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The Stability of Electron Orbital Shells based on a Model of the Riemann-Zeta Function

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It is shown that the atomic number Z is prime at the beginning of the each s^1, p^1, d^1, and f^1 energy levels of electrons, with some fluctuation in the actinide and lanthanide series. The periodic prime number boundary of s^1, p^1, d^1, and f^1 is postulated to occur because of stability of Schrodinger's wave equation due to a fundamental relationship with the Riemann-Zeta function.

1 Introduction

It has been known that random matrix theory, and in particular a Gaussian Unitary Ensemble (GUE), can be used to solve the eigenvalue states of high-Z nuclei which would otherwise be computationally impossible. In 1972, Freeman Dyson and Hugh Montgomery of the University of Michigan realized that the values in the GUE matrix used in predicting energy levels of high-Z nuclei where similar to the spacing of zeros in the Riemann-Zeta function [1]. Prior to these discoveries, the use of approximation in traditional quantum mechanical models was well known and used, such as the Born-Oppenheimer method [2]. These approaches experienced problems at high-Z levels where many interacting factors made approximation difficult. The remaining question as to why the periodicity of zeros from the Riemann-Zeta function would match the spacing of energy levels in high-Z nuclei still remains a mystery, however.

It is the goal of this paper to explain the spacing of energy levels in electron orbital shells s^1, p^1, d^1, and f^1, where these designations represent the first electron to occupy the s, p, d and f shells. The first electron in each of these shells is an important boundary where new electron orbital shells are created in the atomic structure. The newly created shell is dependent upon the interaction of many electrons in the previous orbital shells that are filled much like the many body problems of gravitational masses. The first electron in each of the s, p, d and f shells is therefore hypothesized to represent a prime stability area where a new shell can form within the many-electron atom without significant perturbation to previous shells. With enough computational power the interaction of electrons in any combination of orbital shells can be computed through multiple manipulations of a system of Schrödinger's equations, but even the present numerical methods for this approach will use rough approximations due to the complexity of several non-linear equations and their solutions.

Choudhury and Pitchers [3] have used a configuration-interaction method of computation for many electron atoms where Schrödinger's equation is reduced to a system of linear homogenous equations. They then argue that the energy eigenvalues obtained by truncating this linear set of equations will converge, in the limit, to those of the original system. They show that this approximation holds true for two-electron atoms, but they note that variations start occurring for the three or more electron atom. These approximation methods are difficult enough for two or three electron atoms but for many-electron atoms the approximation methods are uncertain and are likely to introduce errors.

It is therefore proposed that the final result of the many-electron atom be first evaluated from the standpoint of the Riemann-Zeta function so that a simplifying method of working back to a valid system of Schrödinger's equations can hopefully be obtained. To justify this approach, the atoms for each of the newly filled s^1, p^1, d^1, and f^1 shells are examined to show a potential relation between the spacing of the non-trivial zero solutions of the Riemann-Zeta function, where the argument s in the Zeta function that produces the zero lies on the critical line of $\mathrm{Re}[s] = \frac{1}{2}$.

2 The Riemann-Zeta function

The Riemann-Zeta function takes the form:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{n^s} + \dots \qquad (1)$$

Where $\zeta(s)$ is an alternating series function in powers of s as n terms go to infinity. The function $\zeta(s)$ is a single-valued, complex scalar function of s, much like the single complex variable of Schrödinger's wave equation. The addition of several Ψ solutions of Schrödinger's equation from many orbital electrons may be effectively modeled by (1), where additional $1/n^s$ terms in (1) contribute to the overall probability distribution of n interacting shells.

Hadamard and Vallée Poussin independently proved the prime number theorem in 1896 by showing that the Riemann-Zeta function $\zeta(s)$ has no zeros of the form $s = 1 + i\beta$, so that no deeper properties of $\zeta(s)$ are required for the proof of the prime number theorem. Thus the distribution of primes is intimately related to $\zeta(s)$.

Z number	Ln(Z number)	Z for S1 Shell	Ln (Z) from F	Number of S electrons at Z (x axis on graph)	Ln(Z) / (Number of S electrons) - y axis
1	0	1	0	0	undefined
3	1.098612289	3	1.098612289	1	1.098612289
5	1.609437912	11	2.397895273	2	1.198947636
11	2.397895273	19	2.944438979	3	0.98147966
13	2.564949357	37	3.610917913	4	0.902729478
19	2.944438979				
29	3.36729583				
31	3.433987204				
37	3.610917913				

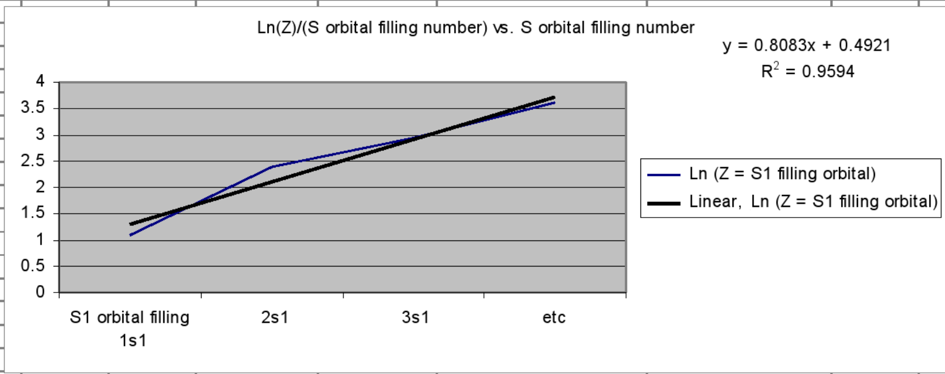

Table 2: $\mathrm{Ln}\,(Z)$ for first s-orbital electron vs. Energy level (n).

3 The Periodic Table

From an analysis of the periodic table it is postulated that the stability of electronic shells s^1, p^1, d^1, and f^1 follow a larger set of zeros which correlate to prime numbers from the Riemann-Zeta function.

If one examines the first and second periods of the periodic table as shown in Table 1, we find that the boundary of filling the first electron in the s, p, d and f shells of each quantum level designated as n is a stable zone that is indicated by a prime atomic number Z (the format in Table 1 is Z: $n\,$Level^1).

Z:	Shell
1:	$1s^1$
3:	$2s^1$
5:	$2p^1$
11:	$3s^1$
13:	$3p^1$
19:	$4s^1$
31:	$4p^1$
37:	$5s^1$
41:	[Kr].$4d^4.5s^1$ (repeat)
43:	[Kr].$4d^6.5s^1$ (repeat)
47:	[Kr].$4d^{10}.5s^1$ (repeat)
57:	$5d^1$
71:	$5d^1$ (repeat)
87:	$7s^1$
89:	$6d^1$
91:	$15f^1$
103:	$7p^1$

Table 1: Prime Atomic Numbers with respect to s^1, p^1, d^1, and f^1 orbitals.

Where (repeat) indicates a repeat of the shell from a previous Z number and where the use of the format [Kr].$4d^{10}.5s^1$ shows the previous electronic formula of Krypton with the additional filling of the d and s shells so as to show the repeated s shell with different d-filling electrons. From the above data, both $Z = 57$ and $Z = 89$ begin a sequence of f shells filling before d shells ($Z = 57$ is the beginning of the Lanthanide series and $Z = 89$ is beginning of the Actinide series). In both the Lanthanide and Actinide series, the d shells that fill after the f shells are primes, explaining why only these d shells (beginning with $Z = 71$ for Lanthanide and $Z = 103$ for Actinide) are filled with primes because they would normally be f shells in the sequence if we looked strictly at observed spectroscopic data.

Notice that the prime Z numbers — 1, 3, 5, 11, 13, 19, 29, 31, 37, 41, 43 47, etc. shows one consecutive set of primes $\{Z = 3, 5\}$, skips a prime $\{Z = 7\}$ then has two more consecutive primes $\{Z = 11, 13\}$. Note that at $Z = 13$ where we have skipped a prime ($Z = 7$), the ratio of $13/\ln(13)$ is 5.0 and $Z = 13$ is the fifth prime Z number with a valid p^1 shell. The sequence then skips one prime $\{Z = 17\}$, then has a valid prime at $\{Z = 19\}$ and it then skips another prime $\{Z = 23\}$, which follows five consecutive primes $\{Z = 29, 31, 37, 41, 43\}$. At this point we have skipped $Z = 7$, 17, and 23 but when we look at $Z = 43$, we take the ratio of $43/\ln(43) = 11.4$ and note that $Z = 43$ is the 11th valid Z prime (with three Z numbers skipped). There appears to be a similar relationship between this data and the prime number theorem of $n/\ln(n)$, but unlike the traditional prime number theorem where all primes are included, this data considers only valid Z primes (where the first shell is filled, s^1, p^1, d^1, f^1) with other primes skipped if they don't fill the first s, p, d, or f shell. From this consideration, the linearity of these values is significant to the periodic table alone. Table 2 shows the first five s-orbital shells filled (to $5s^1$) plotted against the $\mathrm{Ln}(Z)$ where Z is the associated atomic number for the valid s^1 shell. The number of valid s^1 shells also corresponds to the energy level n ($n = 1$ through $n = 5$ for $1s^1 - 5s^1$). The slope in Table 2 for just the s-shell is good with a linear relationship to $R^2 = 0.95$.

This sequence is also hypothesized to be similar to the distribution pattern of primes produced by finding the zeros on the critical line of the Riemann-Zeta function of (1). Based on the results of linearity in Table 2 there may be a relationship between the difference between valid s-shell orbitals (the Z numbers of skipped shells) versus the total number of shells, a further indication that Riemann-Zeta function could explain the prime orbital filling. There is also a similar prime number correlation for the nuclear energy levels where s, p, d, f and g shells begin on prime boundaries [4].

4 Conclusions

It is found by examining the Z number related to the s^1, p^1, d^1, and f^1 shells of the periodic table that Z is prime for the first filling of s, p, d and f orbitals. It is also found that for shell filling of ns^1, the logarithm of the prime number associated with Z is linear with respect to energy level n. This relationship is believed to correlate with the Riemann-Zeta function, a complex scalar function similar to the complex-scalar wave function of Schrödinger. The atomic Z primes that correspond to the s^1, p^1, d^1, and f^1 shells is predicted to follow the distribution of primes that result from the non-trivial zeros of the Riemann-Zeta function.

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In-Depth Development of Classical Electrodynamics

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There is hope that a properly developed Classical Electrodynamics (CED) will be able to play a rôle in a unified field theory explaining electromagnetism, quantum phenomena, and gravitation. There is much work that has to be done in this direction. In this article we propose a move towards this aim by refining the basic principles of an improved CED. Attention is focused on the reinterpretation of the E-M potential. We use these basic principles to obtain solutions that explain the interactions between a constant electromagnetic field and a thin layer of material continuum; between a constant electromagnetic field and a spherical configuration of material continuum (for a charged elementary particle); between a transverse electromagnetic wave and a material continuum; between a longitudinal aether wave (dummy wave) and a material continuum.

1 Introduction

The development of Classical Electrodynamics in the late 19th and early 20th century ran into serious trouble from which Classical Electrodynamics was not able to recover (see R. Feynman's *Lectures on Physics* [1]: Volume 2, Chapter 28). According to R. Feynman, this development "ultimately falls on its face" and "It is interesting, though, that the classical theory of electromagnetism is an unsatisfactory theory all by itself. There are difficulties associated with the *ideas* of Maxwell's theory which are not solved by and not directly associated with quantum mechanics". Further in the book he also writes: "To get a *consistent* picture, we must imagine that something holds the electron together", and "the extra non-electrical forces are also known by the more elegant name, the Poincare stresses". He then concludes: "— there have to be other forces in nature to make a consistent theory of this kind". CED was discredited not only by R. Feynman but also by many other famous physicists. As a result the whole of theoretical physics came to believe in the impossibility of explaining the stability of electron charge by classical means, claiming defect in the classical principles. But this is not true.

We showed earlier [2, 3, 4] and further elaborate here that there is nothing wrong with the basic classical *ideas* that Maxwell's theory is based upon. It simply needs further development. The work [2] opens the way to the natural (without singularities) development of CED. In this work it was shown that Poincare's claim in 1906 that the "material" part of the energy-momentum tensor, "Poincare stresses", has to be of a "nonelectromagnetic nature" (see Jackson, [5]) is incorrect. It was shown that the definite material part is expressed only through current desity (see formula (9) in [2]), and given a static solution: Ideal Particle, IP, see (19). The proper covariance of IP is manifest — the charges actually hold together and the energy inside an IP comes from the interior electric field (positive energy) and the interior charge density (negative energy, see formula (22) of [2]). The total energy inside an IP is zero, which means that the rest mass (total energy) corresponds to the vacuum energy only. The contributions to the "inertial mass" (linear momentum divided by velocity; R. Feynman called it "electromagnetic mass") can be calculated by making a Lorentz transformation and a subsequent integration. The total inertial mass is equal to the rest mass (which is in compliance with covariance) but the contributions are different: 4/3 comes from the vacuum electric field, 2/3 comes from the interior electric field, and -1 comes from the interior charge density. This is the explanation of the "anomalous factor of 4/3 in the inertia" (first found in 1881 by J. J. Thomson [5]).

Let us begin with Maxwell's equations:

$$j^i + \frac{c}{4\pi} F^{ik}_{|k} = 0\,, \qquad j^k_{|k} = 0\,; \tag{1}$$

$$\operatorname{div}\vec{E} = \frac{4\pi}{c}\, j^0, \qquad \oint_S \vec{E}\cdot d\vec{S} = \frac{4\pi}{c}\int_V j^0 dV\,; \tag{1a}$$

$$\operatorname{rot}\vec{H} = \frac{4\pi}{c}\,\vec{j} + \frac{1}{c}\frac{\partial\vec{E}}{\partial t}\,, \qquad \oint_\Gamma \vec{H}\cdot d\vec{\Gamma} = \frac{1}{c}\int_S\left(4\pi\vec{j} + \frac{\partial\vec{E}}{\partial t}\right)\cdot d\vec{S}\,; \tag{1b}$$

$$\frac{1}{c}\frac{\partial j^0}{\partial t} + \operatorname{div}\vec{j} = 0\,, \qquad \frac{\partial}{c\partial t}\int_V j^0 dV = \oint_S \vec{j}\cdot d\vec{S}\,. \tag{1c}$$

The other half of Maxwell's equations is

$$F^{*ik}_{|k} = 0\,, \qquad F^{*ik} \equiv \frac{1}{2}\, e^{iklm} F_{lm}\,; \tag{2}$$

$$\operatorname{div}\vec{H} = 0\,, \qquad \oint_S \vec{H}\cdot d\vec{S} = 0\,; \tag{2a}$$

$$\operatorname{rot}\vec{E} = -\frac{1}{c}\frac{\partial\vec{H}}{\partial t}\,; \quad \oint_\Gamma \vec{E}\cdot d\vec{\Gamma} = -\frac{1}{c}\int_S \frac{\partial\vec{H}}{\partial t}\cdot d\vec{S}\,. \tag{2b}$$

The equations are given in 4D form, 3D form, and in an integral form. Equation (1) represents the interaction law between the electromagnetic field and the current density. Equation (2) applies only to the electromagnetic field. This whole system, wherein equation (1c) is not included, is definite for the 6 unknown components of the electromagnetic field on the condition that the currents (all the components) are given. This is the first order PDE system, the characteristics of which are the wave fronts.

What kind of currents can be given for this system? Not only can continuous fields of currents be prescribed. A jump in a current density is a normal situation. We can even go further and prescribe infinite (but the space integral has to be finite) current density. But in this case we have to check the results. In other words, the system allows that the given current density can contain Dirac's delta-functions if none of the integrals in (1) and (2) goes infinite. But this is not the end. There exists an energy-momentum tensor that gives us the energy density in space. The space integral of that density also has to be finite. Here arises the problem. If we prescribe a point charge (3D delta-function) then the energy integral will be infinite. If we prescribe a charged infinitely thin string (2D delta-function) then the energy will also be infinite. But if we prescribe an infinitely thin surface with a finite surface charge density on it (1-d delta-function) then the energy integral will be finite. It appears that this is the only case that we can allow. But we have to remember that it is possible that a **disruption surface** (where the charge/current density can be infinite) can be present in our physical system. This kind of surface allows the electromagnetic field to have a jump across this surface (this very important fact was ignored in conventional CED — see below). It is also very important to understand that all these delta-functions for the charge distribution are at our discretion: we can prescribe them or we can "hold out". If we choose to prescribe then we are taking on an additional responsibility. The major attempt to discredit CED (to remove any "obstacles" in the way of quantum theory) was right here. The detractors of CED (including celebrated names like R. Feynman in the USA and L. D. Landau in Russia but, remarkably, not A. Einstein) tried to convince us that a point charge is inherent to CED. With it comes the divergence of energy and the radiation reaction problem. This problem is solvable for the extended particle (which has infinite degrees of freedom) but is not solvable for the point particle. This is not an indication that the "classical theory of electromagnetism is an unsatisfactory theory by itself". Rather this means that we should not use the point charge model (or charged string model). Only a charged closed surface model is suitable.

We have another serious problem in conventional electrodynamics. As we have shown below, the variation procedure of conventional CED results in the requirement that the electromagnetic field must be continuous across any disruption surface. That actually implies the impossibility of a surface charge/current on a disruption surface. I changed the variation procedure of CED and arrived at a theory where the electromagnetic interaction (ultimately represented by Maxwell's equation (1)) is the only interaction. The so-called interaction term in the Lagrangian ($A^k j_k$) is abandoned. Also abandoned is the possibility introducing any other interactions (like the "strong" or "weak"). I firmly believe that all the experimental data for elementary particles, quantum phenomena, and gravitation can be explained starting only with the electromagnetic interaction (1).

What is the right expression for the energy-momentum tensor that corresponds to the system described by (1) and (2)? The classical principles require that this expression must be unique. Conventional electrodynamics provides us with the expression: $T^{ik} = \mu c u^i u^k \frac{ds}{dt}$ (for a "material" part containing free particles only: see Landau [6], formula 33.5) that contains density of mass, μ, and velocity only. No charge/current density is included. It seems that the mere presence of charge/current density has to contribute to the energy of the system. To correct the situation we took the simplest possible Lagrangian with charge density:

$$\Lambda = -\frac{1}{16\pi}\, g^{ab} g^{cd} F_{ac} F_{bd} - \frac{2\pi}{k_0^2 c^2}\, g^{ab} j_a j_b \,, \tag{3}$$

where k_0 is a new constant. No interaction term (like $A_k j^k$) is included.

2 Variation of metrics

Let us find the energy-momentum tensor that corresponds to the Lagrangian (3). The metric tensor in classical 4-space is $g_{ik} = \mathrm{diag}\,[1, -1, -1, -1]$ (we assume $c = 1$). Let us consider an arbitrary variation of a metric tensor but on the condition that this variation does not introduce any curvature in space. This variation is:

$$\delta g_{ik} = \xi_{i|k} + \xi_{k|i} \,, \tag{4}$$

where ξ^k is an arbitrary but small vector. One has to use the mathematical apparatus of General Relativity to check that with the variation (3) the Riemann curvature tensor remains zero to first order. Assuming that the covariant components of the physical fields are kept constant (then the contravariant components will be varied as a result of the variation of the metric tensor, but we do not use them — see (3) for an explanation) we can calculate the variation of the action. The variation of the square root of the determinant of the metric tensor is: $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g_{ik}\, \delta g^{ik}$ (this result can be found in textbooks on field theory). The variation of action becomes:

$$\delta S = -\int \left\{ 2\frac{\partial \Lambda}{\partial g^{ik}} - \Lambda g_{ik} \right\} \xi^{i|k} \sqrt{-g}\, d\Omega = \\ = -\int T_{ik}\, \xi^{i|k} \sqrt{-g}\, d\Omega \,, \tag{5}$$

where

$$T_{ik} = -\frac{1}{4\pi}\, g^{ab} F_{ia} F_{kb} + \frac{1}{16\pi}\, F_{ab} F^{ab} g_{ik} - \\ - \frac{4\pi}{k_0^2}\, j_i j_k + \frac{2\pi}{k_0^2}\, j_a j^a g_{ik}\,.$$

If our system consists of two regions that are separated by a closed disruption surface S then the above procedure has to be applied to each region separately. We can write: $T_{ik}\xi^{i|k} = (T_{ik}\xi^i)^{|k} - T_{ik}^{|k}\xi^i$. The 4D volume integrals over divergence (the first term) can be expressed through 3D hypersurface integrals according to the 4D theorem of Gauss. The integral over some remote closed surface becomes zero due to the smallness of T_{ik} on infinity (usually assumed). The integral over a 3D volume at t_1 and t_2 becomes zero due to the assumption: $\xi_i = 0$ at these times. What is left is:

$$\delta S = -\int T_i^k \xi^i_{|k} \sqrt{-g}\, d\Omega = \int_S \left(T^k_{i\,\mathrm{out}} - T^k_{i\,\mathrm{in}}\right) \xi^i dS_k + \\ + \int_{\mathrm{in}} T^k_{i\,|k}\, \xi^i \sqrt{-g}\, d\Omega + \int_{\mathrm{out}} T^k_{i\,|k}\, \xi^i \sqrt{-g}\, d\Omega\,.$$

Since ξ_i are arbitrary small functions (between t_1 and t_2), the requirement $\delta S = 0$ yields:

$$T^{ia}_{|a} = 0\,. \tag{6}$$

This condition has to be fulfilled for the inside and the outside regions separately. And the additional requirement on the disruption surface S,

$$T^{ia} N_a, \tag{6a}$$

is continuous, where N_k is a normal to the surface.

We have found the unique definition of the energy-momentum tensor (5). If we want the action to be minimum with respect to the arbitrary variation of the metric tensor in flat space then (6) and (6a) should be satisfied. Let us rewrite the energy-momentum tensor in 3D form:

$$\left.\begin{aligned} T^{00} &= \frac{1}{8\pi}\left(E^2 + H^2\right) - \frac{2\pi}{k_0^2 c^2}\left[(j^0)^2 + (\vec{j})^2\right] \\ T^{11} &= \frac{1}{8\pi}\left(E^2 + H^2 - 2E_1^2 - 2H_1^2\right) - \\ &\quad - \frac{2\pi}{k_0^2 c^2}\left[(j^0)^2 - (\vec{j})^2 + 2(j^1)^2\right] \\ T^{01} &= \frac{1}{4\pi}\left(E_2 H_3 - E_3 H_2\right) - \frac{4\pi}{k_0^2 c^2}\, j^0 j^1 \\ T^{12} &= -\frac{1}{4\pi}\left(E_1 E_2 + H_1 H_2\right) - \frac{4\pi}{k_0^2 c^2}\, j^1 j^2 \end{aligned}\right\}. \tag{5a}$$

Notice that we have not used Maxwell's or any other field equations so far. It should also be noted that for the energy-momentum tensor (5), (5a) is not defined on the disruption surface itself, despite the fact that there can be a surface charge/current on a surface (infinite volume density but finite surface density).

Going further, **we are definitely stating that Maxwell's equation (1) is a universal law that should be fulfilled in all space without exceptions. It defines the interaction between the electromagnetic field and the field of current density. This law cannot be subjected to any variation procedure**. Maxwell's equation (2) we will confirm later as a result of a variation; see formula (9). Substituting (5) in (6) and using Maxwell's equation (1) and the antisymmetry of F_{ik}, we obtain:

$$\left.\begin{aligned} & j^a\left(\frac{k_0^2 c}{4\pi}\, F_{ai} + j_{a|i} - j_{i|a}\right) = 0 \\ & j^0\left(\frac{k_0^2 c}{4\pi}\,\vec{E} + \nabla j^0 + \frac{1}{c}\frac{\partial \vec{j}}{\partial t}\right) + \\ &\qquad + \vec{j} \times \left(\frac{k_0^2 c}{4\pi}\,\vec{H} - \mathrm{rot}\,\vec{j}\right) = 0 \end{aligned}\right\}. \tag{7}$$

This equation has to be fulfilled for the inside and outside regions separately because (6) is fulfilled separately in these regions. This is important. It is also important to realize that while the conservation of charge is fulfilled everywhere, including a disruption surface, the disruption surface itself is exempt from energy-momentum conservation (no surface energy, no surface tension). This arrangement is in agreement with the fact that we can integrate a delta-function (charge) but we cannot integrate its square (would be energy).

3 A new dynamics

Equation (7) we call a **Dynamics Equation.** It is a nonlinear equation. But it has to be fulfilled inside and outside the particle separately. This will allow us to reduce it to a linear equation inside these regions.

Definition: vacuum is a region of space where all the components of current density are zero.

Equation (7) is automatically satisfied in vacuum ($J^k = 0$). The other possibility ($J^k \neq 0$) will be the interior region of an elementary particle. The boundary between these regions will be a disruption surface. Inside the particle instead of (7) we have:

$$\left.\begin{aligned} & \frac{k_0^2 c}{4\pi}\, F_{ai} + j_{a|i} - j_{i|a} = 0 \\ & \frac{k_0^2 c}{4\pi}\,\vec{E} + \nabla j^0 + \frac{1}{c}\frac{\partial \vec{j}}{\partial t} = 0\,, \quad \frac{k_0^2 c}{4\pi}\,\vec{H} - \mathrm{rot}\,\vec{j} = 0 \end{aligned}\right\}. \tag{7a}$$

All the solutions of equation (7a) are also solutions of the nonlinear equation (7). At present we know nothing about the solutions of (7) that do not satisfy (7a). Inside the elementary particle the dynamics equation (7) or (7a) describes, as we call it, a **Material Continuum**. A Material Continuum cannot be divided into a system of material points. The

Relativistic (or Newtonian) Dynamics Equation of CED, that describes the behavior of the particle as a whole, completely disappears inside the elementary particle. There is no mass, no force, no velocity or acceleration inside the particle. The field of current density j^k defines a kinematic state of the Material Continuum. A world line of current j^k is not a world line of a material point. That allows us to deny any causal connection between the points on this line. In consequence, j^k can be space-like as well as time-like. That is in no contradiction with the fact that the boundary of the particle cannot exceed the speed of light. Equation (7a) is linear and allows superposition of different solutions. Using (1) we can obtain:

$$\left.\begin{aligned} & j^{k|a}_{|a} - k_0^2\, j^k = 0; \qquad j^k + k_0^2\, j^k = 0 \\ & \Delta j^k - \frac{1}{c^2}\frac{\partial^2 j^k}{\partial t^2} + k_0^2\, j^k = 0 \end{aligned}\right\}. \tag{7b}$$

By equation (7) we have obtained something very important, but we are just on the beginning of a difficult and uncertain journey. Now the current density cannot be prescribed arbitrarily. Inside the particle it has to satisfy equation (7b). However, there are no provisions on the surface current density (if a surface current is different from zero then its density is necessarily expressed by a delta-function across the disruption surface).

4 The electromagnetic potential

Now we are going to vary the electromagnetic field F_{ik} in all the space, including a disruption surface. As usual, the variation is kept zero at t_1 and t_2 and also on a remote closed surface, at infinity. In this case the results of variation will be in force on the disruption surface itself. Still, we have to write the variation formulae for each region separately. We claim that equation (1) cannot be subjected to variation. It is the preliminary condition before any variation. In our system we have 10 unknown independent functions (4 functions in J_k and 6 functions in F_{ik}). These functions already have to satisfy 8 equations: 4 equations in (1) and 4 equations in (7). We have only 2 degrees of freedom left. We cannot vary F_{ik} by a straightforward procedure. Let us employ here the Lagrange method of indefinite factors. Let us introduce a modified Lagrangian:

$$\Lambda' = \Lambda + A^a \left(j_a + \frac{1}{4\pi} F^{|b}_{ab} \right), \tag{8}$$

where A^k are 4 indefinite Lagrange factors. Now we have $2+4=6$ degrees of freedom and we use them to vary F_{ik}. We have:

$$\delta S = -\int \left\{ \frac{\partial \Lambda'}{\partial F_{ik}} \delta F_{ik} + \frac{\partial \Lambda'}{\partial F_{ik|l}} \delta F_{ik|l} \right\} dV_4 =$$

$$-\int \left\{ \left(\frac{\partial \Lambda'}{\partial F_{ik|l}} \delta F_{ik} \right)_{|l} + \left[\frac{\partial \Lambda'}{\partial F_{ik}} - \left(\frac{\partial \Lambda'}{\partial F_{ik|l}} \right)_{l} \right] \delta F_{ik} \right\} dV_4 = 0.$$

The first term under integration is divergence and can be transformed to the hypersurface integral according to Gauss theorem. Since the variation is arbitrary, the square brackets term has to be zero in either case. It gives:

$$F_{ik} = A_{k|i} - A_{i|k}\,. \tag{9}$$

If V_4 is the inside region of the particle from t_1 to t_2 then the hypersurface integrals at t_1 and t_2 will be zero, but the hypersurface integral over the closed disruption surface will be

$$\frac{1}{4\pi} \int dt \oint \left(A^i g^{kl} - A^k g^{il} \right)_{\text{in}} \delta F_{ik}\, dS_l\,.$$

If V_4 is the outside vacuum then the hypersurface integrals at t_1 and t_2 will be zero. The hypersurface integral over the remote closed surface will be zero, but the hypersurface integral over the disruption surface will be

$$-\frac{1}{4\pi} \int dt \oint \left(A^i g^{kl} - A^k g^{il} \right)_{\text{out}} \delta F_{ik}\, dS_l\,.$$

These integrals will annihilate if the **potential A^k is continuous across the disruption surface**. The continuity of potential does not preclude the possibility of a surface charge/current and a jump of electromagnetic field as a consequence.

Claim: The variation procedure of conventional CED results in the impossibility of a surface charge/current on a disruption surface. The variation procedure of conventional CED begins with equation (9) replacing the electromagnetic field with a potential. It introduces the interaction term $A^k j_k$ in the Lagrangian and varies the potential δA^k. As a result of the least action it obtains Maxwell's equation (1). But it can be shown that the consideration of a disruption surface will produce the requirement of electromagnetic field continuity. This actually denies the possibility of a single layer surface charge/current (the double layers are not interesting and they will require the jump of potential and infinite electromagnetic field). Therefore, the conventional variation procedure is incorrect.

5 The physical meaning of potential

Now we learned that the electromagnetic potential, which was devoid of a physical meaning, has to be continuous across all the boundaries of disruption. This is a very important result. It allows me to reinterpret the physical meaning of potential. It is true that according to (9) we can add to the potential a gradient of some arbitrary function and the electromagnetic field won't change (gauge invariance). Yes, but this fact can be given another interpretation: **the potential is unique and it actually contains more information about physical reality than the electromagnetic field does**. To make the potential mathematically unique, besides initial data and boundary conditions we need only to impose the conservation equation (formerly Lorenz gauge).

$$A^k_{|k} = 0\,, \qquad A^{k|a}_{|a} = \frac{4\pi}{c}\, j^k, \qquad \Box A^k = -\frac{4\pi}{c}\, j^k. \quad (10)$$

This is true everywhere. Using (1), (7a), and (9) we can conclude that inside a material continuum the potential has to satisfy:

$$\left(A^{k}_{|b}{}^{|b} - k_0^2 A^k\right)^{|i} - \left(A^{i}_{|b}{}^{|b} - k_0^2 A^i\right)^{|k} = 0\,. \quad (7c)$$

If the equation:

$$A^{k}_{|b}{}^{|b} - k_0^2 A^k = 0 \quad \text{or} \quad \Box A^k + k_0^2 A^k = 0\,, \quad (11)$$

$$\Box \equiv \Delta - \frac{\partial^2}{c^2\, \partial t^2}$$

is satisfied then (7c) also satisfied. This type of equation is satisfied by the current density, see (7b). This equation can be called the "Generalized Helmholtz Equation". In static conditions (11) coincides with the Helmholtz equation. Equation (11) differs from the Klein-Gordon equation by the sign before the square of a constant.

The new interpretation of potential: A^0 represents the aether quantity (positive or negative), the 3-vector $\vec{A}$ represents the aether current. All together: the potential uniquely describes the existing physical reality — the aether. In general, the interpretation of potential doubles the interpretation of current.

6 The implications of the re-interpretation of potential

Let us suppose that the potential is equal to a gradient of some function G, which we call a "dummy generator":

$$\left.\begin{aligned} &A^k = g^{ka} G_{|a}\,, \quad A_0 = \frac{1}{c}\frac{\partial G}{\partial t}\,, \quad \vec{A} = -\nabla G \\ &G^{|a}_{|a} = 0\,, \qquad \Delta G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = 0 \end{aligned}\right\}, \quad (12)$$

G has to be the solution of a homogeneous wave equation. However, there are no requirements for G on a disruption surface that we know of at present. But now we won't say that G is devoid of a physical meaning (remember the mistake we made with potential).

What kind of a physical process is described here by the corresponding potential? There is no electromagnetic field and the energy-momentum tensor is equal to zero. These are the "dummy waves" — the longitudinal aether waves. These waves are physically significant only due to the boundary conditions on the disruption surfaces, which they affect. If this is the case, then G can be significant in physical experiment. It can be even unique under the laws (these laws are not completely clear) of another physical realm (realm of electromagnetic potential).

It is difficult to imagine an elementary particle without some oscillating electromagnetic field inside it. If we assume that the oscillating field is present inside the particle then the boundary conditions may require the corresponding oscillating electromagnetic field in vacuum that surrounds the particle. It is easy to show that the energy of this vacuum electromagnetic field will be infinite. However, it is possible that in vacuum only waves of the scalar potential take care of the necessary boundary conditions. Since the potential is not present in the energy-momentum tensor (5), there won't be any energy connected with it. **We are free to suggest that the massive elementary particles are the sources of these waves**. These waves are emitted continuously with an amplitude (or its square) that is proportional to the mass of the particle (this proposition seems to be reasonable). These waves are only outgoing waves. The incoming waves can only be plane incoherent waves (the spherical incoming coherent waves are impossible). We are not considering any incoming waves at this point.

We now show, by some examples, that the concept of the material continuum really works.

7 Obtaining solutions

Fortunately, all the equations for finding the solutions are linear. That allows us to seek a total solution as a superposition of the **particular solutions** which satisfy the equations and the boundary conditions separately. The only unlinear condition is (6a), which has to be fulfilled only on the disruption surface. Only the total solution can be used in (6a).

IP2 (Ideal Particle Second): Let us obtain the simplest static spherically symmetric solution with electric charge and electric field only. We have:

$$\left.\begin{aligned} &A^0_{\text{in}} = \alpha\big(R_0(z) - R_0(z_1) + b z_1\big) \\ &0 \leqslant z \leqslant z_1\,, \qquad j^0 = \frac{k_0^2\, c}{4\pi}\,\alpha R_0(z) \end{aligned}\right\}, \quad (13a)$$

$$\left.\begin{aligned} &A^0_{\text{out}} = \alpha\, b\, \frac{z_1^2}{z}\,, \qquad z_1 \leqslant z < \infty \\ &b \equiv \sqrt{R_0^2(z_1) + R_1^2(z_1)}\,, \quad z = k_0 r \end{aligned}\right\}, \quad (13b)$$

$$\left.\begin{aligned} &E^r_{\text{in}} = \alpha\, k_0\, R_1(z)\,, \qquad 0 \leqslant z \leqslant z_1 \\ &E^r_{\text{out}} = \alpha\, k_0\, b\, \frac{z_1^2}{z^2}\,, \qquad z_1 \leqslant z < \infty \end{aligned}\right\}, \quad (13c)$$

$$Q_{\text{tot}} = \frac{\alpha}{k_0}\, z_1^2\, b\,, \quad Q_{\text{surf}} = \frac{\alpha}{k_0}\, z_1^2\big(b - R_1(z_1)\big), \quad (13d)$$

$$\begin{aligned} mc^2 = \frac{\alpha^2}{2k_0}\big(&-z_1^2\, R_0(z_1)\, R_1(z_1) + z_1^3\, R_0^2(z_1) + \\ &+ z_1^3\, R_1^2(z_1)\big) = \frac{\alpha^2}{2k_0}\big(z_1 - \sin z_1\, \cos z_1\big), \end{aligned} \quad (13e)$$

where $R_0(z)$ and $R_1(z)$ are spherical Bessel functions. In general, the electric field has a jump at the boundary of IP2. The position of the boundary z_1 is arbitrary, but only at $z_1 = n\pi$ (correspond to IP1) the surface charge is zero and the electric field continuous. The first term in the mass expression (with the minus sign) corresponds to the energy of the interior region of the particle. It can be positive or negative, depending on z_1 (at $z_1 = n\pi$ it is zero). The second and third terms together represent the vacuum energy, which is positive. The total energy/mass remains positive at all z_1.

It was confirmed that IP is an unstable "equilibrium". Given a small perturbation it will grow in time. We hope to find a stable solution among the more complicated solutions than IP. The first idea was to introduce a spin in a static solutions. Then we tried to introduce the steady-state oscillating solutions. It was confirmed that there exist oscillating solutions with oscillating potential in vacuum that does not produce any vacuum E-M field. Then we tried to introduce a spin that originates from the oscillating solutions. Also we tried to consider the cylindrically shaped particles that are moving with the speed of light (close to a photon, see [3]). All these attempts indicate that the boundary of a particle that separates the material continuum from vacuum is a key player in any solution.

8 The mechanism of interaction between a constant electric field and a static charge (simplified thin layer model)

The simplest solutions can be obtained in plane symmetry where all the physical quantities depend only on the third coordinate — z. Let us consider symmetry of the type, vacuum — material continuum — vacuum. The thin layer of material continuum from $z = 0$ to $z = a$ (a is of the order of the size of elementary particle) will represent a simplified model of an elementary particle. The boundaries at $z = 0$ and $z = a$ are deemed to be enforced by the particle and the whole deficit of energy or momentum on these boundaries is deemed to go directly to the particle. Actually, if we have a deficit of energy or momentum it means that we are missing a particular solution that brings this deficit to zero, according to (6a).

For further discussion we need to write down the integral form of the energy-momentum conservation:

$$\frac{\partial}{c\,\partial t}\int\limits_V T^{m0}dV = -\oint\limits_\Sigma T^{mq}d\Sigma_q\,, \tag{6b}$$

where V is a 3D volume (which is not moving — it is our choice), and Σ is a 3D closed surface around this volume (obviously also not moving). The index m can correspond to any coordinate, while the index q corresponds only to the terrestrial coordinates (1, 2, 3). If $m = 0$ then the left part of (6b) is the time rate of increase of the energy inside V. T^{0q} is the 3-dimensional Pointing vector (or the flow of energy through a square unit per unit of time). If $m = 3$ (in the plane symmetry only one coordinate is of interest) then the left part of (6b) is the time rate of increase of the linear momentum of the volume V (actually it is a force applied to the volume V). T^{3q} is the 3-vector (in general q can be 1, 2, 3; in our case $q = 3$) of the flow of linear momentum through a square unit per unit of time. It is obvious that when static (or in a steady state) the left part of (6b) must be zero if there is no source/drain of energy/linear momentum inside the said volume.

Suppose the constant electric field in the first vacuum region is E. The scalar potential (aether quantity), the electric field, and the charge density are:

$$\left.\begin{aligned}
&\Phi_1 = -Ez + C_1\,, \qquad E_1 = E\\
&\Phi_2 = -\frac{E}{k_0}\sin k_0 z + C_1\cos k_0 z\\
&C_1 = \frac{4\pi Q + E(1-\cos k_0 a)}{k_0\sin k_0 a}\\
&E_2 = E\cos k_0 z + k_0 C_1 \sin k_0 z\\
&\rho = \frac{k_0^2}{4\pi}\,\Phi_2\,, \quad \Phi_3 = -(E+4\pi Q)(z-a) + C_2\\
&C_2 = \frac{4\pi Q\cos k_0 a - E(1-\cos k_0 a)}{k_0\sin k_0 a}\\
&E_3 = E + 4\pi Q
\end{aligned}\right\}. \tag{14}$$

Here the charge density is the solution of (7b) inside the second region. The potentials are the solutions of (10). All the physical quantities except ρ are continuous on the boundaries. That means that the jumps of the components of the energy-momentum tensor will be due to the jumps of the charge density only. The energy momentum tensor in this symmetry (and this particular case) is:

$$\left.\begin{aligned}
&T^{00} = \frac{1}{8\pi}E^2 - \frac{2\pi}{k_0^2}\rho^2 = T^{11} = T^{22}\\
&T^{03} = 0\,, \quad T^{33} = -\frac{1}{8\pi}E^2 - \frac{2\pi}{k_0^2\,c}\rho^2
\end{aligned}\right\}. \tag{15}$$

There is no energy flow in this system, but there is a flow of linear momentum. In the first vacuum region it is: $T^{33} = -E^2/8\pi$. Then it jumps on the first and on the second boundaries:

$$\left.\begin{aligned}
&T^{33}(z=0+) - T^{33}(z=0-) = -\frac{k_0^2\,C_1^2}{8\pi}\\
&T^{33}(z=a+) - T^{33}(z=a-) = \frac{k_0^2\,C_2^2}{8\pi}\\
&\frac{k_0^2\,C_1^2}{8\pi} - \frac{k_0^2\,C_2^2}{8\pi} = Q\left(E + \frac{4\pi Q}{2}\right)
\end{aligned}\right\}. \tag{16}$$

After that it is: $T^{33} = -(E + 4\pi Q)^2/8\pi$. As we go from left to right the jump on the first boundary is negative. That means that the small volume that includes the first boundary gets negative outside (we always consider the outside normal to the closed surface Σ) flow of linear momentum. That means that the volume itself, according to expression (6b), gets the positive rate of linear momentum, which is the force in the positive direction of the z-axis. The first boundary is pushed in the positive direction of the z-axis. The second boundary is also pushed, but in the negative direction of the z-axis. The difference is exactly equal to the force with which the field acts on a particle; see (16). We see that electric field does not act on a charge *per se* but only on a whole particle and only through its boundaries. This picture is true only at $t = 0$ because the missing particular solution that makes the appearance of "free" sources and drains most definitely will depend on time (the particle will begin to accelerate). This is the actual success of the proposed modification of CED.

9 The mechanism of interaction between a constant electric field and a static spherical charge

Here we will confirm that the thin layer treatment corresponds to the more accurate but more complicated spherical charge treatment. Suppose we have a constant electric field E directed along the z-axis in vacuum. Also we have a sphere of radius r_1 that separates the material continuum inside the sphere, from vacuum. The situation is static at $t = 0$. The potential in general has to satisfy the equation $A^k_{|k} = 0$ (10) everywhere, and equation (7c) inside the material continuum. This last equation, with 3rd derivatives, has to be satisfied strictly inside a material continuum and not on the disruption surface itself (where a single layer of charge/current density is possible and the charge/current density, $j^k = \frac{c}{4\pi} A^{k\,|a}_{|a}$, can be infinite). In vacuum we have

$$A^{k\,|a}_{|a} = 0\,. \tag{17}$$

Let us define a "dummy" potential by:

$$\left.\begin{aligned} & D^k_{|k} = 0\,, \quad D^{i|k} - D^{k|i} = 0\,, \\ & \text{consequently:} \quad D^{k\,|a}_{|a} = 0\,. \end{aligned}\right\} \tag{18}$$

If we have a solution A^k of (10)+(7c) or a solution of (10)+(17) then $A^k + D^k$ will also be the solution of the same equations (it does not matter whether inside the material continuum or in vacuum).

Now we return to our particular case. The solution of (18) that we are interested in would be: $D^0 = \text{const}$. If there is no time dependence then (10) is satisfied for any A^0 if a vector potential is zero. Equation (7c) is a Laplace operator taken from a Helmholtz equation. The solutions of the Helmholtz equation being considered would be: $R_0(k_0 r)$ and $R_1(k_0 r)\cos\theta$ where R_n are the spherical Bessel functions. In vacuum we consider the solutions e/r, (where e is the total charge), $r\cos\theta$, and $(1/r^2)\cos\theta$. So, let us consider the potential

$$\left.\begin{aligned} A^0_{\text{in}} &= \alpha R_0(k_0 r) + \frac{e}{r_1} - \alpha R_0(k_0 r_1) \\ A^0_{\text{out}} &= \frac{e}{r} + E\left(\frac{r_1^3}{r^2} - r\right)\cos\theta \end{aligned}\right\}. \tag{19}$$

It is continuous at $r = r_1$. The corresponding electric field and charge density will be,

$$\left.\begin{aligned} & E_{r\ \text{in}} = \alpha k_0 R_1(k_0 r) \\ & E_{r\ \text{out}} = \frac{e}{r^2} + E\left(1 + 2\frac{r_1^3}{r^3}\right)\cos\theta \\ & E_{\theta\ \text{in}} = 0\,, \quad E_{\theta\ \text{out}} = E\left(\frac{r_1^3}{r^3} - 1\right)\sin\theta \\ & \rho = \frac{\alpha k_0^2}{4\pi} R_0(k_0 r) \end{aligned}\right\}. \tag{20}$$

We see that the radial component of the electric field has a jump while the θ component is continuous. The surface charge density and the total surface charge are:

$$\left.\begin{aligned} 4\pi\rho_{\text{surf}} &= -E_{r\ \text{in}}(r_1) + E_{r\ \text{out}}(r_1) = \\ &= \frac{e}{r_1^2} - \alpha k_0 R_1(k_0 r_1) + 3E\cos\theta \\ Q_{\text{surf tot}} &= e - \alpha k_0 r_1^2 R_1(k_0 r_1) \end{aligned}\right\}. \tag{21}$$

We see that it does not matter what the relation is between the constants α and e, the surface of the particle has a "surface charge polarization" $3E\cos\theta$. Only this polarization will result in the net force on the charge. The polarization in the volume of the particle can be introduced using the solution $R_1(k_0 r)\cos\theta$. But this polarization won't change the net force (it can be introduced with any constant factor). We've made the corresponding calculations that support this statement. We do not present them here, for simplification.

The double radial component of the energy-momentum tensor will be:

$$\left.\begin{aligned} & 8\pi T^{rr} = E_\theta^2 - E_r^2 - \frac{16\pi^2}{k_0^2}\rho^2 \\ & 8\pi T^{rr}_{\text{surf in}} = -\alpha^2 k_0^2 \left(R_0^2(k_0 r_1) + R_1^2(k_0 r_1)\right) \\ & 8\pi T^{rr}_{\text{surf out}} = -\left(\frac{e^2}{r_1^4} + \frac{6e}{r_1^2} E\cos\theta + 9E^2\cos^2\theta\right) \\ & T^{\theta r}_{\text{surf in}} = T^{\theta r}_{\text{surf out}} = 0 \end{aligned}\right\}. \tag{22}$$

The force applied to the surface will be normal to the surface and equal to $T_{\text{surf in}} - T^{rr}_{\text{surf out}}$. This force is zero if $E = 0$. This case corresponds to the true static solution of our equations with (6a) satisfied. This solution enforces the spherical boundary. If E is not zero, then we do not know the actual solution because (6a) is not satisfied. The actual solution will be not static. But we can calculate the force at the moment when E was "turned on". To get the z component of this force we have to multiply the expression on $\cos\theta$. If we integrate this over the spherical surface then all the terms except the one with $\cos\theta$ are zero. The result of integration will be eE. This is exactly the force with which the electric field E acts on a charge e.

10 The transverse electromagnetic wave

Let us consider that the transverse electromagnetic wave is coming from the left and encounters the layer of material continuum. We expect to find the transmitted and reflected waves as well as the radiation pressure. "Behind" the transverse E-M wave we find that the transverse aether wave with only an x component (for x-polarized E-M wave) of the vector potential (aether current) is different from zero:

$$\left.\begin{aligned}
&{}_1A^1 = \Phi_1^+ + \Phi_1^- \\
&\Phi_1^+ = F_1^+ e^{-ikz}, \quad \Phi_1^- = F_1^- e^{ikz} \\
&{}_1E_1 = -ik\cdot {}_1A^1, \quad {}_1H_2 = -ik\cdot\left(\Phi_1^+ - \Phi_1^-\right) \\
&{}_2A^1 = \Phi_2^+ + \Phi_2^- \\
&\Phi_2^+ = F_2^+ e^{-ik'z}, \quad \Phi_2^- = F_2^- e^{ik'z} \\
&k = \frac{\omega}{c}, \quad (k')^2 = k_0^2 + k^2 \\
&{}_2E_1 = -ik\cdot {}_2A^1, \quad {}_2H_2 = -ik'\cdot\left(\Phi_2^+ - \Phi_2^-\right) \\
&j(z,t) = \frac{ck_0^2}{4\pi}\cdot {}_2A^1 \\
&{}_3A^1 = F_3^+ e^{-ikz} \\
&{}_3E_1 = -ik\cdot {}_3A^1, \quad {}_3H_2 = -ik\cdot {}_3A^1
\end{aligned}\right\}, \qquad (23)$$

where the prefixes to the fields always denote the number of the region (we did not attach indexes to the current density j because it is different from zero only in the second region). We assume that all the functions depend on t through the factor $\exp(i\omega t)$. In the first region the given incoming wave F_1^+ and some reflected wave F_1^- are present. In the second region two waves are present. They satisfy the equations:

$${}_2A^{1\prime\prime} + k^2\cdot {}_2A^1 = -\frac{4\pi}{c}\,j\,, \qquad \frac{\partial}{\partial x}\cdot {}_2A^1 = 0\,. \qquad (24)$$

On the boundaries the vector potential (aether current) and its first derivative have to be continuous. We found that

$$\left.\begin{aligned}
&F_1^- = -F_1^+\,\frac{2ik_0^2\sin(k'a)}{D} \\
&F_3^+ e^{-ika} = F_1^+\,\frac{4kk'}{D} \\
&D \equiv (k+k')^2 e^{ik'a} - (k-k')^2 e^{-ik'a} \\
&F_2^+ = F_1^+\,\frac{2k(k+k')}{D}\,e^{ik'a} \\
&F_2^- = F_1^+\,\frac{2k(k'-k)}{D}\,e^{-ik'a}
\end{aligned}\right\}. \qquad (25)$$

Here we found the amplitudes of reflected and transmitted waves and the amplitudes of both waves in the second region (only F_1^+ is considered to be real and given).

We found previously that the energy-momentum tensor in a material continuum has the form (one-dimensional symmetry assumed):

$$\left.\begin{aligned}
&T^{00} = \frac{1}{8\pi}\left(E^2+H^2\right) - \frac{2\pi}{k_0^2c^2}\,j^2 \\
&T^{33} = \frac{1}{8\pi}\left(E^2+H^2\right) + \frac{2\pi}{k_0^2c^2}\,j^2 \\
&T^{11} = \frac{1}{8\pi}\left(-E^2+H^2\right) - \frac{2\pi}{k_0^2c^2}\,j^2 = -T^{22} \\
&T^{03} = \frac{1}{4\pi}\,EH
\end{aligned}\right\}. \qquad (26)$$

Since we use complex numbers — we have to take the real parts of the physical values, multiply them and then take the time average. The result will be the real part of the product of the first complex amplitude on the conjugate of the second complex amplitude. The result in the second region is:

$$\left.\begin{aligned}
&\frac{2\pi}{k_0^2c^2}\,j^2 = F_1^{+2}\,\frac{k_0^2k^2}{\pi|D|^2}\,\times \\
&\qquad\times\left(k_0^2 + 2k^2 + k_0^2\cos 2k'(a-z)\right) \\
&T^{00} = -F_1^{+2}\,\frac{2k^2}{\pi|D|^2}\,\times \\
&\qquad\times\left(k_0^4\cos 2k'(a-z) - k^2\left(k_0^2+2k^2\right)\right) \\
&T^{03} = F_1^{+2}\,\frac{4k^4k'^2}{\pi|D|^2} \\
&T^{33} = F_1^{+2}\,\frac{2k^2k'^2}{\pi|D|^2}\left(k_0^2+2k^2\right)
\end{aligned}\right\}. \qquad (27)$$

The electric and magnetic fields are continuous in this system. The flow of energy appear to be independent of z in the second region. It is continuous on the boundaries (see (26); the currents are not included in T^{03}). This means that it is constant through the whole system. The flow of linear momentum (T^{33}) is positive in the first region and then jumps up on the first boundary due to the jump of the current j. It means that the surface integral in (6b) is positive and the first boundary is losing linear momentum. The surface is pulled in the negative direction of the z-axis. But this pull is less than another pull due to the jump on the second boundary; this can be determined from (27). We consider $k'a \sim \frac{\pi}{4}$, but it will be true for any $k'a$ different from π. Notice also that at $k'a = \pi$ the reflected wave is zero as can be seen from (25). Thus, the material continuum will experience the force (through its boundaries) in the positive direction of the z-axis. The numerical value of this force can be calculated from the jumps and it is equal to the force that we usually calculate from the linear momentum of incident transmitted and reflected waves.

11 The longitudinal aether (dummy) wave

Let us consider a longitudinal aether wave travelling from the left, encountering the layer of material continuum. There are no electromagnetic fields that accompany this wave in vacuum. Not so inside the material continuum. We have:

$$\left.\begin{aligned}
&{}_1A^0 = \Phi_1^+ + \Phi_1^- \\
&\Phi_1^+ = F_1^+ e^{-ikz}, \qquad \Phi_1^- = F_1^- e^{ikz}, \quad k = \frac{\omega}{c} \\
&{}_1A^3 = \Phi_1^+ - \Phi_1^-, \quad {}_2A^0 = \Phi_2^+ + \Phi_2^- \\
&\Phi_2^+ = F_2^+ e^{-ik'z}, \qquad \Phi_2^- = F_2^- e^{ik'z} \\
&{}_2A^3 = \frac{k}{k'}\left(\Phi_2^+ - \Phi_2^-\right), \quad (k')^2 = k_0^2 + k^2 \\
&j^0(z,t) = \frac{c k_0^2}{4\pi}\left(\Phi_2^+ + \Phi_2^-\right) \\
&j^3(z,t) = \frac{c k_0^2 k}{4\pi k'}\left(\Phi_2^+ - \Phi_2^-\right) \\
&E_3 = \frac{i k_0^2}{k'}\left(\Phi_2^+ - \Phi_2^-\right), \quad {}_3A^0 = {}_3A^3 = F_3^+ e^{-ikz}
\end{aligned}\right\} \qquad (28)$$

where we assume that all the functions depend on t through the factor $\exp(i\omega t)$. In the first region the given incoming wave F_1^+ and some reflected wave F_1^- are present (both are dummy waves). In the second region two waves are present. They satisfy the equations:

$$\left.\begin{aligned}
&{}_2A^{0\prime\prime} + k^2 \cdot {}_2A^0 = -\frac{4\pi}{c}\, j^0 \\
&{}_2A^{3\prime\prime} + k^2 \cdot {}_2A^3 = -\frac{4\pi}{c}\, j^3 \\
&i k \cdot {}_2A^0 + {}_2A^{3\prime} = 0\,.
\end{aligned}\right\}. \qquad (29)$$

To define all the waves we have to satisfy the conditions on the boundaries. The scalar potential (aether quantity) and the vector potential (aether current) should be continuous across the boundaries. We found that

$$\left.\begin{aligned}
&\text{on } z = a: \quad F_2^+ e^{-ik'a} = \frac{k + k'}{2k}\, F_3^+ e^{-ika} \\
&\qquad\qquad F_2^- e^{ik'a} = -\frac{k' - k}{2k}\, F_3^+ e^{-ika} \\
&\text{on } z = 0: \quad F_1^- = F_1^+ \frac{2 i k_0^2 \sin(k'a)}{D} \\
&\qquad\qquad F_3^+ e^{-ika} = F_1^+ \frac{4kk'}{D} \\
&D \equiv (k + k')^2 e^{ik'a} - (k' - k)^2 e^{-ik'a} \\
&F_2^+ = F_1^+ \frac{2k'(k + k')}{D}\, e^{ik'a} \\
&F_2^- = -F_1^+ \frac{2k'(k' - k)}{D}\, e^{-ik'a}
\end{aligned}\right\}. \qquad (30)$$

Here we found the amplitudes of reflected and transmitted waves and the amplitudes of both waves in the second region (only F_1^+ is considered to be real).

From (28) we can calculate the derivatives:

$$\left.\begin{aligned}
&{}_1A^{0\prime} = -i k \cdot {}_1A^3, \quad {}_2A^{0\prime} = -\frac{i k'^2}{k} \cdot {}_2A^3 \\
&{}_1A^{3\prime} = -i k \cdot {}_1A^0, \quad {}_2A^{3\prime} = -i k \cdot {}_2A^0
\end{aligned}\right\}. \qquad (28a)$$

We see that the aether current (A^3) has a continuous derivative while the derivative of aether quantity (A^0) has a jump at the boundaries. This means that there are surface charges associated with the boundaries.

We notice from (28) that the electric field, charge density, and current density are different from zero inside the second region. This means that the material continuum produces a kind of physical response to the energy-less dummy waves. We also found previously that the energy-momentum tensor in a material continuum has the form (one-dimensional symmetry assumed),

$$\left.\begin{aligned}
&T^{00} = \frac{1}{8\pi} E^2 - \frac{2\pi}{k_0^2 c^2}\left(c^2\rho^2 + j^2\right) \\
&T^{11} = -\frac{1}{8\pi} E^2 - \frac{2\pi}{k_0^2 c^2}\left(c^2\rho^2 + j^2\right) \\
&T^{22} = T^{33} = \frac{1}{8\pi} E^2 - \frac{2\pi}{k_0^2 c^2}\left(c^2\rho^2 - j^2\right) \\
&T^{01} = -\frac{4\pi}{k_0^2 c^2}\, c\rho j
\end{aligned}\right\}. \qquad (31)$$

To actually calculate a time average of the energy-momentum tensor we have to take the real parts of the physical values, multiply them, and then take the time average. The result will be the real part of the product of the first complex amplitude on the conjugate of the second complex amplitude. The result of calculation is,

$$\left.\begin{aligned} T^{33} &= -(F_1^+)^2\,\frac{2\,k_0^4}{\pi|D|^2}\,(k_0^2+2k^2) \\ T^{03} &= -(F_1^+)^2\,\frac{4\,k_0^2\,k^2\,k'^2}{\pi|D|^2} \\ T^{00} &= (F_1^+)^2\,\frac{2\,k_0^6}{\pi|D|^2}\cos\bigl(2\,k'(a-z)\bigr) \end{aligned}\right\}. \tag{32}$$

The first two time averages of the tensor components appear to be independent of z. The energy density depends on z. All these tensor components are zero in both vacuum regions. This means that all of them jump at the boundaries.

On the first boundary the jump of T^{33} is negative. It means that the first boundary will be pushed to the right. On the second boundary the jump will be positive and the same by its absolute value (because T^{33} is constant inside the second region). The second boundary will be pushed in the negative direction of the z-axis with the same force — we have equilibrium — no "free" force.

On the first boundary the jump of T^{03} is negative. It means that the first boundary will be getting energy. On the second boundary the jump will be positive and the same by its absolute value (because T^{03} is constant inside the second region). The second boundary will be losing the same amount of energy — no "free" energy.

It appears that the particular solutions that we have carry energy and momentum from the second boundary to the first, while the missing particular solution carries them back. If we imagine that the energy and momentum can be lost on the way from the sourse to the drain then we get a free linear momentum directed to the sourse of dummy waves (gravitational force). Also we get a free energy for heating stars. This unconservation proposition can be quite real if we consider that we obtained the conservation of energy-momentum from the requirement of minimum action. In the real physical world the action may has a small jitter around the exact minimum. Obviously this jitter is very small so that it can revile itself only on a cosmic scale.

At the present time we hesitate in proceeding further from these results because the meaning of these results has still to be clarified.

12 De Broglie's waves

Let us suppose, in addition (see Section 6), that the frequency of dummy waves (as well as the intensity) also proportional to the mass of the particle: $\omega = mc^2/\hbar$. The resting particles are present in abundance in the experimental arrangement itself. These resting particles can be partially synchronized in some proximity (the extent of this proximity is not known yet) of any point inside the experimental device. We can expect some standing scalar waves of a dummy generator that can be experienced by the moving particle independently of the direction of motion. In this case we can explain De Broglie's waves as beat frequency waves between the frequency of a resting particle and the Doppler shifted frequency of a moving particle. The rôle of the nonlinear device that is necessary to obtain the beat frequency wave, can be very well played by the boundary of the particle itself. This will explain "the wave properties of particles" by purely classical means, as first proposed in 1993 by Milo Wolff [7].

In the foregoing reformulation of conventional classical electrodynamics, we omitted the interaction term in the Lagrangian/Hamiltonian. Quantum Theory was undermined by this action. One should note that, historically, after the creation of quantum theory, there were attempts to legitimize the electromagnetic potential as a physically measurable value (see R. Feynman, [1]). Still, it is too early to try to find a classical basis for quantum theory, but the direction to go is that of the physical realm of the electromagnetic potential.

13 Conclusion

Probably it is not right to keep the disruption surface devoid from surface energy and surface tension. To introduce that correctly we have to consider some surface Lagrange density and add a surface integral to the action volume integral. That I hope to see in a future development.

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A Model of Electron-Positron Pair Formation

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The elementary electron-positron pair formation process is considered in terms of a revised quantum electrodynamic theory, with special attention to the conservation of energy, spin, and electric charge. The theory leads to a wave-packet photon model of narrow line width and needle-radiation properties, not being available from conventional quantum electrodynamics which is based on Maxwell's equations. The model appears to be consistent with the observed pair production process, in which the created electron and positron form two rays that start within a very small region and have original directions along the path of the incoming photon. Conservation of angular momentum requires the photon to possess a spin, as given by the present theory but not by the conventional one. The nonzero electric field divergence further gives rise to a local intrinsic electric charge density within the photon body, whereas there is a vanishing total charge of the latter. This may explain the observed fact that the photon decays on account of the impact from an external electric field. Such a behaviour should not become possible for a photon having zero local electric charge density.

1 Introduction

During the earliest phase of the expanding universe, the latter is imagined to be radiation-dominated, somewhat later also including particles such as neutrinos and electron-positron pairs. In the course of the expansion the "free" states of highly energetic electromagnetic radiation thus become partly "condensed" into "bound" states of matter as determined by Einstein's energy relation.

The pair formation has for a long time both been studied experimentally [1] and been subject to theoretical analysis [2]. When a high-energy photon passes the field of an atomic nucleus or that of an electron, it becomes converted into an electron and a positron. The orbits of these created particles form two rays which start within a very small volume and have original directions along the path of the incoming photon.

In this paper an attempt is made to understand the elementary electron-positron pair formation process in terms of a revised quantum electrodynamic theory and its application to a wave-packet model of the individual photon [3, 4, 5, 6]. The basic properties of the latter will be described in Section 2, the intrinsic electric charge distribution of the model in Section 3, the conservation laws of pair formation in Section 4, some questions on the vacuum state in Section 5, and the conclusions are finally presented in Section 6.

2 A photon model of revised quantum electrodynamics

The detailed deductions of the photon model have been reported elsewhere [3, 4, 5, 6] and will only be summarized here. The corresponding revised Lorentz and gauge invariant theory represents an extended version which aims beyond Maxwell's equations. Here the electric charge density and the related electric field divergence are nonzero in the vacuum state, as supported by the quantum mechanical vacuum fluctuations and the related zero-point energy. The resulting wave equation of the electric field $\mathbf{E}$ then has the form

$$\left(\frac{\partial^2}{\partial t^2} - c^2\nabla^2\right)\mathbf{E} + \left(c^2\nabla + \mathbf{C}\frac{\partial}{\partial t}\right)(\operatorname{div}\mathbf{E}) = 0\,, \quad (1)$$

which includes the effect of a space-charge current density $\mathbf{j} = \varepsilon_0(\operatorname{div}\mathbf{E})\mathbf{C}$ that arises in addition to the displacement current $\varepsilon_0\,\partial\mathbf{E}/\partial t$. The velocity $\mathbf{C}$ has a modulus equal to the velocity c of light, as expressed by $\mathbf{C}^2 = c^2$. The induction law still has the form

$$\operatorname{curl}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad (2)$$

with $\mathbf{B}$ standing for the magnetic field strength.

The photon model to be discussed here is limited to axisymmetric normal modes in a cylindrical frame (r, φ, z) where $\partial/\partial\varphi = 0$. A form of the velocity vector

$$\mathbf{C} = c\,(0, \cos\alpha, \sin\alpha) \quad (3)$$

is chosen under the condition $0 < |\cos\alpha| \ll 1$, such as not to get into conflict with the Michelson-Morley experiments, i.e. by having phase and group velocities which only differ by a very small amount from c. The field components can be expressed in terms of a generating function

$$G_0 \cdot G = E_z + (\cot\alpha)E_\varphi\,, \qquad G = R(\rho)\,e^{i(-\omega t + kz)}, \quad (4)$$

where G_0 is an amplitude factor, $\rho = r/r_0$ with r_0 as a characteristic radial distance of the spatial profile, and ω and k standing for the frequency and wave number of a normal mode. Such modes are superimposed to form a wave-packet having the spectral amplitude

$$A_k = \left(\frac{k}{k_0^2}\right) \exp\left[-z_0^2(k-k_0)^2\right], \tag{5}$$

where k_0 and $\lambda_0 = \frac{2\pi}{k_0} = \frac{c}{\nu_0}$ are the main wave number and wave length, and $2z_0$ represents the effective axial length of the packet. According to experimental observations, the packet must have a narrow line width, as expressed by $k_0 z_0 \gg 1$. The spectral averages of the field components in the case $|\cos\alpha| \ll 1$ are then

$$\bar{E}_r = -iE_0 R_5\,, \tag{6}$$

$$\bar{E}_\varphi = E_0(k_0 r_0)(\sin\alpha)(\cos\alpha)R_3\,, \tag{7}$$

$$\bar{E}_z = E_0(k_0 r_0)(\cos\alpha)^2 R_4 \tag{8}$$

and

$$\bar{B}_r = -\frac{1}{c}\frac{1}{\sin\alpha}\,\bar{E}_\varphi\,, \tag{9}$$

$$\bar{B}_\varphi = \frac{1}{c}(\sin\alpha)\,\bar{E}_r\,, \tag{10}$$

$$\bar{B}_z = \frac{1}{c}(\cos\alpha)\,\frac{R_8}{R_5}\,\bar{E}_r\,. \tag{11}$$

Here

$$R_3 = \rho^2 D_\rho R\,,\quad R_4 = R - R_3\,,\quad R_5 = \frac{d}{d\rho}(R - R_3)\,, \tag{12}$$

$$R_8 = \left(\frac{d}{d\rho} + \frac{1}{\rho}\right) R_3\,,\qquad D_\rho = \frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} \tag{13}$$

and

$$E_0 = e_0 \bar{f}\,,\quad e_0 = \frac{g_0\sqrt{\pi}}{k_0^2 r_0 z_0}\,,\quad G_0 = g_0(\cos\alpha)^2, \tag{14}$$

$$\bar{f} = \left[\cos(k_0\bar{z}) + i\sin(k_0\bar{z})\right] \exp\left[-\left(\frac{\bar{z}}{2z_0}\right)^2\right], \tag{15}$$

where $\bar{z} = z - c(\sin\alpha)t$.

Choosing the part of the normalized generating function G which is symmetric with respect to the axial centre $\bar{z} = 0$ of the moving wave packet, the components $(\bar{E}_\varphi, \bar{E}_z, \bar{B}_r)$ become symmetric and the components $(\bar{E}_r, \bar{B}_\varphi, \bar{B}_z)$ antisymmetric with respect to the same centre. Then the integrated electric charge and magnetic moment vanish.

The equivalent total mass defined by the electromagnetic field energy and the energy relation by Einstein becomes on the other hand

$$m \cong 2\pi\,\frac{\varepsilon_0}{c^2}\,r_0^2 W_m e_0^2 \int_{-\infty}^{+\infty} f^2\,d\bar{z}\,, \tag{16}$$

$$W_m = \int \rho\,R_5^2\,d\rho\,,$$

where expression (15) has to be replaced by the reduced function

$$f = \left[\sin(k_0\bar{z})\right] \exp\left[-\left(\frac{\bar{z}}{2z_0}\right)^2\right] \tag{17}$$

due to the symmetry condition on G with respect to $\bar{z} = 0$. Finally the integrated angular momentum is obtained from the Poynting vector, as given by

$$s \cong -2\pi\varepsilon_0 \int_{-\infty}^{+\infty}\int r^2\,\bar{E}_r\bar{B}_z\,dr\,d\bar{z} =$$

$$= 2\pi\,\frac{\varepsilon_0}{c}\,(\cos\alpha)\,r_0^3\,W_s\,e_0^2 \int_{-\infty}^{+\infty} f^2\,d\bar{z}\,, \tag{18}$$

$$W_s = -\int \rho^2\,R_5 R_8\,d\rho\,.$$

Even if the integrated (total) electric charge of the photon body as a whole vanishes, there is on account of the nonzero electric field divergence a local nonzero electric charge density

$$\bar{\rho} = e_0 f\,\frac{\varepsilon_0}{r_0}\frac{1}{\rho}\frac{d}{d\rho}(\rho R_5)\,. \tag{19}$$

Due to the factor $\sin(k_0\bar{z})$ this density oscillates rapidly in space as one proceeds along the axial direction. Thus the electric charge distribution consists of two equally large positive and negative oscillating contributions of total electric charge, being mixed up within the volume of the wave packet.

To proceed further the form of the radial function $R(\rho)$ has now to be specified. Since the experiments clearly reveal the pair formation to take place within a small region of space, the incoming photon should have a strongly limited extension in its radial (transverse) direction, thus having the character of “needle radiation”. Therefore the analysis is concentrated on the earlier treated case of a function R which is divergent at $\rho = 0$, having the form

$$R(\rho) = \rho^{-\gamma} e^{-\rho}\,,\qquad \gamma > 0\,. \tag{20}$$

In the radial integrals of equations (16) and (18) the dominant terms then result in $R_8 \cong -R_5$ and

$$W_m = \int_{\rho_m}^{\infty} \rho\,R_5^2\,d\rho = \frac{1}{2}\,\gamma^5 \rho_m^{-2\gamma}\,, \tag{21}$$

$$W_s = \int_{\rho_s}^{\infty} \rho^2\,R_5^2\,d\rho = \frac{1}{2}\,\gamma^5 \rho_s^{-2\gamma+1}\,, \tag{22}$$

where $\rho_m \ll 1$ and $\rho_s \ll 1$ are small nonzero radii at the origin $\rho = 0$. To compensate for the divergence of W_m and W_s when ρ_m and ρ_s approach zero, we now introduce the shrinking parameters

$$r_0 = c_r \cdot \varepsilon\,,\qquad g_0 = c_g \cdot \varepsilon^\beta\,, \tag{23}$$

where c_r and c_g are positive constants and the dimensionless smallness parameter ε is defined by $0 < \varepsilon \ll 1$. From relations

(14)–(18), (21)–(23), the energy relation $mc^2 = h\nu_0$, and the quantum condition of the angular momentum, the result becomes

$$m = \pi^2 \frac{\varepsilon_0}{c^2} \gamma^5 \left(\frac{1}{k_0^2 z_0}\right)^2 c_g^2 \frac{\varepsilon^{2\beta}}{\rho_m^{2\gamma}} J_m = \frac{h}{\lambda_0 c}, \quad (24)$$

$$s = \pi^2 \frac{\varepsilon_0}{c} \gamma^5 \left(\frac{1}{k_0^2 z_0}\right)^2 c_g^2 c_r (\cos\alpha) \frac{\varepsilon^{2\beta+1}}{\rho_s^{2\gamma-1}} J_m = \frac{h}{2\pi} \quad (25)$$

with

$$J_m = \int_{-\infty}^{+\infty} f^2 \, d\bar{z} \cong z_0 \sqrt{2\pi}\,. \quad (26)$$

Here we are free to choose $\beta = \gamma \gg 1$ which leads to

$$\rho_s \cong \rho_m = \varepsilon\,. \quad (27)$$

The lower limits ρ_m, and ρ_s of the integrals (21) and (22) then decrease linearly with ε and with the radius r_0. This forms a "similar" set of geometrical configurations, having a common shape which is independent of ρ_m, ρ_s, and ε in the range of small ε.

Taking $\hat{r} = r_0$ as an effective radius of the configuration (20), combination of relations (23)–(25) finally yields a photon diameter

$$2\hat{r} = \frac{\varepsilon \lambda_0}{\pi |\cos\alpha|} \quad (28)$$

being independent of γ. Thus the individual photon model becomes strongly needle-shaped when $\varepsilon \leqslant |\cos\alpha|$.

It should be observed that the photon spin of expression (25) disappears when $\operatorname{div} \mathbf{E}$ vanishes and the basic relations reduce to Maxwell's equations. This is also the case under more general conditions, due to the behaviour of the Poynting vector and to the requirement of a finite integrated field energy [3, 4, 5, 6].

3 The intrinsic electric charge distribution

We now turn to the intrinsic electric charge distribution within the photon wave-packet volume, representing an important but somewhat speculative part of the present analysis. It concerns the detailed process by which the photon configuration and its charge distribution are broken up to form a pair of particles of opposite electric polarity. Even if electric charges can arise and disappear in the vacuum state due to the quantum mechanical fluctuations, it may be justified as a first step to investigate whether the total intrinsic photon charge of one polarity can become sufficient as compared to the electric charges of the electron and positron.

With the present strongly oscillating charge density in space, the total intrinsic charge of either polarity can be estimated with good approximation from equations (17) and (19). This charge appears only within half of the axial extension of the packet, and its average value differs by the factor $\frac{2}{\pi}$ from the local peak value of its sinusoidal variation. From equation (19) this intrinsic charge is thus given by

$$q = \frac{z_0}{\pi} \int_{\rho_q}^{\infty} 2\pi r \frac{\bar{\rho}}{f} \, dr = 2\sqrt{\pi} z_0 \varepsilon_0 \gamma^3 \frac{1}{k_0^2 z_0} c_g \frac{\varepsilon^\beta}{\rho_q^\gamma}, \quad (29)$$

where the last factor becomes equal to unity when $\beta = \gamma$ and the limit $\rho_q = \varepsilon$ for a similar set of geometrical configurations. Relations (29) and (24) then yield

$$q^2 = \frac{8}{\pi^3} \varepsilon_0 c^2 \gamma z_0 m = \frac{8}{\pi^3} \varepsilon_0 c h \gamma \frac{z_0}{\lambda_0} \cong \cong 45 \times 10^{-38} \gamma \frac{z_0}{\lambda_0} \quad (30)$$

and

$$\frac{q}{e} \cong 4.2 \left(\frac{\gamma z_0}{\lambda_0}\right)^{1/2}. \quad (31)$$

With a large γ and a small line width leading to $\lambda_0 \ll z_0$, the total intrinsic charge thus substantially exceeds the charge of the created particle pair. However, the question remains how much of the intrinsic charge becomes available during the disintegration process of the photon.

A much smaller charge would become available in a somewhat artificial situation where the density distribution of charge is perturbed by a 90 degrees phaseshift of the sinusoidal factor in expression (17). This would add a factor $2\exp\left[-4\pi^2 (z_0/\lambda_0)^2\right]$ to the middle and right-hand members of equation (30), and makes $q \gtrsim e$ only for extremely large values of γ and for moderately narrow line widths.

4 Conservation laws of pair formation

There are three conservation laws to be taken into account in the pair formation process. The first concerns the total energy. Here we limit ourselves to the marginal case where the kinetic energy of the created particles can be neglected as compared to the equivalent energy of their rest masses. Conservation of the total energy is then expressed by

$$mc^2 = \frac{hc}{\lambda_0} = 2m_e c^2. \quad (32)$$

Combination with equation (28) yields an effective photon diameter

$$2\hat{r} = \frac{\varepsilon h}{2\pi m_e c |\cos\alpha|}. \quad (33)$$

With $\varepsilon \leqslant |\cos\alpha|$ we have $2\hat{r} \leqslant 3.9 \times 10^{-13}$ m being equal to the Compton wavelength and representing a clearly developed form of needle radiation.

The second conservation law concerns the preservation of angular momentum. It is satisfied by the spin $\frac{h}{2\pi}$ of the photon in the capacity of a boson particle, as given by expression (25). This angular momentum becomes equal to the sum of the spin $\frac{h}{4\pi}$ of the created electron and positron being fermions. In principle, the angular momenta of the two created particles could also become antiparallel and the spin of the photon zero, but such a situation would contradict all other experience about the photon spin.

The third conservation law deals with the preservation of the electric charge. This condition is clearly satisfied by the vanishing integrated photon charge, and by the opposite polarities of the created particles. In a more detailed picture where the photon disintegrates into the charged particles, it could also be conceived as a splitting process of the positive and negative parts of the intrinsic electric charge distributions of the photon.

Magnetic moment conservation is satisfied by having parallel angular momenta and opposite charges of the electron and positron, and by a vanishing magnetic moment of the photon [5, 6].

5 Associated questions of the vacuum state concept

The main new feature of the revised quantum electrodynamical theory of Section 2 is the introduction of a nonzero electric field divergence in the vacuum, as supported by the existence of quantum mechanical fluctuations. In this theory the values of the dielectric constant and the magnetic permeability of the conventional empty-space vacuum have been adopted. This is because no electrically polarized and magnetized atoms or molecules are assumed to be present, and that the vacuum fluctuations as well as superimposed regular phenomena such as waves take place in a background of empty space.

As in a review by Gross [7], the point could further be made that a "vacuum polarization" screens the point-charge-like electron in such a way that its effective electrostatic force vanishes at large distances. There is, however, experimental evidence for such a screening not to become important at the scale of the electron and photon models treated here. In the vacuum the electron is thus seen to be subject to scattering processes due to its full electrostatic field, and an electrically charged macroscopic object is also associated with such a measurable field. This would be consistent with a situation where the vacuum fluctuations either are small or essentially independent as compared to an external disturbance, and where their positive contributions to the local electric charge largely cancel their negative ones.

To these arguments in favour of the empty-space values of the dielectric constant and the magnetic permeability two additional points can also be added. The first is due to the Heisenberg uncertainty relation which implies that the vacuum fluctuations appear spontaneously during short time intervals and independently of each other. They can therefore hardly have a screening effect such as that due to Debye in a quasi-neutral plasma. The second point is based on the fact that static measurements of the dielectric constant and the magnetic permeability result in values the product of which becomes equal to the inverted square of the measured velocity of light.

6 Conclusions

The basis of the conservation laws in Section 4 is rather obvious, but it nevertheless becomes nontrivial when a comparison is made between conventional quantum electrodynamics based on Maxwell's equations on one hand and the present revised theory on the other. Thereby the following points should be observed:

- The needle-like radiation of the present photon model is necessary for understanding the observed creation of an electron-positron pair which forms two rays that start within a small region, and which have original directions along the path of the incoming photon. Such needle radiation does not come out of conventional theory [3, 4, 5, 6];
- The present revised theory leads to a nonzero spin of the photon, not being available from conventional quantum electrodynamics based on Maxwell's equations; [3, 4, 5, 6]. The present model is thus consistent with a photon as a boson which decays into two fermions;
- The nonzero divergence of the electric field in the present theory allows for a local nonzero electric charge density, even if the photon has a vanishing net charge. This may indicate how the intrinsic electric photon charges can form two charged particles of opposite polarity when the photon structure becomes disintegrated. Such a process is supported by the experimental fact that the photon decays into two charged particles through the impact of the electric field from an atomic nucleus or from an electron. This could hardly occur if the photon body would become electrically neutral at any point within its volume. Apart from such a scenario, the electromagnetic field configuration of the photon may also be broken up by nonlinear interaction with a strong external electric field;
- The present approach to the pair formation process has some similarity with the breaking of the stability of vacuum by a strong external electric field, as being investigated by Fradkin et al. [8].

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Ricci Flow and Quantum Theory

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We show how Ricci flow is related to quantum theory via Fisher information and the quantum potential.

1 Introduction

In [9, 13, 14] we indicated some relations between Weyl geometry and the quantum potential, between conformal general relativity (GR) and Dirac-Weyl theory, and between Ricci flow and the quantum potential. We would now like to develop this a little further. First we consider simple Ricci flow as in [35, 49]. Thus from [35] we take the Perelman entropy functional as $(\mathbf{1A})$ $\mathfrak{F}(g,f) = \int_M (|\nabla f|^2 + R)\exp(-f)dV$ (restricted to f such that $\int_M \exp(-f)dV = 1$) and a Nash (or differential) entropy via $(\mathbf{1B})$ $N(u) = \int_M u\log(u)dV$ where $u = \exp(-f)$ (M is a compact Riemannian manifold without boundary). One writes $dV = \sqrt{\det(g)}\prod dx^i$ and shows that if $g \to g + \mathrm{sh}\,(g, h \in \mathcal{M} = Riem(M))$ then $(\mathbf{1C})$ $\partial_s \det(g)|_{s=0} = g^{ij}h_{ij}\det(g) = (Tr_g h)\det(g)$. This comes from a matrix formula of the following form $(\mathbf{1D})$ $\partial_s \det(A+B)|_{s=0} = (A^{-1}:B)\det(A)$ where $A^{-1}: B = a^{ij}b_{ji} = a^{ij}b_{ij}$ for symmetric B (a^{ij} comes from A^{-1}). If one has Ricci flow $(\mathbf{1E})$ $\partial_s g = -2Ric$ (i.e. $\partial_s g_{ij} = -2R_{ij}$) then, considering $h \sim -2Ric$, one arrives at $(\mathbf{1F})$ $\partial_s dV = -RdV$ where $R = g^{ij}R_{ij}$ (more general Ricci flow involves $(\mathbf{1G})$ $\partial_t g_{ik} = -2(R_{ik} + \nabla_i\nabla_k\phi)$). We use now t and s interchangeably and suppose $\partial_t g = -2Ric$ with $u = \exp(-f)$ satisfying $\Box^* u = 0$ where $\Box^* = -\partial_t - \Delta + R$. Then $\int_M \exp(-f)dV = 1$ is preserved since $(\mathbf{1H})$ $\partial_t \int_M u dV = \int_M (\partial_s u - Ru)dV = -\int_M \Delta u dV = 0$ and, after some integration by parts,

$$\partial_t N = \int_M \left[\partial_t u(\log(u)+1)dV + u\log(u)\partial_t dV\right] = \int_M (|\nabla f|^2 + R)e^{-f}dV = \mathfrak{F}. \tag{1.1}$$

In particular for $R \geqslant 0$, N is monotone as befits an entropy. We note also that $\Box^* u = 0$ is equivalent to $(\mathbf{1I})$ $\partial_t f = -\Delta f + |\nabla f|^2 - R$.

It was also noted in [49] that $\mathfrak{F}$ is a Fisher information functional (cf. [8, 10, 24, 25]) and we showed in [13] that for a given 3-D manifold M and a Weyl-Schrödinger picture of quantum evolution based on [42, 43] (cf. also [4, 5, 6, 8, 9, 10, 11, 12, 16, 17, 51]) one can express $\mathfrak{F}$ in terms of a quantum potential Q in the form $(\mathbf{1J})$ $\mathfrak{F} \sim \alpha\int_M QPdV + \beta\int_M |\vec{\phi}|^2 PdV$ where $\vec{\phi}$ is a Weyl vector and P is a probability distribution associated with a quantum mass density $\hat{\rho} \sim |\psi|^2$. There will be a corresponding Schrödinger equation (SE) in a Weyl space as in [10, 13] provided there is a phase S (for $\psi = |\psi|\exp(iS/\hbar)$) satisfying $(\mathbf{1K})$ $(1/m)\mathrm{div}(P\nabla S) = \Delta P - RP$ (arising from $\partial_t\hat{\rho} - \Delta\hat{\phi} = -(1/m)\mathrm{div}(\hat{\rho}\nabla S)$ and $\partial_t\hat{\rho} + \Delta\hat{\rho} - R\hat{\rho} = 0$ with $\hat{\rho} \sim P \sim u \sim |\psi|^2$). In the present work we show that there can exist solutions S of ($\mathbf{1K}$) and this establishes a connection between Ricci flow and quantum theory (via Fisher information and the quantum potential). Another aspect is to look at a relativistic situation with conformal perturbations of a 4-D semi-Riemannian metric g based on a quantum potential (defined via a quantum mass). Indeed in a simple minded way we could perhaps think of a conformal transformation $\hat{g}_{ab} = \Omega^2 g_{ab}$ (in 4-D) where following [14] we can imagine ourselves immersed in conformal general relativity (GR) with metric $\hat{g}$ and $(\mathbf{1L})$ $\exp(Q) \sim \mathfrak{M}^2/m^2 = \Omega^2 = \hat{\phi}^{-1}$ with $\beta \sim \mathfrak{M}$ where β is a Dirac field and Q a quantum potential $Q \sim (\hbar^2/m^2c^2)(\Box_g\sqrt{\rho})/\sqrt{\rho})$ with $\rho \sim |\psi^2|$ referring to a quantum matter density. The theme here (as developed in [14]) is that Weyl-Dirac action with Dirac field β leads to $\beta \sim \mathfrak{M}$ and is equivalent to conformal GR (cf. also [8, 10, 36, 45, 46, 47] and see [28] for ideas on Ricci flow gravity).

REMARK 1.1. For completeness we recall (cf. [10, 50]) for $\mathfrak{L}_G = (1/2\chi)\sqrt{-g}\,R$

$$\delta\mathfrak{L} = \frac{1}{2\chi}\left[R_{ab} - \frac{1}{2}g_{ab}R\right]\sqrt{-g}\,\delta g^{ab} + \frac{1}{2\chi}g^{ab}\sqrt{-g}\,\delta R_{ab}\,. \tag{1.2}$$

The last term can be converted to a boundary integral if certain derivatives of g_{ab} are fixed there. Next following [7, 9, 14, 27, 38, 39, 40] the Einstein frame GR action has the form

$$S_{GR} = \int d^4x\sqrt{-g}\,(R - \alpha(\nabla\psi)^2 + 16\pi L_M) \tag{1.3}$$

(cf. [7]) whose conformal form (conformal GR) is

$$\hat{S}_{GR} = \int d^4x\sqrt{-\hat{g}}\,e^{-\psi} \times \left[\hat{R} - \left(\alpha - \frac{3}{2}\right)(\hat{\nabla}\psi)^2 + 16\pi e^{-\psi}L_M\right] = \int d^4x\sqrt{-g}\left[\hat{\phi}\hat{R} - \left(\alpha - \frac{3}{2}\right)\frac{(\hat{\nabla}\hat{\phi})^2}{\hat{\phi}} + 16\pi\hat{\phi}^2 L_M\right], \tag{1.4}$$

where $\hat{g}_{ab} = \Omega^2 g_{ab}$, $\Omega^2 = \exp(\psi) = \phi$, and $\hat{\phi} = \exp(-\psi) = \phi^{-1}$. If we omit the matter Lagrangians, and set $\lambda = \frac{3}{2} - \alpha$, (1.4) becomes for $\hat{g}_{ab} \to g_{ab}$

$$\tilde{S} = \int d^4x \sqrt{-g}\, e^{-\psi}\big[R + \lambda(\nabla\psi)^2\big]. \tag{1.5}$$

In this form on a 3-D manifold M we have **exactly** the situation treated in [10, 13] with an associated SE in Weyl space based on (**1K**). ■

2 Solution of (1K)

Consider now $(\mathbf{1K})$ $(1/m)\mathrm{div}(P\nabla S) = \Delta P - RP$ for $P \sim \sim \hat{\rho} \sim |\psi|^2$ and $\int P\sqrt{|g|}d^3x = 1$ (in 3-D we will use here $\sqrt{|g|}$ for $\sqrt{-g}$). One knows that $\mathrm{div}(P\nabla S) = P\Delta S + + \nabla P \cdot \nabla S$ and

$$\left.\begin{aligned} &\Delta\psi = \frac{1}{\sqrt{|g|}}\,\partial_m\big(\sqrt{|g|}\nabla\psi\big), \quad \nabla\psi = g^{mn}\partial_n\psi \\ &\int_M \mathrm{div}\mathbf{V}\,\sqrt{|g|}\,d^3x = \int_{\partial M}\mathbf{V}\cdot\mathbf{ds} \end{aligned}\right\} \tag{2.1}$$

(cf. [10]). Recall also $\int P\sqrt{|g|}\,d^3x = 1$ and

$$\left.\begin{aligned} &Q \sim -\frac{\hbar^2}{8m}\left[\left(\frac{\nabla P}{P}\right)^2 - 2\left(\frac{\Delta P}{P}\right)\right] \\ &<Q>_\psi = \int PQ\,d^3x \end{aligned}\right\}. \tag{2.2}$$

Now in 1-D an analogous equation to (**1K**) would be $(\mathbf{3A})$ $(PS')' = P' - RP = F$ with solution determined via

$$\begin{aligned} &PS' = P' - \int RP + c \Rightarrow \\ &\Rightarrow S' = \partial_x\log(P) - \frac{1}{P}\int RP + cP^{-1} \Rightarrow \\ &\Rightarrow S = \log(P) - \int\frac{1}{P}\int RP + c\int P^{-1} + k\,, \end{aligned} \tag{2.3}$$

which suggests that solutions of (**1K**) do in fact exist in general. We approach the general case in Sobolev spaces à la [1, 2, 15, 22]. The volume element is defined via $\eta = \sqrt{|g|}dx^1 \wedge \cdots \wedge dx^n$ (where $n = 3$ for our purposes) and $* : \wedge^p M \to \wedge^{n-p}M$ is defined via

$$\left.\begin{aligned} &(*\alpha)_{\lambda_{p+1}\cdots\lambda_n} = \frac{1}{p!}\,\eta_{\lambda_1\cdots\lambda_n}\,\alpha^{\lambda_1\cdots\lambda_p} \\ &(\alpha,\beta) = \frac{1}{p!}\,\alpha_{\lambda_1\cdots\lambda_p}\,\beta^{\lambda_1\cdots\lambda_p} \end{aligned}\right\}, \tag{2.4}$$

$*1 = \eta$; $**\alpha = (-1)^{p(n-p)}\alpha$; $*\eta = 1$; $\alpha \wedge (*\beta) = (\alpha,\beta)\eta$. One writes now $<\alpha,\beta> = \int_M(\alpha,\beta)\eta$ and, for (Ω,ϕ) a local chart we have $(\mathbf{2A})$ $\int_M f dV = \int_{\phi(\Omega)}(\sqrt{|g|}f)\circ\phi^{-1}\prod dx^i$ $(\sim \int_M f\sqrt{|g|}\prod dx^i)$. Then one has $(\mathbf{2B})$ $<d\alpha,\gamma> = =<\alpha,\delta\gamma>$ for $\alpha \in \wedge^p M$ and $\gamma \in \wedge^{p+1}M$ where the codifferential δ on p-forms is defined via $(\mathbf{2C})$ $\delta = (-1)^p *^{-1} d*$. Then $\delta^2 = d^2 = 0$ and $\Delta = d\delta + \delta d$ so that $\Delta f = \delta df = = -\nabla^\nu\nabla_\nu f$. Indeed for $\alpha \in \wedge^p M$

$$(\delta\alpha)_{\lambda_1,\cdots,\lambda_{p-1}} = -\nabla^\gamma\alpha_{\gamma,\lambda_1,\cdots,\lambda_{p-1}} \tag{2.5}$$

with $\delta f = 0$ $(\delta : \wedge^p M \to \wedge^{p-1}M)$. Then in particular $(\mathbf{2D})$ $<\Delta\phi,\phi>=<\delta d\phi,\phi>=<d\phi,d\phi>=\int_M\nabla^\nu\phi\nabla_\nu\phi\eta$.

Now to deal with weak solutions of an equation in divergence form look at an operator $(\mathbf{2E})$ $Au = -\nabla(a\nabla u) \sim (-1/\sqrt{|g|})\,\partial_m(\sqrt{|g|}\,ag^{mn}\nabla_n u) = -\nabla_m(a\nabla^m u)$ so that for $\phi \in \mathcal{D}(M)$

$$\begin{aligned} &\int_M Au\phi dV = -\int\big[\nabla_m(ag^{mn}\nabla_n u)\big]\phi dV = \\ &= \int ag^{mn}\nabla_n u\,\nabla_m\phi dV = \int a\nabla^m u\,\nabla_m\phi dV. \end{aligned} \tag{2.6}$$

Here one imagines M to be a complete Riemannian manifold with Soblev spaces $H_0^1(M) \sim H^1(M)$ (see [1, 3, 15, 26, 29, 48]). The notation in [1] is different and we think of $H^1(M)$ as the space of L^2 functions u on M with $\nabla u \in L^2$ and H_0^1 means the completion of $\mathcal{D}(M)$ in the H^1 norm $\|u\|^2 = \int_M[|u|^2 + |\nabla u|^2]dV$. Following [29] we can also assume $\partial M = \emptyset$ with M connected for all M under consideration. Then let $H = H^1(M)$ be our Hilbert space and consider the operator $A(S) = -(1/m)\nabla(P\nabla S)$ with

$$B(S,\psi) = \frac{1}{m}\int P\,\nabla^m S\,\nabla_m\psi\,dV \tag{2.7}$$

for $S,\psi \in H_0^1 = H^1$. Then $A(S) = RP - \Delta P = F$ becomes $(\mathbf{2F})$ $B(S,\psi) = <F,\psi> = \int F\psi\,dV$ and one has $(\mathbf{2G})$ $|B(S,\psi)| \leqslant c\,\|S\|_H\,\|\psi\|_H$ and $|B(S,S)| = \int P(\nabla S)^2\,dV$. Now $P \geqslant 0$ with $\int PdV = 1$ but to use the Lax-Milgram theory we need here $|B(S,S)| \geqslant \beta\|S\|_H^2$ $(H = H^1)$. In this direction one recalls that in Euclidean space for $\psi \in H_0^1(\mathbf{R}^3)$ there follows $(\mathbf{2H})$ $\|\psi\|_{L^2}^2 \leqslant c\,\|\nabla\psi\|_{L^2}^2$ (Friedrich's inequality — cf. [48]) which would imply $\|\psi\|_H^2 \leqslant (c+1)\|\nabla\psi\|_{L^2}^2$. However such Sobolev and Poincaré-Sobolev inequalities become more complicated on manifolds and (**2H**) is in no way automatic (cf. [1, 29, 48]). However we have some recourse here to the definition of P, namely $P = \exp(-f)$, which basically is a conformal factor and $P > 0$ unless $f \to \infty$. One heuristic situation would then be to assume $(\mathbf{2I})$ $0 < \epsilon \leqslant P(x)$ on M (and since $\int\exp(-f)dV = 1$ with $dV = \sqrt{|g|}\prod_1^3 dx^i$ we must then have $\epsilon\int dV \leqslant 1$ or $\mathrm{vol}(M) = \int_M dV \leqslant (1/\epsilon)$). Then from (**2G**) we have $(\mathbf{2J})$ $|B(S,S)| \geqslant \epsilon\|(\nabla S)^2\|$ and for any $\kappa > 0$ it follows: $|B(S,S)| + \kappa\|S\|_{L^2}^2 \geqslant \min(\epsilon,\kappa)\|S\|_{H^1}^2$. This means via Lax-Milgram that the equation

$$A(S) + \kappa S = -\frac{1}{m}\,\nabla(P\nabla S) + \kappa S = F = RP - \Delta P \tag{2.8}$$

has a unique weak solution $S \in H^1(M)$ for any $\kappa > 0$ (assuming $F \in L^2(M)$). Equivalently $(\mathbf{2K})$ $-\frac{1}{m}[P\Delta S + (\nabla P)(\nabla S)] + \kappa S = F$ has a unique weak solution $S \in H^1(M)$. This is close but we cannot put $\kappa = 0$. A different approach following from remarks in [29], pp. 56–57 (corrected in [30], p. 248), leads to an heuristic weak solution of (**1K**). Thus from a result of Yau [53] if M is a complete simply connected 3-D differential manifold with sectional curvature $K < 0$ one has for $u \in \mathcal{D}(M)$

$$\int_M |\psi| dV \leqslant (2\sqrt{-K})^{-1} \int_M |\nabla\psi| dV \Rightarrow$$
$$\Rightarrow \int_M |\psi|^2 dV \leqslant c \int_M |\nabla\psi|^2 dV. \tag{2.9}$$

Hence (**2H**) holds and one has $||\psi||^2_{H^1} \leqslant (1+c)||\nabla\psi||^2$. Morever if M is bounded and simply connected with a reasonable boundary ∂M (e.g. weakly convex) one expects $(\mathbf{2L})$ $\int_M |\psi|^2 dV \leqslant c \int_M |\nabla\psi|^2 dV$ for $\psi \in \mathcal{D}(M)$ (cf. [41]). In either case $(\mathbf{2M})$ $|B(S,S)| \geqslant \epsilon ||(\nabla S)^2|| \geqslant (c+1)^{-1}\epsilon ||S||^2_{H^1_0}$ and this leads via Lax-Milgram again to a sample result

THEOREM 2.1 *Let M be a bounded and simply connected 3-D differential manifold with a reasonable boundary ∂M. Then there exists a unique weak solution of (**1K**) in $H^1_0(M)$.*

REMARK 2.1. One must keep in mind here that the metric is changing under the Ricci flow and assume that estimates involving e.g. K are considered over some time interval. ■

REMARK 2.2. There is an extensive literature concerning eigenvalue bounds on Riemannian manifolds and we cite a few such results. Here $I_\infty(M) \sim \inf_\Omega (A(\partial\Omega)/V(\Omega))$ where Ω runs over (connected) open subsets of M with compact closure and smooth boundary (cf. [18, 19]). Yau's result is $I_\infty(M) \geqslant 2\sqrt{-K}$ (with equality for the 3-D hyperbolic space) and Cheeger's result involves follows $|\nabla\phi||_{L^2} \geqslant (1/2)I_\infty(M)||\phi||_{L^2} \geqslant \sqrt{-K}||\phi||_{L^2}$. There are many other results where e.g. $\lambda_1 \geqslant c(\mathrm{vol}(M))^{-2}$ for M a compact 3-D hyperbolic manifold of finite volume (see [21, 34, 44] for this and variations). There are also estimates for the first eigenvalue along a Ricci flow in [33, 37] and estimates of the form $\lambda_1 \geqslant 3K$ for closed 3-D manifolds with Ricci curvature $R \geqslant 2K$ $(K > 0)$ in [32, 33]. In fact Ling obtains $\lambda_1 \geqslant K + (\pi^2/\tilde{d}^2)$ where $\tilde{d}$ is the diameter of the largest interior ball in nodal domains of the first eigenfunction. There are also estimates $\lambda_1 \geqslant (\pi^2/d^2)$ $(d = \mathrm{diam}(M),\ R \geqslant 0)$ in [31, 52, 54] and the papers of Ling give an excellent survey of results, new and old, including estimates of a similar kind for the first Dirichlet and Neumann eigenvalues. ■

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A Possible General Approach Regarding the Conformability of Angular Observables with the Mathematical Rules of Quantum Mechanics

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The conformability of angular observales (angular momentum and azimuthal angle) with the mathematical rules of quantum mechanics is a question which still rouses debates. It is valued negatively within the existing approaches which are restricted by two amendable presumptions. If the respective presumptions are removed one can obtain a general approach in which the mentioned question is valued positively.

1 Introduction

In the last decades the pair of angular observables $L_z - \varphi$ (angular momentum — azimuthal angle) was and still is regarded as being unconformable to the accepted mathematical rules of Quantum Mechanics (QM) (see [1–24]). The unconformity is identified with the fact that , in some cases of circular motions, for the respective pair the Robertson-Schrödinger uncertainty relation (RSUR) is not directly applicable. That fact roused many debates and motivated various approaches planned to elucidate in an acceptable manner the missing conformability. But so far such an elucidation was not ratified (or admited unanimously) in the scientific literature.

A minute inspection of the things shows that in the main all the alluded approaches have a restricted character due to the presumptions (***P***):

- $\boldsymbol{P}_1$: Consideration of RSUR as a twofold reference element by: (i) proscription of its direct $L_z - \varphi$ descendant, and (ii) substitution of the respective descendant with some RSUR-mimic relations;
- $\boldsymbol{P}_2$: Discussion only of the systems with sharp circular rotations (SCR).

But the mentioned presumptions are amendable because they conflict with the following facts (***F***):

- $\boldsymbol{F}_1$: Mathematically, the RSUR is only a secondary piece, of limited validity, resulting from a generally valid element represented by a Cauchy Schwarz formula (CSF) (see down Section 4);
- $\boldsymbol{F}_2$: From a natural physical viewpoint the $L_z - \varphi$ pair must be considered in connection not only with SCR but also with any orbital (spatial) motions (e.g. with the non-circular rotations (NCR), presented below in Section 3).

The above facts suggest that for the $L_z - \varphi$ problem ought to search new approaches, by removing the mentioned premises $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$. As we know until now such approaches were not promoted in the publications from the main stream of scientific literature. In this paper we propose a possible general approach of the mentioned kind, able to ensure a natural conformability of the $L_z - \varphi$ pair with the prime mathematical rules of QM.

For distiguinshing our proposal from the alluded restricted approaches, in the next Section we present briefly the respective approaches, including their main assertions and a set of unavoidable shortcomings which trouble them destructively. Then, in Section 3, we disclose the existence of two examples of NCR which are in discordance with the same approaches.

The alluded shorcomings and discordances reenforce the interest for new and differently oriented approaches of the $L_z - \varphi$ problem. Such an approach, of general perspective, is argued and detailed below in our Section 4. We end the paper in Section 5 with some associate conclusions.

2 Briefly on the restricted approaches

Certainly, for the history of the $L_z - \varphi$ problem, the first reference element was the Robertson Schrödinger uncertainty relation (RSUR) introduced [25, 26] within the mathematical formalism of QM. In terms of usual notations from QM the RSUR is written as

$$\Delta_\psi A \cdot \Delta_\psi B \geqslant \frac{1}{2} \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle_\psi \right| , \qquad (1)$$

where $\Delta_\psi A$ and $\langle (\dots) \rangle_\psi$ signify the standard deviation of the observable A respectively the mean value of $(\dots)$ in the state described by the wave function ψ, while $\left[\hat{A}, \hat{B} \right]$ denote the commutator of the operators $\hat{A}$ and $\hat{B}$ (for more details about the notations and validity regarding the RSUR 1, see the next Section).

The attempts for application of RSUR (1) to the case with $A = L_z$ and $B = \varphi$, i.e. to the $L_z - \varphi$ pair, evidenced the folloving intriguing facts.

On the one hand, according to the usual procedures of QM [27], the observables L_z and φ should be described by the conjugated operators

$$\hat{L}_z = -i\hbar \frac{\partial}{\partial \varphi}, \qquad \hat{\varphi} = \varphi. \qquad (2)$$

respectively by the commutation relation

$$\left[\hat{L}_z, \hat{\varphi}\right] = -i\hbar\,. \tag{3}$$

So for the alluded pair the RSUR (1) requires for its direct descendant the relation

$$\Delta_\psi L_z \cdot \Delta_\psi\,\varphi \geqslant \frac{\hbar}{2}\,. \tag{4}$$

On the other hand this last relation is explicitly inapplicable in cases of angular states regarding the systems with sharp circular rotations (SCR). The respective inapplicability is pointed out here bellow.

As examples with SCR can be quoted : (i) a particle (bead) on a circle, (ii) a 1D (or fixedaxis) rotator and (iii) non-degenerate spatial rotations. One finds examples of systems with spatial rotations in the cases of a particle on a sphere, of 2D or 3D rotators and of an electron in a hydrogen atom respectively. The mentioned rotations are considered as non-degenerate if all the specific (orbital) quantum numbers have well-defined (unique) values. The alluded SRC states are described by the following wave functions taken in a φ — representation

$$\psi_m(\varphi) = (2\pi)^{-\frac{1}{2}}\, e^{im\varphi} \tag{5}$$

with the stipulations $\varphi \in [0, 2\pi)$ and $m = 0, \pm1, \pm2, \ldots$ The respective stipulations are required by the following facts. Firstly, in cases of SRC the angle φ is a ordinary polar coordinate which must satistfy the corresponding mathematical rules regarding the range of definition [28]. Secondly, from a physical perspective, in the same cases the wave function $\psi(\varphi)$ is enforced to have the property $\psi(0) = \psi(2\pi - 0) := \lim_{\varphi\to 2\pi-0} \psi(\varphi)$.

For the alluded SRC one finds

$$\Delta_\psi L_z = 0, \qquad \Delta_\psi\,\varphi = \frac{\pi}{\sqrt{3}}\,. \tag{6}$$

But these expressions for $\Delta_\psi L_z$ and $\Delta_\psi\varphi$ are incompatible with relation (4).

For avoiding the mentioned incompatibility many publications promoted the conception that in the case of $L_z-\varphi$ pair the RSUR (1) and the associated procedures of QM do not work correctly. Consequently it was accredited the idea that formula (4) must be proscribed and replaced by adjusted $\Delta_\psi L_z - \Delta_\psi\,\varphi$ relations planned to mime the RSUR (1). So, along the years, a lot of such mimic relations were proposed. In the main the respective relations can be expressed in one of the following forms:

$$\frac{\Delta_\psi L_z \cdot \Delta_\psi\varphi}{a\,(\Delta_\psi\varphi)} \geqslant \hbar\left|\langle b\,(\varphi)\rangle_\psi\right|, \tag{7}$$

$$\Delta_\psi L_z \cdot \Delta_\psi f(\varphi) \geqslant \hbar\left|\langle g(\varphi)\rangle_\psi\right|, \tag{8}$$

$$(\Delta_\psi L_z)^2 + \hbar^2\,(\Delta_\psi u(\varphi))^2 \geqslant \hbar^2\,\langle v(\varphi)\rangle_\psi^2\,, \tag{9}$$

$$\Delta_\psi L_z \cdot \Delta_\psi\,\varphi \geqslant \frac{\hbar}{2}\,|1 - 2\pi\,|\psi(2\pi-0)||\,. \tag{10}$$

In (7)–(9) by a, b, f, g, u and v are denoted various adjusting functions (of $\Delta_\psi\varphi$ or of φ), introduced in literature by means of some circumstantial (and more or less fictitious) considerations.

Among the relations (7)–(10) of some popularity is (8) with $f(\varphi) = \sin\varphi$ (or $= \cos\varphi$) respectively $g(\varphi) = [\hat{L}_z, f(\hat{\varphi})]$. But, generally speaking, none of the respective relations is agreed unanimously as a suitable model able to substitute formula (4).

A minute examination of the facts shows that, in essence, the relations (7)–(10) are troubled by shortcomings revealed in the following remarks (***R***):

R$_1$: The relation (10) is correct from the usual perspective of QM (see formulas 18 and 25 in the next Secion). But the respective relation evidently does not mime the RSUR (1) presumed as standard within the mentioned restricted approaches of $L_z-\varphi$ problem;

R$_2$: Each replica from the classes depicted by (7)–(10) were planned to harmonize in a mimic fashion with the same presumed reference element represented by RSUR (1). But, in spite of such plannings, regarded comparatively, the respective replicas are not mutually equivalent;

R$_3$: Due to the absolutely circumstantial considerations by which they are introduced, the relations (7)–(9) are in fact ad hoc formulas without any direct descendence from general mathematics of QM. Consequently the respective relations ought to be appeciated by taking into account sentences such are:

"*In ... science, ad hoc often means the addition of corollary hypotheses or adjustment to a ... scientific theory to save the theory from being falsified by compensating for anomalies not anticipated by the theory in its unmodified form. ... Scientists are often suspicious or skeptical of theories that rely on ... ad hoc adjustments*" [29].

Then, if one wants to preserve the mathematical formalism of QM as a unitary theory, as it is accreditated in our days, the relations (7)–(9) must be regarded as unconvincing and inconvenient (or even prejudicial) elements;

R$_4$: In fact in relations (7)–(9) the angle φ is substituted more or less factitiously with the adjusting functions a, b, f, g, v or u. Then in fact , from a natural perspective of physics, such substitutions, and consequently the respective relations, are only mathematical artifacts. But, in physics, the mathematical artifacts burden the scientific discussions by additions of extraneous entities (concepts, assertions, reasonings, formulas) which are not associated with a true information regarding the real world. Then, for a good efficiency of the discussions, the alluded additions ought to be evaluated by taking into account the principle of parsimony: "*Entities should not be multiplied unneces-*

sarily" (known also [30, 31] as the "*Ockham's Razor*" slogan). Through such an evaluation the relations (7)–(9) appear as unnecessary exercises which do not give real and useful contributions for the elucidation of the $L_z-\varphi$ problem.

In our opinion the facts revealed in this Section offer a minimal but sufficient base for concluding that as regards the $L_z-\varphi$ problem the approaches restricted around the premises $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are unable to offer true and natural solutions.

3 The discordant examples with non-circular rotations

The discussions presented in the previous Section regard the situation of the $L_z-\varphi$ pair in relation with the mentioned SCR. But here is the place to note that the same pair must be considered also in connection with other orbital (spatial) motions which differ from SCR. Such motions are the non-circular rotations (NCR) . As examples of NCR we mention the quantum torsion pendulum (QTP) respectively the degenerate spatial rotations of the systems mentioned in the previous Section (i.e. a particle on a sphere, 2D or 3D rotators and an electron in a hydrogen atom). A rotation (motion) is degenerate if the energy of the system is well-specified while the non-energetic quantum numbers (here of orbital nature) take all permitted values.

From the class of NCR let us firstly refer to the case of a QTP which in fact is a simple quantum oscillator. Indeed a QTP which oscillates around the z-axis is characterized by the Hamiltonian

$$\hat{H} = \frac{1}{2I}\,\hat{L}_z^2 + \frac{1}{2}\,J\omega^2\varphi^2. \tag{11}$$

Note that in this expression φ denotes the azimuthal angle whose range of definition is the interval $(-\infty, \infty)$. In the same exppression appears $\hat{L}_z$ as the z-component of angular momentum operator defined also by (2). The other symbols J and ω in (11) represent the QTP momentum of inertia respectively the frequency of torsional oscillations. The Schrödinger equation associated to the Hamiltonian (11) shows that the QTP have eigenstates described by the wave functions

$$\psi_n(\varphi) = \psi_n(\xi) \propto \exp\left(-\frac{\xi^2}{2}\right)\mathcal{H}_n(\xi)\,, \quad \xi = \varphi\sqrt{\frac{J\omega}{\hbar}}\,, \tag{12}$$

where $n = 0, 1, 2, 3, \ldots$ signifies the oscillation quantum number and $\mathcal{H}_n(\xi)$ stand for Hermite polinomials of ξ. The eigenstates described by (12) have energies $E_n = \hbar\omega(n+\frac{1}{2})$. In the states (12) for the observables $L_z$and φ associated with the operators (2) one obtains the expressions

$$\Delta_\psi L_z = \sqrt{\hbar J\omega\left(n+\frac{1}{2}\right)}\,, \quad \Delta_\psi\varphi = \sqrt{\frac{\hbar}{J\omega}\left(n+\frac{1}{2}\right)}\,, \tag{13}$$

which are completely similar with the corresponding ones for the $x-p$ pair of a rectiliniar oscillator [27]. With the expressions (13) for $\Delta_\psi L_z$ and $\Delta_\psi\varphi$ one finds that in the case of QTP the $L_z-\varphi$ pair satisfies the proscribed formula (4).

From the same class of NCR let us now refer to a degenerate state of a particle on a sphere or of a 2D rotator. In such a state the energy is $E = \hbar^2\, l\,(l+1)/2J$ where the orbital number l has a well-defined value (J = moment of inertia). In the same state the magnetic number m can take all the values $-l, -l+1, \ldots, -1, 0, 1, \ldots, l-1, l$. Then the mentioned state is described by a wave function of the form

$$\psi_l(\theta,\varphi) = \sum_{m=-l}^{l} c_m\, Y_{lm}(\theta,\varphi)\,. \tag{14}$$

Here θ and φ denote polar respectively azimuthal angles ($\theta \in [0,\pi], \varphi \in [0, 2\pi)$), $Y_{lm}\,(\theta,\varphi)$ are the spherical functions and c_m represent complex coefficients which satisfy the normalization condition $\sum_{m=-l}^{l}|c_m|^2 = 1$. With the expressions (2) for the operators $\hat{L}_z$ and $\hat{\varphi}$ in a state described by (14) one obtains

$$(\Delta_\psi L_z)^2 = \sum_{m=-l}^{l}|c_m|^2\,\hbar^2\, m^2 - \left[\sum_{m=-l}^{l}|c_m|^2\,\hbar m\right]^2, \tag{15}$$

$$(\Delta_\psi\,\varphi)^2 = \sum_{m=-l}^{l}\sum_{r=-l}^{l} c_m^*\, c_r\,\left(Y_{lm}, \varphi^2\, Y_{lr}\right) - \left[\sum_{m=-l}^{l}\sum_{r=-l}^{l} c_m^*\, c_r\,\left(Y_{lm}, \varphi Y_{lr}\right)\right]^2, \tag{16}$$

where (f, g) is the scalar product of the functions f and g.

By means of the expressions (15) and (16) one finds that in the case of alluded NCR described by the wave functions (14) it is possible for the proscribed formula (4) to be satisfied. Such a possibility is conditioned by the concrete values of the coefficients c_m.

Now is the place for the following remark

$\boldsymbol{R}_5$: As regards the $L_z-\varphi$ problem, due to the here revealed aspects, the NCR examples exceed the bounds of the presumptions $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ of usual restricted approaches. That is why the mentioned problem requires new approaches of general nature if it is possible.

4 A possible general appoach and some remarks associated with it.

A general approach of the $L_z-\varphi$ problem, able to avoid the shortcomings and discordances revealed in the previous two Sections, must be done by starting from the prime mathematical rules of QM. Such an approach is possible to be obtained as follows. Let us appeal to the usual concepts and notations of QM. We consider a quantum system whose state (of orbital nature) and two observables A_j ($j = 1, 2$) are described by the wave function ψ respectively by the operators $\hat{A}_j$. As usually with (f, g) we denote the scalar product of the functions

f and g . In relation with the mentioned state, the quantities $\langle A_j \rangle_\psi = \left(\psi\,, \hat{A}_j \psi\right)$ and $\delta_\psi \hat{A}_j = \hat{A}_j - \langle \hat{A}_j \rangle_\psi$ represent the mean (expected) value respectively the deviation-operator of the observable A_j regarded as a random variable. Then, by taking $A_1 = A$ and $A_2 = B$, for the two observables can be written the following Cauchy-Schwarz relation:

$$\left(\delta_\psi \hat{A}\psi,\, \delta_\psi \hat{A}\psi\right)\left(\delta_\psi \hat{B}\psi,\, \delta_\psi \hat{B}\psi\right) \geqslant \left|\left(\delta_\psi \hat{A}\psi,\, \delta_\psi B\psi\right)\right|^2. \quad (17)$$

For an observable A_j regarded as a random variable the quantity $\Delta_\psi A_j = \left(\delta_\psi \hat{A}_j \psi,\, \delta_\psi \hat{A}_j \psi\right)^{1/2}$ represents its standard deviation. From (17) it results directly that the standard deviations $\Delta_\psi A$ and $\Delta_\psi B$ of the observables A and B satisfy the relation

$$\Delta_\psi A \cdot \Delta_\psi B \geqslant \left|\left(\delta_\psi \hat{A}\psi,\, \delta_\psi B\psi\right)\right|, \quad (18)$$

which can be called *Cauchy-Schwarz formula* (CSF). Note that CSF (18) (as well as the relation (17) is always valid, i.e. for all observables, systems and states. Add here the important observation that the CSF (18) implies the restricted RSUR (1) only in the cases when the two operators $\hat{A} = \hat{A}_1$ and $\hat{B} = \hat{A}_2$ satisfy the conditions

$$\left(\hat{A}_j \psi, \hat{A}_k \psi\right) = \left(\psi, \hat{A}_j \hat{A}_k \psi\right), \quad j = 1, 2;\; k = 1, 2\,. \quad (19)$$

Indeed in such cases one can write the relation

$$\left(\delta_\psi \hat{A}\psi,\, \delta_\psi \hat{B}\psi\right) = \\ = \frac{1}{2}\left(\psi,\, \left(\delta_\psi \hat{A} \cdot \delta_\psi \hat{B}\psi + \delta_\psi \hat{B} \cdot \delta_\psi \hat{A}\right)\psi\right) - \\ - \frac{i}{2}\left(\psi,\, i\left[\hat{A},\, \hat{B}\right]\psi\right), \quad (20)$$

where the two terms from the right hand side are purely real and imaginary quantities respectively. Therefore in the mentioned cases from (18) one finds

$$\Delta_\psi A \cdot \Delta_\psi B \geqslant \frac{1}{2}\left|\left\langle [\hat{A},\, \hat{B}] \right\rangle_\psi\right|. \quad (21)$$

i.e. the well known RSUR (1). The above general framing of RSUR (1)/(21) shows that for the here investigated question of $L_z - \varphi$ pair it is important to examine the fulfilment of the conditions (19) in each of the considered cases. In this sense the following remarks are of direct interest.

R$_6$: In the cases described by the wave functions (5) for $L_z - \varphi$ pair one finds

$$\left(\hat{L}_z \psi_m,\, \hat{\varphi}\psi_m\right) = \left(\psi_m,\, \hat{L}_z \hat{\varphi}\psi_m\right) + i\hbar\,, \quad (22)$$

i.e. a clear violation in respect with the conditions (19);

R$_7$: In the cases associated with the wave functions (12) and (14) for $L_z - \varphi$ pair one obtains

$$\left(\hat{L}_z \psi_n,\, \hat{\varphi}\psi_n\right) = \left(\psi_n,\, \hat{L}_z \hat{\varphi}\psi_n\right), \quad (23)$$

$$\left(\hat{L}_z \psi_l,\, \hat{\varphi}\psi_l\right) = \left(\psi_l,\, \hat{L}_z \hat{\varphi}\psi_l\right) + \\ + i\hbar\left\{1 + 2\,\mathrm{Im}\left[\sum_{m=-l}^{l}\sum_{r=-l}^{l} c_m^* c_r \hbar m \left(Y_{lm},\, \hat{\varphi} Y_{lr}\right)\right]\right\}, \quad (24)$$

(where Im $[\alpha]$ denotes the imaginary part of α);

R$_8$: For any wave function $\psi(\varphi)$ with $\varphi \in [0, 2\pi)$ and $\psi(2\pi - 0) = \psi(0)$ it is generally true the formula

$$\left|\left(\delta_\psi \hat{L}_z\, \psi,\, \delta_\psi \hat{\varphi}\, \psi\right)\right| \geqslant \frac{\hbar}{2}\left|1 - 2\pi\left|\psi\left(2\pi - 0\right)\right|\right|, \quad (25)$$

which together with CSF (18) confirms relation (10).

The things mentioned above in this Section justify the following remarks

R$_9$: The CSF (18) is an ab origine element in respect with the RSUR (1)/(21). Moreover, (18) is always valid, independently if the conditions (19) are fulfilled or not;

R$_{10}$: The usual RSUR (1)/(21) are valid only in the circumstances strictly delimited by the conditions (19) and they are false in all other situations;

R$_{11}$: Due to the relations (22) in the cases described by the wave functions (5) the conditions (19) are not fulfilled. Consequently in such cases the restricted RSUR (1)/(21) are essentially inapplicable for the pairs $L_z - \varphi$. However one can see that in the respective cases, mathematically, the CSF (18) remains valid as a trivial equality $0 = 0$;

R$_{12}$: In the cases of NCR described by (12) the $L_z - \varphi$ pair satisfies the conditions (19) (mainly due to the relation (23). Therefore in the respective cases the RSUR (1)/(21) are valid for L_z and φ;

R$_{13}$: The fulfilment of the conditions (19) by the $L_z - \varphi$ pair for the NCR associated with (14) depends on the annulment of the second term in the right hand side from (24) (i.e. on the values of the coefficients c_m). Adequately, in such a case, the correctness of the corresponding RSUR (1)/(21) shows the same dependence;

R$_{14}$: The result (25) points out the fact that the adjusted relation (10) is only a secondary piece derivable fom the generally valid CSF (18);

R$_{15}$: The mimic relations (7)–(9) regard the cases with SCR described by the wave functions (5) when φ plays the role of polar coordinate. But for such a role [28] in order to be a unique (univocal) variable φ must be defined naturally only in the range $[0, 2\pi)$. The same range is considered in practice for the normalization of the wave functions (5). Therefore, in the cases under discussion the derivative with respect to φ refers to the mentioned range. Particularly for the extremities of the interval $[0, 2\pi)$ it has to operate with backward respectively forward derivatives. So in the alluded SCR cases the relations (2) and (3) act well, with a natural correctness. The same correctness is shown by the respective relations in connection with the NCR described by the wave functions (12) or (14). In fact, from a more general perspective, the relations (2) and (3) regard the QM operators $\hat{L}_z$ and $\hat{\varphi}$. Therefore they must have unique forms — i.e. expressions which do not depend on the particularities of the considered situations (e.g. systems with SCR or with NCR);

$\boldsymbol{R}_{16}$: The troubles of RSUR (1) regarding $L_z - \varphi$ pair are directly connected with the conditions (19). Then it is strange that in almost all the QM literature the respective conditions are not taken into account adequately. The reason seems to be related with the nowadays dominant Dirac's $<bra|$ and $|ket>$ notations. In the respective notations the terms from the both sides of (19) have a unique representation namely $< \psi|\hat{A}_j\, \hat{A}_k|\psi >$. The respective uniqueness can entail confusion (unjustified supposition) that the conditions (19) are always fulfiled. It is interesting to note that systematic investigations on the confusions/surprises generated by the Dirac's notations were started only recently [32]. Probably that further efforts on the line of such investigations will bring a new light on the conditions (19) as well as on other QM questions.

The ensemble of things presented above in this Section appoints a possible general approach for the discussed $L_z - \varphi$ problem and answer to a number of questions associated with the respective problem. Some significant aspects of the respective approach are noted in the next Section.

5 Conclusions

The facts and arguments discussed in the previous Sections guide to the following conclusions ($\boldsymbol{C}$):

$\boldsymbol{C}_1$: For the $L_z - \varphi$ pair the relations (2)–(3) are always viable in respect with the general CSF (18). That is why, from the QM perspective, for a correct description of questions regarding the respective pair, it is not at all necessary to resort to the mimetic formulas (7)–(10). Eventually the respective formulas can be accounted as ingenious execises of pure mathematical facture. An adequate description of the mentioned kind can be given by taking CSF (18) and associated QM procedures as basic elements;

$\boldsymbol{C}_2$: In respect with the conjugated observables L_z and φ the RSUR (1)/(21) is not adequate for the role of reference element for normality . For such a role the CSF (18) is the most suitable. In some cases of interest the respective CSF degenerates in the trivial equality $0 = 0$;

$\boldsymbol{C}_3$: In reality the usual procedures of QM, illustrated above by the relations (2), (3), (17) and (18), work well and without anomalies in all situations regarding the $L_z - \varphi$ pair. Consequently with regard to the conceptual as well as practical interests of science the mimic relations like (7)–(9) appear as useless inventions.

Now we wish to add the following observations ($\boldsymbol{O}$):

$\boldsymbol{O}_1$: Mathematically the relation (17) is generalisable in the form

$$\det\left[\left(\delta_\psi \hat{A}_j \psi,\, \delta_\psi \hat{A}_k \psi\right)\right] \geqslant 0 \qquad (26)$$

where $\det[\alpha_{jk}]$ denotes the determinant with elements α_{jk} and $j = 1, 2, \ldots, r$; $k = 1, 2, \ldots, r$ with $r \geqslant 2$. Such a form results from the fact that the quantities $\left(\delta_\psi \hat{A}_j \psi,\, \delta_\psi \hat{A}_k \psi\right)$ constitute the elements of a Hermitian and non-negatively defined matrix. Newertheless, comparatively with (17), the generalisation (26) does not bring supplementary and inedited features regarding the conformability of observables $L_z - \varphi$ with the mathematical rules of QM;

$\boldsymbol{O}_2$: We consider [34, 42] that the above considerations about the problem of $L_z - \varphi$ pair can be of some nontrivial interest for a possible revised approach of the similar problem of the pair $N - \phi$ (number-phase) which is also a subject of controversies in recent publications (see [4, 11, 12, 13, 35, 36, 37, 38, 39] and References therein);

$\boldsymbol{O}_3$: Note that we have limited this paper only to mathematical aspects associated with the RSUR (1), without incursions in debates about the interpretations of the respective RSUR. Some opinions about those interpretations and connected questions are given in [40, 41, 42]. But the subject is delicate and probably that it will rouse further debates.

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A Unified Field Theory of Gravity, Electromagnetism, and the Yang-Mills Gauge Field

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In this work, we attempt at constructing a comprehensive four-dimensional unified field theory of gravity, electromagnetism, and the non-Abelian Yang-Mills gauge field in which the gravitational, electromagnetic, and material spin fields are unified as intrinsic geometric objects of the space-time manifold $\mathbb{S}_4$ via the connection, with the generalized non-Abelian Yang-Mills gauge field appearing in particular as a sub-field of the geometrized electromagnetic interaction.

1 Introduction

In our previous work [1], we developed a semi-classical conformal theory of quantum gravity and electromagnetism in which both gravity and electromagnetism were successfully unified and linked to each other through an "external" quantum space-time deformation on the fundamental Planck scale. Herein we wish to further explore the geometrization of the electromagnetic field in [1] which was achieved by linking the electromagnetic field strength to the torsion tensor built by means of a conformal mapping in the evolution (configuration) space. In so doing, we shall in general disregard the conformal mapping used in [1] and consider an arbitrary, very general torsion field expressible as a linear combination of the electromagnetic and material spin fields.

Herein we shall find that the completely geometrized Yang-Mills field of standard model elementary particle physics, which roughly corresponds to the electromagnetic, weak, and strong nuclear interactions, has a more general form than that given in the so-called rigid, local isospace.

We shall not simply describe our theory in terms of a Lagrangian functional due to our unease with the Lagrangian approach (despite its versatility) as a truly fundamental physical approach towards unification. While the meaning of a particular energy functional (to be extremized) is clear in Newtonian physics, in present-day space-time physics the choice of a Lagrangian functional often appears to be non-unique (as it may be concocted arbitrarily) and hence devoid of straightforward, intuitive physical meaning. We shall instead, as in our previous works [1–3], build the edifice of our unified field theory by carefully determining the explicit form of the connection.

2 The determination of the explicit form of the connection for the unification of the gravitational, electromagnetic, and material spin fields

We shall work in an affine-metric space-time manifold $\mathbb{S}_4$ (with coordinates x^μ) endowed with both curvature and torsion. As usual, if we denote the symmetric, non-singular, fundamental metric tensor of $\mathbb{S}_4$ by g, then $g_{\mu\lambda}\, g^{\nu\lambda} = \delta^\nu_\mu$, where δ is the Kronecker delta. The world-line s is then given by the quadratic differential form $ds^2 = g_{\mu\nu}\, dx^\mu dx^v$. (The Einstein summation convention employed throughout this work.)

As in [1], for reasons that will be clear later, we define the electromagnetic field tensor F via the torsion tensor of space-time (the anti-symmetric part of the connection Γ) as follows:

$$F_{\mu\nu} = 2\,\frac{mc^2}{e}\,\Gamma^\lambda_{[\mu\nu]}\, u_\lambda\,,$$

where m is the mass (of the electron), c is the speed of light in vacuum, and e is the electric charge, and where $u^\mu = \frac{dx^\mu}{ds}$ are the components of the tangent world-velocity vector whose magnitude is unity. Solving for the torsion tensor, we may write, under very general conditions,

$$\Gamma^\lambda_{[\mu\nu]} = \frac{e}{2mc^2}\, F_{\mu\nu}\, u^\lambda + S^\lambda{}_{\mu\nu}\,,$$

where the components of the third-rank material spin (chirality) tensor ${}^3S$ are herein given via the second-rank anti-symmetric tensor ${}^2S$ as follows:

$$S^\lambda{}_{\mu\nu} = S^\lambda{}_\mu\, u_\nu - S^\lambda{}_\nu\, u_\mu\,.$$

As can be seen, it is necessary that we specify the following orthogonality condition:

$$S_{\mu\nu}\, u^\nu = 0\,,$$

such that

$$S^\lambda{}_{\mu\nu}\, u_\lambda = 0\,.$$

We note that ${}^3S$ may be taken as the intrinsic angular momentum tensor for microscopic physical objects which may be seen as the points in the space-time continuum itself. This way, ${}^3S$ may be regarded as a microspin tensor describing the internal rotation of the space-time points themselves [2]. Alternatively, ${}^3S$ may be taken as being "purely material" (entirely non-electromagnetic).

The covariant derivative of an arbitrary tensor field T is given via the asymmetric connection Γ by

$$\nabla_\lambda T^{\mu\nu\ldots}_{\rho\sigma\ldots} = \partial_\lambda T^{\mu\nu\ldots}_{\rho\sigma\ldots} + \Gamma^\mu_{\alpha\lambda} T^{\alpha\nu\ldots}_{\rho\sigma\ldots} + \Gamma^\nu_{\alpha\lambda} T^{\mu\alpha\ldots}_{\rho\sigma\ldots} + \cdots - \\ - \Gamma^\alpha_{\rho\lambda} T^{\mu\nu\ldots}_{\alpha\sigma\ldots} - \Gamma^\alpha_{\sigma\lambda} T^{\mu\nu\ldots}_{\rho\alpha\ldots} - \ldots\,,$$

where $\partial_\lambda = \frac{\partial}{\partial x^\lambda}$. Then, as usual, the metricity condition $\nabla_\lambda g_{\mu\nu} = 0$, or, equivalently, $\partial_\lambda g_{\mu\nu} = \Gamma_{\mu\nu\lambda} + \Gamma_{\nu\mu\lambda}$ (where $\Gamma_{\mu\nu\lambda} = g_{\mu\rho}\Gamma^\rho_{\nu\lambda}$), gives us the relation

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right) + \Gamma^\lambda_{[\mu\nu]} - \\ - g^{\lambda\rho} \left(g_{\mu\sigma}\Gamma^\sigma_{[\rho\nu]} + g_{\nu\sigma}\Gamma^\sigma_{[\rho\mu]}\right).$$

Hence we obtain, for the connection of our unified field theory, the following explicit form:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right) + \\ + \frac{e}{2mc^2} \left(F_{\mu\nu} u^\lambda - F^\lambda_{\ \mu} u_\nu - F^\lambda_{\ \nu} u_\mu\right) + \\ + S^\lambda_{\ \mu\nu} - g^{\lambda\rho} \left(S_{\mu\rho\nu} + S_{\nu\rho\mu}\right),$$

where

$$\Delta^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right)$$

are the components of the usual symmetric Levi-Civita connection, and where

$$K^\lambda_{\mu\nu} = \frac{e}{2mc^2} \left(F_{\mu\nu} u^\lambda - F^\lambda_{\ \mu} u_\nu - F^\lambda_{\ \nu} u_\mu\right) + S^\lambda_{\ \mu\nu} - \\ - g^{\lambda\rho} \left(S_{\mu\rho\nu} + S_{\nu\rho\mu}\right)$$

are the components of the contorsion tensor in our unified field theory.

The above expression for the connection can actually be written alternatively in a somewhat simpler form as follows:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho}\right) + \\ + \frac{e}{2mc^2} \left(F_{\mu\nu} u^\lambda - F^\lambda_{\ \mu} u_\nu - F^\lambda_{\ \nu} u_\mu\right) + 2S^\lambda_{\ \mu} u_\nu .$$

At this point, we see that the geometric structure of our space-time continuum is also determined by the electromagnetic field tensor as well as the material spin tensor, in addition to the gravitational (metrical) field.

As a consequence, we obtain the following relations (where the round brackets on indices, in contrast to the square ones, indicate symmetrization):

$$\Gamma^\lambda_{(\mu\nu)} = \Delta^\lambda_{\mu\nu} - \frac{e}{2mc^2} \left(F^\lambda_{\ \mu} u_\nu + F^\lambda_{\ \nu} u_\mu\right) + S^\lambda_{\ \mu} u_\nu + S^\lambda_{\ \nu} u_\mu,$$

$$\varphi_\mu = K^\lambda_{\mu\lambda} = 2\Gamma^\lambda_{[\mu\lambda]} = \frac{e}{mc^2} F_{\mu\lambda} u^\lambda.$$

We also have

$$\gamma_\mu = \Gamma^\lambda_{\mu\lambda} = \Delta^\lambda_{\mu\lambda} + \frac{e}{mc^2} F_{\mu\lambda} u^\lambda,$$

in addition to the usual relation

$$\Gamma^\lambda_{\lambda\mu} = \Delta^\lambda_{\lambda\mu} = \partial_\mu \left(\ln \sqrt{\det(g)}\right).$$

At this point, we may note that the spin vector φ is always orthogonal to the world-velocity vector as

$$\varphi_\mu u^\mu = 0 .$$

In terms of the four-potential A, if we take the electromagnetic field tensor to be a pure curl as follows:

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu = \bar{\nabla}_\nu A_\mu - \bar{\nabla}_\mu A_\nu ,$$

where $\bar{\nabla}$ represents the covariant derivative with respect to the symmetric Levi-Civita connection alone, then we have the following general identities:

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = \bar{\nabla}_\lambda F_{\mu\nu} + \bar{\nabla}_\mu F_{\nu\lambda} + \bar{\nabla}_\nu F_{\lambda\mu} = 0,$$

$$\nabla_\lambda F_{\mu\nu} + \nabla_\mu F_{\nu\lambda} + \nabla_\nu F_{\lambda\mu} = \\ = -2 \left(\Gamma^\rho_{[\mu\nu]} F_{\lambda\rho} + \Gamma^\rho_{[\nu\lambda]} F_{\mu\rho} + +\Gamma^\rho_{[\lambda\mu]} F_{\nu\rho}\right).$$

The electromagnetic current density vector is then given by

$$J^\mu = -\frac{c}{4\pi} \nabla_\nu F^{\mu\nu}.$$

Its fully covariant divergence is then given by

$$\nabla_\mu J^\mu = -\frac{c}{4\pi} \nabla_\mu \left(\Gamma^\mu_{[\rho\sigma]} F^{\rho\sigma}\right).$$

If we further take $J^\mu = \rho_{em} u^\mu$, where ρ_{em} represents the electromagnetic charge density (taking into account the possibility of a magnetic charge), we see immediately that our electromagnetic current is conserved if and only if $\bar{\nabla}_\mu J^\mu = 0$, as follows

$$\nabla_\mu J^\mu = \partial_\mu J^\mu + \Gamma^\mu_{\lambda\mu} J^\lambda = \\ = \bar{\nabla}_\mu J^\mu + \frac{e}{mc^2} F_{\lambda\mu} J^\lambda u^\lambda = \bar{\nabla}_\mu J^\mu.$$

In other words, for the electromagnetic current density to be conserved in our theory, the following conditions must be satisfied (for an arbitrary scalar field Φ):

$$J^\mu = -\frac{c}{4\pi} \Gamma^\mu_{[\rho\sigma]} F^{\rho\sigma},$$

$$\Gamma^\lambda_{[\mu\nu]} = \delta^\lambda_\mu \partial_\nu \Phi - \delta^\lambda_\nu \partial_\mu \Phi .$$

These relations are reminiscent of those in [1]. Note that we have made use of the relation $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\Phi = = 2\Gamma^\lambda_{[\mu\nu]} \nabla_\lambda \Phi$.

Now, corresponding to our desired conservation law for electromagnetic currents, we can alternatively express the connection as

$$\Gamma^\lambda_{\mu\nu} = \Delta^\lambda_{\mu\nu} + 2\left(g^{\lambda\rho} g_{\mu\nu} \partial_\rho \Phi - \delta^\lambda_\nu \partial_\mu \Phi\right).$$

Contracting the above relation, we obtain the simple relation $\Gamma^\lambda_{\mu\lambda} = \Delta^\lambda_{\mu\lambda} - 6\partial_\mu \Phi$. On the other hand, we also have the relation $\Gamma^\lambda_{\mu\lambda} = \Delta^\lambda_{\mu\lambda} + \frac{e}{mc^2} F_{\mu\lambda} u^\lambda$. Hence we see that Φ

is a constant of motion as

$$\partial_\mu \Phi = -\frac{e}{6mc^2} F_{\mu\nu} u^\nu ,$$

$$\frac{d\Phi}{ds} = 0 .$$

These two conditions uniquely determine the conservation of electromagnetic currents in our theory.

Furthermore, not allowing for external forces, the geodesic equation of motion in $\mathbb{S}_4$, namely,

$$\frac{Du^\mu}{Ds} = u^\nu \nabla_\nu u^\mu = 0 ,$$

must hold in $\mathbb{S}_4$ in order for the gravitational, electromagnetic, and material spin fields to be genuine intrinsic geometric objects that uniquely and completely build the structure of the space-time continuum.

Recalling the relation $\Gamma^\lambda_{(\mu\nu)} = \Delta^\lambda_{\mu\nu} - \frac{e}{2mc^2}\left(F^\lambda{}_\mu u_\nu + F^\lambda{}_\nu u_\mu\right) + S^\lambda{}_\mu u_\nu + S^\lambda{}_\nu u_\mu$, we obtain the equation of motion

$$\frac{du^\mu}{ds} + \Delta^\mu_{\nu\rho} u^\nu u^\rho = \frac{e}{mc^2} F^\mu{}_\nu u^\nu ,$$

which is none other than the equation of motion for a charged particle moving in a gravitational field. This simply means that our relation $F_{\mu\nu} = 2\frac{mc^2}{e}\Gamma^\lambda_{[\mu\nu]} u_\lambda$ does indeed indicate a valid geometrization of the electromagnetic field.

In the case of conserved electromagnetic currents, we have

$$\frac{du^\mu}{ds} + \Delta^\mu_{\nu\rho} u^\nu u^\rho = -6\, g^{\mu\nu}\partial_\nu \Phi .$$

3 The field equations of the unified field theory

The (intrinsic) curvature tensor R of $\mathbb{S}_4$ is of course given by the usual relation

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\, V_\lambda = R^\rho{}_{\lambda\mu\nu} V_\rho - 2\Gamma^\rho_{[\mu\nu]} \nabla_\rho V_\lambda ,$$

where V is an arbitrary vector field. For an arbitrary tensor field T, we have the more general relation

$$(\nabla_\nu \nabla_\mu - \nabla_\mu \nabla_\nu)\, T^{\alpha\beta\ldots}_{\rho\sigma\ldots} = R^\lambda{}_{\rho\mu\nu} T^{\alpha\beta\ldots}_{\lambda\sigma\ldots} + R^\lambda{}_{\sigma\mu\nu} T^{\alpha\beta\ldots}_{\rho\lambda\ldots} + \ldots - R^\alpha{}_{\lambda\mu\nu} T^{\lambda\beta\ldots}_{\rho\sigma\ldots} - R^\beta{}_{\lambda\mu\nu} T^{\alpha\lambda\ldots}_{\rho\sigma\ldots} - \ldots - 2\Gamma^\lambda_{[\mu\nu]} \nabla_\lambda T^{\alpha\beta\ldots}_{\rho\sigma\ldots} .$$

Of course,

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\rho_{\lambda\nu} - \partial_\nu \Gamma^\rho_{\lambda\mu} + \Gamma^\sigma_{\lambda\nu}\Gamma^\rho_{\sigma\mu} - \Gamma^\sigma_{\lambda\mu}\Gamma^\rho_{\sigma\nu} .$$

If we define the following contractions:

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} ,$$

$$R = R^\mu{}_\mu ,$$

then, as usual,

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{1}{2}\left(g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\mu\sigma}R_{\nu\rho} - g_{\nu\rho}R_{\mu\sigma}\right) + \frac{1}{6}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right) R ,$$

where C is the Weyl tensor. Note that the generalized Ricci tensor (given by its components $R_{\mu\nu}$) is generally asymmetric.

Let us denote the usual Riemann-Christoffel curvature tensor by $\bar{R}$, i.e.,

$$\bar{R}^\rho{}_{\lambda\mu\nu} = \partial_\mu \Delta^\rho_{\lambda\nu} - \partial_\nu \Delta^\rho_{\lambda\mu} + \Delta^\sigma_{\lambda\nu}\Delta^\rho_{\sigma\mu} - \Delta^\sigma_{\lambda\mu}\Delta^\rho_{\sigma\nu} .$$

The symmetric Ricci tensor and the Ricci scalar are then given respectively by $\bar{R}_{\mu\nu} = \bar{R}^\lambda{}_{\mu\lambda\nu}$ and $\bar{R} = \bar{R}^\mu{}_\mu$.

Furthermore, we obtain the following decomposition:

$$R^\rho{}_{\lambda\mu\nu} = \bar{R}^\rho{}_{\lambda\mu\nu} + \bar{\nabla}_\mu K^\rho_{\lambda\nu} - \bar{\nabla}_\nu K^\rho_{\lambda\mu} + K^\sigma_{\lambda\nu}K^\rho_{\sigma\mu} - K^\sigma_{\lambda\mu}K^\rho_{\sigma\nu} .$$

Hence, recalling that $\varphi_\mu = K^\lambda_{\mu\lambda} = 2\Gamma^\lambda_{[\mu\lambda]}$, we obtain

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\lambda K^\lambda_{\mu\nu} - K^\lambda_{\mu\rho}K^\rho_{\lambda\nu} - \bar{\nabla}_\nu \varphi_\mu + 2K^\lambda_{\mu\nu}\varphi_\lambda ,$$

$$R = \bar{R} - 2\bar{\nabla}_\mu \varphi^\mu - \varphi_\mu \varphi^\mu - K_{\mu\nu\lambda}K^{\mu\lambda\nu} .$$

We then obtain the following generalized Bianchi identities:

$$R^\rho{}_{\lambda\mu\nu} + R^\rho{}_{\mu\nu\lambda} + R^\rho{}_{\nu\lambda\mu} = -2\big(\partial_\nu \Gamma^\rho_{[\lambda\mu]} + \partial_\lambda \Gamma^\rho_{[\mu\nu]} + \partial_\mu \Gamma^\rho_{[\nu\lambda]} + \Gamma^\rho_{\sigma\lambda}\Gamma^\sigma_{[\mu\nu]} + \Gamma^\rho_{\sigma\mu}\Gamma^\sigma_{[\nu\lambda]} + \Gamma^\rho_{\sigma\nu}\Gamma^\sigma_{[\lambda\mu]}\big) ,$$

$$\nabla_\lambda R_{\mu\nu\rho\sigma} + \nabla_\rho R_{\mu\nu\sigma\lambda} + \nabla_\sigma R_{\mu\nu\lambda\rho} = 2\left(\Gamma^\alpha_{[\rho\sigma]} R_{\mu\nu\alpha\lambda} + \Gamma^\alpha_{[\sigma\lambda]} R_{\mu\nu\alpha\rho} + \Gamma^\alpha_{[\lambda\rho]} R_{\mu\nu\alpha\sigma}\right) ,$$

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 2g^{\mu\nu}\Gamma^\lambda_{[\rho\mu]} R^\rho{}_\lambda + \Gamma^\lambda_{[\rho\sigma]} R^{\rho\sigma\nu}{}_\lambda ,$$

in addition to the standard Bianchi identities

$$\bar{R}^\rho{}_{\lambda\mu\nu} + \bar{R}^\rho{}_{\mu\nu\lambda} + \bar{R}^\rho{}_{\nu\lambda\mu} = 0 ,$$

$$\bar{\nabla}_\lambda \bar{R}_{\mu\nu\rho\sigma} + \bar{\nabla}_\rho \bar{R}_{\mu\nu\sigma\lambda} + \bar{\nabla}_\sigma \bar{R}_{\mu\nu\lambda\rho} = 0 ,$$

$$\bar{\nabla}_\mu \left(\bar{R}^{\mu\nu} - \frac{1}{2} g^{\mu\nu} \bar{R}\right) = 0 .$$

(See [2–4] for instance.)

Furthermore, we can now obtain the following explicit expression for the curvature tensor R:

$$R^\rho{}_{\lambda\mu\nu} = \partial_\mu \Delta^\rho_{\lambda\nu} - \partial_\nu \Delta^\rho_{\lambda\mu} + \Delta^\sigma_{\lambda\nu}\Delta^\rho_{\sigma\mu} - \Delta^\sigma_{\lambda\mu}\Delta^\rho_{\sigma\nu} + \frac{e}{2mc^2}\Big\{\left(\partial_\mu F_{\lambda\nu} - \partial_\nu F_{\lambda\mu}\right) u^\rho + \left(\partial_\nu F^\rho{}_\mu - \partial_\mu F^\rho{}_\nu\right) u_\lambda + u_\mu \partial_\nu F^\rho{}_\lambda - u_\nu \partial_\mu F^\rho{}_\lambda + F_{\lambda\nu}\partial_\mu u^\rho - F_{\lambda\mu}\partial_\nu u^\rho + F^\rho{}_\mu \partial_\nu u_\lambda - F^\rho{}_\nu \partial_\mu u_\lambda + \left(\partial_\nu u_\mu - \partial_\mu u_\nu\right) F^\rho{}_\lambda + \left(F_{\lambda\nu}u^\sigma - F^\sigma{}_\lambda u_\nu - F^\sigma{}_\nu u_\lambda\right)\Delta^\rho_{\sigma\mu} + \left(F_{\sigma\mu}u^\rho - F^\rho{}_\sigma u_\mu - F^\rho{}_\mu u_\sigma\right)\Delta^\sigma_{\lambda\nu} - \left(F_{\lambda\mu}u^\sigma - F^\sigma{}_\lambda u_\mu - F^\sigma{}_\mu u_\lambda\right)\Delta^\rho_{\sigma\nu} -$$

$$- \left(F_{\sigma\nu} u^{\rho} - F^{\rho}_{\ \sigma} u_{\nu} - F^{\rho}_{\ \nu} u_{\sigma}\right) \Delta^{\sigma}_{\lambda\mu} + \frac{e}{2\,m\,c^2} \left(F_{\lambda\nu} F_{\sigma\mu} - \right.$$
$$- F_{\lambda\mu} F_{\sigma\nu}\big) u^{\sigma} u^{\rho} + \frac{e}{2\,m\,c^2} \left(F_{\lambda\mu} F^{\rho}_{\ \sigma} - F_{\lambda\sigma} F^{\rho}_{\ \mu}\right) u_{\nu} u^{\sigma} +$$
$$+ \frac{e}{2\,m\,c^2} \left(F_{\lambda\sigma} F^{\rho}_{\ \nu} - F_{\lambda\nu} F^{\rho}_{\ \sigma}\right) u_{\mu} u^{\sigma} + \frac{e}{2mc^2} \left(F^{\sigma}_{\ \mu} F_{\sigma\nu} - \right.$$
$$- F^{\sigma}_{\ \nu} F_{\sigma\mu}\big) u_{\lambda} u^{\rho} + \frac{e}{2mc^2} \left(F_{\sigma\nu} F^{\rho}_{\ \mu} - F_{\sigma\mu} F^{\rho}_{\ \nu}\right) u_{\lambda} u^{\sigma} +$$
$$+ \frac{e}{2mc^2} F^{\sigma}_{\ \lambda} F_{\sigma\nu} u_{\mu} u^{\rho} + \frac{e}{2mc^2} F^{\sigma}_{\ \nu} F^{\rho}_{\ \sigma} u_{\lambda} u_{\mu} -$$
$$- \frac{e}{2mc^2} F^{\sigma}_{\ \lambda} F_{\sigma\mu} u_{\nu} u^{\rho} - \frac{e}{2mc^2} F^{\sigma}_{\ \mu} F^{\rho}_{\ \sigma} u_{\lambda} u_{\nu} +$$
$$+ \frac{e}{2mc^2} \left(F_{\lambda\mu} F^{\rho}_{\ \nu} - F_{\lambda\nu} F^{\rho}_{\ \mu}\right)\Big\} + \Omega^{\rho}_{\ \lambda\mu\nu}\,,$$

where the tensor Ω consists of the remaining terms containing the material spin tensor ${}^2S$ (or ${}^3S$).

Now, keeping in mind that $\Gamma^{\lambda}_{(\mu\nu)} = \Delta^{\lambda}_{\mu\nu} - \frac{e}{2mc^2}\big(F^{\lambda}_{\ \mu} u_{\nu} + F^{\lambda}_{\ \nu} u_{\mu}\big) + S^{\lambda}_{\ \mu} u_{\nu} + S^{\lambda}_{\ \nu} u_{\mu}$ and also $\gamma_{\mu} = \Gamma^{\lambda}_{\mu\lambda} = \Delta^{\lambda}_{\mu\lambda} + \frac{e}{mc^2} F_{\mu\lambda} u^{\lambda}$, and decomposing the components of the generalized Ricci tensor as $R_{\mu\nu} = R_{(\mu\nu)} + R_{[\mu\nu]}$, we see that

$$R_{(\mu\nu)} = \partial_{\lambda} \Gamma^{\lambda}_{(\mu\nu)} - \frac{1}{2}\left(\partial_{\nu}\gamma_{\mu} + \partial_{\mu}\gamma_{\nu}\right) + \Gamma^{\lambda}_{(\mu\nu)} \gamma_{\lambda} - $$
$$- \frac{1}{2}\left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}\right),$$

$$R_{[\mu\nu]} = \partial_{\lambda} \Gamma^{\lambda}_{[\mu\nu]} - \frac{1}{2}\left(\partial_{\nu}\gamma_{\mu} - \partial_{\mu}\gamma_{\nu}\right) + \Gamma^{\lambda}_{[\mu\nu]} \gamma_{\lambda} - $$
$$- \frac{1}{2}\left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}\right).$$

In particular, we note that

$$R_{[\mu\nu]} = \partial_{\lambda} \Gamma^{\lambda}_{[\mu\nu]} - \frac{1}{2}\left(\partial_{\nu}\gamma_{\mu} - \partial_{\mu}\gamma_{\nu}\right) + \Gamma^{\lambda}_{[\mu\nu]} \gamma_{\lambda} - $$
$$- \frac{1}{2}\left(\Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\rho\mu}\right) =$$
$$= \partial_{\lambda}\Gamma^{\lambda}_{[\mu\nu]} + \Gamma^{\rho}_{\lambda\rho}\Gamma^{\lambda}_{[\mu\nu]} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{[\nu\rho]} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{[\mu\rho]} +$$
$$+ \frac{1}{2}\left(\partial_{\nu}\gamma_{\mu} - \partial_{\mu}\gamma_{\nu}\right) = \nabla_{\lambda}\Gamma^{\lambda}_{[\mu\nu]} + \frac{1}{2}\left(\partial_{\nu}\gamma_{\mu} - \partial_{\mu}\gamma_{\nu}\right).$$

Hence we obtain the relation

$$R_{[\mu\nu]} = \frac{e}{2mc^2}\left(F_{\mu\nu}\nabla_{\lambda} u^{\lambda} + \frac{DF_{\mu\nu}}{Ds}\right) + \nabla_{\lambda} S^{\lambda}_{\ \mu\nu} +$$
$$+ \frac{1}{2}\left(\partial_{\nu}\gamma_{\mu} - \partial_{\mu}\gamma_{\nu}\right),$$

where $\frac{DF_{\mu\nu}}{Ds} = u^{\lambda}\nabla_{\lambda} F_{\mu\nu}$. More explicitly, we can write

$$R_{[\mu\nu]} = \frac{e}{2mc^2}\left(F_{\mu\nu}\nabla_{\lambda} u^{\lambda} + \frac{DF_{\mu\nu}}{Ds} + \left(\partial_{\nu} F_{\lambda\mu} - \right.\right.$$
$$\left.- \partial_{\mu} F_{\lambda\nu}\right) u^{\lambda} + F_{\lambda\mu}\partial_{\nu} u^{\lambda} - F_{\lambda\nu}\partial_{\mu} u^{\lambda}\Big) + \nabla_{\lambda} S^{\lambda}_{\ \mu\nu}.$$

It is therefore seen that, in general, the special identity

$$\partial_{\lambda} R_{[\mu\nu]} + \partial_{\mu} R_{[\nu\lambda]} + \partial_{\nu} R_{[\lambda\mu]} = 0$$

holds only when $\nabla_{\mu} u^{\mu} = 0$, $\frac{DF_{\mu\nu}}{Ds} = 0$, and $\nabla_{\lambda} S^{\lambda}_{\ \mu\nu} = 0$.

We are now in a position to generalize Einstein's field equation in the standard theory of general relativity. The usual Einstein's field equation is of course given by

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu}\bar{R} = kT_{\mu\nu}\,,$$
$$\bar{\nabla}_{\mu}\bar{G}^{\mu\nu} = 0\,,$$

where $\bar{G}$ is the symmetric Einstein tensor, T is the energy-momentum tensor, and $k = \pm\,\frac{8\pi G}{c^4}$ is Einstein's coupling constant in terms of the Newtonian gravitational constant G. Taking $c = 1$ for convenience, in the absence of pressure, traditionally we write

$$\bar{G}^{\mu\nu} = k\left(\rho_m u^{\mu} u^{\nu} + \frac{1}{4\pi}\left(F^{\mu}_{\ \lambda} F^{\nu\lambda} - \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right)\right),$$

where ρ_m is the material density and where the second term on the right-hand-side of the equation is widely regarded as representing the electromagnetic energy-momentum tensor.

Now, with the generalized Bianchi identity for the electromagnetic field, i.e., $\nabla_{\lambda} F_{\mu\nu} + \nabla_{\mu} F_{\nu\lambda} + \nabla_{\nu} F_{\lambda\mu} = -2\left(\Gamma^{\rho}_{[\mu\nu]} F_{\lambda\rho} + \Gamma^{\rho}_{[\nu\lambda]} F_{\mu\rho} + \Gamma^{\rho}_{[\lambda\mu]} F_{\nu\rho}\right)$, at hand, and assuming the "isochoric" condition $\frac{D\rho_m}{Ds} = -\,\rho_m \nabla_{\mu} u^{\mu} = 0$ ($\rho_m \neq 0$), we obtain

$$\nabla_{\nu}\bar{G}^{\mu\nu} = k g^{\mu\nu}\left(\Gamma^{\lambda}_{[\rho\sigma]} F_{\nu\lambda} + \Gamma^{\lambda}_{[\sigma\nu]} F_{\rho\lambda} + \Gamma^{\lambda}_{[\nu\rho]} F_{\sigma\lambda}\right) F^{\rho\sigma}.$$

In other words,

$$\nabla_{\nu}\bar{G}^{\mu\nu} = k\left(2 g^{\mu\nu}\Gamma^{\lambda}_{[\sigma\nu]} F_{\rho\lambda} F^{\rho\sigma} - \frac{1}{4\pi} F^{\mu}_{\ \nu} J^{\nu}\right).$$

This is our first generalization of the standard Einstein's field equation, following the traditional ad hoc way of arbitrarily adding the electromagnetic contribution to the purely material part of the energy-momentum tensor.

Now, more generally and more naturally, using the generalized Bianchi identity $\nabla_{\mu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = 2 g^{\mu\nu}\Gamma^{\lambda}_{[\rho\mu]} R^{\rho}_{\ \lambda} + \Gamma^{\lambda}_{[\rho\sigma]} R^{\rho\sigma\nu}_{\ \ \ \ \lambda}$, we can obtain the following fundamental relation:

$$\nabla_{\mu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = \frac{e}{mc^2}\left(F_{\mu}^{\ \nu} R^{\mu}_{\ \lambda} + \frac{1}{2} F_{\mu\rho} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\right) u^{\lambda} +$$
$$+ 2 S_{\mu\lambda}^{\ \ \ \nu} R^{\mu\lambda} + S^{\lambda}_{\ \mu\rho} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\,.$$

Alternatively, we can also write this as

$$\nabla_{\mu}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right) = \frac{e}{mc^2}\left(F_{\mu}^{\ \nu} R^{\mu}_{\ \lambda} + \frac{1}{2} F_{\mu\rho} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\right) u^{\lambda} +$$
$$+ S_{\mu\rho} R^{\rho\mu} u^{\nu} - S^{\lambda}_{\ \rho} u_{\mu} R^{\mu\rho\nu}_{\ \ \ \ \lambda} + \left(S^{\nu}_{\ \mu} R^{\rho\mu} - S^{\lambda}_{\ \mu} R^{\mu\rho\nu}_{\ \ \ \ \lambda}\right) u_{\rho}\,.$$

Now, as a special consideration, let Σ be the "area" of a three-dimensional space-like hypersurface representing matter in $\mathbb{S}_4$. Then, if we make the following traditional choice for the third-rank material spin tensor ${}^3S$:

$$S^{\mu\nu\lambda} = \iiint\limits_{\Sigma} \rho_m \left(x^\lambda T^{\mu\nu} - x^\nu T^{\mu\lambda} \right) d\Sigma\,,$$

where now T is the total asymmetric energy-momentum tensor in our theory, we see that, in the presence of matter, the condition $S^{\mu\nu\lambda} = 0$ implies that

$$T^{[\mu\nu]} = -\frac{1}{2} \left(x^\mu \nabla_\lambda T^{\lambda\nu} - x^\nu \nabla_\lambda T^{\lambda\mu} \right).$$

In this special case, we obtain the simplified expression

$$\nabla_\mu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = \frac{e}{mc^2} \left(F_\mu{}^\nu R^\mu{}_\lambda + \frac{1}{2} F_{\mu\rho} R^{\mu\rho\nu}{}_\lambda \right) u^\lambda.$$

If we further assume that the sectional curvature $\Psi = \frac{1}{12} R$ of $\mathbb{S}_4$ is everywhere constant in a space-time region where the electromagnetic field (and hence the torsion) is absent, we may consider writing $R_{\mu\nu\rho\sigma} = \Psi \left(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho} \right)$ such that $\mathbb{S}_4$ is conformally flat ($C_{\mu\nu\rho\sigma} = 0$), and hence $R_{\mu\nu} = 3\Psi g_{\mu\nu}$ and $R_{[\mu\nu]} = 0$. In this case, we are left with the simple expression

$$\nabla_\nu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) = -\frac{eR}{6mc^2} F^\mu_\nu u^\nu.$$

This is equivalent to the equation of motion

$$\frac{du^\mu}{ds} + \Delta^\mu_{\nu\rho} u^\nu u^\rho = -\frac{6}{R} \nabla_\nu \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right).$$

4 The minimal Lagrangian density of the theory

Using the general results from the preceding section, we obtain

$$R = \bar{R} + \frac{e^2}{4m^2c^4} F_{\mu\nu} F^{\mu\nu} - \frac{e^2}{m^2c^4} F_{\mu\lambda} F^\mu_\nu u^\lambda u^\nu - \frac{2e}{mc^2} \bar{\nabla}_\mu f^\mu + 2S_{\mu\nu} S^{\mu\nu} - K_{\mu(\nu\lambda)} K^{\mu(\nu\lambda)}\,,$$

for the curvature scalar of $\mathbb{S}_4$. Here $f^\mu = F^\mu_\nu u^\nu$ can be said to be the components of the so-called Lorentz force.

Furthermore, we see that

$$K_{\mu(\nu\lambda)} K^{\mu(\nu\lambda)} = \frac{e^2}{m^2c^4} F_{\mu\nu} F^{\mu\nu} + 2S_{\mu\nu} S^{\mu\nu} - \frac{2e}{mc^2} F_{\mu\nu} S^{\mu\nu} - \frac{e^2}{2m^2c^4} F_{\mu\lambda} F^\mu_\nu u^\lambda u^\nu.$$

Hence we obtain

$$R = \bar{R} - \frac{e^2}{2m^2c^4} F_{\mu\nu} F^{\mu\nu} - \frac{2e}{mc^2} \left(\bar{\nabla}_\mu f^\mu + F_{\mu\nu} S^{\mu\nu} \right) - \frac{e^2}{2m^2c^4} F_{\mu\lambda} F^\mu_\nu u^\lambda u^\nu.$$

The last two terms on the right-hand-side of the expression can then be grouped into a single scalar source as follows:

$$\phi = -\frac{2e}{mc^2} \left(\bar{\nabla}_\mu f^\mu + F_{\mu\nu} S^{\mu\nu} \right) - \frac{e^2}{2m^2c^4} F_{\mu\lambda} F^\mu_\nu u^\lambda u^\nu.$$

Assuming that ϕ accounts for both the total (material-electromagnetic) charge density as well as the total energy density, our unified field theory may be described by the following action integral (where the $L = R\sqrt{\det(g)}$ is the minimal Lagrangian density):

$$I = \iiiint R \sqrt{\det(g)}\, d^4x = \iiiint \left(\bar{R} - \frac{e^2}{2m^2c^4} F_{\mu\nu} F^{\mu\nu} + \phi \right) \sqrt{\det(g)}\, d^4x\,.$$

In this minimal fashion, gravity (described by $\bar{R}$) appears as an emergent phenomenon whose intrinsic nature is of electromagnetic and purely material origin since, in our theory, the electromagnetic and material spin fields are nothing but components of a single torsion field.

5 The non-Abelian Yang-Mills gauge field as a sub-torsion field in $\mathbb{S}_4$

In $\mathbb{S}_4$, let there exist a space-like three-dimensional hypersurface Θ_3, with local coordinates X^i (Latin indices shall run from 1 to 3). From the point of view of projective differential geometry alone, we may say that Θ_3i s embedded (immersed) in $\mathbb{S}_4$. Then, the tetrad linking the embedded space Θ_3 to the enveloping space-time $\mathbb{S}_4$ is readily given by

$$\omega^i_\mu = \frac{\partial X^i}{\partial x^\mu}\,, \qquad \omega^\mu_i = \left(\omega^i_\mu \right)^{-1} = \frac{\partial x^\mu}{\partial X^i}\,.$$

Furthermore, let N be a unit vector normal to the hypersurface Θ_3. We may write the parametric equation of the hypersurface Θ_3 as $H(x^\mu, d) = 0$, where d is constant. Hence

$$N^\mu = \frac{g^{\mu\nu} \partial_\nu H}{\sqrt{g^{\alpha\beta} (\partial_\alpha H)(\partial_\beta H)}}\,,$$

$$N_\mu N^\mu = 1\,.$$

In terms of the axial unit vectors a, b, and c spanning the hypersurface Θ_3, we may write

$$N_\mu = \frac{\varepsilon_{\mu\nu\rho\sigma} a^\nu b^\rho c^\sigma}{\varepsilon_{\alpha\beta\lambda\eta} N^\alpha a^\beta b^\lambda c^\eta}\,,$$

where $\varepsilon_{\mu\nu\rho\sigma}$ are the components of the completely anti-symmetric four-dimensional Levi-Civita permutation tensor density.

Now, the tetrad satisfies the following projective relations:

$$\omega^i_\mu N^\mu = 0\,, \qquad \omega^i_\mu \omega^\mu_k = \delta^i_k\,,$$

$$\omega^\mu_i \omega^i_\nu = \delta^\mu_\nu - N^\mu N^\nu.$$

If we denote the local metric tensor of Θ_3 by h, we obtain the following relations:

$$h_{ik} = \omega^{\mu}_{i}\,\omega^{\nu}_{k}\,g_{\mu\nu}\,,$$

$$g_{\mu\nu} = \omega^{i}_{\mu}\,\omega^{k}_{\nu}\,h_{ik} + N_{\mu}N_{\nu}\,.$$

Furthermore, in the hypersurface Θ_3, let us set $\nabla_i = \omega^{\mu}_{i}\nabla_{\mu}$ and $\partial_i = \frac{\partial}{\partial X^i} = \omega^{\mu}_{i}\,\partial_{\mu}$. Then we have the following fundamental expressions:

$$\nabla_{\nu}\,\omega^{i}_{\mu} = Z^{i}{}_{k}\,\omega^{k}_{\nu}\,N_{\mu} = \partial_{\nu}\,\omega^{i}_{\mu} - \omega^{i}_{\sigma}\,\Gamma^{\sigma}_{\mu\nu} + \Gamma^{i}_{kl}\,\omega^{k}_{\mu}\,\omega^{l}_{\nu}\,,$$

$$\nabla_{k}\,\omega^{\mu}_{i} = Z_{ik}\,N^{\mu} = \partial_{k}\,\omega^{\mu}_{i} - \omega^{\mu}_{p}\,\Gamma^{p}_{ik} + \Gamma^{\mu}_{\rho\sigma}\,\omega^{\rho}_{i}\,\omega^{\sigma}_{k}\,,$$

$$\omega^{\mu}_{i}\,\nabla_{\nu}\,\omega^{k}_{\mu} = 0\,,$$

$$\nabla_{i}\,N^{\mu} = -Z^{k}{}_{i}\,\omega^{\mu}_{k}\,,$$

where Z is the extrinsic curvature tensor of the hypersurface Θ_3, which is generally asymmetric in our theory.

The connection of the hypersurface Θ_3 is linked to that of the space-time $\mathbb{S}_4$ via

$$\Gamma^{p}_{ik} = \omega^{p}_{\mu}\,\partial_{k}\,\omega^{\mu}_{i} + \omega^{p}_{\lambda}\,\Gamma^{\lambda}_{\mu\nu}\,\omega^{\mu}_{i}\,\omega^{\nu}_{k}\,.$$

After some algebra, we obtain

$$\Gamma^{\lambda}_{\mu\nu} = \omega^{\lambda}_{i}\,\partial_{\nu}\,\omega^{i}_{\mu} + \omega^{\lambda}_{p}\,\Gamma^{p}_{ik}\,\omega^{i}_{\mu}\,\omega^{k}_{\nu} + N^{\lambda}\,\partial_{\nu}\,N_{\mu} +$$
$$+ N^{\lambda}\,Z_{ik}\,\omega^{i}_{\mu}\,\omega^{k}_{\nu} - N_{\mu}\,Z^{i}{}_{k}\,\omega^{\lambda}_{i}\,\omega^{k}_{\nu}\,.$$

The fundamental geometric relations describing our embedding theory are then given by the following expressions (see [4] for instance):

$$R_{ijkl} = Z_{ik}\,Z_{jl} - Z_{il}\,Z_{jk} + R_{\mu\nu\rho\sigma}\,\omega^{\mu}_{i}\,\omega^{\nu}_{j}\,\omega^{\rho}_{k}\,\omega^{\sigma}_{l} - \omega^{\mu}_{i}\,\Lambda_{\mu jkl}\,,$$

$$\nabla_{l}\,Z_{ik} - \nabla_{k}\,Z_{il} = -R_{\mu\nu\rho\sigma}\,N^{\mu}\,\omega^{\nu}_{i}\,\omega^{\rho}_{k}\,\omega^{\sigma}_{l} - 2\Gamma^{p}_{[kl]}\,Z_{ip} + N^{\mu}\,\Lambda_{\mu ikl}\,,$$

$$\Lambda^{\mu}{}_{ijk} = \left(\partial_{k}\,\partial_{j} - \partial_{j}\,\partial_{k}\right)\omega^{\mu}_{i} + \omega^{\sigma}_{i}\,\Gamma^{\mu}_{\sigma\rho}\left(\partial_{k}\,\omega^{\rho}_{j} - \partial_{j}\,\omega^{\rho}_{k}\right).$$

Actually, these relations are just manifestations of the following single expression:

$$\left(\nabla_{k}\,\nabla_{j} - \nabla_{j}\,\nabla_{k}\right)\omega^{\mu}_{i} = R^{p}{}_{ijk}\,\omega^{\mu}_{p} - R^{\mu}{}_{\nu\rho\sigma}\,\omega^{\nu}_{i}\,\omega^{\rho}_{j}\,\omega^{\sigma}_{k} -$$
$$- 2\Gamma^{p}_{[jk]}\,Z_{ip}\,N^{\mu} + \Lambda^{\mu}{}_{ijk}\,.$$

We may note that $\Gamma^{p}_{[ik]}$ and

$$R^{i}{}_{jkl} = \partial_{k}\,\Gamma^{i}_{jl} - \partial_{l}\,\Gamma^{i}_{jk} + \Gamma^{p}_{jl}\,\Gamma^{i}_{pk} - \Gamma^{p}_{jk}\,\Gamma^{i}_{pl}$$

are the components of the torsion tensor and the intrinsic curvature tensor of the hypersurface Θ_3, respectively.

Now, let us observe that

$$\partial_{\nu}\,\omega^{i}_{\mu} - \partial_{\mu}\,\omega^{i}_{\nu} = 2\left(\omega^{i}_{\lambda}\,\Gamma^{\lambda}_{[\mu\nu]} - \Gamma^{i}_{[kl]}\,\omega^{k}_{\mu}\,\omega^{l}_{\nu} + Z^{i}{}_{k}\,\omega^{k}_{[\nu}\,N_{\mu]}\right).$$

Hence letting

$$F^{i}_{\mu\nu} = 2\omega^{i}_{\lambda}\,\Gamma^{\lambda}_{[\mu\nu]}\,,$$

we arrive at the expression

$$F^{i}_{\mu\nu} = \partial_{\nu}\,\omega^{i}_{\mu} - \partial_{\mu}\,\omega^{i}_{\nu} + 2\Gamma^{i}_{[kl]}\,\omega^{k}_{\mu}\,\omega^{l}_{\nu} + 2Z^{i}{}_{k}\,\omega^{k}_{[\mu}\,N_{\nu]}\,.$$

In addition, we also see that

$$\Gamma^{i}_{[kl]} = \frac{1}{2}\,\omega^{\mu}_{k}\,\omega^{\nu}_{l}\,F^{i}_{\mu\nu} - \frac{1}{2}\,\omega^{\mu}_{k}\,\omega^{\nu}_{l}\left(\partial_{\nu}\,\omega^{i}_{\mu} - \partial_{\mu}\,\omega^{i}_{\nu}\right).$$

Now, with respect to the local coordinate transformation given by $X^i = X^i\left(\bar{X}^A\right)$ in Θ_3, let us invoke the following Cartan-Lie algebra:

$$\left[e_i, e_k\right] = e_i \otimes e_k - e_k \otimes e_i = C^{p}_{ik}\,e_p\,,$$

$$C_{ikl} = h_{ip}\,C^{p}_{kl} = -2\Gamma_{i[kl]} = -\,i\hat{g}\,\in_{ikl}\,,$$

where $e_i = e^{A}_{i}\,\frac{\partial}{\partial \bar{X}^A}$ are the elements of the basis vector spanning Θ_3, C^{p}_{ik} are the spin coefficients, $i = \sqrt{-1}$, $\hat{g}$ is a coupling constant, and $\in_{ikl} = \sqrt{\det(h)}\,\varepsilon_{ikl}$ (where ε_{ikl} are the components of the completely anti-symmetric three-dimensional Levi-Civita permutation tensor density).

Hence we obtain

$$F^{i}_{\mu\nu} = \partial_{\nu}\,\omega^{i}_{\mu} - \partial_{\mu}\,\omega^{i}_{\nu} + i\hat{g}\,\in^{i}{}_{kl}\,\omega^{k}_{\mu}\,\omega^{l}_{\nu} + 2Z^{i}{}_{k}\,\omega^{k}_{[\mu}\,N_{\nu]}\,.$$

At this point, our key insight is to define the gauge field potential as the tetrad itself, i.e.,

$$B^{i}_{\mu} = \omega^{i}_{\mu}\,.$$

Hence, at last, we arrive at the following important expression:

$$F^{i}_{\mu\nu} = \partial_{\nu}\,B^{i}_{\mu} - \partial_{\mu}\,B^{i}_{\nu} + i\hat{g}\,\in^{i}{}_{kl}\,B^{k}_{\mu}\,B^{l}_{\nu} + 2Z^{i}{}_{k}\,B^{k}_{[\mu}\,N_{\nu]}\,.$$

Clearly, $F^{i}_{\mu\nu}$ are the components of the generalized Yang-Mills gauge field strength. To show this, consider the hypersurface $\mathbb{E}_3$ of rigid frames (where the metric tensor is strictly constant) which is a reduction (or, in a way, local infinitesimal representation) of the more general hypersurface Θ_3. We shall call this an "isospace". In it, we have

$$h_{ik} = \delta_{ik}\,,$$

$$\det(h) = 1\,,$$

$$\Gamma^{i}_{kl} = \Gamma_{ikl} = \Gamma_{i[kl]} - \Gamma_{l[ik]} - \Gamma_{k[il]} = \frac{1}{2}\,i\hat{g}\varepsilon_{ikl}\,,$$

$$Z_{ik} = 0\,.$$

Hence we arrive at the familiar expression

$$F_{i\mu\nu} = \partial_{\nu}\,B_{\mu i} - \partial_{\mu}\,B_{\nu i} + i\hat{g}\varepsilon_{ikl}\,B_{\mu k}\,B_{\nu l}\,.$$

In other words, setting $\vec{F}_{\mu\nu} = F_{i\mu\nu}\,e_i$ and $\vec{B}_{\mu} = B_{\mu i}\,e_i$, we obtain

$$\vec{F}_{\mu\nu} = \partial_{\nu}\,\vec{B}_{\mu} - \partial_{\mu}\,\vec{B}_{\nu} - \left[\vec{B}_{\mu}, \vec{B}_{\nu}\right].$$

Finally, let us define the gauge field potential of the second kind via

$$\omega_{\mu ik} = \varepsilon_{ikp}\,B^{p}_{\mu}\,,$$

such that

$$B^i_\mu = \frac{1}{2}\,\varepsilon_{ikl}\,\omega_{\mu kl}\,.$$

Let us then define the gauge field strength of the second kind via

$$R_{ik\mu\nu} = \in_{ikp}\,F^p_{\mu\nu}\,,$$

such that

$$F^p_{\mu\nu} = \frac{1}{2}\,\in^{pik}\,R_{ik\mu\nu}\,.$$

Hence we obtain the general expression

$$R_{ik\mu\nu} = i\,\hat{g}\,\sqrt{\det{(h)}}\,\Big\{\partial_\nu\,\omega_{\mu ik} - \partial_\mu\omega_{\nu ik} + \frac{1}{\sqrt{\det{(h)}}}\,(\omega_{\mu ip}\omega_{\nu kp} - \omega_{\mu kp}\omega_{\nu ip})\Big\} + \sqrt{\det{(h)}}\;\varepsilon_{ikp}\,Z^p_r\,B^r_{[\mu}\,N_{\nu]}\,.$$

We may regard the object given by this expression as the curvature of the local gauge spin connection of the hypersurface Θ_3.

Again, if we refer this to the isospace $\mathbb{E}_3$ instead of the more general hypersurface Θ_3, we arrive at the familiar relation

$$R_{ik\mu\nu} = i\,\hat{g}\,(\partial_\nu\,\omega_{\mu ik} - \partial_\mu\omega_{\nu ik} + \omega_{\mu ip}\omega_{\nu kp} - \omega_{\mu kp}\omega_{\nu ip})\,.$$

6 Conclusion

We have just completed our program of building the structure of a unified field theory in which gravity, electromagnetism, material spin, and the non-Abelian Yang-Mills gauge field (which is also capable of describing the weak force in the standard model particle physics) are all geometrized only in four dimensions. As we have seen, we have also generalized the expression for the Yang-Mills gauge field strength.

In our theory, the (generalized) Yang-Mills gauge field strength is linked to the electromagnetic field tensor via the relation

$$F_{\mu\nu} = 2\,\frac{m\,c^2}{e}\,\Gamma^\lambda_{[\mu\nu]}\,u_\lambda = \frac{m\,c^2}{e}\,F^i_{\mu\nu}\,u_i\,,$$

where $u^i = \omega^i_\mu u^\mu$. This enables us to express the connection in terms of the Yang-Mills gauge field strength instead of the electromagnetic field tensor as follows:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\,g^{\lambda\rho}\,(\partial_\nu\,g_{\rho\mu} - \partial_\rho\,g_{\mu\nu} + \partial_\mu\,g_{\nu\rho}) + \frac{1}{2}\,u_i\,\big(F^i_{\mu\nu}\,u^\lambda - F^{i\lambda}{}_\mu\,u_\nu - F^{i\lambda}{}_\nu\,u_\mu\big) + S^\lambda{}_{\mu\nu} - g^{\lambda\rho}\,(S_{\mu\rho\nu} + S_{\nu\rho\mu})\,,$$

i.e., the Yang-Mills gauge field is nothing but a sub-torsion field in the space-time manifold $\mathbb{S}_4$.

The results which we have obtained in this work may subsequently be quantized simply by following the method given in our previous work [1] since, in a sense, the present work is but a further in-depth classical consideration of the fundamental method of geometrization outlined in the previous theory.

Dedication

I dedicate this work to my patron and source of inspiration, Albert Einstein (1879–1955), from whose passion for the search of the ultimate physical truth I have learned something truly fundamental of the meaning of being a true scientist and independent, original thinker, even amidst the adversities often imposed upon him by the world and its act of scientific institutionalization.

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An Exact Mapping from Navier-Stokes Equation to Schrödinger Equation via Riccati Equation

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In the present article we argue that it is possible to write down Schrödinger representation of Navier-Stokes equation via Riccati equation. The proposed approach, while differs appreciably from other method such as what is proposed by R. M. Kiehn, has an advantage, i.e. it enables us extend further to quaternionic and biquaternionic version of Navier-Stokes equation, for instance via Kravchenko's and Gibbon's route. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In recent years there were some attempts in literature to find out Schrödinger-like representation of Navier-Stokes equation using various approaches, for instance by R. M. Kiehn [1, 2]. Deriving exact mapping between Schrödinger equation and Navier-Stokes equation has clear advantage, because Schrodinger equation has known solutions, while exact solution of Navier-Stokes equation completely remains an open problem in mathematical-physics. Considering wide applications of Navier-Stokes equation, including for climatic modelling and prediction (albeit in simplified form called "geostrophic flow" [9]), one can expect that simpler expression of Navier-Stokes equation will be found useful.

In this article we presented an alternative route to derive Schrödinger representation of Navier-Stokes equation via Riccati equation. The proposed approach, while differs appreciably from other method such as what is proposed by R. M. Kiehn [1], has an advantage, i.e. it enables us to extend further to quaternionic and biquaternionic version of Navier-Stokes equation, in particular via Kravchenko's [3] and Gibbon's route [4, 5]. An alternative method to describe quaternionic representation in fluid dynamics has been presented by Sprössig [6]. Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

2 From Navier-Stokes equation to Schrödinger equation via Riccati

Recently, Argentini [8] argues that it is possible to write down ODE form of 2D steady Navier-Stokes equations, and it will lead to second order equation of Riccati type.

Let ρ the density, μ the dynamic viscosity, and f the body force per unit volume of fluid. Then the Navier-Stokes equation for the steady flow is [8]:

$$\rho\,(v\cdot\nabla v) = -\nabla p + \rho\cdot f + \mu\cdot\Delta v\,. \tag{1}$$

After some necessary steps, he arrives to an ODE version of 2D Navier-Stokes equations along a streamline [8, p. 5] as follows:

$$u_1\cdot\dot{u}_1 = f_1 - \frac{\dot{q}}{\rho} + v\cdot\dot{u}_1\,, \tag{2}$$

where $v = \frac{\mu}{\rho}$ is the kinematic viscosity. He [8, p. 5] also finds a general exact solution of equation (2) in Riccati form, which can be rewritten as follows:

$$\dot{u}_1 - \alpha\cdot u_1^2 + \beta = 0\,, \tag{3}$$

where:

$$\alpha = \frac{1}{2v}\,, \qquad \beta = -\frac{1}{v}\left(\frac{\dot{q}}{\rho} - f_1\right)s - \frac{c}{v}\,. \tag{4}$$

Interestingly, Kravchenko [3, p. 2] has argued that there is neat link between Schrödinger equation and Riccati equation via simple substitution. Consider a 1-dimensional static Schrödinger equation:

$$\ddot{u} + v\cdot u = 0 \tag{5}$$

and the associated Riccati equation:

$$\dot{y} + y^2 = -v\,. \tag{6}$$

Then it is clear that equation (6) is related to (7) by the inverted substitution [3]:

$$y = \frac{\dot{u}}{u}\,. \tag{7}$$

Therefore, one can expect to use the same method (8) to write down the Schrödinger representation of Navier-Stokes equation. First, we rewrite equation (3) in similar form of equation (7):

$$\dot{y}_1 - \alpha\cdot y_1^2 + \beta = 0\,. \tag{8}$$

By using substitution (8), then we get the Schrödinger equation for this Riccati equation (9):

$$\ddot{u} - \alpha\beta\cdot u = 0\,, \tag{9}$$

where variable α and β are the same with (4). This Schrödinger representation of Navier-Stokes equation is remarkably simple and it also has advantage that now it is possible to generalize it further to quaternionic (ODE) Navier-Stokes equation via quaternionic Schrödinger equation, for instance using the method described by Gibbon *et al.* [4, 5].

3 An extension to biquaternionic Navier-Stokes equation via biquaternion differential operator

In our preceding paper [10, 12], we use this definition for biquaternion differential operator:

$$\diamondsuit = \nabla^q + i\,\nabla^q = \left(-i\,\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right) + \\ + i\left(-i\,\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right), \tag{10}$$

where e_1, e_2, e_3 are *quaternion imaginary units* obeying (with ordinary quaternion symbols: $e_1 = i$, $e_2 = j$, $e_3 = k$): $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$ and quaternion *Nabla operator* is defined as [13]:

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}. \tag{11}$$

(Note that (11) and (12) include partial time-differentiation.)

Now it is possible to use the same method described above [10, 12] to generalize the Schrödinger representation of Navier-Stokes (10) to the biquaternionic Schrödinger equation, as follows.

In order to generalize equation (10) to quaternion version of Navier-Stokes equations (QNSE), we use first quaternion Nabla operator (12), and by noticing that $\Delta \equiv \nabla\nabla$, we get:

$$\left(\nabla^q\bar{\nabla}^q + \frac{\partial^2}{\partial t^2}\right)u - \alpha\beta\cdot u = 0. \tag{12}$$

We note that the multiplying factor $\alpha\beta$ in (13) plays similar role just like $V(x) - E$ factor in the standard Schrödinger equation [12]:

$$-\frac{\hbar^2}{2m}\left(\nabla^q\bar{\nabla}^q + \frac{\partial^2}{\partial t^2}\right)u + \big(V(x) - E\big)u = 0. \tag{13}$$

Note: we shall introduce the second term in order to "neutralize" the partial time-differentiation of $\nabla^q\bar{\nabla}^q$ operator.

To get *biquaternion* form of equation (13) we can use our definition in equation (11) rather than (12), so we get [12]:

$$\left(\diamondsuit\bar{\diamondsuit} + \frac{\partial^2}{\partial t^2} - i\,\frac{\partial^2}{\partial T^2}\right)u - \alpha\beta\cdot u = 0. \tag{14}$$

This is an alternative version of *biquaternionic* Schrödinger representation of Navier-Stokes equations. Numerical solution of the new Navier-Stokes-Schrödinger equation (15) can be performed in the same way with [12] using Maxima software package [7], therefore it will not be discussed here.

We also note here that the route to *quaternionize* Schrödinger equation here is rather different from what is described by Gibbon *et al.* [4, 5], where the Schrödinger-equivalent to Euler fluid equation is described as [5, p. 4]:

$$\frac{D^2 w}{Dt^2} - (\nabla Q)w = 0 \tag{15}$$

and its quaternion representation is [5, p. 9]:

$$\frac{D^2 w}{Dt^2} - q_b \otimes w = 0 \tag{16}$$

with Riccati relation is given by:

$$\frac{D_a^q}{Dt + q_a \otimes q_a} = q_b \tag{17}$$

Nonetheless, further observation is of course recommended in order to refute or verify this proposition (15).

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Numerical Solution of Radial Biquaternion Klein-Gordon Equation

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In the preceding article we argue that biquaternionic extension of Klein-Gordon equation has solution containing imaginary part, which differs appreciably from known solution of KGE. In the present article we present numerical /computer solution of radial biquaternionic KGE (radialBQKGE); which differs appreciably from conventional Yukawa potential. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In the preceding article [1] we argue that biquaternionic extension of Klein-Gordon equation has solution containing imaginary part, which differs appreciably from known solution of KGE. In the present article we presented here for the first time a numerical/computer solution of radial biquaternionic KGE (radialBQKGE); which differs appreciably from conventional Yukawa potential.

This biquaternionic effect may be useful in particular to explore new effects in the context of low-energy reaction (LENR) [2]. Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

2 Radial biquaternionic KGE (radial BQKGE)

In our preceding paper [1], we argue that it is possible to write biquaternionic extension of Klein-Gordon equation as follows:

$$\left[\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)+i\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\right]\varphi(x,t) = \\ = -m^2\,\varphi(x,t), \tag{1}$$

or this equation can be rewritten as:

$$\left(\diamondsuit\bar{\diamondsuit}+m^2\right)\varphi(x,t)=0, \tag{2}$$

provided we use this definition:

$$\diamondsuit = \nabla^q + i\,\nabla^q = \left(-i\frac{\partial}{\partial t}+e_1\frac{\partial}{\partial x}+e_2\frac{\partial}{\partial y}+e_3\frac{\partial}{\partial z}\right)+ \\ + i\left(-i\frac{\partial}{\partial T}+e_1\frac{\partial}{\partial X}+e_2\frac{\partial}{\partial Y}+e_3\frac{\partial}{\partial Z}\right), \tag{3}$$

where e_1, e_2, e_3 are *quaternion imaginary units* obeying (with ordinary quaternion symbols: $e_1 = i$, $e_2 = j$, $e_3 = k$):

$$i^2 = j^2 = k^2 = -1\,,\ ij = -ji = k\,, \\ jk = -kj = i\,, \qquad ki = -ik = j\,. \tag{4}$$

and quaternion *Nabla operator* is defined as [1]:

$$\nabla^q = -i\frac{\partial}{\partial t}+e_1\frac{\partial}{\partial x}+e_2\frac{\partial}{\partial y}+e_3\frac{\partial}{\partial z}\,. \tag{5}$$

(Note that (3) and (4) included partial time-differentiation.)

In the meantime, the standard Klein-Gordon equation usually reads [3, 4]:

$$\left(\frac{\partial^2}{\partial t^2}-\nabla^2\right)\varphi(x,t) = -m^2\varphi(x,t)\,. \tag{6}$$

Now we can introduce polar coordinates by using the following transformation:

$$\nabla = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\ell^2}{r^2}\,. \tag{7}$$

Therefore, by substituting (6) into (5), the radial Klein-Gordon equation reads — by neglecting partial-time differentiation — as follows [3, 5]:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)-\frac{\ell(\ell+1)}{r^2}+m^2\right)\varphi(x,t)=0\,, \tag{8}$$

and for $\ell = 0$, then we get [5]:

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+m^2\right)\varphi(x,t)=0\,. \tag{9}$$

The same method can be applied to equation (2) for radial biquaternionic KGE (BQKGE), which for the 1-dimensional situation, one gets instead of (7):

$$\left(\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right)-i\,\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right)+m^2\right)\varphi(x,t)=0\,. \tag{10}$$

In the next Section we will discuss numerical/computer solution of equation (9) and compare it with standard solution of equation (8) using Maxima software package [6]. It can be shown that equation (9) yields potential which differs appreciably from standard Yukawa potential. For clarity, all solutions were computed in 1-D only.

3 Numerical solution of radial biquaternionic Klein-Gordon equation

Numerical solution of the standard radial Klein-Gordon equation (8) is given by:

(%i1) diff(y,t,2)-'diff(y,r,2)+m^2*y;

(%o1) $m^2 \cdot y - \frac{d^2}{d^2 x} y$

(%i2) ode2 (%o1, y , r);

(%o2) $y = \%k_1 \cdot \%\exp(mr) + \%k_2 \cdot \%\exp(-mr)$ (11)

In the meantime, numerical solution of equation (9) for radial biquaternionic KGE (BQKGE), is given by:

(%i3) diff(y,t,2)- (%i+1)*'diff(y,r,2)+m^2*y;

(%o3) $m^2 \cdot y - (i+1)\, \frac{d^2}{d^2 r} y$

(%i4) ode2 (%o3, y , r);

(%o4) $y = \%k_1 \cdot \sin\left(\frac{|m|\, r}{\sqrt{-\%i-1}}\right) + \%k_2 \cdot \cos\left(\frac{|m|\, r}{\sqrt{-\%i-1}}\right)$ (12)

Therefore, we conclude that numerical solution of radial biquaternionic extension of Klein-Gordon equation yields different result compared to the solution of standard Klein-Gordon equation; and it *differs appreciably* from the well-known Yukawa potential [3, 7]:

$$u(r) = -\frac{g^2}{r}\, e^{-mr}. \qquad (13)$$

Meanwhile, Comay puts forth argument that the Yukawa lagrangian density has theoretical inconsistency within itself [3].

Interestingly one can find argument that biquaternion Klein-Gordon equation is nothing more than quadratic form of (modified) Dirac equation [8], therefore BQKGE described herein, i.e. equation (12), can be considered as a plausible solution to the problem described in [3]. For other numerical solutions to KGE, see for instance [4].

Nonetheless, we recommend further observation [9] in order to refute or verify this proposition of new type of potential derived from biquaternion Klein-Gordon equation.

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VC would like to dedicate this article for RFF.

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Spin Transport in Mesoscopic Superconducting-Ferromagnetic Hybrid Conductor

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The spin polarization and the corresponding tunneling magnetoresistance (TMR) for a hybrid ferromagnetic/superconductor junction are calculated. The results show that these parameters are strongly depends on the exchange field energy and the bias voltage. The dependence of the polarization on the angle of precession is due to the spin flip through tunneling process. Our results could be interpreted as due to spin imbalance of carriers resulting in suppression of gap energy of the superconductor. The present investigation is valuable for manufacturing magnetic recording devices and nonvolatile memories which imply a very high spin coherent transport for such junction.

1 Introduction

Spintronics and spin-based quantum information processing explore the possibility to add new functionality to today's electronic devices by exploiting the electron spin in addition to its charge [1]. Spin-polarized tunneling plays an important role in the spin dependent transport of magnetic nanostructures [2]. The spin-polarized electrons injected from ferromagnetic materials into nonmagnetic one such as superconductor, semiconductor create a non equilibrium spin polarization in such nonmagnetic materials [3, 4, 5].

Ferromagnetic-superconductor hybrid systems are an attractive subject research because of the competition between the spin asymmetry characteristic of a ferromagnet and the correlations induced by superconductivity [1, 2, 6]. At low energies electronic transport in mesoscopic ferromagnet-superconductor hybrid systems is determined by Andreev-reflection [7]. Superconducting materials are powerful probe for the spin polarization of the current injected from ferromagnetic material [8, 9, 10]. Superconductors are useful for exploring how the injected spin-polarized quasiparticles are transported. In this case the relaxation time can be measured precisely in the superconducting state where thermal noise effects are small.

The present paper, spin-polarized transport through ferromagnetic/superconductor/ferromagnetic double junction is investigated. This investigation will show how Andreev-reflection processes are sensitive to the exchange field energy in the ferromagnetic leads.

2 The model

A mesoscopic device is modeled as superconductor sandwiched between two ferromagnetic leads via double tunnel barriers. The thickness of the superconductor is smaller than the spin diffusion length and the magnetization of the ferromagnetic leads are aligned either parallel or antiparallel. The spin polarization of the conduction electrons due to Andreev reflection at ferromagnetic/superconductor interface could be determined through the following equation as:

$$P = \frac{\Gamma_\uparrow(E) - \Gamma_\downarrow(E)}{\Gamma_\uparrow(E) + \Gamma_\downarrow(E)}\,, \tag{1}$$

where $\Gamma_\uparrow$(E) and $\Gamma_\downarrow$(E) are the tunneling probabilities of conduction electrons with up-spin and down-spin respectively. Since the present device is described by the following Bogoliubov-deGennes (BdG) equation [11]:

$$\begin{pmatrix} H_0 - h_{ex}(z)\sigma & \Delta(z) \\ \Delta^*(z) & -H_0 - \sigma h_{ex}(z) \end{pmatrix} \psi = E\psi\,, \tag{2}$$

where H_0 is the single particle Hamiltonian and it is expressed as:

$$H_0 = -\frac{\hbar^2}{2m}\nabla^2 - \varepsilon_{nl}\,, \tag{3}$$

in which the energy, ε_{nl}, is expressed of the Fermi velocity v_F, Fermi-momentum P_F, the magnetic field B as [12]:

$$\begin{aligned} \varepsilon_{nl} &= -(\alpha g + k_F D \sin\theta)\,\mu_B B \pm \\ &\pm \left[v_F^2 P_F^2 (1-\sin\theta)^2 + \Delta^2\right]^{1/2}. \end{aligned} \tag{4}$$

In Eq. (4), $\alpha = \pm 1/2$ for spin-up and spin down respectively, μ_B is the Bohr magneton, g is the g-factor for electrons and θ is the precession angle.

The interface between left ferromagnetic/superconductor and superconductor/right ferromagnetic leads are located at $z = -L/2$ and $z = L/2$ respectively. The parameter $h_{ex}(z)$ represents the exchange field and is given by [13]:

$$h_{ex} = \left\{ \begin{array}{cc} h_0 & z \prec -L/2 \\ 0 & -L/2 \prec z \prec L/2 \\ \pm h_0 & z \succ L/2 \end{array} \right\}, \tag{5}$$

where $+h_0$ and $-h_0$ represents the exchange fields for parallel and anti-parallel alignments respectively, the parameter

$\Delta(z)$ is the superconducting gap:

$$\Delta(z) = \left\{ \begin{array}{ll} 0 & z \prec -L/2, L/2 \prec z \\ \Delta & -L/2 \prec z \prec L/2 \end{array} \right\} . \qquad (6)$$

The temperature dependence of the superconducting gap is given by [14]:

$$\Delta = \Delta_0 \tanh\left(1.74\sqrt{\frac{T_c}{T} - 1}\right), \qquad (7)$$

where Δ_0 is the superconducting gap at $T = 0$ and T_c is the superconducting critical temperature. Now, in order to get the tunneling probability $\Gamma_{\uparrow\downarrow}(E)$ for both up-spin and down-spin electrons by solving the Bogoliubov-deGennes Eqn. (2) as: The eigenfunction in the left ferromagnetic lead $(z < -L/2)$ is given by:

$$\begin{aligned} \psi^{FM1}_{\sigma,nl}(r) = \Bigg[& \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{iP^{+}_{\sigma,nl}(z+\frac{L}{2})} + \\ & + a_{\sigma,nl} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{iP^{-}_{\sigma,nl}(z+\frac{L}{2})} + \\ & + b_{\sigma,nl} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-P^{+}_{\sigma,nl}(z+\frac{L}{2})} \Bigg] S_{nl}(x,y) . \end{aligned} \qquad (8)$$

In the superconductor $(-L/2 < z < L/2)$, the eigenfunction is given by:

$$\begin{aligned} \psi^{SC}_{\sigma,nl}(r) = \Bigg[& \alpha_{\sigma,nl} \begin{pmatrix} u_0 \\ \nu_0 \end{pmatrix} e^{ik^{+}_{nl}(z+\frac{L}{2})} + \\ & + \beta_{\sigma,nl} \begin{pmatrix} \nu_0 \\ u_0 \end{pmatrix} e^{-ik^{-}_{nl}(z+\frac{L}{2})} + \\ & + \xi_{\sigma,nl} \begin{pmatrix} u_0 \\ \nu_0 \end{pmatrix} e^{-ik^{+}_{nl}(z-\frac{L}{2})} + \\ & + \eta_{\sigma,nl} \begin{pmatrix} u_0 \\ \nu_0 \end{pmatrix} e^{ik^{-}_{nl}(z-\frac{L}{2})} \Bigg] S_{nl}(x,y) . \end{aligned} \qquad (9)$$

And the eigenfunction in the right ferromagnetic lead $(L/2 < z)$ is given by:

$$\begin{aligned} \psi^{FM2}_{\sigma,nl}(r) = \Bigg[& C_{\sigma,nl} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{iq^{+}_{\sigma,nl}(z-\frac{L}{2})} + \\ & + d_{\sigma,nl} \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iq^{-}_{\sigma,nl}(z-\frac{L}{2})} \Bigg] S_{nl}(x,y) . \end{aligned} \qquad (10)$$

Since the device is rectangular, the eigenfunction in the transverse $(x\&y)$ directions with channels n, l is given by;

$$S_{nl}(x,y) = \sin\left(\frac{n\pi x}{W}\right) \sin\left(\frac{l\pi y}{W}\right), \qquad (11)$$

where W is the width of the junction.

The wave numbers in the Eqs. (8), (9), (10) are given by:

$$P^{\pm}_{\sigma,nl} = \sqrt{\frac{2m}{\hbar^2}\left(\mu_F \pm E \pm \sigma h_{ex}\right)}, \qquad (12)$$

$$k^{\pm}_{\sigma,nl} = \sqrt{\frac{2m}{\hbar^2}\left(\mu_F \pm \Im\right) - \varepsilon_{nl}}, \qquad (13)$$

$$q^{\pm}_{\sigma,nl} = \sqrt{\frac{2m}{\hbar^2}\left(\mu_F \pm E \pm \sigma h_{ex} \pm \varepsilon_{nl}\right)}, \qquad (14)$$

where $\Im = \sqrt{E^2 - \Delta^2}$, and the energy ε_{nl} is given by Eq. (4). For the coherence factors of electron and holes u_0 and ν_0 are related as [11]:

$$u_0^2 = 1 - \nu_0^2 = \frac{1}{2}\left[1 + \frac{\sqrt{E^2 - \Delta^2}}{E}\right]. \qquad (15)$$

The coefficients in Eqs. (8), (9), (10) are determined by applying the boundary conditions at the interfaces and the matching conditions are:

$$\left.\begin{aligned} \psi^{FM1}_{\sigma,nl}\left(z = -\frac{L}{2}\right) = \psi^{SC}_{\sigma,nl}\left(z = -\frac{L}{2}\right) \\ \psi^{SC}_{\sigma,nl}\left(z = \frac{L}{2}\right) = \psi^{FM2}_{\sigma,nl}\left(z = \frac{L}{2}\right) \end{aligned}\right\}, \qquad (16)$$

$$\left.\frac{d\psi^{SC}_{\sigma,nl}}{dz}\right|_{z=-\frac{L}{2}} - \left.\frac{d\psi^{FM!}_{\sigma,nl}}{dz}\right|_{z=-\frac{L}{2}} = \frac{2mV}{\hbar^2}\psi^{FM1}_{\sigma,nl}\left(z = -\frac{L}{2}\right), \qquad (17)$$

$$\left.\frac{d\psi^{FM2}_{\sigma,nl}}{dz}\right|_{z=\frac{L}{2}} - \left.\frac{d\psi^{SC!}_{\sigma,nl}}{dz}\right|_{z=\frac{L}{2}} = \frac{2mV}{\hbar^2}\psi^{FM2}_{\sigma,nl}\left(z = \frac{L}{2}\right). \qquad (18)$$

Eqs. (14), (15), (16) are solved numerically [15] for the tunneling probabilities corresponding to up-spin and down-spin for the tunneled electrons. The corresponding polarization, P, Eq. (1) is determined at different parameters V, θ, which will be discussed in the next section.

3 Results and discussion

Numerical calculations are performed for the present device, in which the superconductor is Nb and the ferromagnetic leads are of any one of ferromagnetic materials. The features of the present results are:

Fig. 1 shows the dependence of the polarization, P, on the bias voltage, V, at different parameters B, E, h and T. From the figure, the polarization has a peak at the value of V near the value of the energy gap Δ_0 for the present superconductor (Nb) ($\Delta_0 = 1.5$ meV) [16]. But for higher values of V, the polarization, P, decreases. As shown from Fig. 1a, the polarization does not change with the magnetic field, B, due to the Zeeman-energy. Some authors [17] observed the effect of magnetic field of values greater than 1 T, in this case the superconductivity will be destroyed (for Nb, $B_c = 0.19$ T).

Now in order to observe the effect of the spin precession on the value of the polarization, P, this can be shown from Fig. 2. The dependence of the polarization, P, on the angle of precession, θ, is strongly varies with the variation of the magnetic field, temperature, exchange field and the energy of

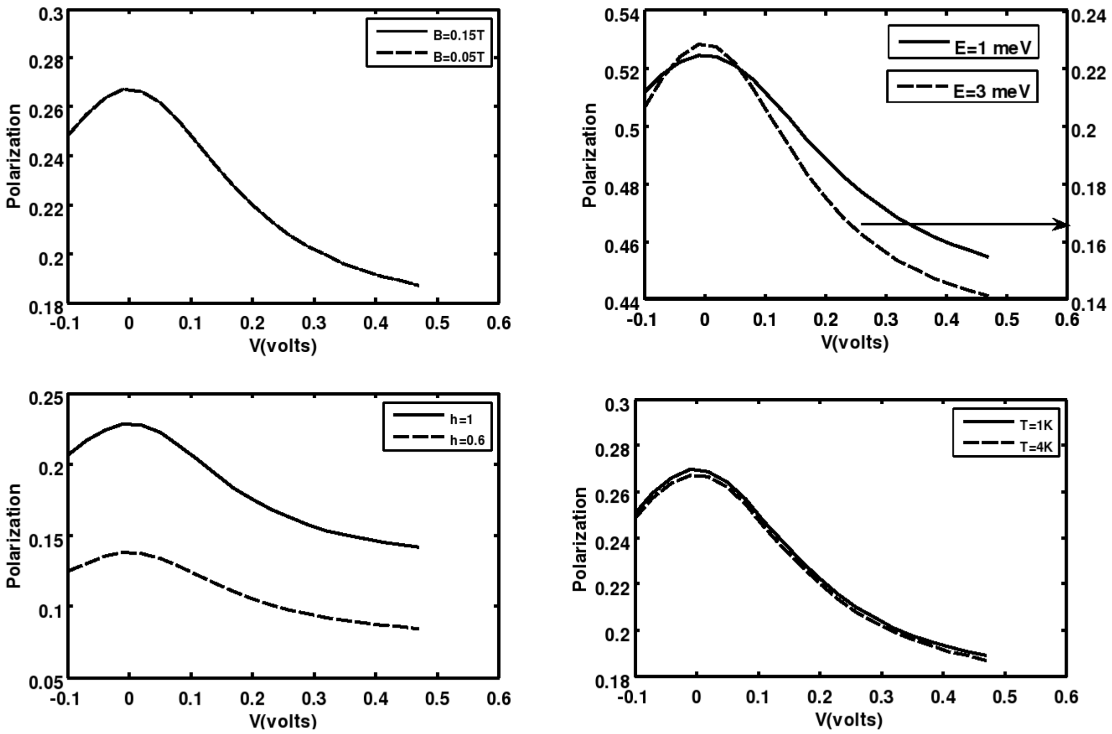

Fig. 1: The dependence of the polarization, P, on the bias voltage, V, at different B, E, h and T.

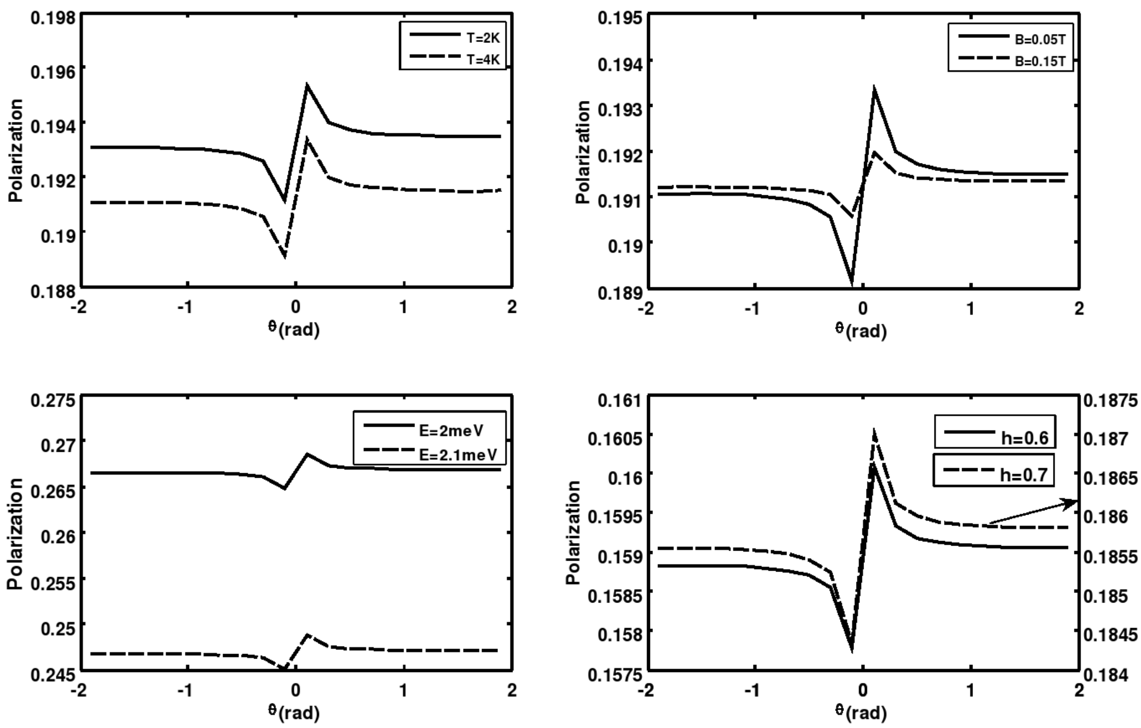

Fig. 2: The dependence of the polarization, P, on the angle of precession at different B, E, h and T.

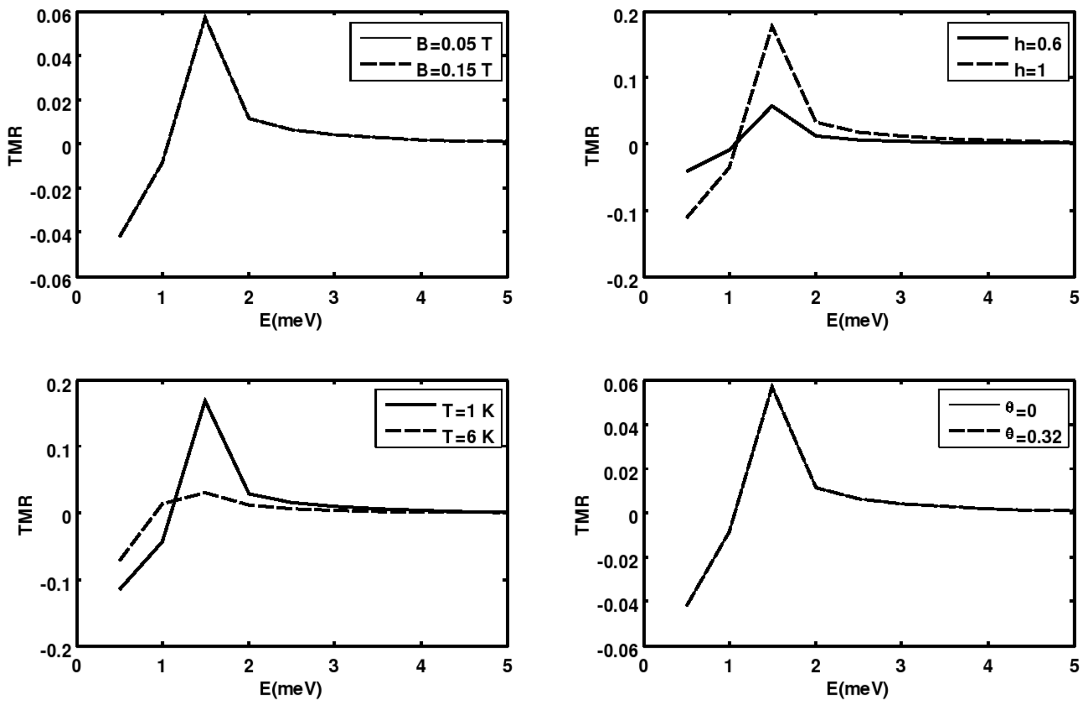

Fig. 3: The variation of the TMR with the energy of the tunneled electrons at different parameters B, T, h and θ.

the tunneled electrons. As shown from Fig. 2, the value of P is minimum at certain values of θ also P is maximum at another values of θ. This trend of the polarization with the angle of the precession is due to the flip of the electron spin when tunneling through the junction.

In order to investigate the spin injection tunneling through such hybrid magnetic system, we calculated the tunnel magnetoresistance (TMR) which is related to the polarization as [18]:

$$TMR = \frac{P^2}{1 - P^2 + \Gamma_s}\,, \tag{19}$$

where Γ_s is the relaxation parameter and is given by [18]:

$$\Gamma_s = \frac{e^2 N(0) R_T A L}{\tau_s}\,, \tag{20}$$

where $N(0)$ is the normal-state density of electrons calculated for both up-spin and down-spin distribution function $f_\sigma(E)$, which is expressed as [18]:

$$f_\sigma(E) \cong f_0(E) - \left(\frac{\partial f_0}{\partial E}\right) \sigma\,\delta\mu\,, \tag{21}$$

where $\sigma = \pm 1$ for both up and down spin of the electrons, $\delta\mu$ is the shift of the chemical potential, τ_s is the spin relaxation time, A is the junction area and R_T is the resistance at the interface of the tunnel junction.

Fig. 3 shows the variation of the TMR with the energy of the tunneled electrons at different parameters B, T, h and θ. A peak is observed for TMR at a certain value which is in the near value of the gap energy Δ_0 for the superconductor (Nb). These results (Fig. 3) show the interplay between the spin polarization of electrons and Andreev-reflection process at the ferromagnetic/superconductor interface [19]. From our results; we can conclude that the spin-polarized transport depends on the relative orientation of magnetization in the two ferromagnetic leads. The spin polarization of the tunneled electrons through the junction gives rise to a nonequilibrium spin density in the superconductor. This is due to the imbalance in the tunneling currents carried by the spin-up and spin-down electrons. The trend of the tunneling magnetoresistance (TMR) is due to the spin-orbit scattering in the superconductor. Our results are found concordant with those in literatures [20, 21, 22].

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Human Perception of Physical Experiments and the Simplex Interpretation of Quantum Physics

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In this paper it is argued that knowledge dividing the usual, unusual, transient and transcendental depends on human perception of the world (macro or micro) and depends too on the inclusion of human consciousness in the system. For the analysis of this problem the idea of "Schrödinger's cat" is employed. Transient and transcendental knowledge of the state of Schrödinger's cat corresponds to the case when the observer's consciousness is included in the system. Here it is possible to speak about the latent parameters of the sub quantum world of which Einstein was convinced. Knowledge of the unusual state of Schrödinger's cat, simultaneously alive and dead, corresponds to a case of the open micro world. The usual knowledge of the state of Schrödinger's cat (alive or dead) corresponds to a case of the open macrocosm. Each world separately divides the objective and illusory.

1 Introduction

Scientific cognition frequently avoids the question of interaction of our consciousness with the external world. However, the celebrated known physicist Wigner [1] maintains that separation of our perception from the laws of a nature is no more than simplification and although we are convinced that it has a harmless character, to nevertheless merely forget about it does not follow.

Purposeful perception is sensation and in order to understand more deeply that sensation it is necessary, in the beginning, to be able to distinguish sensation in a macrocosm (spontaneously) from sensation in a microcosm (through the device). Many scientists believe that information recorded with the help of devices can be equally considered with sentient data. Their belief, harmless at first sight would, should not result in the serious misunderstanding. But actually it is not so.

Sensation in a macrocosm, for example, that of a sunrise, and sensation in a microcosm, for example, some number displayed on an ammeter, are not the same. Perception, by definition, is complete subjective reflection: the phenomena are events resulting from direct influence on sense organs, and in a macrocosm it certainly does not depend on the level of our knowledge. Nobody will argue that a sunrise and other such phenomena, events in a macrocosm, are perceived by all people equally. But in a microcosm this is not so. Perception of the invisible world of electrons is not whole or complete and therefore depends on the level of our scientific knowledge. But that knowledge is connected to our consciousness. It becomes clear then why the consciousness of the observer finds itself a place in quantum physics.

The problematic interpretation of quantum mechanics has been a controversial topic of discussion for more than 80 years. The most important upshot of this for physicists is that this problem is related to the problem of consciousness — an interdisciplinary problem concerning not only physicists, but also philosophers, psychologists, physiologists and biologists. Its solution will result in deeper scientific knowledge. As many scientists have argued, the path to such knowledge should not consider separately the physical phenomena and the phenomena accompanying our thinking. By adhering to this position it is reasonable to conclude that the correct interpretation of the quantum mechanics comprises such knowledge.

Really, the problem of quantum physics, as a choice of one alternative at quantum measurement and a problem of philosophy as to how consciousness functions, is deeply connected with relations between these two. It is quite possible that in solving these two problems, it is likely that experiments in the quantum mechanics will include workings of a brain and consciousness, and it will then be possible to present a new basis for the theory of consciousness

2 Dependence of physical experiment on the state of consciousness

During sensation our brain accepts data and information from an external world. On the basis of these data, during thinking, knowledge is formed. The biological substratum of thinking is the brain. Therefore, knowledge is a product of the brain.

Consciousness, as it is known, is a property of the brain and therefore already concerns the origin of knowledge. Clearly, this relation is either active, i.e. influencing the origin of knowledge, or passive. If active as well as passive, we ask: Does consciousness influence the origin of knowledge? It is possible to answer this because it is known that there are different kinds and levels of consciousness and scientific knowledge which represent various forms and levels of reflection. Considering the definition of knowledge in that it is a reflection of objective characteristics of reality in the consciousness of a person, we are interested with a question:

When and what reflection — passive or active, unequivocal or multiple-valued — takes place?

Passivity or activity of reflection depends on passivity or activity of the consciousness of the observer. Clearly, consciousness is passive if it is not included in the system, being in this case an open system. Consciousness can be active if it is included in the system, being in this case a closed system. Activity or passivity of consciousness is expressed in its ability to influence reflection on reality, i.e. on knowledge. With the contention that active consciousness may influence reflection on reality it is possible to imply that this influence can be directed onto reality as well. Whether or not this is so is however difficult to say. But we know that a closed system should differ from an open one. The difference is expressed in the activity of consciousness, which influences reflection and knowledge.

The unambiguous or the multi-valence nature of reflection does not depend on the activity or passivity of consciousness; it depends on perception, i.e. from integrity of perception. The perception of a macrocosm is complete, but the perception of a microcosm is not complete. Therefore it is clear that reflection on reality in a macrocosm will be unequivocal, but in a microcosm, multiple-valued.

Multiple-valued reflection does not influence knowledge, but, nevertheless, makes knowledge multiple-valued, unclear, and uncertain. It now becomes clear why knowledge of a microcosm results in uncertainties, including the well-known Heisenberg Uncertainties. It is possible that these uncertainties are effects of consciousness, dependent not on the activity of consciousness, but on the impossibility to completely perceive the cognizable world by consciousness.

Thus, in a closed system, reflection is active. In an open system reflection is passive. In a macrocosm it is unequivocal but in a microcosm it is multiple-valued.

For elucidation we shall imagine a mirror; a usual mirror, i.e. a mirror with which we are commonly familiar. Let's assume that this mirror is our consciousness. The mirror is passive, because reflection of objects in it does not depend on itself. Similarly, consciousness is passive, if reflection of reality in it does not depend on itself. Clearly, the passive consciousness appropriate for this mirror is consciousness in an open system, because only in this case is consciousness similar to a mirror that can be counter-posed to a being. If around the mirror there is a bright light, for example, sunlight, the reflection of objects in it will be unequivocal. Perception of these objects will be complete. This case of bright light around of a mirror corresponds to a case of the macrocosm. Really, the macrocosm is our visible world. But now we shall imagine that the mirror is in darkness. Images are absent in the mirror. This case of darkness around the mirror corresponds to a case of the microcosm. The microcosm is our invisible world. Let's now imagine that we want to receive some image from the mirror. For this purpose we artificially illuminate an object. This action corresponds to how we investigate a microcosm with the help of devices. Artificial illumination is not ideal; therefore reflection of objects in the mirror will be multiple-valued. Clearly, perception will not be complete either. Already, as a result, knowledge cannot be unequivocal. The Heisenberg Uncertainties of a microcosm are the proof. Knowledge from these uncertainties is multiple-valued because it is impossible to determine exactly the localization and speed of a micro-particle. So the usual mirror corresponds to passive consciousness. But what mirror will correspond to active consciousness? In this case the system is closed and the mirror should be unusual; the reflection of objects in it depends on itself. Such a mirror includes a mirror, or more exactly, many mirrors; a mirror in a mirror in a mirror.

So consciousness includes consciousness; it is consciousness in consciousness. One could say that such mirror is a distorting mirror, although a word "distorting" is perhaps not the best description. It is a mirror of unusual reflection. Depending on the mirror, reflection in it varies up to the unrecognisable. To make a distorting mirror a person performs an act — alters a usual mirror. To effect this action he must be included in the system — he cannot simply take a usual mirror in his hands. Similar to this action of the person, consciousness is included in the system, can change consciousness, and reflection of reality will depend on it. Therefore knowledge, being this reflection, will depend on consciousness. In this case, consciousness influences processes in the origin of knowledge. Phenomenologically speaking, reflection of objective reality will already be an actual stream of consciousness.

After we have found out in what case some reflection takes place, we shall be able to answer the aforementioned question: Does consciousness influence the origin of knowledge or not?

Passive consciousness can be excluded from being, from what takes place in an open system. In this case, being is determined according to materialist philosophy. In an open system, passive reflection takes place, and consequently knowledge is defined as passive reflection of reality in the consciousness of a person. As remarked above, passive reflection is unequivocal in a macrocosm, and it is multiple-valued in a microcosm. Therefore, in the case of an open system, in a macrocosm, knowledge is passive and unequivocal. In a microcosm it is passive too, but it is multiple-valued. We shall call this knowledge, accordingly, *usual* and *unusual* knowledge respectively — the unusual because knowledge of the microcosm, including the Heisenberg Uncertainties, is for us, unusual.

Thus, in unusual knowledge there is an affection of consciousness. Hence, it is necessary to consider ontological problems in physics. Many physicists adhere to a definition of being according to materialism. Therefore, constructed by them with the help of theories, physical reality characterizes the world, and excludes the consciousness of the observer from consideration. We shall call such a concept of physical reality *usual*. Building on it, the physicists do not take into account questions connected with perception and consciousness, so it is possible to act only in the case of a macrocosm.

For a microcosm, physical reality, as constructed by the physicists, should be entirely different; unusual. We shall call physical reality describing a microcosm, as an open system, *ontological*. In this case, effects of consciousness take place, but the effects are connected not with the activity of consciousness, but with reflection or integrity of perception of the cognizable world.

Answering "yes" to the question: Does consciousness influence the origin of knowledge or not? it is evident that consciousness is active and therefore cannot be excluded from the being participating in the closed system. As we have already seen, in the closed system active reflection takes place, so knowledge is active reflection of reality in the consciousness of a person. In this knowledge there is a place for the effects of consciousness, but they are connected not with perception of the cognizable world, as in case of unusual knowledge, but with the activity of the consciousness of the observer.

Can active consciousness of the observer be consciousness of the person? Certainly not! The system, having captured the consciousness of one person, is not closed, because outside it there is the consciousness of another person in which reality can be reflected. Thus, when we speak of consciousness of the observer in the closed system, i.e. about active consciousness, we mean that it cannot be consciousness of the person. The consciousness of the person is a passive consciousness, i.e. this consciousness of the observer in an open system. Knowledge which takes place in this case is *a simple* knowledge of passive consciousness — the person. Accordingly, this knowledge is *usual* (in case of a macrocosm), or *unusual* (in case of a microcosm).

Knowledge, which takes place in the case when the system is closed, is knowledge of active consciousness. This knowledge is absolute knowledge.

Let's consider absolute knowledge in the case when the closed system is a macrocosm. In this case knowledge is active and unequivocal reflection. We shall call such knowledge *transcendental*. Such a name is justified because transcendental knowledge can be understood by passive consciousness. Clearly, such analysis is possible in a macrocosm because in this case we learn of our world, which, in contrast with the microcosm, is visible, audible, and otherwise sentient. *Transcendental* knowledge concerns scientific knowledge.

In the case of a closed system as a microcosm, knowledge is active, but multiple-valued reflection and so gives rise to latent uncertainties which are not Heisenberg Uncertainties. The paradoxes concerning the laws of the quantum world were explained by Albert Einstein as properties of an unobservable, deeper sub-quantum world; hidden variables. With the help of Bell's inequalities it was proved that latent parameters (hidden variables) do not exist. However, if Heisenberg Uncertainties are open to passive consciousness, i.e. to the consciousness of a person, then the latent parameters are open only to active consciousness. Therefore we also cannot open them. We shall call such knowledge *transient*. Such a name is justified in that it cannot be understood.

Thus, for open systems, knowledge is passive and unequivocal in a macrocosm, passive and multiple-valued in a microcosm. For the closed systems the knowledge is active and unequivocal in a macrocosm, active and multiple-valued in a microcosm. Accordingly, knowledge is divided into the *usual*, *unusual*, *transcendental* and *transient*. Physical reality for these cases are, philosophically speaking, usual, ontological and active.

3 The "Schrödinger cat" experiment

It is known that in a macrocosm a body can be in only one state. Clearly, this knowledge is usual. In a microcosm an elementary particle can be simultaneously in two states. Of course, such knowledge is *unusual*.

However, it has been established that in the result of intensification the superposition of two micro-states turns into superposition of two macro-states. Therefore in a macrocosm there is unusual knowledge. This paradox has been amplified by E. Schrödinger in his mental experiment, known as Schrödinger's cat.

In the paradox of Schrödinger's cat the state of a cat (alive or dead) depends on the act of looking inside the box containing the cat, i.e. depends on the consciousness of the observer. Thus, consciousness becomes an object of quantum physics. We mentioned above that in an open system the consciousness of an observer, being passive, is the consciousness of a person. In an open macrocosm perceived by us unequivocally, the open microcosm is perceived by us as multiple-valued. Frequently it is asked: Where is the border between the macrocosm and the microcosm It is possible to answer that this border is the perception of a person. The state of Schrödinger's cat simultaneously both alive and dead corresponds to an open microcosm. Although we talk about a macro object — a cat — it is connected to a microcosm; it is a microcosm when a person doesn't open the box and look at the cat. As soon as a person looks at the cat in the box, i.e. completely and unequivocally perceive it, the state of the cat is determined, for example, the cat is alive. This state of the cat corresponds to an open macrocosm — to the world which we live.

The state of Schrödinger's cat — simultaneously alive and dead — is the entangled state. In an open system the paradox of Schrödinger's cat is described with the help of the decoherence phenomenon [2]. The open system differs from the closed. In an open system there are some degrees of freedom, including a brain and the consciousness of the observer that by our measurements can give us information. We open the box and find out that the cat is actually alive — it is the decoherence. With a statistical ensemble of Schrödinger cats, we can use probability theory and statistical forecast.

What will be Schrödinger's cat in a closed system? The most interesting theory here is the many-world interpretation of quantum mechanics of Everett and Wheeler [3]. The closed system is the whole world, including the observer. Every component of superposition describes the whole world, and none of them has any advantage. The question here is not: What will be the result of measurement? The question here is not: In what world, of many worlds, does the observer appear? In the Everett-Wheeler theory it depends on the consciousness of the observer. In the terminology of Wheeler such consciousness is called active. Knowledge in this case is knowledge of active consciousness and called by us the transcendental (in a macrocosm) and the transient (in a microcosm).

Recall Einstein's objection to Bohr's probabilistic interpretation of the quantum mechanics: "I do not believe that God plays dice". M. B. Menskii [4] writes "Yes, God does not play dice. He equally accepts all possibilities. In dice plays the consciousness of each observer". The author means, that the consciousness of the person, his mind, builds the forecasts, based on concepts of probability theory. Let's agree that the world, about which Einstein speaks, in which God does not play dice, is a real world. The world in which the person plays dice is a sentient world.

Besides these two worlds there exists, according to Max Plank [5], a third — the world of physical science or the physical picture of world. This world is a bridge for us, and with its help we learn of those worlds. It concerns the aforementioned physical reality. Descriptions of the real and sentient worlds in the world of physical science are the quantum and classical worlds, accordingly.

In physics the classical world is very frequently interpreted as the objective world. The quantum world exists as some mathematical image — a state vector, i.e. the wave function. Therefore it is objectively non-existent, an illusion. Such an interpretation, warns Plank, can result in the opinion that there is only a sentient world. Such an outlook cannot be denied logically, because logic itself cannot pluck anyone from his own sentient world. Plank held that besides logic there is also common sense, which tells us that although we may not directly see some world, that world may still exist. From such a point of view, interpretation of the mutual relations between the worlds will be very different — the quantum world is objective, the classical world is an illusion.

It is possible to interpret these worlds from the new point of view. As we saw above for Schrödinger's cat, the border between quantum and classical worlds is erased. Therefore the real world is both the objective quantum world and objective classical world. Furthermore, the sentient world is both an illusion of the quantum world and an illusion of the classical world. Thus, the quantum and classical world each consist of components — objective and illusory components.

Are there an objective classical world and an illusion of the quantum world in our understanding? The classical world is the world of macroscopic objects and our consciousness sees and perceives this world. For us it should be sentient. Illusion of the classical world satisfies this condition. The quantum world is the world of microscopic objects. This world is invisible to us and so cannot be the sentient world. The objective quantum world satisfies this condition. Thus, although there is an objective classical world and an illusion of the quantum world, these worlds are outside the ambit of our consciousness. It becomes clear now why classical and quantum physics essentially and qualitatively differ from each other. Classical physics studies a physical picture of an illusion of the classical world. Quantum physics studies the physical picture of the objective quantum world.

Thus, our consciousness comprehends the objective quantum world. Following Menskii [4], it can be represented symbolically as some complex volumetric figure, and the illusion of the classical world is only one of the projections of this figure. It will be expedient to present this complex volumetric figure, as a simplex.

4 Simplex interpretation of quantum physics

From functional analysis [6] it is known that a point is zero-dimensional, a line is one-dimensional, a triangle is bi-dimensional, a tetrahedron a three-dimensional simplex. The three-dimensional simplex, a tetrahedron has 4 bi-dimensional sides (triangles), 6 one-dimensional sides (lines) and 4 zero-dimensional sides (points), giving a total of 14 sides.

It is impossible to imagine a four-dimensional simplex in our three-dimensional space.

The parallelepiped or cube is not a simplex because for this purpose it is necessary that all 8 points were in six-measured space. Thus, formed from more than four points, is a complex volumetric figure.

Let's assume in experiment with 100 Schrödinger cats, 80 cats are alive and 20 are dead. Points 20 and 80 are two ends of a simplex. At other moment of time or in another experiment let's assume from 100 cats that 60 are alive and 40 are dead. These two points are also ends of a simplex. We can continue our tests, but we shall stop with these two, and thus, we consider a three-dimensional simplex — a tetrahedron. The ribs of our tetrahedron indicate various probabilities. For example, the rib linking the points 80 live cats and 40 dead cats give $80/120 = 2/3$ of probability of the case in which a cat is alive. In the case 60 live and 20 dead cats, the rib of the simplex shows that the probability is $60/80 = 3/4$, etc. The rib linking the points 20 dead and 40 dead cats and the rib linking the points 80 live and 60 live cats each give a probability of 1. Let's consider the faces of the simplex. In the case of a live cat on one of them the probability changes from 2/3 to 0.8; on another face, from 3/4 to 0.6; on third face, from 2/3 to 0.6; on fourth, from 3/4 to 0.8 etc. As to points of a tetrahedron they specify determinism of an event. For example, the point of 80 live cats specifies that in fact all 80 cats are alive.

We could construct the simplex with various probabilistic ribs and sides because we are observers from outside. In this case we built a physical picture of the real world. Only in this world is the probabilistic interpretation of the quantum mechanics given by Bohr true.

In a physical picture of the sentient world, we cannot construct a simplex. We can only perceptions as projections, i.e. sides of a simplex. After that, classical probability is applied, but it is applied, we shall repeat, not for a whole simplex, but only for one of its sides. This side, perceived by us as the sentient world, is an illusion because it not unique: there exists a set of worlds alternative to it. With a physical picture of the world, we can even count the number of parallel worlds. As our world is three-dimensional and our consciousness exists in it we can count only sides of a three-dimensional simplex — a tetrahedron, which, as shown above, has only 14 sides.

Returning now to the dispute between Einstein and Bohr, in the real and sentient worlds, of course Einstein was right — really, God does not play dice. However, in the physical picture of the world, Bohr had the right to apply probability and statistics.

Usually in a game of dice we mean only the act of throwing dice. However, dice consists of acts before (we build forecasts) and after (realization of one forecast from possible results). This situation can be likened to a court case; there is a hearing of a case, a verdict and a process after the verdict. In the physical picture of the real world, a game of dice by consciousness is a game up to the act of throwing the dice. Our consciousness can only imagine all sides of a three-dimensional simplex, i.e. all alternative results. But the choice of one of them depends on "active" consciousness. In our sentient world, in the act of throwing the dice, we shall see this choice. In the physical picture of the sentient world, a game of dice by consciousness is a game after the act of throwing the dice. Having these outcomes allow us to statistically forecast.

Thus, uncertainty of the real world qualitatively differs from uncertainty of the sentient world. Thus, uncertainty of the sentient world is not present and, as a matter of fact, the finding of the probability of some casual event has no connexion with uncertainty because this probability exists beforehand, a priori, and by doing a series of tests we simply find it. It becomes clear then why quantum statistics essentially differs from the classical.

This simplex, with various probabilistic ribs and sides, we could construct with the help of epistemological analysis. Knowledge which was analyzed in this case is knowledge of active consciousness. In the case when the simplex from a volumetric figure is converted into one of its projections, we see only one of its sides (a point, a line, a triangle). Knowledge appropriate to this case is knowledge of passive consciousness. In a simplex the lines (80, 20) and (60, 40) where points 80, 60 are live, and 20, 40 are dead cats, correspond to *usual* knowledge. In this case we use classical statistics (after we have looked in the box, Schrödinger's cats became simple cats, and we already have data, for example, from 100 cats in one case 80 alive, and in the other case 6, etc.). With the help of this date we find an average and dispersion of a random variable.

But when the ensemble consists not of simple cats, but Schrödinger cats we deal with a microcosm, with a world, the perception of which, is multiple-valued. In this case, for example, the point 80 is already fixed simultaneously and with the point 20, and with the point 40. Therefore the triangle (20, 80, 40) is examined. Similarly, the triangle (40, 60, 20) is also considered. These triangles correspond to *unusual* knowledge. In this case we cannot apply classical statistics. Therefore we use quantum statistics.

There is a question: But what in a simplex will correspond to *transcendental* and *transient* knowledge? We can answer that transcendental knowledge is knowledge of active consciousness in the case of a macrocosm, and corresponds to the entire simplex. Transcendental knowledge can be acquired by us a priori (because we could construct the simplex), but for transient knowledge this is not possible. Knowledge of active consciousness appropriate to transition from a microcosm to macrocosm, i.e. to our world, will be transcendental, and from a microcosm to a microcosm it will be transient. There is no sharp border between macro-world and microcosms, but in fact there is a sharp border between knowledge about them.

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SPECIAL REPORT

On a Geometric Theory of Generalized Chiral Elasticity with Discontinuities

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In this work we develop, in a somewhat extensive manner, a geometric theory of chiral elasticity which in general is endowed with *geometric discontinuities* (sometimes referred to as *defects*). By itself, the present theory generalizes both Cosserat and void elasticity theories to a certain extent via *geometrization* as well as by taking into account the action of the electromagnetic field, i.e., the incorporation of the electromagnetic field into the description of the so-called *microspin* (*chirality*) also forms the underlying structure of this work. As we know, the description of the electromagnetic field as a unified phenomenon requires four-dimensional space-time rather than three-dimensional space as its background. For this reason we embed the three-dimensional material space in four-dimensional space-time. This way, the electromagnetic spin is coupled to the non-electromagnetic microspin, both being parts of the complete microspin to be added to the macrospin in the full description of vorticity. In short, our objective is to generalize the existing continuum theories by especially describing microspin phenomena in a fully geometric way.

1 Introduction

Although numerous generalizations of the classical theory of elasticity have been constructed (most notably, perhaps, is the so-called Cosserat elasticity theory) in the course of its development, we are somewhat of the opinion that these generalizations simply lack geometric structure. In these existing theories, the introduced quantities supposedly describing microspin and irregularities (such as voids and cracks) seem to have been assumed from *without*, rather than from *within*. By our geometrization of microspin phenomena we mean exactly the description of microspin phenomena in terms of intrinsic geometric quantities of the material body such as its curvature and torsion. In this framework, we produce the microspin tensor and the anti-symmetric part of the stress tensor as intrinsic geometric objects rather than alien additions to the framework of classical elasticity theory. As such, the initial microspin variables are not to be freely chosen to be included in the potential energy functional as is often the case, but rather, at first we identify them with the internal properties of the geometry of the material body. In other words, we can not simply adhere to the simple way of adding external variables that are supposed to describe microspin and defects to those original variables of the classical elasticity theory in the construction of the potential energy functional without first discovering and unfolding their underlying internal geometric existence.

Since in this work we are largely concerned with the behavior of *material points* such as their translational and rotational motion, we need to primarily cast the field equations in a manifestly covariant form of the *Lagrangian* system of material coordinates attached to the material body. Due to the presence of geometric discontinuities (*geometric singularities*) and the *local non-orientability* of the material points, the full Lagrangian description is necessary. In other words, the compatibility between the spatial (*Eulerian*) and the material coordinate systems can not in general be directly invoked. This is because the smooth transitional transformation from the Lagrangian to the Eulerian descriptions and vice versa breaks down when geometric singularities and the non-orientability of the material points are taken into account. However, for the sake of accommodating the existence of all imaginable systems of coordinates, we shall assume, at least locally, that the material space lies within the three-dimensional space of spatial (Eulerian) coordinates, which can be seen as a (flat) hypersurface embedded in four-dimensional space-time. With respect to this embedding situation, we preserve the correspondence between the material and spatial coordinate systems in classical continuum mechanics, although not their equality since the field equations defined in the space of material points are in general not independent of the orientation of that local system of coordinates.

At present, due to the limits of space, we shall concentrate ourselves merely on the construction of the field equations of our geometric theory, from which the equations of motion shall follow. We shall not concern ourselves with the over-determination of the field equations and the extraction of their exact solutions. There is no doubt, however, that in the process of investigating particular solutions to the field equations, we might catch a glimpse into the initial states of the microspin field as well as the evolution of the field equations. We'd also like to comment that we have constructed our theory with a relatively small number of variables only, a characteristic which is important in order to prevent superfluous variables from encumbering the theory.

2 Geometric structure of the manifold $\Im_3$ of material coordinates

We shall briefly describe the local geometry of the manifold $\Im_3$ which serves as the space of material (Lagrangian) coordinates (material points)ξ^i $(i = 1, 2, 3)$. In general, in addition to the general non-orientability of its local points, the manifold $\Im_3$ may contain singularities or geometric defects which give rise to the existence of a local material curvature represented by a generally *non-holonomic* (path-dependent) curvature tensor, a consideration which is normally shunned in the standard continuum mechanics literature. This way, the manifold $\Im_3$ of material coordinates, may be defined either as a continuum or a discontinuum and can be seen as a three-dimensional hypersurface of non-orientable points, embedded in the physical four-dimensional space-time of spatial-temporal coordinates $\Re_4$. Consequently, we need to employ the language of general tensor analysis in which the local metric, the local connection, and the local curvature of the material body $\Im_3$ form the most fundamental structural objects of our consideration.

First, the material space $\Im_3$ is spanned by the three curvilinear, covariant (i.e., tangent) basis vectors g_i as $\Im_3$ is embedded in a four-dimensional space-time of physical events $\Re_4$ *for the sake of general covariance*, whose coordinates are represented by y^μ $(\mu = 1, 2, 3, 4)$ and whose covariant basis vectors are denoted by ω_μ. In a neighborhood of local coordinate points of $\Re_4$ we also introduce an enveloping space of spatial (Eulerian) coordinates x^A $(A = 1, 2, 3)$ spanned by locally constant orthogonal basis vectors e_A which form a three-dimensional Euclidean space $\mathbb{E}_3$. (From now on, it is to be understood that small and capital Latin indices run from 1 to 3, and that Greek indices run from 1 to 4.) As usual, we also define the dual, contravariant (i.e., cotangent) counterparts of the basis vectors g_i, e_A, and ω_μ, denoting them respectively as g^i, e^A, and ω^μ , according to the following relations:

$$\langle g^i, g_k \rangle = \delta^i_k,$$

$$\langle e^A, e_B \rangle = \delta^A_B,$$

$$\langle \omega^\mu, \omega_\nu \rangle = \delta^\mu_\nu,$$

where the brackets $\langle\rangle$ denote the so-called projection, i.e., the inner product and where δ denotes the Kronecker delta. From these basis vectors, we define their tetrad components as

$$\gamma^i_A = \langle g^i, e_A \rangle = \frac{\partial \xi^i}{\partial x^A},$$

$$\zeta^i_\mu = \langle g^i, \omega_\mu \rangle = \frac{\partial \xi^i}{\partial y^\mu},$$

$$e^A_\mu = \langle e^A, \omega_\mu \rangle = \frac{\partial x^A}{\partial y^\mu}.$$

Their duals are given in the following relations:

$$\gamma^i_A \gamma^A_k = \zeta^i_\mu \zeta^\mu_k = \delta^i_k\,,$$

$$e^A_\mu e^\mu_B = \delta^A_B\,.$$

(Einstein's summation convention is implied throughout this work.)

The distance between two infinitesimally adjacent points in the (initially undeformed) material body $\Im_3$ is given by the symmetric bilinear form (with $\otimes$ denoting the tensor product)

$$g = g_{ik}\, g^i \otimes g^k\,,$$

called the metric tensor of the material space, as

$$ds^2 = g_{ik}\, d\xi^i d\xi^k\,.$$

By means of projection, the components of the metric tensor of $\Im_3$ are given by

$$g_{ik} = \langle g_i, g_k \rangle\,.$$

Accordingly, for $a, b = 1, 2, 3$, they are related to the four-dimensional components of the metric tensor of $\Im_3$, i.e., $G_{\mu\nu} = \langle \omega_\mu, \omega_v \rangle$, by

$$g_{ik} = \zeta^\mu_i \zeta^\nu_k G_{\mu\nu} = \\ = \zeta^a_i \zeta^b_k G_{ab} + 2k_{(i} b_{k)} + \phi k_i k_k\,,$$

where the round brackets indicate symmetrization (in contrast to the square brackets denoting anti-symmetrization which we shall also employ later) and where we have set

$$k_i = \zeta^4_i = c\frac{\partial t}{\partial \xi^i}\,,$$

$$b_i = G_{4i} = \zeta^a_i G_{4a}\,,$$

$$\phi = G_{44}\,.$$

Here we have obviously put $y^4 = ct$ with $c$the speed of light in vacuum and t time.

Inversely, with the help of the following projective relations:

$$g_i = \zeta^\mu_i \omega_\mu\,,$$

$$\omega_\mu = \zeta^i_\mu g_i + \in n_\mu n\,,$$

we find that

$$G_{\mu\nu} = \zeta^i_\mu \zeta^k_\nu g_{ik} + \in n_\mu n_\nu\,,$$

or, calling the dual components of g, as shown in the relations

$$g_{ir}\, g^{kr} = \delta^k_i\,,$$

$$G_{\mu\rho} G^{\nu\rho} = \delta^\nu_\mu\,,$$

we have

$$\gamma_i^\mu \gamma_\nu^i = \delta_\nu^\mu - \in n^\mu n_\nu \,.$$

Here $\in\ = \pm 1$ and $n^\mu$and n_ν respectively are the contravariant and covariant components of the unit vector field n normal to the hypersurface of material coordinates $\Im_3$, whose canonical form may be given as $\Phi\left(\xi^i, k\right) = 0$ where k is a parameter. (Note that the same 16 relations also hold for the inner product represented by $e_A^\mu e_\nu^A$.) We can write

$$n_\mu = \in^{1/2} \frac{\partial \Phi}{\partial y^\mu} \left(G^{\alpha\beta} \frac{\partial \Phi}{\partial y^\alpha} \frac{\partial \Phi}{\partial y^\beta} \right)^{-1/2} .$$

Note that

$$n_\mu \gamma_i^\mu = n_\mu e_i^\mu = 0 \,,$$

$$n_\mu n^\mu = \in \,.$$

Let now g denote the determinant of the three-dimensional components of the material metric tensor g_{ik}. Then the covariant and contravariant components of the totally anti-symmetric permutation tensor are given by

$$\in_{ijk} = g^{1/2} \varepsilon_{ijk} \,,$$

$$\in^{ijk} = g^{-1/2} \varepsilon^{ijk} \,,$$

where ε_{ijk} are the components of the usual *permutation tensor density*. More specifically, we note that

$$g_i \wedge g_j = \in_{ijk} g^k \,,$$

where the symbol $\wedge$ denotes exterior product, i.e., $g_i \wedge g_j = \left(\zeta_i^\mu \zeta_j^\nu - \zeta_j^\mu \zeta_i^\nu\right) \omega_\mu \otimes \omega_\nu$. In the same manner, we define the four-dimensional permutation tensor as one with components

$$\in_{\alpha\beta\rho\sigma} = G^{1/2} \varepsilon_{\alpha\beta\rho\sigma} \,,$$

$$\in^{\alpha\beta\rho\sigma} = G^{-1/2} \varepsilon^{\alpha\beta\rho\sigma} \,,$$

where $G = \det G_{\mu\nu}$. Also, we call the following simple transitive rotation group:

$$\omega_\alpha \wedge \omega_\beta = - \in\ \in_{\alpha\beta\rho\sigma} n^\rho \omega^\sigma \,,$$

where

$$\in_{ijk} n_\sigma = \zeta_i^\alpha \zeta_j^\beta \zeta_k^\rho \in_{\alpha\beta\rho\sigma} \,,$$

$$n_\sigma = \frac{1}{6} \zeta_i^\alpha \zeta_j^\beta \zeta_k^\rho \in^{ijk} \in_{\alpha\beta\rho\sigma} \,.$$

Note the following identities:

$$\in_{ijk} \in^{pqr} = \delta_{ijk}^{pqr} = \delta_i^p \left(\delta_j^q \delta_k^r - \delta_j^r \delta_k^q\right) + \delta_i^q \left(\delta_j^r \delta_k^p - \delta_j^p \delta_k^r\right) + \\ + \delta_i^r \left(\delta_j^p \delta_k^q - \delta_j^q \delta_k^p\right) \,,$$

$$\in_{ijr} \in^{pqr} = \delta_{ij}^{pq} = \delta_i^p \delta_j^q - \delta_i^q \delta_j^p \,,$$

$$\in_{ijs} \in^{ijr} = \delta_s^r \,,$$

where δ_{ijk}^{pqr} and δ_{ij}^{pq} represent generalized Kronecker deltas. In the same manner, the four-dimensional components of the generalized Kronecker delta, i.e.,

$$\delta_{\mu\nu\gamma\lambda}^{\alpha\beta\rho\sigma} = \det \begin{pmatrix} \delta_\mu^\alpha & \delta_\mu^\beta & \delta_\mu^\rho & \delta_\mu^\sigma \\ \delta_\nu^\alpha & \delta_\nu^\beta & \delta_\nu^\rho & \delta_\nu^\sigma \\ \delta_\gamma^\alpha & \delta_\gamma^\beta & \delta_\gamma^\rho & \delta_\gamma^\sigma \\ \delta_\lambda^\alpha & \delta_\lambda^\beta & \delta_\lambda^\rho & \delta_\lambda^\sigma \end{pmatrix}$$

can be used to deduce the following identities:

$$\in_{\mu\nu\gamma\lambda} \in^{\alpha\beta\rho\sigma} = \in \delta_{\mu\nu\gamma\lambda}^{\alpha\beta\rho\sigma} \,,$$

$$\in_{\mu\nu\gamma\sigma} \in^{\alpha\beta\rho\sigma} = \in \delta_{\mu\nu\gamma}^{\alpha\beta\rho} \,,$$

$$\in_{\mu\nu\rho\sigma} \in^{\alpha\beta\rho\sigma} = 2 \in \delta_{\mu\nu}^{\alpha\beta} \,,$$

$$\in_{\mu\beta\rho\sigma} \in^{\alpha\beta\rho\sigma} = 6 \in \delta_\mu^\alpha \,.$$

Now, for the contravariant components of the material metric tensor, we have

$$g^{ik} = \zeta_a^i \zeta_b^k G^{ab} + 2k^{(i} b^{k)} + \bar{\phi} k^i k^k \,,$$

$$G^{\mu\nu} = \zeta_i^\mu \zeta_k^\nu g^{ik} + \in n^\mu n^\nu \,,$$

where

$$k^i = \frac{1}{c} \frac{\partial \xi^i}{\partial t} \,,$$

$$b^i = G^{4i} = \zeta_a^i G^{4a} \,,$$

$$\bar{\phi} = G^{44} \,.$$

Obviously, the quantities $\frac{\partial \xi^i}{\partial t}$ in k^i are the contravariant components of the local velocity vector field. If we choose an orthogonal coordinate system for the background space-time $\Re_4$, we simply have the following three-dimensional components of the material metric tensor:

$$g_{ik} = \zeta_i^a \zeta_k^b G_{ab} + \phi k_i k_k \,,$$

$$g^{ik} = \zeta_a^i \zeta_b^k G^{ab} + \bar{\phi} k^i k^k \,.$$

In a special case, if the space-time $\Re_4$ is (pseudo-)Euclidean, we may set $\phi = \bar{\phi} = \pm 1$. However, for the sake of generality, we shall not always need to assume the case just mentioned.

Now, the components of the metric tensor of the local Euclidean space of spatial coordinates x^A, $h_{AB} = \langle e_A, e_B \rangle$, are just the components of the *Euclidean Kronecker delta*:

$$h_{AB} = \delta_{AB} \,.$$

Similarly, we have the following relations:

$$g_{ik} = \gamma_i^A \gamma_k^B h_{AB} = \gamma_i^A \gamma_k^A \,,$$

$$h_{AB} = \gamma_A^i \gamma_B^k g_{ik} \,.$$

Now we come to an important fact: from the structure of the material metric tensor alone, we can raise and lower the indices of arbitrary vectors and tensors defined in $\Im_3$, and hence in $\Re_4$, by means of its components, e.g.,

$$A^i = g^{ik} A_k,\, A_i = g_{ik} A^k,\, B^\mu = G^{\mu\nu} B_v,$$
$$B_\mu = G_{\mu\nu} B^v,\ etc.$$

Having introduced the metric tensor, let us consider the transformations among the physical objects defined as acting in the material space $\Im_3$. An arbitrary tensor field T of rank n in $\Im_3$ can in general be represented as

$$\begin{aligned} T &= T^{ij\ldots}_{kl\ldots}\, g_i \otimes g_j \otimes \ldots \otimes g^k \otimes g^l \otimes \cdots = \\ &= T'^{AB\ldots}_{CD\ldots}\, e_A \otimes e_B \otimes \ldots \otimes e^C \otimes e^D \otimes \cdots = \\ &= T''^{\alpha\beta\ldots}_{\mu\nu\ldots}\, \omega_\alpha \otimes \omega_\beta \otimes \ldots \otimes \omega^\mu \otimes \omega^\nu \otimes \ldots . \end{aligned}$$

In other words,

$$T^{ij\ldots}_{kl\ldots} = \gamma^i_A \gamma^j_B \ldots \gamma^C_k \gamma^D_l \ldots T'^{AB\ldots}_{CD\ldots} = \zeta^i_\alpha \zeta^j_\beta \ldots \zeta^\mu_k \zeta^\nu_l \ldots T''^{\alpha\beta\ldots}_{\mu\nu\ldots}\,,$$

$$T'^{AB\ldots}_{CD\ldots} = \gamma^A_i \gamma^B_j \ldots \gamma^k_C \gamma^l_D \ldots T^{ij\ldots}_{kl\ldots} = e^A_\alpha e^B_\beta \ldots e^\mu_C e^\nu_D \ldots T''^{\alpha\beta\ldots}_{\mu\nu\ldots}\,,$$

$$T''^{\alpha\beta\ldots}_{\mu\nu\ldots} = \zeta^k_\mu \zeta^l_\nu \ldots \zeta^\alpha_i \zeta^\beta_j \ldots T^{ij\ldots}_{kl\ldots} = e^C_\mu e^D_\nu \ldots e^\alpha_A e^\beta_B T'^{AB\ldots}_{CD\ldots}\,.$$

For instance, the material line-element can once again be written as

$$ds^2 = g_{ik}\,(\xi^p)\, d\xi^i d\xi^k = \delta_{AB}\, dx^A dx^B = G_{\mu\nu}\,(y^\alpha)\, dy^\mu dy^\nu.$$

We now move on to the notion of *a covariant derivative* defined in the material space $\Im_3$. Again, for an arbitrary tensor field T of $\Im_3$, the covariant derivative of the components of T is given as

$$\nabla_p T^{ij\ldots}_{kl\ldots} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial \xi^p} + \Gamma^i_{rp} T^{rj\ldots}_{kl\ldots} + \Gamma^j_{rp} T^{ir\ldots}_{kl\ldots} + \cdots - \\ - \Gamma^r_{kp} T^{ij\ldots}_{rl\ldots} - \Gamma^r_{lp} T^{ij\ldots}_{kr\ldots} - \ldots\,,$$

such that

$$\nabla_p T = \frac{\partial T}{\partial \xi^p} = \nabla_p T^{ij\ldots}_{kl\ldots}\, g_i \otimes g_j \otimes \ldots \otimes g^k \otimes g^l \otimes \ldots\,,$$

where

$$\frac{\partial g_i}{\partial \xi^k} = \Gamma^r_{ik}\, g_r\,.$$

Here the $n^3 = 27$ quantities Γ^i_{jk} are the components of the connection field Γ, locally given by

$$\Gamma^i_{jk} = \gamma^i_A \frac{\partial \gamma^A_j}{\partial \xi^k}\,,$$

which, in our work, shall be non-symmetric in the pair of its lower indices (jk) in order to describe *both* torsion and discontinuities. If $\bar{\xi}^i$ represent another system of coordinates in the material space $\Im_3$, then locally the components of the connection field Γ are seen to transform *inhomogeneously* according to

$$\Gamma^i_{jk} = \frac{\partial \xi^i}{\partial \bar{\xi}^p} \frac{\partial \bar{\xi}^r}{\partial \xi^j} \frac{\partial \bar{\xi}^s}{\partial \xi^k} \bar{\Gamma}^p_{rs} + \frac{\partial \xi^i}{\partial \bar{\xi}^p} \frac{\partial^2 \bar{\xi}^p}{\partial \xi^k \partial \xi^j}\,,$$

i.e., the Γ^i_{jk} do not transform as components of a local tensor field. Before we continue, we shall note a few things regarding some *boundary conditions* of our material geometry. Because we have assumed that the hypersurface $\Im_3$ is embedded in the four-dimensional space-time $\Re_4$, we must in general have instead

$$\frac{\partial g_i}{\partial \xi^k} = \Gamma^r_{ik}\, g_r + \in K_{ik}\, n\,,$$

where $K_{ik} = \langle \nabla_k g_i, n \rangle = n_\mu \nabla_k \zeta^\mu_i$ are the covariant components of the extrinsic curvature of $\Im_3$. Then the scalar given by $\bar{K} = \in K_{ik} \frac{d\xi^i}{ds} \frac{d\xi^k}{ds}$, which is the Gaussian curvature of $\Im_3$, is arrived at. However our simultaneous embedding situation in which we have also defined an Euclidean space in $\Re_4$ as the space of spatial coordinates embedding the space of material coordinates $\Im_3$, means that the extrinsic curvature tensor, and hence also the Gaussian curvature of $\Im_3$, must vanish and we are left simply with $\frac{\partial g_i}{\partial \xi^k} = \Gamma^r_{ik}\, g_r$. This situation is analogous to the simple situation in which a plane (flat surface) is embedded in a three-dimensional space, where on that plane we define a family of curves which give rise to a system of curvilinear coordinates, however, with discontinuities in the transformation from the plane coordinates to the local curvilinear coordinates and vice versa.

Meanwhile, we have seen that the covariant derivative of the tensor field T is again a tensor field. As such, here we have

$$\nabla_p T^{ij\ldots}_{kl\ldots} = \gamma^i_A \gamma^j_B \ldots \gamma^C_k \gamma^D_l \ldots \gamma^E_p \frac{\partial T'^{AB\ldots}_{CD\ldots}}{\partial x^E}$$

Although a non-tensorial object, the connection field Γ is a fundamental geometric object that establishes comparison of local vectors at different points in $\Im_3$, i.e., in the Lagrangian coordinate system. Now, with the help of the material *metrical condition*

$$\nabla_p\, g_{ik} = 0\,,$$

i.e.,

$$\frac{\partial g_{ik}}{\partial \xi^p} = \Gamma_{ikp} + \Gamma_{kip}\,,$$

where $\Gamma_{ikp} = g_{ir}\ \Gamma^r_{kp}$, one solves for Γ^i_{jk} as follows:

$$\Gamma^i_{jk} = \frac{1}{2} g^{ri} \left(\frac{\partial g_{rj}}{\partial \xi^k} - \frac{\partial g_{jk}}{\partial \xi^r} + \frac{\partial g_{kr}}{\partial \xi^j} \right) + \Gamma^i_{[jk]} - \\ - g^{ri} \left(g_{js} \Gamma^s_{[rk]} + g_{ks} \Gamma^s_{[rj]} \right)\,.$$

From here, we define the following geometric objects:

1. The holonomic (path-independent) Christoffel or Levi-Civita connection, sometimes also called the *elastic connection*, whose components are symmetric in the pair of its lower indices (jk) and given by

$$\left\{{i \atop jk}\right\} = \frac{1}{2} g^{ri} \left(\frac{\partial g_{rj}}{\partial \xi^k} - \frac{\partial g_{jk}}{\partial \xi^r} + \frac{\partial g_{kr}}{\partial \xi^j} \right) .$$

2. The non-holonomic (path-dependent) object, a *chirality tensor* called the torsion tensor which describes *local rotation of material points* in $\Im_3$ and whose components are given by

$$\tau^i_{jk} = \Gamma^i_{[jk]} = \frac{1}{2} \gamma^i_A \left(\frac{\partial \gamma^A_j}{\partial \xi^k} - \frac{\partial \gamma^A_k}{\partial \xi^j} \right) .$$

3. The non-holonomic contorsion tensor, a linear combination of the torsion tensor, whose components are given by

$$\begin{aligned} T^i_{jk} &= \Gamma^i_{[jk]} - g^{ri} \left(g_{js} \Gamma^s_{[rk]} + g_{ks} \Gamma^s_{[rj]} \right) = \\ &= \gamma^i_A \tilde{\nabla}_k \gamma^A_j = \\ &= \gamma^i_A \left(\frac{\partial \gamma^A_j}{\partial \xi^k} - \left\{{r \atop jk}\right\} \gamma^A_r \right) , \end{aligned}$$

which are actually anti-symmetric with respect to the first two indices $i$and$j$.

In the above, we have exclusively introduced a covariant derivative with respect to the holonomic connection alone, denoted by $\tilde{\nabla}_p$. Again, for an arbitrary tensor field T of $\Im_3$, we have

$$\begin{aligned} \tilde{\nabla}_p T^{ij\ldots}_{kl\ldots} &= \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial \xi^p} + \left\{{i \atop rp}\right\} T^{rj\ldots}_{kl\ldots} + \left\{{j \atop rp}\right\} T^{ir\ldots}_{kl\ldots} + \cdots - \\ &- \left\{{r \atop kp}\right\} T^{ij\ldots}_{rl\ldots} - \left\{{r \atop lp}\right\} T^{ij\ldots}_{kr\ldots} - \ldots . \end{aligned}$$

Now we can see that the metrical condition $\nabla_p g_{ik} = 0$ also implies that $\tilde{\nabla}_p g_{ik} = 0$, $\tilde{\nabla}_k \gamma^A_i = T^r_{ik} \gamma^A_r$, and $\nabla_k \gamma^A_i = 0$.

Finally, with the help of the connection field Γ , we derive the third fundamental geometric objects of $\Im_3$, i.e., the local fourth-order curvature tensor of the material space

$$R = R^i_{.jkl} g_i \otimes g^j \otimes g^k \otimes g^l ,$$

where

$$R^i_{.jkl} = \frac{\partial \Gamma^i_{jl}}{\partial \xi^k} - \frac{\partial \Gamma^i_{jk}}{\partial \xi^l} + \Gamma^r_{jl} \Gamma^i_{rk} - \Gamma^r_{jk} \Gamma^i_{rl} .$$

These are given in the relations

$$\left(\nabla_k \nabla_j - \nabla_j \nabla_k \right) F_i = R^r_{.ijk} F_r - 2 \Gamma^r_{[jk]} \nabla_r F_i ,$$

where F_i are the covariant components of an arbitrary vector field F of $\Im_3$. Correspondingly, for the contravariant components F^i we have

$$\left(\nabla_k \nabla_j - \nabla_j \nabla_k \right) F^i = -R^i_{.rjk} F^r - 2 \Gamma^r_{[jk]} \nabla_r F^i .$$

The *Riemann-Christoffel curvature tensor* $\tilde{R}$ here then appears as the part of the curvature tensor R built from the symmetric, holonomic Christoffel connection alone, whose components are given by

$$\tilde{R}^i_{.jkl} = \frac{\partial}{\partial \xi^k} \left\{{i \atop jl}\right\} - \frac{\partial}{\partial \xi^l} \left\{{i \atop jk}\right\} + \left\{{r \atop jl}\right\} \left\{{i \atop rk}\right\} - \left\{{r \atop jk}\right\} \left\{{i \atop rl}\right\} .$$

Correspondingly, the components of the symmetric Ricci tensor are given by

$$\begin{aligned} \tilde{R}_{ik} = \tilde{R}^r_{.irk} &= \frac{\partial}{\partial \xi^r} \left\{{r \atop ik}\right\} - \frac{\partial^{2\,e} \log\, (g)^{1/2}}{\partial \xi^k \partial \xi^i} + \\ &+ \left\{{s \atop ik}\right\} \frac{\partial^e \log\, (g)^{1/2}}{\partial \xi^s} - \left\{{s \atop ir}\right\} \left\{{r \atop sk}\right\} , \end{aligned}$$

where we have used the relations

$$\left\{{k \atop ik}\right\} = \frac{\partial^e \log\, (g)^{1/2}}{\partial \xi^i} = \Gamma^k_{ki} .$$

Then the Ricci scalar is simply $\tilde{R} = \tilde{R}^i_{.i}$, an important geometric object which shall play the role of the *microspin* (*chirality*) *potential* in our generalization of classical elasticity theory developed here.

Now, it is easily verified that

$$\left(\tilde{\nabla}_k \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_k \right) F_i = \tilde{R}^r_{.ijk} F_r$$

and

$$\left(\tilde{\nabla}_k \tilde{\nabla}_j - \tilde{\nabla}_j \tilde{\nabla}_k \right) F^i = -\tilde{R}^i_{.rjk} F^r .$$

The remaining parts of the curvature tensor R are then the remaining non-holonomic objects J and Q whose components are given as

$$J^i_{.jkl} = \frac{\partial T^i_{jl}}{\partial \xi^k} - \frac{\partial T^i_{jk}}{\partial \xi^l} + T^r_{jl} T^i_{rk} - T^r_{jk} T^i_{rl}$$

and

$$Q^i_{.jkl} = \left\{{r \atop jl}\right\} T^i_{rk} + T^r_{jl} \left\{{i \atop rk}\right\} - \left\{{r \atop jk}\right\} T^i_{rl} - T^r_{jk} \left\{{i \atop rl}\right\} .$$

Hence, we write

$$R^i_{.jkl} = \tilde{R}^i_{.jkl} + J^i_{.jkl} + Q^i_{.jkl} .$$

More explicitly,

$$R^i_{.jkl} = \tilde{R}^i_{.jkl} + \tilde{\nabla}_k T^i_{jl} - \tilde{\nabla}_l T^i_{jk} + T^r_{jl} T^i_{rk} - T^r_{jk} T^i_{rl} .$$

From here, we define the two important contractions of the components of the curvature tensor above. We have the generalized Ricci tensor whose components are given by

$$R_{ik} = R^r_{.irk} = \tilde{R}_{ik} + \tilde{\nabla}_r T^r_{ik} - T^r_{is} T^s_{rk} - \tilde{\nabla}_k \omega_i + T^r_{ik} \omega_r ,$$

where the $n = 3$ quantities

$$\omega_i = T^k_{ik} = 2 \Gamma^k_{[ik]}$$

define the components of the microspin vector. Furthermore, with the help of the relations $g^{rs}T^i_{rs} = -2g^{ik}\Gamma^s_{[ks]} = -\omega^i$, the generalized Ricci scalar is

$$R = R^i_{.i} = \tilde{R} - 2\tilde{\nabla}_i\omega^i - \omega_i\omega^i - T_{ijk}T^{ikj}\,.$$

It is customary to give the fully covariant components of the Riemann-Christoffel curvature tensor. They can be expressed somewhat more conveniently in the following form (when the g_{ik} are continuous):

$$\tilde{R}_{ijkl} = \frac{1}{2}\left(\frac{\partial^2 g_{il}}{\partial\xi^k\partial\xi^j} + \frac{\partial^2 g_{jk}}{\partial\xi^l\partial\xi^i} - \frac{\partial^2 g_{ik}}{\partial\xi^l\partial\xi^j} - \frac{\partial^2 g_{jl}}{\partial\xi^k\partial\xi^i}\right) + \\ + g_{rs}\left(\left\{{r \atop il}\right\}\left\{{s \atop jk}\right\} - \left\{{r \atop ik}\right\}\left\{{s \atop jl}\right\}\right).$$

In general, when the g_{ik} are continuous, all the following symmetries are satisfied:

$$\tilde{R}_{ijkl} = -\tilde{R}_{jikl} = -\tilde{R}_{ijlk}\,,$$

$$\tilde{R}_{ijkl} = \tilde{R}_{klij}\,.$$

However, for the sake of generality, we may as well drop the condition that the g_{ik} are continuous in their second derivatives, i.e., with respect to the material coordinates ξ^i such that we can define further more non-holonomic, anti-symmetric objects extracted from R such as the tensor field V whose components are given by

$$V_{ik} = R^r_{.rik} = -\gamma^l_A\left(\frac{\partial}{\partial\xi^k}\left(\frac{\partial\gamma^A_l}{\partial\xi^i}\right) - \frac{\partial}{\partial\xi^i}\left(\frac{\partial\gamma^A_l}{\partial\xi^k}\right)\right).$$

The above relations are equivalent to the following $\frac{1}{2}n(n-1) = 3$ equations for the components of the material metric tensor:

$$\frac{\partial}{\partial\xi^l}\left(\frac{\partial g_{ij}}{\partial\xi^k}\right) - \frac{\partial}{\partial\xi^k}\left(\frac{\partial g_{ij}}{\partial\xi^l}\right) = -\left(R_{ijkl} + R_{jikl}\right),$$

which we shall denote simply by $\|g_{ij,kl}\|$. When the g_{ik} possess such discontinuities, we may define the *discontinuity potential* by

$$\eta_i = \left\{{k \atop ik}\right\} = \frac{\partial^e \log\,(g)^{1/2}}{\partial\xi^i}\,.$$

Hence we have

$$V_{ik} = \frac{\partial\eta_k}{\partial\xi^i} - \frac{\partial\eta_i}{\partial\xi^k}\,.$$

From the expression of the determinant of the material metric tensor, i.e.,

$$g = \varepsilon_{ijk}\,g_{1i}\,g_{2j}\,g_{3k}$$

we see, more specifically, that a discontinuum with arbitrary geometric singularities is characterized by the following discontinuity equations:

$$\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right)g = \\ = \varepsilon_{ijk}\,g_{2j}\,g_{3k}\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right)g_{1i} + \\ + \varepsilon_{ijk}\,g_{1i}\,g_{3k}\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right)g_{2j} + \\ + \varepsilon_{ijk}\,g_{1i}\,g_{2j}\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right)g_{3k}\,.$$

In other words,

$$\left(\frac{\partial^2}{\partial\xi^s\partial\xi^r} - \frac{\partial^2}{\partial\xi^r\partial\xi^s}\right)g = -\varepsilon_{ijk}\left(R_{1irs} + R_{i1rs}\right)g_{2j}\,g_{3k} - \\ - \varepsilon_{ijk}\left(R_{2jrs} + R_{j2rs}\right)g_{1i}\,g_{3k} - \\ - \varepsilon_{ijk}\left(R_{3krs} + R_{k3rs}\right)g_{1i}\,g_{2j}\,.$$

It is easy to show that in three dimensions the components of the curvature tensor R obey the following decomposition:

$$R_{ijkl} = W_{ijkl} + g_{ik}R_{jl} + g_{jl}R_{ik} - g_{il}R_{jk} - g_{jk}R_{il} + \\ + \frac{1}{2}\left(g_{il}\,g_{jk} - g_{ik}\,g_{jl}\right)R\,,$$

i.e.,

$$R^{ij}_{..kl} = W^{ij}_{..kl} + \delta^i_k R_{jl} + \delta^j_l R^i_{.k} - \delta^i_l R_{jk} - \delta^j_k R^i_{.l} + \\ + \frac{1}{2}\left(\delta^i_l\delta^j_k - \delta^i_k\delta^j_l\right)R\,,$$

where W_{ijkl} (and $W^{ij}_{..kl}$) are the components of the Weyl tensor W satisfying $W^r_{.irk} = 0$, whose symmetry properties follow exactly those of R_{ijkl}. Similarly, for the components of the Riemann-Christoffel curvature tensor $\tilde{R}$ we have

$$\tilde{R}^{ij}_{..kl} = \tilde{W}^{ij}_{..kl} + \delta^i_k\tilde{R}_{jl} + \delta^j_l\tilde{R}^i_{.k} - \delta^i_l\tilde{R}_{jk} - \delta^j_k\tilde{R}^i_{.l} + \\ + \frac{1}{2}\left(\delta^i_l\delta^j_k - \delta^i_k\delta^j_l\right)\tilde{R}\,.$$

Later, the above equations shall be needed to generalize the components of the elasticity tensor of classical continuum mechanics, i.e., by means of the components

$$\frac{1}{2}\left(\delta^i_k\delta^j_l - \delta^i_l\delta^j_k\right)\tilde{R}\,.$$

Furthermore with the help of the relations

$$R^i_{.jkl} + R^i_{.klj} + R^i_{.ljk} = -2\left(\frac{\partial\Gamma^i_{[jk]}}{\partial\xi^l} + \frac{\partial\Gamma^i_{[kl]}}{\partial\xi^j} + \frac{\partial\Gamma^i_{[lj]}}{\partial\xi^k} + \right. \\ \left. + \Gamma^i_{rj}\Gamma^r_{[kl]} + \Gamma^i_{rk}\Gamma^r_{[lj]} + \Gamma^i_{rl}\Gamma^r_{[jk]}\right),$$

we derive the following identities:

$$\nabla_p R_{ijkl} + \nabla_k R_{ijlp} + \nabla_l R_{ijpk} = 2\left(\Gamma^r_{[kl]}R_{ijrp} + \right. \\ \left. + \Gamma^r_{[lp]}R_{ijrk} + \Gamma^r_{[pk]}R_{ijrl}\right),$$

$$\nabla_i \left(R^{ik} - \frac{1}{2} g^{ik} R \right) = 2 g^{ik} \Gamma^s_{[ri]} R^r_{.s} + \Gamma^r_{[ij]} R^{ijk}_{\dots r} \,.$$

From these more general identities, we then derive the simpler and more specialized identities:

$$\tilde{R}_{ijkl} + \tilde{R}_{iklj} + \tilde{R}_{iljk} = 0 \,,$$

$$\tilde{\nabla}_p \tilde{R}_{ijkl} + \tilde{\nabla}_k \tilde{R}_{ijlp} + \tilde{\nabla}_l \tilde{R}_{ijpk} = 0 \,,$$

$$\tilde{\nabla}_i \left(\tilde{R}^{ik} - \frac{1}{2} g^{ik} \tilde{R} \right) = 0 \,,$$

often referred to as the Bianchi identities.

We are now able to state the following about the sources of the curvature of the material space $\Im_3$: there are actually two sources that generate the curvature which can actually be sufficiently represented by the Riemann-Christoffel curvature tensor alone. The first source is the torsion represented by $\Gamma^i_{[jk]}$ which makes the hypersurface $\Im_3$ *non-orientable* as any field shall in general depend on the twisted path it traces therein. As we have said, this torsion is the source of microspin, i.e., point-rotation. The torsion tensor enters the curvature tensor as an integral part and hence we can equivalently attribute the source of microspin to the Riemann-Christoffel curvature tensor as well. The second source is the possible discontinuities in regions of $\Im_3$ which, as we have seen, render the components of the material metric tensor $g_{ik} = \gamma^A_i \gamma^B_k \delta_{AB}$ discontinuous at least in their second derivatives with respect to the material coordinates ξ^i. This is explicitly shown in the following relations:

$$R^i_{.jkl} = -\gamma^i_A \left(\frac{\partial}{\partial \xi^l} \left(\frac{\partial \gamma^A_j}{\partial \xi^k} \right) - \frac{\partial}{\partial \xi^k} \left(\frac{\partial \gamma^A_j}{\partial \xi^l} \right) \right) =$$

$$= -\gamma^i_A \left(\nabla_l K^A_{jk} - \nabla_k K^A_{jl} \right) + \Omega^i_{.jkl} \,,$$

where

$$K^A_{ij} = \frac{\partial \gamma^A_i}{\partial \xi^j} = \frac{1}{2} \left(\frac{\partial \gamma^A_i}{\partial \xi^j} + \frac{\partial \gamma^A_j}{\partial \xi^i} \right) + \gamma^A_k \Gamma^k_{[ij]}$$

and

$$\Omega^i_{.jkl} = \gamma^i_A \left(\Gamma^r_{jk} K^A_{rl} - \Gamma^r_{jl} K^A_{rk} - 2 \Gamma^r_{[kl]} K^A_{jr} \right) \,.$$

Another way to cognize the existence of the curvature in the material space $\Im_3$ is as follows: let us inquire into the possibility of "parallelism" in the material space $\Im_3$. Take now a "parallel" vector field ${}^pB$ such that

$$\nabla_k {}^pB_i = 0 \,,$$

i.e.,

$$\frac{\partial\, {}^pB_i}{\partial \xi^k} = \Gamma^r_{ik}\, {}^pB_r \,.$$

Then in general we obtain the following non-integrable equations of the form

$$\frac{\partial}{\partial \xi^l} \left(\frac{\partial\, {}^pB_i}{\partial\, \xi^k} \right) - \frac{\partial}{\partial\, \xi^k} \left(\frac{\partial\, {}^pB_i}{\partial\, \xi^l} \right) = -R^r_{.ikl}\, {}^pB_r$$

showing that not even the "parallel" vector field ${}^pB$ is path-independent. Hence even though parallelism may be possibly defined in our geometry, *absolute parallelism* is obtained if and only if the integrability condition $R^i_{.jkl} = 0$ holds, i.e., if the components of the Riemann-Christoffel curvature tensor are given by

$$\tilde{R}^i_{.jkl} = \tilde{\nabla}_l T^i_{jk} - \tilde{\nabla}_k T^i_{jl} + T^r_{jk} T^i_{rl} - T^r_{jl} T^i_{rk} \,.$$

In other words, in the presence of torsion (microspin) the above situation concerning absolute parallelism is only possible if the material body is free of geometric defects, also known as singularities.

The relations we have been developing so far of course account for arbitrary nonorientability conditions as well as geometric discontinuities of the material space $\Im_3$. Consequently, we see that the holonomic field equations of classical continuum mechanics shall be obtained whenever we drop the assumptions of non-orientability of points and geometric discontinuities of the material body. We also emphasize that *geometric non-linearity* of the material body has been fully taken into account. A material body then becomes linear if and only if we neglect any quadratic and higher-order terms involving the connection field Γ of the material space $\Im_3$.

3 Elements of the generalized kinematics: deformation analysis

Having described the internal structure of the material space $\Im_3$, i.e., the material body, we now move on to the dynamics of the continuum/discontinuum $\Im_3$ when it is subject to an external displacement field. Our goal in this kinematical section is to generalize the notion of a material derivative with respect of the material motion. We shall deal with the external displacement field in the direction of motion of $\Im_3$ which brings $\Im_3$ from its initially undeformed configuration to the deformed configuration ${}^*\Im_3$. We need to generalize the structure of the external displacement (i.e., *external diffeomorphism*) to include two kinds of microspin of material points: the *non-electromagnetic microspin* as well as the *electromagnetic microspin* which is generated, e.g., by electromagnetic polarization.

In this work, in order to geometrically describe the mechanics of the so-called Cosserat continuum as well as other generalized continua, we define the external displacement field ψ as being generally complex according to the decomposition

$$\psi^i = u^i + i \varphi^i \,,$$

where the diffeomorphism $\Im_3 \xrightarrow{\psi} {}^*\Im_3$ is given by

$${}^*\xi^i = \xi^i + \psi^i \,.$$

Here u^i are the components of the usual displacement field u in the neighborhood of points in $\Im_3$, and φ^i are the components of the microspin "point" displacement field φ satisfying

$$\nabla_k \varphi_i + \nabla_i \varphi_k = 0\,,$$

which can be written as an exterior ("Lie") derivative:

$$L_\varphi g_{ik} = 0\,.$$

This just says that the components of the material metric tensor remain invariant with respect to the action of the field φ. We shall elaborate on the notion of exterior differentiation in a short while.

The components of the displacement gradient tensor D are then

$$\begin{aligned} D_{ik} &= \nabla_k \psi_i = \\ &= \frac{1}{2}\left(\nabla_k \psi_i + \nabla_i \psi_k\right) + \frac{1}{2}\left(\nabla_k \psi_i - \nabla_i \psi_k\right) = \\ &= \frac{1}{2}\left(\nabla_k u_i + \nabla_i u_k\right) + \frac{1}{2}\left(\nabla_k u_i - \nabla_i u_k\right) + \\ &+ \frac{1}{2}\, i\left(\nabla_k \varphi_i - \nabla_i \varphi_k\right) = \\ &= \varepsilon_{ik} + \omega_{ik}\,. \end{aligned}$$

Accordingly,

$$\varepsilon_{ik} = \frac{1}{2}\left(\nabla_k u_i + \nabla_i u_k\right) = \frac{1}{2} L_u g_{ik} \overset{lin}{=} \frac{1}{2}\left({}^*g_{ik} - g_{ik}\right)$$

are the components of the *linear strain tensor* and

$$\omega_{ik} = \Omega_{ik} + \phi_{ik}$$

are the components of the *generalized spin* (*vorticity*) *tensor*, where

$$\Omega_{ik} = \frac{1}{2}\left(\nabla_k u_i - \nabla_i u_k\right)$$

are the components of the *ordinary macrospin tensor*, and

$$\phi_{ik} = \frac{1}{2}\, i\left(\nabla_k \varphi_i - \nabla_i \varphi_k\right)$$

are the components of the microspin tensor describing rotation of material points on their own axes due to torsion, or, in the literature, the so-called *distributed moment*. At this point, it may be that the internal rotation of material points is analogous to the spin of electrons if the material point themselves are seen as charged point-particles. However, we know that electrons possess internal spin due to internal structural reasons while the material points also rotate partly due to *externally induced couple stress* giving rise to torsion. For this reason we split the components of the microspin φ into two parts:

$$\varphi_i = \phi_i + eA_i\,,$$

where ϕ_i describe non-electromagnetic microspin and eA_i describe pure electron spin with e being the electric charge and A_i, up to a constant of proportionality, being the material components of the electromagnetic vector potential A:

$$A_i = q\omega_i^\mu A_\mu\,,$$

where q is a parametric constant and A_μ are the components of the four-dimensional electromagnetic vector potential in the sense of Maxwellian electrodynamics. Inversely, we have

$$A_\mu = \frac{1}{q}\left(\omega_\mu^i A_i + \in N n_\mu\right)\,,$$

where $N = qn_\mu A^\mu$. The correspondence with classical electrodynamics becomes complete if we link the *electromagnetic microspin tensor* f represented by the components

$$f_{ik} = \frac{1}{2}\, ie\left(\nabla_k A_i - \nabla_i A_k\right)$$

to the electromagnetic field tensor $F = F_{\mu\nu}\,\omega^\mu \otimes \omega^\nu$ through

$$f_{ik} = \frac{\bar\gamma}{e}\,\omega_i^\mu \omega_k^\nu F_{\mu\nu}\,,$$

where $\bar\gamma = \frac{1}{2} iqe^2$. The four-dimensional components of the electromagnetic field tensor in canonical form are

$$F_{\mu\nu} = \frac{\partial A_\mu}{\partial y^\nu} - \frac{\partial A_\nu}{\partial y^\mu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & B^3 & B^2 \\ E^2 & -B^3 & 0 & B^1 \\ E^3 & B^2 & -B^1 & 0 \end{pmatrix},$$

where $E = \left(E^1, E^2, E^3\right)$ and $B = \left(B^1, B^2, B^3\right)$ are the electric and magnetic fields, respectively. In three-dimensional vector notation, $E = -\frac{1}{c}\frac{\partial \vec A}{\partial t} - \vec\nabla\phi$ and $B = \mathrm{curl}\vec A$, where $\bar{\mathrm{A}} = A^\mu \omega_\mu = \left(\vec A, \phi\right)$. They satisfy Maxwell's equations in the Lorentz gauge $\mathrm{div}\,\vec A = 0$, i.e.,

$$\frac{1}{c}\frac{\partial E}{\partial t} = \mathrm{curl} B - \frac{4\pi}{c}\, j\,,$$

$$\mathrm{div} E = -\nabla^2\phi = 4\pi\rho_e\,,$$

$$\frac{1}{c}\frac{\partial B}{\partial t} = -\mathrm{curl} E\,,$$

$$\mathrm{div} B = 0\,,$$

where j is the electromagnetic current density vector and ρ_e is the electric charge density. In addition, we can write

$$\nabla_\nu F^{\mu\nu} = \frac{4\pi}{c}\, j^\mu\,,$$

i.e., $\nabla_\nu F^{A\nu} = \frac{4\pi}{c} j^A$ and $j^4 = \rho_e$. The inverse transformation relating f_{ik} to F_{ik} is then given by

$$F_{\mu\nu} = \frac{e}{\bar\gamma}\,\omega_\mu^i \omega_\nu^k f_{ik} + \hat F_{\mu\nu}\,,$$

where

$$\hat F_{\mu\nu} = -\in\left(n_\mu F_{\nu\sigma} - n_\nu F_{\mu\sigma}\right) n^\sigma\,,$$

$$\hat F_{\mu\nu} n^\nu = \in^2 F_{\mu\nu} n^\nu = F_{\mu\nu} n^\nu\,.$$

This way, the components of the generalized vorticity tensor once again are

$$\omega_{ik} = \frac{1}{2}\left(\nabla_k u_i - \nabla_k u_i\right) + \frac{1}{2} i \left(\nabla_k \phi_i - \nabla_i \phi_k\right) + \\ + \frac{1}{2} i e \left(\nabla_k A_i - \nabla_i A_k\right) = \\ = \frac{1}{2}\left(\frac{\partial u_i}{\partial \xi^k} - \frac{\partial u_k}{\partial \xi^i}\right) + \frac{1}{2} i \left(\frac{\partial \phi_i}{\partial \xi^k} - \frac{\partial \phi_k}{\partial \xi^i}\right) + \\ + \frac{1}{2} i e \left(\frac{\partial A_i}{\partial \xi^k} - \frac{\partial A_k}{\partial \xi^i}\right) - \Gamma^r_{[ik]}\left(u_r + i\phi_r + ieA_r\right) = \\ = \Omega_{ik} + \varpi_{ik} + \frac{\bar{\gamma}}{e}\,\omega^\mu_i \omega^\nu_k F_{\mu\nu}\,,$$

where

$$\varpi_{ik} = \frac{1}{2} i \left(\nabla_k \phi_i - \nabla_i \phi_k\right)$$

are the components of the non-electromagnetic microspin tensor. Thus we have now seen, in our generalized deformation analysis, how the microspin field is incorporated into the vorticity tensor.

Finally, we shall now produce some basic framework for equations of motion applicable to arbitrary tensor fields in terms of exterior derivatives. We define the exterior derivative of an arbitrary vector field (i.e., a rank-one tensor field) of $\Im_3$, say W, with respect to the so-called *Cartan basis* as the totally anti-symmetric object

$$L_U W = 2 U_{[i} W_{k]}\, g^i \otimes g^k\,,$$

where U is the velocity vector in the direction of motion of the material body $\Im_3$, i.e., $U^i = \frac{\partial \psi^i}{\partial t}$. If we now take the local basis vectors as *directional derivatives*, i.e., the *Cartan coordinate basis vectors* $g_i = \frac{\partial}{\partial \xi^i} = \partial_i$ and $g^i = d\xi^i$, we obtain for instance, in component notation,

$$\left(L_U W\right)_i = L_U W_i = U^k \partial_k W_i + W_k \partial_i U^k\,.$$

Using the exterior product, we actually see that

$$L_U W = U \wedge W = U \otimes W - W \otimes U\,.$$

Correspondingly, for W^i, we have

$$\left(L_U W\right)^i = L_U W^i = U^k \partial_k W^i - W^k \partial_k U^i\,.$$

The *exterior material derivative* is then a direct generalization of the ordinary material derivative (e.g., as we know, for a scalar field ρ it is given by $\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial \xi^i} U^i$) as follows:

$$\frac{DW_i}{Dt} = \frac{\partial W_i}{\partial t} + L_U W_i = \frac{\partial W_i}{\partial t} + \left(U \wedge V\right)_i = \frac{\partial W_i}{\partial t} + \\ + U^k \partial_k W_i + W_k \partial_i U^k\,,$$

$$\frac{DW^i}{Dt} = \frac{\partial W^i}{\partial t} + L_U W^i = \frac{\partial W^i}{\partial t} + \left(U \wedge V\right)^i = \frac{\partial W^i}{\partial t} + \\ + U^k \partial_k W^i - W^k \partial_i U^k\,.$$

Finally, we obtain the generalized material derivative of the components of an arbitrary tensor field T of $\Im_3$ as

$$\frac{DT^{ij\ldots}_{kl\ldots}}{Dt} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + U^m \partial_m T^{ij\ldots}_{kl\ldots} + T^{ij\ldots}_{ml\ldots} \partial_k U^m + \\ + T^{ij\ldots}_{km\ldots} \partial_l U^m + \cdots - T^{mj\ldots}_{kl\ldots} \partial_m U^i - T^{im\ldots}_{kl\ldots} \partial_m U^j - \ldots\,,$$

or alternatively as

$$\frac{DT^{ij\ldots}_{kl\ldots}}{Dt} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + U^m \nabla_m T^{ij\ldots}_{kl\ldots} + T^{ij\ldots}_{ml\ldots} \nabla_k U^m + \\ + T^{ij\ldots}_{km\ldots} \nabla_l U^m + \cdots - T^{mj\ldots}_{kl\ldots} \nabla_m U^i - T^{im\ldots}_{kl\ldots} \nabla_m U^j - \\ - \cdots + 2\Gamma^m_{[kp]} T^{ij\ldots}_{ml\ldots} U^p + 2\Gamma^m_{[lp]} T^{ij\ldots}_{km\ldots} U^p + \cdots - \\ - 2\Gamma^i_{[mp]} T^{mj\ldots}_{kl\ldots} U^p - 2\Gamma^j_{[mp]} T^{im\ldots}_{kl\ldots} U^p - \ldots\,.$$

Written more simply,

$$\frac{DT^{ij\ldots}_{kl\ldots}}{Dt} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + L_U T^{ij\ldots}_{kl\ldots} = \frac{\partial T^{ij\ldots}_{kl\ldots}}{\partial t} + \left(U \wedge T\right)^{ij\ldots}_{kl\ldots}\,.$$

For a scalar field Θ, we have simply

$$\frac{D\Theta}{Dt} = \frac{\partial \Theta}{\partial t} + U^k \partial_k \Theta\,,$$

which is just the ordinary material derivative.

Now, with the help of the Cartan basis vectors, the torsion tensor can be expressed directly in terms of the permutation tensor as

$$\Gamma^i_{[jk]} = -\frac{1}{2} g^{ip} \in_{pjk}\,.$$

Hence from the generalized material derivative for the components of the material metric tensor g (defined with respect to the Cartan basis), i.e.,

$$\frac{Dg_{ik}}{Dt} = \frac{\partial g_{ik}}{\partial t} + L_U g_{ik} = \frac{\partial g_{ik}}{\partial t} + \left(U \wedge g\right)_{ik}\,,$$

we find especially that

$$\frac{Dg_{ik}}{Dt} = \frac{\partial g_{ik}}{\partial t} + \nabla_k U_i + \nabla_i U_k\,,$$

with the help of the metrical condition $\nabla_p g_{ik} = 0$. Similarly, we also find

$$\frac{Dg^{ik}}{Dt} = \frac{\partial g^{ik}}{\partial t} - \left(\nabla^k U^i + \nabla^i U^k\right)\,.$$

Note also that

$$\frac{D\delta^i_k}{Dt} = 0\,.$$

The components of the velocity gradient tensor are given by

$$L_{ik} = \nabla_k U_i = \frac{1}{2}\left(\nabla_k U_i + \nabla_i U_k\right) + \frac{1}{2}\left(\nabla_k U_i - \nabla_i U_k\right)\,,$$

where, following the so-called Helmholtz decomposition theorem, we can write

$$U_i = \nabla_i \alpha + \frac{1}{2} g_{il} \in^{ljk} \left(\nabla_j \beta_k - \nabla_k \beta_j\right),$$

for a scalar field α and a vector field β. However, note that in our case we obtain the following generalized identities:

$$\text{div curl}\, U = -\frac{1}{2} \in^{ijk} \left(R^{l}_{.kij} U_l - 2\Gamma^{l}_{[ij]} \nabla_l A_k\right),$$

$$\text{curl grad}\, \alpha = \in^{ijk} \Gamma^{l}_{[ij]} \nabla_l \alpha\,,$$

which must hold throughout unless a constraint is invoked. We now define the generalized shear scalar by

$$\theta = \nabla_i U^i = \nabla_i \nabla^i \alpha + \frac{1}{2} \in^{ijk} \left(\nabla_i \nabla_j - \nabla_j \nabla_i\right) \beta_k = $$
$$= \nabla^2 \alpha - \frac{1}{2} \in^{ijk} R^{l}_{.kij} \beta_l + \in^{ijk} \Gamma^{l}_{[ij]} \nabla_l \beta_k\,.$$

In other words, the shear now depends on the *microspin field* generated by curvature and torsion tensors.

Meanwhile, we see that the "contravariant" components of the *local acceleration vector* will simply be given by

$$a^i = \frac{DU^i}{Dt} = \frac{\partial U^i}{\partial t} + U^k \partial_k U^i - U^k \partial_k U^i = $$
$$= \frac{\partial U^i}{\partial t}\,.$$

However, we also have

$$a_i = \frac{DU_i}{Dt} = \frac{\partial U_i}{\partial t} + \left(\nabla_k U_i + \nabla_i U_k\right) U^k,$$

for the "covariant" components.

Furthermore, we have

$$a_i = \frac{\partial U_i}{\partial t} + \left(\frac{D g_{ik}}{Dt} - \frac{\partial g_{ik}}{\partial t}\right) U^k.$$

Now, define the *local acceleration covector* through

$$\hat{a}^i = g^{ik} a_i = $$
$$= g^{ik} \frac{\partial U_k}{\partial t} + \left(\nabla_k U^i + \nabla^i U_k\right) U^k = $$
$$= \frac{\partial U^i}{\partial t} - U_k \frac{D g^{ik}}{Dt}\,,$$

such that we have

$$a^i - \hat{a}^i = U_k \frac{D g^{ik}}{Dt}\,.$$

Hence we see that the sufficient condition for the two local acceleration vectors to coincide is

$$\frac{D g_{ik}}{Dt} = 0\,.$$

In other words, in such a situation we have

$$\frac{\partial g_{ik}}{\partial t} = -\left(\nabla_k U_i + \nabla_i U_k\right).$$

In this case, a purely rotational motion is obtained only when the material motion is *rigid*, i.e., when $\frac{\partial g_{ik}}{\partial t} = 0$ or, in other words, when the condition

$$L_U g_{ik} = L_{(ik)} = \nabla_k U_i + \nabla_i U_k = 0$$

is satisfied identically. Similarly, a purely translational motion is obtained when $L_{[ik]} = 0$, which describes a potential motion, where we have $U_i = \nabla_i \alpha$. However, as we have seen, in the presence of torsion even any potential motion of this kind is still obviously path-dependent as the relations $\in^{ijk} \Gamma^{l}_{[ij]} \nabla_l \alpha \neq 0$ hold in general.

We now consider the path-dependent displacement field δ tracing a loop ℓ, say, from point P_1 to point P_2 in $\Re_4$ with components:

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \oint_{P_1 - P_2} \left(\varepsilon^{i}_{.k} + \omega^{i}_{.k} - \psi^l \Gamma^{i}_{lk}\right) d\xi^{.k}$$

Let us observe that

$$\psi^k \Gamma^{i}_{kl} = \psi^k \gamma^{i}_{A} \frac{\partial \gamma^{A}_{k}}{\partial \xi^l} = -\psi^k \gamma^{A}_{k} \frac{\partial \gamma^{i}_{A}}{\partial \xi^l} = $$
$$= -\psi^A \frac{\partial \gamma^{i}_{A}}{\partial \xi^l} = -\left(\frac{\partial}{\partial \xi^l} \left(\gamma^{i}_{A} \psi^A\right) - \gamma^{i}_{A} \frac{\partial \psi^A}{\partial \xi^l}\right) = $$
$$= \gamma^{i}_{A} \frac{\partial \psi^A}{\partial \xi^l} - \frac{\partial \psi^i}{\partial \xi^l}\,.$$

Now since $\psi^i = \delta \xi^i$, and using $\frac{\partial \delta f}{\partial \xi^i} = \delta \frac{\partial f}{\partial \xi^i}$ for an arbitrary function f, we have

$$\psi^k \Gamma^{i}_{kl} = \gamma^{i}_{A} \delta \left(\frac{\partial x^A}{\partial x^B}\right) \gamma^{B}_{l} - \delta \left(\frac{\partial \xi^i}{\partial \xi^l}\right) = 0\,,$$

and we are left with

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \oint_{P_1 - P_2} \left(\varepsilon^{i}_{.k} + \omega^{i}_{.k}\right) d\xi^k\,.$$

Assuming that the ε_{ik} are continuous, we can now derive the following relations:

$$D\psi^i = \nabla_k \psi^i d\xi^k = \frac{\partial \psi^i}{\partial \xi^k} d\xi^k\,,$$

$$\nabla_l \omega^{i}_{.k} d\xi^k = \left(\nabla_l \varepsilon^{i}_{.k} - \nabla^i \varepsilon_{lk}\right) d\xi^k = \left(\frac{\partial \varepsilon^{i}_{.k}}{\partial \xi^l} - \frac{\partial \varepsilon_{lk}}{\partial \xi_i}\right) d\xi^k\,.$$

With the help of the above relations and by direct partial integration, we then have

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \omega^{i}_{.k} \xi^k \Big|_{P_1}^{P_2} - \oint_{P_1 - P_2} \left(\nabla_l \psi^{\,i}_{.k} - \nabla_k \psi^{i}_{.l}\right) \delta \xi^k d\xi^l\,,$$

where

$$\psi^i_{.k} = \varepsilon^i_{.k} - \xi^j \left(\nabla_j \varepsilon^i_{.k} - \nabla^i \varepsilon_{jk} \right) .$$

It can be seen that

$$\nabla_l \psi^i_{.k} - \nabla_k \psi^i_{.l} = -Z^i_{.jkl} \xi^j ,$$

where we have defined another non-holonomic tensor Z with the components

$$Z^i_{.jkl} = \nabla_l \nabla_j \varepsilon^i_{.k} + \nabla_k \nabla^i \varepsilon_{jl} - \nabla_k \nabla_j \varepsilon^i_{.l} - \nabla_l \nabla^i \varepsilon_{jk} .$$

Now, the linearized components of the Riemann-Christoffel curvature tensor are given by

$$\tilde{R}_{ijkl} \overset{lin}{=} \frac{1}{2} \left(\frac{\partial^2 g_{il}}{\partial \xi^k \partial \xi^j} + \frac{\partial^2 g_{jk}}{\partial \xi^l \partial \xi^i} - \frac{\partial^2 g_{ik}}{\partial \xi^l \partial \xi^j} - \frac{\partial^2 g_{jl}}{\partial \xi^k \partial \xi^i} \right) .$$

Direct calculation gives

$$\delta_\psi \tilde{R}_{ijkl} \overset{lin}{=} \frac{1}{2} \left(\nabla_k \nabla_j \delta_\psi g_{il} + \nabla_l \nabla_i \delta_\psi g_{jk} - \nabla_l \nabla_j \delta_\psi g_{ik} - \right.$$
$$\left. - \nabla_k \nabla_i \delta_\psi g_{jl} \right) .$$

However, $\delta_\psi g_{ik} = \varepsilon_{ik}$, and hence we obtain

$$\delta_\psi \tilde{R}_{ijkl} \overset{lin}{=} \frac{1}{2} \left(\nabla_k \nabla_j \varepsilon_{il} + \nabla_l \nabla_i \varepsilon_{jk} - \nabla_l \nabla_j \varepsilon_{ik} - \right.$$
$$\left. - \nabla_k \nabla_i \varepsilon_{jl} \right) .$$

In other words,

$$Z_{ijkl} \overset{lin}{=} -2 \delta_\psi \tilde{R}_{ijkl} .$$

Obviously the Z_{ijkl} possess almost the same fundamental symmetries as the components of the Riemann-Christoffel curvature tensor, i.e., $Z_{ijkl} = - Z_{jikl} = - Z_{ijlk}$ as well as the general asymmetry $Z_{ijkl} \neq Z_{klij}$ as

$$Z_{ijkl} - Z_{klij} = \left(R^r_{.ijl} + R^r_{.jli} \right) \varepsilon_{rk} + \left(R^r_{.klj} + R^r_{.lkj} \right) \varepsilon_{ir} +$$
$$+ \left(R^r_{.jik} + R^r_{.ikj} \right) \varepsilon_{rl} + \left(R^r_{.lik} + R^r_{.kli} \right) \varepsilon_{jr} -$$
$$- 2 \left(\Gamma^r_{[jl]} \nabla_r \varepsilon_{ik} + \Gamma^r_{[ik]} \nabla_r \varepsilon_{jl} + \Gamma^r_{[kj]} \nabla_r \varepsilon_{il} + \right.$$
$$\left. + \Gamma^r_{[li]} \nabla_r \varepsilon_{jk} \right) .$$

When the tensor Z vanishes we have, of course, a set of integrable equations giving rise to the integrability condition for the components of the strain tensor, which is equivalent to the vanishing of the field δ. That is, to the first order in the components of the strain tensor, if the condition

$$\delta_\psi \tilde{R}_{ijkl} = 0$$

is satisfied identically.

Finally, we can write (still to the first order in the components of the strain tensor)

$$\delta^i = \oint_{P_1 - P_2} d\psi^i = \left(\omega^i_{.k} \xi^k \right) \Big|^{P_2}_{P_1} + \frac{1}{2} \oint_{P_1 - P_2} Z^i_{.jkl} \xi^j \, dS^{kl} =$$
$$= \left(\omega^i_{.k} \xi^k \right) \Big|^{P_2}_{P_1} - \oint_{P_1 - P_2} \delta_\psi \tilde{R}^i_{.jkl} \xi^j \, dS^{kl} ,$$

where

$$dS^{ik} = d\xi^i \delta \xi^k - d\xi^k \delta \xi^i$$

are the components of an infinitesimal closed surface in $\Im_3$ spanned by the displacements $d\xi$ and $\delta\xi$ in 2 preferred directions.

Ending this section, let us give further in-depth investigation of the local translational-rotational motion of points on the material body. Define the unit velocity vector by

$$\hat{U}^i = \frac{\zeta^i_4}{\sqrt{g_{kl} \zeta^k_4 \zeta^l_4}} = \frac{d\xi^i}{ds} ,$$

such that

$$\zeta^i_4 = \frac{\partial \xi^i}{\partial t} = \left(g_{kl} \zeta^k_4 \zeta^l_4 \right)^{1/2} \frac{d\xi^i}{ds} ,$$

i.e.,

$$ds = \left(g_{ik} \zeta^i_4 \zeta^k_4 \right)^{1/2} \frac{\partial t}{\partial \xi^l} \, d\xi^l = \left(U_i U^i \right)^{1/2} dt = U \, dt .$$

Then the local equations of motion along arbitrary curves on the hypersurface of material coordinates $\Im_3 \subset \Re_4$ can be described by the quadruplet of unit space-time vectors $(\hat{U}, \hat{V}, \hat{W}, \in n)$ orthogonal to each other where the first three unit vectors (i.e., $\hat{U}, \hat{V}, \hat{W}$) are exclusively defined as local tangent vectors in the hypersurface $\Im_3$ and n is the unit normal vector to the hypersurface $\Im_3$. These equations of motion are derived by generalizing the ordinary Frenet equations of orientable points of a curve in three-dimensional Euclidean space to four-dimensions as well as to include effects of microspin generated by geometric torsion. Setting

$$\hat{U} = u^\mu \omega_\mu = \hat{U}^i g_i ,$$
$$\hat{V} = v^\mu \omega_\mu = \hat{V}^i g_i ,$$
$$\hat{W} = w^\mu \omega_\mu = \hat{W}^i g_i ,$$
$$n = n^\mu \omega_\mu ,$$

we obtain, in general, the following set of equations of motion of the material points on the material body:

$$\frac{\delta u^\mu}{\delta s} = k v^\mu ,$$
$$\frac{\delta v^\mu}{\delta s} = \tau w^\mu - k u^\mu ,$$
$$\frac{\delta w^\mu}{\delta s} = \tau v^\mu + \lambda n^\mu ,$$

$$\frac{\delta n^{\mu}}{\delta s} = \lambda w^{\mu} \,,$$

where the operator

$$\frac{\delta}{\delta s} = \hat{U}^{i} \nabla_{i} = u^{\mu} \nabla_{\mu}$$

represents the *absolute covariant derivative* in $\Im_3 \subset \Re_4$. In the above equations we have defined the following invariants:

$$k = \left(G_{\mu\nu} \frac{\delta u^{\mu}}{\delta s} \frac{\delta u^{\nu}}{\delta s} \right)^{1/2} = \left(g_{ik} \frac{\delta \hat{U}^{i}}{\delta s} \frac{\delta \hat{U}^{k}}{\delta s} \right)^{1/2} ,$$

$$\tau = \in_{\mu\nu\rho\sigma} u^{\mu} v^{\nu} \frac{\delta v^{\rho}}{\delta s} n^{\sigma} = \in \in_{ijk} \hat{U}^{i} \hat{V}^{j} \frac{\delta \hat{V}^{k}}{\delta s} ,$$

$$\lambda = \left(G_{\mu\nu} \frac{\delta n^{\mu}}{\delta s} \frac{\delta n^{\nu}}{\delta s} \right)^{1/2} .$$

In our case, however, the vanishing of the extrinsic curvature of the hypersurface $\Im_3$ means that the direction of the unit normal vector n is fixed. Consequently, we have

$$\lambda = 0 \,,$$

and our equations of motion can be written as

$$\hat{U}^{k} \tilde{\nabla}_{k} \hat{U}^{i} = k \hat{V}^{i} - T^{i}_{kl} \hat{U}^{k} \hat{U}^{l} \,,$$

$$\hat{U}^{k} \tilde{\nabla}_{k} \hat{V}^{i} = \tau \hat{W}^{i} - k \hat{U}^{i} - T^{i}_{kl} \hat{V}^{k} \hat{U}^{l} \,,$$

$$\hat{U}^{k} \tilde{\nabla}_{k} \hat{W}^{i} = \tau \hat{V}^{i} - T^{i}_{kl} \hat{W}^{k} \hat{U}^{l}$$

in three-dimensional notation. In particular, we note that, just as the components of the contorsion tensor T^{i}_{jk}, the scalar τ measures the twist of any given curve in $\Im_3$ due to microspin.

Furthermore, it can be shown that the gradient of the unit velocity vector can be decomposed accordingly as

$$\nabla_{k} \hat{U}_{i} = \alpha_{ik} + \beta_{ik} + \frac{1}{4} h_{ik} \hat{\theta} + \hat{U}_{k} \hat{A}_{i} \,,$$

where

$$h_{ik} = g_{ik} - \hat{U}_{i} \hat{U}_{k} \,,$$

$$\alpha_{ik} = \frac{1}{4} h^{r}_{i} h^{s}_{k} \left(\nabla_{r} \hat{U}_{s} + \nabla_{s} \hat{U}_{r} \right) = \frac{1}{4} h^{r}_{i} h^{s}_{k} \left(\tilde{\nabla}_{r} \hat{U}_{s} + \tilde{\nabla}_{s} \hat{U}_{r} \right) - \frac{1}{2} h^{r}_{i} h^{s}_{k} T^{l}_{(rs)} \hat{U}_{l} \,,$$

$$\beta_{ik} = \frac{1}{4} h^{r}_{i} h^{s}_{k} \left(\nabla_{r} \hat{U}_{s} - \nabla_{s} \hat{U}_{r} \right) = \frac{1}{4} h^{r}_{i} h^{s}_{k} \left(\tilde{\nabla}_{r} \hat{U}_{s} - \tilde{\nabla}_{s} \hat{U}_{r} \right) - \frac{1}{2} h^{r}_{i} h^{s}_{k} T^{l}_{[rs]} \hat{U}_{l} \,,$$

$$\hat{\theta} = \nabla_{i} \hat{U}^{i} \,,$$

$$\hat{A}_{i} = \frac{\delta \hat{U}_{i}}{\delta s} \,.$$

Note that

$$h_{ik} \hat{U}^{k} = \alpha_{ik} \hat{U}^{k} = \beta_{ik} \hat{U}^{k} = 0 \,.$$

Setting $\bar{\lambda} = \left(g_{ik} \zeta^{i}_{4} \zeta^{k}_{4} \right)^{-1/2}$ such that $\hat{U}^{i} = \bar{\lambda} U^{i}$, we obtain in general

$$\bar{\lambda} \nabla_{k} U_{i} = \frac{1}{4} \bar{\lambda} h^{r}_{i} h^{s}_{k} \left(\nabla_{r} U_{s} + \nabla_{s} U_{r} \right) + \frac{1}{4} \bar{\lambda} h^{r}_{i} h^{s}_{k} \left(\nabla_{r} U_{s} - \nabla_{s} U_{r} \right) + \frac{1}{2} \bar{\lambda} \nabla_{i} U_{k} + \frac{1}{4} g_{ik} \frac{\delta \bar{\lambda}}{\delta s} + \frac{1}{4} \bar{\lambda} g_{ik} \nabla_{l} U^{l} + \bar{\lambda} U_{i} U_{k} \frac{\delta \bar{\lambda}}{\delta s} + \bar{\lambda}^{2} U_{k} \frac{\delta U_{i}}{\delta s} - \frac{1}{4} \bar{\lambda}^{2} U_{i} U_{k} \frac{\delta \bar{\lambda}}{\delta s} - \frac{1}{2} \bar{\lambda}^{3} U_{i} \frac{\delta U_{k}}{\delta s} - \frac{1}{4} \bar{\lambda}^{3} U_{i} U_{k} \nabla_{l} U^{l} \,.$$

Again, the vanishing of the extrinsic curvature of the hypersurface $\Im_3$ gives $\frac{\delta \bar{\lambda}}{\delta s} = 0$. Hence we have

$$\nabla_{k} U_{i} = \frac{1}{4} h^{r}_{i} h^{s}_{k} \left(\nabla_{r} U_{s} + \nabla_{s} U_{r} \right) + \frac{1}{4} h^{r}_{i} h^{s}_{k} \left(\nabla_{r} U_{s} - \nabla_{s} U_{r} \right) + \frac{1}{2} \nabla_{i} U_{k} + \frac{1}{4} g_{ik} \nabla_{l} U^{l} + \bar{\lambda} U_{k} \frac{\delta U_{i}}{\delta s} - \frac{1}{2} \bar{\lambda}^{2} U_{i} \frac{\delta U_{k}}{\delta s} - \frac{1}{4} \bar{\lambda}^{2} U_{i} U_{k} \nabla_{l} U^{l} \,,$$

for the components of the velocity gradient tensor.

Meanwhile, with the help of the identities

$$\hat{U}^{j} \nabla_{k} \nabla_{j} \hat{U}_{i} = \nabla_{k} \left(\hat{U}^{j} \nabla_{j} \hat{U}_{i} \right) - \left(\nabla_{k} \hat{U}_{j} \right) \left(\nabla^{j} \hat{U}_{i} \right) = \nabla_{k} \hat{A}_{i} - \left(\nabla_{k} \hat{U}_{j} \right) \left(\nabla^{j} \hat{U}_{i} \right) ,$$

$$\hat{U}^{j} \left(\nabla_{k} \nabla_{j} - \nabla_{j} \nabla_{k} \right) \hat{U}_{i} = R^{l}_{.ijk} \hat{U}_{l} \hat{U}^{j} - 2 \Gamma^{l}_{[jk]} \hat{U}^{j} \nabla_{l} \hat{U}_{i} \,,$$

we can derive the following equation:

$$\frac{\delta \hat{\theta}}{\delta s} = \nabla_{i} \left(\frac{\delta \hat{U}^{i}}{\delta s} \right) - \left(\nabla_{i} \hat{U}^{k} \right) \left(\nabla_{k} \hat{U}^{i} \right) - R_{ik} \hat{U}^{i} \hat{U}^{k} + 2 \Gamma^{l}_{[ik]} \hat{U}^{i} \nabla_{l} \hat{U}^{k} \,.$$

Hence we obtain

$$\frac{\delta \theta}{\delta s} = \nabla_{i} \left(\frac{\delta U^{i}}{\delta s} \right) - \bar{\lambda} \left(\nabla_{i} U^{k} \right) \left(\nabla_{k} U^{i} \right) - 2 \frac{\delta U^{i}}{\delta s} \nabla_{i} \left({}^{e} \log \lambda \right) - \bar{\lambda} R_{ik} U^{i} U^{k} + 2 \bar{\lambda} \Gamma^{l}_{[ik]} U^{i} \nabla_{l} U^{k} \,,$$

for the rate of shear with respect to the local arc length of the material body.

4 Generalized components of the elasticity tensor of the material body $\Im_3$ in the presence of microspin and geometric discontinuities (defects)

As we know, the most general form of the components of a fourth-rank isotropic tensor is given in terms of spatial coordinates by

$$I_{ABCD} = C_1 \delta_{AB} \delta_{CD} + C_2 \delta_{AC} \delta_{BD} + C_3 \delta_{AD} \delta_{BC} \,,$$

where C_1, C_2, and C_3 are constants. In the case of anisotropy, C_1, C_2, andC_3 are no longer constant but still remain invariant with respect to the change of the coordinate system. Transforming these to material coordinates, we have

$$I^{ij}_{..kl} = C_1 g^{ij} g_{kl} + C_2 \delta^i_k \delta^j_l + C_3 \delta^i_l \delta^j_k .$$

On reasonably relaxing the ordinary symmetries, we now generalize the components of the fourth-rank elasticity tensor with the addition of a geometrized part describing microspin and geometric discontinuities as follows:

$$C^{ij}_{..kl} = \alpha g^{ij} g_{kl} + \beta \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right) + \gamma \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right) ,$$

where

$$\alpha = \frac{2}{15} A^{ik}_{.i.k} - \frac{1}{15} A^{ki}_{.i.k} ,$$

$$\beta = \frac{1}{10} A^{ki}_{..ik} - \frac{1}{10} A^{ik}_{.i.k} ,$$

$$\gamma = \frac{1}{2} \eta \tilde{R} ,$$

where η is a non-zero constant, and where

$$A^{ij}_{..kl} = \alpha g^{ij} g_{kl} + \beta \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right)$$

are, of course, the components of the ordinary, non-microspin (non-micropolar) elasticity tensor obeying the symmetries $A^{ij}_{..kl} = A^{ji}_{..kl} = A^{ij}_{..lk} = A_{kl..}^{\;\;\;ij}$. Now if we define the remaining components by

$$B^{ij}_{..kl} = \frac{1}{2} \eta \left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right) \tilde{R} ,$$

with $B^{ij}_{..kl} = - B^{ji}_{..kl} = - B^{ij}_{..lk} = B_{kl..}^{\;\;\;ij}$, then we have relaxed the ordinary symmetries of the elasticity tensor. Most importantly, we note that our choice of the Ricci curvature scalar $\tilde{R}$ (rather than the more general curvature scalar R of which $\tilde{R}$ is a component) to enter our generalized elasticity tensor is meant to accommodate very general situations such that in the absence of geometric discontinuities the above equations will in general still hold. This corresponds to the fact that the existence of the Ricci curvature tensor $\tilde{R}$ is primarily due to microspin while geometric discontinuities are described by the full curvature tensor R as we have seen in Section 2.

Now with the help of the decomposition of the Riemann-Christoffel curvature tensor, we obtain

$$C^{ij}_{..kl} = \alpha g^{ij} g_{kl} + \beta \left(\delta^i_k \delta^j_l + \delta^i_l \delta^j_k\right) + \eta \left(\tilde{W}^{ij}_{..kl} + \delta^i_k \tilde{R}^j_{.l} + \right.$$
$$\left. + \delta^j_l \tilde{R}^i_{.k} - \delta^i_l \tilde{R}^j_{.k} - \delta^j_k \tilde{R}^i_{.l} - \tilde{R}^{ij}_{..kl}\right)$$

for the components of the generalized elasticity tensor. Hence for linear elastic continua/discontinua, with the help of the potential energy functional $\bar{F}$, i.e., the one given by

$$\bar{F} = \frac{1}{2} C_{ij..}^{\;\;\;kl} D^{ij} D_{kl} ,$$

such that

$$\sigma_{ij} = \frac{\partial \bar{F}}{\partial D^{ij}} ,$$

i.e.,

$$\sigma_{(ij)} = \frac{\partial \bar{F}}{\partial \varepsilon^{ij}} ,$$

$$\sigma_{[ij]} = \frac{\partial \bar{F}}{\partial \omega^{ij}} ,$$

we obtain the following constitutive relations:

$$\sigma_{ij} = C_{ij..}^{\;\;\;kl} D_{kl} ,$$

relating the components of the stress tensor σ to the components of the displacement gradient tensor D. Then it follows, as we have expected, that the stress tensor becomes asymmetric. Since $B_{(ij)..}^{\;\;\;\;\;kl} = 0$, we obtain

$$\sigma_{(ij)} = C_{(ij)..}^{\;\;\;\;\;kl} D_{kl} = A_{ij..}^{\;\;\;kl} \varepsilon_{kl} = \alpha g_{ij} \varepsilon^k_{.k} + 2\beta \varepsilon_{ij} ,$$

for the components of the symmetric part of the stress tensor, in terms of the components strain tensor and the dilation scalar $\kappa = \varepsilon^i_{.i}$. Correspondingly, since $A_{[ij]..}^{\;\;\;\;\;kl} = 0$, the components of the anti-symmetric part of the stress tensor are then given by

$$\sigma_{[ij]} = C_{[ij]..}^{\;\;\;\;\;kl} D_{kl} = B_{ij..}^{\;\;\;kl} \omega_{kl} =$$
$$= \eta \left(\tilde{W}_{ij..}^{\;\;\;kl} - \tilde{R}_{ij..}^{\;\;\;kl}\right) + \eta \left(D_{i.}^{\;k} \tilde{R}_{jk} + D^k_{.j} \tilde{R}_{ik} - \right.$$
$$\left. - D^k_{.i} \tilde{R}_{jk} - D_{j.}^{\;k} \tilde{R}_{ik}\right) =$$
$$= \eta \left(\tilde{W}_{ij..}^{\;\;\;kl} - \tilde{R}_{ij..}^{\;\;\;kl}\right) \omega_{kl} + 2\eta \left(\omega_{i.}^{\;k} \tilde{R}_{jk} - \omega_{j.}^{\;k} \tilde{R}_{ik}\right) =$$
$$= \eta \omega_{ij} \tilde{R} ,$$

in terms of the components of the generalized vorticity tensor. We can now define the *geometrized microspin potential* by the scalar

$$S = \eta \tilde{R} = \eta \left(R + 2\tilde{\nabla}_i \omega^i + \omega_i \omega^i + T_{ijk} T^{ikj}\right) .$$

Then, more specifically, we write

$$\sigma_{[ij]} = S \left(\Omega_{ij} + \varpi_{ij} + \frac{\bar{\gamma}}{e} \omega^\mu_i \omega^\nu_j F_{\mu\nu}\right) .$$

From the above relations, we see that when the electromagnetic contribution vanishes, we arrive at a *geometrized Cosserat elasticity theory*. As we know, the standard Cosserat elasticity theory does not consider effects generated by the electromagnetic field. Various continuum theories which can be described as conservative theories often take into consideration electrostatic phenomena since the electric field is simply described by a gradient of a scalar potential which corresponds to their conservative description of force and stress. But that proves to be a limitation especially because magnetic effects are still neglected.

As usual, should we consider thermal effects, then we would define the components of the thermal stress t by

$$t_{ik} = -\mu_T g_{ik} \Delta T \,,$$

where μ_T is the thermal coefficient and ΔT is the temperature increment. Hence the components of the generalized stress tensor become

$$\sigma_{ik} = C_{ik..}^{\;\;\;rs} D_{rs} - \mu_T g_{ik} \Delta T \,.$$

Setting now

$$\bar{\mu}_T = -\frac{1}{3}\frac{\mu_T}{(\alpha + 2\beta)} \,,$$

we can alternatively write

$$\sigma_{ik} = C_{ik..}^{\;\;\;rs} \left(D_{rs} + \bar{\mu}_T g_{rs} \Delta T \right) .$$

Finally, we shall obtain

$$\sigma_{ik} = \left(\alpha \varepsilon^{r}_{.r} - \mu_T \Delta T \right) g_{ik} + 2\beta \varepsilon_{ik} + \eta \omega_{ik} \tilde{R} \,,$$

i.e., as before

$$\sigma_{(ij)} = \left(\alpha \varepsilon^{r}_{.r} - \mu_T \Delta T \right) g_{ij} + 2\beta \varepsilon_{ij} \,,$$

$$\sigma_{[ij]} = \eta \omega_{ij} \tilde{R} = \frac{1}{2} \eta \in_{ijk} S^k \tilde{R} \,,$$

where S^k are the components of the generalized vorticity vector S.

We note that, as is customary, in order to accord with the standard physical description of continuum mechanics, we need to set

$$\alpha = G = \frac{E}{2\,(1+\nu)} \,,$$

$$\beta = G \left(\frac{2\nu}{1 - 2\nu} \right) ,$$

where G is the shear modulus, E is Young's modulus, and ν is Poisson's ratio.

Extending the above description, we shall have a glimpse into the more general non-linear constitutive relations given by

$$\sigma_{ij} = C_{ij..}^{\;\;\;kl} D_{kl} + K_{ij..mn}^{\;\;\;kl} D_{kl} D^{mn} + \ldots \,,$$

where the dots represent terms of higher order. Or, up to the second order in the displacement gradient tensor, we have

$$\sigma_{ij} = C_{ij..}^{\;\;\;kl} D_{kl} + K_{ij..mn}^{\;\;\;kl} D_{kl} D^{mn} \,.$$

Here the K_{ijklmn} are the components of the sixth-rank, isotropic, non-linear elasticity tensor whose most general form appears to be given by

$$\begin{aligned} K_{ijklmn} = {} & A_1 g_{ij} g_{kl} g_{mn} + A_2 g_{ij} g_{km} g_{nl} + A_3 g_{ij} g_{kn} g_{lm} + \\ & + A_4 g_{kl} g_{im} g_{jn} + A_5 g_{kl} g_{in} g_{jm} + A_6 g_{mn} g_{ik} g_{jl} + \\ & + A_7 g_{mn} g_{il} g_{jk} + A_8 g_{im} g_{jk} g_{nl} + A_9 g_{im} g_{jl} g_{kn} + \\ & + A_{10} g_{in} g_{jk} g_{lm} + A_{11} g_{in} g_{jl} g_{km} + A_{12} g_{jm} g_{ik} g_{nl} + \\ & + A_{13} g_{jm} g_{il} g_{kn} + A_{14} g_{jn} g_{ik} g_{lm} + A_{15} g_{jn} g_{il} g_{km} \,, \end{aligned}$$

where $A_1, A_2, \ldots, A_{15}$ are invariants. In a similar manner as in the generalized linear case, we shall call the following symmetries:

$$K_{ijklmn} = K_{klijmn} = K_{klmnij} = K_{mnijkl} \,.$$

Hence, we can bring the K_{ijklmn} into the form

$$\begin{aligned} K_{ijklmn} = {} & B_1 g_{ij} g_{kl} g_{mn} + B_2 g_{ij} \left(g_{km} g_{nl} + g_{kn} g_{lm} \right) + \\ & + B_3 g_{ij} \left(g_{km} g_{nl} - g_{kn} g_{lm} \right) + B_4 g_{kl} \left(g_{im} g_{jn} + \right. \\ & \left. + g_{in} g_{jm} \right) + B_5 g_{kl} \left(g_{im} g_{jn} - g_{in} g_{jm} \right) + \\ & + B_6 g_{mn} \left(g_{ik} g_{jl} + g_{il} g_{jk} \right) + B_7 g_{mn} \left(g_{ik} g_{jl} - \right. \\ & \left. - g_{il} g_{jk} \right) + B_8 g_{im} \left(g_{jk} g_{nl} + g_{jl} g_{kn} \right) + \\ & + B_9 g_{im} \left(g_{jk} g_{nl} - g_{jl} g_{kn} \right) + B_{10} g_{jm} \left(g_{ik} g_{nl} + \right. \\ & \left. + g_{il} g_{kn} \right) + B_{11} g_{jm} \left(g_{ik} g_{nl} - g_{il} g_{kn} \right) , \end{aligned}$$

where, again, $B_1, B_2, \ldots, B_{11}$ are invariants. As in the generalized linear case, relating the coefficients B_3, B_5, B_7, B_9, and B_{11} to the *generator of microspin* in our theory, i.e., the Riemann-Christoffel curvature tensor, we obtain

$$\begin{aligned} K_{ijklmn} = {} & \lambda_1 g_{ij} g_{kl} g_{mn} + \lambda_2 g_{ij} \left(g_{km} g_{nl} + g_{kn} g_{lm} \right) + \\ & + \lambda_3 g_{kl} \left(g_{im} g_{jn} + g_{in} g_{jm} \right) + \lambda_4 g_{mn} \left(g_{ik} g_{jl} + \right. \\ & \left. + g_{il} g_{jk} \right) + \lambda_5 g_{im} \left(g_{jk} g_{nl} + g_{jl} g_{kn} \right) + \\ & + \lambda_6 g_{jm} \left(g_{ik} g_{nl} + g_{il} g_{kn} \right) + \frac{1}{2} \kappa_1 g_{ij} \left(g_{km} g_{nl} - \right. \\ & \left. - g_{kn} g_{lm} \right) \tilde{R} + \frac{1}{2} \kappa_2 g_{kl} \left(g_{im} g_{jn} - g_{in} g_{jm} \right) \tilde{R} + \\ & + \frac{1}{2} \kappa_3 g_{mn} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) \tilde{R} + \frac{1}{2} \kappa_4 g_{im} \left(g_{jk} g_{nl} - \right. \\ & \left. - g_{jl} g_{kn} \right) \tilde{R} + \frac{1}{2} \kappa_5 g_{jm} \left(g_{ik} g_{nl} - g_{il} g_{kn} \right) \tilde{R} \,, \end{aligned}$$

where we have set $B_1 = \lambda_1$, $B_2 = \lambda_2$, $B_4 = \lambda_3$, $B_6 = \lambda_4$, $B_8 = \lambda_5$, $B_{10} = \lambda_6$ and where, for constant $\kappa_1, \kappa_2, \ldots, \kappa_5$, the five quantities

$$K_1 = B_3 = \frac{1}{2} \kappa_1 \tilde{R} \,,$$

$$K_2 = B_5 = \frac{1}{2} \kappa_2 \tilde{R} \,,$$

$$K_3 = B_7 = \frac{1}{2} \kappa_3 \tilde{R} \,,$$

$$K_4 = B_9 = \frac{1}{2} \kappa_4 \tilde{R} \,,$$

$$K_5 = B_{11} = \frac{1}{2} \kappa_5 \tilde{R}$$

form a set of *additional microspin potentials*. Hence we see that in the non-linear case, at least there are in general six microspin potentials instead of just one as in the linear case.

Then the constitutive equations are readily derivable by means of the third-order potential functional

$$^{*}\bar{F} = \frac{1}{2} C_{ij..}^{\;\;\;kl} D^{ij} D_{kl} + \frac{1}{3} K_{ij..mn}^{\;\;\;kl} D^{ij} D_{kl} D^{mn}$$

through

$$\sigma_{ij} = \frac{\partial\, ^{*}\bar{F}}{\partial D^{ij}} = {}^{(1)}\sigma_{ij} + {}^{(2)}\sigma_{ij} \,,$$

where (1) indicates the linear part and (2) indicates the non-linear part. Note that this is true whenever the K_{ijklmn} in general possess the above mentioned symmetries. Direct, but somewhat lengthy, calculation gives

$$\begin{aligned} {}^{(2)}\sigma_{ij} &= K_{ij..mn}^{\;\;\;kl} D_{kl} D^{mn} = \\ &= \left(\lambda_1 \left(\varepsilon^{k}_{.k}\right)^2 + 2\lambda_2 \varepsilon_{kl} \varepsilon^{kl} + \kappa_1 \omega_{kl} \omega^{kl} \tilde{R}\right) g_{ij} + \\ &+ \varepsilon^{k}_{.k} \left(\left(2\lambda_3 + 2\lambda_4\right) \varepsilon_{ij} + \left(\kappa_2 + \kappa_3\right) \omega_{ij} \tilde{R}\right) + \\ &+ D^{k}_{i.} \left(2\lambda_5 \varepsilon_{jk} + \kappa_4 \omega_{jk} \tilde{R}\right) + D^{k}_{j.} \left(2\lambda_6 \varepsilon_{ik} + \kappa_5 \omega_{ik} \tilde{R}\right). \end{aligned}$$

Overall, we obtain, for the components of the stress tensor, the following:

$$\begin{aligned} \sigma_{ij} &= \left(\alpha \varepsilon^{k}_{.k} - \mu_T \Delta T\right) g_{ij} + 2\beta \varepsilon_{ij} + \eta \omega_{ij} \tilde{R} + \\ &+ \left(\lambda_1 \left(\varepsilon^{k}_{.k}\right)^2 + 2\lambda_2 \varepsilon_{kl} \varepsilon^{kl} + \kappa_1 \omega_{kl} \omega^{kl} \tilde{R}\right) g_{ij} + \\ &+ \varepsilon^{k}_{.k} \left(\left(2\lambda_3 + 2\lambda_4\right) \varepsilon_{ij} + \left(\kappa_2 + \kappa_3\right) \omega_{ij} \tilde{R}\right) + \\ &+ D^{k}_{i.} \left(2\lambda_5 \varepsilon_{jk} + \kappa_4 \omega_{jk} \tilde{R}\right) + D^{k}_{j.} \left(2\lambda_6 \varepsilon_{ik} + \kappa_5 \omega_{ik} \tilde{R}\right). \end{aligned}$$

5 Variational derivation of the field equations. Equations of motion

We shall now see that our theory can best be described, in the linear case, independently by 2 Lagrangian densities. We give the first Lagrangian density as

$$\begin{aligned} \bar{L} = \sqrt{g}\Big(&\sigma^{ik} \left(\nabla_k \psi_i - D_{ik}\right) + \frac{1}{2} C^{ij}_{..kl} D_{ij} D^{kl} - \\ &- \mu_T D^{i}_{.i} \Delta T + U^i \left(\nabla_i \psi_k\right) \left(f \psi^k - \rho_m U^k\right)\Big), \end{aligned}$$

where ρ_m is the material density and f is a scalar potential. From here we then arrive at the following invariant integral:

$$\begin{aligned} I = \int_{vol} \Big(&\sigma^{ik} \left(\nabla_{(k} \psi_{i)} - \varepsilon_{ik}\right) + \sigma^{ik} \left(\nabla_{[k} \psi_{i]} - \omega_{ik}\right) + \\ &+ \frac{1}{2} A^{ij}_{..kl} \varepsilon_{ij} \varepsilon^{kl} + \frac{1}{2} B^{ij}_{..kl} \omega_{ij} \omega^{kl} - \mu_T \varepsilon^{i}_{.i} \Delta T + \\ &+ U^i \left(\nabla_i \psi_k\right) \left(f \psi^k - \rho_m U^k\right)\Big)\, dV \,, \end{aligned}$$

where $dV = \sqrt{g}\, d\xi^1 d\xi^2 d\xi^3$.

Writing $\bar{L} = \sqrt{g}\, L$, we then have

$$\begin{aligned} \delta I = \int_{vol} \Big(&\frac{\partial L}{\partial \sigma^{ik}} \delta\sigma^{ik} + \frac{\partial L}{\partial \varepsilon^{ik}} \delta\varepsilon^{ik} + \frac{\partial L}{\partial \omega^{ik}} \delta\omega^{ik} + \\ &+ \frac{\partial L}{\partial \left(\nabla_i \psi_k\right)} \delta\left(\nabla_i \psi_k\right)\Big)\, dV = 0\,. \end{aligned}$$

Now

$$\begin{aligned} \int_{vol} \frac{\partial L}{\partial \left(\nabla_i \psi_k\right)} \delta\left(\nabla_i \psi_k\right) dV &= \int_{vol} \nabla_i \left(\frac{\partial L}{\partial \left(\nabla_i \psi_k\right)} \delta\psi_k\right) dV - \\ &- \int_{vol} \nabla_i \left(\frac{\partial L}{\partial \left(\nabla_i \psi_k\right)}\right) \delta\psi_k\, dV = \\ &= - \int_{vol} \nabla_i \left(\frac{\partial L}{\partial \left(\nabla_i \psi_k\right)}\right) \delta\psi_k\, dV, \end{aligned}$$

since the first term on the right-hand-side of the first line is an absolute differential that can be transformed away on the boundary of integration by means of the divergence theorem. Hence we have

$$\begin{aligned} \delta I = \int_{vol} \Big(&\frac{\partial L}{\partial \sigma^{ik}} \delta\sigma^{ik} + \frac{\partial L}{\partial \varepsilon^{ik}} \delta\varepsilon^{ik} + \frac{\partial L}{\partial \omega^{ik}} \delta\omega^{ik} - \\ &- \nabla_i \left(\frac{\partial L}{\partial \left(\nabla_i \psi_k\right)}\right) \delta\psi_k\Big)\, dV = 0\,, \end{aligned}$$

where each term in the integrand is independent of the others. Note also that the variations $\delta\sigma^{ik}$, $\delta\varepsilon^{ik}$, $\delta\omega^{ik}$, and$\delta\psi_k$ are arbitrary.

From $\frac{\partial L}{\partial \sigma^{ik}} = 0$, we obtain

$$\varepsilon_{ik} = \nabla_{(k}\, \psi_{i)} \,,$$

$$\omega_{ik} = \nabla_{[k}\, \psi_{i]} \,,$$

i.e., the components of the strain and vorticity tensors, respectively.

From $\frac{\partial L}{\partial \varepsilon^{ik}} = 0$, we obtain

$$\sigma^{(ik)} = A^{ik}_{..rs} \varepsilon^{rs} - \mu_T g^{ik} \Delta T \,,$$

i.e., the symmetric components of the stress tensor.

From $\frac{\partial L}{\partial \omega^{ik}} = 0$, we obtain

$$\sigma^{[ik]} = B^{ik}_{..rs} \omega^{rs} = \eta \omega^{ik} \tilde{R} \,,$$

i.e., the anti-symmetric components of the stress tensor.

Finally, from the fourth variation we now show in detail that it yields the equations of motion. We first see that

$$\frac{\partial L}{\partial \left(\nabla_i \psi_k\right)} = \sigma^{ik} + U^i \left(f \psi^k - \rho_m U^k\right) \,.$$

Hence

$$\begin{aligned} \nabla_i \left(\frac{\partial L}{\partial \left(\nabla_i \psi_k\right)}\right) &= \nabla_i \sigma^{ik} + \nabla_i \left(f U^i\right) \psi^k + f U^i \nabla_i \psi^k - \\ &- \nabla_i \left(\rho_m U^i\right) U^k - \rho_m U^i \nabla_i U^k \,. \end{aligned}$$

Define the "extended" shear scalar and the mass current density vector, respectively, through

$$l = \nabla_i \left(f U^i \right) ,$$

$$J^i = \rho_m U^i .$$

Now we readily identify the force per unit mass f and the body force per unit mass b, respectively, by

$$f^i = U^k \nabla_k U^i = \frac{\delta U^i}{\delta t} ,$$

$$b^i = \frac{1}{\rho_m} \left(l \psi^i + f \left(1 - \nabla_k J^k \right) U^i \right) = \\ = \frac{1}{\rho_m} \left(l \psi^i + f \left(1 + \frac{\partial \rho_m}{\partial t} \right) U^i \right) ,$$

where we have used the relation

$$\frac{D \rho_m}{D t} = - \rho_m \nabla_i U^i ,$$

i.e., $\frac{\partial \rho_m}{\partial t} + \nabla_i \left(\rho_m U^i \right) = 0$, derivable from the four-dimensional conservation law $\nabla_\mu \left(\rho_m \, {}^*U^\mu \right) = 0$ where ${}^*U^\mu = \left(U^i, c \right)$.

Hence we have (for arbitrary $\delta \psi_k$)

$$\int_{vol} \left(\nabla_i \sigma^{ik} + \rho_m b^k - \rho_m f^k \right) \delta \psi_k \, dV = 0 ,$$

i.e., the equations of motion

$$\nabla_i \sigma^{ik} = \rho_m \left(f^k - b^k \right) .$$

Before we move on to the second Lagrangian density, let's discuss briefly the so-called couple stress, i.e., the couple per unit area also known as the distributed moment. We denote the couple stress tensor by the second-rank tensor field M. In analogy to the linear constitutive relations relating the stress tensor σ to displacement gradient tensor D, we write

$$M_{ik} = D_{ik..}^{\;\;\;rs} N_{rs} ,$$

where

$$D_{ijkl} = E_{ijkl} + F_{ijkl}$$

are assumed to possess the same symmetry properties as C_{ijkl} (i.e., E_{ijkl} have the same symmetry properties as A_{ijkl} while F_{ijkl}, representing the chirality part, have the same symmetry properties as B_{ijkl}).

Likewise,

$$N_{ik} = N_{(ik)} + N_{[ik]} = X_{ik} + Y_{ik}$$

are comparable to $D_{ik} = D_{(ik)} + D_{[ik]} = \varepsilon_{ik} + \omega_{ik}$.

As a boundary condition, let us now define a completely anti-symmetric third-rank spin tensor as follows:

$$J^{ikl} = J^{[ikl]} = \frac{1}{2} \in^{ikl} \psi ,$$

where ψ is a scalar function such that the spin tensor of our theory (which contains both the macrospin and microspin tensors) can be written as a gradient, i.e.,

$$S_i = \eta \in_{ijk} \tilde{R} \omega^{jk} = \nabla_i \psi ,$$

such that whenever we desire to subject the above to the integrability condition $\in_{ijk} \nabla^j S^k = 0$, we have $\in^{ijk} \Gamma^l_{[jk]} S_l = 0$, resulting in $Y_{ik} = 0$.

In other words,

$$\psi = \psi_0 + \eta \int \in_{ijk} \tilde{R} \omega^{ij} d\xi^k ,$$

where ψ_0 is constant, acts as a scalar generator of spin.

As a consequence, we see that

$$\nabla_l J^{ikl} = \frac{1}{2} \in^{ikl} \nabla_l \psi = \frac{1}{2} \in^{ikl} S_l = \\ = \frac{1}{2} \eta \tilde{R} \in^{ikl} \in_{pql} \omega^{pq} \\ = \frac{1}{2} \eta \tilde{R} \left(\delta^i_p \delta^k_q - \delta^i_q \delta^k_p \right) \omega^{pq} \\ = \eta \tilde{R} \omega^{ik} ,$$

i.e.,

$$\nabla_l J^{ikl} = \sigma^{[ik]} .$$

Taking the divergence of the above equations and using the relations $2 \nabla_{[k} \nabla_{i]} \psi = -2 \Gamma^l_{[ik]} \nabla_l \psi$, we obtain the following divergence equations:

$$\nabla_k \sigma^{[ik]} = \frac{1}{2} \in^{ikl} \Gamma^r_{[kl]} S_r ,$$

coupling the components of the spin vector to the components of the torsion tensor. Furthermore, we obtain

$$\nabla_k \omega^{ik} = \frac{1}{2} \left(\eta \tilde{R} \right) \in^{ikl} \Gamma^r_{[kl]} S_r - \omega^{ik} \frac{\partial^{\,e} \log \left(\eta \tilde{R} \right)}{\partial \xi^k} .$$

We now form the second Lagrangian density of our theory as

$$\bar{H} = \sqrt{g} \Big(M^{ik} \left(\nabla_k S_i - N_{ik} \right) + \frac{1}{2} D^{ij}_{..kl} N_{ij} N^{kl} - \\ - \in^k_{.rs} \left(\nabla_i S_k \right) J^{rsi} + U^i \nabla_i S_k \left(h S^k - I \rho_m V^k \right) \Big),$$

where h is a scalar function (not to be confused with the scalar function f), I is the moment of inertia, and V^i are the components of the angular velocity field.

Hence the action integral corresponding to this is

$$J = \int_{vol} \Big(M^{ik} \left(\nabla_{(k} S_{i)} - X_{ik} \right) + M^{ik} \left(\nabla_{[k} S_{i]} - Y_{ik} \right) + \\ + \frac{1}{2} E^{ij}_{..kl} X_{ij} X^{kl} + \frac{1}{2} F^{ij}_{..kl} Y_{ij} Y^{kl} - \in^k_{.rs} \left(\nabla_i S_k \right) J^{rsi} + \\ + U^i \left(\nabla_i S_k \right) \left(h S^k - I \rho_m V^k \right) \Big) dV .$$

As before, writing $\bar{H} = \sqrt{g}H$ and performing the variation $\delta J = 0$, we have

$$\delta J = \int_{vol} \left(\frac{\partial H}{\partial M^{ik}}\, \delta M^{ik} + \frac{\partial H}{\partial X^{ik}}\, \delta X^{ik} + \frac{\partial H}{\partial Y^{ik}}\, \delta Y^{ik} - \right.$$
$$\left. - \nabla_i \left(\frac{\partial H}{\partial \left(\nabla_i S_k \right)} \right) \delta S_k \right) dV = 0\,,$$

with arbitrary variations δM^{ik}, δX^{ik}, δY^{ik}, and δS_k.

From $\frac{\partial H}{\partial M^{ik}} = 0$, we obtain

$$X_{ik} = \nabla_{(k}\, S_{i)}\,,$$
$$Y_{ik} = \nabla_{[k}\, S_{i]}\,.$$

From $\frac{\partial H}{\partial X^{ik}} = 0$, we obtain

$$M^{(ik)} = E^{ik}_{..rs} X^{rs}\,.$$

From $\frac{\partial H}{\partial Y^{ik}} = 0$, we obtain

$$M^{[ik]} = F^{ik}_{..rs} Y^{rs}\,.$$

Again, we shall investigate the last variation

$$- \int_{vol} \nabla_i \left(\frac{\partial H}{\partial \left(\nabla_i S_k \right)} \right) \delta S_k dV = 0$$

in detail.

Firstly,

$$\frac{\partial H}{\partial \left(\nabla_i S_k \right)} = M^{ik} - \in^{k}_{.rs}\, J^{rsi} + U^i \left(hS^k - I\rho V^k \right)\,.$$

Then we see that

$$\nabla_i \left(\frac{\partial H}{\partial \left(\nabla_i S_k \right)} \right) = \nabla_i M^{ik} - \in{}^{k}_{.rs} \sigma^{[rs]} + \nabla_i \left(hU^i \right) S^k +$$
$$+ hU^i \nabla_i S^k - I \nabla_i \left(\rho_m U^i \right) V^k -$$
$$- I \rho_m U^i \nabla_i V^k\,.$$

We now define the angular force per unit mass α by

$$\alpha^i = U^k \nabla_k V^i = \frac{\delta V^i}{\delta t}\,,$$

and the angular body force per unit mass β by

$$\beta^i = \frac{1}{\rho_m} \left(\bar{l} S^i + h \frac{\delta S^i}{\delta t} - I \left(\nabla_k J^k \right) V^i \right)\,,$$

where $\bar{l} = \nabla_i \left(h\; U^i \right)$.

We have

$$\int_{vol} \left(\nabla_i M^{ik} - \in^{k}_{.rs}\; \sigma^{[rs]} + \rho_m \beta^k - I \rho_m \alpha^k \right) \delta S_k dV = 0\,.$$

Hence we obtain the equations of motion

$$\nabla_i M^{ik} = \in^{k}_{.rs}\; \sigma^{[rs]} + \rho_m \left(I \alpha^k - \beta^k \right)\,.$$

6 Concluding remarks

At this point we see that we have reproduced the field equations and the equations of motion of Cosserat elasticity theory by our variational method, and hence we have succeeded in showing parallels between the fundamental equations of Cosserat elasticity theory and those of our present theory. However we must again emphasize that our field equations as well as our equations of motion involving chirality are *fully geometrized*. In other words, we have succeeded in generalizing various extensions of the classical elasticity theory, especially the Cosserat theory and the so-called void elasticity theory by ascribing both microspin phenomena and geometric defects to the action of geometric torsion and to the source of local curvature of the material space. As we have seen, it is precisely this curvature that plays the role of a fundamental, intrinsic differential invariant which explains microspin and defects throughout the course of our work.

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Geodetic Precession of the Spin in a Non-Singular Gravitational Potential

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Using a non-singular gravitational potential which appears in the literature we analytically derived and investigated the equations describing the precession of a body's spin orbiting around a main spherical body of mass M. The calculation has been performed using a non-exact *Schwarzschild* solution, and further assuming that the gravitational field of the Earth is more than that of a rotating mass. General theory of relativity predicts that the direction of the gyroscope will change at a rate of 6.6 arcsec/year for a gyroscope in a 650 km high polar orbit. In our case a precession rate of the spin of a very similar magnitude to that predicted by general relativity was calculated resulting to a $\frac{\Delta S_{geo}}{S_{geo}} = -5.570{\times}10^{-2}$.

1 Introduction

A new non-singular gravitational potential appears in the literature that has the following form (Williams [1])

$$V(r) = -\frac{GM}{r}\, e^{-\frac{\lambda}{r}}\,, \tag{1}$$

where the constant λ appearing in the potential above is defined as follows:

$$\lambda = \frac{GM}{c^2} = \frac{R_{grav}}{2}\,, \tag{2}$$

and G is the Newtonian gravitational constant, M is the mass of the main body that produces the potential, and c is the speed of light. In this paper we wish to investigate the differences that might exist in the results

2 Geodetic precession

One of the characteristics of curved space is that parallel transport of a vector alters its direction, which suggests that we can probably detect the curvature of the space-time near the Earth by actually examining parallel transport. From non gravitational physics we know that if a gyroscope is suspended in frictionless gimbals the result is a parallel transport of its spin direction, which does not help draw any valuable conclusion immediately. Similarly in gravitational physics the transport of such gyroscope will also result in parallel transport of the spin. To find the conditions under which parallel transport of gyroscope can happen, we start with Newton's equation of motion for the spin of a rigid body. A rigid body in a gravitational field is subject to a tidal torque that results to a spin rate of change given by [2]:

$$\frac{dS^n}{dt} = \varepsilon^{kln} R_{k0s0}\left(-I^s_l + \frac{1}{3}\,\delta^s_l I^r_r\right)\,, \tag{3}$$

where $n, k, l, s, r = 1, 2, 3$. Here R_{k0s0} is the Riemann tensor evaluated in the rest frame of the gyroscope, the presence of which signifies that this particular equation of motion does not obey the principle of minimal coupling, and that the gyroscope spin transport does not imitate parallel transport [1] and the quantity ε^{kqp} is defined as follows $\varepsilon^{123} = \varepsilon^{231} = \varepsilon^{312} = 1$ and $\varepsilon^{321} = \varepsilon^{213} = \varepsilon^{132} = -1$. For a spherical gyroscope we have that $I^s_l \propto \delta^s_l$, then the tidal torque in the equation (3) becomes zero and the equation reads $\frac{dS^n}{dt} = 0$. I^s_l is the moment of inertia tensor defined in the equation below:

$$I^s_l = \int \left(r^2\delta^s_l - x^s x_l\right) dM\,, \tag{4}$$

where δ^s_l is the Kronecker delta. This Newtonian equation remains in tact when we are in curved spacetime, and in a reference frame that freely falls along a geodesic line. Thus the Newtonian time t must now interpreted as the proper time τ measured along the geodesic. In the freely falling reference frame the spin of the gyroscope remains constant in magnitude and direction, which means that it moves by parallel transport.

If now an extra non gravitational force acts on the gyroscope and as a result the gyroscope moves into a world line that is different from a geodesic, then we can not simply introduce local geodesic coordinates at every point on of this world line which makes the equation of motion for the spin $\frac{dS^n}{dt} \neq 0$. In flat space-time the precession of an accelerated gyroscope is called *Thomas Precession*. In a general coordinate system the spin vector in parallel obeys the equation:

$$\frac{dS^\mu}{d\tau} = -\Gamma^\mu_{\nu\lambda} S^\nu \frac{dx^\lambda}{d\tau} = -\Gamma^\mu_{\nu\lambda} S^\nu \dot{x}^\lambda = -\Gamma^\mu_{\nu\lambda} S^\nu v^\lambda\,, \tag{5}$$

where $\Gamma^\mu_{\nu\lambda}$ are the Christoffel symbols of the second kind, and S^μ are the spin vector components (here $\mu, \nu, \lambda = 0, 1, 2, 3$).

Alternative theories of gravitation have also been proposed that predict different magnitudes for this effect [3, 4].

3 Gyroscope in orbit

In order to examine the effect of the new non singular gravitational potential has on the gyroscope let us assume a gyroscope in a circular orbit of radius r around the Earth. In real life somebody measures the change of the gyroscope spin relative to the fixed stars, which is also equivalent of finding this change with respect to a fixed coordinate system at infinity. We can use Cartesian coordinates since they are more convenient in calculating this change of spin direction than polar coordinates. The reason for this is that in Cartesian coordinates any change of the spin can be directly related to the curvature of the space-time, where in polar coordinates there is a contribution from both coordinate curvature and curvature of the space-time [2].

Next let us in a similar way to that of *linear theory* and following Ohanian and Ruffini [2] we write the line element ds^2 in the following way:

$$ds^2 \cong c^2\left(1-\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)dt^2 - \left(1-\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)^{-1}\left(dx^2+dy^2+dz^2\right) \tag{6}$$

further assume that our gyroscope is in orbit around the Earth and let the orbit be located in the $x-y$ plane as shown in Figure 1.

In a circular orbit all points are equivalent and if we know the rate of the spin change at one point we can calculate the rate of change of the spin at any point. For that let us write the line interval in the following way:

$$ds^2 \cong c^2\left(1-\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)dt^2 - \left(1+\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)\left(dx^2+dy^2+dz^2\right), \tag{7}$$

which implies that:

$$g_{00} = \left(1-\frac{2GM}{rc^2}\,e^{-\lambda/r}\right), \qquad g_{11}=g_{22}=g_{33} = -\left(1+\frac{2GM}{rc^2}\,e^{-\lambda/r}\right). \tag{8}$$

4 The spin components

To evaluate the spatial components of the spin we will use equation (5), and the right hand symbols must be calculated. For that we need the four-velocity $v^\beta \approx (v_t,\, v_x,\, v_y,\, v_z) = (1,\, 0,\, v,\, 0)$. We also need the S^0 component of the spin, and for that we note that in the rest frame of the gyroscope

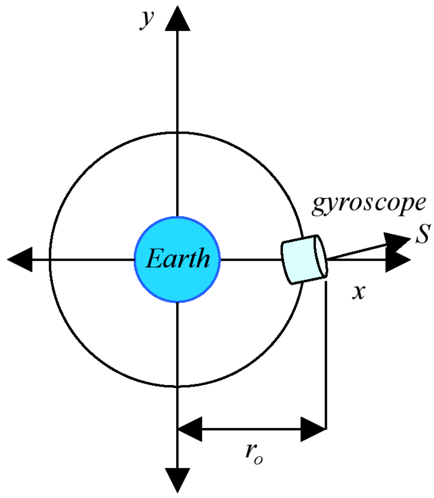

Fig. 1: A gyroscope above a satellite orbiting the Earth, and where its orbital plane coincides with the $x-y$ plane, having at an instant coordinates $x=r_0$, $y=z=0$, and where S is the spin vector of the gyroscope.

$S'^0=0$ and $v'^\beta=(1,\,0,\,0,\,0)$ and therefore $g'_{\mu\nu}S'^\mu v'^\nu=0$, and also in our coordinate system we will also have that $g_{\mu\nu}S^\mu v^\nu=0$, using the latter we have that:

$$S^0 = -\frac{1}{g_{00}}\left[S^1 g_{11}\frac{dx^1}{d\tau}+S^2 g_{22}\frac{dx^2}{d\tau}+S^3 g_{33}\frac{dx^3}{d\tau}\right], \qquad S^0 = -\frac{1}{g_{00}}\left[S_x g_{11}\frac{dx}{d\tau}+S_y g_{22}\frac{dy}{d\tau}+S_z g_{33}\frac{dz}{d\tau}\right], \tag{9}$$

substituting for the metric coefficients we obtain:

$$S^0 = \frac{\left(1+\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)}{\left(1-\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)}\,vS_y \cong \left(1+\frac{2GM}{rc^2}\,e^{-\lambda/r}\right)^2 vS_y\,. \tag{10}$$

Next letting $\mu=1$ and summing over $\nu=0,1,2,3$ the component of the spin equation becomes:

$$\frac{dS^1}{d\tau} = -\Gamma^1_{0\lambda}S^0v^\lambda - \Gamma^1_{1\lambda}S^1v^\lambda - \Gamma^1_{2\lambda}S^2v^\lambda - \Gamma^1_{3\lambda}S^3v^\lambda\,, \tag{11}$$

summing over $\lambda=0,1,2,3$ again we obtain:

$$\begin{aligned}\frac{dS^1}{d\tau} = &-\Gamma^1_{00}S^0v^0-\Gamma^1_{01}S^0v^1-\Gamma^1_{02}S^0v^2-\Gamma^1_{03}S^0v^3-\\ &-\Gamma^1_{10}S^1v^0-\Gamma^1_{11}S^1v^1-\Gamma^1_{12}S^1v^2-\Gamma^1_{13}S^1v^3-\\ &-\Gamma^1_{20}S^2v^0-\Gamma^1_{21}S^2v^1-\Gamma^1_{22}S^2v^2-\Gamma^1_{23}S^2v^3-\\ &-\Gamma^1_{30}S^3v^0-\Gamma^1_{31}S^3v^1-\Gamma^1_{32}S^3v^2-\Gamma^1_{33}S^3v^3.\end{aligned} \tag{12}$$

Next we will calculate the Cristoffel symbols of the second kind for that we use:

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}\,g^{\sigma\lambda}\left(\frac{\partial g_{\mu\sigma}}{\partial x^\nu}+\frac{\partial g_{\nu\sigma}}{\partial x^\mu}-\frac{\partial g_{\mu\nu}}{\partial x^\sigma}\right). \tag{13}$$

Since $\Gamma^{\sigma}_{\mu\nu} = 0$ if $\mu \neq \nu \neq \sigma$ equation (12) further simplifies to:

$$\begin{aligned} \frac{dS^1}{d\tau} &= -\Gamma^1_{00} S^0 v^0 - \Gamma^1_{01} S^0 v^1 - \Gamma^1_{10} S^1 v^0 - \\ &- \Gamma^1_{11} S^1 v^1 - \Gamma^1_{12} S^1 v^2 - \Gamma^1_{13} S^1 v^3 - \\ &- \Gamma^1_{21} S^2 v^1 - \Gamma^1_{22} S^2 v^2 - \Gamma^1_{31} S^3 v^1 - \Gamma^1_{33} S^3 v^3 . \end{aligned} \tag{14}$$

The only non-zero Christoffel symbols calculated at $r = r_0$ are:

$$\Gamma^0_{01} = \Gamma^0_{10} = \frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\lambda/r_0}}{\left(1 - \frac{2GM}{r_0 c^2}\, e^{-\lambda/r_0}\right)} , \tag{15}$$

$$\Gamma^1_{00} = \frac{GM}{c^2 r_0^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\lambda/r_0}}{\left(1 - \frac{2GM}{r_0 c^2}\, e^{-\lambda/r_0}\right)} , \tag{16}$$

$$\Gamma^1_{11} = -\frac{GM}{c^2 r_0^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\lambda/r_0}}{\left(1 - \frac{2GM}{r_0 c^2}\, e^{-\lambda/r_0}\right)} , \tag{17}$$

$$\Gamma^1_{22} = \frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}}{\left(1 + \frac{2GM}{r_0 c^2}\, e^{-\frac{\lambda}{r_0}}\right)} , \tag{18}$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}}{\left(1 + \frac{2GM}{r_0 c^2}\, e^{-\frac{\lambda}{r_0}}\right)} , \tag{19}$$

$$\Gamma^1_{33} = \frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}}{\left(1 + \frac{2GM}{r_0 c^2}\, e^{-\frac{\lambda}{r_0}}\right)} , \tag{20}$$

$$\Gamma^3_{13} = \Gamma^3_{31} = -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}}{\left(1 + \frac{2GM}{r_0 c^2}\, e^{-\frac{\lambda}{r_0}}\right)} . \tag{21}$$

Thus equation (14) further becomes:

$$\frac{dS_x}{d\tau} = -\Gamma^1_{00} S^0 - \Gamma^1_{22} S^2 v^2 , \tag{22}$$

substituting we obtain:

$$\begin{aligned} \frac{dS_x}{d\tau} &= -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}}{\left(1 + \frac{2GM}{r_0 c^2} e^{-\frac{\lambda}{r_0}}\right)} \times \\ &\times \left[1 + \left(1 + \frac{2GM}{r_0 c^2}\, e^{-\frac{\lambda}{r_0}}\right)^2\right] v S_y . \end{aligned} \tag{23}$$

Expanding in powers of $\frac{\lambda}{r}$ to first order we can rewrite (23) as follows:

$$\begin{aligned} \frac{dS_x}{d\tau} &= -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r_0}\right)^2}{\left(1 + \frac{2GM}{r_0 c^2}\left(1 - \frac{\lambda}{r_0}\right)\right)} \times \\ &\times \left[1 + \left(1 + \frac{2GM}{r_0 c^2}\left(1 - \frac{\lambda}{r_0}\right)\right)^2\right] v S_y , \end{aligned} \tag{24}$$

$$\begin{aligned} \frac{dS_x}{d\tau} &\cong -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{2\lambda}{r_0}\right)}{\left(1 + \frac{2GM}{r_0 c^2}\left(1 - \frac{\lambda}{r_0}\right)\right)} \times \\ &\times \left[1 + \left(1 + \frac{2GM}{r_0 c^2}\left(1 - \frac{\lambda}{r_0}\right)\right)^2\right] v S_y , \end{aligned} \tag{25}$$

keeping only $\frac{1}{c^2}$ terms and omitting the rest higher powers $\frac{G^2 M^2}{c^4}$ equation (25) can be simplified to:

$$\frac{dS_x}{d\tau} \cong -\frac{2GM}{r_0^2 c^2}\left(1 - \frac{2\lambda}{r_0}\right) v S_y . \tag{26}$$

Similarly the equation for the S_Y component of the spin becomes:

$$\frac{dS_y}{d\tau} = -\Gamma^2_{12} v S_x - \Gamma^2_{20} S_y - \Gamma^2_{22} v S_y - \Gamma^2_{32} v S_y , \tag{27}$$

which becomes:

$$\frac{dS_y}{d\tau} = -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{\lambda}{r}\right) e^{-\frac{\lambda}{r_0}}}{\left(1 + \frac{2GM}{r_0 c^2} e^{-\frac{\lambda}{r_0}}\right)} , \tag{28}$$

can be approximated to:

$$\begin{aligned} \frac{dS_y}{d\tau} &= -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{2\lambda}{r_0}\right)}{\left(1 + \frac{2GM}{r_0 c^2}\left(1 - \frac{\lambda}{r_0}\right)\right)} = \\ &= -\frac{GM}{r_0^2 c^2} \frac{\left(1 - \frac{2\lambda}{r_0}\right)}{\left(1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}\right)} v S_x . \end{aligned} \tag{29}$$

Finally the equation for the S_z component becomes:

$$\begin{aligned} \frac{dS^3}{d\tau} &= -\Gamma^3_{00} S^0 v^0 - \Gamma^3_{01} S^0 v^1 - \Gamma^3_{02} S^0 v^2 - \\ &- \Gamma^3_{03} S^0 v^3 - \Gamma^3_{10} S^1 v^0 - \Gamma^3_{11} S^1 v^1 - \\ &- \Gamma^3_{12} S^1 v^2 - \Gamma^3_{13} S^1 v^3 - \Gamma^3_{20} S^2 v^0 - \\ &- \Gamma^3_{21} S^2 v^1 - \Gamma^3_{22} S^2 v^2 - \Gamma^3_{23} S^2 v^3 - \\ &- \Gamma^3_{30} S^3 v^0 - \Gamma^3_{31} S^3 v^1 - \Gamma^3_{32} S^3 v^2 - \Gamma^3_{33} S^3 v^3 , \end{aligned} \tag{30}$$

which finally becomes:

$$\frac{dS_z}{d\tau} = 0 . \tag{31}$$

Equations (23), (28), (31) are valid at the chosen $x = r_0$, $y = z = 0$ point. These equations can also be written in a form that is valid at any point of the orbit, if we just recognize that all of them can be combined in the following single 3-D equa-

$$S_x(t) = S_0 \left\{ \cosh\left(\frac{\sqrt{2}\,GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right) - \right.$$
$$\left. - \sqrt{2\left(1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}\right)} \sinh\left(\frac{\sqrt{2}\,GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right) \right\} \tag{43}$$

$$S_y(t) \cong S_0 \left\{ \cosh\left(\frac{\sqrt{2}\,GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right) - \frac{1}{\sqrt{2}} \sinh\left(\frac{\sqrt{2}\,GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right) \right\} \tag{44}$$

tion in the following way [2]:

$$\frac{dS}{d\tau} = -2v \cdot S\nabla V + vS \cdot \nabla V\,, \tag{32}$$

where $V = -\frac{GM}{r_0} e^{-\lambda/r_0}$ is the non singular potential used. Below in order to compare we can write down the same equations for the spin components in the case of the Newtonian potential.

$$\frac{dS_x}{d\tau} = -\frac{GM}{r_0^2 c^2}\, vS_y \left[\left(1 + \frac{2GM}{r_0 c^2}\right) + \frac{1}{\left(1 + \frac{2GM}{r_0 c^2}\right)} \right] \cong$$
$$\cong -\frac{2GM}{r_0^2 c^2}\, vS_y\,, \tag{33}$$

$$\frac{dS_y}{d\tau} = \frac{GM}{r_0^2 c^2 \left(1 + \frac{2GM}{r_0 c^2}\right)}\, vS_x \cong \frac{GM}{r_0^2 c^2}\, vS_x\,, \tag{34}$$

$$\frac{dS_z}{d\tau} = 0\,. \tag{35}$$

5 Non-singular potential solutions

To find the components of the precessing spin let us now solve the system of equations (26) (29) (31) solving we obtain:

$$S_x(t) = C_1 \cosh\left\{ \frac{\sqrt{2}GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right\} +$$
$$+ C_2\sqrt{2} \sinh\left\{ \frac{\sqrt{2}GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right\}, \tag{36}$$

$$S_y(t) = C_2 \cosh\left\{ \frac{\sqrt{2}GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right\} +$$
$$+ \frac{C_1}{\sqrt{2}} \sinh\left\{ \frac{\sqrt{2}GM\,(2\lambda - r_0)}{c^2 r_0^3 \sqrt{1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}}}\, vt \right\}, \tag{37}$$

$$S_z(t) = \text{const} = D_0\,, \tag{38}$$

since the motion is not relativistic we have that $dt = d\tau$, and the orbital velocity of the gyroscope is $v = \sqrt{\frac{GM}{r_0}}$.

6 Newtonian gravity solutions

Next we can compare the solutions in (36), (37), (38) with those of the system (33), (34), (35) which are:

$$S_x(t) = C_1 \cos\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2}\, vt \right) -$$
$$- C_2\sqrt{2} \sin\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2}\, vt \right), \tag{39}$$

$$S_y(t) = C_2 \cos\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2}\, vt \right) +$$
$$+ \frac{C_1}{\sqrt{2}} \sin\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2}\, vt \right), \tag{40}$$

$$S_z(t) = \text{const} = D_0\,.$$

If we now assume the initial conditions $t = 0$, $S_x(0) = S_y(0) = S_0$ we obtain the final solution:

$$S_x(t) = S_0 \left\{ \cos\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2} vt \right) - \right.$$
$$\left. - \sqrt{2} \sin\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2}\, vt \right) \right\}, \tag{41}$$

$$S_y(t) = S_0 \left\{ \cos\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2} \sqrt{\frac{GM}{r_0}}\, t \right) - \right.$$
$$\left. - \frac{1}{\sqrt{2}} \sin\left(\frac{\sqrt{2}\,GM}{c^2 r_0^2} \sqrt{\frac{GM}{r_0}}\, t \right) \right\}. \tag{42}$$

Similarly from the solutions of the non-singular Newtonian potential we obtain (43) and (44).

Since numerically $c^2 r_0^3 \gg 2GMr_0^2 - 2GM\lambda r_0$ the above equations take the form (45) and (46).

7 Numerical results for an Earth satellite

Let us now assume a satellite in a circular orbit around the Earth, at an orbital height $h = 650$ km or an orbital radius

$$S_x(t) = S_0 \left\{ \cosh\left(\frac{2\lambda GM}{c^2 r_0^3} \left(1 - \frac{r_0}{2\lambda}\right) \sqrt{\frac{2GM}{r_0\left(1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}\right)}}\, t \right) - \right.$$
$$\left. - \sqrt{2\left(1 + \frac{2GM}{r_0 c^2} - \frac{2\lambda GM}{r_0^2 c^2}\right)} \sinh\left(\frac{2\lambda GM}{c^2 r_0^2} \left(1 - \frac{r_0}{2\lambda}\right) \sqrt{\frac{2GM}{r_0\left(1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}\right)}}\, t \right) \right\} \tag{45}$$

$$S_y(t) = S_0 \left\{ \cosh\left(\frac{2\lambda GM}{c^2 r_0^3} \left(1 - \frac{r_o}{2\lambda}\right) \sqrt{\frac{2GM}{r_0\left(1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}\right)}}\, t \right) - \right.$$
$$\left. - \frac{1}{\sqrt{2}} \sinh\left(\frac{2\lambda GM}{c^2 r_0^3} \left(1 - \frac{r_0}{2\lambda}\right) \sqrt{\frac{2GM}{r_0\left(1 + \frac{2GM}{r_0 c^2} - \frac{2GM\lambda}{r_0^2 c^2}\right)}}\, t \right) \right\} \tag{46}$$

$V(r)$	$\Delta S_x/S_x$	$\Delta S_y/S_y$	Geodetic precession S (arcsec/year)	$\Delta S_{geo}/S_g$
Newtonian	-4.30×10^{-5}	2.00×10^{-5}	-6.6	
Non-Singular	-4.80×10^{-5}	2.40×10^{-5}	-6.289	-5.570×10^{-2}

Table 1: Changes of the spin components and final geodetic precession of an orbiting the Earth satellite t an altitude $h = 650$ km.

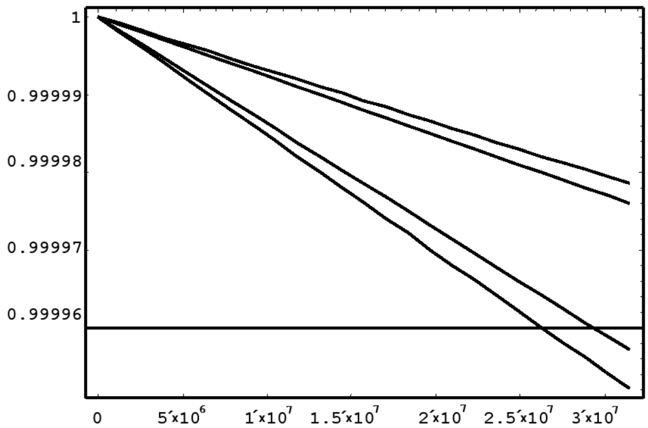

Fig. 2: Gyroscope spin components (S_x, S_y). Newtonian and non-singular potential change in the gyro pin components for a satellite orbiting the earth for a year. Abscissa axis means time.

$r_0 = 7.028\times10^6$ m, then $\lambda = 4.372\times10^{-3}$ m, $v = 7.676$ km/s, $t = 1$ year $= 3.153\times10^7$ s using (41) and (42) we obtain:

Newtonian potential

$$S_x = 0.999957\, S_0\,, \qquad S_y = 1.000020\, S_0\,, \tag{47}$$

and from (45) and (46) we obtain:

Non-Singular Potential

$$S_x = 0.999952\, S_0\,, \qquad S_y = 0.999976\, S_0\,. \tag{48}$$

For a gyroscope in orbit around the Earth we can write an expression for the geodetic precession in such a non-singular potential to be equal to:

$$S_{geo} = \frac{3}{2}\,\nabla\Phi \times v = \frac{3GM}{2c^2 r_0^2}\sqrt{\frac{GM}{r_0}}\left(1 - \frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}\,, \tag{49}$$

substituting values for the parameters above we obtain that:

$$S_{geo} = 1.01099\times10^{-12}\ \text{rad/s}\,, \tag{50}$$

$$S_{geo} = 6.289\ \text{arcsec/year}\,. \tag{51}$$

8 Conclusions

We have derived the equations for the precession of the spin in a case of a non-singular potential and we have compared them with those of the Newtonian potential. In the case of the non-singular gravitational potential both components of the spin are very slow varying functions of time. In a hypothetically large amount of time of the order of $\sim 10^5$ years or more spin components S_x and S_y of the non-singular potential appear to diverge in opposite directions, where those of the Newtonian potential exhibit a week periodic motion in time.

In the case of the non-singular potential we found that $\frac{\Delta S_x}{S_x} = -4.80\times10^{-5}$ and $\frac{\Delta S_y}{S_y} = -2.40\times10^{-5}$ where in the case of the Newtonian potential we have that $\frac{\Delta S_x}{S_x} = -4.30\times10^{-5}$ and $\frac{\Delta S_y}{S_y} = 2.00\times10^{-5}$. The calculation has been performed using a non-exact *Schwarzschild* solution. On the other hand the gravitational field of the Earth is not an exact *Schwarzschild* field, but rather the field of a rotating mass. Compared to the Newtonian result, the non-singular potential modifies the original equation of the geodetic precession by the term $\left(1-\frac{\lambda}{r_0}\right) e^{-\frac{\lambda}{r_0}}$ which at the orbital altitude of $h = 650$ km contributes to a spin reduction effect of the order of 9.99×10^{-1}. If such a type of potential exists its effect onto a gyroscope of a satellite orbiting at $h = 650$ km could probably be easily detected.

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A Note on Computer Solution of Wireless Energy Transmit via Magnetic Resonance

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In the present article we argue that it is possible to find numerical solution of coupled magnetic resonance equation for describing wireless energy transmit, as discussed recently by Karalis (2006) and Kurs *et al.* (2007). The proposed approach may be found useful in order to understand the phenomena of magnetic resonance. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In recent years there were some new interests in methods to transmit energy without wire. While it has been known for quite a long-time that this method is possible theoretically (since Maxwell and Hertz), until recently only a few researchers consider this method seriously.

For instance, Karalis et al [1] and also Kurs et al. [2] have presented these experiments and reported that efficiency of this method remains low. A plausible way to solve this problem is by better understanding of the mechanism of magnetic resonance [3].

In the present article we argue that it is possible to find numerical solution of coupled magnetic resonance equation for describing wireless energy transmit, as discussed recently by Karalis (2006) and Kurs *et al.* (2007). The proposed approach may be found useful in order to understand the phenomena of magnetic resonance.

Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

2 Numerical solution of coupled-magnetic resonance equation

Recently, Kurs *et al.* [2] argue that it is possible to represent the physical system behind wireless energy transmit using coupled-mode theory, as follows:

$$a_m(t) = (i\omega_m - \Gamma_m)\, a_m(t) + + \sum_{n\neq m} i\kappa_{nm} a_n(t) - F_m(t)\,. \tag{1}$$

The simplified version of equation (1) for the system of two resonant objects is given by Karalis *et al.* [1, p. 2]:

$$\frac{da_1}{dt} = -\,i\,(\omega_1 - i\Gamma_1)\, a_1 + i\kappa a_2\,, \tag{2}$$

and

$$\frac{da_2}{dt} = -\,i\,(\omega_2 - i\Gamma_2)\, a_2 + i\kappa a_1\,. \tag{3}$$

These equations can be expressed as linear 1st order ODE as follows:

$$f'(t) = -\,i\,\alpha\, f(t) + i\,\kappa\, g(t) \tag{4}$$

and

$$g'(t) = -\,i\beta\, g(t) + i\,\kappa\, f(t)\,, \tag{5}$$

where

$$\alpha = (\omega_1 - i\,\Gamma_1) \tag{6}$$

and

$$\beta = (\omega_2 - i\Gamma_2) \tag{7}$$

Numerical solution of these coupled-ODE equations can be found using Maxima [4] as follows. First we find test when parameters (6) and (7) are set up to be 1. The solution is:

```
(%i5) 'diff(f(x),x)+%i*f=%i*b*g(x);
(%o5) 'diff(f(x),x,1)+%i*f=%i*b*g(x)
(%i6) 'diff(g(x),x)+%i*g=%i*b*f(x);
(%o6) 'diff(g(x),x,1)+%i*g=%i*b*f(x)
(%i7) desolve([%o5,%o6],[f(x),g(x)]);
```

The solutions for $f(x)$ and $g(x)$ are:

$$f(x) = \frac{\left[i\,g(0)\,b - i\,f(x)\right]\sin(bx)}{b} - - \frac{\left[g(x) - f(0)\,b\right]\cos(bx)}{b} + \frac{g(x)}{b}\,, \tag{8}$$

$$g(x) = \frac{\left[i\,f(0)\,b - i\,g(x)\right]\sin(bx)}{b} - - \frac{\left[f(x) - g(0)\,b\right]\cos(bx)}{b} + \frac{f(x)}{b}\,. \tag{9}$$

Translated back to our equations (2) and (3), the solutions for $\alpha = \beta = 1$ are given by:

$$a_1(t) = \frac{\left[i\,a_2(0)\kappa - i\,a_1\right]\sin(\kappa t)}{\kappa} - - \frac{\left[a_2 - a_1(0)\,\kappa\right]\cos(\kappa t)}{\kappa} + \frac{a_2}{\kappa} \tag{10}$$

$$f(x) = e^{-(ic-ia)t/2}\left[\frac{\left[2i\,f(0)\,c + 2i\,g(0)\,b - f(0)(ic - i\,a)\right]\sin\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}} + \right.$$
$$\left. + \frac{f(0)\cos\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}}\right] \tag{13}$$

$$g(x) = e^{-(ic-ia)t/2}\left[\frac{\left[2i\,f(0)\,c + 2i\,g(0)\,a - g(0)(ic - i\,a)\right]\sin\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}} + \right.$$
$$\left. + \frac{g(0)\cos\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}}\right] \tag{14}$$

$$a_1(t) = e^{-(i\beta-i\,\alpha)t/2}\left(\frac{\left[2i\,a_1(0)\beta + 2i\,a_2(0)\,\kappa - (i\beta - i\alpha)\,a_1\right]\sin\left(\frac{\xi}{2}\,t\right)}{\xi} - \frac{a_1(0)\cos\left(\frac{\xi}{2}\,t\right)}{\xi}\right) \tag{15}$$

$$a_2(t) = e^{-(i\beta-i\alpha)t/2}\left(\frac{\left[2i\,a_2(0)\beta + 2i\,a_1(0)\,\kappa - (i\beta - i\,\alpha)\,a_2\right]\sin\left(\frac{\xi}{2}\,t\right)}{\xi} - \frac{a_2(0)\cos\left(\frac{\xi}{2}\,t\right)}{\xi}\right) \tag{16}$$

and

$$a_2(t) = \frac{\left[i\,a_1(0)\kappa - i\,a_2\right]\sin(\kappa t)}{\kappa} - $$
$$- \frac{\left[a_1 - a_2(0)\,\kappa\right]\cos(\kappa t)}{\kappa} + \frac{a_1}{\kappa}\,. \tag{11}$$

Now we will find numerical solution of equations (4) and (5) when $\alpha \neq \beta \neq 1$. Using Maxima [4], we find:

```
(%i12) 'diff(f(t),t)+%i*a*f(t)=%i*b*g(t);
(%o12) 'diff(f(t),t,1)+%i*a*f(t)=%i*b*g(t)
(%i13) 'diff(g(t),t)+%i*c*g(t)=%i*b*f(t);
(%o13) 'diff(g(t),t,1)+%i*c*g(t)=%i*b*f(t)
(%i14) desolve([%o12,%o13],[f(t),g(t)]);
```

and the solution is found to be quite complicated: these are formulae (13) and (14).

Translated back these results into our equations (2) and (3), the solutions are given by (15) and (16), where we can define a new "ratio":

$$\xi = \sqrt{\beta^2 - 2\alpha\beta + 4\kappa^2 + \alpha^2}\,. \tag{12}$$

It is perhaps quite interesting to remark here that there is no "distance" effect in these equations.

Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

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VC would like to dedicate this article to R.F.F.

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Structure of Even-Even $^{218\text{-}230}$Ra Isotopes within the Interacting Boson Approximation Model

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A good description of the excited positive and negative parity states of radium nuclei ($Z = 88$, $N = 130$–142) is achieved using the interacting boson approximation model (IBA-1). The potential energy surfaces, energy levels, parity shift, electromagnetic transition rates $B(E1)$, $B(E2)$ and electric monopole strength $X(E0/E2)$ are calculated for each nucleus. The analysis of the eigenvalues of the model Hamiltonian reveals the presence of an interaction between the positive and negative parity bands. Due to this interaction the $\Delta I = 1$ staggering effect, between the energies of the ground state band and the negative parity state band, is produced including beat patterns.

1 Introduction

The existence of stable octupole deformation in actinide nuclei has encouraged many authors to investigate these nuclei experimentally and theoretically but until now no definitive signatures have been established. Different models have been considered, but none has provided a complete picture of octupole deformation.

Cluster model has been applied to $^{221\text{-}226}$Ra by many authors [1–7]. The intrinsic multipole transition moment and parity splitting were calculated. Also, the half-lives of cluster emission are predicted. In general, cluster model succeeded in reproducing satisfactory the properties of normal deformed ground state and super deformed excited bands in a wide range of even-even nuclei.

A proposed formalism of the collective model [8, 9, 10] have been used in describing the strong parity shift observed in low-lying spectra of ^{224,226}Ra and ^{224,226}Th with octupole deformations together with the fine rotational band structure developed at higher angular momenta. Beat staggering patterns are obtained also for $^{218\text{-}226}$Ra and ^{224,226}Th.

The mean field model [11] and the analytic quadrupole octupole axially symmetric ($AQOA$) model [12] have been applied to ^{224,226}Ra and ^{226}Ra nuclei respectively, and found useful for the predictions of the decay properties where the experimental data are scarce.

Spdf interacting boson model [13] has been applied to the even-even $^{218\text{-}228}$Ra isotopes and an explanation of how the octupole deformation can arise in the rotational limit. The discussion of the properties of the fractional symmetric rigid rotor spectrum [14] and the results of its application to the low excitation energy of the ground state band of $^{214\text{-}224}$Ra show an agreement with the experimental data.

The aim of the present paper is to calculate and analyze the complete spectroscopic properties of the low-lying positive and negative parity excited states in $^{218\text{-}230}$Ra isotopes using IBA-1 Hamiltonian. The potential energy surfaces, levels energy, parity shift, electromagnetic transition rates and electric monopole strength $X(E0/E2)$ are calculated.

2 (IBA-1) model

2.1 Level energies

The IBA-1 model describes the low-lying energy states of the even-even radium nuclei as a system of interacting s-bosons and d-bosons. The π and ν bosons are treated as one boson. Introducing creation $\left(s^\dagger d^\dagger\right)$ and annihilation $\left(s\tilde{d}\right)$ operators for s and d bosons, the most general Hamiltonian [15] which includes one-boson term in boson-boson interaction has been used in calculating the levels energy is:

$$\begin{aligned} H = {} & EPS \cdot n_d + PAIR \cdot (P \cdot P) + \\ & + \frac{1}{2} ELL \cdot (L \cdot L) + \frac{1}{2} QQ \cdot (Q \cdot Q) + \\ & + 5OCT \cdot (T_3 \cdot T_3) + 5HEX \cdot (T_4 \cdot T_4) , \end{aligned} \tag{1}$$

where

$$P \cdot p = \frac{1}{2} \left[\begin{array}{l} \left\{ (s^\dagger s^\dagger)_0^{(0)} - \sqrt{5}(d^\dagger d^\dagger)_0^{(0)} \right\} x \\ \left\{ (ss)_0^{(0)} - \sqrt{5}(\tilde{d}\tilde{d})_0^{(0)} \right\} \end{array} \right]_0^{(0)} , \tag{2}$$

$$L \cdot L = -10\sqrt{3} \left[(d^\dagger \tilde{d})^{(1)} x \, (d^\dagger \tilde{d})^{(1)} \right]_0^{(0)} , \tag{3}$$

$$Q \cdot Q = \sqrt{5} \left[\begin{array}{l} \left\{ (S^\dagger \tilde{d} + d^\dagger s)^{(2)} - \frac{\sqrt{7}}{2} (d^\dagger \tilde{d})^{(2)} \right\} x \\ \left\{ (s^\dagger \tilde{d} + +\tilde{d} s)^{(2)} - \frac{\sqrt{7}}{2} (d^\dagger \tilde{d})^{(2)} \right\} \end{array} \right]_0^{(0)} , \tag{4}$$

$$T_3 \cdot T_3 = -\sqrt{7} \left[(d^\dagger \tilde{d})^{(2)} x \, (d^\dagger \tilde{d})^{(2)} \right]_0^{(0)} , \tag{5}$$

$$T_4 \cdot T_4 = 3 \left[(d^\dagger \tilde{d})^{(4)} x \, (d^\dagger \tilde{d})^{(4)} \right]_0^{(0)} . \tag{6}$$

nucleus	EPS	$PAIR$	ELL	QQ	OCT	HEX	$E2SD(eb)$	$E2DD(eb)$
$_{218}$Ra	0.3900	0.000	0.0005	−0.0090	0.0000	0.0000	0.2020	−0.5957
$_{220}$Ra	0.3900	0.000	0.0005	−0.0420	0.0000	0.0000	0.1960	−0.5798
$_{222}$Ra	0.0650	0.0000	0.0100	−0.0650	0.0000	0.0000	0.1960	−0.5798
$_{224}$Ra	0.2000	0.0000	0.0060	−0.0450	0.0000	0.0000	0.1640	−0.4851
$_{226}$Ra	0.0700	0.0000	0.0060	−0.0450	0.0000	0.0000	0.1660	−0.4910
$_{228}$Ra	0.0600	0.0000	0.0060	−0.0380	0.0000	0.0000	0.1616	−0.4780
$_{230}$Ra	0.0580	0.0000	0.0060	−0.0502	0.0000	0.0000	0.1560	−0.4615

Table 1: Table 1. Parameters used in IBA-1 Hamiltonian (all in MeV).

In the previous formulas, n_d is the number of boson; $P{\cdot}P$, $L\cdot L$, $Q\cdot Q$, $T_3\cdot T_3$ and $T_4\cdot T_4$ represent pairing, angular momentum, quadrupole, octupole and hexadecupole interactions between the bosons; EPS is the boson energy; and $PAIR$, ELL, QQ, OCT, HEX is the strengths of the pairing, angular momentum, quadrupole, octupole and hexadecupole interactions.

2.2 Transition rates

The electric quadrupole transition operator [15] employed in this study is given by:

$$T^{(E2)} = E2SD\cdot(s^\dagger\tilde{d}+d^\dagger s)^{(2)} + \\ +\frac{1}{\sqrt{5}}\,E2DD\cdot(d^\dagger\tilde{d})^{(2)}\,. \qquad (7)$$

The reduced electric quadrupole transition rates between $I_i\to I_f$ states are given by

$$B(E_2,I_i-I_f)=\frac{[<I_f\parallel T^{(E_2)}\parallel I_i>]^2}{2I_i+1}\,. \qquad (8)$$

3 Results and discussion

3.1 The potential energy surface

The potential energy surfaces [16], $V(\beta,\gamma)$, for radium isotopes as a function of the deformation parameters β and γ have been calculated using:

$$\begin{aligned} &E_{N_\Pi N_\nu}(\beta,\gamma) = <N_\pi N_\nu;\beta\gamma\,|H_{\pi\nu}|N_\pi N_\nu;\beta\gamma> = \\ &=\zeta_d(N_\nu N_\pi)\beta^2(1+\beta^2)+\beta^2(1+\beta^2)^{-2}\times \\ &\times\left\{kN_\nu N_\pi[4-(\bar{X}_\pi\bar{X}_\nu)\beta\cos 3\gamma]\right\}+ \\ &+\left\{[\bar{X}_\pi\bar{X}_\nu\beta^2]+N_\nu(N_\nu-1)\left(\frac{1}{10}c_0+\frac{1}{7}c_2\right)\beta^2\right\}, \end{aligned} \qquad (9)$$

where

$$\bar{X}_\rho=\left(\frac{2}{7}\right)^{0.5}X_\rho \quad \rho=\pi \text{ or } \upsilon\,. \qquad (10)$$

The calculated potential energy surfaces for radium series of isotopes presented in Fig. 1 show that ^{218}Ra is a vibrational-like nucleus where the deformation β is zero. ^{220}Ra nucleus started to deviate from vibrational-like and a slight prolate deformation appeared. $^{222\text{-}230}$Ra nuclei show more deformation on the prolate and oblate sides, but the deformation on the prolate side is deeper.

3.2 Energy spectra

IBA-1 model has been used in calculating the energy of the positive and negative parity low -lying levels of radium series of isotopes. A comparison between the experimental spectra [17–23] and our calculations, using the values of the model parameters given in Table 1 for the ground and octupole bands, is illustrated in Fig. 2. The agreement between the theoretical and their correspondence experimental values for all the nuclei are slightly higher but reasonable. The most striking is the minimum observed in the negative parity states, Fig. 3, at $N=136$ which interpreted as ^{224}Ra is the most deformed nucleus in this chain of isotopes.

3.3 Electromagnetic transitions rates

Unfortunately there is no enough measurements of $B(E1)$ or $B(E2)$ rates for these series of nuclei. The only measured $B(E2,0_1^+\to 2_1^+)$'s are presented, in Table 2a, for comparison with the calculated values. The parameters $E2SD$ and $E2DD$ used in the present calculations are determined by normalizing the calculated values to the experimentally known ones and displayed in Tables 2a and 2b.

For calculating $B(E1)$ and $B(E2)$ transition rates of intraband and interaband we did not introduce any new parameters. The calculated values some of it are presented in Fig. 4 and Fig. 5 which show bending in the two figures at $N=136$ which support what we have seen in Fig. 3 as ^{224}Ra is the most octupole deformed nucleus.

3.4 Electric monopole transitions

The electric monopole transitions, $E0$, are normally occurring between two states of the same spin and parity by transferring energy and zero unit of angular momentum. The strength of the electric monopole transition, $X_{if'f}(E0/E2)$,

I_i^+ I_f^+	^{218}Ra	^{220}Ra	^{222}Ra	^{224}Ra	^{226}Ra	^{228}Ra	^{230}Ra
0_1 Exp. 2_1	1.10(20)	———	4.54(39)	3.99(15)	5.15(14)	5.99(28)	——-
0_1 Theor. 2_1	1.1222	2.4356	4.5630	3.9633	5.1943	5.9933	6.6861
2_1 0_1	0.224	0.4871	0.9126	0.7927	1.0389	1.1987	1.3372
2_2 0_1	0.0005	0.0028	0.0001	0.0014	0.0002	0.0001	0.0001
2_2 0_2	0.086	0.2509	0.5978	0.5287	0.7444	0.8878	1.0183
2_3 0_1	0.000	0.0058	0.0001	0.0015	0.0001	0.0001	0.0001
2_3 0_2	0.173	0.0854	0.0141	0.0075	0.0122	0.0122	0.0118
2_3 0_3	0.022	0.0476	0.0001	0.0011	0.0001	0.0000	0.0000
2_4 0_3	0.010	0.1322	0.3662	0.3013	0.5326	0.6481	0.7627
2_4 0_4	0.152	0.0707	0.0006	0.0041	0.0001	0.0000	0.0000
2_2 2_1	0.300	0.0819	0.0003	0.0034	0.0003	0.0003	0.0001
2_3 2_1	0.0001	0.0023	0.0002	0.0022	0.0002	0.0002	0.0001
2_3 2_2	0.088	0.4224	0.0690	0.0730	0.0398	0.0348	0.0302
4_1 2_1	0.368	0.7474	1.2490	1.0973	1.4449	1.6752	1.8756
4_1 2_2	0.0318	0.0337	0.0004	0.0051	0.0004	0.0004	0.0002
4_1 2_3	0.0715	0.0331	0.0000	0.0002	0.0000	0.0000	0.0000
6_1 4_1	0.4194	0.7924	1.2673	1.1380	1.5138	1.7714	1.9970
6_1 4_2	0.0463	0.0270	0.0004	0.0057	0.0005	0.0004	0.0002
6_1 4_3	0.0514	0.0249	0.0001	0.0009	0.0000	0.0000	0.0000
8_1 6_1	0.3749	0.7217	1.626	1.0830	1.4672	1.7430	1.9864
8_1 6_2	0.0529	0.0205	0.0004	0.0051	0.0006	0.0005	0.0002
8_1 6_3	0.0261	0.0170	0.0001	0.0018	0.0001	0.0001	0.0000
10_1 8_1	0.2346	0.5600	0.9737	0.9649	1.3492	1.6406	1.9005
10_1 8_2	0.0553	0.0161	0.0003	0.0041	0.0005	0.0005	0.0002

Table 2: Table 2a. Values of the theoretical reduced transition probability, $B(E2)$ (in $e^2\,b^2$).

I_i^- I_f^+	^{218}Ra	^{220}Ra	^{222}Ra	^{224}Ra	^{226}Ra	^{228}Ra	^{230}Ra
1_1 0_1	0.0008	0.0605	0.1942	0.1886	0.2289	0.2612	0.3033
1_1 0_2	0.1203	0.0979	0.0222	0.0293	0.0195	0.0190	0.0183
3_1 2_1	0.1117	0.1921	0.3352	0.3301	0.3927	0.4378	0.4937
3_1 2_2	0.0451	0.0325	0.0245	0.0330	0.0243	0.0238	0.0228
3_1 2_3	0.0025	0.0095	0.0001	0.0001	0.0000	0.0000	0.0001
3_1 4_1	0.0015	0.0094	0.0419	0.0458	0.0791	0.0883	0.0926
3_1 4_2	0.0007	0.0040	0.0073	0.0100	0.0099	0.0090	0.0074
5_1 4_1	0.2397	0.3169	0.4358	0.4349	0.5032	0.5493	0.6043
5_1 4_2	0.0531	0.0267	0.0187	0.0260	0.0214	0.0214	0.0205
5_1 4_3	0.0017	0.0027	0.0006	0.0005	0.0002	0.0002	0.0003
7_1 6_1	0.3839	0.4454	0.5349	0.5388	0.6033	0.6476	0.6996
7_1 6_2	0.0476	0.0204	0.0121	0.0187	0.0168	0.0173	0.0169
9_1 8_1	0.5429	0.5785	0.6398	0.6479	0.7041	0.7452	0.7936
9_1 8_2	0.0295	0.0139	0.0070	0.0129	0.0122	0.0131	0.0132

Table 3: Table 2b. Values of the theoretical reduced transition probability, $B(E1)$ (in $\mu\, e^2 b$).

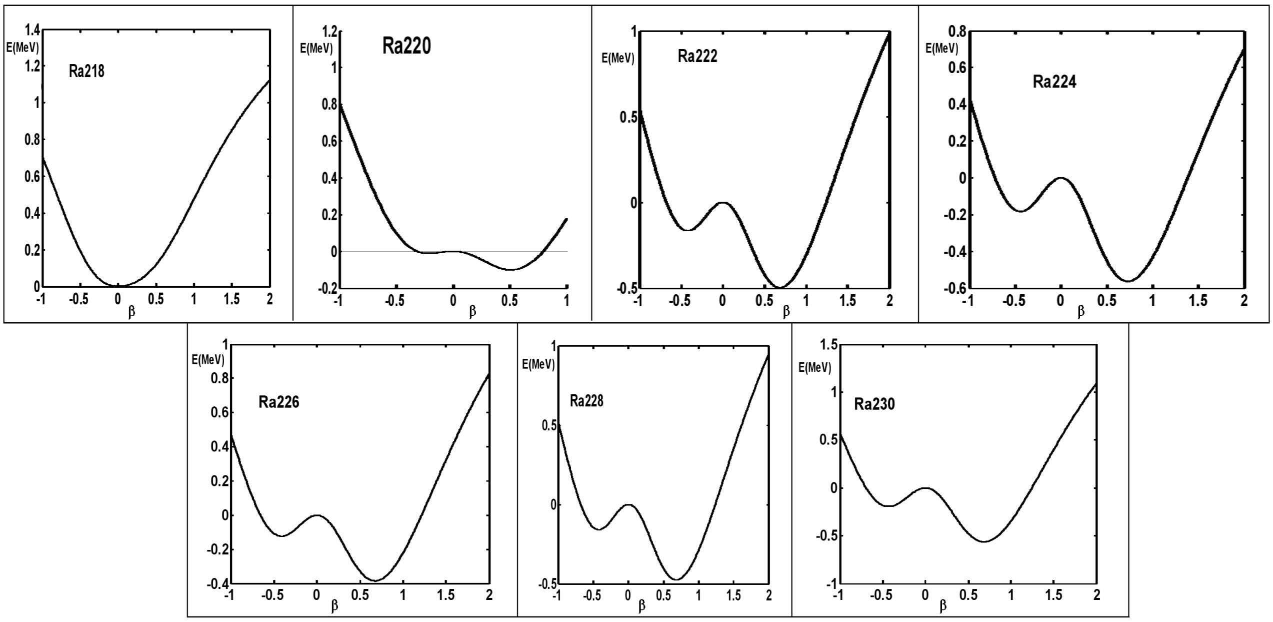

Fig. 1: The potential energy surfaces for $^{218-230}$Ra nuclei.

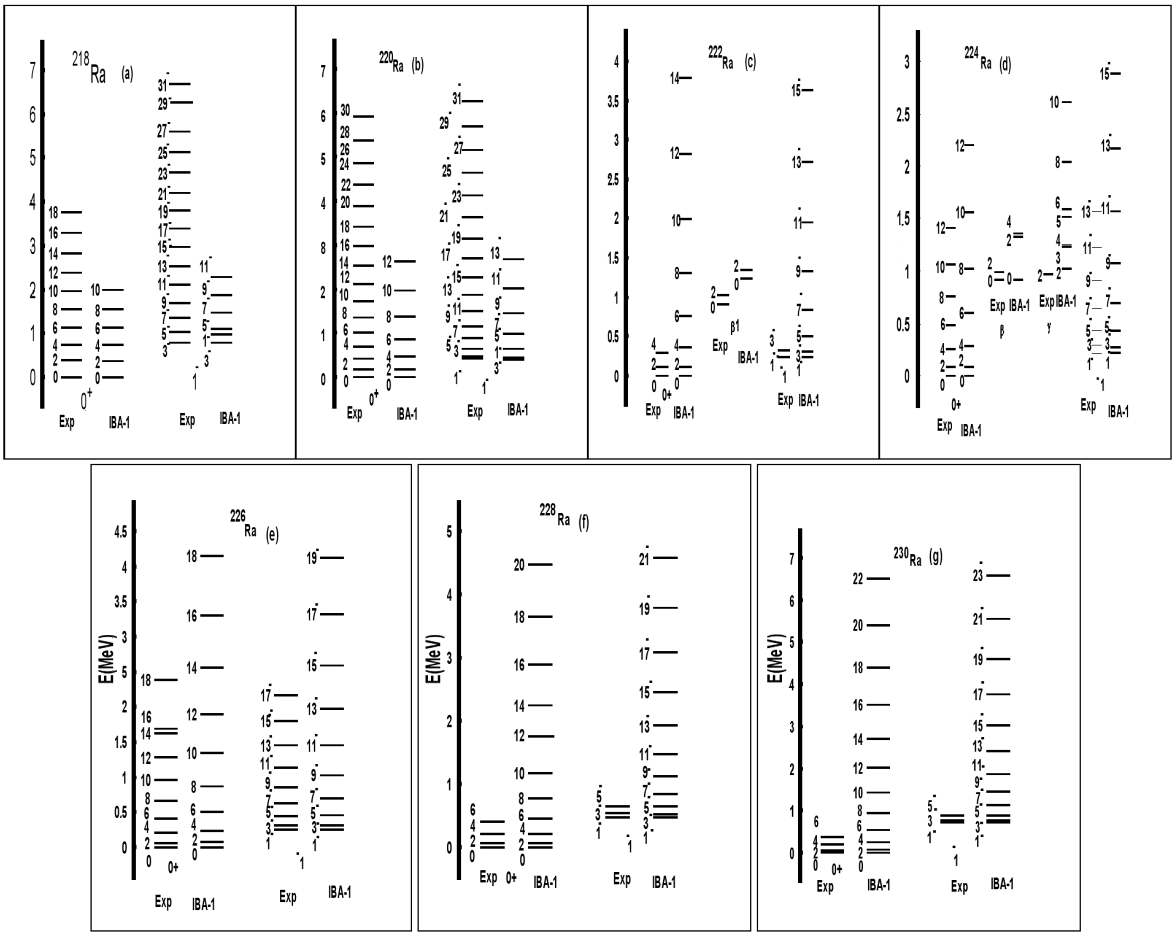

Fig. 2: Comparison between experimental (Exp.) and theoretical (IBA-1) energy levels in $^{218-230}$Ra, (a–g).

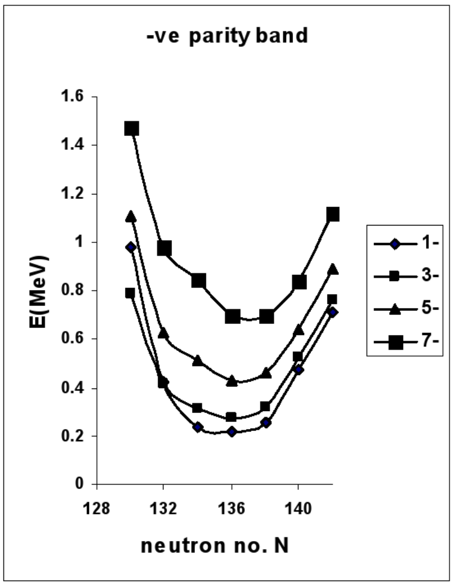

Fig. 3: Energy versus neutron numbers N for the $-$ve parity band in $^{218\text{-}230}$Ra.

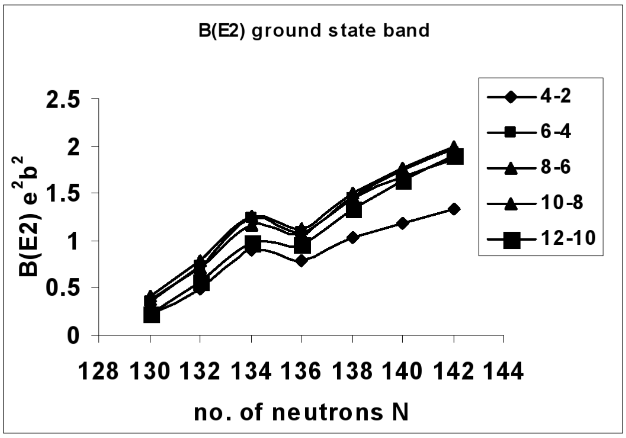

Fig. 4: The calculated $B(E2)$'s for the ground state band of Ra isotopes.

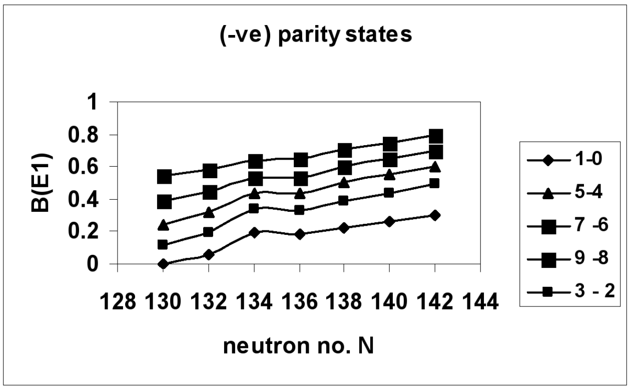

Fig. 5: The calculated $B(E1)$'s for the $(-$ve$)$ parity band.

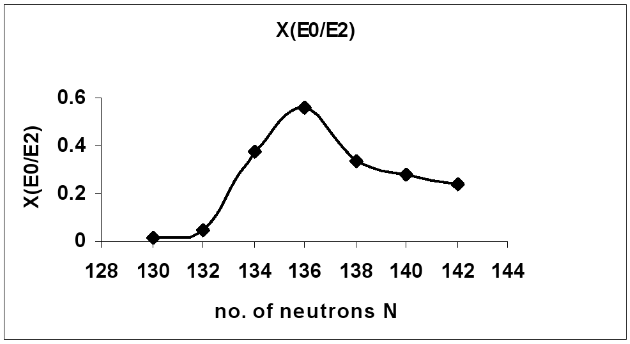

Fig. 6: The calculated $X(E0/E2,\ 2_2^+ \rightarrow 0_1^+)$ versus N for $^{218\text{-}230}$Ra isotopes.

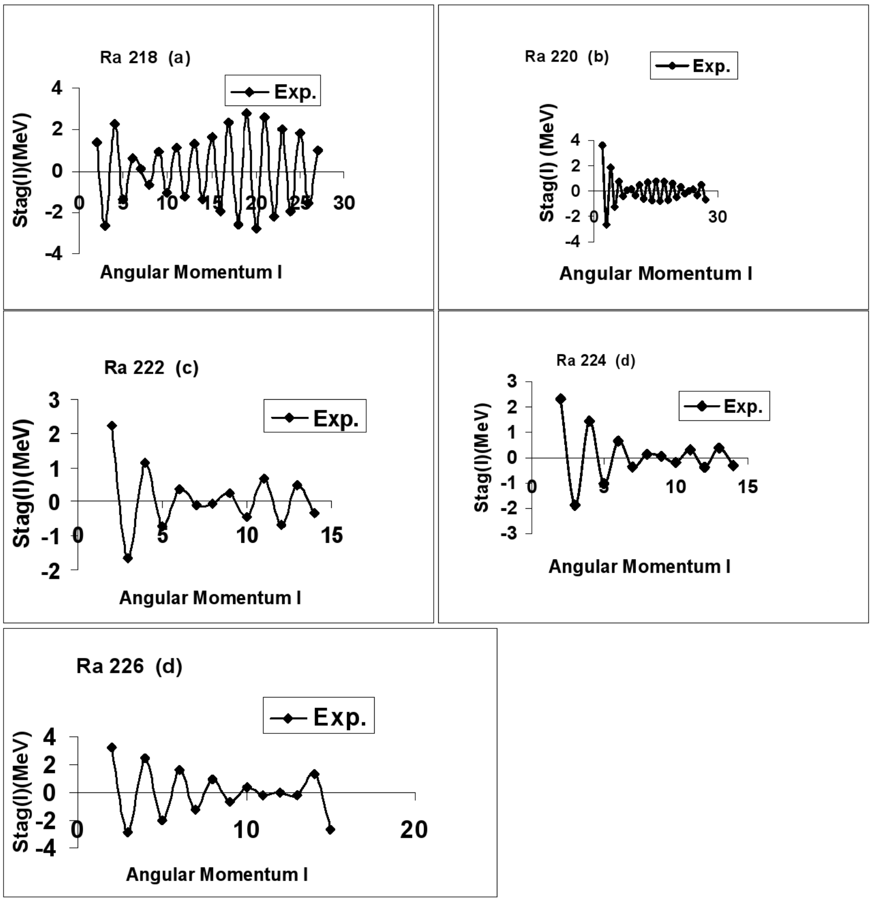

Fig. 7: $\Delta I = 1$, staggering patterns for the ground state and octupole bands of $^{218\text{-}230}$Ra isotope.

I_i^+	I_f^+	$I_{'f}^+$	^{218}Ra	^{220}Ra	^{222}Ra	^{224}Ra	^{226}Ra	^{228}Ra	^{230}Ra
0_2	0_1	2_1	0.016	0.046	0.376	0.562	0.335	0.279	0.243
0_3	0_1	2_1	0.125	——	——	0.058	0.081	—-	0.500
0_3	0_1	2_2	0.007	0.058	—-	0.009	0.003	——	0.230
0_3	0_1	2_3	0.015	0.002	—-	0.333	0.005	——	0.0005
0_3	0_2	2_1	—-	——	10.00	1.705	0.702	2.000	9.000
0_3	0_2	2_2	——	0.029	0.008	0.027	0.027	0.189	4.153
0_3	0_2	2_3	———	0.001	——	9.666	0.054	0.049	0.103
0_4	0_1	2_2	0.009	0.008	1.200	—-	1.700	0.391	0.094
0_4	0_1	2_3	0.009	1.000	—-	1.230	0.459	1.636	—-
0_4	0_1	2_4	0.005	0.018	0.027	4.000	1.307	0.129	5.000
0_4	0_2	2_2	0.018	0.042	1.400	——	0.1000	——	0.037
0_4	0_2	2_3	0.019	5.000	——	0.769	0.027	——	——
0_4	0_2	2_4	0.011	0.093	0.031	2.500	0.076	——	2.000
0_4	0_3	2_1	——	——	0.250	—-	0.333	——-	——
0_4	0_3	2_2	—-	——	0.066	——	0.001	——	——
0_4	0_3	2_3	——	——	——	——	0.027	——	——
0_4	0_3	2_4	——	——	0.001	—-	0.076	——	——-

Table 3. Theoretical $X_{if'f}$ $(E0/E2)$ ratios for $E0$ transitions in Ra isotopes.

[24] can be calculated using equations (11, 12) and presented in Table 3. Fig. 6 shows also that ^{224}Ra has strong electric monopole strength than the other radium isotopes which is in agreement with the previous explanations.

$$X_{if'f}(E0/E2) = \frac{B(E0, I_i - I_f)}{B(E2, I_i - I_{'f})}\,, \qquad (11)$$

$$X_{if'f}(E0/E2) = (2.54\times10^9)\,A^{3/4} \times \times \frac{E_\gamma^5(\text{MeV})}{\Omega_{KL}}\,\alpha(E2)\,\frac{T_e(E0, I_i - I_f)}{T_e(E_2, I_i - I_{'f})}\,. \qquad (12)$$

3.5 The staggering

A presence of an odd-even staggering effect has been observed for $^{218\text{-}230}$Ra series of isotopes [8, 9, 10, 25]. Odd-even staggering patterns between the energies of the ground state band and the $(-ve)$ parity octupole band have been calculated, $\Delta I = 1$, using staggering function as in equations (13, 14) using the available experimental data [17–23].

$$Stag(I) = 6\Delta E(I) - 4\Delta E(I-1) - 4\Delta E(I+1) + + \Delta E(I+2) + \Delta E(I-2)\,, \qquad (13)$$

with

$$\Delta E(I) = E(I+1) - E(I)\,. \qquad (14)$$

The calculated staggering patterns are illustrated in Fig. 7, where we can see the beat patterns of the staggering behavior which show an interaction between the ground state and the octupole bands.

3.6 Conclusions

The IBA-1 model has been applied successfully to $^{218\text{-}230}$Ra isotopes and we have got:

1. The ground state and octupole bands are successfully reproduced;
2. The potential energy surfaces are calculated and show vibrational characters to ^{218,220}Ra and rotational behavior to $^{222\text{-}230}$Ra isotopes where they are prolate deformed nuclei;
3. Electromagnetic transition rates $B(E1)$ and $B(E2)$ are calculated;
4. The strength of the electric monopole transitions are calculated and show with the other calculated data that ^{224}Ra is the most octupole deformed nucleus;
5. Staggering effect have been observed and beat patterns obtained which show an interaction between the ground state and octupole bands;

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Covariance, Curved Space, Motion and Quantization

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Weak external forces and non-inertial motion are equivalent with the free motion in a curved space. The Hamilton-Jacobi equation is derived for such motion and the effects of the curvature upon the quantization are analyzed, starting from a generalization of the Klein-Gordon equation in curved spaces. It is shown that the quantization is actually destroyed, in general, by a non-inertial motion in the presence of external forces, in the sense that such a motion may produce quantum transitions. Examples are given for a massive scalar field and for photons.

Newton's law. We start with Newton's law

$$m\,\frac{dv_\alpha}{dt} = f_\alpha\,, \tag{1}$$

for a particle of mass m, with usual notations. I wish to show here that it is equivalent with the motion of a free particle of mass m in a curved space, *i.e.* it is equivalent with

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{jk} u^j u^k = 0\,, \tag{2}$$

again with usual notations.*

Obviously, the spatial coordinates of equation (1) are euclidean, and equation (1) is a non-relativistic limit. It follows that the metric we should look for may read

$$ds^2 = (1+h)\,c^2 dt^2 + 2cdt\,g_{0\alpha} dx^\alpha + g_{\alpha\beta}\,dx^\alpha dx^\beta, \tag{3}$$

where $g_{\alpha\beta} = -\delta^\beta_\alpha (= \delta_{\alpha\beta})$, while functions h, $g_{0\alpha} \ll 1$ are determined such that equation (2) goes into equation (1) in the non-relativistic limit $\frac{v_\alpha}{c} \ll 1$ and for a correspondingly weak force f_α. Such a metric, which recovers Newton's law in the non-relativistic limit, is not unique. The metric given by equation (3) can be written as

$$g_{ij} = \begin{pmatrix} 1+h & g_{10} & g_{20} & g_{30} \\ g_{01} & -1 & 0 & 0 \\ g_{02} & 0 & -1 & 0 \\ g_{03} & 0 & 0 & -1 \end{pmatrix}, \tag{4}$$

(where $g_{0\alpha} = g_{\alpha 0} = g_\alpha$). We perform the calculations up to the first order in h, g_α and $\frac{v_\alpha}{c}$. The distance given by (3) becomes then $ds = cdt\left(1 + \frac{h}{2}\right)$ and the velocities read

$$u^0 = \frac{dx^0}{ds} = 1 - \frac{h}{2}\,, \qquad u^\alpha = \frac{dx^\alpha}{ds} = \frac{v_\alpha}{c}\,. \tag{5}$$

It is the Christoffel's symbols (affine connections)

$$\Gamma^i_{jk} = \frac{1}{2}\,g^{im}\left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m}\right) \tag{6}$$

which require more calculations. First, the contravariant metric is $g^{00} = 1 - h$, $g^{0\alpha} = g^{\alpha 0}$, $g_{0\alpha} = g_{\alpha 0}$, $g^\alpha_\beta = -\delta^\alpha_\beta$, such that $g_{im} g^{mj} = g^{jm} g_{mi} = \delta^j_i$. By making use of (6) we get

$$\left.\begin{aligned} &\Gamma^0_{00} = \frac{1}{2c}\frac{\partial h}{\partial t}\,, \qquad \Gamma^0_{0\alpha} = \Gamma^0_{\alpha 0} = \frac{1}{2}\frac{\partial h}{\partial x^\alpha} \\ &\Gamma^0_{\alpha\beta} = \Gamma^0_{\beta\alpha} = \frac{1}{2}\left(\frac{\partial g_{0\alpha}}{\partial x^\beta} + \frac{\partial g_{0\beta}}{\partial x^\alpha}\right) \\ &\Gamma^\alpha_{\beta 0} = \Gamma^\alpha_{0\beta} = \frac{1}{2}\left(\frac{\partial g_{0\beta}}{\partial x^\alpha} - \frac{\partial g_{0\alpha}}{\partial x^\beta}\right) \\ &\Gamma^\alpha_{00} = \frac{1}{2}\frac{\partial h}{\partial x^\alpha} - \frac{1}{c}\frac{\partial g_{0\alpha}}{\partial t}\,, \qquad \Gamma^\alpha_{\beta\gamma} = 0 \end{aligned}\right\}. \tag{7}$$

Now, the first equation in (2) has $\frac{du^0}{ds} = -\frac{1}{2c}\frac{\partial h}{\partial t}$ and $\Gamma^0_{jk} u^j u^k = \frac{1}{2c}\frac{\partial h}{\partial t}$, so it is satisfied identically in this approximation. The remaining equations in (2) read

$$\frac{dv_\alpha}{dt} = c^2\left(\frac{\partial g_{0\alpha}}{c\partial t} - \frac{1}{2}\frac{\partial h}{\partial x^\alpha}\right). \tag{8}$$

*The geometry of the curved spaces originates probably with Gauss (1830). It was given a sense by Riemann (*Uber die Hypothesen welche der Geometrie zugrunde liegen*, 1854), Grassman (1862), Christoffel (1869), thereafter Klein (*Erlanger Programm, Programm zum Eintritt in die philosophische Fakultät in Erlangen*, 1872), Ricci and Levi-Civita (1901). It was Einstein (1905,1916), Poincare (1905), Minkowski (1907), Sommerfeld (1910), (Kottler, 1912), Weyl (*Raum, Zeit und Materie*, 1918), Hilbert (1917) who made the connection with the physical theories. It is based on point (local) coordinate transforms, cogredient (contravariant) and contragredient (covariant) tensors and the distance element. It is an absolute calculus, as it does not depend on the point, *i.e.* the reference frame. It may be divided into the motion of a particle, the motion of the fields, the motion of the gravitational field, and their applications, especially in cosmology and cosmogony. As the curved space is universal for gravitation, so it is for the non-inertial motion, which we focus upon here. The body which creates the gravitation and the corresponding curved space is here the moving observer for the non-inertial motion, beside forces. It could be very well that the world and the motion are absolute, but they depend on subjectivity, though it could be an universal subjectivity (inter-subjectivity). See W. Pauli, *Theory of Relativity*, Teubner, Leipzig, (1921).

By comparing this with Newton's equation (1) we get the functions h and $g_{0\alpha}$ as given by

$$\frac{\partial g_{0\alpha}}{c\partial t} - \frac{1}{2}\frac{\partial h}{\partial x^{\alpha}} = \frac{f_{\alpha}}{mc^2}\,. \tag{9}$$

As it is well-known for a static gravitational potential Φ, the force is given by $f_{\alpha} = -\,m\,\frac{\partial \Phi}{\partial x^{\alpha}}$, so that $h = \frac{2\Phi}{c^2}$ and also $g_{0\alpha} = \text{const.}$*

Translations. Suppose that the force $\mathbf{f}$ is given by a static potential φ, such that $\mathbf{f} = -\frac{\partial \varphi}{\partial \mathbf{r}}$. Then $h = \frac{2\varphi}{mc^2}$ and $\mathbf{g} = \text{const.}$

Let us perform a translation

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}(t')\,, \qquad t = t'. \tag{10}$$

Then, Newton's equation $m\frac{d\mathbf{v}}{dt} = \mathbf{f}$ given by (1) becomes

$$m\,\frac{d\mathbf{v}'}{dt'} = \mathbf{f}' - m\,\frac{d\mathbf{V}}{dt'}\,, \tag{11}$$

where $\mathbf{f}'$ is the force in the new coordinates and $\mathbf{V} = \frac{d\mathbf{R}}{dt'}$ is the translation velocity. The inertial force $-\,m\frac{d\mathbf{V}}{dt'}$ appearing in (11) is accounted by the $\mathbf{g}$ in the metric of the curved space. Indeed, equation (9) gives

$$\mathbf{g} = -\frac{\mathbf{V}}{c}\,, \tag{12}$$

up to a constant. The constant reflects the principle of inertia. We may put it equal to zero. The time-dependent $\mathbf{g}$ and $\mathbf{V}$ represent a non-inertial motion. Such a non-inertial motion is therefore equivalent with a free motion in a curved space. Of course, this statement is nothing else but the principle of equivalence, or the general principle of relativity. It is however noteworthy that the non-inertial curved space depends on the observer, through the velocity $\mathbf{V}$, by virtue of the reciprocity of the motion.

Rotations. A rotation of angular frequency Ω about some axis is an orthogonal transformation of coordinates defined locally by

$$d\mathbf{r}' = d\mathbf{r} + \big(\Omega \times \mathbf{r}\big)\,dt\,, \tag{13}$$

such that the velocity is $\mathbf{v}' = \mathbf{v} + \Omega \times \mathbf{r}$ and

$$\begin{aligned} d\mathbf{v}' &= d\mathbf{v} + \big(\dot{\Omega} \times \mathbf{r}\big)dt + (\Omega \times \mathbf{v})\,dt + \\ &+ \big[\Omega \times (\mathbf{v} + \Omega \times \mathbf{r})\big]dt = \\ &= d\mathbf{v} + \big(\dot{\Omega} \times \mathbf{r}\big)dt + 2\,(\Omega \times \mathbf{v})\,dt + \big[\Omega \times (\Omega \times \mathbf{r})\big]dt\,. \end{aligned}$$

It is easy to see that in Newton's law for a particle of mass m there appears a force related to the non-uniform rotation ($\dot{\Omega}$), the Coriolis force $\sim \Omega \times \mathbf{v}$ and the centrifugal force $\sim \Omega^2$. The lagrangian $L = \frac{1}{2}\,mv'^2 - \varphi$, where φ is a potential, leads to the hamiltonian

$$\begin{aligned} H &= \frac{mv^2}{2} - \frac{m}{2}\big(\Omega \times \mathbf{r}\big)^2 + \varphi = \\ &= \frac{1}{2m}\,p^2 - \Omega\big(\mathbf{r} \times \mathbf{p}\big) + \varphi = \frac{1}{2m}\,p^2 - \Omega\mathbf{L} + \varphi\,, \end{aligned} \tag{14}$$

where $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is the angular momentum. We can see that neither the Coriolis force nor the centrifugal potential appear anymore in the hamiltonian. Instead, it contains the angular momentum.

The local coordinate transformation (13) leads to a distance given by

$$\begin{aligned} ds^2 = \left[1 + h - \frac{(\Omega \times \mathbf{r})^2}{c^2}\right](dx^0)^2 - \\ - \frac{2}{c}\big(\Omega \times \mathbf{r}\big)\,d\mathbf{r}\,dx^0 - d\mathbf{r}^2, \end{aligned} \tag{15}$$

where a static potential $\sim h$ is introduced as before, related to the potential φ in (14). It can be checked, through more laborious calculations, that the free motion in the curved space given by (15) is equivalent with the non-relativistic equations of motion given by (14).

As it is well-known, a difficulty appears however in the above metric, related to the unbounded increase with $\mathbf{r}$ of the $\Omega \times \mathbf{r}$. Therefore, we drop out the square of this term in the g_{00}-term above, and keep only the first-order contributions in $\Omega \times \mathbf{r}$ in the subsequent calculations. As one can see, this approximation does not affect the hamiltonian (14). With this approximation, the metric given by (15) is identical with the metric given by equation (4), with the identification

$$\mathbf{g} = -\frac{1}{c}\,(\Omega \times \mathbf{r})\,. \tag{16}$$

Coordinate transformations. The translation given by (10) or the rotations given by (13) correspond to local coordinate transformations. As it is well-known, we can define such transformations in general, through suitable matrices (vierbeins). They take locally the infinitesimal coordinates in a flat space into infinitesimal coordinates in a curved space. For instance, the coordinate transformation corresponding to our metric given by equation (3) is given by

$$\left.\begin{aligned} dt &= \frac{(1+h)dt' + (g + \beta\Delta)\frac{dx'}{c}}{\sqrt{(1+h)(1-\beta^2)}} \\ dx &= \frac{c\beta(1+h)\,dt' + (\beta g + \Delta)dx'}{\sqrt{(1+h)(1-\beta^2)}} \end{aligned}\right\}, \tag{17}$$

$dy = dy'$, $dz = dz'$, where $\Delta = \sqrt{1+h+g^2}$, while $\mathbf{g}$ is along $dx = dx^1$, $\beta = \frac{V}{c}$ and the velocity V is $V = \frac{dx}{dt}$ for $dx' = 0$ ($dy = dx^2$, $dz = dx^3$). The inverse of this transformation is

$$\left.\begin{aligned} dt' &= \frac{g\big(\beta dt - \frac{dx}{c}\big) + \Delta\big(dt - \frac{\beta dx}{c}\big)}{\Delta\,\sqrt{(1+h)(1-\beta^2)}} \\ dx' &= \sqrt{1+h}\,\frac{dx - c\beta\,dt}{\Delta\,\sqrt{1-\beta^2}} \end{aligned}\right\}. \tag{18}$$

*With regard to equation (3), this was for the first time when Einstein "suspected the time" (1905).

All the square roots in these equations must exist, which imposes certain restrictions upon h and β (reality conditions; in particular, $1+h>0$ and $1-\beta^2>0$).

In the local transformations given above it is assumed that there exist global transformations $x^i(x')$ and $x'^i(x)$, where x, x' stand for all x^i and, respectively, x'^i, because the coefficients in these transformations are functions of x or, respectively, x'. This restricts appreciably the derivation of metrics by means of (global) coordinate transformations, because in general, as it is well-known, the 10 elements of a metric cannot be obtained by 4 functions $x^i(x')$. Conversely, we can diagonalize the curved metric at any point, such as to reduce it to a locally flat metric (tangent space), but the flat coordinates (axes) will not, in general, be the same for all the points; they depend, in general, on the point.

One can see from (17) that in the flat limit $h,\ g\to 0$ the above transformations become the Lorentz transformations, as expected. Therefore, we may have corrections to the flat relativistic motion by first-order contributions of the parameters h and $\mathbf{g}$. Indeed, in this limit, the transformation (18) becomes

$$\left.\begin{aligned} dt &= \frac{\left(1+\frac{h}{2}\right)dt' + (g+\beta)\frac{dx'}{c}}{\sqrt{1-\beta^2}} \\ dx &= \frac{c\beta\left(1+\frac{h}{2}\right)dt' + (g\beta+1)\,dx'}{\sqrt{1-\beta^2}} \end{aligned}\right\} . \tag{19}$$

which include corrections to the Lorentz transformations, due to the curved space.

The metric given by (3) provides the proper time

$$d\tau = \sqrt{1+h}\,dt\,, \tag{20}$$

corresponding to $dx^\alpha = 0$. The metric given by (3) can also be written as

$$\begin{aligned} ds^2 &= c^2(1+h)\left[dt + \frac{1}{c\,(1+h)}\,\mathbf{g}\,d\mathbf{r}\right]^2 - \\ &\quad - \left[d\mathbf{r}^2 + \frac{1}{1+h}(\mathbf{g}\,d\mathbf{r})^2\right], \end{aligned} \tag{21}$$

hence the length given by

$$dl^2 = d\mathbf{r}^2 + \frac{1}{1+h}\,(\mathbf{g}\,d\mathbf{r})^2 \tag{22}$$

and the time

$$dt' = \sqrt{1+h}\cdot\left[dt + \frac{1}{c\,(1+h)}\,\mathbf{g}\,d\mathbf{r}\right], \tag{23}$$

corresponding to the length dl. The difference $\Delta t = \frac{\mathbf{g}d\mathbf{r}}{c(1+h)}$ between the two times, $dt_1 = \frac{d\tau}{\sqrt{g_{00}}} = dt$ in the proper time (20) and $dt_2 = \frac{dt'}{\sqrt{g_{00}}} = dt + \frac{\mathbf{g}d\mathbf{r}}{c(1+h)}$ in the time given by (23), gives the difference in the synchronization of two simultaneous events, infinitesimally separated. The difference in time depends on the path followed to reach a point starting from another point.

We limit ourselves to the first order in h, $\mathbf{g}$, and put $\mathbf{g} = -\frac{\mathbf{V}}{c}$, in order to investigate corrections to the motion under the action of a weak force in a flat space moving with a non-uniform velocity $\mathbf{V}$ with respect to the observer. We will do the calculations basically for translations but a similar analysis can be made for rotations, using equation (16). For the observer, such a motion is then a free motion in a curved space with metric (3). The proper time is then $d\tau = \left(1+\frac{h}{2}\right)dt$, the time given by (23) becomes $dt' = \left(1+\frac{h}{2}\right)dt + \frac{\mathbf{g}d\mathbf{r}}{c}$ and the length is given by $dl^2 = d\mathbf{r}^2$, as for a three-dimensional euclidean space.

Hamilton-Jacobi equation. Let us assume that we have a particle moving freely in a flat space. We denote its contravariant momentum by $(P_0 = \frac{E_0}{c}, \mathbf{P})$ and the corresponding covariant momentum by $(P_0 - \mathbf{P})$, such that $P_0^2 - P^2 = m^2c^2$, where E_0 is the energy of the particle, and $P_0, \mathbf{P}$ are constant.

We can use the coordinate transformation given by (18) to get the momentum of the particle in the curved space. We prefer to write it down in its covariant form, using the metric (4). We get

$$\left.\begin{aligned} p_0 &= (1+h)p^0 + gp^1 = \sqrt{1+h}\,\frac{P_0 - \beta P_1}{\sqrt{1-\beta^2}} \\ p_1 &= gp^0 - p^1 = \frac{(g+\beta\Delta)P_0 - (g\beta+\Delta)P_1}{\sqrt{(1+h)(1-\beta^2)}} \end{aligned}\right\} . \tag{24}$$

Then, it seems that we would have already an integral of motion for the motion in the curved space, by using the definition $p_i = mc\frac{du_i}{ds}$. However, this is not true, because the p_i are at point x' in the curved space, while the coefficients in the transformation (18) are at point x in the flat space. To know the global coordinate transformations $x(x')$ and $x'(x)$ would amount to solve in fact the equations of motion.

We can revert the above transformations for P_0 and P_1, and make use of $P_0^2 - P^2 = m^2c^2$, with $p_2 = -P_2$, $p_3 = -P_3$ for $g = -\beta$. We get

$$\left(p_0 + gp_1\right)^2 - \Delta^2\left(p^2 + m^2c^2\right) = 0\,, \tag{25}$$

or

$$\left(E - c\mathbf{g}\mathbf{p}\right)^2 - c^2\left(1+h+g^2\right)\left(p^2+m^2c^2\right) = 0\,, \tag{26}$$

where E is the energy of the particle and $\mathbf{p}$ denotes its three-dimensional momentum. This is the relation between energy and momentum for the motion in the curved space. It gives the Hamilton-Jacobi equation.

Indeed, $p_i = -\frac{\partial S}{\partial x^i}$ and, obviously, for a free particle, $p_i p^i$ is a constant; we put $p_i p^i = m^2c^2$ and get $g^{ij}p_i p_j = m^2c^2$ or

$$\left(\frac{\partial S}{\partial t} + c\mathbf{g}\frac{\partial S}{\partial \mathbf{r}}\right)^2 - c^2\left(1+h+g^2\right)\left[\left(\frac{\partial S}{\partial \mathbf{r}}\right)^2 + m^2c^2\right] = 0. \tag{27}$$

In the limit $h=\frac{2\varphi}{mc^2}\to 0$ and $\mathbf{g}=-\frac{\mathbf{V}}{c}\to 0$ it describes the relativistic motion of a particle under the action of the (weak) force $\mathbf{f}=-\frac{\partial\varphi}{\partial\mathbf{r}}$ and for an observer moving with a (small) velocity $\mathbf{V}$. One can check directly that the coordinate transformations given by equation (19) takes the free Hamilton-Jacobi equation $\left(\frac{\partial S}{\partial t}\right)^2 - c^2\left[\left(\frac{\partial S}{\partial\mathbf{r}}\right)^2 + m^2c^2\right]=0$ into the "interacting" Hamilton-Jacobi equation (27), as expected.

The eikonal equation. Waves move through $k_i dx^i = -d\Phi$, where $k_i = -\frac{\partial\Phi}{\partial x^i} = \left(\frac{\omega}{c},\mathbf{k}\right)$, ω is the frequency, $\mathbf{k}$ is the wavevector and Φ is called the eikonal. In a flat space k_i are constant, and the wave propagates along a straight line, such that $k_i k^i = 0$, *i.e.* $\frac{\omega^2}{c^2} - k^2 = 0$ and $\Phi = -\omega t + \mathbf{kr}$. This is a light ray. In a curved space $k_i k^i = 0$ reads $g^{ij}k_i k_j = 0$, and for g^{ij} slightly departing from the flat metric we have the geometric approximation to the wave propagation. It is governed by the eikonal equation $g^{ij}\left(\frac{\partial\Phi}{\partial x^i}\right)\left(\frac{\partial\Phi}{\partial x^j}\right)=0$, or

$$\left(\frac{1}{c}\frac{\partial\Phi}{\partial t} + \mathbf{g}\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 - \left(1+h+g^2\right)\left(\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 = 0\,, \qquad (28)$$

which is the Hamilton-Jacobi equation (27) for $m=0$.

We neglect the g^2-contributions to this equation and notice that the first term may not depend on the time (h is a function of the coordinates only). It follows then that the first term in the above equation can be put equal to $\frac{\omega_0}{c}$,

$$\frac{1}{c}\frac{\partial\Phi}{\partial t} + \mathbf{g}\frac{\partial\Phi}{\partial\mathbf{r}} = -\frac{\omega_0}{c}\,, \qquad (29)$$

where ω_0 is the frequency of the wave in the flat space, and

$$\left(\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 = k^2 = \frac{1}{1+h}\left(\frac{\omega_0}{c}\right)^2 = \frac{1}{1+h}\,k_0^2\,, \qquad (30)$$

where $\mathbf{k}_0$ is the wavevector in the flat space. Within our approximation equation (29) becomes

$$\frac{\partial\Phi}{c\partial t} = -\frac{\omega_0}{c} - \mathbf{g}\mathbf{k}_0\,. \qquad (31)$$

We measure the frequency ω corresponding to the proper time, *i.e.* $\frac{\omega}{c} = -\frac{\partial\Phi}{c\partial\tau}$, where $d\tau = \sqrt{1+h}\,dt$ for our metric, so the measured frequency of the wave is given by

$$\frac{\omega}{c} = -\frac{\partial\Phi}{c\partial\tau} = -\frac{1}{\sqrt{1+h}}\frac{\partial\Phi}{c\partial t} = \frac{1}{\sqrt{1+h}}\frac{\omega_0}{c} + \mathbf{g}\mathbf{k}_0\,. \qquad (32)$$

There exists, therefore, a shift in frequency

$$\frac{\Delta\omega}{\omega_0} = -\frac{h}{2} + \frac{c\,\mathbf{g}\mathbf{k}_0}{\omega_0}\,. \qquad (33)$$

The first term in equation (33) is due to the static forces (like the gravitational potential, for instance), while the second term is analogous to the (longitudinal) Doppler effect, for $\mathbf{g}=-\frac{\mathbf{V}}{c}$.

By (30) we have

$$\left(\frac{\partial\Phi}{\partial\mathbf{r}}\right)^2 = (1-h)\,k_0^2\,. \qquad (34)$$

We assume that h depends only on the radius r, and write the above equation in spherical coordinates; Φ does not depend on θ, and we put $\theta = \frac{\pi}{2}$;

$$\left(\frac{\partial\Phi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\Phi}{\partial\varphi}\right)^2 = (1-h)\,k_0^2\,; \qquad (35)$$

the solution is of the form

$$\Phi = \Phi_r(r) + M\varphi\,, \qquad (36)$$

where M is a constant and

$$\Phi_r(r) = \int_\infty^r dr\cdot\sqrt{(1-h)\,k_0^2 - \frac{M^2}{r^2}}\,; \qquad (37)$$

the trajectory is given by $\frac{\partial\Phi}{\partial M} = \text{const}$,* hence

$$\varphi = -\int_\infty^r dr\cdot\frac{M}{r^2\sqrt{(1-h)\,k_0^2 - \frac{M^2}{r^2}}}\,. \qquad (38)$$

For $h=0$ we get $r\sin\varphi = \frac{M}{k_0}$, which is a straight line passing at distance $\frac{M}{k_0}$ from the centre. The deviation angle is

$$\Delta\varphi = -\frac{k_0^2}{2}\int_\infty^r dr\,\frac{h\,M}{r^2\left(k_0^2 - \frac{M^2}{r^2}\right)^{3/2}}\,. \qquad (39)$$

Therefore, the light ray is bent by the static forces in a curved space.† One can also define the refractive index $\mathbf{n}$ of the curved space, by $\mathbf{k} = \mathbf{n}\frac{\omega}{c}$. Its magnitude is related to $\mathbf{g}\mathbf{k}_0$, while its direction is associated to the inhomogeneity h of the space.

It is worth noting, by (31), that the time-dependent part of the eikonal is given by

$$\Phi_t(t) = -\omega_0 t + \mathbf{k_0}\mathbf{R}(t)\,, \qquad (40)$$

for $\mathbf{g}=-\frac{\mathbf{V}}{c}$, *i.e.* the eikonal corresponding to a translation, as expected. A similar solution of the Hamilton-Jacobi equation can be obtained for massive particles.

Quantization. Suppose that we have a free motion. Then we know its solution, *i.e.* the dependence of the coordinates, say some x, on some parameter, which may be called some

*Constant M is a generalized moving freely coordinate; therefore, the force acting upon it vanishes, $\frac{\partial L}{\partial M}=0$, or $\frac{d(\partial S/\partial M)}{dt}=0$, *i.e.* $\frac{\partial S}{\partial M} = \text{const}$.

†The metric given by (3) for $h=\frac{2\varphi}{c^2}$ differs from the metric created by a gravitational point mass m with $\varphi = \frac{Gm}{r}$; they coincide only in the non-relativistic limit. The deviation angle given by (39) for a gravitational potential is smaller by a factor of 4 than the deviation angle in the gravitational potential of a point mass.

time t. Suppose further that we have a motion under the action of some forces. Then, we know the dependence of its coordinates, say some x', on some parameter, which may be the same t as in the former case. Then, we may establish a correspondence between x and x', *i.e.* a global coordinate transformation. It follows that the motion under the action of the forces is a global coordinate transformation applied to the free motion. Similarly, two distinct motions are put in relation to each other by such global coordinate transformations.

This line of thought, due to Einstein, lies at the basis of both the special theory of relativity and the general theory of relativity.

Indeed, it has beeen noticed that the equations of the electromagnetic field are invariant under Lorentz transformations of the coordinates, which leave the distance given by $s^2 = c^2t^2 - \mathbf{r}^2$ invariant. These transformations are an expression of the principle of inertia, and this invariance is the principle of relativity. As such, the Lorentz transformations are applicable to the motion of particles, starting, for instance, from a particle at rest. Let $x = \frac{c\beta\tau}{\sqrt{1-\beta^2}}$, $t = \frac{\tau}{\sqrt{1-\beta^2}}$ be these Lorentz transformations where τ is the time of the particle at rest. We may apply these transformations to the momentum $\mathbf{p} = \frac{\partial S}{\partial \mathbf{r}}$ and $p_0 = -\frac{\partial S}{c\,\partial t} = \frac{E}{c}$, where E is the energy of the particle. Then, we get immediately $\mathbf{p} = \frac{\mathbf{v}E}{c^2}$ and $E = \frac{E_0}{\sqrt{1-\beta^2}}$. The non-relativistic limit is recovered for $E_0 = mc^2$, the "inertia of the energy". The equations of motion are $\frac{d\mathbf{p}}{dt} = \mathbf{f}$, and we can see that indeed, there appear additional, "dynamic forces", depending on relativistic $\frac{v^2}{c^2}$-terms, in comparison with Newton's law. In adition, we get the Hamilton-Jacobi equation $E^2 - c^2\left(p^2 + m^2c^2\right) = 0$. This is the whole theory of special relativity.

The situation is similar in the general theory of relativity, except for the fact that in a curved space we have not the global coordinate transformations, in general, as in a flat space. However, the Hamilton-Jacobi equation gives access to the action function, which may provide a relationship between some integrals of motion. Action S depends on some constants of integration, say M. Then, these constants can be viewed as freely-moving generalized coordinates, so $\frac{\partial S}{\partial M} = \text{const}$, because the force $\frac{\partial L}{\partial M} = \frac{d(\partial S/\partial M)}{dt}$ vanishes. Equation $\frac{\partial S}{\partial M} = \text{const}$ provides the equation of the trajectory. Of course, this is based upon the assumption that the motion is classical, *i.e.* non-quantum, in the sense that there exists a trajectory. For instance, the solution of the Hamilton-Jacobi equation for a free particle is $S = -Et + \mathbf{pr}$, where E and $\mathbf{p}$ are constants such that $E = \sqrt{m^2c^4 + c^2p^2}$. By $\frac{\partial S}{\partial E} = \text{const}$ we get $-t + \frac{E}{c^2p^2}\,\mathbf{pr} = \text{const}$, which is the trajectory of a free particle.

For a classical motion it is useless to attempt to solve the motion in a curved space produced by a non-inertial motion, like non-uniform translations, because it is much simpler to solve the motion in the absence of the non-inertial motion and then get the solution by a coordinate transformation, like a non-uniform translation for instance. For a quantum motion, however, the things change appreciably.

The Hamilton-Jacobi equation admits another kind of motion too, the quantum motion. Obviously, for a free particle, the classical action given above is the phase of a wave. Then, it is natural to introduce a wavefunction ψ through $S = -i\hbar\ln\psi$, where $\hbar$ turns out to be Planck's constant. The classical motion is recovered in the limit $\hbar \to 0$, $\mathrm{Re}\psi = \text{finite}$ and $\mathrm{Im}\psi \to \infty$, such that $S = \text{finite}$. With this transformation we have $\mathbf{p} = -i\hbar\frac{\partial\psi/\partial\mathbf{r}}{\psi}$ and $E = i\hbar\frac{\partial\psi/\partial t}{\psi}$, which means that momentum and energy are eigenvalues of their corresponding operators, $-i\hbar\frac{\partial}{\partial\mathbf{r}}$ and $i\hbar\frac{\partial}{\partial t}$, respectively.* It follows that the physical quantities have not well-defined values anymore, in contrast to the classical motion. In particular, there is no trajectory of the motion. Instead, they have mean values and deviations, *i.e.* they have a statistical meaning, and the measurement process has to be defined in such terms. It turns out that the wavefunction squared is just the density of probability for the motion to be in some quantum state, and for a defined motion this probability must be conserved.

Klein-Gordon equation. With the substitution $E \to i\hbar\frac{\partial}{\partial t}$ and $\mathbf{p} \to -i\hbar\frac{\partial}{\partial\mathbf{r}}$ in the Hamilton-Jacobi equation in the flat space we get the Klein-Gordon equation

$$\frac{\partial^2\psi}{\partial t^2} - c^2\,\frac{\partial^2\psi}{\partial\mathbf{r}^2} + \frac{m^2c^4}{\hbar^2}\,\psi = 0\,. \qquad (41)$$

A similar quantization for the Hamilton-Jacobi equation given by (27) encounters difficulties, since the operators $1 + h + g^2$ and $p^2 + m^2c^2$ do not commute with each other, nor with the operator $E - c\mathbf{g}\mathbf{p}$.† We may neglect the g^2-term in $1 + h + g^2$, and write the Hamilton-Jacobi equation (27) as

$$\frac{1}{1+h}\left(E - c\mathbf{g}\mathbf{p}\right)^2 = c^2\left(p^2 + m^2c^2\right), \qquad (42)$$

where the two operators in the left side of this equation commute now, up to quantities of the order of hg (or higher), which we neglect. With these approximations, the quantization rules can now be applied, and we get an equation which can be written as

$$\left(\frac{\partial}{\partial t} + c\mathbf{g}\,\frac{\partial}{\partial\mathbf{r}}\right)^2\psi - c^2(1+h)\left[\frac{\partial^2\psi}{\partial\mathbf{r}^2} - \frac{m^2c^2}{\hbar^2}\,\psi\right] = 0. \quad (43)$$

It can be viewed as describing the quantum motion of a particle under the action of a weak force $-\frac{mc^2}{2}\frac{\partial h(\mathbf{r})}{\partial\mathbf{r}}$, as seen by an observer moving with the small velocity $-c\mathbf{g}(t)$. It can

*Einstein's (1905) quantization of energy and de Broglie's (1923) quantization of momentum follow immediately by this assumption, which gives a meaning to the Bohr-Sommerfeld quantization rules (Bohr, 1913, Sommerfeld, 1915). The quantum operators was first seen as matrices by Heisenberg, Born, Jordan, Pauli (1925-1926).

†We recall that h is a function of the coordinates only, $h(\mathbf{r})$, and $\mathbf{g}$ is a function of the time only, $\mathbf{g}(t)$.

be derived directly from (41) by the coordinate transformations (19), in the limit $h, \mathbf{g} \to 0$.* It is worth noting, however, that there is still a slight inaccuracy in deriving this equation, arising from the fact that the operator $(1+h)(p^2+m^2c^2)$ is not hermitean. It reflects the indefineteness in writing $(1+h)(p^2+m^2c^2)$ or $(p^2+m^2c^2)(1+h)$ when passing from (42) to (43). This indicates the ambiguities in quantizing the relativistic motion, and they are remedied by the theory of the quantal fields, as it is shown below.

The above equation can be written more conveniently as

$$\left(i\hbar\frac{\partial}{\partial t} - c\mathbf{g}\mathbf{p}\right)^2 \psi - c^2(1+h)\left(p^2+m^2c^2\right)\psi = 0\,, \quad (44)$$

where $\mathbf{p} = -i\hbar\frac{\partial}{\partial \mathbf{r}}$ and $i\hbar\frac{\partial}{\partial t}$ stands for the energy E.

We introduce the operator

$$\begin{aligned} H^2 &= c^2(1+h)\left(p^2+m^2c^2\right) = \\ &= c^2\left(p^2+m^2c^2\right) + c^2h\left(p^2+m^2c^2\right), \end{aligned} \quad (45)$$

which is time-independent, and treat the h-term as a small perturbation. It is easy to see, in the first-order of the perturbation theory, that the wavefunctions are labelled by momentum $\mathbf{p}$, and are plane waves with a weak admixture of plane waves of the order of h; we denote them by $\varphi(\mathbf{p})$. Similarly, in the first-order of the perturbation theory, the eigenvalues of H^2 can be written as $E^2(p) = c^2(1+\bar{h})(p^2+m^2c^2)$, where $\bar{h} = \frac{1}{V}\int d\mathbf{r}\cdot h$, V being the volume of normalization. We have, therefore, $H^2\varphi(\mathbf{p}) = E^2(p)\,\varphi(\mathbf{p})$. Now, we look for a time-dependent solution of equation (44) $\left(i\hbar\frac{\partial}{\partial t} - c\mathbf{g}\mathbf{p}\right)^2\psi = H^2\psi$, which can also be written as $\left(i\hbar\frac{\partial}{\partial t} - c\mathbf{g}\mathbf{p}\right)\psi = H\psi$, where ψ is a superposition of eigenfunctions

$$\psi = \sum_{\mathbf{p}} c_{\mathbf{p}}(t)\; e^{-iE(p)t/\hbar}\; \varphi(\mathbf{p})\,. \quad (46)$$

We get

$$\dot{c}_{\mathbf{p}'} = -\frac{i}{\hbar}\sum_{\mathbf{p}} c_{\mathbf{p}}\, e^{-i[E(p)-E(p')]/\hbar}\, c\mathbf{g}\mathbf{p}_{\mathbf{p}'\mathbf{p}}\,, \quad (47)$$

where $\mathbf{p}_{\mathbf{p}'\mathbf{p}}$ is the matrix element of the momentum $\mathbf{p}$ betwen the states $\varphi(\mathbf{p}')$ and $\varphi(\mathbf{p})$. We assume $c_{\mathbf{p}} = c^0_{\mathbf{p}} + c^1_{\mathbf{p}}$, such as $c^0_{\mathbf{p}'} = 0$ for all $\mathbf{p}' \neq \mathbf{p}$ and $c^0_{\mathbf{p}} = 1$, and get

$$\dot{c}^1_{\mathbf{p}'} = -\frac{i}{\hbar}\, e^{-i[E(p)-E(p')]/\hbar}\, c\mathbf{g}\mathbf{p}_{\mathbf{p}'\mathbf{p}}\,, \quad (48)$$

which can be integrated straightforwardly. The square $|c^1_{\mathbf{p}'}|^2$ gives the transition probability from state $\varphi(\mathbf{p})$ in state $\varphi(\mathbf{p}')$.

It follows that an observer in a non-uniform translation might see quantum transitions between the states of a relativistic particle, providing the frequencies in the Fourier expansion of $\mathbf{g}(t)$ match the difference in the energy levels. In the zeroth-order of the perturbation theory the eigenfunctions $\varphi(\mathbf{p})$ are plane waves, and the matrix elements $\mathbf{p}_{\mathbf{p}'\mathbf{p}}$ of the momentum vanish, so there are no such transitions to this order. In general, if the total momentum is conserved, as for free or interacting particles, these transitions do not occur. In the first order of the perturbation theory for the external force represented by h the matrix elements of the momentum do not vanish, in general, and we may have transitions, as an effect of a non-uniform translation. Within this order of the perturbation theory the matrix elements of the momentum are of the order of h, and the transition amplitudes given by (48) are of the order of gh. We can see that the time-dependent term of the order of gh neglected in deriving equation (44) produces corrections to the transition amplitides of the order of gh^2, so its neglect is justified.

In general, the solution of the second-order differential equation (43) can be approached by using the Fourier transform. Then, it reduces to a homogeneous matricial equation, where labels are the frequency and the wavevector $(\omega, \mathbf{k})$, conveniently ordered. The condition of a non-trivial solution is the vanishing of the determinant of such an equation. This gives a set of conditions for the ordered points $(\omega, \mathbf{k})$ in the $(\omega, \mathbf{k})$-space, but these conditions do not provide anymore an algebraic connection between the frequency ω and the wavevector $\mathbf{k}$. This amounts to saying that for a given ω the wavevectors are not determined, and, conversely, for a given wavevector $\mathbf{k}$ the frequencies are not determined, *i.e.* the quantum states do not exist in fact, anymore. The particle exhibits quantum transitions, which make its quantum state undetermined. The same conclusion can also be seen by introducing a non-uniform translation in the phase of a plane wave, expanding the plane wave with respect to this translation, under certain restrictions, and then using the time Fourier expansion of the translation. The frequency of the original plane wave changes correspondingly, which indicates indeed that there are quantum transitions. One may say that for a curved space as the one represented by the metric given here, the quantization question has no meaning anymore, or it has the meaning given here.

In the non-relativistic limit, the above Klein-Gordon equation becomes

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi = \left(mc^2 + \frac{p^2}{2m} + \varphi\right)\psi + c\mathbf{g}\mathbf{p}\psi\,, \quad (49)$$

which is Schrödinger's equation up to the rest energy mc^2, and one can see more directly the perturbation $c\mathbf{g}\mathbf{p} = -\mathbf{V}\mathbf{p}$. It is worth noting that the derivation of Schrödinger's equation holds irrespectively of the ambiguities related to the quantization of the Hamilton-Jacobi equation. It follows, that under the conditions mentioned above, *i.e.* in the presence of a (non-trivial) external field φ, an observer in a non-uniform translation may observe quantum transitions in the non-

*It has to be compared with the Klein-Gordon equation written as $\left(i\hbar\frac{\partial}{\partial t} - e\varphi\right)^2\psi - c^2\left[\left(i\hbar\frac{\partial}{\partial \mathbf{r}} + \frac{e\mathbf{A}}{c}\right)^2 + m^2c^2\right] = 0$ for a particle with charge e in the electromagnetic field $(\varphi, \mathbf{A})$, which, hystorically, was first considered for the Hydrogen atom (Schrödinger, Klein, Gordon, 1926). There, the forces come by the electromagnetic gauge field.

relativistic limit, due to the non-inertial motion.* Obviously, the frequency of this motion must match the quantum energy gaps, for such transitions to be observed.

Similar considerations hold for the metric corresponding to rotations. It is the hamiltonian (14) which is subjected to quantization in that case, so we may have quantum transitions between the states of the particle, providing these states do not conserve the angular momentum. This requires a force, as the one given by a potential φ. The $\Omega \times \mathbf{r}$ is exactly the rotation velocity $\mathbf{V}$, so we can apply directly the formalism developed above for a non-uniform translation to a non-uniform rotation. The only difference is that the $\mathbf{g}$ for rotations depends on the spatial coordinates too, beside its time dependence. The $\mathbf{g}$-interaction gives rise to terms of the type $\Omega\mathbf{L}$, and the evaluation of the matrix elements in the interacting terms becomes more cumbersome. It is worth keeping in mind the condition $\Omega r \ll c$ in such evaluations.

The difficulties encountered above with the quantization of the Klein-Gordon equation in curved spaces remain for a corresponding Dirac equation. It is impossible, in general, to get a Dirac equation for equation (43), because the operators $\left(1-\frac{h}{2}\right)(E-c\mathbf{g}\mathbf{p})$ and $\alpha c\mathbf{p}+\beta mc^2$ (with α and β the Dirac matrices), which represent the square roots of the two sides of equation (42), do not commute anymore. Nevertheless, if we limit ourselves to the first order of the perturbation theory, we can see that the operator H^2 defined above reduces to $c^2(p^2+m^2)$ providing we redefine the energy levels such as to include the factor $1+\bar{h}$. Within this approximation, we get the Dirac equation

$$\left(i\hbar\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)\psi=(\alpha c\mathbf{p}+\beta mc^2)\psi\,, \tag{50}$$

where ψ contains now a weak admixture of plane waves, of the order of h. It is worth noting that this equation is the Dirac equation corresponding to (41), subjected to the translation $\mathbf{r}=\mathbf{r}'+\mathbf{R}$, and $t=t'$. The non-uniform translation in the left side of equation (50) gives now quantum transitions.

As it is well-known, there remain problems with the quantization of the Klein-Gordon equation, which are not solved by the Dirac equation. These problems find for themselves a natural solution with the quantum fields.

A scalar field in a curved space. Let

$$S=\int dx^0 d\mathbf{r}\,\sqrt{-g}\,\left[(\partial_i\psi)(\partial^i\psi)+\frac{m^2c^2}{\hbar^2}\,\psi^2\right] \tag{51}$$

be the lagrangian for the (real) scalar field ψ, where $g=-\Delta^2=-(1+h+g^2)$ is the determinant of the metric given by (4).† It is easy to see that the principle of least action for ψ in a flat space leads to the Klein-Gordon equation (41). For the metric given by (4), and neglecting g^2-terms, we get a generalized Klein-Gordon equation

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+c\mathbf{g}\,\frac{\partial}{\partial\mathbf{r}}\right)\frac{1}{\sqrt{1+h}}\left(\frac{\partial}{\partial t}+c\mathbf{g}\,\frac{\partial}{\partial\mathbf{r}}\right)\psi-\\&\quad-c^2\,\frac{\partial}{\partial\mathbf{r}}\sqrt{1+h}\,\frac{\partial}{\partial\mathbf{r}}\,\psi+\sqrt{1+h}\,\frac{m^2c^4}{\hbar^2}\,\psi=0\,.\end{aligned} \tag{52}$$

We can apply the same perturbation approach to this equation as we did for equation (42). Doing so, we get equation (44) and an additional term $i\,\frac{c^2\hbar}{2}\frac{\partial h}{\partial\mathbf{r}}\,\mathbf{p}$, which yields no difficulties in the perturbation approach. The resulting equation reads

$$\begin{aligned}&\left(i\hbar\,\frac{\partial}{\partial t}-c\mathbf{g}\mathbf{p}\right)^2\psi-c^2(1+h)\left(p^2+m^2c^2\right)\psi+\\&\qquad+\frac{i\,c^2\hbar}{2}\left(\frac{\partial h}{\partial\mathbf{r}}\right)\mathbf{p}\psi=0\,.\end{aligned} \tag{53}$$

It is worth noting that in the limit $\mathbf{g}\to 0$ this is an exact equation. The qualitative conclusions derived above for equation (44), as regards the quantum transitions produced by the non-uniform translation, remain valid, though, we have now a language of fields. It follows that a quantum particle, either relativistic or non-relativistic, in a curved space of the form analyzed herein becomes a wave packet from a plane wave (or even forms a bound state), as a consequence of the forces, and, at the same time, it may suffer quantum transitions, due to the time-dependent metric (as if in a non-inertial translation for instance). This gives no meaning to the problem of the quantization in curved spaces, or it gives the meaning discussed here.

The density L of lagrangian in the action $S=\int dt\,d\mathbf{r}\cdot L$ given by (51) gives the momentum $\Pi=\frac{\partial L}{\partial(\partial\psi/\partial t)}$ and the hamiltonian density $\Pi\,\frac{\partial\psi}{\partial t}-L$. The quantized field reads

$$\psi=\sum_{\mathbf{p}}\frac{c\hbar}{2\sqrt{\varepsilon}}\left(a_{\mathbf{p}}\,e^{-i\varepsilon t/\hbar+i\mathbf{p}\mathbf{r}/\hbar}+a_{\mathbf{p}}^{+}\,e^{i\varepsilon t/\hbar-i\mathbf{p}\mathbf{r}/\hbar}\right), \tag{54}$$

and

$$\Pi=-i\sum_{\mathbf{p}}\frac{\sqrt{\varepsilon}}{c}\left(a_{\mathbf{p}}\,e^{-i\varepsilon t/\hbar+i\mathbf{p}\mathbf{r}/\hbar}-a_{\mathbf{p}}^{+}\,e^{i\varepsilon t/\hbar-i\mathbf{p}\mathbf{r}/\hbar}\right), \tag{55}$$

where $\varepsilon=c\sqrt{m^2c^2+p^2}$ and $\left[\psi(t,\mathbf{r}),\Pi(t,\mathbf{r}')\right]=i\hbar\delta(\mathbf{r}-\mathbf{r}')$ with usual commutation relations for the bosonic operators $a_{\mathbf{p}}$, $a_{\mathbf{p}}^{+}$ and a normalization of one $\mathbf{p}$-state in a unit volume. The hamiltonian is obtained by integrating its density given

*A suitable unitary transformation of the wavefunction — for instance, $\exp\left(-i\,\frac{\mathbf{R}\mathbf{p}}{\hbar}\right)$ — can produce such an interaction in the time-dependent left side of the Schrödinger equation, but, at the same time, it produces an equivalent interaction in the hamiltonian, such that the Schrödinger equation is left unchanged. Such unitary transformations are related to symmetries (Wigner's theorem, 1931) and they are different from a change of coordinates.

†In general, the action for fields must be written by replacing the flat metric η_{ij} by the curved metric g_{ij} (including $\sqrt{-g}$ in the elementary volume of integration) and replacing the derivatives ∂_i by covariant derivatives D_i. The latter requirement can produce technical difficulties, in general. However, for a scalar field or for the electromagnetic field the D_i has the same effect as ∂_i, so the former are superfluous.

above over the whole space. It can be written as $H = H_0 + H_{1h} + H_{1g}$, where

$$H_0 = \int d\mathbf{r} \cdot \left[\frac{1}{4} c^2 \Pi^2 + \left(\frac{\partial \psi}{\partial \mathbf{r}} \right)^2 + \frac{m^2 c^2}{\hbar^2} \psi^2 \right] = \sum_{\mathbf{p}} \frac{\varepsilon}{2} \left(a_{\mathbf{p}} a^+_{\mathbf{p}} + a^+_{\mathbf{p}} a_{\mathbf{p}} \right) \tag{56}$$

is the free hamiltonian,

$$H_{1h} = \int d\mathbf{r} \times \times \left(\sqrt{1+h} - 1 \right) \left[\frac{1}{4} c^2 \Pi^2 + \left(\frac{\partial \psi}{\partial \mathbf{r}} \right)^2 + \frac{m^2 c^2}{\hbar^2} \psi^2 \right] \tag{57}$$

is the interacting part due to the external field h and

$$H_{1g} = -\frac{c}{2} \int d\mathbf{r} \cdot \left[\Pi \left(\mathbf{g} \frac{\partial \psi}{\partial \mathbf{r}} \right) + \left(\mathbf{g} \frac{\partial \psi}{\partial \mathbf{r}} \right) \Pi \right] = -\frac{c}{2} \sum_{\mathbf{p}} (\mathbf{g}\mathbf{p}) (a_{\mathbf{p}} a^+_{\mathbf{p}} + a^+_{\mathbf{p}} a_{\mathbf{p}}) \tag{58}$$

is the time-dependent interaction. Perturbation theory can now be applied systematically to the first-order of $\mathbf{g}$ and all the orders of h, with the same results as those described above: the quanta will scatter both their wavevectors and their energy. Similar field theories can be set up for charged particles, or for particles with spin $\frac{1}{2}$ and for photons, moving in a curved space given by the metric (4).

Electromagnetic field in curved spaces. Photons. The action for the electromagnetic field is

$$S = -\frac{1}{16\pi c} \int dx^0 d\mathbf{r} \cdot \sqrt{-g}\, F_{ij} F^{ij}, \tag{59}$$

where the electromagnetic fields F_{ij} are given by the potentials A_i through $F_{ij} = \partial_i A_j - \partial_j A_i$. This leads immediately to the first pair of Maxwell equations (the free equations) $\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0$ and the principle of least action gives the second pair of Maxwell equations

$$\partial_j \left(\sqrt{-g}\, F^{ij} \right) = 0. \tag{60}$$

In the presence of charges and currents the right side of equation (60) contains the current, conveniently defined. The antisymmetric tensor F_{ij} consists of a vector and a three-tensor in spatial components, the latter being representable by another vector, its dual. Let these vectors be denoted by $\mathbf{E}$ and $\mathbf{B}$. Similarly, by raising or lowering the suffixes we can define other two vectors, related to the former pair of vectors, and denoted by $\mathbf{D}$ and $\mathbf{H}$. Then, the Maxwell equations obtained above take the usual form of Maxwell equations in matter, namely $\mathrm{curl}\, \mathbf{E} = -\frac{1}{c\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\mathbf{B})}{\partial t}$, $\mathrm{div}\, \mathbf{B} = 0$ (the free equations) and $\mathrm{div}\, \mathbf{D} = 4\pi\rho$, $\mathrm{curl}\, \mathbf{H} = \frac{1}{c\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\mathbf{D})}{\partial t} + \frac{4\pi}{c} \rho \mathbf{v}$, where ρ is the density of charge divided by $\sqrt{\gamma}$ and $\gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}$ is the spatial metric (div and curl are conveniently defined in the curved space). For our metric, and neglecting g^2, the matrix γ reduces to the euclidean metric of the space ($\gamma = 1$).

We use $A_0 = 0$, $F_{0\alpha} = \partial_0 A_\alpha$ and $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. We define an electric field $\mathbf{E} = \mathrm{grad}\, \mathbf{A}$ and a magnetization field $\mathbf{B} = -\mathrm{curl}\, \mathbf{A}$. Then, neglecting g^2, equation (60) can be written as

$$\mathrm{div} \left[\frac{1}{\Delta} (\mathbf{E} + g \times \mathbf{B}) \right] = 0 \tag{61}$$

and

$$\frac{\partial}{c\,\partial t} \left[\frac{1}{\Delta} (\mathbf{E} + \mathbf{g} \times \mathbf{B}) \right] = \mathrm{curl} \left[\Delta \mathbf{B} + \frac{1}{\Delta} \mathbf{g} \times \mathbf{E} \right], \tag{62}$$

where $\Delta = \sqrt{1+h}$. One can see that we may have a displacement field $\mathbf{D} = \frac{\mathbf{E}+\mathbf{g}\times\mathbf{B}}{\Delta}$ and a magnetic field $\mathbf{H} = \Delta\mathbf{B} + \frac{\mathbf{g}\times\mathbf{E}}{\Delta}$, and the Maxwell equations $\mathrm{div}\, \mathbf{D} = 0$, $\frac{\partial \mathbf{D}}{c\,\partial t} = \mathrm{curl}\, \mathbf{H}$ without charges.

Equations (61) and (62) can be solved by the perturbation theory, for small values of h and $\mathbf{g}$, starting with free electromagnetic waves as the unperturbed solution. Doing so, we arrive immediately at the result that the solution must be a wave packet, and the frequencies are not determined anymore, in the sense that either for a given wavevector we have many frequencies or for a given frequency we have many wavevectors. This can be most conveniently expressed in terms of photons which suffer quantum transitions.

The quantization of the electromagnetic field in a curved space proceeds in the usual way. The action given by (59) can be written as

$$S = \frac{1}{8\pi} \int dt d\mathbf{r} \cdot \Delta \left(\mathbf{D}^2 - \mathbf{B}^2 \right) = \frac{1}{8\pi} \int dt d\mathbf{r} \cdot \frac{1}{\Delta} \left[\mathbf{E}^2 + 2\mathbf{E}(\mathbf{g} \times \mathbf{B}) - \Delta^2 \mathbf{B}^2 \right], \tag{63}$$

which exhibits the well-known density of lagrangian in the limit $h, \mathbf{g} \to 0$. We change now to the covariant vector potential $\mathbf{A} \to -\mathbf{A}$, such that $\mathbf{E} = -\frac{\partial \mathbf{A}}{c\,\partial t}$ and $\mathbf{B} = \mathrm{curl}\, \mathbf{A}$. Leaving aside the factor $\frac{1}{8\pi}$, the momentum is given by $\Pi = \frac{\partial L}{\partial(\partial \mathbf{A}/\partial t)} = \frac{2}{\Delta c^2} \left(\frac{\partial \mathbf{A}}{\partial t} - \mathbf{g} \times \mathbf{B} \right)$. The vector potential is represented as

$$\mathbf{A}_\alpha = \sum_{\alpha\mathbf{p}} \frac{c\hbar}{2\sqrt{\varepsilon}} \left[a_{\alpha\mathbf{p}} \mathbf{e}^\alpha e^{-i\varepsilon t/\hbar + i\mathbf{p}\mathbf{r}/\hbar} + h\,c \right] \tag{64}$$

and the momentum by

$$\Pi_\alpha = -i \sum_{\alpha\mathbf{p}} \frac{\sqrt{\varepsilon}}{c} \left[a_{\alpha\mathbf{p}} \mathbf{e}^\alpha e^{-i\varepsilon t/\hbar + i\mathbf{p}\mathbf{r}/\hbar} - h\,c \right], \tag{65}$$

where $\mathbf{e}^\alpha$ is the polarization vector along the direction α, perpendicular to $\mathbf{p} = \hbar\mathbf{k}$ (we assume the transversality condition $\mathrm{div}\mathbf{A} = 0$), $\varepsilon = \hbar\omega = cp$, while ω is the frequency and $\mathbf{k}$ is the wavevector. The commutation relations are the usual bosonic ones, and we get the hamiltonian $H = H_0 + H_{1h} + H_{1g}$, given by

$$\left.\begin{aligned} H_0 &= \int d\mathbf{r}\cdot\left(\frac{1}{4}\,c^2\Pi^2 + B^2\right) = \\ &\qquad = \sum_{\alpha\mathbf{p}} \frac{\varepsilon}{2}\left(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}} + a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}}\right) \\ H_{1h} &= \int d\mathbf{r}\cdot\left(\sqrt{1+h}-1\right)\left(\frac{1}{4}\,c^2\Pi^2 + \mathbf{B}^2\right) \\ H_{1g} &= -\frac{1}{2}\sum_{\alpha\mathbf{p}} \mathbf{g}\,\mathbf{p}\left(a^+_{\alpha\mathbf{p}}a_{\alpha\mathbf{p}} + a_{\alpha\mathbf{p}}a^+_{\alpha\mathbf{p}}\right) \end{aligned}\right\}. \qquad (66)$$

Systematic calculations can now be performed within the perturbation theory, and we can see that quantum transitions between the photonic states may appear, starting with the hg-order of the perturbation theory. Therefore, an observer moving with a non-uniform velocity is able to see a "blue shift" in the frequency of the photons "acted" by a force like the gravitational one.* The shift occurs obviously at the expense of the energy of the observer's motion.†

Other fields. A similar approach can be used for other fields in a curved space. In particular, it can be applied to spin-1/2 Dirac fields, with similar conclusions, though, technically, it is more cumbersome to write down the action for spinors in curved spaces. It can be speculated upon the question of quantizing the gravitational field in a similar manner. Indeed, weak perturbations of the flat metric can be represented as gravitational waves, which can be quantized by using the gravitational action $\int dx^0 d\mathbf{r}\cdot\sqrt{-g}\,R$, where R is the curvature of the space.‡ Now, we may suppose that these gravitons move in a curved space with the metric g. We may use the same gravitational action as before, where g is now the metric of the space and R contains the graviton field. Or, alternately, we expand $g = g_0 + \delta g$, where g_0 is the background part and δg is the graviton part. We get a field theory of gravitons interacting with the underlying curved space, and we get quantum transitions of the gravitons, which gives a meaning to the quantization of the gravity, in the sense that either it is not possible or the gravitons suffer quantum transitions. The space and time (the gravitons) are then scattered statistically by matter (which in turn suffers a similar process) or by the non-inertial motion.

Conclusion. The quantum motion implies, basically, delocalized waves, like plane wave, both in space and time. The general theory of relativity, gravitation or curved space as the one discussed here, arising from weak static forces and non-inertial motion, imply localized field, both in space and time. Consequently, the quantization is destroyed in those situations involved by the latter case, in the sense that quanta are scattered both in energy and the wavevector, and we have to deal there with transition amplitudes and probabilities, *i.e.* with a statistical perspective. The basic equations for the classical motion in these cases become meaningful only with scattered quanta. This shows indeed that the quantization is both necessary and illusory. The basic aspect of the natural world is its statistical character in terms of quanta.

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*This is similar with the Unruh effect (1976).

†It is worth investigating the change in the equilibrium distribution of the black-body radiation as a consequence of the non-uniform translation in a gravitational field. The frequency shift amounts to a change of temperature, which increases, most likely, by $\frac{\Delta T}{T} \sim \left(g\,\bar{h}\right)^2$, with temporal and spatial averages (for the quantization of the black-body radiation see Fermi, 1932). In this respect, the effect discussed here, though related to the Unruh effect, is different. The Unruh effect assumes rather that the external non-uniform translation, as a macroscopic motion, consists of a coherent vacuum, so equilibrium photons can be created; the related increase in temperature is rather the measurement made by the observer of its own motion.

‡Though there are difficulties in establishing a relativistically-invariant quantum theory for particles with helicity 2, like the gravitons. Another related difficulty is the general non-localizability of the gravitational energy.

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LETTERS TO PROGRESS IN PHYSICS

Where is the Science?

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The important rôle played in society today by scientific research is highlighted, and the related various social, economical and political conditionings of science are discussed. It is suggested that the exclusive emphasis upon the multiple technological applications of science, the use and abuse of scientific research, may lead to the very disappearance of science, transforming scientific research into a routine and almost ritualistic activity, empty of any real content. This may already be seen in the inadequate way present day society tackles the fundamental problems we are confronted with, issues such as the environment, conflict, life and the thinking process.

Science is used and misused today in a great variety of ways, in all of the utmost relevance to human life and activity. Worldwide policy has found it useful for science to be employed by the military, and developed nations spend generously on this application of science. New, sophisticated, powerful weaponry is produced today, by an application of scientific achievements. It has also been found beneficial to put science to work for a more comfortable life; highly-developed technologies, industry, manufacturing, farming, agriculture, commerce, services, transport and communications are science-based today. Education, culture, civilization, a highly-qualified work force are produced on the basis of science. Everything that matters to humans, namely wealth, fame and pleasure, is achieved on an ever larger scale today by using science. Modern science is viewed as an immensely beneficial resource, whose rôle in society is to be tapped more and more for the greatest of profit. In this respect, everybody talks now only of "technology transfer", "competitiveness", "innovation", "leadership", and last but not least, of "intellectual leadership", through science. Science is everywhere "oriented" on our epoch towards the military, warfare, technology, industry, economy, education, etc, etc. There is no more "simply science"; it is everywhere determined, oriented, conditioned.

Scientists should feel well and flattered by the great interest shown by society in their art and trade. The fact is that science has provided much for society, through mechanical constructions, thermal machines, electricity, nuclear energy, materials, electronics, and it is natural for society to try to control, accelerate and harness all this in the process of profiting by the use and abuse of science.

Yet nobody is satisfied with such a policy, all around the world. Taxpayers want more and more from science, and the scientists are more and more incapable of responding to their high demands. In addition, politicians stirr up heavily this conflictual issue. The reason for such a failure resides in the inadequacy of this type of science policy.

Indeed, science is not funded, according to this policy, unless it produces something immediately relevant to society, i.e. something useful for the military, for industry, the economy, education, etc. Scientific research, which is the way science advances, is only desired for its applications. Yet all these outlets for science, in various areas of activity and interest, are not science; they are only its applications. Science policy today greatly confuses science with its applications. By laying emphasis exclusively on applications we will end up having no science at all.

Science is a resource, like any other, and yet a bit special. Of course, scientific knowledge does not fade, or degrade, by repeated use, it is not wasted or dissipated by using it. Newton's laws do not vanish by being repeatedly used. But people who have scientific knowledge, and who at least endeavour to maintain it, if not advancing it, i.e. those we call scientists, disappear, if not properly cultivated. We have a lot of applications of science, a serious endeavour for technology transfer, great expectations from using this science, but where is the science? We have no science anymore by such policy which provides exclusively for scientific applications, irrespective of how desirable and benefical they might be.

A very deeply-rooted fallacy is to think that scientists are in universities. This is profoundly wrong. In universities we have professors who teach science to young people. They need to acquire scientific fuel for this teaching process, from elsewhere. We cannot say reasonably that teachers in universities do both science and teaching contemporaneously, because they then do either half of each or half of neither. It is more appropriate to emphasize the exclusive educational task of the universities, and provide separately for scientists, in distinct laboratories, institutes, etc. The great advances in science and in its applications made by the former Soviet

Union and the USA in the last half of the past century were achieved precisely because these States cultivated distinctly science and scientists, and did not mix up science with teaching or production.

Of course, these things are related, and it is desirable and profitable to cultivate such naturally beneficial relations. How are we going to strenghten the relations between universitaries, scientists and high-tech entrepreneurs? Simply by doing precisely what we need to: by providing for close relationships between such people, encouraging their meetings, discussions, talks, cooperation, etc. The main cause of the difficulties and dissatisfaction today with the "failure" of science in society is due precisely to the vanishing relationship between scientists, technologists, entrepreneurs, and teachers. We need to urgently provide for such close contacts, but we have to be very careful not to mix things up: to keep the distinction between these socio-professional categories. It is a scientific fact that distinctiveness and variety produce force and motion, whilst admixture increases only the potential of ineffectiveness, resulting in only a restful peace.

If we are going to cultivate, by our policies, the distinction between scientists, teachers, professors, technologists, entrepreneurs, to provide for close collaborative relationships between all them, keeping at the same time the distinction, and not to mistake science and scientific research for teaching or production, then we will be more scientific in our endeavours, and will be more fortunate in our expectations.

We are yet pretty unscientific with respect to basic issues. For instance, nowadays we set for science the mission of reducing, or circumventing, the degradation of the environment, without noticing that every human activity degrades the environment. Indeed, even the mental processes degrade their environment; brains in this case. Life is an organized process whereby entropy is diminished, and therefore it is a great fluctuation, but at the same time we increase also the environmental entropy, including that of our own body, just by living, and the increase is greater than the decrease, and the process goes to equilibrium. We will end with a more balanced world, where life will become extinct, because the fluctuations diminish near equlibrium. We would think of finding a solution for preserving life by creating artificially another similar fluctuation, then with a greater spending of energy. The inherent limitations of such an artificial process will then pose serious issues regarding how, who and how many would be going to live that artificial life. This may present a serious problem for science and technology, and for the future of our society. Another is the process of thinking, for many believe that we should think the thinking process in order to understand what we are tinking. First, they assume erroneously that there exists a conscience, or a consciousness, i.e. a state or process of thinking the thinking process, which is false. Anyone who thinks is not conscious of what he or she is doing, there is no double thinking; consciousness is identical to thinking itself. Thinking is a natural process, associated with the complexity of the human brain, and so we do not think of thinking, because it is impossible, we just do it. To think is just to be. Such sorts of things we only learn through science, so, providing in our policies for properly cultivating science will greatly enhance our chances of responding to truly relevant questions. Besides, life and the thinking process may be manipulated and controlled by others, but never in those who are doing that. But full power is illusory. We may destroy science in others but never in ourselves. The need for scientific knowledge is essential for survival.

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LETTERS TO PROGRESS IN PHYSICS

On the Current Situation Concerning the Black Hole Problem

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This paper reviews a new solution which concerns the black hole problem. The new solution, by S. J. Crothers, doesn't eliminate the line-element of the classical "black hole solution" produced by the founders of the problem, but represents the gravitational collapse condition in terms of physical observable quantities accessible to a real observer whose location is in the real Schwazschild space itself, not with the quantities in an abstract flat space tangential to it at the point of observation (as it was in the classical solution). Besides, Schwarzschild space is only a very particular case of Einstein spaces of type I. There are minor studies on the physical conditions of gravitational collapse in other spaces of type I, and nothing on Einstein spaces of type II and type III (of which there are hundreds). Einstein spaces (empty spaces, without distributed matter, wherein Ricci's tensor is proportional to the fundamental metric tensor), are spaces filled by an electromagnetic field, dust, or other substances, of which there are many. As a result, studies on the physical conditions of gravitational collapse are only in their infancy.

In a series of pioneering papers, starting in 1979, Leonard S. Abrams (1924–2001) discussed [1] the physical sense of the black hole solution. Abrams claimed that the correct solution for the gravitational field in a Schwarzschild space (an empty space filled by a spherically symmetric gravitational field produced by a spherical source mass) shouldn't lead to a black hole as a physical object. Such a statement has profound consequences for astrophysics.

It is certain that if there is a formal error in the black hole solution, committed by the founders of this theroy, in the period from 1915–1920's, a long list of research produced during the subsequent decades would be brought into question. Consequently, Abrams' conclusion has attracted the attention of many physicists. Since millions of dollars have been invested by governments and private organizations into astronomical research connected with black holes, this discussion ignited the scientific community.

Leonard S. Abrams' professional reputation is beyond doubt. As a result, it is particularly noteworthy to observe that Stephen J. Crothers [2], building upon the work of Abrams, was able to deduce solutions for the gravitational field in a Schwarzschild metric space produced in terms of a physical observable (proper) radius. Crothers' solutions fully verify the initial arguments of Abrams. Therefore, the claim that the correct solution for the gravitational field in a Schwarzschild space does not lead to a black hole as a physical object requires serious attention.

Herein, it is important to give a clarification of Crothers' solution from the viewpoint of a theoretical physicist whose professional field is the General Theory of Relativity. The historical aspect of the black hole problem will not be discussed as this has been sufficiently addressed in the scientific literature and, especially, in a historical review [3]. The technical details of Crothers' solution will also not be reanalyzed: his calculations were reviewed by many professional relativists prior to publication in *Progress in Physics*. These reviewers had a combined forty years of professional employment in this field and it is thus extremely unlikely that a formal error exists within Crothers' work. Rather, our attention will be focused only upon clarification of the new result in comparison to the classical solution in Schwarzschild space. In other words, the main objective is to answer the question: what have Abrams and Crothers achieved?

In this letter, two important items must be highlighted:

1. The new solution, by Crothers, doesn't eliminate the classical "black hole solution" (i.e. the line-element thereof) produced by the founders of the black hole problem, but represents the perspective of a real observer whose location is in the real Schwazschild space itself (inhomogeneous and curved), not by quantities in an abstract flat space tangential to it at the point of observation (as it was previously, in the classical solution). Consequently, the new solution opens a doorway to new research on the specific physical conditions accompanying gravitational collapse in Schwarzschild space. This can now be studied in a reasonable manner both through a purely theoretical approach and with the methods of numerical relativity (computers);
2. Schwarzschild space is only a very particular case related to Einstein spaces of type I. There are minor studies on the physical conditions of gravitational collapse in other spaces of type I, but nothing on it in relation to Einstein spaces of type II and type III (of which there are hundreds). Besides Einstein spaces (empty spaces, without distributed matter, wherein Ricci's tensor is proportional to the fundamental metric tensor $R_{\alpha\beta} \sim k\, g_{\alpha\beta}$), there are spaces filled by an electromagnetic field, dust, or other substances, of which there are

many. As a result, studies on the physical conditions of gravitational collapse are only in their infancy.

First, the corner-stone of Crothers' solution is that it was produced in terms of the physical observed (proper) radius which is dependent on the properties of the space itself, while the classical solution was produced in terms of the coordinate radius determined in the tangentially flat space (it can be chosen at any point of the inhomogeneous, curved space). For instance, when one makes a calculation at such a proper radius where the gravitational collapse condition $g_{00} = 0$ occurs, the calculation result manifests in what might be really measurable on the surface of collapse from the perspective of a real observer who has a real reference body which is located in this space, and is bearing not on the ideal, but on real physical standards whereto this observer compares his measurements. This is in contrast to the classic procedure of calculation oriented to the coordinate quantities measurable by an "abstract" observer who has an "ideal" reference body which, in common with its ideal physical standards, is located in the flat space tangential to the real space at the point of observation, not the real space which is inhomogeneous and curved.

In the years 1910–1920's people had no clear understanding of physical observable quantities in General Relativity. Later, in the years 1930–1940's, many researchers such as Einstein, Lichnerowicz, Cattaneo and others, were working on methods for determination of physical observable quantities in the inhomogeneous curved space of General Relativity. For instance, Landau and Lifshitz, in §84 of their famous book, *The Classical Theory of Fields*, first published in 1939, introduced observable time and the observable three-dimensional interval. But they all limited themselves to only a few particular cases and did not arrive at general mathematical methods to define physical observable quantities in pseudo-Riemannian spaces. The complete mathematical apparatus for calculating physical observable quantities in four-dimensional pseudo-Riemannian space, that is a strict solution to the problem of physical observable quantities in General Relativity, was only constructed in the 1940's, by Abraham Zelmanov (1913–1987), and first published in 1944 in his doctoral dissertation [4].

Therefore David Hilbert and the other founders of the black hole problem*, who did their work during the period 1916–1920's, worked in the circumstances of the gravitational collapse condition $g_{00} = 0$ in Schwarzschild space in terms of the coordinate radius (which isn't the same as the real distance in this space). As a result, they concluded that the spherical mass which produces the gravitational field in Schwarzschild space, with the increase of its density, becomes a a "self-closed" object surrounded by the gravitational collapse surface of the condition $g_{00} = 0$ so that all events can occur only inside it (this means a singular break in the surface of collapse).

By the new solution, which was obtained by Crothers in terms of the proper radius, there is no observable singular break under any physical conditions: so a real spherical body of a Schwarzschild metric cannot become a "self-closed" object observable as a "black hole" in the space.

This new solution, in common with the classical solution, means that we have two actual pictures of gravitational collapse, drawn by two observers who are respectively located in different spaces: (1) a real observer located in the same Schwarzschild space where the gravitational collapse occurs; (2) an "abstract" observer whose location is in the flat space tangential to the Schwarzschild space at the point of observation.

So, the new solution doesn't eliminate the classical "black hole solution" (i.e. the line-element thereof), but represents the same phenomenon of gravitational collapse in a Schwarzschild space from another perspective, related to real observation and experiment.

Second, Schwarzschild space is only a very particular case of Einstein spaces of Type I. Einstein spaces [5] are empty spaces without distributed matter, wherein Ricci's tensor is proportional to the fundamental metric tensor ($R_{\alpha\beta} \sim k\, g_{\alpha\beta}$). There are three known kinds of Einstein spaces, and there are many spaces related to each kind (hundreds, as expected, and nobody knows exactly how many). There are almost no studies of the gravitational collapse condition $g_{00} = 0$ in most other Einstein spaces of Type I. There are no studies at all of the collapse condition in Einstein spaces of Type II and Type III. Besides that, General Relativity has many spaces beyond Einstein spaces: spaces filled by distributed matter such as an electromagnetic field, dust, or other substances, of which there are many. Such spaces are closer to real observation and experiment than Schwarzschild space, so it would be very interesting to study the collapse condition in spaces beyond Einstein spaces.

This is why Schwarzschild (empty) space is good for basic considerations, where there are no sharp boundaries for the physical conditions therein. However, such a space becomes unusable under some ultimate physical conditions, which are smooth in the real Universe due to the influences of many other space bodies and fields. Gravitational collapse as the ultimate condition in Scwarzschild space leads to black holes outside a real physical space, with the consequence that the black hole solution in Schwarzschild space has no real meaning (despite the fact that it can be formally obtained). Mathematical curiosities are always interesting, but if these things have no real meaning, then one must make it clear in the end. Consequently, the current mathematical treatment of black holes in Schwarzschild space does not have physical validity in nature, as Crothers explains.

These results are not amazing: many solutions to Ein-

*Karl Schwarzschild died in 1916, and had no relation to the black hole solution. He only deduced the metric of a space filled by the spherically symmetric gravitational field produced by a spherical mass therein (such a space is known as a space with a Schwarzschild metric or, alternatively, a Schwarzschild space).

stein's equation have no validity in the physical world. Therefore the collapse condition in a real case, which could be met in the real Universe filled by fields and substance, should be a subject of numerical relativity which produces approximate solutions to Einstein's equations with the use of computers, not an exact theory of the phenomenon.

As a result we see that studies on the physical conditions of gravitational collapse are only beginning. New solutions, given in terms of physical observable quantities, do not close the gravitational collapse problem, but open new horizons for studies by both exact theory and numerical methods of General Relativity.

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