NEUTROSOPHIC LATTICE, NEUTROSOPHIC OPERATORS, AND TOTAL ORDER ON THE NEUTROSOPHIC TRIPLETS

— As Answers to Rivieccio's Critics —

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Preface

This volume brings together three interconnected contributions to the mathematical foundations of neutrosophic theory, written to consolidate the algebraic and order-theoretic apparatus of neutrosophic logic and to respond directly to recent critical remarks raised by Umberto Rivieccio [13] concerning the operators and ordering structures used in neutrosophic reasoning.

The first chapter, "Neutrosophic, Refined Neutrosophic, Over/Under/Off-Neutrosophic, and Refined Over/Under/Off-Neutrosophic Lattices," develops a full lattice-theoretic treatment of the set of neutrosophic triplets (T, I, F). Grounded in the Score–Accuracy–Certainty (SAC) total order, it constructs chain lattices on the unit cube, on the extended over/under/off domain, and on their refined (n-valued) counterparts, together with a geometric projection theorem that interprets the lexicographic SAC order as a foliation of the neutrosophic cube into score planes, accuracy lines, and certainty points. The chapter closes by situating these new lattices among classical structures — Boolean, fuzzy, intuitionistic fuzzy, Łukasiewicz–Moisil, product, and concept lattices — and by surveying applications ranging from multi-criteria decision making to quantum state ordering and theory selection.

The second chapter, "Neutrosophic and Plithogenic Operators: Negation, Intersection, Union, Implication, Equivalence," returns to first principles and re-derives the basic connectives of neutrosophic and plithogenic logic from triangular norms and conorms. By making explicit how t-norms act on the truth and falsehood components while t-conorms (or their averages, in the plithogenic case) act on indeterminacy, the chapter gives a uniform and transparent construction of negation, conjunction, disjunction, implication, and equivalence for both neutrosophic and plithogenic settings.

The third chapter, "The Score, Accuracy, and Certainty Functions Determine a Total Order on the Set of Neutrosophic Triplets (T, I, F)," reproduced here from Neutrosophic Sets and Systems, Vol. 38 (2020), supplies the technical backbone on which the first chapter's lattice constructions rest. It proves, for single-valued, interval-valued, and subset-valued neutrosophic triplets, that the score, accuracy, and certainty functions taken together determine a genuine total order, and it shows the equivalence of this approach with the possibility-degree method of Xu and Da and with the interval neutrosophic functions of Zhang, Wang, and Chen.

Read together, the three chapters answer a natural methodological question that Rivieccio's critique [13] brings into focus: on what grounds can one impose a total, lattice-compatible order on inherently three-dimensional, only partially ordered neutrosophic data, and how do the resulting operators and order relations connect back to the classical algebraic structures — Boolean algebras, fuzzy lattices, intuitionistic fuzzy sets, and Łukasiewicz–Moisil algebras — that a rigorous generalization must remain compatible with. The Score–Accuracy–Certainty order, proved total in Chapter 3, is the device that makes this possible; Chapter 1 shows what it buys us in lattice theory and geometry; Chapter 2 shows how it interacts with the logical connectives built from t-norms and t-conorms.

It is our hope that this compact volume will be useful both to readers encountering neutrosophic algebra for the first time and to specialists seeking a self-contained reference for the SAC order, its lattice consequences, and the operator calculus built upon it.

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Chapter 1

Neutrosophic, Refined Neutrosophic, Over/Under/Off-Neutrosophic, and Refined Over/Under/Off-Neutrosophic Lattices

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Abstract

We build a Neutrosophic Lattice on the set of triplets (T, I, F), where T, I, F are in [0, 1], under ∧, ∨, and ¬. On the set (T, I, F) there is a total order using the Score, Accuracy, and Certainty Functions. We ground the construction in the linked Score–Accuracy–Certainty source, then give a clean lattice definition with meet, join, negation, and order properties.

Keywords: Neutrosophic Lattice, Refined Neutrosophic Lattice, Over/Under/Off-Neutrosophic Lattice, Refined Over/Under/Off-Neutrosophic Lattice

1. Introduction

We develop quite a few new ideas:

- Neutrosophic and Refined Neutrosophic Lattices,
- Over/Under/Off-Neutrosophic Lattices,
- SAC and RSAC chain orders,
- Geometric Projection and Foliation Theorems,

- Neutrosophic Cube and OUO-Cube geometry,
- Connections with classical lattice theory,
- Several promising applications and future research directions.

Define $M = [0,1]^3$, where $x = (T, I, F)$.

Use the total order from Score–Accuracy–Certainty (SAC):

$$s(T,I,F) = (2 + T - I - F)/3, \quad a(T,I,F) = T - F, \quad c(T,I,F) = T.$$

For $x, y \in M$, define $x \leq_{SAC} y$ lexicographically by:

$$x \leq_{SAC} y \iff (s(x), a(x), c(x)) \leq_{lex} (s(y), a(y), c(y)).$$

This is a total order on neutrosophic triplets, as proved in [1].

2. Lexicographic Order

There is a deeper geometric interpretation behind the lexicographic order. Suppose we have the neutrosophic cube $N = [0,1]^3$. Each point $x = (T, I, F)$ is mapped by $\Phi(x) = (S, A, C)$. The lexicographic order means that the SAC-space is explored in three successive geometric layers:

Level 1: Score Hyperplanes

First compare $S = (2 + T - I - F)/3$. The equation $T - I - F = k$ defines a plane inside the neutrosophic cube. Thus, all points having the same score lie on the same plane. So, the score partitions the cube into parallel planes.

Level 2: Accuracy Lines

Inside each score plane, A = T − F creates another family of parallel lines. Thus, if two points belong to the same score plane, accuracy determines their ordering within that plane.

Level 3: Certainty Direction

C = T. Inside each intersection, provides the final tie-breaker.

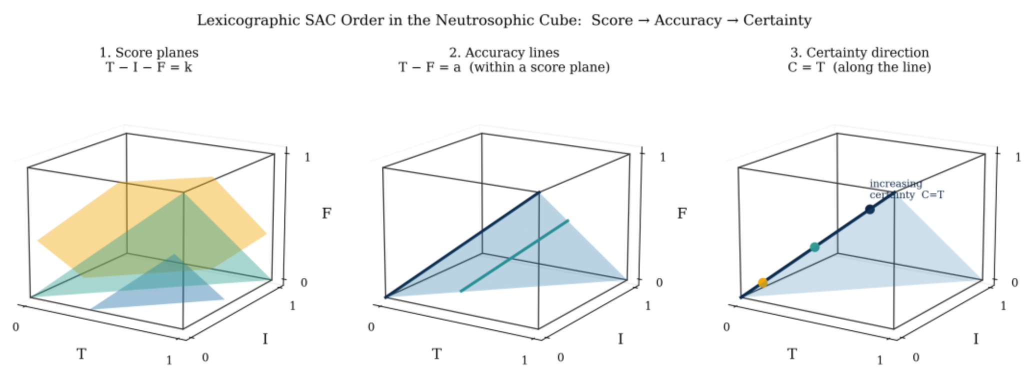

Figure 1.1 — Lexicographic Order in the Neutrosophic Cube under SAC = (Score, Accuracy, Certainty): Level 1 compares score across parallel planes T − I − F = k; Level 2 compares accuracy along parallel lines T − F = a within a score plane; Level 3 compares certainty T along a line of constant score and accuracy. Summary: Lexicographic order = compare Score (planes) → then Accuracy (lines in the plane) → then Certainty (along the line), transforming the 3-D neutrosophic cube into a totally ordered (chain) lattice under the SAC order.

3. Neutrosophic Lattice

Define $\mathcal{L}N = (M, \wedge, \vee, \neg, \leq SAC)$. For any x, y ∈ M:

$$x \wedge y = \min_{\leq SAC}\{x, y\}, \quad x \vee y = \max_{\leq SAC}\{x, y\}.$$

Thus:

$$x \wedge y = \{ x \text{ if } x \leq SAC\, y;\ \ y \text{ if } y \leq SAC\, x \}, \quad x \vee y = \{ y \text{ if } x \leq SAC\, y;\ \ x \text{ if } y \leq SAC\, x \}.$$

Negation may be defined by the usual neutrosophic exchange of truth and falsehood:

$$\neg(T, I, F) = (F, I, T). \quad \text{Therefore,} \quad \neg\neg(T, I, F) = (T, I, F).$$

Because ≤SAC is total, every pair has a meet and a join. Hence (M, ∧, ∨) is a lattice; in fact, it is a chain lattice. The bottom and top elements are:

$$0N = (0, 1, 1), \quad 1N = (1, 0, 0), \quad \text{since } s(0,1,1) = 0,\ s(1,0,0) = 1.$$

So the complete construction is:

$$\mathcal{L}N = ([0,1]^3, \wedge, \vee, \neg, \leq SAC)$$

where ∧ and ∨ are determined by the Score–Accuracy–Certainty total order.

4. Over/Under/Off-Neutrosophic Lattice

Let's build an Over/Under/Off-Neutrosophic Lattice (when values of T, I, F are outside of [0, 1], see [2]), under the same operators ∧, ∨, ¬, also totally ordered under the Score–Accuracy–Certainty functions. Define the extended universe:

$$M\Psi,\Omega = \{(T, I, F) : T, I, F \in [\Psi, \Omega]\}, \quad \text{where } \Omega < 0 < 1 < \Psi.$$

Here: $T > 1$, $I > 1$, $F > 1$ are over-components, and $T < 0$, $I < 0$, $F < 0$ are under-components. If both over- and under-components occur, we have an off-neutrosophic case. This follows Smarandache's over/under/off extension, where components may be greater than 1 or less than 0.

Use the same functions and define the total order lexicographically:

$$x \leq_{SAC} y \iff (s(x), a(x), c(x)) \leq_{lex} (s(y), a(y), c(y)).$$

The original Score–Accuracy–Certainty method gives a total order on neutrosophic triplets by comparing score first, then accuracy, then certainty.

5. Over/Under/Off-Neutrosophic Lattice

$$\mathcal{L}_{OUO} = (M_{\Psi,\Omega}, \wedge, \vee, \neg, \leq_{SAC})$$

For $x, y \in M_{\Psi,\Omega}$,

$$x \wedge y = \min_{\leq SAC} \{x, y\}, \quad x \vee y = \max_{\leq SAC} \{x, y\}.$$

Equivalently, $x \wedge y = \{ x$ if $x \leq_{SAC} y$; y if $y \leq_{SAC} x \}$, $x \vee y = \{ y$ if $x \leq_{SAC} y$; x if $y \leq_{SAC} x \}$. Negation: $\neg(T, I, F) = (F, I, T)$; then $\neg\neg(T, I, F) = (T, I, F)$. The least and greatest elements are:

$$0_{OUO} = (\Psi, \Omega, \Omega), \quad 1_{OUO} = (\Omega, \Psi, \Psi).$$

Therefore ₤OUO is a total-order lattice, i.e. a chain lattice, extending the ordinary neutrosophic lattice from $[0,1]^3$ to the over/under/off domain $[\Psi,\Omega]^3$.

Theorem 1 (Over/Under/Off-Neutrosophic Chain Lattice)

Let $M\Psi,\Omega = [\Psi,\Omega]^3$, $\Psi < 0 < 1 < \Omega$, and define the SAC order $x \leq SAC\ y \Leftrightarrow (s(x), a(x), c(x)) \leq lex\ (s(y), a(y), c(y))$ as above. Then $(M\Psi,\Omega, \leq SAC)$ is a totally ordered set.

Proof. Since the lexicographic order on $\mathbb{R}^3$ is total, for any two neutrosophic triplets x, y, either $(s(x), a(x), c(x)) \leq lex\ (s(y), a(y), c(y))$ or the reverse holds. Hence either $x \leq SAC\ y$ or $y \leq SAC\ x$. Therefore, the order is total. ■

Corollary 1

Every pair of elements has a unique meet and join: $x \wedge y = \min{\leq SAC}\{x, y\}$, $x \vee y = \max{\leq SAC}\{x, y\}$. Hence $(M\Psi,\Omega, \wedge, \vee)$ is a lattice.

Theorem 2

The lattice is distributive. Proof. Every totally ordered lattice (chain) is distributive. Therefore $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, and $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Theorem 3

The lattice is complete. Proof. For every subset $A \subseteq M\Psi,\Omega$, the set $\{(s(x), a(x), c(x)) : x \in A\}$ is totally ordered. Therefore, every subset possesses a supremum and an infimum under the SAC ordering. Hence $\mathcal{L}$OUO is a complete lattice.

6. Over/Under/Off-Neutrosophic Negation

Define $\neg(T, I, F) = (F, I, T)$. Then:

1. Involution: $\neg(\neg x) = x$.

2. Order Reversal: If $x = (T, I, F)$, then $s(\neg x) = (2 + F - I - T)/3$. One can prove that under the SAC ordering: $x \leq_{SAC} y \Rightarrow \neg y \leq_{SAC} \neg x$. Thus, negation acts as an antiautomorphism of the lattice.

Example

Let $A = (1.3, 0.2, -0.1)$ (over-truth), $B = (0.7, 1.4, 0.5)$ (over-indeterminacy), $C = (-0.2, 0.3, 1.5)$ (under-truth and over-falsehood). Scores: $s(A) = (2 + 1.3 - 0.2 + 0.1)/3 = 1.0667$, $s(B) = (2 + 0.7 - 1.4 - 0.5)/3 = 0.2667$, $s(C) = (2 - 0.2 - 0.3 - 1.5)/3 = 0$. Hence $C <_{SAC} B <_{SAC} A$. Therefore $A \wedge B = B$, $A \vee B = A$, $B \wedge C = C$, $B \vee C = B$.

This gives a rigorous Over/Under/Off-Neutrosophic Complete Distributive Chain Lattice, extending the ordinary neutrosophic lattice to oversets, undersets, and offsets while preserving the Score–Accuracy–Certainty ranking introduced in this work.

7. Refined Neutrosophic Lattice

Let's build a Refined Neutrosophic Lattice (based on the refined neutrosophic set) and similarly a Refined Neutrosophic Over/Under/Off-Set Lattice. We formulate both constructions in parallel: the refined lattice on split components and the extended over/under/off refined lattice, using the same SAC-style lexicographic idea.

Let $R = [0,1]^n$ where an element is $x = (T1, \ldots, Tp; I1, \ldots, Ir; F1, \ldots, Fs)$, with $p + r + s = n$. This follows the refined neutrosophic set idea, where T, I, F are split into subcomponents $T1,\ldots,Tp$, $I1,\ldots,Ir$, $F1,\ldots,Fs$. Define the Refined Score, Accuracy, and Certainty Functions:

$$SR(x) = [\, r + s + \Sigma Tj - \Sigma Ik - \Sigma F\ell \,] / n, \quad AR(x) = \Sigma Tj - \Sigma F\ell, \quad CR(x) = \Sigma Tj.$$

Define the total order:

$$x \leq RSAC\ y \iff (SR(x), AR(x), CR(x)) \leq lex\ (SR(y), AR(y), CR(y)).$$

Then define $x \wedge y = \min_{\leq RSAC}\{x,y\}$, $x \vee y = \max_{\leq RSAC}\{x,y\}$.
Negation:

$$\neg(T_1,\ldots,T_p; I_1,\ldots,I_r; F_1,\ldots,F_s) = (F_1,\ldots,F_s; I_1,\ldots,I_r; T_1,\ldots,T_p).$$

Hence $\mathcal{L}R = ([0,1]^n, \wedge, \vee, \neg, \leq RSAC)$ is a Refined Neutrosophic Chain Lattice.

8. Refined Over/Under/Off-Neutrosophic Lattice

Let $R\Psi,\Omega = [\Psi,\Omega]^n$, $\Psi < 0 < 1 < \Omega$. An element is again $x = (T_1,\ldots,T_p; I_1,\ldots,I_r; F_1,\ldots,F_s)$, but now each component may be < 0, $\in [0,1]$, or > 1, so it can represent refined oversets, undersets, and offsets. Using the same SR, AR, CR functions and the same RSAC order, define $x \wedge y$, $x \vee y$, and $\neg x = (F_1,\ldots,F_s; I_1,\ldots,I_r; T_1,\ldots,T_p)$ as above. Thus $\mathcal{L}ROUO = ([\Psi,\Omega]^n, \wedge, \vee, \neg, \leq RSAC)$ is a Refined Over/Under/Off-Neutrosophic Chain Lattice.

This gives four related structures: $\mathcal{L}N$ the ordinary neutrosophic lattice, $\mathcal{L}OUO$ the over/under/off-neutrosophic lattice, $\mathcal{L}R$ the refined neutrosophic lattice, and $\mathcal{L}ROUO$ the refined over/under/off-neutrosophic lattice.

9. Connections with other lattices

The lattices $\mathcal{L}$N, $\mathcal{L}$OUO, $\mathcal{L}$R, and $\mathcal{L}$ROUO are related to several well-known lattice classes.

9.1. Chain (Linear) Lattices

Since the SAC (or RSAC) order is total, $x \leq y$ or $y \leq x$ for every pair x, y. Therefore, each of our neutrosophic lattices is a chain lattice (totally ordered lattice). This immediately implies: distributivity, modularity, completeness (under suitable completeness assumptions), and semidistributivity. Thus $\mathcal{L}$R and $\mathcal{L}$ROUO belong to the classical family of chain lattices.

9.2. Relation with Boolean Lattices

Consider the crisp neutrosophic points (1,0,0) and (0,0,1). The sublattice generated by these two elements is isomorphic to the two-element Boolean algebra B2. Hence Boolean lattices appear as special cases inside neutrosophic lattices. More generally, crisp neutrosophic values (1,0,0), (0,1,0), (0,0,1) generate finite ordered substructures.

9.3. Relation with Fuzzy Lattices

For $I = 0$, $F = 1 - T$, a neutrosophic value reduces to a fuzzy value. Then $(T, 0, 1-T)$ corresponds to the fuzzy membership degree T.

Therefore, the fuzzy lattice ([0,1], min, max) embeds into $\mathcal{L}N$. Hence: Fuzzy Lattice ⊂ Neutrosophic Lattice.

9.4. Relation with Intuitionistic Fuzzy Lattices

For $T + F \leq 1$, and $I = 1 - T - F$, one obtains the intuitionistic fuzzy set of Krassimir Atanassov. Thus, the Intuitionistic Fuzzy Lattice is a sublattice of the neutrosophic lattice.

9.5. Relation with Łukasiewicz–Moisil Lattices

The refined neutrosophic lattice ($T_1,\ldots,T_p$, $I_1,\ldots,I_r$, $F_1,\ldots,F_s$) can be viewed as an n-valued extension because each component contributes an independent truth level. Hence there is a connection with LMn-algebras. The difference is that LM-algebras have one truth dimension, whereas refined neutrosophics have three families: T_i, I_j, F_k. One may regard refined neutrosophic lattices as a three-axis generalization of LM-lattices.

9.6. Relation with Product Lattices

Without SAC ordering, one could define the coordinate order $(T,I,F) \leq_c (T',I',F')$ iff $T \leq T'$, $I \leq I'$, $F \leq F'$. Then $[0,1]^3$ becomes the product lattice $[0,1] \times [0,1] \times [0,1]$. Our SAC lattice is different: it collapses the multidimensional structure into a ranking through (S, A, C). Therefore: Product Lattice ⟶ SAC Chain Lattice. This is a remarkable feature of our construction.

9.7. Relation with Neutrosophic Cubes

Geometrically, (T, I, F) is a point in the Neutrosophic Cube. The SAC function defines a projection π: (T,I,F) ↦ (S,A,C), which transforms the cube into a totally ordered chain. Thus, the lattice can be interpreted geometrically as: Neutrosophic Cube →π Ordered Chain. This geometric interpretation is developed into a theorem below.

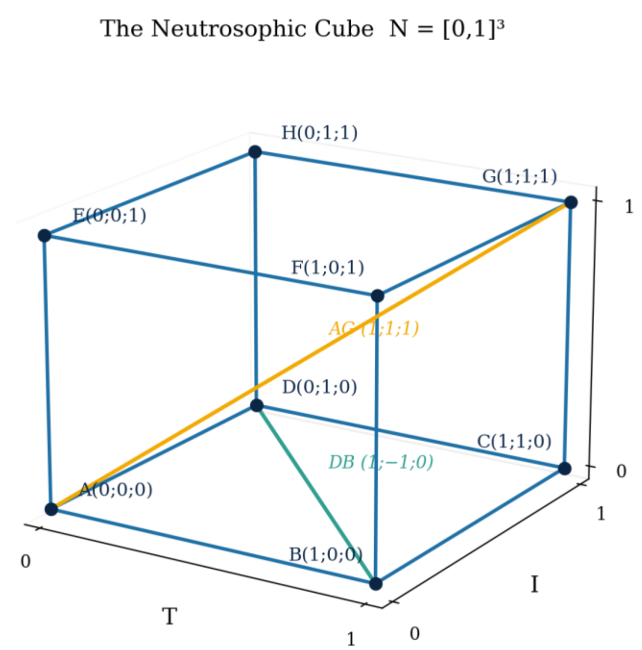

Figure 1.2 — The Neutrosophic Cube with vertices B(1;0;0), C(1;1;0), D(0;1;0), E(0;0;1), F(1;0;1), G(1;1;1), H(0;1;1), illustrating diagonal vectors such as →DB(1;−1;0) and →AG(1;1;1) used in the SAC projection.

10. Theorem: Geometric Projection Theorem for the Neutrosophic SAC Lattice

Let $N = [0,1]^3$ be the neutrosophic cube, whose points are $x = (T, I, F)$. Define the SAC projection $\Phi: N \to \mathbb{R}^3$ by $\Phi(T,I,F) = (S, A, C)$, where $S = (2 + T - I - F)/3$, $A = T - F$, $C = T$. On $\mathbb{R}^3$, use the lexicographic order $(S,A,C) \leq_{lex} (S',A',C')$. Then the relation $x \leq_{SAC} y \Leftrightarrow \Phi(x) \leq_{lex} \Phi(y)$ turns the neutrosophic cube into a totally ordered lattice. That is, $([0,1]^3, \wedge, \vee, \leq_{SAC})$ is a chain lattice. Moreover, the meet and join are geometrically obtained by comparing the SAC projections:

$$x \wedge y = \Phi^{-1}(\min_{\leq lex}\{\Phi(x), \Phi(y)\}), \quad x \vee y = \Phi^{-1}(\max_{\leq lex}\{\Phi(x), \Phi(y)\}).$$

Proof. The map $\Phi(T,I,F) = (S,A,C)$ assigns to each point of the neutrosophic cube a point in the SAC-space. Since the lexicographic order on $\mathbb{R}^3$ is total, for any two points $\Phi(x), \Phi(y) \in \mathbb{R}^3$, we have either $\Phi(x) \leq_{lex} \Phi(y)$ or $\Phi(y) \leq_{lex} \Phi(x)$. Therefore, for any two neutrosophic triplets x, y, either $x \leq_{SAC} y$ or $y \leq_{SAC} x$. Hence the SAC order is total. Since every totally ordered set becomes a lattice by defining $x \wedge y = \min\{x,y\}$, $x \vee y = \max\{x,y\}$, the neutrosophic cube under SAC ordering is a lattice. Thus, the geometric cube $[0,1]^3$ is transformed by the SAC projection into a totally ordered chain lattice. ■

11. Geometric Meaning

The neutrosophic cube is three-dimensional: (T, I, F). The SAC map gives a new coordinate system: $(T,I,F) \mapsto (S,A,C)$. Therefore, the lattice order is not simply coordinate-wise. It is a ranking order induced by geometric projection: Neutrosophic Cube ⟶ SAC Ordered Space ⟶ Chain Lattice.

12. Refined Version

For $x = (T_1,\ldots,T_p; I_1,\ldots,I_r; F_1,\ldots,F_s)$, define $SR(x) = (r + s + \Sigma T_j - \Sigma I_k - \Sigma F_\ell)/(p+r+s)$, $AR(x) = \Sigma T_j - \Sigma F_\ell$, $CR(x) = \Sigma T_j$. Then $\Phi R(x) = (SR(x), AR(x), CR(x))$. Thus, the refined neutrosophic hypercube $[0,1]^{(p+r+s)}$ is projected into SAC-space, giving $[0,1]^{(p+r+s)} \to^{\Phi R} \mathbb{R}^3 \to^{\leq lex}$ Chain Lattice, where "lex" means lexicographic (dictionary) order. This is the Refined Neutrosophic Geometric Projection Theorem.

13. Relation with Concept Lattices (Formal Concept Analysis)

In Formal Concept Analysis, concepts form a lattice. A refined neutrosophic concept could be represented by $(T_1,\ldots,T_p; I_1,\ldots,I_r; F_1,\ldots,F_s)$. Using RSAC ranking, concepts become linearly ordered. Hence one obtains what might be called a Refined

Neutrosophic Concept Lattice, which could be useful in AI and decision-making.

14. New Observation: Quotient of the Product Lattice

This is the most interesting theoretical connection. Start from the coordinate product lattice $[\Psi,\Omega]^n$. Define the mapping $\Phi(x) = (SR(x), AR(x), CR(x))$. Then the RSAC lattice is essentially the image of the refined neutrosophic space under Φ, ordered lexicographically. Therefore: Refined Neutrosophic SAC Lattice $= \Phi([\Psi,\Omega]^n)$, which can be viewed as a quotient/ranking lattice induced by the three invariants SR, AR, CR.

This connection does not seem to have been studied and could form the central theoretical contribution of a study of Neutrosophic Lattices Induced by Score–Accuracy–Certainty Orders and Their Connections with Boolean, Fuzzy, Intuitionistic Fuzzy, Łukasiewicz–Moisil, Product, and Concept Lattices. This last connection — the SAC lattice as a quotient/ranking of the refined neutrosophic cube — is the most original direction. These lattices have many potential applications because they combine: (1) neutrosophic representation (truth–indeterminacy–falsehood), (2) total ranking (SAC order), (3) lattice structure (meet/join), (4) geometric interpretation (cube/hypercube foliations). This combination is rather unique.

15. Applications

15.1. Multi-Criteria Decision Making (MCDM)

This is perhaps the most immediate application. Suppose several alternatives are evaluated by (T, I, F). For example, alternative A = (0.8, 0.1, 0.2), B = (0.7, 0.2, 0.1), C = (0.9, 0.4, 0.3). The SAC lattice provides a total ranking, a best alternative A ∨ B, and a worst alternative A ∧ B. Thus, the lattice becomes a decision structure.

15.2. Artificial Intelligence

In AI many conclusions are partially supported, partially contradicted, and partially unknown. Represent each conclusion by (T,I,F). The lattice provides: join — choose the more credible conclusion x ∨ y; meet — choose the less credible conclusion x ∧ y. Thus, reasoning becomes lattice-based.

15.3. Explainable AI (XAI)

The geometric foliation theorem is particularly attractive. A decision can be explained as: (1) higher score plane; (2) if tied, higher accuracy line; (3) if tied, higher certainty position. Hence every ranking has a transparent geometric explanation.

15.4. Information Fusion

Suppose several sensors — radar, camera, infrared, human operator — report (T,I,F). The lattice can rank the reliability of observations. The join identifies the most reliable report; the meet identifies the least reliable report.

15.5. Medical Diagnosis

A disease hypothesis may have (T,I,F) representing (evidence for, uncertainty, evidence against). The SAC lattice orders competing diagnoses. Refined lattices are even more natural: (T_1,T_2,T_3; I_1,I_2; F_1,F_2,F_3), where symptoms, laboratory tests, and imaging results are separated.

15.6. Legal Reasoning

Court cases are rarely true or false. Instead we have (T,I,F) representing supporting evidence, missing evidence, and contradictory evidence. The lattice provides a formal way to rank legal hypotheses.

15.7. Knowledge Bases

In databases, some facts are confirmed, some disputed, some unknown. The lattice structure naturally organizes knowledge: join $x \vee y$ selects the more supported fact; meet $x \wedge y$ selects the less supported fact.

15.8. Refined Scientific Theories

Representing a scientific theory as ($T_1,\ldots,T_p$; $I_1,\ldots,I_r$; $F_1,\ldots,F_s$) — with truth components such as experimental support, mathematical consistency, and predictive power; indeterminacy components such as missing data and measurement uncertainty; and falsehood components such as contradictory experiments and internal inconsistencies — the refined lattice ranks competing theories. This seems very suitable for contradictory theories, plithogeny, and neutrosophy.

15.9. Quantum Mechanics

A quantum state may be represented neutrosophically by (T,I,F), where T = evidence for a state, I = superposition uncertainty, F = evidence against the state. Then the SAC lattice orders possible states. For work on neutrosophic qubits, nubits, and neutrosophic quantum computing, this becomes a foundational ordering structure.

15.10. Attractors, Repellers, and Neutrals

Recent work introduced refined neutrosophic attractors, repellers, and neutrals. Each attractor can be represented by $(T_1,\ldots,T_p; I_1,\ldots,I_r; F_1,\ldots,F_s)$. The lattice then orders attractors according to their strength; similarly for repellers and neutrals. One can define $A_1 \vee A_2$ as the dominant attractor and $A_1 \wedge A_2$ as the weaker attractor, leading naturally to a Refined Neutrosophic Attractor Lattice, a Refined Neutrosophic Repeller Lattice, and a Refined Neutrosophic Neutral Lattice. These structures do not seem to have been investigated before.

15.11. A New Application: Theory Selection Lattice

Perhaps the most original application: let each theory be represented by (T,I,F) or by its refined version — e.g., the Copenhagen interpretation, the De Broglie–Bohm interpretation, Neutrosophic Quantum Mechanics, the Infinitesimally Punctured Wave (IPW), and the Finitesimally Punctured Wave (FPW). The SAC lattice produces an ordered hierarchy of theories, and the geometric foliation theorem gives a visual explanation of why one theory ranks above another. This creates what could be called a Neutrosophic Theory Selection Lattice for evaluating competing scientific models under uncertainty, contradiction, and indeterminacy.

Among all applications, the most publishable, in order, appear to be: (1) Theory Selection Lattice (science and philosophy of science); (2) Refined Attractor/Repeller/Neutral Lattices; (3) Neutrosophic Quantum State Lattices; (4) MCDM and AI decision lattices; (5) Medical diagnosis lattices. The first two seem especially aligned with the current research program and could each support a standalone paper.

16. Refined Neutrosophic Geometric Lattice Theorem

Consider the refined neutrosophic hypercube $[0,1]^{(p+r+s)}$. The hypersurfaces $\Sigma T_j - \Sigma I_k - \Sigma F_\ell = K$ (score), $\Sigma T_j - \Sigma F_\ell = A$ (accuracy), $\Sigma T_j = C$ (certainty), induce a nested geometric stratification of the hypercube. The RSAC order then transforms this stratified hypercube into a complete distributive chain lattice. This theorem elegantly connects geometry, lattice theory, order theory, and refined neutrosophic sets, and appears not to have been developed in the literature before.

A natural further construction, unexplored as yet, is the Neutrosophic Over/Under/Off-Cube (with values for T, I, F outside of [0, 1]). One may define $COUO = [\Psi,\Omega]^3$, $\Psi < 0 < 1 < \Omega$, where $(T,I,F) \in [\Psi,\Omega]^3$. Then: the ordinary neutrosophic cube is $[0,1]^3$; over-neutrosophic regions have $T > 1$, $I > 1$, $F > 1$; under-neutrosophic regions have $T < 0$, $I < 0$, $F < 0$; and off-neutrosophic regions mix over and under components. Geometrically $[0,1]^3 \subset [\Psi,\Omega]^3$: the ordinary neutrosophic cube becomes the central unit cube, surrounded by over-, under-, and off-neutrosophic zones. This leads to the theorem that the Neutrosophic Over/Under/Off-Cube under SAC order forms a complete chain lattice, with score planes $T - I - F = k$ extended beyond the classical unit cube.

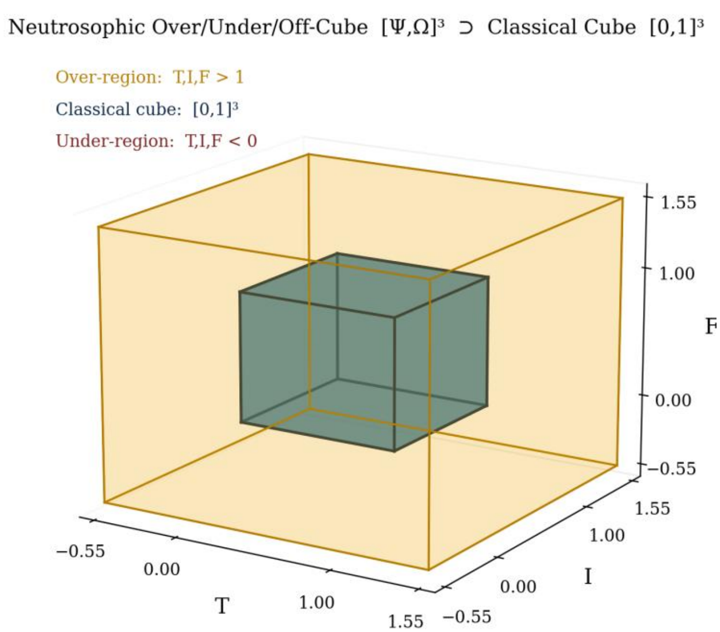

Figure 1.3 — Conceptual diagrams of the Neutrosophic Over/Under/Off-Cube $[\Psi,\Omega]^3$ containing the classical unit cube $[0,1]^3$ at its centre, together with a coordinate-axis view (+x,+y,+z / −x,−y,−z) and a nested-region view showing the OVER-REGION, the classical NeutroCube, and the UNDER/OFF region, bounded by $\Psi < 0 < 1 < \Omega$.

17. Conceptual Structure

Schematically, the outer cube $[\Psi,\Omega]^3$ encloses the classical unit cube $[0,1]^3$, which in turn is surrounded by the over-region above Ω and the under/off region below Ψ, with $\Psi < 0 < 1 < \Omega$. The inner cube is $[0,1]^3$, while the outer cube is $[\Psi,\Omega]^3$.

18. Coordinate Interpretation

Axes: T, I, F. The central cube contains ordinary neutrosophic values. Over-Neutrosophic Zones have, for example, $T > 1$, $I > 1$, $F > 1$, such as (1.3, 0.4, 0.2) or (0.8, 1.2, 0.1); these lie outside the classical cube.

19. Under-Neutrosophic Zones

Examples: $T < 0$, $I < 0$, $F < 0$, such as (−0.2, 0.3, 0.4).

20. Off-Neutrosophic Zones

Mixed cases: (1.4, −0.3, 0.7), (0.5, 1.2, −0.4). These occupy regions that are neither purely over nor purely under.

21. SAC Geometry Inside the OUO-Cube

The score hypersurfaces remain $T - I - F = k$, but now they continue beyond the unit cube and sweep the entire OUO-cube. Thus, the foliation theorem becomes $[\Psi,\Omega]^3 \supset \mathcal{F}S \supset \mathcal{F}A \supset \mathcal{F}C$.

The SAC lattice is then generated by the nested foliation of the OUO-cube.

22. Future Figures and Directions

Future illustrative work could include: a large transparent outer cube $[\Psi,\Omega]^3$; a colored inner cube $[0,1]^3$; green regions labeled Overset; blue regions labeled Underset; orange regions labeled Offset; and several translucent score planes. Such a figure would visually introduce the Neutrosophic Over/Under/Off-Cube, analogously to how the classical Neutrosophic Cube visualizes (T,I,F)-space, and could become a standard illustration for future work on over/under/off-neutrosophic geometry.

The central ideas developed in this chapter are: (1) the Neutrosophic Chain Lattice $\mathcal{L}$N; (2) the Over/Under/Off-Neutrosophic Chain Lattice $\mathcal{L}$OUO; (3) the Refined Neutrosophic Chain Lattice $\mathcal{L}$R; (4) the Refined Over/Under/Off-Neutrosophic Chain Lattice $\mathcal{L}$ROUO; (5) SAC and RSAC orders; (6) the Geometric Projection Theorem; (7) the Foliation Theorem; (8) Neutrosophic Cube geometry; (9) Over/Under/Off-Cube geometry; and (10) applications to AI, MCDM, quantum theory, theory selection, and attractor/repeller systems.

The most original ideas are: (A) the Neutrosophic Over/Under/Off-Cube $[\Psi,\Omega]^3$ as an extension of the classical neutrosophic cube; (B) the Refined Neutrosophic Hypercube [0,1]^(p+r+s) with RSAC foliation; and (C) the Refined Over/Under/Off-Hypercube [Ψ,Ω]^(p+r+s). This may become a whole new branch of Neutrosophic Geometry, following a possible hierarchy: Neutrosophic Point → Neutrosophic Cube → Refined Hypercube → OUO-Cube → OUO-Hypercube, with associated orders, lattices, metrics, topologies, manifolds, attractors, and dynamical systems. In fact, the notion of a Neutrosophic OUO-Manifold modeled locally on $[\Psi,\Omega]^n$ would be a natural continuation of the geometric ideas developed here.

23. Conclusion

This chapter developed several new ideas: Neutrosophic and Refined Neutrosophic Lattices; Over/Under/Off-Neutrosophic Lattices; SAC and RSAC chain orders; Geometric Projection and Foliation Theorems; Neutrosophic Cube and OUO-Cube geometry; connections with classical lattice theory; and several promising applications and future research directions. The idea of the Neutrosophic Over/Under/Off-Cube and its refined hypercube extension is especially rich, with much still to explore: metrics, topology, manifolds, dynamical systems, attractors/repellers, and perhaps even a neutrosophic geometric category built upon these objects.

These constructions also bear directly on the methodological concerns raised by Rivieccio [13], who questioned the algebraic soundness and the order-theoretic grounding of the operators used in neutrosophic logic. By deriving the SAC total order from first principles (Chapter 3) and showing that it induces a genuine, complete, distributive chain lattice — compatible with Boolean, fuzzy, intuitionistic fuzzy, and Łukasiewicz–Moisil algebras as special or related cases — the present chapter supplies exactly the kind of rigorous algebraic footing that such critiques call for.

Chapter 2

Neutrosophic and Plithogenic Operators: Negation, Intersection, Union, Implication, Equivalence

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Introduction

First of all, we need to recall the t-norms (triangular norms) and t-conorms (triangular conorms) that the neutrosophic and plithogenic operators utilize. Triangular norms and conorms are generalizations of the logical conjunction and logical disjunction to fuzzy logic [Hájek (1998)], and they are used to combine uncertain information.

The t-norms and t-conorms below are well-known in the literature; they are taken here from Mirko Navara (2007), Triangular norms and conorms, Scholarpedia, 2(3):2398.

Triangular Norms

Definition

A triangular norm (abbreviation t-norm) is a binary operation T on the interval [0, 1] satisfying the following conditions:

- T(x, y) = T(y, x) (commutativity)
- T(x, T(y, z)) = T(T(x, y), z) (associativity)

- $y \leq z \Rightarrow T(x, y) \leq T(x, z)$ (monotonicity)
- $T(x, 1) = x$ (neutral element 1)

Examples

- $TM(x, y) = \min(x, y)$ (minimum or Gödel t-norm)
- $TP(x, y) = x \cdot y$ (product t-norm)
- $TL(x, y) = \max(x + y - 1, 0)$ (Łukasiewicz t-norm)

No t-norm can attain greater values than TM. There are many parametrized families of t-norms. The Frank t-norms are defined for all $r > 0$, $r \neq 1$ by

$$TFr(x, y) = \log r\,[\,1 + (r^x - 1)(r^y - 1) / (r - 1)\,].$$

The limit elements of this family are the above t-norms $TF0 = TM$, $TF1 = TP$, and $TF\infty = TL$. The only t-norms which are rational functions are the Hamacher t-norms, defined for all $r > 0$ by

$$THr(x, y) = xy / [\,r + (1 - r)(x + y - xy)\,]$$

and for $r = 0$ by $THr(x, y) = xy / (x + y - xy)$, with $TH0(0,0) = 0$.

Triangular Conorms

Definition

The dual notion to a triangular norm is a triangular conorm (abbreviation t-conorm, also s-norm), S. Its neutral element is 0 instead of 1, all other conditions remain unchanged:

- $S(x, y) = S(y, x)$ (commutativity)
- $S(x, S(y, z)) = S(S(x, y), z)$ (associativity)
- $y \leq z \Rightarrow S(x, y) \leq S(x, z)$ (monotonicity)
- $S(x, 0) = x$ (neutral element 0)

Examples of t-conorms

- $SM(x, y) = \max(x, y)$ (maximum or Gödel t-conorm)
- $SP(x, y) = x + y - x{\cdot}y$ (product t-conorm, probabilistic sum)
- $SL(x, y) = \min(x + y, 1)$ (Łukasiewicz t-conorm, bounded sum)

No t-conorm can attain smaller values than SM. If T is a t-norm, then $S(x, y) = 1 - T(1-x, 1-y)$ is a t-conorm, and vice versa; we obtain a dual pair (T, S) of a t-norm and a t-conorm (instead of the standard fuzzy negation $x \mapsto 1-x$, another strong fuzzy negation can be used in the duality formula).

The classification and representations of t-conorms are dual to those of t-norms. Each continuous Archimedean t-conorm S has a (non-unique) additive generator, an increasing bijection $s : [0,1] \to [0, B]$ ($B \in]0, \infty]$) such that

$$S(x, y) = s^{-1}(s(x) + s(y)) \text{ if } s(x) + s(y) \leq B, \text{ and } 1 \text{ otherwise.}$$

Neutrosophic Operators Employ the Above t-Norms and t-Conorms

The neutrosophic triplets T, I, F are considered in the following way: the truth T of positive quality, the falsity F of negative quality (opposed to T), and the indeterminacy I of partially negative quality. That is why, when applying the t-norm on the T's, one needs to apply the opposite, i.e. the t-conorm, on the F's, and vice versa.

For most cases, since the indeterminacy I is being partially of negative quality, it is considered mostly closer to F than to T; that is why one also applies the same t-norm (or t-conorm) as on the F component.

Neutrosophic Negation

$$\neg\{ N(T, I, F) \} = \neg N(F, I, T).$$

Neutrosophic Intersection (Conjunction)

$$N1(T1,I1,F1) \wedge N2(T2,I2,F2) = N(\, tnorm(T1,T2), tconorm(I1,I2), tconorm(F1,F2)\,).$$

Neutrosophic Union (Disjunction)

$$N1(T1,I1,F1) \vee N2(T2,I2,F2) = N(\, tconorm(T1,T2), tconorm(I1,I2), tnorm(F1,F2)\,).$$

Neutrosophic Implication

$$N1 \rightarrow N2 = \neg N1 \vee N2 = (F1,I1,T1) \vee (T2,I2,F2) = N(\, tconorm(F1,T2), tconorm(I1,I2), tnorm(T1,F2)\,).$$

Neutrosophic Equivalence

$$N1 \leftrightarrow N2 = (N1 \rightarrow N2) \wedge (N2 \rightarrow N1).$$

Plithogenic Operators

They are similar to the Neutrosophic Operators, the only distinction being with respect to the indeterminacy I, which is considered between falsehood and truth, or partially false and partially true. Therefore, instead of taking either only tconorm(I1,I2), or only tnorm(I1,I2), we take their average: 0.5[tconorm(I1,I2) + tnorm(I1,I2)].

Plithogenic Negation

$$\neg\{ N(T,I,F) \} =_P \neg N(F,I,T).$$

Plithogenic Intersection (Conjunction)

$$N1 \wedge_P N2 =_P N(tnorm(T1,T2), 0.5[tconorm(I1,I2)+tnorm(I1,I2)], tconorm(F1,F2)).$$

Plithogenic Union (Disjunction)

$$N1 \vee_P N2 =_P N(tconorm(T1,T2), 0.5[tconorm(I1,I2)+tnorm(I1,I2)], tnorm(F1,F2)).$$

Plithogenic Implication

$$N1 \rightarrow_P N2 =_P \neg N1 \vee_P N2 =_P N(tconorm(F1,T2), 0.5[tconorm(I1,I2)+tnorm(I1,I2)], tnorm(T1,F2)).$$

Plithogenic Equivalence

$$N1 \leftrightarrow_P N2 =_P (N1 \rightarrow_P N2) \wedge_P (N2 \rightarrow_P N1).$$

Conclusion

The general Neutrosophic and Plithogenic operators (conjunction, disjunction, negation, implication, equivalence) were presented in this chapter, based on t-norms and t-conorms, providing a uniform algebraic basis for the logical connectives used throughout neutrosophic and plithogenic reasoning.

Chapter 3

The Score, Accuracy, and Certainty Functions Determine a Total Order on the Set of Neutrosophic Triplets (T, I, F)

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Abstract

In this paper we prove that the Single-Valued (and respectively Interval-Valued, as well as Subset-Valued) Score, Accuracy, and Certainty Functions determine a total order on the set of neutrosophic triplets (T, I, F). This total order is needed in the neutrosophic decision-making applications.

Keywords: single-valued neutrosophic triplet numbers; single-valued neutrosophic score function; single-valued neutrosophic accuracy function; single-valued neutrosophic certainty function.

1. Introduction

We reveal the easiest to use single-valued neutrosophic score, accuracy, and certainty functions that exist in the literature and the algorithm for how to use them all together. We present Xu and Da's Possibility Degree that an interval is greater than or equal to another interval, and we prove that this method is equivalent to the intervals' midpoints comparison. Also, Hong-yu Zhang et al.'s interval-valued neutrosophic score, accuracy, and certainty functions are listed, which we simplify. Numerical examples are provided.

2. Single-Valued Neutrosophic Score, Accuracy, and Certainty Functions

We firstly present the most known and used in the literature single-valued score, accuracy, and certainty functions. Let M be the set of single-valued neutrosophic triplet numbers,

$$M = \{(T, I, F), \text{ where } T, I, F \in [0, 1], 0 \leq T + I + F \leq 3\}. \quad (1)$$

Let N = (T, I, F) ∈ M be a generic single-valued neutrosophic triplet number. Then: T = truth (or membership) represents the positive quality of N; I = indeterminacy represents a negative quality of N, hence 1–I represents a positive quality of N; F = falsehood (or nonmembership) also represents a negative quality of N, hence 1–F represents a positive quality of N.

We present the three most used and best functions in the literature.

2.1. The Single-Valued Neutrosophic Score Function

$$s : M \to [0, 1], \quad s(T, I, F) = [T + (1-I) + (1-F)] / 3 = (2 + T - I - F) / 3. \quad (2)$$

This represents the average of positiveness of the single-valued neutrosophic components T, I, F.

2.2. The Single-Valued Neutrosophic Accuracy Function

$$a : M \to [-1, 1], \quad a(T, I, F) = T - F. \quad (3)$$

2.3. The Single-Valued Neutrosophic Certainty Function

$$c : M \to [0, 1], \quad c(T, I, F) = T. \quad (4)$$

3. Algorithm for Ranking the Single-Valued Neutrosophic Triplets

Let (T_1, I_1, F_1) and (T_2, I_2, F_2) be two single-valued neutrosophic triplets from M, i.e. $T_1, I_1, F_1, T_2, I_2, F_2 \in [0, 1]$. Apply the Neutrosophic Score Function.

1. If $s(T_1,I_1,F_1) > s(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) > (T_2,I_2,F_2)$.

2. If $s(T_1,I_1,F_1) < s(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) < (T_2,I_2,F_2)$.

3. If $s(T_1,I_1,F_1) = s(T_2,I_2,F_2)$, then apply the Neutrosophic Accuracy Function:

3.1 If $a(T_1,I_1,F_1) > a(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) > (T_2,I_2,F_2)$.

3.2 If $a(T_1,I_1,F_1) < a(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) < (T_2,I_2,F_2)$.

3.3 If $a(T_1,I_1,F_1) = a(T_2,I_2,F_2)$, then apply the Neutrosophic Certainty Function:

3.3.1 If $c(T_1,I_1,F_1) > c(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) > (T_2,I_2,F_2)$.

3.3.2 If $c(T_1,I_1,F_1) < c(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) < (T_2,I_2,F_2)$.

3.3.3 If $c(T_1,I_1,F_1) = c(T_2,I_2,F_2)$, then $(T_1,I_1,F_1) \equiv (T_2,I_2,F_2)$, i.e. $T_1 = T_2$, $I_1 = I_2$, $F_1 = F_2$.

3.1. Theorem

The single-valued neutrosophic score, accuracy, and certainty functions all together form a total order relationship on M. That is, for any two single-valued neutrosophic triplets (T_1,I_1,F_1) and (T_2,I_2,F_2) we have: (a) either $(T_1,I_1,F_1) > (T_2,I_2,F_2)$; (b) or $(T_1,I_1,F_1) < (T_2,I_2,F_2)$; (c) or $(T_1,I_1,F_1) \equiv (T_2,I_2,F_2)$, which means that $T_1 = T_2$, $I_1 = I_2$, $F_1 = F_2$. Therefore, on the set of single-valued neutrosophic triplets M, the score, accuracy, and certainty functions altogether form a total order relationship.

Proof. Firstly we apply the score function. The only problematic case is when we get equality $s(T_1,I_1,F_1) = s(T_2,I_2,F_2)$, i.e. $T_1-I_1-F_1 = T_2-I_2-F_2$. Secondly we apply the accuracy function; again the only problematic case is $a(T_1,I_1,F_1) = a(T_2,I_2,F_2)$, i.e.

$T_1 - F_1 = T_2 - F_2$. Thirdly, we apply the certainty function; similarly the only problematic case is $c(T_1,I_1,F_1) = c(T_2,I_2,F_2)$, i.e. $T_1 = T_2$.

For the most problematic case, we obtain the following linear algebraic system of 3 equations in 6 variables:

$$T_1 - I_1 - F_1 = T_2 - I_2 - F_2; \quad T_1 - F_1 = T_2 - F_2; \quad T_1 = T_2.$$

Since $T_1 = T_2$, replacing this into the second equation gives $F_1 = F_2$. Now, replacing both $T_1 = T_2$ and $F_1 = F_2$ into the first equation gives $I_1 = I_2$. Therefore the two neutrosophic triplets are identical: $(T_1,I_1,F_1) \equiv (T_2,I_2,F_2)$, i.e. $T_1 = T_2$, $I_1 = I_2$, $F_1 = F_2$. In conclusion, for any two single-valued neutrosophic triplets, either one is bigger than the other, or both are equal (identical). ■

4. Definition of Neutrosophic Negative Score Function

We introduced in 2017 [1] the Average Negative Quality Neutrosophic Function of a single-valued neutrosophic triplet, defined as:

$$s^{-}: [0,1]^3 \to [0,1], \quad s^{-}(t, i, f) = [(1-t) + i + f] / 3. \quad (6)$$

4.1. Theorem

The average positive quality (score) neutrosophic function and the average negative quality neutrosophic function are complementary to each other, i.e. $s^+(t,i,f) + s^-(t,i,f) = 1$. (7)

Proof. $s^+(t,i,f) + s^-(t,i,f) = (2+t-i-f)/3 + (1-t+i+f)/3 = 1$. (8)

The Neutrosophic Accuracy Function has been defined by $h : [0,1]^3 \to [-1,1]$, $h(t,i,f) = t - f$. (9) We also introduced [1] the Extended Accuracy Neutrosophic Function, defined as $he : [0,1]^3 \to [-2,1]$, $he(t,i,f) = t - i - f$, (10) which varies from the worst negative quality (−2) to the best positive quality (+1).

4.2. Theorem

If $s(T1,I1,F1) = s(T2,I2,F2)$, $a(T1,I1,F1) = a(T2,I2,F2)$, and $c(T1,I1,F1) = c(T2,I2,F2)$, then $T1 = T2$, $I1 = I2$, $F1 = F2$, i.e. the two neutrosophic triplets are identical: $(T1,I1,F1) \equiv (T2,I2,F2)$.
Proof: This results from the proof of Theorem 3.1.

5. Xu and Da's Possibility Degree

Xu and Da [3] defined in 2002 the possibility degree $P(\cdot)$ that an interval is greater than another interval: $[a1,a2] \geq [b1,b2]$ for $a1,a2,b1,b2 \in [0,1]$ and $a1 \leq a2$, $b1 \leq b2$, in the following way:

$$P([a1,a2] \geq [b1,b2]) = \max\{ 1 - \max((b2-a1)/(a2-a1+b2-b1), 0), 0 \},$$

where $a2 - a1 + b2 - b1 \neq 0$.

5.1. Properties

1) $P([a1,a2] \geq [b1,b2]) \in [0,1]$; 2) $P([a1,a2] \approx [b1,b2]) = 0.5$; 3) $P([a1,a2] \geq [b1,b2]) + P([b1,b2] \geq [a1,a2]) = 1$.

5.2. Example

Let [0.4, 0.7] and [0.3, 0.6] be two intervals. Then $P([0.4,0.7] \geq [0.3,0.6]) = \max\{1 - \max(0.2/0.6, 0), 0\} = 0.4/0.6 \approx 0.66 > 0.50$, therefore $[0.4,0.7] \geq [0.3,0.6]$. The opposite: $P([0.3,0.6] \geq [0.4,0.7]) = \max\{1 - \max(0.4/0.6, 0), 0\} = 0.2/0.6 \approx 0.33 < 0.50$, therefore $[0.3,0.6] \leq [0.4,0.7]$. We see that $P([0.4,0.7]\geq[0.3,0.6]) + P([0.3,0.6]\geq[0.4,0.7]) = 0.4/0.6 + 0.2/0.6 = 1$.

Another method of ranking two intervals is the midpoint method.

6. Midpoint Method

Let A = [a1,a2] and B = [b1,b2] be two intervals included in or equal to [0,1], with $mA = (a1+a2)/2$ and $mB = (b1+b2)/2$ the midpoints of A and B respectively. Then: 1) If $mA < mB$ then $A <$

B. 2) If mA > mB then A > B. 3) If mA = mB then A =N B, i.e. A is neutrosophically equal to B.

6.1. Example

Take A = [0.4, 0.7], mA = 0.55; and B = [0.3, 0.6], mB = 0.45. Since mA = 0.55 > 0.45 = mB, we have A > B. Let C = [0.1, 0.7] and D = [0.3, 0.5]; then mC = 0.4 = mD, so C =N D. Verifying with Xu and Da's possibility degree method gives P([0.1,0.7]≥[0.3,0.5]) = 0.5 and P([0.3,0.5]≥[0.1,0.7]) = 0.5, thus [0.1,0.7] =N [0.3,0.5].

6.2. Corollary

The possibility method for two intervals having the same midpoint always gives 0.5. For example, P([0.3,0.5]≥[0.2,0.6]) = 0.5 and P([0.2,0.6]≥[0.3,0.5]) = 0.5, so neither of the intervals [0.3,0.5] and [0.2,0.6] is bigger than the other; we may consider [0.3,0.5] =N [0.2,0.6], neutrosophically equal (or equivalent).

7. Normalized Hamming Distance between Two Intervals

Consider the Normalized Hamming Distance between two intervals [a1,a2] and [b1,b2], h : int([0,1]) × int([0,1]) → [0,1], defined as h([a1,b1],[a2,b2]) = ½(|a1−b1| + |a2−b2|).

7.1. Theorem

7.1.1. The Normalized Hamming Distance between two intervals having the same midpoint and the negative-ideal interval [0,0] is the same. 7.1.2. The Normalized Hamming Distance between two intervals having the same midpoint and the positive-ideal interval [1,1] is also the same (Jun Ye [4,5]).

Proof. Let A = [m−a, m+a] and B = [m−b, m+b] be two intervals from [0,1] with the same midpoint m. Then h([m−a,m+a],[0,0]) = m and h([m−b,m+b],[0,0]) = m; likewise h([m−a,m+a],[1,1]) = 1−m and h([m−b,m+b],[1,1]) = 1−m.

8. Xu and Da's Possibility Degree Method Is Equivalent to the Midpoint Method

8.1. Theorem

The Xu and Da Possibility Degree Method is equivalent to the Midpoint Method in ranking two intervals included in [0,1].

Proof. Let A = [m1−a, m1+a] and B = [m2−b, m2+b], with m1, m2 ∈ [0,1] the midpoints of A and B respectively, and a, b ∈ [0,1], A, B ⊆ [0,1].

Case 1: m1 < m2. By the Midpoint Method, A < B. Applying Xu and Da's method, P(A≥B) = max{1 −

max((m2−m1+a+b)/(2a+2b), 0), 0}. (i) If a+b+m1−m2 ≤ 0, then P(A≥B) = 0, hence A < B. (ii) If a+b+m1−m2 > 0, then P(A≥B) = (a+b+m1−m2)/(2a+2b), and one shows this is < 0.5 exactly when m1 < m2, which holds by assumption. So A < B in both sub-cases, agreeing with the Midpoint Method.

Case 2: m1 = m2. Then P(A≥B) = P(B≥A) = 0.5, so again A =N B, agreeing with the Midpoint Method.

Case 3: m1 > m2. By a symmetric argument, P(A≥B) > 0.5, so A > B, again agreeing with the Midpoint Method. ■

8.2. Consequence

All intervals included in [0,1] with the same midpoint are considered neutrosophically equal. C(m) = {[m−a, m+a] : m, a, m−a, m+a ∈ [0,1]} represents the class of all neutrosophically equal intervals included in [0,1] whose midpoint is m. (i) If m = 0 or m = 1, there is only one interval centered at 0 (namely [0,0]) and only one centered at 1 (namely [1,1]). (ii) If m ∉ {0,1}, there are infinitely many intervals from [0,1] centered at m.

8.3. Consequence

Remarkably, we can rank an interval [a,b] ⊆ [0,1] with respect to a number n ∈ [0,1], since the number may be transformed into an interval [n,n] as well. For example [0.2, 0.8] > 0.4, since the midpoint of [0.2,0.8] is 0.5 and the midpoint of [0.4,0.4] is 0.4, and 0.5 > 0.4. Similarly, 0.7 > (0.5, 0.8).

9. Interval(-Valued) Neutrosophic Score, Accuracy, and Certainty Functions

Let T, I, F ⊆ [0,1] be three open, semi-open/semi-closed, or closed intervals. Let TL = inf T and TU = sup T; IL = inf I and IU = sup I; FL = inf F and FU = sup F, all in [0,1], with TL ≤ TU, IL ≤ IU, FL ≤ FU. We consider all possible types of intervals but, for simplicity of notation, write only [a,b], understanding all types. Then A = ([TL,TU], [IL,IU], [FL,FU]) is an Interval Neutrosophic Triplet.

Hong-yu Zhang, Jian-qiang Wang, and Xiao-hong Chen [2] in 2014 defined the Interval Neutrosophic Score, Accuracy, and Certainty Functions as follows (int([0,1]) denotes the set of all such intervals included in or equal to [0,1]):

9.1. Zhang Interval Neutrosophic Score Function

$$SZhangint(A) = [\, TL + 1-IU + 1-FU\,,\ TU + 1-IL + 1-FL\,]. \quad (11)$$

9.2. Zhang Interval Neutrosophic Accuracy Function

$$aZhangint(A) = [\, \min\{TL-FL,\ TU-FU\},\ \max\{TL-FL,\ TU-FU\}\,]. \quad (12)$$

9.3. Zhang Interval Neutrosophic Certainty Function

$$cZhangint(A) = [\ TL,\ TU\ ]. \quad (13)$$

9′. New Interval Neutrosophic Score, Accuracy, and Certainty Functions

Since comparing/ranking two intervals is equivalent to comparing/ranking their midpoints, we simplify the Zhang Interval Neutrosophic Score, Accuracy, and Certainty functions as follows, using the superscript FS for our own versions:

10.1. New Interval Neutrosophic Score Function

$$sFSint(\ [TL,TU],[IL,IU],[FL,FU]\ ) =$$
$$[TL+TU+(1-IL)+(1-IU)+(1-FL)+(1-FU)]\ /\ 6 =$$
$$(4+TL+TU-IL-IU-FL-FU)/6,$$

which is the average of six positivenesses.

10.2. New Interval Neutrosophic Accuracy Function

$$aFSint(\ [TL,TU],[IL,IU],[FL,FU]\ ) = (TL+TU-FL-FU)/2,$$

the average of differences between positiveness and negativeness.

10.3. New Interval Neutrosophic Certainty Function

$$cFSint(\ [TL,TU],[IL,IU],[FL,FU]\ ) = (TL+TU)/2,$$

the average of two positivenesses.

10.4. Theorem

Let Mint = $\{(T,I,F) : T,I,F \subseteq [0,1]$ and T,I,F are intervals$\}$ be the set of interval neutrosophic triplets. The New Interval Neutrosophic Score, Accuracy, and Certainty Functions determine a total order relationship on Mint.

Proof. Let P1 = ([T1L,T1U],[I1L,I1U],[F1L,F1U]) and P2 = ([T2L,T2U],[I2L,I2U],[F2L,F2U]) be two elements of Mint. Applying sFSint, if the scores differ the order follows immediately; if they coincide, one obtains T1L+T1U−I1L−I1U−F1L−F1U = T2L+T2U−I2L−I2U−F2L−F2U, and the accuracy function aFSint is applied next; if that too coincides, one obtains T1L+T1U−F1L−F1U = T2L+T2U−F2L−F2U, and the certainty function cFSint is applied last. If the certainty values also coincide, T1L+T1U = T2L+T2U, and combining all three equalities via subtraction yields F1L+F1U = F2L+F2U and I1L+I1U = I2L+I2U as well, so the corresponding intervals [T1L,T1U] and [T2L,T2U], [I1L,I1U] and [I2L,I2U], [F1L,F1U] and [F2L,F2U] each share a common midpoint and are therefore pairwise neutrosophically equal. Hence P1 =N P2, and the three functions taken together always yield a determinate ranking. ■

10.5. Theorem

The ranking of intervals defined by Xu and Da, being equivalent to the ranking of intervals' midpoints, is also equivalent to the algorithm by Hong-yu Zhang et al. for ranking interval neutrosophic triplets, which in turn is equivalent to the algorithm presented here. That is, for any interval neutrosophic triplet P, sZhangint(P) ~ sFSint(P), aZhangint(P) ~ aFSint(P), and cZhangint(P) ~ cFSint(P), where ~ denotes equivalence of the induced rankings. The proof proceeds by expressing each Zhang function's governing midpoint in terms of TL, TU, IL, IU, FL, FU and checking it coincides, after simplification, with the corresponding FS function.

11. Subset Neutrosophic Score, Accuracy, and Certainty Functions

Let Msubset = {(Tsubset, Isubset, Fsubset), where the subsets Tsubset, Isubset, Fsubset ⊆ [0,1]}. We approximate each subset by the smallest closed interval that includes it: TL = inf(Tsubset), TU = sup(Tsubset), so Tsubset ⊆ [TL,TU]; similarly for I and F. Then Msubset is approximated by the set of triples of nested intervals ([TL,TU],[IL,IU],[FL,FU]) with TL ≤ TU, IL ≤ IU, FL ≤ FU, all in [0,1].

11.1. Definition of Subset Neutrosophic Score, Accuracy, and Certainty Functions

The formulas for the Subset Neutrosophic Score, Accuracy, and Certainty Functions coincide with those for the Interval Neutrosophic Score, Accuracy, and Certainty Functions given above (both the Zhang and the FS versions).

11.2. Theorem

Let N be the Interval Neutrosophic Triplet N = ([TL,TU],[IL,IU],[FL,FU]) with TL≤TU, IL≤IU, FL≤FU, all subsets of [0,1]. If each interval collapses to a single point — TL=TU=T, IL=IU=I, FL=FU=F — then sFSint(N) = s(N), aFSint(N) = a(N), and cFSint(N) = c(N).

Proof. sFSint(N) = (4+TL+TU−IL−IU−FL−FU)/6 = (4+2T−2I−2F)/6 = (2+T−I−F)/3 = s(N). Likewise aFSint(N) = (TL+TU−FL−FU)/2 = (2T−2F)/2 = T−F = a(N), and cFSint(N) = (TL+TU)/2 = (2T)/2 = T = c(N). ■

12. Conclusion

The most used and easiest functions for ranking the Neutrosophic Triplets (T, I, F), which together provide a total order, are the Single-Valued Neutrosophic Score, Accuracy, and Certainty Functions:

$$s(T,I,F) = (2+T-I-F)/3, \quad a(T,I,F) = T-F, \quad c(T,I,F) = T,$$

and their Interval-Valued generalizations:

$$sFSint([TL,TU],[IL,IU],[FL,FU]) = (4+TL+TU-IL-IU-FL-FU)/6,$$

$$aFSint([TL,TU],[IL,IU],[FL,FU]) = (TL+TU-FL-FU)/2,$$

$$cFSint([TL,TU],[IL,IU],[FL,FU]) = (TL+TU)/2.$$

All these functions are very much used in decision-making applications.

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NEUTROSOPHIC LATTICE, NEUTROSOPHIC OPERATORS, AND TOTAL ORDER ON THE NEUTROSOPHIC TRIPLETS

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This volume gathers three foundational contributions to neutrosophic algebra and order theory: a full lattice-theoretic construction on neutrosophic, refined, and over/under/off-neutrosophic triplets built on the Score–Accuracy–Certainty total order; a t-norm/t-conorm based derivation of the neutrosophic and plithogenic logical operators; and the original proof that the Score, Accuracy, and Certainty functions together determine a total order on the set of neutrosophic triplets (T, I, F). Together they respond to Umberto Rivieccio's critical remarks on neutrosophic operators and provide a self-contained algebraic and geometric foundation for neutrosophic decision-making, artificial intelligence, and theory-selection applications.

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