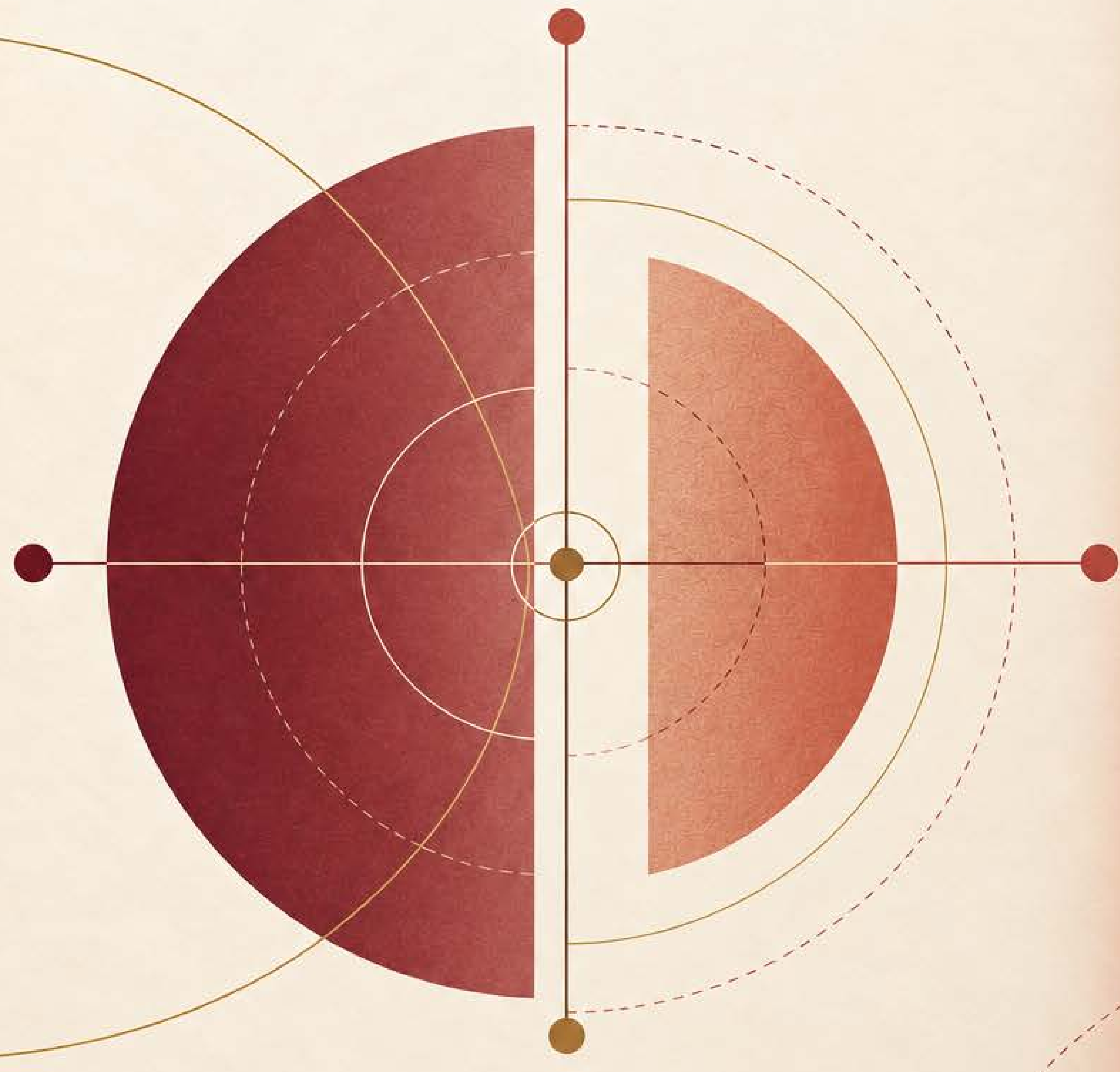
Polar Concepts in Uncertainty Theory: Fuzzy, Neutrosophic, and Uncertain Set Frameworks



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Neutrosophic Science International Association (NSIA)
Publishing House
2026

Editor:

Neutrosophic Science International Association (NSIA)
Publishing House
https://fs.unm.edu/NSIA/

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ISBN 978-197250254-9

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Chapter 1

Introduction

1.1 Uncertainty-Aware Sets and Numbers

Classical set theory provides a fundamental mathematical framework for rigorous reasoning, formal representation, and mathematical modeling [1]. However, many real-world systems involve information that cannot be adequately represented by purely crisp descriptions. Typical sources of uncertainty include vagueness, imprecision, incomplete knowledge, inconsistency, ambiguity, and conflicting evidence. These limitations have motivated the development of numerous extensions of classical set theory designed to represent and process different forms of uncertain information.

Representative uncertainty-oriented frameworks include fuzzy sets [2, 3], hyperfuzzy sets [4–6], intuitionistic fuzzy sets [7, 8], hesitant fuzzy sets [9, 10], picture fuzzy sets [11, 12], single-valued neutrosophic sets [13–15], double-valued neutrosophic sets [16–18], neutrosophic offsets [19–21], plithogenic sets [22–24], and soft sets [25, 26]. Although these models differ substantially in their mathematical formulations and interpretations, they share the common objective of providing expressive mechanisms for representing uncertain information beyond the binary membership structure of classical set theory.

In particular, many uncertainty models describe an element through several numerical or structural components. Such multidimensional representations make it possible to distinguish different aspects of uncertainty and provide a natural foundation for further extensions involving multiple viewpoints, opposing properties, hierarchical criteria, or interacting uncertainty dimensions.

1.2 Polarity and Multipolar Structures

An important direction in uncertainty modeling is the introduction of *polarity*. Rather than describing an element from only a single perspective, polarity-based frameworks permit information to be represented through multiple, complementary, contrasting, or opposing viewpoints. Such representations are particularly useful when an object must be evaluated simultaneously with respect to several attributes, criteria, agents, directions, states, or counter-properties.

Several polarity-based extensions of fuzzy and neutrosophic sets have already been investigated. Representative examples include bipolar fuzzy sets [27–31], tripolar fuzzy sets [32], bipolar neutrosophic sets [33–37], m-polar fuzzy sets [38, 39], and m-polar neutrosophic sets [40, 41]. By introducing several poles or evaluative dimensions into a common framework, these models considerably broaden the expressive capacity of uncertainty representation.

The notion of polarity, however, is not restricted to fuzzy sets, neutrosophic sets, or uncertainty theory. Concepts involving poles, opposite states, directional components, orthogonality, polarization, dual structures, and related mechanisms appear throughout mathematics, physics, engineering, and other scientific disciplines. Consequently, polarity may be viewed as a broad structural principle rather than merely as a specialized mechanism for introducing multiple fuzzy membership components.

This broader viewpoint is important because not all polar constructions have the same mathematical meaning. In some models, poles act primarily as indices for several evaluation criteria. In others, they encode genuine opposition, geometric direction, hierarchical organization, network structure, temporal change, or physically motivated polarization. Accordingly, a unified treatment of polarity should distinguish these different mathematical roles while retaining a sufficiently general framework in which they can be compared.

1.3 Types of Polarity

The term *polarity* is used in several mathematically distinct senses across uncertainty theory and related disciplines. To clarify these different roles, the polarity-oriented constructions considered in this book can be broadly organized according to the conceptual classification presented in Table 1.1.

The categories in Table 1.1 are intended as a conceptual taxonomy rather than a strict partition. A particular polarity-based model may naturally belong to more than one category when it combines several structural mechanisms.

The classification in Table 1.1 emphasizes that polarity, as used throughout this book, is considerably broader than the simple introduction of several numerical membership components. In *index polarity*, poles mainly distinguish several evaluation dimensions or viewpoints. In *oppositional polarity*, the essential feature is the relationship between a property and an opposite, complementary, or counter-property.

Geometric polarity introduces directional, orthogonal, vectorial, or spatial structure, whereas *network-structured polarity* explicitly organizes the pole system by means of graphs, trees, or related discrete structures. *Context-dependent polarity* allows the relevant pole system to vary according to the object or surrounding context, while *dynamic polarity* incorporates temporal evolution or history dependence. In *heterogeneous polarity*, different poles may even be governed by different uncertainty models.

The classification also includes polarity notions whose origins lie outside the conventional multi-polar fuzzy-set setting. *Potential-theoretic polarity* incorporates concepts such as semipolarity and pluripolarity, whereas *physical polarization* transfers mathematical structures motivated by physics and engineering, including channel-based, circuit-based, hysteretic, and Stokes-type representations, into uncertainty modeling.

Thus, the common term *polarity* encompasses several distinct but related mathematical mechanisms. Their systematic treatment provides a basis for understanding both their differences and the structural relationships that connect them.

Table 1.1: Representative types of polarity in uncertainty-oriented frameworks.

Type of Polarity	Basic Interpretation	Representative Concepts
Index Polarity	Poles primarily serve as indices for multiple perspectives, criteria, attributes, agents, or evaluation channels.	Bipolar Uncertain Set; Tripolar Uncertain Set; Quadripolar Uncertain Set; Pentapolar Uncertain Set; Multipolar Uncertain Set; Iterative Multipolar Uncertain Set; Hyperpolar Uncertain Set; Free-Polar Uncertain Set
Oppositional Polarity	Poles represent opposite, complementary, counter-oriented, or mutually contrasting states or properties.	Bipolar Uncertain Set; Tripolar Uncertain Set; Self-Polar Uncertain Set; Antipolar Uncertain Set
Geometric Polarity	Polarity is expressed through geometric, vectorial, directional, orthogonal, projective, or spatial relationships.	Cross-Polar Uncertain Set; Antipolar Uncertain Set; Pluripolar Uncertain Set
Network-Structured Polarity	Poles possess an explicit relational or hierarchical organization described by a graph, tree, directed network, or related discrete structure.	Tree-Polar Uncertain Set; Graph-Polar Uncertain Set; Circuit-Polar Uncertain Set
Context-Dependent Polarity	The active poles, their number, or their interpretation may depend on the element, context, state, or external conditions.	Variable-Polar Uncertain Set; Adaptive-Polar Uncertain Set
Dynamic Polarity	Polar information varies over time or depends on temporal evolution, history, or state transitions.	Dynamic-Polar Uncertain Set; Hysteretic-Polar Uncertain Set
Heterogeneous Polarity	Different poles may employ different uncertainty models, degree-domains, or internal mathematical representations.	Heterogeneous-Polar Uncertain Set
Potential-Theoretic Polarity	Polarity is derived from mathematical notions originating in potential theory or pluripotential theory rather than solely from a finite family of indexed poles.	Semipolar Uncertain Set; Pluripolar Uncertain Set
Physical Polarization	Polarity is motivated by physical or engineering notions involving polarization, channels, circuit orientation, memory effects, or measurable polarized states.	BEC-Polarized Uncertain Set; Circuit-Polar Uncertain Set; Hysteretic-Polar Uncertain Set; Stokes-Polar Uncertain Set

1.4 Our Contributions

Motivated by the preceding observations, this book develops a unified framework for studying polarity-based extensions of Uncertain Sets. Rather than restricting polarity to a fixed number of independent membership components, we investigate several forms of polarity that differ in their indexing systems, internal structures, dependence mechanisms, and mathematical origins.

In particular, the book considers Bipolar, Tripolar, Multipolar, Iterative Multipolar, Hyperpolar, Variable-Polar, Dynamic-Polar, Tree-Polar, Heterogeneous-Polar, Graph-Polar, Adaptive-Polar, Semipolar, Pluripolar, Self-Polar, Antipolar, Cross-Polar, BEC-Polarized, Circuit-Polar, Hysteretic-Polar, Stokes-Polar, and Free-Polar Uncertain Sets. For these constructions, we provide mathematical definitions, representative examples, representation results, and structure-dependent invariants within the typed Uncertain Set framework.

A central feature of this approach is that the uncertainty degree itself is not restricted to an ordinary fuzzy membership value. Instead, the underlying uncertain model may be selected from a broader class of uncertainty representations. Consequently, the proposed constructions can be specialized to fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and other uncertainty-oriented settings whenever the corresponding degree-domain satisfies the required assumptions.

The framework therefore permits uncertainty to be represented from multiple, opposing, hierarchical, geometric, structural, context-dependent, heterogeneous, and dynamically changing perspectives. It also provides a common setting in which polarity concepts originating from different mathematical and scientific areas can be formulated and compared.

Accordingly, the principal objective of this book is not merely to increase the number of polar components, but to establish a broader mathematical view of polarity as an organizing principle for uncertainty representation. The resulting framework provides a foundation for further theoretical developments involving polarity-based sets, algebraic structures, graphs, hyperstructures, topological structures, decision-making models, and other uncertainty-aware mathematical systems.

Chapter 2

Preliminaries

In this chapter, we briefly review the basic definitions, terminology, and notation used throughout this book.

2.1 Fuzzy Sets

Fuzzy sets extend classical (crisp) sets by replacing binary membership with a graded membership degree in $[0,1]$, thereby allowing partial membership to be represented mathematically [3]. This fundamental paradigm has led to numerous well-known generalizations, including bipolar fuzzy sets [30, 42], complex fuzzy sets [43, 44], and Pythagorean fuzzy sets [45, 46].

Definition 2.1.1 (Fuzzy Set)**.** [3, 47] Let Y be a nonempty set. A *fuzzy set* on Y is a function

$$\mu\colon Y \longrightarrow [0,1],$$

where $\mu(y)$ represents the membership degree of y in the set.

A *fuzzy relation* on Y is a mapping

$$\delta\colon Y \times Y \longrightarrow [0,1],$$

that is, a fuzzy set defined on $Y \times Y$. We say that δ is a *fuzzy relation on* μ if, for all $y, z \in Y$,

$$\delta(y,z) \le \min\{\mu(y), \mu(z)\}.$$

2.2 Neutrosophic Sets

Neutrosophic set theory extends graded membership by representing each element through three components associated with truth, indeterminacy, and falsity [13, 19, 48, 49]. These components are generally treated independently, which makes the framework suitable for representing incomplete, indeterminate, and potentially inconsistent information.

The original theory of neutrosophic sets admits more general neutrosophic-valued components. In this book, we mainly use the single-valued formulation, in which all three components take values in the ordinary unit interval $[0,1]$.

Definition 2.2.1 (Single-Valued Neutrosophic Set)**.** [50, 51] Let $X \neq \emptyset$ be a set. A *single-valued neutrosophic set* A on X is specified by three functions

$$T_A,\ I_A,\ F_A \colon X \longrightarrow [0,1],$$

where, for every $x \in X$,

$$T_A(x), \qquad I_A(x), \qquad F_A(x)$$

represent, respectively, the truth-membership, indeterminacy-membership, and falsity-membership degrees of x with respect to A.

Equivalently, A may be represented by the mapping

$$\mu_A \colon X \longrightarrow [0,1]^3, \qquad \mu_A(x) = \big(T_A(x), I_A(x), F_A(x)\big).$$

Since

$$T_A(x), I_A(x), F_A(x) \in [0,1],$$

it follows automatically that

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$$

for every $x \in X$. No additional normalization condition such as

$$T_A(x) + I_A(x) + F_A(x) = 1$$

is required.

Example 2.2.2 (Single-Valued Neutrosophic Set for Medical Diagnosis)**.** Consider a preliminary medical assessment in which three patients are evaluated with respect to the possibility of having a particular disease. Let

$$X = \{x_1, x_2, x_3\},$$

where x_1, x_2, x_3 represent the three patients.

Suppose that the available clinical information is summarized by the single-valued neutrosophic membership values

$$\mu_A(x_1) = \big(0.80,\ 0.15,\ 0.20\big),$$
$$\mu_A(x_2) = \big(0.45,\ 0.50,\ 0.35\big),$$

and

$$\mu_A(x_3) = \big(0.20,\ 0.25,\ 0.75\big).$$

Equivalently,

$$T_A(x_1) = 0.80, \qquad I_A(x_1) = 0.15, \qquad F_A(x_1) = 0.20,$$
$$T_A(x_2) = 0.45, \qquad I_A(x_2) = 0.50, \qquad F_A(x_2) = 0.35,$$

and

$$T_A(x_3) = 0.20, \qquad I_A(x_3) = 0.25, \qquad F_A(x_3) = 0.75.$$

Here, $T_A(x_i)$ represents the degree of evidence supporting the presence of the disease, $I_A(x_i)$ represents the degree of indeterminacy caused by incomplete or conflicting medical information, and $F_A(x_i)$ represents the degree of evidence supporting the absence of the disease.

For example,

$$\mu_A(x_2) = (0.45, 0.50, 0.35)$$

indicates that the available evidence provides a truth-membership degree of 0.45, an indeterminacy-membership degree of 0.50, and a falsity-membership degree of 0.35.

Notice that

$$0.45 + 0.50 + 0.35 = 1.30,$$

which is allowed in the single-valued neutrosophic framework because the three components are not required to sum to 1.

Thus,

$$A = \{\langle x_1, 0.80, 0.15, 0.20\rangle, \langle x_2, 0.45, 0.50, 0.35\rangle, \langle x_3, 0.20, 0.25, 0.75\rangle\}$$

is a Single-Valued Neutrosophic Set representing uncertain medical assessment information.

2.3 Uncertain Sets

An Uncertain Set assigns to each element an admissible degree taken from a prescribed uncertainty model. In this way, it provides a general framework capable of representing fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and other uncertainty-based settings [52–54].

Definition 2.3.1 (Typed Uncertain-Model Hierarchy)**.** [54] An *uncertain model* is specified at an explicitly declared type. The notation $\mathrm{Dom}(M)$ always denotes a nonempty set of admissible degrees; no embedding $\mathrm{Dom}(M) \subseteq [0,1]^k$ is assumed unless it is stated as part of the model. The following types are used in this book.

(U1) A *carrier model* is

$$M_0 = (\mathrm{Dom}(M)), \qquad \mathrm{Dom}(M) \neq \emptyset.$$

This is the minimum type required for an Uncertain Set.

(U2) A *scored model* is

$$M_s = (\mathrm{Dom}(M), s_M), \qquad s_M : \mathrm{Dom}(M) \longrightarrow [0,1].$$

The score is additional structure; it is not determined by the carrier.

(U3) An *encoded model* is

$$M_e = (\mathrm{Dom}(M), \Phi_M), \qquad \Phi_M : [0,1] \longrightarrow \mathrm{Dom}(M).$$

A *coherent scored–encoded model* is

$$M_{se} = (\mathrm{Dom}(M), s_M, \Phi_M)$$

with

$$s_M \circ \Phi_M = \mathrm{id}_{[0,1]}\,.$$

Thus scalar information is not changed by encoding and rescoring.

(U4) An *ordered model* is

$$M_{\le} = (\mathrm{Dom}(M), \le_M, 0_M, 1_M),$$

where $(\mathrm{Dom}(M), \le_M)$ is a bounded poset. A *lattice model* is

$$M_{\mathrm{lat}} = (\mathrm{Dom}(M), \le_M, \vee_M, \wedge_M, 0_M, 1_M),$$

where the degree-domain is a bounded lattice. A *De Morgan model* is

$$M_{\mathrm{DM}} = (\mathrm{Dom}(M), \le_M, \vee_M, \wedge_M, N_M, 0_M, 1_M),$$

where N_M is an order-reversing involution satisfying the De Morgan laws.

(U5) An *algebraic model* is a carrier model equipped with a typed signature Σ_M of total operations on $\mathrm{Dom}(M)$ and a specified axiom set $\mathcal{A}_M$. Every use of a model-dependent operation later in the book must declare the corresponding model type.

The types are capability refinements of (U1). For example, a model may simultaneously be coherent scored–encoded and De Morgan; forgetting the extra maps or operations returns its carrier model. This convention avoids applying scores, encoders, orders, or lattice operations to a model for which they have not been supplied.

Typical carriers include

$$[0,1], \qquad \{(\mu,\nu)\in[0,1]^2 : \mu+\nu\le 1\}, \qquad [0,1]^3,$$

but the hierarchy also permits interval-valued degrees, hesitant degrees (finite nonempty subsets of $[0,1]$), signed bipolar degrees, function-valued degrees, and plithogenic degree spaces. Such models are therefore not forced into a fixed finite Cartesian cube.

Definition 2.3.2 (Uncertain Set (U-Set))**.** [54] Let X be a nonempty universe, and let $M_0 = (\mathrm{Dom}(M))$ be a carrier model in the sense of Definition 2.3.1. An *Uncertain Set of type M*, briefly called a *U-Set*, on X is a pair

$$\mathcal{U} = (X, \mu_M),$$

where

$$\mu_M : X \longrightarrow \mathrm{Dom}(M)$$

is called the *uncertainty-degree function* or *membership map* of $\mathcal{U}$.

For each $x \in X$, the value

$$\mu_M(x) \in \mathrm{Dom}(M)$$

records the degree, or collection of degrees, associated with the membership of x under the uncertainty model M.

Lemma 2.3.3 (Function-Space and Dependent-Product Lemma)**.** *Let $D \neq \emptyset$. For every set I, the function space D^I is nonempty. More generally, if $D_i \neq \emptyset$ for every $i \in I$, then the dependent product*

$$\prod_{i\in I} D_i$$

is the precise carrier for an I-indexed family of typed values (with nonemptiness requiring the usual choice principle when I is infinite). Restrictions, sections, evaluations, and currying of maps between these carriers are well-defined whenever their displayed domains and codomains match.

Lemma 2.3.4 (Typed Transport Lemma)**.** *Let $M_{se} = (\mathrm{Dom}(M), s_M, \Phi_M)$ be a coherent scored–encoded model. If $q : Y \to [0,1]$, then*

$$\Phi_M \circ q : Y \longrightarrow \mathrm{Dom}(M)$$

is an Uncertain Set and

$$s_M \circ \Phi_M \circ q = q.$$

If an ordered, lattice, De Morgan, or algebraic operation is declared as part of the type of M, its pointwise extension to $\mathrm{Dom}(M)^I$ has the same equations. No such extension is used for a carrier model alone.

Lemma 2.3.5 (Finite-Product Preservation Lemma)**.** *Let $m \ge 1$. Finite products preserve nonemptiness and preserve bounded-poset, bounded-lattice, De Morgan, and equational-algebraic structure under componentwise operations. In particular, if M has one of these types, then $\mathrm{Dom}(M)^m$ has the corresponding product type. Moreover, for arbitrary sets I, J, there are canonical bijections*

$$(D^I)^J \cong D^{J\times I} \qquad \textit{and} \qquad \prod_{j\in J} D^{I_j} \cong D^{\bigsqcup_{j\in J} I_j}.$$

All routine existence and codomain checks in the sequel are instances of Lemmas 2.3.3–2.3.5. They are not repeated as separate well-definedness theorems. Section-specific results are stated only when they assert an additional invariant, representation, or structural property.

Remark 2.3.6. Several familiar uncertainty structures arise as special cases of the U-Set framework:

- If M is the fuzzy model with
$$\mathrm{Dom}(M) = [0, 1],$$
then
$$\mu_M : X \longrightarrow [0, 1]$$
is an ordinary fuzzy membership function, and $\mathcal{U}$ becomes a fuzzy set.
- If M is a neutrosophic model, then
$$\mu_M(x) = \big(T(x), I(x), F(x)\big),$$
and $\mathcal{U}$ becomes a neutrosophic set.
- Appropriate choices of M similarly recover intuitionistic fuzzy sets, picture fuzzy sets, plithogenic sets, and many other related uncertainty models.

As indicated above, the U-Set framework admits a wide variety of specializations. For reference, Table 2.1 presents the finite-vector subfamily of uncertainty sets for which a particular realization $\mathrm{Dom}(M) \subseteq [0, 1]^k$ has been declared (cf. [55, 56]). It is a catalogue of one typed subfamily, not a restriction on Definition 2.3.1.

Table 2.1: A catalogue of uncertainty-set families (U-Sets) organized by the dimension k of the degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ [56].

k	note	Representative U-Set model(s) whose degree-domain is a subset of $[0,1]^k$
1		Fuzzy Set [3, 57–59]; N-Fuzzy Set [60–62]; Shadowed Set [63–65]
2		Intuitionistic Fuzzy Set [8, 66]; Interval-Valued Fuzzy Set [67, 68]; Pythagorean Fuzzy Set [69, 70]; Fermatean Fuzzy Set [71, 72]; q-Rung Orthopair Fuzzy Set [73,74], Vague Set [75,76]; Bipolar Fuzzy Set (two-component description) [31,42]; Variable Fuzzy Set [77–79]; Paraconsistent Fuzzy Set [80, 81]; Bifuzzy Set [82, 83]
3		Single-Valued Neutrosophic Set [15, 50, 84]; Picture Fuzzy Set [12, 85]; Cubic Set [86,87]; Spherical Fuzzy Set [88,89]; T-Spherical Fuzzy Set [90,91]; Tripolar Fuzzy Set (three-component formalisms) [32, 92–94]; Neutrosophic Vague Set [95, 96]
4		Quadripartitioned Neutrosophic Set [97, 98]; Interval-Valued Intuitionistic Fuzzy Set [99, 100]; Interval-Valued Pythagorean Fuzzy Set [101, 102]; Linear Diophantine Fuzzy Set [103,104]; Double-Valued Neutrosophic Set [16,105,106]; Dual Hesitant Fuzzy Set [107, 108]; Quadripolar Fuzzy Set [109]; Ambiguous Set [110–112]; Turiyam Neutrosophic Set [113–116]
5		Pentapartitioned Neutrosophic Set [117–119]; Triple-Valued Neutrosophic Set [120–122]
6		Hexapartitioned Neutrosophic Set; Quadruple-Valued Neutrosophic Set [21, 121] Intuitionistic fuzzy Cubic Set [123–126]; Interval-Valued Neutrosophic Set [127, 128];
7		Heptapartitioned Neutrosophic Set [129, 130]; Quintuple-Valued Neutrosophic Set [121, 131, 132]
8		Octapartitioned Neutrosophic Set [133, 134]
9		Nonapartitioned Neutrosophic Set [133, 134] ; Neutrosophic Cubic Set [135–139]
n	$(n \geq 1)$	Multi-valued (Fuzzy) Sets [140]; MultiFuzzy Set [141]; Multipolar Fuzzy Sets [142, 143]; n-Refined Fuzzy Set [144, 145]
$2n$	$(n \geq 1)$	n-Refined Intuitionistic Fuzzy Set [145]; Multipolar Intuitionistic Fuzzy Sets [146–148], Multi-Intuitionistic Fuzzy Set [141]
$3n$	$(n \geq 1)$	n-Refined Neutrosophic Set [145, 149]; Multipolar Neutrosophic Sets [41, 150]; Multi-Neutrosophic Set [141, 151]

Reading guide. Within the U-Set framework [54], each model M is determined by a degree-domain $\mathrm{Dom}(M) \subseteq [0,1]^k$ together with a membership map $\mu_M : X \to \mathrm{Dom}(M)$. Accordingly, the table groups representative families by the ambient dimension k, that is, by the number of numerical components assigned to each element.

(a) A common viewpoint in the literature is that neutrosophic sets form a broad umbrella encompassing several earlier multi-component fuzzy models and their generalizations; see [11].

(b) Ambiguous sets are often described as subclasses of certain four-component neutrosophic families; see [97, 98, 112].

(c) Turiyam neutrosophic sets are reported as subclasses of quadripartitioned neutrosophic sets; see [152].

Chapter 3

Polar Uncertain Set

This chapter examines various forms of polarity in Uncertain Sets.

The concepts investigated in this chapter generalize uncertainty representation through several distinct forms of polarity. Depending on the model, polarity may be represented by a fixed number of components, an arbitrary family of poles, a hierarchy, a graph, a time-dependent structure, a context-dependent selector, or a geometric or physics-inspired transformation. Table 3.1 summarizes the principal mathematical differences among the Polar Uncertain Set frameworks considered in this chapter.

Table 3.1: Comparison of the principal Polar Uncertain Set frameworks considered in this chapter.

Concept	**Mathematical representation**	**Pole or structural organization**	**Main distinguishing feature**
Bipolar Uncertain Set Def. 3.2.2	$\mu_A^+ : X \to \mathrm{Dom}(M)$, $\mu_A^- : X \to \mathrm{Dom}(M)$.	Two distinguished components: a property and its corresponding counter-property.	Provides the basic two-pole uncertainty representation. Both components take values in the same uncertainty domain $\mathrm{Dom}(M)$. In the fuzzy case, the counter-polar component may be represented in the signed range $[-1, 0]$.
Tripolar Uncertain Set Def. 3.3.2	$\mu_A^+ : X \to \mathrm{Dom}(M)$, $\mu_A^* : X \to \mathrm{Dom}(M)$, $\mu_A^- : X \to \mathrm{Dom}(M)$, $s_M\big(\mu_A^+(x)\big) + s_M\big(\mu_A^*(x)\big) \le 1$.	Three distinguished components: positive, neutral, and counter-polar.	Introduces an explicit neutral component in addition to positive and counter-polar information. A score map $s_M : \mathrm{Dom}(M) \to [0, 1]$ controls the admissibility condition.

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Table 3.1 – *continued from the previous page*

Concept	**Mathematical representation**	**Pole or structural organization**	**Main distinguishing feature**
Quadripolar Uncertain Set Def. 3.4.1	$\mu_A : X \to \mathrm{Dom}(M)^4$, $\mu_A(x) = \big(\mu_A^{(i)}(x)\big)_{i=1}^{4}$.	A fixed collection of four distinguished poles.	Represents each element through four simultaneous uncertainty perspectives. It is the $m = 4$ specialization of a Multipolar Uncertain Set when no additional relations among the four poles are imposed.
Pentapolar Uncertain Set Def. 3.5.1	$\mu_A : X \to \mathrm{Dom}(M)^5$, $\mu_A(x) = \big(\mu_A^{(i)}(x)\big)_{i=1}^{5}$.	A fixed collection of five distinguished poles.	Represents each element through five simultaneous uncertainty perspectives. It is the $m = 5$ specialization of a Multipolar Uncertain Set when no additional relations among the five poles are imposed.
Multipolar Uncertain Set Def. 3.6.2	$\mu_A : X \to \mathrm{Dom}(M)^m$, $m \geq 2$, $\mu_A(x) = \big(\mu_A^{(i)}(x)\big)_{i=1}^{m}$.	A fixed finite collection of m poles.	Assigns one admissible uncertainty value to every element under each of m distinct poles. It provides the general finite-dimensional multipolar extension of an ordinary Uncertain Set.
Iterative Multipolar Uncertain Set Def. 3.7.2	$\mathcal{D}^{(0)}_{M,m} = \mathrm{Dom}(M)$, $\mathcal{D}^{(r+1)}_{M,m} = \big(\mathcal{D}^{(r)}_{M,m}\big)^m$, $\mu_A^{(r)} : X \to \mathcal{D}^{(r)}_{M,m}$.	Recursively nested multipolar levels.	Iterates the multipolar construction itself. The first order satisfies $\mathcal{D}^{(1)}_{M,m} = \mathrm{Dom}(M)^m$, while higher orders encode nested multipolar uncertainty.
Hyperpolar Uncertain Set Def. 3.8.2	$H : X \to \Delta_M$, $\Delta_M = \prod_{j=1}^{n} \mathrm{Dom}(M)^{\delta_j}$.	A finite family of polarity blocks, where the j-th block has dimension δ_j.	Allows each major polarity to contain a multidimensional family of uncertainty values. Different polarity blocks need not have the same internal dimension.
Variable-Polar Uncertain Set Def. 3.9.2	$\mu_{A,x} : P_x \to \mathrm{Dom}(M)$, $A \in \prod_{x\in X} \mathrm{Dom}(M)^{P_x}$.	The pole set P_x depends on the element x.	Both the number and identity of the relevant poles may vary between elements. Hence, different elements need not be evaluated using the same collection of perspectives.

Continued on the next page

Table 3.1 – *continued from the previous page*

Concept	Mathematical representation	Pole or structural organization	Main distinguishing feature
Dynamic-Polar Uncertain Set Def. 3.10.2	$\mu_A : X\times T\times P \to \mathrm{Dom}(M)$, $\mu_{A,\sigma(t)}(x) = \mathcal{E}_t\big(\mu_{A,t}(x)\big)$.	A pole set P, a discrete time system, and typed evolution operators.	Admissible time sections are trajectories of the declared evolution law rather than independent copies indexed only by time.
Tree-Polar Uncertain Set Def. 3.11.2	$A = (\mathcal{T}_P, \mathrm{Agg}, \mu_A)$, $\mu_A : X \times P \to \mathrm{Dom}(M)$. For internal p, $\mu_A(x,p) = \mathrm{Agg}_{M,p}\big(\boldsymbol{\mu}_{x,p}\big)$, $\boldsymbol{\mu}_{x,p} = \big(\mu_A(x,q)\big)_{q\in \mathrm{Ch}(p)}$.	Poles are vertices of a finite rooted tree.	Parent values are constrained by typed child aggregators. Consequently, the root and the tree relation affect the admissible degree assignments.
Heterogeneous-Polar Uncertain Set Def. 3.12.2	$\mu_A : X \to \prod_{p\in P} \mathrm{Dom}(M_p)$.	Each pole p is equipped with its own uncertainty model M_p.	Permits different poles to employ mathematically different uncertainty models and degree-domains. One pole may, for example, be fuzzy while another is interval-valued or neutrosophic.
Graph-Polar Uncertain Set Def. 3.13.2	$A = (G_P, \mathcal{C}, \mu_A)$, $\mu_A : X \times P \to \mathrm{Dom}(M)$, $\big(\mu_A(x,p),\mu_A(x,q)\big) \in C_e$.	Poles are vertices of a finite graph G_P.	Each edge carries a typed compatibility relation. The graph therefore defines constraints on the admissible flattened pole assignments.
Adaptive-Polar Uncertain Set Def. 3.14.2	$\Pi : X \times C \to \mathcal{P}^{+}_{\mathrm{fin}}(P^*)$, $\mu_A : \Omega_\Pi \to \mathrm{Dom}(M)$, $\mu_{A,x,c'} = \mathcal{U}^M_{x,c,c'}\big(\mu_{A,x,c}\big)$.	Active poles are selected from a master pole set P^* according to the element and context.	The selector changes the active carrier, while update maps relate the uncertainty assignments across context transitions.
Semipolar Uncertain Set Def. 3.15.2	$A^M_\alpha = \big\{x \in E : s_M(\mu_A(x)) \geq \alpha\big\}$, $A^M_\alpha \in \mathfrak{S}(\mathbb{X})$.	Polarity is induced through semipolar subsets associated with a Hunt process $\mathbb{X}$.	Unlike ordinary multipolar constructions, this model does not introduce several coordinate poles. Instead, potential-theoretic semipolarity is transferred through positive α-cuts.

Continued on the next page

Table 3.1 – *continued from the previous page*

Concept	**Mathematical representation**	**Pole or structural organization**	**Main distinguishing feature**
Pluripolar Uncertain Set Def. 3.16.2	$u \in \mathrm{PSH}(\Omega)$, $u \le 0$, $\mu_{A_u}(z) = \Phi_M\big(1 - e^{u(z)}\big)$.	Polarity is derived from a plurisubharmonic function on a complex domain $\Omega \subseteq \mathbb{C}^n$.	Its full-membership core is the pluripolar set $\{u = -\infty\}$. Nontrivial α-cuts are sublevel sets and need not themselves be pluripolar.
Self-Polar Uncertain Set Def. 3.17.3	$\pi_P : P \to H$, $\pi_H : H \to P$, $\mu_H \circ \pi_P = \mu_P$.	An incidence-reversing duality between point and hyperplane sorts.	Degree assignments agree under a genuine projective polarity. A bare involution on a single untyped set is treated separately as involution invariance.
Antipolar Uncertain Set Def. 3.18.2	$\mu_A : X \times P \to \mathrm{Dom}(M)$, $Q_A(x) = \sum_{p \in P} s_M\big(\mu_A(x,p)\big)v(p) = 0$.	The finite pole set P has a fixed-point-free involution τ, with $v(\tau(p)) = -v(p)$.	Models mutually opposite polar directions whose weighted uncertainty contributions cancel. Its defining condition is zero net polarization for every element.
Cross-Polar Uncertain Set Def. 3.19.3	$z_A : X \to \overline{B}_{\mathcal{H}}$, $z_A^{\mathrm{co}} = P_u z_A$, $z_A^{\mathrm{cross}} = z_A - P_u z_A$.	Orthogonal decomposition with respect to a distinguished unit direction u in a Hilbert space.	Separates each state into a co-polar component parallel to the preferred direction and a cross-polar component orthogonal to it.
BEC-Polarized Uncertain Profile Def. 3.20.2	$a_x = s_M(\eta_1(x))$, $b_x = s_M(\eta_2(x))$, $\mu_A^{\mathrm{good}}(x) = \Phi_M(a_x b_x)$, $\mu_A^{\mathrm{bad}}(x) = \Phi_M\big(a_x + b_x - a_x b_x\big)$.	Two independent binary erasure-channel profiles represented by scored degrees.	This is the exact BEC specialization of one polar-code kernel step. Without erasure-channel semantics, the formulas give a product/probabilistic-sum transformation.
Circuit-Polar Uncertain Set Def. 3.21.2	$\eta_A : X \times V \to \mathrm{Dom}(M)$, $\Delta_A(x,e) = \sum_{w \in V} b_{we}\, s_M\big(\eta_A(x,w)\big)$, $i_A(x,e) = g(e)\Delta_A(x,e)$, $B\, i_A(x,\cdot) = j_A(x,\cdot)$.	A finite directed graph, incidence matrix, conductances, and balanced node injections.	The incidence gradient is coupled with Ohm's law and Kirchhoff conservation, distinguishing the model from an arbitrary directed graph gradient.

Continued on the next page

Table 3.1 – *continued from the previous page*

Concept	Mathematical representation	Pole or structural organization	Main distinguishing feature
Hysteretic-Polar Uncertain Set Def. 3.22.3	$\eta_A : X\times\mathcal{H}(U) \to \mathrm{Dom}(M)$, $p_A : X \times \mathcal{H}(U) \to [-1,1]$, $q_A^{\pm}(x,h) = \dfrac{m_A(x,h)\big(1 \pm p_A(x,h)\big)}{2}$.	Polarity depends on the entire finite input history $h \in \mathcal{H}(U)$.	Introduces memory dependence. Two histories having the same terminal input may produce different uncertainty and polarity values.
Stokes-Polar Uncertain Set Def. 3.23.3	$S_A : X \to \mathbb{S}$, $S_A(x) = (s_0, s_1, s_2, s_3)$, $D_A(x) = \dfrac{\sqrt{s_1^2+s_2^2+s_3^2}}{s_0}$, $\mu_A(x) = \Psi_M\big(s_0(x), D_A(x)\big)$.	A four-component Stokes representation satisfying $s_1^2+s_2^2+s_3^2 \le s_0^2$.	Transfers the Stokes polarization formalism to uncertainty theory. The total intensity and normalized degree of polarization are jointly encoded into an admissible uncertainty value.
Free-Polar Uncertain Set Def. 3.24.1	$\mu_A : X \times P \to \mathrm{Dom}(M)$, or equivalently $\widehat{\mu}_A : X \to \mathrm{Dom}(M)^P$.	An arbitrary nonempty pole set P, with no restriction on its cardinality.	Provides the cardinality-free version of multipolar uncertainty. If $\lvert P\rvert = m < \infty$, it is equivalent, up to indexing, to an m-Polar Uncertain Set; if $\lvert P\rvert = 1$, it reduces to an ordinary Uncertain Set.

3.1 Status of Concepts and Representation Results

To distinguish notions recalled from the literature from the constructions introduced or systematically formulated in this book, we use the following status convention.

Here "book-specific" identifies the formal definition used in this book; it does not assert priority beyond the cited literature. Whenever a fuzzy or external mathematical notion is recalled, it is explicitly separated from its typed U-Set lift.

Theorem 3.1.1 (Representation and Non-Collapse Theorem)**.** *Let $D = \mathrm{Dom}(M) \neq \emptyset$, let $[m] = \{1,\dots,m\}$, and let X be a universe. Define*

$$\mathcal{D}^{(0)}_{M,m} = D, \qquad \mathcal{D}^{(r+1)}_{M,m} = (\mathcal{D}^{(r)}_{M,m})^m.$$

(i) *An m-polar carrier is canonically*

$$D^m \cong D^{[m]}.$$

Table 3.2: Status of the principal ingredients and constructions considered in this book.

Status	**Concepts**
Existing notions recalled from the literature	Fuzzy, bipolar fuzzy, tripolar fuzzy, and m-polar fuzzy sets, including their 4-polar and 5-polar specializations; semipolar and pluripolar subsets in potential theory; projective polarity; channel polarization for binary erasure channels; oriented incidence matrices and resistive circuits; hysteresis; and Stokes parameters.
Book-specific typed lifts	Bipolar, Tripolar, Quadripolar, Pentapolar, Multipolar, Iterative Multipolar, Hyperpolar, Variable-Polar, Dynamic-Polar, Tree-Polar, Heterogeneous-Polar, Graph-Polar, Adaptive-Polar, Semipolar, Pluripolar, Self-Polar, Antipolar, Cross-Polar, BEC-Polarized, Circuit-Polar, Hysteretic-Polar, Stokes-Polar, and Free-Polar Uncertain Sets of a declared uncertain-model type.
Book-specific terminology or organization	The common typed uncertain-model hierarchy; the systematic organization of polarity-based Uncertain Sets; the structural-coherence specifications imposed on the corresponding models; and the representation and reduction results developed below.

(ii) *The r-fold iterative carrier is canonically*

$$\mathcal{D}^{(r)}_{M,m} \cong D^{[m]^r} \cong D^{m^r}.$$

Thus nesting alone does not create a new carrier invariant.

(iii) *For finite block sizes δ_j,*

$$\prod_{j=1}^{n} D^{\delta_j} \cong D^{\bigsqcup_{j=1}^{n}[\delta_j]} \cong D^{\sum_{j=1}^{n}\delta_j}.$$

Thus a hyperpolar carrier becomes distinct only when its block partition or blockwise operations are retained as structure.

(iv) *There are canonical currying bijections*

$$D^{X\times P} \cong (D^P)^X, \qquad D^{X\times T\times P} \cong (D^{T\times P})^X.$$

Accordingly, Free-, Tree-, Graph-, and Dynamic-Polar maps are reindexings of families unless their pole or transition structures impose additional coherence conditions.

(v) *Variable and adaptive carriers are dependent products,*

$$\prod_{x\in X} D^{P_x} \quad \textit{and} \quad D^{\Omega_\Pi},$$

respectively. Their distinct content lies in the declared variation or update policy, not merely in the existence of the displayed function space.

Proof. Parts (i)–(iii) are the canonical product bijections of Lemma 2.3.5; part (iv) is currying; and part (v) is the dependent-product description of Lemma 2.3.3. A structural condition cuts out a specified subspace of the flattened carrier and is therefore retained by structure-preserving, but not arbitrary, bijections. □

3.2 Bipolar Uncertain Set

A Bipolar Fuzzy Set assigns positive and negative membership degrees to each element, representing both a property and its corresponding counter-property within one unified framework [30, 31].

Definition 3.2.1 (Bipolar Fuzzy Set)**.** Let $X \neq \emptyset$ be a universe. A *bipolar fuzzy set* A on X is characterized by two functions

$$\mu_A^+ : X \longrightarrow [0,1], \qquad \mu_A^- : X \longrightarrow [-1,0],$$

where, for each $x \in X$, $\mu_A^+(x)$ represents the degree to which x satisfies a positive property, while $\mu_A^-(x)$ represents the degree to which x satisfies the corresponding negative property.

Equivalently,

$$A = \left\{ \left(x, \mu_A^+(x), \mu_A^-(x)\right) \;\middle|\; x \in X \right\}.$$

A Bipolar Uncertain Set assigns two uncertainty values to each element, representing a property and its corresponding counter-property within a single uncertainty framework.

Definition 3.2.2 (Bipolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. A *Bipolar Uncertain Set* of type M on X is a pair of mappings

$$\mu_A^+, \mu_A^- : X \longrightarrow \mathrm{Dom}(M),$$

where, for each $x \in X$,

$$\mu_A^+(x)$$

represents the uncertainty value associated with a property, while

$$\mu_A^-(x)$$

represents the uncertainty value associated with its corresponding counter-property.

Equivalently,

$$A = \left\{ \left(x, \mu_A^+(x), \mu_A^-(x)\right) \;\middle|\; x \in X \right\}.$$

Example 3.2.3 (Bipolar Uncertain Set for Apartment Selection)**.** Consider a person choosing an apartment for daily commuting. Let

$$X = \{a_1, a_2, a_3, a_4\}$$

be four candidate apartments.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0,1].$$

For each apartment a_i, let

$$\mu_A^+(a_i)$$

represent the degree to which the apartment is considered *suitable for the tenant*, taking into account factors such as location, rent, and convenience, while

$$\mu_A^-(a_i)$$

represents the degree to which it satisfies the corresponding counter-property, namely *unsuitability for the tenant.*

Suppose that the evaluations are

Apartment	μ_A^+	μ_A^-
a_1	0.85	0.20
a_2	0.70	0.45
a_3	0.40	0.75
a_4	0.60	0.35

Then the corresponding Bipolar Uncertain Set is

$$A = \{(a_1, 0.85, 0.20), (a_2, 0.70, 0.45), (a_3, 0.40, 0.75), (a_4, 0.60, 0.35)\}.$$

For example, a_1 has a high degree of suitability and a relatively low degree of unsuitability, whereas a_3 has a lower suitability degree and a higher counter-property degree. Hence, the Bipolar Uncertain Set represents both favorable and unfavorable evaluations of each alternative within a unified framework.

Remark 3.2.4. If M is the fuzzy model with

$$\mathrm{Dom}(M) = [0, 1],$$

then

$$\mu_A^+, \mu_A^- : X \to [0, 1].$$

Define the signed counter-polar membership by

$$\widetilde{\mu}_A^-(x) = -\mu_A^-(x).$$

Then

$$\widetilde{\mu}_A^- : X \to [-1, 0],$$

and the representation becomes

$$\mu_A^+ : X \to [0, 1], \qquad \widetilde{\mu}_A^- : X \to [-1, 0],$$

which recovers the usual Bipolar Fuzzy Set representation.

The finite-vector specializations of Bipolar Uncertain Sets can be classified according to a declared realization of the uncertainty degree-domain. Suppose, only for the following catalogue, that

$$\mathrm{Dom}(M) \subseteq [0, 1]^k.$$

Since a Bipolar Uncertain Set is represented by two mappings

$$\mu_A^+, \mu_A^- : X \longrightarrow \mathrm{Dom}(M),$$

each element is described by two k-dimensional uncertainty values. Hence, when the degree-domain admits a finite-dimensional numerical representation, the resulting bipolar representation has $2k$ numerical components.

Table 3.3 summarizes representative special cases according to this dimensional viewpoint.

For example, when the underlying uncertain model is fuzzy,

$$\mathrm{Dom}(M) = [0, 1],$$

each polar component is one-dimensional, and the corresponding Bipolar Uncertain Set has two scalar components. When the underlying model is intuitionistic fuzzy or vague, each polar component is represented by two numerical degrees, resulting in four scalar components in the bipolar representation. Similarly, single-valued neutrosophic, spherical fuzzy, and picture fuzzy models use three numerical components per pole and therefore lead to six-component bipolar representations.

Hesitant fuzzy and plithogenic frameworks require separate treatment. A hesitant fuzzy degree is generally set-valued rather than a fixed scalar vector of prescribed dimension, whereas the dimensionality of a plithogenic representation depends on the particular underlying attribute and uncertainty structure. Accordingly, these cases are more naturally regarded as variable-dimensional or model-dependent specializations of the general Bipolar Uncertain Set framework.

Table 3.3: Dimension-based classification of representative Bipolar Uncertain Sets.

Dimension of each pole k	Total bipolar dimension	Representative Bipolar Uncertain Set	References
1	2	Bipolar Fuzzy Set	[30, 31]
2	4	Bipolar Vague Set	[153, 154]
2	4	Bipolar Intuitionistic Fuzzy Set	[155, 156]
3	6	Bipolar Neutrosophic Set	[37, 157–160]
3	6	Bipolar Spherical Fuzzy Set	[161, 162]
3	6	Bipolar Picture Fuzzy Set	[163–165]
Variable / set-valued	Variable	Bipolar Hesitant Fuzzy Set	[166, 167]
Model-dependent	Model-dependent	Bipolar Plithogenic Set	[168]

3.3 Tripolar Uncertain Set

A Tripolar Fuzzy Set assigns positive, neutral, and negative membership degrees, enabling three-way representation of uncertainty and opposing information simultaneously [32, 93, 94]. Although Tripolar Fuzzy Sets are less widely recognized than Bipolar Fuzzy Sets, a growing number of studies on tripolar frameworks have been published in recent years (cf. [169–172]). It should also be noted that the interpretation and formulation of tripolarity may differ across the literature.

Definition 3.3.1 (Tripolar Fuzzy Set)**.** [32] Let $X \neq \emptyset$ be a universe. A *tripolar fuzzy set* A on X is characterized by three functions

$$\mu_A^+ : X \longrightarrow [0,1], \qquad \mu_A^* : X \longrightarrow [0,1], \qquad \mu_A^- : X \longrightarrow [-1,0],$$

such that, for every $x \in X$,

$$0 \leq \mu_A^+(x) + \mu_A^*(x) \leq 1.$$

Here, $\mu_A^+(x)$ represents the degree to which x satisfies the positive property, $\mu_A^*(x)$ represents the degree to which x satisfies the irrelevant or neutral property, and $\mu_A^-(x)$ represents the degree to which x satisfies the implicit counter-property.

Equivalently,

$$A = \left\{ \left(x, \mu_A^+(x), \mu_A^*(x), \mu_A^-(x)\right) \;\middle|\; x \in X \right\}.$$

A Tripolar Uncertain Set assigns positive, neutral, and counter-polar uncertainty values to each element, providing a three-way representation under a prescribed uncertain model.

Definition 3.3.2 (Tripolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let

$$M_s = (\mathrm{Dom}(M), s_M)$$

be a scored uncertain model.

A *Tripolar Uncertain Set* of type M on X is a triple of mappings

$$\mu_A^+, \mu_A^*, \mu_A^- : X \longrightarrow \mathrm{Dom}(M)$$

such that, for every $x \in X$,

$$s_M\big(\mu_A^+(x)\big) + s_M\big(\mu_A^*(x)\big) \le 1.$$

Here,

$$\mu_A^+(x), \qquad \mu_A^*(x), \qquad \mu_A^-(x)$$

represent the positive, neutral, and counter-polar uncertainty values of x, respectively.

Example 3.3.3 (Tripolar Uncertain Set for Restaurant Selection)**.** Consider a customer selecting a restaurant for dinner. Let

$$X = \{r_1, r_2, r_3, r_4\}$$

be four candidate restaurants.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0,1]$$

with the scalar evaluation map

$$s_M(t) = t.$$

For each restaurant r_i, let

$$\mu_A^+(r_i)$$

represent the degree of a favorable evaluation,

$$\mu_A^*(r_i)$$

represent the degree of a neutral or indeterminate evaluation, and

$$\mu_A^-(r_i)$$

represent the degree associated with the counter-property, namely an unfavorable evaluation.

Suppose that the evaluations are given by

Restaurant	μ_A^+	μ_A^*	μ_A^-
r_1	0.70	0.20	0.15
r_2	0.55	0.30	0.40
r_3	0.35	0.40	0.65
r_4	0.60	0.25	0.30

For every $r_i \in X$,

$$s_M\big(\mu_A^+(r_i)\big) + s_M\big(\mu_A^*(r_i)\big) \le 1.$$

For example,

$$0.70 + 0.20 = 0.90 \le 1$$

for r_1, and

$$0.35 + 0.40 = 0.75 \le 1$$

for r_3.

Hence,

$$A = \{(r_1, 0.70, 0.20, 0.15), (r_2, 0.55, 0.30, 0.40), (r_3, 0.35, 0.40, 0.65), (r_4, 0.60, 0.25, 0.30)\}$$

is a Tripolar Uncertain Set.

In this example, r_1 receives a relatively high favorable evaluation and a low unfavorable evaluation, whereas r_3 has a lower favorable degree and a higher counter-polar degree. Thus, the model simultaneously represents positive, neutral, and opposing evaluations of each restaurant.

Remark 3.3.4. Let M be the fuzzy model with

$$\mathrm{Dom}(M) = [0, 1]$$

and

$$s_M(t) = t.$$

Then a Tripolar Uncertain Set is determined by

$$\mu_A^+, \mu_A^*, \mu_A^- : X \to [0, 1]$$

with

$$\mu_A^+(x) + \mu_A^*(x) \leq 1.$$

If the counter-polar component is written in signed form as

$$\widetilde{\mu}_A^-(x) = -\mu_A^-(x),$$

then

$$\widetilde{\mu}_A^- : X \to [-1, 0],$$

and the representation becomes the corresponding Tripolar Fuzzy Set form with positive, neutral, and negative polar components.

Representative concepts that can be expressed within the framework of Tripolar Uncertain Sets are summarized in Table 3.4.

Table 3.4: Dimension-based classification of representative Tripolar Uncertain Sets.

Dimension of each pole k	**Total tripolar dimension**	**Representative Tripolar Uncertain Set**	**References**
1	3	Tripolar Fuzzy Set	[32, 93, 94]
2	6	Tripolar Intuitionistic Fuzzy Set	[173, 174]
3	9	Tripolar Neutrosophic Set	[174, 175]

3.4 Quadripolar Uncertain Set

A Quadripolar Uncertain Set assigns four distinguished uncertainty values to each element, thereby representing uncertain information from four simultaneous poles, perspectives, criteria, or evaluative directions. Unlike a Tripolar Uncertain Set, whose three components may be given specific positive, neutral, and counter-polar interpretations, the four poles of a Quadripolar Uncertain Set are treated abstractly unless a particular application specifies additional semantic or algebraic relationships among them.

Definition 3.4.1 (Quadripolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let

$$M_0 = (\mathrm{Dom}(M))$$

be a carrier uncertain model.

Let

$$P_4 = \{p_1, p_2, p_3, p_4\}$$

be a fixed set of four distinguished poles.

A *Quadripolar Uncertain Set* of type M on X is a mapping

$$\mu_A : X \times P_4 \longrightarrow \mathrm{Dom}(M).$$

Equivalently, it is specified by four mappings

$$\mu_A^{(1)}, \mu_A^{(2)}, \mu_A^{(3)}, \mu_A^{(4)} : X \longrightarrow \mathrm{Dom}(M),$$

where

$$\mu_A^{(i)}(x) = \mu_A(x, p_i), \qquad i \in \{1, 2, 3, 4\}.$$

Thus, for every $x \in X$,

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \mu_A^{(2)}(x), \mu_A^{(3)}(x), \mu_A^{(4)}(x)\big) \in \mathrm{Dom}(M)^4.$$

The values

$$\mu_A^{(1)}(x), \qquad \mu_A^{(2)}(x), \qquad \mu_A^{(3)}(x), \qquad \mu_A^{(4)}(x)$$

represent the four uncertainty values associated with the four distinguished poles of x.

No relation among the four components is required unless an additional constraint is explicitly included as part of the quadripolar model.

Example 3.4.2 (Quadripolar Uncertain Set for Restaurant Selection)**.** Consider a customer selecting a restaurant for dinner. Let

$$X = \{r_1, r_2, r_3, r_4\}$$

be four candidate restaurants.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0, 1].$$

Suppose that four evaluation poles are considered:

$$P_4 = \{p_{\text{food}}, p_{\text{service}}, p_{\text{price}}, p_{\text{atmosphere}}\}.$$

For each restaurant r_i, define

$$\mu_A^{(1)}(r_i), \quad \mu_A^{(2)}(r_i), \quad \mu_A^{(3)}(r_i), \quad \mu_A^{(4)}(r_i)$$

as its uncertainty-adjusted degrees with respect to food quality, service quality, price attractiveness, and atmosphere, respectively.

Suppose that

Restaurant	$\mu_A^{(1)}$	$\mu_A^{(2)}$	$\mu_A^{(3)}$	$\mu_A^{(4)}$
r_1	0.85	0.70	0.55	0.80
r_2	0.65	0.80	0.75	0.60
r_3	0.75	0.50	0.90	0.45
r_4	0.80	0.75	0.60	0.85

Then

$$\mu_A(r_1) = (0.85, 0.70, 0.55, 0.80),$$
$$\mu_A(r_2) = (0.65, 0.80, 0.75, 0.60),$$
$$\mu_A(r_3) = (0.75, 0.50, 0.90, 0.45),$$

and

$$\mu_A(r_4) = (0.80, 0.75, 0.60, 0.85).$$

Hence,

$$A = \left\{ \begin{array}{l} (r_1, 0.85, 0.70, 0.55, 0.80), \\ (r_2, 0.65, 0.80, 0.75, 0.60), \\ (r_3, 0.75, 0.50, 0.90, 0.45), \\ (r_4, 0.80, 0.75, 0.60, 0.85) \end{array} \right\}$$

is a Quadripolar Uncertain Set.

For example, r_1 receives relatively high evaluations for food quality and atmosphere, whereas r_3 receives its highest evaluation for price attractiveness. Thus, the quadripolar representation allows four distinct uncertainty perspectives to be recorded simultaneously.

Remark 3.4.3 (Relation to Multipolar Uncertain Sets)**.** A Quadripolar Uncertain Set is precisely the $m = 4$ specialization of a Multipolar Uncertain Set.

Indeed,

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)^4$$

is the special case of

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)^m$$

obtained by setting

$$m = 4.$$

Therefore, without additional relations among its four poles, quadripolarity introduces a distinguished four-pole interpretation but does not introduce a carrier structure beyond the corresponding four-dimensional product model.

Remark 3.4.4 (Fuzzy Specialization)**.** Let M be the fuzzy model with

$$\mathrm{Dom}(M) = [0, 1].$$

Then a Quadripolar Uncertain Set is determined by

$$\mu_A^{(1)}, \mu_A^{(2)}, \mu_A^{(3)}, \mu_A^{(4)} : X \longrightarrow [0, 1],$$

or equivalently by

$$\mu_A : X \longrightarrow [0, 1]^4.$$

Thus, every element $x \in X$ is represented by a four-dimensional membership vector

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \mu_A^{(2)}(x), \mu_A^{(3)}(x), \mu_A^{(4)}(x)\big).$$

This may be regarded as the fuzzy specialization of the Quadripolar Uncertain Set framework. No normalization condition such as

$$\sum_{i=1}^{4} \mu_A^{(i)}(x) \le 1$$

is imposed unless it is explicitly required by the particular quadripolar model under consideration.

Table 3.5: Dimension-based classification of representative Quadripolar Uncertain Sets.

Dimension of each pole k	**Total quadripolar dimension**	**Representative specialization**
1	4	Quadripolar fuzzy-type Uncertain Set (cf. [109])
2	8	Quadripolar intuitionistic-fuzzy-type Uncertain Set
3	12	Quadripolar neutrosophic-type Uncertain Set
k	$4k$	Quadripolar k-dimensional Uncertain Set

Representative finite-dimensional specializations that can be expressed within the framework of Quadripolar Uncertain Sets are summarized in Table 3.5.

Remark 3.4.5 (Flattening Representation)**.** Suppose that the degree-domain of M has a declared finite-dimensional realization

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Then

$$\mathrm{Dom}(M)^4 \subseteq [0,1]^{4k}.$$

Consequently, every finite-dimensional Quadripolar Uncertain Set admits the canonical flattened representation

$$\mu_A : X \longrightarrow [0,1]^{4k}.$$

In particular, the total scalar dimension of a quadripolar construction is four times the scalar dimension of the underlying uncertainty model.

3.5 Pentapolar Uncertain Set

A Pentapolar Uncertain Set assigns five distinguished uncertainty values to each element, thereby representing uncertain information from five simultaneous poles, perspectives, criteria, or evaluative directions.

The five poles are treated abstractly unless a particular application specifies additional semantic, algebraic, ordinal, or oppositional relationships among them.

Definition 3.5.1 (Pentapolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let

$$M_0 = (\mathrm{Dom}(M))$$

be a carrier uncertain model.

Let

$$P_5 = \{p_1, p_2, p_3, p_4, p_5\}$$

be a fixed set of five distinguished poles.

A *Pentapolar Uncertain Set* of type M on X is a mapping

$$\mu_A : X \times P_5 \longrightarrow \mathrm{Dom}(M).$$

Equivalently, it is specified by five mappings

$$\mu_A^{(1)}, \mu_A^{(2)}, \mu_A^{(3)}, \mu_A^{(4)}, \mu_A^{(5)} : X \longrightarrow \mathrm{Dom}(M),$$

where

$$\mu_A^{(i)}(x) = \mu_A(x, p_i), \qquad i \in \{1, 2, 3, 4, 5\}.$$

Thus, for every $x \in X$,

$$\mu_A(x) = \left(\mu_A^{(1)}(x), \mu_A^{(2)}(x), \mu_A^{(3)}(x), \mu_A^{(4)}(x), \mu_A^{(5)}(x)\right) \in \mathrm{Dom}(M)^5.$$

The values

$$\mu_A^{(1)}(x), \qquad \mu_A^{(2)}(x), \qquad \mu_A^{(3)}(x), \qquad \mu_A^{(4)}(x), \qquad \mu_A^{(5)}(x)$$

represent the five uncertainty values associated with the five distinguished poles of x.

No relation among the five components is required unless an additional constraint is explicitly imposed as part of the pentapolar model.

Example 3.5.2 (Pentapolar Uncertain Set for Restaurant Selection)**.** Consider a customer selecting a restaurant for dinner. Let

$$X = \{r_1, r_2, r_3, r_4\}$$

be four candidate restaurants.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0, 1].$$

Suppose that five evaluation poles are considered:

$$P_5 = \{p_{\text{food}}, p_{\text{service}}, p_{\text{price}}, p_{\text{atmosphere}}, p_{\text{location}}\}.$$

For each restaurant r_i, let

$$\mu_A^{(1)}(r_i), \quad \mu_A^{(2)}(r_i), \quad \mu_A^{(3)}(r_i), \quad \mu_A^{(4)}(r_i), \quad \mu_A^{(5)}(r_i)$$

represent its uncertainty-adjusted evaluations with respect to food quality, service quality, price attractiveness, atmosphere, and location convenience, respectively.

Suppose that

Restaurant	$\mu_A^{(1)}$	$\mu_A^{(2)}$	$\mu_A^{(3)}$	$\mu_A^{(4)}$	$\mu_A^{(5)}$
r_1	0.85	0.70	0.55	0.80	0.75
r_2	0.65	0.80	0.75	0.60	0.90
r_3	0.75	0.50	0.90	0.45	0.60
r_4	0.80	0.75	0.60	0.85	0.70

Then

$$\mu_A(r_1) = (0.85, 0.70, 0.55, 0.80, 0.75),$$
$$\mu_A(r_2) = (0.65, 0.80, 0.75, 0.60, 0.90),$$

$$\mu_A(r_3) = (0.75, 0.50, 0.90, 0.45, 0.60),$$

and

$$\mu_A(r_4) = (0.80, 0.75, 0.60, 0.85, 0.70).$$

Hence,

$$A = \begin{Bmatrix} (r_1, 0.85, 0.70, 0.55, 0.80, 0.75), \\ (r_2, 0.65, 0.80, 0.75, 0.60, 0.90), \\ (r_3, 0.75, 0.50, 0.90, 0.45, 0.60), \\ (r_4, 0.80, 0.75, 0.60, 0.85, 0.70) \end{Bmatrix}$$

is a Pentapolar Uncertain Set.

For example, r_2 receives its highest evaluation for location convenience, whereas r_3 receives its highest evaluation for price attractiveness. Thus, the pentapolar representation allows five distinct uncertainty perspectives to be represented simultaneously.

Remark 3.5.3 (Relation to Multipolar Uncertain Sets)**.** A Pentapolar Uncertain Set is precisely the $m = 5$ specialization of a Multipolar Uncertain Set.

Indeed,

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)^5$$

is the special case of

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)^m$$

obtained by setting

$$m = 5.$$

Therefore, without additional relations among its five poles, pentapolarity introduces a distinguished five-pole interpretation but does not introduce a carrier structure beyond the corresponding five-fold product model.

Remark 3.5.4 (Fuzzy Specialization)**.** Let M be the fuzzy model with

$$\mathrm{Dom}(M) = [0, 1].$$

Then a Pentapolar Uncertain Set is determined by

$$\mu_A^{(1)}, \mu_A^{(2)}, \mu_A^{(3)}, \mu_A^{(4)}, \mu_A^{(5)} : X \longrightarrow [0, 1],$$

or equivalently by

$$\mu_A : X \longrightarrow [0, 1]^5.$$

Thus, every element $x \in X$ is represented by the five-dimensional membership vector

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \mu_A^{(2)}(x), \mu_A^{(3)}(x), \mu_A^{(4)}(x), \mu_A^{(5)}(x)\big).$$

This may be regarded as the fuzzy specialization of the Pentapolar Uncertain Set framework.

No normalization condition such as

$$\sum_{i=1}^{5} \mu_A^{(i)}(x) \le 1$$

or

$$\sum_{i=1}^{5} \mu_A^{(i)}(x) = 1$$

is imposed unless it is explicitly required by the particular pentapolar model under consideration.

Representative finite-dimensional specializations that can be expressed within the framework of Pentapolar Uncertain Sets are summarized in Table 3.6.

Table 3.6: Dimension-based classification of representative Pentapolar Uncertain Sets.

Dimension of each pole k	**Total pentapolar dimension**	**Representative specialization**
1	5	Pentapolar fuzzy-type Uncertain Set
2	10	Pentapolar intuitionistic-fuzzy-type Uncertain Set
3	15	Pentapolar neutrosophic-type Uncertain Set
k	$5k$	Pentapolar k-dimensional Uncertain Set

Remark 3.5.5 (Flattening Representation)**.** Suppose that the degree-domain of M has a declared finite-dimensional realization

$$\mathrm{Dom}(M) \subseteq [0,1]^k.$$

Then

$$\mathrm{Dom}(M)^5 \subseteq [0,1]^{5k}.$$

Consequently, every finite-dimensional Pentapolar Uncertain Set admits the canonical flattened representation

$$\mu_A : X \longrightarrow [0,1]^{5k}.$$

In particular, the total scalar dimension of a pentapolar construction is five times the scalar dimension of the underlying uncertainty model.

3.6 Multipolar Uncertain Set

A Multipolar Fuzzy Set assigns multiple independent fuzzy membership degrees to each element, representing several perspectives, attributes, or agents simultaneously [176–180].

Definition 3.6.1 (Multipolar Fuzzy Set)**.** (cf. [143, 181]) Let $X \neq \emptyset$ be a universe and let $m \in \mathbb{N}$. A *multipolar fuzzy set* (or m-polar fuzzy set) A on X is characterized by a mapping

$$\mu_A : X \longrightarrow [0,1]^m.$$

Equivalently,

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \mu_A^{(2)}(x), \dots, \mu_A^{(m)}(x)\big), \qquad x \in X,$$

where

$$\mu_A^{(i)} : X \longrightarrow [0,1], \qquad i = 1, 2, \dots, m,$$

and $\mu_A^{(i)}(x)$ represents the membership degree of x with respect to the i-th pole.

Thus,

$$A = \left\{ \Big(x, \mu_A^{(1)}(x), \mu_A^{(2)}(x), \dots, \mu_A^{(m)}(x)\Big) \;\Big|\; x \in X \right\}.$$

A Multipolar Uncertain Set assigns multiple uncertainty values to each element, representing poles, perspectives, criteria, or agents within one framework.

Definition 3.6.2 (Multipolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, let $m \geq 2$, and let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. A *Multipolar Uncertain Set* of type M on X is a mapping

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)^m.$$

Equivalently, for every $x \in X$,

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \ldots, \mu_A^{(m)}(x)\big),$$

where

$$\mu_A^{(i)} : X \longrightarrow \mathrm{Dom}(M), \qquad i = 1, \ldots, m.$$

Thus, each element x is assigned one admissible uncertainty value for each of the m poles.

Example 3.6.3 (Multipolar Uncertain Set for Smartphone Selection)**.** Consider a customer comparing several smartphones. Let

$$X = \{s_1, s_2, s_3, s_4\}$$

be a set of four candidate smartphones.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0, 1]$$

and let

$$m = 4.$$

Suppose that the four poles represent the following evaluation criteria:

$$p_1 = \text{performance}, \qquad p_2 = \text{battery life}, \qquad p_3 = \text{camera quality}, \qquad p_4 = \text{affordability}.$$

Define

$$\mu_A : X \longrightarrow [0, 1]^4$$

by

$$\begin{aligned} \mu_A(s_1) &= (0.90, 0.75, 0.85, 0.40), \\ \mu_A(s_2) &= (0.70, 0.90, 0.65, 0.75), \\ \mu_A(s_3) &= (0.80, 0.60, 0.95, 0.35), \end{aligned}$$

and

$$\mu_A(s_4) = (0.60, 0.80, 0.55, 0.90).$$

Equivalently, the corresponding Multipolar Uncertain Set is

$$A = \left\{ \begin{array}{l} (s_1, (0.90, 0.75, 0.85, 0.40)), \\ (s_2, (0.70, 0.90, 0.65, 0.75)), \\ (s_3, (0.80, 0.60, 0.95, 0.35)), \\ (s_4, (0.60, 0.80, 0.55, 0.90)) \end{array} \right\}.$$

For example,

$$\mu_A(s_1) = (0.90, 0.75, 0.85, 0.40)$$

indicates that smartphone s_1 is evaluated highly in performance and camera quality, moderately highly in battery life, but relatively low in affordability.

Similarly,

$$\mu_A(s_4) = (0.60, 0.80, 0.55, 0.90)$$

indicates that s_4 is highly affordable and has good battery life, although its performance and camera quality receive lower evaluations.

Thus, the Multipolar Uncertain Set provides a unified representation of several uncertainty-based evaluations of each smartphone from multiple criteria or perspectives.

Remark 3.6.4. If

$$\mathrm{Dom}(M) = [0, 1],$$

then a Multipolar Uncertain Set reduces to an m-polar fuzzy set, since

$$\mu_A : X \longrightarrow [0, 1]^m.$$

More generally, choosing intuitionistic fuzzy, neutrosophic, plithogenic, or other uncertainty domains yields the corresponding multipolar extensions.

Representative concepts that can be expressed within the framework of m-Polar Uncertain Sets are summarized in Table 3.7.

Table 3.7: Dimension-based classification of representative m-Polar Uncertain Sets.

Dimension of each pole k	**Total m-polar dimension**	**Representative m-Polar Uncertain Set**	**References**
1	m	m-Polar Fuzzy Set [181–183]	—
2	$2m$	m-Polar Intuitionistic Fuzzy Set	[148, 184–186]
2	$2m$	m-Polar Vague Set	—
3	$3m$	m-Polar Neutrosophic Set	[41, 150, 187–189]
3	$3m$	m-Polar Picture Fuzzy Set	[56, 190, 191]
3	$3m$	m-Polar Spherical Fuzzy Set	[192, 193]
Variable / set-valued	Variable	m-Polar Hesitant Fuzzy Set	[194]

3.7 Iterative Multipolar Uncertain Set

An Iterative m-Polar Fuzzy Set recursively nests multipolar membership structures, representing hierarchical fuzzy evaluations across successive levels of polarity information.

Definition 3.7.1 (Iterative m-Polar Fuzzy Set)**.** Let $X \neq \emptyset$ be a universe, let $m \geq 2$, and let $r \in \mathbb{N}_0$. Define the sequence of membership spaces recursively by

$$\mathcal{P}_m^{(0)} = [0, 1], \qquad \mathcal{P}_m^{(r+1)} = \left(\mathcal{P}_m^{(r)}\right)^m.$$

An *r-th order iterative m-polar fuzzy set A* on X is characterized by a mapping

$$\mu_A^{(r)} : X \longrightarrow \mathcal{P}_m^{(r)}.$$

In particular,

$$\mathcal{P}_m^{(1)} = [0, 1]^m,$$

so that the first-order case is an ordinary m-polar fuzzy set, while

$$\mathcal{P}_m^{(2)} = ([0,1]^m)^m, \qquad \mathcal{P}_m^{(3)} = (([0,1]^m)^m)^m,$$

and so on.

Equivalently, for every $x \in X$,

$$\mu_A^{(r)}(x) = (\mu_{A,i_1,\ldots,i_r}(x))_{(i_1,\ldots,i_r)\in\{1,\ldots,m\}^r},$$

where

$$\mu_{A,i_1,\ldots,i_r}(x) \in [0,1].$$

Thus,

$$A = \left\{ \left(x, (\mu_{A,i_1,\ldots,i_r}(x))_{(i_1,\ldots,i_r)\in\{1,\ldots,m\}^r} \right) \;\middle|\; x \in X \right\}.$$

Definition 3.7.2 (Iterative Multipolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, let $m \geq 2$, and let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. Define recursively

$$\mathcal{D}_{M,m}^{(0)} = \mathrm{Dom}(M), \qquad \mathcal{D}_{M,m}^{(r+1)} = \left(\mathcal{D}_{M,m}^{(r)}\right)^m, \qquad r \in \mathbb{N}_0.$$

An *r-th order Iterative Multipolar Uncertain Set* of type M on X is a mapping

$$\mu_A^{(r)} : X \longrightarrow \mathcal{D}_{M,m}^{(r)}.$$

In particular,

$$\mathcal{D}_{M,m}^{(1)} = \mathrm{Dom}(M)^m,$$

so the first-order case is a Multipolar Uncertain Set, whereas

$$\mathcal{D}_{M,m}^{(2)} = (\mathrm{Dom}(M)^m)^m,$$

and higher orders recursively represent nested multipolar uncertainty.

Example 3.7.3 (Iterative Multipolar Uncertain Set for Project Evaluation)**.** Consider a company evaluating several candidate projects. Let

$$X = \{p_1, p_2, p_3\}$$

be the set of projects.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0,1],$$

and let

$$m = 2, \qquad r = 2.$$

Then

$$\mathcal{D}_{M,2}^{(1)} = [0,1]^2$$

and

$$\mathcal{D}_{M,2}^{(2)} = \left([0,1]^2\right)^2.$$

Suppose that each project is evaluated by two committees:

$$C_1 = \text{Technical Committee}, \qquad C_2 = \text{Business Committee}.$$

Each committee evaluates two aspects. The Technical Committee considers

$$\text{feasibility} \quad \text{and} \quad \text{innovation},$$

whereas the Business Committee considers

$$\text{cost efficiency} \quad \text{and} \quad \text{market potential}.$$

Define

$$\mu_A^{(2)} : X \longrightarrow \big([0,1]^2\big)^2$$

by

$$\mu_A^{(2)}(p_1) = \big((0.85, 0.90), (0.65, 0.80)\big),$$

$$\mu_A^{(2)}(p_2) = \big((0.70, 0.60), (0.90, 0.75)\big),$$

and

$$\mu_A^{(2)}(p_3) = \big((0.60, 0.95), (0.55, 0.85)\big).$$

For example,

$$\mu_A^{(2)}(p_1) = \big((0.85, 0.90), (0.65, 0.80)\big)$$

means that the Technical Committee assigns p_1 the first-order multipolar evaluation

$$(0.85, 0.90)$$

for feasibility and innovation, respectively, while the Business Committee assigns

$$(0.65, 0.80)$$

for cost efficiency and market potential.

Thus, the second-order Iterative Multipolar Uncertain Set represents a nested evaluation in which several groups each provide several uncertainty-based assessments for the same project. Higher iterative orders can similarly represent additional hierarchical levels, such as departments, regional offices, or organizational layers.

Remark 3.7.4. The construction contains several important special cases:

$$r = 0 \quad \Longrightarrow \quad \mu_A^{(0)} : X \to \mathrm{Dom}(M),$$

which is an ordinary Uncertain Set, and

$$r = 1 \quad \Longrightarrow \quad \mu_A^{(1)} : X \to \mathrm{Dom}(M)^m,$$

which is a Multipolar Uncertain Set.

If

$$\mathrm{Dom}(M) = [0,1],$$

the construction reduces to the corresponding Iterative m-Polar Fuzzy Set.

3.8 Hyperpolar Uncertain Set

A Hyperpolar Fuzzy Set assigns multidimensional fuzzy membership vectors to multiple polarities, enabling structured representation of heterogeneous multi-perspective uncertainty information [195].

Definition 3.8.1 (Hyperpolar Fuzzy Set)**.** [195] Let $X \neq \emptyset$ be a universe and let $n \in \mathbb{N}$. For each $k \in \{1, \ldots, n\}$, let

$$\delta_k \in \mathbb{N}$$

denote the dimension of the k-th polarity, and define

$$\Delta := \prod_{k=1}^{n} [0,1]^{\delta_k}.$$

A *hyperpolar fuzzy set* H on X is a mapping

$$H : X \longrightarrow \Delta.$$

Equivalently, for each $x \in X$,

$$H(x) = \big(H_1(x), H_2(x), \ldots, H_n(x)\big),$$

where

$$H_k(x) = \big(\mu_{k,1}(x), \mu_{k,2}(x), \ldots, \mu_{k,\delta_k}(x)\big) \in [0,1]^{\delta_k}.$$

Thus, each $H_k(x)$ represents the multidimensional fuzzy membership information associated with the k-th polarity.

Definition 3.8.2 (Hyperpolar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, let $M_0 = (\mathrm{Dom}(M))$ be a carrier model, and let $n \geq 1$.

For each $k \in \{1, \ldots, n\}$, let

$$\delta_k \geq 1$$

denote the dimension of the k-th polarity, and define

$$\Delta_M := \prod_{k=1}^{n} \mathrm{Dom}(M)^{\delta_k}.$$

A *Hyperpolar Uncertain Set* of type M on X is a mapping

$$H : X \longrightarrow \Delta_M.$$

Equivalently, for each $x \in X$,

$$H(x) = \big(H_1(x), \ldots, H_n(x)\big),$$

where

$$H_k(x) = \big(d_{k,1}(x), \ldots, d_{k,\delta_k}(x)\big) \in \mathrm{Dom}(M)^{\delta_k}.$$

Thus, each polarity contains a finite multidimensional collection of admissible uncertainty values determined by the model M.

Example 3.8.3 (Hyperpolar Uncertain Set for Employee Evaluation)**.** Consider a company evaluating several employees for promotion. Let

$$X = \{e_1, e_2, e_3\}$$

be the set of employees.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0, 1].$$

Suppose that the company uses three major polarities:

$$P_1 = \text{Technical Ability}, \qquad P_2 = \text{Teamwork}, \qquad P_3 = \text{Leadership}.$$

Assume that the corresponding polarity dimensions are

$$\delta_1 = 3, \qquad \delta_2 = 2, \qquad \delta_3 = 3.$$

Thus,

$$\Delta_M = [0, 1]^3 \times [0, 1]^2 \times [0, 1]^3.$$

The first polarity evaluates

programming skill, problem-solving ability, technical knowledge.

The second polarity evaluates

cooperation, communication.

The third polarity evaluates

decision making, responsibility, strategic thinking.

Define

$$H : X \longrightarrow \Delta_M$$

by

$$H(e_1) = \big((0.90, 0.85, 0.88), (0.75, 0.80), (0.70, 0.78, 0.72)\big),$$
$$H(e_2) = \big((0.75, 0.80, 0.70), (0.92, 0.88), (0.82, 0.85, 0.80)\big),$$

and

$$H(e_3) = \big((0.85, 0.90, 0.82), (0.68, 0.72), (0.90, 0.92, 0.88)\big).$$

For example,

$$H_1(e_1) = (0.90, 0.85, 0.88)$$

represents the three uncertainty-based evaluations of employee e_1 under the Technical Ability polarity, while

$$H_2(e_1) = (0.75, 0.80)$$

represents the evaluations under Teamwork.

Similarly,

$$H_3(e_3) = (0.90, 0.92, 0.88)$$

shows that employee e_3 receives high evaluations across the three dimensions of Leadership.

Hence, the Hyperpolar Uncertain Set allows each major evaluation polarity to contain its own multidimensional collection of uncertainty values, providing a structured representation of complex employee assessment.

Remark 3.8.4. If

$$\mathrm{Dom}(M) = [0, 1],$$

then

$$\Delta_M = \prod_{k=1}^{n} [0, 1]^{\delta_k},$$

and the Hyperpolar Uncertain Set reduces to a Hyperpolar Fuzzy Set. Thus, Hyperpolar Fuzzy Sets constitute a special case of Hyperpolar Uncertain Sets.

3.9 Variable-Polar Uncertain Set

Let $X \neq \emptyset$ be a universe. For a nonempty set P, we write

$$[0, 1]^P = \{f : P \to [0, 1]\}.$$

Thus, an element of $[0, 1]^P$ represents a family of fuzzy membership degrees indexed by the set of poles P.

We also denote by

$$\mathcal{P}^+_{\mathrm{fin}}(P) = \{Q \subseteq P : 1 \leq |Q| < \infty\}$$

the family of all nonempty finite subsets of P.

Definition 3.9.1 (Variable-Polar Fuzzy Set)**.** Let

$$\mathcal{P} = \{P_x : x \in X\}$$

be a family of nonempty finite sets, where P_x is the set of poles associated with $x \in X$.

A *Variable-Polar Fuzzy Set* on X with respect to $\mathcal{P}$ is a family

$$A = \{\mu_{A,x} : x \in X\},$$

where

$$\mu_{A,x} : P_x \longrightarrow [0, 1]$$

for every $x \in X$.

Equivalently,

$$A \in \prod_{x \in X} [0, 1]^{P_x}.$$

Hence, the number of poles is allowed to depend on the element x. If

$$|P_x| = m \qquad \text{for every } x \in X,$$

then the structure reduces to an ordinary fixed m-polar fuzzy representation.

A Variable-Polar Uncertain Set allows the collection of poles associated with an element to vary across the universe while assigning an admissible uncertainty value to each active pole.

Definition 3.9.2 (Variable-Polar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. Let

$$\mathcal{P} = \{P_x : x \in X\}$$

be a family of nonempty finite pole sets.

A *Variable-Polar Uncertain Set* of type M on X with respect to $\mathcal{P}$ is a family

$$A = \{\mu_{A,x} : x \in X\},$$

where

$$\mu_{A,x} : P_x \longrightarrow \mathrm{Dom}(M)$$

for every $x \in X$.

Equivalently,

$$A \in \prod_{x \in X} \mathrm{Dom}(M)^{P_x}.$$

Thus, both the number and indexing of poles may depend on the element x, while every pole receives an admissible uncertainty value from $\mathrm{Dom}(M)$.

Example 3.9.3 (Variable-Polar Uncertain Set for Personalized Medical Evaluation)**.** Consider a hospital evaluating several patients with different sets of relevant clinical risk factors. Let

$$X = \{p_1, p_2, p_3\}$$

be the set of patients.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0, 1].$$

Because different patients may require different clinical criteria, assign a separate pole set to each patient:

$$P_{p_1} = \{\text{blood pressure}, \text{blood glucose}, \text{cholesterol}\},$$

$$P_{p_2} = \{\text{blood pressure}, \text{heart rate}\},$$

and

$$P_{p_3} = \{\text{blood glucose}, \text{cholesterol}, \text{body mass index}, \text{smoking risk}\}.$$

Define

$$\mu_{A,p_1} : P_{p_1} \longrightarrow [0, 1]$$

by

$$\mu_{A,p_1}(\text{blood pressure}) = 0.80,$$

$$\mu_{A,p_1}(\text{blood glucose}) = 0.65,$$

$$\mu_{A,p_1}(\text{cholesterol}) = 0.70.$$

Similarly, let

$$\mu_{A,p_2}(\text{blood pressure}) = 0.55, \qquad \mu_{A,p_2}(\text{heart rate}) = 0.75,$$

and

$$\mu_{A,p_3}(\text{blood glucose}) = 0.85,$$

$$\mu_{A,p_3}(\text{cholesterol}) = 0.60,$$
$$\mu_{A,p_3}(\text{body mass index}) = 0.70,$$
$$\mu_{A,p_3}(\text{smoking risk}) = 0.90.$$

Thus,

$$A = \{\mu_{A,p_1}, \mu_{A,p_2}, \mu_{A,p_3}\}$$

is a Variable-Polar Uncertain Set.

For example, patient p_1 is evaluated using three clinically relevant poles, whereas patient p_2 is evaluated using only two. Patient p_3 requires four poles because additional risk factors are relevant to the assessment.

Hence, the Variable-Polar Uncertain Set allows both the number and the meaning of the uncertainty poles to vary from one patient to another, making it suitable for personalized evaluation problems.

Remark 3.9.4. If

$$\mathrm{Dom}(M) = [0,1],$$

then a Variable-Polar Uncertain Set reduces to a Variable-Polar Fuzzy Set.

If, in addition,

$$|P_x| = m \qquad \text{for every } x \in X,$$

with a common identification of the pole sets with $\{1, \dots, m\}$, then

$$\prod_{x\in X} [0,1]^{P_x} \cong \left([0,1]^m\right)^X,$$

which recovers the ordinary fixed m-polar fuzzy representation.

3.10 Dynamic-Polar Uncertain Set

A time index by itself produces only a family of static polar sets. The following definition therefore includes an explicit evolution law.

Definition 3.10.1 (Transition-Coherent Dynamic-Polar Fuzzy Set)**.** Let $X \neq \emptyset$, let $P \neq \emptyset$ be a finite pole set, and let (T, T°, σ) be a discrete time system, where $T^\circ \subseteq T$ and $\sigma : T^\circ \to T$ is a successor map. For every $t \in T^\circ$, prescribe a total evolution operator

$$\mathcal{E}_t : [0,1]^P \longrightarrow [0,1]^P.$$

A *Transition-Coherent Dynamic-Polar Fuzzy Set* is a map

$$\mu_A : X \times T \times P \longrightarrow [0,1]$$

whose time sections

$$\mu_{A,t}(x) = \big(\mu_A(x,t,p)\big)_{p\in P} \in [0,1]^P$$

satisfy the evolution equation

$$\mu_{A,\sigma(t)}(x) = \mathcal{E}_t\big(\mu_{A,t}(x)\big) \qquad (x \in X,\ t \in T^\circ).$$

It is *dynamically nontrivial* if some admissible trajectory has two distinct time sections.

Definition 3.10.2 (Transition-Coherent Dynamic-Polar Uncertain Set)**.** Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model, let $X, P \neq \emptyset$ with P finite, and let (T, T°, σ) be a discrete time system. For every $t \in T^\circ$, prescribe a typed evolution operator

$$\mathcal{E}_t : \mathrm{Dom}(M)^P \longrightarrow \mathrm{Dom}(M)^P.$$

A *Transition-Coherent Dynamic-Polar Uncertain Set* of type M is a map

$$\mu_A : X \times T \times P \longrightarrow \mathrm{Dom}(M)$$

such that

$$\mu_{A,\sigma(t)}(x) = \mathcal{E}_t\big(\mu_{A,t}(x)\big), \qquad \mu_{A,t}(x) = \big(\mu_A(x,t,p)\big)_{p \in P}.$$

Thus the admissible objects form the trajectory subspace

$$\mathrm{Dyn}(X, T, P, M, \mathcal{E}) = \left\{\mu \in \mathrm{Dom}(M)^{X \times T \times P} : \mu_{\sigma(t)}(x) = \mathcal{E}_t(\mu_t(x))\right\},$$

not the entire flattened function space.

Example 3.10.3 (A Three-Step Traffic Trajectory)**.** Let $X = \{x\}$, $P = \{c, a\}$, and $T = \{t_0, t_1, t_2\}$, with $\sigma(t_0) = t_1$ and $\sigma(t_1) = t_2$. In the fuzzy model set

$$\mathcal{E}_{t_0}(u, v) = (\min\{1, u + 0.10\}, \max\{0, v - 0.10\})$$

and

$$\mathcal{E}_{t_1}(u, v) = (0.8u, \min\{1, v + 0.20\}).$$

Starting from $\mu_{A,t_0}(x) = (0.60, 0.40)$, coherence forces

$$\mu_{A,t_1}(x) = (0.70, 0.30), \qquad \mu_{A,t_2}(x) = (0.56, 0.50).$$

The three tables are therefore linked by a declared evolution rule rather than being independent observations.

Proposition 3.10.4 (Trajectory Determination)**.** *If $T = \{t_0, \dots, t_n\}$ is a successor chain, then a transition-coherent Dynamic-Polar Uncertain Set is uniquely determined by its initial section μ_{A,t_0} and the operators $\mathcal{E}_{t_i}$. It exists for every initial section because the evolution operators are total.*

Proof. Apply the recurrence successively. Typedness at every step follows from the codomain of $\mathcal{E}_{t_i}$ and Lemma 2.3.3. □

3.11 Tree-Polar Uncertain Set

A rooted tree must constrain the polar degrees; merely recording the tree next to an arbitrary map does not use its edges or root.

Definition 3.11.1 (Tree-Coherent Polar Fuzzy Set)**.** Let $\mathcal{T}_P = (P, E_P, r)$ be a finite rooted tree, oriented away from its root. For each internal vertex p, let $\mathrm{Ch}(p)$ be its nonempty child set and prescribe an aggregation map

$$\mathrm{Agg}_p : [0,1]^{\mathrm{Ch}(p)} \longrightarrow [0,1].$$

A *Tree-Coherent Polar Fuzzy Set* on X is a map

$$\mu_A : X \times P \longrightarrow [0,1]$$

satisfying, for every internal vertex p,

$$\mu_A(x, p) = \mathrm{Agg}_p\left((\mu_A(x,q))_{q \in \mathrm{Ch}(p)}\right) \qquad (x \in X).$$

In particular, the root value is computed from its descendants through the tree, rather than supplied independently.

Definition 3.11.2 (Tree-Coherent Polar Uncertain Set)**.** Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model and let $\mathcal{T}_P = (P, E_P, r)$ be a finite rooted tree. For every internal vertex p, prescribe a typed aggregation map

$$\mathrm{Agg}_{M,p} : \mathrm{Dom}(M)^{\mathrm{Ch}(p)} \longrightarrow \mathrm{Dom}(M).$$

A *Tree-Coherent Polar Uncertain Set* of type M on X is the structure

$$A = (\mathcal{T}_P, (\mathrm{Agg}_{M,p})_p, \mu_A), \qquad \mu_A : X \times P \longrightarrow \mathrm{Dom}(M),$$

subject to

$$\mu_A(x,p) = \mathrm{Agg}_{M,p}\left((\mu_A(x,q))_{q\in\mathrm{Ch}(p)}\right)$$

for every $x \in X$ and every internal p. The tree and aggregation family are part of the structure and must be preserved by an isomorphism.

Example 3.11.3 (Hierarchical Course Evaluation)**.** Let the root r represent overall quality and let its children p_1, p_2 represent teaching and design. In the fuzzy model take

$$\mathrm{Agg}_r(a,b) = \frac{a+b}{2}.$$

For a course x, leaf values $\mu_A(x,p_1) = 0.80$ and $\mu_A(x,p_2) = 0.60$ force

$$\mu_A(x,r) = 0.70.$$

Changing an edge or the aggregator changes the admissible set of evaluations.

Proposition 3.11.4 (Leaf Determination)**.** *For a finite rooted tree with total aggregation maps, every assignment of typed degrees to the leaves extends uniquely to a Tree-Coherent Polar Uncertain Set.*

Proof. Proceed upward from the leaves. At each internal vertex the values of all children have already been determined, so the prescribed total aggregation map gives one typed parent value. Finiteness terminates the recursion at the root. $\square$

3.12 Heterogeneous-Polar Uncertain Set

A Heterogeneous-Polar Fuzzy Set allows different poles to use different admissible fuzzy-degree domains, thereby supporting heterogeneous membership representations within a single polar framework.

Definition 3.12.1 (Heterogeneous-Polar Fuzzy Set)**.** Let $X \neq \emptyset$ be a universe, and let

$$P = \{p_1, \dots, p_m\}, \qquad m \geq 1,$$

be a finite nonempty set of poles.

For every $p \in P$, let

$$\emptyset \neq D_p \subseteq [0,1]$$

be an admissible fuzzy-degree domain, and denote the indexed family of degree-domains by

$$\mathcal{D} = (D_p)_{p\in P}.$$

A *Heterogeneous-Polar Fuzzy Set* on X with respect to $\mathcal{D}$ is a mapping

$$\mu_A : X \longrightarrow \prod_{p\in P} D_p.$$

Equivalently, for every $x \in X$,

$$\mu_A(x) = \big(\mu_{A,p}(x)\big)_{p \in P},$$

where

$$\mu_{A,p} : X \longrightarrow D_p$$

for every $p \in P$.

Thus, different poles may employ different admissible fuzzy-degree domains.

A Heterogeneous-Polar Uncertain Set generalizes this construction by allowing each pole to employ its own uncertainty model and associated degree-domain.

Definition 3.12.2 (Heterogeneous-Polar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let

$$P = \{p_1, \dots, p_m\}, \qquad m \geq 1,$$

be a finite nonempty set of poles.

For each $p \in P$, let

$$M_{p,0} = (\mathrm{Dom}(M_p))$$

be a carrier model. The carriers may have different mathematical types; no common finite-vector realization is required. Denote the indexed family of uncertain models by

$$\mathcal{M} = (M_p)_{p \in P}.$$

A *Heterogeneous-Polar Uncertain Set* on X with respect to $\mathcal{M}$ is a mapping

$$\mu_A : X \longrightarrow \prod_{p \in P} \mathrm{Dom}(M_p).$$

Equivalently, for every $x \in X$,

$$\mu_A(x) = \big(\mu_{A,p}(x)\big)_{p \in P},$$

where

$$\mu_{A,p} : X \longrightarrow \mathrm{Dom}(M_p)$$

for every $p \in P$.

Thus, each pole may employ a different uncertainty model, possibly with a different degree-domain dimension k_p.

Example 3.12.3 (Heterogeneous-Polar Uncertain Set for Hospital Evaluation)**.** Consider a health authority evaluating several hospitals. Let

$$X = \{h_1, h_2, h_3\}$$

be the set of hospitals, and let

$$P = \{p_1, p_2, p_3\}$$

be the set of evaluation poles, where

$$p_1 = \text{Service Quality}, \qquad p_2 = \text{Waiting-Time Reliability}, \qquad p_3 = \text{Patient-Safety Assessment}.$$

Suppose that different uncertainty models are appropriate for these three criteria.

For p_1, use the fuzzy model M_{p_1} with

$$\mathrm{Dom}(M_{p_1}) = [0,1].$$

For p_2, use an interval-valued model M_{p_2} with

$$\mathrm{Dom}(M_{p_2}) = \mathbb{I} = \{[a,b] : 0 \le a \le b \le 1\}.$$

For p_3, use a single-valued neutrosophic model M_{p_3} with

$$\mathrm{Dom}(M_{p_3}) = [0,1]^3,$$

whose components represent truth, indeterminacy, and falsity degrees.

Hence the heterogeneous degree-domain is

$$[0,1] \times \mathbb{I} \times [0,1]^3.$$

Define

$$\mu_A : X \longrightarrow [0,1] \times \mathbb{I} \times [0,1]^3$$

by

$$\mu_A(h_1) = (0.85,\ [0.70, 0.80],\ (0.90, 0.15, 0.10)),$$

$$\mu_A(h_2) = (0.72,\ [0.55, 0.70],\ (0.75, 0.25, 0.20)),$$

and

$$\mu_A(h_3) = (0.90,\ [0.65, 0.85],\ (0.82, 0.30, 0.12)).$$

For example,

$$\mu_{A,p_1}(h_1) = 0.85$$

is the fuzzy degree associated with the service quality of hospital h_1, whereas

$$\mu_{A,p_2}(h_1) = [0.70, 0.80]$$

is an interval-valued assessment of its waiting-time reliability. Furthermore,

$$\mu_{A,p_3}(h_1) = (0.90, 0.15, 0.10)$$

represents the truth, indeterminacy, and falsity degrees associated with its patient-safety assessment.

Therefore, μ_A is a Heterogeneous-Polar Uncertain Set in which different poles employ different uncertainty models according to the type of information being represented.

Remark 3.12.4. Suppose that

$$M_p = M \qquad \text{for every } p \in P.$$

Then

$$\prod_{p \in P} \mathrm{Dom}(M_p) = \mathrm{Dom}(M)^P.$$

If

$$|P| = m < \infty,$$

then, after choosing a bijection

$$P \longrightarrow \{1, \dots, m\},$$

the space $\mathrm{Dom}(M)^P$ can be identified with

$$\mathrm{Dom}(M)^m.$$

Hence the Heterogeneous-Polar Uncertain Set reduces, up to the chosen indexing of the poles, to an m-Polar Uncertain Set.

If every M_p is a fuzzy model with

$$\mathrm{Dom}(M_p) = D_p \subseteq [0, 1],$$

then the construction reduces to a Heterogeneous-Polar Fuzzy Set.

In particular, if

$$D_p = [0, 1] \qquad \text{for every } p \in P$$

and $|P| = m$, then the construction yields the usual m-polar fuzzy representation.

Remark 3.12.5 (Graph-Polar Uncertain Set versus m-Polar Uncertain Graph)**.** A Graph-Polar Uncertain Set and an m-Polar Uncertain Graph are conceptually different constructions.

In a *Graph-Polar Uncertain Set*, the graph

$$G_P = (P, E_P)$$

is defined on the set of poles P, and the uncertainty mapping

$$\mu_A : X \times P \longrightarrow \mathrm{Dom}(M)$$

assigns uncertainty degrees to elements of X with respect to graphically related poles. Thus, the graph describes relationships among the poles themselves.

By contrast, an *m-Polar Uncertain Graph* starts with a graph

$$G = (V, E)$$

and assigns m-polar uncertainty values directly to its vertices and edges, typically through mappings

$$\mu : V \longrightarrow \mathrm{Dom}(M)^m, \qquad \nu : E \longrightarrow \mathrm{Dom}(M)^m.$$

Hence, the former uses a graph to organize the pole structure, whereas the latter equips an underlying graph with multipolar uncertain membership information.

3.13 Graph-Polar Uncertain Set

The graph version is defined by edge compatibility, so two graphs on the same pole set need not determine the same admissible polar assignments.

Definition 3.13.1 (Graph-Coherent Polar Fuzzy Set)**.** Let $G_P = (P, E_P)$ be a finite simple graph. For every edge $e = \{p, q\}$, prescribe a symmetric compatibility relation

$$C_e \subseteq [0, 1]^2.$$

A *Graph-Coherent Polar Fuzzy Set* on X is a map

$$\mu_A : X \times P \longrightarrow [0, 1]$$

such that

$$(\mu_A(x, p), \mu_A(x, q)) \in C_e \qquad (x \in X,\ e = \{p, q\} \in E_P).$$

It is *structurally nontrivial* if at least one C_e is a proper subset of $[0, 1]^2$.

Definition 3.13.2 (Graph-Coherent Polar Uncertain Set)**.** Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model and let $G_P = (P, E_P)$ be a finite simple graph. A typed edge specification is a family

$$\mathcal{C} = (C_e)_{e \in E_P}, \qquad C_e \subseteq \mathrm{Dom}(M) \times \mathrm{Dom}(M),$$

where each C_e is symmetric. A *Graph-Coherent Polar Uncertain Set* is

$$A = (G_P, \mathcal{C}, \mu_A), \qquad \mu_A : X \times P \longrightarrow \mathrm{Dom}(M),$$

such that

$$(\mu_A(x, p), \mu_A(x, q)) \in C_{\{p,q\}}$$

for all $x \in X$ and $\{p, q\} \in E_P$. Nonemptiness is an admissibility condition on $\mathcal{C}$, not an automatic consequence of $\mathrm{Dom}(M) \neq \emptyset$.

Example 3.13.3 (Threshold-Compatible Criteria)**.** In the fuzzy model, assign to every edge e a tolerance $\lambda_e \in [0, 1]$ and put

$$C_e = \{(a, b) \in [0, 1]^2 : |a - b| \leq \lambda_e\}.$$

Adjacent criteria must then have compatible evaluations. Removing an edge removes its constraint, while decreasing λ_e strengthens it.

Proposition 3.13.4 (Constraint-Subspace Representation)**.** *The carrier of Graph-Coherent Polar Uncertain Sets is the constraint subspace*

$$\left\{\mu \in \mathrm{Dom}(M)^{X \times P} : (\mu(x, p), \mu(x, q)) \in C_{\{p,q\}} \text{ for every } x \text{ and } \{p, q\} \in E_P\right\}.$$

It equals the entire flattened space exactly when the edge constraints impose no restriction (for example, when every $C_e = \mathrm{Dom}(M)^2$).

3.14 Adaptive-Polar Uncertain Set

An active-pole selector describes context dependence. Adaptation additionally requires a rule connecting consecutive contexts.

Definition 3.14.1 (Policy-Coherent Adaptive-Polar Fuzzy Set)**.** Let $X, C, P^* \neq \emptyset$, let $K \subseteq C \times C$ be a context transition relation, and let

$$\Pi : X \times C \longrightarrow \mathcal{P}^{+}_{\mathrm{fin}}(P^*)$$

be an active-pole selector. For every $x \in X$ and $(c, c') \in K$, prescribe an update map

$$\mathcal{U}_{x,c,c'} : [0,1]^{\Pi(x,c)} \longrightarrow [0,1]^{\Pi(x,c')}.$$

Writing

$$\Omega_\Pi = \{(x, c, p) : p \in \Pi(x, c)\},$$

a *Policy-Coherent Adaptive-Polar Fuzzy Set* is a map $\mu_A : \Omega_\Pi \to [0, 1]$ satisfying

$$\mu_{A,x,c'} = \mathcal{U}_{x,c,c'}(\mu_{A,x,c}) \qquad ((c, c') \in K),$$

where $\mu_{A,x,c} = (\mu_A(x, c, p))_{p \in \Pi(x,c)}$.

Definition 3.14.2 (Policy-Coherent Adaptive-Polar Uncertain Set)**.** Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model and retain (X, C, P^*, K, Π) from Definition 3.14.1. For each x and $(c, c') \in K$, prescribe a typed update map

$$\mathcal{U}^M_{x,c,c'} : \mathrm{Dom}(M)^{\Pi(x,c)} \longrightarrow \mathrm{Dom}(M)^{\Pi(x,c')}.$$

A *Policy-Coherent Adaptive-Polar Uncertain Set* of type M is a map

$$\mu_A : \Omega_\Pi \longrightarrow \mathrm{Dom}(M)$$

whose context sections satisfy

$$\mu_{A,x,c'} = \mathcal{U}^M_{x,c,c'}(\mu_{A,x,c}) \qquad (x \in X,\ (c, c') \in K).$$

For composable transitions, a *consistent policy* additionally satisfies

$$\mathcal{U}^M_{x,c',c''} \circ \mathcal{U}^M_{x,c,c'} = \mathcal{U}^M_{x,c,c''}.$$

Example 3.14.3 (Context-Aware Triage Policy)**.** Let $C = \{c_{\text{routine}}, c_{\text{emergency}}\}$ with one transition from routine to emergency. Suppose

$$\Pi(x, c_{\text{routine}}) = \{\text{blood pressure}, \text{temperature}\}$$

and

$$\Pi(x, c_{\text{emergency}}) = \{\text{blood pressure}, \text{temperature}, \text{respiration}\}.$$

In the fuzzy model one possible update is

$$\mathcal{U}_{x,c_{\text{routine}},c_{\text{emergency}}}(a, b) = (a, b, \max\{a, b\}).$$

The newly activated respiratory pole is therefore determined by the declared policy rather than assigned independently.

Proposition 3.14.4 (Policy-Path Determination)**.** *Along any directed context path, the value at the initial context and the update maps uniquely determine every later section. If the policy is consistent, the result is independent of how a composable path is parenthesized.*

3.15 Semipolar Fuzzy Set

A semipolar set in probabilistic potential theory is a countable union of totally thin sets, capturing exceptional sets relevant to Hunt processes and balayage theory [196–198]. A Semipolar Fuzzy Set is a fuzzy set whose every positive alpha-cut forms a semipolar subset of an underlying Hunt-process state space at all levels.

Definition 3.15.1 (Semipolar Fuzzy Set)**.** Let $E \neq \emptyset$ be the state space of a Hunt process $\mathbb{X}$, and let

$$\mathfrak{S}(\mathbb{X})$$

denote the family of semipolar subsets of E.

For a fuzzy set
$$\mu_A : E \longrightarrow [0,1],$$
define its α-cut by
$$A_\alpha = \{x \in E : \mu_A(x) \geq \alpha\}, \qquad \alpha \in (0,1].$$

The fuzzy set A is called a *Semipolar Fuzzy Set* with respect to $\mathbb{X}$ if
$$A_\alpha \in \mathfrak{S}(\mathbb{X})$$
for every
$$\alpha \in (0,1].$$

Definition 3.15.2 (Semipolar Uncertain Set)**.** Let $E \neq \emptyset$ be the state space of a Hunt process $\mathbb{X}$, and let
$$\mathfrak{S}(\mathbb{X})$$
denote the family of semipolar subsets of E.

Let $M_s = (\mathrm{Dom}(M), s_M)$ be a scored uncertain model.

For an Uncertain Set
$$\mu_A : E \longrightarrow \mathrm{Dom}(M),$$
define its α-cut by
$$A_\alpha^M = \left\{x \in E : s_M\big(\mu_A(x)\big) \geq \alpha\right\}, \qquad \alpha \in (0,1].$$

The Uncertain Set A is called a *Semipolar Uncertain Set* with respect to $(\mathbb{X}, M, s_M)$ if
$$A_\alpha^M \in \mathfrak{S}(\mathbb{X})$$
for every
$$\alpha \in (0,1].$$

Example 3.15.3 (Semipolar Uncertain Set for Pollutant Hotspot Monitoring)**.** Consider the random movement of a pollutant particle in a two-dimensional environment. Let
$$E = \mathbb{R}^2,$$
and let $\mathbb{X}$ be standard planar Brownian motion, used here as a simplified stochastic model of particle dispersion.

Suppose that three particular monitoring locations are identified as potential contamination hotspots:
$$z_1,\ z_2,\ z_3 \in E.$$

Take the fuzzy uncertainty model
$$\mathrm{Dom}(M) = [0,1]$$
with scalar evaluation map
$$s_M(t) = t.$$

Define the Uncertain Set
$$\mu_A : E \longrightarrow [0,1]$$

by

$$\mu_A(z_1) = 0.90, \qquad \mu_A(z_2) = 0.70, \qquad \mu_A(z_3) = 0.40,$$

and

$$\mu_A(x) = 0$$

for every

$$x \in E \setminus \{z_1, z_2, z_3\}.$$

The values represent uncertain degrees of contamination risk at the three monitoring locations.

For $\alpha \in (0, 1]$, the corresponding α-cuts are

$$A_\alpha^M = \begin{cases} \{z_1, z_2, z_3\}, & 0 < \alpha \leq 0.40, \\ \{z_1, z_2\}, & 0.40 < \alpha \leq 0.70, \\ \{z_1\}, & 0.70 < \alpha \leq 0.90, \\ \emptyset, & 0.90 < \alpha \leq 1. \end{cases}$$

For planar Brownian motion, each singleton

$$\{z_i\}$$

is a polar set. Therefore every finite set appearing above is polar, and hence semipolar. Consequently,

$$A_\alpha^M \in \mathfrak{S}(\mathbb{X})$$

for every

$$\alpha \in (0, 1].$$

Thus A is a Semipolar Uncertain Set. In this interpretation, the model represents uncertainty concentrated at a finite collection of rarely hit spatial locations within a stochastic diffusion process.

Remark 3.15.4. If M is the fuzzy model with

$$\mathrm{Dom}(M) = [0, 1]$$

and

$$s_M(t) = t,$$

then

$$A_\alpha^M = \{x \in E : \mu_A(x) \geq \alpha\},$$

and the Semipolar Uncertain Set reduces to a Semipolar Fuzzy Set.

3.16 Pluripolar Fuzzy and Uncertain Sets

A pluripolar subset of a domain in $\mathbb{C}^n$ is locally contained in the negative-infinity locus of a nontrivial plurisubharmonic function (cf. [199–203]). The word "pluripolar" below refers specifically to the full-membership core; it does not assert that every nontrivial level cut is pluripolar.

Definition 3.16.1 (Pluripolar Fuzzy Set)**.** Let $\Omega \subseteq \mathbb{C}^n$ be a nonempty domain and let $u \in \mathrm{PSH}(\Omega)$ satisfy $u \le 0$ and $u \not\equiv -\infty$ on every connected component. Define

$$\mu_{A_u}(z) = 1 - \exp(u(z)), \qquad \exp(-\infty) = 0.$$

The pair (u, μ_{A_u}) is a *Pluripolar Fuzzy Set.* Its full-membership core is

$$\mathrm{Core}(A_u) = \{z : \mu_{A_u}(z) = 1\} = \{z : u(z) = -\infty\},$$

which is pluripolar. For $0 < \alpha < 1$, however,

$$[A_u]_\alpha = \{z : \mu_{A_u}(z) \ge \alpha\} = \{z : u(z) \le \log(1-\alpha)\}$$

is only a sublevel set and need not be pluripolar.

Definition 3.16.2 (Pluripolar Uncertain Set)**.** Let $M_{se} = (\mathrm{Dom}(M), s_M, \Phi_M)$ be a coherent scored–encoded model, and let u satisfy the hypotheses of Definition 3.16.1. Define

$$\mu_{A_u}(z) = \Phi_M\big(1 - \exp(u(z))\big).$$

Then

$$s_M(\mu_{A_u}(z)) = 1 - \exp(u(z)),$$

and the distinguished top degree is $d_M^\top = \Phi_M(1)$. The pair (u, μ_{A_u}) is called a *Pluripolar Uncertain Set* of type M.

Proposition 3.16.3 (Exact Pluripolar Core)**.** *For every Pluripolar Uncertain Set,*

$$\{z : \mu_{A_u}(z) = d_M^\top\} = \{z : u(z) = -\infty\}.$$

Proof. Coherence gives $s_M \circ \Phi_M = \mathrm{id}$, hence Φ_M is injective. Therefore $\Phi_M(1 - \exp u) = \Phi_M(1)$ holds exactly when $\exp u = 0$, equivalently when $u = -\infty$. $\square$

Example 3.16.4 (Why Only the Core Is Pluripolar)**.** On the unit ball

$$\Omega = \{z \in \mathbb{C}^2 : \|z\| < 1\},$$

take $u(z) = \log \|z\|^2$, with $u(0) = -\infty$. Then

$$\mu_{A_u}(z) = 1 - \|z\|^2$$

and $\mathrm{Core}(A_u) = \{0\}$ is pluripolar. For $0 < \alpha < 1$,

$$[A_u]_\alpha = \{z : \|z\|^2 \le 1 - \alpha\}$$

has nonempty interior and is not pluripolar. This example is the reason for the term *Pluripolar* rather than a claim about all α-cuts.

3.17 Self-Polar Fuzzy and Uncertain Sets

Projective polarity is an incidence-reversing duality between point and hyperplane sorts, not merely an involution of an untyped set (cf. [204–206]).

Definition 3.17.1 (Self-Dual Incidence Space and Polarity)**.** Let $\mathcal{I} = (P, H, \mathbf{I})$ be an incidence space with point set P, hyperplane set H, and incidence relation $\mathbf{I} \subseteq P \times H$. A *polarity* is a pair of inverse bijections

$$\pi_P : P \longrightarrow H, \qquad \pi_H : H \longrightarrow P$$

such that

$$p \,\mathbf{I}\, h \iff \pi_H(h) \,\mathbf{I}\, \pi_P(p).$$

Thus the duality reverses the two incidence sorts and preserves incidence.

Definition 3.17.2 (Self-Polar Fuzzy Set)**.** Let $(\mathcal{I}, \pi_P, \pi_H)$ be as in Definition 3.17.1. A *Self-Polar Fuzzy Set* is a pair

$$\mu_P : P \to [0,1], \qquad \mu_H : H \to [0,1]$$

satisfying

$$\mu_H(\pi_P(p)) = \mu_P(p) \qquad (p \in P).$$

Equivalently, $\mu_P(\pi_H(h)) = \mu_H(h)$ for every $h \in H$.

Definition 3.17.3 (Self-Polar Uncertain Set)**.** Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. A *Self-Polar Uncertain Set* of type M on $(\mathcal{I}, \pi_P, \pi_H)$ is a pair

$$\mu_P : P \to \mathrm{Dom}(M), \qquad \mu_H : H \to \mathrm{Dom}(M)$$

such that

$$\mu_H \circ \pi_P = \mu_P \qquad \text{and hence} \qquad \mu_P \circ \pi_H = \mu_H.$$

Isomorphisms must preserve both incidence and the two typed degree maps.

Proposition 3.17.4 (Classification by One Incidence Sort)**.** *Restriction to the point sort gives a canonical bijection between Self-Polar Uncertain Sets and arbitrary maps $P \to \mathrm{Dom}(M)$. The inverse sends μ_P to $\mu_H = \mu_P \circ \pi_H$.*

Example 3.17.5 (A Self-Dual Triangle)**.** Let $P = \{p_1, p_2, p_3\}$, $H = \{h_1, h_2, h_3\}$, and declare $p_i \,\mathbf{I}\, h_j$ exactly when $i \neq j$. The maps $\pi_P(p_i) = h_i$ and $\pi_H(h_i) = p_i$ form a polarity. Choosing degrees $a_i \in \mathrm{Dom}(M)$ and setting

$$\mu_P(p_i) = a_i, \qquad \mu_H(h_i) = a_i$$

produces a Self-Polar Uncertain Set.

Remark 3.17.6. If Y is only a set with an involution $\tau : Y \to Y$, the condition $\mu \circ \tau = \mu$ defines an *involution-invariant* uncertain set. Without point/hyperplane sorts and incidence reversal, it is not called a projective self-polar set here.

3.18 Antipolar Fuzzy Set

Antipolar order describes arrangements of local electric dipoles or displacements that alternate in direction, producing cancellation of macroscopic polarization while preserving ordered local polar structure (cf. [207–209]). An Antipolar Fuzzy Set assigns degrees to paired opposite poles so their weighted directional contributions cancel, producing zero net fuzzy polarization for every element state.

Definition 3.18.1 (Antipolar Fuzzy Set)**.** Let $X \neq \emptyset$, and let $P \neq \emptyset$ be a finite set of poles equipped with a fixed-point-free involution

$$\tau : P \longrightarrow P$$

such that

$$\tau^2 = \mathrm{id}_P$$

and

$$\tau(p) \neq p$$

for every $p \in P$.

Let

$$v : P \longrightarrow \mathbb{S}^{d-1}$$

satisfy

$$v(\tau(p)) = -v(p)$$

for every $p \in P$.

For a mapping

$$\mu_A : X \times P \longrightarrow [0, 1],$$

define the net fuzzy polarization by

$$Q_A(x) = \sum_{p \in P} \mu_A(x, p) v(p).$$

The fuzzy set A is called an *Antipolar Fuzzy Set* if

$$Q_A(x) = 0$$

for every

$$x \in X.$$

Definition 3.18.2 (Antipolar Uncertain Set)**.** Let $X \neq \emptyset$, and let $P \neq \emptyset$ be a finite set of poles equipped with a fixed-point-free involution

$$\tau : P \longrightarrow P$$

satisfying

$$\tau^2 = \mathrm{id}_P, \qquad \tau(p) \neq p$$

for every $p \in P$.

Let

$$v : P \longrightarrow \mathbb{S}^{d-1}$$

satisfy

$$v(\tau(p)) = -v(p)$$

for every $p \in P$.

Let $M_s = (\mathrm{Dom}(M), s_M)$ be a scored uncertain model.

For an uncertainty-degree function

$$\mu_A : X \times P \longrightarrow \mathrm{Dom}(M),$$

define the net uncertain polarization by

$$Q_A(x) = \sum_{p \in P} s_M\big(\mu_A(x, p)\big) v(p).$$

The Uncertain Set A is called an *Antipolar Uncertain Set* if

$$Q_A(x) = 0$$

for every $x \in X$.

Example 3.18.3 (Antipolar Uncertain Set for Local Dipole Balance in a Material)**.** Consider several microscopic regions of an antipolar material,

$$X = \{x_1, x_2, x_3\},$$

where each region may contain local electric dipoles oriented along opposite spatial directions.

Let

$$P = \{p_{+x}, p_{-x}, p_{+y}, p_{-y}\}$$

be the set of possible dipole-orientation poles. Define the involution

$$\tau(p_{+x}) = p_{-x}, \qquad \tau(p_{-x}) = p_{+x},$$

and

$$\tau(p_{+y}) = p_{-y}, \qquad \tau(p_{-y}) = p_{+y}.$$

Take

$$d = 2$$

and define

$$v(p_{+x}) = (1, 0), \qquad v(p_{-x}) = (-1, 0),$$
$$v(p_{+y}) = (0, 1), \qquad v(p_{-y}) = (0, -1).$$

Hence,

$$v(\tau(p)) = -v(p)$$

for every $p \in P$.

Let the uncertainty model be the fuzzy model

$$\mathrm{Dom}(M) = [0, 1]$$

with

$$s_M(t) = t.$$

Suppose that the uncertainty degrees for region x_1 are

$$\mu_A(x_1, p_{+x}) = 0.80, \qquad \mu_A(x_1, p_{-x}) = 0.80,$$

and

$$\mu_A(x_1, p_{+y}) = 0.50, \qquad \mu_A(x_1, p_{-y}) = 0.50.$$

Then

$$\begin{aligned} Q_A(x_1) &= 0.80(1, 0) + 0.80(-1, 0) \\ &\quad + 0.50(0, 1) + 0.50(0, -1) \\ &= (0, 0). \end{aligned}$$

Similarly, let

$$\mu_A(x_2, p_{+x}) = \mu_A(x_2, p_{-x}) = 0.60,$$
$$\mu_A(x_2, p_{+y}) = \mu_A(x_2, p_{-y}) = 0.75,$$

and

$$\mu_A(x_3, p_{+x}) = \mu_A(x_3, p_{-x}) = 0.90,$$
$$\mu_A(x_3, p_{+y}) = \mu_A(x_3, p_{-y}) = 0.30.$$

For each $x_i \in X$, the opposite directional contributions cancel pairwise, and therefore

$$Q_A(x_i) = (0, 0).$$

Hence,

$$Q_A(x) = 0 \qquad \text{for every } x \in X,$$

so A is an Antipolar Uncertain Set.

This example represents a material in which local electric dipoles may be strongly present but occur with balanced uncertainty degrees in opposite directions, resulting in zero net macroscopic polarization.

Remark 3.18.4. A sufficient condition for antipolarity is

$$s_M\big(\mu_A(x,p)\big) = s_M\big(\mu_A(x,\tau(p))\big)$$

for every $x \in X$ and $p \in P$. Indeed, each opposite pair then satisfies

$$s_M\big(\mu_A(x,p)\big)v(p) + s_M\big(\mu_A(x,\tau(p))\big)v(\tau(p)) = 0.$$

Remark 3.18.5. If

$$\mathrm{Dom}(M) = [0,1]$$

and

$$s_M(t) = t,$$

then

$$Q_A(x) = \sum_{p \in P} \mu_A(x,p)v(p),$$

and the Antipolar Uncertain Set reduces to an Antipolar Fuzzy Set.

3.19 Cross-Polar Fuzzy Set

Cross-polarization refers to the component of an electromagnetic field orthogonal to a prescribed reference polarization and is commonly used to quantify undesired polarization components in antenna radiation and scattering (cf. [210–212]). Motivated by this idea, a Cross-Polar Fuzzy Set separates a Hilbert-space-valued representation into co-polar and orthogonal cross-polar components and associates fuzzy membership degrees with their respective magnitudes.

Definition 3.19.1 (Cross-Polar Fuzzy Set)**.** Let $X \neq \emptyset$, and let $\mathcal{H}$ be a real or complex Hilbert space. Fix a unit vector

$$u \in \mathcal{H}, \qquad \|u\| = 1.$$

Let

$$P_u : \mathcal{H} \longrightarrow \mathrm{span}\{u\}$$

denote the orthogonal projection onto $\mathrm{span}\{u\}$.

Let

$$z_A : X \longrightarrow \overline{B}_{\mathcal{H}}$$

be a mapping into the closed unit ball

$$\overline{B}_{\mathcal{H}} = \{z \in \mathcal{H} : \|z\| \leq 1\}.$$

For every $x \in X$, define the *co-polar component* by

$$z_A^{\mathrm{co}}(x) = P_u z_A(x)$$

and the *cross-polar component* by

$$z_A^{\mathrm{cross}}(x) = z_A(x) - P_u z_A(x).$$

The corresponding fuzzy membership degrees are defined by

$$\mu_A^{\mathrm{co}}(x) = \|z_A^{\mathrm{co}}(x)\|$$

and

$$\mu_A^{\mathrm{cross}}(x) = \|z_A^{\mathrm{cross}}(x)\|.$$

The triple

$$A = \left(z_A, \mu_A^{\mathrm{co}}, \mu_A^{\mathrm{cross}}\right)$$

is called a *Cross-Polar Fuzzy Set*.

If the inner product on $\mathcal{H}$ is taken to be linear in the first argument, then

$$P_u z = \langle z, u \rangle u.$$

Proposition 3.19.2 (Orthogonal Polar Decomposition)**.** *Under the assumptions of Definition 3.19.1, the mappings*

$$\mu_A^{\mathrm{co}}, \mu_A^{\mathrm{cross}} : X \longrightarrow [0, 1]$$

are well-defined fuzzy membership functions.

Moreover, for every $x \in X$,

$$z_A^{\mathrm{co}}(x) \perp z_A^{\mathrm{cross}}(x),$$

and

$$\left(\mu_A^{\mathrm{co}}(x)\right)^2 + \left(\mu_A^{\mathrm{cross}}(x)\right)^2 = \|z_A(x)\|^2 \leq 1.$$

Proof. Fix $x \in X$. Since P_u is the orthogonal projection onto $\mathrm{span}\{u\}$,

$$z_A^{\mathrm{co}}(x) = P_u z_A(x) \in \mathrm{span}\{u\},$$

whereas

$$z_A^{\mathrm{cross}}(x) = z_A(x) - P_u z_A(x) \in \mathrm{span}\{u\}^{\perp}.$$

Therefore,

$$z_A^{\mathrm{co}}(x) \perp z_A^{\mathrm{cross}}(x).$$

By the orthogonal decomposition theorem,

$$z_A(x) = z_A^{\mathrm{co}}(x) + z_A^{\mathrm{cross}}(x),$$

and hence, by the Pythagorean identity,

$$\|z_A(x)\|^2 = \|z_A^{\mathrm{co}}(x)\|^2 + \|z_A^{\mathrm{cross}}(x)\|^2.$$

Since

$$z_A(x) \in \overline{B}_{\mathcal{H}},$$

we have

$$\|z_A(x)\| \leq 1.$$

Consequently,

$$0 \leq \|z_A^{\mathrm{co}}(x)\| \leq 1$$

and

$$0 \leq \|z_A^{\mathrm{cross}}(x)\| \leq 1.$$

Thus,

$$\mu_A^{\mathrm{co}}(x), \mu_A^{\mathrm{cross}}(x) \in [0, 1].$$

Finally,

$$\big(\mu_A^{\mathrm{co}}(x)\big)^2 + \big(\mu_A^{\mathrm{cross}}(x)\big)^2 = \|z_A(x)\|^2 \leq 1.$$

Hence the Cross-Polar Fuzzy Set is well-defined. □

Definition 3.19.3 (Cross-Polar Uncertain Set)**.** Let $X \neq \emptyset$, and let $\mathcal{H}$ be a real or complex Hilbert space. Fix a unit vector

$$u \in \mathcal{H}, \qquad \|u\| = 1,$$

and let

$$P_u : \mathcal{H} \longrightarrow \mathrm{span}\{u\}$$

be the orthogonal projection onto $\mathrm{span}\{u\}$.

Let

$$M_e = (\mathrm{Dom}(M), \Phi_M)$$

be an encoded uncertain model.

Let

$$z_A : X \longrightarrow \overline{B}_{\mathcal{H}}, \qquad \overline{B}_{\mathcal{H}} = \{z \in \mathcal{H} : \|z\| \leq 1\}.$$

For each $x \in X$, define the *co-polar component* and *cross-polar component* by

$$z_A^{\mathrm{co}}(x) = P_u z_A(x)$$

and

$$z_A^{\mathrm{cross}}(x) = z_A(x) - P_u z_A(x),$$

respectively.

Define the corresponding uncertainty degrees by

$$\mu_A^{\mathrm{co}}(x) = \Phi_M\big(\|z_A^{\mathrm{co}}(x)\|\big)$$

and

$$\mu_A^{\mathrm{cross}}(x) = \Phi_M\big(\|z_A^{\mathrm{cross}}(x)\|\big).$$

The structure

$$A = (z_A, \mu_A^{\mathrm{co}}, \mu_A^{\mathrm{cross}})$$

is called a *Cross-Polar Uncertain Set* of type M.

Example 3.19.4 (Cross-Polar Uncertain Set for Antenna Polarization Evaluation)**.** Consider a wireless communication system in which several antennas are evaluated according to their co-polarized and cross-polarized signal components. Let

$$X = \{a_1, a_2, a_3\}$$

be a set of three antennas.

Take the real Hilbert space

$$\mathcal{H} = \mathbb{R}^2$$

with the usual inner product, and choose the desired polarization direction

$$u = (1, 0), \qquad \|u\| = 1.$$

The orthogonal projection onto span$\{u\}$ is therefore

$$P_u(x_1, x_2) = (x_1, 0).$$

Let the fuzzy uncertainty model be

$$\mathrm{Dom}(M) = [0, 1]$$

with the identity encoding map

$$\Phi_M(t) = t.$$

Suppose that the normalized polarization signals are

$$z_A(a_1) = (0.80, 0.30),$$

$$z_A(a_2) = (0.60, 0.60),$$

and

$$z_A(a_3) = (0.90, 0.10).$$

Indeed,

$$\|z_A(a_i)\| \leq 1, \qquad i = 1, 2, 3,$$

so all three signals belong to $\overline{B}_{\mathcal{H}}$.

For a_1,

$$z_A^{\mathrm{co}}(a_1) = P_u(0.80, 0.30) = (0.80, 0),$$

whereas

$$z_A^{\mathrm{cross}}(a_1) = (0.80, 0.30) - (0.80, 0) = (0, 0.30).$$

Hence,

$$\mu_A^{\mathrm{co}}(a_1) = 0.80, \qquad \mu_A^{\mathrm{cross}}(a_1) = 0.30.$$

Similarly,

$$\mu_A^{\mathrm{co}}(a_2) = 0.60, \qquad \mu_A^{\mathrm{cross}}(a_2) = 0.60,$$

and

$$\mu_A^{\mathrm{co}}(a_3) = 0.90, \qquad \mu_A^{\mathrm{cross}}(a_3) = 0.10.$$

Thus, the corresponding uncertainty evaluations are

Antenna	μ_A^{co}	μ_A^{cross}
a_1	0.80	0.30
a_2	0.60	0.60
a_3	0.90	0.10

In this example, antenna a_3 has a strong component aligned with the desired polarization direction and a relatively small orthogonal component, whereas antenna a_2 exhibits a larger cross-polar component.

Hence,

$$A = (z_A, \mu_A^{\mathrm{co}}, \mu_A^{\mathrm{cross}})$$

is a Cross-Polar Uncertain Set representing aligned and orthogonal polarization components within a unified uncertainty framework.

Proposition 3.19.5 (Encoded Orthogonal Polar Decomposition). *Under the assumptions of Definition 3.19.3, the mappings*

$$\mu_A^{\mathrm{co}}, \mu_A^{\mathrm{cross}} : X \longrightarrow \mathrm{Dom}(M)$$

are well-defined.

Moreover, for every $x \in X$,

$$z_A^{\mathrm{co}}(x) \perp z_A^{\mathrm{cross}}(x),$$

and

$$\|z_A^{\mathrm{co}}(x)\|^2 + \|z_A^{\mathrm{cross}}(x)\|^2 = \|z_A(x)\|^2 \leq 1.$$

Furthermore, the class of Cross-Polar Uncertain Sets of type M is nonempty.

Proof. Fix $x \in X$. By the defining property of the orthogonal projection P_u,

$$P_u z_A(x) \in \mathrm{span}\{u\},$$

and

$$z_A(x) - P_u z_A(x) \in \mathrm{span}\{u\}^{\perp}.$$

Therefore,

$$z_A^{\mathrm{co}}(x) \perp z_A^{\mathrm{cross}}(x).$$

By the Pythagorean identity,

$$\|z_A(x)\|^2 = \|z_A^{\mathrm{co}}(x)\|^2 + \|z_A^{\mathrm{cross}}(x)\|^2.$$

Since

$$z_A(x) \in \overline{B}_{\mathcal{H}},$$

we have

$$\|z_A(x)\| \leq 1.$$

Consequently,

$$\|z_A^{\mathrm{co}}(x)\|, \|z_A^{\mathrm{cross}}(x)\| \in [0, 1].$$

Because

$$\Phi_M : [0, 1] \longrightarrow \mathrm{Dom}(M),$$

it follows that

$$\mu_A^{\mathrm{co}}(x) = \Phi_M(\|z_A^{\mathrm{co}}(x)\|) \in \mathrm{Dom}(M)$$

and

$$\mu_A^{\mathrm{cross}}(x) = \Phi_M(\|z_A^{\mathrm{cross}}(x)\|) \in \mathrm{Dom}(M).$$

Hence both uncertainty-degree mappings are well-defined.

To establish nonemptiness, define

$$z_0(x) = 0 \qquad \text{for every } x \in X.$$

Since

$$0 \in \overline{B}_{\mathcal{H}},$$

this defines an admissible mapping

$$z_0 : X \longrightarrow \overline{B}_{\mathcal{H}}.$$

Moreover,

$$P_u z_0(x) = 0,$$

so

$$z_0^{\mathrm{co}}(x) = z_0^{\mathrm{cross}}(x) = 0.$$

Therefore,

$$\mu_0^{\mathrm{co}}(x) = \mu_0^{\mathrm{cross}}(x) = \Phi_M(0) \in \mathrm{Dom}(M).$$

Thus at least one Cross-Polar Uncertain Set of type M exists. □

Remark 3.19.6. If M is the fuzzy model with

$$\mathrm{Dom}(M) = [0, 1]$$

and

$$\Phi_M(t) = t,$$

then

$$\mu_A^{\mathrm{co}}(x) = \|z_A^{\mathrm{co}}(x)\|$$

and

$$\mu_A^{\mathrm{cross}}(x) = \|z_A^{\mathrm{cross}}(x)\|.$$

Hence the Cross-Polar Uncertain Set reduces to a Cross-Polar Fuzzy Set.

3.20 BEC-Polarized Fuzzy and Uncertain Profiles

Channel polarization combines communication channels into synthetic channels of different reliability [213, 214]. The simple product/probabilistic-sum formulas below are exact for independent binary erasure channels (BECs); for arbitrary fuzzy memberships they are only a t-norm/t-conorm transform and are not identified with channel polarization.

Definition 3.20.1 (BEC-Polarized Erasure Profile)**.** Let $\varepsilon_1, \varepsilon_2 : X \to [0, 1]$ be erasure-probability profiles of two independent BECs. One polarization step produces the bad and good synthetic-channel profiles

$$\varepsilon^{\mathrm{bad}}(x) = \varepsilon_1(x) + \varepsilon_2(x) - \varepsilon_1(x)\varepsilon_2(x)$$

and

$$\varepsilon^{\mathrm{good}}(x) = \varepsilon_1(x)\varepsilon_2(x).$$

For identical inputs this is the usual BEC specialization of Arıkan's $2{\times}2$ polarization kernel. Recursion on blocks of length 2^n is defined by repeated application of this step.

Definition 3.20.2 (BEC-Polarized Uncertain Profile)**.** Let $M_{se} = (\mathrm{Dom}(M), s_M, \Phi_M)$ be a coherent scored–encoded model. Let $\eta_1, \eta_2 : X \to \mathrm{Dom}(M)$ be uncertainty profiles whose scores

$$a_x = s_M(\eta_1(x)), \qquad b_x = s_M(\eta_2(x))$$

are explicitly interpreted as independent BEC erasure probabilities. Define

$$\mu_A^{\mathrm{bad}}(x) = \Phi_M(a_x + b_x - a_x b_x), \qquad \mu_A^{\mathrm{good}}(x) = \Phi_M(a_x b_x).$$

Coherence guarantees that rescoring the outputs recovers the two synthetic erasure probabilities exactly.

Proposition 3.20.3 (BEC Polarization Identities)**.** *For $a, b \in [0,1]$,*

$$0 \le ab \le \min\{a,b\} \le \max\{a,b\} \le a + b - ab \le 1$$

and

$$ab + (a + b - ab) = a + b.$$

Consequently, the good synthetic BEC has no larger erasure probability and the bad synthetic BEC has no smaller erasure probability than either input.

Remark 3.20.4. If a and b are generic fuzzy membership degrees with no channel semantics, the same algebraic pair may be called the *product–probabilistic-sum transform*. The channel interpretation requires the BEC and independence assumptions in the definitions above.

3.21 Circuit-Polar Fuzzy and Uncertain Sets

A signed incidence difference is a graph gradient. A circuit model additionally requires a constitutive law and Kirchhoff conservation.

Definition 3.21.1 (Circuit-Polar Fuzzy Set)**.** Let $G = (V, E)$ be a finite directed graph with oriented incidence matrix $B = (b_{we})$, and let $g : E \to (0, \infty)$ be an edge-conductance map. For every $x \in X$, let

$$v_A(x, \cdot) : V \to [0,1]$$

be a fuzzy vertex-potential and let $j_A(x, \cdot) : V \to \mathbb{R}$ be a prescribed injection satisfying $\sum_{w \in V} j_A(x, w) = 0$. Define

$$\Delta_A(x, e) = \sum_{w \in V} b_{we} v_A(x, w), \qquad i_A(x, e) = g(e)\Delta_A(x, e).$$

The tuple (G, g, j_A, v_A) is a *Circuit-Polar Fuzzy Set* when Kirchhoff's current law holds:

$$\sum_{e \in E} b_{we} i_A(x, e) = j_A(x, w) \qquad (x \in X,\ w \in V).$$

The normalized edge contrast

$$\rho_A(x, e) = \frac{1 + \Delta_A(x, e)}{2}$$

belongs to $[0,1]$, because vertex potentials lie in $[0,1]$.

Definition 3.21.2 (Circuit-Polar Uncertain Set)**.** Let $M_{se} = (\mathrm{Dom}(M), s_M, \Phi_M)$ be a coherent scored–encoded model. Retain (G, B, g, j_A) from Definition 3.21.1 and let

$$\eta_A : X \times V \longrightarrow \mathrm{Dom}(M).$$

Set

$$v_A(x, w) = s_M(\eta_A(x, w)), \quad \Delta_A(x, e) = \sum_{w \in V} b_{we} v_A(x, w), \quad i_A(x, e) = g(e)\Delta_A(x, e).$$

The tuple is a *Circuit-Polar Uncertain Set* if

$$B i_A(x, \cdot) = j_A(x, \cdot)$$

for every x. Its encoded edge-contrast degree is

$$\rho_A^M(x,e) = \Phi_M\left(\frac{1+\Delta_A(x,e)}{2}\right).$$

Thus the typed uncertain construction preserves both Ohm's law and Kirchhoff conservation instead of using the incidence difference alone.

Proposition 3.21.3 (Circuit Energy Identity)**.** *Every Circuit-Polar Fuzzy or scored Uncertain Set satisfies*

$$\sum_{w\in V} v_A(x,w)j_A(x,w) = \sum_{e\in E} g(e)\Delta_A(x,e)^2 \geq 0.$$

Proof. In matrix form, $\Delta = B^{\mathsf{T}}v$, $i = G\Delta$, and $j = Bi$. Hence $v^{\mathsf{T}}j = v^{\mathsf{T}}BGB^{\mathsf{T}}v = \Delta^{\mathsf{T}}G\Delta$. $\square$

Example 3.21.4 (One Resistive Edge)**.** Let $u \to v$ be one edge of conductance 1, with $v_A(x,u) = 0.20$ and $v_A(x,v) = 0.80$. Then

$$\Delta_A(x,e) = 0.60, \quad i_A(x,e) = 0.60, \quad j_A(x,u) = -0.60, \quad j_A(x,v) = 0.60,$$

and $\rho_A(x,e) = 0.80$. Both node-balance equations hold.

3.22 Hysteretic-Polar Fuzzy Set

A Hysteretic-Polar Uncertain Set represents uncertainty whose intensity and polarity may depend on the history of preceding input states, thereby capturing path-dependent polar uncertainty.

Definition 3.22.1 (Hysteretic-Polar Fuzzy Set)**.** Let $X \neq \emptyset$ be a universe and let $U \neq \emptyset$ be a set of input states.

Define the set of finite nonempty histories by

$$\mathcal{H}(U) = \bigcup_{n\geq 1} U^n.$$

For

$$h = (u_1,\ldots,u_n) \in \mathcal{H}(U),$$

write

$$\operatorname{end}(h) = u_n.$$

A *Hysteretic-Polar Fuzzy Set* on X is a pair of mappings

$$m_A : X \times \mathcal{H}(U) \longrightarrow [0,1]$$

and

$$p_A : X \times \mathcal{H}(U) \longrightarrow [-1,1],$$

where $m_A(x,h)$ represents the fuzzy membership intensity and $p_A(x,h)$ represents the signed polarization state determined by the history h.

The associated positive and negative polar membership functions are defined by

$$\mu_A^+(x,h) = \frac{m_A(x,h)(1+p_A(x,h))}{2},$$

and

$$\mu_A^-(x,h) = \frac{m_A(x,h)(1 - p_A(x,h))}{2}.$$

The structure is called *strictly hysteretic* if there exist $x \in X$ and histories $h_1, h_2 \in \mathcal{H}(U)$ such that

$$\mathrm{end}(h_1) = \mathrm{end}(h_2),$$

but

$$\big(m_A(x,h_1), p_A(x,h_1)\big) \neq \big(m_A(x,h_2), p_A(x,h_2)\big).$$

Proposition 3.22.2 (Hysteretic Polar-Mass Decomposition)**.** *Let*

$$m_A : X \times \mathcal{H}(U) \longrightarrow [0,1]$$

and

$$p_A : X \times \mathcal{H}(U) \longrightarrow [-1,1].$$

Then the associated functions

$$\mu_A^+, \mu_A^- : X \times \mathcal{H}(U) \longrightarrow [0,1]$$

are well-defined. Moreover,

$$\mu_A^+(x,h) + \mu_A^-(x,h) = m_A(x,h)$$

for every

$$(x,h) \in X \times \mathcal{H}(U).$$

Proof. Fix

$$(x,h) \in X \times \mathcal{H}(U).$$

Since

$$p_A(x,h) \in [-1,1],$$

we have

$$0 \leq \frac{1 + p_A(x,h)}{2} \leq 1$$

and

$$0 \leq \frac{1 - p_A(x,h)}{2} \leq 1.$$

Because

$$m_A(x,h) \in [0,1],$$

it follows that

$$0 \leq \mu_A^+(x,h) \leq 1$$

and

$$0 \leq \mu_A^-(x,h) \leq 1.$$

Thus both polar membership functions are well-defined.

Furthermore,

$$\begin{aligned} \mu_A^+(x,h) + \mu_A^-(x,h) &= \frac{m_A(x,h)(1 + p_A(x,h))}{2} \\ &\quad + \frac{m_A(x,h)(1 - p_A(x,h))}{2} \\ &= m_A(x,h). \end{aligned}$$

Hence the total fuzzy membership intensity is preserved under the positive-negative polar decomposition. ∎

Definition 3.22.3 (Hysteretic-Polar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe and let $U \neq \emptyset$ be a set of input states. Define the set of finite nonempty histories by

$$\mathcal{H}(U) = \bigcup_{n \geq 1} U^n.$$

For

$$h = (u_1, \ldots, u_n) \in \mathcal{H}(U),$$

write

$$\mathrm{end}(h) = u_n.$$

Let

$$M_{se} = (\mathrm{Dom}(M), s_M, \Phi_M)$$

be a coherent scored–encoded uncertain model.

A *Hysteretic-Polar Uncertain Set* of type M is determined by the mappings

$$\eta_A : X \times \mathcal{H}(U) \longrightarrow \mathrm{Dom}(M)$$

and

$$p_A : X \times \mathcal{H}(U) \longrightarrow [-1, 1].$$

Define the scalar uncertainty intensity by

$$m_A(x, h) = s_M\big(\eta_A(x, h)\big).$$

The positive and negative scalar polar components are

$$q_A^+(x, h) = \frac{m_A(x, h)(1 + p_A(x, h))}{2}$$

and

$$q_A^-(x, h) = \frac{m_A(x, h)(1 - p_A(x, h))}{2}.$$

The corresponding polar uncertainty degrees are

$$\mu_A^+(x, h) = \Phi_M\big(q_A^+(x, h)\big)$$

and

$$\mu_A^-(x, h) = \Phi_M\big(q_A^-(x, h)\big).$$

The structure is called *strictly hysteretic* if there exist $x \in X$ and $h_1, h_2 \in \mathcal{H}(U)$ such that

$$\mathrm{end}(h_1) = \mathrm{end}(h_2),$$

but

$$\big(\eta_A(x, h_1), p_A(x, h_1)\big) \neq \big(\eta_A(x, h_2), p_A(x, h_2)\big).$$

Example 3.22.4 (Hysteretic-Polar Uncertain Set for Magnetic Hysteresis)**.** Consider a ferromagnetic material whose magnetic response depends not only on the current external magnetic field but also on its previous field history. Let

$$X = \{x\}$$

consist of one magnetic sample, and let the normalized magnetic-field states be

$$U = \{-1, 0, 1\},$$

where -1, 0, and 1 represent negative, zero, and positive applied magnetic fields, respectively.

Take the fuzzy uncertainty model

$$\mathrm{Dom}(M) = [0, 1],$$

with

$$s_M(t) = t \qquad \text{and} \qquad \Phi_M(t) = t.$$

Consider the two field histories

$$h_1 = (-1, 0) \qquad \text{and} \qquad h_2 = (1, 0).$$

Both histories end at the same current field:

$$\mathrm{end}(h_1) = \mathrm{end}(h_2) = 0.$$

Suppose that the uncertainty intensity associated with the magnetic state is

$$\eta_A(x, h_1) = 0.80, \qquad \eta_A(x, h_2) = 0.80.$$

Hence,

$$m_A(x, h_1) = m_A(x, h_2) = 0.80.$$

Because of magnetic hysteresis, assume that the polarization depends on the preceding field direction:

$$p_A(x, h_1) = -0.50, \qquad p_A(x, h_2) = 0.50.$$

For the first history,

$$q_A^+(x, h_1) = \frac{0.80(1 - 0.50)}{2} = 0.20,$$

and

$$q_A^-(x, h_1) = \frac{0.80(1 + 0.50)}{2} = 0.60.$$

Therefore,

$$\mu_A^+(x, h_1) = 0.20, \qquad \mu_A^-(x, h_1) = 0.60.$$

For the second history,

$$q_A^+(x, h_2) = \frac{0.80(1 + 0.50)}{2} = 0.60,$$

and

$$q_A^-(x, h_2) = \frac{0.80(1 - 0.50)}{2} = 0.20.$$

Hence,

$$\mu_A^+(x, h_2) = 0.60, \qquad \mu_A^-(x, h_2) = 0.20.$$

Although

$$\mathrm{end}(h_1) = \mathrm{end}(h_2),$$

we have

$$\big(\eta_A(x,h_1), p_A(x,h_1)\big) = (0.80, -0.50) \neq (0.80, 0.50) = \big(\eta_A(x,h_2), p_A(x,h_2)\big).$$

Thus, this is a strictly Hysteretic-Polar Uncertain Set. The example captures the physical fact that a ferromagnetic material can exhibit different residual magnetic states at the same present magnetic field depending on the field history that preceded it.

Proposition 3.22.5 (Encoded Hysteretic Polar-Mass Decomposition)**.** *Under the assumptions of Definition 3.22.3, the mappings*

$$q_A^+, q_A^- : X \times \mathcal{H}(U) \longrightarrow [0,1]$$

and

$$\mu_A^+, \mu_A^- : X \times \mathcal{H}(U) \longrightarrow \mathrm{Dom}(M)$$

are well-defined.

Moreover, for every

$$(x,h) \in X \times \mathcal{H}(U),$$

we have

$$q_A^+(x,h) + q_A^-(x,h) = m_A(x,h) = s_M\big(\eta_A(x,h)\big).$$

The class of Hysteretic-Polar Uncertain Sets is nonempty.

Proof. Fix

$$(x,h) \in X \times \mathcal{H}(U).$$

Since

$$\eta_A(x,h) \in \mathrm{Dom}(M)$$

and

$$s_M : \mathrm{Dom}(M) \to [0,1],$$

we have

$$m_A(x,h) \in [0,1].$$

Also,

$$p_A(x,h) \in [-1,1],$$

so

$$0 \leq \frac{1 + p_A(x,h)}{2} \leq 1$$

and

$$0 \leq \frac{1 - p_A(x,h)}{2} \leq 1.$$

Therefore,

$$q_A^+(x,h), q_A^-(x,h) \in [0,1].$$

Since

$$\Phi_M : [0,1] \to \mathrm{Dom}(M),$$

it follows that
$$\mu_A^+(x,h) = \Phi_M\big(q_A^+(x,h)\big) \in \mathrm{Dom}(M)$$
and
$$\mu_A^-(x,h) = \Phi_M\big(q_A^-(x,h)\big) \in \mathrm{Dom}(M).$$
Hence all induced mappings are well-defined.

Furthermore,
$$\begin{aligned} q_A^+(x,h) + q_A^-(x,h) &= \frac{m_A(x,h)(1+p_A(x,h))}{2} \\ &\quad + \frac{m_A(x,h)(1-p_A(x,h))}{2} \\ &= m_A(x,h) \\ &= s_M\big(\eta_A(x,h)\big). \end{aligned}$$
Thus the total scalar uncertainty intensity is preserved by the positive-negative polar decomposition.

Finally, choose any
$$d_0 \in \mathrm{Dom}(M)$$
and define
$$\eta_0(x,h) = d_0, \qquad p_0(x,h) = 0$$
for every $(x,h) \in X \times \mathcal{H}(U)$. These mappings satisfy all requirements of the definition, so at least one Hysteretic-Polar Uncertain Set exists. □

Remark 3.22.6. If
$$\mathrm{Dom}(M) = [0,1], \qquad s_M(t) = t, \qquad \Phi_M(t) = t,$$
then
$$m_A(x,h) = \eta_A(x,h),$$
and
$$\mu_A^+(x,h) = \frac{m_A(x,h)(1+p_A(x,h))}{2},$$
$$\mu_A^-(x,h) = \frac{m_A(x,h)(1-p_A(x,h))}{2}.$$
Hence the Hysteretic-Polar Uncertain Set reduces to a Hysteretic-Polar Fuzzy Set.

3.23 Stokes-Polar Fuzzy and Uncertain Sets

Stokes parameters provide a classical representation of polarization through an intensity component and three directional polarization components [215]. Motivated by this representation, we introduce Stokes-Polar Fuzzy and Uncertain Sets by combining a normalized Stokes state with fuzzy or typed uncertainty information.

Definition 3.23.1 (Stokes-Polar Fuzzy Set)**.** Define the normalized Stokes domain by
$$\mathbb{S} = \left\{(s_0,s_1,s_2,s_3) \in [0,1] \times \mathbb{R}^3 : s_1^2 + s_2^2 + s_3^2 \le s_0^2\right\}.$$

A *Stokes-Polar Fuzzy Set* on a nonempty universe X is a mapping
$$S_A : X \longrightarrow \mathbb{S}.$$

For every $x \in X$, write

$$S_A(x) = \bigl(s_0(x), s_1(x), s_2(x), s_3(x)\bigr),$$

where $s_0(x)$ represents the normalized total fuzzy intensity, while

$$\mathbf{s}_A(x) = \bigl(s_1(x), s_2(x), s_3(x)\bigr)$$

is the corresponding Stokes-type polarization vector.

The *degree of fuzzy polarization* is defined by

$$D_A(x) = \begin{cases} \dfrac{\sqrt{s_1(x)^2 + s_2(x)^2 + s_3(x)^2}}{s_0(x)}, & s_0(x) > 0, \\ 0, & s_0(x) = 0. \end{cases}$$

Proposition 3.23.2 (Normalized Stokes Bound)**.** *The normalized Stokes domain $\mathbb{S}$ is nonempty. Hence, for every nonempty universe X, the class*

$$\mathrm{SPF}(X) = \mathbb{S}^X$$

of all Stokes-Polar Fuzzy Sets on X is nonempty.

Moreover, for every Stokes-Polar Fuzzy Set S_A and every $x \in X$,

$$0 \le D_A(x) \le 1.$$

Proof. Since

$$(0,0,0,0) \in \mathbb{S},$$

the set $\mathbb{S}$ is nonempty. Consequently, the constant mapping

$$S_0 : X \longrightarrow \mathbb{S}, \qquad S_0(x) = (0,0,0,0),$$

shows that

$$\mathrm{SPF}(X) \neq \emptyset.$$

Now let

$$S_A(x) = \bigl(s_0(x), s_1(x), s_2(x), s_3(x)\bigr) \in \mathbb{S}.$$

By definition,

$$s_1(x)^2 + s_2(x)^2 + s_3(x)^2 \le s_0(x)^2.$$

If $s_0(x) > 0$, then

$$0 \le \frac{\sqrt{s_1(x)^2 + s_2(x)^2 + s_3(x)^2}}{s_0(x)} \le 1,$$

and therefore

$$0 \le D_A(x) \le 1.$$

If $s_0(x) = 0$, the defining inequality implies

$$s_1(x) = s_2(x) = s_3(x) = 0,$$

and hence, by definition,

$$D_A(x) = 0.$$

Thus,

$$D_A(x) \in [0,1]$$

for every $x \in X$. □

Definition 3.23.3 (Stokes-Polar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let

$$M_0 = (\mathrm{Dom}(M))$$

be a carrier uncertain model.

Suppose that M is additionally equipped with a declared total *Stokes encoding map*

$$\Psi_M : [0,1]^2 \longrightarrow \mathrm{Dom}(M).$$

Let

$$S_A : X \longrightarrow \mathbb{S}$$

be a normalized Stokes-state mapping, where

$$S_A(x) = \big(s_0(x), s_1(x), s_2(x), s_3(x)\big).$$

Define the degree of polarization by

$$D_A(x) = \begin{cases} \dfrac{\sqrt{s_1(x)^2 + s_2(x)^2 + s_3(x)^2}}{s_0(x)}, & s_0(x) > 0, \\ 0, & s_0(x) = 0. \end{cases}$$

The associated uncertainty-degree mapping is defined by

$$\mu_A : X \longrightarrow \mathrm{Dom}(M), \qquad \mu_A(x) = \Psi_M\big(s_0(x), D_A(x)\big).$$

The pair

$$A = (S_A, \mu_A)$$

is called a *Stokes-Polar Uncertain Set of type M*.

Thus, $S_A(x)$ retains the normalized intensity and directional polarization information, while $\mu_A(x)$ encodes the intensity and degree of polarization as an admissible uncertainty value in $\mathrm{Dom}(M)$.

Example 3.23.4 (Stokes-Polar Uncertain Set for Optical Surface Inspection)**.** Consider an industrial optical inspection system evaluating reflected light from three manufactured surfaces,

$$X = \{x_1, x_2, x_3\}.$$

Suppose that a polarimeter provides a normalized Stokes state for each surface. Take

$$\mathrm{Dom}(M) = [0,1]^2$$

and define the Stokes encoding map by

$$\Psi_M(a, b) = (a, b).$$

Let

$$S_A(x_1) = (0.80, 0.48, 0, 0),$$

$$S_A(x_2) = (0.90, 0.54, 0.36, 0),$$

and

$$S_A(x_3) = (0.60, 0, 0.30, 0.30).$$

For x_1,

$$0.48^2 \le 0.80^2,$$

so

$$S_A(x_1) \in \mathbb{S}.$$

Moreover,

$$D_A(x_1) = \frac{\sqrt{0.48^2}}{0.80} = 0.60,$$

and therefore

$$\mu_A(x_1) = \Psi_M(0.80, 0.60) = (0.80, 0.60).$$

For x_2,

$$0.54^2 + 0.36^2 = 0.4212 \le 0.90^2,$$

and hence

$$S_A(x_2) \in \mathbb{S}.$$

Its degree of polarization is

$$D_A(x_2) = \frac{\sqrt{0.54^2 + 0.36^2}}{0.90} \approx 0.721.$$

Thus,

$$\mu_A(x_2) = \Psi_M(0.90, 0.721) = (0.90, 0.721).$$

Similarly, for x_3,

$$0.30^2 + 0.30^2 = 0.18 \le 0.60^2,$$

and

$$D_A(x_3) = \frac{\sqrt{0.30^2 + 0.30^2}}{0.60} \approx 0.707.$$

Hence,

$$\mu_A(x_3) = \Psi_M(0.60, 0.707) = (0.60, 0.707).$$

The resulting encoded uncertainty values are summarized by

Surface	s_0	D_A
x_1	0.80	0.600
x_2	0.90	0.721
x_3	0.60	0.707

Here, s_0 represents the normalized reflected-light intensity, while D_A measures the normalized degree of polarization. The Stokes-Polar Uncertain Set therefore combines intensity and polarization information within a single typed uncertainty representation.

Proposition 3.23.5 (Encoded Stokes Bound)**.** *Under the assumptions of Definition 3.23.3, for every normalized Stokes-state mapping*

$$S_A : X \longrightarrow \mathbb{S},$$

the degree of polarization satisfies

$$D_A(x) \in [0,1] \qquad \text{for every } x \in X.$$

Consequently,

$$\big(s_0(x), D_A(x)\big) \in [0,1]^2,$$

and hence

$$\mu_A(x) = \Psi_M\big(s_0(x), D_A(x)\big) \in \mathrm{Dom}(M).$$

Therefore,

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)$$

is well-defined. Moreover, the class of Stokes-Polar Uncertain Sets of type M is nonempty.

Proof. Let

$$S_A(x) = \big(s_0(x), s_1(x), s_2(x), s_3(x)\big) \in \mathbb{S}.$$

By the defining condition of $\mathbb{S}$,

$$s_1(x)^2 + s_2(x)^2 + s_3(x)^2 \le s_0(x)^2.$$

If $s_0(x) > 0$, then

$$0 \le \frac{\sqrt{s_1(x)^2 + s_2(x)^2 + s_3(x)^2}}{s_0(x)} \le 1,$$

so

$$D_A(x) \in [0,1].$$

If $s_0(x) = 0$, then

$$s_1(x) = s_2(x) = s_3(x) = 0,$$

and therefore

$$D_A(x) = 0.$$

Thus,

$$D_A(x) \in [0,1]$$

in every case.

Since $S_A(x) \in \mathbb{S}$, we also have

$$s_0(x) \in [0,1].$$

Consequently,

$$\big(s_0(x), D_A(x)\big) \in [0,1]^2.$$

Because the Stokes encoding map is total,

$$\Psi_M : [0,1]^2 \longrightarrow \mathrm{Dom}(M),$$

it follows that

$$\mu_A(x) = \Psi_M\big(s_0(x), D_A(x)\big) \in \mathrm{Dom}(M).$$

Hence

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)$$

is well-defined.

Finally, the constant mapping

$$S_0 : X \longrightarrow \mathbb{S}, \qquad S_0(x) = (0, 0, 0, 0),$$

is admissible. Its associated polarization degree is

$$D_0(x) = 0,$$

and therefore

$$\mu_0(x) = \Psi_M(0, 0) \in \mathrm{Dom}(M).$$

Thus at least one Stokes-Polar Uncertain Set of type M exists. □

Remark 3.23.6. If

$$\mathrm{Dom}(M) = [0, 1]$$

and the Stokes encoding map is chosen as

$$\Psi_M(a, b) = b,$$

then

$$\mu_A(x) = D_A(x).$$

Hence the construction reduces to a fuzzy membership representation obtained directly from the normalized degree of polarization.

3.24 Free-Polar Uncertain Set

A Free-Polar Uncertain Set assigns uncertainty degrees to elements with respect to an arbitrary nonempty collection of poles, allowing finite, countably infinite, or uncountable polar representations within a unified uncertainty framework.

Definition 3.24.1 (Free-Polar Uncertain Set)**.** Let $X \neq \emptyset$ be a universe, and let $P \neq \emptyset$ be an arbitrary set of poles, with no restriction on its cardinality.

Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. A *Free-Polar Uncertain Set* of type M on X, indexed by P, is a mapping

$$\mu_A : X \times P \longrightarrow \mathrm{Dom}(M).$$

Equivalently, μ_A may be regarded as a mapping

$$\widehat{\mu}_A : X \longrightarrow \mathrm{Dom}(M)^P,$$

where

$$\widehat{\mu}_A(x)(p) = \mu_A(x, p)$$

for every $x \in X$ and $p \in P$.

Thus, each element $x \in X$ is assigned a P-indexed family

$$\big(\mu_A(x, p)\big)_{p \in P}$$

of admissible uncertainty degrees.

Example 3.24.2 (Free-Polar Uncertain Set for Online Product Evaluation)**.** Consider an online marketplace in which several laptop computers are evaluated according to a potentially unrestricted collection of user-defined criteria. Let

$$X = \{\ell_1, \ell_2, \ell_3\}$$

represent three laptop models.

Let P be the set of all evaluation poles submitted by users. For example,

$$P \supseteq \{p_{\text{price}}, p_{\text{battery}}, p_{\text{performance}}, p_{\text{portability}}, p_{\text{durability}}\}.$$

The cardinality of P is not prescribed in advance, and new evaluation poles may be added whenever additional criteria are introduced.

For this example, consider the interval-valued uncertain model $M_{\mathbb{I}}$ with degree-domain

$$\mathrm{Dom}(M_{\mathbb{I}}) = \mathbb{I} = \{[a, b] : 0 \le a \le b \le 1\}.$$

Define

$$\mu_A : X \times P \longrightarrow \mathbb{I}.$$

Suppose, for example, that the uncertainty evaluations of ℓ_1 are

$$\mu_A(\ell_1, p_{\text{price}}) = [0.65, 0.80],$$

$$\mu_A(\ell_1, p_{\text{battery}}) = [0.80, 0.92],$$

$$\mu_A(\ell_1, p_{\text{performance}}) = [0.85, 0.95],$$

and

$$\mu_A(\ell_1, p_{\text{portability}}) = [0.70, 0.88].$$

For the second laptop, suppose that

$$\mu_A(\ell_2, p_{\text{price}}) = [0.82, 0.93],$$

$$\mu_A(\ell_2, p_{\text{battery}}) = [0.68, 0.82],$$

and

$$\mu_A(\ell_2, p_{\text{performance}}) = [0.72, 0.86].$$

For instance,

$$\mu_A(\ell_1, p_{\text{battery}}) = [0.80, 0.92]$$

indicates that the uncertain degree associated with satisfactory battery performance of laptop ℓ_1 is represented by the interval from 0.80 to 0.92.

Thus,

$$\mu_A : X \times P \longrightarrow \mathrm{Dom}(M_{\mathbb{I}})$$

defines a Free-Polar Uncertain Set of type $M_{\mathbb{I}}$.

This example illustrates that the collection of relevant poles need not be fixed or finite, while each polar evaluation remains an admissible uncertainty degree under the selected uncertain model.

Remark 3.24.3 (Finite-Polar Special Case)**.** Suppose that

$$|P| = m < \infty.$$

After choosing a bijection

$$\varphi : P \longrightarrow \{1, \dots, m\},$$

every Free-Polar Uncertain Set

$$\mu_A : X \times P \longrightarrow \mathrm{Dom}(M)$$

can be identified with the mapping

$$\widetilde{\mu}_A : X \longrightarrow \mathrm{Dom}(M)^m$$

defined by

$$\widetilde{\mu}_A(x) = \big(\mu_A(x, \varphi^{-1}(1)), \dots, \mu_A(x, \varphi^{-1}(m))\big).$$

Hence, for finite P, the Free-Polar Uncertain Set is equivalent, up to an indexing of the poles, to an m-Polar Uncertain Set.

Remark 3.24.4 (Further Special Cases)**.** If

$$|P| = 1,$$

then a Free-Polar Uncertain Set is naturally identified with an ordinary Uncertain Set of type M.

If

$$\mathrm{Dom}(M) = [0, 1],$$

then

$$\mu_A : X \times P \longrightarrow [0, 1]$$

gives a Free-Polar Fuzzy Set.

If

$$\mathrm{Dom}(M) = \mathbb{I} = \{[a, b] : 0 \le a \le b \le 1\},$$

then the construction yields the interval-valued Free-Polar Uncertain Set used in Example 3.24.2.

The pole set P may also be countably infinite or uncountable.

Chapter 4

Polarity with Structure

This chapter introduces and discusses polarity-based concepts equipped with additional mathematical structures.

4.1 Classical Structure

A *Hyperstructure* is organized around the powerset and serves as a vehicle for modeling relations among elements of a set [216–218]. Owing to its flexibility, the hyperstructure framework has been investigated across several areas, including mathematics and chemistry [219, 220]. A *Superhyperstructure* advances this idea by utilizing the n-th powerset to encode multi-layered hierarchical interactions, thereby enabling deeper abstraction and greater structural complexity [221]. Because of this wide scope, superhyperstructures have likewise been explored in mathematics, chemistry, and related disciplines [222, 223]. Prominent instances include constructs such as the *SuperHyperGraph* [23, 224–227]. We next record the n-th powerset, which underpins these structures.

Definition 4.1.1 (Iterated Nonempty Power Set)**.** [221, 228] Let $H \neq \emptyset$ be a set. Define

$$\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}.$$

Recursively, let

$$\mathcal{P}^{*0}(H) = H$$

and

$$\mathcal{P}^{*n}(H) = \mathcal{P}^*\left(\mathcal{P}^{*(n-1)}(H)\right), \qquad n \geq 1.$$

Thus, $\mathcal{P}^{*n}(H)$ is the n-fold iterated nonempty power set of H.

Definition 4.1.2 (Classical Structure)**.** Let $H \neq \emptyset$ be a set. A *Classical Structure* on H is a system

$$\mathfrak{C} = (H, \Omega, \mathcal{A}),$$

where

$$\Omega = \{\omega_i : i \in I\}$$

is a family of finitary operations such that, for each $i \in I$,

$$\omega_i : H^{m_i} \longrightarrow H, \qquad m_i \geq 1,$$

and $\mathcal{A}$ is a specified family of axioms imposed on these operations.

Definition 4.1.3 (HyperStructure)**.** Let $H \neq \emptyset$ be a set. A *HyperStructure* on H is a system

$$\mathfrak{H} = (H, \Omega_H, \mathcal{A}_H),$$

where

$$\Omega_H = \{\omega_i^H : i \in I\}$$

is a family of hyperoperations such that, for each $i \in I$,

$$\omega_i^H : H^{m_i} \longrightarrow \mathcal{P}^*(H), \qquad m_i \geq 1,$$

and $\mathcal{A}_H$ is a specified family of hyperaxioms.

Thus, unlike a classical operation, a hyperoperation may assign a nonempty set of possible outputs to each input tuple.

Definition 4.1.4 (SuperHyperStructure)**.** Let $H \neq \emptyset$ be a set. A *SuperHyperStructure* based on H is a system

$$\mathfrak{S} = (H, \Omega_{SH}, \mathcal{A}_{SH}),$$

where

$$\Omega_{SH} = \{\omega_i^{SH} : i \in I\}$$

is a family of SuperHyperOperations such that, for each $i \in I$,

$$\omega_i^{SH} : (\mathcal{P}^{*r_i}(H))^{m_i} \longrightarrow \mathcal{P}^{*n_i}(H),$$

where

$$m_i \geq 1, \qquad r_i \geq 1, \qquad n_i \geq 1,$$

and $\mathcal{A}_{SH}$ is a specified family of SuperHyperAxioms.

Hence, a SuperHyperStructure permits operations whose inputs and outputs belong to iterated levels of the nonempty power-set hierarchy.

Remark 4.1.5. The three notions can be distinguished by the codomains of their operations:

$$H^m \longrightarrow H \qquad \text{(Classical Structure)},$$

$$H^m \longrightarrow \mathcal{P}^*(H) \qquad \text{(HyperStructure)},$$

and

$$(\mathcal{P}^{*r}(H))^m \longrightarrow \mathcal{P}^{*n}(H) \qquad \text{(SuperHyperStructure)}.$$

Accordingly, Classical Structures are single-valued, HyperStructures are set-valued, and SuperHyperStructures permit set-valued operations across iterated levels of the power-set hierarchy.

4.2 Multipolar Structure

A Multipolar Structure assigns multiple pole-indexed outputs to each input tuple through operations, together with axioms governing their coordinated behavior. Throughout this section, for sets P and Y, we use the notation

$$Y^P = \{f : P \longrightarrow Y\}$$

for the set of all P-indexed families of elements of Y.

We also define the iterated nonempty power-set hierarchy by

$$\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\},$$

$$\mathcal{P}^{*0}(H) = H,$$

and, recursively,

$$\mathcal{P}^{*(n+1)}(H) = \mathcal{P}^*\big(\mathcal{P}^{*n}(H)\big), \qquad n \geq 0.$$

Definition 4.2.1 (Multipolar Structure)**.** Let $H \neq \emptyset$ be a set, and let

$$P = \{p_1, \dots, p_k\}, \qquad k \geq 1,$$

be a finite nonempty set of poles.

A *Multipolar Structure* on H with respect to P is a system

$$\mathfrak{M} = (H, P, \Omega_M, \mathcal{A}_M),$$

where

$$\Omega_M = \{\omega_i^M : i \in I\}$$

is a family of multipolar operations such that, for every $i \in I$,

$$\omega_i^M : H^{m_i} \longrightarrow H^P, \qquad m_i \geq 1,$$

and $\mathcal{A}_M$ is a specified family of axioms imposed on these operations.

For every $i \in I$ and $p \in P$, define the p-th polar component by

$$\omega_{i,p} = \mathrm{ev}_p \circ \omega_i^M,$$

where

$$\mathrm{ev}_p : H^P \longrightarrow H, \qquad \mathrm{ev}_p(f) = f(p).$$

Thus,

$$\omega_{i,p} : H^{m_i} \longrightarrow H.$$

Equivalently, one may write

$$\omega_i^M = \big(\omega_{i,p}\big)_{p \in P},$$

meaning that

$$\omega_i^M(x_1, \dots, x_{m_i})(p) = \omega_{i,p}(x_1, \dots, x_{m_i})$$

for every

$$(x_1, \dots, x_{m_i}) \in H^{m_i}$$

and $p \in P$.

Hence, each input tuple is assigned one output in H for every pole $p \in P$. When $k \geq 2$, the structure is genuinely multipolar.

4.3 Multipolar HyperStructure

A Multipolar HyperStructure assigns, for each input tuple, multiple pole-indexed nonempty output sets, enabling simultaneous representation of several perspectives within a unified hyperstructural algebraic framework across distinct polar components.

Definition 4.3.1 (Multipolar HyperStructure)**.** Let $H \neq \emptyset$ be a set, and let

$$P = \{p_1, \dots, p_k\}, \qquad k \geq 1,$$

be a finite nonempty set of poles.

A *Multipolar HyperStructure* on H with respect to P is a system

$$\mathfrak{H}_M = (H, P, \Omega_{MH}, \mathcal{A}_{MH}),$$

where

$$\Omega_{MH} = \{\omega_i^{MH} : i \in I\}$$

is a family of multipolar hyperoperations such that, for every $i \in I$,

$$\omega_i^{MH} : H^{m_i} \longrightarrow \big(\mathcal{P}^*(H)\big)^P, \qquad m_i \geq 1,$$

and $\mathcal{A}_{MH}$ is a specified family of hyperaxioms.

For every $i \in I$ and $p \in P$, the corresponding polar hyperoperation is

$$\omega_{i,p}^H = \mathrm{ev}_p \circ \omega_i^{MH},$$

and hence

$$\omega_{i,p}^H : H^{m_i} \longrightarrow \mathcal{P}^*(H).$$

Equivalently,

$$\omega_i^{MH} = \big(\omega_{i,p}^H\big)_{p \in P},$$

where

$$\omega_i^{MH}(x_1, \dots, x_{m_i})(p) = \omega_{i,p}^H(x_1, \dots, x_{m_i})$$

for every $p \in P$.

Therefore, for each input tuple and each pole p, the corresponding polar component assigns a nonempty subset of H. When $k \geq 2$, several pole-dependent hyperoutputs are represented simultaneously.

4.4 Multipolar SuperHyperStructure

A Multipolar SuperHyperStructure extends multipolar and hyperstructural operations to iterated levels of the nonempty power-set hierarchy, thereby allowing several pole-dependent higher-order outputs to be represented simultaneously.

Definition 4.4.1 (Multipolar SuperHyperStructure)**.** Let $H \neq \emptyset$ be a set, and let

$$P = \{p_1, \dots, p_k\}, \qquad k \geq 1,$$

be a finite nonempty set of poles.

A *Multipolar SuperHyperStructure* based on H with respect to P is a system

$$\mathfrak{S}_M = (H, P, \Omega_{MSH}, \mathcal{A}_{MSH}),$$

where

$$\Omega_{MSH} = \{\omega_i^{MSH} : i \in I\}$$

is a family of multipolar SuperHyperOperations such that, for every $i \in I$,

$$\omega_i^{MSH} : (\mathcal{P}^{*r_i}(H))^{m_i} \longrightarrow (\mathcal{P}^{*n_i}(H))^P,$$

with

$$m_i \geq 1, \qquad r_i \geq 1, \qquad n_i \geq 1,$$

and $\mathcal{A}_{MSH}$ is a specified family of SuperHyperAxioms.

For every $i \in I$ and $p \in P$, define

$$\omega_{i,p}^{SH} = \mathrm{ev}_p \circ \omega_i^{MSH}.$$

Then

$$\omega_{i,p}^{SH} : (\mathcal{P}^{*r_i}(H))^{m_i} \longrightarrow \mathcal{P}^{*n_i}(H).$$

Equivalently,

$$\omega_i^{MSH} = \left(\omega_{i,p}^{SH}\right)_{p \in P},$$

meaning that

$$\omega_i^{MSH}(A_1, \ldots, A_{m_i})(p) = \omega_{i,p}^{SH}(A_1, \ldots, A_{m_i})$$

for every

$$(A_1, \ldots, A_{m_i}) \in (\mathcal{P}^{*r_i}(H))^{m_i}$$

and $p \in P$.

Thus, each pole determines one SuperHyperOutput belonging to the n_i-th iterated nonempty powerset level.

Remark 4.4.2. The three constructions can be distinguished by the forms of their operations:

$$H^m \longrightarrow H^P \qquad \text{(Multipolar Structure)},$$

$$H^m \longrightarrow \left(\mathcal{P}^*(H)\right)^P \qquad \text{(Multipolar HyperStructure)},$$

and

$$(\mathcal{P}^{*r}(H))^m \longrightarrow (\mathcal{P}^{*n}(H))^P \qquad \text{(Multipolar SuperHyperStructure)}.$$

If

$$P = \{p\},$$

then there are canonical bijections

$$H^P \cong H,$$

$$\left(\mathcal{P}^*(H)\right)^P \cong \mathcal{P}^*(H),$$

and

$$\left(\mathcal{P}^{*n}(H)\right)^P \cong \mathcal{P}^{*n}(H).$$

Consequently, the corresponding constructions reduce, up to these canonical identifications, to the ordinary Classical Structure, HyperStructure, and SuperHyperStructure, respectively.

For

$$|P| \geq 2,$$

the constructions represent genuinely multipolar structures.

Example 4.4.3 (A Multipolar SuperHyperStructure)**.** Let

$$H = \{a, b, c, d\}, \qquad P = \{p_1, p_2\},$$

and let

$$\mathcal{P}^*(H) = \mathcal{P}(H) \setminus \{\emptyset\}.$$

Consider the binary multipolar SuperHyperOperation

$$\omega^{MSH} : (\mathcal{P}^*(H))^2 \longrightarrow (\mathcal{P}^*(H))^P .$$

Thus,

$$m = 2, \qquad r = 1, \qquad n = 1.$$

For every

$$A, B \in \mathcal{P}^*(H),$$

define the two polar components by

$$\omega^{SH}_{p_1}(A, B) = A \cup B$$

and

$$\omega^{SH}_{p_2}(A, B) = (A \cap B) \cup \{d\}.$$

Equivalently, define

$$\omega^{MSH}(A, B) : P \longrightarrow \mathcal{P}^*(H)$$

by

$$\omega^{MSH}(A, B)(p_1) = A \cup B$$

and

$$\omega^{MSH}(A, B)(p_2) = (A \cap B) \cup \{d\}.$$

Since $A, B \neq \emptyset$,

$$A \cup B \neq \emptyset.$$

Moreover,

$$d \in (A \cap B) \cup \{d\},$$

and therefore

$$(A \cap B) \cup \{d\} \neq \emptyset.$$

Hence

$$\omega^{SH}_{p_j}(A, B) \in \mathcal{P}^*(H), \qquad j = 1, 2,$$

for all

$$A, B \in \mathcal{P}^*(H).$$

Thus,

$$\omega^{MSH}(A, B) \in (\mathcal{P}^*(H))^P ,$$

and the operation is well-defined.

For example, take

$$A = \{a, b\}, \qquad B = \{b, c\}.$$

Then

$$\omega^{MSH}(A, B)(p_1) = A \cup B = \{a, b, c\},$$

whereas

$$\omega^{MSH}(A,B)(p_2) = (A \cap B) \cup \{d\} = \{b,d\}.$$

Since

$$P = \{p_1, p_2\},$$

we may identify this P-indexed function with the ordered pair

$$\omega^{MSH}(A,B) \cong \big(\{a,b,c\}, \{b,d\}\big).$$

Thus, the same pair of SuperHyperInputs produces two distinct pole-dependent SuperHyperOutputs. This example illustrates the essential feature of a Multipolar SuperHyperStructure: a single SuperHyperOperation can simultaneously encode several higher-order outputs indexed by different poles.

4.5 Free-Polar Structure

A Free-Polar Structure assigns outputs across an arbitrary collection of poles, allowing finite or infinite polar representations within algebraic systems.

Definition 4.5.1 (Free-Polar Structure)**.** Let $H \neq \emptyset$ be a set, and let $P \neq \emptyset$ be an arbitrary set of poles, with no restriction on its cardinality.

A *Free-Polar Structure* on H with respect to P is a system

$$\mathfrak{F} = (H, P, \Omega_F, \mathcal{A}_F),$$

where

$$\Omega_F = \{\omega_i^F : i \in I\}$$

is a family of free-polar operations such that, for each $i \in I$,

$$\omega_i^F : H^{m_i} \longrightarrow H^P, \qquad m_i \geq 1,$$

and $\mathcal{A}_F$ is a specified family of axioms satisfied by these operations.

Equivalently,

$$\omega_i^F = \big(\omega_{i,p}\big)_{p \in P},$$

where

$$\omega_{i,p} : H^{m_i} \longrightarrow H$$

for every $p \in P$.

Thus, each input tuple is assigned a P-indexed family of outputs, and the set of poles may be finite, countably infinite, or uncountable.

Remark 4.5.2. A Free-Polar Structure does not prescribe the cardinality of the pole set P. In particular,

$$|P| = 1$$

gives an ordinary single-output structure, while

$$2 \leq |P| < \infty$$

gives a finite Multipolar Structure. Infinite pole sets are also admissible.

Chapter 5

Some Multipolar Concepts

This chapter introduces several representative multipolar concepts, including graph-based and other related mathematical structures.

5.1 m-polar Uncertain Graph

An m-Polar Fuzzy Graph assigns multiple fuzzy membership degrees to vertices and edges, representing several perspectives or criteria simultaneously within a unified graph-based mathematical framework [229–232]. Bipolar Fuzzy Graphs [233–235] can be regarded as special cases of m-Polar Fuzzy Graphs. In addition, Bipolar Neutrosophic Graphs [160, 236, 237] are known as closely related polarity-based graph structures.

Definition 5.1.1 (m-Polar Fuzzy Graph)**.** [238, 239] Let

$$G^* = (V, E)$$

be a simple graph, and let $m \in \mathbb{N}$. An *m-polar fuzzy graph* on G^* is a triple

$$\mathcal{G} = (G^*, \mu, \nu),$$

where

$$\mu : V \longrightarrow [0,1]^m$$

is an m-polar fuzzy membership function on the vertices, and

$$\nu : E \longrightarrow [0,1]^m$$

is an m-polar fuzzy membership function on the edges such that, for every $\{u, v\} \in E$ and every $i \in \{1, \ldots, m\}$,

$$\pi_i\big(\nu(\{u, v\})\big) \leq \min\left\{\pi_i\big(\mu(u)\big), \pi_i\big(\mu(v)\big)\right\},$$

where

$$\pi_i : [0,1]^m \longrightarrow [0,1]$$

denotes the projection onto the i-th coordinate.

Equivalently, writing

$$\mu(v) = \big(\mu_1(v), \ldots, \mu_m(v)\big) \quad \text{and} \quad \nu(\{u, v\}) = \big(\nu_1(u, v), \ldots, \nu_m(u, v)\big),$$

the defining condition is

$$\nu_i(u, v) \leq \min\{\mu_i(u), \mu_i(v)\}, \qquad i = 1, \ldots, m.$$

An m-Polar Uncertain Graph extends this idea by replacing scalar fuzzy membership degrees with admissible degrees from a prescribed uncertainty model.

Definition 5.1.2 (m-Polar Uncertain Graph)**.** Let

$$G^* = (V, E)$$

be a simple graph, let $m \geq 1$, and let

$$M_{\text{lat}} = \big(\text{Dom}(M), \leq_M, \vee_M, \wedge_M, 0_M, 1_M\big)$$

be a bounded-lattice uncertain model.

An *m-Polar Uncertain Graph* of type M on G^* is a triple

$$\mathcal{G}_M = (G^*, \mu_A, \nu_A),$$

where

$$\mu_A : V \longrightarrow \text{Dom}(M)^m$$

and

$$\nu_A : E \longrightarrow \text{Dom}(M)^m$$

satisfy, for every $\{u, v\} \in E$ and $i = 1, \ldots, m$,

$$\nu_A^{(i)}(u, v) \leq_M \mu_A^{(i)}(u) \wedge_M \mu_A^{(i)}(v).$$

Equivalently, under the componentwise order and meet on $\text{Dom}(M)^m$,

$$\nu_A(\{u, v\}) \leq_m \mu_A(u) \wedge_m \mu_A(v).$$

Remark 5.1.3. If

$$\text{Dom}(M) = [0, 1], \qquad \wedge_M = \min,$$

then

$$\mu_A : V \to [0, 1]^m, \qquad \nu_A : E \to [0, 1]^m,$$

and the condition becomes

$$\nu_A^{(i)}(u, v) \leq \min\{\mu_A^{(i)}(u), \mu_A^{(i)}(v)\},$$

so the construction reduces to the corresponding m-Polar Fuzzy Graph.

5.2 m-polar Uncertain ideals

Various types of ideals incorporating uncertainty have been studied, including Fuzzy Ideals and Neutrosophic Ideals [240, 241]. An m-Polar Fuzzy Ideal assigns multiple fuzzy membership degrees to elements of an algebraic structure while satisfying ideal-type closure conditions across every polar component simultaneously [182, 242, 243].

Definition 5.2.1 (m-Polar Fuzzy Ideal)**.** Let E be a ternary semiring and let $m \in \mathbb{N}$. An *m-polar fuzzy set* on E is a mapping

$$\mu : E \longrightarrow [0, 1]^m, \qquad \mu(x) = \big(\mu_1(x), \ldots, \mu_m(x)\big).$$

The set μ is called an *m-polar fuzzy ideal* of E if, for all $x, y, z \in E$ and every $i \in \{1, \ldots, m\}$,

$$\mu_i(x + y) \geq \min\{\mu_i(x), \mu_i(y)\},$$

and

$$\mu_i(xyz) \geq \max\{\mu_i(x), \mu_i(y), \mu_i(z)\}.$$

An m-Polar Uncertain Ideal assigns multiple uncertainty degrees to elements of an algebraic structure while satisfying ideal-type closure conditions at every uncertainty pole.

Definition 5.2.2 (m-Polar Uncertain Ideal)**.** Let E be a ternary semiring, let $m \geq 1$, and let

$$M_{\mathrm{lat}} = \big(\mathrm{Dom}(M), \leq_M, \vee_M, \wedge_M, 0_M, 1_M\big)$$

be a bounded-lattice uncertain model.

An *m-Polar Uncertain Set* on E is a mapping

$$\mu_A : E \longrightarrow \mathrm{Dom}(M)^m,$$

written as

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \ldots, \mu_A^{(m)}(x)\big).$$

The mapping μ_A is called an *m-Polar Uncertain Ideal* of E if, for all $x, y, z \in E$ and every $i \in \{1, \ldots, m\}$,

$$\mu_A^{(i)}(x + y) \geq_M \mu_A^{(i)}(x) \wedge_M \mu_A^{(i)}(y),$$

and

$$\mu_A^{(i)}(xyz) \geq_M \mu_A^{(i)}(x) \vee_M \mu_A^{(i)}(y) \vee_M \mu_A^{(i)}(z).$$

Equivalently, under the componentwise lattice operations on $\mathrm{Dom}(M)^m$,

$$\mu_A(x + y) \geq_m \mu_A(x) \wedge_m \mu_A(y),$$

and

$$\mu_A(xyz) \geq_m \mu_A(x) \vee_m \mu_A(y) \vee_m \mu_A(z).$$

Remark 5.2.3. When

$$m = 1,$$

the construction reduces to an Uncertain Ideal of type M.

If

$$\mathrm{Dom}(M) = [0, 1], \qquad \wedge_M = \min, \qquad \vee_M = \max,$$

then the defining conditions become

$$\mu_A^{(i)}(x + y) \geq \min\{\mu_A^{(i)}(x), \mu_A^{(i)}(y)\},$$

and

$$\mu_A^{(i)}(xyz) \geq \max\{\mu_A^{(i)}(x), \mu_A^{(i)}(y), \mu_A^{(i)}(z)\},$$

for every $i = 1, \ldots, m$, yielding the corresponding m-polar fuzzy ideal formulation.

5.3 m-polar Uncertain topology

In the field of topological spaces, related concepts such as Fuzzy Topology [244, 245] and Neutrosophic Topology [246–250] have also been introduced and studied. An m-Polar Uncertain Topology is a family of multipolar uncertain sets containing null and universal elements and closed under componentwise arbitrary unions and finite intersections (cf. [251, 252]).

Definition 5.3.1 (m-Polar Fuzzy Topology)**.** Let $X \neq \emptyset$ be a set and let $m \in \mathbb{N}$. Denote by

$$\mathcal{F}_m(X) = \{\mu : X \longrightarrow [0,1]^m\}$$

the family of all m-polar fuzzy sets on X.

For $\mu, \nu \in \mathcal{F}_m(X)$, define their union and intersection componentwise by

$$(\mu \vee \nu)_i(x) = \max\{\mu_i(x), \nu_i(x)\},$$

and

$$(\mu \wedge \nu)_i(x) = \min\{\mu_i(x), \nu_i(x)\},$$

for every $x \in X$ and $i = 1, \ldots, m$.

A family

$$\tau \subseteq \mathcal{F}_m(X)$$

is called an *m-polar fuzzy topology* on X if:

1. $\mathbf{0}, \mathbf{1} \in \tau$, where
$$\mathbf{0}(x) = (0, \ldots, 0), \qquad \mathbf{1}(x) = (1, \ldots, 1);$$
2. for every family $\{\mu_\lambda\}_{\lambda \in \Lambda} \subseteq \tau$,
$$\bigvee_{\lambda \in \Lambda} \mu_\lambda \in \tau;$$
3. for every finite family $\mu_1, \ldots, \mu_n \in \tau$,
$$\bigwedge_{j=1}^{n} \mu_j \in \tau.$$

The pair

$$(X, \tau)$$

is called an *m-polar fuzzy topological space*, and the members of τ are called *m-polar fuzzy open sets*.

An m-Polar Uncertain Topology is a family of multipolar uncertain sets containing the null and universal uncertain sets and closed under arbitrary unions and finite intersections.

Definition 5.3.2 (m-Polar Uncertain Topology)**.** Let $X \neq \emptyset$, let $m \geq 1$, and let

$$M_{\text{clat}} = \big(\mathrm{Dom}(M), \leq_M, \vee_M, \wedge_M, 0_M, 1_M\big)$$

be a complete-lattice uncertain model.

An *m-Polar Uncertain Set* on X is a mapping

$$\mu_A : X \longrightarrow \mathrm{Dom}(M)^m.$$

Let

$$\mathbf{0}_M = (0_M, \ldots, 0_M), \qquad \mathbf{1}_M = (1_M, \ldots, 1_M).$$

Define the null and universal m-polar uncertain sets by

$$\mu_{\emptyset_M}(x) = \mathbf{0}_M, \qquad \mu_{X_M}(x) = \mathbf{1}_M$$

for every $x \in X$.

A family

$$\mathcal{T} \subseteq \left(\mathrm{Dom}(M)^m\right)^X$$

is called an *m-Polar Uncertain Topology* on X if:

1. $$\mu_{\emptyset_M}, \mu_{X_M} \in \mathcal{T};$$

2. for every family
$$\{\mu_{A_\lambda}\}_{\lambda \in \Lambda} \subseteq \mathcal{T},$$
the pointwise join
$$\bigvee_{\lambda \in \Lambda} \mu_{A_\lambda} \in \mathcal{T},$$
where
$$\left(\bigvee_{\lambda \in \Lambda} \mu_{A_\lambda}\right)(x) = \bigvee_{\lambda \in \Lambda} \mu_{A_\lambda}(x);$$

3. for every finite family
$$\mu_{A_1}, \ldots, \mu_{A_n} \in \mathcal{T},$$
the pointwise meet
$$\bigwedge_{j=1}^{n} \mu_{A_j} \in \mathcal{T},$$
where
$$\left(\bigwedge_{j=1}^{n} \mu_{A_j}\right)(x) = \bigwedge_{j=1}^{n} \mu_{A_j}(x).$$

Here, all joins and meets in $\mathrm{Dom}(M)^m$ are taken componentwise. The pair
$$(X, \mathcal{T})$$
is called an *m-Polar Uncertain Topological Space*.

Remark 5.3.3. When
$$m = 1,$$
the construction reduces to an ordinary Uncertain Topology.

If
$$\mathrm{Dom}(M) = [0, 1],$$
with the usual order,
$$\vee_M = \max, \qquad \wedge_M = \min,$$
then
$$\mu_A : X \to [0, 1]^m,$$
and the construction yields the corresponding m-polar fuzzy topological framework.

5.4 m-polar Uncertain Group

An m-Polar Uncertain Group assigns multiple uncertainty values to group elements while preserving group-operation and inversion compatibility independently across all uncertainty poles simultaneously within a framework (cf. [242, 253]).

Definition 5.4.1 (Bipolar Fuzzy Group)**.** Let

$$(G, \cdot)$$

be a group with identity element e. A *Bipolar Fuzzy Group* (or *Bipolar Fuzzy Subgroup*) of G is a pair

$$A = (\mu_A^+, \mu_A^-),$$

where

$$\mu_A^+ : G \longrightarrow [0,1], \qquad \mu_A^- : G \longrightarrow [-1,0],$$

such that, for all $x, y \in G$,

$$\mu_A^+(xy) \geq \min\{\mu_A^+(x), \mu_A^+(y)\},$$
$$\mu_A^-(xy) \leq \max\{\mu_A^-(x), \mu_A^-(y)\},$$

and

$$\mu_A^+(x^{-1}) \geq \mu_A^+(x), \qquad \mu_A^-(x^{-1}) \leq \mu_A^-(x).$$

Here, μ_A^+ and μ_A^- represent the positive and negative membership degrees of the elements of G, respectively.

An m-Polar Uncertain Group assigns multiple uncertainty values to group elements while preserving compatibility with the group operation and inversion at every uncertainty pole.

Definition 5.4.2 (m-Polar Uncertain Group)**.** Let

$$(G, \cdot)$$

be a group with identity element e, let $m \geq 1$, and let

$$M_{\text{lat}} = \big(\mathrm{Dom}(M), \leq_M, \vee_M, \wedge_M, 0_M, 1_M\big)$$

be a bounded-lattice uncertain model.

An *m-Polar Uncertain Group* of type M on G is a mapping

$$\mu_A : G \longrightarrow \mathrm{Dom}(M)^m$$

such that, writing

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \ldots, \mu_A^{(m)}(x)\big),$$

the following conditions hold for every

$$x, y \in G \qquad \text{and} \qquad i = 1, \ldots, m:$$

$$\mu_A^{(i)}(xy) \geq_M \mu_A^{(i)}(x) \wedge_M \mu_A^{(i)}(y),$$

and

$$\mu_A^{(i)}(x^{-1}) \geq_M \mu_A^{(i)}(x).$$

Equivalently, under the componentwise order and meet on $\mathrm{Dom}(M)^m$,

$$\mu_A(xy) \geq_m \mu_A(x) \wedge_m \mu_A(y)$$

and

$$\mu_A(x^{-1}) \geq_m \mu_A(x).$$

Proposition 5.4.3 (Invariance under Inversion)**.** *For every m-Polar Uncertain Group A and every $x \in G$,*

$$\mu_A(x^{-1}) = \mu_A(x).$$

Proof. By definition,

$$\mu_A(x^{-1}) \geq_m \mu_A(x).$$

Applying the same condition to x^{-1}, we obtain

$$\mu_A\big((x^{-1})^{-1}\big) \geq_m \mu_A(x^{-1}).$$

Since

$$(x^{-1})^{-1} = x,$$

it follows that

$$\mu_A(x) \geq_m \mu_A(x^{-1}).$$

Therefore,

$$\mu_A(x^{-1}) = \mu_A(x).$$

□

Remark 5.4.4. When

$$m = 1,$$

the construction reduces to an ordinary Uncertain Group of type M.

If

$$\mathrm{Dom}(M) = [0,1], \qquad \wedge_M = \min, \qquad \vee_M = \max,$$

then

$$\mu_A : G \to [0,1]^m,$$

and the defining conditions become

$$\mu_A^{(i)}(xy) \geq \min\{\mu_A^{(i)}(x), \mu_A^{(i)}(y)\},$$

and

$$\mu_A^{(i)}(x^{-1}) \geq \mu_A^{(i)}(x)$$

for every $i = 1, \ldots, m$, yielding the corresponding m-polar fuzzy-group formulation.

5.5 m-polar Uncertain Lattice

An m-Polar Uncertain Lattice assigns multiple uncertainty values to lattice elements while preserving join and meet compatibility across every uncertainty pole within one unified framework.

Definition 5.5.1 (Bipolar Fuzzy Lattice)**.** (cf. [254–256]) Let

$$(L, \vee, \wedge)$$

be a lattice.

A *Bipolar Fuzzy Lattice* on L is a bipolar fuzzy set

$$A = (\mu_A^+, \mu_A^-),$$

where

$$\mu_A^+ : L \longrightarrow [0,1], \qquad \mu_A^- : L \longrightarrow [-1,0],$$

such that, for all $x, y \in L$,

$$\mu_A^+(x \vee y) \geq \min\{\mu_A^+(x), \mu_A^+(y)\},$$
$$\mu_A^+(x \wedge y) \geq \min\{\mu_A^+(x), \mu_A^+(y)\},$$

and

$$\mu_A^-(x \vee y) \leq \max\{\mu_A^-(x), \mu_A^-(y)\},$$
$$\mu_A^-(x \wedge y) \leq \max\{\mu_A^-(x), \mu_A^-(y)\}.$$

Equivalently, A is a bipolar fuzzy subset of L that is closed under the lattice operations $\vee$ and $\wedge$ in the bipolar fuzzy sense.

An m-Polar Uncertain Lattice assigns multiple uncertainty values to lattice elements while preserving the lattice structure under each uncertainty pole.

Definition 5.5.2 (m-Polar Uncertain Lattice)**.** Let

$$(L, \vee, \wedge)$$

be a lattice, let $m \geq 1$, and let

$$M_{\text{lat}} = \big(\text{Dom}(M), \leq_M, \vee_M, \wedge_M, 0_M, 1_M\big)$$

be a bounded-lattice uncertain model.

An *m-Polar Uncertain Lattice* of type M on L is a mapping

$$\mu_A : L \longrightarrow \text{Dom}(M)^m$$

such that, writing

$$\mu_A(x) = \big(\mu_A^{(1)}(x), \ldots, \mu_A^{(m)}(x)\big),$$

the conditions

$$\mu_A^{(i)}(x \vee y) \geq_M \mu_A^{(i)}(x) \wedge_M \mu_A^{(i)}(y)$$

and

$$\mu_A^{(i)}(x \wedge y) \geq_M \mu_A^{(i)}(x) \wedge_M \mu_A^{(i)}(y)$$

hold for every

$$x, y \in L \qquad \text{and} \qquad i = 1, \ldots, m.$$

Equivalently, using the componentwise order and meet on $\text{Dom}(M)^m$,

$$\mu_A(x \vee y) \geq_m \mu_A(x) \wedge_m \mu_A(y)$$

and

$$\mu_A(x \wedge y) \geq_m \mu_A(x) \wedge_m \mu_A(y).$$

Remark 5.5.3. When

$$m = 1,$$

the construction reduces to an ordinary Uncertain Lattice.

If

$$\text{Dom}(M) = [0, 1], \qquad \wedge_M = \min, \qquad \vee_M = \max,$$

then

$$\mu_A : L \to [0, 1]^m,$$

and the defining conditions become

$$\mu_A^{(i)}(x \vee y) \geq \min\{\mu_A^{(i)}(x), \mu_A^{(i)}(y)\},$$

and

$$\mu_A^{(i)}(x \wedge y) \geq \min\{\mu_A^{(i)}(x), \mu_A^{(i)}(y)\},$$

for every $i = 1, \ldots, m$, yielding the corresponding m-polar fuzzy lattice formulation.

5.6 m-polar Uncertain automata

An m-Polar Uncertain Automaton assigns multipolar uncertainty values to state transitions, enabling multiple perspectives of transition behavior within an automata simultaneously.

Definition 5.6.1 (Bipolar Fuzzy Automaton)**.** [257–259] A *Bipolar Fuzzy Automaton* is a triple

$$\mathcal{A} = (Q, \Sigma, \delta),$$

where $Q \neq \emptyset$ is a finite set of states, $\Sigma \neq \emptyset$ is a finite input alphabet, and

$$\delta = (\delta^+, \delta^-)$$

is a bipolar fuzzy transition relation with

$$\delta^+ : Q \times \Sigma \times Q \longrightarrow [0, 1]$$

and

$$\delta^- : Q \times \Sigma \times Q \longrightarrow [-1, 0].$$

For

$$p, q \in Q \qquad \text{and} \qquad a \in \Sigma,$$

the values

$$\delta^+(p, a, q)$$

and

$$\delta^-(p, a, q)$$

represent, respectively, the positive and negative degrees of the transition from state p to state q under the input symbol a.

An m-Polar Uncertain Automaton assigns multipolar uncertainty values to state transitions, enabling multiple perspectives of transition behavior within an automata-theoretic framework.

Definition 5.6.2 (m-Polar Uncertain Automaton)**.** Let $Q \neq \emptyset$ be a finite set of states, let $\Sigma \neq \emptyset$ be a finite input alphabet, and let $m \geq 1$.

Let $M_0 = (\mathrm{Dom}(M))$ be a carrier model. An *m-Polar Uncertain Automaton* of type M is a triple

$$\mathcal{A}_M^{(m)} = (Q, \Sigma, \delta_M),$$

where

$$\delta_M : Q \times \Sigma \times Q \longrightarrow \mathrm{Dom}(M)^m$$

is an m-polar uncertain transition function.

For

$$p, q \in Q, \qquad a \in \Sigma,$$

write

$$\delta_M(p, a, q) = \big(\delta_M^{(1)}(p, a, q), \ldots, \delta_M^{(m)}(p, a, q)\big),$$

where

$$\delta_M^{(i)} : Q \times \Sigma \times Q \longrightarrow \mathrm{Dom}(M), \qquad i = 1, \ldots, m.$$

Thus, each possible transition

$$p \xrightarrow{a} q$$

is assigned m admissible uncertainty values, one for each pole.

Remark 5.6.3. When

$$m = 1,$$

the construction reduces to an ordinary Uncertain Automaton with transition function

$$\delta_M : Q \times \Sigma \times Q \longrightarrow \mathrm{Dom}(M).$$

If

$$\mathrm{Dom}(M) = [0, 1],$$

then

$$\delta_M : Q \times \Sigma \times Q \longrightarrow [0, 1]^m,$$

which gives the corresponding m-polar fuzzy transition representation.

5.7 m-Polar Uncertain Soft Sets

A Soft Set is a parameterized family of subsets of a universe and provides a flexible framework for representing uncertainty, vagueness, and approximate information [26, 260, 261]. Several related extensions have also been studied, including HyperSoft Sets [24, 262–266], SuperHyperSoft Sets [**?**, 267–269], TreeSoft Sets [**?**, **?**, 270, 271], ContraSoft Sets [272], and Soft Expert Sets [273, 274].

Within the typed uncertain-model framework, an Uncertain Soft Set assigns an Uncertain Set to each active parameter. Thus, parameterized uncertainty can be represented without requiring the underlying uncertainty degrees to belong to a fixed finite-dimensional Cartesian cube.

Definition 5.7.1 (Uncertain Soft Set)**.** [275] Let $U \neq \emptyset$ be a universe, let $E \neq \emptyset$ be a parameter set, and let

$$M_0 = (\mathrm{Dom}(M))$$

be a carrier uncertain model.

Define the class of all Uncertain Sets of type M on U by

$$\mathrm{Unc}_M(U) = \{\mu : U \longrightarrow \mathrm{Dom}(M)\} = \mathrm{Dom}(M)^U.$$

For a nonempty subset $A \subseteq E$, an *Uncertain Soft Set of type M* over U is a pair

$$(F, A),$$

where

$$F : A \longrightarrow \mathrm{Unc}_M(U).$$

Equivalently, every parameter $e \in A$ determines an Uncertain Set

$$F(e) : U \longrightarrow \mathrm{Dom}(M).$$

Thus, for every $e \in A$ and $u \in U$,

$$F(e)(u) \in \mathrm{Dom}(M).$$

Representative specializations of Uncertain Soft Sets include Fuzzy Soft Sets [276, 277], Intuitionistic Fuzzy Soft Sets [278, 279], Vague Soft Sets [280], Neutrosophic Soft Sets [281–283], Hesitant Fuzzy Soft Sets [284–286], Picture Fuzzy Soft Sets [287–289], Spherical Fuzzy Soft Sets [290–292], and Plithogenic Soft Sets [293], whenever the corresponding degree-domain is chosen as $\mathrm{Dom}(M)$.

An m-Polar Uncertain Soft Set extends this construction by assigning, under every active parameter, an m-tuple of admissible uncertainty degrees to each element of the universe.

Definition 5.7.2 (m-Polar Uncertain Soft Set)**.** Let $U \neq \emptyset$ be a universe, let $E \neq \emptyset$ be a parameter set, let

$$M_0 = (\mathrm{Dom}(M))$$

be a carrier uncertain model, and let $m \geq 1$.

By Lemma 2.3.5, the natural m-polar carrier is

$$\mathrm{Dom}(M)^m.$$

Define the class of all m-Polar Uncertain Sets of type M on U by

$$\mathrm{Unc}_M^{(m)}(U) = \{\mu : U \longrightarrow \mathrm{Dom}(M)^m\} = \left(\mathrm{Dom}(M)^m\right)^U.$$

For a nonempty subset $A \subseteq E$, an *m-Polar Uncertain Soft Set of type M* over U is a pair

$$(F, A),$$

where

$$F : A \longrightarrow \mathrm{Unc}_M^{(m)}(U).$$

Equivalently, for every $e \in A$,

$$F(e) : U \longrightarrow \mathrm{Dom}(M)^m.$$

Writing

$$F(e)(u) = \big(\mu_{e,1}(u), \ldots, \mu_{e,m}(u)\big),$$

we have

$$\mu_{e,i}(u) \in \mathrm{Dom}(M)$$

for every

$$e \in A, \qquad u \in U, \qquad i \in \{1, \ldots, m\}.$$

Thus, each active parameter determines an m-Polar Uncertain Set on U.

Remark 5.7.3 (Reduction to Ordinary Uncertain Soft Sets)**.** If

$$m = 1,$$

then there is the canonical identification

$$\mathrm{Dom}(M)^1 \cong \mathrm{Dom}(M).$$

Consequently,

$$\mathrm{Unc}_M^{(1)}(U) \cong \mathrm{Unc}_M(U),$$

and a 1-Polar Uncertain Soft Set reduces to an ordinary Uncertain Soft Set of type M.

Remark 5.7.4 (Fuzzy Specialization)**.** If M is the fuzzy carrier model with

$$\mathrm{Dom}(M) = [0,1],$$

then

$$F(e) : U \longrightarrow [0,1]^m$$

for every $e \in A$. Hence the construction reduces to the corresponding m-polar fuzzy soft-set framework.

Remark 5.7.5 (Finite-Dimensional Specialization)**.** If, for a particular uncertainty model,

$$\mathrm{Dom}(M) = D \subseteq [0,1]^r$$

for some $r \geq 1$, then

$$\mathrm{Dom}(M)^m = D^m \subseteq [0,1]^{rm}.$$

Thus the finite-dimensional formulation based on D^m is recovered as a special case of Definition 5.7.2, rather than being imposed on the general theory.

Related concepts include m-polar Fuzzy Soft Sets [142, 294, 295] and m-polar Neutrosophic Soft Sets [296–298].

5.8 m-polar Uncertain Rough Set

Classical rough approximation uses an equivalence relation [299–301]. In the typed setting, reflexivity is the minimum condition needed for the approximation sandwich.

Definition 5.8.1 (Reflexive Uncertain Rough Set)**.** Let $X \neq \emptyset$ be finite and let

$$M_{\mathrm{DM}} = (\mathrm{Dom}(M), \leq_M, \vee_M, \wedge_M, N_M, 0_M, 1_M)$$

be a bounded De Morgan model. Define

$$a \Rightarrow_M b = N_M(a) \vee_M b.$$

An M-valued relation $R_M : X \times X \to \mathrm{Dom}(M)$ is *reflexive* if

$$R_M(x,x) = 1_M \qquad (x \in X).$$

For a reflexive R_M and an Uncertain Set $\mu_A : X \to \mathrm{Dom}(M)$, define

$$\mu_{\underline{R}_M(A)}(x) = \bigwedge_{y \in X} \big(R_M(x,y) \Rightarrow_M \mu_A(y)\big)$$

and

$$\mu_{\overline{R}_M(A)}(x) = \bigvee_{y \in X} \big(R_M(x,y) \wedge_M \mu_A(y)\big).$$

The pair $(\underline{R}_M(A), \overline{R}_M(A))$ is the *Reflexive Uncertain Rough Set* of A. If R_M is also symmetric and $\wedge_M$-transitive, it is called an uncertain equivalence (or similarity) rough set.

Definition 5.8.2 (Reflexive m-Polar Uncertain Rough Set)**.** Let $m \geq 1$, put $\mathrm{Dom}_m(M) = \mathrm{Dom}(M)^m$, and extend the De Morgan operations componentwise as in Lemma 2.3.5. An m-polar relation

$$R_m : X \times X \longrightarrow \mathrm{Dom}_m(M)$$

is reflexive if

$$R_m(x,x) = \mathbf{1}_M = (1_M, \ldots, 1_M).$$

For $\mu_A : X \to \mathrm{Dom}_m(M)$, define

$$\mu_{\underline{R}_m(A)}(x) = \bigwedge_{y \in X} \big(R_m(x, y) \Rightarrow_m \mu_A(y)\big)$$

and

$$\mu_{\overline{R}_m(A)}(x) = \bigvee_{y \in X} \big(R_m(x, y) \wedge_m \mu_A(y)\big),$$

where all operations are componentwise. This pair is the *Reflexive m-Polar Uncertain Rough Set* of A.

Proposition 5.8.3 (Approximation Sandwich)**.** *For every reflexive uncertain relation,*

$$\underline{R}_M(A) \leq_M A \leq_M \overline{R}_M(A)$$

pointwise. The same inequalities hold componentwise in the m-polar case.

Proof. In the lower meet, the term $y = x$ equals $N_M(1_M) \vee_M \mu_A(x) = \mu_A(x)$, so the meet is no larger than $\mu_A(x)$. In the upper join, the term $y = x$ equals $1_M \wedge_M \mu_A(x) = \mu_A(x)$, so the join is no smaller. Apply the argument coordinatewise for m poles. □

Remark 5.8.4 (Necessity of Reflexivity)**.** Without reflexivity, the terms "lower" and "upper" need not be ordered. For the fuzzy De Morgan model, take $X = \{x\}$, $R(x, x) = 0$, and $\mu_A(x) = 1$. Then

$$\mu_{\underline{R}(A)}(x) = 1, \qquad \mu_{\overline{R}(A)}(x) = 0.$$

The reflexivity condition in the revised definitions excludes this counterexample.

Remark 5.8.5 (Pawlak Specialization)**.** If $\mathrm{Dom}(M) = \{0, 1\}$, $N_M(a) = 1 - a$, and R_M is the characteristic function of an equivalence relation, the construction reduces to Pawlak lower and upper approximations. With $\mathrm{Dom}(M) = [0, 1]$, $\vee_M = \max$, $\wedge_M = \min$, and a reflexive fuzzy relation, it gives the corresponding componentwise fuzzy rough approximation.

Chapter 6

Conclusion

In this book, we introduced and systematically investigated several polarity-based extensions of uncertain sets within a unified mathematical setting. In particular, we considered multipolar, iterative, structural, dynamic, context-dependent, and other polarity-oriented formulations, and provided their fundamental definitions, representative examples, common typed-construction lemmas, and structure-sensitive properties. These models extend conventional uncertainty representations by allowing information to be described through multiple, opposing, hierarchical, network-structured, heterogeneous, and dynamically changing perspectives.

The proposed framework also clarifies that polarity can play different mathematical roles. Depending on the model, poles may function as indices of multiple criteria, represent opposite or counter-properties, form hierarchical or graph-based structures, vary with time or context, or arise from geometric and physically motivated notions of polarization. In this sense, the study provides a broad foundation for organizing and comparing different polarity-based uncertainty models within a common framework.

As future work, further theoretical developments should investigate algebraic properties, structural relationships, reduction principles, equivalence conditions, and transformations among the proposed models. It would also be valuable to develop computational algorithms and practical implementations for handling polarity-based uncertain information, together with complexity analyses and numerical experiments for evaluating their behavior. Furthermore, detailed case studies in areas such as decision-making, network analysis, engineering systems, information processing, and other uncertainty-aware applications may help clarify the practical advantages, limitations, and applicability of the proposed frameworks.

Disclaimer

Funding

This study was conducted without any financial support from external organizations or grants.

Acknowledgments

We would like to express our sincere gratitude to everyone who provided valuable insights, support, and encouragement throughout this research. We also extend our thanks to the readers for their interest and to the authors of the referenced works, whose scholarly contributions have greatly influenced this study. Lastly, we are deeply grateful to the publishers and reviewers who facilitated the dissemination of this work.

Data Availability

Since this research is purely theoretical and mathematical, no empirical data or computational analysis was utilized. Researchers are encouraged to expand upon these findings with data-oriented or experimental approaches in future studies.

Ethical Statement

As this study does not involve experiments with human participants or animals, no ethical approval was required.

Conflicts of Interest

The authors declare that they have no conflicts of interest related to the content or publication of this book.

Code Availability

No code or software was developed for this study.

Use of Generative AI and AI-Assisted Tools

The authors used generative AI and AI-assisted tools only for limited language-editing tasks, such as checking English grammar and improving readability. The mathematical definitions, results, interpretations, and final verification remain the responsibility of the authors.

Disclaimer (Others)

This work presents theoretical ideas and frameworks that have not yet been empirically validated. Readers are encouraged to explore practical applications and further refine these concepts. Although care has been taken to ensure accuracy and appropriate citations, any errors or oversights are unintentional. The perspectives and interpretations expressed herein are solely those of the authors and do not necessarily reflect the viewpoints of their affiliated institutions.

Appendix (List of Tables)

Bibliography

[1] Thomas Jech. *Set theory: The third millennium edition, revised and expanded*. Springer, 2003.

[2] Orhan Dalkılıç, Naime Demirtaş, and Abdullah Demirtaş. Evaluating prostate cancer diagnosis using the adaptive neural fuzzy inference system (anfis): A comparative analysis of diagnostic accuracy. *Turkish Journal of Science and Technology*, 20(2):583–593, 2025.

[3] Lotfi A Zadeh. Fuzzy sets. *Information and control*, 8(3):338–353, 1965.

[4] Young Bae Jun, Kul Hur, and Kyoung Ja Lee. Hyperfuzzy subalgebras of bck/bci-algebras. *Annals of Fuzzy Mathematics and Informatics*, 2017.

[5] Seok-Zun Song, Seon Jeong Kim, and Young Bae Jun. Hyperfuzzy ideals in bck/bci-algebras. *Mathematics*, 5(4):81, 2017.

[6] Jayanta Ghosh and Tapas Kumar Samanta. Hyperfuzzy sets and hyperfuzzy group. *Int. J. Adv. Sci. Technol*, 41:27–37, 2012.

[7] Uttam Mondal, Madhumangal Pal, and Kalyani Das. A study on bipolar intuitionistic fuzzy soft graphs and its application. *TWMS Journal of Applied and Engineering Mathematics*, 2025.

[8] Krassimir T Atanassov. Circular intuitionistic fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 39(5):5981–5986, 2020.

[9] AbdUlazeez Alkouri, Eman A AbuHijleh, Ghada Alafifi, Eman Almuhur, and Fadi MA Al-Zubi. More on complex hesitant fuzzy graphs. *AIMS Mathematics*, 8(12):30429–30444, 2023.

[10] Vicenç Torra. Hesitant fuzzy sets. *International journal of intelligent systems*, 25(6):529–539, 2010.

[11] Florentin Smarandache. Neutrosophic set is a generalization of intuitionistic fuzzy set, inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set), pythagorean fuzzy set, spherical fuzzy set, and q-rung orthopair fuzzy set, while neutrosophication is a generalization of regret theory, grey system theory, and three-ways decision (revisited). *Journal of New Theory*, 29:1–31, 2019.

[12] Bui Cong Cuong. Picture fuzzy sets. *Journal of Computer Science and Cybernetics*, 30:409, 2015.

[13] Said Broumi, Mohamed Talea, Assia Bakali, and Florentin Smarandache. Single valued neutrosophic graphs. *Journal of New theory*, 10:86–101, 2016.

[14] Pu Ji, Jian qiang Wang, and Hong yu Zhang. Frank prioritized bonferroni mean operator with single-valued neutrosophic sets and its application in selecting third-party logistics providers. *Neural Computing and Applications*, 30:799–823, 2018.

[15] Haibin Wang, Florentin Smarandache, Yanqing Zhang, and Rajshekhar Sunderraman. *Single valued neutrosophic sets*. Infinite study, 2010.

[16] Qaisar Khan, Peide Liu, and Tahir Mahmood. Some generalized dice measures for double-valued neutrosophic sets and their applications. *Mathematics*, 6(7):121, 2018.

[17] Raed Hatamleh, Nasir Odat, Hamza Ali Abujabal, Faria Khan, Arif Mehmood Khattak, Alaa M. Abd El-latif, Husham M. Attaalfadeel, and Abdelhalim Hasnaoui. Fermatean double-valued neutrosophic soft topological spaces. *European Journal of Pure and Applied Mathematics*, 2025.

[18] Lin Wei. An integrated decision-making framework for blended teaching quality evaluation in college english courses based on the double-valued neutrosophic sets. *J. Intell. Fuzzy Syst.*, 45:3259–3266, 2023.

[19] Florentin Smarandache. *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off-Logic, Probability, and Statistics*. Infinite Study, 2016.

[20] Florentin Smarandache. *Introduction of Over/Under/Off Masses*. Infinite Study, 2026.

[21] Fang Sun, Mifeng Ren, Yujing Shi, and Xuanbai Feng. Quadruple-valued neutrosophic offset for maintenance platforms design quality evaluation of digital monitoring and remote operation in new energy power generation systems. *Neutrosophic Sets and Systems*, 88:940–953, 2025.

[22] Florentin Smarandache. *Plithogenic set, an extension of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets-revisited*. Infinite study, 2018.

[23] Florentin Smarandache. *Extension of HyperGraph to n-SuperHyperGraph and to Plithogenic n-SuperHyperGraph, and Extension of HyperAlgebra to n-ary (Classical-/Neutro-/Anti-) HyperAlgebra*. Infinite Study, 2020.

[24] Florentin Smarandache. Extension of soft set to hypersoft set, and then to plithogenic hypersoft set. *Neutrosophic sets and systems*, 22(1):168–170, 2018.

[25] Maha Noorwali, Raed Hatamleh, Ahmed Salem Heilat, Haitham Ali Qawaqneh, Arif Mehmood Khattak, Aqeedat Hussain, Jamil J. Hamja, and Cris L. Armada. Artificial intelligence enhanced framework for complex double valued neutrosophic soft sets and emotional affinity evaluation via cotangent similarity. *European Journal of Pure and Applied Mathematics*, 2025.

[26] Dmitriy Molodtsov. Soft set theory-first results. *Computers & mathematics with applications*, 37(4-5):19–31, 1999.

[27] Muhammad Gulistan, Naveed Yaqoob, Ahmed Elmoasry, and Jawdat Alebraheem. Complex bipolar fuzzy sets: An application in a transport's company. *J. Intell. Fuzzy Syst.*, 40:3981–3997, 2021.

[28] Ubaid ur Rehman, Tahir Mahmood, Jabbar Ahmmad, Hafiz Muhammad Waqas, and Dragan Pamucar. Selection of optimal combined heat and power systems for industrial energy needs by using bipolar fuzzy rough mcdm based on schweizer sklar operators. *International Journal of Computational Intelligence Systems*, 2026.

[29] Ömer Faruk Görçün, Abhijit Saha, and Fatih Ecer. Data-driven analysis of green logistics strategies for the logistics and transportation companies: A bipolar fuzzy methodology. *Business Strategy and the Environment*, 2026.

[30] Muhammad Akram. Bipolar fuzzy graphs. *Information sciences*, 181(24):5548–5564, 2011.

[31] Wen-Ran Zhang. Bipolar fuzzy sets and relations: a computational framework for cognitive modeling and multiagent decision analysis. *NAFIPS/IFIS/NASA '94. Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference. The Industrial Fuzzy Control and Intellige*, pages 305–309, 1994.

[32] M Murali Krishna Rao and B Venkateswarlu. Tripolar fuzzy interior ideals of a γ-semiring. *Asia Pacific Journal of Management*, 5(2):192–207, 2018.

[33] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *Journal of Intelligent & Fuzzy Systems*, 33(6):4077–4087, 2017.

[34] TA Akhila and G Deepa. Matrix energy of multi-valued and hesitant bipolar neutrosophic structures: A unified framework for multi-criteria decision-making under bipolarity and hesitation. *Research in Mathematics*, 13(1):2700120, 2026.

[35] Lazim Abdullah. An approach using border approximation area in bipolar neutrosophic settings for optimal alternative of agricultural sustainability. *AIUB Journal of Science and Engineering*, 24(3):216–227, 2026.

[36] Raja Muhammad Hashim, Muhammad Gulistan, Musaed Alhussein, Khursheed Aurangzeb, and Adnan Khurshid. A decision–making system for managing renewable energy alternatives and strategy using triangular neutrosophic bipolar fuzzy topsis. *International Journal of Information Technology & Decision Making*, 25(05):1579–1607, 2026.

[37] Irfan Deli, Mumtaz Ali, and Florentin Smarandache. Bipolar neutrosophic sets and their application based on multi-criteria decision making problems. *2015 International Conference on Advanced Mechatronic Systems (ICAMechS)*, pages 249–254, 2015.

[38] Muhammad Riaz, Muhammad Tahir Hamid, Deeba Afzal, Dragan Pamucar, and Yuming Chu. Multi-criteria decision making in robotic agri-farming with q-rung orthopair m-polar fuzzy sets. *PLoS ONE*, 16, 2021.

[39] Juanjuan Chen, Shenggang Li, Shengquan Ma, and Xueping Wang. m-polar fuzzy sets: an extension of bipolar fuzzy sets. *The scientific world journal*, 2014(1):416530, 2014.

[40] M Sivakumar, Rabıyathul Basarıya, Abdul Rajak, M Senthil, T Vetriselvi, G Raja, and R Rajavarman. Transforming arabic text analysis: Integrating applied linguistics with m-polar neutrosophic set mood change and depression on social media. *International Journal of Neutrosophic Science (IJNS)*, 25(2), 2025.

[41] Khalid Naeem and Bijan Divvaz. Information measures for madm under m-polar neutrosophic environment. *Granular Computing*, 8(3):597–616, 2023.

[42] Juanjuan Lu, Linli Zhu, and Wei Gao. Structured representation of fuzzy data by bipolar fuzzy hypergraphs. In *International Conference on Machine Learning for Cyber Security*, pages 663–676. Springer, 2022.

[43] AM Ismayil and N Azhagendran. Isomorphism on complex fuzzy graph. *Indian Journal of Science and Technology*, 17:86–92, 2024.

[44] Saima Anis, Madad Khan, Muhammad Mazhar, Muhammad Naveed, and Kostaq Hila. Complex fuzzy dynamical graphs and their applications in signals processing. *International Journal of Analysis and Applications*, 23:9–9, 2025.

[45] Mani Parimala, Arwa Almunajam, Muthusamy Karthika, and Ibtesam Alshammari. Pythagorean fuzzy digraphs and its application in healthcare center. *Journal of Mathematics*, 2021(1):5459654, 2021.

[46] Florentin Smarandache and Said Broumi. *Neutrosophic graph theory and algorithms*. IGI Global, 2019.

[47] Lotfi A Zadeh. Fuzzy logic, neural networks, and soft computing. In *Fuzzy sets, fuzzy logic, and fuzzy systems: selected papers by Lotfi A Zadeh*, pages 775–782. World Scientific, 1996.

[48] Florentin Smarandache and Maissam Jdid. *An Overview of Neutrosophic and Plithogenic Theories and Applications*. Infinite Study, 2023.

[49] Florentin Smarandache. *Neutrosophic Physics: More Problems, More Solutions*. North-European Scientific Publishers, Hanko, Finland, 2010.

[50] Florentin Smarandache. A unifying field in logics: Neutrosophic logic. In *Philosophy*, pages 1–141. American Research Press, 1999.

[51] Mustafa Hasan Hadi and LAA Al-Swidi. The neutrosophic axial set theory. *Neutrosophic Sets and Systems, vol. 51/2022: An International Journal in Information Science and Engineering*, page 295, 2022.

[52] Takaaki Fujita and Florentin Smarandache. *A Survey of Fuzzy and Uncertain Concepts in Applied Mathematics*. Infinite Study, 2025.

[53] Takaaki Fujita and Florentin Smarandache. *A comprehensive survey of set-theoretic concepts related to fuzzy, neutrosophic, and uncertain sets*. Infinite Study, 2026.

[54] Takaaki Fujita and Florentin Smarandache. A unified framework for u-structures and functorial structure: Managing super, hyper, superhyper, tree, and forest uncertain over/under/off models. *Neutrosophic Sets and Systems*, 91:337–380, 2025.

[55] Takaaki Fujita and Florentin Smarandache. *A Comprehensive and Updated Survey of Fuzzy, Neutrosophic, and Uncertain Sets*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[56] Takaaki Fujita and Florentin Smarandache. *A dynamic survey of fuzzy, intuitionistic fuzzy, neutrosophic, plithogenic, and extensional sets*. Neutrosophic Science International Association (NSIA), 2025.

[57] Mohammed A. Al Shumrani, Muhammad Gulistan, and Tahir Abbas. Multi-criteria decision making approach for solar energy implementation using n-cubic fuzzy interaction aggregation operators. *Scientific Reports*, 16, 2025.

[58] Sheikh Rashid, Muhammad Gulistan, Tahir Abbas, Kholood Mohammad Alsager, and Muhammad Rahim. Multi-attribute decision-making methods to low-carbon green supply chain management in n-cubic fuzzy aggregation operators. *International Journal of Fuzzy Systems*, 28:1629 – 1641, 2025.

[59] John N Mordeson and Premchand S Nair. *Fuzzy graphs and fuzzy hypergraphs*, volume 46. Physica, 2012.

[60] P. R. Kavyasree and B. Surender Reddy. N-cubic sets applied to linear spaces. *Kragujevac Journal of Mathematics*, 2022.

[61] Gundeti Soujanya and B. Surender Reddy. N-cubic picture fuzzy linear spaces. *Indian Journal Of Science And Technology*, 2024.

[62] Young Bae Jun, Seok Zun Song, and Seon Jeong Kim. N-hyper sets. *Mathematics*, 2018.

[63] Witold Pedrycz. Shadowed sets: representing and processing fuzzy sets. *IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)*, 28(1):103–109, 1998.

[64] Mingjie Cai, Qingguo Li, and Guangming Lang. Shadowed sets of dynamic fuzzy sets. *Granular Computing*, 2:85 – 94, 2016.

[65] Gianpiero Cattaneo and Davide Ciucci. Shadowed sets and related algebraic structures. *Fundamenta Informaticae*, 55(3-4):255–284, 2003.

[66] Krassimir T Atanassov. *On intuitionistic fuzzy sets theory*, volume 283. Springer, 2012.

[67] RA Borzooei and Hossein Rashmanlou. Cayley interval-valued fuzzy graphs. *UPB Scientific Bulletin, Series A: Applied Mathematics and Physics*, 78(3):83–94, 2016.

[68] Madhumangal Pal, Sovan Samanta, and Hossein Rashmanlou. Some results on interval-valued fuzzy graphs. *International Journal of Computer Science and Electronics Engineering*, 3(3):205–211, 2015.

[69] Xindong Peng. New operations for interval-valued pythagorean fuzzy set. *Scientia Iranica*, 26:1049–1076, 2018.

[70] Weijian Ren and Yuhong Du. Development and application of an interval-valued pythagorean fuzzy set-based grade evaluation method to foreign fiber content. *Journal of Natural Fibers*, 20, 2023.

[71] Manish Kumar Gunjan, Amal Kumar Adak, and Davood Darvishi Salookolaei. Fermatean fuzzy artinian and noetherian rings. *Asian Research Journal of Mathematics*, 21(5):179–190, 2025.

[72] Fei Gao, Meihong Han, Siyang Wang, and Jie Gao. A novel fermatean fuzzy bwm-vikor based multi-criteria decision-making approach for selecting health care waste treatment technology. *Eng. Appl. Artif. Intell.*, 127:107451, 2024.

[73] Ping Wang, Jie Wang, Guiwu Wei, and Cun Wei. Similarity measures of q-rung orthopair fuzzy sets based on cosine function and their applications. *Mathematics*, 2019.

[74] Muhmamet Deveci, Dragan Pamucar, Ilgin Gokasar, Mario Köppen, and Brij Bhooshan Gupta. Personal mobility in metaverse with autonomous vehicles using q-rung orthopair fuzzy sets based opa-rafsi model. *IEEE Transactions on Intelligent Transportation Systems*, 24:15642–15651, 2023.

[75] Muhammad Akram, A Nagoor Gani, and A Borumand Saeid. Vague hypergraphs. *Journal of Intelligent & Fuzzy Systems*, 26(2):647–653, 2014.

[76] Humberto Bustince and P Burillo. Vague sets are intuitionistic fuzzy sets. *Fuzzy sets and systems*, 79(3):403–405, 1996.

[77] Chen Shouyu and Guo Yu. Variable fuzzy sets and its application in comprehensive risk evaluation for flood-control engineering system. *Fuzzy Optimization and Decision Making*, 5(2):153–162, 2006.

[78] Wen-chuan Wang, Dong-mei Xu, Kwok-wing Chau, and Guan-jun Lei. Assessment of river water quality based on theory of variable fuzzy sets and fuzzy binary comparison method. *Water resources management*, 28(12):4183–4200, 2014.

[79] Yunhai Fang, Xilai Zheng, Hui Peng, Huan Wang, and Jia Xin. A new method of the relative membership degree calculation in variable fuzzy sets for water quality assessment. *Ecological indicators*, 98:515–522, 2019.

[80] Vasile Patrascu. A new penta-valued logic based knowledge representation. *arXiv preprint arXiv:1502.05562*, 2015.

[81] Vasile Patrascu. Similarity, cardinality and entropy for bipolar fuzzy set in the framework of penta-valued representation. *arXiv preprint arXiv:1506.02060*, 2015.

[82] Yong-Uk Cho, Young-Bae Jun, and Seok-Zun Song. Bifuzzy ideals of pseudo mv-algebras. *JOURNAL OF APPLIED MATHEMATICS AND COMPUTING*, 22(1/2):475, 2006.

[83] Tadeusz Gerstenkorn and Jacek Mańko. Bifuzzy probabilistic sets. *Fuzzy sets and systems*, 71(2):207–214, 1995.

[84] Raja Muhammad Hashim, Muhammad Gulistan, Musaed A. Alhussein, Khursheed Aurangzeb, and Adnan Khurshid. A decision-making system for managing renewable energy alternatives and strategy using triangular neutrosophic bipolar fuzzy topsis. *International Journal of Information Technology & Decision Making*, 2025.

[85] Sankar Das, Ganesh Ghorai, and Madhumangal Pal. Picture fuzzy tolerance graphs with application. *Complex & Intelligent Systems*, 8(1):541–554, 2022.

[86] G Muhiuddin, Sun Shin Ahn, Chang Su Kim, and Young Bae Jun. Stable cubic sets. *J. Comput. Anal. Appl*, 23(5):802–819, 2017.

[87] Madad Khan, Young Bae Jun, Muhammad Gulistan, and Naveed Yaqoob. The generalized version of jun's cubic sets in semigroups. *Journal of Intelligent & Fuzzy Systems*, 28(2):947–960, 2015.

[88] Muhammad Akram, Danish Saleem, and Talal Al-Hawary. Spherical fuzzy graphs with application to decision-making. *Mathematical and Computational Applications*, 25(1):8, 2020.

[89] Fatma Kutlu Gündoğdu and Cengiz Kahraman. Spherical fuzzy sets and spherical fuzzy topsis method. *Journal of intelligent & fuzzy systems*, 36(1):337–352, 2019.

[90] Khushbakhat Asif, Muhammad Kamran Jamil, Hanen Karamti, Muhammad Azeem, and Kifayat Ullah. Randić energies for t-spherical fuzzy hamacher graphs and their applications in decision making for business plans. *Computational and Applied Mathematics*, 42:1–32, 2023.

[91] Abhishek Guleria and Rakesh Kumar Bajaj. T-spherical fuzzy graphs: operations and applications in various selection processes. *Arabian Journal for Science and Engineering*, 45(3):2177–2193, 2020.

[92] Takaaki Fujita. A mathematical study of tripolar intuitionistic fuzzy graphs. *International Journal of Computer Application (IJCA)*, 16(4):16/1–16/11, 2026.

[93] M Murali Krishna Rao, B Venkateswarlu, and Y Adi Narayana. Tripolar fuzzy soft ideals and tripolar fuzzy soft interior ideals over semiring. *Italian journal of pure and applied Mathematics*, page 731743, 2019.

[94] M Murali Krishna Rao. Tripolar fuzzy interior ideals of γ-semigroup. *Ann. Fuzzy Math. Inform*, 15(2):199–206, 2018.

[95] Shawkat Alkhazaleh. Neutrosophic vague set theory. *Critical Review*, 10:29–39, 2015.

[96] Ashraf Al-Quran, Hazwani Hashim, and Lazim Abdullah. A hybrid approach of interval neutrosophic vague sets and dematel with new linguistic variable. *Symmetry*, 12(2):275, 2020.

[97] R Radha, A Stanis Arul Mary, and Florentin Smarandache. Quadripartitioned neutrosophic pythagorean soft set. *International Journal of Neutrosophic Science (IJNS) Volume 14, 2021*, page 11, 2021.

[98] Satham Hussain, Jahir Hussain, Isnaini Rosyida, and Said Broumi. Quadripartitioned neutrosophic soft graphs. In *Handbook of Research on Advances and Applications of Fuzzy Sets and Logic*, pages 771–795. IGI Global, 2022.

[99] Mouhamed Bayane Bouraima, Ertugrul Ayyıldız, Melike Erdogan, and Dragan Pamucar. An interval-valued intuitionistic fuzzy group decision model for evaluation of cross-border railway development. *Cognitive Computation*, 18(1):22, 2026.

[100] Ravindar Raj Chinnappan and Dhanasekar Sundaram. Image enhancement framework combining interval-valued intuitionistic fuzzy sets and fractional sobel operator. *Artificial Intelligence Review*, 58(9):291, 2025.

[101] Yukun Hu, Hong Wang, Chunhui Li, Ang Xu, Li Yi, Chao Liang, Bingyang Xi, and Fang Dai. An integrated approach for design concept evaluation based on interval-valued pythagorean fuzzy set and consensus model. *Kybernetes*, 55(8):3611–3636, 2026.

[102] Yoojeong Oh, Hans Pasman, Safyan Akram Khan, and Sunhwa Park. Hydrogen storage selection for saudi arabia: A multi-criteria decision making under interval-valued pythagorean fuzzy environment. *International Journal of Hydrogen Energy*, 105:1281–1293, 2025.

[103] Abdul Wahab Mustafa, Zia Bashir, Jawad Ali, and Muhammed I Syam. A unified framework of information measures for complex linear diophantine fuzzy sets with application to chronic kidney disease. *Scientific Reports*, 2026.

[104] Masooma Raza Hashmi, Muhammad Riaz, Arshid Mahmood, Muhammad Ajmaeen, and Muhammad Aslam. Optimizing performance metrics with new distance and similarity measures using control parameters of linear diophantine fuzzy sets. *Engineering Applications of Artificial Intelligence*, 162:112557, 2025.

[105] Ilanthenral Kandasamy. Double-valued neutrosophic sets, their minimum spanning trees, and clustering algorithm. *Journal of Intelligent systems*, 27(2):163–182, 2018.

[106] Qiuyan Zhao and Wentao Li. Incorporating intelligence in multiple-attribute decision-making using algorithmic framework and double-valued neutrosophic sets: Varied applications to employment quality evaluation for university graduates. *Neutrosophic Sets and Systems*, 76:59–78, 2025.

[107] Bin Zhu and Zeshui Xu. Some results for dual hesitant fuzzy sets. *Journal of Intelligent & Fuzzy Systems*, 26(4):1657–1668, 2014.

[108] José Carlos R Alcantud, Gustavo Santos-García, Xindong Peng, and Jianming Zhan. Dual extended hesitant fuzzy sets. *Symmetry*, 11(5):714, 2019.

[109] M Balamurugan, Khalil H Hakami, Moin A Ansari, Anas Al-Masarwah, and K Loganathan. Quadri-polar fuzzy fantastic ideals in bci-algebras: A topsis framework and application. *European Journal of Pure and Applied Mathematics*, 17(4):3129–3155, 2024.

[110] Florentin Smarandache. Ambiguous set is a subclass of the double refined indeterminacy neutrosophic set, and of the refined neutrosophic set in general. *Neutrosophic Sets & Systems*, 58, 2023.

[111] Pritpal Singh et al. Ambiguous set theory: A new approach to deal with unconsciousness and ambiguousness of human perception. *Full Length Article*, 5(1):52–2, 2023.

[112] Florentin Smarandache. Ambiguous set brings nothing new. *nidus idearum*, page 24, 2024.

[113] Takaaki Fujita and Florentin Smarandache. Antipodal turiyam neutrosophic graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:1–13, 2024.

[114] Takaaki Fujita. Review of rough turiyam neutrosophic directed graphs and rough pentapartitioned neutrosophic directed graphs. *Neutrosophic Optimization and Intelligent Systems*, 5:48–79, 2025.

[115] Takaaki Fujita and Florentin Smarandache. Pythagorean, fermatean, and complex turiyam neutrosophic graphs. *SciNexuses*, 2:39–63, 2025.

[116] Prem Kumar Singh, Naveen Surathu, Ghattamaneni Surya Prakash, et al. Turiyam based four way unknown profile characterization on social networks. *Full Length Article*, 10(2):27–7, 2024.

[117] Rama Mallick and Surapati Pramanik. Pentapartitioned neutrosophic set and its properties. *Neutrosophic Sets and Systems*, 35:49, 2020.

[118] Radha R. and Stanis Arul Mary. Pentapartitioned neutrosophic pythagorean set. *International Research Journal on Advanced Science Hub*, 2021.

[119] Suman Das, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. *Iraqi Journal of Science*, 2022.

[120] Jiarong Jia. Triple-valued neutrosophic off for public digital cultural service level evaluation in craft and art museums. *Neutrosophic Sets and Systems*, 88(1):44, 2025.

[121] Takaaki Fujita. Triple-valued neutrosophic set, quadruple-valued neutrosophic set, quintuple-valued neutrosophic set, and double-valued indetermsoft set. *Neutrosophic Systems with Applications*, 25(5):3, 2025.

[122] Hongxin Wang. Professional identity formation in traditional chinese medicine students: An educational perspective using triple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88:83–92, 2025.

[123] M. Priyadharshini and D. Jayanthi. Similarity measures on cubic intuitionistic fuzzy sets. *Notes on Intuitionistic Fuzzy Sets*, 2025.

[124] Priyadharshini Manoharan, Jayanthi Duraisamy, and Saranya Manoharan. Cardinality and relative cardinality on cubic intuitionistic fuzzy sets. *International Journal of Information Technology*, 16:4059 – 4068, 2024.

[125] Harish Garg and Gagandeep Kaur. Novel distance measures for cubic intuitionistic fuzzy sets and their applications to pattern recognitions and medical diagnosis. *Granular Computing*, 5:169 – 184, 2018.

[126] Muhammad Sajid, Khuram Ali Khan, Atiqe Ur Rahman, Sanaa Ahmed Bajri, Alhanouf Alburaikan, and Hamiden Abd El-Wahed Khalifa. A novel algorithmic multi-attribute decision-making framework for solar panel selection using modified aggregations of cubic intuitionistic fuzzy hypersoft set. *Heliyon*, 10, 2024.

[127] Maha Mohammed Saeed, Sami Ullah Khan, Fatima Suriyya, Arif Mehmood, and Jamil J Hamja. Interval-valued complex neutrosophic sets and complex neutrosophic soft topological spaces. *International Journal of Analysis and Applications*, 23:132–132, 2025.

[128] Samaneh Salari and Ali Karimi. Introducing an integrated approach for fire safety assessment in healthcare facilities by interval valued neutrosophic-ahp and fuzzy inference system. *Heliyon*, 11(2), 2025.

[129] Raed Hatamleh, Nasir Odat, Abdallah Al-Husban, Arif Mehmood Khattak, Alaa M Abd El-latif, Husham M Attaalfadeel, Walid Abdelfattah, and Ahmad M Abdel-Mageed. Mapping love: A heptapartitioned neutrosophic machine learning study of university students' romantic sensations. *European Journal of Pure and Applied Mathematics*, 18(3):6484–6484, 2025.

[130] Maha Mohammed Saeed, Raed Hatamleh, Hamza Ali Abujabal, Yahya Khan, Abdallah Al-Husban, Amy A Laja, Sulfaisa MM Pangilan, Cris L Armada, Jamil J Hamja, and Arif Mehmood Khattak. Heptapartitioned neutrosophic soft topologies and machine learning techniques for exploring romantic feelings. *European Journal of Pure and Applied Mathematics*, 18(3):6221–6221, 2025.

[131] Wenwen Meng. Quintuple-valued neutrosophic offset for quality evaluation of cross-border e-commerce talent training based on artificial intelligence. *Neutrosophic Sets and Systems*, 88(1):54, 2025.

[132] Yuan Sun. Assessing the performance of basic education informatization in the digital age using quintuple-valued neutrosophic set. *Neutrosophic Sets and Systems*, 88(1):18, 2025.

[133] Takaaki Fujita, Raed Hatamleh, and Ahmed Heilat. Advanced partitioned neutrosophic offsets, oversets, and undersets: Modeling "neither agree nor disagree" and beyond. *Statistics, Optimization & Information Computing*, 2026.

[134] Takaaki Fujita. Advanced partitioned neutrosophic sets: Formalization of hexa-, hepta-, octa-, nona-, and deca-partitioned structures. *Abhath Journal of Basic and Applied Sciences*, 4(2):40–60, 2025.

[135] CV Malavika, Kalaivani Chandran, Sofia Jennifer John, and Florentin Smarandache. Python based tool for neutrosophic cubic sets and their operations. *Journal of Fuzzy Extension and Applications*, 7(2):592–614, 2026.

[136] Majid Khan, Muhammad Gulistan, and Mohammed M. Ali Al-Shamiri. The approach of induced generalized neutrosophic cubic shapley choquet integral aggregation operators via the codas method to solve distance-based multicriteria decision-making problems. *Journal of Mathematics*, 2022, 2022.

[137] Majid Khan, Muhammad Gulistan, Musaed A. Alhussein, Khursheed Aurangzeb, and Adnan Khurshid. Navigating ambiguity: A novel neutrosophic cubic shapley normalized weighted bonferroni mean aggregation operator with application in the investment environment. *Heliyon*, 10, 2024.

[138] Yi-Ming Li, Majid Khan, Adnan Khurshid, Muhammad Gulistan, Ateeq ur Rehman, Mumtaz Ali, Shahab Abdulla, and Aitazaz Ahsan Farooque. Designing pentapartitioned neutrosophic cubic set aggregation operator-based air pollution decision-making model. *Complex & Intelligent Systems*, pages 1–18, 2023.

[139] Adil Darvesh, Khalid Naeem, and Syed Bilal Hussain Shah. Scrutiny of neutrosophic cubic fuzzy data with different parametric values using bonferroni weighted mean: A robust mcdm approach. *Decision making advances*, 3(1):245–265, 2025.

[140] Giovanni Panti. Multi-valued logics. In *Quantified representation of uncertainty and imprecision*, pages 25–74. Springer, 1998.

[141] Fujita Takaaki and Arif Mehmood. Iterative multifuzzy set, iterative multineutrosophic set, iterative multisoft set, and multiplithogenic sets. *Neutrosophic Computing and Machine Learning*, 41:1–30, 2025.

[142] Hilah Awad Alharbi and Kholood Mohammad Alsager. Fermatean m-polar fuzzy soft rough sets with application to medical diagnosis. *AIMS Mathematics*, 10(6):14314–14346, 2025.

[143] Ganesh Ghorai and Madhumangal Pal. Results of m-polar fuzzy graphs with application. *Journal of Uncertain Systems*, 12(1):47–55, 2018.

[144] Muhammad Arshad, Muhammad Saeed, and Atiqe Ur Rahman. Convexity cum concavity on refined fuzzy set with some properties. *Uncertainty Discourse and Applications*, 1(1):140–150, 2024.

[145] Florentin Smarandache. n-valued refined neutrosophic logic and its applications to physics. *Infinite study*, 4:143–146, 2013.

[146] Rajab Ali Borzooei, Hee Sik Kim, Young Bae Jun, and Sun Shin Ahn. On multipolar intuitionistic fuzzy b-algebras. *Mathematics*, 2020.

[147] Lian Chen, Aysha Khan, Maryam Akhoundi, AliAsghar Talebi, G. Muhiuddin, and Syed Bahman Hosseinzadeh Sadati. A study on m-polar interval-valued intuitionistic fuzzy graphs with application in management. *Journal of Mathematics*, 2022.

[148] K Sankar and D Ezhilmaran. Morphism of m-polar intuitionistic fuzzy graphs. *Journal of Computational and Theoretical Nanoscience*, 15(6):2277–2282, 2018.

[149] Vakkas Uluçay. Some concepts on interval-valued refined neutrosophic sets and their applications. *Journal of Ambient Intelligence and Humanized Computing*, 12(7):7857–7872, 2021.

[150] Basavaraj V Hiremath, Durga Nagarajan, Hossein Rashmanlou, Farshid Mofidnakhaei, et al. m-polar quadripartitioned neutrosophic graphs with applications in decision-making for mobile network selection. *Neutrosophic Sets & Systems*, 82:458, 2025.

[151] Florentin Smarandache. *Introduction to the multineutrosophic set*. Infinite Study, 2023.

[152] Florentin Smarandache. *Nidus Idearum. Scilogs, XIII: Structure/NeutroStructure/AntiStructure*. Infinite Study, 2024.

[153] A Iampan, T Eswarlal, et al. Bipolar vague relations and their application to real-world problems. *IAENG International Journal of Computer Science*, 51(9):1316, 2024.

[154] U Venkata Kalyani and T Eswarlal. Bipolar vague cosets. *Adv. Math., Sci. J*, 9:6777–6787, 2020.

[155] Abd Ulazeez MJS Alkouri and Zeyad A Alshboul. Environmental impact assessment using bipolar complex intuitionistic fuzzy sets. *Soft Computing*, 29(8):3795–3809, 2025.

[156] Deva Nithyanandham and Augustin Felix. Bipolar intuitionistic fuzzy matrices and its determinant. *TWMS Journal of Applied and Engineering Mathematics*, 2024.

[157] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and Rajkumar Verma. Computing minimum spanning tree in interval valued bipolar neutrosophic environment. *Collected Papers. Volume VII: On Neutrosophic Theory and Applications*, 7(5):143, 2022.

[158] R Balaji, Murugan Palanikumar, and Aiyared Iampan. Characterization of bipolar neutrosophic sets to novel concept of complex q bipolar neutrosophic sets using bisemirings. *International Journal of Neutrosophic Science,(2)*, pages 233–33, 2025.

[159] Aliya Fahmi, Aziz Khan, and Thabet Abdeljawad. A novel approach to group decision-making using generalized bipolar neutrosophic sets. *PLoS One*, 20(6):e0317746, 2025.

[160] Muhammad Akram, Musavarah Sarwar, Wieslaw A Dudek, Muhammad Akram, Musavarah Sarwar, and Wieslaw A Dudek. Bipolar neutrosophic graph structures. *Graphs for the Analysis of Bipolar Fuzzy Information*, pages 393–446, 2021.

[161] Jeonghwan Gwak, Harish Garg, Naeem Jan, and Bushra Akram. A new approach to investigate the effects of artificial neural networks based on bipolar complex spherical fuzzy information. *Complex & Intelligent Systems*, 9(4):4591–4614, 2023.

[162] Adem Yolcu. Bipolar spherical fuzzy soft topology with applications to multi-criteria group decision-making in buildings risk assessment. *Symmetry*, 14(11):2362, 2022.

[163] Waheed Ahmad Khan, Khurram Faiz, and Abdelghani Taouti. Bipolar picture fuzzy sets and relations with applications. *Songklanakarin Journal of Science & Technology*, 44(4):987, 2022.

[164] Waheed Ahmad Khan, Hajra Begum, Trung Tuan Nguyen, Minh Hoan Pham, and Hai Van Pham. A study of competitions in different fields through graphs under bipolar picture fuzzy environment. *Plos one*, 20(10):e0329192, 2025.

[165] Naeem Jan, Bushra Akram, Abdul Nasir, Mohsin S Alhilal, Amerah Alabrah, and Naziha Al-Aidroos. An innovative approach to investigate the effects of artificial intelligence based on complex bipolar picture fuzzy information. *Scientific Programming*, 2022(1):1460544, 2022.

[166] Swarup Jana and Sahidul Islam. Solving multi-objective optimization problem in bipolar hesitant fuzzy environment. *RAIRO-Operations Research*, 59(6):4023–4049, 2025.

[167] Nikita Prajapati and Gautam Patel. Extending hesitant fuzzy graphs to bipolar hesitant frameworks for capturing positive and negative uncertainty. *International Journal of Computer Information Systems and Industrial Management Applications*, 18(8s):678–688, 2026.

[168] Rakhal Das, Suman Das, and Florentin Smarandache. *Neutrosophic Super Hyper Set: Theory, Algebraic Structures, Topology, and Interdisciplinary Applications*. Infinite Study, 2026.

[169] Sarapee Chairat and Somsak Lekkoksung. On tripolar fuzzy soft sets and their application in ordered semigroups. *International Journal of Analysis and Applications*, 24:232–232, 2026.

[170] Anothai Phukhaengsi, Pannawit Khamrot, and Thiti Gaketem. Tripolar fuzzy (m, n)-quasi-ideals of ordered semigroups. *International Journal of Analysis and Applications*, 24:156–156, 2026.

[171] Tanaphong Prommai, Pannawit Khamrot, and Thiti Gaketem. Tripolar fuzzy quasi-ideals in semigroups. *IAENG International Journal of Computer Science*, 53(5):1770, 2026.

[172] Pannawit Khamrot, Aiyared Iampan, and Thiti Gaketem. Tripolar fuzzy interior ideals in semigroups. *IAENG International Journal of Applied Mathematics*, 56(1):249–256, 2026.

[173] Takaaki Fujita. A mathematical study of tripolar intuitionistic fuzzy graphs. *International Journal of Computer Application (IJCA)*, 16(4):16/1–16/11, 2026.

[174] Florentin Smarandache. *Nidus idearum. Scilogs, III: Viva la Neutrosophia!* Pons, Brussels, 2017.

[175] Takaaki Fujita. A mathematical study of tripolar neutrosophic graphs. *International Journal of Research in Engineering & Science (IJRES)*, 16(4):13/1–13/8, 2026.

[176] Ganesh Ghorai and Madhumangal Pal. A study on m-polar fuzzy planar graphs. *International Journal of Computing Science and Mathematics*, 7(3):283–292, 2016.

[177] OSMAN Kazanci and S Hoskova-Mayerova. Multipolar (m, n)-fuzzy set and its applications to semihypergroups. *Journal of Algebraic Hyperstructures and Logical Algebras*, 6(2):177–195, 2025.

[178] Deivanai Jaisankar and Sujatha Ramalingam. A multipolar fuzzy outerplanar graph approach to one-way road network construction. *Journal of Computational Analysis & Applications*, 34(8), 2025.

[179] Zainab Usmani, Ahsan Mahboob, and Mohammad Aasim Khan. Exploring multipolar fuzzy hyperfilters in ordered semihypergroups. *New Mathematics and Natural Computation*, pages 1–20, 2025.

[180] Ganesh Ghorai and Madhumangal Pal. Faces and dual of m-polar fuzzy planar graphs. *Journal of Intelligent & Fuzzy Systems*, 31(3):2043–2049, 2016.

[181] Krishna Rani Maity and Madhumangal Pal. A novel approach on m-polar fuzzy set with dombi power aggregation operators to solve multi-attribute decision-making problems. *TWMS Journal of Applied and Engineering Mathematics*, 15(9):2313–2330, 2025.

[182] Damla Yılmaz, Hasret Yazarlı, and Young Bae Jun. m-polar fuzzy d-ideals on d-algebras. *Journal of Mathematical Extension*, 19, 2025.

[183] Zainab Usmani, Ghulam Muhiuddin, Ahsan Mahboob, and Mohammad Aasim Khan. Interval valued m-polar fuzzy ideals in ordered semigroups: A structural analysis. *Journal of Fuzzy Extension and Applications*, 7(2):332–346, 2026.

[184] Muhammad Saqlain and Muhammad Saeed. From ambiguity to clarity: unraveling the power of similarity measures in multi-polar interval-valued intuitionistic fuzzy soft sets. *Decision Making Advances*, 2(1):48–59, 2024.

[185] Iuliana Carmen BĂRBĂCIORU. On some operations of m-polar intuitionistic fuzzy graphs. *Annals of'Constantin Brancusi'University of Targu-Jiu. Engineering Series/Analele Universităţii Constantin Brâncuşi din Târgu-Jiu. Seria Inginerie*, (4):27, 2020.

[186] Rajab Ali Borzooei, Hee Sik Kim, Young Bae Jun, and Sun Shin Ahn. On multipolar intuitionistic fuzzy b-algebras. *Mathematics*, 8(6):907, 2020.

[187] Asma Mahmood, Mujahid Abbas, and Ghulam Murtaza. Multi-valued multi-polar neutrosophic sets with an application in multi-criteria decision-making. *Neutrosophic sets and systems*, 53(1):32, 2023.

[188] Atiqa Siraj, Khalid Naeem, and Broumi Said. Pythagorean m-polar fuzzy neutrosophic metric spaces. *Neutrosophic Sets and Systems*, 53(1):33, 2023.

[189] Asmaa Ghasoob Raoof, Qays Hatem Imran, S Khalil, Ali Younis Shakir, et al. On new classes of supra soft connectedness via m-polar neutrosophic topological spaces. *Neutrosophic Sets and Systems*, 94(1):12, 2025.

[190] Warud Nakkhasen, Atthchai Chada, and Teerapan Jodnok. m-polar picture fuzzy bi-ideals and their applications in semigroups. *Symmetry*, 17(12):2051, 2025.

[191] Shovan Dogra and Madhumangal Pal. m-polar picture fuzzy ideal of a bck algebra. *International Journal of Computational Intelligence Systems*, 13(1):409–420, 2020.

[192] Rukhsana Kausar, Shaista Tanveer, Muhammad Riaz, Dragan Pamucar, and Cirovic Goran. Topological data analysis of m-polar spherical fuzzy information with lam and sir models. *Symmetry*, 14(10):2216, 2022.

[193] Muhammad Riaz, Maryam Saba, Muhammad Abdullah Khokhar, and Muhammad Aslam. Medical diagnosis of nephrotic syndrome using m-polar spherical fuzzy sets. *International Journal of Biomathematics*, 15(02):2150094, 2022.

[194] Muhammad Akram and Arooj Adeel. Introducing hesitancy: Topsis and electre-i models. In *Multiple Criteria Decision Making Methods with Multi-polar Fuzzy Information: Algorithms and Applications*, pages 157–235. Springer, 2023.

[195] Takaaki Fujita and Arif Mehmood. Extending classical uncertainty models via hyperpolar structures: Fuzzy, neutrosophic, and soft set perspectives. *Galoitica: Journal of Mathematical Structures and Applications*, 12(2):24–39, 2025.

[196] Wolfhard Hansen and Ivan Netuka. Hunt's hypothesis (h) and triangle property of the green function. *Expositiones Mathematicae*, 34(1):95–100, 2016.

[197] Ze-Chun Hu and Wei Sun. Hunt's hypothesis (h) for the sum of two independent lévy processes. *Communications in Mathematics and Statistics*, 6(2):227–247, 2018.

[198] Ze-Chun Hu and Wei Sun. Hunt's hypothesis (h) and getoor's conjecture for lévy processes. *Stochastic Processes and their Applications*, 122(6):2319–2328, 2012.

[199] Eric Bedford and BA Taylor. A new capacity for plurisubharmonic functions. *Acta Mathematica*, 149:1–40, 1982.

[200] B Drinovec Drnovšek. Proper discs in stein manifolds avoiding complete pluripolar sets. *Math. Res. Lett*, 11(5-6):575–581, 2004.

[201] Mihnea Coltoiu. Complete locally pluripolar sets. *Journal für die reine und angewandte Mathematik*, 405:108–112, 1990.

[202] Pham Hiep. Pluripolar hulls and complete pluripolar sets. *Potential Analysis*, 2008.

[203] Nguyen Quang Dieu and Tang Van Long. A new class of pluripolar sets. In *Annales Polonici Mathematici*, volume 90, pages 229–245. Instytut Matematyczny Polskiej Akademii Nauk, 2007.

[204] Benoît Kloeckner. Polarit\'es d\'efinies par un triangle. *arXiv preprint arXiv:1302.1813*, 2013.

[205] Haifei Huang, Hui Zhang, and Yiu-ming Cheung. The common self-polar triangle of concentric circles and its application to camera calibration. In *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*, pages 4065–4072, 2015.

[206] Yang Guo. A minimal closed-form solution to the conic based on self-polar triangle. *arXiv preprint arXiv:1801.02751*, 2018.

[207] Charles Kittel. Theory of antiferroelectric crystals. *Physical Review*, 82(5):729, 1951.

[208] Kenkichi Okada. Phenomenological theory of antiferroelectric transition. iii. phase diagram and bias effects of first-order transition. *Journal of the Physical Society of Japan*, 37(5):1226–1232, 1974.

[209] Pablo Vales-Castro, Krystian Roleder, Lei Zhao, Jing-Feng Li, Dariusz Kajewski, and Gustau Catalan. Flexoelectricity in antiferroelectrics. *Applied Physics Letters*, 113(13), 2018.

[210] Arthur Ludwig. The definition of cross polarization. *IEEE Transactions on Antennas and Propagation*, 21(1):116–119, 1973.

[211] JE Roy and L Shafai. Generalization of the ludwig-3 definition for linear copolarization and cross polarization. *IEEE Transactions on Antennas and Propagation*, 49(6):1006–1010, 2001.

[212] Nafati A Aboserwal, Jorge L Salazar, Javier A Ortiz, José D Díaz, Caleb Fulton, and Robert D Palmer. Source current polarization impact on the cross-polarization definition of practical antenna elements: Theory and applications. *IEEE Transactions on Antennas and Propagation*, 66(9):4391–4406, 2018.

[213] Erdal Arikan. Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels. *IEEE Transactions on information Theory*, 55(7):3051–3073, 2009.

[214] Erdal Arikan and Emre Telatar. On the rate of channel polarization. In *2009 IEEE International Symposium on Information Theory*, pages 1493–1495. IEEE, 2009.

[215] George Gabriel Stokes. On the composition and resolution of streams of polarized light from different sources. *Transactions of the Cambridge Philosophical Society*, 9:399, 1851.

[216] M Al Tahan and Bijan Davvaz. Weak chemical hyperstructures associated to electrochemical cells. *Iranian Journal of Mathematical Chemistry*, 9(1):65–75, 2018.

[217] Gulay Oguz and Bijan Davvaz. Soft topological hyperstructure. *J. Intell. Fuzzy Syst.*, 40:8755–8764, 2021.

[218] P Kamporoudi and T Vougiouklis. Cyclicity in some classes of hv-groups. *Journal of Algebraic Hyperstructures and Logical Algebras*, 1(3):91–97, 2020.

[219] Fakhry Asad Agusfrianto, Sonea Andromeda, and Mariam Hariri. Hyperstructures in chemical hyperstructures of redox reactions with three and four oxidation states. *JTAM (Jurnal Teori dan Aplikasi Matematika)*, 8(1):50, 2024.

[220] Sang-Cho Chung. Chemical hyperstructures for ozone depletion. *Journal of the Chungcheong Mathematical Society*, 32(4):491–508, 2019.

[221] Florentin Smarandache. *SuperHyperFunction, SuperHyperStructure, Neutrosophic SuperHyperFunction and Neutrosophic SuperHyperStructure: Current understanding and future directions.* Infinite Study, 2023.

[222] Marzieh Rahmati and Mohammad Hamidi. Extension of g-algebras to superhyper g-algebras. *Neutrosophic Sets and Systems*, 55:557–567, 2023.

[223] Prabakaran Raghavendran and Tharmalingam Gunasekar. Optimizing organ transplantation success using neutrosophic superhyperstructure and artificial intelligence. *New Trends in Neutrosophic Theories and Applications Volume IV*, page 117, 2025.

[224] T. Fujita and F. Smarandache. Competition super-hypergraphs: Revealing hierarchical competition in real-world networks. *Journal of Algebra and Applied Mathematics*, 23(2):97–116, 2025.

[225] Masoud Ghods, Zahra Rostami, and Florentin Smarandache. Introduction to neutrosophic restricted superhypergraphs and neutrosophic restricted superhypertrees and several of their properties. *Neutrosophic Sets and Systems*, 50:480–487, 2022.

[226] Takaaki Fujita and Florentin Smarandache. *Representing higher-order networks: A survey of graph-based frameworks*. Infinite Study, 2026.

[227] Florentin Smarandache. *Introduction to the n-SuperHyperGraph-the most general form of graph today*. Infinite Study, 2022.

[228] Florentin Smarandache. Foundation of superhyperstructure & neutrosophic superhyperstructure. *Neutrosophic Sets and Systems*, 63(1):21, 2024.

[229] Muhammad Akram, Neha Waseem, and Wieslaw A Dudek. Certain types of edge m-polar fuzzy graphs. *Iranian Journal of fuzzy systems*, 14(4):27–50, 2017.

[230] Uttam Mondal, Tanmoy Mahapatra, Qin Xin, and Madhumangal Pal. Generalized m-polar fuzzy planar graph and its application. *IEEE Access*, 11:138399–138413, 2023.

[231] Ghulam Muhiuddin, Tanmoy Mahapatra, Madhumangal Pal, Ohoud Alshahrani, and Ahsan Mahboob. Integrity on m-polar fuzzy graphs and its application. *Mathematics*, 11(6):1398, 2023.

[232] Muhammad Akram, Saba Siddique, and Majed G Alharbi. Clustering algorithm with strength of connectedness for m-polar fuzzy network models. *Mathematical Biosciences and Engineering*, 19(1):420–455, 2022.

[233] Sovan Samanta and Madhumangal Pal. Irregular bipolar fuzzy graphs. *ArXiv*, abs/1209.1682, 2012.

[234] Hai-Long Yang, Sheng-Gang Li, Wen-Hua Yang, and You Lu. Notes on “bipolar fuzzy graphs”. *Information Sciences*, 242:113–121, 2013.

[235] Basheer Ahamed Mohideen. Perfect bipolar fuzzy graphs. *J. Math. Comput. Sci*, 19(2):120–128, 2019.

[236] Said Broumi, Assia Bakali, Mohamed Talea, and Florentin Smarandache. Generalized bipolar neutrosophic graph of type 1. *Infinite Study*, 2017.

[237] Said Broumi, Assia Bakali, Mohamed Talea, Florentin Smarandache, and Rajkumar Verma. Computing minimum spanning tree in interval valued bipolar neutrosophic environment. *International Journal of Modeling and Optimization*, 7(5):300–304, 2017.

[238] Abdulaziz Mohammed Alanazi, Ghulam Muhiuddin, Tanmoy Mahapatra, Zaid Bassfar, and Madhumangal Pal. Inverse graphs in m-polar fuzzy environments and their application in robotics manufacturing allocation problems with new techniques of resolvability. *Symmetry*, 15(7):1387, 2023.

[239] Abdulaziz M Alanazi, Ghulam Muhiuddin, Bashair M Alenazi, Tanmoy Mahapatra, and Madhumangal Pal. Utilizing m-polar fuzzy saturation graphs for optimized allocation problem solutions. *Mathematics*, 11(19):4136, 2023.

[240] Zaman T Samer. On the algebraic structure of neutrosophic ideal-open sets in topological spaces. *Sphinx Journal of Mathematical Sciences*, 1(1):37–47, 2026.

[241] Amr Elrawy, Florentin Smarandache, and Susan F El-Deken. Introduction and application of the neutrosophic ideal in gamma-near-ring. *Neutrosophic Sets and Systems*, 83(1):27, 2025.

[242] Shahida Bashir, Talal Alharbi, Rabia Mazhar, Issra Khalid, Muneeb ul Hassan Afzal, and Nauman Riaz Chaudhry. An efficient approach to study multi-polar fuzzy ideals of semirings. *Scientific Reports*, 14(1):2446, 2024.

[243] Shahida Bashir, Mohammed M Ali Al-Shamiri, Shahzeen Khalid, and Rabia Mazhar. Regular and intra-regular ternary semirings in terms of m-polar fuzzy ideals. *Symmetry*, 15(3):591, 2023.

[244] Nasim Abdolmaleki, M. Ahmadi, Hadi Tabatabaee Malazi, and Sebastiano Milardo. Fuzzy topology discovery protocol for sdn-based wireless sensor networks. *Simul. Model. Pract. Theory*, 79:54–68, 2017.

[245] Kimfung Liu, Wenzhong Shi, and Hua Zhang. A fuzzy topology-based maximum likelihood classification. *Isprs Journal of Photogrammetry and Remote Sensing*, 66:103–114, 2011.

[246] Anil Kr Ram and Anupam K Singh. A neutrosophic topology for neutrosophic automata. *SuperHyperTopologies and SuperHyperStructures with their Applications*, page 253, 2025.

[247] Carlos Granados. Neutrosophic statistical convergence in neutrosophic topology. *Afrika Matematika*, 36(1):39, 2025.

[248] B Kalaiselvi, K Sivakumar, S Chandrasekar, P Kalarani, and A Kesavan. Advancing neutrosophic topology through gamma generalized alpha closed sets. *Neutrosophic Sets and Systems*, 98(1):2, 2026.

[249] Karrar Saad Hamzah and Samer R Yaseen. On core-dominated closed sets in neutrosophic topology. *Gulf Journal of Mathematics*, 23(1), 2026.

[250] K Lalithambigai and P Gnanachandra. Induced neutrosophic topologies from neutrosophic graph coloring: A rigorous approach to approximation under indeterminacy. *Boletim da Sociedade Paranaense de Matemática*, 43(3), 2025.

[251] Khalid Naeem, Muhammad Riaz, Xindong Peng, and Deeba Afzal. Pythagorean m-polar fuzzy topology with topsis approach in exploring most effectual method for curing from covid-19. *International Journal of Biomathematics*, 13(08):2050075, 2020.

[252] Muhammad Riaz, Khadija Akmal, Yahya Almalki, SA Alblowi, M Riaz, K Akmal, Y Almalki, and SA Alblowi. Cubic m-polar fuzzy topology with multi-criteria group decision-making. *AIMS Mathematics*, 7(7):13019–13052, 2022.

[253] Zainab Zainab, Ghulam Muhiuddin, Ahsan Mahboob Mahboob, and Mohammad Aasim Khan. Interval valued m-polar fuzzy ideals in ordered semigroups: A structural analysis. *Journal of Fuzzy Extension and Applications*, 7(2):332–346, 2026.

[254] Wen-Ran Zhang. Yinyang bipolar lattices and l-sets for bipolar knowledge fusion, visualization, and decision. *International Journal of Information Technology & Decision Making*, 4(04):621–645, 2005.

[255] U Venkata Kalyani, Aiyared Iampan, and T Eswarlal. Bipolar fuzzy magnified translation of a lattice. *International Journal of Analysis and Applications*, 22:195–195, 2024.

[256] U Venkata Kalyani, T Eswarlal, J KaviKumar, and A Iampan. Bipolar fuzzy sublattices and ideals. *International Journal of Analysis and Applications*, 20:45–45, 2022.

[257] Bikky Kumar and Anupam K Singh. An operator-oriented bipolar fuzzy automata. *New Mathematics and Natural Computation*, pages 1–20, 2026.

[258] Bikky Kumar, Anupam K Singh, and Anil Kumar Ram. Properties of bipolar fuzzy automata. *New Mathematics and Natural Computation*, 20(03):583–600, 2024.

[259] Anupam K Singh. Bipolar fuzzy preorder, alexandrov bipolar fuzzy topology and bipolar fuzzy automata. *New Mathematics and Natural Computation*, 15(03):463–477, 2019.

[260] Pradip Kumar Maji, Ranjit Biswas, and A Ranjan Roy. Soft set theory. *Computers & mathematics with applications*, 45(4-5):555–562, 2003.

[261] Jinta Jose, Bobin George, and Rajesh K Thumbakara. Soft graphs: A comprehensive survey. *New Mathematics and Natural Computation*, pages 1–52, 2024.

[262] Atiqe Ur Rahman, Florentin Smarandache, Muhammad Saeed, and Khuram Ali Khan. Development of some new hybrid structures of hypersoft set with possibility-degree settings. In *Neutrosophic and Plithogenic Inventory Models for Applied Mathematics*, pages 337–386. IGI Global Scientific Publishing, 2025.

[263] Sikkanan Sudha, Ravi Priya, Nivetha Martin, Said Broumi, and Florentin Smarandache. Combined plithogenic hypersoft sets in decision making on supplier selection with different mcdm approaches. *Engineering Review: Međunarodni časopis namijenjen publiciranju originalnih istraživanja s aspekta analize konstrukcija, materijala i novih tehnologija u području strojarstva, brodogradnje, temeljnih tehničkih znanosti, elektrotehnike, računarstva i građevinarstva*, 44(4-SI 2024):122–140, 2024.

[264] Takaaki Fujita and Florentin Smarandache. *A Comprehensive Survey of Soft Graphs and Rough Graphs*. Neutrosophic Science International Association (NSIA) Publishing House, 2026.

[265] Atiqe Ur Rahman, Muhammad Saeed, Ebenezer Bonyah, and Muhammad Arshad. Graphical exploration of generalized picture fuzzy hypersoft information with application in human resource management multiattribute decision-making. *Mathematical Problems in Engineering*, 2022(1):6435368, 2022.

[266] Atiqe Ur Rahman, Diaa S. Metwally, M. M. Abd El-Raouf, M. A. El-Qurashi, Hamiden Abd El-Wahed Khalifa, and Khadiga Wadi Nahar Tajer. Application in multi-attribute decision-making using possibility single-valued neutrosophic hypersoft graphs. *Journal of Statistics Applications & Probability*, 15(1):69–96, 2026.

[267] Santiago Salomon Chicaiza-Toapaxi, Anthony Alexander Ortega-Toapanta, and Héctor Luis Laurencio-Alfonso. Analysis of hydraulic behavior in pressurized systems with different automation mechanisms using neutrosophic superhypersoft sets. *Neutrosophic Sets and Systems*, 89(1):6, 2025.

[268] Mona Mohamed, Ahmed M AbdelMouty, Khalid Mohamed, and Florentin Smarandache. Superhypersoft-driven evaluation of smart transportation in centroidous-moosra: Real-world insights for the uav era. *Neutrosophic Sets and Systems*, 78:149–163, 2025.

[269] T Kiruthika, M Karpagadevi, S Krishnaprakash, and G Deepa. Superhypersoft sets using python and its applications in neutrosophic superhypersoft sets under topsis method. *Neutrosophic Sets and Systems*, 91(1):41, 2025.

[270] Takaaki Fujita, Ajoy Kanti Das, Arif Mehmood, Suman Das, and Volkan Duran. Decision analytics applications of the relationship between treesoft graphs and forestsoft graphs. *Applied Decision Analytics*, 2(1):73–92, 2026.

[271] M Myvizhi, Ahmed A Metwaly, and Ahmed M Ali. Treesoft approach for refining air pollution analysis: A case study. *Neutrosophic Sets and Systems*, 68(1):17, 2024.

[272] Takaaki Fujita, Raed Hatamleh, and Ahmed Salem Heilat. Contrasoft set and contrarough set with using upside-down logic. *Statistics, Optimization & Information Computing*, 14(6):3589–3632, 2025.

[273] Noreen Mushtaq, Wajid Ali, Afshan Qayyum, Faiza Tufail, and Amal Kumar Adak. An algebraic framework for generalized interval-valued intuitionistic fuzzy soft expert sets. *Intelligent Systems Research and Applications Journal*, 2:399–427, 2026.

[274] Hazwani Hashim, Nur Syahida Mohd Noor, Noor Azzah Awang, Ashraf Al-Quran, and Lazim Abdullah. An integrated topsis model with neutrosophic vague soft expert sets for uncertain decision making. *International Journal of Analysis and Applications*, 24:218–218, 2026.

[275] Takaaki Fujita and Florentin Smarandache. A dynamic survey of soft set theory and its extensions. *arXiv preprint arXiv:2602.21268*, 2026.

[276] Qonita Qurratu Aini, Imam Mukhlash, Fatia Fatimah, and Dian Winda Setyawati. Energy of intuitionistic fuzzy soft set for multi-criteria decision-making. In *International Conference on Intelligent and Fuzzy Systems*, pages 399–406. Springer, 2025.

[277] Orhan Dalkılıç. A stability-oriented virtual fuzzy soft set framework inspired by topological perspectives for decision analysis. *International Journal of Systems Science*, pages 1–17, 2026.

[278] Sania Saleem, Maria Riaz, Fatima Razaq, and Muhammad Saeed. Dynamic intuitionistic fuzzy soft set-based waspas model for multi-criteria decision making in renewable energy selection. *Argumentation Based Systems Journal*, 2:208–235, 2026.

[279] Muhammad Saeed and Fatima Razaq. A dynamic circular intuitionistic fuzzy soft set framework for temporal multi-criteria decision making. *Journal of Innovative Research in Mathematical and Computational Sciences*, 5(1), 2026.

[280] Esraa Masadeh, Khaleed Alhazaymeh, Abedallah Al-shboul, Waleed Mohammed Abdelfattah, Rana Muhammad Zulqarnain, Imran Siddique, and Hijaz Ahmad. An extended topsis technique in cubic vague soft set relations with its application in renewable energy sources. *European Journal of Pure and Applied Mathematics*, 18(4):7017–7017, 2025.

[281] Mithun Datta, Kalyani Debnath, and Surapati Pramanik. Pentapartitioned neutrosophic soft set with interval membership. *Neutrosophic Sets and Systems*, 79(1):37, 2025.

[282] Irfan Deli. Refined neutrosophic sets and refined neutrosophic soft sets: theory and applications. In *Handbook of research on generalized and hybrid set structures and applications for soft computing*, pages 321–343. IGI Global, 2016.

[283] Mumtaz Ali, Le Hoang Son, Irfan Deli, and Nguyen Dang Tien. Bipolar neutrosophic soft sets and applications in decision making. *J. Intell. Fuzzy Syst.*, 33:4077–4087, 2017.

[284] Fuqiang Wang, Xihua Li, and Xiaohong Chen. Hesitant fuzzy soft set and its applications in multicriteria decision making. *Journal of Applied Mathematics*, 2014(1):643785, 2014.

[285] Xi Wen. Weighted hesitant fuzzy soft set and its application in group decision making. *Granular Computing*, 8(6):1583–1605, 2023.

[286] Sreelekshmi C Warrier, Terry Jacob Mathew, Nellimala Abdul Shukoor, and Vijayakumar Varadarajan. Hesitant fuzzy hyper soft set for decision making. In *AIP Conference Proceedings*, volume 3134, page 020018. AIP Publishing LLC, 2024.

[287] Muhammad Saeed and Muhammad Imran Harl. Fundamentals of picture fuzzy hypersoft set with application. *Neutrosophic Sets Syst*, 53(1):382–400, 2023.

[288] Muhammad Jabir Khan, Supak Phiangsungnoen, Wiyada Kumam, et al. Applications of generalized picture fuzzy soft set in concept selection. *Thai Journal of Mathematics*, 18(1):296–314, 2020.

[289] Ali Asghar, Khuram A Khan, Marwan A Albahar, and Abdullah Alammari. An optimized multi-attribute decision-making approach to construction supply chain management by using complex picture fuzzy soft set. *PeerJ Computer Science*, 9:e1540, 2023.

[290] Fathima Perveen PA, Jacob John Sunil, KV Babitha, and Harish Garg. Spherical fuzzy soft sets and its applications in decision-making problems. *Journal of Intelligent & Fuzzy Systems*, 37(6):8237–8250, 2019.

[291] Tie Hou, Zheng Yang, Yanling Wang, Hongliang Zheng, Li Zou, and Luis Martínez. Linguistic interval-valued spherical fuzzy soft set and its application in decision making. *Applied Sciences*, 14(3):973, 2024.

[292] Reefan Mosallam Almozaini and Kholood Mohammad Alsager. Interval-valued spherical fuzzy soft rough sets: A hybrid framework for multi-criterion group decision-making. *Symmetry*, 18(7):1090, 2026.

[293] Imam Mukhlash, Qonita Qurratu Aini, Muhammad Luthfi Shahab, Osman Anis, and Madeleine Al-Tahan. Plithogenic soft set models based on neutrosophic and intuitionistic fuzzy sets for multi-attribute time series forecasting. *International Journal of Fuzzy Logic and Intelligent Systems*, 26(1):116–127, 2026.

[294] Muhammad Akram, Ghous Ali, and José Carlos R Alcantud. Parameter reduction analysis under interval-valued m-polar fuzzy soft information. *Artificial Intelligence Review*, 54(7):5541–5582, 2021.

[295] Mohammed M Khalaf. Algorithms and optimal choice for power plants based on m-polar fuzzy soft set decision making criterions. *Acta Electronica Malaysia*, 4(1):11–23, 2020.

[296] Muhammad Saeed, Muhammad Saqlain, Asad Mehmood, Khushbakht Naseer, and Sonia Yaqoob. Multi-polar neutrosophic soft sets with application in medical diagnosis and decision-making. *Neutrosophic Sets and Systems*, 33(1):13, 2020.

[297] Rana Muhammad Zulqarnain, Imran Siddique, Aiyared Iampan, and Ebenezer Bonyah. Algorithms for multipolar interval-valued neutrosophic soft set with information measures to solve multicriteria decision-making problem. *Computational Intelligence and Neuroscience*, 2021(1):7211399, 2021.

[298] Limin Wu. Multipolar interval-valued neutrosophic soft set for integrating sustainability into logistics: A performance-based evaluation of green supply chains. *Neutrosophic Sets and Systems*, 88(1):65, 2025.

[299] Zdzisław Pawlak. Rough sets. *International journal of computer & information sciences*, 11:341–356, 1982.

[300] Zdzislaw Pawlak, S. K. Michael Wong, Wojciech Ziarko, et al. Rough sets: probabilistic versus deterministic approach. *International Journal of Man-Machine Studies*, 29(1):81–95, 1988.

[301] Zdzisław Pawlak and Andrzej Skowron. Rudiments of rough sets. *Information sciences*, 177(1):3–27, 2007.

Polar Concepts in Uncertainty Theory: Fuzzy, Neutrosophic, and Uncertain Set Frameworks

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Abstract

Fuzzy sets, neutrosophic sets, and uncertain sets have been generalized and studied from various perspectives, including bipolar, tripolar, and m-polar frameworks. In this book, we introduce and investigate several polarity-based extensions of uncertain sets within a typed mathematical setting. We distinguish recalled notions from book-specific lifts, prove representation results identifying constructions that flatten to ordinary function spaces, and impose explicit coherence conditions on structural, dynamic, and physically motivated models. The resulting framework clarifies which constructions add genuine structure and which are alternative indexings of the same degree data.

Keywords: Fuzzy Set, MultiPolar Fuzzy Set, Uncertain Set, MultiPolar Uncertain Set

ABSTRACT

This book develops a unified mathematical framework for polarity-based representations in uncertainty theory. Moving beyond conventional bipolar, tripolar, and multipolar models, it examines polarity as an organizing principle for expressing multiple viewpoints, opposition, hierarchy, geometric orientation, network structure, contextual variation, heterogeneity, and temporal change.

The study introduces typed polar uncertain sets and investigates representation results, coherence conditions, structural invariants, and applications to graphs, algebraic systems, topology, lattices, automata, soft sets, and rough sets. The resulting framework provides a rigorous foundation for comparing polarity-based models and developing new methods in uncertainty representation and decision analysis.

ISBN 978-197250254-9

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