These quantities play a central role in defining the **information metric** of the neutrosophic statistical manifold.

Geometric Interpretation

Geometrically, each neutrosophic probability distribution corresponds to a point in a functional space composed of three coupled density functions (see Figure 1) . The statistical manifold can therefore be interpreted as a **three-layer information geometry**, where each layer corresponds to one epistemic component.
More precisely:

- the **truth layer** represents distributions describing positive epistemic evidence,
- the **indeterminacy layer** captures unresolved uncertainty,
- the **falsity layer** represents opposing evidence.

The geometry of the manifold describes how these three components evolve as model parameters change.
This structure generalizes classical statistical manifolds by incorporating additional epistemic dimensions while preserving the underlying geometric framework of information theory [1].

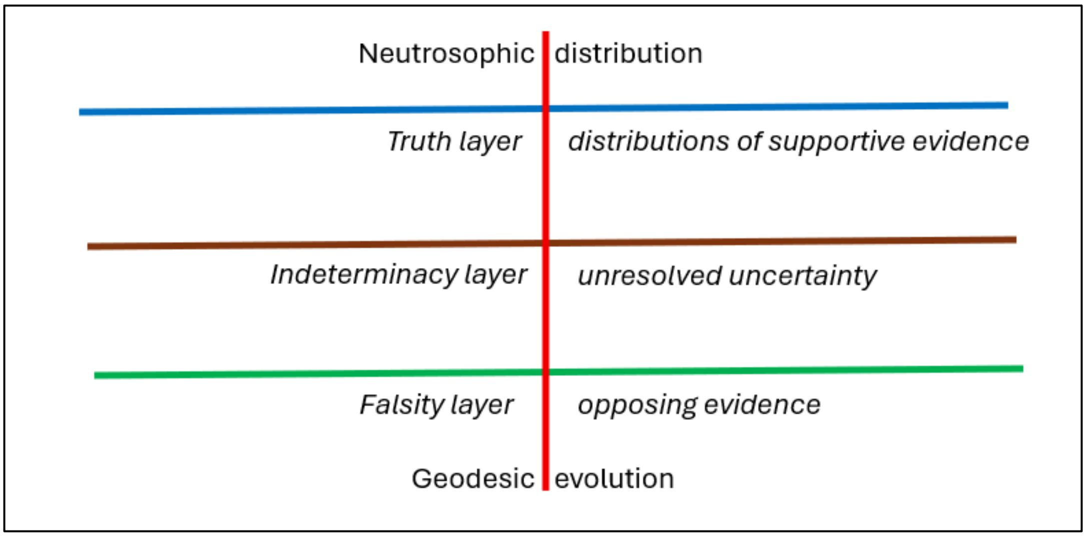

Figure 1. Geometric interpretation of the continuum neutrosophic statistical manifold.

Each neutrosophic probability distribution corresponds to a point represented by a triple of coupled density functions (p_T, p_I, p_F). The manifold can be viewed as a three-layer information geometry composed of truth, indeterminacy, and falsity layers. Geodesic trajectories on the manifold describe the evolution of epistemic states as model parameters change.

Reduction to Classical Statistical Manifolds

The neutrosophic statistical manifold reduces to the classical statistical manifold when the indeterminacy and falsity components vanish: $p_I(\omega \mid \theta) = 0, p_F(\omega \mid \theta) = 0$.

In this case the density triple reduces to $p_N(\omega \mid \theta) = (p_T(\omega \mid \theta), 0, 0)$ and the manifold $\mathcal{M}_N$ becomes the classical manifold of probability distributions studied in information geometry.

Thus, continuum neutrosophic information geometry extends the classical theory while maintaining compatibility with existing statistical frameworks.

Role in Neutrosophic Information Geometry

The neutrosophic statistical manifold provides the geometric foundation for analyzing the structure of continuum neutrosophic probability distributions. In subsequent sections we will define divergence measures and information metrics on this manifold, allowing the study of geodesic curves, statistical inference, and the evolution of uncertainty within the neutrosophic framework.

Neutrosophic Fisher Information Metric and Dual Affine Connections

A central concept in information geometry is the **Fisher information metric**, which defines a Riemannian metric on the statistical manifold of probability distributions [1, 8]. This metric measures the sensitivity of probability distributions to changes in model parameters and plays a fundamental role in statistical inference.

In the continuum neutrosophic framework, probability distributions are represented by **triples of density functions** corresponding to truth, indeterminacy, and falsity components. Consequently, the information metric must account for contributions from all three epistemic layers.

Neutrosophic Fisher Information Metric

Let $p_N(\omega \mid \theta) = (p_T(\omega \mid \theta), p_I(\omega \mid \theta), p_F(\omega \mid \theta))$ be a parametric family of continuum neutrosophic densities defined on the statistical manifold $\mathcal{M}_N$.

The **neutrosophic Fisher information metric** is defined as $g_{ij}^{(N)}(\theta) = g_{ij}^{(T)}(\theta) + g_{ij}^{(I)}(\theta) + g_{ij}^{(F)}(\theta)$, where each term corresponds to the Fisher information associated with one epistemic component.

The truth component is given by $g_{ij}^{(T)}(\theta) = \int_{\Omega} \frac{\partial \log\, p_T(\omega|\theta)}{\partial \theta_i} \frac{\partial \log\, p_T(\omega|\theta)}{\partial \theta_j} p_T(\omega \mid \theta)\, d\omega$

with analogous expressions for the indeterminacy and falsity components:

$$g_{ij}^{(I)}(\theta) = \int_{\Omega} \frac{\partial \log\, p_I(\omega \mid \theta)}{\partial \theta_i} \frac{\partial \log\, p_I(\omega \mid \theta)}{\partial \theta_j} p_I(\omega \mid \theta)\, d\omega$$

$$g_{ij}^{(F)}(\theta) = \int_{\Omega} \frac{\partial \log\, p_F(\omega \mid \theta)}{\partial \theta_i} \frac{\partial \log\, p_F(\omega \mid \theta)}{\partial \theta_j} p_F(\omega \mid \theta)\, d\omega.$$

Thus, the neutrosophic Fisher metric is obtained by summing the Fisher metrics of the three epistemic components.

Interpretation

The neutrosophic Fisher metric quantifies how rapidly the epistemic structure of the probability distribution changes with respect to parameter variations.

More specifically:

- the **truth information** component measures sensitivity of supportive evidence,
- the **indeterminacy information** component measures sensitivity of unresolved uncertainty,
- the **falsity information** component measures sensitivity of contradictory evidence.

Together these components define the **local geometry of the neutrosophic statistical manifold**.

Dual Affine Connections

In classical information geometry, the statistical manifold possesses a pair of dual affine connections associated with exponential and mixture coordinate systems [1].

These connections can be generalized to the neutrosophic setting.

Let $\nabla^{(e)}, \nabla^{(m)}$ denote the **exponential** and **mixture** connections defined on the neutrosophic statistical manifold.

For the truth component, the exponential connection coefficients are defined by $\Gamma_{ijk}^{(e,T)} = \int_{\Omega} \frac{\partial^2 \log\, p_T}{\partial \theta_i \partial \theta_j} \frac{\partial \log\, p_T}{\partial \theta_k} p_T\, d\omega.$

Analogous expressions hold for the indeterminacy and falsity components.

The global neutrosophic connection coefficients are obtained by summing these contributions:

$$\Gamma_{ijk}^{(e,N)} = \Gamma_{ijk}^{(e,T)} + \Gamma_{ijk}^{(e,I)} + \Gamma_{ijk}^{(e,F)}.$$

A similar construction applies to the mixture connection $\nabla^{(m)}$.

Geometric Structure

The statistical manifold $\left(\mathcal{M}_N, g^{(N)}, \nabla^{(e)}, \nabla^{(m)}\right)$ thus forms a **dualistic geometric structure**, extending the classical theory of statistical manifolds to neutrosophic probability distributions.

This structure enables the analysis of:

- geodesic paths between neutrosophic distributions,
- information flow in epistemic systems,
- optimal statistical estimators under triple-valued uncertainty.

Reduction to Classical Fisher Geometry

If the indeterminacy and falsity components vanish, $p_I = 0, p_F = 0$, the neutrosophic Fisher metric reduces to the classical Fisher information metric $g_{ij} = \int_{\Omega} \frac{\partial \log p}{\partial \theta_i} \frac{\partial \log p}{\partial \theta_j} p(\omega)\, d\omega$
studied in information geometry [8].
Thus, classical information geometry appears as a special case of the neutrosophic framework.

Neutrosophic Divergence and Information Distance

A central concept in information geometry is the use of **divergence functions** to measure the difference between probability distributions. In classical information theory, the most widely used divergence is the **Kullback–Leibler (KL) divergence** [9], which quantifies the information loss incurred when one distribution is used to approximate another [10]. Divergence functions play a fundamental role in defining the geometric structure of statistical manifolds because they generate both the Fisher information metric and the associated dual affine connections [1]. In the continuum neutrosophic framework, probability distributions are represented by triples of density functions describing truth, indeterminacy, and falsity components. Consequently, divergence between neutrosophic distributions must account for differences across all three epistemic dimensions.

Definition of Neutrosophic Divergence

Let $P_N = (p_T, p_I, p_F)$ and $Q_N = (q_T, q_I, q_F)$ be two continuum neutrosophic probability distributions defined on the same measurable space.
The **neutrosophic divergence** between P_N and Q_N is defined as $D_N(P_N \parallel Q_N) = D_T + D_I + D_F$, where each component measures the divergence between the corresponding density functions.
The truth divergence is defined as $D_T = \int_{\Omega} p_T\,(\omega) \log \frac{p_T(\omega)}{q_T(\omega)}\, d\omega$ with analogous expressions for the indeterminacy and falsity components:

$$D_I = \int_{\Omega} p_I\,(\omega) \log \frac{p_I(\omega)}{q_I(\omega)}\, d\omega$$

$$D_F = \int_{\Omega} p_F\,(\omega) \log \frac{p_F(\omega)}{q_F(\omega)}\, d\omega.$$

Thus, the neutrosophic divergence is obtained by summing the divergences of the three epistemic components.

Properties of Neutrosophic Divergence

The neutrosophic divergence inherits several important properties from the classical Kullback–Leibler divergence [9].

Non-negativity $D_N(P_N \parallel Q_N) \geq 0$ *for all neutrosophic distributions* P_N *and* Q_N.

Identity of indiscernibles $D_N(P_N \parallel Q_N) = 0$
if and only if $p_T = q_T, p_I = q_I, p_F = q_F$.

Asymmetry
In general $D_N(P_N \parallel Q_N) \neq D_N(Q_N \parallel P_N)$ which reflects the directional nature of information divergence.
These properties make neutrosophic divergence a natural generalization of classical information-theoretic distance measures.

Relation to the Fisher Information Metric

In information geometry, the Fisher information metric arises as the second-order approximation of the divergence between nearby distributions [8].
Let $P_N(\theta)$ be a parametric neutrosophic distribution and consider a nearby distribution
$P_N(\theta + d\theta)$.

Expanding the neutrosophic divergence to second order yields $D_N(P_N(\theta) \parallel P_N(\theta + d\theta)) = \frac{1}{2} g_{ij}^{(N)} d\theta_i d\theta_j + O(\parallel d\theta \parallel^3)$, where $g_{ij}^{(N)}$ is the neutrosophic Fisher information metric.
Thus, the divergence function generates the Riemannian geometry of the neutrosophic statistical manifold.

Symmetrized Information Distance

For certain applications it is convenient to define a symmetric divergence measure.
The **symmetrized neutrosophic divergence** may be defined as

$$D_N^{(sym)}(P,Q) = \frac{1}{2}[D_N(P \parallel Q) + D_N(Q \parallel P)].$$

This quantity provides a symmetric measure of information distance between neutrosophic probability distributions.

Reduction to Classical Divergence

When the indeterminacy and falsity components vanish, $p_I = q_I = 0, p_F = q_F = 0$, the neutrosophic divergence reduces to the classical Kullback–Leibler divergence

$$D(P \parallel Q) = \int p(\omega) \log \frac{p(\omega)}{q(\omega)}\, d\omega.$$

Thus, classical information geometry appears as a special case of the neutrosophic divergence framework.

Geometric Interpretation

Within the neutrosophic statistical manifold, the divergence function defines the **information distance** between two epistemic states. Each state corresponds to a triple density distribution describing supportive, indeterminate, and contradictory evidence.

The divergence therefore measures how strongly the epistemic structure of one distribution differs from another. In this sense, the neutrosophic divergence provides a geometric description of the **evolution of uncertainty** across the statistical manifold.

Geodesics and Epistemic Evolution

In information geometry, the geometry of a statistical manifold provides insight into how probability distributions evolve under parameter changes. In particular, **geodesic curves** describe the shortest or most natural paths between distributions on the manifold [1, 8]. These curves play an important role in statistical inference, learning algorithms, and information-theoretic analysis.
Within the continuum neutrosophic framework, probability distributions are represented by triples of density functions corresponding to truth, indeterminacy, and falsity components. Consequently, geodesic curves describe the smooth evolution of these epistemic components across the statistical manifold.

Geodesic Curves on the Neutrosophic Statistical Manifold

Let $p_N(\omega \mid \theta) = (p_T(\omega \mid \theta), p_I(\omega \mid \theta), p_F(\omega \mid \theta))$ be a parametric family of continuum neutrosophic distributions defined on the statistical manifold $\mathcal{M}_N$.
A **geodesic curve** on the manifold is a path $\theta(t), t \in [0,1]$ that satisfies the geodesic equation associated with the neutrosophic Fisher metric $g_{ij}^{(N)}$.
The geodesic equation takes the form
$\frac{d^2\theta^k}{dt^2} + \Gamma_{ij}^k \frac{d\theta^i}{dt}\frac{d\theta^j}{dt} = 0$, where Γ_{ij}^k are the Christoffel symbols associated with the neutrosophic Fisher metric.
This equation determines the trajectory along which the neutrosophic probability distribution evolves most naturally within the statistical manifold.

Evolution of Epistemic Components

Along a geodesic path, the three density components evolve simultaneously:

$$p_T(\omega \mid \theta(t)), p_I(\omega \mid \theta(t)), p_F(\omega \mid \theta(t)).$$

This evolution can be interpreted as a continuous transformation of the epistemic state of the system.

More specifically:

- changes in p_T describe variation in **supportive evidence**,
- changes in p_I represent evolution of **indeterminate information**,
- changes in p_F reflect variation in **contradictory evidence**.

The combined evolution of these three components defines a trajectory in the neutrosophic statistical manifold.

Exponential and Mixture Geodesics

Information geometry distinguishes between two important types of geodesics corresponding to the dual affine connections introduced earlier [1]:

Exponential Geodesics

These correspond to linear paths in exponential coordinates and are typically associated with parametric statistical models.

Mixture Geodesics

These correspond to linear interpolation between probability distributions.

In the neutrosophic framework, mixture geodesics may be written as: $p_N^{(t)} = (1-t)p_N^{(1)} + tp_N^{(2)}$, where $p_N^{(1)}$ and $p_N^{(2)}$ are two neutrosophic distributions.

This representation corresponds to gradual interpolation between two epistemic states.

Geometric Interpretation of Uncertainty Evolution

Geodesic paths provide a geometric description of how uncertainty evolves as information is updated or model parameters change.

For example:

- **learning processes** correspond to trajectories on the statistical manifold,
- **Bayesian updates** can be interpreted as movement toward regions of lower divergence,
- **information fusion** corresponds to combining epistemic components along mixture geodesics.

In this sense, the neutrosophic statistical manifold provides a geometric framework for analyzing the dynamics of epistemic uncertainty.

Reduction to Classical Information Geometry

If the indeterminacy and falsity components vanish, $p_I = 0, p_F = 0,$ the neutrosophic statistical manifold reduces to the classical statistical manifold studied in information geometry.

In this case the geodesic equation reduces to the standard Fisher–Rao geodesics governing classical probability distributions [8].

Thus, classical information geometry appears as a special case of the neutrosophic geometric framework.

Applications of Continuum Neutrosophic Information Geometry

The geometric framework developed in the previous sections provides new analytical tools for studying systems characterized by complex uncertainty structures. By representing neutrosophic probability distributions as points on a statistical manifold equipped with an information metric and divergence structure, continuum neutrosophic information geometry allows the analysis of epistemic uncertainty using geometric concepts such as distance, curvature, and geodesic trajectories.

Several application domains can benefit from this approach.

Machine Learning and Statistical Learning

Information geometry has played an important role in machine learning by providing geometric interpretations of learning algorithms, optimization procedures, and parameter estimation [4]. Gradient-based learning methods, natural gradient descent, and variational inference all rely on geometric properties of statistical manifolds.

In many machine learning problems, training data may contain ambiguous labels, missing information, or contradictory observations. Classical probabilistic models often struggle to represent these situations because they rely on a single probability distribution.

Continuum neutrosophic information geometry allows machine learning models to represent uncertainty using three epistemic components. Learning algorithms may then be interpreted as trajectories on the neutrosophic statistical manifold that minimize divergence between model predictions and observed data. The neutrosophic Fisher metric can also be used to define **natural gradient learning algorithms** adapted to triple-valued uncertainty structures.

Sensor Fusion and Information Integration

Modern sensor networks often combine data from multiple sources with varying levels of reliability. Measurements from different sensors may contain noise, partial information, or contradictory signals.

Neutrosophic information geometry provides a natural framework for analyzing such systems. Each sensor observation can be represented as a neutrosophic probability distribution describing supportive evidence, indeterminate information, and contradictory evidence. Information fusion can then be interpreted as moving along geodesic paths on the neutrosophic statistical manifold that minimize divergence between multiple information sources.

This geometric interpretation complements existing information fusion frameworks such as the Dezert–Smarandache theory of evidence [11].

Bayesian Inference

Bayesian inference describes how probability distributions evolve when new information becomes available. In information geometry, Bayesian updates can be interpreted as movements on the statistical manifold that minimize information divergence between prior and posterior distributions [1].

Within the continuum neutrosophic framework, prior and posterior beliefs may be represented by neutrosophic distributions. The updating process then corresponds to a trajectory on the neutrosophic statistical manifold reflecting the incorporation of new evidence.

This approach is particularly useful in situations where observations are incomplete or conflicting, since indeterminacy and falsity components can explicitly represent unresolved information.

Complex Systems and Socio-Ecological Modeling

Many real-world systems involve complex interactions between natural processes, human decision-making, and uncertain observations. Examples include ecological monitoring, climate systems, and socio-economic networks. Such systems frequently involve multiple layers of uncertainty, including stochastic variability, incomplete knowledge, and conflicting data sources. Continuum neutrosophic information geometry provides a framework for analyzing these systems by modeling the evolution of epistemic uncertainty on a statistical manifold.

Geometric quantities such as divergence and curvature may help identify regions of high uncertainty or instability in complex systems.

Information-Theoretic Analysis of Uncertainty

Entropy and divergence measures are widely used to analyze the information content of statistical models [10]. In the neutrosophic framework, these measures can be extended to capture the contributions of truth, indeterminacy, and falsity components.

The geometric interpretation provided by neutrosophic information geometry enables the study of **information flow, uncertainty propagation, and learning dynamics** in systems characterized by multi-dimensional uncertainty structures.

Conclusion

This paper introduced **Continuum Neutrosophic Information Geometry**, a differential-geometric framework for analyzing probability distributions characterized by triple-valued epistemic uncertainty. By extending classical information geometry to the neutrosophic setting, we developed a geometric structure capable of representing uncertainty that includes not only stochastic variability but also indeterminacy and contradictory information.

The study established the fundamental elements of this framework. First, continuum neutrosophic probability densities were defined as triples of density functions corresponding to truth, indeterminacy, and falsity components. These densities were used to construct **neutrosophic statistical manifolds**, whose points represent continuum neutrosophic probability distributions.

Second, the paper introduced the **neutrosophic Fisher information metric**, which generalizes the classical Fisher metric by incorporating contributions from all three epistemic components. This metric defines the local geometry of the neutrosophic statistical manifold and provides a measure of the sensitivity of neutrosophic probability distributions to parameter variations.

Third, a **neutrosophic divergence measure** was defined as an extension of the Kullback–Leibler divergence. This divergence generates the information metric and provides a natural measure of distance between neutrosophic distributions. Using this structure, geodesic curves were introduced to describe the smooth evolution of epistemic states on the statistical manifold.

Together, these elements establish a geometric framework for analyzing the structure and dynamics of continuum neutrosophic probability distributions. The resulting theory extends classical information geometry while preserving compatibility with established principles of statistical inference and information theory[1, 8].

The framework also opens new possibilities for analyzing complex systems characterized by incomplete, ambiguous, or conflicting information. Potential applications include machine learning, sensor fusion, Bayesian inference, and the modeling of socio-ecological systems where multiple layers of uncertainty coexist.

From a broader perspective, the present work contributes to the development of a **continuum neutrosophic mathematical program** that integrates logical, probabilistic, statistical, and geometric aspects of uncertainty modeling. Previous studies introduced continuum neutrosophic sets, probability measures, and statistical models. The geometric perspective developed here provides a unifying structure that links these elements within a coherent theoretical framework.

Future research may explore several promising directions, including the study of curvature properties of neutrosophic statistical manifolds, the development of natural gradient algorithms for neutrosophic learning systems, and applications in large-scale data analysis where uncertainty cannot be adequately represented by single probability distributions.

In summary, **continuum neutrosophic information geometry** provides a new mathematical perspective on uncertainty, combining concepts from neutrosophic theory, probability, and information geometry. This framework may serve as a foundation for further theoretical developments and practical applications in the analysis of complex uncertain systems.

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Chapter-11

Neutrosophic Bishop Graphs: Geometric and Computational Reasoning Under Uncertainty

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ABSTRACT

Neutrosophic Bishop Graphs (NBGs), a graph-theoretic framework for making decisions in situations with ambiguity, inconsistency and multi-criteria dependence are presented in this study. Neutrosophic Bishop Graphs (NBGs), model inference as path-based propagation of truth, indeterminacy and falsehood, building on Neutrosophic logic and a geometric constraint inspired by diagonal movement in chess. Formal definitions, dominance-based inference, aggregate semantics, a worked diagnostic case and comprehensive theoretical analysis are all provided. Specifically, we prove NP-completeness of precise inference, establish monotonicity properties and derive fixed-parameter tractability results and ε-approximation guarantees. The framework provides a computationally valid and accessible substitute for fuzzy and probabilistic decision models.

Keywords: Neutrosophic Graphs, Neutrosophic logic, Neutrosophic Bishop Graphs, Uncertainty Modelling.

INTRODUCTION

The expectations of comprehensive, regular and uninterrupted information are not often fulfilled by reasoning in intricate decision systems. Evidence is frequently lacking, conflicting and context-dependent in fields involving specialized systems, standardized medical diagnosis, error analysis and reasoning in law. The structure of uncertainty and disagreement is often obscured by classical decision frameworks, whether they are fuzzy [1], probabilistic or logical which flatten these epistemic subtleties into a single scalar variable. By clearly differentiating between degrees of truth, indeterminacy and falsity, Neutrosophic logic [2] offers a more comprehensive epistemic basis.

Logic by itself, yet, is unable to explain how these epistemic values spread via organized systems of relationships. Conversely, graph-based reasoning [3], which usually depends on numeric edges values, offers a suitable paradigm for relational assessment. Neutrosophic Bishop Graphs which combine geometric constraints, graph theory and Neutrosophic logic are proposed in this chapter. The main point is that valid inference frequently necessitates simultaneous change across several decision dimensions, which are represented as diagonal movement in a decision-space. Neutrosophic Bishop Graphs provide for expressive, comprehensible and computationally rigorous decision reasoning by combining this restriction with Neutrosophic edge valuations.

BACKGROUND

The fundamental ideas of uncertainty modeling presented by fuzzy [1]and intuitionistic frameworks [4] serve as the basis for the development of Neutrosophic graph theory. Zadeh's groundbreaking work [1, 5] developed linguistic variables and fuzzy sets to deal with ambiguity, and Atanassov expanded this to intuitionistic fuzzy sets by adding non-membership. Building on these constraints, Smarandache [2, 6] developed Neutrosophic sets, which provide a progressively sophisticated framework for managing inconsistent and indeterminate data by incorporating truth, indeterminacy, and falsity components. Theoretical development and applications of neutrosophic sets and their extensions [7-10] and different hybrid sets [11-14] were presented in the studies [15-23]. Fuzzy and intuitionistic aggregation techniques were reinforced by later theoretical advances by Klir and Yuan [24] and Xu and Yager [25] which served as the foundation for sophisticated uncertainty models.

By introducing single -valued, bipolar and interval-valued Neutrosophic graphs, Broumi et al. [26-28] formulized the shift from Fuzzy to Neutrosophic graphs and greatly improved the depiction of uncertain relationships. These structures were later extended into pentapartitioned and quadripartitioned Neutrosophic graphs by Das et al. [29] and Muhiuddin et al. [30] and overview of Fermatean Neutrosophic graphs was given by Raut et al. [31], showing their adaptability in simulating higher order uncertainty. The increasing complexity of graph models is demonstrated by recent developments such as complex t-Neutrosophic graphs for biodiversity preservation by Kaviyarasu et al. [32], complement properties of Pythagorean co-Neutrosophic graphs by Govindan et al. [33] and irregular Pythagorean Neutrosophic graphs by Chellamani and Ajay [34].

Additionally, Fujita [35, 36] focused on theoretical generalization while revisiting bipolar and interval-valued Neutrosophic graphs and helping to unify uncertain combinatorics. The structural and operational characteristics of Neutrosophic graphs, Such as operations on interval-valued graphs [28], connectivity indices [37], dominating path colouring and chromatic numbers [38] and the topological structures with Neutrosophic bridges using computational tools like MATLAB [39], have also been thoroughly studied.

Classical graph theory and complexity such as NP-Completeness theory by Garey and Johnson [40], computational complexity by Papadimitriou [41], parameterized complexity frameworks by Cygan et al. [42] and Downey and Fellows [43], combinatorial optimization techniques by Korte and Vygen [44] and approximation algorithms by Hassin [45] serve as the foundation for algorithmic and computational viewpoints. Practical applications, such as Prim's technique for minimum spanning tree evaluation in Neutrosophic graphs, have been made possible by these theoretical bases. Neutrosophic graphs have been used extensively in real-world applications such as computer network analysis [37], biodiversity conservation [32], election analysis utilizing Wiener index in Fermatean Neutrosophic graphs [46] and earthquake response systems.

Also, when intelligent systems that are faced with uncertainty, Reals Probabilistic reasoning frameworks [47] supplement Neutrosophic methods. The transition from Fuzzy to Neutrosophic graph models is highlighted in comprehensive evaluations like Vetrivel et al. [48] and their relevance in engineering and decision-making contexts is shown by Abd et al. [39] and Al-Omeri et al.[46, 49].Although there are still issues with computational efficiency, standardization and large-scale real-world implementation, the literature generally shows a progressive shift from classical and fuzzy graphs theories toward highly generalized Neutrosophic structures capable of modeling complex, uncertain and indeterminate systems.

Neutrosophic Bishop Graph Model

This section establishes Neutrosophic Bishop Graph Model and its fundamental mathematical properties of Neutrosophic Bishop Inference.

Definition 1. A Neutrosophic Bishop Graph is a tuple $\mathcal{G} = (V, E, \mu)$,where:

- $V$is a finite set of vertices representing states, hypotheses, or decisions,
- $E \subseteq V \times V$is a set of edges,
- $\mu : E \to [0,1]^3$assigns to each edge ea neutrosophic weight

$\mu(e) = (T_e, I_e, F_e)$, denoting degrees of truth, indeterminacy, and falsity.

Definition 2. Let $\phi : V \to \mathbb{R}^k$ be an embedding of vertices into a k-dimensional decision space. An edge $(v_i, v_j) \in E$is bishop-admissible if the vector $\phi(v_i) - \phi(v_j)$ has at least two non-zero components. This constraint enforces multi-criteria inference, distinguishing NBGs from ordinary weighted graphs.

Definition 3. Given a path $P = (v_0, v_1, \dots, v_n)$,its neutrosophic valuation is defined as $\mu(P) = (T_P, I_P, F_P)$, where

$$T_P = \min_{e \in P} T_e,$$
$$I_P = \max_{e \in P} I_e,$$
$$F_P = \max_{e \in P} F_e.$$

These operators reflect weakening confidence, accumulating uncertainty, and persistent contradiction.

Definition 4. Given two paths $P_1$and P_2, P_1dominates P_2if $T_{P_1} \geq T_{P_2}, I_{P_1} \leq I_{P_2}, F_{P_1} \leq F_{P_2}$, with at least one strict inequality. Dominated paths are never rationally preferable.

Example 1.
Consider a simplified medical diagnosis scenario with vertices: {Fever,Fatigue,Inflammation,Diagnosis}. Edges encode neutrosophic causal relations derived from expert knowledge. Two distinct inference paths connect Fever to Diagnosis. Although both paths support the diagnosis, one exhibits higher truth and lower indeterminacy, and therefore dominates the other. Crucially, the inferior path is not discarded because it is "wrong," but because it is epistemically weaker. Contradiction and uncertainty remain explicit, enabling transparent explanation of the decision.

Proposition 1. For any Neutrosophic path P, extending the path cannot increase its truth value.

Proof. By definition,$T_P = \min_{e \in P} T_e$. Adding an additional edge introduces a new candidate for the minimum, which cannot increase the result.

Proposition 2. Indeterminacy along a path is non-decreasing under path extension.

Proof: Since $I_P = \max_{e \in P} I_e$, adding edges can only maintain or increase the maximum.

Proposition 3. Between any two connected vertices in a finite NBG, at least one non-dominated path exists.

Proof: The set of simple paths is finite. Dominance defines a partial order on this set, which must contain at least one maximal element.

Definition 5. Given thresholds (α, β, γ), determine whether there exists a path Psuch that

$T_P \geq \alpha, I_P \leq \beta, F_P \leq \gamma.$

Proposition 4. The Neutrosophic Bishop Path Decision Problem is NP-complete.

Proof: Verification path is polynomial. NP-hardness follows from a polynomial reduction from the multi-constrained path problem, which is known to be NP-complete.

Definition 6. For $\varepsilon > 0$, path P_1ε-dominates P_2if

$$
\begin{aligned}
T_{P_1} &\geq T_{P_2} - \varepsilon,\\
I_{P_1} &\leq I_{P_2} + \varepsilon,\\
F_{P_1} &\leq F_{P_2} + \varepsilon.
\end{aligned}
$$

Proposition 5. For fixed ε, neutrosophic bishop inference admits a fully polynomial-time approximation scheme.

Proof: Each valuation component lies in $[0, 1]$. Discretizing the space into $O(1/\varepsilon^3)$buckets bound the number of retained path states per vertex, yielding polynomial runtime in input size and $1/\varepsilon$.

Proposition 6. Neutrosophic bishop inference is fixed-parameter tractable with respect to maximum path length k.

Proof: All paths of length at most $k$can be enumerated in $O(m^k)$time, which is polynomial for fixed k.

Proposition 7. If the underlying graph has bounded treewidth τ, the problem is fixed-parameter tractable in τ.

Proof: Dynamic programming over a tree decomposition aggregates neutrosophic values locally and combines them efficiently across bags.

Proposition 8. Let $P = (v_0, \ldots, v_k)$be a neutrosophic bishop path, and let $P' = (v_0, \ldots, v_k, v_{k+1})$be its extension by one admissible edge. Then $T_{P'} \leq T_P$.

Proof: By Definition 3, the truth value of a path is defined as: $T_P = \min_{e \in P} T_e$.
The extended path P'contains all edges of P, plus one additional edge $e_{k+1} = (v_k, v_{k+1})$.

Hence, $T_{P'} = \min\left(\min_{e\in P} T_e, T_{e_{k+1}}\right)$. Since the minimum of a set augmented by an additional element cannot exceed the minimum of the original set, we conclude that $T_{P'} \leq T_P$.
Therefore, extending a path cannot increase its truth value.
Proposition 9. For any path extension $P \subseteq P'$, $I_{P'} \geq I_P$ and $F_{P'} \geq F_P$.
Proof: From Definition 3, $I_P = \max_{e\in P} I_e, F_P = \max_{e\in P} F_e$.
The extension P' introduces a new edge e_{k+1}.

Thus, $I_{P'} = \max(I_P, I_{e_{k+1}}), F_{P'} = \max(F_P, F_{e_{k+1}})$.
Since the maximum of a set cannot decrease when an element is added, both inequalities follow directly:

$I_{P'} \geq I_P, F_{P'} \geq F_P$. Hence, uncertainty and contradiction are non-decreasing along inference chains.

Proposition 10. Between any two connected vertices in a finite neutrosophic bishop graph, there exists at least one non-dominated path.

Proof: Let $\mathcal{P}$ denote the set of all simple bishop-admissible paths between two fixed vertices s and t. Since the graph is finite, $\mathcal{P}$ is finite. Define a binary relation $\preceq$ on $\mathcal{P}$ by: $P_1 \preceq P_2 \iff T_{P_1} \leq T_{P_2}, I_{P_1} \geq I_{P_2}, F_{P_1} \geq F_{P_2}$. This relation is reflexive, antisymmetric and transitive. Hence, $(\mathcal{P}, \preceq)$ is a partially ordered set. Every finite poset contains at least one maximal element. A maximal element under $\preceq$ is precisely a *non-dominated path.*

Proposition 11. The Neutrosophic Bishop Path Decision Problem is NP-complete.

Proof:

Membership in NP

Given a candidate path P, its Neutrosophic valuation (T_P, I_P, F_P) can be computed by scanning the edges of P, which takes time linear in $|P| \leq |V|$. Checking threshold constraints is constant time. Hence, the problem is in NP.

NP-Hardness

We reduce from the Multi-Constrained Path Problem (MCPP), known to be NP-complete. Given an instance of MCPP with constraints c_1, c_2, c_3, construct an NBG where each original edge maps to a bishop-admissible edge, constraints map to Neutrosophic components: $T_e = 1 - c_1(e)$, $I_e = c_2(e)$, $F_e = c_3(e)$. Thresholds (α, β, γ) are chosen accordingly.

A feasible constrained path exists in MCPP if and only if a path satisfying Neutrosophic thresholds exists in the constructed NBG. Thus, the decision problem is NP-hard. Since the problem is both in NP and NP-hard, it is NP-complete.

Proposition 12. For any fixed $\varepsilon > 0$, ε-approximate neutrosophic bishop inference can be solved in polynomial time.

Proof: Each Neutrosophic component lies in [0, 1]. Partition this interval into $\left\lceil\frac{1}{\varepsilon}\right\rceil$ equal-width buckets.

Thus, the total number of distinct ε-equivalence classes is bounded by $O\left(\frac{1}{\varepsilon^3}\right)$. At each vertex, we retain at most one path per ε-class, discarding ε-dominated paths. Path extension generates polynomial many, and pruning ensures the state space remains polynomial bounded. Therefore, total runtime is polynomial in $|V|, |E|, \frac{1}{\varepsilon}$. Hence, ε-approximate inference is polynomial-time solvable.

Proposition 13. The NBG inference problem is fixed-parameter tractable with respect to maximum path length k.

Proof: Any admissible inference path of length at most k contains at most k edges. The total number of such paths is bounded by $O(|E|^k)$. For fixed k, this quantity is polynomial in input size. Neutrosophic aggregation for each path takes $O(k)$ time. Thus, the total runtime is $O(f(k) \cdot \text{poly}(|V|))$, with $f(k) = |E|^k$. Hence, the problem is fixed-parameter tractable in k.

Proposition 14. If the underlying graph has bounded treewidth τ, then neutrosophic bishop inference is fixed-parameter tractable in τ.

Proof: Let $(T,\mathcal{B})$be a tree decomposition of width τ. Each bag contains at most $\tau + 1$vertices. Dynamic programming proceeds bottom-up, storing for each bag all feasible Neutrosophic summaries of partial paths crossing the bag. Since each summary is a triple discretized by ε (or exact for bounded bags), the number of states per bag is bounded by a function of τ alone. Thus, runtime is: $O(|V| \cdot g(\tau))$,for computable g.Hence, the problem is FPT in treewidth.

Results and Discussions

Neutrosophic Bishop Graphs provide a principled way to preserve epistemic nuance while enabling rational decision making. Unlike probabilistic or fuzzy models, NBGs do not suppress contradiction or uncertainty but incorporate them structurally into inference. The complexity results clarify both the limitations of exact reasoning and the regimes in which efficient inference is possible. This balance between expressiveness and tractability is essential for real-world expert systems.

Conclusion

Neutrosophic Bishop Graphs were introduced in this chapter as a cohesive framework for computational, logical and geometric reasoning under uncertainty. By using formal definitions, proofs and complexity analysis, we were able to show that Neutrosophic Bishop Graphs are useful in both theory and practice. In situations that are unpredictable and inconsistent, the framework provides additional avenues for reliable and explainable decision-making.

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Chapter 12

Plithogenic Cognitive Maps with Extended Plithogenic Representations & Applications

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ABSTRACT

Plithogenic Cognitive Maps (PCM) represent a decision-making approach designed for the Plithogenic environment. This study introduces a new concept called Extended Plithogenic Cognitive Maps (EPCM), combining Extended Plithogenic sets with Cognitive Maps. EPCM aims to enhance the decision-making process by taking into account both superior and inferior attribute values, thereby seeking more optimal solutions. The extension from PCM to EPCM is a step toward creating a more inclusive decision-making tool. To demonstrate the effectiveness of EPCM, a decision-making problem in the manufacturing domain is addressed, specifically examining the degree of association among factors promoting IoT-based manufacturing. EPCM is proposed as a comprehensive tool for deriving optimal solutions, and also suggestions are made to combine it with decision-making tools as part of further research extensions.

Keywords: Extended plithogenic sets, plithogenic cognitive maps, decision-making, optimal solutions

INTRODUCTION

Managerial decision-making constitutes a systematic process designed to tackle identified challenges and optimize the overall performance of business operations. The selection process involves cognitive processing of knowledge, expertise and beliefs of the decision -makers. This approach intends to effectively evaluate the alternatives over the criteria. The decision-making systems minimizes the complexity in evaluating or in prioritizing the alternatives in the contexts of integrated manufacturing systems. The existing manufacturing system's inclusion of Internet of Things (IoT) features adds complexity to managerial decision-making. The prevalence of automation is evident in the proliferation of new and hybrid manufacturing systems. Businesses across different scales are actively integrating IoT into their routine operations through viable promotion strategies. Nonetheless, sectors implementing these strategies for the promotion of IoT-based manufacturing systems must be cognizant of the consequential and inter relational impacts between these strategies.

This juncture raises several research questions. Should a business firm be attentive to the inter relational impacts between promotion strategies? If so, how should these inter associational impacts be studied? What aspects of these strategies need to be focused on? Certainly, a firm aware of these inter associations will be more cost-effective and energy-efficient, thereby enhancing overall productivity. Decision-making models based on Plithogenic Cognitive maps emerge as the optimal choice for facilitating an intensive study of factors related to the core aspects of the problem, particularly concerning superior and inferior attributes. Plithogenic Cognitive maps, a generalized version of cognitive maps, consider the representations of Plithogenic sets introduced by Smarandache. These sets, represented as quintuples (P, a, V, d, c), comprise a set P, an attribute a, a set of attribute values V, the incidental degree d, and the degree of discrepancy c. While Plithogenic cognitive maps are typically based on superior factors, business sectors must focus on both acceptance (superior aspect) and denial (inferior aspect) to fulfill objectives related to profit maximization and loss minimization. This paper proposes an extended Plithogenic cognitive decision model based on the representations of extended Plithogenic sets developed by Sudha et al. [19]. The suggested decision-making tool is designed to evaluate the interrelational impacts between promotion strategies for IoT-based manufacturing. This model is more inclusive, enabling decision-makers to consider two extreme attributes related to profit and loss in contemporary business endeavours.

The rest of the content is outlined as follows: Section 2 elaborates the state of the art of Plithogenic cognitive maps. Section 3 outlines the methodology of the proposed method. Section 4 applies the proposed method in managerial decision-making. Section 5 discourses the results, and the last section presents the conclusion of the work with future scope.

BACKGROUND

This section outlines the contribution of researchers in the field of Plithogenic cognitive maps, existing research gaps and novel contributions of this paper.

Cognitive maps developed by Robert Axelrod are basically graphical structures which depicts the factors of the decision making problem as nodes and the relational impacts as edges respectively. Kosko discussed Fuzzy cognitive maps by integrating the representations of fuzzy sets with cognitive maps. FCMs are extended to intuitionistic fuzzy cognitive maps and neutrosophic cognitive maps with representations of intuitionistic fuzzy sets and neutrosophic set respectively. Plithogenic cognitive maps are the generalization of cognitive maps discussed under the environments of crisp, fuzzy, intuitionistic and neutrosophic. The edge weights determine the type of Plithogenic cognitive maps.

Plithogenic Cognitive Maps (PCM) are widely applied in several decision making scenarios. Sujatha et al. [10] devised a diagnostic model for COVID-19. Priya et al. [13] took a distinct approach, employing Plithogenic cognitive analysis to delve into the evolution of spiritual intelligence among smart age youth. Nivetha et al. [16] discoursed sub-cognitive Plithogenic maps considering mediating effects of the factors in COVID - 19 diagnostic model. Priya et al. [12] developed induced plithogenic based cognitive maps and applied in online learning systems addressing the challenges in the existing educational sector. Gomathy et al. [14] Contributed to the development of Plithogenic combined disjoint block based fuzzy cognitive maps. Angel et al. [15] extended Plithogenic based decision making model with linguistic discrepancy degrees to address academic stress factors.

Fuzzy Cogntiive Maps are characterized as cognitive structures which are applied in complex and conflicting business situations. FCMs are highly preferred decision models that are employed to devise strategic planning and decision making using cause-effect relationships. Lukkanen [8] investigates and compares different business scenarios. Xirogianni and Glykas [1] discusses the implications of FCM in performance analysis. The complex relationships are explored in industrial decision making. Stakias et al. [3] explicated FCMs applications in social and business networks. The interconnected elements are analysed using FCM dynamics. Ntarlas and Groumpos [4] discourses the various applications FCMs in diverse dimensions of business scenario.

Glykas [5] demonstrated that FCMs shall be applied not only as a tool to analyse the factors but also as a means of devising strategic frameworks. Ross et al. [6] explored the strategic business models with FCM framework. Sachan et al. [7] discussed how FCM models facilitate the decision makers in blockchain integrated loan management system. From the aforementioned contributions, it is very evident that FCMs empower decision-makers to gain a holistic understanding of intricate business systems. FCMs also facilitate collaborative decision-making by providing a shared visualization of key factors, fostering communication among stakeholders. Overall, the versatility of FCMs makes them a valuable tool in business management, aiding in both strategic thinking and operational decision-making processes. Further extensions on FCM are made and NCM are introduced. NCM models are designed for the betterment of the decision making. Pramanik et al. [20] performed a study on the problems of construction workers in West Bengal based on NCM. Mondal et al. [21] performed a study on the problems of Hijras in West Bengal based on NCM.

Research gaps and Significant Contributions

The following research gaps are identified as follows

- Fuzzy Cognitive map-based decision-making models are used in business management however the other kinds of cognitive maps are not employed to the best of our knowledge.
- Plithogenic cognitive maps are not applied to address business related problems.
- The cognitive maps are not discussed based on extended plithogenic sets.

The significant contributions made in this paper are

- Construction of Plithogenic cognitive map models based on extended plithogenic sets
- Determining the interassociational impacts of contemporary business problems using Plithogenic based cognitive map model.
- Developing a new genre of decision making model.

Methodology of Extended Plithogenic Cognitive Maps

This section presents the steps involved in extended Plithogenic cognitive maps. The extended Plithogenic sets developed by Nivetha and Smarandache is a septuple of the form (P,a,V,d_D,c_D,d_R,c_R) which includes the incidental degree and degrees of discrepancy with respect to both the superior and inferior attribute values. The steps of this proposed method are based on the steps involved in the method proposed by Angel et al. [15] with few modifications.

Step 1: The decision making problem is clearly outlined. The factors Pi subjected to the problem are determined and the sub-factors Gi subjected to the core factors are also decided. In this case the sub-factors are assumed to be the attributes and each of the attributes possess attribute values Gij.

Step 2: The plithogenic connection matrix between the factors is determined with values as incidental degree in either crisp form of {-1,0,1}, fuzzy of the form [-1,1], intuitionistic of the form $\rho([0,1])^2$ and neutrosophic of the form $\rho([0,1])^3$

Step 3: The discrepancy degrees, considering both superior and inferior attribute values, are determined. Initially, an instantaneous vector Z(t) = (a1, a2, ..., an) is considered, with ai representing values of 0 or 1, denoting OFF or ON positions of factors. For instance, the vector (10000..00) signifies the ON position of the first factor, transmitted to the connection matrix. The resulting vector is obtained through plithogenic operators, incorporating the discrepancy degree of factors Fi concerning superior and inferior attributes. This resultant vector is then updated using a threshold function, assigning 1 if ai's values exceed 1, and 0 if they are less than 1.

The following describes the Plithogenic Operators:

$$u \wedge_P v = (1-c)[u \wedge_F v] + c[u \vee_F v]$$

where c represents the degree of discrepancy and

$\wedge_F$ represents the t_{norm} given by $u \wedge_F v = uv$

$\vee_F$ represents the t_{conorm} given by $u \vee_F v = u + v - uv$

Step 4: The Step 3 is iteratively performed until consecutive resultant vectors become equal

i.e $|Z(t+1) - Z(t)| \le \epsilon$

The resultant vector is the fixed point of the Plithogenic cognitive map

Step 5: The resultant vectors obtained with respect to each of the superior and inferior attribute values of each attributes are compared to determine the interdependence effect between the factors of the problem.

Application of Extended Plithogenic Cognitive Maps in Managerial Decision – Making

This section applies the proposed method of extended Plithogenic cognitive maps in managerial decision making.

Definition of the Problem:

Here is a clear-cut outline of the decision making problem. A business firm decides to promote the IoT based manufacturing system. The promotion strategies are considered as the factors pertaining to three attributes of Quality, Cost and Reliability.

Framing of the Factors and Attribute Values

The factors associated with the problem are as follows

F1 Progressive Automation of Manufacturing activities

F2 Engaging Manpower with Business Analytic Tools

F3 Digitalization of Supply chain management

F4 Augmenting Technology based infrastructure

F5 Building data security features

The superior and inferior attributes values with respect to each of the attributes are presented in Table 1

Table 1. Superior and Inferior Attribute Values

Attributes	**Attribute Values**	**Superior Attribute Value**	**Inferior Attribute Value**
Quality (G1)	Low (G11), Medium (G12), High (G13)	High (G13)	Low (G11)
Cost (G2)	Cheap (G21), Economic (G22), Expensive (G23)	Cheap (G21)	Expensive (G23)
Reliability (G3)	Meagre (G31), Moderate (G32), Extreme (G33)	Extreme (G33)	Meagre (G31)

Description of the Factors

The description of the factors are as follows

F1 Progressive Automation of Manufacturing Activities:

Presently, the manufacturing industries are automated using robotic technology. Automation is augmented into the production system to streamline the whole process effectively. The automated elements contribute to enhanced precision and overall performance of the system.

F2 Engaging Manpower with Business Analytic Tools:

The factor of integrating business analytic tool eases the routine operations of a manufacturing systems. The engagement of manpower with such effective analytic tools will facilitate the managerial to draw data driven insights for further strategic planning and improved decision making.

F3 Digital transformation of Supply Chain Administration:

The aspect of digitalization is gaining momentum in the industrial domains. This factor attributes to enhance the visibility and efficiency of the entire supply chain. This includes the adoption of advanced technologies integrated with IoT, blockchain and other features to optimize inventory management resulting in improved supply chain performance.

F4 Augmenting Technology-Based Infrastructure:

The technology -based infrastructure has to be augmented to support various business functions. The components of cloud computing, IT architects and other flexible tools have to be integrated into the production systems to get adapted to the evolving needs of the business.

F5 Building Data Security Features:

Ensuring data security is essential in the organizational systems. The data is a resource to any business entity and it has to be fortified to battle cyber threats. The implementation of efficient data security systems and mechanisms are required to uphold data integrity.

4.4 Formulation of Plithogenic Connection Matrix

The Plithogenic connection matrix with fuzzy incidental degrees constructed based on the experts of the opinion in the field of management as in Table 2. This Plithogenic connection matrix comprises fuzzy values.

Table 2. Plithogenic Connection Matrix

	F1	F2	F3	F4	F5
F1	0	0.7	0.8	0.5	0.5
F2	0.5	0	0.7	0.3	0.2
F3	0.8	0.3	0	0.5	0.5
F4	0.8	0.2	0.7	0	0.8
F5	0.7	0.3	0.5	0.7	0

The discrepancy degrees of the factors with respect to each of the superior and inferior attribute values are presented in the Table 3.

Table 3. Discrepancy degrees of the Factors with respect to the Superior and Inferior Attribute Values

Factors	Superior Attribute Values			Inferior Attribute Values		
	G13	G21	G33	G11	G23	G31
F1	0.5	0.5	0.8	0.5	0.5	0.2
F2	0.7	0.2	0.7	0.3	0.8	0.5
F3	0.5	0.5	0.5	0.5	0.5	0.5
F4	0.8	0.5	0.7	0.2	0.3	0.3
F5	0.3	0.3	0.5	0.5	0.7	0.5

The Step 3 and Step 4 is applied recurrently to determine the fixed point of ExPCM as presented in Table 4

Table 4. Fixed points of ExPCM

ON Position of Vectors	Superior Attribute Values	F1	F2	F3	F4	F5	Inferior Attribute Values	F1	F2	F3	F4	F5
(1 0 0 0 0)	G13	1	0.91	0.9	0.9	0.8	G11	1	0.79	0.9	0.61	0.75
	G21	1	0.76	0.9	0.75	0.71	G23	1	0.94	0.9	0.7	0.85
	G33	1	0.91	0.9	0.85	0.83	G31	1	0.85	0.9	0.65	0.75
	G13	0.83	1	0.85	0.86	0.77	G11	0.83	1	0.85	0.56	0.68

(0 1 0 0 0)	G21	0.83	1	0.85	0.68	0.66	G23	0.83	1	0.85	0.69	0.82
	G33	0.92	1	0.86	0.83	0.82	G31	0.74	1	0.85	0.62	0.71
(0 0 1 0 0)	G13	0.9	0.87	1	0.9	0.8	G11	0.9	0.73	1	0.61	0.75
	G21	0.9	0.7	1	0.75	0.71	G23	0.9	0.9	1	0.7	0.85
	G33	0.96	0.89	1	0.85	0.83	G31	0.84	0.77	1	0.65	0.75
(0 0 0 1 0)	G13	0.9	0.87	0.85	1	0.86	G11	0.9	0.73	0.85	1	0.9
	G21	0.9	0.7	0.85	1	0.86	G23	0.9	0.9	0.85	1	0.94
	G33	0.96	0.89	0.88	1	0.9	G31	0.84	0.77	0.85	1	0.9
(0 0 0 0 1)	G13	0.87	0.86	0.84	0.94	1	G11	0.85	0.7	0.83	0.76	1
	G21	0.85	0.67	0.83	0.85	1	G23	0.85	0.88	0.83	0.79	1
	G33	0.94	0.89	0.87	0.91	1	G31	0.76	0.73	0.78	0.79	1

Discussions

From the above table 4, the fixed point of ExPCM is obtained with special reference to superior and inferior attribute values. The values in the table shall be interpreted as follows. For instance the factor Progressive Automation of Manufacturing activities (F1) when posed in ON position, has effects over the other factors say F2,F3,F4,F5 with respect to both the superior and inferior attribute values of the attributes quality, cost and reliability. The interdependence effect between the factors with respect to the superior attribute values of quality and reliability is high in comparison with respect to the inferior attribute values of quality and reliability, whereas it is just the reverse in the case of the attribute cost.

The interdependence effects between the factors analysed with respect to the attribute values help the decision makers to know the intensity of the association between the factors in a more specific manner based on the managerial circumstances and considerations. If the decision makers are more concerned on a combination of high quality, cost and reliability the interdependence effect shall also be determined. Similarly, the different combinations of attribute values shall be considered and the interdependence effect shall be obtained.

This proposed extended Plithogenic cognitive maps is more efficient in comparison with Plithogenic cognitive maps with linguistic degrees of discrepancy. The latter is a modified version of conventional Plithogenic Cognitive Maps.

The same procedure is repeated with the conventional Plithogenic sets with linguistic degrees of discrepancy as discussed by Angel et al [15]. The Table 5 gives the fixed point of Plithogenic cognitive maps with linguistic degrees of discrepancy in which only the attributes are considered but not the attribute values.

Table 5. Fixed Points of PCM with Discrepancy Degrees of Attributes

ON Position of Vectors	Attributes	F1	F2	F3	F4	F5
(1 0 0 0 0)	G1	1	0.76	0.9	0.85	0.83
	G2	1	0.85	0.86	0.65	0.65
	G3	1	0.85	0.84	0.75	0.67

(0 1 0 0 0)	G1	0.83	1	0.85	0.83	0.82
	G2	0.86	1	0.79	0.58	0.6
	G3	0.71	1	0.76	0.65	0.6
(0 0 1 0 0)	G1	0.9	0.7	1	0.85	0.83
	G2	0.94	0.82	1	0.65	0.65
	G3	0.86	0.78	1	0.75	0.67
(0 0 0 1 0)	G1	0.90	0.7	0.85	1	0.9
	G2	0.94	0.82	0.82	1	0.86
	G3	0.86	0.78	0.76	1	0.84
(0 0 0 0 1)	G1	0.86	0.67	0.83	0.91	1
	G2	0.91	0.81	0.8	0.79	1
	G3	0.79	0.75	0.7	0.85	1

The associational impacts between the factors with respect to the attributes state the influence of each of the factors in a generic sense. There is no specification with respect to the attribute values. However, the nature of the attributes is not considered and henceforth this lacks specification. In the earlier method, the associational impacts between the factors with respect to specific cases cannot be determined and this limitation is overcome by the method proposed in this research work. The consideration of attribute values in finding the associational impacts is more compatible and comprehensive in nature.

Industrial Implications

The proposed model of extended Plithogenic cognitive maps has several managerial implications as it facilitates paradigm shift in handling complexity. The strategic evolution in industrial management is also signified. The inferences drawn from these decision models facilitates in comprehending the inter-dependencies among the factors of the study. The conflicting factors existing in the production scenarios and the fluctuating trends shall be well handled using such robust decision making model. As the industries are hurdled with multiple challenges, the extended plithogenic cognitive maps are proposed to be an effective decision making model to address diverse perspectives. The industries are expected to respond to the demands of environmental sustainability and hence this model supports in building resilience in an industrial system.

Conclusion

This paper proposes a novel decision model with the extension of plithogenic cognitive maps. The newly evolved model using extended plithogenic sets is an additional value. The proposed approach provides a comprehensive outlook of handling conflicting and intricate industrial challenges. The decision makers are provided with opportunities to get adapted into dynamic environments. The decision making problem discussed in this work demonstrates the efficacy of the developed approach in dealing with the factors associated with IoT manufacturing systems. The model shall be further extended with generalized plithogenic cognitive maps to address the same problem with different sets of attributes and attribute values. The problem shall be further explored in other areas of industrial decision making focusing on sustainability, optimality and resource efficacy.

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Chapter 13

Neutrosophy and Plithogeny Applied in Medicine:Toward a Multi-Valued Logic Framework for Clinical Decision-Making and Biomedical Uncertainty

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ABSTRACT

Medicine is a discipline defined by uncertainty. Diagnoses are rarely absolute: a symptom may simultaneously suggest multiple conditions, a treatment may be effective for one patient and harmful for another, and clinical evidence is frequently incomplete, contradictory, or indeterminate. Classical binary logic, where a proposition is either true or false, is inadequate for capturing this complexity. This paper explores how two advanced philosophical and mathematical frameworks developed by Florentin Smarandache, neutrosophy and plithogeny, can be systematically applied to medical science to address its inherent ambiguities.

Neutrosophy introduces a tripartite truth-value system (truth, indeterminacy, falsity) that transcends both classical and fuzzy logic, enabling richer representations of diagnostic uncertainty, contradictory evidence, and unknown factors. Plithogeny, as its extension, models entities characterized by multiple attributes simultaneously, accommodating the multidimensional nature of disease, patient profiles, and biomedical data. We examine these frameworks in the contexts of diagnostic reasoning, clinical decision support systems, medical ontologies, epidemiology, and multi-criteria drug evaluation. This chapter argues that neutrosophic and plithogenic tools offer transformative potential for advancing precision medicine, artificial intelligence in healthcare, and evidence-based clinical practice.

Keywords: Neutrosophy, plithogeny, neutrosophic logic, medical diagnosis, clinical uncertainty, plithogenic set, evidence-based medicine, artificial intelligence in medicine, fuzzy logic, indeterminacy

INTRODUCTION

Medical practice fundamentally operates in conditions of partial information and uncertainty [1]. A physician seldom encounters a textbook case where symptoms align perfectly with a single diagnosis; more often, multiple diagnoses must be weighed, probabilities estimated, and decisions made despite incomplete or conflicting evidence. Classical propositional logic, which assigns a truth value of either true (1) or false (0) to any statement, fails to adequately represent this reality. Even fuzzy logic, which allows for partial truth values on a continuous scale from 0 to 1. does not capture the full complexity of medical reasoning, as it does not model indeterminacy as a distinct epistemic category.

Neutrosophy, a new branch of philosophy developed by Florentin Smarandache [2], addresses this gap by introducing a three-component truth-value: T (degree of truth), I (degree of indeterminacy), and F (degree of falsity), each independently ranging within [0, 1]. This framework generalizes classical logic [3] , fuzzy logic [4] , and intuitionistic fuzzy logic [5], making it uniquely suited to represent the ambiguities inherent in medicine. A symptom can be 'somewhat indicative of disease X' (partial truth), 'inconclusive with respect to disease Y' (indeterminacy), and 'unlikely to indicate disease Z' (partial falsity) simultaneously.

Plithogeny, also introduced by Smarandache [6], represents a further generalization. Where neutrosophy focuses on truth/indeterminacy/falsity as a tripartite structure, plithogeny models entities as characterized by multiple, potentially conflicting attributes simultaneously accommodating the rich multidimensionality of biological systems, patient phenotypes, and complex medical data. Together, these frameworks constitute a powerful mathematical and philosophical toolkit for reimagining how medicine processes knowledge and makes decisions.

The rest of the Chapter is organized as follows. First, an overview of neutrosophic principles and their mathematical formalism is presented. Next, plithogeny and plithogenic set theory are introduced. This is followed by a discussion of medical applications of neutrosophy, including diagnostic reasoning and clinical decision support. Subsequently, plithogenic applications in medicine, such as multi-attribute patient classification and drug evaluation, are explored. Finally, future directions and challenges are outlined, and the paper concludes with a summary of key findings.

Neutrosophy: Foundations and Formalism

Origins and Definition

Neutrosophy (from Latin neuter- neutral, and Greek sophia -wisdom) was established by Florentin Smarandache as a branch of philosophy studying the origin, nature, and scope of neutralities and their interactions with different ideational spectra [7]. Its etymological essence captures what sets it apart from prior philosophical traditions: the systematic inclusion of the neutral.

The fundamental thesis of neutrosophy [7] states that any idea <A> is T% true, I% indeterminate, and F% false, where T, I, and F are standard or non-standard subsets included in the non-standard interval $[0^{-}, 1^{+}]$. This departure from binary logic is not merely mathematical elegance, it reflects the ontological reality that most propositions in complex disciplines such as medicine occupy intermediate or contested epistemic territory.

Between any idea <A> and its opposite <Anti-A>, neutrosophy posits a continuum of neutralities <Neut-A>. This differs fundamentally from Hegelian dialectics [8, 9], which recognizes only thesis and antithesis. Neutrosophy [10] introduces a 'pluradic pyramidal scheme' in which ideas evolve through multiple states, mixing with neutral concepts before settling into a new form. In medicine, this captures how clinical knowledge evolves not in sudden reversals but in incremental refinements punctuated by periods of uncertainty.

Neutrosophic Logic and Set Theory

Neutrosophic logic [10] generalizes fuzzy logic by allowing the truth-value of a statement to be an ordered triple (T, I, F) rather than a single scalar. This enables the representation of phenomena such as contradictory evidence, where T and F are both high; unknown factors, where I is high; and certainty, where T approaches 1 and both I and F approach 0. Each component is independent, meaning that their sum need not equal 1, a critical difference from probability theory.

A neutrosophic set N on a universal set U assign to each element $x \in U$ a triple (T_A(x), I_A(x), F_A(x)), where each component is a real standard or non-standard subset of $[0^{-}, 1^{+}]$. In single-valued neutrosophic sets (SVNS) [11], these components are constrained to [0, 1] with $T + I + F \leq 3$, enabling practical computational implementation. Operations such as union, intersection, and complement are defined accordingly, forming a rich algebraic structure applicable to data classification and decision-making [12-22].

Neutrosophic Probability and Statistics

Beyond logic and sets, Smarandache [23] extended neutrosophy into probability theory. Neutrosophic probability assigns to each event a triple (chance of occurrence, chance of indeterminate occurrence, chance of non-occurrence) rather than a single probability value. This is particularly relevant in medicine, where clinical trials yield probabilistic estimates that are always subject to measurement error, incomplete randomization, and population heterogeneity, all forms of indeterminacy.

Neutrosophic statistics [23], by extension, concerns itself with data sets that are imprecise, uncertain, or partially known. This includes handling missing data, contradictory laboratory results, and patient self-reports that are inconsistent over time, ubiquitous challenges in clinical research.

Plithogeny: From Neutrosophy to Multi-Attribute Frameworks

Definition and Etymology

Plithogeny (from Greek plithos, plurality/multitude, and genesis, origin/creation) is a generalization of neutrosophy introduced by Smarandache [24] in 2017. Whereas neutrosophy models propositions as having three truth-values (T, I, F), plithogeny models entities as characterized by multiple attributes simultaneously, each with their own appurtenance degree functions and contradiction (dissimilarity) degree functions.

The plithogenic set [25] extends classical set [26], fuzzy set [4], intuitionistic fuzzy set [5] , and neutrosophic sets [7] by introducing an attribute value spectrum for each defining characteristic of elements. This allows the representation of the full range of possible states an entity can occupy with respect to a given attribute, weighted by a contradiction degree function that measures how far any given value is from a designated dominant value.

Plithogenic Set Formal Structure

Formally, a plithogenic set P is defined over a universe of discourse U by: a non-empty set of attribute values $V = \{v_1, v_2, ..., v_n\}$; an attribute value appurtenance degree function $d: U \times V \rightarrow P([0,1]^s)$, where $s \in \{1, 2, 3\}$ depending on whether the set is fuzzy (s=1), intuitionistic fuzzy (s=2), or neutrosophic (s=3); and an attribute value contradiction degree function $c: V \times V \rightarrow [0, 1]$ that measures the dissimilarity between any two attribute values.

The plithogenic aggregation operators, intersection, union, and complement combine information from all attribute values using a weighted formula that incorporates the contradiction degree function. This ensures that dominant attribute values contribute more strongly to aggregated decisions, while highly dissimilar (contradictory) values receive differential weighting. The result is a mathematically rigorous framework for multi-criteria decision analysis.

Plithogenic Logic and Probability

Plithogenic logic [24,25] extends neutrosophic logic [7] by allowing the truth-value of a proposition to vary across an entire spectrum of attribute values rather than just a single triple (T, I, F). This reflects the reality that a proposition can be evaluated differently depending on context, perspective, or measurement method, all modeled as distinct attribute values. Plithogenic probability generalizes neutrosophic probability by assigning plithogenic truth-value distributions to events, capturing the full complexity of scenarios where outcomes depend on multiple interacting factors.

Applications of Neutrosophy in Medicine

Diagnostic Reasoning Under Uncertainty

Medical diagnosis [27, 28, 29] is perhaps the most natural domain for neutrosophic logic application. Consider a patient presenting with fatigue, mild fever, and joint pain. A physician must assess: is this rheumatoid arthritis, systemic lupus erythematosus, viral infection, or another condition? Classical binary assessment (disease present / disease absent) is clinically unrealistic. Fuzzy logic allows partial truth but cannot distinguish between 'partially true due to genuine partial fit' and 'partially true due to missing information', the latter being classic indeterminacy.

With neutrosophic logic, the physician can formally represent: 'Evidence for rheumatoid arthritis: $T= 0.6$, $I= 0.25$, $F= 0.15$' and 'Evidence for viral arthritis: $T=0.5$, $I=0.35$, $F=0.15$.' The separate indeterminacy component captures what is not yet known, awaiting laboratory results, imaging, or further history, and can guide targeted investigation. This richer representation supports more transparent and auditable clinical reasoning.

Clinical decision support systems (CDSS) built on neutrosophic logic have been proposed [30] and studied for conditions including cancer screening, sepsis identification, and cardiac risk stratification. In each domain, the ability to explicitly model indeterminacy reduces premature diagnostic closure, a well-documented source of medical error, and supports appropriate escalation of uncertainty to specialists or further testing.

Medical Ontologies and Knowledge Representation

Medical ontologies such as SNOMED CT and ICD-11 represent clinical concepts in hierarchical structures. However, many clinical concepts are inherently imprecise: 'mild hypertension,' 'probable dementia,' 'borderline diabetes.' These terms resist binary categorization. Neutrosophic set theory provides a natural enrichment of medical ontologies by allowing each concept membership to be represented as (T, I, F) triples, capturing graduated degrees of category membership, diagnostic confidence, and definitional ambiguity.

Furthermore, neutrosophy's fundamental theory that every idea tends to be neutralized by opposing ideas — has epistemological value in medicine. It formalizes how clinical guidelines are always provisional: a treatment endorsed as the standard of care (T-value high) may have accumulated evidence of harm in specific populations (rising F-value) and unresolved questions for subpopulations (high I-value). Neutrosophic ontologies can dynamically represent this evolving knowledge state.

Neutrosophic Epidemiology

Epidemiological studies are foundational to evidence-based medicine but are inherently probabilistic and subject to indeterminacy. Confounding variables, incomplete follow-up, measurement error, and population heterogeneity all introduce indeterminate components into epidemiological estimates. Neutrosophic probability, by assigning (T, I, F) triples to event occurrence, provides a more complete characterization of epidemiological risk than classical probability alone.

For example, in a cohort study examining the association between a dietary factor and cardiovascular disease, the neutrosophic probability of a participant developing disease might be represented as (0.12, 0.06, 0.82), where 0.12 is the classical probability estimate, 0.06 reflects measurement indeterminacy from dietary recall bias, and 0.82 represents non-occurrence. This representation makes explicit the epistemic limitations of the study, enabling more nuanced meta-analytic synthesis.

Neutrosophic Approaches in Pharmacology

Drug development and pharmacological evaluation involve inherent uncertainty at every stage: from target identification (is this protein truly implicated in the disease mechanism?) to clinical efficacy (does the drug work in the full target population?) and safety (are observed adverse events causally linked to the drug?). Each of these questions involves not just probabilistic answers but irreducible indeterminacies.

Neutrosophic frameworks have been proposed for evaluating drug safety signals in pharmacovigilance databases, where adverse event reports are often incomplete, inconsistently coded, and subject to reporting bias. A neutrosophic safety signal for a drug-event pair, for instance, (T=0.55, I=0.30, F=0.15), explicitly conveys both the degree of evidence for a causal association and the degree of unresolved uncertainty, guiding regulatory decision-making more transparently than simple frequentist p-values.

Applications of Plithogeny in Medicine

Multi-Attribute Patient Classification

Patients are not one-dimensional entities. They are characterized simultaneously by genetic factors, comorbidities, lifestyle attributes, socioeconomic determinants, medication histories, and psychosocial profiles. Classical classification schemes such as staging systems for cancer or risk scores for cardiovascular disease, aggregate these attributes into single scalars, inevitably losing information. Plithogenic set theory preserves multi-attribute complexity by modeling each patient as an element of a plithogenic set, with appurtenance degrees defined for every relevant attribute [31, 32].

For example, in oncology, a patient with colorectal cancer might be characterized across attribute dimensions including: tumor histology (dominant value: adenocarcinoma), molecular profile (dominant value: microsatellite stable), stage (dominant value: Stage IIIB), comorbidity burden (dominant value: moderate), and treatment history (dominant value: chemotherapy-naive). Each attribute has a contradiction degree function measuring how different a given patient's value is from the dominant, enabling nuanced risk stratification that preserves individualized information while enabling population-level pattern recognition.

Multi-Criteria Drug and Treatment Evaluation

Therapeutic decision-making routinely requires balancing multiple competing criteria: efficacy, safety, cost, patient preference, and compatibility with existing treatments. Multi-criteria decision analysis (MCDA) methods are increasingly used in health technology assessment, but classical MCDA models require precise numerical inputs and do not handle interdependence among criteria or linguistic imprecision in evaluator judgments.

Plithogenic MCDA addresses these limitations. Each treatment option is modeled as an element of a plithogenic set defined over criteria attributes. Appurtenance degrees capture how well each treatment aligns with each criterion, while contradiction degree functions model how conflicting criteria trade off against each other. Plithogenic aggregation operators synthesize this multi-attribute information into a composite recommendation that preserves uncertainty information rather than collapsing it prematurely. This approach has been applied to antimicrobial treatment selection and immunotherapy sequencing decisions.

Plithogenic Medical Imaging and AI

Medical imaging including radiography, computed tomography, magnetic resonance imaging, and pathological image analysis generates multi-attribute data. A lesion on a scan is characterized simultaneously by its size, shape, intensity distribution, border characteristics, enhancement pattern, and location. Machine learning models trained on such data typically output a single probability score; plithogenic models can instead output a multi-attribute assessment that maps these dimensions independently, providing richer information for radiologists and clinicians.

Deep learning architectures augmented with plithogenic set operations can represent the uncertainty of each attribute assessment independently, enabling uncertainty quantification in AI-assisted diagnosis that is both mathematically rigorous and clinically interpretable. Preliminary studies have demonstrated this approach for pulmonary nodule characterization and diabetic retinopathy grading, showing that plithogenic models achieve comparable or superior performance to classical deep learning while providing richer uncertainty characterization.

Epidemiological Modeling with Plithogenic Probability

Infectious disease epidemiology routinely grapples with multi-attribute uncertainty. During the emergence of a novel pathogen, epidemiologists must simultaneously assess transmissibility (by multiple routes), severity (across demographic subgroups), immune evasion (relative to existing immunity), and environmental persistence — each with its own uncertainty profile. Plithogenic probability provides a natural framework for this multi-dimensional uncertainty quantification.

In pandemic preparedness modeling, plithogenic distributions over key epidemiological parameters can capture correlated uncertainties (e.g., higher transmissibility may covary with lower immune evasion in some evolutionary scenarios) while maintaining the independence of distinct epistemic components. This enables more realistic sensitivity analyses and scenario planning, informing public health interventions with clearer representations of what is known, what is unknown, and what is contested.

Future Directions and Challenges

Integration with Clinical Data Systems

The practical implementation of neutrosophic and plithogenic frameworks in clinical medicine requires integration with electronic health record (EHR) systems, clinical decision support tools, and medical AI platforms. Current EHR systems are designed around structured, predominantly binary or categorical data. Adapting these systems to accommodate neutrosophic (T, I, F) triples or plithogenic multi-attribute representations will require new data standards, interface designs, and clinician training.

Promising avenues include hybrid systems where neutrosophic/plithogenic representations exist in a dedicated reasoning layer that communicates with underlying EHR data in standard formats, translating uncertainty information into actionable clinical alerts or summaries. Standards bodies such as HL7 FHIR may be extended to include uncertainty quantification extensions compatible with neutrosophic representations.

Educational and Cultural Challenges

The adoption of neutrosophic and plithogenic thinking in medicine also requires a cultural shift. Medical education has historically emphasized pattern recognition and decisive action, sometimes at the expense of explicit uncertainty communication. Physicians trained in classical logic may find the tripartite truth-value system counterintuitive. However, increasing emphasis on diagnostic humility, shared decision-making, and evidence-based uncertainty communication in modern medical education creates a favorable context for these frameworks.

Validation and Empirical Research

A critical need for the field is rigorous empirical validation of neutrosophic and plithogenic medical applications. While theoretical justifications are strong, clinical validation studies are required to demonstrate that CDSS built on these frameworks improve patient outcomes, reduce diagnostic error, or enhance clinician confidence relative to classical or fuzzy logic alternatives. Such studies should employ prospective cohort designs with clinically meaningful endpoints and include assessment of the interpretability and usability of uncertainty outputs by clinicians.

Conclusion

Medicine is, at its core, a discipline of managed uncertainty. From the first clinical encounter to the final therapeutic decision, physicians navigate a landscape of partial information, conflicting evidence, and irreducible unknowns. Classical and even fuzzy logical frameworks, while useful, are insufficient to capture the full depth of this epistemic complexity.

Neutrosophy, through its tripartite (T, I, F) truth-value structure, provides a philosophically grounded and mathematically rigorous means of representing diagnostic uncertainty, contradictory clinical evidence, and genuine indeterminacy. Plithogeny, as its multi-attribute extension, opens further vistas for modeling the multidimensional nature of patients, diseases, and therapeutic contexts. Together, they constitute a comprehensive and transformative framework for rethinking medical knowledge representation, clinical decision-making, and biomedical AI.

The application of these frameworks to medical diagnosis, pharmacology, epidemiology, clinical imaging, and multi-criteria treatment evaluation demonstrates both conceptual richness and practical promise. As precision medicine increasingly demands individualized, data-driven, uncertainty-aware clinical reasoning, neutrosophic and plithogenic tools stand ready to provide the logical and mathematical infrastructure this evolution requires.

We anticipate that future interdisciplinary collaboration among philosophers, mathematicians, biomedical informaticists, and clinicians will catalyze the translation of these powerful frameworks from theoretical innovation to clinical practice, ultimately serving patients who deserve decisions made with the fullest and most honest representation of what medicine knows, what it suspects, and what it does not yet know.

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Chapter 14

RNNWAA Operator Based MCGDM Strategy in Rough Neutrosophic Environment

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ABSTRACT

The purpose of the chapter is to develop a new decision-making strategy to the Rough Neutrosophic Number (RNN) environment which we call the RNNWAA strategy. The RNN is derived from the hybrid concept of theories of rough sets and neutrosophic sets. It comprehensively deals with uncertainty, incompleteness and inconsistent information. To demonstrate the applicability of the developed strategy, a multi criteria group decision making problem is solved.

Keywords: Fuzzy set, neutrosophic set, single valued neutrosoophic set, rough set, rough neutrosopohic set, rough neutrosophic sets, MCDM, MCGDM

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INTRODUCTION

Smarandache established the concept of Neutrosophic Set (NS) [1] by extending the Fuzzy Set (FS) [2]) and Intuitionistic FS [3] to handle uncertainty comprehensively. Single-Valued NSs (SVNSs) [4] were grounded as a subclass of the NS for dealing with real–life decision masking problems. Numerous studies have detailed the theoretical enhancements and diverse applications of NSs and SVNSs [5-14]. SVNSs have been deeply studied for Multi Attribute Decision Making (MADM) and Multi Attribute Group Decision Making (MCGDM) [15-41]. Pawlak grounded the Rough Set (RS) [42] to handle uncertainty and incomplete information. Broumi et al. [43] developed Rough NS (RNS) by integrating the RS [43] and SVNS [4] to deal with uncertainty comprehensively. A comprehensive overview of RNSs has been presented in [44, 45].

MADM strategy selects the best option form the feasible options subject to a given conflicting criterion. Several MADM methods have been established in the Rough Neutrosophic Sets (RNN) environment. Mondal and Pramanik [46] integrated the Grey Relational Analysis (GRA) [47]and MADM in the RNN setting. Using trigonometric Hamming similarity measures [48], several MADM strategies have been proposed in the RNN setting. TOPSIS [49], correlation coefficient measure based MADM strategy [50], projection and bidirectional projection measure based MADM strategies [51] were further developed in RNN setting. Using averaging operators, Mondal, Pramanik, and Giri [52] established four MADM strategies. Pramanik and Smarandache developed MABAC [53] and COPRAS [54] in the RNN setting.

Research gap: No studies have been proposed using Rough Neutrosophic Number Weighted Arithmetic Average (RNNWAA) operator based MCGDM in the RNN environment.

Motivation: The research gap motivates us to develop the RNNWAA operator based MCGDM strategy in RNN settings.

The rest of the paper is presented as follows. Some operators of the RNSs are presented in Section II. RNNWAA method is developed in Section III. Section IV solves an illustrative example of an MADM problem using the RNNWAA strategy. Section V concludes the chapter by stating the future research direction.

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Rough Neutrosophic Number Weighted Arithmetic Average (RNNWAA) Operator

We propose Rough Neutrosophic Number Weighted Arithmetic Average (RNNWAA)operator, a score function for RNS, best and worse rough neutrosophic values of two RNSs in this section.
Let X be a non-null set and R be an equivalence relation on Y. Assume that P is a NS in Y with the membership function T_P, indeterminacy function I_P and non-membership function F_P. Let $\left\langle \underline{N}(P_j), \overline{N}(P_j) \right\rangle$ be a collection of RNNs where, j = 1, 2, …n is the Decision Makers (DMs). The RNNWAA operator is presented as follows:
RNNWAA$(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{n} w_k\, x_k$
$[\{\left(1-\prod_{k=1}^{n}(1-\underline{Tj})\right)^{wk}, \left(\prod_{k=1}^{n}(\underline{Ij})\right)^{wk}, \left(\prod_{k=1}^{n}(\underline{Fj})\right)^{wk}\}, \{\left(1-\prod_{k=1}^{n}(1-\overline{Tj})\right)^{wk}, \left(\prod_{k=1}^{n}(\overline{Ij})\right)^{wk}, \left(\prod_{k=1}^{n}(\overline{Fj})\right)^{wk}\}]$
(1)
and w_k is the weight vector of DM and $\sum_{k=0}^{n} w_k = 1$.

The RNNWAA operator satisfies the following properties.

P1. Idempotent law: If $x_i = x$ for i = 1, 2, …, n then,

RNNWAA$(x, x, \ldots, x) = \sum_{k=0}^{n} w_k\, x = x\sum_{k=0}^{n} w_k = x.1 = x$

P2. Boundedness: The operator is bounded.

Proof.
Let $x_{\min} = \min(x_1, x_2, \ldots, x_n)$, $x_{\max} = \max(x_1, x_2, \ldots, x_n)$ for i = 1, 2, …, n then,
$x_{\min} \leq$ RNNWAA$(x_1, x_2, \ldots, x_n) \leq x_{\max}$
Hence, the operator is bounded.
P3. Monotonicity: If $x_i \geq x_i^*$ for i = 1, 2, …, n then, RNNWAA$(x_1, x_2, \ldots, x_n) \leq$ RNNWAA$(x_1^*, x_2^*, \ldots, x_n^*)$

Proof.
RNNWAA$(x_1, x_2, \ldots, x_n)$ − RNNWAA$(x_1^*, x_2^*, \ldots, x_n^*)$
$= \sum_{k=0}^{n} w_k\, x_k - \sum_{k=0}^{n} w_k\, x_k *$
$\geq$ (0, 0, 0), (0, 0, 0)>
⇨ $\sum_{k=0}^{n} w_k\, x_k \geq \sum_{k=0}^{n} w_k\, x_k *$
Hence, the operator is monotonic.

P4. Commutativity: If $(x_1^0, x_2^0, \ldots, x_n^0)$ be any permutation of $(x_1, x_2, \ldots, x_n)$ then,

RNNWAA$(x_1, x_2, \ldots, x_n)$ = RNNWAA$(x_1^0, x_2^0, \ldots, x_n^0)$
Proof.
RNNWAA$(x_1, x_2, \ldots, x_n) = \sum_{k=0}^{n} w_k\, x_k$, $\sum_{k=0}^{n} w_k = 1$
Since $\{x_1^0, x_2^0, \ldots, x_n^0\} = \{x_1, x_2, \ldots, x_n\}$,
RNNWAA$(x_1^0, x_2^0, \ldots, x_n^0) = \sum_{k=0}^{n} w_k\, x_k$, $\sum_{k=0}^{n} w_k = 1$
⇨ RNNWAA$(x_1, x_2, \ldots, x_n)$ = RNNWAA$(x_1^0, x_2^0, \ldots, x_n^0)$ for any order of set elements, value of the aggregation operator remains unaltered.

Best, Worst and Zero RNNs

We define best worst and zero rough neutrosophic values in decision making situation as follows:
B(RNN) = <(1, 0, 0), (1, 0, 0)>
W(RNN) = <(0, 1, 1), (0, 1, 1)>
Z(RNN) = <(0, 0, 0), (0, 0, 0)>

Score function

The score function of an RNS which is defined as follows:

$$S(j) = \frac{2\underline{Tj}-\underline{Ij}-\underline{Fj}+2\overline{Tj}-\overline{Ij}-\overline{Fj}}{4} \qquad (2)$$

where, j = $(\underline{Tj},\ \underline{Ij},\ \underline{Fj}), (\overline{Tj},\ \overline{Ij},\ \overline{Fj})$

Theorem 1. The score function S(j) is bounded.

Proof. Since

$0 \le \underline{Tj} \le 1$, $0 \le \underline{Ij} \le 1$, $0 \le \underline{Fj} \le 1$, $0 \le \overline{Tj} \le 1$, $0 \le \overline{Ij} \le 1$, $0 \le \overline{Fj} \le 1$

⇨ $0 \le 2\underline{Tj} + 2\overline{Tj} \le 2 + 2$ & $0 \le 2\underline{Tj} + 2\overline{Tj} \le 4$

⇨ $0 \le \underline{Ij} + \underline{Fj} + \overline{Ij} + \overline{Fj} \le 1 + 1 + 1 + 1$

⇨ $0 \le \underline{Ij} + \underline{Fj} + \overline{Ij} + \overline{Fj} \le 4$

⇨ $-4 \le 2\underline{Tj} + 2\overline{Tj} - \underline{Ij} - \underline{Fj} - \overline{Ij} - \overline{Fj} \le 4$

⇨ $-1 \le \frac{2\underline{Tj}+2\overline{Tj}-\underline{Ij}-\underline{Fj}-\overline{Ij}-\overline{Fj}}{4} \le 1$

⇨ $-1 \le S(j) \le 1$

Theorem 2. Monotonicity: If j ≤ j* for i = 1, 2, …, n then, S(j) ≤ S(j*)

Proof.
S(j) − S(j*) ≤ 0 when W(RNS) ≤ j ≤ j*≤ B(RNS)
⇨ S(j) ≤ S(j*)
Hence, S(j) is monotonic.

Subtraction Addition and Multiplication Operation of Two RNNs

Subtraction, addition and multiplication of two RNSs $x\left\langle(\underline{T}_x,\underline{I}_x,\underline{F}_x),(\overline{T}_x,\overline{I}_x,\overline{F}_x)\right\rangle$ and $y\left\langle(\underline{T}_y,\underline{I}_y,\underline{F}_y),(\overline{T}_y,\overline{I}_y,\overline{F}_y)\right\rangle$ are defined as follows:

$$x-y=\left\langle\left(\left|\underline{T}_x-\underline{T}_y\right|,\max(\underline{I}_x,\underline{I}_y),\left|\underline{F}_x-\underline{F}_y\right|\right),\left(\left|\overline{T}_x-\overline{T}_y\right|,\max(\overline{I}_x,\overline{I}_y),\left|\overline{F}_x-\overline{F}_y\right|\right)\right\rangle \qquad (3)$$

$$x\oplus y=\left\langle\left(\underline{T}_x+\underline{T}_y-\underline{T}_x\underline{T}_y,\ \underline{I}_x\underline{I}_y,\ \underline{F}_x\underline{F}_y\right),\ \left(\overline{T}_x+\overline{T}_y-\overline{T}_x\overline{T}_y,\ \overline{I}_x\overline{I}_y,\ \overline{F}_x\overline{F}_y\right)\right\rangle \qquad (4)$$

$$x\otimes y=\left\langle\left(\underline{T}_x\underline{T}_y,\underline{I}_x+\underline{I}_y-\underline{I}_x\underline{I}_y,\ \underline{F}_x+\underline{F}_y-\underline{F}_x\underline{F}_y\right),\ \left(\overline{T}_x\overline{T}_y,\ \overline{T}_x+\overline{T}_y-\overline{T}_x\overline{T}_y\ ,\overline{F}_x+\overline{F}_y-\overline{F}_x\overline{F}_y\right)\right\rangle \qquad (5)$$

RNNWAA Operator Based MCGDM Strategy

In this section, we develop the RNNWAA strategy. Consider an MCGDM problem having m alternatives and n attributes The steps of the strategy are as follows:

Step 1: Formulate the RN decision matrix ($D_i×_j$) with respect to expert (E_j), where the benefit and cost variables are b_{mn} and c_{mn} respectively. The linguistic scale used to construct RN decision matrix is presented in Table 1.

Table 1. Scale for RN decision matrix $D_i×_j$

Linguistic terms		$<(\underline{T},\underline{I},\underline{F}),(\overline{T},\overline{I},\overline{F})>$
PL	Positively low	<(0.1, 0.8, 0.9), (0.2, 0.8, 0.9]>
VL	Very low	<(0.2, 0.7, 0.8), (0.3, 0.7, 0.8)>
L	Low	<(0.3, 0.6, 0.7), (0.4, 0.6, 0.7)>
UA	Under average	<(0.4, 0.5, 0.6), (0.5, 0.5, 0.6)>
A	Average	<(0.5, 0.4, 0.5), (0.6, 0.4, 0.5)>
AA	Above average	<(0.6, 0.3, 0.4), (0.7, 0.3, 0.4)>
H	High	<(0.7, 0.2, 0.3), (0.8, 0.2, 0.3)>
VH	Very high	<(0.8, 0.1, 0.2), (0.8, 0.1, 0.2)>
PH	Positively high	<(0.9, 0.1, 0.2), (0.9, 0.1, 0.1)>

Using the linguistic scale, RN decision matrix with respect to expert j is given in Table 2.

Table 2 RN decision matrix with respect to expert j

	C_1	C_2	$\cdots$	C_n
A_1	$\left\langle(\underline{T}_{11}\underline{I}_{11}\underline{F}_{11}),(\overline{T}_{11}\overline{I}_{11}\overline{F}_{11})\right\rangle_j$	$\left\langle(\underline{T}_{12}\underline{I}_{12}\underline{F}_{12}),(\overline{T}_{12}\overline{I}_{12}\overline{F}_{12})\right\rangle_j$	$\cdots$	$\left\langle(\underline{T}_{1n}\underline{I}_{1n}\underline{F}_{1n}),(\overline{T}_{1n}\overline{I}_{1n}\overline{F}_{1n})\right\rangle_j$
A_2	$\left\langle(\underline{T}_{21}\underline{I}_{21}\underline{F}_{21}),(\overline{T}_{21}\overline{I}_{21}\overline{F}_{21})\right\rangle_j$	$\left\langle(\underline{T}_{22}\underline{I}_{22}\underline{F}_{22}),(\overline{T}_{22}\overline{I}_{22}\overline{F}_{22})\right\rangle_j$	$\cdots$	$\left\langle(\underline{T}_{2n}\underline{I}_{2n}\underline{F}_{2n}),(\overline{T}_{2n}\overline{I}_{2n}\overline{F}_{2n})\right\rangle_j$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
A_m	$\left\langle(\underline{T}_{m1}\underline{I}_{m1}\underline{F}_{m1}),(\overline{T}_{m1}\overline{I}_{m1}\overline{F}_{m1})\right\rangle_j$	$\left\langle(\underline{T}_{m2}\underline{I}_{m2}\underline{F}_{m2}),(\overline{T}_{m2}\overline{I}_{m2}\overline{F}_{m2})\right\rangle_j$	$\cdots$	$\left\langle(\underline{T}_{mn}\underline{I}_{mn}\underline{F}_{mn}),(\overline{T}_{mn}\overline{I}_{mn}\overline{F}_{mn})\right\rangle_j$

Here $\left\langle\overline{T}_{ij},\overline{I}_{ij},\overline{F}_{ij}\right\rangle$ *and* $\left\langle\underline{T}_{ij},\underline{I}_{ij},\underline{F}_{ij}\right\rangle$ is the rough neutrosophic sets of i-th row and j-th column.

Step 2: Aggregate the decision matrix for obtaining average RN decision matrix. We use Eq. (1) for aggregation operation. Aggregated RN decision matrix is constructed as in Table 3. In here,
$A(xj_{mn})=\left\langle(\underline{T^{Ag}}_j,\underline{I^{Ag}}_j,\underline{F^{Ag}}_j),(\overline{T^{Ag}}_j,\overline{I^{Ag}}_j,\overline{F^{Ag}}_j)\right\rangle$ notation is used to express aggregated RN sets.

Table 3 Aggregated RN decision matrix $(D^{Ag})_i×_j$

$$\begin{array}{c|cccc}
 & C_1 & C_2 & \cdots & C_n \\
\hline
A_1 & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{11}, \underline{I^{Ag}}_{11}, \underline{F^{Ag}}_{11}\right), \\ \left(\overline{T^{Ag}}_{11}, \overline{I^{Ag}}_{11}, \overline{F^{Ag}}_{11}\right)\end{matrix} \right\rangle & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{12}, \underline{I^{Ag}}_{12}, \underline{F^{Ag}}_{12}\right), \\ \left(\overline{T^{Ag}}_{12}, \overline{I^{Ag}}_{12}, \overline{F^{Ag}}_{12}\right)\end{matrix} \right\rangle & \cdots & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{1n}, \underline{I^{Ag}}_{1n}, \underline{F^{Ag}}_{1n}\right), \\ \left(\overline{T^{Ag}}_{1n}, \overline{I^{Ag}}_{1n}, \overline{F^{Ag}}_{1n}\right)\end{matrix} \right\rangle \\
A_2 & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{21}, \underline{I^{Ag}}_{21}, \underline{F^{Ag}}_{21}\right), \\ \left(\overline{T^{Ag}}_{21}, \overline{I^{Ag}}_{21}, \overline{F^{Ag}}_{21}\right)\end{matrix} \right\rangle & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{22}, \underline{I^{Ag}}_{22}, \underline{F^{Ag}}_{22}\right), \\ \left(\overline{T^{Ag}}_{22}, \overline{I^{Ag}}_{22}, \overline{F^{Ag}}_{22}\right)\end{matrix} \right\rangle & \cdots & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{2n}, \underline{I^{Ag}}_{2n}, \underline{F^{Ag}}_{2n}\right), \\ \left(\overline{T^{Ag}}_{2n}, \overline{I^{Ag}}_{2n}, \overline{F^{Ag}}_{2n}\right)\end{matrix} \right\rangle \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_m & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{m1}, \underline{I^{Ag}}_{m1}, \underline{F^{Ag}}_{m1}\right), \\ \left(\overline{T^{Ag}}_{m1}, \overline{I^{Ag}}_{m1}, \overline{F^{Ag}}_{m1}\right)\end{matrix} \right\rangle & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{m2}, \underline{I^{Ag}}_{m2}, \underline{F^{Ag}}_{m2}\right), \\ \left(\overline{T^{Ag}}_{m2}, \overline{I^{Ag}}_{m2}, \overline{F^{Ag}}_{m2}\right)\end{matrix} \right\rangle & \cdots & \left\langle \begin{matrix}\left(\underline{T^{Ag}}_{mn}, \underline{I^{Ag}}_{mn}, \underline{F^{Ag}}_{mn}\right), \\ \left(\overline{T^{Ag}}_{mn}, \overline{I^{Ag}}_{mn}, \overline{F^{Ag}}_{mn}\right)\end{matrix} \right\rangle
\end{array}$$

Here $\left\langle \left(\underline{T^{Ag}}_{ij}, \underline{I^{Ag}}_{ij}, \underline{F^{Ag}}_{ij}\right), \left(\overline{T^{Ag}}_{ij}, \overline{I^{Ag}}_{ij}, \overline{F^{Ag}}_{ij}\right)\right\rangle$ is the aggregated rough neutrosophic sets of i-th row and j-th column.

Step 3: Construct the average solution matrix of attribute weights with regards to experts (j). The weights are shown as follows:

$W = [w_j]_{1xm}$

Step 4: Construct the matrix of average Attribute Weights (AV) with regards to scores that are determined in Table 4. The constructed attributes weight matrix is shown as below:

In here, $AV_n = \left\langle \left(\underline{T^{AV}}_{ij}, \underline{I^{AV}}_{ij}, \underline{F^{AV}}_{ij}\right), \left(\overline{T^{AV}}_{ij}, \overline{I^{AV}}_{ij}, \overline{F^{AV}}_{ij}\right)\right\rangle$

where the average weight of attribute x_{mn} with respect to scores taken from the decision matrix.

Table 4 Average attribute weights with regards to scores

Attributes	*Types*	*Averageattributeweights*
C_1	cost	$\left\langle \left(\underline{T^{AV}}, \underline{I^{AV}}, \underline{F^{AV}}\right), \left(\overline{T^{AV}}, \overline{I^{AV}}, \overline{F^{AV}}\right)\right\rangle$
C_2	cost	$\left\langle \left(\underline{T^{AV}}, \underline{I^{AV}}, \underline{F^{AV}}\right), \left(\overline{T^{AV}}, \overline{I^{AV}}, \overline{F^{AV}}\right)\right\rangle$
⋮	⋮	⋮
Cn	*benefit*	$\left\langle \left(\underline{T^{AV}}, \underline{I^{AV}}, \underline{F^{AV}}\right), \left(\overline{T^{AV}}, \overline{I^{AV}}, \overline{F^{AV}}\right)\right\rangle$

Step 5: Calculate the Positive Distance from Average (PDA) and Negative Distance from Average (NDA) according to benefit and cost attributes, respectively. As we mentioned above benefit and cost variables are shown as b_{mn} and c_{mn}.

$$PDA = [pda_{mn}]_{mxn} \qquad (6)$$

$$NDA = [nda_{mn}]_{mxn} \qquad (7)$$

Where pda_{mn} and nda_{mn} denote the positive and negative distance performance value of the n^{th} alternative from average solution in terms of m^{th} criterion respectively.

Step 6: Calculate the weighted sum of positive and negative distances for all alternatives as follows:

$$sp_n = \sum\left(w_j \times pda_{mn}\right) \qquad (8)$$

$$sn_n = \sum\left(w_j \times nda_{mn}\right) \qquad (9)$$

Step 7: Normalize the sp_n and sn_n values. The normalize values of sp_n and sn_n for all alternatives are calculated as follows:

$$nsp_n = \frac{sp_n}{\sum\left(sp_n\right)} \qquad (10)$$

$$nsn_n = \frac{sn_n}{\sum(sn_n)} \qquad (11)$$

Step 8: Calculate the appraisal score as_n for all alternatives as follows:

$$as_n = \frac{1}{2}\left(nsp_n + nsn_n\right) \qquad (12)$$

Step 9: Rank the alternatives according to the decreasing values of appraisal score as_n.

Illustrative Example

In this section, we solve a multi-attribute group decision making problem using the proposed RNNWAA method.

A. Problem Definition

A road construction company plans to purchase an excavator among A_1, A_2, and A_3 models proposed by experts. Experts (DM_1, DM_2, DM_3) provided attributes such as annual maintenance cost (C_1), price (C_2), working weight (C_3), fuel consumption rate (C_4), the complexity level of working with excavator by the operator (C_5), and bucket capacity (C_6). The purpose is to choose the best excavator model.

B. Problem Solution

In this sub-section, we present the solution of the problem using the proposed method.

Step 1: Constructed decision matrices with respect to experts are given in Table 5.5:

Table 5 Decision matrices with respect to experts

Attribute	Type	DM_1 (weight 0.35)			DM_2 (weight 0.45)			DM_3 (weight 0.20)		
		A_1	A_2	A_3	A_1	A_2	A_3	A_1	A_2	A_3
C_1	cost	AA	AA	A	UA	A	UA	A	A	UA
C_2	cost	UA	AA	H	H	H	AA	VH	H	H
C_3	benefit	L	AA	A	L	UA	A	A	H	A
C_4	cost	H	VH	L	H	VH	AA	H	H	L
C_5	cost	PL	H	A	L	UA	UA	PL	L	A
C_6	benefit	L	UA	H	UA	A	AA	L	AA	A

Step 2: Aggregate the decision matrices to obtain the average RN decision matrix. We use Eq. (1) for aggregation operation. Aggregated RN decision matrix is constructed as in Table 5.6.

Table 6 Aggregated RN decision matrix $(D^{Ag})_{3\times 6}$

$$\begin{array}{c|cccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
\hline
A_1 & \left\langle\begin{matrix}(0.50,0.42,0.52),\\(0.60,0.42,0.52)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.64,0.31,0.42),\\(0.72,0.31,0.41)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.35,0.57,0.67),\\(0.45,0.57,0.67)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.70,0.20,0.30),\\(0.80,0.20,0.30)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.20,0.73,0.84),\\(0.30,0.73,0.84)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.35,0.56,0.66),\\(0.45,0.56,0.66)\end{matrix}\right\rangle \\
A_2 & \left\langle\begin{matrix}(0.54,0.37,0.47),\\(0.64,0.37,0.47)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.67,0.24,0.34),\\(0.77,0.24,0.34)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.55,0.38,0.48),\\(0.65,0.38,0.48)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.78,0.12,0.30),\\(0.80,0.12,0.22)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.51,0.44,0.54),\\(0.62,0.44,0.54)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.49,0.42,0.52),\\(0.59,0.42,0.52)\end{matrix}\right\rangle \\
A_3 & \left\langle\begin{matrix}(0.44,0.47,0.57),\\(0.54,0.47,0.57)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.66,0.25,0.35),\\(0.76,0.25,0.35)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.50,0.40,0.50),\\(0.60,0.40,0.50)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.46,0.49,0.59),\\(0.56,0.49,0.59)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.46,0.45,0.55),\\(0.56,0.45,0.55)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.58,0.34,0.44),\\(0.69,0.34,0.44)\end{matrix}\right\rangle
\end{array}$$

Step 3: Construct the average solution matrix of attribute weights with regards to experts (j). The attribute weights are determined by Table 8.

Table 7 Average attribute weights

	C_1	C_2	C_3	C_4	C_5	C_6
DM_1	A	AA	AA	UA	A	H
DM_2	AA	H	VH	A	UA	AA
DM_3	H	VH	A	AA	A	A
Average weights	[(0.59, 0.32, 0.42), (0.69, 0.32, 0.42)]	[(0.69, 0.22, 0.34), (0.77, 0.22, 0.32)]	[(0.69, 0.24, 0.38), (0.74, 0.24, 0.34)]	[(0.49, 0.42, 0.52), (0.59, 0.42, 0.52)]	[(0.46, 0.29, 0.55), (0.56, 0.45, 0.55)]	[(0.62, 0.29, 0.39), (0.72, 0.29, 0.39)]

Step 4: Compute the Positive Distance from Average (PDA) and Negative Distance from Average (NDA) respectively (See Table 8 and Table 9) .

Table 8 PDA decision matrix

$$\begin{array}{c|cccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
\hline
A_1 & \left\langle\begin{matrix}(0.50,1.00,0.48),\\(0.60,1.00,0.48)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.64,1.00,0.58),\\(0.72,1.00,0.59)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.35,1.00,0.33),\\(0.45,1.00,0.33)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.70,1.00,0.70),\\(0.80,1.00,0.70)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.20,1.00,0.16),\\(0.30,1.00,0.16)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.35,1.00,0.34),\\(0.45,1.00,0.34)\end{matrix}\right\rangle \\
A_2 & \left\langle\begin{matrix}(0.54,1.00,0.53),\\(0.64,1.00,0.53)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.67,1.00,0.66),\\(0.77,1.00,0.66)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.55,1.00,0.52),\\(0.65,1.00,0.52)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.78,1.00,0.70),\\(0.80,1.00,0.78)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.51,1.00,0.46),\\(0.62,1.00,0.46)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.49,1.00,0.48),\\(0.59,1.00,0.48)\end{matrix}\right\rangle \\
A_3 & \left\langle\begin{matrix}(0.44,1.00,0.43),\\(0.54,1.00,0.43)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.66,1.00,0.65),\\(0.76,1.00,0.65)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.50,1.00,0.50),\\(0.60,1.00,0.50)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.46,1.00,0.41),\\(0.56,1.00,0.41)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.46,1.00,0.45),\\(0.56,1.00,0.45)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.58,1.00,0.56),\\(0.69,1.00,0.56)\end{matrix}\right\rangle
\end{array}$$

Table 9 NDA decision matrix

$$\begin{array}{c|cccccc}
 & C_1 & C_2 & C_3 & C_4 & C_5 & C_6 \\
\hline
A_1 & \left\langle\begin{matrix}(0.50,1.00,0.48),\\(0.60,1.00,0.48)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.64,1.00,0.58),\\(0.72,1.00,0.59)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.35,1.00,0.33),\\(0.45,1.00,0.33)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.70,1.00,0.70),\\(0.80,1.00,0.70)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.20,1.00,0.16),\\(0.30,1.00,0.16)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.35,1.00,0.34),\\(0.45,1.00,0.34)\end{matrix}\right\rangle \\
A_2 & \left\langle\begin{matrix}(0.54,1.00,0.53),\\(0.64,1.00,0.53)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.67,1.00,0.66),\\(0.77,1.00,0.66)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.55,1.00,0.52),\\(0.65,1.00,0.52)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.78,1.00,0.70),\\(0.80,1.00,0.78)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.51,1.00,0.46),\\(0.62,1.00,0.46)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.49,1.00,0.48),\\(0.59,1.00,0.48)\end{matrix}\right\rangle \\
A_3 & \left\langle\begin{matrix}(0.44,1.00,0.43),\\(0.54,1.00,0.43)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.66,1.00,0.65),\\(0.76,1.00,0.65)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.50,1.00,0.50),\\(0.60,1.00,0.50)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.46,1.00,0.41),\\(0.56,1.00,0.41)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.46,1.00,0.45),\\(0.56,1.00,0.45)\end{matrix}\right\rangle & \left\langle\begin{matrix}(0.58,1.00,0.56),\\(0.69,1.00,0.56)\end{matrix}\right\rangle
\end{array}$$

Step 5: Compute the weighted sum of positive and negative distances for all alternatives (see Table 10).

Table 10 rough neutrosophic sp_n and sn_n values

sp_n			sn_n		
A_1	A_2	A_3	A_1	A_2	A_3

[(0.90, 0.05, 0.17), (0.89, 0.06, 0.16)]	[(0.86, 0.09, 0.20), (0.84, 0.11, 0.18)]	[(0.94, 0.08, 0.14), (0.92, 0.10, 0.11)]	[(0.87, 0.12, 0.23), (0.90, 0.13, 0.25)]	[(0.84, 0.17, 0.27), (0.86, 0.19, 0.28)]	[(0.91, 0.15, 0.17), (0.94, 0.16, 0.20)]

Step 6: Compute the weighted sum of positive and negative distances for all alternatives (see Table 11):

Table 11 sp_n and sn_n score values

sp_n (score value)			sn_n (score value)		
A_1	A_2	A_3	A_1	A_2	A_3
0.7850	0.7050	0.8225	0.7275	0.6225	0.7550

Step 7: Normalize the sp_n and sn_n values.

The normalize values of sp_n and sn_n for all alternatives are calculated and presented in Table 12.

Table 12 nsp_n and nsn_n score values

nsp_n (score value)			nsn_n (score value)		
A_1	A_2	A_3	A_1	A_2	A_3
0.3395	0.3049	0.3557	0.3456	0.2957	0.3587

Step 8: Compute the appraisal score as_n for all alternatives (see Table 13).

Table 13 as_n score values

as_n values		
A_1	A_2	A_3
0.34255	0.30030	0.35720

Step 9: Rank the alternatives based on the decreasing values of appraisal score as_n .

The obtained ranking is as follows:

$A_3 > A_1 > A_2$.

Conclusion

In this study, we have proposed rough neutrosophic RNNWAA operator and established its basic properties. We have also developed an MCGDM strategy base don the developed RNNWAA operator. We have presented an illustrative example, namely selection of best excavator in construction field. The concept presented in this chapter can be applied for other MCGDM problems such as teacher selection [55], quality-brick selection [56], weaver selection [57, 58] , e-car selection [59] and so on in RNS environment.

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Chapter 15

A Trapezoidal Neutrosophic MADM Strategy Based on MARCOS for Solving Car Selection Problems

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ABSTRACT

In this chapter, we develop the Measurement of Alternatives and Ranking according to COmpromise Solution (MARCOS) strategy within a trapezoidal neutrosophic number environment. Trapezoidal neutrosophic numbers (TrNNs) are efficient to present the degrees of truth, indeterminacy, and falsity. This number allows for flexible and comprehensive modeling of uncertainty. The MARCOS strategy assesses the utility of each alternative with respect to both ideal and anti-ideal solutions. This strategy also generates a compromise-based ranking. To illustrate the applicability of the proposed strategy, a car selection problem is discussed. A score function is used to defuzzify the trapezoidal neutrosophic number. The results exhibit that the proposed framework offers a sturdy and efficient solution under trapezoidal neutrosophic environments for Multi-Attribute Decision-Making (MADM) problems.

Keywords: MADM, TrNN, MARCOS method, Car selection.

INTRODUCTION

Multi-Attribute Decision-Making (MADM) is a extensively used strategy for selecting the most suitable alternative from a set of feasible alternative based on predefined attribute. Its ability to handle complex decision scenarios has attracted significant attention, specially in response to the increasing complexity of real-world problems [1, 2, 3].

Florentin Smarandache introduced the concept of Neutrosophic Sets (NSs)[4] to handle uncertainty. NSs extends classical and fuzzy set theories by incorporating three independent membership degrees: "truth" (ς), "indeterminacy" (σ), and "falsity" (φ), with respect to the condition $0^- \leq \varsigma + \sigma + \varphi \leq 3^+$. Neutrosophic theory has been extensively explored across various domains and has proven effective in addressing ambiguity, uncertainty, and incomplete knowledge in decision-making problems [5, 6, 7].

Trapezoidal Neutrosophic Numbers (TrNNS) [8, 9] have been widely adopted to model uncertain and indeterminate information using truth, indeterminacy, and falsity membership degree.

MADM strategies have been successfully employed in numerous fields, including engineering [10], healthcare [11], and logistics [12]. The primary objective of MADM strategy is to prioritize alternatives by systematically ranking them from the most to the least preferred option.

Many decision decision making strategies have been proposed in the literature, including TOPSIS [13], VIKOR [1], GRA [14], TODIM [2], MULTIMOORA [15], WASPAS [16], COPRAS [17], and PROMETHEE [18]. Among these, the Measurement of Alternatives and Ranking according to COmpromise Solution (MARCOS) strategy, introduced by Stević et al. [19], has emerged as a modern and effective multi-criteria decision-making (MCDM) strategy. MARCOS strategy identify most suitable alternatives by determining their utility degrees relative to both ideal and anti-ideal solutions. This strategies owing to its systematic structure and reliable ranking mechanism, and has gained considerable attention and it applied in areas such as risk analysis, sustainability assessment, logistics management, and supplier selection.

If we compare MARCOS with other strategies like TOPSIS and VIKOR, we show that it shares similarities with these classical strategies. These strategies rank alternatives based on their distance from ideal and anti-ideal solutions. In VIKOR approach we also introduced with compromise solution. In the both the approaches involves constructing a normalized decision matrix, defining ideal and anti-ideal alternatives, computing utility measures, and aggregating them into an overall utility function. The highest value of the alternative is considered the most desirable.

Martin et al. [20] defined the Neutrosophic MARCOS strategy which is an expansion of the traditional MARCOS strategy. In their expansion, both alternatives and attribute weights are represented using single-valued triangular neutrosophic number. So many studies have been done to examine the evolution of MARCOS approach. Demir et al. [21] developed an extensive bibliometric analysis. In this study, they [21] examines 115 pertinent articles taken from the Scopus database spanning from the years from 2020 to 2024. Puška et al. [22] introduced MARCOS method for ranking project management programs. Ecer and Pamučar [23] developed novel intuitionistic fuzzy MARCOS method. Also, they [22] do a sensitive analysis to assess insurance companies performance during the COVID-19 pandemic.

Furthermore, MARCOS strategy work very well in fuzzy and neutrosophic frameworks to better handle uncertainty and imprecision in expert evaluations. Tang et al. [24] applied a single-valued neutrosophic MARCOS method to assess English teaching systems, demonstrating how the neutrosophic environment can capture vagueness and incomplete information in decision matrices.

Advantage of MARCOS approach is that it frequently outperforms or exhibits competitive stability against classical techniques. To comparative studies in hybrid modeling contexts like combining MARCOS with fuzzy logic or other weighting systems like CRITIC. These advancements show that the MARCOS method is now a crucial and adaptable tool in contemporary MADM research. The MARCOS approach's capacity to handle real-world decision-making issues involving uncertainty, imprecision, and incomplete information is further strengthened by the addition of fuzzy and neutrosophic environments.

Motivated by these research gaps, the present study aims to develop the MARCOS strategy within a trapezoidal neutrosophic framework.

The main objectives of this study are as follows:

(i) To develop the MARCOS strategy in a TrNN environment.

(ii) To demonstrate the applicability of the proposed approach through a car selection problem.

Background

Definition 1 Assume that β is a trapezoidal neutrosophic number (TrNN) [8, 9]. The corresponding truth-membership, indeterminacy-membership, and falsity-membership functions are defined as follows:

$$T_\beta(x) = \begin{cases} \dfrac{(x-c_{11})t_\beta}{c_{12}-c_{11}}, & c_{11} \le x < c_{12}, \\ t_\beta, & c_{12} \le x \le c_{13}, \\ \dfrac{(c_{14}-x)t_\beta}{c_{14}-c_{13}}, & c_{13} < x \le c_{14}, \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

$$I_\beta(x) = \begin{cases} \dfrac{(c_{12}-x)+(x-c_{11})i_\beta}{c_{12}-c_{11}}, & c_{11} \le x < c_{12}, \\ i_\beta, & c_{12} \le x \le c_{13}, \\ \dfrac{x-c_{13}+(c_{14}-x)i_\beta}{c_{14}-c_{13}}, & c_{13} < x \le c_{14}, \\ 0, & \text{otherwise,} \end{cases} \tag{2}$$

$$F_\beta(x) = \begin{cases} \dfrac{c_{12} - x + (x - c_{11}) f_\beta}{c_{12} - c_{11}}, & c_{11} \le x < c_{12}, \\ 1, & c_{12} \le x \le c_{13}, \\ \dfrac{x - c_{13} + (c_{14} - x) f_\beta}{c_{14} - c_{13}}, & c_{13} < x \le c_{14}, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

A trapezoidal neutrosophic number (TNN) is a particular type of fuzzy number, denoted as

$$\beta = \langle (c_{11}, c_{12}, c_{13}, c_{14}); T_\beta, I_\beta, F_\beta \rangle, \tag{4}$$

which is designed to represent uncertain, inconsistent, and incomplete information by defining a trapezoidal membership shape for truth (T), indeterminacy (I), and falsity (F). It extends trapezoidal fuzzy numbers by adding independent constraints on I and F within $[0, 1]$.

Definition 2 Let $\beta_1 = \langle (c_{11}, c_{12}, c_{13}, c_{14}); t_1, i_1, f_1 \rangle$ and $\beta_2 = \langle (c_{21}, c_{22}, c_{23}, c_{24}); t_2, i_2, f_2 \rangle$ be two trapezoidal neutrosophic numbers (TrNNs). Then the operational laws for TrNN [8] are as follows:

(i) $\beta_1 \oplus \beta_2 = \Big\langle (c_{11} + c_{21},\ c_{12} + c_{22},\ c_{13} + c_{23},\ c_{14} + c_{24});\ t_1 + t_2 - t_1 t_2,\ i_1 i_2,\ f_1 f_2 \Big\rangle.$

(ii)$\beta_1 \otimes \beta_2 = \Big\langle (c_{11} c_{21},\ c_{12} c_{22},\ c_{13} c_{23},\ c_{14} c_{24});\ t_1 t_2,\ i_1 + i_2 - i_1 i_2,\ f_1 + f_2 - f_1 f_2 \Big\rangle.$

(iii) $\lambda \beta_1 = \Big\langle (\lambda c_{11},\ \lambda c_{12},\ \lambda c_{13},\ \lambda c_{14});\ 1 - (1 - t_1)^\lambda,\ i_1^\lambda,\ f_1^\lambda \Big\rangle, \lambda > 0.$

(iv) $\beta_1^\lambda = \Big\langle (c_{11}^\lambda,\ c_{12}^\lambda,\ c_{13}^\lambda,\ c_{14}^\lambda);\ t_1^\lambda,\ 1 - (1 - i_1)^\lambda,\ 1 - (1 - f_1)^\lambda \Big\rangle.$

Definition 3 The score function [9] of TrNN $\beta = \langle (c_{11}, c_{12}, c_{13}, c_{14}); t, i, f \rangle$ is defined as

$$Sc(\beta) = \frac{1}{16}(c_{11} + c_{12} + c_{13} + c_{14})(2 + t - i - f), -1 \le Sc(\beta) \le 1 \tag{5}$$

Definition 4 The accuracy function [9] $Ac(\beta)$ is defined as

$$Ac(\beta) = \frac{1}{16}(c_{11} + c_{12} + c_{13} + c_{14})(2 + t - f), -1 \le Ac(\beta) \le 1 \tag{6}$$

Definition 5: For any two TrNNs $\beta_{11}, \beta_{12} \in TrNNs$, the generalized concept of comparability [9] is established as follows:
I. If $S(\beta_{11}) > S(\beta_{12})$, then $\beta_{11} > \beta_{12}$.
II. If $Sc(\beta_{11}) < Sc(\beta_{12})$, then $\beta_{11} < \beta_{12}$.
III. If $S(\beta_{11}) = S(\beta_{12})$, then:
(a) If $A(\beta_{11}) > A(\beta_{12})$, then $\beta_{11} > \beta_{12}$.
(b) If $A(\beta_{11}) = A(\beta_{12})$, then $\beta_{11} = \beta_{12}$.

MARCOS Strategy for Trapezoidal Neutrosophic Numbers

Assume that $\hat{Q} = \langle \hat{Q}_1, \hat{Q}_2, \cdots, \hat{Q}_n \rangle$ denotes a set of n attributes, and $\hat{T} = \langle \hat{T}_1, \hat{T}_2, \cdots, \hat{T}_m \rangle$ represents a set of m alternatives. $w_i (i = 1, 2, ..., n)$ is the weight of the attributes. The alternatives are ranked by the MARCOS method according to how close they are to the ideal and anti-ideal solutions. The following is a description of the MARCOS approach for the TrNNs framework:

Step 1: Constructing the decision matrix.
The decision matrix is formulated based on the decision-maker's assessment of the alternatives with respect to the stipulated conflicting attributes. Then the decision matrix is constructed as follows:

$$A = (\alpha_{ij})_{m\times n} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{m1} & \alpha_{m2} & \cdots & \alpha_{mn} \end{pmatrix} \tag{7}$$

Step 2: Identifying the ideal and anti-ideal alternatives.
First, the score values of the decision matrix are computed prior to identifying the ideal and anti-ideal solutions. The ideal solution consists of alternatives that attain the highest values for benefit criteria and the lowest values for cost criteria. Conversely, the anti-ideal solution comprises alternatives that have the lowest values for benefit criteria and the highest values for cost criteria.

$$A_{ID} = \begin{cases} \max(\alpha_{ij}), & \text{for benefit criteria} \\ \min(\alpha_{ij}), & \text{for cost criteria} \end{cases} \tag{8}$$

$$A_{AID} = \begin{cases} \min(\alpha_{ij}), & \text{for benefit criteria} \\ \max(\alpha_{ij}), & \text{for cost criteria} \end{cases} \tag{9}$$

Step 3: Normalizing the decision matrix.
The component r_{ij}^k of the entry $\tilde{r}_{ij} = \left(\left[r_{ij}^1, r_{ij}^2, r_{ij}^3, r_{ij}^4\right]; t_{ij}, i_{ij}, f_{ij}\right)$ in the normalised decision matrix $R = (\tilde{r}_{ij})_{m\times n}$, is considered as follows:

(i) For benefit type attribute:

$$\tilde{r}_{ij} = \left(\left[\frac{\alpha_{ij}^1}{u_j^+}, \frac{\alpha_{ij}^2}{u_j^+}, \frac{\alpha_{ij}^3}{u_j^+}, \frac{\alpha_{ij}^4}{u_j^+}\right]; t_{ij}, i_{ij}, f_{ij}\right) \tag{10}$$

(ii) For cost type attribute:

$$\tilde{r}_{ij} = \left(\left[\frac{u_j^-}{\alpha_{ij}^4}, \frac{u_j^-}{\alpha_{ij}^3}, \frac{u_j^-}{\alpha_{ij}^2}, \frac{u_j^-}{\alpha_{ij}^1}\right]; t_{ij}, i_{ij}, f_{ij}\right) \tag{11}$$

where

$$u_j^+ = \max\{\alpha_{ij}^4 \mid i = 1, 2, \ldots, m\}$$

and

$$u_j^- = \min\{\alpha_{ij}^1 \mid i = 1, 2, \ldots, m\}, \qquad j = 1, 2, \ldots, n.$$

In the same procedure, we normalize the ideal and anti- ideal solutions.

Step 4: Computing the weighted normalized matrix.
Prior to evaluating the weighted normalised matrix, each element's score value must be determined. The following formula is employed to compute the score value of each standardized matrix element, as well as to determine the ideal and anti-ideal alternatives:

$$n_{ij} = Sc(\tilde{r}_{ij}) = \frac{1}{16}(r_{ij}^1 + r_{ij}^2 + r_{ij}^3 + r_{ij}^4)(2 + t_{ij} - i_{ij} - f_{ij}), -1 \le Sc(\tilde{r}_{ij}) \le 1 \tag{12}$$

The weighted normalized matrix for each alternative, as well as for the ideal and anti-ideal alternatives, is computed using the following formula.

$$\varpi_{ij} = n_{ij} \times w_j \tag{13}$$

Step 5: Determining the degree of utility.
Determine the utility degree of each alternative, along with those of the ideal and anti-ideal alternatives.

$$\hat{Q}_i = \sum_j \varpi_{ij} \tag{14}$$

Step 6: Calculation of utility ratios.
The utility ratio of each alternative is calculated using the following formula.

$$\hat{R}_i = \frac{\hat{Q}_i}{\hat{Q}_{AID}}, \quad \hat{R}_i^* = \frac{\hat{Q}_i}{\hat{Q}_{ID}} \tag{15}$$

Step 7: Determining the total utility function.
The overall utility function of each alternative is determined as follows:

$$f(\hat{R}_i) = \frac{\hat{R}_i + \hat{R}_i^*}{2} \tag{16}$$

Step 8: Sorting the options in order of priority.
The values of $f(\hat{R}_i)$ are used to rank the alternatives in descending order; a higher $f(\hat{R}_i)$ denotes a superior alternative.

Figure 1 provides a summary of the steps mentioned above.

Illustrative Example

The applicability and efficacy of the suggested MADM strategies are illustrated in this section through a practical numerical example. The selection of an appropriate car is an important and complex decision, particularly for middle-income families and budget-constrained buyers. Affordability, fuel efficiency, comfort, safety, and long-term maintenance costs are just a few of the many, frequently contradictory factors that must be carefully considered when making a decision.

Such choices are typically made in real-world situations by a number of decision-makers, such as family members, each of whom has distinct priorities and preferences. For instance, one member may prioritize lower price, while another may emphasize safety or comfort. Consequently, it is possible to formulate the car selection process as an MADM problem. We apply the proposed methods to determine the optimal choice among several small cars in order to show their feasibility.

Four options are taken into consideration in this study:

- $\hat{T}_1$: Maruti Suzuki Alto K10
- $\hat{T}_2$: Maruti Suzuki S-Presso
- $\hat{T}_3$: Renault Kwid
- $\hat{T}_4$: Tata Tiago

These cars are practical options in the small car category and fall into the affordable segment (base models cost about less than six lakh).

To evaluate these alternatives, we consider five attributes:

- $\hat{Q}_1$: Price
- $\hat{Q}_2$: Fuel Efficiency (km/l)
- $\hat{Q}_3$: Safety Rating
- $\hat{Q}_4$: Comfort Level

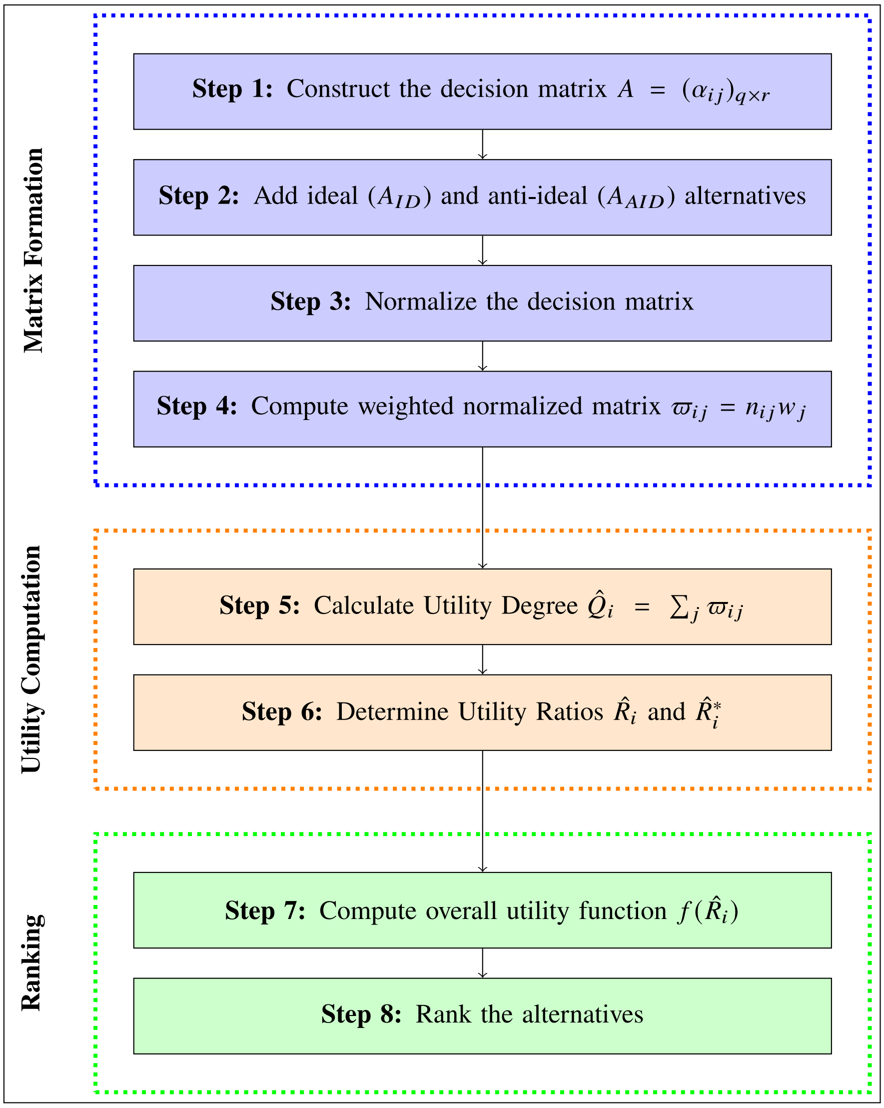

Figure 1: Flowchart of MARCOS strategy in trapezoidal neutrosophic environment

- $\hat{Q}_5$: Maintenance Cost

All of these attributes, Fuel Efficiency ($\hat{Q}_2$), Safety Rating ($\hat{Q}_3$), and Comfort Level ($\hat{Q}_4$) are benefit-type attributes, in that case we preferred higher values. On the other hand, Price ($\hat{Q}_1$) and Maintenance Cost ($\hat{Q}_5$) are cost-type attributes, where we preferred lower values.
A normalization technique is used to bring benefit-type and cost-type attribute onto a common scale. The objective of MADM is to evaluate and rank the vehicle alternatives based on the given attribute.

MARCOS Strategy for TrNNs

The decision matrix is first constructed as shown below, and the computational steps described in Section 4 are then applied.
Step 1: The formulated decision matrix is given in Table 1.
Step 2: Utilize Eqs. (9) and (10) to evaluate ideal and anti-ideal solutions; the results are presented in Table 2.
Step 3: Using Eqs. (10) and (11), the defined matrix is standardized, which is shown in Table 3.
Step 4: The score value of the standardized decision matrix is evaluated using Eq. (12), which is given in Table 4.
The weighted normalized matrix is determine using Eq. (13), which is presented in Table 5.
Step 5: The utility degree of each of the alternative is calculated using Eq. (14), as shown in Table 6.
Step 6: The utility ratios are computed using Eq. (15); the outcomes are presented in Table 7.
Step 7: Overall utility functions are determined using Eq. (16); the corresponding values are shown in Table 7.
Step 8: The alternatives are ordered as follows: $\hat{T}_1 > \hat{T}_2 > \hat{T}_4 > \hat{T}_3$
From Table 7, it is concluded that $\hat{T}_1$ is the best alternative.

Conclusion

This chapter presents a decision-making framework by extending the MARCOS strategy within a TrNN environment. The MACROS method defines the value of each alternative with respect to the best and worst possible solutions and creates a compromise ranking of the alternatives [19, 23].

The proposed approach is illustrated by a car selection problem, where several alternatives are compared with respect to several criteria. To convert trapezoidal neutrosophic data into comparable numerical values for the final ranking procedure, an appropriate score function is used. The results show that the developed trapezoidal neutrosophic MARCOS model efficiently handles ambiguous and imprecise data while producing consistent and logical rankings.

Future research may focus on extending the proposed methodology to various practical decision-making fields such as healthcare [25, 26], finance [27], green supplier selection [28], supply chain management [29, 30], teacher selection[31, 32, 33], quality brick selection [34, 35], weaver selection [36, 37], e-car selection [38], and school choice [39]. Additionally, hybrid models that combine the MACROS strategy with optimisation methods or neutrosophic aggregation operators may improve its suitability in challenging decision-making situations.

Table 1: Decision matrix

Alt	$\hat{Q}_1$	$\hat{Q}_2$	$\hat{Q}_3$	$\hat{Q}_4$	$\hat{Q}_5$
$\hat{T}_1$	$(\langle .75, .65, .32, .15\rangle; .5,.2,.3)$	$(\langle .82, .53, .29, .20\rangle; 0.6,0.2,0.4)$	$(\langle .69, .55, .31, .15\rangle);0.8,0.3,0.1$	$(\langle .72, .51, .36, .2\rangle;0.7,0.4,0.2)$	$(\langle .69, .52, .35, .25\rangle;0.6,0.4,0.2)$
$\hat{T}_2$	$(\langle .86, .40, .31, .12\rangle;0.6,0.2,0.1)$	$(\langle .82, .48, .31, .15\rangle;0.7,0.4,0.2)$	$(\langle .85, .45, .29, .21\rangle;0.8,0.5,0.3)$	$(\langle .79, .41, .32, .22\rangle;0.6,0.4,0.3)$	$(\langle .78, .42, .31, .25\rangle;0.7,0.5,0.2)$
$\hat{T}_3$	$(\langle .71, .45, .35, .15\rangle;0.8,0.3,0.4)$	$(\langle .68, .48, .2, .10\rangle;0.7,0.5,0.2)$	$(\langle .55, .41, .35, .2\rangle;0.8,0.4,0.3)$	$(\langle .68, .41, .23, .15\rangle;0.6,0.3,0.1)$	$(\langle .69, .41, .32, .23\rangle;0.9,0.2,0.4)$
$\hat{T}_4$	$(\langle .68, .48, .32, .19\rangle;0.6,0.4,0.2)$	$(\langle .69, .54, .35, .12\rangle;0.8,0.4,0.2)$	$(\langle .55, .41, .32, .3\rangle;0.6,0.2,0.2)$	$(\langle .68, .51, .3, .25\rangle;0.7,0.4,0.2)$	$(\langle .76, .56, .37, .15\rangle;0.8,0.5,0.2)$

Table 2: Ideal and anti-ideal solutions of the decision matrix

Alt	$\hat{Q}_1$	$\hat{Q}_2$	$\hat{Q}_3$	$\hat{Q}_4$	$\hat{Q}_5$
Ideal	((⟨.75, .65, .32, .15⟩; .5,.2,.3)	((⟨.82, .53, .29, .20⟩; 0.6,0.2,0.4)	((⟨.69, .55, .31, .15⟩);0.8,0.3,0.1	((⟨.72, .51, .36, .2⟩;0.7,0.4,0.2)	((⟨.69, .52, .35, .25⟩;0.6,0.4,0.2)
Anti-ideal	((⟨.86, .40, .31, .12⟩;0.6,0.2,0.1)	((⟨.82, .48, .31, .15⟩;0.7,0.4,0.2)	((⟨.85, .45, .29, .21⟩;0.8,0.5,0.3)	((⟨.79, .41, .32, .22⟩;0.6,0.4,0.3)	((⟨.78, .42, .31, .25⟩;0.7,0.5,0.2)

Table 3: Standardized decision matrix

Alt	$\hat{Q}_1$	$\hat{Q}_2$	$\hat{Q}_3$	$\hat{Q}_4$	$\hat{Q}_5$
$\hat{T}_1$	((⟨.8000, .3750, .1846, .1600⟩; .5,.2,.3)	((⟨1.0000, .6463, .3537, .2439⟩; 0.6,0.2,0.4)	((⟨.8118, .6471, .3647, .1765⟩);0.8,0.3,0.1	((⟨.9114, .6456, .4557, .2532⟩;0.7,0.4,0.2)	((⟨.6000, .4286, .2885, .2174⟩;0.6,0.4,0.2)
$\hat{T}_2$	((⟨1.0000, .3871, .3000, .1395⟩;0.6,0.2,0.1)	((⟨1.0000, .5854, .3780, .1829⟩;0.7,0.4,0.2)	((⟨1.0000, .5294, .3412, .2471⟩;0.8,0.5,0.3)	((⟨1.0000, .5190, .4051, .2785⟩;0.6,0.4,0.3)	((⟨.6000, .4839, .3571, .1923⟩;0.7,0.5,0.2)
$\hat{T}_3$	((⟨.8000, .3429, .2667, .1690⟩;0.8,0.3,0.4)	((⟨.8293, .5854, .2439, .1220⟩;0.7,0.5,0.2)	((⟨.6471, .4824, .4118, .2353⟩;0.8,0.4,0.3)	((⟨.8608, .5190, .2911, .1899⟩;0.6,0.3,0.1)	((⟨.6522, .4688, .3659, .2174⟩;0.9,0.2,0.4)
$\hat{T}_4$	((⟨.6316, .3750, .2500, .1765⟩;0.6,0.4,0.2)	((⟨.8415, .6585, .4268, .1463⟩;0.8,0.4,0.2)	((⟨.6471, .4824, .3765, .3529⟩;0.6,0.2,0.2)	((⟨.8608, .6456, .3797, .3165⟩;0.7,0.4,0.2)	((⟨1.0000, .4054, .2679, .1974⟩;0.8,0.5,0.2)
Ideal	((⟨1.0000, .5938, .3958, .2794⟩;0.8,0.3,0.4)	((⟨1.0000, .6463, .3537, .2439⟩;0.7,0.5,0.2)	((⟨1.0000, .7971, .4493, .2174⟩;0.8,0.4,0.3)	((⟨1.0000, .7083, .5000, .2778⟩;0.6,0.3,0.1)	((⟨1.0000, .8065, .4524, .2436⟩;0.9,0.2,0.4)
Anti-ideal	((⟨1.0000, .3871, .3000, .1395⟩;0.6,0.4,0.2)	((⟨1.0000, .7059, .2941, .1471⟩;0.8,0.4,0.2)	((⟨1.0000, .7455, .6364, .3636⟩;0.6,0.2,0.2)	((⟨1.0000, .6029, .3382, .2206⟩;0.7,0.4,0.2)	((⟨.8000, .3243, .2143, .1579⟩;0.8,0.5,0.2)

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Table 4: Score value of standardized decision matrix

Alt	$\hat{Q}_1$	$\hat{Q}_2$	$\hat{Q}_3$	$\hat{Q}_4$	$\hat{Q}_5$
$\hat{T}_1$	3.0392	5.1610	4.8000	4.7582	3.0688
$\hat{T}_2$	4.2013	4.5073	4.2353	4.1848	3.2666
$\hat{T}_3$	3.3149	3.5610	3.7306	4.0937	3.9196
$\hat{T}_4$	2.8661	4.5610	4.0894	4.6253	3.9283
Ideal	4.5380	5.1610	5.9130	5.2208	5.0048
Anti-ideal	4.2013	4.2941	5.7655	4.7559	3.1427

Table 5: Weighted normalized decision matrix

Alt	$\hat{Q}_1$	$\hat{Q}_2$	$\hat{Q}_3$	$\hat{Q}_4$	$\hat{Q}_5$
$\hat{T}_1$	0.6990	0.9290	1.2000	0.9516	0.4296
$\hat{T}_2$	0.9663	0.8113	1.0588	0.8370	0.4573
$\hat{T}_3$	0.7624	0.6410	0.9326	0.8187	0.5487
$\hat{T}_4$	0.6592	0.8210	1.0224	0.9251	0.5500
Ideal	1.0437	0.9290	1.4783	1.0442	0.7007
Anti-ideal	0.9663	0.7729	1.4414	0.9512	0.4400

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Table 6: Utility degree of each alternative

Alt	$\hat{T}_1$	$\hat{T}_2$	$\hat{T}_3$	$\hat{T}_4$	Ideal	Anti-Ideal
Utility degree	4.2093	4.1307	3.7035	3.9776	5.1958	4.5717

Table 7: Utility degrees and utility functions of alternatives

Alternative	K_i	K_i^*	$f(K_i)$
$\hat{T}_1$	0.8101	0.9207	0.8654
$\hat{T}_2$	0.7950	0.9035	0.8493
$\hat{T}_3$	0.7128	0.8101	0.7614
$\hat{T}_4$	0.7655	0.8700	0.8178

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Chapter 16

Novel Einstein Aggregation Operators and Score Function for Multi-Criteria Group Decision-Making in Pentapartitioned Neutrosophic Number Environment

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ABSTRACT

Smarandache introduced the concept of Neutrosophic Sets (NS) to capture indeterminacy in real-world decision-making. Building on his Five-Symbol Valued Logic (FSVL), Pentapartitioned Neutrosophic Numbers (PNNs) emerged as a significant tool to handle this indeterminacy more precisely through its five-component structure of truth, contradiction, ignorance, unknown and falsity. To integrate PNNs into decision-making algorithms, aggregation operations play the most prominent role. These operations simultaneously synthesize group opinions and assimilate criteria information within the decision-making framework.

We present in this paper a novel aggregation operator for PNNs utilizing the Einstein triangular norm and conorm, a subclass of the Archimedean family. Initially, we define binary operations of addition, multiplication, scalar multiplication, and power operations for PNN structures. We have verified these operations for the corresponding properties of commutativity, associativity, and scalar distributivity. Next, a new aggregation operator as Pentapartitioned Neutrosophic Einstein Weighted Arithmetic Aggregation (PNEWAA) is established based on those initial definitions. The operator is verified for the properties of consistency, symmetry and idempotency. We also develop a new operational framework for MCDM incorporating the PNEWAA operator and utilizing a novel score function. The score function has also been validated through the properties of boundedness, monotonicity and transitivity which are essential to produce consistent ranking. Based on the proposed methodology two numerical illustrations are provided encompassing group as well as single decision-making process.

Keywords: Pentapartitioned Neutrosophic Number(PNN), Pentapartitioned Neutrosophic Einstein Weighted Arithmetic Aggregation (PNEWAA) operator, Multi-criteria Decision-Making (MCDM), Score function.

INTRODUCTION

Fuzzy Set (FS)[1] introduced the concept of membership functions to capture real-world uncertainties. FS was extended to Intuitionistic FS (IFS) [2] using the concept of non-membership functions. However, a formal measure of inconsistency was initially introduced through the definition of Neutrosophic Sets (NS) [3]which incorporated the factor of "Indeterminacy." NS was modified to Single-Valued NS (SVNS) [4], which was further modified to Quadripartitioned SVNS (QSVNS) [5] using the principles of n-valued logic [6] and four-valued logic [7]. The concept of Pentapartitioned Neutrosophic Sets (PNS) [8] emerged later in 2020 as a more comprehensive measure of uncertainty. The Pentapartitioned Neutrosophic Number (PNN) [9] is an elemental representation of PNS with a five-component structure representing Truth, Contradiction, Ignorance, Unknown, and Falsity. Details development and applications of NSs and SVNSs and various hybrid NSs can be found in the studies [10-19].

Multi-Criteria Decision-Making (MCDM) and Multi-Criteria Group Decision-Making (MCGDM) deal with an algorithmic framework to identify the most suitable alternative with respect to some pre-defined set of criteria. The existing literature indexes several MCDM frameworks[20, 21] involving methods like TOPSIS [21-25] and TODIM [26] and also including their extensions in different fuzzy or neutrosophic frameworks. In both MCGDM and MCDM, the aggregation process plays a substantial role in aggregating group opinions and assimilating criteria information. After the development of PNN frameworks, some prominent developments were noted. These include tangent similarity measures [27], dice similarity measures [28], hyperbolic tangent similarity [29], hyperbolic sine similarity [30], and exponential similarity measures [31] with integrated MCDM framework.

To incorporate appropriate aggregation strategies while maintaining the PNN frameworks, researchers mainly relied on the Archimedean family of triangular norms and conorms. Notable developments included weighted arithmetic operators with ARAS [32] and weighted geometric operators [9] using algebraic aggregation strategies. Some studies included Dombi-based aggregation operators [33]involving both arithmetic and geometric aggregation separately with MCDM. Apart from these evidence shows the usage of Hamacher operators [34] in PNN environment. A recent study in 2026 depicted PNN-COCOSO [35] approach, using the algebraic aggregation validating it for both MCDM and MCGDM scenarios

All the aforementioned operators have certain limitations associated with them which develop the urge for introduction of anew operator. While the algebraic aggregation lacks the competency of handling non-linear situations, Dombi and Hamacher operators involve high parameter dependency, leading to subjective bias in decision-making. To overcome this, researchers focussed toward the usage of Einstein operators [36], which belong to the Archimedean family and are free from parameter dependency. Einstein operators with additive generators were initially designed for IFS [37]and was later extended to Single-Valued Neutrosophic Trapezoidal Linguistic (SVNTL) [38] environments, as well as Neutrosophic Rough Sets (NRS) [39]. Another instance in the literature presented the application of Einstein-based operations for Simplified Neutrosophic Numbers (SNN) [40] with new operational laws and aggregation strategies. A recent 2024 study also depicted the usage of Einstein operations in Complex PNN (CPNN) [41] environments within an MCDM framework to solve an oil-centric agricultural problem.

Currently, the literature does not show any evidence of Einstein operators integrated within the standard PNN environment which motivates the present study. Although Einstein operations exist for complex PNN setups, standard non-complex frameworks remain unaddressed. Moreover, it is also noted that many existing PNN studies [9, 32, 33] utilize score functions to rank alternatives without formally verifying the mathematical properties required to ensure the stability and consistency of the rankings.

To address these issues, we develop novel Einstein operators specifically for standard PNN frameworks. We establish binary operations for addition, multiplication, scalar multiplication, and power. All these operations are verified for the properties of commutativity, associativity, and scalar distributivity. Using this foundational operating structure a new aggregation operator is designed. The aggregation operator is also validated through the properties of closure, symmetry and idempotency. To rank the PNN alternatives reliably, we also introduce a novel score function and validate its boundedness, monotonicity, and transitivity.

BACKGROUND

Definition 1. [4] Single-Valued Neutrosophic Set (SVNS)

A SVNS $\widehat{\mathbb{N}}(x)$defined over a universe of discourse $\mathbb{X}$, is denoted as

$$\widehat{\mathbb{N}}(x) = \langle x \in \mathbb{X} : T(x), I(x), F(x) \rangle,$$

where,$T(x), I(x), F(x)$ respectively represents the Truth, Indeterminacy and Falsity membership functions. Each of the components satisfies the conditions,

$$T(x), I(x), F(x) \in [0,1] \quad and \quad 0 \leq T(x) + I(x) + F(x) \leq 3, \qquad \forall x \in \mathbb{X}.$$

Every single element having a set of fixed values $\langle T, I, F \rangle$ is known as a Single-Valued Neutrosophic Number (SVNN).

Definition 2. [8] Pentapartitioned Neutrosophic Set (PNS)

The Indeterminacy component of SVNS is divided into three independent components of Contradiction, Ignorance and Unknown which together with Truth and Falsity generates a 5-tuple structure called a Pentapartitioned Neutrosophic Set (PNS).

A PNS $\widehat{\mathbb{A}}(x)$defined over a universe of discourse $\mathbb{X}$, is denoted as

$$\widehat{\mathbb{A}}(x) = \langle x \in \mathbb{X} : \tau(x), \eta(x), \zeta(x), \kappa(x), \xi(x)\rangle,$$

where,$\tau(x), \eta(x), \zeta(x), \kappa(x), \xi(x)$ respectively represents the Truth, Contradiction, Ignorance, Unknown and Falsity membership functions. Each of the components satisfies the conditions,

$$\tau(x), \eta(x), \zeta(x), \kappa(x), \xi(x) \in [0,1] \ \ and \ \ 0 \le \tau(x) + \eta(x) + \zeta(x) + \kappa(x) + \xi(x) \le 5, \qquad \forall x \in \mathbb{X}.$$

The elemental version of a PNS is denoted as PNN as defined below.

Definition 3. [9] Pentapartitioned Neutrosophic Number (PNN)

A PNN $\mathbb{P}$is an element of PNS [8] with five independent components and denoted as

$$\mathbb{P} = \langle \tau, \eta, \zeta, \kappa, \xi\rangle,$$

where,$\tau, \eta, \zeta, \kappa, \xi$ respectively represents the membership degrees of Truth, Contradiction, Ignorance, Unknown and Falsity. Each of the components are assigned a fixed and unique value for a PNN and they satisfy the conditions

$$\tau, \eta, \zeta, \kappa, \xi \in [0,1], \qquad and \qquad 0 \le \tau + \eta + \zeta + \kappa + \xi \le 5.$$

Definition 4. [9] Normalization of a PNN

For any PNN $\mathbb{P} = \langle \tau, \eta, \zeta, \kappa, \xi\rangle$, normalization is performed with respect to the nature of criteria as follows:

$$\begin{cases} \text{Benefit criteria} = (\mathbb{P})' = \langle \tau, \eta, \zeta, \kappa, \xi\rangle \\ \text{Cost criteria} = (\mathbb{P})' = \langle \xi, \kappa, 1 - \zeta, \eta, \tau\rangle \end{cases} \tag{1}$$

$(\mathbb{P})'$is the normalized PNN of $\mathbb{P}$.

NEW PROPOSED SCORE FUNCTION

Definition 5. Let $\mathbb{A} = \{\tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1\}$ be a PNN, then score of $\mathbb{A}$ is defined as,

$$Sc(\mathbb{A}) = \left(\frac{\tau_1 + \eta_1}{2}\right) - \left(\frac{\zeta_1 + \kappa_1 + \xi_1}{3}\right) \tag{2}$$

Score function provides a crisp value for a PNN and helps in ranking of PNN during decision-making scenarios.

Proposition-1: The score function is bounded.

Proof: Since $\mathbb{A} = \{\tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1\}$ is a PNN,$0 \le \tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1 \le 1$.

$$\Rightarrow 0 \le \tau_1 \le 1, 0 \le \eta_1 \le 1 \ \ 0 \le \zeta_1 \le 1 \ , 0 \le \kappa_1 \le 1 \ and \ 0 \le \xi_1 \le 1$$

$\Rightarrow 0 \le \tau_1 + \eta_1 \le 2$ and $0 \le \zeta_1 + \kappa_1 + \xi_1 \le 3$

$\Rightarrow 0 \le \frac{\tau_1 + \eta_1}{2} \le 1$ and $0 \le \frac{\zeta_1 + \kappa_1 + \xi_1}{3} \le 1 \Rightarrow 0 \ge -\frac{\zeta_1 + \kappa_1 + \xi_1}{3} \ge -1$

$\Rightarrow 0 \le \frac{\tau_1 + \eta_1}{2} \le 1$ and $-1 \le -\frac{\zeta_1 + \kappa_1 + \xi_1}{3} \le 0$

Therefore, adding the inequalities we have,

$$-1 \le \left(\frac{\tau_1 + \eta_1}{2}\right) - \left(\frac{\zeta_1 + \kappa_1 + \xi_1}{3}\right) \le 1 \Rightarrow -1 \le Sc(\mathbb{A}) \le 1$$

Hence the function is bounded in $[-1,1]$.

Proposition-2: The score function is monotonic.

Proof: Let $\mathbb{A} = \{\tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1\}$ and $\mathbb{B} = \{\tau_2, \eta_2, \zeta_2, \kappa_2, \xi_2\}$ be two PNN such that,

$\tau_1 \ge \tau_2, \eta_1 \ge \eta_2 \ and \ \zeta_1 \le \zeta_2, \kappa_1 \le \kappa_2, \xi_1 \le \xi_2$.

Hence, $(\tau_1-\tau_2)\geq 0,\ (\eta_1-\eta_2)\geq 0,\ (\zeta_2-\zeta_1)\geq 0,\ (\kappa_2-\kappa_1)\geq 0,\ (\xi_2-\xi_1)\geq 0$

Now, $Sc(\mathbb{A})-Sc(\mathbb{B})=\left(\left(\frac{\tau_1+\eta_1}{2}\right)-\left(\frac{\zeta_1+\kappa_1+\xi_1}{3}\right)\right)-\left(\left(\frac{\tau_2+\eta_2}{2}\right)-\left(\frac{\zeta_2+\kappa_2+\xi_2}{3}\right)\right)$

Or, $Sc(\mathbb{A})-Sc(\mathbb{B})=\frac{(\tau_1-\tau_2)+(\eta_1-\eta_2)}{2}+\frac{(\zeta_2-\zeta_1)+(\kappa_2-\kappa_1)+(\xi_2-\xi_1)}{3}\geq 0$

$$\Rightarrow Sc(\mathbb{A})-Sc(\mathbb{B})\geq 0 \Rightarrow Sc(\mathbb{A})\geq Sc(\mathbb{B})$$

Hence the function is monotonic.

Proposition-3: The score function is transitive.

Proof: Let $\mathbb{A}=\{\tau_1,\eta_1,\zeta_1,\kappa_1,\xi_1\}$, $\mathbb{B}=\{\tau_2,\eta_2,\zeta_2,\kappa_2,\xi_2\}$ and $\mathbb{C}=\{\tau_3,\eta_3,\zeta_3,\kappa_3,\xi_3\}$ be any three PNN such that,

$$\tau_1\geq\tau_2\geq\tau_3,\qquad \eta_1\geq\eta_2\geq\eta_3\ \ and\ \ \zeta_1\leq\zeta_2\leq\zeta_3,\ \kappa_1\leq\kappa_2\leq\kappa_3,\ \xi_1\leq\xi_2\leq\xi_3$$

Since,$\tau_1\geq\tau_2,\eta_1\geq\eta_2,\zeta_1\leq\zeta_2,\zeta_1\leq\zeta_2,\zeta_1\leq\zeta_2\rightarrow Sc(\mathbb{A})\geq Sc(\mathbb{B})$ (using proposition-2)

Similarly, $\tau_2\geq\tau_3,\eta_2\geq\eta_3,\zeta_2\leq\zeta_3,\zeta_2\leq\zeta_3,\zeta_2\leq\zeta_3\rightarrow Sc(\mathbb{B})\geq Sc(\mathbb{C})$.

Hence, we can say, $Sc(\mathbb{A})\geq Sc(\mathbb{B})\geq Sc(\mathbb{C})$, or the score function follows the transitive property.

Definition 6. [33] **Accuracy function for PNN**

Let $\mathbb{A}=\{\tau_1,\eta_1,\zeta_1,\kappa_1,\xi_1\}$ be a PNN, then the accuracy of $\mathbb{A}$ is defined as,

$$Ac(\mathbb{A})=(\tau_1+\eta_1)-(\kappa_1+\xi_1)$$

Accuracy function also provides a crisp value for a PNN. The above function is bounded in [-2,2].

Definition 7. [33] **Ranking rules for PNN**

Let $\mathbb{A}=\{\tau_1,\eta_1,\zeta_1,\kappa_1,\xi_1\}$ and$\mathbb{B}=\{\tau_2,\eta_2,\zeta_2,\kappa_2,\xi_2\}$ be two PNN such that

(i) If $Sc(\mathbb{A})>Sc(\mathbb{B})$, then $\mathbb{A}>\mathbb{B}$.
(ii) If $Sc(\mathbb{A})=Sc(\mathbb{B})$, but $Ac(\mathbb{A})>Ac(\mathbb{B})$, then $\mathbb{A}>\mathbb{B}$
(iii) If $Sc(\mathbb{A})=Sc(\mathbb{B})$, but $Ac(\mathbb{A})=Ac(\mathbb{B})$, then $\mathbb{A}=\mathbb{B}$

This rule helps in ranking of alternatives in PNN format during an MCDM framework.

Einstein based aggregation in PNN environment

In fuzzy logic or neutrosophic environments aggregation of the elements is to be performed to compile the information necessary for MCDM. Mainly with reference to PNN when we add two numbers, we have to ensure the result should follow the same boundary structure of PNN as [0,1]. For example, when we take two PNN as $A=\langle 0.7,0.2,0.1,0.3,0.2\rangle$,$B=\langle 0.8,0.3,0.1,0.4,0.1\rangle$ and add them component-wise then the combined truth value becomes 1.5, which is outside the definition range for PNN as [0,1]. Hence, to achieve this aggregation we rely on Archimedean Triangular-norms (T-norm) for multiplication or intersection and Triangular-conorm (or S-norm) for addition or union.

Definition 8. [36] Archimedean norms and generators

- A continuous T-norm ($T(x,x)$) is said to be Archimedean if it satisfies

$T(x,x)<x$, $\forall x\in(0,1)$. T-norm is generated by a continuous and strictly decreasing function $f{:}\,[0,1]\rightarrow[0,\infty)$ where $f(1)=0$ and following the relation $T(x,y)=f^{-1}\{f(x)+f(y)\}$.

- A continuous S-norm ($S(x,x)$) is said to be Archimedean if it satisfies

$S(x,x)>x$,$\forall x\in(0,1)$. S-norm is generated by a continuous and strictly increasing function $g{:}\,[0,1]\rightarrow[0,\infty)$ where $g(0)=0$ and following the relation $S(x,y)=g^{-1}\{g(x)+g(y)\}$.
$f(x)$ and $g(x)$ are said to be the additive generators.

Definition 9. [36] Einstein triangular norm and conorm

Einstein norm and conorm are subclass of the Archimedean family, with additive generators as

$f(x) = \ln\frac{(2-x)}{x}$ and $g(x) = \ln\frac{(1-x)}{(1+x)}$.

Using these generators the Einstein norms and conorms are defined for any two real numbers $x, y \in [0,1]$as,

- Einstein T-conorm (Einstein sum or S-norm) is denoted as S_E, and defined as $S_E = \frac{x+y}{1+xy}$.
- Einstein T-norm (Einstein product) is denoted as T_E, and defined as $T_E = \frac{xy}{1+(1-x)(1-y)}$. (3)
- Both these operations are binary mapping $[0,1] \times [0,1] \rightarrow [0,1]$ and are commutative, associative and monotonic within the specified interval.

Definition 10. [40] Einstein operators for Simplified Neutrosophic Number (SNN)
For any two SNN, $A = \langle T_1, I_1, F_1\rangle$ and $B = \langle T_2, I_2, F_2\rangle$ the Einstein operations are defined as,

- Addition: $A \oplus \mathrm{B} = \langle\frac{T_1+T_2}{1+T_1T_2}, \frac{I_1I_2}{1+(1-I_1)(1-I_2)}, \frac{F_1F_2}{1+(1-F_1)(1-F_2)}\rangle$
- Multiplication: $A \otimes \mathrm{B} = \langle\frac{T_1T_2}{1+(1-T_1)(1-T_2)}, \frac{I_1+I_2}{1+I_1I_2}, \frac{F_1+F_2}{1+F_1F_2}\rangle$
- Scalar multiplication:$\lambda A = \langle\frac{(1+T_1)^\lambda-(1-T_1)^\lambda}{(1+T_1)^\lambda+(1-T_1)^\lambda}, \frac{2I_1{}^\lambda}{(2-I_1)^\lambda+I_1{}^\lambda}, \frac{2F_1{}^\lambda}{(2-F_1)^\lambda+F_1{}^\lambda}\rangle, \lambda > 0$
- Power operation: $A^\lambda = \langle\frac{2T_1{}^\lambda}{(2-T_1)^\lambda+T_1{}^\lambda}, \frac{(1+I_1)^\lambda-(1-I_1)^\lambda}{(1+I_1)^\lambda+(1-I_1)^\lambda}, \frac{(1+F_1)^\lambda-(1-F_1)^\lambda}{(1+F_1)^\lambda+(1-F_1)^\lambda}\rangle, \lambda > 0$

New proposed Einstein-based binary operations in PNN environment

Extending the aggregation process of SNN we define the binary operations of PNN based on Einstein operators as presented below.

Definition 11. For any two PNN$\mathbb{A} = \{\tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1\}$, $\mathbb{B} = \{\tau_2, \eta_2, \zeta_2, \kappa_2, \xi_2\}$the binary operations of addition ($\oplus$), multiplication ($\otimes$), scalar multiplication and power operation are defined using Einstein t-conorm and t-norm (Eq. (3)) as,

Addition:$\mathbb{A} \oplus \mathbb{B} = \langle\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}, \frac{\eta_1+\eta_2}{1+\eta_1\eta_2}, \frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)}, \frac{\kappa_1\kappa_2}{1+(1-\kappa_1)(1-\kappa_2)}, \frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)}\rangle$

$$= \langle\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}, \frac{\eta_1+\eta_2}{1+\eta_1\eta_2}, \frac{\zeta_1\zeta_2}{2-\zeta_1-\zeta_2+\zeta_1\zeta_2}, \frac{\kappa_1\kappa_2}{2-\kappa_1-\kappa_2+\kappa_1\kappa_2}, \frac{\xi_1\xi_2}{2-\xi_1-\xi_2+\xi_1\xi_2}\rangle \qquad (4)$$

Multiplication:$\mathbb{A} \otimes \mathbb{B} = \langle\frac{\tau_1\tau_2}{1+(1-\tau_1)(1-\tau_2)}, \frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)}, \frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}, \frac{\kappa_1+\kappa_2}{1+\kappa_1\kappa_2}, \frac{\xi_1+\xi_2}{1+\xi_1\xi_2}\rangle$

$$= \langle\frac{\tau_1\tau_2}{2-\tau_1-\tau_2+\tau_1\tau_2}, \frac{\eta_1\eta_2}{2-\eta_1-\eta_2+\eta_1\eta_2}, \frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}, \frac{\kappa_1+\kappa_2}{1+\kappa_1\kappa_2}, \frac{\xi_1+\xi_2}{1+\xi_1\xi_2}\rangle \qquad (5)$$

Scalar multiplication:

$$\lambda\mathbb{A} = \langle\frac{(1+\tau_1)^\lambda-(1-\tau_1)^\lambda}{(1+\tau_1)^\lambda+(1-\tau_1)^\lambda}, \frac{(1+\eta_1)^\lambda-(1-\eta_1)^\lambda}{(1+\eta_1)^\lambda+(1-\eta_1)^\lambda}, \frac{2\zeta_1{}^\lambda}{(2-\zeta_1)^\lambda+\zeta_1{}^\lambda}, \frac{2\kappa_1{}^\lambda}{(2-\kappa_1)^\lambda+\kappa_1{}^\lambda}, \frac{2\xi_1{}^\lambda}{(2-\xi_1)^\lambda+\xi_1{}^\lambda}\rangle, \lambda > 0 \qquad (6)$$

Power operation:

$$\mathbb{A}^\lambda = \langle\frac{2\tau_1{}^\lambda}{(2-\tau_1)^\lambda+\tau_1{}^\lambda}, \frac{2\eta_1{}^\lambda}{(2-\eta_1)^\lambda+\eta_1{}^\lambda}, \frac{(1+\zeta_1)^\lambda-(1-\zeta_1)^\lambda}{(1+\zeta_1)^\lambda+(1-\zeta_1)^\lambda}, \frac{(1+\kappa_1)^\lambda-(1-\kappa_1)^\lambda}{(1+\kappa_1)^\lambda+(1-\kappa_1)^\lambda}, \frac{(1+\xi_1)^\lambda-(1-\xi_1)^\lambda}{(1+\xi_1)^\lambda+(1-\xi_1)^\lambda}\rangle, \lambda > 0 \qquad (7)$$

Proposition-4: All these PNN Einstein operations are closed in [0,1] and monotonic

Proof: Since every element of PNN lie in [0,1], it follows from Einstein structure (Eq. (3)) they are closed in [0,1] and monotonic

Proposition-5: The PNN operations satisfy the following properties.

(i) **Commutative**
(a) $\mathbb{A} \oplus \mathbb{B} = \mathbb{B} \oplus \mathbb{A}$
(b) $\mathbb{A} \otimes \mathbb{B} = \mathbb{B} \otimes \mathbb{A}$

(ii) **Associative**
(c) $\{\mathbb{A} \oplus \mathbb{B}\} \oplus \mathbb{C} = \mathbb{A} \oplus \{\mathbb{B} \oplus \mathbb{C}\}$
(d) $\{\mathbb{A} \otimes \mathbb{B}\} \otimes \mathbb{C} = \mathbb{A} \otimes \{\mathbb{B} \otimes \mathbb{C}\}$

(iii) **Distributive**
(e) $\lambda\{\mathbb{A} \oplus \mathbb{B}\} = \lambda\mathbb{A} \oplus \lambda\mathbb{B}, \lambda > 0$
(f) $\{\lambda_1 + \lambda_2\}\mathbb{A} = \lambda_1\mathbb{A} \oplus \lambda_2\mathbb{A}, \lambda_1, \lambda_2 > 0$

Proof: Let $\mathbb{A} = \{\tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1\}$, $\mathbb{B} = \{\tau_2, \eta_2, \zeta_2, \kappa_2, \xi_2\} and\ \mathbb{C} = \{\tau_3, \eta_3, \zeta_3, \kappa_3, \xi_3\}$ be three PNNs.

(a) $\mathbb{A} \oplus \mathbb{B} = \langle\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}, \frac{\eta_1+\eta_2}{1+\eta_1\eta_2}, \frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)}, \frac{\kappa_1\kappa_2}{1+(1-\kappa_1)(1-\kappa_2)}, \frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)}\rangle$

$$= \langle \frac{\tau_2+\tau_1}{1+\tau_2\tau_1}, \frac{\eta_2+\eta_1}{1+\eta_2\eta_1}, \frac{\zeta_2\zeta_1}{1+(1-\zeta_2)(1-\zeta_1)}, \frac{\kappa_2\kappa_1}{1+(1-\kappa_2)(1-\kappa_1)}, \frac{\xi_2\xi_1}{1+(1-\xi_2)(1-\xi_1)} \rangle = \mathbb{B} \oplus \mathbb{A} \text{(Eq. (4))}$$

(b) Follows similarly.

(c) $\mathbb{A} \oplus \mathbb{B} = \langle \frac{\tau_1+\tau_2}{1+\tau_1\tau_2}, \frac{\eta_1+\eta_2}{1+\eta_1\eta_2}, \frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)}, \frac{\kappa_1\kappa_2}{1+(1-\kappa_1)(1-\kappa_2)}, \frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)} \rangle$ (See Eq.(4))

Then, $\{\mathbb{A} \oplus \mathbb{B}\} \oplus \mathbb{C} = \langle \frac{\tau_1+\tau_2}{1+\tau_1\tau_2}, \frac{\eta_1+\eta_2}{1+\eta_1\eta_2}, \frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)}, \frac{\kappa_1\kappa_2}{1+(1-\kappa_1)(1-\kappa_2)}, \frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)} \rangle \oplus \langle \tau_3, \eta_3, \zeta_3, \kappa_3, \xi_3 \rangle$

$$\begin{bmatrix} \frac{\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}+\tau_3}{1+\left\{\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}\right\}\tau_3}, \frac{\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}+\eta_3}{1+\left\{\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}\right\}\eta_3}, \frac{\frac{\zeta_1\zeta_2\zeta_3}{1+(1-\zeta_1)(1-\zeta_2)}}{1+\left(1-\frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)}\right)(1-\zeta_3)}, \\ \frac{\frac{\kappa_1\kappa_2\kappa_3}{1+(1-\kappa_1)(1-\kappa_2)}}{1+\left(1-\frac{\kappa_1\kappa_2}{1+(1-\kappa_1)(1-\kappa_2)}\right)(1-\kappa_3)}, \frac{\frac{\xi_1\xi_2\xi_3}{1+(1-\xi_1)(1-\xi_2)}}{1+\left(1-\frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)}\right)(1-\xi_3)} \end{bmatrix}$$

This, on simplification gives,

$$\begin{bmatrix} \frac{\tau_1+\tau_2+\tau_3+\tau_1\tau_2\tau_3}{1+\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1}, \frac{\eta_1+\eta_2+\eta_3+\eta_1\eta_2\eta_3}{1+\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1}, \frac{\zeta_1\zeta_2\zeta_3}{4-2(\zeta_1+\zeta_2+\zeta_3)+(\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1)}, \\ \frac{\kappa_1\kappa_2\kappa_3}{4-2(\kappa_1+\kappa_2+\kappa_3)+(\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1)}, \frac{\xi_1\xi_2\xi_3}{4-2(\xi_1+\xi_2+\xi_3)+(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1)} \end{bmatrix} \tag{8}$$

Now for, $\mathbb{A} \oplus \{\mathbb{B} \oplus \mathbb{C}\} = \langle \tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1 \rangle \oplus \langle \frac{\tau_2+\tau_3}{1+\tau_2\tau_3}, \frac{\eta_2+\eta_3}{1+\eta_2\eta_3}, \frac{\zeta_2\zeta_3}{1+(1-\zeta_2)(1-\zeta_3)}, \frac{\kappa_2\kappa_3}{1+(1-\kappa_2)(1-\kappa_3)}, \frac{\xi_2\xi_3}{1+(1-\xi_2)(1-\xi_3)} \rangle$

$$= \begin{bmatrix} \frac{\tau_1+\frac{\tau_2+\tau_3}{1+\tau_2\tau_3}}{1+\tau_1\left(\frac{\tau_2+\tau_3}{1+\tau_2\tau_3}\right)}, \frac{\eta_1+\frac{\eta_2+\eta_3}{1+\eta_2\eta_3}}{1+\eta_1\left(\frac{\eta_2+\eta_3}{1+\eta_2\eta_3}\right)}, \frac{\frac{\zeta_1\zeta_2\zeta_3}{1+(1-\zeta_2)(1-\zeta_3)}}{1+(1-\zeta_1)\left(1-\frac{\zeta_2\zeta_3}{1+(1-\zeta_2)(1-\zeta_3)}\right)}, \\ \frac{\frac{\kappa_1\kappa_2\kappa_3}{1+(1-\kappa_2)(1-\kappa_3)}}{1+(1-\kappa_1)\left(1-\frac{\kappa_2\kappa_3}{1+(1-\kappa_2)(1-\kappa_3)}\right)}, \frac{\frac{\xi_1\xi_2\xi_3}{1+(1-\xi_2)(1-\xi_3)}}{1+(1-\xi_1)\left(1-\frac{\xi_2\xi_3}{1+(1-\xi_2)(1-\xi_3)}\right)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\tau_1+\tau_2+\tau_3+\tau_1\tau_2\tau_3}{1+\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1}, \frac{\eta_1+\eta_2+\eta_3+\eta_1\eta_2\eta_3}{1+\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_2}, \frac{\zeta_1\zeta_2\zeta_3}{(2-\zeta_2-\zeta_3-\zeta_2\zeta_3)+(1-\zeta_1)(2-\zeta_2-\zeta_3)}, \\ \frac{\kappa_1\kappa_2\kappa_3}{(2-\kappa_2-\kappa_3-\kappa_2\kappa_3)+(1-\kappa_1)(2-\kappa_2-\kappa_3)}, \frac{\xi_1\xi_2\xi_3}{(2-\xi_2-\xi_3-\xi_2\xi_3)+(1-\xi_1)(2-\xi_2-\xi_3)} \end{bmatrix}$$

This further simplifies to,

$$\begin{bmatrix} \frac{\tau_1+\tau_2+\tau_3+\tau_1\tau_2\tau_3}{1+\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1}, \frac{\eta_1+\eta_2+\eta_3+\eta_1\eta_2\eta_3}{1+\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1}, \frac{\zeta_1\zeta_2\zeta_3}{4-2(\zeta_1+\zeta_2+\zeta_3)+(\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1)}, \\ \frac{\kappa_1\kappa_2\kappa_3}{4-2(\kappa_1+\kappa_2+\kappa_3)+(\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1)}, \frac{\xi_1\xi_2\xi_3}{4-2(\xi_1+\xi_2+\xi_3)+(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1)} \end{bmatrix} \tag{9}$$

From Eqs. (8)-(9), we have $\{\mathbb{A} \oplus \mathbb{B}\} \oplus \mathbb{C} = \mathbb{A} \oplus \{\mathbb{B} \oplus \mathbb{C}\}$.

(d) $\mathbb{A} \otimes \mathbb{B} = \langle \frac{\tau_1\tau_2}{1+(1-\tau_1)(1-\tau_2)}, \frac{\eta_1\eta_2}{1+(1-\eta_1)(1-\eta_2)}, \frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}, \frac{\kappa_1+\kappa_2}{1+\kappa_1\kappa_2}, \frac{\xi_1+\xi_2}{1+\xi_1\xi_2} \rangle$ (See Eq. (5))

$$\{\mathbb{A} \otimes \mathbb{B}\} \otimes \mathbb{C} = \left[\frac{\frac{\tau_1\tau_2\tau_3}{1+(1-\tau_2)(1-\tau_3)}}{1+(1-\tau_3)\left(1-\frac{\tau_2\tau_1}{1+(1-\tau_2)(1-\tau_1)}\right)}, \frac{\frac{\eta_1\eta_2\eta_3}{1+(1-\eta_2)(1-\eta_3)}}{1+(1-\eta_3)\left(1-\frac{\eta_2\eta_1}{1+(1-\eta_2)(1-\eta_1)}\right)}, \frac{\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}+\zeta_3}{1+\left\{\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}\right\}\zeta_3}, \frac{\frac{\kappa_1+\kappa_2}{1+\kappa_1\kappa_2}+\kappa_3}{1+\left\{\frac{\kappa_1+\kappa_2}{1+\kappa_1\kappa_2}\right\}\kappa_3}, \frac{\frac{\xi_1+\xi_2}{1+\xi_1\xi_2}+\xi_3}{1+\left\{\frac{\xi_1+\xi_2}{1+\xi_1\xi_2}\right\}\xi_3} \right]$$

$$= \begin{bmatrix} \frac{\tau_1\tau_2\tau_3}{4-2(\tau_1+\tau_2+\tau_3)+(\tau_1\tau_2+\tau_2\tau_3+\tau_3x_1)}, \frac{\eta_1\eta_2\eta_3}{4-2(\eta_1+\eta_2+\eta_3)+(\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1)}, \\ \frac{\zeta_1+\zeta_2+\zeta_3+\zeta_1\zeta_2\zeta_3}{1+\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1}, \frac{\kappa_1+\kappa_2+\kappa_3+\kappa_1\kappa_2\kappa_3}{1+\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1}, \frac{\xi_1+\xi_2+\xi_3+\xi_1\xi_2\xi_3}{1+\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1} \end{bmatrix} \tag{10}$$

$$Now, \ \mathbb{B} \otimes \mathbb{C} = \left[\frac{\tau_2\tau_3}{1+(1-\tau_2)(1-\tau_3)}, \frac{\eta_2\eta_3}{1+(1-\eta_2)(1-\eta_3)}, \frac{\zeta_2+\zeta_3}{1+\zeta_2\zeta_3}, \frac{\kappa_2+\kappa_3}{1+\kappa_2\kappa_3}, \frac{\xi_2+\xi_3}{1+\xi_2\xi_3} \right]$$

Then

$$\mathbb{A}\otimes\{\mathbb{B}\otimes\mathbb{C}\}=\left[\frac{\frac{\tau_1\tau_2\tau_3}{1+(1-\tau_2)(1-\tau_3)}}{1+(1-\tau_1)\left(1-\frac{\tau_2\tau_3}{1+(1-\tau_2)(1-\tau_3)}\right)},\frac{\frac{\eta_1\eta_2\eta_3}{1+(1-\eta_2)(1-\eta_3)}}{1+(1-\eta_1)\left(1-\frac{\eta_2\eta_3}{1+(1-\eta_2)(1-\eta_3)}\right)},\frac{\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}+\zeta_3}{1+\left\{\frac{\zeta_1+\zeta_2}{1+\zeta_1\zeta_2}\right\}\zeta_3},\frac{\frac{\kappa_3+\kappa_2}{1+\kappa_3\kappa_2}+\kappa_1}{1+\left\{\frac{\kappa_3+\kappa_2}{1+\kappa_3\kappa_2}\right\}\kappa_1},\frac{\frac{\xi_3+\xi_2}{1+\xi_3\xi_2}+\xi_1}{1+\left\{\frac{\xi_3+\xi_2}{1+\xi_3\xi_2}\right\}\xi_1}\right]$$

On simplifying we get,

$$\left[\begin{array}{c}\frac{\tau_1\tau_2\tau_3}{4-2(\tau_1+\tau_2+\tau_3)+(\tau_1\tau_2+\tau_2\tau_3+\tau_3\tau_1)},\frac{\eta_1\eta_2\eta_3}{4-2(\eta_1+\eta_2+\eta_3)+(\eta_1\eta_2+\eta_2\eta_3+\eta_3\eta_1)},\\ \frac{\zeta_1+\zeta_2+\zeta_3+\zeta_1\zeta_2\zeta_3}{1+\zeta_1\zeta_2+\zeta_2\zeta_3+\zeta_3\zeta_1},\frac{\kappa_1+\kappa_2+\kappa_3+\kappa_1\kappa_2\kappa_3}{1+\kappa_1\kappa_2+\kappa_2\kappa_3+\kappa_3\kappa_1},\frac{\xi_1+\xi_2+\xi_3+\xi_1\xi_2\xi_3}{1+\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1}\end{array}\right] \tag{11}$$

Hence proves the result (from Eqs.(10) and (11)).

(e) Using Eq. (6), we have,

$$\lambda\mathbb{A}=\langle\frac{(1+\tau_1)^\lambda-(1-\tau_1)^\lambda}{(1+\tau_1)^\lambda+(1-\tau_1)^\lambda},\frac{(1+\eta_1)^\lambda-(1-\eta_1)^\lambda}{(1+\eta_1)^\lambda+(1-\eta_1)^\lambda},\frac{2{\zeta_1}^\lambda}{(2-\zeta_1)^\lambda+{\zeta_1}^\lambda},\frac{2{\kappa_1}^\lambda}{(2-\kappa_1)^\lambda+{\kappa_1}^\lambda},\frac{2{\xi_1}^\lambda}{(2-\xi_1)^\lambda+{\xi_1}^\lambda}\rangle$$

$\lambda\mathbb{B}=\langle\frac{(1+\tau_2)^\lambda-(1-\tau_2)^\lambda}{(1+\tau_2)^\lambda+(1-\tau_2)^\lambda},\frac{(1+\eta_2)^\lambda-(1-\eta_2)^\lambda}{(1+\eta_2)^\lambda+(1-\eta_2)^\lambda},\frac{2{\zeta_2}^\lambda}{(2-\zeta_2)^\lambda+{\zeta_2}^\lambda},\frac{2\kappa_2{}^\lambda}{(2-\kappa_2)^\lambda+\kappa_2{}^\lambda},\frac{2{\xi_2}^\lambda}{(2-\xi_2)^\lambda+{\xi_2}^\lambda}\rangle$, hence, we have

$$\boldsymbol{\lambda}\mathbb{A}\oplus\boldsymbol{\lambda}\mathbb{B}=\left[\begin{array}{c}\frac{\frac{(1+\tau_1)^\lambda-(1-\tau_1)^\lambda}{(1+\tau_1)^\lambda+(1-\tau_1)^\lambda}+\frac{(1+\tau_2)^\lambda-(1-\tau_2)^\lambda}{(1+\tau_2)^\lambda+(1-\tau_2)^\lambda}}{1+\left\{\frac{(1+\tau_1)^\lambda-(1-\tau_1)^\lambda}{(1+\tau_1)^\lambda+(1-\tau_1)^\lambda}\right\}\left\{\frac{(1+\tau_2)^\lambda-(1-\tau_2)^\lambda}{(1+\tau_2)^\lambda+(1-\tau_2)^\lambda}\right\}},\\ \frac{\frac{(1+\eta_1)^\lambda-(1-\eta_1)^\lambda}{(1+\eta_1)^\lambda+(1-\eta_1)^\lambda}+\frac{(1+\eta_2)^\lambda-(1-\eta_2)^\lambda}{(1+\eta_2)^\lambda+(1-\eta_2)^\lambda}}{1+\left\{\frac{(1+\eta_1)^\lambda-(1-\eta_1)^\lambda}{(1+\eta_1)^\lambda+(1-\eta_1)^\lambda}\right\}\left\{\frac{(1+\eta_2)^\lambda-(1-\eta_2)^\lambda}{(1+\eta_2)^\lambda+(1-\eta_2)^\lambda}\right\}},\\ \frac{\left\{\frac{2{\zeta_1}^\lambda}{(2-\zeta_1)^\lambda+{\zeta_1}^\lambda}\right\}\left\{\frac{2{\zeta_2}^\lambda}{(2-\zeta_2)^\lambda+{\zeta_2}^\lambda}\right\}}{2-\left\{\frac{2{\zeta_1}^\lambda}{(2-\zeta_1)^\lambda+{\zeta_1}^\lambda}\right\}-\left\{\frac{2{\zeta_2}^\lambda}{(2-\zeta_2)^\lambda+{\zeta_2}^\lambda}\right\}+\left\{\frac{2{\zeta_1}^\lambda}{(2-\zeta_1)^\lambda+{\zeta_1}^\lambda}\right\}\left\{\frac{2{\zeta_2}^\lambda}{(2-\zeta_2)^\lambda+{\zeta_2}^\lambda}\right\}}\\ \frac{\left\{\frac{2{\kappa_1}^\lambda}{(2-\kappa_1)^\lambda+{\kappa_1}^\lambda}\right\}\left\{\frac{2{\kappa_2}^\lambda}{(2-\kappa_2)^\lambda+{\kappa_2}^\lambda}\right\}}{2-\left\{\frac{2{\kappa_1}^\lambda}{(2-\kappa_1)^\lambda+{\kappa_1}^\lambda}\right\}-\left\{\frac{2{\kappa_2}^\lambda}{(2-\kappa_2)^\lambda+{\kappa_2}^\lambda}\right\}+\left\{\frac{2{\kappa_1}^\lambda}{(2-\kappa_1)^\lambda+{\kappa_1}^\lambda}\right\}\left\{\frac{2{\kappa_2}^\lambda}{(2-\kappa_2)^\lambda+{\kappa_2}^\lambda}\right\}},\\ \frac{\left\{\frac{2{\xi_1}^\lambda}{(2-\xi_1)^\lambda+{\xi_1}^\lambda}\right\}\left\{\frac{2{\xi_2}^\lambda}{(2-\xi_2)^\lambda+{\xi_2}^\lambda}\right\}}{2-\left\{\frac{2{\xi_1}^\lambda}{(2-\xi_1)^\lambda+{\xi_1}^\lambda}\right\}-\left\{\frac{2{\xi_2}^\lambda}{(2-\xi_2)^\lambda+{\xi_2}^\lambda}\right\}+\left\{\frac{2{\xi_1}^\lambda}{(2-\xi_1)^\lambda+{\xi_1}^\lambda}\right\}\left\{\frac{2{\xi_2}^\lambda}{(2-\xi_2)^\lambda+{\xi_2}^\lambda}\right\}}\end{array}\right]$$

$$=\left[\begin{array}{c}\frac{[(1+\tau_1)^\lambda(1+\tau_2)^\lambda-(1-\tau_1)^\lambda(1-\tau_2)^\lambda]}{[(1+\tau_1)^\lambda(1+\tau_2)^\lambda+(1-\tau_1)^\lambda(1-\tau_2)^\lambda]},\frac{[(1+\eta_1)^\lambda(1+\eta_2)^\lambda-(1-\eta_1)^\lambda(1-\eta_2)^\lambda]}{[(1+\eta_1)^\lambda(1+\eta_2)^\lambda+(1-\eta_1)^\lambda(1-\eta_2)^\lambda]},\\ \frac{2(\zeta_1\zeta_2)^\lambda}{\{4-2\zeta_1-2\zeta_2+\zeta_1\zeta_2\}^\lambda+(\zeta_1\zeta_2)^\lambda},\frac{2(\kappa_1\kappa_2)^\lambda}{\{4-2\kappa_1-2\kappa_2+\kappa_1\kappa_2\}^\lambda+(\kappa_1\kappa_2)^\lambda},\\ \frac{2(\xi_1\xi_2)^\lambda}{\{4-2\xi_1-2\xi_2+\xi_1\xi_2\}^\lambda+(\xi_1\xi_2)^\lambda}\end{array}\right] \tag{12}$$

For $\boldsymbol{\lambda}\{\mathbb{A}\oplus\mathbb{B}\}=\boldsymbol{\lambda}\left[\frac{\tau_1+\tau_2}{1+\tau_1\tau_2},\frac{\eta_1+\eta_2}{1+\eta_1\eta_2},\frac{\zeta_1\zeta_2}{1+(1-\zeta_1)(1-\zeta_2)},\frac{\kappa_1\kappa_2}{1+(1-\kappa_1)(1-\kappa_2)},\frac{\xi_1\xi_2}{1+(1-\xi_1)(1-\xi_2)}\right]$

$$=\left[\begin{array}{c}\frac{\left(1+\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}\right)^\lambda-\left(1-\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}\right)^\lambda}{\left(1+\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}\right)^\lambda+\left(1-\frac{\tau_1+\tau_2}{1+\tau_1\tau_2}\right)^\lambda},\frac{\left(1+\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}\right)^\lambda-\left(1-\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}\right)^\lambda}{\left(1+\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}\right)^\lambda+\left(1-\frac{\eta_1+\eta_2}{1+\eta_1\eta_2}\right)^\lambda},\\ \frac{2\left\{\frac{\zeta_1\zeta_2}{2-\zeta_1-\zeta_2+\zeta_1\zeta_2}\right\}^\lambda}{\left(2-\left\{\frac{\zeta_1\zeta_2}{2-\zeta_1-\zeta_2+\zeta_1\zeta_2}\right\}\right)^\lambda+\left\{\frac{\zeta_1\zeta_2}{2-\zeta_1-\zeta_2+\zeta_1\zeta_2}\right\}^\lambda},\\ \frac{2\left\{\frac{\kappa_1\kappa_2}{2-\kappa_1-\kappa_2+\kappa_1\kappa_2}\right\}^\lambda}{\left(2-\left\{\frac{\kappa_1\kappa_2}{2-\kappa_1-\kappa_2+\kappa_1\kappa_2}\right\}\right)^\lambda+\left\{\frac{\kappa_1\kappa_2}{2-\kappa_1-\kappa_2+\kappa_1\kappa_2}\right\}^\lambda},\\ \frac{2\left\{\frac{\xi_1\xi_2}{2-\xi_1-\xi_2+\xi_1\xi_2}\right\}^\lambda}{\left(2-\left\{\frac{\xi_1\xi_2}{2-\xi_1-\xi_2+\xi_1\xi_2}\right\}\right)^\lambda+\left\{\frac{\xi_1\xi_2}{2-\xi_1-\xi_2+\xi_1\xi_2}\right\}^\lambda}\end{array}\right]\text{(Using Eqs. (4), (6))}$$

Further simplification provides,

$$=\left[\begin{matrix}\frac{[(1+\tau_1)(1+\tau_2)]^{\lambda}-[(1-\tau_1)(1-\tau_2)]^{\lambda}}{[(1+\tau_1)(1+\tau_2)]^{\lambda}+[(1-\tau_1)(1-\tau_2)]^{\lambda}},\frac{[(1+\eta_1)(1+\eta_2)]^{\lambda}-[(1-\eta_1)(1-\eta_2)]^{\lambda}}{[(1+\eta_1)(1+\eta_2)]^{\lambda}+[(1-\eta_1)(1-\eta_2)]^{\lambda}},\\ \frac{2(\zeta_1\zeta_2)^{\lambda}}{\{4-2\zeta_1-2\zeta_2+\zeta_1y_3\}^{\lambda}+(\zeta_1\zeta_2)^{\lambda}},\frac{2(\kappa_1\kappa_2)^{\lambda}}{\{4-2\kappa_1-2\kappa_2+\kappa_1\kappa_2\}^{\lambda}+(\kappa_1\kappa_2)^{\lambda}},\\ \frac{2(\xi_1\xi_2)^{\lambda}}{\{4-2\xi_1-2\xi_2+\xi_1\xi_2\}^{\lambda}+(\xi_1\xi_2)^{\lambda}}\end{matrix}\right] \tag{13}$$

Hence, $\boldsymbol{\lambda}\{\mathbb{A}\oplus\mathbb{B}\}=\boldsymbol{\lambda}\mathbb{A}\oplus\boldsymbol{\lambda}\mathbb{B}$(using Eqs. (12)-(13)).

(f)

Using Eq. (6) we get,

$$\boldsymbol{\lambda_1}\mathbb{A}=\langle\frac{(1+\tau_1)^{\lambda_1}-(1-\tau_1)^{\lambda_1}}{(1+\tau_1)^{\lambda_1}+(1-\tau_1)^{\lambda_1}},\frac{(1+\eta_1)^{\lambda_1}-(1-\eta_1)^{\lambda_1}}{(1+\eta_1)^{\lambda_1}+(1-\eta_1)^{\lambda_1}},\frac{2{\zeta_1}^{\lambda_1}}{(2-\zeta_1)^{\lambda_1}+{\zeta_1}^{\lambda_1}},\frac{2{\kappa_1}^{\lambda_1}}{(2-\kappa_1)^{\lambda_1}+{\kappa_1}^{\lambda_1}},\frac{2{\xi_1}^{\lambda_1}}{(2-\xi_1)^{\lambda_1}+{\xi_1}^{\lambda_1}}\rangle$$

and, $\boldsymbol{\lambda_2}\mathbb{A}=\langle\frac{(1+\tau_1)^{\lambda_2}-(1-\tau_1)^{\lambda_2}}{(1+\tau_1)^{\lambda_2}+(1-\tau_1)^{\lambda_2}},\frac{(1+\eta_1)^{\lambda_2}-(1-\eta_1)^{\lambda_2}}{(1+\eta_1)^{\lambda_2}+(1-\eta_1)^{\lambda_2}},\frac{2{\zeta_1}^{\lambda_2}}{(2-\zeta_1)^{\lambda_2}+{\zeta_1}^{\lambda_2}},\frac{2{\kappa_1}^{\lambda_2}}{(2-\kappa_1)^{\lambda_2}+{\kappa_1}^{\lambda_2}},\frac{2{\xi_1}^{\lambda_2}}{(2-\xi_1)^{\lambda_2}+{\xi_1}^{\lambda_2}}\rangle$

Now $\boldsymbol{\lambda_1}\mathbb{A}\oplus\boldsymbol{\lambda_2}\mathbb{A}$ (Eq. (4))

$$=\langle\frac{(1+\tau_1)^{\lambda_1}-(1-\tau_1)^{\lambda_1}}{(1+\tau_1)^{\lambda_1}+(1-\tau_1)^{\lambda_1}},\frac{(1+\eta_1)^{\lambda_1}-(1-\eta_1)^{\lambda_1}}{(1+\eta_1)^{\lambda_1}+(1-\eta_1)^{\lambda_1}},\frac{2{\zeta_1}^{\lambda_1}}{(2-\zeta_1)^{\lambda_1}+{\zeta_1}^{\lambda_1}},\frac{2{\kappa_1}^{\lambda_1}}{(2-\kappa_1)^{\lambda_1}+{\kappa_1}^{\lambda_1}},\frac{2{\xi_1}^{\lambda_1}}{(2-\xi_1)^{\lambda_1}+{\xi_1}^{\lambda_1}}\rangle\oplus$$

$$\langle\frac{(1+\tau_1)^{\lambda_2}-(1-\tau_1)^{\lambda_2}}{(1+\tau_1)^{\lambda_2}+(1-\tau_1)^{\lambda_2}},\frac{(1+\eta_1)^{\lambda_2}-(1-\eta_1)^{\lambda_2}}{(1+\eta_1)^{\lambda_2}+(1-\eta_1)^{\lambda_2}},\frac{2{\zeta_1}^{\lambda_2}}{(2-\zeta_1)^{\lambda_2}+{\zeta_1}^{\lambda_2}},\frac{2{\kappa_1}^{\lambda_2}}{(2-\kappa_1)^{\lambda_2}+{\kappa_1}^{\lambda_2}},\frac{2{\xi_1}^{\lambda_2}}{(2-\xi_1)^{\lambda_2}+{\xi_1}^{\lambda_2}}\rangle$$

$$=\left[\begin{matrix}\dfrac{\frac{(1+\tau_1)^{\lambda_1}-(1-\tau_1)^{\lambda_1}}{(1+\tau_1)^{\lambda_1}+(1-\tau_1)^{\lambda_1}}+\frac{(1+\tau_1)^{\lambda_2}-(1-\tau_1)^{\lambda_2}}{(1+\tau_1)^{\lambda_2}+(1-\tau_1)^{\lambda_2}}}{1+\left\{\frac{(1+\tau_1)^{\lambda_1}-(1-\tau_1)^{\lambda_1}}{(1+\tau_1)^{\lambda_1}+(1-\tau_1)^{\lambda_1}}\right\}\left\{\frac{(1+\tau_1)^{\lambda_2}-(1-\tau_1)^{\lambda_2}}{(1+\tau_1)^{\lambda_2}+(1-\tau_1)^{\lambda_2}}\right\}},\\ \dfrac{\frac{(1+\eta_1)^{\lambda_1}-(1-\eta_1)^{\lambda_1}}{(1+\eta_1)^{\lambda_1}+(1-\eta_1)^{\lambda_1}}+\frac{(1+\eta_1)^{\lambda_2}-(1-\eta_1)^{\lambda_2}}{(1+\eta_1)^{\lambda_2}+(1-\eta_1)^{\lambda_2}}}{1+\left\{\frac{(1+\eta_1)^{\lambda_1}-(1-\eta_1)^{\lambda_1}}{(1+\eta_1)^{\lambda_1}+(1-\eta_1)^{\lambda_1}}\right\}\left\{\frac{(1+\eta_1)^{\lambda_2}-(1-\eta_1)^{\lambda_2}}{(1+\eta_1)^{\lambda_2}+(1-\eta_1)^{\lambda_2}}\right\}},\\ \dfrac{\left\{\frac{2{\zeta_1}^{\lambda_1}}{(2-\zeta_1)^{\lambda_1}+{\zeta_1}^{\lambda_1}}\right\}\left\{\frac{2{\zeta_1}^{\lambda_2}}{(2-\zeta_1)^{\lambda_2}+{\zeta_1}^{\lambda_2}}\right\}}{2-\left\{\frac{2{\zeta_1}^{\lambda_1}}{(2-\zeta_1)^{\lambda_1}+{\zeta_1}^{\lambda_1{}_1}}\right\}-\left\{\frac{2{\zeta_1}^{\lambda_2}}{(2-\zeta_1)^{\lambda_2}+{\zeta_1}^{\lambda_2}}\right\}+\left\{\frac{2{\zeta_1}^{\lambda_1}}{(2-\zeta_1)^{\lambda_1}+{\zeta_1}^{\lambda_1}}\right\}\left\{\frac{2{\zeta_1}^{\lambda_2}}{(2-\zeta_1)^{\lambda_2}+{\zeta_1}^{\lambda_2}}\right\}},\\ \dfrac{\left\{\frac{2{\kappa_1}^{\lambda_1}}{(2-\kappa_1)^{\lambda_1}+{\kappa_1}^{\lambda_1}}\right\}\left\{\frac{2{\kappa_1}^{\lambda_2}}{(2-\kappa_1)^{\lambda_2}+{\kappa_1}^{\lambda_2}}\right\}}{2-\left\{\frac{2{\kappa_1}^{\lambda_1}}{(2-\kappa_1)^{\lambda_1}+{\kappa_1}^{\lambda_1}}\right\}-\left\{\frac{2{\kappa_1}^{\lambda_2}}{(2-\kappa_1)^{\lambda_2}+{\kappa_1}^{\lambda_2}}\right\}+\left\{\frac{2{\kappa_1}^{\lambda_1}}{(2-\kappa_1)^{\lambda_1}+{\kappa_1}^{\lambda_1}}\right\}\left\{\frac{2{\kappa_1}^{\lambda_2}}{(2-\kappa_1)^{\lambda_2}+{\kappa_1}^{\lambda_2}}\right\}},\\ \dfrac{\left\{\frac{2{\xi_1}^{\lambda_1}}{(2-\xi_1)^{\lambda_1}+{\xi_1}^{\lambda_1}}\right\}\left\{\frac{2{\xi_1}^{\lambda_2}}{(2-\xi_1)^{\lambda_2}+{\xi_1}^{\lambda_2}}\right\}}{2-\left\{\frac{2{\xi_1}^{\lambda_1}}{(2-\xi_1)^{\lambda_1}+q^{\lambda_1}}\right\}-\left\{\frac{2{\xi_1}^{\lambda_2}}{(2-\xi_1)^{\lambda_2}+{\xi_1}^{\lambda_2}}\right\}+\left\{\frac{2{\xi_1}^{\lambda_1}}{(2-\xi_1)^{\lambda_1}+{\xi_1}^{\lambda_1}}\right\}\left\{\frac{2{\xi_1}^{\lambda_2}}{(2-\xi_1)^{\lambda_2}+{\xi_1}^{\lambda_2}}\right\}}\end{matrix}\right]$$

Which on simplifying provides,

$$\left\langle\begin{matrix}\frac{(1+\tau_1)^{\lambda_1}(1+\tau_1)^{\lambda_2}+(1-\tau_1)^{\lambda_1}(1-\tau_1)^{\lambda_2}}{(1+\tau_1)^{\lambda_1}(1+\tau_1)^{\lambda_2}-(1-\tau_1)^{\lambda_1}(1-\tau_1)^{\lambda_2}},\frac{(1+\eta_1)^{\lambda_1}(1+\eta_1)^{\lambda_2}+(1-\eta_1)^{\lambda_1}(1-\eta_1)^{\lambda_2}}{(1+\eta_1)^{\lambda_1}(1+\eta_1)^{\lambda_2}-(1-\eta_1)^{\lambda_1}(1-\eta_1)^{\lambda_2}},\\ \frac{2{\zeta_1}^{\lambda_1}{\zeta_1}^{\lambda_2}}{(2-\zeta_1)^{\lambda_1}(2-\zeta_1)^{\lambda_2}+{\zeta_1}^{\lambda_1}{\zeta_1}^{\lambda_2}},\frac{2{\kappa_1}^{\lambda_1}{\kappa_1}^{\lambda_2}}{(2-\kappa_1)^{\lambda_1}(2-\kappa_1)^{\lambda_2}+{\kappa_1}^{\lambda_1}{\kappa_1}^{\lambda_2}},\frac{2{\xi_1}^{\lambda_1}{\xi_1}^{\lambda_2}}{(2-\xi_1)^{\lambda_1}(2-\xi_1)^{\lambda_2}+{\xi_1}^{\lambda_1}{\xi_1}^{\lambda_2}}\end{matrix}\right\rangle$$

$$= \left\langle \begin{matrix} \frac{(1+\tau_1)^{\lambda_1+\lambda_2}-(1-\tau_1)^{\lambda_1+\lambda_2}}{(1+\tau_1)^{\lambda_1+\lambda_2}+(1-\tau_1)^{\lambda_1+\lambda_2}}, \frac{(1+\eta_1)^{\lambda_1+\lambda_2}-(1-\eta_1)^{\lambda_1+\lambda_2}}{(1+\eta_1)^{\lambda_1+\lambda_2}+(1-\eta_1)^{\lambda_1+\lambda_2}}, \\ \frac{2{\zeta_1}^{\lambda_1+\lambda_2}}{(2-\zeta_1)^{\lambda_1+\lambda_2}+{\zeta_1}^{\lambda_1+\lambda_2}}, \frac{2{\kappa_1}^{\lambda_1+\lambda_2}}{(2-\kappa_1)^{\lambda_1+\lambda_2}+{\kappa_1}^{\lambda_1+\lambda_2}}, \frac{2{\xi_1}^{\lambda_1+\lambda_2}}{(2-\xi_1)^{\lambda_1+\lambda_2}+{\xi_1}^{\lambda_1+\lambda_2}} \end{matrix} \right\rangle = \{\boldsymbol{\lambda_1} + \boldsymbol{\lambda_2}\}\mathbb{A} \text{ (Using Eq. (10)).}$$

New proposed Einstein aggregation operator for PNN

Definition 12. The Pentapartitioned Neutrosophic Einstein Weighted Arithmetic Aggregation (PNEWAA) operator is defined as $PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \dots, \mathbb{A}_n) = \oplus_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w_i}\mathbb{A}_i = w_1\mathbb{A}_1 \oplus w_2\mathbb{A}_2 \oplus w_3\mathbb{A}_3 \oplus \dots \oplus w_n\mathbb{A}_n$, where $\mathbb{A}_i = \langle \tau_i, \eta_i, \zeta_i, \kappa_i, \xi_i \rangle, i = 1,2, \dots, n$ denotes the PNNs and $W^T = [w_1, w_2, \dots, w_n]$ denotes the corresponding weight vector, such that $\sum_{i=1}^{n} w_i = 1$ and $0 \le w_i \le 1$.

Theorem 1: If $\mathbb{A}_i = \langle \tau_i, \eta_i, \zeta_i, \kappa_i, \xi_i \rangle$ be a collection of PNN for $i = 1,2, \dots, n$. Let $w_i, i = 1,2,3 \dots\dots n$ be some scalars such that $\sum_{i=1}^{n} w_i = 1 \; and \; 0 \le w_i \le 1, \forall n \in N$. Then

$$PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \dots, \mathbb{A}_n) = \oplus_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{w_i}\mathbb{A}_i = w_1\mathbb{A}_1 \oplus w_2\mathbb{A}_2 \oplus w_3\mathbb{A}_3 \oplus \dots \oplus w_n\mathbb{A}_n =$$

$$\left\langle \begin{matrix} \frac{\prod_{i=1}^{n}(1+\tau_i)^{w_i}-\prod_{i=1}^{n}(1-\tau_i)^{w_i}}{\prod_{i=1}^{n}(1+\tau_i)^{w_i}+\prod_{i=1}^{n}(1-\tau_i)^{w_i}}, \frac{\prod_{i=1}^{n}(1+\eta_i)^{w_i}-\prod_{i=1}^{n}(1-\eta_i)^{w_i}}{\prod_{i=1}^{n}(1+\eta_i)^{w_i}+\prod_{i=1}^{n}(1-\eta_i)^{w_i}}, \\ \frac{2\prod_{i=1}^{n}{\zeta_i}^{w_i}}{\prod_{i=1}^{n}(2-\zeta_i)^{w_i}+\prod_{i=1}^{n}{\zeta_i}^{w_i}}, \frac{2\prod_{i=1}^{n}{\kappa_i}^{w_i}}{\prod_{i=1}^{n}(2-\kappa_i)^{w_i}+\prod_{i=1}^{n}{\kappa_i}^{w_i}}, \frac{2\prod_{i=1}^{n}{\xi_i}^{w_i}}{\prod_{i=1}^{n}(2-\xi_i)^{w_i}+\prod_{i=1}^{n}{\xi_i}^{w_i}} \end{matrix} \right\rangle \qquad (14)$$

Proof: Considering, $\mathbb{A}_i = \langle \tau_i, \eta_i, \zeta_i, \kappa_i, \xi_i \rangle$

for n=1, we have, $PNEWAA(\mathbb{A}_1) = \oplus_{\mathrm{i}=1}^{\mathrm{n}=1} \mathrm{w_i}\mathbb{A}_i = w_1\mathbb{A}_1$

now for n=2, we have $PNEWAA(\mathbb{A}_1, \mathbb{A}_2) = \oplus_{\mathrm{i}=1}^{\mathrm{n}=2} \mathrm{w_i}\mathbb{A}_i = w_1\mathbb{A}_1 \oplus w_2\mathbb{A}_2$

Using scalar multiplication (Eq. (10)), $= w_1\{\tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1\} + w_2\{\tau_2, \eta_2, \zeta_2, \kappa_2, \xi_2\}$

$$= \left\langle \frac{(1+\tau_1)^{w_1}-(1-\tau_1)^{w_1}}{(1+\tau_1)^{w_1}+(1-\tau_1)^{w_1}}, \frac{(1+\eta_1)^{w_1}-(1-\eta_1)^{w_1}}{(1+\eta_1)^{w_1}+(1-\eta_1)^{w_1}}, \frac{2{\zeta_1}^{w_1}}{(2-\zeta_1)^{w_1}+{\zeta_1}^{w_1}}, \frac{2{\kappa_1}^{w_1}}{(2-\kappa_1)^{w_1}+{\kappa_1}^{w_1}}, \frac{2{\xi_1}^{w_1}}{(2-\xi_1)^{w_1}+{\xi_1}^{w_1}} \right\rangle \oplus$$

$$\left\langle \frac{(1+\tau_2)^{w_2}-(1-\tau_2)^{w_2}}{(1+\tau_2)^{w_2}+(1-\tau_2)^{w_2}}, \frac{(1+\eta_2)^{w_2}-(1-\eta_2)^{w_2}}{(1+\eta_2)^{w_2}+(1-\eta_2)^{w_2}}, \frac{2{\zeta_2}^{w_2}}{(2-\zeta_2)^{w_2}+{\zeta_2}^{w_2}}, \frac{2{\kappa_2}^{w_2}}{(2-\kappa_2)^{w_2}+{\kappa_2}^{w_2}}, \frac{2{\xi_2}^{w_2}}{(2-\xi_2)^{w_2}+{\xi_2}^{w_2}} \right\rangle$$

$$= \begin{bmatrix} \frac{\frac{(1+\tau_1)^{w_1}-(1-\tau_1)^{w_1}}{(1+\tau_1)^{w_1}+(1-\tau_1)^{w_1}} + \frac{(1+\tau_2)^{w_2}-(1-\tau_2)^{w_2}}{(1+\tau_2)^{w_2}+(1-\tau_2)^{w_2}}}{1+\left\{\frac{(1+\tau_2)^{w_2}-(1-\tau_2)^{w_2}}{(1+\tau_2)^{w_2}+(1-\tau_2)^{w_2}}\right\}\left\{\frac{(1+\tau_1)^{w_1}-(1-\tau_1)^{w_1}}{(1+\tau_1)^{w_1}+(1-\tau_1)^{w_1}}\right\}}, \frac{\frac{(1+\eta_1)^{w_1}-(1-\eta_1)^{w_1}}{(1+\eta_1)^{w_1}+(1-\eta_1)^{w_1}} + \frac{(1+\eta_2)^{w_2}-(1-\eta_2)^{w_2}}{(1+\eta_2)^{w_2}+(1-\eta_2)^{w_2}}}{1+\left\{\frac{(1+\eta_2)^{w_2}-(1-\eta_2)^{w_2}}{(1+\eta_2)^{w_2}+(1-\eta_2)^{w_2}}\right\}\left\{\frac{(1+\eta_1)^{w_1}-(1-\eta_1)^{w_1}}{(1+\eta_1)^{w_1}+(1-\eta_1)^{w_1}}\right\}}, \\ \frac{\left\{\frac{2{\zeta_1}^{w_1}}{(2-\zeta_1)^{w_1}+{\zeta_1}^{w_1}}\right\}\left\{\frac{2{\zeta_2}^{w_2}}{(2-\zeta_2)^{w_2}+{\zeta_2}^{w_2}}\right\}}{2-\left\{\frac{2{\zeta_1}^{w_1}}{(2-\zeta_1)^{w_1}+{\zeta_1}^{w_1}}\right\}-\left\{\frac{2{\zeta_2}^{w_2}}{(2-\zeta_2)^{w_2}+{\zeta_2}^{w_2}}\right\}+\left\{\frac{2{\zeta_1}^{w_1}}{(2-\zeta_1)^{w_1}+{\zeta_1}^{w_1}}\right\}\left\{\frac{2{\zeta_2}^{w_2}}{(2-\zeta_2)^{w_2}+{\zeta_2}^{w_2}}\right\}}, \\ \frac{\left\{\frac{2{\kappa_1}^{w_1}}{(2-\kappa_1)^{w_1}+{\kappa_1}^{w_1}}\right\}\left\{\frac{2{\kappa_2}^{w_2}}{(2-\kappa_2)^{w_2}+{\kappa_2}^{w_2}}\right\}}{2-\left\{\frac{2{\kappa_1}^{w_1}}{(2-\kappa_1)^{w_1}+{\kappa_1}^{w_1}}\right\}-\left\{\frac{2{\kappa_2}^{w_2}}{(2-\kappa_2)^{w_2}+{\kappa_2}^{w_2}}\right\}+\left\{\frac{2{\kappa_1}^{w_1}}{(2-\kappa_1)^{w_1}+{\kappa_1}^{w_1}}\right\}\left\{\frac{2{\kappa_2}^{w_2}}{(2-\kappa_2)^{w_2}+{\kappa_2}^{w_2}}\right\}}, \\ \frac{\left\{\frac{2{\xi_1}^{w_1}}{(2-\xi_1)^{w_1}+{\xi_1}^{w_1}}\right\}\left\{\frac{2{\xi_2}^{w_2}}{(2-\xi_2)^{w_2}+{\xi_2}^{w_2}}\right\}}{2-\left\{\frac{2{\xi_1}^{w_1}}{(2-\xi_1)^{w_1}+{\xi_1}^{w_1}}\right\}-\left\{\frac{2{\xi_2}^{w_2}}{(2-\xi_2)^{w_2}+{\xi_2}^{w_2}}\right\}+\left\{\frac{2{\xi_1}^{w_1}}{(2-\xi_1)^{w_1}+{\xi_1}^{w_1}}\right\}\left\{\frac{2{\xi_2}^{w_2}}{(2-\xi_2)^{w_2}+{\xi_2}^{w_2}}\right\}} \end{bmatrix} \qquad \text{(using Eq. (4))}$$

On simplification this provides,

$$\begin{bmatrix} \frac{(1+\tau_1)^{w_1}(1+\tau_2)^{w_2}-(1-\tau_1)^{w_1}(1-\tau_2)^{w_2}}{(1+\tau_1)^{w_1}(1+\tau_2)^{w_2}+(1-\tau_1)^{w_1}(1-\tau_2)^{w_2}}, \frac{(1+\eta_1)^{w_1}(1+\eta_2)^{w_2}-(1-\eta_1)^{w_1}(1-\eta_2)^{w_2}}{(1+\eta_1)^{w_1}(1+\eta_2)^{w_2}+(1-\eta_1)^{w_1}(1-\eta_2)^{w_2}}, \\ \frac{2{\zeta_1}^{w_1}{\zeta_2}^{w_2}}{(2-\zeta_1)^{w_1}(2-\zeta_2)^{w_2}+{\zeta_1}^{w_1}{\zeta_2}^{w_2}}, \frac{2{\kappa_1}^{w_1}{\kappa_2}^{w_2}}{(2-\kappa_1)^{w_1}(2-\kappa_2)^{w_2}+{\kappa_1}^{w_1}{\kappa_2}^{w_2}}, \frac{2{\xi_1}^{w_1}{\xi_2}^{w_2}}{(2-\xi_1)^{w_1}(2-\xi_2)^{w_2}+{\xi_1}^{w_1}{\xi_2}^{w_2}} \end{bmatrix} \qquad (15)$$

Now using the PNEWAA operator for n=2 (Eq. (14)), we get

$$\left[\begin{array}{c}\frac{\prod_{i=1}^{2}(1+\tau_i)^{w_i}-\prod_{i=1}^{2}(1-\tau_i)^{w_i}}{\prod_{i=1}^{2}(1+\tau_i)^{w_i}+\prod_{i=1}^{2}(1-\tau_i)^{w_i}},\frac{\prod_{i=1}^{2}(1+\eta_i)^{w_i}-\prod_{i=1}^{2}(1-\eta_i)^{w_i}}{\prod_{i=1}^{2}(1+\eta_i)^{w_i}+\prod_{i=1}^{2}(1-\eta_i)^{w_i}},\\ \frac{2\prod_{i=1}^{2}{\zeta_i}^{w_i}}{\prod_{i=1}^{2}(2-\zeta_i)^{w_i}+\prod_{i=1}^{2}{\zeta_i}^{w_i}},\frac{2\prod_{i=1}^{2}{\kappa_i}^{w_i}}{\prod_{i=1}^{2}(2-\kappa_i)^{w_i}+\prod_{i=1}^{2}{\kappa_i}^{w_i}},\frac{2\prod_{i=1}^{2}{\xi_i}^{w_i}}{\prod_{i=1}^{2}(2-\xi_i)^{w_i}+\prod_{i=1}^{2}{\xi_i}^{w_i}}\end{array}\right]$$

This finally simplifies to,

$$\left[\begin{array}{c}\frac{(1+\tau_1)^{w_1}(1+\tau_2)^{w_2}-(1-\tau_1)^{w_1}(1-\tau_2)^{w_2}}{(1+\tau_1)^{w_1}(1+\tau_2)^{w_2}+(1-\tau_1)^{w_1}(1-\tau_2)^{w_2}},\frac{(1+\eta_1)^{w_1}(1+\eta_2)^{w_2}-(1-\eta_1)^{w_1}(1-\eta_2)^{w_2}}{(1+\eta_1)^{w_1}(1+\eta_2)^{w_2}+(1-\eta_1)^{w_1}(1-\eta_2)^{w_2}},\\ \frac{2{\zeta_1}^{w_1}{\zeta_2}^{w_2}}{(2-\zeta_1)^{w_1}(2-\zeta_2)^{w_2}+{\zeta_1}^{w_1}{\zeta_2}^{w_2}},\frac{2{\kappa_1}^{w_1}{\kappa_2}^{w_2}}{(2-\kappa_1)^{w_1}(2-\kappa_2)^{w_2}+{\kappa_1}^{w_1}{\kappa_2}^{w_2}},\frac{2{\xi_1}^{w_1}{\xi_2}^{w_2}}{(2-\xi_1)^{w_1}(2-\xi_2)^{w_2}+{\xi_1}^{w_1}{\xi_2}^{w_2}}\end{array}\right] \quad (16)$$

Hence (using Eq. (15)-(16)), the relation holds for $n = 2$. Therefore, the relation holds for$n = 1,2$. Assuming, the relation holds for $n = k$, from Eq. (14), we have

$$PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \ldots., \mathbb{A}_k) = \oplus_{i=1}^{k} \mathrm{w_i}\mathbb{A}_i = w_1\mathbb{A}_1 \oplus w_2\mathbb{A}_2 \oplus w_3\mathbb{A}_3 \oplus \ldots \oplus w_k\mathbb{A}_k$$

$$= \left[\begin{array}{c}\frac{\prod_{i=1}^{k}(1+\tau_i)^{w_i}-\prod_{i=1}^{k}(1-\tau_i)^{w_i}}{\prod_{i=1}^{k}(1+\tau_i)^{w_i}+\prod_{i=1}^{k}(1-\tau_i)^{w_i}},\frac{\prod_{i=1}^{k}(1+\eta_i)^{w_i}-\prod_{i=1}^{k}(1-\eta_i)^{w_i}}{\prod_{i=1}^{k}(1+\eta_i)^{w_i}+\prod_{i=1}^{k}(1-\eta_i)^{w_i}},\\ \frac{2\prod_{i=1}^{k}{\zeta_i}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_i)^{w_i}+\prod_{i=1}^{k}{\zeta_i}^{w_i}},\frac{2\prod_{i=1}^{n}{\kappa_i}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_i)^{w_i}+\prod_{i=1}^{k}{\kappa_i}^{w_i}},\frac{2\prod_{i=1}^{k}{\xi_i}^{w_i}}{\prod_{i=1}^{k}(2-\xi_i)^{w_i}+\prod_{i=1}^{k}{\xi_i}^{w_i}}\end{array}\right] \quad (17)$$

Now considering $\mathbb{A}_{k+1} = \langle \tau_{k+1}, \eta_{k+1}, \zeta_{k+1}, \kappa_{k+1}, \xi_{k+1} \rangle$, we get

$$PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \ldots., \mathbb{A}_k, \mathbb{A}_{k+1}) = PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \ldots., \mathbb{A}_k) \oplus w_{k+1}\mathbb{A}_{k+1}$$

$= \sum_{i=1}^{k} \mathrm{w_i}\mathbb{A}_i \oplus w_{k+1}\mathbb{A}_{k+1}$ where, (Using Eq. (6))

$$w_{k+1}\mathbb{A}_{k+1} = \left[\begin{array}{c}\frac{(1+\tau_{k+1})^{w_{k+1}}-(1-\tau_{k+1})^{w_{k+1}}}{(1+\tau_{k+1})^{w_{k+1}}+(1-\tau_{k+1})^{w_{k+1}}},\frac{(1+\eta_{k+1})^{w_{k+1}}-(1-\eta_{k+1})^{w_{k+1}}}{(1+\eta_{k+1})^{w_{k+1}}+(1-\eta_{k+1})^{w_{k+1}}},\frac{2{\zeta_{k+1}}^{w_{k+1}}}{(2-\zeta_{k+1})^{w_{k+1}}+{\zeta_{k+1}}^{w_{k+1}}},\\ \frac{2{\kappa_{k+1}}^{w_{k+1}}}{(2-\kappa_{k+1})^{w_{k+1}}+{\kappa_{k+1}}^{w_{k+1}}},\frac{2{\xi_{k+1}}^{w_{k+1}}}{(2-\xi_{k+1})^{w_{k+1}}+{\xi_{k+1}}^{w_{k+1}}}\end{array}\right]$$

(18)

Therefore, we have (Eq. (17)-(18)), $PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \ldots., \mathbb{A}_k) \oplus w_{k+1}\mathbb{A}_{k+1} =$

$$= \left[\begin{array}{c}\frac{\prod_{i=1}^{k}(1+\tau_i)^{w_i}-\prod_{i=1}^{k}(1-\tau_i)^{w_i}}{\prod_{i=1}^{k}(1+\tau_i)^{w_i}+\prod_{i=1}^{k}(1-\tau_i)^{w_i}},\frac{\prod_{i=1}^{k}(1+\eta_i)^{w_i}-\prod_{i=1}^{k}(1-\eta_i)^{w_i}}{\prod_{i=1}^{k}(1+\eta_i)^{w_i}+\prod_{i=1}^{k}(1-\eta_i)^{w_i}},\\ \frac{2\prod_{i=1}^{k}{\zeta_i}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_i)^{w_i}+\prod_{i=1}^{k}{\zeta_i}^{w_i}},\frac{2\prod_{i=1}^{n}{\kappa_i}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_i)^{w_i}+\prod_{i=1}^{k}{\kappa_i}^{w_i}},\frac{2\prod_{i=1}^{k}{\xi_i}^{w_i}}{\prod_{i=1}^{k}(2-\xi_i)^{w_i}+\prod_{i=1}^{k}{\xi_i}^{w_i}}\end{array}\right] \oplus$$

$$\left[\begin{array}{c}\frac{(1+\tau_{k+1})^{w_{k+1}}-(1-\tau_{k+1})^{w_{k+1}}}{(1+\tau_{k+1})^{w_{k+1}}+(1-\tau_{k+1})^{w_{k+1}}},\frac{(1+\eta_{k+1})^{w_{k+1}}-(1-\eta_{k+1})^{w_{k+1}}}{(1+\eta_{k+1})^{w_{k+1}}+(1-\eta_{k+1})^{w_{k+1}}},\frac{2{\zeta_{k+1}}^{w_{k+1}}}{(2-\zeta_{k+1})^{w_{k+1}}+{\zeta_{k+1}}^{w_{k+1}}},\\ \frac{2{\kappa_{k+1}}^{w_{k+1}}}{(2-\kappa_{k+1})^{w_{k+1}}+{\kappa_{k+1}}^{w_{k+1}}},\frac{2{\xi_{k+1}}^{w_{k+1}}}{(2-\xi_{k+1})^{w_{k+1}}+{\xi_{k+1}}^{w_{k+1}}}\end{array}\right]$$

On simplifying we get,

$$
=\left[\begin{array}{c}
\dfrac{\left\{\frac{\prod_{i=1}^{k}(1+\tau_i)^{w_i}-\prod_{i=1}^{k}(1-\tau_i)^{w_i}}{\prod_{i=1}^{k}(1+\tau_i)^{w_i}+\prod_{i=1}^{k}(1-\tau_i)^{w_i}}\right\}+\left\{\frac{(1+\tau_{k+1})^{w_{k+1}}-(1-\tau_{k+1})^{w_{k+1}}}{(1+\tau_{k+1})^{w_{k+1}}+(1-\tau_{k+1})^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{i=1}^{k}(1+\tau_i)^{w_i}-\prod_{i=1}^{k}(1-\tau_i)^{w_i}}{\prod_{i=1}^{k}(1+\tau_i)^{w_i}+\prod_{i=1}^{k}(1-\tau_i)^{w_i}}\right\}\left\{\frac{(1+\tau_{k+1})^{w_{k+1}}-(1-\tau_{k+1})^{w_{k+1}}}{(1+\tau_{k+1})^{w_{k+1}}+(1-\tau_{k+1})^{w_{k+1}}}\right\}},\\
\dfrac{\left\{\frac{\prod_{i=1}^{k}(1+\eta_i)^{w_i}-\prod_{i=1}^{k}(1-\eta_i)^{w_i}}{\prod_{i=1}^{k}(1+\eta_i)^{w_i}+\prod_{i=1}^{k}(1-\eta_i)^{w_i}}\right\}+\left\{\frac{(1+\eta_{k+1})^{w_{k+1}}-(1-\eta_{k+1})^{w_{k+1}}}{(1+\eta_{k+1})^{w_{k+1}}+(1-\eta_{k+1})^{w_{k+1}}}\right\}}{1+\left\{\frac{\prod_{i=1}^{k}(1+\eta_i)^{w_i}-\prod_{i=1}^{k}(1-\eta_i)^{w_i}}{\prod_{i=1}^{k}(1+\eta_i)^{w_i}+\prod_{i=1}^{k}(1-\eta_i)^{w_i}}\right\}\left\{\frac{(1+\eta_{k+1})^{w_{k+1}}-(1-\eta_{k+1})^{w_{k+1}}}{(1+\eta_{k+1})^{w_{k+1}}+(1-\eta_{k+1})^{w_{k+1}}}\right\}},\\
\dfrac{\left\{\frac{2\prod_{i=1}^{k}\zeta_i{}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_i)^{w_i}+\prod_{i=1}^{k}\zeta_i{}^{w_i}}\right\}\left\{\frac{2\zeta_{k+1}{}^{w_{k+1}}}{(2-\zeta_{k+1})^{w_{k+1}}+\zeta_{k+1}{}^{w_{k+1}}}\right\}}{\left[2-\left\{\frac{2\prod_{i=1}^{k}\zeta_i{}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_i)^{w_i}+\prod_{i=1}^{k}\zeta_i{}^{w_i}}\right\}-\left\{\frac{2\zeta_{k+1}{}^{w_{k+1}}}{(2-\zeta_{k+1})^{w_{k+1}}+\zeta_{k+1}{}^{w_{k+1}}}\right\}\right]+\left\{\frac{2\prod_{i=1}^{k}\zeta_i{}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_i)^{w_i}+\prod_{i=1}^{k}\zeta_i{}^{w_i}}\right\}\left\{\frac{2\zeta_{k+1}{}^{w_{k+1}}}{(2-\zeta_{k+1})^{w_{k+1}}+\zeta_{k+1}{}^{w_{k+1}}}\right\}},\\
\dfrac{\left\{\frac{2\prod_{i=1}^{k}\kappa_i{}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_i)^{w_i}+\prod_{i=1}^{k}\kappa_i{}^{w_i}}\right\}\left\{\frac{2\kappa_{k+1}{}^{w_{k+1}}}{(2-\kappa_{k+1})^{w_{k+1}}+\kappa_{k+1}{}^{w_{k+1}}}\right\}}{\left[2-\left\{\frac{2\prod_{i=1}^{k}\kappa_i{}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_i)^{w_i}+\prod_{i=1}^{k}\kappa_i{}^{w_i}}\right\}-\left\{\frac{2\kappa_{k+1}{}^{w_{k+1}}}{(2-\kappa_{k+1})^{w_{k+1}}+\kappa_{k+1}{}^{w_{k+1}}}\right\}\right]+\left\{\frac{2\prod_{i=1}^{k}\kappa_i{}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_i)^{w_i}+\prod_{i=1}^{k}\kappa_i{}^{w_i}}\right\}\left\{\frac{2\kappa_{k+1}{}^{w_{k+1}}}{(2-\kappa_{k+1})^{w_{k+1}}+\kappa_{k+1}{}^{w_{k+1}}}\right\}},\\
\dfrac{\left\{\frac{2\prod_{i=1}^{k}\xi_i{}^{w_i}}{\prod_{i=1}^{k}(2-\xi_i)^{w_i}+\prod_{i=1}^{k}\xi_i{}^{w_i}}\right\}\left\{\frac{2\xi_{k+1}{}^{w_{k+1}}}{(2-\xi_{k+1})^{w_{k+1}}+\xi_{k+1}{}^{w_{k+1}}}\right\}}{\left[2-\left\{\frac{2\prod_{i=1}^{k}\xi_i{}^{w_i}}{\prod_{i=1}^{k}(2-\xi_i)^{w_i}+\prod_{i=1}^{k}\xi_i{}^{w_i}}\right\}-\left\{\frac{2\xi_{k+1}{}^{w_{k+1}}}{(2-\xi_{k+1})^{w_{k+1}}+\xi_{k+1}{}^{w_{k+1}}}\right\}\right]+\left\{\frac{2\prod_{i=1}^{k}\xi_i{}^{w_i}}{\prod_{i=1}^{k}(2-\xi_i)^{w_i}+\prod_{i=1}^{k}\xi_i{}^{w_i}}\right\}\left\{\frac{2\xi_{k+1}{}^{w_{k+1}}}{(2-\xi_{k+1})^{w_{k+1}}+\xi_{k+1}{}^{w_{k+1}}}\right\}}
\end{array}\right]
$$

$$
=\left[\begin{array}{c}
\dfrac{2\prod_{i=1}^{k}(1+\tau_i)^{w_i}(1+\tau_{k+1})^{w_{k+1}}-2\prod_{i=1}^{k}(1-\tau_i)^{w_i}(1-\tau_{k+1})^{w_{k+1}}}{2\prod_{i=1}^{k}(1+\tau_i)^{w_i}\,(1+\tau_{k+1})^{w_{k+1}}+2\prod_{i=1}^{k}(1-\tau_i)^{w_i}(1-\tau_{k+1})^{w_{k+1}}},\\
\dfrac{2\prod_{i=1}^{k}(1+\eta_i)^{w_i}\,(1+\eta_{k+1})^{w_{k+1}}-2\prod_{i=1}^{k}(1-\eta_i)^{w_i}(1-\eta_{k+1})^{w_{k+1}}}{2\prod_{i=1}^{k}(1+\eta_i)^{w_i}\,(1+\eta_{k+1})^{w_{k+1}}+2\prod_{i=1}^{k}(1-\eta_i)^{w_i}(1-\eta_{k+1})^{w_{k+1}}},\\
\dfrac{4\prod_{i=1}^{k}\zeta_i{}^{w_i}\,\zeta_{k+1}{}^{w_{k+1}}}{2[\prod_{i=1}^{k}(2-\zeta_i)^{w_i}(2-\zeta_{k+1})^{w_{k+1}}+\prod_{i=1}^{k}\zeta_i{}^{w_i}\,\zeta_{k+1}{}^{w_{k+1}}]},\\
\dfrac{4\prod_{i=1}^{k}\kappa_i{}^{w_i}\,\kappa_{k+1}{}^{w_{k+1}}}{2[\prod_{i=1}^{k}(2-\kappa_i)^{w_i}(2-\kappa_{k+1})^{w_{k+1}}+\prod_{i=1}^{k}\kappa_i{}^{w_i}\,\kappa_{k+1}{}^{w_{k+1}}]},\\
\dfrac{4\prod_{i=1}^{k}\xi_i{}^{w_i}\,z_{k+1}{}^{w_{k+1}}}{2[\prod_{i=1}^{k}(2-\xi_i)^{w_i}(2-\xi_{k+1})^{w_{k+1}}+\prod_{i=1}^{k}\xi_i{}^{w_i}\,\xi_{k+1}{}^{w_{k+1}}]}
\end{array}\right]
$$

$$
\left\langle\begin{array}{c}
\frac{\prod_{i=1}^{k+1}(1+\tau_i)^{w_i}-\prod_{i=1}^{k+1}(1-\tau_i)^{w_i}}{\prod_{i=1}^{k+1}(1+\tau_i)^{w_i}+\prod_{i=1}^{k+1}(1-\tau_i)^{w_i}},\frac{\prod_{i=1}^{k+1}(1+\eta_i)^{w_i}-\prod_{i=1}^{k+1}(1-\eta_i)^{w_i}}{\prod_{i=1}^{k+1}(1+\eta_i)^{w_i}+\prod_{i=1}^{k+1}(1-\eta_i)^{w_i}},\\
\frac{2\prod_{i=1}^{k+1}\zeta_i{}^{w_i}}{[\prod_{i=1}^{k+1}(2-\zeta_i)^{w_i}+\prod_{i=1}^{k+1}\zeta_i{}^{w_i}]},\frac{2\prod_{i=1}^{k+1}\kappa_i{}^{w_i}}{[\prod_{i=1}^{k+1}(2-\kappa_i)^{w_i}+\prod_{i=1}^{k+1}\kappa_i{}^{w_i}]},\frac{2\prod_{i=1}^{k+1}\xi_i{}^{w_i}}{[\prod_{i=1}^{k+1}(2-\xi_i)^{w_i}+\prod_{i=1}^{k+1}\xi_i{}^{w_i}]}
\end{array}\right\rangle=
$$

$PNEWAA(\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3,\dots,\mathbb{A}_k,\mathbb{A}_{k+1})$ (Using Eq. (14)).

Which shows the relation holds for $n = k + 1$.

Hence, the relation holds for all $n \in N$ by mathematical induction.

Proposition 6: The PNEWAA operator satisfies the following properties:

(i) **Consistency:** $PNEWAA(\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3,\dots,\mathbb{A}_k) \in PNN$
(ii) **Idempotency:** $PNEWAA(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\dots,\mathbb{A}_1)(k\ times) = \mathbb{A}_1$
(iii) **Symmetry:** $PNEWAA(\mathbb{A}_1,\mathbb{A}_2,\mathbb{A}_3,\dots,\mathbb{A}_{k-1},\mathbb{A}_k) = PNEWAA(\mathbb{A}_k,\mathbb{A}_{k-1},\dots.,\mathbb{A}_3,\mathbb{A}_2,\mathbb{A}_1)$

Proof:

(i) This is evident from the definition.

(ii) Let us consider collection of PNNs as $(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\dots,\mathbb{A}_1)(k\ times)$ where, $\mathbb{A}_1 = \langle\tau_1,\eta_1,\zeta_1,\kappa_1,\xi_1\rangle\ and$ with corresponding weights as w_i, such that $0 \le w_i \le 1, \sum_{i=1}^{k} w_i = 1$.

$$PNEWAA(\mathbb{A}_1,\mathbb{A}_1,\mathbb{A}_1,\dots,\mathbb{A}_1)(k\ times) = w_1\mathbb{A}_1 + w_2\mathbb{A}_1 + w_3\mathbb{A}_1 + \cdots \dots\dots\dots.. + w_k\mathbb{A}_1$$

$$= \left[\begin{array}{c} \frac{\prod_{i=1}^{k}(1+\tau_1)^{w_i}-\prod_{i=1}^{k}(1-\tau_1)^{w_i}}{\prod_{i=1}^{k}(1+\tau_1)^{w_i}+\prod_{i=1}^{k}(1-\tau_1)^{w_i}}, \frac{\prod_{i=1}^{k}(1+\eta_1)^{w_i}-\prod_{i=1}^{k}(1-\eta_1)^{w_i}}{\prod_{i=1}^{k}(1+\eta_1)^{w_i}+\prod_{i=1}^{k}(1-\eta_1)^{w_i}}, \\ \frac{2\prod_{i=1}^{k}{\zeta_1}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_1)^{w_i}+\prod_{i=1}^{k}{\zeta_1}^{w_i}}, \frac{2\prod_{i=1}^{k}{\kappa_1}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_1)^{w_i}+\prod_{i=1}^{k}{\kappa_1}^{w_i}}, \frac{2\prod_{i=1}^{k}{\xi_1}^{w_i}}{\prod_{i=1}^{k}(2-\xi_1)^{w_i}+\prod_{i=1}^{k}{\xi_1}^{w_i}} \end{array}\right] \quad \text{(Eq. (14))}$$

Simplifying we get,

$$\left[\begin{array}{c} \frac{(1+\tau_1)^{\sum_{i=1}^{k}w_i}-(1-\tau_1)^{\sum_{i=1}^{k}w_i}}{(1+\tau_1)^{\sum_{i=1}^{k}w_i}+(1-\tau_1)^{\sum_{i=1}^{k}w_i}}, \frac{(1+\eta_1)^{\sum_{i=1}^{k}w_i}-(1-\eta_1)^{\sum_{i=1}^{k}w_i}}{(1+\eta_1)^{\sum_{i=1}^{k}w_i}+(1-\eta_1)^{\sum_{i=1}^{k}w_i}}, \frac{2{\zeta_1}^{\sum_{i=1}^{k}w_i}}{(2-\zeta_1)^{\sum_{i=1}^{k}w_i}+{\zeta_1}^{\sum_{i=1}^{k}w_i}}, \\ \frac{2{\kappa_1}^{\sum_{i=1}^{k}w_i}}{(2-\kappa_1)^{\sum_{i=1}^{k}w_i}+{\kappa_1}^{\sum_{i=1}^{k}w_i}}, \frac{2{\xi_1}^{\sum_{i=1}^{k}w_i}}{(2-\xi_1)^{\sum_{i=1}^{k}w_i}+{\xi_1}^{\sum_{i=1}^{k}w_i}} \end{array}\right]$$

$$= \left[\frac{(1+\tau_1)-(1-\tau_1)}{(1+\tau_1)+(1-\tau_1)}, \frac{(1+\eta_1)-(1-\eta_1)}{(1+\eta_1)+(1-\eta_1)}, \frac{2\zeta_1}{(2-\zeta_1)+\zeta_1}, \frac{2\kappa_1}{(2-\kappa_1)+\kappa_1}, \frac{2\xi_1}{(2-\xi_1)+\xi_1}\right] \; since, \; \sum_{i=1}^{k} w_i = 1$$

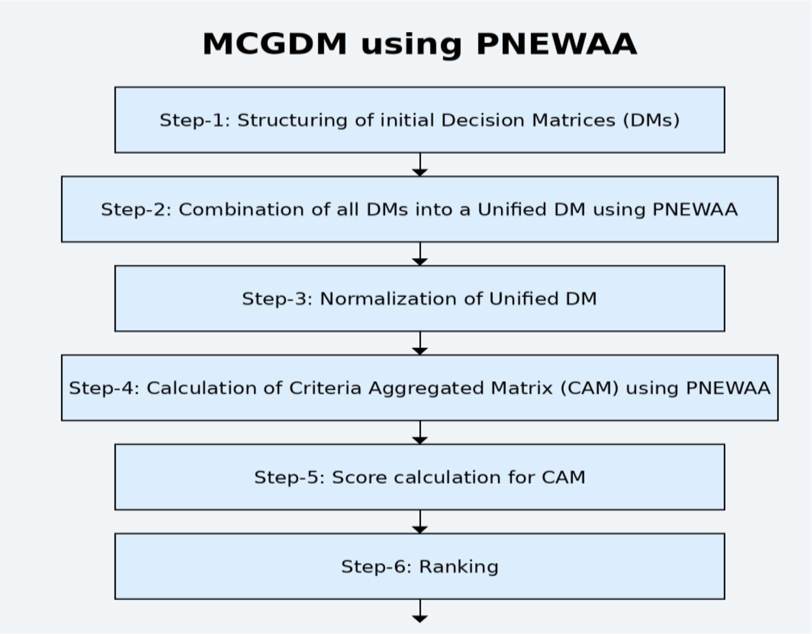
$= \langle \tau_1, \eta_1, \zeta_1, \kappa_1, \xi_1 \rangle = \mathbb{A}_1$.

(iii) $PNEWAA(\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \dots, \mathbb{A}_{k-1}, \mathbb{A}_k) = w_1\mathbb{A}_1 \oplus w_2\mathbb{A}_2 \oplus w_3\mathbb{A}_3 \oplus \dots \dots \dots \dots . \oplus w_{k-1}\mathbb{A}_{k-1} \oplus w_k\mathbb{A}_k = w_k\mathbb{A}_k \oplus w_{k-1}\mathbb{A}_{k-1} \oplus \dots \dots \dots \oplus w_3\mathbb{A}_3 \oplus w_2\mathbb{A}_2 \oplus w_1\mathbb{A}_1 = PNEWAA(\mathbb{A}_k, \mathbb{A}_{k-1}, \dots, \mathbb{A}_3, \mathbb{A}_2, \mathbb{A}_1)$

Since $\oplus$ is commutative.

Methodology

Here we consider a decision-making problem, with m alternatives $(\mathcal{A}\_\mathcal{L}_1, \mathcal{A}\_\mathcal{L}_2, \dots, \mathcal{A}\_\mathcal{L}_m)$ and n criteria $(\mathcal{C}\_\mathcal{T}_1, \mathcal{C}\_\mathcal{T}_2, \dots, \mathcal{C}\_\mathcal{T}_n)$, where all the elements are in PNN format as $x_{ij} = \langle \tau_{ij}, \eta_{ij}, \zeta_{ij}, \kappa_{ij}, \xi_{ij} \rangle$.The criteria are associated with weight vector as $W = [w_1, w_2, \dots, w_n]^T$ such that, $\sum_{i=1}^{n} w_i = 1$ and $0 < w_i < 1$. The proposed methodology for identification of the most suitable alternative is depicted in the flowchart and demonstrated below.



Step-1: Structuring of initial DMs$(DM_1, DM_2, \dots, DM_k)$with m alternatives $(\mathcal{A}\_\mathcal{L}_1, \mathcal{A}\_\mathcal{L}_2, \dots, \mathcal{A}\_\mathcal{L}_m)$ and n criteria $(\mathcal{C}\_\mathcal{T}_1, \mathcal{C}\_\mathcal{T}_2, \dots, \mathcal{C}\_\mathcal{T}_n)$, and with respect to k decision makers.

Step-2: We combine all the initial DMs$(DM_1, DM_2, \dots, DM_k)$to develop a single Unified Decision Matrix (UDM), using PNEWAA operator (Eq. (14)), considering the weights of the decision makers. Let this UDM be denoted as, $\widetilde{\mathcal{M}} = \langle \delta_{ij} \rangle_{m\times n}$, where all the elements are calculated as

$$\delta_{ij} = \langle \begin{array}{c} \frac{\prod_{i=1}^{k}(1+\tau_i)^{w_i} - \prod_{i=1}^{k}(1-\tau_i)^{w_i}}{\prod_{i=1}^{n}(1+\tau'_i)^{w_i} + \prod_{i=1}^{n}(1-\tau'_i)^{w_i}}, \frac{\prod_{i=1}^{k}(1+\eta_i)^{w_i} - \prod_{i=1}^{k}(1-\eta_i)^{w_i}}{\prod_{i=1}^{n}(1+\eta'_i)^{w_i} + \prod_{i=1}^{n}(1-\eta'_i)^{w_i}}, \\ \frac{2\prod_{i=1}^{k}{\zeta_i}^{w_i}}{\prod_{i=1}^{k}(2-\zeta_i)^{w_i} + \prod_{i=1}^{k}{\zeta_i}^{w_i}}, \frac{2\prod_{i=1}^{k}{\kappa'_i}^{w_i}}{\prod_{i=1}^{k}(2-\kappa_i)^{w_i} + \prod_{i=1}^{k}{\kappa'_i}^{w_i}}, \frac{2\prod_{i=1}^{k}{\xi_i}^{w_i}}{\prod_{i=1}^{k}(2-\xi_i)^{w_i} + \prod_{i=1}^{k}{\xi_i}^{w_i}} \end{array} \rangle. \quad (19)$$

Where, w_i denotes the decision makers weights.

Note: In case of single decision-maker (MCDM), we avoid the process of aggregation and move to Step-3 from Step-1.

Step-3: We normalize (using Eq. (1)) the UDM(Eq. (19)) to obtain a normalized matrix as $\widetilde{\mathcal{N}} = \langle \mathscr{Y}_{ij} \rangle_{m\times n}$, where $\langle \mathscr{Y}_{ij} \rangle_{m\times n} = \langle {\tau'}_{ij}, {\eta'}_{ij}, {\zeta'}_{ij}, {\kappa'}_{ij}, {\xi'}_{ij} \rangle$.

Step-4: We construct the Criteria Aggregated Matrix (CAM) from UDM by combining all the criteria corresponding to every alternative with respect to their weights. The CAM is denoted by $\widetilde{\mathcal{P}} = \langle z_{ij} \rangle_{m\times 1}$, which is a column matrix constructed using PNEWAA operator (Eq. (14)), whose entries are calculated by Eq. (20).

$$z_{ij} = \langle \begin{array}{c} \frac{\prod_{i=1}^{n}(1+\tau'_i)^{w'_i} - \prod_{i=1}^{n}(1-\tau'_i)^{w'_i}}{\prod_{i=1}^{n}(1+\tau'_i)^{w'_i} + \prod_{i=1}^{n}(1-\tau'_i)^{w'_i}}, \frac{\prod_{i=1}^{n}(1+\eta'_i)^{w'_i} - \prod_{i=1}^{n}(1-\eta'_i)^{w'_i}}{\prod_{i=1}^{n}(1+\eta'_i)^{w'_i} + \prod_{i=1}^{n}(1-\eta'_i)^{w'_i}}, \\ \frac{2\prod_{i=1}^{n}{\zeta'_i}^{w'_i}}{\prod_{i=1}^{n}(2-\zeta'_i)^{w'_i} + \prod_{i=1}^{n}{\zeta_i}^{w'_i}}, \frac{2\prod_{i=1}^{n}{\kappa'_i}^{w'_i}}{\prod_{i=1}^{n}(2-\kappa'_i)^{w'_i} + \prod_{i=1}^{n}{\kappa'_i}^{w'_i}}, \frac{2\prod_{i=1}^{n}{\xi'_i}^{w'_i}}{\prod_{i=1}^{n}(2-\xi'_i)^{w'_i} + \prod_{i=1}^{n}{\xi'_i}^{w'_i}} \end{array} \rangle \quad (20)$$

Where, w'_i represents the criteria weights.

Step-5: Next, we calculate the score for every alternative (Eq. (2)) of CAM.

Step-6: Then we rank in descending order based on score to identify the best alternative.

Illustration of the proposed methodology

In this section we consider two case studies, one from MCGDM and MCDM to illustrate the efficacy of the proposed methodology.

Illustration-1 (MCGDM): We consider a benchmark problem (Green Supplier problem) [32, 35] from existing literature with pre-validated datasets to verify the workflow of the aforesaid methodology.

Problem outline:

In the problem four potential suppliers are considered for evaluation which act as our alternatives out of which the best is to be selected based on predefined criteria. Three decision makers with different weights assigned are selected from different departments to provide rating for the alternatives based on some pre-defined criteria with associated weights. Every decision maker is also assigned a fixed weight on the basis of their proficiency and expertise.

Alternatives:

$\mathcal{A}\_\mathcal{L}_1$(Supplier1), $\mathcal{A}\_\mathcal{L}_2$(Supplier2), $\mathcal{A}\_\mathcal{L}_3$(Supplier3), $\mathcal{A}\_\mathcal{L}_4$(Supplier4).

Criteria:

$\mathcal{C}\_\mathcal{T}_1$(Quality of product) , $\mathcal{C}\_\mathcal{T}_2$(factors controlling pollution), $\mathcal{C}\_\mathcal{T}_3$(Environment friendly).

Note- Since the criteria measures the product quality, pollution control efficiency and environment

Friendly factors they are considered to be of benefit type.

Decision maker's weights:$W = [w_1, w_2, w_3]^T = [0.25,0.41,0.34]$, where $w_1 = 0.25$, $w_2 = 0.41$, $w_3 = 0.34$ denotes the weights of the decision makers $(Decision\ maker_1, Decision\ maker_2\ , Decision\ maker_3)$.

Criteria Weights:$W' = [w'_1, w'_2, w'_3]^T = [0.28,0.31,0.41]$, where $w'_1 = 0.28$, $w'_2 = 0.31$, $w'_3 = 0.41$ denotes the weights of $\mathcal{C}\_\mathcal{T}_1$, $\mathcal{C}\_\mathcal{T}_2$, $\mathcal{C}\_\mathcal{T}_3$.

Methodology

Step-1: The initial decision matrices (DM_1, DM_2, DM_3) from three decision maker are constructed (see Table 1, Table 2 and Table 3).

Table-1: DM_1

	$\mathcal{C}\_\mathcal{T}_1$	$\mathcal{C}\_\mathcal{T}_2$	$\mathcal{C}\_\mathcal{T}_3$
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.48, 0.32, 0.45, 0.21, 0.27\rangle$	$\langle 0.53, 0.52, 0.58, 0.56, 0.34\rangle$	$\langle 0.74, 0.75, 0.56, 0.45, 0.43\rangle$
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.74, 0.42, 0.42, 0.37, 0.28\rangle$	$\langle 0.45, 0.22, 0.71, 0.58, 0.29\rangle$	$\langle 0.54, 0.55, 0.56, 0.38, 0.31\rangle$
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.71, 0.53, 0.70, 0.65, 0.75\rangle$	$\langle 0.53, 0.45, 0.75, 0.58, 0.69\rangle$	$\langle 0.74, 0.53, 0.62, 0.32, 0.48\rangle$
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.51, 0.57, 0.64, 0.62, 0.48\rangle$	$\langle 0.62, 0.45, 0.42, 0.81, 0.55\rangle$	$\langle 0.87, 0.42, 0.25, 0.59, 0.34\rangle$

Table-2: DM_2

	$\mathcal{C}\_\mathcal{T}_1$	$\mathcal{C}\_\mathcal{T}_2$	$\mathcal{C}\_\mathcal{T}_3$
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.68, 0.62, 0.35, 0.51, 0.36\rangle$	$\langle 0.35, 0.72, 0.52, 0.62, 0.34\rangle$	$\langle 0.64, 0.82, 0.92, 0.55, 0.43\rangle$
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.64, 0.38, 0.58, 0.75, 0.48\rangle$	$\langle 0.85, 0.52, 0.82, 0.42, 0.39\rangle$	$\langle 0.74, 0.55, 0.85, 0.23, 0.41\rangle$
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.61, 0.87, 0.28, 0.47, 0.35\rangle$	$\langle 0.36, 0.71, 0.22, 0.58, 0.50\rangle$	$\langle 0.79, 0.55, 0.56, 0.42, 0.26\rangle$
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.22, 0.62, 0.44, 0.79, 0.25\rangle$	$\langle 0.82, 0.46, 0.78, 0.40, 0.32\rangle$	$\langle 0.65, 0.56, 0.35, 0.23, 0.34\rangle$

Table-3: DM_3

	$\mathcal{C}\_\mathcal{T}_1$	$\mathcal{C}\_\mathcal{T}_2$	$\mathcal{C}\_\mathcal{T}_3$
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.48, 0.65, 0.45, 0.31, 0.47\rangle$	$\langle 0.53, 0.62, 0.43, 0.30, 0.32\rangle$	$\langle 0.64, 0.82, 0.42, 0.25, 0.53\rangle$
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.84, 0.26, 0.45, 0.37, 0.38\rangle$	$\langle 0.85, 0.52, 0.21, 0.50, 0.39\rangle$	$\langle 0.64, 0.35, 0.56, 0.48, 0.31\rangle$
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.81, 0.72, 0.36, 0.27, 0.35\rangle$	$\langle 0.53, 0.65, 0.33, 0.40, 0.29\rangle$	$\langle 0.64, 0.43, 0.32, 0.32, 0.38\rangle$
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.41, 0.77, 0.29, 0.52, 0.22\rangle$	$\langle 0.92, 0.75, 0.52, 0.39, 0.23\rangle$	$\langle 0.77, 0.73, 0.46, 0.39, 0.34\rangle$

Step-2: Using Eq. (19) we compute the UDM as shown in Table 4.

Table-4: UDM

	$\mathcal{C}\_\mathcal{T}_1$	$\mathcal{C}\_\mathcal{T}_2$	$\mathcal{C}\_\mathcal{T}_3$
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.571, 0.567, 0.407, 0.35, 0.369\rangle$	$\langle 0.460, 0.643, 0.536, 0.456, 0.333\rangle$	$\langle 0.668, 0.804, 0.641, 0.406, 0.462\rangle$
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.746, 0.351, 0.492, 0.505, 0.390\rangle$	$\langle 0.787, 0.453, 0.527, 0.489, 0.363\rangle$	$\langle 0.663, 0.487, 0.672, 0.322, 0.348\rangle$
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.714, 0.763, 0.433, 0.342, 0.431\rangle$	$\langle 0.464, 0.634, 0.418, 0.441, 0.366\rangle$	$\langle 0.709, 0.506, 0.383, 0.320, 0.406\rangle$
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.363, 0.667, 0.401, 0.650, 0.284\rangle$	$\langle 0.832, 0.576, 0.590, 0.482, 0.340\rangle$	$\langle 0.618, 0.594, 0.354, 0.354, 0.390\rangle$

Step-3: Here normalization is not required as we deal with benefit type criteria. Hence the normalized matrix $\langle \mathscr{y}_{ij}\rangle_{m\times n} = \langle \tau'_{ij}, \eta'_{ij}, \zeta'_{ij}, \kappa'_{ij}, \xi'_{ij}\rangle$is same as represented in Table 4.

Steps-4 and 5: CAM is calculated (using Eq. (20)), with respective score values (Eq. (2)) of the alternatives (see (Table5)).

Table-5: CAM

	Criteria Aggregated Matrix (CAM)	
	PNN - values	**Score Values**
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.5827, 0.7025, 0.5369, 0.4040, 0.3930\rangle$	0.1979
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.7291, 0.4400, 0.5736, 0.4175, 0.3640\rangle$	0.1328
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.6466, 0.6301, 0.4076, 0.3011, 0.4000\rangle$	0.22488
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.6490, 0.6102, 0.4324, 0.4665, 0.3238\rangle$	0.2220

Step-6: Based on obtained scores of the alternatives the final ranks are as $\mathcal{A}\_\mathcal{L}_3 > \mathcal{A}\_\mathcal{L}_4 > \mathcal{A}\_\mathcal{L}_1 > \mathcal{A}\_\mathcal{L}_2$.

Result-This shows $\mathcal{A}\_\mathcal{L}_3$(**Supplier$_3$**) is the best alternative.

Illustration-2(MCDM): Here we consider a simulated case study of a healthcare organization for selected of the optimal service provider from four available alternatives based on three predefined criteria to deal with the situation of single decision-maker.

Problem outline:

A healthcare organization needs to select the most optimal Cloud Service Provider (CSP) to host sensitive patient data. The decision-making committee evaluates four potential CSP alternatives ($\mathcal{A}\_\mathcal{L}_1$, $\mathcal{A}\_\mathcal{L}_2$, $\mathcal{A}\_\mathcal{L}_3$, $\mathcal{A}\_\mathcal{L}_4$) against three criteria ($\mathcal{C}\_\mathcal{T}_1$, $\mathcal{C}\_\mathcal{T}_2$, $\mathcal{C}\_\mathcal{T}_3$).

Alternatives:

$\mathcal{A}\_\mathcal{L}_1$: Global Tech Cloud (An established, global industry leader).

$\mathcal{A}\_\mathcal{L}_2$: Innovate Host (A newly launched, highly innovative cloud startup).

$\mathcal{A}\_\mathcal{L}_3$: Secure Net Systems (A niche provider specializing strictly in healthcare).

$\mathcal{A}\_\mathcal{L}_4$: Value Cloud (A budget-friendly, basic storage provider).

Criteria:

$\mathcal{C}\_\mathcal{T}_1$: Data Security & Compliance **(Benefit Type)** – Evaluates encryption standards and HIPAA compliance.

$\mathcal{C}\_\mathcal{T}_2$: System Reliability & Uptime **(Benefit Type)** – Evaluates server stability and latency.

$\mathcal{C}\_\mathcal{T}_3$: Implementation & Maintenance Cost **(Cost Type)** – Evaluates the financial burden of the service.

Criteria Weights and justification: In the context of healthcare data management, System Reliability is assigned the highest weight as uninterrupted access to patient records is the highest priority for any healthcare service provider. Data Security is the next essential component due to strict legal compliance mandates of confidentiality of patient information. Implementation Cost is assigned the lowest relative weight, as it cannot prioritized compared to system reliability and security. Hence, we have

$W' = [w'_1, w'_2, w'_3]^T = [0.35, 0.40, 0.25]$, where, $w'_1 = 0.35$, $w'_2 = 0.40$, $w'_3 = 0.25$ denotes the weights of $\mathcal{C}\_\mathcal{T}_1$, $\mathcal{C}\_\mathcal{T}_2$, $\mathcal{C}\_\mathcal{T}_3$.

Table-6: DM

	$\mathcal{C}\_\mathcal{T}_1$	$\mathcal{C}\_\mathcal{T}_2$	$\mathcal{C}\_\mathcal{T}_3$
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.8, 0.2, 0.1, 0.1, 0.1\rangle$	$\langle 0.85, 0.1, 0.1, 0.2, 0.1\rangle$	$\langle 0.8, 0.2, 0.1, 0.2, 0.1\rangle$
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.6, 0.3, 0.4, 0.3, 0.2\rangle$	$\langle 0.7, 0.5, 0.2, 0.2, 0.3\rangle$	$\langle 0.5, 0.4, 0.3, 0.2, 0.4\rangle$
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.9, 0.1, 0.1, 0.1, 0.1\rangle$	$\langle 0.8, 0.2, 0.1, 0.1, 0.2\rangle$	$\langle 0.9, 0.1, 0.1, 0.1, 0.1\rangle$
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.4, 0.4, 0.2, 0.3, 0.6\rangle$	$\langle 0.5, 0.3, 0.1, 0.4, 0.5\rangle$	$\langle 0.2, 0.2, 0.1, 0.2, 0.8\rangle$

Justification of Initial PNN Decision Matrix:

The above simulated values of the PNN as $\langle \tau_{ij}, \eta_{ij}, \zeta_{ij}, \kappa_{ij}, \xi_{ij}\rangle$ is synthetically generated considering the structure of the alternatives. Main considerations are whether the supplier is an established brand or new, whether it is cost effective as a new brand should be or expensive based on a global established organisation. Also, whether the alternative is reliable having high truth or high ignorance due to unknown brand. The PNN values are justified as presented below.

- **Evaluation against $\mathcal{C}\_\mathcal{T}_1$ (Data Security - Benefit Criterion):**

$\mathcal{A}\_\mathcal{L}_1$ (Global Tech):$\langle 0.8, 0.2, 0.1, 0.1, 0.1\rangle$ Being a global leader, its security infrastructure is highly trusted with better truth (τ_{ij}= 0.8) and minimal falsity (ξ_{ij}= 0.1).

$\mathcal{A}\_\mathcal{L}_2$ (Innovate Host):$\langle 0.7, 0.3, 0.2, 0.2, 0.2\rangle$: Novel organisation yields moderate trust (τ_{ij}= 0.6), due to advanced technologies but there is significant Ignorance (ζ_{ij}= 0.4) regarding its long-term reliability which is not tested.

$\mathcal{A}\_\mathcal{L}_3$ (Secure Net):$\langle 0.9, 0.1, 0.1, 0.1, 0.1\rangle$: Specializing exclusively in healthcare compliance results in the highest level of absolute truth/trust (τ_{ij}= 0.9).

$\mathcal{A}\_\mathcal{L}_4$ (Value Cloud):$\langle 0.4, 0.4, 0.2, 0.3, 0.6\rangle$: Past reports of minor data leaks generate high Falsity (ξ_{ij}= 0.6) and strong Contradiction (η_{ij}= 0.4) among experts regarding its safety for sensitive data.

- **Evaluation against $\mathcal{C}\_\mathcal{T}_2$ (System Reliability - Benefit Criterion):**

$\mathcal{A}\_\mathcal{L}_1$ (Global Tech):$\langle 0.85, 0.1, 0.1, 0.2, 0.1\rangle$Established brand providing high reliability offering lifetime service agreement hence high truth (τ_{ij}= 0.85).

$\mathcal{A}\_\mathcal{L}_2$ (Innovate Host):$\langle 0.7, 0.5, 0.2, 0.2, 0.3\rangle$Fast baseline servers, new technology earns good truth but user reviews show heavy Contradiction (η_{ij}= 0.5) regarding latency during peak operational hours.

$\mathcal{A}\_\mathcal{L}_3$ (Secure Net): $\langle 0.8, 0.2, 0.1, 0.1, 0.2\rangle$Highly reliable, showing good truth value (τ_{ij}= 0.8), as specialized in the field, with low contradiction.

$\mathcal{A}\_\mathcal{L}_4$ (Value Cloud):$\langle 0.5, 0.3, 0.1, 0.4, 0.5\rangle$As budget friendly option reported with frequent micro-outages results in a high Unknown (κ_{ij}= 0.4) factor for emergency situations.

- **Evaluation against $\mathcal{C}\_\mathcal{T}_3$ (Implementation Cost - Cost Criterion):**

$\mathcal{A}\_\mathcal{L}_1$ (Global Tech):$\langle 0.8, 0.2, 0.1, 0.2, 0.1\rangle$: Premium branding and a highly expensive option resulting high truth values (τ_{ij}= 0.8).

$\mathcal{A}\_\mathcal{L}_2$ (Innovate Host):$\langle 0.5, 0.4, 0.3, 0.2, 0.4\rangle$: Moderate base pricing as newly launched service, but high Contradiction (η_{ij}= 0.4) in long-term budget or maintenance forecasting.

$\mathcal{A}\_\mathcal{L}_3$ (SecureNet):$\langle 0.9, 0.1, 0.1, 0.1, 0.1\rangle$: The niche healthcare specialization makes it definitively the most expensive alternative (τ_{ij}= 0.9).

$\mathcal{A}\_\mathcal{L}_4$ (Value Cloud):$\langle 0.2, 0.2, 0.1, 0.2, 0.8\rangle$: A highly budget-friendly provider; therefore, the assertion that it is costly is strongly false (ξ_{ij}= 0.8).

Methodology

Step-1: The initial decision matrix (DM) from the decision maker is displayed in Table 6.

Step-2: Since a single decision-maker is involved, no aggregation is required. Hence, the UDM is same as represented in Table 6.

Step-3: Here the third criteria is of cost type, so we normalize (Eq. (1)) the UDM. The normalized matrix $\langle \psi_{ij}\rangle_{m\times n} = \langle \tau'_{ij}, \eta'_{ij}, \zeta'_{ij}, \kappa'_{ij}, \xi'_{ij}\rangle$is displayed in Table 7.

Table-7:

	$\mathcal{C}\_\mathcal{T}_1$	$\mathcal{C}\_\mathcal{T}_2$	$\mathcal{C}\_\mathcal{T}_3$
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.8, 0.2, 0.1, 0.1, 0.1\rangle$	$\langle 0.85, 0.1, 0.1, 0.2, 0.1\rangle$	$\langle 0.1, 0.2, 0.9, 0.2, 0.8\rangle$
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.6, 0.3, 0.4, 0.3, 0.2\rangle$	$\langle 0.7, 0.5, 0.2, 0.2, 0.3\rangle$	$\langle 0.4, 0.2, 0.7, 0.4, 0.5\rangle$
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.9, 0.1, 0.1, 0.1, 0.1\rangle$	$\langle 0.8, 0.2, 0.1, 0.1, 0.2\rangle$	$\langle 0.1, 0.1, 0.9, 0.1, 0.9\rangle$
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.4, 0.4, 0.2, 0.3, 0.6\rangle$	$\langle 0.5, 0.3, 0.1, 0.4, 0.5\rangle$	$\langle 0.8, 0.2, 0.9, 0.2, 0.2\rangle$

Steps-4 and 5: CAM (Eq. (20)) along with respective score values (Eq. (2)) of the alternatives (Table 8).

Table-8:

	Criteria Aggregated Matrix (CAM)	
	PNN - values	**Score Values**
$\mathcal{A}\_\mathcal{L}_1$	$\langle 0.7221, 0.1604, 0.1892, 0.1576, 0.1806\rangle$	0.26541
$\mathcal{A}\_\mathcal{L}_2$	$\langle 0.6905, 0.1966, 0.2179, 0.2396, 0.1956\rangle$	0.22581
$\mathcal{A}\_\mathcal{L}_3$	$\langle 0.5111, 0.2491, 0.3911, 0.2307, 0.4238\rangle$	0.03157
$\mathcal{A}\_\mathcal{L}_4$	$\langle 0.6273, 0.3171, 0.5007, 0.3016, 0.3061\rangle$	0.10277

Step-6: Considering the scores of the alternatives we obtain the final ranks are $\mathcal{A}\_\mathcal{L}_1 > \mathcal{A}\_\mathcal{L}_2 > \mathcal{A}\_\mathcal{L}_4 > \mathcal{A}\_\mathcal{L}_3$. $\mathcal{A}_\mathcal{L}_1$stands out to be the best alternative.

Result-This shows $\mathcal{A}\_\mathcal{L}_1$(**Global Tech**) is the best alternative.

Conclusion

This study is conducted to develop a novel decision-making framework through the introduction of new aggregation operators with a newly developed score function. In this study we first designed a novel score function to obtain consistent alternative rankings in decision-making. The score function has been verified to satisfy the properties of boundedness, monotonicity, and transitivity. Secondly, we formulated new Einstein-based binary operations for the PNN environment including PNN addition, PNN multiplication, PNN scalar multiplication, and PNN power operations. All these operations are mathematically validated for commutativity, associativity, and scalar distributivity properties.

Based on these foundational structures, we developed the PNWEAA operator to aggregate PNNs against normalized weight vectors. This new operator has been validated for the properties of consistency, symmetry and idempotency, establishing it as the core of our MCGDM framework. To demonstrate the practical utility of this complete framework, we applied it to two distinct scenarios. First, we solved the benchmark "Green supplier selection problem," yielding a ranking of $\mathcal{A}\_\mathcal{L}_3 > \mathcal{A}\_\mathcal{L}_4 > \mathcal{A}\_\mathcal{L}_1 > \mathcal{A}\_\mathcal{L}_2$ that perfectly aligns with existing literature [35]. Second, we simulated a healthcare supplier selection problem to confirm the methodology's effectiveness for a single decision-maker, resulting in a ranking of $\mathcal{A}\_\mathcal{L}_1 > \mathcal{A}\_\mathcal{L}_2 > \mathcal{A}\_\mathcal{L}_4 > \mathcal{A}\_\mathcal{L}_3$.

Future Research Directions

Although this study primarily focusses on usage of Einstein operators for arithmetic aggregation it creates a foundational structure to develop a new operator for geometric aggregation for future researchers. Moreover, it leads to future considerations where both these operators can also be combined simultaneously to design more robust decision-making frameworks. The proposed strategy can be effectively employed in realistic decision-making scenarios, including teacher selection [42-44], quality assessment of bricks [45,46], weaver selection in Khadi institution [47, 48], institutional selection [49], and e-car selection [50].

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Chapter 17

A novel approach to MCDM problems using Fermatean Neutrosophic Hamacher aggregation Operators

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ABSTRACT

In real-life situations, making the right decision can significantly impact one's lifespan. Cropland selection for agriculture is a popular Multi-Criteria Decision Making (MCDM) problem. Therefore, many researchers are interested in conducting research in the agricultural field. The main aim of farmer optimizing land and water resources and enhancing agricultural systems. However, making the right decisions in crop land selection is a challenging task due to the uncertainty of available information. Fermatean Neutrosophic sets have the potential to deal with uncertain or incomplete data due to its structure. In this paper, we introduced fundamental Hamacher operations for FNSs and based on these operations we defined an aggregation operator in Fermatean Neutrosophic sets (FNSs) context and proved its basic properties. We have designed a general structure for MCDM problems, followed by some steps. A real-life example of Crop Land Selection is also demonstrated to highlight the effectiveness of the proposed operator and MCDM method.

Keywords: Neutrosophic sets, Fermatean neutrosophic sets, aggregation operator, decision making, crop land selection.

INTRODUCTION

The invention of the neutrosophic set (NS) [1] by F. Smarandache is a significant exploration in research. Neutrosophic set is the generalization of Intuitionistic fuzzy set [2] (IFS). IFSs contain only two independent components: truth and falsity, whereas NSs contain three independent components: truth, falsity, and indeterminacy which are held in the $]\,^{-}0, 1^{+}[$. In 2010 Wang et. al [3] introduced Single Values Neutrosophic Set (SVNS) on the basis of standard interval [0,1]. Due to its three components, any incomplete and uncertain information can be easily handled by the SVNSs. The attractive structure of SVNSs has gained significant attention from many researchers, who are now focusing their work in the SVNS environment [4-11], Senapati [12] in 2024 develop Aczel-Alsina aggregation operator in SVNSs and proved its necessary properties and construct a strategy for MADM issues in SVNS environment. Ye [13] introduced two aggregation operators for SVNSs, applying them to solve MCDM problems. Nancy and Garg [14] proposed some operations for SVNNs under Frank norm operations and developed averaging and geometric aggregation operators for decision-making problems. Liu et al. [15] proposed a MAGDM strategy based on power average operators in SVNSs and grey relational analysis (GRA) for the Ranking according to Compromise Solution (MARCOS) method. Liu et al. [16] integrated SVNS with Dombi extended power aggregation operators, proposing two weighted operators and applying them to an intelligent transportation system.

Due to the importance of various neutrosophic sets in different aspects, many researchers have applied different neutrosophic sets to decision-making problems [17-42].

Senapati et al. [43,44] introduced the concept of the Fermatean fuzzy set, where the sum of the cubes of membership and non-membership degrees is constrained to be less than one. An extensively study of Fermatean Fuzzy Set and its applications is illustrated in [45 - 47]. Antony and Jansi [48] introduced the concept of Fermatean Neutrosophic Fuzzy Sets. Later, Broumi et al. [49] defined Complex Fermatean Neutrosophic sets. Roopadevi et al. [50] introduced the Spherical Fermatean Neutrosophic Set (SFNS) and applied for selecting the best material for a structural engineering project. The concept of Fermatean neutrosophic has extensive applications in various fields [51-53]. In [54] a new hesitancy-based distance measure was developed for Complex Fermatean Neutrosophic Sets, allowing effective assessment of uncertainty in complex decision-making situations. Althuniyan et al. [55] proposed the Fermatean neutrosophic vague soft set (FNVSS).

Tan et al. [56] developed hesitant fuzzy Hamacher aggregation operators for multi-attribute decision-making. Senapati and Yager [57] introduced four new types of weighted aggregation operators for FFS, namely, Fermatean fuzzy weighted average (FFWA) operator, Fermatean fuzzy weighted geometric (FFWG) operator, Fermatean fuzzy weighted power average (FFWPA) operator, and Fermatean fuzzy weighted power geometric (FFWPG) operator. Liu et. al [57] proposed the generalized neutrosophic number Hamacher weighted averaging (GNNHWA) operator, generalized neutrosophic number Hamacher ordered weighted averaging (GNNHOWA) operator, and generalized neutrosophic number Hamacher hybrid averaging (GNNHHA) operator. Mallick et al. [58] introduced two types Hamacher aggregative operators and applied real life decision-making problem. Wang et al [59] introduced interactive Hamacher operation, and presented power aggregation operators. Asif et al. [60] proposed several Pythagorean fuzzy Hamacher interactive aggregation operators named as FHIWA, PFHIOWA, PFHIWG, and PFHIOWG and applied them to decision-making problems.

Despite the various applications of Hamacher aggregation operators and Fermatean neutrosophic sets in the decision-making field, no study has yet combined Hamacher aggregation operators with Fermatean neutrosophic sets.

Contribution of the study

The purpose of this study is to design a method that recommends the best alternative from a collection of feasible options. For this decision-making purpose, at first, we develop a Fermatean neutrosophic Hamacher aggregation operator and proves its related impotent properties. We also investigate the sensitivity analysis for this operator. A multi-step MCDM approach was developed and demonstrated through a numerical example focusing on crop land location selection based on key criteria.

Structure of this study:

This rest of the paper is organized as follows. In Background section presents a brief overview of fundamental concepts related to FNSs and their operations. Next, a novel aggregation operator is presented by extending the Fermatean fuzzy Hamacher aggregation operator, along with a discussion of its fundamental properties. Then, an MCDM strategy is developed based on the proposed aggregation operator in the FNS setting. After that, an illustrative problem of land selection for agriculture is provided. The paper is then concluded. Finally, emerging research directions stemming from this study are provided.

BACKGROUND

This section provides some fundamental definitions required in this paper

Definition 1. Neutrosophic sets (NS)[1]

Let X be a universe. $\mathcal{A}$ neutrosophic sets $\mathcal{A}$ over X is defined by

$$\mathcal{A} = \{\prec u, (T_{\mathcal{A}}(u), I_{\mathcal{A}}(x), F(x)) \succ : x \in X\}$$

Here, $T_{\mathcal{A}}(x)$, $I_{\mathcal{A}}(x)$, and $F_{\mathcal{A}}(x)$represent the truth-membership, indeterminacy-membership, and falsity-membership functions, respectively, and are defined as follows

$$T_{\mathcal{A}}: X \to]^{-}0, 1^{+}[, \qquad I_{\mathcal{A}}: X \to]^{-}0, 1^{+}[, \qquad F_{\mathcal{A}}: X \to]^{-}0, 1^{+}[$$

such that $0^{-} \leq T_{\mathcal{A}}(x) + I_{\mathcal{A}}(x) + F_{\mathcal{A}}(x) \leq 3^{+}$.

Definition 2. Fermatean fuzzy set ($FF-set$) [44]

An FF-set Adefined on the universe of discourse $X$is a structure expressed in the following form:

$$A=\{(x,T_A(x),F_A(x))|x\in X\}$$

Here, $T_A(x):X\rightarrow[0,1]$denotes the membership degree, while $F_A(x):X\rightarrow[0,1]$represents the non-membership degree of an element $x\in X$with respect to the set A, subject to the following conditions:

$$0\leq(T_A(x))^3+(F_A(x))^3\leq1$$

In [48], Antony et al. introduced the concept of the Fermatean neutrosophic set, incorporating additional forms of uncertainty along with a neutral membership measure. The definitions are presented below

Definition 3. Fermatean neutrosophic set [48]

Fermatean neutrosophic set ($FN-set$) A on a universe of discourse X is a structure having the form as:

$$A=\{(x,T_A(x),I_A(x),F_A(x))|x\in X\}$$

Let $T_A(x)$: X ⟶ [0,1] represent the degree of membership, $I_A(x)$: X ⟶ [0,1] represent the degree of indeterminacy, and $F_A(x)$: X ⟶ [0,1] represent the degree of non-membership of an element x ∈ X in the set A. These functions satisfy the following conditions:

$$0\leq[T_A(x)]^3+[F_A(x)]^3\leq1,\text{and } 0\leq[I_A(x)]^3\leq1.$$

Consequently, the combined constraint can be expressed as

$$0\leq[T_A(x)]^3+[I_A(x)]^3+[F_A(x)]^3\leq2,\forall x\in X.$$

Here, $T_A(x)$ and $F_A(x)$ are regarded as dependent components, while $I_A(x)$ acts as an independent component.

Definition 4. Pythagorean fuzzy set [61]

A Pythagorean fuzzy set (PFS) Adefined on a universe of discourse Xcan be expressed as

$$A=\{(x,T_A(x),F_A(x))\mid x\in X\},$$

where the function $T_A(x):X\rightarrow[0,1]$represents the degree of membership of an element $x$in the set A, and $F_A(x):X\rightarrow[0,1]$denotes the degree of non-membership. These functions satisfy the condition

$$0\leq(T_A(x))^2+(F_A(x))^2\leq1.$$

Definition 5. Pythagorean neutrosophic set [62]

A Pythagorean neutrosophic set (PN-set) Adefined on a universe of discourse Xcan be written as

$$A=\{(x,T_A(x),I_A(x),F_A(x))\mid x\in X\},$$

where $T_A(x):X\rightarrow[0,1]$, $I_A(x):X\rightarrow[0,1]$, and $F_A(x):X\rightarrow[0,1]$denote the degrees of membership, indeterminacy, and non-membership of an element $x\in X$with respect to the set A, respectively.
These functions satisfy the conditions

$$0\leq(T_A(x))^2+(F_A(x))^2\leq1,0\leq(I_A(x))^2\leq1,$$

which together imply $0\leq(T_A(x))^2+(I_A(x))^2+(F_A(x))^2\leq2$.
In this framework, the membership and non-membership degrees $T_A(x)$and $F_A(x)$are interrelated (dependent), whereas the indeterminacy degree $I_A(x)$is treated as an independent component.

Definition 6. For FNS, $A_1=(T_1,I_1,F_1)$, the rating function of A_1 can be defined as follows:

$$rt(A_1)=\frac{1+T_1^3-F_1^3-I_1^3}{2}\in[-1,1], \quad (1)$$

Case 1: Maximum Score = 1
To get $rt=1$:

$$\frac{1+T^3-F^3-I^3}{2}=1\Rightarrow1+T^3-F^3-I^3=2\Rightarrow T^3=1,\ F^3=0,\ I^3=0$$

Values:

$$(T,I,F)=(1,0,0)$$
$$rt=\frac{1+1-0-0}{2}=1$$

Case 2: Minimum Score = -1
To get $rt=-1$*:*

$$\frac{1+T^3-F^3-I^3}{2}=-1 \Rightarrow 1+T^3-F^3-I^3=-2 \Rightarrow T^3=0,\, F^3=1,\, I^3=1$$

Values:

$$(T,I,F)=(0,1,1)$$

Check:

$$rt=\frac{1+0-1-1}{2}=-1$$

In the following, we examine a new relation for Fermatean neutrosophic elements.

Definition 7.

Let $A_1=(T_1,I_1,F_1)$, and A₂ = (T₂, I₂, F₂) be any two FNSs and let $rt(A_1)$ and $rt(A_2)$ be the respective ratings of A_1 and A_2, then

1. $rt(A_1)<rt(A_2) \Rightarrow A_1<A_2$;
2. $rt(A_1)>rt(A_2) \Rightarrow A_1>A_2$;
3. $rt(A_1)=rt(A_2) \Rightarrow A_1 \sim A_2$.

Definition 8. Let A = (T, I, F) be an FNSs. The accuracy function (briefly as), of A_1 can be defined as follows:

$Acc(A_1)=T^3+I^3 \in [0,2]$. (2)

Using the analogy of rating function and accuracy function, we give a complete criterion for the ranking of FNSs in the following.

Definition 9. Let $A_1=(T_1,I_1,F_1)$, and $A_2=(T_2,I_2,\ F_2)$, be any two FNSs and let $rt(A_1)$ and $Acc(A_1)(i=1{,}2)$ be the respective ratings and accuracies of F_1 and F_2, then

4. $rt(A_1)<rt(A_1) \Rightarrow A_1<A_2$;
5. $rt(A_1)>rt(A_2) \Rightarrow A_1>A_2$;
6. If $rt(A_1)=rt(A_2)$, then
   a. $Acc(A_1)<Acc(A_2) \Rightarrow A_1<A_2$;
   b. $Acc(A_1)>Acc(A_2) \Rightarrow A_1>A_2$;
   c. $Acc(A_1)=Acc(A_2) \Rightarrow A_1 \sim A_2$.

Some operation of Fermatean neutrosophic set

Definition 10.

Let $A=(T,\mathrm{I},F)$, $A_1=(T_1,I_1,F_1)$, and $A_2=(T_2,I_2,F_2)$ be any three FNSs [48], then their set operations are defined as in the following:

1. $A_1\cap A_2=(\min\{T_1,T_2\},\max\{I_1,I_2\},\max F_1,F_2)$;
2. $A_1\cup A_2=(\max\{T_1,T_2\},\min\{I_1,I_2\},\min F_1,F_2)$;
3. $A_1\subseteq A_2$ if and only if $T_1\le T_2$, $I_1\ge I_2$, and $F_1\ge F_2$;
4. $A^c=(1-\mathrm{F},1-\mathrm{I},1-T^c)$.

Hamacher operations

Hamacher introduced a generalized framework for both T-norm and T-conorm, known as Hamacher operations, which include the Hamacher product and Hamacher sum. These formulations serve as standard models for T-norm and T-conorm, as outlined in the following definition.

Definition 11. Hamacher T-norm and T-conorm [63, 64].

$$\Gamma_\gamma(m,\mathrm{n})=\frac{mn}{\gamma+(1-\gamma)(m+n-\mathrm{mn})},\gamma>0$$
$$\Gamma_\gamma^*(m,n)=\frac{m+n-\mathrm{mn}-(1-\gamma)mn}{1-(1-\gamma)mn},\gamma>0$$

when $\gamma=2$, these will become $\Gamma(m,n)=\frac{mn}{1+(1-\mathrm{m})(1-n)}$, and $\Gamma^*(m,n)=\frac{m+n}{1+\mathrm{mn}}$,these correspond to the Einstein T-norm and Einstein T-conorm, respectively.

Fermatean Neutrosophic Hamacher operator

The operational rules of FNSs based on Hamacher T-norm and T-conorm

Based on definition 10 and definition 11, we establish the operational rules of FNSs.

In this section, utilizing the notion of HT-norm and HT-conform, we explain Hamacher operations with respect to FNSs. We propose the Hamacher arithmetic aggregation operators with FNSs. In this regard, we define Fermatean neutrosophic Hamacher weighted averaging (FNHWA) operator.

Let A₁ = (T₁, I₁, F₁) and A₂ = (T₂, I₂, F₂) be two FNSs, and γ, η ≥ 0, then the operational rules based on Hamacher T-norm and T-conorm are defined as follows:

Definition 12. Let $A_j = (T_j, I_j, F_j)(j = 1,2)$ be an array of FNSs, $\lambda > 0, \kappa > 0$, then, the fundamental Hamacher operations for FNSs are introduced as

1. $$A_1 \oplus A_2 = \left(\sqrt[3]{\frac{T_1^3+T_2^3-T_1^3T_2^3-(1-\lambda)T_1^3T_2^3}{1-(1-\lambda)T_1^3T_2^3}}, \sqrt[3]{\frac{I_1^3+I_2^3-I_1^3I_2^3-(1-\lambda)I_1^3I_2^3}{1-(1-\lambda)I_1^3I_2^3}} \; \frac{F_1F_2}{\sqrt[3]{\lambda+(1-\lambda)(F_1^3+F_2^3-F_1^3F_2^3)}} \right)$$

2. $$A_1 \otimes A_2 = \left(\frac{T_1T_2}{\sqrt[3]{\lambda+(1-\lambda)(T_1^3+T_2^3-T_1^3T_2^3)}}, \frac{I_1I_2}{\sqrt[3]{\lambda+(1-\lambda)(I_1^3+I_2^3-I_1^3I_2^3)}}, \sqrt[3]{\frac{F_1^3+F_2^3-F_1^3F_2^3-(1-\lambda)F_1^3F_2^3}{1-(1-\lambda)F_1^3F_2^3}} \right)$$

3. $$\kappa(A_1) = \left(\sqrt[3]{\frac{\left[1+(\lambda-1)T_1^3\right]^{\kappa}-(1-T_1^3)^{\kappa}}{\left[1+(\lambda-1)T_1^3\right]^{\kappa}+(\lambda-1)(1-T_1^3)^{\kappa}}}, \frac{\sqrt[3]{\lambda}\, F_1^{\kappa}}{\sqrt[3]{\left[1+(\lambda-1)(1-F_1^3)\right]^{\kappa}+(\lambda-1)F_1^{3\kappa}}}, \sqrt[3]{\frac{\left[1+(\lambda-1)F_1^3\right]^{\kappa}-(1-F_1^3)^{\kappa}}{\left[1+(\lambda-1)F_1^3\right]^{\kappa}+(\lambda-1)(1-F_1^3)^{\kappa}}} \right)$$

4. $$(A_1)^{\kappa} = \left(\frac{\sqrt[3]{\lambda}\, T_1^{\kappa}}{\sqrt[3]{\left[1+(\lambda-1)(1-T_1^3)\right]^{\kappa}+(\lambda-1)T_1^{3\kappa}}}, \sqrt[3]{\frac{\left[1+(\lambda-1)I_1^3\right]^{\kappa}-(1-I_1^3)^{\kappa}}{\left[1+(\lambda-1)I_1^3\right]^{\kappa}+(\lambda-1)(1-I_1^3)^{\kappa}}}, \frac{\sqrt[3]{\lambda}\, F_1^{\kappa}}{\sqrt[3]{\left[1+(\lambda-1)(1-F_1^3)\right]^{\kappa}+(\lambda-1)F_1^{3\kappa}}} \right)$$

Fermatean Neutrosophic Hamacher Arithmetic Aggregation Operators

Here, we develop arithmetic aggregation operators with Fermatean neutrosophic numbers based on Hamacher operations, called FNHWA operator, and study the basic characteristics of these proposed operators in detail.

Definition 13. Let $A_r = (T_r, I_r, F_r)(r = 1, \dots, p)$ be a family of FNSs. Then Fermatean neutrosophic Hamacher weighted averaging (FNHWA) operator is a mapping FNHWA: $A^p \rightarrow A$ such that:

$$FNHWA_w(A_1, A_2, \dots, A_p) = \oplus_{r=1}^{p} (w_r A_r)$$

where $w = (w_1, w_2, \dots, w_p)^T$ represents the weight vector of $A_r(r = 1, \dots, p)$ for the conditions $w_r > 0$ and $\sum_{r=1}^{p} \mathrm{w}_r = 1$.

In the following theorem we use the Hamacher operations on FNSs to prove that the aggregation value of a family of FNSs is again an FNS.

In the following theorem we use the Hamacher operations on FNSs to prove that the aggregation value of a family of FNSs is again an FNS.

Theorem 1. Let $A_r = (\mathrm{T}_r, \mathrm{I}_r, \mathrm{F}_r)$ $(r = 1, \dots, p)$ be a family of FNSs, then the aggregation value of this family by the FNHWA operator is also an FNSs, and $\mathrm{FNHWA}_w(A_1, A_2, \dots, A_p) = \oplus_{r=1}^{p}(w_r A_r)$

$$= \left(\sqrt[3]{\frac{\prod_{r=1}^{p}(1+(\lambda-1)T_r^3)^{w_r} - \prod_{r=1}^{p}(1-T_r^3)^{w_r}}{\prod_{r=1}^{p}(1+(\lambda-1)T_r^3)^{w_r} + (\lambda-1)\prod_{r=1}^{p}(1-T_r^3)^{w_r}}}, \sqrt[3]{\frac{\prod_{r=1}^{p}(1+(\lambda-1)I_r^3)^{w_r} - \prod_{r=1}^{p}(1-I_r^3)^{w_r}}{\prod_{r=1}^{p}(1+(\lambda-1)I_r^3)^{w_r} + (\lambda-1)\prod_{r=1}^{p}(1-I_r^3)^{w_r}}}, \right.$$

$$\left. \frac{\sqrt{\lambda}\prod_{r=1}^{p} F_r^{w_r}}{\prod_{r=1}^{p}\left(1+(\lambda-1)(1-F_r^3)\right)^{w_r} + (\lambda-1)\prod_{r=1}^{p} F_r^{3w_r}} \right) \tag{3}$$

where $w = (w_1, w_2, \ldots, w_p)$ represents the weight vector of $A_r (r = 1, \ldots, p)$ with $w_r > 0$ and $\sum_{r=1}^{p} w_r = 1$.

Proof. This proof can easily be shown by mathematical induction on p. Thus

If $p = 1$, then $w_1 = 1$ and from the left part of Equation (3), we get

$$\mathrm{FNHWA}_w(A_1, A_2, \ldots, A_p) = A_1 = (T_1, I_1, F_1) \tag{4}$$

and for the right part of Equation (6), we obtain

$$\left(\sqrt[3]{\frac{1+(\lambda-1)T_1^3 - (1-T_1^3)}{1+(\lambda-1)T_1^3 + (\lambda-1)(1-T_1^3)}}, \sqrt[3]{\frac{1+(\lambda-1)I_1^3 - (1-I_1^3)}{1+(\lambda-1)I_1^3 + (\lambda-1)(1-I_1^3)}} \right.$$

$$\left. \frac{\sqrt{\lambda}F_1}{\left(1+(\lambda-1)(1-F_1^3)\right) + (\lambda-1)F_1^3} \right) = (T_1, I_1, F_1).$$

Thus, Equation (3) holds for $p = 1$.

(ii) Assume that Equation (3) holds true for $p = k$, where $k \in \mathbb{N}$, Equation (3) then becomes

$$FNHWA_w(A_1, A_2, \ldots, A_k) = \bigoplus_{r=1}^{k} (w, A_r)$$

$$= \sqrt[3]{\frac{\prod_{r=1}^{k}(1+(\lambda-1)T_r^3)^{w_r} - \prod_{r=1}^{k}(1-T_r^3)^{w_r}}{\prod_{r=1}^{k}(1+(\lambda-1)T_r^3)^{w_r} + (\lambda-1)\prod_{r=1}^{k}(1-T_r^3)^{w_r}}}, \sqrt[3]{\frac{\prod_{r=1}^{k}(1+(\lambda-1)I_r^3)^{w_r} - \prod_{r=1}^{k}(1-I_r^3)^{w_r}}{\prod_{r=1}^{k}(1+(\lambda-1)I_r^3)^{w_r} + (\lambda-1)\prod_{r=1}^{k}(1-I_r^3)^{w_r}}},$$

$$\sqrt[3]{\frac{\sqrt{2}\lambda\prod_{r=1}^{k} F_r^{w_r}}{\sqrt{2}\lambda\prod_{r=1}^{k}\left(1+(\lambda-1)(1-F_r^3)\right)^{w_r} + (\lambda-1)\prod_{r=1}^{k} F_r^{3w_r}}} \tag{5}$$

Now for $p = k + 1$, we consider the following equation.

$$FNHWA_w(A_1, A_2, \ldots, A_{k+1}) = \bigoplus_{r=1}^{k} (w, A_r) \oplus (w, A_{k+1})$$

$$= \sqrt[3]{\frac{\prod_{r=1}^{k}(1+(\lambda-1)T_r^3)^{w_r} - \prod_{r=1}^{k}(1-T_r^3)^{w_r}}{\prod_{r=1}^{k}(1+(\lambda-1)T_r^3)^{w_r} + (\lambda-1)\prod_{r=1}^{k}(1-T_r^3)^{w_r}}}, \sqrt[3]{\frac{\prod_{r=1}^{k}(1+(\lambda-1)I_r^3)^{w_r} - \prod_{r=1}^{k}(1-I_r^3)^{w_r}}{\prod_{r=1}^{k}(1+(\lambda-1)I_r^3)^{w_r} + (\lambda-1)\prod_{r=1}^{k}(1-I_r^3)^{w_r}}},$$

$$\sqrt[3]{\frac{\sqrt{2}\lambda\prod_{r=1}^{k} F_r^{w_r}}{\sqrt{2}\lambda\prod_{r=1}^{k}\left(1+(\lambda-1)(1-F_r^3)\right)^{w_r} + (\lambda-1)\prod_{r=1}^{k} F_r^{3w_r}}} \oplus (T_{k+1}, I_{k+1}, F_{k+1})$$

$$= \left(\sqrt[3]{\frac{\prod_{r=1}^{k}\left(1+(\lambda-1)T_r^3\right)^{w_r} - \prod_{r=1}^{k}\left(1-T_r^3\right)^{w_r}}{\prod_{r=1}^{k}\left(1+(\lambda-1)T_r^3\right)^{w_r} + (\lambda-1)\prod_{r=1}^{k}\left(1-T_r^3\right)^{w_r}}}, \sqrt[3]{\frac{\prod_{r=1}^{k}\left(1+(\lambda-1)I_r^3\right)^{w_r} - \prod_{r=1}^{k}\left(1-I_r^3\right)^{w_r}}{\prod_{r=1}^{k}\left(1+(\lambda-1)I_r^3\right)^{w_r} + (\lambda-1)\prod_{r=1}^{k}\left(1-I_r^3\right)^{w_r}}}, \right.$$

$$\left.\sqrt[3]{\frac{\sqrt{2}\lambda\prod_{r=1}^{k}F_r^{w_r}}{\sqrt{2}\lambda\prod_{r=1}^{k}\left(1+(\lambda-1)(1-F_r^3)\right)^{w_r}+(\lambda-1)\prod_{r=1}^{k}F_r^{3w_r}}}\right)$$

$$+\left(\sqrt[3]{\frac{\left(1+(\lambda-1)T_{k+1}^3\right)^{w_{k+1}}-\left(1-T_{k+1}^3\right)^{w_{k+1}}}{\left(1+(\lambda-1)T_{k+1}^3\right)^{w_{k+1}}+(\lambda-1)\left(1-T_{k+1}^3\right)^{w_{k+1}}}},\sqrt[3]{\frac{\left(1+(\lambda-1)I_{k+1}^3\right)^{w_{k+1}}-\left(1-I_{k+1}^3\right)^{w_{k+1}}}{\left(1+(\lambda-1)I_{k+1}^3\right)^{w_{k+1}}+(\lambda-1)\left(1-I_{k+1}^3\right)^{w_{k+1}}}},\right.$$

$$\left.\sqrt[3]{\frac{\sqrt{2}\lambda F_{K+1}^{w_{K+1}}}{\left(1+(\lambda-1)(1-F_{k+1}^3)\right)^{w_{k+1}}+(\lambda-1)F_{k+1}^{3w_{k+1}}}}\right)$$

$$=\left(\sqrt[3]{\frac{\prod_{r=1}^{k+1}(1+(\lambda-1)T_r^3)^{w_r}-\prod_{r=1}^{k+1}(1-T_r^3)^{w_r}}{\prod_{r=1}^{k+1}(1+(\lambda-1)T_r^3)^{w_r}+(\lambda-1)\prod_{r=1}^{k+1}(1-T_r^3)^{w_r}}},\sqrt[3]{\frac{\prod_{r=1}^{k+1}(1+(\lambda-1)I_r^3)^{w_r}-\prod_{r=1}^{k+1}(1-I_r^3)^{w_r}}{\prod_{r=1}^{k+1}(1+(\lambda-1)I_r^3)^{w_r}+(\lambda-1)\prod_{r=1}^{k+1}(1-I_r^3)^{w_r}}},\right.$$

$$\left.\sqrt[3]{\frac{\sqrt{2}\lambda\prod_{r=1}^{k+1}F_r^{w_r}}{\sqrt{2}\lambda\prod_{r=1}^{k+1}\left(1+(\lambda-1)(1-F_r^3)\right)^{w_r}+(\lambda-1)\prod_{r=1}^{k+1}F_r^{3w_r}}}\right).$$

Therefore, Equation (3) holds for $p = k + 1$. It follows from steps (i) and (ii) that the result stands for any $p \in N$.

Theorem 2. Idempotency Property

If $A_r = (\mathrm{T}_r, \mathrm{I}_r, \mathrm{F}_r)\ (r = 1, \dots, p)$

be a family of FNSs, such that they all equal, i.e., if $A_r = (T, I, F)$for all r, then

$$FNHWA_w(A_1, A_2, \dots, A_p) = A.$$

Proof. Since $A_r = (\mathrm{T}_r, \mathrm{I}_r, \mathrm{F}_r) = (T, I, F)$ for $r = 1, \dots, p$, then by using Equation (6), we have

$$FNHWA_w(A_1, A_2, \dots, A_p) = \bigoplus_{r=1}^{p} \quad (w, A_r)$$

$$=\sqrt[3]{\frac{\prod_{r=1}^{p}(1+(\lambda-1)T_r^3)^{w_r}-\prod_{r=1}^{p}(1-T_r^3)^{w_r}}{\prod_{r=1}^{p}(1+(\lambda-1)T_r^3)^{w_r}+(\lambda-1)\prod_{r=1}^{p}(1-T_r^3)^{w_r}}},\sqrt[3]{\frac{\prod_{r=1}^{p}(1+(\lambda-1)I_r^3)^{w_r}-\prod_{r=1}^{p}(1-I_r^3)^{w_r}}{\prod_{r=1}^{p}(1+(\lambda-1)I_r^3)^{w_r}+(\lambda-1)\prod_{r=1}^{p}(1-I_r^3)^{w_r}}},$$

$$\sqrt[3]{\frac{\sqrt{2}\lambda\prod_{r=1}^{p}F_r^{w_r}}{\sqrt{2}\lambda\prod_{r=1}^{p}\left(1+(\lambda-1)(1-F_r^3)\right)^{w_r}+(\lambda-1)\prod_{r=1}^{p}F_r^{3w_r}}}$$

$$=\left(\sqrt[3]{\frac{1+(\lambda\text{-}1)T^3-(1\text{-}T^3)}{1+(\lambda\text{-}1)T^3+(\lambda\text{-}1)(1\text{-}T^3)}}\ \sqrt[3]{\frac{1+(\lambda\text{-}1)I^3-(1\text{-}I^3)}{1+(\lambda\text{-}1)I^3+(\lambda\text{-}1)(1\text{-}I^3)}}\ \sqrt[3]{\frac{\sqrt{2}\lambda F}{1+(\lambda\text{-}1)(1\text{-}F^3)+(\lambda\text{-}1)F^3}}\right)$$

Thus, $FNHWA_w(A_1, A_2, \dots, A_p) = A$.

Similarly, we can prove some other properties of FNHWA operator given in the following:

Theorem 3. Boundedness Property

Let $A_r = (\mathrm{T}_r, \mathrm{I}_r, \mathrm{F}_r)\ (r = 1, \dots, p)$ be a family of FFNs, and $A^- = \min_{1 \le r \le p} A_r$ and $A^+ = \max_{1 \le r \le p} A_r$. Then

$$A^- \le FNHWA_w(A_1, A_2, \dots, A_p) \le A^+$$

Theorem 4. Monotonicity Property

Let $A_r = (\mathrm{T}_r, \mathrm{I}_r, \mathrm{F}_r)$ $(r = 1, \dots, p)$ and $A'_r = (T'_r, I'_r, A'_r)$ $(r = 1, \dots, p)$ be two families of FNNs, such that $A_r \leq A'_r$ for all r, then

$$FNHWA_w(A_1, A_2, \dots, A_p) \leq FNHWA_w(A'_1, A'_2, \dots, A'_p).$$

Theorem 5. Commutativity Property

Let $A_r = (\mathrm{T}_r, \mathrm{I}_r, \mathrm{F}_r)$ $(r = 1, \dots, p)$ and $A'_r = (T'_r, I'_r, A'_r)$ $(r = 1, \dots, p)$ be two families of FFNs then

$$FNHWA_w(A_1, A_2, \dots, A_p) = FNHWA_w(A'_1, A'_2, \dots, A'_p)$$

where A'_r is any permutation of A_r for $r = 1, 2, \dots, p$.

Now, we discuss two special types of FNHWA in the following: (1) If $\lambda = 1$, then FNHWA is equivalent to the FNWA operator:

$$FNWA_w(A_1, A_2, \dots, A_p) = \bigoplus_{r=1}^{p} (w, A_r) = \left(\sqrt[3]{1 - \prod_{r=1}(1\text{-}T_r^3)^{w_r}}, \sqrt[3]{1 - \prod_{r=1}(1\text{-}I_r^3)^{w_r}}, \prod_{r=1} F_r^{\ w_r}\right)$$

(2) If $\lambda = 2$, then FNHWA becomes the Fermatean neutrosophic Einstein weighted average (FNEWA) operator:

$$FNWA_w(A_1, A_2, \dots, A_p) = \bigoplus_{r=1}^{p} (w, A_r) =$$

$$\left(\sqrt[3]{\frac{\prod_{r=1}^{p}(1+T_r^3)^{w_r} - \prod_{r=1}^{p}(1\text{-}T_r^3)^{w_r}}{\prod_{r=1}^{p}(1+T_r^3)^{w_r} + \prod_{r=1}^{p}(1\text{-}T_r^3)^{w_r}}}, \sqrt[3]{\frac{\prod_{r=1}^{p}(1+I_r^3)^{w_r} - \prod_{r=1}^{p}(1\text{-}I_r^3)^{w_r}}{\prod_{r=1}^{p}(1+I_r^3)^{w_r} + \prod_{r=1}^{p}(1\text{-}I_r^3)^{w_r}}}, \frac{\sqrt{2}\prod_{r=1}^{p} F_r^{\ w_r}}{\prod_{r=1}^{p}(2\text{-}F_r^3)^{w_r} + \prod_{r=1}^{p} F_r^{\ w_r}}\right)$$

An MCDM technique constructed by the proposed aggregation operator in the FNS setting

This part presents a systematic framework for assessing and prioritizing various options under Fermatean neutrosophic sets, facilitating effective decision analysis. Let us assume that $\{\mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_3, \dots, \mathcal{L}_n\}$, represents n-alternatives or options, which are ranked based on the criteria $\{\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \dots, \mathcal{C}_m.\}$. The decision-maker expresses opinions for alternatives based on attributes in terms of FNSs. Now, we describe the MCDM framework using the following steps based on the proposed operators:

Step 1. Formation of alternatives versus criteria decision matrix

The interrelationship of the alternatives $\mathcal{L}_i$ $(i = 1,2,3,..,n)$ and their corresponding criteria $\mathcal{C}_j (j = 1,2,3, \dots, m)$ is captured in terms of FNSs and can be illustrated through a matrix representation M^1.

M^1: Alternatives versus criteria decision matrix

$$M^1 = \left[\begin{array}{ccccc} & \mathcal{C}_1 & \mathcal{C}_2 & \cdots & \mathcal{C}_m \\ \mathcal{L}_1 & (T_{11,}I_{11,}F_{11}) & (T_{12}, I_{12}, F_{12}) & \cdots & (T_{1\mathrm{m}}, I_{1\mathrm{m}}, F_{1\mathrm{m}}) \\ \mathcal{L}_2 & (T_{21}, I_{21}, F_{21}) & (T_{22}\ , I_{22}, F_{22}) & \cdots & (T_{2\mathrm{m}}, I_{2\mathrm{m}}, F_{2\mathrm{m}}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathcal{L}_n & (T_{\mathrm{n}1}, I_{\mathrm{n}1}, F_{\mathrm{n}1}) & (T_{\mathrm{n}2}\ , I_{\mathrm{n}2}, F_{\mathrm{n}2}) & \cdots & (T_{\mathrm{nm}}, I_{\mathrm{nm}}, F_{\mathrm{nm}}) \end{array}\right]$$

Step 2. Aggregate the decision matrix using proposed operator

Utilizing the proposed operator (Eq. 3), we aggregate the original decision matrix ($\boldsymbol{M^1}$) and derive a new decision matrix ($\boldsymbol{M^2}$) that synthesizes the information in a structured form.

$\boldsymbol{M^2}$: Aggregated decision matrix

$$\boldsymbol{M^2} = \left[\begin{array}{ll} & \mathcal{C}_{\text{Agg}} \\ \boldsymbol{\mathcal{L}}_1 & \left(T_{11}^{\text{Agg}}, I_{11}^{\text{Agg}}, F_{11}^{\text{Agg}}\right) \\ \boldsymbol{\mathcal{L}}_2 & \left(T_{21}^{\text{Agg}}, I_{21}^{\text{Agg}}, F_{21}^{\text{Agg}}\right) \\ \ldots & \ldots \\ \boldsymbol{\mathcal{L}}_n & \left(T_{\text{n1}}^{\text{Agg}} I_{\text{n1}}^{\text{Agg}} F_{\text{n1}}^{\text{Agg}}\right) \end{array}\right]$$

Step 3. Calculate rating Values from Aggregate decision matrix using proposed Score function

Eq. (1) is employed to compute the rating values of alternatives from matrix $\boldsymbol{M^2}$, resulting in real-valued rating that quantify their performance.

Step 4. Calculate Accuracy Values from Aggregate decision matrix using proposed accuracy function

Eq. (2) is employed to compute the score values of alternatives from matrix $\boldsymbol{M^2}$, resulting in real-valued accuracy that quantify their performance.

Step 5. Prepare ranking order based on rating, accuracy values

Alternatives are ranked according to the obtained rating and accuracy measures. Alternatives with higher rating values are considered the most preferable. When two alternatives have identical rating values, their accuracy values are then used to differentiate them.

Step 6. End

Illustrative Example of Crop-Land Selection

Agricultural systems are vulnerable to climate change, with notable effects on water resources and crop output. Climate change poses serious threats to agricultural production, potentially escalating the likelihood of food scarcity and famine. The growing population's demands and natural resource degradation pose significant threats to sustainable agriculture.

Advancements in agricultural technologies boost productivity, yet they gradually exhaust natural resources. Introducing technology in farming requires careful consideration of its ecological consequences. A plant needs rain for specific time frame, and the soil must contain moisture substance etc. Therefore, for selecting appropriate agricultural crop land is a challenging thing.

Knowledgeable producers select farmland by examining soil conditions, water availability, and local climate to ensure optimal crop growth and limited environmental effects.

Properly chosen agricultural land supports efficient production, reducing input use and mitigating negative effects on the environment.

Determining the appropriate soil and climate conditions is a necessary prerequisite to starting agricultural production that reduce the farmer loss and increase the productivity. Here, I adapted [68] following criteria for choosing appropriate Crop land for planting.

$(\mathcal{C}_1)$ Climate, $(\mathcal{C}_2)$Soil quality, $(\mathcal{C}_3)$ Water availability, $(\mathcal{C}_4)$Transportation, $(\mathcal{C}_5)$Energy access, $(\mathcal{C}_6)$Biodiversity

Step 1. Formation of alternatives versus criteria decision matrix

The decision maker provided the rating values of criteria in terms of Fermatean neutrosophic number with respect to the land, and represents these values in matrix form for easy calculation and understanding. The following matrix $\boldsymbol{M^2}$is presented below:

$$\left[\begin{array}{lllllll} & \mathcal{C}_1 & \mathcal{C}_2 & \mathcal{C}_3 & \mathcal{C}_4 & \mathcal{C}_5 & \mathcal{C}_6 \\ \mathcal{L}_1 & (0.9, 0.4, 0.6) & (0.7, 0.3, 0.8) & (0.8, 0.4, 0.6) & (0.6, 0.2, 0.9) & (0.8, 0.3, 0.6) & (0.9, 0.5, 0.4) \\ \mathcal{L}_2 & (0.8, 0.1, .6) & (0.5, 0.2, 0.9) & (0.8, 0.3, 0.6) & (0.8, 0.3, 0.6) & (0.7, 0.4, 0.7) & (0.9, 0.5, 0.4) \\ \mathcal{L}_3 & (0.6, 0.3, 0.8) & (0.7, 0.3, 0.7) & (0.8, 0.3, 0.6) & (0.4, 0.3, 0.9) & (0.6, 0.4, 0.8) & (0.8, 0.4, 0.6) \\ \mathcal{L}_4 & (0.7, 0.3, 0.7) & (0.8, 0.4, 0.6) & (0.7, 0.4, 0.7) & (0.8, 0.3, 0.6) & (0.7, 0.3, 0.7) & (0.5, 0.3, 0.9) \end{array}\right]$$

Step 2. Aggregated decision matrix using proposed operator

By using proposed operators (Eq. 3), the aggregate decision matrix of $\boldsymbol{M}^{\mathbf{2}}$and new decision matrix are obtained, synthesizing the information in a structured form.

We have accurately computed the aggregation operator with equal weights $w_r = \frac{1}{6}$ $p = 6$ criteria,

Three cases: $\lambda = 1$, $\lambda = 2$, and $\lambda = 1.5$ (for $1 < \lambda < 2$)

All values are numerically evaluated and rounded to 4 decimal places.

Aggregated Matrix (λ = 1)

$$\begin{bmatrix} \mathcal{L}_1 & (0.8163, 0.3763, 0.6295) \\ \mathcal{L}_2 & (0.7871, 0.3502, 0.6156) \\ \mathcal{L}_3 & (0.6914, 0.3405, 0.7249) \\ \mathcal{L}_4 & (0.7217, 0.3405, 0.6934) \end{bmatrix}$$

Aggregated Matrix (λ = 2)

$$\begin{bmatrix} \mathcal{L}_1 & (0.8105, 0.3747, 0.4667) \\ \mathcal{L}_2 & (0.7807, 0.3479, 0.4531) \\ \mathcal{L}_3 & (0.6841, 0.3401, 0.5261) \\ \mathcal{L}_4 & (0.7178, 0.3401, 0.5008) \end{bmatrix}$$

Aggregated Matrix (λ = 1.5) (1 < λ < 2)

$$\begin{bmatrix} \mathcal{L}_1 & (0.8128, 0.3755, 0.5291) \\ \mathcal{L}_2 & (0.7834, 0.3490, 0.5151) \\ \mathcal{L}_3 & (0.6873, 0.3403, 0.6013) \\ \mathcal{L}_4 & (0.7195, 0.3403, 0.5733) \end{bmatrix}$$

Step 3. Rating values of aggregate decision matrix

Eq. (1) is employed to compute the rating values of alternatives from aggregated decision matrix, that quantify their performance.

For $\boldsymbol{\lambda} = \mathbf{1}$

Calculations

$$\mathcal{L}_1 \colon rt = \frac{1+0.8163^3-0.6295^3-0.3763^3}{2} = 0.6202$$

$$\mathcal{L}_2 \colon rt = 0.5933,\ \mathcal{L}_3 \colon rt = 0.3770,\ \mathcal{L}_4 \colon rt = 0.4307$$

$$\begin{bmatrix} \mathcal{L}_1 & 0.6202 \\ \mathcal{L}_2 & 0.5933 \\ \mathcal{L}_3 & 0.3770 \\ \mathcal{L}_4 & 0.4307 \end{bmatrix}$$

For $\boldsymbol{\lambda} = \mathbf{2}$

$$\mathcal{L}_1 \colon rt = 0.7074, \mathcal{L}_2 \colon rt = 0.6816,\ \ \mathcal{L}_3 \colon rt = 0.5282, \mathcal{L}_4 \colon rt = 0.5736$$

$$\begin{bmatrix} \mathcal{L}_1 & 0.7074 \\ \mathcal{L}_2 & 0.6816 \\ \mathcal{L}_3 & 0.5282 \\ \mathcal{L}_4 & 0.5736 \end{bmatrix}$$

For $\mathbf{1} < \boldsymbol{\lambda} < \mathbf{2}$(λ = 1.5)

$$\mathcal{L}_1 \colon rt = 0.6685, \mathcal{L}_2 \colon rt = 0.6423, \mathcal{L}_3 \colon rt = 0.4668, \mathcal{L}_4 \colon rt = 0.5135$$

$$\begin{bmatrix} \mathcal{L}_1 & 0.6685 \\ \mathcal{L}_2 & 0.6423 \\ \mathcal{L}_3 & 0.4668 \\ \mathcal{L}_4 & 0.5135 \end{bmatrix}$$

Step 4. Calculate accuracy values from aggregate decision matrix using proposed accuracy function

Since all rating values are different for the alternatives, it is not necessary to calculate accuracy values in this situation

Step 5. Prepare ranking order based on score, accuracy and values

The ranking order of land based on the rating values for different $\lambda$is as follows:
Ranking for $\boldsymbol{\lambda = 1}$
Ranking Order: $\mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_3$
Ranking for $\lambda = \mathbf{2}$
Ranking Order: $\mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_3$
Ranking for $\mathbf{1} < \boldsymbol{\lambda} < \mathbf{2}$**(λ = 1.5)**
Ranking Order: $\mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_3$

Final Conclusion

Ranking is identical for all considered cases

$\boxed{\mathcal{L}_1 > \mathcal{L}_2 > \mathcal{L}_4 > \mathcal{L}_3}$

Best alternative: $\mathcal{L}_1$
Worst alternative: $\mathcal{L}_3$

The decision model is stable with respect to λ variation.

Conclusion

In recent times, aggregation operators have emerged as a prominent area of research within decision-making contexts. Selecting suitable cropland is a crucial decision-making issue in agriculture. The decision-making method with an aggregation operator, which has been successfully applied in the agricultural field, is an interesting topic of research. In this study, we have proposed a new aggregation operator based on FNSs and proved its necessary properties. We structured a general MCDM method to select cropland for agriculture. The introduced approach is tested on a cropland selection example based on standard criteria. The results obtained for different values of λ indicate that the proposed operator works stably and reliably. This proves that the operator can efficiently handle the complexity and uncertainty of real problems. The proposed MCDM technique requires that each alternative be evaluated against the same set of criteria. This is a necessary condition because the aggregation operators used in this technique are designed to handle a uniform number of criteria across all alternatives, enabling the calculation of overall aggregation values. It also has application potential in practical decision-making, especially in teachers' election processes, school management decisions, weather forecasting, and investment evaluation.

Future Research Directions

Future research should refine FNS models, expand their use across industries, and benchmark them against other advanced methods to improve robust, real-world decision-making. The developed strategy can also be employed to deal with other decision-making problems such as agricultural land selection [65], brick selection [66, 67], weaver selection [68, 69], teacher selection [70, 71], school choice [72], supply chain [73], green supplier selection [74], healthcare [75], e-car selection [76], etc.

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Chapter 18

VIKOR Strategies in Rough and Interval Rough Neutrosophic Environments

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ABSTRACT

In this paper, two new Multi-Attribute Decision Making (MADM) strategies in rough neutrosophic environment have been developed by extending the VIKOR (Vlse Kriterijumska Optimizacija I Kompromisno Resenje) strategy. In the decision-making situation, ratings are prescribed in the form of rough neutrosophic values. Firstly, we propose the VIKOR strategies in rough neutrosophic set environment and interval rough neutrosophic set environment. Then the decision- making problem of selecting the best suitable laptop is solved by the proposed strategies. The alternatives are ranked by using the developed VIKOR strategies. Finally, we compare the obtained solution with those obtained from existing MADM strategies.

Keywords: Rough set**,** Rough neutrosophic set, Interval rough neutrosophic set, MADM, VIKOR.

INTRODUCTION

Broumi, Smarandache and Dhar [1, 2] grounded the Rough Neutrosophic Set (RNS) by combining the Rough Set (RS) [3] and the Neutrosophic Set (NS) [4] to cope with uncertainty. Broumi and Smarandache proposed the Interval RNS (IRNS) [5] by combining the RS [3] and Interval NS (INS) [6]. The theoretical development and applications of NS and RNS can found in the studies [7, 8., 9, 10, 11, 12]. Pramanik [13] presented an overview of RNS to draw the attentions of the researchers for conducting research with RNSs. Different decision-making strategies [14-26] have been proposed under RNS [14-22] and IRNS [23-26] environments.

Opricovic [27] developed the basic ideas of VlseKriterijumska Optimizacija I Kompromisno Resenje (VIKOR) strategy for Multi-Criteria Decision Making (MCDM). Sayadi et. al. [28] grounded the VIKOR strategy for interval data. In 2004, Opricovic and Tzeng [29] presented a comparative analysis between the TOPSIS and VIKOR. The VIKOR strategy [30] is extended by providing a stability analysis. Sanayei et al. [31] proposed the VIKOR strategy in Fuzzy Set (FS) environment dealing with hierarchy MADM. Liao and Xu [32] grounded the VIKOR strategy in hesitant FS environment. Zhang and We [33] extended the VIKOR strategy in hesitant FS environment. Pramanik and Mallick [34] grounded the VIKOR strategy in trapezoidal NS environment .The VIKOR [35] based Multi Attribute Group Decision Making (MAGDM) strategy was presented under Neutrosophic Cubic Set (NCS) environment. Pramanik and Dalapati [36] extended the VIKOR strategy by incorporating compromise solution in NCS environment. Pramanik, Dalapati, Alam and Roy [37] developed the VIKOR strategy for MAGDM in bipolar NS environment. Mardani et al. [38] presented a systematic review of VIKOR strategy.

Research gap:

No studies have been conducted on MADM using VIKOR strategy under RNS environment and IRNS environment.

Motivation of the study:

Since no studies have been developed in the literature on VIKOR strategy under RNS and IRNS environments, we receive much motivation to develop VIKOR strategy under RNS and IRNS environment.

The objectives of the study:

1. To develop a VIKOR strategy for MADM under RNS Environment.
2. To establish a VIKOR strategy for MADM under IRNS Environment.

The main contributions of the study:

1. This study develops a new VIKOR strategy for MADM under RNS environment
2. This study develops a new VIKOR strategy for MADM under IRNS environment
3. This study presents two illustrative examples to show the feasibility and applicability of the developed VIKOR strategies.

BACKGROUND

This Section presents relevant definitions related to RNSs and IRNSs.

Definition 1. Let S be a non-null set and R^1 be an equivalence relation on S. Let W be an NS in S. The lower and the upper approximations of W in the approximation (S, R^1) are respectively defined as:

$$\underline{K(W)} =<< x^1, T_{\underline{K(W)}}(x^1), I_{\underline{K(W)}}(x^1), F_{\underline{K(W)}}(x^1) >/s \in [x^1]_{R1}, x^1 \in S>$$

and

$$\overline{K(W)} =<< x^1, T_{\overline{K(W)}}(x^1), I_{\overline{K(W)}}(x^1), F_{\overline{K(W)}}(x^1) >/s \in [x^1]_{R1}, x^1 \in S>$$

Where T$_W$= truth Membership Function (MF), I$_W$ = Indeterminacy MF and F$_W$= Falsity MF ,

$T_{\underline{K(W)}}(x^1) = \wedge z \in [x^1]_{R1} T_W(S), I_{\underline{K(W)}}(x^1) = \wedge z \in [x^1]_{R1} I_W(S), F_{\underline{K(W)}}(x^1) = \wedge z \in [x^1]_{R1} F_W(S)$ and

$T_{\overline{K(W)}}(x^1) = \vee z \in [x^1]_{R1} T_W(S), I_{\overline{K(W)}}(x^1) = \vee z \in [x^1]_{R1} I_W(S), F_{\overline{K(W)}}(x^1) = \vee z \in [x^1] F_W(S)$.

So,

$$0 \le T_{\underline{K(W)}}(x^1) + I_{\underline{K(W)}}(x^1) + F_{\underline{K(W)}}(x^1) \le 3$$

and

$$0 \le T_{\overline{K(W)}}(x^1) + I_{\overline{K(W)}}(x^1) + F_{\overline{K(W)}}(x^1) \le 3$$

Here $\vee$ *and* $\wedge$ reflect "max" and "min" operators respectively, T$_W$(s), I$_W$(s) and F$_W$(s) are the degrees of truth, indeterminacy and falsity membership of S with respect to Q.

Here, NS mappings, $\underline{K}, \overline{K}$: K(S)-> K(S) indicate the lower and upper rough NS approximation operators respectively. The pair $(\underline{K(W)}, \overline{K(W)})$ is termed as the RNS in (S, R^1) [1].

Definition 2. If K(W) = $(\underline{K(W)}, \overline{K(W)})$ is an RNS in (S, R^1) , the complement of K(W) is the RNS denoted by ~(K(W)) = $((\underline{K(W)})^c, (\overline{K(W)})^c)$,where $(\underline{K(W)})^c$ and $(\overline{K(W}$ are the complements of NSs $\underline{K(W)}$and $\overline{K(W)}$ respectively [2].

Definition 3. Let R^1 BE2 an Equivalence Relation (ER) on the universal set U^1. The pair (U^1, R^1) is referred to a Pawlak approximation space. An Equivalence Class (EC) of R^1 containing x^1 is denoted by $[x^1]_{R1}$ for $X^1 \in$ U^1, the lower and upper approximation of X^1 w.r.t. (U^1, R1) are denoted by R^{1*} X^1 and R^{1*} X^1 respectively and

R^{1*} $X^1 = \{ x^1 \in$ U^1 : $[x^1]_{R1} \subseteq X^1 \}$,

R^1 * $X^1 = \{ x^1 \in$ U^1 : $[x^1]_{R1} \cap X^1 \neq Ø \}$.

Now if R^{1*} X^1 = R^{1*} X^1, then X^1 is definable; otherwise, X^1 is an RS [3].

Definition 4 : Assume that, (U^1, R^1) is a Pawlak Approximation Space (AS) for an INS

A = {< x^1 , [$T_A^L(x^1)$, $T_A^U(x^1)$], [$I_A^L(x^1)$, $I_A^U(x^1)$], [$F_A^L(x^1)$, $F_A^U(x^1)$]> : x^1 ∈ U^1}

The lower approximation $\underline{A}_{R1}$ and the upper approximation $\overline{A}_{R1}$ of A in the Pawlak AS (U, R^1) are presented as:

$$\underline{A}_{R1} = \{< x^1 , [\wedge_{s\in[x^1]_{R1}} \{T_A^L(s)\}, \wedge_{s\in[x^1]_{R1}} \{T_A^U(s)\}], [\vee_{s\in[x^1]_{R1}} \{I_A^L(s)\}, \vee_{s\in[x^1]_{R1}} \{I_A^U(s)\}],$$

$$[\vee_{s\in[x^1]_{R1}} \{F_A^L(s)\}, \vee_{s\in[x^1]_{R1}} \{F_A^U(s)\}]> : x^1 \in U1\}$$

$$\overline{A}_{R1} = \{< x^1 , [\vee_{s\in[x^1]_{R1}} \{T_A^L(s)\}, \vee_{s\in[x^1]_{R1}} \{T_A^U(s)\}], [\wedge_{s\in[x^1]_{R1}} \{I_A^L(s)\}, \wedge_{s\in[x^1]_{R1}} \{I_A^U(s)\}],$$

$$[\wedge_{s\in[x^1]_{R1}} \{F_A^L(s)\}, \wedge_{s\in[x^1]_{R1}} \{F_A^U(s)\}]> : x^1 \in U1\}$$

Here, ∧ *and* ∨ reflects respectively “min” and “max” operators. R^1 denotes an ER for INS A. Here [x^1] $_{R1}$ is the EC of the element x^1 . Here,

$$[\wedge_{s\in[x^1]_{R1}} \{T_A^U(s)\}, \wedge_{s\in[x^1]_{R1}} \{T_A^L(s)\}] \subset [0,1],$$

$$[\vee_{s\in[x^1]_{R1}} \{I_A^U(s)\}, \vee_{s\in[x^1 R1} \{I_A^L(s)\}] \subset [0,1],$$

$$[\vee_{s\in[x^1]_{R1}} \{F_A^U(s)\}, \vee_{s\in[x^1]_{R1}} \{F_A^L(s)\}] \subset [0,1].$$

$$\text{and}0 \le \wedge_{s\in[x^1]_{R1}} \{T_A^U(s)\} + \vee_{s\in[x^1]_{R1}} \{I_A^U(s)\} + \vee_{s\in[x^1]_{R1}} \{F_A^U(s)\} \le 3$$

Then $\underline{A}_{R1}$ is an INS [5].

Similarly, we have

$$[\vee_{s\in[x^1]_{R1}} \{T_A^L(s)\}, \vee_{s\in[x^1]_{R1}} \{T_A^U(s)\}] \subset [0,1],$$

$$[\wedge_{s\in[x^1]_{R1}} \{I_A^L(s)\}, \wedge_{s\in[x^1]_{R1}} \{I_A^U(s)\}] \subset [0,1],$$

$$[\wedge_{s\in[x^1]_{R1}} \{F_A^L(s)\}, \wedge_{s\in[x^1]_{R1}} \{F_A^U(s)\}] \subset [0,1]\text{and}$$

$$0 \le \vee_{s\in[x^1]_{R1}} \{T_A^U(s)\} + \wedge_{s\in[x^1]_{R1}} \{I_A^U(s)\} + \wedge_{y\in[x^1]_{R1}} \{F_A^U(s)\}] \le 3$$

Then $\underline{A}$ R^1 is an INS [5].

If $\underline{A}_{R1} = \overline{A}_{R1}$ then A is a definable set, otherwise A is an IRNS. Here, $\underline{A}_{R1}$ and $\overline{A}_{R1}$ are called the lower and upper approximations of INS w.r.t. AS (U^1, R^1) respectively. $\underline{A}_{R1}$ and $\overline{A}_{R1}$ are denoted by $\underline{A}$ and $\overline{A}$ respectively.

Definition 4: *Euclidean distance between RNSs*

Assume that there are two RNSs

$$M_1 = \{< x_i, (\underline{T_{iM_1}}, \underline{I_{iM_1}}, \underline{F_{iM_1}}), (\overline{T_{iM_1}}, \overline{I_{iM_1}}, \overline{F_{iM_1}}) >: i = 1,2,...,n_1\}$$

and

$$N_1 = \{< x_i, (\underline{T_{iN_1}}, \underline{I_{iN_1}}, \underline{F_{iN_1}}), (\overline{T_{iN_1}}, \overline{I_{iN_1}}, \overline{F_{iN_1}}) >: i = 1,2,...,n_1\}$$

Following [23], the normalized Euclidean distance between M_1 and N_1 denoted by $E(M_1, N_1)$ is defined as:

$$E(M_1, N_1) = [\frac{1}{6n_1}\sum_{i=1}^{n_1}\{(\underline{T_{iM_1}} - \underline{T_{iN_1}})^2 + (\underline{I_{iM_1}} - \underline{I_{iN_1}})^2 + (\underline{F_{iM_1}} - \underline{F_{iN_1}})^2 + (\overline{T_{iM_1}} - \overline{T_{iN_1}})^2 + (\overline{I_{iM_1}} - \overline{I_{iN_1}})^2 + (\overline{I_{iM_1}} - \overline{I_{iN_1}})^2\}]^{0.5} \quad (1)$$

Definition 5. Euclidean distance between IRNSs

Assume that there are two IRNSs

$$P_1 = \{< x_i, ([\underline{T_{iP_1}^-}, \underline{T_{iP_1}^+}], [\underline{I_{iP_1}^-}, \underline{I_{iP_1}^+}], [\underline{F_{iP_1}^-}, \underline{F_{iP_1}^+}], [\overline{T_{iP_1}^-}, \overline{T_{iP_1}^+}], [\overline{I_{iP_1}^-}, \overline{I_{iP_1}^+}], [\overline{F_{iP_1}^-}, \overline{F_{iP_1}^+}]) >: i = 1,2,...,n_1\}$$

and

$$Q_1 = \{< x_i, ([\underline{T^-_{iQ_1}}, \underline{T^+_{iQ_1}}], [\underline{I^-_{iQ_1}}, \underline{I^+_{iQ_1}}], [\underline{F^-_{iQ_1}}, \underline{F^+_{iQ_1}}], [\overline{T^-_{iQ_1}}, \overline{T^+_{iQ_1}}], [\overline{I^-_{iQ_1}}, \overline{I^+_{iQ_1}}], [\overline{F^-_{iQ_1}}, \overline{F^+_{iQ_1}}]) >: i = 1, 2, ..., n_1\}$$

Then the normalized Euclidean distance [23] between P_1 and Q_1 denoted by $E(P_1, Q_1)$ is defined as

$$E(P_1, Q_1) = [\frac{1}{12n_1}\sum_{i=1}^{n_1}\{(\underline{T^-_{iP_1}} - \underline{T^-_{iQ_1}})^2 + (\underline{T^+_{iP_1}} - \underline{T^+_{iQ_1}})^2 + (\underline{I^-_{iP_1}} - \underline{I^-_{iQ_1}})^2 + (\underline{I^+_{iP_1}} - \underline{I^+_{iQ_1}})^2 + (\underline{F^-_{iP_1}} - \underline{F^-_{iQ_1}})^2$$
$$+ (\underline{F^+_{iP_1}} - \underline{F^F_{iQ_1}})^2 + (\overline{T^-_{iP_1}} - \overline{T^-_{iQ_1}})^2 + (\overline{T^+_{iP_1}} - \overline{T^+_{iQ_1}})^2 + (\overline{I^-_{iP_1}} - \overline{I^-_{iQ_1}})^2 + (\overline{I^+_{iP_1}} - \overline{I^+_{iQ_1}})^2 + (\overline{F^-_{iP_1}} - \overline{F^-_{iQ_1}})^2 + (\overline{F^+_{iP_1}} - \overline{F^+_{iQ_1}})^2\}]^{0.5} \tag{2}$$

VIKOR Method for MADM Problems in RNS Environment

Assume that $A = \{A_1, ..., A_m\}$ be the set of alternatives and $C = \{C_1, ..., C_n\}$ be the set of criteria.
Let $W = (w_1, \ldots, w_n)^T$ denote the weight vector of criteria with $w_j \geq 0$ ($j = 1, 2, \cdots, n$) and $\sum_{j=1}^{n} w_j = 1$.
Algorithm:

Step 1: The Decision Maker (DM) formulates the decision matrix as

$$D = < Z_{ij} >_{n\times m} = \begin{bmatrix} z_{11}\ z_{12} \cdots\ z_{1m} \\ z_{21}\ z_{22} \cdots\ z_{2m} \\ \vdots\ \ \vdots\ \ \cdots\ \vdots \\ z_{n1}\ z_{n2} \cdots\ z_{nm} \end{bmatrix}$$

where $z_{ij} = < (\underline{T_{ij}}, \underline{I_{ij}}, \underline{F_{ij}}), (\overline{T_{ij}}, \overline{I_{ij}}, \overline{F_{ij}}) >$

with $0 \leq \underline{T_{ij}} + \underline{I_{ij}} + \underline{F_{ij}} \leq 3$ and $0 \leq \overline{T_{ij}} + \overline{I_{ij}} + \overline{F_{ij}} \leq 3$ for $i = 1, 2, ..., n$ and $j = 1, 2, ..., m$.

Step 2: Calculate the Positive Ideal Solution (PIS) and the Negative Ideal Solution (NIS)

$$z_{j^*} = \{(\max_j \underline{T_{ij}}, \min_j \underline{I_{ij}}, \min_j \underline{F_{ij}}), (\min_j \overline{T_{ij}}, \max_j \overline{I_{ij}}, \max_j \overline{F_{ij}})\} \tag{3}$$

$$z_{j^-} = \{(\min_j \underline{T_{ij}}, \max_j \underline{I_{ij}}, \max_j \underline{F_{ij}}), (\max_j \overline{T_{ij}}, \min_j \overline{I_{ij}}, \min_j \overline{F_{ij}})\} \tag{4}$$

Step 3: Compute the Euclidean distances

Using Eq. (1), the Euclidean distances between alternative A_r ($r = 1, 2, \ldots, m$) and Y^* , the Euclidean Distance between Y^- on Y^* are calculated.

Step 4: Compute the Utility Value (UV) S_i and the Regret Value(RV) R_i ($i = 1, 2, \cdots, m$)

$$S_i = \sum_{j=1}^{n} w_j \frac{d(z_j^*, z_{rj})}{d(z_j^*, z_j^-)}, \tag{5}$$

and $R_i = max[\, w_j \frac{d(z_j^*, z_{rj})}{d(z_j^*, z_j^-)}]$. (6)

Step 5: Calculate the values Q_i' ($i = 1, 2, \cdots, m$)

$$Q_i' = v\frac{(S_i - S^*)}{(S^- - S^*)} + (1 - v)\frac{(R_i - R^*)}{(R^- - R^*)}. \tag{7}$$

Step 6: Rank the alternatives by S, R and Q'

The alternatives are ranked according to the decreasing values of S, R and Q'.

Step 7: Determine the compromise solution

Procedure to obtain A^{1s}t as a compromise solution

Obtain alternative A^{1st} as a compromise solution that has the most optimal ranking based on the minimum value of Q' if it meets the following condition:

Condition: Acceptable advantage: Q' (A^{2nd}) − Q' (A^{1st}) ≥ D(Q'), where A^{2nd} ranks second in the ordered list by Q'; D(Q') = 1/(m− 1); m = no. of alternatives.

VIKOR Method for MADM Problems in IRNS Environment

In this section, the VIKOR strategy is developed to solve an MADM problem under IRNS environment. Assume that $A=\{A_1, \ldots, A_m\}$ be the set of alternatives and C = $\{C_1, \ldots, C_n\}$ be the set of criteria.
Let W = $(w_1, \ldots, w_n)^T$ denote the weight vector of criteria with $w_j \geq 0$ (j = 1, 2, ⋯,n) and $\sum_{j=1}^{n} w_j = 1$.
Algorithm:

Step 1: Formulate the decision matrix

$$D = < Z_{ij} >_{n\times m} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1m} \\ z_{21} & z_{22} & \cdots & z_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ z_{n1} & z_{n2} & \cdots & z_{nm} \end{bmatrix}$$

where $z_{ij} = <([\underline{T_{iP_1}^-}, \underline{T_{iP_1}^+}], [\underline{I_{iP_1}^-}, \underline{I_{iP_1}^+}], [\underline{F_{iP_1}^-}, \underline{F_{iP_1}^+}], [\overline{T_{iP_1}^-}, \overline{T_{iP_1}^+}], [\overline{I_{iP_1}^-}, \overline{I_{iP_1}^+}], [\overline{F_{iP_1}^-}, \overline{F_{iP_1}^+}]) >$

with $0 \leq \vee_{y\in[x]_R} \{T_A^U(y)\} + \wedge_{y\in[x]_R} \{I_A^U(y)\} + \wedge_{y\in[x]_R} \{F_A^U(y)\} \leq 3$

for i = 1, 2, ..., n and j = 1, 2, ..., m.

Step 2: Calculate the PIS and the NIS

$$\begin{aligned} z_j^* = <& ([\max_l(\underline{T_{lj}^-}), \max_l(\underline{T_{lj}^+})], [\min_l(\underline{I_{lj}^-}), \min_l(\underline{I_{lj}^+})], [\min_l(\underline{F_{lj}^-}), \min_l(\underline{F_{lj}^+})]), \\ & ([\min_l(T_{lj}^-), \min_l(T_{lj}^+)], [\max_l(I_{lj}^-), \max_l(I_{lj}^+)], [\max_l(F_{lj}^-), \max_l(F_{lj}^+)]) >. \end{aligned} \tag{8}$$

$$\begin{aligned} z_j^- = <& ([\min_l(\underline{T_{lj}^-}), \min_l(\underline{T_{lj}^+})], [\max_l(\underline{I_{lj}^-}), \max_l(\underline{I_{lj}^+})], [\max_l(\underline{F_{lj}^-}), \max_l(\underline{F_{lj}^+})]), \\ & ([\max_l(T_{lj}^-), \max_l(T_{lj}^+)], [\min_l(I_{lj}^-), \min_l(I_{lj}^+)], [\min_l(F_{lj}^-), \min_l(F_{lj}^+)]) >. \\ l = 1, 2, &\ldots, n \end{aligned} \tag{9}$$

Step 3: Compute the Euclidean distances

Using Eq.(2), the Euclidean distances between the alternative A_r (r = 1, 2, … , m) and Y^* ,and the Euclidean Distance between Y^- and Y^* are calculated.
Step 4: Compute the UV and the RV

$$S_i = \sum_{j=1}^{n} w_j \frac{d(z_j^*, z_{rj})}{d(z_j^*, z_j^-)}, \text{ i = 1, 2, …,m} \tag{10}$$

and $R_i = max[w_j \frac{d(z_j^*, z_{rj})}{d(z_j^*, z_j^-)}]$, i = 1, 2, …, m. (11)

Step 5: Calculate the values Q_i' (i = 1, 2, ⋯, m).

$$Q_i' = v \frac{(S_i - S^*)}{(S^- - S^*)} + (1 - v) \frac{(R_i - R^*)}{(R^- - R^*)}. \tag{12}$$

Step 6: Rank the alternatives by S, R and Q'.

The alternatives are ranked according to the decreasing values of S, R and Q'.

Step 7: Determine the compromise solution

Obtain alternative A^{1st} as a compromise solution that has the most optimal ranking based on the minimum value of Q' if it meets the condition below:

Condition: Acceptable advantage: Q' (A^{2nd}) − Q' (A^{1st}) ≥ DQ', where A^{2nd} ranks second in the ordered list by Q'; D(Q') = 1/(m− 1); m = no. of alternatives.

Illustrative Example for RNS Environment

Consider the following MADM problem presented by Pramanik, Roy, and Roy [18]:

*"*Assume that a decision maker intends to select the most suitable laptop for random use from the three initially chosen laptops (A_1', A_2', A_3') by considering four attributes namely: features C_1, reasonable price C_2, customer care C_3, risk factor C_4. The weight vector of the four attribute is w = {0.35, 0.25, 0.25, 0.15*}".*

The problem is solved using the following six steps:

Step1: Formulate the decision matrix

The DM constructs the decision matrix (see Table 1) w.r.t the feasible alternatives and prescribed criteria via rough neutrosophic numbers.

Table 1 : The decision matrix

	C₁	**C₂**	**C₃**	**C₄**
A_1'	<(.6, .3, .3), (.8, .1, .1)>	<(6, .4, .4), (.8, .2, .2)>	<(.6, .4, .4), (.8, .2, .2)>	<(.7, .4, .4), (.9, .2, .2)>
A_2'	<(.7, .3, .3), (.9, .1, .3)>	<(.6, .3, .3), (.8, .3, .3)>	<(.6, .2, .2), (.8, .4 .2)>	<(.7, .3, .3), (.9, .3, .3)>
A_3'	<(.6, .2, .2), (.8, .0, .2)>	<(.7, 3, .3), (.9, .1, .1)>	<(.7, .4, .6), (.9, .2, .4)>	<(.6, .3, .2), (.8, .1, .2])>

Step 2: Determine the PIS and NIS

Here three benefit type attributes C_1, C_2, C_3 and one cost type attribute C_4. We calculate the positive and negative ideal alternatives (see table 2):

Table 2 : The PIS and NIS

	C₁	**C₂**	**C₃**	**C₄**
Y^+	<(.7, .2, .2), (.8, .1, .3)>	<(.7, .3, .3), (.8, .3, .3)>	<(.7, .2, .2), (.8, .4, .4)>	<(.7, .3, .2) (.8, .3, .3)>
Y^-	<(.6, .3, .3), (.9, .0, .1)>	<(.6, .4, .4), (.9, .1, .1)>	<(.6, .4, .6), (.9, .2, .2)>	<(.6, .4, .4), (.9, .1, .2])>

Step 3: Calculate the Euclidean distances

The Euclidean distances (see Table 3) are calculated using Eq. (4).

Table 3 : Euclidean distance of each alternative and the NIS from the PIS

	C₁	**C₂**	**C₃**	**C₄**
A_1'	0.108012	0.091287	0.168325	0.115470
A_2'	0.070711	0.040825	0.0912.871	0.057735
A_3'	0.070711	0.122474	0.204124	0.100000
Y^-	0.122474	0.141421	0.223607	0.141421

Step 4: Calculate the values S_i and R_i

We calculate the UVs and RVs using Eqs. (8) and (9). We obtain:

$S_1 = 0.780713$, $S_2 = 0.437543$, $S_3 = 0.752864$.

$R_1 = 0.308671$, $R_2 = 0.202075$, $R_3 = 0.228217$.

Step 5: Calculate the values Q_i

We calculate the Q_i using Eq. (10). The results are as follows :
$Q_1 = 0$, $Q_2 = 1$, $Q_3 = 0.417954$.

Step 6: Rank the alternatives

Ranking order is shown in table 4. The best alternative is A_1' with Q(A_1') = 0. Alternative A_3' can be compromised solution as DQ = 0.5 and Q(A_3') - Q(A_1') = 0.417954 < DQ.

Table 4: Ranking order based on the values of S, R, Q

	A_1'	A_2'	A_3'	Ranking order
S	0.780713	0.437543	0.752864	$A_2' > A_3' > A_1'$
R	0.308671	0.202075	0.228217	$A_2' > A_3' > A_1'$
Q	0	1	0.417954	$A_1' > A_3' > A_2'$

Step 7: Determine the compromise solution

The preference ranking order based on Q in decreasing order and alternative with the best position is A_1' with A_1' = 0, and the second-best position is A_3', with $A_3', = 0.42$. Therefore,
A_3' -$A_1' = 0.42 - 0 = 0.42 \ < 0.5$
Since, $\frac{1}{m-1} = \frac{1}{3-1} = 0.5$
Therefore, we obtain the compromise solution as follows:
A_3' -$A_1' = 0.42 - 0 = 0.42 \ < 0.5$
A_2' -$A_1' = 1 - 0 = 1 > 0.5$
So, A_1', A_3' are compromise solution.

A Numerical Example for IRNS Environment

Consider the following MADM problem discussed by Pramanik et al. [19]:

"A decision maker intends to select the most suitable laptop for random use from the three initially chosen laptops (A_1', A_2', A_3') by considering four attributes namely: features C_1, reasonable price C_2, customer care C_3, risk factor C_4. The weight vector of the four attribute is w = {0.35, 0.25, 0.25, 0.15}".
The problem is solved using the following six steps:

Step 1: Construct the decision matrix

The DM constructs the decision matrix (see Table 5) with respect to the three alternatives and four criteria.

Table 5 : The decision matrix

	C1	**C2**	**C3**	**C4**
A_1'	<([.6, .7], [.3, .5], [.3, .4]), ([.8, .9], [.1, .3], [.1, .2])>	<([.5, .7], [.3, .4], [.1, .2]), ([.7, .9], [.3, .5], [.3, .4])>	<([.5, .6], [.4, .5], [.4, .6]), ([.7, .8], [.2, .4], [.3, .4])>	<([.8, .9], [.3, .4], [.5, .6]), ([.7, .8], [.3, .5], [.3, .5])>
A_2'	<([.7, .8], [.2, .3], [.0, .2]), ([.7, .9], [.1, .2], [.1, .2])>	<([.6, .7], [.1, .2], [.0, .2]), ([.6, .7], [.1, .3], [.1, .3])>	<([.5, .7], [.2, .3], [.1, .2]), ([.6, .9], [.3, .5], [.2 .4])>	<([.7, .8], [.3, .5], [.1, .3]), ([.5, .7], [.5, .6], [.2, .3])>
A_3'	<([.6, .7], [.3, .4], [.0, .3]), ([.6, .9], [.1, .2], [.1, .2])>	<([.5, .7], [.2, .4], [.2, .4]), ([.6, .8], [.1, .3], [.1, .2])>	<([.6, .8], [.2, .4], [.3, .4]), ([.6, .8], [.2, .5], [.3, .5])>	<([.4, .7], [.2, .4], [.4, .5]), ([.5, .8], [.2, .5], [.0, .2])>

Step 2: Determine the PIS and NIS

Here three benefit type attributes C_1, C_2, C_3 and one cost type attribute C_4. We calculate the positive and negative ideal alternatives (see Table 6) as follows:

Table 6 : The PIS and NIS

	C_1	**C_2**	**C_3**	**C_4**
Y^+	<([.7, .8], [.2, .3], [.0, .2]), ([.6, .9], [.1, .3], [.1, .2])>	<([.6, .7], [.1, .2], [.0, .2]), ([.6, .7], [.3, .5], [.3, .4])>	<([.6, .8], [.2, .3], [.1, .2]), ([.6, .8], [.3, .5], [.3, .5])>	<([.8, .9], [.2, .4], [.1, .3]), ([.5, .7], [.5, .6], [.3, .5])>
Y^-	<([.6, .7], [.3, .5], [.3, .4]), ([.8, .9], [.1, .2], [.1, .2])>	<([.5, .7], [.3, .4], [.2, .4]), ([.7, .9], [.1, .3], [.1, .2])>	<([.5, .6], [.4, .5], [.4, .6]), ([.7, .9], [.2, .4], [.2 .4])>	<([.4, .7], [.3, .5], [.5, .6]), ([.7, .8], [.2, .5], [.0, .2])>

Step 3: Calculate the Euclidean distances

Using Eq. (4), the Euclidean distances (see table 7) are calculated.

Table 7: Euclidean distances of each alternative and the NIS from the PIS

	C_1	**C_2**	**C_3**	**C_4**
A_1'	0.040825	0.111803	0.193649	0.189297
A_2'	0.008333	0.1040833	0.064550	0.086602
A_3'	0.020412	0.160727	0.091287	0.227303
Y^-	0.041667	0.051370	0.055277	0.074536

Step 4: Calculate S_i and R_i

Utilizing Eqs. (8) and (9), we obtain:
$S_1 = 2.143796$, $S_2 = 1.042755$, $S_3 =1.823960$.
$R_1 = 0.875812$, $R_2 = 0.506537$, $R_3 = 0.782203$.

Step 5: Calculate $Q_i (i = 1, 2, 3)$

We calculate the VIKOR values Q_i using Eq. (10). The results are as follows :
$Q_1 = 0$, $Q_2 = 1$, $Q_3 = 0.271989$.

Step 6: Rank the alternatives

Ranking order is shown in table 8. The best alternative is A_1 with $Q(A_1') = 0$. Alternative A_3' can be compromised solution as $DQ = 0.5$ and $Q(A_3') - Q(A_1') = 0.271989 < DQ$.

Table 7: Ranking order based on the values of S, R, Q

	A_1'	A_2'	A_3'	Ranking order
S	2.143796	1.042755	1.823960	$A_2' > A_3' > A_1'$
R	0.875812	0.506537	0.782203	$A_2' > A_3' > A_1'$
Q	0	1	0.271989	$A_1' > A_3' > A_2'$

Step 7: Determine the compromise solution

The preference ranking order based on Q in decreasing order and the alternative with the best position is A_1' with $A_1' = 0$, and the second-best position is A_3', with $A_3', = 0.27$. Therefore,

$A_3{'}$ -$A_1{'} = 0.27 - 0 = 0.27 \; < 0.5$
Since, $\frac{1}{m-1} = \frac{1}{3-1} = 0.5$
Therefore, we obtain the compromise solution as follows:
$A_3{'}$ -$A_1{'} = 0.27 - 0 = 0.27 \; < 0.5$
$A_2{'}$ -$A_1{'} = 1 - 0 = 1 > 0.5$
So, $A_1{'}$, $A_3{'}$ are compromise solution.

For comparison, the considered problem has been solved using existing strategies in IRNS environment and the comparison is shown in Table 9.

Table 9: Comparison among the proposed strategy and existing strategies

Method	Ranking
Bidirectional projection measure [19]	$A_1' > A_2' > A_3'$
Hamming similarity measure [17]	$A_1' > A_3' > A_2'$
The proposed strategy	$A_1' > A_3' > A_2'$

Conclusion

In this study two new VIKOR strategies are developed to cope with MADM problems under RNS environment and IRNS environment. The developed strategies are applied to solve the problem of selecting the best suitable laptop for random use to reflect the feasibility and applicability of the proposed strategies. The alternatives are ranked by using the proposed VIKOR strategies based on the Euclidean distance. The obtained results are compared with other existing strategies, namely, bidirectional projection strategy, Hamming similarity measure strategy.

Future Research Directions

The developed strategies can be implemented to cope with real-life decision making problems like brick selection [39, 40], weaver selection [41, 42], teacher selection [43-45,], school choice [46], supply chain [47], green supplier selection [48], healthcare [49], e-car selectin [50], etc.

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Chapter 19

A Neutrosophic Open Set–Based Multi-Criteria Decision Framework for Sustainable Lithium-Ion Battery Waste Management under Uncertainty

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ABSTRACT

The rapid growth of lithium-ion battery deployment in electric vehicles and renewable energy systems has generated critical sustainability challenges associated with end-of-life management. Decision-making in battery waste management is inherently complex due to conflicting criteria, evolving technologies, regulatory diversity, and incomplete life-cycle data. This study proposes a novel *neutrosophic open set–based* multi-criteria decision-making (MCDM) framework that explicitly models truth, indeterminacy, and falsity information within a topological structure. Unlike conventional fuzzy and intuitionistic approaches, the proposed model incorporates neutrosophic topology to define flexible sustainability regions rather than rigid evaluation classes. A weighted aggregation function is developed and its mathematical properties are formally established. Additional results—including a monotonicity lemma, a convexity proposition, and a consistency theorem—strengthen the theoretical foundation. A numerical case study comparing recycling, landfill disposal, and pyrometallurgical recovery demonstrates the robustness and transparency of the framework. Sensitivity analysis confirms ranking stability under varying decision-maker preferences. The proposed methodology provides a resilient and interpretable decision-support tool for sustainable lithium-ion battery waste management under uncertainty.

Keywords: Lithium-ion battery; Neutrosophic topology; Neutrosophic open set; Sustainable waste management; Multi-criteria decision-making; Uncertainty modelling; Aggregation function.

INTRODUCTIO

The accelerated electrification of transportation and renewable energy systems has substantially increased the production and disposal of lithium-ion batteries (LIBs). By 2030, annual global LIB waste is projected to exceed 11 million tonnes [1]. Improper disposal may lead to heavy metal contamination, fire hazards, and severe environmental degradation [2]. Selecting an appropriate end-of-life management strategy is therefore a critical sustainability challenge requiring systematic, mathematically principled decision support.

Uncertainty in Sustainable Decision-Making

Multi-criteria decision-making (MCDM) is widely used to structure complex environmental evaluation problems. Classical methods such as AHP, TOPSIS, and VIKOR operate under crisp or probabilistic assumptions that inadequately capture the full spectrum of uncertainty present in LIB waste management. Fuzzy set theory, introduced by Zadeh [3], extended MCDM to imprecise environments, but the binary membership–non-membership constraint limits its expressiveness. Atanassov's intuitionistic fuzzy sets [4] addressed this by adding explicit non-membership, yet they still collapse *indeterminacy* into the complement of truth and falsity, preventing independent modelling of conflicting expert opinions or missing data.

Smarandache's neutrosophic set theory [5] resolves this limitation by independently assigning a truth-membership (T), an indeterminacy-membership (I), and a falsity-membership (F) to every element. Single-valued neutrosophic sets (SVNS), introduced by Wang et al. [6], made the theory computationally tractable and opened the way for neutrosophic MCDM frameworks[7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 17, 19, 20, 21, 22, 23].

Neutrosophic Topology and Its Applications

Salama and Alblowi [24] established the foundational framework of neutrosophic topological spaces. This structure has since been investigated extensively. Das and Pramanik [25] introduced generalised neutrosophic b-open sets and characterised their relationship to standard neutrosophic open sets. Das and Pramanik [26] further developed neutrosophic Φ-open sets and neutrosophic Φ-continuous functions, providing new tools for continuity analysis in neutrosophic topology. Das and Tripathy [27] introduced and studied neutrosophic simply b-open sets, establishing their separation and continuity properties. Topology on quadripartitioned neutrosophic sets—a framework in which the indeterminacy component is split into two independent parts—was developed by Das, Das, and Granados [28]. Das, Das, and Pramanik [29] extended this programme to ultra neutrosophic sets, which permit membership values beyond the standard unit interval. Topology on rough pentapartitioned neutrosophic sets was investigated by Das, Das, and Tripathy [30], combining roughness with the five-component neutrosophic structure. Pentapartitioned neutrosophic topological spaces were introduced and studied by Das and Tripathy [31], providing the topological foundation for the five-component generalisation. Single-valued neutrosophic rough continuous mappings and compactness via rough topological spaces were established by Tripathy, Das, and Das [32]. The ρ-level sets of neutrosophic sets and their infinite extendable graph structures were investigated by Das and Rakshit [33]. Das, Datta, Nath et al. [34] developed the quadripartitioned neutrosophic quasi-coincident topological space. Neutrosophic d-filters of d-algebras were introduced by Das, Das, Granados, and Hasan [35].

Neutrosophic MCDM and Similarity Measures

Neutrosophic similarity measures were applied to COVID-19 diagnosis by Das, Mukherjee, and Tripathy [36], demonstrating the practical power of neutrosophic evaluation functions. Mallick and Pramanik [37] developed pentapartioned neutrosophic sets. Pentapartitioned neutrosophic Q-ideals of Q-algebras were studied by Das, Das, Granados, and Mukherjee [38], providing algebraic foundations for multi-valued neutrosophic reasoning. Single-valued pentapartitioned neutrosophic Dice similarity measures and their application to material selection were developed by Saha, Majumder, Das, Das, and Tripathy [39]. Single-valued pentapartitioned neutrosophic soft sets were introduced by Shil, Das, Das, and Pramanik [40], extending neutrosophic soft computing to the five-component framework. Pentapartitioned neutrosophic probability distributions were introduced by Das, Shil, Das, Khaled, and Salama [41], providing a probabilistic layer over the topological one. Cylindrical (α, β)-cuts of normalised pentapartitioned neutrosophic sets for decision making were developed by Das, Datta, Bal, and Nath [42], connecting the topological and decision-theoretic strands. Neutrosophic fuzzy matrices and their algebraic operations were studied by Das, Smarandache, and Tripathy [43], establishing the algebraic backbone for neutrosophic computation. Neutrosophic multiset topological spaces were introduced by Das and Tripathy [44], extending the classical topological programme to multiset universes.

Battery Waste Management Literature

Life-cycle assessment (LCA) of LIB recycling by Dunn [2] showed that recycling recovers up to 60% of embedded energy while avoiding critical metal loss. Li et al. [1] provided a comprehensive survey of sustainable LIB management strategies—including second-life reuse, hydrometallurgy, and pyrometallurgy—and highlighted the gap between technical feasibility and regulatory deployment. Despite the rich literature on neutrosophic MCDM and LIB sustainability, *no existing work* integrates neutrosophic topology as a structural decision layer for LIB waste management. This paper fills that gap.

Contributions of This Paper

The principal contributions are:

(a) Integration of neutrosophic topology into MCDM via the notion of neutrosophic open sets as flexible sustainability regions.

(b) Development of a mathematically bounded weighted aggregation function $D(S_i) \in [-1, 1]$.

(c) Formal establishment of the mathematical properties of the framework: boundedness, monotonicity, convexity, and weight-consistency.

(d) A complete numerical case study with five expert evaluations and four criteria.

(e) Sensitivity analysis over four preference scenarios demonstrating ranking robustness.

(f) Application to the practically important problem of sustainable LIB waste management.

BACKGROUND

We recall the essential definitions underpinning the proposed framework.

Single-Valued Neutrosophic Set

Definition 1 ([6])**.** *Let X be a non-empty universe of discourse. A* single-valued neutrosophic set *(SVNS) A on X is*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\},$$

where $T_A, I_A, F_A : X \to [0,1]$ are the truth-membership*,* indeterminacy-membership*, and* falsity-membership *functions, respectively, satisfying $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ for all $x \in X$.*

Definition 2. *For SVNSs A and B on X:*

(1) $A \subseteq B$ *iff* $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ *for all* $x \in X$.

(2) $A^c = \langle x, F_A(x), 1 - I_A(x), T_A(x) \rangle$.

(3) $(A \cap B)(x) = \langle \min(T_A, T_B), \max(I_A, I_B), \max(F_A, F_B) \rangle$.

(4) $(A \cup B)(x) = \langle \max(T_A, T_B), \min(I_A, I_B), \min(F_A, F_B) \rangle$.

Neutrosophic Topology

Definition 3 ([24])**.** *A* neutrosophic topology τ_N *on* X *is a family of neutrosophic subsets of* X *satisfying:*

(1) $\emptyset_N$ *and* $X_N \in \tau_N$*;*

(2) If $A, B \in \tau_N$ *then* $A \cap B \in \tau_N$ *(finite intersections);*

(3) If $\{A_i : i \in J\} \subseteq \tau_N$ *then* $\bigcup_{i \in J} A_i \in \tau_N$ *(arbitrary unions).*

The pair (X, τ_N) *is a* neutrosophic topological space *(NTS). Elements of* τ_N *are called* neutrosophic open sets*; their complements are* neutrosophic closed sets*.*

Definition 4. *The* neutrosophic interior *and* closure *of a SVNS* A *in a NTS* (X, τ_N) *are, respectively,* $\mathrm{Nint}(A) = \bigcup\{U \in \tau_N : U \subseteq A\}$ *and* $\mathrm{Ncl}(A) = \bigcap\{F : F^c \in \tau_N,\ A \subseteq F\}$.

Remark 1. *In this framework, each decision criterion* C_j *induces a neutrosophic open set* $U_j \in \tau_N$ *representing a* flexible sustainability region*: a strategy belongs to* U_j *to a degree* $(T_{U_j}, I_{U_j}, F_{U_j})$ *rather than unconditionally. This is a decisive conceptual advantage over crisp and even fuzzy classification, since real sustainability criteria exhibit overlapping and graded membership. The topological structure—closed under unions and intersections—then guarantees that combined criteria also define valid sustainability regions. This generalises the topological approach to MCDM developed in [25, 28, 29, 31].*

Proposed Neutrosophic Open Set–Based MCDM Framework

Decision Space and Criteria

Let the decision universe be $X = \{S_1, S_2, S_3\}$, where:

- S_1: Mechanical and chemical recycling,
- S_2: Landfill disposal,
- S_3: Pyrometallurgical recovery.

Four sustainability criteria are considered:

- C_1: Economic feasibility,
- C_2: Environmental safety,
- C_3: Technological maturity,
- C_4: Regulatory compliance.

Each criterion C_j induces a neutrosophic open set $U_j = \langle T_{U_j}(x), I_{U_j}(x), F_{U_j}(x) \rangle$ for each strategy $x = S_i$. The union $\bigcup_{j=1}^{4} U_j$ and pairwise intersections $U_j \cap U_k$ remain neutrosophic open sets by the topology axioms, so composite sustainability assessments are structurally consistent.

Criterion-Weighted Aggregation Function

For each strategy S_i, define the criterion-wise mean truth, indeterminacy, and falsity values as:

$$\bar{T}_i = \frac{1}{4}\sum_{j=1}^{4} T_{ij}, \qquad \bar{I}_i = \frac{1}{4}\sum_{j=1}^{4} I_{ij}, \qquad \bar{F}_i = \frac{1}{4}\sum_{j=1}^{4} F_{ij}.$$

The *neutrosophic aggregation decision score* is:

$$D(S_i) = \alpha\,\bar{T}_i - \beta\,\bar{F}_i - \gamma\,\bar{I}_i, \tag{1}$$

subject to

$$\alpha + \beta + \gamma = 1, \quad \alpha, \beta, \gamma \geq 0. \tag{2}$$

Remark 2. *The three weights in (2) allow the decision-maker to encode sustainability priorities: α rewards positive performance, β penalises environmental/regulatory failure, and γ penalises incomplete or conflicting information. The constraint $\alpha + \beta + \gamma = 1$ normalises the score to the interval $[-1, 1]$ (established in Theorem 1 below).*

Mathematical Properties of the Framework

Theorem 1 (Boundedness)**.** *For any strategy S_i and weights satisfying (2),*

$$-1 \;\leq\; D(S_i) \;\leq\; 1.$$

Proof. Since $\bar{T}_i, \bar{I}_i, \bar{F}_i \in [0, 1]$ and $\alpha + \beta + \gamma = 1$ with all non-negative:

Upper bound: $D(S_i) \leq \alpha \cdot 1 - \beta \cdot 0 - \gamma \cdot 0 = \alpha \leq 1$. Equality holds when $\bar{T}_i = 1$, $\bar{F}_i = 0$, $\bar{I}_i = 0$ and $\alpha = 1$.

Lower bound: $D(S_i) \geq \alpha \cdot 0 - \beta \cdot 1 - \gamma \cdot 1 = -(\beta + \gamma) = -(1 - \alpha) \geq -1$. Equality holds when $\bar{T}_i = 0$, $\bar{F}_i = 1$, $\bar{I}_i = 1$ and $\alpha = 0$. □ □

Lemma 1 (Monotonicity)**.** *Let S_i and S_k be two strategies such that $\bar{T}_i \geq \bar{T}_k$, $\bar{F}_i \leq \bar{F}_k$, and $\bar{I}_i \leq \bar{I}_k$. Then $D(S_i) \geq D(S_k)$ for any weights (α, β, γ) satisfying (2).*

Proof.

$$D(S_i) - D(S_k) = \alpha(\bar{T}_i - \bar{T}_k) - \beta(\bar{F}_i - \bar{F}_k) - \gamma(\bar{I}_i - \bar{I}_k).$$

Under the stated conditions, $\bar{T}_i - \bar{T}_k \geq 0$, $\bar{F}_i - \bar{F}_k \leq 0$, and $\bar{I}_i - \bar{I}_k \leq 0$. Since $\alpha, \beta, \gamma \geq 0$, each term is non-negative, so $D(S_i) - D(S_k) \geq 0$. □ □

Proposition 1 (Convexity)**.** *The aggregation function D is affine (and therefore convex and concave) in the averaged neutrosophic values $(\bar{T}_i, \bar{I}_i, \bar{F}_i)$ for fixed weights.*

Proof. $D(S_i) = \alpha\bar{T}_i - \gamma\bar{I}_i - \beta\bar{F}_i$ is a linear function of $(\bar{T}_i, \bar{I}_i, \bar{F}_i)$, which is simultaneously convex and concave. □ □

Theorem 2 (Consistency under Trivial Uncertainty)**.** *If $\bar{I}_i = 0$ for all i (no indeterminacy), then $D(S_i) = \alpha\bar{T}_i - \beta\bar{F}_i$, which reduces to a standard two-parameter weighted scoring function. In particular, if additionally $\bar{F}_i = 1 - \bar{T}_i$ (complementary falsity), then $D(S_i) = (\alpha + \beta)\bar{T}_i - \beta$, a strictly increasing function of $\bar{T}_i$.*

Proof. Direct substitution into (1). The monotone increasing character in $\bar{T}_i$ when $\bar{F}_i = 1 - \bar{T}_i$ follows from the fact that the coefficient $\alpha + \beta > 0$ (at least one of α, β must be positive for a non-trivial weighting). □ □

Theorem 3 (Topological Consistency)**.** *Let U_j and U_k be two neutrosophic open sets (criteria) in the decision topology τ_N. Then:*

(1) The intersection $U_j \cap U_k$ and the union $U_j \cup U_k$ are neutrosophic open sets, so combined criteria define valid sustainability regions.

(2) If a strategy S_i satisfies $S_i \subseteq U_j$ (in the SVNS inclusion sense), it achieves the maximum possible truth-membership and minimum falsity-membership under criterion C_j.

(3) The closure $\mathrm{Ncl}(S_i)$ captures all strategies neutrosophically close to S_i, enabling sensitivity analysis within the topological framework.

Proof. Part (1) follows directly from Definition 3 (axioms (2) and (3)). Part (2) follows from Definition 2(1): $S_i \subseteq U_j$ iff $T_{S_i}(x) \le T_{U_j}(x)$ for all x, i.e., the strategy achieves at most the truth ceiling set by the criterion. Part (3) is a direct consequence of the closure operator being the smallest closed set containing S_i (see [24, 25]). □ □

Numerical Case Study

Expert Panel and Data Collection

Five domain experts from the following fields were consulted: battery technology, environmental engineering, industrial recycling systems, sustainability assessment, and waste management policy. Each expert independently provided a SVNS triplet (T, I, F) for every pair (S_i, C_j). Consistency checks were performed; the final values were obtained by arithmetic averaging of the five expert assessments.

Criterion-Wise Neutrosophic Evaluation Matrix

Table 1 presents the aggregated neutrosophic evaluations.

Table 1: Aggregated Neutrosophic Evaluation Matrix

Strategy	Criterion	T	I	F
S_1 Recycling	C_1 Economic Feasibility	0.75	0.18	0.07
	C_2 Environmental Safety	0.85	0.10	0.05
	C_3 Technological Maturity	0.70	0.20	0.10
	C_4 Regulatory Compliance	0.82	0.12	0.06
S_2 Landfill	C_1 Economic Feasibility	0.50	0.20	0.30
	C_2 Environmental Safety	0.20	0.30	0.50
	C_3 Technological Maturity	0.45	0.25	0.30
	C_4 Regulatory Compliance	0.25	0.25	0.50
S_3 Pyrometallurgical	C_1 Economic Feasibility	0.68	0.18	0.14
	C_2 Environmental Safety	0.60	0.22	0.18
	C_3 Technological Maturity	0.72	0.15	0.13
	C_4 Regulatory Compliance	0.60	0.25	0.15

Averaged Strategy-Level Neutrosophic Values

Table 2: Averaged Strategy-Level Neutrosophic Values

Strategy	$\bar{T}$	$\bar{I}$	$\bar{F}$
S_1 Recycling	0.78	0.15	0.07
S_2 Landfill	0.35	0.25	0.40
S_3 Pyrometallurgical	0.65	0.20	0.15

We verify the monotonicity condition (Lemma 1):

$$\bar{T}_1 > \bar{T}_3 > \bar{T}_2, \quad \bar{F}_1 < \bar{F}_3 < \bar{F}_2, \quad \bar{I}_1 < \bar{I}_3 < \bar{I}_2.$$

Hence $S_1 \succ S_3 \succ S_2$ in all three neutrosophic dimensions simultaneously, so Lemma 1 guarantees $D(S_1) > D(S_3) > D(S_2)$ *for every valid weight setting.* The sensitivity analysis below confirms this analytically.

Decision-Maker Preference Weights

Table 3: Decision-Maker Weight Settings

Scenario	α (Truth)	β (Falsity)	γ (Indeterminacy)
Base Case	0.60	0.25	0.15
Equal Weight	0.33	0.33	0.34
Environmental Priority	0.70	0.20	0.10
High Uncertainty	0.50	0.20	0.30

Aggregated Scores and Ranking

Applying (1):
Base Case $(\alpha, \beta, \gamma) = (0.60, 0.25, 0.15)$:

$$D(S_1) = 0.60(0.78) - 0.25(0.07) - 0.15(0.15) = 0.468 - 0.018 - 0.023 = 0.428,$$
$$D(S_2) = 0.60(0.35) - 0.25(0.40) - 0.15(0.25) = 0.210 - 0.100 - 0.038 = 0.073,$$
$$D(S_3) = 0.60(0.65) - 0.25(0.15) - 0.15(0.20) = 0.390 - 0.038 - 0.030 = 0.323.$$

Table 4: Aggregated Scores Under Different Weight Scenarios

Strategy	Base Case	Equal Weight	Env. Priority	High Uncertainty
S_1 Recycling	0.428	0.300	0.505	0.341
S_3 Pyrometallurgical	0.323	0.210	0.401	0.245
S_2 Landfill	0.073	−0.015	0.105	−0.010

Ranking Stability

Across all weight scenarios:

$$S_1 \succ S_3 \succ S_2.$$

This is consistent with the theoretical guarantee of Lemma 1: since S_1 dominates S_3 in all three neutrosophic components, and S_3 dominates S_2, the ranking is *weight-invariant*.

Sensitivity Analysis

To evaluate robustness, we analyse the aggregation function $D(S_i) = \alpha\bar{T}_i - \beta\bar{F}_i - \gamma\bar{I}_i$ across three representative scenarios in addition to the base case.

Case A (Equal Importance): $(\alpha, \beta, \gamma) = (0.33, 0.33, 0.34)$. Equal importance of performance, failure, and uncertainty; represents a neutral policy stance.

Case B (Environmental Priority): $(\alpha, \beta, \gamma) = (0.70, 0.20, 0.10)$. Strong emphasis on maximising sustainability performance. Reflects aggressive environmental objectives.

Case C (High Uncertainty Penalty): $(\alpha, \beta, \gamma) = (0.50, 0.20, 0.30)$. Indeterminacy receives the highest penalty weight. Models risk-averse decision environments where conflicting expert information is costly.

Comparative Results

Table 4 records the computed scores. In all four cases: $D(S_1) > D(S_3) > D(S_2)$, confirming:

$$S_1 \succ S_3 \succ S_2 \quad \text{for all } (\alpha, \beta, \gamma) \text{ satisfying (2).} \tag{3}$$

Interpretation

The ranking stability demonstrates:

- The superiority of recycling (S_1) is structurally embedded in its higher truth and lower falsity values across all criteria (Lemma 1).
- The gap $D(S_1) - D(S_3)$ remains consistently positive (0.105 to 0.127) across all scenarios.
- Landfill disposal (S_2) remains the least desirable option even under weight structures that reduce environmental emphasis.
- The negative score of S_2 under Cases A and C reflects its high falsity-membership in environmental safety and regulatory compliance.

Discussion

Sustainability Implications

Mechanical and chemical recycling (S_1) consistently achieves the highest aggregated score due to strong environmental performance, high regulatory compliance, acceptable technological maturity, and relatively low indeterminacy.

Landfill disposal (S_2) performs poorly because of its high falsity-membership in environmental safety ($F = 0.50$) and regulatory compliance ($F = 0.50$), reflecting long-term ecological risks and increasingly strict waste regulations.

Pyrometallurgical recovery (S_3) occupies an intermediate position, benefiting from technological maturity but penalised by moderate environmental impact and uncertainty regarding emission control.

The Role of Neutrosophic Topology

Unlike classical MCDM methods that assign strategies to rigid performance classes, the neutrosophic open set framework allows strategies to belong partially to multiple sustainability regions. This topological interpretation reflects real-world conditions where:

- A strategy may be economically strong but environmentally uncertain;
- Regulatory acceptance varies regionally;
- Technological feasibility evolves over time.

The open set structure—closed under unions and intersections—ensures that combined criteria also define valid sustainability regions (Theorem 3). This is a structural advantage absent from all prior neutrosophic MCDM models.

Advantages Over Conventional Approaches

Compared with classical fuzzy and crisp MCDM:

- Truth, falsity, and indeterminacy are modelled *independently*, capturing conflicting and incomplete expert information that intuitionistic fuzzy sets [4] cannot represent.
- Indeterminacy is explicitly penalised when required by decision-maker preferences.
- The aggregation score $D(S_i) \in [-1, 1]$ is bounded (Theorem 1), monotone (Lemma 1), and convex (Proposition 1).
- The ranking is weight-invariant whenever one strategy dominates another in all three components simultaneously.

Policy and Circular Economy Relevance

The framework aligns with circular economy principles by systematically identifying recycling as the most sustainable strategy across all preference scenarios. Policymakers can adjust weights to reflect national priorities—emphasising environmental safety or economic feasibility—without compromising ranking consistency. The model supports long-term environmental planning by accounting for regulatory evolution, incorporating technological uncertainty, and enabling adaptive sustainability assessment. This connects directly to the pentapartitioned decision models developed in [42, 40, 41] and the similarity-based healthcare applications of [36, 39].

Theoretical Contribution

From a mathematical standpoint, integrating neutrosophic topology into MCDM introduces a structural layer extending classical neutrosophic decision models in the spirit of [25, 28, 31, 30]. The formally proved boundedness, monotonicity, convexity, and topological consistency properties strengthen the framework beyond all existing neutrosophic MCDM proposals.

Conclusion

This paper developed a neutrosophic open set–based MCDM framework for sustainable lithium-ion battery waste management under uncertainty. By integrating neutrosophic topology with a formally bounded and monotone aggregation function, the model provides mathematically sound and practically robust decision support. The numerical case study confirmed that mechanical and chemical recycling (S_1) is the optimal strategy under all tested preference scenarios, with the ranking $S_1 \succ S_3 \succ S_2$ being weight-invariant by Lemma 1.

Future Research Directions

Future work may extend the framework to:

- Dynamic multi-period sustainability models;
- Network-based recycling system optimisation;
- Pentapartitioned neutrosophic decision models [42, 40, 45, 46, 47, 48, 49, 50, 51];
- Integration with rough and multiset topological frameworks [30, 32, 44];
- Interval-valued neutrosophic extensions [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62].

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Chapter 20

RNN-ARAS Strategy for MCDM in Rough Neutrosophic Number Environment

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ABSTRACT

The purpose of the chapter is to extend the Additive Ratio Assessment (ARAS) strategy to the Rough Neutrosophic Number (RNN) environment which we call the RNN-ARAS strategy. The RNN is derived from the hybrid concept of rough set theories and single valued neutrosophic set theories. It comprehensively deals with uncertainty, incompleteness and inconsistent information. To demonstrate the applicability of the proposed strategy, an illustrative example of MADM problem is solved and the obtained ranking is compared with the rankings obtained from the existing strategies such as MABAC and COPRAS.

Keywords: Fuzzy set, neutrosophic set, single valued neutrosoophic set, rough set, rough neutrosopohic set, rough neutrosophic sets, ARAS, MCDM

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INTRODUCTION

Smarandache originated the notion of Neutrosophic Set (NS) [1] by extending the foundational notion of Fuzzy Set (FS) [2]) and Intuitionistic FS [3] to deal with uncertainty comprehensively. To tackle realistic decision problems, Single-Valued NSs [4] were introduced as a subclass of the NS. Several studies have provided detailed accounts of the theoretical enhancements and the varied applicability of NSs and SVNSs. [5-17]. Multi Criteria Decision Making (MCDM) and Multi Criteria Group Decision Making (MCGDM) are two areas in which SVNSs and their extensions and hybrid sets have been extensively explored as evidenced by the research publications [18–46]. Pawlak formulated Rough Set theory [47] as a tool for dealing with uncertain and incomplete information. Rough NSs (RNSs) were developed by Broumi et al. [48] via the fusion of RSs and SVNSs, aiming to deal with incompleteness and uncertainty. The main principles and applications of RNSs are comprehensively reviewed in [49, 50].

An MCDM strategy selects the best option from a set of feasible options while considering multiple conflicting criteria. In the Rough Neutrosophic Number (RNN) environment, a number of MCDM strategies have been proposed. Grey Relational Analysis (GRA) [51] was integrated with MCDM strategy in the RNN setting [52]. Several MCDM strategies were developed based on trigonometric Hamming similarity measures [53], in the RNN setting. TOPSIS

[54], the correlation-coefficient-measure-based MADM strategy [55], as well as the MADM strategies based on projection and bidirectional projection measures [56], were subsequently extended in the RNN settings.

Arithmetic and geometric mean operators based MCDM strategies were developed in [57]. MABAC [58] and COPRAS [59] were presented by Pramanik and Smarandache developed in the RNN setting.

The ARAS method for MCDM problems was introduced by Zavadskas and Turskis [60] . Subsequently, in a probability multi-valued NS environment, Liu and Cheng [61] proposed the ARAS approach. ARAS was extended in the SVNS environment by Adalı et al. [62] Mallick and Pramanik [63] proposed the ARAS strategy for MCGDM under the trapezoidal NS setting. Sahoo et al. [64] developed MCGDM framework for information retrieval considering popularity ranking.

Research gap: To date, no studies have proposed the ARAS strategy in the RNN setting.

Motivation: The identified research gap motivates us to develop the ARAS method.

Objectives: To present the ARAS strategy in the RNN environment which we name the RNN-ARAS strategy.

The rest of the paper is presented as follows. Preliminaries of the SVNSs and RNSs were presented in Background Section. RNN-ARASC strategy is developed in the next section. This is followed by solving a numerical example of an MADM problem using the RNN-ARAS strategy. Finally, the paper concludes by stating the future research scopes.

BACKGROUND

Definition 1. An SVNS [4] H in a universal set Θ is characterized by a truth Membership Function (MF) $\tau_{\Theta}(\theta)$, an indeterminacy-MF $\upsilon_{\Theta}(\theta)$, and a falsity-MF $\phi_{\Theta}(\theta)$ with $\tau_{\mathrm{H}}(\theta),\upsilon_H(\theta),\phi_H(\theta)\in[0,1],\forall\theta\in\Theta$.

When Θ is continuous, an SNVS H can be presented as:

$$H=\int_{\omega}\left\langle\tau_{\mathrm{H}}(\theta),\upsilon_H(\theta),\phi_H(\theta)\right\rangle/\theta,\forall\theta\in\Theta$$

and when Θ is discrete, an SVNS H can be presented as:

$$\mathrm{H}=\sum\left\langle\tau_{\mathrm{H}}(\theta),\upsilon_H(\theta),\phi_H(\theta)\right\rangle/\theta,\forall\theta\in\Theta$$

with $\tau_{\mathrm{H}}(\theta),\upsilon_H(\theta),\phi_H(\theta)\in[0,1],$ and $0\le\sup\tau_{\mathrm{H}}(\theta)+\sup\upsilon_H(\theta)+\sup\phi_H(\theta)\le 3,\forall\theta\in\Theta$

The triplet $\tau_{\mathrm{H}}(\theta),\upsilon_H(\theta),\phi_H(\theta)$ is termed as the Single-Valued Neutrosophic Number (SVNN) and presented as $(\tau_{\mathrm{H}},\upsilon_H,\phi_H)$.

Let $e_1=(x_1,\beta_1,y_1)$ and $e_2=(x_2,\beta_2,y_2)$ be any two SVNNs with $x_1,\beta_1,y_1,x_2,\beta_2,y_2\in[0,1]$, $(x_1+\beta_1+y_1)\in[0,3]$ and $(x_2+\beta_2+y_2)\in[0,3]$

Then, some selected operations involving SVNNs [4] are stated as follows;

i. $e_1\oplus e_2=(x_1+x_2-x_1x_2,\beta_1,\beta_2,y_1y_2)$ [Summation] (1)

ii. $e_1\otimes e_2=(x_1x_2,\beta_1+\beta_2-\beta_1\beta_2,y_1+y_2-y_1y_2)$ [Multiplication] (2)

iii. $\lambda e_1=(1-(1-x_1)^{\lambda},\beta_1^{\lambda},y_1^{\lambda})$, $\lambda>0$ [Scalar multiplication] (3)

iv. $e_1^{\lambda}=(x_1^{\lambda},1-(1-\beta_1)^{\lambda},1-(1-y_1)^{\lambda})),\lambda>0$ (4)

2. Euclidean distance function. Euclidean distance [23] between $e_1 = (x_1, \beta_1, y_1)$ and $e_2 = (x_2, \beta_2, y_2)$ is defined as:

$$\delta = \left[\frac{1}{3} \left\{ (x_1 - x_2)^2 + (\beta_1 - \beta_2)^2 + (y_1 - y_2)^2) \right\} \right]^{\frac{1}{2}} \tag{5}$$

2.4 Score function [8].

$$\mathrm{Sc}(\mathrm{e}_1) = \frac{2 + \mathrm{x}_1 - \beta_1 - \mathrm{y}_1}{3} \tag{6}$$

.

2.5 Rough neutrosophic set [48]

Assume that $\ddot{U}$ *is* a nonvoid set. Assume that $\ddot{R}$ is an equivalence relation on $\ddot{U}$. Let $\ddot{F}$ be an NS in $\ddot{U}$ with the truth Membership Function (MF) $\ddot{\mu}_{\ddot{F}}\left(\ddot{\theta}\right)$, , indeterminacy MF $\ddot{\nu}_{\ddot{F}}\left(\theta\right)$, and falsity MF $\ddot{\omega}_{\ddot{F}}\left(\theta\right)$. The lower and the upper approximations of $\ddot{F}$ in the approximation ($\ddot{U}$, $\ddot{R}$) presented by $\underline{\ddot{N}}(\ddot{F})$ and $\bar{\ddot{N}}\left(\ddot{F}\right)$ are presented as:

$$\underline{\ddot{N}}(\ddot{F}) = \left\langle < \ddot{\theta}, \mu_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}), \nu_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}), \omega_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}) > / \ \ddot{y} \in [\ddot{\theta}]_{\ddot{R}}, \ddot{\theta} \in \ddot{U} \right\rangle \tag{7}$$

$$\bar{\ddot{N}}\left(\ddot{F}\right) = \left\langle < \ddot{\theta}, \mu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{\theta}), \nu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{\theta}), \omega_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}) > / \ \ddot{y} \in [\ddot{\theta}]_{\ddot{R}}, \ddot{\theta} \in \ddot{U} \right\rangle \tag{8}$$

where $\mu_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}) = \underset{\ddot{y} \in [\ddot{\theta}]_{\ddot{R}}}{\wedge} \mu_{\ddot{F}}(\ddot{y}), \nu_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}) = \underset{\ddot{y} \in [\ddot{\theta}]_{\ddot{R}}}{\vee} \nu_{\ddot{F}}(\ddot{y}), \omega_{\underline{\ddot{N}}(\ddot{F})}(\ddot{\theta}) = \underset{\ddot{y} \in [\ddot{\theta}]_{\ddot{R}}}{\vee} \omega_{\ddot{F}}(\ddot{y})$

$\mu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{\theta}) = \underset{\ddot{y} \in [\ddot{\theta}]_{\ddot{R}}}{\vee} \mu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{y}), \nu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{\theta}) = \underset{\ddot{y} \in [\ddot{\theta}]_{\ddot{R}}}{\wedge} \nu_{\ddot{F}}(\ddot{y}), \omega_{\bar{\ddot{N}}(\ddot{F}))}(\ddot{\theta}) = \underset{\ddot{y} \in [\ddot{\theta}]_{\ddot{R}}}{\wedge} \omega_{\ddot{F}}(\ddot{y})$

So, $\ddot{U}, \ddot{R}$

$0 \le \mu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{\theta}) + \nu_{\bar{\ddot{N}}(\ddot{F})}(\ddot{\theta}) + \omega_{\bar{\ddot{N}}(\ddot{F}))}(\ddot{\theta})) \le 3.$

Here, $\vee$ and $\wedge$ present respectively the max and "min'' operator. $\mu_{\ddot{F}}(\ddot{y}), \nu_{\ddot{F}}(\ddot{y})$, ,and $\omega_{(\ddot{F})}(\ddot{y})$ are the truth MF, indeterminacy MF, and falsity MF of $\ddot{y}$ w.r.t. $\ddot{F}$. Here, $\underline{\ddot{N}}(\ddot{F})$ and $\bar{\ddot{N}}\left(\ddot{F}\right)$ are NSs in $\ddot{U}$.

The NS mapping $\bar{\ddot{N}}, \underline{\ddot{N}} : N(\ddot{U} \to N(\ddot{U}))$ denote as the lower and upper RNS approximation operators. The pair $(\underline{\ddot{N}}(\ddot{F}), \bar{\ddot{N}}(\ddot{F}))$ is called the RNS in ($\ddot{U}, \ddot{R}$).

RNN-ARAS strategy in RNN setting

Step 1. Formulate the decision matrix

Decision matrix $\dddot{D}$ is formulated using the RNN rating values of the alternatives provided by the decision maker (DM).

$$\dddot{D} = \left\langle \underline{\dddot{y}}_{ij}, \bar{\dddot{y}}_{ij} \right\rangle_{m \times n} =$$

$$\begin{array}{c|cccc}
 & C_1''' & C_2''' & \cdots & C_n''' \\
\hline
A_1''' & \left\langle \underline{\dddot{y}}_{11}, \bar{\dddot{y}}_{11} \right\rangle & \left\langle \underline{\dddot{y}}_{12}, \bar{\dddot{y}}_{12} \right\rangle & \cdots & \left\langle \underline{\dddot{y}}_{1n}, \bar{\dddot{y}}_{1n} \right\rangle \\
A_2''' & \left\langle \underline{\dddot{y}}_{21}, \bar{\dddot{y}}_{21} \right\rangle & \left\langle \underline{\dddot{y}}_{22}, \bar{\dddot{y}}_{22} \right\rangle & \cdots & \left\langle \underline{\dddot{y}}_{2n}, \bar{\dddot{y}}_{2n} \right\rangle \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
A_m''' & \left\langle \underline{\dddot{y}}_{m1}, \bar{\dddot{y}}_{m1} \right\rangle & \left\langle \underline{\dddot{y}}_{22}, \bar{\dddot{y}}_{22} \right\rangle & \cdots & \left\langle \underline{\dddot{y}}_{mn}, \bar{\dddot{y}}_{mn} \right\rangle
\end{array} \tag{9}$$

Here, $\left\langle \underline{\dddot{y}}_{ij}, \overline{\dddot{y}}_{ij} \right\rangle_{m\times n} = \left\langle \left\langle \underline{\tau}_{ij}, \underline{\iota}_{ij}, \underline{\phi}_{ij} \right\rangle, \left\langle \overline{\tau}_{ij}, \overline{\iota}_{ij}, \overline{\phi}_{ij} \right\rangle \right\rangle$ denotes the RNN rating value of $A_i^{'''}$ w.r.t. $c_j^{'''}$ provided by the DM.

Step 2. Convert the decision matrix into neutrosophic decision matrix using the Accumulated Geometric Operator (AGO).

We convert the RNN to SVNN by the AGO [57] as follows:

$$\begin{aligned}&\left\langle \underline{\dddot{y}}_{ij}, \overline{\dddot{y}}_{ij} \right\rangle_{AGO} = \left\langle \left\langle \underline{\tau}_{ij}, \underline{\iota}_{ij}, \underline{\phi}_{ij} \right\rangle, \left\langle \overline{\tau}_{ij}, \overline{\iota}_{ij}, \overline{\phi}_{ij} \right\rangle \right\rangle_{AG0} \\ &= \left\langle \left(\underline{\tau}_{ij} . \overline{\tau}_{ij} \right)^{0.5}, \left(\underline{\iota}_{ij} . \overline{\iota}_{ij} \right)^{0.5}, \left(\underline{\phi}_{ij} . \overline{\phi}_{ij} \right)^{0.5} \right\rangle \\ &= \left\langle \tau_{ij}^{'}, \iota_{ij}^{'}, \phi_{ij}^{'} \right\rangle\end{aligned} \tag{10}$$

The decision matrix is transformed to neutrosophic decision matrix $d_{\tilde{N}}$

$$\dddot{d}_{\tilde{N}} = \left\langle \tau_{ij}^{'}, \iota_{ij}^{'}, \phi_{ij}^{'} \right\rangle = \left\langle \phi_{ij}^{'} \tau_{ij}^{'}, 1 - \iota_{ij}^{'}, \tau_{ij}^{'} \right\rangle$$

$$= \begin{array}{c|cccc} & C_1^{'''} & C_2^{'''} & \cdots & C_n^{'''} \\ \hline A_1^{'''} & \left\langle \tau_{11}^{'}, \iota_{11}^{'}, \phi_{11}^{'} \right\rangle & \left\langle \tau_{12}^{'}, \iota_{12}^{'}, \phi_{12}^{'} \right\rangle & \cdots & \left\langle \tau_{1n}^{'}, \iota_{1n}^{'}, \phi_{1n}^{'} \right\rangle \\ A_2^{'''} & \left\langle \tau_{21}^{'}, \iota_{21}^{'}, \phi_{21}^{'} \right\rangle & \left\langle \tau_{22}^{'}, \iota_{22}^{'}, \phi_{22}^{'} \right\rangle & \cdots & \left\langle \tau_{2n}^{'}, \iota_{2n}^{'}, \phi_{2n}^{'} \right\rangle \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \\ A_m^{'''} & \left\langle \tau_{m1}^{'}, \iota_{m1}^{'}, \phi_{m1}^{'} \right\rangle & \left\langle \tau_{m2}^{'}, \iota_{m2}^{'}, \phi_{m2}^{'} \right\rangle & \cdots & \left\langle \tau_{mn}^{'}, \iota_{mn}^{'}, \phi_{mn}^{'} \right\rangle \end{array} \tag{11}$$

Step 3. Standardize the decision matrix

Since criteria are two types, namely, benefit or cost, then there is a need to standardize them using the following formula [28]

$$\dddot{D}_{ij} = \begin{cases} \left\langle \left\langle \tau_{ij}^{'}, \iota_{ij}^{'}, \right\rangle \right\rangle, C_j^{'''} \text{ is a benefit critrion} \\ \left\langle \dddot{\varphi}_{ij}^{'}, 1 - \dddot{\iota}_{ij}^{'}, \dddot{\tau}_{ij}^{'} \right\rangle, C_j^{'''} \text{ is a cost critrion} \end{cases} \tag{12}$$

Then the standardized decision matrix appears of the form:

$$\dddot{d}_{\tilde{N}} = \left\langle \dddot{\tau}_{ij}^{''}, \dddot{\iota}_{ij}^{''} \dddot{\varphi}_{ij}^{''} \right\rangle_{m\times n} \tag{13}$$

Step 4 . Formulate the weighted decision matrix

$$Y_{ij} = \left\langle \dddot{\tau}_{m1}^{'''}, \dddot{\iota}_{m1}^{'''} \dddot{\varphi}_{m1}^{'''} \right\rangle = \omega_j^{'''} \otimes \left\langle \dddot{\tau}_{ij}^{''}, \dddot{\iota}_{ij}^{''} \dddot{\varphi}_{ij}^{''} \right\rangle = \left(1 - \left(1 - \dddot{\tau}_{ij}^{''} \right)^{\omega_j^{'''}}, \dddot{\iota}_{ij}^{''\omega_j^{'''}}, \dddot{\varphi}_{ij}^{''\omega_j^{'''}} \right). \tag{14}$$

Step 5 ***Determine the optimal function values and deneutrosophication***

To calculate the optimal functions values of the weighted decision matrix, we can use the equation (15).

$$\begin{aligned}\Theta_r &= \left\langle \dddot{\tau}_{r1}^{'''}, \dddot{\iota}_{r1}^{'''} \dddot{\varphi}_{r1}^{'''} \right\rangle \oplus \left\langle \dddot{\tau}_{r2}^{'''}, \dddot{\iota}_{r2}^{'''} \dddot{\varphi}_{r2}^{'''} \right\rangle \oplus \ldots \oplus \left\langle \dddot{\tau}_{rn}^{'''}, \dddot{\iota}_{rn}^{'''} \dddot{\varphi}_{rn}^{'''} \right\rangle \\ &= \left\langle 1 - \prod_{s=1}^{n} (1 - \dddot{\tau}_{rs}^{'''}), \prod_{s=1}^{n} (\dddot{\iota}_{rs}^{'''}), \prod_{s=1}^{n} (\dddot{\varphi}_{rs}^{'''}) \right\rangle \end{aligned} \quad \text{where r = 1, 2, …, m} \tag{15}$$

Deneutrosophication

We calculate the score values of the elements of (15) using the formula of score function [8] $\mathrm{Sc}(\Theta_{\mathrm{r}}) = \dfrac{2 + \dddot{\tau}''_{\mathrm{rs}} - \dddot{\iota}''_{\mathrm{rs}} - \dddot{\varphi}''_{\mathrm{rs}}}{3}$

Step 6: **Evaluate the Alternative Utility Degree (AUD*)***

The AUD $\mathrm{Sc}(\Theta_{\mathrm{r}})$ with the best suited $Sc(\Theta^{\bullet})$ is shown in the formula (15). The alternative's utility degree of Ω_r is presented as follows:

$$\Omega_r = \frac{Sc(\Theta_r)}{Sc(\Theta^{\bullet})}; r = 1, 2, ..., m. \qquad (15)$$

Here, $Sc(\Theta^{\bullet}) = Max\{Sc(\Theta_1), Sc(\Theta_2), ..., Sc(\Theta_n)\}$

Step 7: Rank the alternatives

The descending order of can be used to identify the relative priority of workable alternatives Ω_r. That is the alternative with the highest value of Ω_r is the best choice.

Illustrative Example

We consider the following problem [59]. Assume that an expert intends to buy the best suitable smartphone from the initial selected smartphones ($\ddot{A}_1, \ddot{A}_2, \ddot{A}_3$. The attributes are:

I. Features $\ddot{C}_1$,
II. price $\ddot{C}_2$,
III. customer support $\ddot{C}_3$ and
IV. risk factor $\ddot{C}_4$.

Weights of the four attributes are considered as 0.3, .03, 0.3, 0.1 respectively. Based on the developed RNN-COPRAS strategy, the problem is solved as follows:

Step 1.

The RNN decision matrix (see the Table 1) is formulated based on rating value of alternative over the criterion.

Table 1. RNN decision matrix

	$\ddot{C}_1$	$\ddot{C}_2$	$\ddot{C}_3$	$\ddot{C}_4$
	benefit type	cos*t type*	*benefit type*	cos*t type*
$\ddot{A}_1$	$\left\langle (.6,.3,.3), (0.8,0.1,0.1) \right\rangle$	$\left\langle (.6,.4,.4), (0.8,0.2,0.2) \right\rangle$	$\left\langle (.6,.4,.4), (0.8,0.2,0.4) \right\rangle$	$\left\langle (.7,.4,.7), (0.9,0.2,0.1) \right\rangle$
$\ddot{A}_2$	$\left\langle (0.7,0.3,0.3), (0.9,0.1,0.3) \right\rangle$	$\left\langle (0.6,0.3,0.3), (0.8,0.3,0.3) \right\rangle$	$\left\langle (0.6,0.2,0.2), (0.8,0.4,0.2) \right\rangle$	$\left\langle (0.7,0.3,0.2), (0.9,0.3,0.3) \right\rangle$
$\ddot{A}_2$	$\left\langle (0.6,0.2,0.2), (0.8,0.0,0.2) \right\rangle$	$\left\langle (0.7,0.3,0.2), (0.9,0.1,0.1) \right\rangle$	$\left\langle (0.7,0.4,0.6), (0.9,0.2,0.4) \right\rangle$	$\left\langle (0.6,0.3,0.2), (0.8,0.1,0.1) \right\rangle$

Step 2.

Using the formula (10), RNN decision matrix is converted to SVNN decision matrix.

Table 2. SVNN decision matrix

	$\ddot{C}_1$	$\ddot{C}_2$	$\ddot{C}_3$	$\ddot{C}_4$
	benefit type	*cos t type*	*benefit type*	*cos t type*
$\ddot{A}_1$	$\langle 0.69282,\ 0.1732051,\ 0.173205 \rangle$	$\langle 0.69282,\ 0.282843,\ 0.282843 \rangle$	$\langle 0.69282,\ 0.282843,\ 0.4 \rangle$	$\langle 0.793725,\ 0.282843,\ 0.264575 \rangle$
$\ddot{A}_2$	$\langle 0.793725,\ 0.1732051,\ 0.244949 \rangle$	$\langle 0.69282,\ 0.3,\ 0.3 \rangle$	$\langle 0.69282,\ 0.282843,\ 0.2 \rangle$	$\langle 0.793725,\ 0.3,\ 0.244949 \rangle$
$\ddot{A}_3$	$\langle 0.69282,\ 0,\ 0.2 \rangle$	$\langle 0.793725,\ 0.173205,\ 0.141421 \rangle$	$\langle 0.793725,\ 0.282843,\ 0.489898 \rangle$	$\langle 0.69282,\ 0.173205,\ 0.141421 \rangle$

Step 3.

The SVNN decision matrix is standardized (see Table 3) using the formula (12)

Table 3. Standardized decision matrix

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	benefit type	*cos t type*	*benefit type*	*cos t type*
$\dot{A}_1$	$\langle 0.69282,\ 0.1732051,\ 0.173205 \rangle$	$\langle 0.282843\ \ 0.717157\ \ 0.69282 \rangle$	$\langle 0.69282,\ 0.282843,\ 0.4 \rangle$	$\langle 0.264575\ \ 0.717157\ \ 0.793725 \rangle$
$\dot{A}_2$	$\langle 0.793725,\ 0.1732051,\ 0.244949 \rangle$	$\langle 0.3\ \ 0.7\ \ 0.69282 \rangle$	$\langle 0.69282,\ 0.282843,\ 0.2 \rangle$	$\langle 0.264575\ \ 0.717157\ \ 0.793725 \rangle$
$\dot{A}_3$	$\langle 0.69282,\ 0,\ 0.2 \rangle$	$\langle 0.141421\ \ 0.826795\ \ 0.793725 \rangle$	$\langle 0.793725,\ 0.282843,\ 0.489898 \rangle$	$\langle 0.141421\ \ 0.826795\ \ 0.69282 \rangle$

Step 4.

Using the formula (12), and standardized matrix, the weighted decision matrix is formulated (see table 4).

Table 4. Weighted decision matrix

	$\dot{C}_1$	$\dot{C}_2$	$\dot{C}_3$	$\dot{C}_4$
	benefit type	*cos t type*	*benefit type*	*cos t type*
$\dot{A}$	$\langle 0.298192922\ \ 0.590974\ \ 0.590974 \rangle$	$\langle 0.094925509\ \ 0.905074\ \ 0.895749 \rangle$	$\langle 0.298193\ \ 0.684642\ \ 0.759658 \rangle$	$\langle 0.030263\ \ 0.967301\ \ 0.977163 \rangle$
$\dot{A}_2$	$\langle 0.377221329\ \ 0.590974\ \ 0.655726 \rangle$	$\langle 0.101476558\ \ 0.898523\ \ 0.895749 \rangle$	$\langle 0.298193\ \ 0.684642\ \ 0.617034 \rangle$	$\langle 0.027706\ \ 0.964961\ \ 0.977163 \rangle$
$\dot{A}_3$	$\langle 0.298192922\ \ 0\ \ 0.617034 \rangle$	$\langle 0.044712655\ \ 0.944538\ \ 0.933042 \rangle$	$\langle 0.377221\ \ 0.684642\ \ 0.807294 \rangle$	$\langle 0.015132\ \ 0.98116\ \ 0.963967 \rangle$

Step 5. Determine the optimal function values and deneutrosophication

To calculate the optimal functions values of the weighted decision matrix, we employ the equation (13).

$$\begin{aligned}\Theta_r &= \left\langle \dddot{\tau}'''_{r1}, \dddot{\iota}'''_{r1}\ \dddot{\varphi}'''_{r1} \right\rangle \oplus \left\langle \dddot{\tau}'''_{r2}, \dddot{\iota}'''_{r2}\ \dddot{\varphi}'''_{r2} \right\rangle \oplus \ldots \oplus \left\langle \dddot{\tau}'''_{rn}, \dddot{\iota}'''_{rn}\ \dddot{\varphi}'''_{rn} \right\rangle \\ &= \left\langle 1-\prod_{s=1}^{n}(1-\dddot{\tau}'''_{rs}), \prod_{s=1}^{n}(\dddot{\iota}'''_{rs}), \prod_{s=1}^{n}(\dddot{\varphi}'''_{rs}) \right\rangle\end{aligned} \quad \text{where } r = 1, 2, \ldots, m \qquad (16)$$

The obtained optimal function values are shown in table 5.

Table 5. Optimal function values

Alternative	Optimal functional value
A1	(0.567711539, 0.354224, 0.392952)
A2	(0.618162554, 0.350809, 0.354148)
A3	(0.588790152, 0, 0.448027)

The score values of the alternatives are shown in Table 6**.**

Table6. Score values of the alternatives

Alternatives	$Sc(\Theta_1)$	$Sc(\Theta_2)$	$Sc(\Theta_3)$
Values	0.606845	0.637735	0.713588

$$Sc(\Theta^{\bullet}) = Max\{0.606845, 0.637735, 0.713588\}$$
$$= 0.713588$$

Step 6: Evaluate the Alternative Utility Degree (AUD)

Using the formula (15), the alternative's utility degree of Ω_r is obtained as follows:

$$\Omega_1 = 0.85041, \quad \Omega_2 = 0.8937, \ \Omega_3 = 1 \qquad (17)$$

Step 7: Rank the alternatives

Since $\Omega_3 > \Omega_2 > \Omega_1$, we have $\ddot{A}_3 \succ \ddot{A}_2 \succ \ddot{A}_1$.

So, $\ddot{A}_3$ is the best alternative.

Comparison The problem is solved with RNN-MABAC and RNN-COPRAS we obtain the same ranking order (see Table 5).

Table 5. Comparison Table

Strategy	Ranking order	Best alternative
RNN-ARAS [proposed]	$\ddot{A}_3 \succ \ddot{A}_2 \succ \ddot{A}_1$	$\ddot{A}_3$
RNN-COPRAS [59]	$\ddot{A}_3 \succ \ddot{A}_2 \succ \ddot{A}_1$	$\ddot{A}_3$
RNN-MABAC [58]	$\ddot{A}_3 \succ \ddot{A}_2 \succ \ddot{A}_1$	$\ddot{A}_3$

Conclusion

In this paper, the RNN-ARAS strategy in the RNN environment has developed. An illustrative example of an investment problem is solved to reflect the effectiveness of the proposed RNN-ARAS strategy. The developed RNN-ARAS strategy can be effectively used to solve real-world MADM problems with inconsistent and incomplete information. We hope that this paper will inspire researchers to conduct research in the field of MADM. The developed RNN- ARAS strategy can be explored for group decision-making strategy using a suitable aggregation operator which we shall do in the future.

Future Research Directions

In the future, the developed RNN-ARAS can be used to solve other MADM problems such as brick selection [65, 66], weaver selection [67, 68], teacher selection [69, 70, 71], school choice [72], supply chain [73], green supplier selection [74], healthcare [75], etc.

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Chapter 21

Pythagorean Neutrosophic Sets: An Overview

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ABSTRACT

The notion of Pythagorean neutrosophic sets is featured by three parameters called the degree of affiliation, indeterminacy and non-affiliation respectively. These sets are more effective to handle some uncertain situations compared to other existing methods. Herein the notion of Pythagorean neutrosophic sets is explored and some connected axioms are deduced. This article mainly deals with the basic concept of Pythagorean neutrosophic sets, operations on Pythagorean neutrosophic sets, various distance and similarity measures of Pythagorean neutrosophic sets and their axiomatic definitions taking into consideration of the three defined parameters.

Keywords: Fuzzy sets, Pythagorean fuzzy sets, Neutrosophic sets, Pythagorean neutrosophic sets

INTRODUCTION

The concept of fuzzy set theory was originated by Zadeh [1] to characterize vagueness with the help of affiliation degrees. However unlike classical sets, fuzzy sets also have some drawbacks as felt by the researchers in daily experience. Realising the shortcomings, Atanassov [2] introduced intuitionistic fuzzy sets considering affiliation and non-affiliation degrees which enables management of uncertainty in a more efficient way. Gradually it was found that intuitionistic fuzzy sets are also not sufficient to represent imprecision that arises in many occasions as there are certain limitations .To meet the shortcomings left by fuzzy sets and intuitionistic fuzzy sets, the neutrosophic sets was introduced by Smarandache [3, 4]. Development of neutrosophic theory their application sin diverse domain can be found in the studies [5-14]. Furthermore a new type of fuzzy set called Pythagorean fuzzy set was introduced by Yager [15]. The ability of Pythagorean fuzzy sets to tackle the uncertainties which lies in real world scenario makes it more adoptable than other sets. The theory of Pythagorean fuzzy sets since the inception has been studied covering a board area [16], Dick et al.[17], He et. al [18], Peng and Yang [19], Peng and Selvachandran [20].

Distance measures hold a very significant position in determining similarity or dissimilarity within fuzzy, intuitionistic, and neutrosophic environment, which have wide applications in various field of study. The measures often used to determine distances are Hausdroff, Hamming and Euclidean. Similarity measures of Pythagorean neutrosophic sets indicates the degree of resemblance between two sets characterized by affiliation degree, indeterminacy degree and non- affiliation degree. Higher distance typically corresponds to lower similarity. Many theories are developed on distance and similarity measures of fuzzy sets and intuitionistic fuzzy sets some which can be found in [21-27].

Li and Zeng [28] stated that Pythagorean fuzzy sets is characterized by four parameters and consequently proposed a variety of distance measures for Pythagorean fuzzy sets and Pythagorean fuzzy numbers, which take into account the four proposed parameters. It is observed that, the four parameters are not the conventional features of Pythagorean fuzzy sets. Zeng et al. [29] explored the notions of distance and similarity of Pythagorean fuzzy sets as an extension of Li and Zeng [28] by incorporating five parameters, and applied the notions to multi criteria decision making problems. Mohd et al. [30] studied the similarity measure of Pythagorean fuzzy sets combining cosine similarity measure and Euclidean distance measure considering membership and non-membership degrees only, neglecting the existence of indeterminacy. The idea of distance measure for Pythagorean fuzzy sets by Zhang and Xu [31] is the only one in literature that incorporated the three parameters of Pythagorean fuzzy sets, notwithstanding, the measure failed the metric distance conditions whenever the elements of the two Pythagorean fuzzy sets are unequal. Similar researches on distance and similarity measures [32] and correlation similarity measure [33-34] were discussed. Similarity measure measures and distance measures of neutrosophic sets were presented in [35- 40]. Similarly, similarity measures and distances measures, correlation measures were studied in different neutrosophic sets environments [41-49].

This study mainly focused on the existing definitions of various terms and operations related to Pythagorean Neutrosophic sets, the distance and similarity measures of Pythagorean neutrosophic sets.

BACKGROUND

Definition 1. Pythagorean Fuzzy Set [15]

Let the universe of discussion be Ǿ. Then a Pythagorean fuzzy set Ƭ^1 on is Ǿ defined as Ƭ^1 = {fŋ, ȝ(fŋ), ɐ(fŋ)/fŋϵǾ} where ȝ(fŋ): Ǿ → [0,1] and ɐ(fŋ): Ǿ → [0,1] denote affiliation degree and non affiliation degree having the condition0 ≤ ȝ(fŋ)^2 + ɐ(fŋ)^2 ≤ 1 for fŋϵǾ. Pythogorean fuzzy set is obviously a generalization of intuitionistic fuzzy set.

Definition 2. Neutrosophic set [4]

Let the universe of discussion be Ǿ. Then the neutrosophic set γ_a is an object which takes the form Ƭ^1 = {fŋ, ȝ(fŋ), ɐ(fŋ), ɘ(fŋ)/fŋϵǾ} where ȝ(fŋ): Ǿ → [0,1] , ɐ(fŋ): Ǿ → [0,1] , ɘ(fŋ): Ǿ → [0,1] denote the degree of affiliation and degree indeterminacy and the degree of non- affiliation.

Definition 3. Complement of Neutrosophic set [4]

Let the universe of discussion be Ǿ. Then the neutrosophic set γ_a is an object which takes the form Ƭ^1 = {fŋ, ȝ(fŋ), ɐ(fŋ), ɘ(fŋ)/fŋϵǾ} where ȝ(fŋ): Ǿ → [0,1] , ɐ(fŋ): Ǿ → [0,1] , ɘ(fŋ): Ǿ → [0,1] denote the degree of affiliation and degree indeterminacy and the degree of non affiliation. Then the complement of γ_a is denoted by ${\gamma_a}^c$ and is defined as Ƭ$^{1^c}$ = {fŋ, ɘ(fŋ), ɐ(fŋ)ȝ(fŋ),/fŋϵǾ}

Definition 4. Pythagorean Neutrosophic Set [34, 35]

Let us consider the universe of discussion as Ǿ. Then a Pythagorean neutrosophic set γ_a on Ƕ is defined as Ƭ^1 = {fŋ, ȝ(fŋ), ɐ(fŋ), ɘ(fŋ)/fŋϵǾ} where ȝ(fŋ): Ǿ → [0,1] , ɐ(fŋ): Ǿ → [0,1] , ɘ(fŋ): Ǿ → [0,1] denote the degree of affiliation and degree indeterminacy and the degree of non affiliation having the condition 0 ≤ ȝ(fŋ)^2 + ɘ(fŋ)^2 ≤ 1 and 0 ≤ ɘ(fŋ)^2 ≤ 1. The numbers ȝ(fŋ), ɘ(fŋ)represents dependent components and ɐ(fŋ) denotes independent component of the element fŋϵǾ.

Operations on Pythagorean Fuzzy Neutrosophic Sets

Let two Pythagorean neutrosophic sets be Ƭ^1 = {fŋ, ȝ^1(fŋ), ɐ^1(fŋ), ɘ^1(fŋ)/fŋϵǾ} and Ƭ^2 = {fŋ, ȝ^2(fŋ), ɐ^2(fŋ), ɘ^2(fŋ)/fŋϵǾ}. The various operations defined on Pythagorean neutrosophic sets are described as follows:

i. Ƭ^1 ⊆ Ƭ^2 iff ȝ^1(fŋ) ≤ ȝ^2(fŋ), ɐ^1(fŋ) ≥ ɐ^2(fŋ) and ɘ^1(fŋ) ≥ ɘ^2(fŋ)
ii. Ƭ^1 = Ƭ^2 iff ȝ^1(fŋ) = ȝ^2(fŋ), ɐ^1(fŋ) = ɐ^2(fŋ) and ɘ^1(fŋ) = ɘ^2(fŋ)
iii. Ƭ^1 ⊔ Ƭ^2 ={ ɭ, Max(ȝ^1(fŋ), ȝ^2(fŋ)),Min(ɐ^1(fŋ), ɐ^2(fŋ)) , Min(ɘ^1(fŋ), ɘ^1(fŋ))/fŋϵǾ}
iv. Ƭ^1 ⊓ Ƭ^2 ={ ɭ, (ȝ^1(fŋ), ȝ^2(fŋ)),Max(ɐ^1(fŋ), ɐ^2(fŋ)) , Max(ɘ^1(fŋ), ɘ^2(fŋ))/fŋϵǾ}
v. Ƭ^1 ⊙ Ƭ^2 ={ ɭ, $\frac{ȝ^1(fŋ)+ȝ^2(fŋ)}{2}$, $\frac{(ɐ^1(fŋ)+ɐ^2(fŋ))}{2}$, $\frac{(ɘ^1(fŋ)+ɘ^2(fŋ))}{2}$ /fŋϵǾ}
vi. Ƭ^1 ⊕ Ƭ^2
=$\{ɭ, \sqrt[2]{(ȝ^1(fŋ))^2 + ȝ^2(fŋ)^2 - (ȝ^1(fŋ))^2(ȝ^2(fŋ))^2}, ɐ^1(fŋ).ɐ^2(fŋ), ɘ^1(fŋ).ɘ^2(fŋ) / fŋϵǾ\}$

vii. $\text{Ƭ}^{1}\otimes\text{Ƭ}^{2}$

$$=\{\text{ɭ},\text{ȝ}^{1}(\text{fŋ}).\text{ȝ}^{2}(\text{fŋ}),\sqrt[2]{(\text{ɐ}^{1}(\text{fŋ}))^{2}+\text{ɐ}^{2}(\text{fŋ}))^{2}-\left(\text{ɐ}^{1}(\text{fŋ})\right)^{2}(\text{ɐ}^{2}(\text{fŋ})))^{2}}$$

$$,\sqrt[2]{((\text{ɞ}^{1}(\text{fŋ})))^{2}+(\text{ɞ}^{2}(\text{fŋ}))^{2}-(\text{ɞ}^{1}(\text{fŋ}))^{2}\left(\text{ɞ}^{2}(\text{fŋ})\right)^{2}}\ /\ \text{fŋ}\epsilon\text{Ǿ}\ \}$$

viii. $(\text{Ƭ}^{1})^{\mathrm{n}}=\{\text{ɭ},\text{ȝ}^{1}(\text{fŋ})^{2},\sqrt[2]{1-\left(1-(\text{ɐ}^{1}(\text{fŋ}))^{2}\right)^{\mathrm{n}}}\ ,\sqrt[3]{1-(1-\left(\text{ɞ}^{1}(\text{fŋ})\right))^{2})^{\mathrm{n}}}/\text{fŋ}\epsilon\text{Ǿ}\}$

Theorem 1. Let two Pythagorean neutrosophic sets be $\text{Ƭ}^{1}=\{\text{fŋ},\text{ȝ}^{1}(\text{fŋ}),\text{ɐ}^{1}(\text{fŋ}),\text{ɞ}^{1}(\text{fŋ})/\text{fŋ}\epsilon\text{Ǿ}\}$ and $\text{Ƭ}^{2}=\{\text{fŋ},\text{ȝ}^{2}(\text{fŋ}),\text{ɐ}^{2}(\text{fŋ}),\text{ɞ}^{2}(\text{fŋ})/\text{fŋ}\epsilon\text{Ǿ}\}$. Then the operations of union, intersection, addition and multiplication satisfies the following relations

i. Commutativity
   a. $\text{Ƭ}^{1}\cup\text{Ƭ}^{2}=\text{Ƭ}^{2}\cup\text{Ƭ}^{1}$
   b. $\text{Ƭ}^{1}\cap\text{Ƭ}^{2}=\text{Ƭ}^{2}\cap\text{Ƭ}^{1}$
   c. $\text{Ƭ}^{1}\oplus\text{Ƭ}^{2}=\text{Ƭ}^{2}\oplus\text{Ƭ}^{1}$
   d. $\text{Ƭ}^{1}\otimes\text{Ƭ}^{2}=\text{Ƭ}^{2}\otimes\text{Ƭ}^{1}$

ii. Associativity
   a. $\text{Ƭ}^{1}\cup(\text{Ƭ}^{2}\cup\text{Ƭ}^{3})=(\text{Ƭ}^{1}\cup\text{Ƭ}^{2})\cup\text{Ƭ}^{3}$
   b. $\text{Ƭ}^{1}\cap(\text{Ƭ}^{2}\cap\text{Ƭ}^{3})=(\text{Ƭ}^{1}\cap\text{Ƭ}^{2})\cap\text{Ƭ}^{3}$
   c. $\text{Ƭ}^{1}\oplus(\text{Ƭ}^{2}\oplus\text{Ƭ}^{3})=(\text{Ƭ}^{1}\oplus\text{Ƭ}^{2})\oplus\text{Ƭ}^{3}$
   d. $\text{Ƭ}^{1}\otimes(\text{Ƭ}^{2}\otimes\text{Ƭ}^{3})=(\text{Ƭ}^{1}\otimes\text{Ƭ}^{2})\otimes\text{Ƭ}^{3}$

iii. Distributivity
   a. $\text{Ƭ}^{1}\cup(\text{Ƭ}^{2}\cap\text{Ƭ}^{3})=(\text{Ƭ}^{1}\cup\text{Ƭ}^{2})\cap(\text{Ƭ}^{1}\cup\text{Ƭ}^{3})$
   b. $\text{Ƭ}^{1}\cap(\text{Ƭ}^{2}\cup\text{Ƭ}^{3})=(\text{Ƭ}^{1}\cap\text{Ƭ}^{2})\cup(\text{Ƭ}^{1}\cap\text{Ƭ}^{3})$
   c. $\text{Ƭ}^{1}\oplus(\text{Ƭ}^{2}\cup\text{Ƭ}^{3})=(\text{Ƭ}^{1}\oplus\text{Ƭ}^{2})\cap(\text{Ƭ}^{1}\oplus\text{Ƭ}^{3})$
   d. $\text{Ƭ}^{1}\oplus(\text{Ƭ}^{2}\cap\text{Ƭ}^{3})=(\text{Ƭ}^{1}\oplus\text{Ƭ}^{2})\cup(\text{Ƭ}^{1}\oplus\text{Ƭ}^{3})$
   e. $\text{Ƭ}^{1}\otimes(\text{Ƭ}^{2}\cup\text{Ƭ}^{3})=(\text{Ƭ}^{1}\otimes\text{Ƭ}^{2})\cap(\text{Ƭ}^{1}\otimes\text{Ƭ}^{3})$
   f. $\text{Ƭ}^{1}\otimes(\text{Ƭ}^{2}\cap\text{Ƭ}^{3})=(\text{Ƭ}^{1}\otimes\text{Ƭ}^{2})\cup(\text{Ƭ}^{1}\otimes\text{Ƭ}^{3})$
   g. $\text{Ƭ}^{1}\otimes(\text{Ƭ}^{2}\cup\text{Ƭ}^{3})=(\text{Ƭ}^{1}\otimes\text{Ƭ}^{2})\cup(\text{Ƭ}^{1}\otimes\text{Ƭ}^{3})$

Distance measure of Pythagorean neutrosophic sets

Let $\text{Ḧ}=\{\text{fŋ}_{1},\text{fŋ}_{2},\text{fŋ}_{3},\ldots\ldots,\text{fŋ}_{n}\}$ be a discrete finite set. Let us consider two Pythagorean neutrosophic sets as $\text{Ƭ}^{1}=\{\text{fŋ},\text{ȝ}^{1}\left(\text{fŋ}_{\iota}\right),\text{ɐ}^{1}\left(\text{fŋ}_{\iota}\right),\text{ɞ}^{1}\left(\text{fŋ}_{\iota}\right)/\text{fŋ}_{\iota}\epsilon\text{Ḧ}\}$ and $\text{Ƭ}^{2}=\{\text{fŋ},\text{ȝ}^{2}(\text{fŋ}_{i}),\text{ɐ}^{2}(\text{fŋ}_{i}),\text{ɞ}^{2}(\text{fŋ}_{i})/\text{fŋ}_{i}\epsilon\text{Ḧ}\}$. Then the various distance measures of Pythagorean neutrosophic fuzzy sets can be described as follows:

Hausdroff distance between two Pythagorean neutrosophic sets

$$\mathcal{D}_{H}(\text{Ƭ}^{1},\text{Ƭ}^{2})=\frac{1}{2n}\sum_{\iota=1}^{n}\max\left\{\left|\left(\text{ȝ}^{1}\left(\text{fŋ}_{\iota}\right)\right)^{2}-\left(\text{ȝ}^{2}\left(\text{fŋ}_{\iota}\right)\right)^{2}\right|,\left|\left(\text{ɐ}^{1}\left(\text{fŋ}_{\iota}\right)\right)^{2}-\left(\text{ɐ}^{2}\left(\text{fŋ}_{\iota}\right)\right)^{2}\right|,\left|(\text{ɞ}^{1}\left(\text{fŋ}_{\iota}\right))^{2}-(\text{ɞ}^{2}\left(\text{fŋ}_{\iota}\right))^{2}\right|\right\}$$

Weighted Hausdroff distance between two Pythagorean neutrosophic sets

$$\mathcal{D}_{H}(\text{Ƭ}^{1},\text{Ƭ}^{2})=\tfrac{1}{2n}\textstyle\sum_{\iota=1}^{n}\mathcal{Y}_{\iota}\,[\max\left\{\left|\left(\text{ȝ}^{1}\left(\text{fŋ}_{\iota}\right)\right)^{2}-\left(\text{ȝ}^{2}\left(\text{fŋ}_{\iota}\right)\right)^{2}\right|,\left|\left(\text{ɐ}^{1}\left(\text{fŋ}_{\iota}\right)\right)^{2}-\left(\text{ɐ}^{2}\left(\text{fŋ}_{\iota}\right)\right)^{2}\right|,\left|(\text{ɞ}^{1}(_{\iota}))^{2}-(\text{ɞ}^{2}\left(\text{fŋ}_{\iota}\right))^{2}\right|\right\}]$$

Hamming distance between two Pythagorean neutrosophic sets

$$\mathcal{D}_{H\hat{d}}(\mathcal{T}^1,\mathcal{T}^2)=\left|\left(\text{ȝ}^1\left(\text{fŋ}_\iota\right)\right)^2-\left(\text{ȝ}^2\left(\text{fŋ}_\iota\right)\right)^2\right|+\left|\left(\text{ɐ}^1\left(\text{fŋ}_\iota\right)\right)^2-\left(\text{ɐ}^2\left(\text{fŋ}_\iota\right)\right)^2\right|+\left|(\text{ɞ}^1\left(\text{fŋ}_\iota\right))^2-(\text{ɞ}^2\left(\text{fŋ}_\iota\right))^2\right|$$

Normalized Hamming distance between two Pythagorean neutrosophic sets

$$\mathcal{D}_{\tilde{\eta}H\hat{d}}(\mathcal{T}^1,\mathcal{T}^2)=\left\{\frac{1}{2n}\sum_{\iota=1}^{n}[\left|\left(\text{ȝ}^1\left(\text{fŋ}_\iota\right)\right)^2-\left(\text{ȝ}^2\left(\text{fŋ}_\iota\right)\right)^2\right|+\left|\left(\text{ɐ}^1\left(\text{fŋ}_\iota\right)\right)^2-\left(\text{ɐ}^2\left(\text{fŋ}_\iota\right)\right)^2\right| \right.$$
$$\left. +\left|(\text{ɞ}^1\left(\text{fŋ}_\iota\right))^2-(\text{ɞ}^2\left(\text{fŋ}_\iota\right))^2\right|]\right\}$$

Euclidean distance between two Pythagorean neutrosophic sets

$$\mathrm{D}_{\check{E}\hat{d}}(\mathcal{T}^1,\mathcal{T}^2)=[\left|(\text{ȝ}^1(\text{fŋ}_\iota))^2-(\text{ȝ}^2(\text{fŋ}_\iota))^2\right|^2+\left|(\text{ɐ}^1(\text{fŋ}_\iota))^2-(\text{ɐ}^2(\text{fŋ}_\iota))^2\right|^2+\left|(\text{ɞ}^1(\text{fŋ}_\iota))^2-(\text{ɞ}^2(\text{fŋ}_\iota))^2\right|^2]^{\frac{1}{2}}$$

Normalized Euclidean distance between two Pythagorean neutrosophic sets

$$\mathrm{D}_{\tilde{\eta}\check{E}\hat{d}}(\mathcal{T}^1,\mathcal{T}^2)=\left\{\frac{1}{2n}\sum_{\iota=1}^{n}[\left|(\text{ȝ}^1(\text{fŋ}_\iota))^2-(\text{ȝ}^2(\text{fŋ}_\iota))^2\right|^2+\left|(\text{ɐ}^1(\text{fŋ}_\iota))^2-(\text{ɐ}^2(\text{fŋ}_\iota))^2\right|^2+\left|(\text{ɞ}^1(\text{fŋ}_\iota))^2-(\text{ɞ}^2(\text{fŋ}_\iota))^2\right|^2]^{\frac{1}{2}}\right\}$$

Then the generalized weighted distance measure of Pythagorean neutrosophic sets is defined as $\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^2)=\left\{\frac{1}{2}\sum_{\iota=1}^{n}\omega_\iota[\left|(\text{ȝ}^1(\text{fŋ}_\iota))^2-(\text{ȝ}^2(\text{fŋ}_\iota))^2\right|^{\text{ꞵ}}+\left|(\text{ɐ}^1(\text{fŋ}_\iota))^2-(\text{ɐ}^2(\text{fŋ}_\iota))^2\right|^{\text{ꞵ}}+\left|(\text{ɞ}^1(\text{fŋ}_\iota))^2-(\text{ɞ}^2(\text{fŋ}_\iota))^2\right|^{\text{ꞵ}}]^{\frac{1}{\text{ꞵ}}}\right\}$

where $\text{ꞵ} > 0$.

When $\text{ꞵ} = 1$ then weighted Hamming distance for Pythagorean neutrosophic sets can be obtained as

$$\mathrm{D}_1(\mathcal{T}^1,\mathcal{T}^2)=\left\{\frac{1}{2}\sum_{\iota=1}^{n}\omega_\iota[\left|\left(\text{ȝ}^1(\text{fŋ}_\iota)\right)^2-\left(\text{ȝ}^2(\text{fŋ}_\iota)\right)^2\right|+\left|\left(\text{ɐ}^1(v)\right)^2-\left(\text{ɐ}^2(\text{fŋ}_\iota)\right)^2\right|+\left|(\text{ɞ}^1(\text{fŋ}_\iota))^2-(\text{ɞ}^2(\text{fŋ}_\iota))^2\right|]\right\}$$

When $\text{ꞵ} = 2$ then weighted Euclidean distance for Pythagorean neutrosophic sets can be obtained as

$$\mathrm{D}_2(\mathcal{T}^1,\mathcal{T}^2)=\left\{\frac{1}{3}\sum_{\iota=1}^{n}\omega_\iota[\left|(\text{ȝ}^1(\text{fŋ}_\iota))^2-(\text{ȝ}^2(\text{fŋ}_\iota))^2\right|^2+\left|(\text{ɐ}^1(\text{fŋ}_\iota))^2-(\text{ɐ}^2(\text{fŋ}_\iota))^2\right|^2+\left|(\text{ɞ}^1(\text{fŋ}_\iota))^2-(\text{ɞ}^2(\text{fŋ}_\iota))^2\right|^2]^{\frac{1}{2}}\right\}$$

Proposition 1. The distance measure defined above satisfies the following properties

a. $\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^2)\geq 0$
b. $\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^2)=0$ iff $\mathcal{T}^1=\mathcal{T}^2$
c. $\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^2)=\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^2,\mathcal{T}^1)$
d. If $\mathcal{T}^1\sqsubseteq\mathcal{T}^2\sqsubseteq\mathcal{T}^3$ where $\mathcal{T}^3$ is also a Pythagorean neutrosophic set then $\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^2)\geq\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^3)$ and $\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^1,\mathcal{T}^3)\geq\mathrm{D}_{\text{ꞵ}}(\mathcal{T}^2,\mathcal{T}^3)$

Similarity Measure of Pythagorean Neutrosophic Sets

It is well known that similarity measures can be obtained from distance measures. Thus the proposed distance measures can be used to define similarity measures discussed in this paper. On the basis of the connection that generally exists between similarity measures and distance measures, the following similarity measures can be defined

Definition 5. Assume that $Ꞇ^1, Ꞇ^2, Ꞇ^3$ be three Pythagorean neutrosophic sets defined on the universe of discussion $Ǿ$. If $D_{ꞵ}(Ꞇ^1, Ꞇ^2)$ is the distance measure between $Ꞇ^1$ *and* $Ꞇ^2$ then the similarity measure $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2)$ between them can be defined as $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2) = 1 - D_{ꞵ}(Ꞇ^1, Ꞇ^2)$

Proposition 2. The similarity measure defined above satisfies the following properties

a. $0 \le Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2) \le 1$
b. $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2) = 1$ iff $Ꞇ^1 = Ꞇ^2$
c. $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2) = Ƨ_{ꞵ}(Ꞇ^2, Ꞇ^1)$
d. $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^3) + Ƨ_{ꞵ}(Ꞇ^2, Ꞇ^3) \ge Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2)$

Proposition 3. If the similarity measure between $Ꞇ^1$ and $Ꞇ^2$ is $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2)$ then the distance measure is defined as $1 - Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2)$.

Proposition 4. If $Ꞇ^1$ and $Ꞇ^2$ be two Pythagorean neutrosophic sets then

a. $D_{ꞵ}(Ꞇ^1, Ꞇ^2) = D_{ꞵ}({Ꞇ^1}^{c}, {Ꞇ^2}^{c})$
b. $Ƨ_{ꞵ}(Ꞇ^1, Ꞇ^2) = Ƨ_{ꞵ}({Ꞇ^1}^{c}, {Ꞇ^2}^{c})$

Proof of (a).

From definition

$$D_{ꞵ}(Ꞇ^1, Ꞇ^2) = \left\{ \frac{1}{2} \sum_{ι=1}^{n} ɰ_{ι} \left[\left| (ʓ^1(fŋ_{ι}))^2 - (ʓ^2(fŋ_{ι}))^2 \right|^{ꞵ} + \left| (ɐ^1(fŋ_{ι}))^2 - (ɐ^2(fŋ_{ι}))^2 \right|^{ꞵ} + \left| (ɞ^1(fŋ_{ι}))^2 - (ɞ^2(fŋ_{ι}))^2 \right|^{ꞵ} \right]^{\frac{1}{ꞵ}} \right\}$$

where $ꞵ > 0$.

Again consider $Ꞇ^1$ and $Ꞇ^2$ be two Pythagorean neutrosophic sets then from the definition of complement of neutrosophic sets, it can be seen that
${Ꞇ^1}^{c} = \{fŋ, ɞ^1(fŋ), ɐ^1(fŋ), ʓ^1(fŋ), / fŋ \epsilon Ǿ\}$ and
${Ꞇ^2}^{c} = \{fŋ, ɞ^2(fŋ_i), ɐ^2(fŋ_i), ʓ^2(fŋ_i) / fŋ_i \epsilon Ḧ\}$
Then

$$D_{ꞵ}(Ꞇ^1, Ꞇ^2) = \left\{ \frac{1}{2} \sum_{ι=1}^{n} ɰ_{ι} \left[\left| (ʓ^1(fŋ_{ι}))^2 - (ʓ^2(fŋ_{ι}))^2 \right|^{ꞵ} + \left| (ɐ^1(fŋ_{ι}))^2 - (ɐ^2(fŋ_{ι}))^2 \right|^{ꞵ} + \left| (ɞ^1(fŋ_{ι}))^2 - (ɞ^2(fŋ_{ι}))^2 \right|^{ꞵ} \right]^{\frac{1}{ꞵ}} \right\}$$

$$D_{ꞵ}({Ꞇ^1}^{c}, {Ꞇ^2}^{c}) = \left\{ \frac{1}{2} \sum_{ι=1}^{n} ɰ_{ι} \left[\left| (ɞ^1(fŋ_{ι}))^2 - (ɞ^2(fŋ_{ι}))^2 \right|^{ꞵ} + \left| (ɐ^1(fŋ_{ι}))^2 - (ɐ^2(fŋ_{ι}))^2 \right|^{ꞵ} + \left| (ʓ^1(fŋ_{ι}))^2 - (ʓ^2(fŋ_{ι}))^2 \right|^{ꞵ} \right]^{\frac{1}{ꞵ}} \right\}$$

Hence $D_{ꞵ}(Ꞇ^1, Ꞇ^2) = D_{ꞵ}({Ꞇ^1}^{c}, {Ꞇ^2}^{c})$
Proof of (b).
From definition of similarity

$\text{Ȣ}_{\beta}(\mathcal{T}^1,\mathcal{T}^2) = 1 - \mathcal{D}_{\beta}(\mathcal{T}^1,\mathcal{T}^2)$

Then

$$\text{Ȣ}_{\beta}(\mathcal{T}^1,\mathcal{T}^2) = 1 - \left\{\frac{1}{2}\sum_{\text{ɩ}=1}^{n} \text{ɰ}_{\text{ɩ}}\left[\left|(\text{ȝ}^1(\text{ﬁŋ}_{\text{ɩ}}))^2 - (\text{ȝ}^2(\text{ﬁŋ}_{\text{ɩ}}))^2\right|^{\beta} + \left|(\text{ɐ}^1(\text{ﬁŋ}_{\text{ɩ}}))^2 - (\text{ɐ}^2(\text{ﬁŋ}_{\text{ɩ}}))^2\right|^{\beta} + \left|(\text{ɘ}^1(\text{ﬁŋ}_{\text{ɩ}}))^2 - (\text{ɘ}^2(\text{ﬁŋ}_{\text{ɩ}}))^2\right|^{\beta}\right]^{\frac{1}{\beta}}\right\}$$

Again since

$\mathcal{D}_{\beta}(\mathcal{T}^1,\mathcal{T}^2) = \mathcal{D}_{\beta}({\mathcal{T}^1}^{c},{\mathcal{T}^2}^{c})$

Then

$$\begin{aligned}\text{Ȣ}_{\beta}({\mathcal{T}^1}^{c},{\mathcal{T}^2}^{c}) &= 1 - \mathcal{D}_{\beta}({\mathcal{T}^1}^{c},{\mathcal{T}^2}^{c})\\ &=1 - \mathcal{D}_{\beta}(\mathcal{T}^1,\mathcal{T}^2)\\ &=\text{Ȣ}_{\beta}(\mathcal{T}^1,\mathcal{T}^2)\end{aligned}$$

Proposition 5. If $\mathcal{T}^1 \sqsubseteq \mathcal{T}^2 \sqsubseteq \mathcal{T}^3$ where $\mathcal{T}^1, \mathcal{T}^2, \mathcal{T}^3$ are Pythagorean neutrosophic sets then

a. $\mathcal{D}_{\beta}(\mathcal{T}^1,\mathcal{T}^3) \geq \mathcal{D}_{\beta}(\mathcal{T}^1,\mathcal{T}^2)$ and $\mathcal{D}_{\beta}(\mathcal{T}^1,\mathcal{T}^3) \geq \mathcal{D}_{\beta}(\mathcal{T}^2,\mathcal{T}^3)$
b. $\text{Ȣ}_{\beta}(\mathcal{T}^1,\mathcal{T}^3) \leq \text{Ȣ}_{\beta}(\mathcal{T}^1,\mathcal{T}^2)$ and $\text{Ȣ}_{\beta}(\mathcal{T}^1,\mathcal{T}^3) \leq \text{Ȣ}_{\beta}(\mathcal{T}^2,\mathcal{T}^3)$

Conclusions

Many operations and properties associated with Pythagorean neutrosophic sets are outlined in this paper. Distance measures between neutrosophic sets are topic of interest for many researchers working in the field of decision making and likewise the distance measure between Pythagorean fuzzy sets can also be applied in various field. Many distance measures are discussed herein and these measures are applied in finding similarity and dissimilarity between Pythagorean neutrosophic sets.

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The field of neutrosophic set theory and its applications has been rapidly expanding, particularly since the introduction of the journal "Neutrosophic Sets and Systems." New theories, techniques, and algorithms are being developed at a very high rate. One of the most notable trends in neutrosophic theory is its hybridization with other set theories such as rough set theory, bipolar set theory, soft set theory, hesitant fuzzy set theory, and more.

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