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Primeness of Supersubdivision of Some Graphs

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Abstract: A graph with n vertices is said to admit a prime labeling if it's vertices are labeled with distinct integers $1, 2, \cdots, n$ such that for edge xy , the labels assigned to x and y are relatively prime. The graph that admits a prime labeling is said to be prime. G. Sethuraman has introduced concept of supersubdivision of a graph. In the light of this concept, we have proved that supersubdivision by $K_{2,2}$ of star, cycle and ladder are prime.

Key Words: Star, ladder, cycle, subdivision of graphs, supersubdivision of graphs, prime labeling, Smarandachely prime labeling.

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§1. Introduction

We consider finite undirected graphs without loops, also without multiple edges. G Sethuraman and P. Selvaraju [2] have introduced supersubdivision of graphs and proved that there exists a graceful arbitrary supersubdivision of $C_n, n \geq 3$ with certain conditions. Alka Kanetkar has proved that grids are prime [1]. Some results on prime labeling for some cycle related graphs were established by S.K. Vaidya and K.K.Kanani [6]. It was appealing to study prime labeling of supersubdivisions of some families of graphs.

§2. Definitions

Definition 2.1(Star) *A star S_n is the complete bipartite graph $K_{1,n}$ a tree with one internal node and n leaves, for $n > 1$.*

Definition 2.2(Ladder) *A ladder L_n is defined by $L_n = P_n \times P_2$ here P_n is a path of length n , $\times$ denotes Cartesian product. L_n has $2n$ vertices and $3n - 2$ edges.*

Definition 2.3(Cycle) *A cycle is a graph with an equal number of vertices and edges where vertices can be placed around circle so that two vertices are adjacent if and only if they appear*

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consecutively along the circle. The cycle is denoted by C_n.

Definition 2.4(Subdivision of a Graph) *Let G be a graph with p vertices and q edges. A graph H is said to be a subdivision of G if H is obtained by subdividing every edge of G exactly once. H is denoted by $S(G)$. Thus, $|V| = p + q$ and $|E| = 2q$.*

Definition 2.5(Supersubdivision of a Graph) *Let G be a graph with p vertices and q edges. A graph H is said to be a supersubdivision of G if it is obtained from G by replacing every edge e of G by a complete bipartite graph $K_{2,m}$. H is denoted by $SS(G)$. Thus, $|V| = p + mq$ and $|E| = 2mq$.*

Definition 2.6(Prime Labelling) *A prime labeling of a graph is an injective function $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ such that for every pair of adjacent vertices u and v, $gcd\left(f\left(u\right), f\left(v\right)\right) = 1$ i.e.labels of any two adjacent vertices are relatively prime. A graph is said to be prime if it has a prime labeling.*

Generally, a labeling is called Smarandachely prime on a graph H by Smarandachely denied axiom ([5], [8]) *if there is such a labeling $f : V(G) \to \{1, 2, \cdots, |V(G)|\}$ on G that for every edge uv not in subgraphs of G isomorphic to H, $gcd\left(f\left(u\right), f\left(v\right)\right) = 1$.*

For a complete bipartite graph $K_{2,m}$, we call the part consisting of two vertices, the 2 *vertices part* of $K_{(2,m)}$ and the part consisting of m vertices, the *m-vertices part* of $K_{2,m}$ in this paper.

§3. Main Results

Theorem 3.1 *A supersubdivision of S_n, i.e. $SS\left(S_n\right)$ is prime for $m = 2$.*

Proof Let u be the internal node i.e.centre vertex. Let $v_1, v_2, \cdots, v_n$ be endpoints. Let $v_i^1, v_i^2, i = 1, 2, \cdots, n$ be vertices of graph $K_{2,2}$ replacing edge uv_i. Here, $|V| = 3n + 1$.

Let $f : V \to \{1, 2, \ldots, 3n + 1\}$ be defined as follows:

$$
\begin{aligned}
&f\left(u\right) = 1, && \\
&f\left(v_i\right) = 3i, && i = 1, 2, \cdots, n,\\
&f\left(v_i^1\right) = 3i - 1, && i = 1, 2, \cdots, n,\\
&f\left(v_i^2\right) = 3i + 1, && i = 1, 2, \cdots, n.
\end{aligned}
$$

As $f\left(u\right) = 1$, $gcd\left(f\left(u\right), f\left(v_i^1\right)\right) = 1$ and $gcd\left(f\left(u\right), f\left(v_i^2\right)\right) = 1$.

As successive integers are coprime, $gcd\left(f\left(v_i^1\right), f\left(v_i\right)\right) = (3i - 1, 3i) = 1$ and $gcd\left(f\left(v_i^2\right), f\left(v_i\right)\right) = (3i + 1, 3i) = 1$. Thus $SS\left(S_n\right)$ is prime. □

Let C_n be a cycle of length n. Let $c_1, c_2, \cdots, c_n$ be the vertices of cycle. Let $c_{i,i+1}^k$, $k = 1, 2$ be the vertices of the bipartite graph that replaces the edge c_ic_{i+1} for $i = 1, 2, \cdots, n - 1$ Let $c_{n,1}^k$, $k = 1, 2$ be the vertices of the bipartite graph that replaces the edge c_nc_1. To illustrate these notations a figure is shown below.

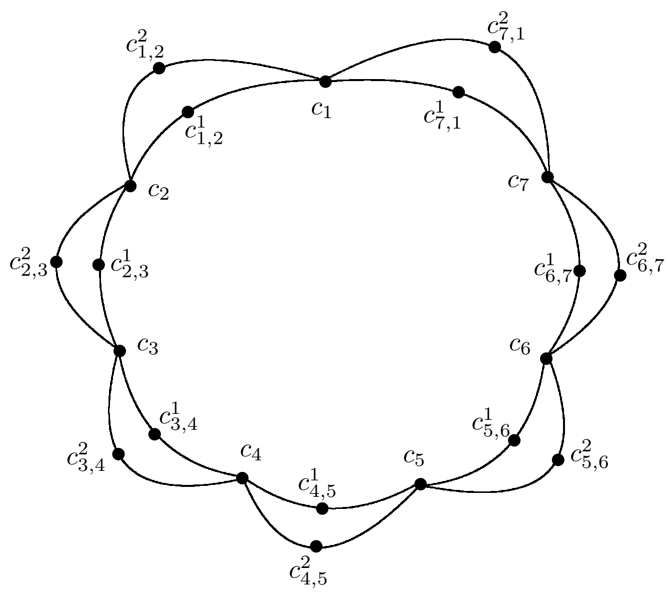

Fig.1 Graph with $n = 7$ with general vertex labels

Theorem 3.2 *A supersubdivision of C_n, i.e. $SS(C_n)$ is prime for $m = 2$.*

Proof Let $p_1, p_2, \cdots, p_k$ be primes such that $3 \leq p_1 < p_2 < p_3 \cdots < p_k < 3n$ such that if p is any prime from 3 to $3n$ then $p = p_i$ for some i between 1 to k.

Define $S_2 = \{S_{2_i}/S_{2_i} = 2^i, i \in \mathbb{N} \ \ such \ that \ \ S_{2_i} \leq 3n\}$. Choose greatest i such that $p_i \leq n$ and denote it by l. Let $S_{p_1} = \{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i, \ \ i \in \{2, 3, \cdots, n\} \backslash \{p_l, p_{l-1}, \cdots, p_{l-(n-k-2)}\}$. Define $f : V \to \{1, 2, \ldots, 3n\}$ using following algorithm.

Case 1. $n = 3$ to 8.

In this case, $k = n$.

Step 1. $f(c_r) = p_r \qquad for \quad r = 1, 2, \cdots, k$ and $f\left(c_{1,2}^1\right) = 1$.

Step 2. Choose greatest i, such that $2p_i < 3n$ and denote it by r. Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be $S_{p_j} = \left\{S_{p_{j_i}}/S_{p_{j_i}} = p_j \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{3n}{p_j} \right\rceil \right\}\right\}$.

Step 3. For $i = 2, 3, \cdots, n$, $k = 1, 2$. Label $c_{i,i+1}^k$ using elements of S_{p_j} in increasing order starting from $j = 1, 2, \cdots, r$ and then by elements of S_2 in increasing order.

Step 4. Choose greatest i such that $2^i \leq 3n$. Label $c_{n,1}^k, \ \ k = 1, 2$ as $2^{i-1}, 2^{i-2}$.

Step 5. Label $c_{1,2}^2$ as 2^i.

Case 2. $n = 9$ to 11

In this case, $k + 1 = n$.

Step 1. $f(c_r) = p_r \qquad for \quad r = 1, 2, \ldots, k$ and $f(c_n) = 1$.

Step 2. Choose greatest i, such that $2p_i < 3n$ and denote it by r. Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be $S_{p_j} = \left\{S_{p_{j_i}}/S_{p_{j_i}} = p_j \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{3n}{p_j} \right\rceil \right\}\right\}$.

Step 3. For $i = 2, 3, \cdots, n$ and $k = 1, 2$, label $c^k_{i,i+1}$ using elements of S_{p_j} in increasing order starting from $j = 1, 2, \ldots, r$ and then by elements of S_2 in increasing order.

Step 4. Choose greatest i such that $2^i \leq 3n$. Label $c^k_{n,1}, \quad k = 1, 2$ as $2^{i-2}, 2^{i-3}$.

Step 5. Label $c^k_{1,2}, k = 1, 2$ as 2^i and 2^{i-1}.

Case 3. $n \geq 12$.

Step 1. $f(c_r) = p_r \quad for \quad r = 1, 2, \cdots, k$.

Step 2. $f(c_{k+1}) = 1$.

For $j = 1, 2, \cdots, n-k-2$, $f(c_{n-j}) = 3p_{l-j}$.

Step 3. Choose greatest i, such that $2p_i < 3n$ and denote it by r. Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be

$$S_{p_j} = \left\{ S_{p_{j_i}} / S_{p_{j_i}} = p_j \times i, i \in \left\{ 2, 3, \cdots, \left\lceil \frac{3n}{p_j} \right\rceil \right\} \setminus \bigcup_{r=1}^{j-1} \{k \times p_r / k \in \mathbb{N} \right\} \right\}.$$

Step 4. For $i = 2, 3, \cdots, n$ and $k = 1, 2$. Label $c^k_{i,i+1}$ using elements of S_{p_j} in increasing order starting from $j = 1, 2, \ldots, r$ and then by elements of S_2 in increasing order.

Step 5. Choose greatest i such that $2^i \leq 3n$. Label $c^k_{n,1}, \quad k = 1, 2$ as $2^{i-2}, 2^{i-3}$.

Step 6. Label $c^k_{1,2}, k = 1, 2$ as 2^i and 2^{i-1}.

In this case, labels of vertices $c_1, c_2, \cdots, c_k$ are prime . Vertices c_{k+1}, to c_n get labels which are multiples by 3 of $p_l, p_{l-1}, \cdots, p_{l-(n-k-2)}$. Apart from these labels and 3 itself, we have $k-1$ more multiples of 3. Thus $k-1$ vertices of the type $c^j_{i,i+1}, \quad 2 \leq i \leq \left\lceil \frac{k-1}{2} \right\rceil, j = 1, 2$ will get labels as multiples of 3. And hence are relatively prime to labels of corresponding $c'_i s$. Similarly, for multiples of $5, 7$ and so on. Thus, $SS(C_n)$ is prime. □

Theorem 3.3 *A supersubdivision of L_n, i.e. $SS(L_n)$ is prime for $m = 2$.*

Proof Let $u_1, u_2, \cdots .u_n$ and $v_1, v_2, \cdots, v_n$ be the vertices of the two paths in L_n. Let $u_i u_{i+1}, v_i v_{i+1}$ for $i = 1, 2, \cdots, n-1$ and $u_i v_i$ for $i = 1, 2, \cdots, n-1, n$ be the edges of L_n. Let $x^k_i, k = 1, 2$ be the vertices of bipartite graph $K_{2,2}$ replacing the edge $u_i u_{i+1}, i = 1, 2, \cdots, n-1$. Let $y^k_i, k = 1, 2, \cdots, m$ be the vertices of the bipartite graph $K_{2,2}$ replacing the edge $v_{n-i} v_{n-i-1}, i = 1, 2, \cdots, n-1$. Let $w^k_i, k = 1, 2$ be the vertices of the bipartite graph $K_{2,2}$ replacing the edge $u_i v_i$ for $i = 1, 2, \cdots, n-1, n$.

Thus, $|V| = 2n + 2n + 2(n-1) + 2(n-1) = 8n - 4$. Let $p_1, p_2, \cdots, p_k$ be primes such that $3 \leq p_1 < p_2 < p_3 \cdots < p_k < 3n$ such that if p is any prime between 3 to $3n$ then $p = p_i$ for some i between 1 to k. Choose greatest i, such that $2p_i < 8n - 4$ and denote it by r.

Define S_{p_j} for $j = 2, 3, \cdots, r$ such that $S_{p_{j_{i-1}}} < S_{p_{j_i}}$ to be

$$S_{p_j} = \left\{ S_{p_{j_i}} / S_{p_{j_i}} = p_j \times i, i \in \left\{ 2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil \right\} \setminus \bigcup_{r=1}^{j-1} \{k \times p_r / k \in \mathbb{N}\} \right\}.$$

Define $S_2 = \left\{ S_{2_i} / S_{2_i} = 2^i, i \in \mathbb{N} \text{ such that } S_{2_i} \leq 3n \right\}$ and a labeling from $V \to \{1, 2, \cdots, 8n - 4\}$ as follows.

Case 1. $n = 2$.

In this case, $k = 2n$. Let $X = \{w_2^1, w_2^2, y_1^1, y_1^2, w_1^1, w_1^2, x_1^2\}$ be an ordered set. Define S_{p_1} such that $S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i = 3 \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil\right\}\right\}$.

Step 1. $f(u_r) = p_r$ for $r = 1, 2$.

Step 2. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1$.

Step 3. $f(x_1^1) = 1$.

Step 4. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Case 2. $n = 3$ and 6.

In this case, $k = 2n+1$. Let $X = \{x_2^1, x_2^2, x_3^1, \cdots, x_{n-1}^1, x_{n-1}^2, y_1^1, y_1^2, y_2^1, \cdots, y_{n-1}^1, y_{n-1}^2, w_1^1, w_1^2, w_2^1, \cdots, w_n^1, w_n^2\}$ be an ordered set. Define S_{p_1} such that

$$S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i = 3 \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil\right\}\right\}.$$

Step 1. $f(u_r) = p_r$ for $r = 1, 2, \cdots, n$.

Step 2. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1, \cdots, n-1$.

Step 3. $f(x_1^1) = 1$ and $f(x_1^2) = p_k$.

Step 4. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Case 3. $n = 4, 5$ and 7 to 11.

In this case, $k = 2n$. Let $X = \{x_2^1, x_2^2, x_3^1, \cdots, x_{n-1}^1, x_{n-1}^2, y_1^1, y_1^2, y_2^1, \cdots, y_{n-1}^1, y_{n-1}^2, w_1^1, w_1^2, w_2^1, \cdots, w_n^1, w_n^2, x_1^2\}$ be an ordered set. Define S_{p_1} such that

$$S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i = 3 \times i, i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{p_j} \right\rceil\right\}\right\}.$$

Step 1. $f(u_r) = p_r$ for $r = 1, 2, \cdots, n$.

Step 2. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1, \ldots, n-1$.

Step 3. $f(x_1^1) = 1$.

Step 4. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Case 4. $n \geq 12$.

Let $X = \{x_2^1, x_2^2, x_3^1, \cdots, x_{n-1}^1, x_{n-1}^2, y_1^1, y_1^2, y_2^1, \cdots, y_{n-1}^1, y_{n-1}^2, w_n^1, w_n^2, w_{n-1}^1, \cdots, w_1^1, w_1^2\}$ be an ordered set. Choose greatest i, such that $p_i \leq \left\lceil \frac{8n-4}{3} \right\rceil$ and denote it by l.

Step 1. $f(u_r) = p_r$ for $r = 1, 2, \cdots, n$.

Step 2. $f(v_r) = 3p_{l-(r-1)}$ for $r = 1, 2, \cdots, 2n-k$.

Step 3. $f(v_{n-r}) = p_{n+r+1}$ for $r = 0, 1, \cdots, n-(2n-k+1)$.

Step 4. $S_{p_1} = \left\{S_{p_{1_i}}/S_{p_{1_i}} = p_1 \times i, \quad i \in \left\{2, 3, \cdots, \left\lceil \frac{8n-4}{3} \right\rceil\right\}\right\} \setminus \left\{p_l, p_{l-1}, \cdots, p_{l-(2n-k-1)}\right\}$.

Step 5. Label elements of X in order by using elements of S_{p_j} in increasing order starting with $j = 1, 2, \cdots, r$ and then using elements of S_2 in increasing order.

Step 6. Choose greatest i such that $2^i \leq 3n$. Label x_1^1, x_1^2 as 2^i and 2^{i-1}.

In the above labeling, vertices $u_i's$ and $v_i's$ receive prime labels. Vertices $x_i's$, $y_i's$,$w_i's$ adjacent to $u_i's$,$v_i's$ are labeled with numbers which are multiples of 3 followed by multiples of 5 and so on. Since $m = 2$(small), labels are not multiples of respective primes. Thus $SS(L_n)$ prime. □

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