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# On Geometry of Equiform Smarandache Ruled Surfaces Via Equiform Frame in Minkowski 3-Space

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## Abstract

In this paper, some geometric properties of equiform Smarandache ruled surfaces in Minkowski space $E_1^3$ using an equiform frame are investigated. Also, we give the sufficient conditions that make these surfaces are equiform developable and equiform minimal related to the equiform curvatures and when the equiform base curve contained in a plane or general helix. Finally, we provide an example, such as these surfaces.

**Keywords:** Ruled surfaces; Equiform frame; Minkowski 3-space; Smarandache curve

**MSC 2010 No.:** 53B30, 53C40, 53C50

## 1. Introduction

The theory of ruled surface is a branch of the classical differential geometry which has been developed by several researchers. In general, the rulings of the ruled surface are the set of a family of straight lines that depend on a parameter that is mentioned see Do Carmo (2016); Struik (1988); Barbosa and Colares (1986). One of the most interesting points is to study of ruled surfaces with different moving frames (see, for example, Hu et. al. (2020); Ibrahim Al-Dayel and Solouma

(2021); Lam (2020); Ouarab et al. (2018); Ouarab et al. (2020); Ouarab (2021); Solouma and Ibrahim AL-Dayel (2021); Emad Solouma and Mohamed Abdelkawy (2022)).

In Euclidean and Lorentzian geometry, the Smarandache curve is the curve whose position vector is made by Frenet frame vectors on another regular curve (Ashbacher (1997); Bishop (1975); Iseri (2002); Mao (2006)). Many researchers (such as Cetin et al. (2014); Emad Solouma (2021); Solouma (2017); Solouma and Mahmoud (2017); Solouma and Mahmoud (2019); Solouma (2021); Turgut and Yılmaz (2008); Taskopru and Tosun (2014); Yılmaz and Turgut (2010)) studied Smarandache curves in Minkowski and the Euclidean spaces.

In this work, we introduce the definitions of a special kind of ruled surfaces called equiform Smarandache ruled surfaces via the equiform frame in Minkowski 3- space. The main results are presented in theorems that we concert the sufficient and necessary conditions for those ruled surfaces to be equiform developable and equiform minimally. Finally, an illustration-based example is provided.

## 2. Preliminaries

In Minkowski space $E_1^3$ the Lorentzian product is defined as:

$$\mathcal{H} = -de_1^2 + de_2^2 + de_3^2,$$

where $(e_1, e_2, e_3)$ is the $E_1^3$ rectilinear coordinate system. An arbitrary $u \in E_1^3$ vector is one of the following; spacelike if $\mathcal{H}(u,u) > 0$ or $u = 0$, timelike if $\mathcal{H}(u,u) < 0$ and zero if $\mathcal{H}(u,u) = 0$ and $u \neq 0$. Likewise, a curve $\xi = \xi(\varrho)$ can be spacelike, timelike or zero if its $\xi'(\varrho)$ is spacelike, timelike or null. Let $\varphi = \varphi(\varrho)$ is a spacelike curve with a timelike principal normal. If $\{t, n, b\}$ denotes the moving Frenet frame of the spacelike curve $\varphi$, then $\{t, n, b\}$ has the following properties:

$$\begin{aligned} \dot{t}(\varrho) &= \kappa(\varrho) n(\varrho), \\ \dot{n}(\varrho) &= \kappa(\varrho) t(\varrho) + \tau(\varrho)\, b(\varrho), \\ \dot{b}(\varrho) &= \tau(\varrho) n(\varrho), \end{aligned} \tag{1}$$

where $\left(\cdot = \frac{d}{d\varrho}\right)$, $\mathcal{H}(t,t) = -\mathcal{H}(n,n) = \mathcal{H}(b,b) = 1$ and $\mathcal{H}(t,n) = \mathcal{H}(t,b) = \mathcal{H}(n,b) = 0$.

For a spacelike curve $\zeta: I \to E_1^3$ with a timelike principal normal in $E_1^3$. The equiform parameter of $\zeta$ by $\vartheta = \int \kappa d\varrho$. Then $\sigma = \frac{d\varrho}{d\vartheta}$, where $\sigma = \frac{1}{\kappa}$. We recall that $\{T, N, B\}$ is the moving equiform Frenet frame with the equiform tangent $T(\vartheta) = \sigma\, t(\varrho)$, the equiform principal normal $N(\vartheta) = \sigma\, n(\varrho)$ and the equiform binormal $B(\vartheta) = \sigma\, b(\varrho)$. The equiform curvatures of $\zeta = \zeta(\vartheta)$ are defined by $k_1(\vartheta) = \dot{\sigma} = \frac{d\sigma}{d\varrho}$ and $k_2(\vartheta) = \left(\frac{\tau}{\kappa}\right)$. As a result, the $\zeta$ equiform Frenet frame is given as:

$$\begin{aligned} T'(\vartheta) &= k_1(\vartheta)\, T(\vartheta) + N(\vartheta), \\ N'(\vartheta) &= -T(\vartheta) + k_1(\vartheta) N(\vartheta) + k_2(\vartheta) B(\vartheta), \\ B'(\vartheta) &= k_1(\vartheta)\, N(\vartheta) + k_2(\vartheta)\, B(\vartheta), \end{aligned} \tag{2}$$

for $\left(' = \frac{d}{d\vartheta}\right)$, $\mathcal{H}(T,T) = -\mathcal{H}(B,B) = \mathcal{H}(N,N) = \sigma^2$, and $\mathcal{H}(T,B) = \mathcal{H}(N,B) = \mathcal{H}(T,N) = 0$.

Let $\zeta = \zeta(\vartheta)$ be a regular equiform spacelike curve in $\mathrm{E}_1^3$ via equiform frame $\{T, N, B\}$. Then $TN$, $TB$ and $NB$- equiform Smarandache curves of $\zeta$ are defined, respectively, as follows (Solouma (2021)):

$$\begin{aligned} \varphi(\vartheta^*(\vartheta)) &= \frac{1}{\sqrt{2}\sigma}\left(T(\vartheta) + N(\vartheta)\right), \\ \psi(\vartheta^*(\vartheta)) &= \frac{1}{\sqrt{2}\sigma}\left(T(\vartheta) + B(\vartheta)\right), \\ \omega(\vartheta^*(\vartheta)) &= \frac{1}{\sqrt{2}\sigma}\left(N(\vartheta) + B(\vartheta)\right). \end{aligned}$$

The Lorentzian sphere with the origin center in the $\mathrm{E}_1^3$ space and a radius of $\epsilon > 0$ is defined as

$$S_1^2 = \{x \in \mathrm{E}_1^3 : \mathcal{H}(x,x) = \epsilon^2\}.$$

A ruled surface $\Gamma$ in $\mathrm{E}_1^3$ can be represented as

$$\Gamma(\varrho, \upsilon) = \varphi(\varrho) + \upsilon X(\varrho), \tag{3}$$

where $\varphi(\varrho)$ is the base curve and $X(\varrho)$ is a space curve that represents the direction of a straight line.

The unit normal vector field $\mathbb{N}$ on $\Gamma$ can be defined by

$$\mathbb{N} = \frac{\Gamma_\varrho \times \Gamma_\upsilon}{\|\Gamma_\varrho \times \Gamma_\upsilon\|}, \tag{4}$$

where $\Gamma_\varrho = \frac{\partial \Gamma}{\partial \varrho}$ and $\Gamma_\upsilon = \frac{\partial \Gamma}{\partial \upsilon}$. The components of $\Gamma$'s first and second fundamental forms are given by, and respectively,

$$E = \|\Gamma_\varrho\|^2, \quad F = \langle \Gamma_\varrho, \Gamma_\upsilon \rangle, \quad G = \|\Gamma_\upsilon\|^2,$$

$$e = \langle \Gamma_{\varrho\varrho}, \mathbb{N} \rangle, \quad f = \langle \Gamma_{\varrho\upsilon}, \mathbb{N} \rangle, \quad g = \langle \Gamma_{\upsilon\upsilon}, \mathbb{N} \rangle.$$

The Gaussian and mean curvatures of $\Gamma$ respectively are given by

$$\mathrm{K} = \frac{e\,g - f^2}{EG - F^2}, \tag{5}$$

$$\mathrm{H} = \frac{E\,g + G\,e - 2F\,f}{2(EG - F^2)}. \tag{6}$$

A ruled surface is developable if and only if $\mathrm{K} = 0$ and minimal if and only if $\mathrm{H} = 0$.

## 3. Main results

In this section, we define the equiform Smarandache ruled surfaces within Mikowski 3-space $\mathrm{E}_1^3$ referring to the equiform frame $\{T, N, B\}$. Also, we investigate the necessary and sufficient conditions that make these surfaces have $\mathrm{K} = 0$ and $\mathrm{H} = 0$.

### 3.1 $TN$-equiform Smarandache ruled surface

**Definition 3.1.**

For a regular equiform spacelike curve $\zeta = \zeta(\vartheta)$ in $\mathrm{E}_1^3$ via the frame (2). The $TN$-equiform Smarandache ruled surface is given by

$$\Lambda = \Lambda(\vartheta, \upsilon) = \frac{1}{\sqrt{2}\sigma}\left(T(\vartheta) + N(\vartheta)\right) + \upsilon B(\vartheta). \tag{7}$$

**Theorem 3.1.**

Let $\Lambda = \Lambda(\vartheta, \upsilon)$ is $TN$-equiform Smarandache ruled surface in $\mathrm{E}_1^3$ defined by (7). Then, we have
1. If $k_1 = 1$, then $\Lambda$ is equiform developable surface and $\mathrm{H}_\Lambda$ given by the formula

$$\mathrm{H}_\Lambda = \frac{\sigma\upsilon k_2(\sqrt{2}\sigma\upsilon k_2+2)+2k_2\left(k_2+\sqrt{2}\sigma\upsilon\right)}{\sqrt{2}(\sqrt{2}\sigma\upsilon k_2+2)^{\frac{3}{2}}}.$$

2. If $\zeta(\varrho)$ is a plane curve ($k_2 = 0$), then $\Lambda$ is equiform developable surface and $\mathrm{H}_\Lambda$ satisfying

$$\mathrm{H}_\Lambda = \frac{(k_1+1)\left[k_1'+k_1(k_1-2)\right]+(k_1-1)\left(k_1^2+k_1'+2k_1+1\right)}{4\sqrt{2}(k_1)^{\frac{3}{2}}}.$$

***Proof:***

Let $\Lambda(\vartheta, \upsilon) = \frac{1}{\sqrt{2}\sigma}\left(T(\vartheta) + N(\vartheta)\right) + \upsilon B(\vartheta)$ be $TN$-equiform Smarandache ruled surface recording by the equiform frame $\{T, N, B\}$ in $\mathrm{E}_1^3$. Taking the first derivative of $\Lambda(\vartheta, \upsilon)$ with respect to $\vartheta$ and $\upsilon$, we get

$$\begin{aligned} \Lambda_\vartheta &= \left[\frac{k_1-1}{\sqrt{2}\sigma}\right]T(\vartheta) + \left[\frac{k_1+1}{\sqrt{2}\sigma} + \upsilon k_2\right]N(\vartheta) + \left[\frac{k_2}{\sqrt{2}\sigma} + \upsilon k_1\right]B(\vartheta), \\ \Lambda_\upsilon &= B(\vartheta). \end{aligned} \tag{8}$$

From (8), The components of $\Lambda$'s first fundamental form and the unit normal vector field are given by:

$$\begin{aligned} E_\Lambda &= \frac{1}{2}\left[(k_1 - 1)^2 - \left(k_1 + \sqrt{2}\sigma\upsilon k_2\right)^2 + \left(k_2 + \sqrt{2}\sigma\upsilon k_1\right)^2\right], \\ F_\Lambda &= \frac{\sigma}{\sqrt{2}}\left[k_2 + \sqrt{2}\sigma\upsilon k_1\right], \\ G_\Lambda &= \sigma^2. \end{aligned} \tag{9}$$

$$\mathbb{N}_\Lambda = \frac{\left(k_1+\sqrt{2}\sigma\upsilon k_2+1\right)T(\vartheta)-(k_1-1)N(\vartheta)}{\sigma\sqrt{\left(k_1+\sqrt{2}\sigma\upsilon k_2+1\right)^2-(k_1-1)^2}}. \tag{10}$$

Another time, we can differentiate (7) with respect to $\vartheta$ and $\upsilon$, respectively, and use (2) to get

$$\begin{aligned} \Lambda_{\vartheta\vartheta} &= \varepsilon_1 T(\vartheta) + \varepsilon_2\, N(\vartheta) + \varepsilon_3\, B(\vartheta), \\ \Lambda_{\vartheta\upsilon} &= k_2\, N(\vartheta) + k_1\, B(\vartheta), \\ \Lambda_{\upsilon\upsilon} &= 0. \end{aligned} \tag{11}$$

where

$$\begin{aligned} \varepsilon_1 &= \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_1(k_1-2) + \sqrt{2}\sigma\upsilon k_2 + 1\right], \\ \varepsilon_2 &= \frac{1}{\sqrt{2}\sigma}\left[k_1^2 + k_2^2 + k_1' + 2k_1 + \sqrt{2}\sigma\upsilon(k_1k_2 + k_2')\right], \\ \varepsilon_3 &= \frac{1}{\sqrt{2}\sigma}\left[k_2' + k_2(2k_1+1) + \sqrt{2}\sigma\upsilon(k_1^2 + k_2^2 + k_2')\right]. \end{aligned} \tag{12}$$

From (10) and (11), the components of Λ's second fundamental form are given by:

$$\begin{aligned} e_\Lambda &= \frac{\sigma\left\{\varepsilon_1\left[k_1+\sqrt{2}\sigma\upsilon k_2+1\right]+\varepsilon_2(k_1-1)\right\}}{\sqrt{\left(k_1+\sqrt{2}\sigma\upsilon k_2+1\right)^2-(k_1-1)^2}}, \\ f_\Lambda &= \frac{\sigma k_2(k_1-1)}{\sqrt{\left(k_1+\sqrt{2}\sigma\upsilon k_2+1\right)^2-(k_1-1)^2}}, \\ g_\Lambda &= 0. \end{aligned} \tag{13}$$

So, from (9) and (11), the equiform Gaussian and mean curvatures of $TN$-equiform Smarandache ruled surface Λ given by:

$$\begin{aligned} \mathrm{K}_\Lambda &= \frac{k_2^2(k_1-1)^2}{\left[\left(k_1+\sqrt{2}\sigma\upsilon k_2+1\right)^2-(k_1-1)^2\right]^2}, \\ \mathrm{H}_\Lambda &= \frac{2\sqrt{2}k_2\left(k_2+\sqrt{2}\sigma\upsilon k_2\right)-2\sigma\left\{\varepsilon_1\left[k_1+\sqrt{2}\sigma\upsilon k_2+1\right]+\varepsilon_2(k_1-1)\right\}}{\left[\left(k_1+\sqrt{2}\sigma\upsilon k_2+1\right)^2-(k_1-1)^2\right]^{\frac{3}{2}}}. \end{aligned} \tag{14}$$

Consequently, from (14) we complete our proof. ■

**Corollary 3.2.**

Let $\Lambda = \Lambda(\vartheta, \upsilon)$ is $TN$-equiform Smarandache ruled surface in $\mathrm{E}_1^3$ defined by (7). Then Λ is equiform minimal surface if and only if the equiform curvatures satisfy the following differential equation

$$2\sqrt{2}k_2\left(k_2+\sqrt{2}\sigma\upsilon k_2\right) - 2\sigma\left\{\varepsilon_1\left[k_1+\sqrt{2}\sigma\upsilon k_2+1\right]+\varepsilon_2(k_1-1)\right\} = 0,$$

where $\varepsilon_1$ and $\varepsilon_2$ are given by (12).

***Proof:***

Let $\Lambda(\vartheta, \upsilon)$ be $TN$-equiform Smarandache ruled surface defined by (7) via the equiform frame $\{T, N, B\}$ in $\mathrm{E}_1^3$. From (14), the equiform surface $\Lambda(\vartheta, \upsilon)$ is equiform minimal surface if and only if $\mathrm{H}_\Lambda = 0$ which mean that

$$2\sqrt{2}k_2\left(k_2 + \sqrt{2}\sigma\upsilon k_2\right) - 2\sigma\left\{\varepsilon_1\left[k_1 + \sqrt{2}\sigma\upsilon k_2 + 1\right] + \varepsilon_2(k_1 - 1)\right\} = 0,$$

where $\varepsilon_1$ and $\varepsilon_2$ are given by (12) which complete our proof. ■

## 3.2 $TB$-equiform Smarandache ruled surface

**Definition 3.2.**

For a regular equiform spacelike curve $\zeta = \zeta(\vartheta)$ in $\mathrm{E}_1^3$ via the frame (2). The $TB$-equiform Smarandache ruled surface is given by

$$\Theta = \Theta(\vartheta, \upsilon) = \frac{1}{\sqrt{2}\sigma}\left(T(\vartheta) + B(\vartheta)\right) + \upsilon N(\vartheta). \tag{15}$$

**Theorem 3.3.**

Let $\Theta = \Theta(\vartheta, \upsilon)$ is $TB$-equiform Smarandache ruled surface in $\mathrm{E}_1^3$ defined by (15). Then, we have

1. If $\zeta(\varrho)$ has non-zero constant curvature ($k_1 = 0$), then $\Theta$ is equiform developable surface and

$$\mathrm{H}_\Theta = \frac{-\sqrt{2}k_2(k_2+1) + \sigma\upsilon k_2'}{2\sigma^2\upsilon^2(k_2+1)^{\frac{3}{2}}}.$$

2. If $\zeta(\varrho)$ is a general helix ($k_2 = 1$), then $\Theta$ is equiform developable surface and

$$\mathrm{H}_\Theta = -\frac{k_1' + k_1\left(k_1 + \sqrt{2}\sigma\upsilon\right)}{\left(k_1 - \sqrt{2}\sigma\upsilon\right)^2}.$$

***Proof:***

We can study the $\mathrm{K}_\Theta$ and $\mathrm{H}_\Theta$ of $TB$-equiform Smarandache ruled surface via the equiform frame $\{T, N, B\}$. The velocity vectors of (15) are given by

$$\begin{aligned} \Theta_\vartheta &= \left[\frac{k_1 - \sqrt{2}\sigma\upsilon}{\sqrt{2}\sigma}\right]T(\vartheta) + \left[\frac{k_2 + \sqrt{2}\sigma\upsilon k_1 + 1}{\sqrt{2}\sigma}\right]N(\vartheta) + \left[\frac{k_1 + \sqrt{2}\sigma\upsilon k_2}{\sqrt{2}\sigma}\right]B(\vartheta), \\ \Theta_\upsilon &= N(\vartheta). \end{aligned} \tag{16}$$

Now, using (16), we get the quantities of the first fundamental form and the unit normal vector field of $\Theta$ are given by:

$$\begin{aligned} E_\Theta &= \frac{1}{2}\left[\left(k_1 - \sqrt{2}\sigma\upsilon\right)^2 - \left(k_2 + \sqrt{2}\sigma\upsilon k_1 + 1\right)^2 + \left(k_1 + \sqrt{2}\sigma\upsilon k_2\right)^2\right], \\ F_\Theta &= -\frac{\sigma}{\sqrt{2}}\left[k_2 + \sqrt{2}\sigma\upsilon k_1 + 1\right], \\ G_\Theta &= -\sigma^2. \end{aligned} \tag{17}$$

$$\mathbb{N}_\Theta = \frac{(k_1+\sqrt{2}\sigma\upsilon k_2)T(\vartheta)+(k_1-\sqrt{2}\sigma\upsilon)B(\vartheta)}{\sigma\sqrt{(k_1+\sqrt{2}\sigma\upsilon k_2)^2+(k_1-\sqrt{2}\sigma\upsilon)^2}}. \tag{18}$$

Differentiating (16) with respect to $\vartheta$ and $\upsilon$ respectively and using (2) we get

$$\begin{aligned} \Theta_{\vartheta\vartheta} &= \mu_1 T(\vartheta) + \mu_2 N(\vartheta) + \mu_3 B(\vartheta), \\ \Theta_{\vartheta\upsilon} &= -T(\vartheta) + k_1 N(\vartheta) + k_2 B(\vartheta), \\ \Theta_{\upsilon\upsilon} &= 0, \end{aligned} \tag{19}$$

where

$$\begin{aligned} \mu_1 &= \frac{1}{\sqrt{2}\sigma}\left[k_1' - k_2 + k_1(k_1 - \sqrt{2}\sigma\upsilon) - \sqrt{2}\sigma\upsilon k_1 + 1\right], \\ \mu_2 &= \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_2' + k_1(k_2 + \sqrt{2}\sigma\upsilon k_1 + 2) + k_2(k_1 + \sqrt{2}\sigma\upsilon k_2) + \sqrt{2}\sigma\upsilon(k_1' - 1)\right], \\ \mu_3 &= \frac{1}{\sqrt{2}\sigma}\left[k_1' + k_2(k_2 + \sqrt{2}\sigma\upsilon k_1 + 1) + k_1(k_1 + \sqrt{2}\sigma\upsilon k_2) + \sqrt{2}\sigma\upsilon k_2'\right]. \end{aligned} \tag{20}$$

From (18) and (19), the quantities of the second fundamental form of Θ are given by:

$$\begin{aligned} e_\Theta &= \frac{\sigma\{\mu_1(k_1+\sqrt{2}\sigma\upsilon k_2)+\mu_2(k_1+\sqrt{2}\sigma\upsilon)\}}{\sqrt{(k_1+\sqrt{2}\sigma\upsilon k_2)^2+(k_1-\sqrt{2}\sigma\upsilon)^2}}, \\ f_\Theta &= \frac{\sigma k_1(k_2-1)}{\sqrt{(k_1+\sqrt{2}\sigma\upsilon k_2)^2+(k_1-\sqrt{2}\sigma\upsilon)^2}}, \\ g_\Theta &= 0. \end{aligned} \tag{21}$$

Then, from (17) and (21), the equiform $K_\Theta$ and $H_\Theta$ of $TB$-equiform Smarandache ruled surface Θ given by

$$\begin{aligned} K_\Theta &= \frac{2k_1^2(k_2-1)^2}{\left[(k_1+\sqrt{2}\sigma\upsilon k_2)^2+(k_1-\sqrt{2}\sigma\upsilon)^2\right]^2}, \\ H_\Theta &= \frac{2\sqrt{2}k_1(k_1-1)(k_2+\sqrt{2}\sigma\upsilon k_1+1)-2\sigma\{\mu_1[k_1+\sqrt{2}\sigma\upsilon k_2]+\mu_3(k_1+\sqrt{2}\sigma\upsilon)\}}{\left[(k_1+\sqrt{2}\sigma\upsilon k_2)^2+(k_1-\sqrt{2}\sigma\upsilon)^2\right]^{\frac{3}{2}}}, \end{aligned} \tag{22}$$

which complete our proof. ■

**Corollary 3.4.**

Let $\Theta = \Theta(\vartheta, \upsilon)$ is $TB$-equiform Smarandache ruled surface in $E_1^3$ defined by (15). Then Θ is equiform minimal surface if and only if the equiform curvatures satisfy the following differential equation

$$2\sqrt{2}k_1(k_1-1)(k_2+\sqrt{2}\sigma\upsilon k_1+1) - 2\sigma\{\mu_1[k_1+\sqrt{2}\sigma\upsilon k_2] + \mu_3(k_1+\sqrt{2}\sigma\upsilon)\} = 0,$$

where $\mu_1$ and $\mu_3$ are given by (20).

***Proof:***

Let $\Theta = \Theta(\vartheta, \upsilon)$ be $TB$-equiform Smarandache ruled surface defined by (15) in $E_1^3$ via the equiform frame $\{T, N, B\}$. Then, $H_\Theta = 0$ implies that $\Theta(\vartheta, \upsilon)$ is equiform minimal surface. From (22), we have

$$2\sqrt{2}k_1(k_1 - 1)\left(k_2 + \sqrt{2}\sigma\upsilon k_1 + 1\right) - 2\sigma\left\{\mu_1\left[k_1 + \sqrt{2}\sigma\upsilon k_2\right] + \mu_3\left(k_1 + \sqrt{2}\sigma\upsilon\right)\right\} = 0,$$

for $\mu_1$ and $\mu_3$ are given by (20). This complete the proof. ■

## 3.3 $NB$-equiform Smarandache ruled surface

**Definition 3.2.**

For a regular equiform spacelike curve $\zeta = \zeta(\vartheta)$ in $E_1^3$ via the frame (2). The $NB$-equiform Smarandache ruled surface is given by

$$\Upsilon = \Upsilon(\vartheta, \upsilon) = \frac{1}{\sqrt{2}\sigma}\left(N(\vartheta) + B(\vartheta)\right) + \upsilon T(\vartheta). \tag{23}$$

**Theorem 3.5.**

Let $\Upsilon = \Upsilon(\vartheta, \upsilon)$ is $NB$-equiform Smarandache ruled surface in $E_1^3$ defined by (23). If $k_1 + k_2 = 0$, then $\Upsilon$ is equiform developable surface satisfying

$$H_\Upsilon = \frac{k_2}{2\sigma^2\upsilon^2}.$$

***Proof:***

We compute the equiform Gaussian and the equiform mean curvatures of $NB$-equiform Smarandache ruled surface given by (23) via the equiform frame $\{T, N, B\}$. The $\Upsilon$'s velocity vectors are given by

$$\begin{aligned} \Upsilon_\vartheta &= \left[\frac{\sqrt{2}\sigma\upsilon k_1 - 1}{\sqrt{2}\sigma}\right]T(\vartheta) + \left[\frac{k_1 + k_2 + \sqrt{2}\sigma\upsilon}{\sqrt{2}\sigma}\right]N(\vartheta) + \left[\frac{k_1 + k_2}{\sqrt{2}\sigma}\right]B(\vartheta), \\ \Upsilon_\upsilon &= T(\vartheta). \end{aligned} \tag{24}$$

By using (24), we get the components of the first fundamental form and the unit normal vector field of $\Upsilon$ are given by:

$$\begin{aligned} E_\Upsilon &= \frac{1}{2}\left[\left(\sqrt{2}\sigma\upsilon k_1 - 1\right)^2 - \left(k_1 + k_2 + \sqrt{2}\sigma\upsilon\right)^2 + (k_1 + k_2)^2\right], \\ F_\Upsilon &= \frac{\sigma}{\sqrt{2}}\left[\sqrt{2}\sigma\upsilon k_1 - 1\right], \\ G_\Upsilon &= \sigma^2, \end{aligned} \tag{25}$$

$$\mathbb{N}_\Upsilon = \frac{(k_1 + k_2)N(\vartheta) - \left(k_1 + k_2 + \sqrt{2}\sigma\upsilon\right)B(\vartheta)}{\sqrt{2}\sigma\sqrt{\sigma\upsilon(k_1 + k_2 + \sigma\upsilon)}}. \tag{26}$$

Using (2) and differentiating (24) again with respect to $\vartheta$ and $\upsilon$ respectively, we get

$$\begin{aligned} \Upsilon_{\vartheta\vartheta} &= \alpha_1 T(\vartheta) + \alpha_2 N(\vartheta) + \alpha_3 B(\vartheta), \\ \Upsilon_{\vartheta\upsilon} &= k_1 T(\vartheta) + N(\vartheta), \\ \Upsilon_{\upsilon\upsilon} &= 0, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \alpha_1 &= \tfrac{1}{\sqrt{2}\sigma}\left[k_1\left(\sqrt{2}\sigma\upsilon k_1 - 1\right) + \sqrt{2}\sigma\upsilon(k_1' - 1) - k_1 - k_2\right], \\ \alpha_2 &= \tfrac{1}{\sqrt{2}\sigma}\left[k_1' + k_2' + k_1\left(k_1 + k_2 + \sqrt{2}\sigma\upsilon\right) + k_2(k_1 + k_2) + \sqrt{2}\sigma\upsilon k_1 - 1\right], \\ \alpha_3 &= \tfrac{1}{\sqrt{2}\sigma}\left[k_1' + k_2' + k_1\left(k_1 + k_2 + \sqrt{2}\sigma\upsilon\right) + k_2(k_1 + k_2)\right]. \end{aligned} \tag{28}$$

From (26) and (27), the quantities of the second fundamental form of $\Upsilon$ are given by:

$$\begin{aligned} e_\Upsilon &= -\frac{\sigma\{\alpha_2(k_1+k_2)+\alpha_3\left(k_1+k_2+\sqrt{2}\sigma\upsilon\right)\}}{\sqrt{2}\sqrt{\sigma\upsilon(k_1+k_2+\sigma\upsilon)}}, \\ f_\Upsilon &= -\frac{\sigma(k_1+k_2)}{\sqrt{2}\sigma\sqrt{\sigma\upsilon(k_1+k_2+\sigma\upsilon)}}, \\ g_\Upsilon &= 0. \end{aligned} \tag{29}$$

Then, from (25) and (29), the equiform Gaussian curvature $\mathrm{K}_\Upsilon$ and the equiform mean curvature $\mathrm{H}_\Upsilon$ of $\Upsilon$ are given by

$$\begin{aligned} \mathrm{K}_\Upsilon &= \frac{(k_1+k_2)^2}{2\sigma^2\upsilon^2(k_1+k_2+\sigma\upsilon)^2}, \\ \mathrm{H}_\Upsilon &= \frac{\sqrt{2}\sigma\{\alpha_2(k_1+k_2)+\alpha_3\left(k_1+k_2+\sqrt{2}\sigma\upsilon\right)\}+2(k_1+k_2)\left(\sqrt{2}\sigma\upsilon k_1-1\right)}{2[\sigma\upsilon(k_1+k_2+\sigma\upsilon)]^{\frac{3}{2}}}. \end{aligned} \tag{30}$$

As a consequence of the above results, we complete the proof. ■

**Corollary 3.4.**

Let $\Upsilon = \Upsilon(\vartheta, \upsilon)$ is $NB$-equiform Smarandache ruled surface in $\mathrm{E}_1^3$ defined by (23). Then $\Upsilon$ is equiform minimal surface if and only if the equiform curvatures satisfy the following differential equation

$$\sigma\left\{\alpha_2(k_1 + k_2) + \alpha_3\left(k_1 + k_2 + \sqrt{2}\sigma\upsilon\right)\right\} + \sqrt{2}(k_1 + k_2)\left(\sqrt{2}\sigma\upsilon k_1 - 1\right) = 0,$$

where $\alpha_2$ and $\alpha_3$ are given by (28).

***Proof:***

Let $\Upsilon = \Upsilon(\vartheta, \upsilon)$ be $NB$-equiform Smarandache ruled surface defined by (23) in $\mathrm{E}_1^3$ via the equiform frame $\{T, N, B\}$. As the above way, the equiform mean curvature $\mathrm{H}_\Upsilon$ of $\Upsilon$ is given by (30). Then, $\mathrm{H}_\Upsilon = 0$ means the equiform surface $\Upsilon(\vartheta, \upsilon)$ is equiform minimal surface. Then, from (30), we have

$$\sigma\left\{\alpha_2(k_1 + k_2) + \alpha_3\left(k_1 + k_2 + \sqrt{2}\sigma\upsilon\right)\right\} + \sqrt{2}(k_1 + k_2)\left(\sqrt{2}\sigma\upsilon k_1 - 1\right) = 0,$$

where $\alpha_2$ and $\alpha_3$ are given by (28) which complete the proof. ■

## 3.4 Example

Consider the case of a regular spacelike curve $\varphi(\varrho)$ with a timelike principal normal in $E_1^3$ (see Figure 1)

$$\varphi(\varrho) = \left(\frac{\varrho}{\sqrt{2}} \cosh\left(\sqrt{2}\ln\varrho\right), \frac{\varrho}{\sqrt{2}} \sinh\left(\sqrt{2}\ln\varrho\right), \frac{\varrho}{\sqrt{2}}\right). \tag{31}$$

Then, the Frenet apparatus are given as the following

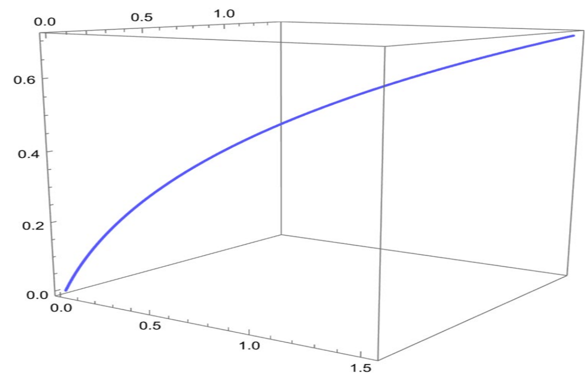


**Figure 1:** Spacelike curve $\varphi = \varphi(\varrho)$

$$t(\varrho) = \left(\frac{1}{\sqrt{2}} \cosh\left(\sqrt{2}\ln\varrho\right) + \sinh\left(\sqrt{2}\ln\varrho\right), \frac{1}{\sqrt{2}} \sinh\left(\sqrt{2}\ln\varrho\right) + \cosh\left(\sqrt{2}\ln\varrho\right), \frac{1}{\sqrt{2}}\right),$$
$$n(\varrho) = \left(\sqrt{2}\cosh\left(\sqrt{2}\ln\varrho\right) + \sinh\left(\sqrt{2}\ln\varrho\right), \sqrt{2}\sinh\left(\sqrt{2}\ln\varrho\right) + \cosh\left(\sqrt{2}\ln\varrho\right), 0\right),$$
$$\kappa = \frac{1}{\varrho}, \quad \sigma = \varrho, \quad k_1 = 1,$$
$$b(\varrho) = \left(\frac{1}{\sqrt{2}} \cosh\left(\sqrt{2}\ln\varrho\right) + \sinh\left(\sqrt{2}\ln\varrho\right), \frac{1}{\sqrt{2}} \sinh\left(\sqrt{2}\ln\varrho\right) + \cosh\left(\sqrt{2}\ln\varrho\right), \frac{-1}{\sqrt{2}}\right),$$
$$\tau = \frac{1}{\varrho}, \quad k_2 = 1.$$

Then, the equiform parameter is $\vartheta = \int_0^{\varrho} \kappa \, d\varrho = \ln\varrho$, so we have $\varrho = \sigma = e^{\vartheta}$. Now, the equiform spacelike curve $\zeta(\vartheta)$ is define as (see Figure 2)

$$\zeta(\vartheta) = \left(\frac{e^{\vartheta}}{\sqrt{2}} \cosh\left(\sqrt{2}\vartheta\right), \frac{e^{\vartheta}}{\sqrt{2}} \sinh\left(\sqrt{2}\vartheta\right), \frac{e^{\vartheta}}{\sqrt{2}}\right). \tag{32}$$

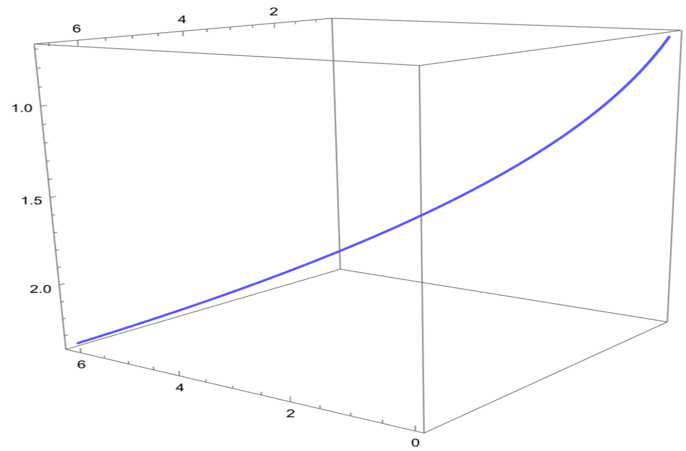


**Figure 2:** Equiform spacelike curve $\zeta = \zeta(\vartheta)$

It is easy to show that the vectors of equiform Frenet frame are given as:

$$T(\varrho) = e^{\vartheta}\left(\frac{1}{\sqrt{2}}\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\vartheta) + \cosh(\sqrt{2}\vartheta), \frac{1}{\sqrt{2}}\right),$$
$$N(\varrho) = e^{\vartheta}\left(\sqrt{2}\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \sqrt{2}\sinh(\sqrt{2}\vartheta) + \cosh(\sqrt{2}\vartheta), 0\right),$$
$$B(\varrho) = e^{\vartheta}\left(\frac{1}{\sqrt{2}}\cosh(\sqrt{2}\vartheta) + \sinh(\sqrt{2}\vartheta), \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\vartheta) + \cosh(\sqrt{2}\vartheta), \frac{-1}{\sqrt{2}}\right).$$

Thus, the equiform Smarandache ruled surfaces $\Lambda(\vartheta, \upsilon)$, $\Theta(\vartheta, \upsilon)$ and $\Upsilon(\vartheta, \upsilon)$ are respectively given as (see Figures 3, 4 and 5)

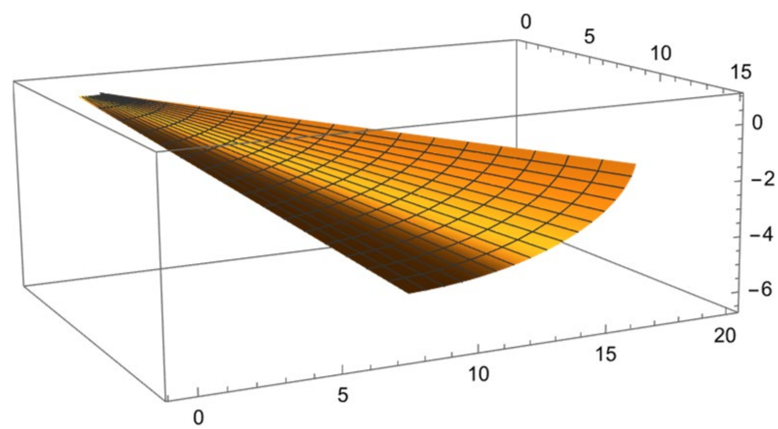


**Figure 3:** $TN$-equiform Smarandache developable ruled surface $\Lambda(\vartheta, \upsilon)$

$$\Lambda(\vartheta, \upsilon) = \left(\left(\frac{3+\sqrt{2}\upsilon e^{\vartheta}}{2}\right)\cosh(\sqrt{2}\vartheta) + \left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\sinh(\sqrt{2}\vartheta), \left(\frac{3+\sqrt{2}\upsilon e^{\vartheta}}{2}\right)\sinh(\sqrt{2}\vartheta) + \left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\cosh(\sqrt{2}\vartheta), \frac{1-\upsilon e^{\vartheta}}{\sqrt{2}}\right). \tag{33}$$

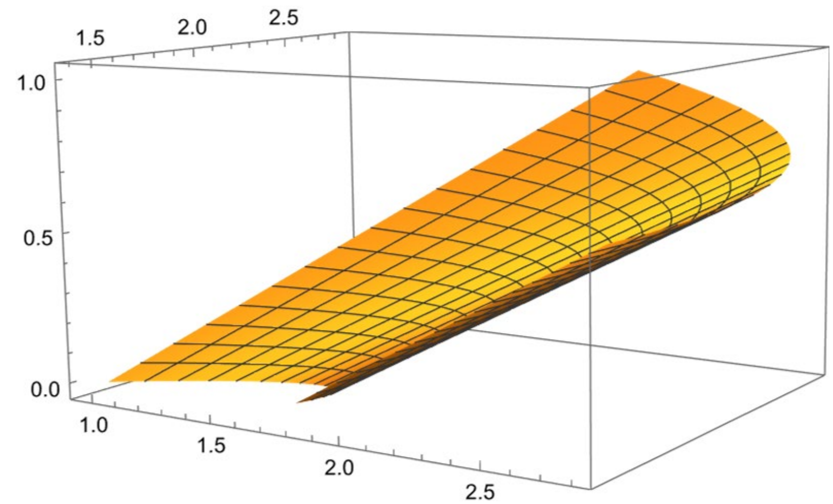


**Figure 4:** $TB$-equiform Smarandache developable ruled surface $\Theta(\vartheta, \upsilon)$

$$\Theta(\vartheta,\upsilon) = \left(1+\sqrt{2}\upsilon e^{\vartheta}\right)\left(\cosh\left(\sqrt{2}\vartheta\right)+\sinh\left(\sqrt{2}\vartheta\right), \sinh\left(\sqrt{2}\vartheta\right)+\left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\cosh\left(\sqrt{2}\vartheta\right), 0\right). \tag{34}$$

$$\Upsilon(\vartheta,\upsilon) = \left(\left(\frac{3+\sqrt{2}\upsilon e^{\vartheta}}{2}\right)\cosh\left(\sqrt{2}\vartheta\right)+\left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\sinh\left(\sqrt{2}\vartheta\right), \left(\frac{3+\sqrt{2}\upsilon e^{\vartheta}}{2}\right)\sinh\left(\sqrt{2}\vartheta\right) + \left(\frac{1+\sqrt{2}\upsilon e^{\vartheta}}{\sqrt{2}}\right)\cosh\left(\sqrt{2}\vartheta\right), \frac{\upsilon e^{\vartheta}-1}{\sqrt{2}}\right). \tag{35}$$

## 4. Conclusion

Using an equiform frame, various geometric characteristics of equiform Smarandache ruled surfaces in Minkowski space Minkowski 3-space are studied. We also provide the necessary requirements for these surfaces to be equiform developable and equiform minimal in relation to equiform curvatures, as well as when the equiform base curve is located in a plane or general helix.

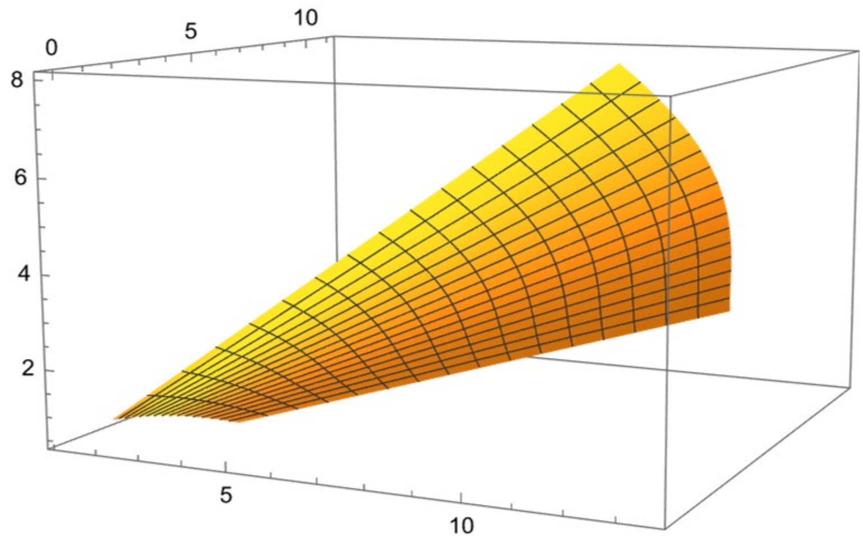


**Figure 5:** $NB$-equiform Smarandache ruled surface $\Upsilon(\vartheta, \upsilon)$

# REFERENCES

Ashbacher, C. (1997). Smarandache geometries, Smarandache Notions Journal, Vol. 8, No. 1, pp. 212–215.

Aydin, M.E. and Ergut, M. (2013). The equiform differential geometry of curves in 4-dimensional Galilean space $G_4$, Stud. Univ. Babes-Bolyai Math., Vol. 58, No. 3, pp. 399–406.

Bishop, L.R. (1975). There is more than one way to frame a curve, Amer. Math. Monthly, Vol. 82, No. 3, pp. 246–251.

Barbosa, J.L.M. and Colares, A.G. (1986). *Minimal Surfaces in* $R^3$, Springer Verlag, Berlin heidelberg.

Cetin, M., Tuncer, Y. and Karacan, M.K. (2014). Smarandache curves according to Bishop frame in Euclidean 3- space, General Mathematics Notes, Vol. 20, No. 2, pp. 50–66.

Do Carmo, M.P. (2016). *Differential Geometry of Curves and Surfaces*, Revised and Updated Second Edition, Courier Dover Publications.

Hu, G., Cao, H., Wu, J. and Wei, G. (2020). Construction of developable surfaces using generalized C-Bézier bases with shape parameters, Computational and Applied Mathematics, Vol. 39, No. 3, pp. 1–32.

Iseri, H. (2002). *Smarandache Manifolds*, American Res. Press, Mansfield University, PA.

Ibrahim Al-Dayel and Solouma, E.M. (2021). Characteristic properties of type-2 Smarandache ruled surfaces according to the type-2 Bishop frame in $E^3$, Advances in Mathematical Physics, Vol. 2021, pp. 1–7.

López, R. (2014). Differential geometry of curves and surfaces in Lorentz-Minkowski space, International Electronic Journal of Geometry, Vol. 7, No. 1, pp. 44-107.

Lam, W.Y. (2020). Minimal surfaces from infinitesimal deformations of circle packings, Advances in Mathematics, Vol. 362, pp. 106–139.

Mao, L. (2006). Smarandache Geometries & Map Theory with Applications, Chinese Branch Xiquan House, Academy of Mathematics and Systems Chinese Academy of Sciences, Beijing, China.

O'Neill, B. (1983). *Semi-Riemannian Geometry with Application to Relativity*, Academic Press.

Ouarab, S., Chahdi, A.O., and Izid, M. (2018). Ruled surfaces with alternative moving frame in Euclidean 3-space, International Journal of Mathematical Sciences and Engineering Applications, Vol. 12, No. 2, pp. 43–58.

Ouarab, S., Chahdi, A.O. and Izid, M. (2020). Ruled surface generated by a curve lying on a regular surface and its characterizations, Journal for Geometry and Graphics, Vol. 24, No. 2, pp. 257– 267.

Ouarab, S. (2021). Smarandache ruled surfaces according to Frenet-Serret frame of a regular curve in $E^3$, Abstract and Applied Analysis, Vol. 2021, pp. 1–8.

Struik, D.J. (1988). *Lectures on Classical Differential Geometry*, Dover, AddisonWesley, 2nd edition.

Solouma, E.M. (2017). Type-2 spacelike Bishop frame and an application to spherical image in Minkowski space–time, Int. J. Appl. Comput. Math, Vol. 3, No. 4, pp. 3575–3591.

Solouma, E.M. (2021). Equiform spacelike Smarandache curves of anti-Eqiform Salkowski curve according to Equiform frame, Int. J. of Mathematical Analysis, Vol. 15, No. 1, pp. 43–59.

Solouma, E. (2021). Generalized Smarandache curves of spacelike and equiform spacelike curves via timelike second binormal in $\mathfrak{R}_1^4$, Applications and Applied Mathematics: An International Journal, Vol. 15, No. 1, pp. 1–9.

Solouma, E and Abdelkawy, M. (2022). Family of ruled surfaces generated by equiform Bishop spherical image in Minkowski 3-space, AIMS Mathematics, Vol. 8, No. 2, pp. 4372–4389.

Solouma, E.M. and Mahmoud, W.M. (2017). Special Smarandache curves according to Bishop frame in Euclidean space-time, International J.Math. Combin., Vol. 1, pp. 1–9.

Solouma, E.M. and Mahmoud, W.M. (2019). On spacelike equiform Bishop Smarandache curves on $S_1^2$, Journal of the Egyptian Mathematical Society, Vol. 27, No. 1, pp. 1–17.

Solouma, E.M. and Ibrahim Al-Dayel (2021). Harmonic evolute surface of tubular surfaces via B-Darboux frame in Euclidean 3-space, Advances in Mathematical Physics, Vol. 2021, pp. 1–7.

Turgut, M. and Yılmaz, S. (2008). Smarandache curves in Minkowski space-time, International Journal of Mathematical Combinatorics, Vol. 3, pp. 51–55.

Taskopru, K. and Tosun, M. (2014). Smarandache curves according to Sabban frame on, Boletim da Sociedade Paraneanse de Matematica, Vol. 32, No. 1, pp. 51–59.

Yılmaz, S. and Turgut, A. (2010). A new version of Bishop frame and an application to spherical images, J. Math. Anal. Appl., Vol. 371, pp. 764 –776.