Journal of the Egyptian Mathematical Society 25 (2017) 382–390

Contents lists available at ScienceDirect

Journal of the Egyptian Mathematical Society

journal homepage: www.elsevier.com/locate/joems

Original Article

Computation of Smarandache curves according to Darboux frame in Minkowski 3-space

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ARTICLE INFO

Article history:
Received 6 April 2017
Revised 9 May 2017
Accepted 28 May 2017
Available online 7 June 2017

MSC:
53A04
53A35
53C50

Keywords:
Smarandache curves
Timelike curves
Timelike surfaces
Darboux frame
Minkowski 3-space

ABSTRACT

In this paper, we study Smarandache curves according to Darboux frame in the three-dimensional Minkowski space E_1^3. Using the usual transformation between Frenet and Darboux frames, we investigate some special Smarandache curves for a given timelike curve lying fully on a timelike surface. Finally, we defray a computational example to confirm our main results.

1. Introduction

In the study of the fundamental theory and the characterizations of space curves, the corresponding relations between the curves are very interesting and important problem.

Among all space curves, Smarandache curves have special emplacement regarding their properties, because of this they deserve especial attention in Euclidean geometry as well as in other geometries. It is known that Smarandache geometry is a geometry which has at least one Smarandache denied axiom [1]. An axiom is said to be Smarandache denied, if it behaves in at least two different ways within the same space. Smarandache geometries are connected with the theory of relativity and the parallel universes. Smarandache curves are the objects of Smarandache geometry. By definition, if the position vector of a curve δ is composed by Frenet frame's vectors of another curve β, then the curve δ is called a Smarandache curve [2]. In three-dimensional curve theory, each unit speed curve with at least four continuous derivatives, one can associate three mutually orthogonal unit vector fields **T, N** and **B** (Frenet frame vectors) are respectively, the tangent, the principal normal and the binormal vector fields.

In the light of the existing studies in the field of geometry, many interesting results on Smarandache curves have been obtained by many mathematicians [3–9]. Turgut and Yilmaz have introduced a particular circumstance of such curves, they entitled it Smarandache $\mathbf{TB}_2$ curves in the space E_4^1 [2]. They studied special Smarandache curves which are defined by the tangent and second binormal vector fields. In [4], the author has illustrated certain special Smarandache curves in the Euclidean space.

Special Smarandache curves in such a manner that Smarandache curves $\mathbf{TN}_1$, $\mathbf{TN}_2$, $\mathbf{N}_1\mathbf{N}_2$ and $\mathbf{TN}_1\mathbf{N}_2$ with respect to Bishop frame in Euclidean 3-space have been seeked for by Çetin et al.[6]. Furthermore, they worked differential geometric properties of these special curves and checked out first and second curvatures of these curves. Also, they get the centers of the curvature spheres and osculating spheres of Smarandache curves.

Recently, H.S. Abdel-Aziz and M. Khalifa Saad have studied special Smarandache curves of an arbitrary curve such as **TN, TB** and **TNB** with respect to Frenet frame in the three-dimensional Galilean and pseudo-Galilean spaces [3,7].

The main goal of this article is to introduce and describe some special Smarandache curves in E_1^3 for a given timelike surface and a timelike curve lying fully on it with reference to its Darboux

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http://dx.doi.org/10.1016/j.joems.2017.05.004

frame. We looking forward to that our results will be helpful to researchers who are specialized on mathematical modeling and other applications.

2. Basic concepts

Let us first recall the basic notions from Lorentz geometry [10].

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in \mathbb{R}\ \}$be a 3-dimensional vector space, and let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The Lorentz scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y}\ \rangle_L = x_1y_1 + x_2y_2 - x_3y_3.$$

$E_1^3 = (\mathbb{R}^3, \langle \mathbf{x}, \mathbf{y}\ \rangle_L)$ is called 3-dimensional Lorentzian space, Minkowski 3-space or 3-dimensional Semi-Euclidean space. The arbitrary vector $\mathbf{x}$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike, timelike and lightlike (null) if $\langle \mathbf{x}, \mathbf{x} \rangle_L > 0$ or $\mathbf{x} = 0$, $\langle \mathbf{x}, \mathbf{x} \rangle_L < 0$ and $\langle \mathbf{x}, \mathbf{x} \rangle_L = 0$ and $\mathbf{x} \neq 0$, respectively. Similarly, a curve r, locally parameterized by $r = r(s) : I \subset \mathbb{R} \longrightarrow E_1^3$ where s is pseudo arclength parameter, is called a spacelike curve if $\langle r'(s), r'(s)\rangle_L > 0$, timelike if $\langle r'(s), r'(s)\rangle_L < 0$ and lightlike if $\langle r'(s), r'(s)\ \rangle_L = 0$ and $r'(s) \neq 0$ for all $s \in I$. The two vectors $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3) \in E_1^3$ are orthogonal if and only if $\langle \mathbf{x}\ , \mathbf{y}\ \rangle_L = 0$. Also, for any $\mathbf{x}, \mathbf{y} \in E_1^3$, Lorentzian vector product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\mathbf{x} \times_L \mathbf{y} = \begin{vmatrix} e_1 & e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

The norm of a vector $\mathbf{x} \in E_1^3$ is given by $\|\mathbf{x}\|_L = \sqrt{|\langle\ \mathbf{x}, \mathbf{x}\ \rangle_L|}$. We denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet frame along the curve $r(s)$ in the Minkowski space E_1^3, where the vectors **T, N, B** are called the vectors of the tangent, principal normal and the binormal of r, respectively. Consider now the following definition that we are needed throughout this study.

Definition 1. A surface M in the Minkowski 3-space E_1^3 is said to be spacelike, timelike surface if, respectively the induced metric on the surface is a positive definite Riemannian metric, Lorentz metric. In other words, the normal vector on the spacelike (timelike) surface is a timelike (spacelike) vector [10].

3. Smarandache curves of a timelike curve on a timelike surface

Consider a timelike curve $r = r(s)$ in E_1^3, parameterized by its arc length s and lying fully on an oriented timelike surface M. Let **T, N, B** be the tangent, principal normal and binormal vector fields along $r(s)$. Then, Frenet frame is given by

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{1}$$

where a prime denotes differentiation with respect to s. For this frame the following are satisfying

$$\langle \mathbf{T}, \mathbf{T}\rangle = -1,\ \langle \mathbf{B}, \mathbf{B}\rangle = \langle\ \mathbf{N}, \mathbf{N}\ \rangle = 1,$$
$$\langle \mathbf{T}, \mathbf{N}\ \rangle = \langle \mathbf{T}, \mathbf{B}\rangle = \langle \mathbf{N}, \mathbf{B}\rangle = 0.$$

The coefficients κ and τ are the curve's curvature and torsion.

Let $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$ be the Darboux frame of $r(s)$ with $\mathbf{T}$ as the tangent vector of r and $\mathbf{U}$ is the unit normal to the surface M and $\mathbf{P} = \mathbf{T} \times \mathbf{U}$, then the usual transformation between Frenet and Darboux frames takes the form [10,11]:

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}, \tag{2}$$

where θ is the angle between the vectors $\mathbf{P}$ and $\mathbf{N}$. Therefore, the Darboux frame of $r(s)$ is given as follows

$$\begin{pmatrix} \mathbf{T}' \\ \mathbf{P}' \\ \mathbf{U}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa_g & \kappa_N \\ \kappa_g & 0 & -\tau_g \\ \kappa_N & \tau_g & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{P} \\ \mathbf{U} \end{pmatrix}, \tag{3}$$

where κ_N, κ_g and τ_g are the normal curvature, geodesic curvature and geodesic torsion, respectively. In the differential geometry of surfaces, for a curve $r = r(s)$ lying on a surface M the following are well-known [11]

(i) $r(s)$ is a geodesic curve if and only if $\kappa_g = 0$.
(ii) $r(s)$ is an asymptotic line if and only if $\kappa_N = 0$.
(iii) $r(s)$ is a principal line if and only if $\tau_g = 0$.

Definition 2. A regular curve in Minkowski 3-space, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache curve [2].

In the following, we investigate Smarandache curves **TP**, **TU**, **PU** and **TPU** and study some of their properties which represent the main results.

3.1. **TP***–Smarandache curves*

From the Definition 2, the **TP**– Smarandache curve of r can be defined by

$$\alpha(\bar{s}) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{P}). \tag{4}$$

Differentiating Eq. (4) with respect to s, we obtain

$$\alpha' = \frac{d\alpha}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}(\kappa_g\mathbf{T} + \kappa_g\mathbf{P} + (\kappa_N - \tau_g)\mathbf{U}),$$

with the parameterization

$$\frac{d\bar{s}}{ds} = \frac{1}{\sqrt{2}}(\kappa_N - \tau_g), \tag{5}$$

and then

$$\bar{\mathbf{T}}_\alpha = \frac{1}{(\kappa_N - \tau_g)}(\kappa_g\mathbf{T} + \kappa_g\mathbf{P} + (\kappa_N - \tau_g)\mathbf{U}). \tag{6}$$

Differentiating Eq. (6) with respect to s and using Eq. (5), we get

$$\frac{d\bar{\mathbf{T}}_\alpha}{d\bar{s}} = \frac{-\sqrt{2}}{(\kappa_N - \tau_g)^3}(\omega_1\mathbf{T} + \omega_2\mathbf{P} + \omega_3\mathbf{U}),$$

where

$$\omega_1 = \left(\tau_g'\ \kappa_g - \kappa_N'\ \kappa_g + (\kappa_N - \tau_g)\left(\kappa_g' + {\kappa_g}^2 + {\kappa_N}^2 - \kappa_N\tau_g\right)\right),$$

$$\omega_2 = \left(\tau_g'\ \kappa_g - \kappa_N'\ \kappa_g + (\kappa_N - \tau_g)\left(\kappa_g' + {\kappa_g}^2 + (\kappa_N - \tau_g)\tau_g\right)\right),$$

$$\omega_3 = \kappa_g(\kappa_N - \tau_g)^2.$$

The curvature and torsion of α are given as follows

$$\bar{\kappa}_\alpha = \left\| \frac{d\bar{\mathbf{T}}_\alpha}{d\bar{s}} \right\| = \frac{1}{(\kappa_N - \tau_g)^3}\sqrt{2\left(-\omega_1^2 + \omega_2^2 - \omega_3^2\right)}, \tag{7}$$

and

$$\bar{\mathbf{N}}_\alpha = \frac{\omega_1\mathbf{T} + \omega_2\mathbf{P} + \omega_3\mathbf{U}}{\sqrt{-\omega_1^2 + \omega_2^2 - \omega_3^2}}.$$

On the other hand, we express

$$\bar{\mathbf{B}}_\alpha = -\left(\bar{\mathbf{T}} \times \bar{\mathbf{N}}\right) = \frac{-1}{(\kappa_N - \tau_g)\sqrt{-\omega_1^2 + \omega_2^2 - \omega_3^2}}$$

$$\begin{vmatrix} \mathbf{T} & \mathbf{P} & -\mathbf{U} \\ \kappa_g & \kappa_g & (\kappa_N - \tau_g) \\ \omega_1 & \omega_2 & \omega_3 \end{vmatrix}.$$

So, the binormal vector of α is

$$\bar{\mathbf{B}}_\alpha = \frac{-1}{(\kappa_N - \tau_g)\sqrt{-\omega_1^2 + \omega_2^2 - \omega_3^2}}(\bar{\omega}_1\mathbf{T} + \bar{\omega}_2\mathbf{P} + \bar{\omega}_3\mathbf{U}),$$

where

$$\bar{\omega}_1 = (\tau_g - \kappa_N)\omega_2 + \kappa_g\omega_3,$$
$$\bar{\omega}_2 = (\kappa_N - \tau_g)\omega_1 - \kappa_g\omega_3,$$
$$\bar{\omega}_3 = \kappa_g\omega_1 - \kappa_g\omega_2.$$

We consider the derivatives α'', α''' with respect to s as follows

$$\alpha'' = \frac{1}{\sqrt{2}}\Big\{(\kappa_g' - \tau_g\kappa_N + \kappa_N^2 + \kappa_g^2)\mathbf{T} + (\kappa_g' - \tau_g^2 + \tau_g\kappa_N + \kappa_g^2)\mathbf{P} + (-\tau_g' + \kappa_N' - \kappa_g\tau_g + \kappa_g\kappa_N)\mathbf{U}\Big\},$$

$$\alpha''' = \frac{1}{\sqrt{2}}(\lambda_1\mathbf{T} + \lambda_2\mathbf{P} + \lambda_3\mathbf{U}),$$

where

$$\lambda_1 = -2\tau_g'\,\kappa_N + \kappa_N'(3\kappa_N - \tau_g) + \kappa_g\left(3\kappa_g' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_g'',$$

$$\lambda_2 = \tau_g'(\kappa_N - 3\tau_g) + 2\kappa_N'\tau_g + \kappa_g\left(3\kappa_g' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_g'',$$

$$\lambda_3 = -\tau_g'\kappa_g + \kappa_N'\kappa_g + (\kappa_N - \tau_g)\left(2\kappa_g' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) - \tau_g'' + \kappa_N''.$$

In the light of the above, the torsion of α is given by

$$\bar{\tau}_\alpha = -\frac{(\kappa_N - \tau_g)}{2\sqrt{2}\ \bar{\kappa}_\alpha^2}\left(\frac{\chi_1}{\chi_2}\right), \tag{8}$$

where

$$\chi_1 = \begin{matrix} -3\tau_g'^2\kappa_g - 3\kappa_N'^2\kappa_g - \kappa_N'(\kappa_N - \tau_g) \\ \times\left(3\kappa_g' + 5{\kappa_g}^2 + (\kappa_N - \tau_g)\tau_g\right) \\ +\tau_g'\left(6\kappa_N'\kappa_g + (\kappa_N - \tau_g)\left(3\kappa_g' + 5{\kappa_g}^2 + {\kappa_N}^2 - \kappa_N\tau_g\right)\right) \\ +(\kappa_N - \tau_g) \end{matrix},$$

$$\chi_2 = \left\{\begin{matrix} 5\kappa_g'\kappa_g(\kappa_N - \tau_g) + 2{\kappa_g}^3(\kappa_N - \tau_g) + (\kappa_N - \tau_g)\kappa_g'' \\ +\kappa_g\left(2(\kappa_N - \tau_g)^2(\kappa_N + \tau_g) - \tau_g'' + \kappa_N''\right) \end{matrix}\right\}.$$

Thus we can state the following Corollary.

Corollary 1. *Let $\alpha(\bar{s})$ be a timelike curve lies on a timelike surface M in Minkowski 3-space E_1^3, then*

1. *If α is a geodesic curve, then the equations*

$$\bar{\kappa}_\alpha = \sqrt{\frac{2\left({\tau_g}^2 - {\kappa_N}^2\right)}{(\kappa_N - \tau_g)^2}},$$

$$\bar{\tau}_\alpha = -\frac{(\kappa_N - \tau_g)^5\left(\tau_g'\kappa_N - \kappa_N'\tau_g\right)}{4\sqrt{2}\left({\tau_g}^2 - {\kappa_N}^2\right)},$$

are hold.

2. *If α is an asymptotic line, the following hold*

$$\bar{\kappa}_\alpha = \sqrt{\frac{2\left(2\tau_g'\,\kappa_g - \tau_g\left(2\kappa_g' + 3{\kappa_g}^2 - {\tau_g}^2\right)\right)}{{\tau_g}^3}},$$

$$\bar{\tau}_\alpha = \frac{\tau_g^4\begin{pmatrix} -3\tau_g'^2\kappa_g - \tau_g'\left(3\kappa_g' + 5{\kappa_g}^2\right)\tau_g \\ +\tau_g\left(\tau_g\left(\kappa_g\left(5\kappa_g' + 2{\kappa_g}^2 - 2{\tau_g}^2\right) + \kappa_g''\right) + \kappa_g\tau_g''\right) \end{pmatrix}}{4\sqrt{2}\left(2\tau_g'\,\kappa_g - \tau_g\left(2\kappa_g' + 3{\kappa_g}^2 - {\tau_g}^2\right)\right)}.$$

3. *If α is a principal line, the following hold*

$$\bar{\kappa}_\alpha = \sqrt{\frac{4\kappa_N'\,\kappa_g - 2\kappa_N\left(2\kappa_g' + 3{\kappa_g}^2 + {\kappa_N}^2\right)}{{\kappa_N}^3}},$$

$$\bar{\tau}_\alpha = \frac{\kappa_N^4\begin{pmatrix} 3\kappa_N'^2\kappa_g + \kappa_N'\left(3\kappa_g' + 5{\kappa_g}^2\right)\kappa_N \\ -\kappa_N\left(\kappa_N\left(\kappa_g\left(5\kappa_g' + 2\left({\kappa_g}^2 + {\kappa_N}^2\right)\right) + \kappa_g''\right) + \kappa_g\kappa_N''\right) \end{pmatrix}}{2\sqrt{2}\left(4\kappa_N'\,\kappa_g - 2\kappa_N\left(2\kappa_g' + 3{\kappa_g}^2 + {\kappa_N}^2\right)\right)}.$$

3.2. **TU**–*Smarandache curves*

Let $r = r(s)$ be a timelike curve lying on an oriented timelike surface M in Minkowski 3-space E_1^3. Using Definition 2, the **TU**–Smarandache curve of r is given by

$$\beta(\bar{s}) = \frac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{U}), \tag{9}$$

it leads to

$$\bar{\mathbf{T}}_\beta = \frac{1}{\sqrt{-2{\kappa_N}^2 + (\kappa_g + \tau_g)^2}}(\kappa_N\mathbf{T} + (\kappa_g + \tau_g)\mathbf{P} + \kappa_N\mathbf{U}). \tag{10}$$

By taking the derivative of Eq. (10) with respect to s, we have

$$\frac{d\bar{\mathbf{T}}_\beta}{d\bar{s}} = \frac{\sqrt{2}}{\left(-2{\kappa_N}^2 + (\kappa_g + \tau_g)^2\right)^2}(\epsilon_1\mathbf{T} + \epsilon_2\mathbf{P} + \epsilon_3\mathbf{U}),$$

where

$$\epsilon_1 = \begin{pmatrix} {\kappa_g}^4 + 3{\kappa_g}^3\tau_g + \kappa_N'(\kappa_g + \tau_g)^2 - {\kappa_g}^2\left({\kappa_N}^2 - 3{\tau_g}^2\right) \\ +\kappa_N\left(-2{\kappa_N}^3 - (\kappa_g' + \tau_g')\tau_g + \kappa_N{\tau_g}^2\right) \\ +\kappa_g\left(-(\kappa_g' + \tau_g')\kappa_N + {\tau_g}^3\right) \end{pmatrix},$$

$$\epsilon_2 = \kappa_N\left(-2(\kappa_g' + \tau_g')\kappa_N + (\kappa_g + \tau_g)\left(2\kappa_N' - 2{\kappa_N}^2 + (\kappa_g + \tau_g)^2\right)\right),$$

$$\epsilon_3 = \begin{pmatrix} -\kappa_g'\,\kappa_N(\kappa_g + \tau_g) - \tau_g'\,\kappa_N(\kappa_g + \tau_g) + \kappa_N'(\kappa_g + \tau_g)^2 \\ -\left(-{\kappa_N}^2 + \tau_g(\kappa_g + \tau_g)\right)\left(-2{\kappa_N}^2 + (\kappa_g + \tau_g)^2\right) \end{pmatrix}.$$

Therefore, from aforementioned equations, the curvature functions $\bar{\kappa}_\beta$, $\bar{\tau}_\beta$ are expressed as follows

$$\bar{\kappa}_\beta = \left\|\frac{d\bar{\mathbf{T}}_\beta}{d\bar{s}}\right\| = \frac{1}{\left(-2{\kappa_N}^2 + (\kappa_g + \tau_g)^2\right)^2}\sqrt{2\left(-\epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2\right)},$$

$$\bar{\mathbf{N}}_\beta = \frac{\epsilon_1\mathbf{T} + \epsilon_2\mathbf{P} + \epsilon_3\mathbf{U}}{\sqrt{-\epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2}}.$$

Besides, the binormal vector of β is

$$\bar{\mathbf{B}}_\beta = \frac{-1}{\sqrt{-2{\kappa_N}^2 + (\kappa_g + \tau_g)^2}\sqrt{-\epsilon_1^2 + \epsilon_2^2 - \epsilon_3^2}}(\bar{\epsilon}_1\mathbf{T} + \bar{\epsilon}_2\mathbf{P} + \bar{\epsilon}_3\mathbf{U}),$$

where

$$\bar{\epsilon}_1 = \kappa_N\epsilon_3 + (\kappa_g + \tau_g)\epsilon_1,$$
$$\bar{\epsilon}_2 = \kappa_N\epsilon_1 - \kappa_N\epsilon_3,$$
$$\bar{\epsilon}_3 = (\kappa_g + \tau_g)\epsilon_1 - \kappa_N\epsilon_2.$$

After differentiate β with respect to s, we get

$$\beta'' = \frac{1}{\sqrt{2}}\left\{\begin{matrix} (\kappa_N' + \tau_g\kappa_g + \kappa_N^2 + \kappa_g^2)\mathbf{T} + (\kappa_g' + \tau_g' + \tau_g\kappa_N + \kappa_N\kappa_g)\mathbf{P} \\ +(\kappa_N' - \kappa_g\tau_g - \tau_g^2 + \kappa_N^2)\mathbf{U} \end{matrix}\right\},$$

similarly,

$$\beta''' = \frac{1}{\sqrt{2}}(\mu_1\mathbf{T} + \mu_2\mathbf{P} + \mu_3\mathbf{U}),$$

where

$$\mu_1 = 2\tau_g'\kappa_g + \kappa_g'(3\kappa_g + \tau_g) + \kappa_N\left(3\kappa_N' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_N'',$$

$$\mu_2 = (\kappa_g' + \tau_g')\kappa_N + (\kappa_g + \tau_g)\left(2\kappa_N' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_g'' + \tau_g'',$$

$$\mu_3 = -2\kappa_g'\tau_g - \tau_g'(\kappa_g + 3\tau_g) + \kappa_N\left(3\kappa_N' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_N''.$$

Following that, the torsion of β is obtained as

$$\bar{\tau}_\beta = \frac{1}{2\sqrt{2}\bar{\kappa}_\beta^2}(\xi_1 + \xi_2 - \xi_3), \tag{11}$$

where

$$\xi_1 = \begin{matrix} -\kappa_N\left(2\kappa_N' + {\kappa_g}^2 + 2{\kappa_N}^2 - {\tau_g}^2\right) \\ \left((\kappa_g' + \tau_g')\kappa_N + (\kappa_g + \tau_g)\left(2\kappa_N' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right)\right. \\ \left. + \kappa_g'' + \tau_g''\right), \end{matrix}$$

$$\xi_2 = \begin{matrix} \left((\kappa_g' + \tau_g')\kappa_N + (\kappa_g + \tau_g)\left(\kappa_N' + 2{\kappa_N}^2 - \tau_g(\kappa_g + \tau_g)\right)\right) \\ \left(2\tau_g'\kappa_g + \kappa_g'(3\kappa_g + \tau_g) + \kappa_N\left(3\kappa_N' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_N''\right), \end{matrix}$$

$$\xi_3 = \begin{matrix} \left(-(\kappa_g' + \tau_g')\kappa_N + (\kappa_g + \tau_g)\left(\kappa_N' + \kappa_g(\kappa_g + \tau_g)\right)\right) \\ \left(-2\kappa_g'\tau_g - \tau_g'(\kappa_g + 3\tau_g) + \kappa_N\left(3\kappa_N' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_N''\right). \end{matrix}$$

Thus we can give the following Corollary.

Corollary 2. *Let $\beta(\bar{s})$ be a timelike curve lies on M in Minkowski 3-space E_1^3, then*

1. *If β is a geodesic curve, the following are satisfied*

$$\bar{\kappa}_\beta = \sqrt{\frac{2\bar{\chi}_1}{\left({\tau_g}^2 - 2{\kappa_N}^2\right)^3}},$$

where

$$\bar{\chi}_1 = \begin{matrix} 4{\kappa_N}^6 - 2\tau_g'^2{\kappa_N}^2 - 2\kappa_N'^2{\tau_g}^2 - 8{\kappa_N}^4{\tau_g}^2 - 2\tau_g'\kappa_N{\tau_g}^3 \\ + 5{\kappa_N}^2{\tau_g}^4 - {\tau_g}^6 + 2\kappa_N'\tau_g\left(2\tau_g'\kappa_N + {\tau_g}^3\right) \end{matrix},$$

$$\bar{\tau}_\beta = \frac{\bar{\chi}_2}{4\sqrt{2}\ \bar{\chi}_3},$$

where

$$\bar{\chi}_2 = \begin{pmatrix} \kappa_N\left({\tau_g}^2 - 2\kappa_N' - 2{\kappa_N}^2\right) \\ \left(\tau_g'\kappa_N + \tau_g\left(2\kappa_N' + {\kappa_N}^2 - {\tau_g}^2\right) + \tau_g''\right) \\ + \left(\tau_g'\kappa_N + \tau_g\left(\kappa_N' + 2{\kappa_N}^2 - {\tau_g}^2\right)\right) \\ \left(\kappa_N\left(3\kappa_N' + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_N''\right) \\ -\left(\kappa_N'\tau_g - \tau_g'\kappa_N\right)\left(-3\tau_g'\tau_g + \kappa_N\left(3\kappa_N' + {\kappa_N}^2 - {\tau_g}^2\right)\right. \\ \left. + \kappa_N''\right) \end{pmatrix},$$

$$\bar{\chi}_3 = \frac{1}{\left({\tau_g}^2 - 2{\kappa_N}^2\right)^3} \begin{pmatrix} 4{\kappa_N}^6 - 2\tau_g'^2{\kappa_N}^2 - 2\kappa_N'^2{\tau_g}^2 \\ -8{\kappa_N}^4{\tau_g}^2 - 2\tau_g'\kappa_N{\tau_g}^3 \\ + 5{\kappa_N}^2{\tau_g}^4 - {\tau_g}^6 \\ + 2\kappa_N'\tau_g\left(2\tau_g'\kappa_N + {\tau_g}^3\right) \end{pmatrix}.$$

2. *If β is an asymptotic line, then*

$$\bar{\kappa}_\beta = \sqrt{\frac{-2\left({\kappa_g}^6 + 4{\kappa_g}^5\tau_g + 7{\kappa_g}^4{\tau_g}^2 + 8{\kappa_g}^3{\tau_g}^3 + 7{\kappa_g}^2{\tau_g}^4 + 4\kappa_g{\tau_g}^5 + {\tau_g}^6\right)}{(\kappa_g + \tau_g)^6}},$$

$$\bar{\tau}_\beta = \frac{(\kappa_g + \tau_g)^9\left(\tau_g'\kappa_g - \kappa_g'\tau_g\right)}{-4\sqrt{2}\left({\kappa_g}^6 + 4{\kappa_g}^5\tau_g + 7{\kappa_g}^4{\tau_g}^2 + 8{\kappa_g}^3{\tau_g}^3 + 7{\kappa_g}^2{\tau_g}^4 + 4\kappa_g{\tau_g}^5 + {\tau_g}^6\right)}.$$

3. *If β is a principal line, the following are clarified*

$$\bar{\kappa}_\beta = \sqrt{\frac{2\begin{bmatrix} 4{\kappa_N}^6 - 2\kappa_N'^2{\kappa_g}^2 - {\kappa_g}^6 + 2\kappa_g'{\kappa_g}^3\kappa_N - 2\kappa_g'^2{\kappa_N}^2 \\ + {\kappa_g}^4{\kappa_N}^2 - 2\kappa_N'\kappa_g\left({\kappa_g}^3 - 2\kappa_g'\kappa_N\right) \end{bmatrix}}{\left({\kappa_g}^2 - 2{\kappa_N}^2\right)^3}},$$

$$\bar{\tau}_\beta = \frac{\left({\kappa_g}^2 - 2{\kappa_N}^2\right)^3\ \Sigma_1}{4\sqrt{2}\begin{pmatrix} 4{\kappa_N}^6 - 2\kappa_N'^2{\kappa_g}^2 - {\kappa_g}^6 + 2\kappa_g'{\kappa_g}^3\kappa_N - 2\kappa_g'^2{\kappa_N}^2 \\ + {\kappa_g}^4{\kappa_N}^2 - 2\kappa_N'\kappa_g\left({\kappa_g}^3 - 2\kappa_g'\kappa_N\right) \end{pmatrix}},$$

where

$$\Sigma_1 = \begin{pmatrix} 3\kappa_g'^2\kappa_g\kappa_N \\ -\kappa_N\left(4\kappa_N'^2\kappa_g + 7\kappa_N'{\kappa_g}^3 + 2{\kappa_g}^3\left({\kappa_g}^2 + {\kappa_N}^2\right) + \left(2\kappa_N' + {\kappa_g}^2 + 2{\kappa_N}^2\right)\kappa_g''\right) \\ -\kappa_g\left({\kappa_g}^2 - 2{\kappa_N}^2\right)\kappa_N'' + \kappa_g'\left(\kappa_N'\left(3{\kappa_g}^2 + 4{\kappa_N}^2\right) + \kappa_N\left(7{\kappa_g}^2\kappa_N + 2\kappa_N''\right)\right) \end{pmatrix}.$$

3.3. **PU**–*Smarandache curves*

Assume that $\gamma = \gamma(\bar{s})$ is a timelike curve lying fully on M in E_1^3. Let $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$ be Darboux frame of γ. Then by Definition 2, the **PU**–Smarandache curve of γ is identified by

$$\gamma(\bar{s}) = \frac{1}{\sqrt{2}}(\mathbf{P} + \mathbf{U}).$$

and

$$\bar{\mathbf{T}}_\gamma = \frac{((\kappa_g + \kappa_N)\mathbf{T} + \tau_g\mathbf{P} - \tau_g\mathbf{U})}{\sqrt{-(\kappa_g + \kappa_N)^2}}. \tag{12}$$

Differentiating Eq. (12) with respect to s, we have

$$\frac{d\bar{\mathbf{T}}_\gamma}{d\bar{s}} = \frac{\sqrt{2}}{(\kappa_g + \kappa_N)^3}(\zeta_1\mathbf{T} + \zeta_2\mathbf{P} + \zeta_3\mathbf{U}),$$

where

$$\zeta_1 = \tau_g\left(\kappa_N^2 - \kappa_g^2\right),$$

$$\zeta_2 = \left((\kappa_g' + \kappa_N')\tau_g - (\kappa_g + \kappa_N)\left(\tau_g' + \kappa_g(\kappa_g + \kappa_N) - {\tau_g}^2\right)\right),$$

$$\zeta_3 = \left(-(\kappa_g' + \kappa_N')\tau_g + (\kappa_g + \kappa_N)\left(\tau_g' - \kappa_N(\kappa_g + \kappa_N) + {\tau_g}^2\right)\right).$$

The curvature of γ is determined by

$$\bar{\kappa}_\gamma = \left\| \frac{d\bar{\mathbf{T}}_\gamma}{d\bar{s}} \right\| = \frac{1}{(\kappa_g + \kappa_N)^3}\sqrt{2\left(-\zeta_1^2 + \zeta_2^2 - \zeta_3^2\right)}.$$

Further, we define the principal normal and the binormal vectors as follows

$$\bar{\mathbf{N}}_\gamma = \frac{\zeta_1\mathbf{T} + \zeta_2\mathbf{P} + \zeta_3\mathbf{U}}{\sqrt{-\zeta_1^2 + \zeta_2^2 - \zeta_3^2}},$$

$$\bar{\mathbf{B}}_\gamma = \frac{-1}{\sqrt{-(\kappa_g + \kappa_N)^2}\sqrt{-\zeta_1^2 + \zeta_2^2 - \zeta_3^2}}\left(\bar{\zeta}_1\mathbf{T} + \bar{\zeta}_2\mathbf{P} + \bar{\zeta}_3\mathbf{U}\right),$$

where

$$\bar{\zeta}_1 = \tau_g\zeta_3 + \tau_g\zeta_2,$$
$$\bar{\zeta}_2 = -\tau_g\zeta_1 - (\kappa_g + \boldsymbol{\tau}_{\mathbf{g}})\zeta_3,$$
$$\bar{\zeta}_3 = \tau_g\zeta_1 - (\kappa_g + \boldsymbol{\tau}_{\mathbf{g}})\zeta_2.$$

If we differentiate γ' to get γ'', γ''', then we obtain the torsion of γ as follows

$$\gamma'' = \frac{1}{\sqrt{2}}\Big\{(\kappa_g' + \kappa_N' + \tau_g\kappa_g - \tau_g\kappa_N)\mathbf{T} + (\tau_g' + \kappa_g^2 + \kappa_N\kappa_g - {\tau_g}^2)\mathbf{P}$$
$$+ (-\tau_g' + \kappa_N\kappa_g + \kappa_N^2 - {\tau_g}^2)\mathbf{U}\Big\},$$

$$\gamma''' = \frac{1}{\sqrt{2}}(\nu_1\mathbf{T} + \nu_2\mathbf{P} + \nu_3\mathbf{U}),$$

where

$$\nu_1 = 2\tau_g'(\kappa_g - \kappa_N) + (\kappa_g' - \kappa_N')\tau_g + (\kappa_g + \kappa_N)\left({\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right)$$
$$+ \kappa_g'' + \kappa_N'',$$

$$\nu_2 = 2\kappa_N'\kappa_g + \kappa_g'(3\kappa_g + \kappa_N) + \tau_g\left(-3\tau_g' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \tau_g'',$$

$$\nu_3 = 2\kappa_g'\kappa_N + \kappa_N'(\kappa_g + 3\kappa_N) - \left(3\tau_g' + {\kappa_g}^2 + {\kappa_N}^2\right)\tau_g + {\tau_g}^3 - \tau_g'',$$

$$\bar{\tau}_\gamma = \frac{1}{2\sqrt{2}\bar{\kappa}_\gamma^2}(\varkappa_1 + \varkappa_2 + \varkappa_3), \tag{13}$$

where

$$\varkappa_1 = \begin{pmatrix} -\left(-\tau_g'(\kappa_g + \kappa_N) + \kappa_N(\kappa_g + \kappa_N)^2 \right.\\ \left. -(\kappa_g' + \kappa_N')\tau_g - 2\kappa_g{\tau_g}^2\right) \\ \left(2\kappa_N'\kappa_g + \kappa_g'(3\kappa_g + \kappa_N)\right. \\ \left. +\tau_g\left(-3\tau_g' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \tau_g''\right) \end{pmatrix},$$

$$\varkappa_2 = \begin{pmatrix} -\left(\tau_g'(\kappa_g + \kappa_N) - (\kappa_g' + \kappa_N')\tau_g\right. \\ \left. +\kappa_g\left((\kappa_g + \kappa_N)^2 - 2{\tau_g}^2\right)\right) \\ \left(-2\kappa_g'\kappa_N - \kappa_N'(\kappa_g + 3\kappa_N)\right. \\ \left. +\tau_g\left(3\tau_g' + {\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \tau_g''\right) \end{pmatrix},$$

$$\varkappa_3 = \begin{pmatrix} -\left(2\tau_g' + {\kappa_g}^2 - {\kappa_N}^2\right)\tau_g \\ \left(2\tau_g'(\kappa_g - \kappa_N) + (\kappa_g' - \kappa_N')\tau_g\right. \\ \left. +(\kappa_g + \kappa_N)\left({\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_g'' + \kappa_N''\right) \end{pmatrix}.$$

From the above calculations, we can introduce the following result:

Corollary 3. *Let $\gamma(\bar{s})$ be a timelike curve lies on M in Minkowski 3-space E_1^3, then*

1. *If γ is a geodesic curve, the curvature and torsion of γ are, respectively*

$$\bar{\kappa}_\gamma = \sqrt{\frac{2{\kappa_N}^3\left(2\tau_g' - {\kappa_N}^2\right) - 4\kappa_N'{\kappa_N}^2\tau_g - 2\kappa_N\left(4\tau_g' - {\kappa_N}^2\right){\tau_g}^2 + 8\kappa_N'{\tau_g}^3}{{\kappa_N}^5}},$$

$$\bar{\tau}_\gamma = \frac{1}{2\sqrt{2}\ \varkappa_4}\begin{pmatrix} -2{\tau_g'}^2\kappa_N\tau_g - 3{\kappa_N'}^2\kappa_N\tau_g \\ +\kappa_N'\tau_g\left({\kappa_N}^2\tau_g - 2{\tau_g}^3 + 2\tau_g''\right) \\ +\tau_g'\left(\kappa_N'\left(3{\kappa_N}^2 + 2{\tau_g}^2\right)\right. \\ \left. +\tau_g\left(-{\kappa_N}^3 + 2\kappa_N{\tau_g}^2 - 2\kappa_N''\right)\right) \\ +{\kappa_N}^2\left(-\kappa_N\tau_g'' + \tau_g\kappa_N''\right) \end{pmatrix},$$

where

$$\varkappa_4 = \left(\frac{2{\kappa_N}^3\left(2\tau_g' - {\kappa_N}^2\right) - 4\kappa_N'{\kappa_N}^2\tau_g - 2\kappa_N\left(4\tau_g' - {\kappa_N}^2\right){\tau_g}^2 + 8\kappa_N'{\tau_g}^3}{{\kappa_N}^5}\right).$$

2. *If γ is an asymptotic line, we get*

$$\bar{\kappa}_\gamma = \sqrt{\frac{2{\kappa_g}^3\left(2\tau_g' + {\kappa_g}^2\right) - 4\kappa_g'{\kappa_g}^2\tau_g - 2\kappa_g\left(4\tau_g' + 3{\kappa_g}^2\right){\tau_g}^2 + 8\kappa_g'{\tau_g}^3}{{\kappa_g}^5}},$$

$$\bar{\tau}_\gamma = \frac{1}{2\sqrt{2}}\left(\frac{\varkappa_5}{\varkappa_6}\right).$$

where

$$\varkappa_5 = \begin{pmatrix} 3{\kappa_g'}^2\kappa_g\tau_g \\ +\tau_g\left\{-\kappa_g\begin{pmatrix} 10{\tau_g'}^2 + 7\tau_g'{\kappa_g}^2 + 2{\kappa_g}^4 \\ -2\left(\tau_g' + 3{\kappa_g}^2\right){\tau_g}^2 + 4{\tau_g}^4 \end{pmatrix} - \left(2\tau_g' + {\kappa_g}^2\right)\kappa_g''\right\} \\ -\kappa_g\left({\kappa_g}^2 - 4{\tau_g}^2\right)\tau_g'' + \kappa_g'\left(\tau_g'\left(3{\kappa_g}^2 - 2{\tau_g}^2\right)\right. \\ \left. +\tau_g\left(7{\kappa_g}^2\tau_g - 2{\tau_g}^3 + 2\tau_g''\right)\right) \end{pmatrix},$$

$$\varkappa_6 = \left(\frac{2{\kappa_g}^3\left(2\tau_g' + {\kappa_g}^2\right) - 4\kappa_g'{\kappa_g}^2\tau_g - 2\kappa_g\left(4\tau_g' + 3{\kappa_g}^2\right){\tau_g}^2 + 8\kappa_g'{\tau_g}^3}{{\kappa_g}^5}\right)$$

3. *If γ is a principal line, the following hold*

$$\bar{\kappa}_\gamma = \sqrt{\frac{2\left({\kappa_g}^2 - {\kappa_N}^2\right)}{(\kappa_g + \kappa_N)^2}},$$

$$\bar{\tau}_\gamma = \frac{(\kappa_g + \kappa_N)^3\left(\kappa_N'\kappa_g - \kappa_g'\kappa_N\right)}{2\sqrt{2}\left(\frac{2\left({\kappa_g}^2 - {\kappa_N}^2\right)}{(\kappa_g + \kappa_N)^2}\right)}.$$

3.4. **TPU**–*Smarandache curves*

Let $r = r(s)$ be a timelike curve lying on a timelike surface M in Minkowski 3-space E_1^3 and $\{\mathbf{T}, \mathbf{P}, \mathbf{U}\}$ be the Darboux frame of $r(s)$. According to the definition of Smarandache curve, the **TPU**–Smarandache curve of r is expressed as

$$\delta(\bar{s}) = \frac{1}{\sqrt{3}}(\mathbf{T} + \mathbf{P} + \mathbf{U}).$$

This implies to

$$\bar{\mathbf{T}}_\delta = \frac{((\kappa_g + \kappa_N)\mathbf{T} + (\kappa_g + \tau_g)\mathbf{P} + (\kappa_N - \tau_g)\mathbf{U})}{\sqrt{-2(\kappa_g + \kappa_N)(\kappa_N - \tau_g)}}. \tag{14}$$

Differentiate Eq. (14) with respect to s and use Eq. (3), we obtain

$$\frac{d\bar{\mathbf{T}}_\delta}{d\bar{s}} = \frac{\sqrt{3}}{4(\kappa_g + \kappa_N)^2(\kappa_N - \tau_g)^2}(\xi_1\mathbf{T} + \xi_2\mathbf{P} + \xi_3\mathbf{U}),$$

where

$$\xi_1 = (\kappa_g + \kappa_N)\left(-\tau_g'(\kappa_g + \kappa_N) + \kappa_N'(\kappa_g + \tau_g)\right. \\ \left. -(\kappa_N - \tau_g)\left(\kappa_g' + 2\left({\kappa_g}^2 + {\kappa_N}^2 + (\kappa_g - \kappa_N)\tau_g\right)\right)\right),$$

$$\xi_2 = -(\kappa_g + \kappa_N)\begin{pmatrix} \tau_g'(\kappa_g + \kappa_N)(\kappa_g + 2\kappa_N - \tau_g) \\ -\kappa_N'(\kappa_g + 2\kappa_N - \tau_g)(\kappa_g + \tau_g) + (\kappa_N - \tau_g) \\ \left(\kappa_g'(\kappa_g + 2\kappa_N - \tau_g)\right. \\ \left. +2(\kappa_g + \kappa_N)(\kappa_g + \kappa_N - \tau_g)(\kappa_g + \tau_g)\right) \end{pmatrix},$$

$$\xi_3 = (\kappa_N - \tau_g)\left(\tau_g'(\kappa_g + \kappa_N) - \kappa_N'(\kappa_g + \tau_g)\right. \\ \left. -(\kappa_N - \tau_g)\left(-\kappa_g' + 2(\kappa_g + \kappa_N)(\kappa_g + \kappa_N + \tau_g)\right)\right).$$

Then, the curvature and principal normal vector of δ are, respectively

$$\bar{\kappa}_\delta = \left\|\frac{d\bar{\mathbf{T}}_\delta}{d\bar{s}}\right\| = \frac{1}{4(\kappa_g + \kappa_N)^2(\kappa_N - \tau_g)^2}\sqrt{3\left(-\xi_1^2 + \xi_2^2 - \xi_3^2\right)}, \tag{15}$$

and

$$\bar{\mathbf{N}}_\delta = \frac{\xi_1\mathbf{T} + \xi_2\mathbf{P} + \xi_3\mathbf{U}}{\sqrt{-\xi_1^2 + \xi_2^2 - \xi_3^2}}.$$

Also, the binormal vector of δ is given by

$$\bar{\mathbf{B}}_\delta = \frac{-1}{\sqrt{-2(\kappa_g + \kappa_N)(\kappa_N - \tau_g)}\sqrt{-\xi_1^2 + \xi_2^2 - \xi_3^2}}\left(\bar{\xi}_1\mathbf{T} + \bar{\xi}_2\mathbf{P} + \bar{\xi}_3\mathbf{U}\right),$$

where

$$\bar{\xi}_1 = (\kappa_g + \tau_g)\xi_3 - (\kappa_N - \tau_g)\xi_2,$$
$$\bar{\xi}_2 = (\kappa_N - \tau_g)\xi_1 - (\kappa_g + \kappa_N)\xi_3,$$
$$\bar{\xi}_3 = (\kappa_g + \tau_g)\xi_1 - (\kappa_g + \kappa_N)\xi_2.$$

The derivatives δ'', $\delta'''$of δ are

$$\delta'' = \frac{1}{\sqrt{3}}\left\{\begin{array}{c} (\kappa_g' + \kappa_N' + \kappa_N^2 + \kappa_g^2 + \tau_g\kappa_g - \tau_g\kappa_N)\mathbf{T} \\ +(\kappa_g' + \tau_g' + \kappa_g^2 + \kappa_N\kappa_g + \kappa_N\tau_g - {\tau_g}^2)\mathbf{P} \\ +(-\tau_g' + \kappa_N' + \kappa_N\kappa_g + \tau_g\kappa_g + \kappa_N^2 - {\tau_g}^2)\mathbf{U} \end{array}\right\},$$

$$\delta''' = \frac{1}{\sqrt{3}}(\eta_1\mathbf{T} + \eta_2\mathbf{P} + \eta_3\mathbf{U}),$$

where

$$\eta_1 = 2\tau_g'(\kappa_g - \kappa_N) + 3\kappa_N'\kappa_N - \kappa_N'\tau_g + \kappa_g'(3\kappa_g + \tau_g) \\ +(\kappa_g + \kappa_N)\left({\kappa_g}^2 + {\kappa_N}^2 - {\tau_g}^2\right) + \kappa_g'' + \kappa_N'',$$

$$\eta_2 = \kappa_g'(3\kappa_g + \kappa_N) + \tau_g'(\kappa_N - 3\tau_g)$$

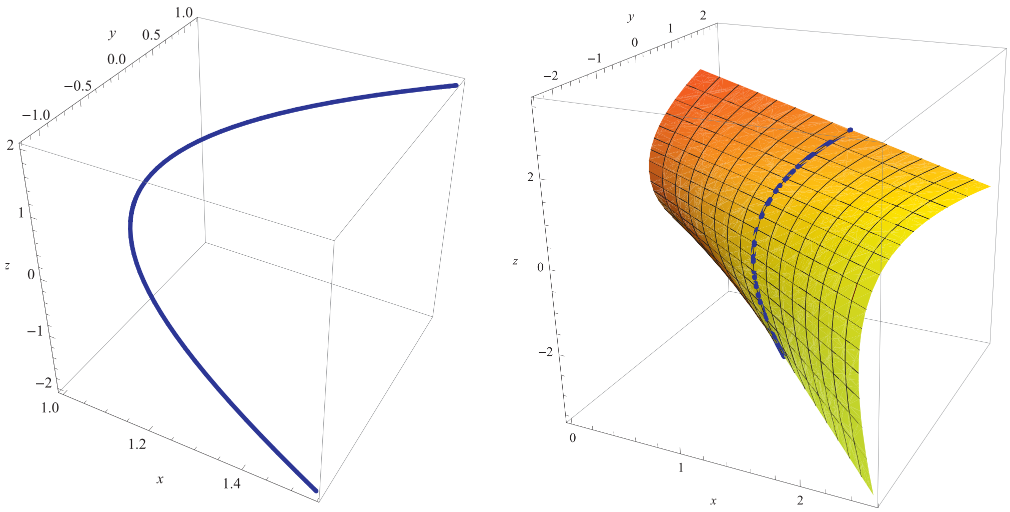

Fig. 1. The timelike curve $r(u)$ on the timelike surface M.

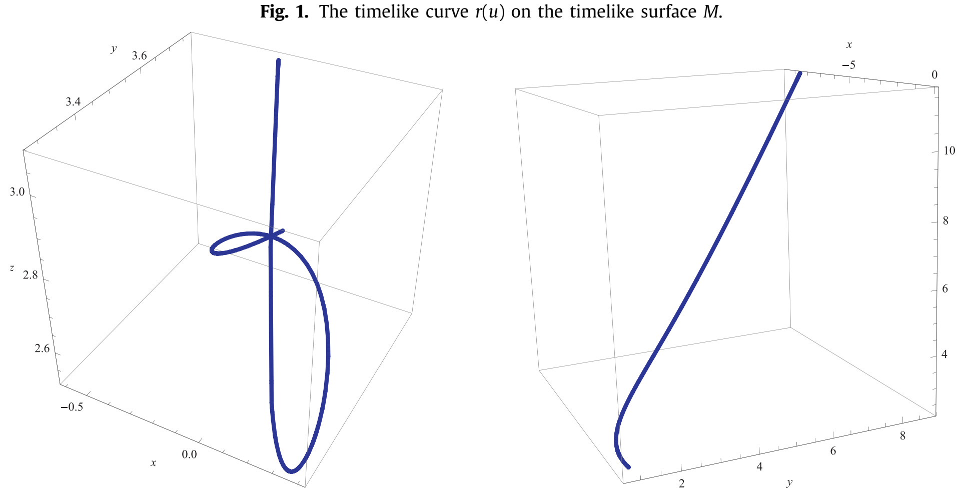

Fig. 2. **TP** and **TU**− Smarandache curves.

$$+(\kappa_g+\tau_g)\left(2\kappa_N'+\kappa_g{}^2+\kappa_N{}^2-\tau_g{}^2\right)+\kappa_g''+\tau_g'',$$

$$\begin{aligned}\eta_3 =\ &\kappa_N'(\kappa_g+3\kappa_N)-\tau_g'(\kappa_g+3\tau_g)\\ &+(\kappa_N-\tau_g)\left(2\kappa_g'+\kappa_g{}^2+\kappa_N{}^2-\tau_g{}^2\right)-\tau_g''+\kappa_N''.\end{aligned}$$

In the light of these derivatives, the curve's torsion of δ can be computed as follows

$$\bar{\tau}_\delta = \frac{\varkappa_7+\varkappa_8}{3\sqrt{3}\bar{\kappa}_\delta^2((\kappa_N-\tau_g)(\varkappa_9+2\varkappa_{10}))}, \tag{16}$$

where

$$\varkappa_7 = \left(\begin{array}{c} -\kappa_N'^2(3\kappa_g+4\kappa_N-\tau_g)(\kappa_g+\tau_g) \\ -\tau_g'^2\left(3\kappa_g{}^2-4\kappa_g\kappa_N+\kappa_N{}^2+2(5\kappa_g+\kappa_N)\tau_g\right) \\ -2\kappa_N'(\kappa_N-\tau_g)\left\{\begin{array}{c}\kappa_g'(5\kappa_g-2\kappa_N+3\tau_g) \\ +(\kappa_g+\tau_g)\left(\begin{array}{c}2\kappa_g(\kappa_g-2\kappa_N)\\ +(3\kappa_g+\kappa_N)\tau_g-\tau_g{}^2\end{array}\right) \\ +\kappa_g''+\tau_g''\end{array}\right\}\end{array}\right),$$

$$\varkappa_8 = 2\tau_g'\left(\begin{array}{c}\kappa_N'\left(3\kappa_g{}^2+2\kappa_N{}^2+6\kappa_g\tau_g+\tau_g{}^2\right) \\ +(\kappa_N-\tau_g)\left\{\begin{array}{c}\kappa_g'(5\kappa_g+\tau_g) \\ +(\kappa_g+\kappa_N)(2\kappa_g-\tau_g)(\kappa_g-\kappa_N+\tau_g) \\ +\kappa_g''+\kappa_N''\end{array}\right\}\end{array}\right),$$

$$\varkappa_9 = \left(\begin{array}{c}\kappa_g'^2(\kappa_N-\tau_g) \\ +2\kappa_g'\left\{-(\kappa_N-\tau_g)\left(\begin{array}{c}-2\kappa_g{}^2+\kappa_g(4\kappa_N-\tau_g)\\ +\tau_g(\kappa_N+\tau_g)\end{array}\right)-\tau_g''+\kappa_N''\right\}\end{array}\right),$$

$$\varkappa_{10} = \left(\begin{array}{c}2\kappa_g{}^4(\kappa_N-\tau_g)+2\kappa_g{}^3(\kappa_N-\tau_g)\tau_g \\ +2\kappa_g{}^2\left((\kappa_N-\tau_g)^2(\kappa_N+\tau_g)-\tau_g''+\kappa_N''\right) \\ +\kappa_N\left(-\kappa_N\left(\kappa_g''+\tau_g''\right)+\tau_g\left(\kappa_g''+\kappa_N''\right)\right) \\ +\kappa_g\left\{\begin{array}{c}2\kappa_N{}^3\tau_g-2\kappa_N{}^2\tau_g{}^2+\kappa_N\left(-2\tau_g{}^3-\tau_g''+\kappa_N''\right)\\ +2\tau_g\left(\tau_g{}^3-\tau_g''+\kappa_N''\right)\end{array}\right\}\end{array}\right).$$

Corollary 4. *Let $\delta(\bar{s})$ be a timelike curve lies on M in Minkowski 3-space E_1^3, then*

1. *If δ is a geodesic curve, the curvature and torsion can be expressed as follows*

$$\bar{\kappa}_\delta = \sqrt{\frac{3\ \Lambda_1}{8\ \kappa_N{}^3(\kappa_N-\tau_g)^3}},$$

$$\bar{\tau}_\delta = \frac{\Lambda_2}{3\sqrt{3}\ \bar{\kappa}_\delta^2},$$

where

$$\Lambda_1 = \left(\begin{array}{c}\tau_g'^2\kappa_N{}^2-2\tau_g'\kappa_N \\ \left(\kappa_N'-2(\kappa_N-\tau_g)^2\right)\tau_g+\kappa_N'^2\tau_g{}^2-4\kappa_N'(\kappa_N-\tau_g)^2\tau_g{}^2 \\ -4\kappa_N{}^2(\kappa_N-\tau_g)^3(\kappa_N+\tau_g)\end{array}\right),$$

$$\Lambda_2=\begin{pmatrix} {\kappa'_N}^2\tau_g(-4\kappa_N+\tau_g)-\tau_g'^2\kappa_N(\kappa_N+2\tau_g) \\ -2\kappa'_N(\kappa_N-\tau_g)\left((\kappa_N-\tau_g){\tau_g}^2+\tau''_g\right) \\ +2\kappa_N(\kappa_N-\tau_g)\left(-\kappa_N\tau''_g+\tau_g\kappa''_N\right)+2\tau'_g \\ \left(\kappa'_N\left(2{\kappa_N}^2+{\tau_g}^2\right)+(\kappa_N-\tau_g)\left(\kappa_N(\kappa_N-\tau_g)\tau_g+\kappa''_N\right)\right) \end{pmatrix}.$$

2. *If δ is an asymptotic line, we have*

$$\bar{\kappa}_\delta=\frac{1}{2}\sqrt{\frac{3}{2}}\sqrt{\frac{\Lambda_3}{{\kappa_g}^3{\tau_g}^3}},$$

$$\bar{\tau}_\delta=\frac{\Lambda_4}{3\sqrt{3}\bar{\kappa}_\delta^2},$$

where

$$\Lambda_3=\begin{pmatrix} \tau_g'^2{\kappa_g}^2-2\tau'_g\kappa_g\tau_g\left(\kappa'_g+2{\kappa_g}^2-2{\tau_g}^2\right) \\ +{\tau_g}^2\left(\kappa_g'^2+4{\kappa_g}^2(\kappa_g+\tau_g)^2+4\kappa'_g\left({\kappa_g}^2-{\tau_g}^2\right)\right) \end{pmatrix},$$

$$\Lambda_4=\begin{pmatrix} -\tau_g'^2\kappa_g(3\kappa_g+10\tau_g)-2\tau'_g\tau_g(\kappa_g(2\kappa_g-\tau_g)(\kappa_g+\tau_g) \\ +\kappa'_g(5\kappa_g+\tau_g)+\kappa''_g) \\ +\tau_g\left(\kappa'_g+2\kappa_g(\kappa_g+\tau_g)\right)\left(\tau_g\left(\kappa'_g+2{\kappa_g}^2-2{\tau_g}^2\right)+2\tau''_g\right) \end{pmatrix}.$$

3. *If δ is a principal line, we obtain*

$$\bar{\kappa}_\delta=\sqrt{\frac{3}{8}}\sqrt{\frac{\Lambda_5}{{\kappa_N}^3(\kappa_g+\kappa_N)^3}},$$

$$\bar{\tau}_\delta=\frac{1}{3\sqrt{3}}\begin{pmatrix} -{\kappa'_N}^2\kappa_g(3\kappa_g+4\kappa_N) \\ +2\kappa'_N\kappa_N\left(-5\kappa'_g\kappa_g-2{\kappa_g}^3+2\left(\kappa'_g+2{\kappa_g}^2\right)\kappa_N-\kappa''_g\right) \\ +\Lambda_6\ \kappa_N \end{pmatrix},$$

where

$$\Lambda_5=\begin{pmatrix} {\kappa'_N}^2{\kappa_g}^2+{\kappa_N}^2\left(\left(\kappa'_g+2{\kappa_g}^2\right)^2+4\kappa'_g\kappa_g\kappa_N\right. \\ \left.-8{\kappa_g}^2{\kappa_N}^2-8\kappa_g{\kappa_N}^3-4{\kappa_N}^4\right) \\ -2\kappa'_N\kappa_g\kappa_N\left(\kappa'_g+2\kappa_g(\kappa_g+\kappa_N)\right) \end{pmatrix},$$

$$\Lambda_6=\begin{pmatrix} \kappa_N\left(\kappa_g'^2+4\kappa'_g\kappa_g(\kappa_g-2\kappa_N)\right. \\ \left.+4{\kappa_g}^2\left({\kappa_g}^2+{\kappa_N}^2\right)-2\kappa_N\kappa''_g\right) \\ +2\left(\kappa'_g+\kappa_g(2\kappa_g+\kappa_N)\right)\kappa''_N \end{pmatrix}.$$

4. Computational example

In this example, we construct some special Smarandache curves (**TP, TU, PU** and **TPU**) of a timelike curve which lies on a timelike surface. Moreover, using Mathematica program, we compute their differential geometric properties (Figs. 1–3).

Suppose we are given a timelike ruled surface represented as

$$M(u,v)=r(u)+vQ(u),$$

where the timelike base curve is given by

$$r(u)=\left(u,\sqrt{2}\cosh u,\sqrt{2}\sinh u\right),$$

and

$$Q(u)=\left(\sqrt{2}\cosh u,1,\sqrt{2}\sinh u\right),$$

is the ruling vector of M.

The triples of Darboux frame can be computed as follows

$$\mathbf{T}=\left(1,\sqrt{2}\sinh u,\sqrt{2}\cosh u\right),$$

$$\mathbf{P}=\frac{(A1,A2,A3)}{\Delta_1},$$

where

$$\Delta_1=\sqrt{\begin{matrix}\left(1+2v+\cosh(2u)-\sqrt{2}\sinh(u)\right)^2+\left(\sqrt{2}(1+v)\cosh(u)-2\sinh^2(u)\right)^2 \\ -\left(1+\sqrt{2}v\sinh(u)-\sinh(2u)\right)^2\end{matrix}},$$

$$\begin{aligned} A1&=2\sqrt{2}(1+v)\cosh(u)+\sinh(u)\left(\sqrt{2}-2\cosh(u)+2v\sinh(u)\right),\\ A2&=v+\frac{\cosh(u)}{\sqrt{2}}+(1+v)\cosh(2u)-\frac{\cosh(3u)}{\sqrt{2}}\\ &\quad-\sqrt{2}v\sinh(u)+\sinh(2u),\\ A3&=1+2v+\cosh(2u)+\frac{\sinh(u)}{\sqrt{2}}+(1+v)\sinh(2u)-\frac{\sinh(3u)}{\sqrt{2}}, \end{aligned}$$

and

$$\mathbf{U}=\frac{(B1,B2,B3)}{\Delta_1},$$

where

$$\begin{aligned} B1&=-\sqrt{2}(1+v)\cosh(u)+2\sinh(u)^2,\\ B2&=1+2v+\cosh(2u)-\sqrt{2}\sinh(u),\\ B3&=-1-\sqrt{2}v\sinh(u)+\sinh(2u). \end{aligned}$$

According to Eq. (3), the geodesic curvature κ_g, the normal curvature κ_N and the geodesic torsion τ_g of the curve $r(u)$ are

$$\begin{aligned}\kappa_g=\langle\mathbf{T}',\mathbf{P}\rangle=&\frac{1}{\sqrt{\Delta_2}}\left(-2\sinh(u)\left(2v+\sqrt{2}\sinh(u)\right)\right.\\ &\left.+\cosh(u)\left(2+4v-2\sqrt{2}v\sinh(u)\right)\right),\end{aligned}$$

$$\kappa_N=\langle\mathbf{U}',\mathbf{T}\rangle$$

$$=\frac{1}{2(\Delta_2)^{3/2}}\begin{pmatrix} 4\sqrt{2}\left(-4+v\left(-2+3v+6v^2\right)\right) \\ -4(10+3v(7+4v))\cosh(u) \\ +3\sqrt{2}v(7+4v(5+2v))\cosh(2u) \\ -2(16+v(43+24v))\cosh(3u) \\ +4\sqrt{2}(4+3v)\cosh(4u) \\ -2(4+3v)\cosh(5u) \\ -\sqrt{2}v\cosh(6u)-4(5+3v(2+v(5+4v))) \\ \sinh(u)+\sqrt{2}(17 \\ +12v(5+4v))\sinh(2u)-2\left(9+22v+6v^2\right)\sinh(3u) \\ +2\sqrt{2}(8+11v)\sinh(4u)+2(-3+2v)\sinh(5u) \\ +\sqrt{2}\sinh(6u) \end{pmatrix},$$

$$\tau_g=\langle\mathbf{P}',\mathbf{U}\rangle=\frac{1}{\Delta_2}\begin{pmatrix} 1-2v-\sqrt{2}\left(-4+v+4v^2\right) \\ \cosh(u)+2(3+5v)\cosh(2u) \\ +\sqrt{2}(-2+v)\cosh(3u)+\cosh(4u) \\ -\sqrt{2}(7+4v(2+v))\sinh(u) \\ +2(2+v(3+2v))\sinh(2u) \\ -\sqrt{2}(3+4v)\sinh(3u)-v\sinh(4u) \end{pmatrix}.$$

where

$$\Delta_2=\begin{pmatrix} 5+12v(1+v)+2\sqrt{2}\cosh(u) \\ +4(1+3v)\cosh(2u)-2\sqrt{2}\cosh(3u) \\ +\cosh(4u)-2\sqrt{2}(1+6v)\sinh(u)+4\sinh(2u) \\ -2\sqrt{2}\sinh(3u) \end{pmatrix}$$

If we choose $u=0$ and $v=0$, the curvatures (κ_g, κ_N and τ_g) are

$$\kappa_g=\sqrt{\frac{2}{5}},\ \ \kappa_N=-2\sqrt{\frac{2}{5}},\ \ \tau_g=\frac{1}{10}\left(8+2\sqrt{2}\right).$$

TP– Smarandache curve

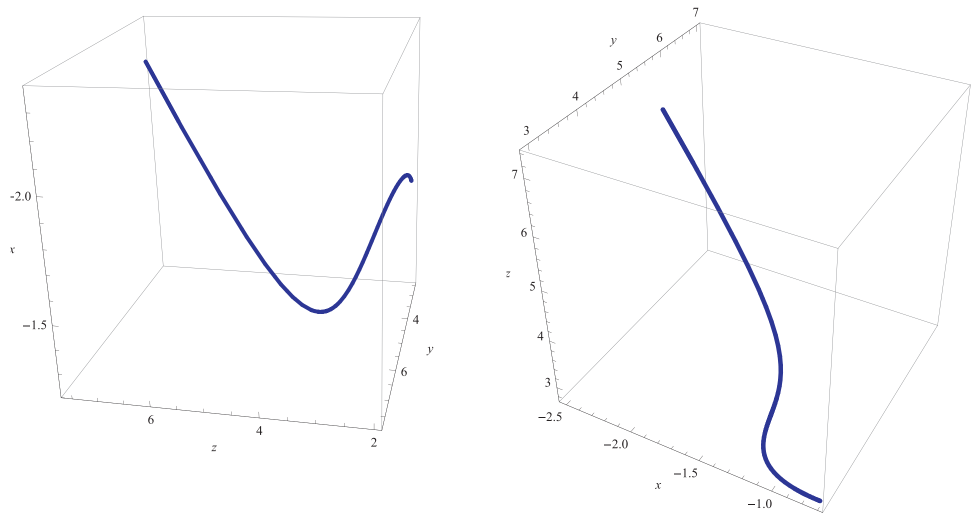

Fig. 3. **PU** and **TPU**– Smarandache curves.

The **TP**–Smarandache curve can be computed as

$$\alpha_{\mathbf{TP}} = (\alpha_1, \alpha_2, \alpha_3),$$

where

$$\alpha_1 = \frac{1}{\sqrt{2}}\left\{1 + \frac{2\sqrt{2}(1+v)\cosh(u) + \sinh(u)\left(\sqrt{2} - 2\cosh(u) + 2v\sinh(u)\right)}{\Delta_1}\right\},$$

$$\alpha_2 = \sqrt{2}\sinh(u) + \frac{1}{\sqrt{2}}\left\{\frac{v + \frac{\cosh(u)}{\sqrt{2}} + (1+v)\cosh(2u) - \frac{\cosh(3u)}{\sqrt{2}} - \sqrt{2}v\sinh(u) + \sinh(2u)}{\Delta_1}\right\},$$

$$\alpha_3 = \frac{1}{\sqrt{2}}\left\{\cosh(u) + \frac{1 + 2v + \cosh(2u) + \frac{\sinh(u)}{\sqrt{2}} + (1+v)\sinh(2u) - \frac{\sinh(3u)}{\sqrt{2}}}{\Delta_1}\right\}.$$

Therefore, we get ($u = 0$ and $v = 0$)

$\bar{\kappa}_\alpha = 0.793, \quad \bar{\tau}_\alpha = 0.455.$

TU– Smarandache curve

For this curve, we get

$$\beta_{\mathbf{TU}} = (\beta_1, \beta_2, \beta_3),$$

where

$$\beta_1 = \frac{1}{\sqrt{2}}\left(1 + \frac{-\sqrt{2}(1+v)\cosh(u) + 2\sinh^2(u)}{\Delta_1}\right),$$

$$\beta_2 = \frac{1}{\sqrt{2}}\left(\sqrt{2}\sinh(u) + \frac{1 + 2v + \cosh(2u) - \sqrt{2}\sinh(u)}{\Delta_1}\right),$$

$$\beta_3 = \frac{1}{\sqrt{2}}\left(\sqrt{2}\cosh(u) + \frac{-1 - \sqrt{2}v\sinh(u) + \sinh(2u)}{\Delta_1}\right),$$

after some calculations, we obtain ($u = 0$ and $v = 0$)

$\bar{\kappa}_\beta = 2.965, \quad \bar{\tau}_\beta = 0.199.$

PU– Smarandache curve

The **PU**–Smarandache curve can be computed as

$$\gamma_{\mathbf{PU}} = (\gamma_1, \gamma_2, \gamma_3),$$

$$\gamma_1 = \left(\frac{\sqrt{2}(1+v)\cosh(u) + \sinh(u)\left(\sqrt{2} - 2\cosh(u) + 2(1+v)\sinh(u)\right)}{\sqrt{2}\ \Delta_1}\right),$$

$$\gamma_2 = \left(\frac{2 + 6v + \sqrt{2}\cosh(u) + 2(2+v)\cosh(2u) - \sqrt{2}\cosh(3u) - 2\sqrt{2}(1+v)\sinh(u) + 2\sinh(2u)}{2\sqrt{2}\Delta_1}\right),$$

$$\gamma_3 = \left(\frac{4v + 2\cosh(2u) + \sqrt{2}(1-2v)\sinh(u) + 2(2+v)\sinh(2u) - \sqrt{2}\sinh(3u)}{2\sqrt{2}\ \Delta_1}\right).$$

In this case, we get ($u = 0$ and $v = 0$)

$\bar{\kappa}_\gamma = 2.204, \quad \bar{\tau}_\gamma = 0.113.$

TPU– Smarandache curve

The **TPU**–Smarandache curve is given by

$$\delta_{\mathbf{TPU}} = \frac{(\delta_1, \delta_2, \delta_3)}{\left(2\sqrt{3}\Delta_1\right)},$$

$$\delta_1 = \left(\begin{array}{c} 2\left(\sqrt{2}(1+v)\cosh(u) + \sqrt{2}\sinh(u) + 2\sinh(u)^2 + 2v\sinh(u)^2\right) \\ +2\Delta_1 - 2\sinh(2u) \end{array}\right),$$

$$\delta_2 = \left(\begin{array}{c} \sqrt{2}\cosh(u) + 2(2+v)\cosh(2u) - \sqrt{2}\cosh(3u) \\ +2\left(\begin{array}{c} 1 + 3v + \sinh(2u) + \sinh(u) \\ -\sqrt{2}(1+v) + \Delta_2 \end{array}\right) \end{array}\right),$$

$$\delta_3 = \left(\begin{array}{c} 4v + 2\cosh(2u) + \sqrt{2}(1-2v)\sinh(u) + 2(2+v)\sinh(2u) \\ -\sqrt{2}\sinh(3u) + 2\Delta_2\cosh(u) \end{array}\right),$$

it follows that ($u = 0$ and $v = 0$)

$\bar{\kappa}_\delta = 1.319, \quad \bar{\tau}_\delta = 0.549.$

5. Conclusion

In the present paper, we have studied special curves called Smarandache curves according to Darboux frame in the three-dimensional Minkowski space E_1^3. These curves are composed using Frenet frame vectors of another curve. Moreover, some results for the meaning curves are obtained. Finally, for confirming our results, a computational example is given and plotted.

Acknowledgment

The authors are very grateful to referees for the useful suggestions and remarks for the revised version.

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